Chapter 5

Inverse Spectral Theory for Symmetric and Self-adjoint Operators

We prove that an operator measure in general is non-orthogonal and unbounded**.** We prove that two orthogonal spectral measures are unitarily equivalent. In accordance with the stieltjes inversion formula the spectral measure admits an analytic continuation [90]. We discuss and prove a sharp estimate that a strictly monotone function on each component interval of the inverse function is analytic and also Strictly monotone. We prove that a non-orthogonal spectral measure in a gab of any self-adjoint extensions can be calculated, if exist a boundary triple and have various examples [93].

Section (5-1): Inverse Spectral Problem for Direct Sum of Symmetric Operators

Let S be a densely defined symmetric operator in Hilbert space H with deficiency indices $n_+(S) = n_-(S) \leq \infty$. We recall that abounded open interval $J = (\alpha, \beta)$ is called a gap for S if

$$\|2S - (\alpha, \beta)\| \geq (\alpha - \beta)\|f\|, f \in dom\ s \quad , \tag{1}$$

if $\alpha = -\infty$, then (1) turns into $(Sf, f) \geq \beta\|f\|^2$ for all $f \in dom\ S$, meaning that $(-\infty, \beta)$, is a gap for A if S is semi bounded below with the lower bound .

Theorem (5-1-1) [131]:

Let $\{S_k\}_{k=1}^{\infty}$ be a family of closed symmetric operators S_k, defined in the separable Hilbert space H such that the operators S_k are unitarily equivalent to a closed symmetric operator A in H with equal positive deficiency indices. If there exists a boundary triple $\Pi_0 = \{H_0, \Gamma_0^0, \Gamma_1^0\}$ for A such that the corresponding Weyl function $M(.)$ is monotone with respect to open set $J \subseteq \rho(A_0)$, $A_0 = A^*|\ker(\Gamma_0^0)$, then for any auxiliary self-adjoint operator in some separable Hilbert space the closed symmetric operator admits a self-adjoin extension such that the spectral, parts and are unitarily equivalent i.e. $S_J^{\%} \cong R_J$ [95.109,110].

The following result is known as a generalized Nuimark dilation theorem.

Proposition (5-1-2) [131]:

If $\sum(.) : B(R) \to [H]$ is a bounded operator measure, then there exist a Hilbert space abounded operator $k \in [H, K]$ and an orthogonal measure $E(.) = B(R) \to [k]$ (an orthogonal dilation) such that

$$\sum(\delta) = k^* E(\delta) k , \delta \in B(R) \quad (2)$$

If the orthogonal dilation is minima i.e.,

$$span\{E(\delta) ran(k) : \delta \in B(R)\} = K \quad , \quad (3)$$

then it is uniquely determined up to unitary equivalence that is if one has two bounded operator $k \in [H, k]$ and $K' \in [H, K]$ as well as two minimal orthogonal dilation $E(.) = B(R) \to [K]$ and $E'(.) : B(R) \to [K']$ obeying $\sum(\delta) = K^* E(\delta) K$ $= K'^* E'(\delta) K', \delta \in (R) B(R)$, then there exists an isometry $v : K' \to K$ such that $E'(\delta) = v^* E(\delta) v , \delta \in B(R)$.

Definition (5-1-3) [131]:

We call $E(.)$ satisfying (2) and (3) the minimal orthogonal measure associated to $\Sigma(.)$, or the minimal orthogonal dilation of $\Sigma(.)$.

Every operator measure $\Sigma(.)$ admits the Lebesque Jordan decomposition

$\sum = \sum^{ac} + \sum^{s} , \sum^{s} = \sum^{sc} + \sum^{pp}$ where , , and are the absolutely continuous, singular, singular continuous and pure point components (measure) of $\Sigma(.)$, respectively. Non-topological supports of mutually disjoint, therefore if an operator measure Σ is orthogonal, $\Sigma(.) = E_T(.)$, then the ortho-projections $P^{\tau} = E_T^{\tau}(R)(\tau \in \{ac, sc, pp\})$ are pair wise orthogonal. Every subspace $h_T^{\tau} = p^{\tau}h$ reduces the operator $T = T^*$ and the Lebesgue-Jordan decomposition yields

$$\begin{aligned} h &= h_T^{ac} \oplus h_T^{sc} \oplus h_T^{pp} \\ T &= T^{ac} \oplus T^{sc} \oplus T^{pp} \end{aligned} \qquad (4)$$

Where $T^{\tau} = P^{\tau}T \upharpoonright h_T^{\tau}, \quad T \in \{ac, sc, pp\}$. Now we show Nevanlinna functions:

Let H be a separable Hilbert space, we recall that an operator-valued function $F : c_+ \to [H]$ is said to be a Neranlinna (or Herglotz or R_H) one if it is holomerphic and takes values in the set of dissipative operators on H i.e.,

$$\bar{S}m(F(z)) = \frac{F(z) - F(z)^*}{2!} \geq 0, z \in C_+$$

Usually one considers a continuation of F in C by setting $F(z) =$

$F(\bar{z}), z \in C_-$. Bounded operator $k \in [H, K]$ obeying $\ker(K) = \ker l \sum_F^0 (R)$ and

$\sum_F^0 (\delta) = k^* E_F(\delta) k, \delta \in B(R)$. By

$$\sum_F(\delta) = \int_{\delta} (1 + t^2) d \sum_F^0 (t), \delta \in B_b(R) \qquad (5)$$

One defines and operator measure which in general is non-orthogonal and unbounded. It is called the unbounded spectral measure of $F(.)$. Using Σ_F the representation [118],

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{1}{1 + t^2} \right) d \sum_F (t), z \in C_+ \cup C_- \qquad (6)$$

To show this we have **[5]:**

From this representation $F(z)=C_0+C_1 z+\int_{-\infty}^{+\infty}\frac{1+tz}{t-z}d\sum_F^0(t), z\in C_+\cup C_-$. To prove representation (6) use equation (5)

$$\sum_F(\delta)=\int_\delta (1+t^2)\,d\sum_F^0(t),\ \delta\in B_b(R)$$

so $d\sum_F(\delta)=(1+t^2)\,d\sum_F^0(t)$, which implies that $d\sum_F^0(t)=\frac{1}{1+t^2}d\sum_F(t)$, put this in the representation above we have

$$F(z)=C_0+C_1 z+\int_{-\infty}^{+\infty}\frac{1+tz}{t-z}\left(\frac{1}{1+t^2}\right)d\sum_F(t)=C_0+C_1 z+\int_{-\infty}^{+\infty}\frac{1+tz}{(t-z)(1+t^2)}\left(\frac{1}{1+t^2}\right)d\sum_F(t)$$

To analysis this component we use this $\frac{1+tz}{(t-z)(1+t^2)}=\frac{A}{t-z}+\frac{Bt}{1+t^2}+\frac{c}{1+t^2}=1+tz$

and $A(1+t^2)+Bt(t-z)+C(t-z)=1+tz$ put $t=z$ we get $A(1+z^2)=1+z^2$, so $A=1$ at $t=0, A-Cz=1$ implies that $c=0$ since $A=1, c=0$. Our equation become $1+t^2+B_t(t-z)+0=1+tz$, $Bt(t-z)=1+tz-1-t^2=-t(t-z)$, $Bt=-t\left(\frac{t-z}{t-z}\right)$, $B=-1$. Substituted $A, B, \text{and } C$ the equation

$$\frac{1+tz}{(t-z)(1+t^2)}=\frac{A}{t-z}+\frac{Bt}{1+t^2}+\frac{C}{1+t^2}=1+tz$$

We get the following

$$F(z)=C_0+C_1+\int_{-\infty}^{+\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sum_F(t) \qquad (7)$$

$z\in C_+\cup C_-$

Which complete the proof. From representation

$$F(z)=C_0+C_1 z=\int_{-\infty}^{\infty}\frac{1-tz}{t-z}d\sum_F{}^0(t), z\in C_+\cup C_-$$

F determines uniquely the unbounded spectral measure $\sum_F(\cdot)$ by means of the Stieltjes inversion formula, which is given by

$$\sum_F((a,b)) = s-\lim_{\delta\to+0} s-\lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{a+\delta}^{b-\delta} Sm\big(F(x+i\varepsilon)\big)\,dx \tag{8}$$

By supp (F) we denote the topological (minimal closed) support of the spectral measure Σ_F. Since supp (F) is closed the set $O_F = R\setminus \operatorname{supp}(F)$ is open. The Nevanlinna function $F(.)$ admits an analytic continuation to O_F given by

$$F(\lambda) = C_0 + C_1\lambda + \int_{-\infty}^{+\infty}\left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\sum_F(t),\ \lambda\in O_F$$

Using this representation we immediately find that $F(.)$ is monotone on each component interval Δ of O_F i.e., $F(\lambda) \le F(\mu),\ \lambda<\mu,\ \lambda,\mu\in\Delta$ In general, this relation is not satisfied if λ and μ belong to different component interval.

Definition (5-1-4) [131]:

Let $F(.)$ be a Nevanlinna function, the Nevanlinna function is monotone with respect to the open set $J \le O_F$ if for any two component intervals J_1 and J_2 of J one has $F(\lambda_1) \le F(\lambda_2)$ for all $\lambda_1 \in J_1$ and $\lambda_2 \in J_2$ or $F(\lambda_1) \ge F(\lambda_2)$ for all $\lambda_1 \in J_1$ and $\lambda_2 \in J_2$.

Let $L \in N \cup \infty$ be the number of component interval of J. obviously if $F(.)$ is monotone with respect to J and $L<\infty$, then there exists an enumeration $\{J_k\}_{k=1}^{L}$ of the components of J such that

$$F(\lambda_1) \le F(\lambda_2) \le .. \le F(\lambda_L)$$

Holds for $\{\lambda_1,\lambda_2,...,\lambda_L\} \in J_1 \times J_2 \times .. \times J_L$. If $L=\infty$, then it can happen that such an enumeration does not exist. If $F(.)$ is a scalar Nevanlinna function, then $F(.)$ is monotone with respect to J if and only if the condition $F(J_1) \cap F(J_2) = 0$ is satisfied for any two component intervals J_1 and J_2 of J.

Definition (5-1-5) [131]:

A triple $\Pi = \{H, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space and linear mappings $\Gamma_i : dom(A^*) \to H, i = 0,1$. Called a boundary triple for the adjoint operator of if the following two conditions are satisfied:

(i) The second Green's formula takes place

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), f, g \in dom(A^*)$$

(ii) The mapping $\Gamma = \{\Gamma_0, \Gamma_1\} : dom(A^*) \to H \oplus H$, $\Gamma f = \{\Gamma_0 f, \Gamma_1 f\}$ is surjective the above definition allows one to describe the set Ext_A in the following way.

Proposition (5-1-6) [131]:

Let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple for then the mapping established abijective correspondence $\tilde{A} \to \theta = \Gamma(dom(\tilde{A}))$ between the set Ext_A of self-adjoint linear relations in . By proposition (5-1-6) the following definition is natural [144,145].

Definition (5-1-7) [131]:

Let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple for . We put $A_\theta = \tilde{A}$, if $\theta = \Gamma(dom(\tilde{A}))$ that is $A_\theta = A^* | D_\theta$,

$$dom(A_\theta) = D_\theta = \{f \in dom(A^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \theta\} \qquad (9)$$

If $\theta = G(B)$ is the graph of an operator $B = B^* \in C(H)$, then $dom(A_\theta)$ is determined by the equation $dom(A_B) = D_B = \ker(\Gamma_1 - B\Gamma_0)$. We set $A_B = A_\theta$

Let us recall the basic facts on Weyl functions:

Definition (5-1-8) [131]:

Let be a densely defined closed symmetric operator and $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple for . The unique mapping $M(.) = \rho(A_0) \to [H]$ defined by $\Gamma_1 f_z = M(z)\Gamma_0 f_z, f_z \in N_z = \ker(A^* - z), z \in C_+$

Is called the Weyl function corresponding to the boundary triple .

Proposition (5-1-9) [131]:

Let be a simple closed symmetric operator and let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triple for with Weyl function $M(\lambda)$. Suppose that is self-adjoint linear relation in and $\lambda\in\rho(A_0)$ then

(i) $\delta(A_0)=\operatorname{supp}(M)$

(ii) $\lambda\in\rho(A_\Theta)$ if and only if $\Theta\in\rho(\Theta-M(\lambda))$

(iii) $\lambda\in\delta_\tau(A_\Theta)$ if and only if $O\in\delta_\tau(\Theta-M(\lambda)).\tau\in\{p,c\}$

We need the following simple proposition.

Proposition (5-1-10) [131]:

Let be a closed symmetric operator and let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triple for

(i) If is simple and $\Pi_1=\{H_1,\Gamma_0^1,\Gamma_1^1\}$ is another boundary triple for such that $\ker(\Gamma_0)=\ker(\Gamma_1^1)$, then the Weyl functions $M(.)$ and $M_1(.)$ of and Π_1, respectively are related by $M_1(z)=k^*M(z)k+D$, $z\in C_+\cup C_-$. Where $D=D^*\in[H]$ and $k\in[H_1,H]$ is boundedly invertible.

(ii) If $\Theta=G(B),B=B^*\in H$, then the Weyl function $M_B(.)$ corresponding to the boundary triple $\Pi_B=\{H,\Gamma_0^B,\Gamma_1^B\}=\{H,B\Gamma_0-\Gamma_1,\Gamma_0\}$ is given by $M_B(z)=(B-M(z))^{-1},z\in £_+\cup £_-$

Definition (5-1-11) [131]:

Let be a densely defined closed symmetric operator and let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triple for . The mapping $\rho(A_0)\ni z\longrightarrow\gamma(z)\in[H\,N_z]$

$$\gamma(z)=(\Gamma_0|N_z)^{-1}:H\longrightarrow N_z,z\in\rho(A_0)$$

is called the –filed of the boundary triple . One can easily have

$$\gamma(z)=(A_0-z_0)(A_0-z_0)^{-1}\gamma(z_0),\ z,z_0\in\rho(A_0) \tag{10}$$

The -field and the Weyl function $M(\cdot)$ are related by

$$M(z)-M(z_0)^*=(z-\bar{z}_0)\gamma(z_0)^*\gamma(z)$$

Lemma (5-1-12) [131]:

Let be a simple densely defined closed symmetric operator on a separable Hilbert space with equal deficiency indies. Further let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple for with Weyl function $M(\cdot)$. If $E_{A_0}(\cdot)$ is the orthogonal spectral measure of define on and $E_M(\cdot)$ the associated minimal orthogonal spectral dilation of $\Sigma_M^0(\cdot)$ defined on such that $E_{A_0}(\delta)=\quad W^*E_M(\delta)W$ for any Borel set $\delta\in B(R)$.

Proof: By (10) one obtains

$$S(M(x+iy)h,h)=y(\gamma(x+iy)h,(x+iy)h)h\in\mathcal{H} \tag{11}$$

To show this we have **[5]:**

$$Sm(M(z)h,h)=\frac{(M(z)h,h)-(M(z)h,h)^*}{2i}$$

Where

$$z=x+iy=|h|\left[(M(z),1)-(M(z),1)\right]/2i$$
$$=|h|\left[(z-\bar{z}_0)\gamma(z_0)^*\gamma(z)+M(z_0)-(z-\bar{z}_0)^*\gamma(z)^*-M(z_0)^*\right]/2i$$

Multiply and divided by $(z-\bar{z}_0)\gamma(z_0)^*$

$$=\frac{|h|}{2i}\left[\frac{(z-\bar{z}_0)\gamma(z_0)^*\gamma(z)}{(z-\bar{z}_0)\gamma(z_0)^*}-\frac{(z-\bar{z}_0)\gamma(z_0)\gamma(z_0)^*}{(z-\bar{z}_0)\gamma(z_0)^*}\right]$$

$$=\frac{|h|}{2i}\left[\gamma(z)-\gamma^*(z)\right]=\frac{|h|}{2i}\left[\gamma(z)-\gamma^*(z)\right]$$

$$=\frac{|h|}{2i}\left[\gamma(z)-(\bar{z}_0)\gamma(\bar{z}_0)^*\gamma(z)\right]$$

$$=|h|\left[\frac{\gamma(z)-\gamma^*(z)}{2}(z-\bar{z}_0)\gamma^*(z_0)\right]$$

$$=\frac{|h|}{2i}\left[(z-\bar{z}_0)\gamma^*(\bar{z}_0)\gamma(z)-\gamma^*(z)\gamma^*(z_0)(z-\bar{z}_0)\right]$$

$$=\frac{|h|}{2i}\gamma^*(z_0)\left[(z-\bar{z}_0)\gamma(z)-(z-\bar{z}_0)\gamma^*(z)\right]$$

Where $\gamma^*(z_0)/i2=y$ $\quad =|h|y\left[\gamma(z),\gamma(z)\right]=y(\gamma(z)h,\gamma(z)h)$

Since $z=x+iy$, we get

$$Sm(M(x+iy)h,h)=y(\gamma(x+iy)h,\gamma(x+iy)h)$$

Which is the prove of (11). Further, it follows from (10) that

$$\gamma(x+iy)=\left[I+(x+i(y-1))(A_0-x-iy)^{-1}\right]\gamma(i) \qquad (12)$$

To prove (12) we use (10) **[5]:**

$$\gamma(z)=(A_0-z)(A_0-z)^{-1}\gamma(z_0)$$

$$\gamma(z)=A_0(A_0-z)^{-1}\gamma z_0-z_0(A_0-z)^{-1}\gamma(z_0)$$

$$=A_0\frac{1}{A_0}\left(I-\frac{z}{A_0}\right)^{-1}\gamma(z_0)-z_0(A_0-z)^{-1}\gamma(z_0)$$

$$=\left[(I-ZA_0^{-1})^{-1}-Z_0(A_0-z)^{-1}\right]\gamma(z_0)$$

$$=\left[\left(I+\sum_{n=1}^{\infty}Z^n\left\|A_0^{-1}\right\|^n-Z_0(A_0-Z)^{-1}\right)\right]\gamma(Z_0)$$

$$=\left[I+\sum_{n=1}^{\infty}Z^{n+1}\left\|A_0^{n+1}\right\|-Z_0(A_0-Z)^{-1}\right]\gamma(Z_0)$$

Since $A_0=A^*$ is self adjoint spectrum and $\left\|A_0^{n+1}\right\|=1$, so

$$\gamma(z)=\left[I+\sum_{n=0}^{\infty}Z^{n+1}-Z_0(A_0-Z)^{-1}\right]\gamma(z_0)$$

$$=\left[I+\sum_{n=0}^{\infty}Z^n.Z-Z_0(A_0-Z)^{-1}\right]\gamma(z_0)$$

But $\sum_{n=0}^{\infty} z^n = (A_0 - z)^{-1}$

Hence $\gamma(z) = \left[I + Z\,(A_0 - z)^{-1} - Z_0 \left(A_0 (A_0 - Z)^{-1} \right) \right] \gamma(Z_0)$

$$= \left[I + (z - z_0)(A_0 - Z)^{-1} \right] \gamma(Z_0)$$

Let $x = 0, y = 1 \Rightarrow z_0 = 0 + i$

Therefore $\gamma(z) = \left[I + (z - i)(A_0 - z)^{-1} \right] \gamma(i)$

Since $z = x + iy$

$$\gamma(x + iy) = \left[I + (x + iy - i)\left(A_0 - (x + iy)\right)^{-1} \right] \gamma(i)$$

$$= \left[I + (x + i(y - 1))(A_0 - x - iy)^{-1} \right] \gamma(i)$$

Which is the proof of (12). Inserting (12) into (11) one gets

$$Sm\left(M\,(x + iy)\,h, h \right) = y \int_{-\infty}^{+\infty} \frac{1 + t^2}{(t - x)^2 + y^2}\, d\left(E_{A_0}(t)\, \gamma(i)\, h, \gamma(i)\, h \right), h \in \mathrm{H}$$

On the other hand we obtain that $d\left(\sum_M (t)\, h, h \right) = (1 + t^2)\, d\left(E_{A_0}(t)\, \gamma(i)\, h, \gamma(i)\, h \right)$,

inserting in the above representation we get

$$Sm\left(M\,(x + iy)\,h, h \right) = \int_{-\infty}^{+\infty} \frac{d\left(\sum_M (t)\, h, h \right)}{(t - x)^2 + y^2} \quad , \quad h \in \mathrm{H}$$

Applying the stieltjes inversion formula (8) we find

$$\left(\sum_M ((a,b))\, h, h \right) = \int_{(a,b)} (1 + t^2)\, d\left(E_{A_0}(t)\, \gamma(i)\, h, h \right), h \in \mathrm{H}$$

Which yields

$$\sum_M^0 ((a,b)) = \gamma(i)^* E_{A_0}((a,b))\, \gamma(i) \tag{13}$$

for any bounded open interval $(a,b) \subseteq R$. Since is simple it follows from (12) that

$$\left\{ (A_0 - \lambda)^{-1} \tan\left(\gamma(i) \right) : \lambda \in C_+ \cup C_- \right\} = h \tag{14}$$

By (13) and (14), $E_{A_0}(.)$ is a minimal orthogonal dilation of $\Sigma_M^0(.)$. By proposition (5-1-2) we find that the spectral measure $E_{A_0}(.)$ and $E_M(.)$ are unitarily equivalent.

Definition (5-1-13) [131]:

Let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triple for with corresponding Weyl function $M(.)$. We will call $\Sigma_M^0(.)$ the bounded non-orthogonal spectral measure of the extension $A_0=(A^*|\ker(\Gamma_0))$.

Corollary (5-1-14) [131]:

Let be a simple densely defined closed symmetric operator in a separable Hilbert space with equal deficiency indices. Further, let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triple for and $M(.)$ the corresponding Weyl function, then

$$\delta(A_0)=\operatorname{supp}(M)=\operatorname{supp}\left(\sum_M\right);\ \delta_T(A_0)=\operatorname{supp}\left(\sum_M^\tau\right)$$. Where $\tau\in\{ac,s,sc,pp\}$

Remark (5-1-15) [131]:

$M_B(.)$ of the form $M_B(z)=(B-M(z))^{-1}=(B-m(z).I_H)^{-1}$ is the Weyl function of the generalized boundary triple . Being a Wyle function. $M_B(.)$ admits the representation

$$M_B(z)=C_0+\int_{-\infty}^{+\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sum_B(t),\quad z\in C_+\cup C_- \tag{15}$$

Where $\sum_B(.)=\sum_{MB}(.)$ is the (unbounded) non-orthogonal spectral measure of $M_B(.)$. In accordance with the Stieltjes inversion formula (8) the spectral measure can be re-obtained by

$$\sum_B(a,b)=s-\lim_{\delta\to 0}s-\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(M_B(x+i\varepsilon)-M_B(x-i\varepsilon)\right)dx \tag{16}$$

With $M(z) = M(\bar{z})^*$. We get

$$M_B(x+i\varepsilon) - M_B(x-i\varepsilon) = \int_{-\infty}^{+\infty} \left(\lambda - m(x+i\varepsilon)\right)^{-1} - \int_{-\infty}^{+\infty} \left(\lambda - m(x+i\varepsilon)\right)^{-1} dE_B(\lambda) \tag{17}$$

Where $z = x + i\varepsilon$ and $z^{-*} = x - i\varepsilon$. The representation admits this

$$M_B(x+i\varepsilon) - M_B(x-i\varepsilon) = \int_{-\infty}^{+\infty} \left(\left(\lambda - m(x+i\varepsilon)\right)^{-1} - \left(\lambda - m(x-i\varepsilon)\right)^{-1} \right) dE_B(\lambda)$$

By taking the integration both sides of equation (16) which leads to the expression **[5]:**

$$\frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(M_B(x+i\varepsilon) - M_B(x-i\varepsilon) \right) dx$$

$$= \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \int_{-\infty}^{+\infty} \left(\left(\lambda - m(x+i\varepsilon)\right)^{-1} - \left(\lambda - m(x-i\varepsilon)\right)^{-1} \right) dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(\left(\lambda - m(x+i\varepsilon)\right)^{-1} - \left(\lambda - m(x-i\varepsilon)\right)^{-1} \right) dE_B(\lambda)$$

Put $= \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(\left(\lambda - m(x+i\varepsilon)\right)^{-1} - \left(\lambda - m(x-i\varepsilon)\right)^{-1} \right) dx = k_\Delta(\lambda, \delta, t)$

We get the following

$$= \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(\left(M_B(x+i\varepsilon)\right)^{-1} - \left(M_B(x-i\varepsilon)\right)^{-1} \right) dx = \int_{-\infty}^{+\infty} k_\Delta(\lambda, \delta, t)\, dE_B(\lambda), \varepsilon > 0 \tag{18}$$

and

$$k_\Delta(\lambda, \delta, \varepsilon) = \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(\left(\lambda - m(x+i\varepsilon)\right)^{-1} - \left(\lambda - m(x+i\varepsilon)\right)^{-1} \right) dE_B(\lambda) \tag{19}$$

$\lambda \varepsilon R, \Delta = (a,b) \subseteq R$ and $\varepsilon > 0$ with $m(z) = \overline{m(\bar{z})}, z \varepsilon C_-$ we denote by the family of the component intervals $\Delta_L = (a_L, b_L)$ of $O_m = R \setminus Supp(m)$.

Further the function $M(.)$ admits an analytic continuation to [illegible] such that

$$m(x) = C_0 + \int_{-\infty}^{+\infty} \left(\frac{1}{t-x} - \frac{t}{1+t^2} \right) d_\mu(t), x \in O_m$$

Hence the function $m(.)$ restricted to O_m is analytic. Moreover one easily verifies that for every component interval Δ of O_m

$$m(x) < m(y), x < y, x, \quad y \varepsilon \Delta$$

Therefore for every component interval Δ of O_m the set $\Delta' = m(\Delta)$ is gain an open interval. Thus $O'_m = m(O_m)$ is also open and the union of the sets $O' = m(\Delta)$ where the union is taken over all component intervals Δ of O_m.

Lemma (5-1-16) [5]:

Let $m(.)$ be a scalar Nevalinna function. If $\Delta = (a,b)$ is contained in a component interval Δ of O_m then $C_\Delta(\delta) = \underset{\lambda \varepsilon R, \varepsilon \varepsilon (0,1]}{Sup} \left| k_\Delta(\lambda, \delta, \varepsilon) \right| < \infty$, for each

$$\delta \in \left(0, \frac{b-a}{2} \right) \qquad (20)$$

Proof: we have

$$m(x+i\varepsilon)=m(x)-\varepsilon^2 T_0(\varepsilon,x)+\overline{z}\,\varepsilon T_1(\varepsilon,x),\ x\in O_m \tag{21}$$

Where

$$T_0(\varepsilon,x)=\int_{-\infty}^{+\infty}\frac{1}{y-x}\cdot\frac{1}{(y-x)^2+\varepsilon^2}d_\mu(y) \tag{22}$$

and

$$T_1(\varepsilon,x)=\int_{-\infty}^{+\infty}\frac{1}{(y-x)^2\,\varepsilon^2}d_\mu(y) \tag{23}$$

using (21) and (22) we find constant $x_0(\delta),k_1(\delta)$ and $w_1(\delta)$ such that

$|T_0(\varepsilon,x)|\le x_0(\delta)$ and $0<w_1(\delta)\le T_1(t,x)\le x_1(\delta)$,

$$x\in(a+\delta,b-\delta) \tag{24}$$

For $\varepsilon\in[0,1]$ further we get from (20)

$$\begin{aligned}P(\lambda,x,\varepsilon)&=\frac{1}{\lambda-m(x+i\varepsilon)}-\frac{1}{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)}\\&=\frac{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)-\lambda+m(x+i\varepsilon)}{(\lambda-m(x+i\varepsilon))(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))}\end{aligned} \tag{25}$$

From (20) we get

$P(\lambda,x,\varepsilon)=\dfrac{\varepsilon^2T_0(\varepsilon,x)}{(\lambda-m(x+i\varepsilon))(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))}$, $\lambda\in R,\ x\in O_m,\in>0$. Since both $m(x)$ and $T_0(\in,x)$ are real for $x\in O_m$ we have from (20) that $|\lambda-m(x+i\varepsilon)|\ge\varepsilon T_1(\varepsilon,x)$ and $|\lambda-m(x)-i\varepsilon T_1|\ge\varepsilon T_1(\varepsilon,x),\lambda\in R$. In view of (36) these inequalities yield

$$|p(\lambda,x,\varepsilon)|\le\left|\frac{T_0(t,x)}{T_1(t,x)^2}\right|,\lambda\in R,x\in O_m,\varepsilon>0 \tag{26}$$

Combining (23) with (25) we obtain the estimate **[5]:**

$$|P(\lambda,x,\varepsilon)|\le\frac{x_0(\delta)}{w_1(\delta)^2},\lambda\in R,x\in(a+\delta,b-\delta),\varepsilon\in(0,1] \tag{27}$$

We set **[5]:**

$$r_{\Delta}(\lambda,\delta,\varepsilon)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{1}{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)}-\frac{1}{\lambda-m(x)+i\varepsilon T_1(\varepsilon,x)}\right)dx$$

for $\lambda\in R$ and $\varepsilon>0$. By the representation

$$r_{\Delta}(\lambda,\delta,\varepsilon)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\frac{\lambda-m(x)+i\varepsilon T_1(\varepsilon,x)-\lambda+m(x)+i\varepsilon T_1(\varepsilon,x)}{(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))(\lambda-m(x)+i\varepsilon T_1(\varepsilon,x))}dx$$

$$=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{2i\,\varepsilon T_1(\varepsilon,x)}{\left(\lambda-m(x)^2+\varepsilon^2T_1(\varepsilon,x)^2\right)}\right)dx$$

$$=\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\left(\frac{\varepsilon T_1(\varepsilon,x)}{\left(\lambda-m(x)^2+\varepsilon^2T_1(\varepsilon,x)^2\right)}\right)dx$$

and the estimate (23) we obtain that $T_1(\varepsilon,x)=x_1(\delta)$ and $T_1(\varepsilon,x)^2=w_1^2(\delta)$ put this in the above equation we get

$$r_{\Delta}(\lambda,\delta,\varepsilon)=\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\frac{\varepsilon x_1(\delta)}{\left(\lambda-m(x)^2+\varepsilon^2 w_1^2(\delta)\right)}dx\,,\lambda\in R\,,\varepsilon\in(0,1] \tag{28}$$

Form this equation

$$m(x)=C_0+\int_{-\infty}^{+\infty}\left(\frac{1}{t-x}-\frac{t}{1+t^2}\right)d\mu(t)\,,x\in O_m$$

The derivation $m'(x),x\in O_m$, admits the representation

$$m'(x)=\int_{-\infty}^{+\infty}\frac{1}{(1-x)^2}d\mu(t)\,,x\in O_m \tag{29}$$

Obviously, there exist constants $w_z(\delta)$ and $x_2(\delta)$ such that

$$0<w_2(\delta)\le m'(x)\le x_2(\delta),x\in(a+\delta,b-\delta) \tag{30}$$

By combining the equation (27) and equation (29) where $0<w_2(\delta)\le m'(x)$, $x\in(a+\delta,b-\delta)$ we have the following

$$r_{\Delta}(\lambda,\delta,\varepsilon)\le\frac{x_1(\delta)}{\pi w_2(\delta)}\int_{a+\delta}^{b-\delta}\frac{\varepsilon.m'(x)}{(\lambda-m(x))^2+\varepsilon^2 w_1^2(\delta)}dx\,,\quad \lambda,R\,,\varepsilon\in(0,1]\quad .$$

Using the substitution $y = m(x)$ we derive that $\frac{dy}{dx} = m'(x)$ so $dx = \frac{dy}{m'(x)}$ in the equation we get

$$r_{\Delta}(\lambda,\delta,\varepsilon) \le \frac{x_1(\delta)}{\pi w_2(\delta)} \int_{a+\delta}^{b-\delta} \frac{\varepsilon . m'(x)}{(\lambda - m(x))^2 + \varepsilon^2 w_1^2(\delta)} \frac{dy}{m'(x)}$$

$$\le \frac{x_1(\delta)}{\pi w_2(\delta)} \int_{m(a+\delta)}^{m(b-\delta)} \frac{\varepsilon}{(\lambda - y)^2 + \varepsilon^2 w_1^2(\delta)} dy, \lambda \in R, \varepsilon \in (0,1]$$

Finally, we get

$$r_{\Delta}(\lambda,\delta,\varepsilon) \le \frac{x_1}{w_1 w_2}, \lambda \in R, \varepsilon \in (0,1] \qquad (31)$$

Obviously we have

$$k_{\Delta}(\lambda,\delta,\varepsilon) = \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (\rho)(\lambda,\delta,\varepsilon) - \overline{\rho(\lambda,\delta,\varepsilon)} dx + r_{\Delta}(\lambda,\delta,\varepsilon), \lambda \in R, \varepsilon > 0$$

Hence we find the estimate

$$|k_{\Delta}(\lambda,\delta,\varepsilon)| \le \frac{1}{\pi} \int_{a+\delta}^{b-\delta} |\rho(\lambda,\delta,\varepsilon)| dx + r_{\Delta}(\lambda,\delta,\varepsilon), \lambda \in R, \varepsilon > 0$$

Taking into account equation $|\rho(\lambda,\delta,\varepsilon)| \le \frac{x_0(\delta)}{w_1(\delta)^2}$ and the equation

$r_{\Delta}(\lambda,\delta,\varepsilon) \le \frac{x_1}{w_1 w_2}$ we arrive at the estimate $|k_{\Delta}(\lambda,\delta,\varepsilon)| \le \frac{x_0}{\pi w_1(\delta)}(b-a)$

$+\frac{x_1(\delta)}{w_1(\delta) w_2(\delta)}, \lambda \in R, \varepsilon \in (0,1]$. Which proves (19).

Since the function is strictly monotone on each component interval of the inverse function $\varphi_i(.)$ exists there. The function $\varphi_i(.)$ is analytic and also strictly monotone, its first derivative $\varphi_i'(.)$ exists, it is analytic and non-negative.

Lemma (5-1-17) [131]:

Suppose that $m(.)$ is a scalar Nevanlinna function, let $\Delta=(a,b)$ be contained is some component interval of $O_m = R \setminus \operatorname{supp}(m)$, then (with defined as in (18)).

$$\lim_{\varepsilon\to+0} k_{\Delta}(\lambda,\delta,\varepsilon) = \theta_L(\lambda,\delta) = \begin{cases} 0 & \lambda\in R\setminus\left[m(a+\delta), m(b-\delta)\right] \\ \frac{1}{2}\varphi'_L & \lambda\in\left\{m(a+\delta), m(b-\delta)\right\} \\ \varphi'_L(\lambda) & \lambda\in\left(m(a+\delta), m(b-\delta)\right) \end{cases} \tag{32}$$

For $\delta\in(0,(b-a)/2)$ and

$$\lim_{\varepsilon\to+0}\lim_{\varepsilon\to+0} k_{\Delta}(\lambda,\delta,\varepsilon) = \theta_L(\lambda,\delta) = \begin{cases} 0 & \lambda\in R\setminus\left(m(a), m(b)\right) \\ \varphi'_L(\lambda) & \lambda\in\left(m(a), m(b)\right) \end{cases} \tag{33}$$

Proof [5]:

At first let us show that

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\rho(\lambda,x,\varepsilon)\,dx = 0, \quad \lambda\in R \tag{34}$$

by (24) one immediately gets that

$$\lim_{\varepsilon\to 0}\rho(\lambda,x,\varepsilon) = \lim_{\varepsilon\to 0}\left(\frac{1}{\lambda-m(x+i\varepsilon)} - \frac{1}{\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)}\right)$$

$$= \lim_{\varepsilon\to 0}\rho\left(\frac{\varepsilon^2 T_0(\varepsilon,x)}{\left(\lambda-m(x+i\varepsilon)\right)\left(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x)\right)}\right) = 0,\ \lambda\in R, x\in O_m, \varepsilon>0$$

Which implies that $\lim_{\varepsilon\to 0}\rho(\lambda,x,\varepsilon)=0$ by lemma (5-1-16). Now (33) is implied by (26) and the Lebesque dominated convergence theorem. Next we set Lebesque

$$T_3(t,x) = \int_{-\infty}^{+\infty}\frac{1}{(y-x)^2+\varepsilon^2}\cdot\frac{1}{(y-x)^2}\,d\mu(y),\ x\in O_m, t\geq 0 \tag{35}$$

Obviously there is a constant $x_3(\delta)>0$ such that

$$0\leq \tau_3(\varepsilon,x)\leq x_3(\delta),\ x\in(a+\delta,b-\delta),\ \varepsilon\in[0,1] \tag{36}$$

Let

$$\rho_0(\lambda,x,t)=\frac{1}{\lambda-m(x)-i\varepsilon\tau_1(\varepsilon,x)}-\frac{1}{\lambda-m(x)-i\varepsilon T_1(0,x)},\lambda\in R,x\in O_m \tag{37}$$

For $\varepsilon>0$, it follows from (20) , (35) and (37)

That

$$\rho_0(\lambda,x,\varepsilon)=\frac{-i\varepsilon^3\tau_3(\varepsilon,x)}{(\lambda-m(x)-i\varepsilon T_1(\varepsilon,x))(\lambda-m(x)-i\varepsilon T_1(0,x))} \tag{38}$$

for $\varepsilon>0$, since $\lambda\in R$ and $m(x)$ is real for $x\in O_m$ we get from (38)

$$|\rho_0(\lambda,x,\varepsilon)|\le\in\frac{\tau_3(\varepsilon,x)}{\tau_1(\varepsilon,x)T_1(0,x)},\lambda\in R,x\in O_m,\varepsilon>0$$ where

$$\tau_1(\varepsilon,x)=\lambda-m(x)-i\varepsilon\tau_1(\varepsilon,x),$$
$$\tau_1(0,x)=\lambda-m(x)-i\varepsilon\tau_1(0,x),$$

by using (23) and (36) we obtain the estimate **[5]:**

$$|\rho_0(\lambda,x,\varepsilon)|\le\frac{\varepsilon\tau_3(\delta)}{w_1(\delta)^2},\lambda\in R,x\in(a+\delta,b-\delta),\varepsilon\in(0,1]$$

Which immediately yields

$$\lim_{\varepsilon\to0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\rho_0(\lambda,x,\varepsilon)\,dx=0,\lambda\in R,\delta>0 \tag{39}$$

Finally, let us introduce

$$q_\Delta(\lambda,\delta,\varepsilon)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{1}{\lambda-m(x)-i\varepsilon\tau_1(0,x)}-\frac{1}{\lambda-m(x)+i\varepsilon\tau_1(0,x)}\right)dx \tag{40}$$

For $\lambda\in R$ and $\varepsilon>0$. Using the representation

$$q_\Delta(\lambda,\delta,\varepsilon)=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{(\lambda-m(x)+i\varepsilon\tau_1(0,x))-(\lambda-m(x)-i\varepsilon\tau_1(0,x))}{(\lambda-m(x)-i\varepsilon\tau_1(0,x))(\lambda-m(x)+i\varepsilon\tau_1(0,x))}\right)dx$$
$$=\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{2i\varepsilon\tau_1(0,x)}{(\lambda-m(x))^2+\varepsilon^2\tau_1(0,x)^2}\right)dx$$

Form the equation (20) $\tau_1(0,x)=\int_{-\infty}^{+\infty}\frac{1}{(y-x)^2}d\mu$ and the equation

$m'(x)=\int_{-\infty}^{+\infty}\frac{1}{(t-x)}d\mu\geq(y),x\in O_m$. We get this relation $m'(x)=\tau_1(0,x),x\in O_m$ from the equation (20) and equation (28) we get after change of variable $y=m(x)$ that

$$q_{\triangle}(\lambda,\delta,\varepsilon)=\frac{1}{\pi}\int_{m(a+\delta)}^{(b-\delta)}\frac{\varepsilon m'(x)}{(\lambda-m(x))^2+\varepsilon^2\tau_1(0,x)^2}dx$$

$$=\frac{1}{\pi}\int_{m(a+\delta)}^{(b-\delta)}\frac{\varepsilon m'(x)}{(\lambda-y)^2+\varepsilon^2\tau_1(0,\varphi_L(y))^2}\frac{dx}{m'(x)},\lambda\in R,\varepsilon>0$$

$$=\frac{1}{\pi}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\varepsilon}{(\lambda-y)^2+\varepsilon^2\tau_1(0,\varphi_L(y))^2}dx$$

where $x=\varphi_l(y)$

By $\tau_1(0,,\varphi_l(y))=m'(\varphi_l(y))=1/\varphi_l'(y),y\in\triangle_L$, we finally obtain that

$$q_{\triangle}(\lambda,\delta,\varepsilon)=\frac{1}{\pi}\int_{m(a+\delta)}^{m(b-\delta)}\frac{t\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda-y)^2+\varepsilon^2}dy,\ y\in R,\varepsilon>0 \tag{41}$$

Next we prove the relation

$$\lim_{\varepsilon\to 0}q_{\triangle}(\lambda,\delta,\varepsilon)=\theta_L(\lambda,\delta),\delta\in(0,(b-a)/2),\lambda\in R \tag{42}$$

We consider only the case when $\lambda\in(m(a+\delta),m(b-\delta))$. The other cases can be treated in a similar way.

Noting that $\varphi_l'(\lambda)>0$ choose an arbitrary $C\in(0,\varphi_l'(\lambda))$. Since is continuous we can choose $\eta>0$ such that $m(a+\delta)<\lambda-\eta<\lambda+\eta<m(b+a)$ and

$$0<\varphi_l'(\lambda)-C\leq\varphi_l'(y)\leq\varphi_l'(\lambda)+C,\lambda-\eta<y\leq\lambda+\eta \tag{43}$$

Let $a,b>0$. The change of variables $x=b(y-\lambda)/\varepsilon$ yields **[5]:**

$$\int_{\lambda-\eta}^{\lambda+\eta}\frac{a^2\varepsilon}{b^2(\lambda-y)^2+\varepsilon^2}dy=\frac{a^2}{\varepsilon}\int_{-\frac{b\eta}{\varepsilon}}^{\frac{b\eta}{\varepsilon}}\frac{1}{1+x^2}.\frac{\varepsilon}{b}dx\to\frac{\pi a^2}{b}\ \text{ as }\ \varepsilon\to 0 \tag{44}$$

Setting $a = \varphi_i'(\lambda) - C$ and $b = \varphi_i' - C$ in (43) and using (44) we obtain

$$\pi \frac{\left(\varphi_i'(\lambda) - C\right)^2}{\varphi_i'(\lambda) + C} \le \liminf_{\varepsilon \to 0} \int_{\lambda-\eta}^{\lambda+\eta} \frac{\varepsilon \varphi_i'(y)^2}{\varphi_i'(y)^2 (\lambda - y)^2 + \varepsilon^2} dy \tag{45}$$

$$\liminf_{\varepsilon \to 0} \int_{\lambda-\eta}^{\lambda+\eta} \frac{\varepsilon \varphi_i'(y)^2}{\varphi_i'(y)^2 (\lambda - y)^2 + \varepsilon^2} dy \le \pi \frac{\left(\varphi_i'(\lambda) - C\right)^2}{\varphi_i'(\lambda) + C}$$

Setting $G = (m(a+\delta), m(b-a)) \setminus (\lambda - \eta, \lambda + \eta)$ and applying the Lebesgue dominated convergence theorem we get

$$\lim_{\varepsilon \to 0} \int_G \frac{\varepsilon \varphi_i'(y)^2}{\varphi_i'(y)^2 (\lambda - y)^2 + \varepsilon^2} dy = 0 \tag{45}$$

By (44) and (45)

$$\begin{aligned} \pi \frac{\left(\varphi_i'(\lambda) - C\right)^2}{\varphi_i'(\lambda) + C} &\le \liminf_{\varepsilon \to 0} \int_{m(a+\delta)}^{m(b-\delta)} \frac{\varepsilon \varphi_i'(y)^2}{\varphi_i'(y)^2 (\lambda - y)^2 + \varepsilon^2} dy \\ &\le \liminf_{\varepsilon \to 0} \int_{m(a+\delta)}^{m(b-\delta)} \frac{\varepsilon \varphi_i'(y)^2}{\varphi_i'(y)^2 (\lambda - y)^2 + \varepsilon^2} dy \le \frac{\left(\varphi_i'(\lambda) + c\right)^2}{\varphi_i'(\lambda) - c} \end{aligned} \tag{46}$$

Since (46) holds for every $C \in (0, \varphi_i'(\lambda))$, (46) in combination with (40) imply (41) combining (18), (26), (36) and (39) we derive the representation

$$\begin{aligned} k_\Delta(\lambda, \delta, \varepsilon) &= \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(\rho(\lambda, x, \varepsilon) - \rho\overline{(\lambda, x, \varepsilon)} \right) + \\ &\frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(\rho_0(\lambda, x, \varepsilon) - \rho_0\overline{(\lambda, x, \varepsilon)} \right) + q_\Delta(\lambda, x, \varepsilon) \end{aligned} \tag{47}$$

Where $\lambda \in R$ and $\varepsilon > 0$. Now combining the relation (33), (38) and (41) with (37) we arrive at (41). The relation (32) immediately follows from (31). Now we are ready to calculate a non-orthogonal spectral measure Σ_B^0 in a gap of any self-adjoint extension $A_B = A_B^* \in E_{Xt_A}$ if only A admits a boundary triple of a scalar-type Weyl function.

Section (5-2): Gaps and Examples

The symmetric operator admits a boundary triple $\Pi = \{H, \Gamma_0, \Gamma_1\}$ and is of scalar-type. On the spectrum $\delta(A_B)$ of the operator , outside the gaps $O_m^c = R \setminus O_m = Supp(m)$. We obtain results on the absolutely continuous spectrum [116,146].

Theorem (5-2-1) [131]:

Let $m(.)$ be a scalar Nevanlinna function in with the integral representation $F(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{1}{1+t^2} \right) d\sum_F (t), z \in C_+ \cup C_-$ and the imaginary part $\upsilon(z) = S(m(z))$ with admits the representation

$$\upsilon(x, y) = C_1 y + \int_R \frac{y d\mu(t)}{(t-x)^2 + y^2}, \int_R \frac{d\mu(t)}{1+t^2} < \infty$$

Where $\upsilon(x, y) = \upsilon(x + iy), z = x + iy \in C_+$. Then

(i) For any $x \in R$ the $\lim \upsilon(x + i0) = \lim \upsilon(x + iy)$ exists and is finite if and only if the symmetric derivation $D_\mu(x) = \lim_{\varepsilon \to 0} \frac{\mu(x+\varepsilon) - \mu(x-t)}{2t}$ exists and is finite. In case one has $\upsilon(x + i0) = \pi D_\mu(x)$.

(ii) If the symmetric derivative $D_\mu(x)$ exists and is infinite, then $\upsilon(z) \to +\infty$ as $z \to x$.

(iii) For each $x \in ¡$ one has $Sm(z - x)\upsilon(z) \to \mu(\{x\})$ as $z \to x$.

(iv) $\upsilon(z)$ converges to a finite constant as $z \to x$ if and only if the derivative $\mu'(t) = \frac{d\mu(t)}{dt}$ exists at $t = x$ and is finite.

Moreover, one has $\upsilon(x_0 + i_0) = \pi\mu'(x)$. The symbol $\to$ means that the limit $\lim_{r \to 0} \upsilon(x + re^{i\theta}), x \in ¡$ exist uniformly in $\theta \in [\varepsilon, \pi - \varepsilon]$ for each $\varepsilon \in (0, \pi/2)$.

Theorem (5-2-2) [131]:

Let be a simple symmetric operator in with infinite deficiency indices. Further, let $\Pi=\{H,\Gamma_0,\Gamma_1\}$ be a boundary triple for with scalar-type Weyl function $M(.)$ i.e., $M(z)=m(z)I_H$ and let $B=B^*\in C(H)$.

(i) Then $\delta_{ac}(A_B)\supset\delta_{ac}(A_0), A_0=A^*/\ker(\Gamma_0)$.

(ii) If the operator is purely absolutely continuous, then the self-adjoint extension is purely absolutely continuous to .

Proof:

By corollary (5-1-14) we get that $\delta_{ac}(A_0)=Supp_{ac}(\mu)$ where is the random measure of the representation

$$\Omega_{ac}(m)=\Big[x\in R:\exists m(x+io)=\lim_{y\to 0} m(x+iy)\ and\ 0<\upsilon(x,0)=Sm(m(x+i0))<\infty\Big]$$

Notice that the limit $m(x+io)=\lim_{y\to 0} m(x+iy)$ exists for almost all $x\in R$.

Further, let us introduce the set $cl_{ac}(x)=\{x\in R: mes(x-\varepsilon,x+\varepsilon)$ $I\ x>0\ for\ all\ \varepsilon>0)$. We get $cl_{ac}(\Omega_{ac}(m))=Supp_{ac}(\mu)$. By remark (5-1-15) the Weyl function $M_B(.)$ of the extension is given by $M_B(z)=(B-M(z))^{-1}$ $=(B-m(z).I_H)^{-1}, z\in C_+$. Let us introduce the scalar-function

$$M_{B,h}(z)=(M_B(z)h,h)=\left((B-m(z)I_H)^{-1}h,h\right)=\int_R\frac{d(E_B(t)h,h)}{t-m(z)}, z\in C_+ \tag{48}$$

For $h\in H$. If $z=x+iy$ and $m(z)=u(x,y)+i\upsilon(x,y)$, then we get from (48)

$$F_{B,n}(z)=S(m_{B,h}(z))=\int_R\frac{\upsilon(x,y)\,d(E_B(t)h,h)}{(t-u(x,y))^2+\upsilon(x+y)^2} \tag{49}$$

Let $x \in \Omega_{ac}(m)$. Notice the limits $\upsilon(x,0) = \lim_{y \to 0} \upsilon(x,y) > 0$ and $u(x,0) = \lim_{y \to 0} u(x,y)$ exists if $x \in \Omega_{ac}(m)$. If $y_0 > 0$ is small enough, then

$$\frac{\upsilon(x,y)}{(t-u(x,y))^2 + \upsilon^2(x,y)} \le \frac{1}{\upsilon(x,y)} \le \frac{2}{\upsilon(x,0)}, y \in [0, y_0), x \in \Omega_{ac}(m) \tag{50}$$

Taking in to account (50) and applying the Lebesque dominated convergence theorem we obtain from (48) that

$$\begin{aligned} &F_{B,h}(x+i0) = \lim_{y \to 0} F_{B,h}(x,y) = \\ &\upsilon(x,0) \int_R \frac{d(E_B(t)h,h)}{(t-u(x,0))^2 + \upsilon(x,0)}, x \in \Omega_{ac}(m) \end{aligned} \tag{51}$$

Since $\upsilon(x,0) > 0$ for $x \in \Omega_{ac}(m)$ we fined

$$0 < F_{B,h}(x,i0) < \infty, x \in \Omega_{ac}(m)$$

Furthermore we have

$$G_{B,h}(z) = \mathrm{Re}(M_{B,h}(z)) = \int_R \frac{(t-u(x,y))\, d(E_B(t)h,h)}{(t-u(x,y))^2 + \upsilon(x,y)^2}$$

Since $\dfrac{|t-u(x,y)|}{(t-u(x,y))^2 + \upsilon(x,y)^2} \le \dfrac{1}{\sqrt{(t-u(x,y))^2 + \upsilon(x,y)^2}} \le \dfrac{\sqrt{2}}{\upsilon(x,0)}$

For $x \in \Omega_{ac}(m)$ and $y \in (0, y_0)$. A gain by the Lebesque dominated convergence theorem we find

$$G_{B,h}(x+i0) = \lim_{y \to 0} G_{B,h}(x+iy) = \int_R \frac{(t-u(x,0))\, d(E_B(t)h,h)}{\sqrt{(t-u(x,y))^2 + \upsilon(x,0)^2}} \le \frac{\sqrt{2}}{\upsilon(x,0)}$$

Which shows that $x \in \Omega_{ac}(m)$ implies $x \in \Omega_{ac}(M_{B,h})$ for every $h \in H$ where

$$\Omega_{ac}(M_{B,h}) = \left\{ x \in R : \exists M_{B,h}(x+io) = \lim_{y \to 0} M_{B,h}(x+iy) \, and \; 0 < \Im m(M_{B,h}(x+io)) < \infty \right\}$$

Since $\Omega_{ac}(m) \subseteq \Omega_{ac}(M_{B,h})$ one gets $\sigma_{ac}(A_0) = Supp_{ac}(\mu) = cl_{ac}(\Omega_{ac}(m))$

$\subseteq cl_{ac}(\Omega_{ac}(M_{B,h}))$ for each $h \in H$.

If $B = B^{ac}$ then the measure $\rho_h(.) = (E_B(.)h, h)$ is absolutely continuous for any $h \in H$, that is $d\rho_h(t) = \rho_h'(t)dt$, where $\rho_h'(.) \in L^1(R)$ for any $h \in H$. One rewrite (48) as

$$F_{B,h}(z) = \int_R \frac{\upsilon(x,y)\,\rho_h'(t)\,dt}{\left(t - u(x,y)\right)^2 + \upsilon(x+y)} \qquad (52)$$

and the subset $H_\infty = \{h \in H : \rho_h' \in L^\infty(R) \cap L^1(R)\}$ is dense in $H = H^{ac}(B)$. For $h \in H_\infty$ we obtain from (48) that

$$C_\infty(h) = \underset{0<y<1}{Sup}\ \underset{x \in R}{Sup}\ \Im m F_{B,h}(x+iy) \le \|\rho_h'\|_{L^\infty} \underset{v>0}{Sup}\ \underset{u \in R}{Sup} \int_R \frac{\upsilon ds}{(s-u)^2 + \upsilon^2} \qquad (53)$$

Corollary (5-2-3) [131]:

Let A be a simple symmetric operator with infinite deficiency indices. Further, let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ a boundary triple for A^* with scalar-type Weyl function $M(.)$. If $\tilde{A} \in E_{xt_A}$ A, then $\delta_{ac}(A_0) \subseteq \delta_{ac}(\tilde{A})$.

Corollary (5-2-4) [131]:

Shows that under the assumption of a scalar-type Weyl function the absolutely continuous spectrum of any extension always contains $\delta_{ac}(A_0)$. The above result implies the following corollary.

Corollary (5-2-5) [131]:

Let A be a simple symmetric operator with infinite deficiency indices on the separable Hilbert space H. Further, let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* with scalar-type Weyl function $M(.) = m(.)I_H$ which is monotone with respect to the open set $J \subseteq O_m \subset \rho(A_0)$. Then for any operator T on some separable Hilbert space there is a self-adjoint extension $\tilde{A}$ such that $\tilde{A}_J \cong R_J^{ac}$ and $\tilde{A}$ is absolutely continuous [111,130].

Theorem (5-2-6) [131]:

Let be a simple symmetric operator in with infinite deficiency indices. Further, let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple for with scalar-type Weyl function $M(.)$. i.e., $M(z) = m(z) I_H$ and let $B = B^* \in C(H)$

(i) If is singular i.e., $B^* = B$, then the absolutely continuous parts and is unitarily equivalent, in particular $\sigma_{ac}(A_B) = \sigma_{ac}(A_0)$.

(ii) If and are singular, then is singular.

(iii) If is pure point and the spectrum of consist of isolated eigenvalues, then is pure point.

Proposition (5-2-7) [131]:

Let be a simple symmetric operator in with infinite deficiency indices. Further, let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple for with scalar-type Weyl function $M(.)$, i.e., $M(z) = m(z) I_H$, and $Supp^+(\mu) = \{x \in \sup p(\mu) : \exists$

$D_\mu(x)\ and\ D_\mu(x) > 0\}$ where is the radon measure of representation

$$m(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) \quad .$$

If $B \in C(H)$, then

$$E_{A_B}^T (Supp^+(\mu)) = 0, T \in \{s, pp, sc\} \quad (54)$$

In particular, it holds

(i) $\sigma_p(A_B) \cap \sup p(\mu) \le Supp(\mu) \setminus \sup p^+(\mu)$ and

(ii) $E_{A_B}^{ac}(\sup p(\mu)) = 0$ provided $\mathrm{supp}(\mu) \setminus \mathrm{supp}^+(\mu)$ is either finite or countable.

Proof:

We set $Supp_\infty^+(\mu) = \{x \in \mathrm{supp}^+(\mu) : D_\mu(x) = \infty\}$. By theorem (5-2-1) we derive that the limit $\lim_{y\to 0}(\upsilon(x, y))$ exists and is finite for

$x \in \sup p^+(\mu) \setminus \sup p_\infty^+(\mu)$ and

$$\upsilon(x,0)=\lim_{y\to 0}\upsilon(x,y)=\pi D_{\mu}(x)>0 \quad , \quad x\in \sup p^{+}(\mu)\setminus \sup p^{+}_{\infty}(\mu) \tag{55}$$

by proposition (5-1-5) there exists an operator $B=B^{*}\in C(H)$ such that $\tilde{A}=A_B=A^{*}\upharpoonright \ker(\Gamma_1-B\Gamma_0)$. We consider the generalized Weyl function $M_B(z)=(B-M(z))^{-1}$ and define $F_{B,h}$ by (48). Following the line of reasoning of theorem (5-2-12) we obtain

$$0<F_{B,h}(x+io)<\infty, x\in \operatorname{supp}^{+}(\mu)\setminus \operatorname{supp}^{+}_{\infty}(\mu), h\in H \tag{56}$$

Further, let $x\in \operatorname{supp}^{+}_{\infty}(\mu)$. By theorem (5-2-1) (ii) and (iii) we fined

$\upsilon(x,0)=\lim_{y\to 0}\upsilon(x,y)=\infty$ and $\lim_{y\to 0} y\,\upsilon(x,y)=\mu(\{x\})$.

Therefore for every $y_0>0$ there exists $N=N(y_0)$ such that $\upsilon(x,y)\geq N$ for $y\in(0,y_0)$. Hence

$$\frac{\upsilon(x,y)}{\left(t-u(x_0,y)\right)^2+\upsilon^2(x,y)}\leq\frac{1}{N}$$

By Lebesgue dominated theorem we obtain from (48) that

$$\lim_{y\to 0}F_{B,h}(x,iy)=0,\ x\in \operatorname{supp}^{+}_{\infty}(\mu),\ h\in H \tag{57}$$

Let $\sum_B(.)$ be the unbounded non-orthogonal spectral measure of the Weyl function $M_B(z)=(B-M(z))^{-1}, z\in C_{+}$ and $\sum_{B,h}(.)=\left(\sum_B(.)\,h,h\right), h\in H$. If

$$\delta''_s\left(\sum_{B,h}\right)=\{x\in R: F_{B,h}(z)\to\infty\ as\ z\to x\}, h\in H$$

then we fined from (55) and (56) that $\delta''_s\left(\sum_{B,h}\right)\cap \operatorname{supp}^{+}(\mu)=0$. Let $\tau=\{h_k\}_{k=1}^{\infty}$ be a total set in H. Setting $\delta''_s\left(\sum_B,\tau\right)=\bigcup_{k=1}^{\infty}\delta''_s\left(\sum_{B,h_k}\right)$. One gets $\delta''_s\left(\sum_B,\tau\right)\cap \operatorname{supp}(\mu^{+})=0$, and we gets

$$E_{A_B}^{s}\left(\sup p^{+}(\mu)\right)=E_{A_B}\left(\operatorname{supp}^{+}(\mu)\,\mathsf{I}\;\delta_s\left(\sum_B ,\tau\right)\right)=0$$

Which proves (53) for $\tau=s$. Similarly, setting

$$\delta_{pp}\left(\sum_{B,h}\right)=\left\{x\in \mathsf{i}\;:\lim_{z\to x}(z-x)\,F_{B,h}(z)>0\right\},\ h\in H$$

and

$$\delta_{pp}\left(\sum_B ,\tau\right)=\bigcup_{k=1}^{\infty}\delta_{pp}\left(\sum\nolimits_{B,h_k}\right)$$

We verity $\delta_{pp}\left(\sum_B ,\tau\right)\subseteq\delta_s\left(\sum_B ,\tau\right)$, one proves (53) for $\tau=pp$. Finally, setting

$\delta_{sc}\left(\sum_{B,h}\right)=\left\{x\in R: F_{B,h}(z)\to\infty\ and\ (z-x)\,F_{B,h}(z)\to 0\ as\ z\to x\right\}$, $h\in H$, and

$$\delta_{sc}\left(\sum\nolimits_B ,\tau\right)=\bigcup_{k=1}^{\infty}\delta_{sc}\left(\sum\nolimits_{B,h_k}\right)\setminus\delta_{pp}\left(\sum\nolimits_B ,\tau\right)$$

We obtain $\delta_{sc}\left(\sum_{B,h}\right)\subseteq\delta_s\left(\sum_{B,h}\right)$ which yield (53) for $\tau=sc$

(i) We have $\delta_p(A_B)=\delta_{pp}\left(\sum_B ,\tau\right)$ which yields $\delta_p(A_B)\,\mathsf{I}\ \operatorname{supp}(\mu)$

$\subset\operatorname{supp}(\mu)\setminus\operatorname{supp}^{+}(\mu)$.

(ii) We have $E_{A_B}^{sc}\left(\operatorname{supp}(\mu)\right)=E_{A_B}^{sc}\left(\operatorname{supp}^{+}(\mu)\right)+E_{A_B}^{sc}\left(\operatorname{supp}(\mu)\setminus\operatorname{supp}^{+}(\mu)\right)$

$=E_{A_B}^{sc}\left(\operatorname{supp}(\mu)\setminus\operatorname{supp}^{+}(\mu)\right)$.

Since by assumption $\operatorname{supp}(\mu)\setminus\operatorname{supp}^{+}(\mu)$ is countable we obtain

$E_{A_B}^{sc}\left(\operatorname{supp}(\mu)\setminus\operatorname{supp}^{+}(\mu)\right)=0$, which shows $E_{A_B}^{sc}\left(\operatorname{supp}(\mu)\right)=0$.

We consider several examples in order to illustrate the previous results [148].

Example (5-2-8) [131]:

Let $h=L^2((0,1))$. By i we denote the closed symmetric operator

$(Af)(x)=$

$-i\dfrac{d}{dx}f(x), x\in(0,1), f\in dom(A) \quad =\{f\in W_2'((0,1)): f(0)=f(1)=0\}$. Which is simple and has deficiency indices $(1,1)$. We note that A^* is given by $(A^*f)(x)=$ $-i\dfrac{d}{dx}f(x), f\in dom(A^*)=W_2'((0,1))$. A straight forward computation shows that $\Pi=\{H,\Gamma_0,\Gamma_1\}$ where $H=C$

$$\Gamma_0 f=\frac{f(0)-f(1)}{\sqrt{2}}, \Gamma_1 f=i\,\frac{f(0)+f(1)}{\sqrt{2}}, f\in dom(A^*)=W_2'((0,1)) \qquad (58)$$

forms a boundary triple for A. The operator $A_0=A^*/\ker(\Gamma_0)$ is given by

$$(A_0 f)(x)=-i\frac{d}{dx}f(x), x\in(0,1), f\in dom(A_0)$$
$$=\{w_2'(0,1): f(0)=f(1)\}$$

The spectrum of A_0 is discrete. It consist of isolated eigenvalues we have $\delta(A_0)=\{\lambda\}_{L\in z}$ with $\lambda_L=2L\pi$. Obviously we have $\rho(A_0)U_{L\in z}\Delta_L$ where $\Delta_L=(2L\pi,2(L+1)\pi)$.

Trivially the open intervals Δ_L are gaps of the operator $A_0=A_0^*$. Hence they are gaps of the symmetric operator A. The extension $A_1=A^*\uparrow\ker(\Gamma_1)$ has the domain $(A_1), dom(A_1)=\{f\in W^{(1,2)}((0,1)): f(0)=-f(1)\}$. Its spectrum is discrete and consists of the eigenvalues $\lambda_L=(2L+1)\pi, L\in Z$. Any other extension of A is given by a real constant $\theta\in R$ and the boundary triple $\Pi_\theta=\{C,\Gamma_0^\theta,\Gamma_1^\theta\}$, where $\Gamma_1^\theta=\Gamma_0$ and $\Gamma_0^\theta=\theta\Gamma_0-\Gamma_1$. The domain $dom(A_\theta)$ of the self-adjoint extension $A_\theta=A^*\uparrow\ker(\Gamma_0^\theta)$ can be alternatively described by

$$dom(A_\theta)=\{f\in W_2^1((0,1)): (\theta-i)(\theta+i)^{-1}f(0)=f(1)\}$$

the spectrum of A_θ is also describe and consists of eigenvalues. Setting $\theta=-\cot(T/2)$, $T\in(0,2\pi)$ one easily verifies that $\lambda_L^{(\theta)}=T+L, L\in ¢$.

In other words any extension of , which is different from , has an eigenvalues in the gaps $\Delta_L, L \in ¢$. i.e., it does not preserve the gaps . It is easily seen that the Weyl function corresponding to the boundary triple $\Pi = \{H, \Gamma_0, \Gamma_1\}$ other form (57) is

$$m(z) = -\frac{\cos(z/2)}{\sin(z/2)} = -\cot(z/2), z \in C_+ \cup C_-$$

The open set $O_m = i \setminus \sup p(m)$ coincides with $P(A_0) \mid i$. i.e., $O_m = U_{L \in z} \Delta_L$.
The Weyl function admits an extension to which is given by $m(\lambda) = -\cot(\lambda/2).\lambda \in O_m$. Obviously the Weyl function $m(.)$ is increasing on each open interval . However, choosing $J = O_m$ one easily verifies that the Weyl function $m(.)$ is not monotone with respect to . The lack of monotonicity is related to the fact that there does not exist an extension of which has only an eigenvalue in one gap as we have seen above.

Let us consider the closed symmetric operator $S = \oplus_{k=1}^{\infty} S_k$ on the Hilbert space $R = \oplus_{k=1}^{\infty} R_k$ where the operators are unitarily equivalent to defined above. Obviously the operator is unitarily equivalent to the operator defined on $h = L^2((0, \infty))$

$$(Cf)(x) = -i \frac{d}{dx} f(x), f \in dom(C) = \{W_2^1(i_+) : f(k) = 0, k \in \{0\} \cup N\} \quad .$$

We note that $O_m = U_{L \in z}(2\pi L, 2\pi(L+1))$ and $\varphi_L(t) = -2\mathrm{arecot}(t) + 2\pi(L+1), L \in i$.
By the associated non-orthogonal spectral measure $\Sigma_B^0(.)$ and $\sum_B(.)$ of the Weyl function $M_B(z) = (B - m(z), I)^{-1}$ are given by

$$\begin{aligned} &\sum\nolimits_B^0(\delta) = \varphi_L'(B)\left(1+\varphi_L(B)^2\right)^{-1} E_B(m(\delta)) = \\ &\left(1+2\pi(L+1) - 2are\cot-(B)^2\right)^{-1} E_B(-\cot(\delta/2)) \end{aligned} \tag{59}$$

and

$$\sum_B(\delta) = \varphi_L^{-1}(B)\, E_B\left(-\cot(\delta/2)\right) = 2\left(1+B^2\right)^{-2} E_B\left(-\cot(\delta/2)\right)\ \delta \in B(\triangle_L) \qquad (60)$$

It follows from (59) that the measure $\Sigma_B(\cdot)$ is periodic:

$$\sum(\delta + 2\pi L) = \sum(\delta), \delta \in B(\triangle_0), L \in ¢$$

Having in mind this fact one obtains that for any $L \in ¢$ the operator $\delta_B E_{s_B}(2\pi L, 2\pi(L+1))$ is unitarily equivalent to the operator $\delta_B E_{s_B}((0, 2\pi))$.

Example (5-2-9) [131]:

Let $\mathfrak{h} = L^2(\mathbb{R}_+)$ and let be a closed symmetric operator in defined by

$$\begin{aligned} (\delta_1 f)(x) &= -\frac{d^2}{dx^2} f(x), f \in dom(s_1) = W_2^2(\mathbb{R}_+) \\ &= \left\{ f \in W_2^2(\mathbb{R}_+) : f(0) = f'(0) = 0 \right\} \end{aligned} \qquad (61)$$

Obviously $\delta_1 \geq 0$. Setting

$$\Gamma_0'(\theta) f = f'(0) - \theta f(0), \Gamma_1'(\theta) f = -f(0), f \in dom\left(\delta_1^*\right) = W_2^2(\mathbb{R}_+)\ 0 \in \mathbb{R}$$

We obtain the boundary triple $\Pi_1^\theta = \{£, \Gamma_0^*(\theta), \Gamma_1^*(\theta)\}$. It is clear that the extension is non-negative if $\theta \geq 0$. The corresponding Weyl function is $m_\theta(\lambda) = \left(\theta - i\sqrt{\lambda}\right)^{-1}$. It is regular in $£ \setminus \mathbb{R}_+$ if $\theta \geq 0$, where the branch of $\sqrt{\cdot}$ is fixed by the condition $\sqrt{1} = 1$. The Weyl function $m_\theta(\cdot)$ admits the following integeral representation

$$m_\theta(\lambda) = \left(\theta - i\sqrt{z}\right)^{-1} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(t-z)\left(t+\theta^2\right)} dt, \theta \geq 0$$

and the corresponding spectral measure is given by $d\mu\theta = \pi^{-1} t^{\frac{1}{2}} \left(t + \theta^2\right)^{-1} dt$.

Clearly $m_\theta(\cdot)$ is holomorphic within $(-\infty, 0)$ such that $m_\theta(-\infty, 0) = (0, \theta^{-1})$. The inverse function $\varphi_\theta(\cdot) : (0, \theta^{-1}) \to (-\infty, 0)$ is given by $\varphi_\theta(\zeta) = -\left(\zeta^{-1} - \theta\right)^2$,

$\zeta \in (0, \theta^{-1})$. We set $\Delta = (-\infty, 0)$ and $\Delta' = m_\theta(\Delta) = (0, \theta^{-1})$. Notice that

$\varphi_0(\zeta) = 2(\zeta^{-1} - \theta)\zeta^{-2}$.

Let $h = \oplus_{k=1}^{\infty} h_k, A = \oplus_{k=1}^{\infty} S_k$ and $\Pi = \{H, \Gamma_0, \Gamma_1\} = \oplus_{k=1}^{\infty} \Pi_k^{\rho}$ where $h_k = h_1$, function $M(.)$ is of scalar-type, i.e., $M(z) = m_\theta(z) I_H$. Further, let $B = B^* \in C(H)$. To the self-adjoint extension it corresponding the Weyl function $M_B(z) = (B - m_\theta(z) I_H)^{-1}$. Let $\Sigma_B(.)$ be the unbounded non-orthogonal spectral measure of the Weyl function $M_B(.)$. It follows from (60)

$$\sum_B(\delta) = 2(B_{\Delta'}^{-1} - \theta) B_{\Delta'}^{-2} E_B(m_\theta(\delta)), B_{m_\theta(\Delta)} = B_{\Delta'} = BE_B(\Delta'), \delta \in B(\Delta) \qquad (62)$$

Let $\delta = (x, 0), x < 0$. Since $m_\theta((x,0)) = \left(\left(\theta + \sqrt{|x|} \right)^{-1}, \theta^{-1} \right)$ for $x < 0$ we get from (62)

$$\sum_B((x,0)) = 2(B_{\Delta'}^{-1} - \theta) B_{\Delta'}^{-2} E_B \left(\left(\theta + \sqrt{|x|} \right)^{-1}, \theta^{-1} \right) x < 0 \qquad (63)$$

we note that $\sum_B(x) \in [H]$ for every $x < 0$, while B may be unbounded.

Further, starting with (50) we can explicitly calculate the non-orthogonal spectral measure $\sum_B(.)$ outside the $\text{gap}\ \Delta = (-\infty, 0) = \left(\sum_B(.)\, h, h \right)$. Setting $\sum_{B,h}(.)$ and $F_{B,h}(z)$ $3m(M_B(z)\, h, h)$ we easily derive from (51) and the Fatous theorem that

$$\pi \frac{d \sum_{B,h}(x)}{dx} = F_{B,h}(x + io) = \int_R \frac{\sqrt{x}\, d(E_B(t)\, h, h)}{(t\theta - 1)^2 + xt^2}, x > 0, h \in H \qquad (64)$$

Where $\sum_{B,h}(x) = \sum_{B,h}((0,x)), x > 0$. straight forward computation shows that $Supp^+(\mu_0) = (0, \infty)$. By proposition (5-2-7) we have $E_{AB}^T((0,\infty)) = 0, T = s, pp, sc$. Hence $\delta_T(A_B) \subseteq (-\infty, 0], T = s, p, sc$. Since $\delta_{ac}(A_0) = [0, \infty)$ we obtain from theorem (5-2-2) that $E_{AB}(.) \supseteq [0, \infty)$. Therefore, the orthogonal spectral measure $E_{AB}(.)$ of is absolutely continuous on $(0,\infty)$ which yields that $\sum_B(.)$ is absolutely continuous on $(0,\infty)$. i.e., $\sum_B^{ac}(\delta)=\sum_B(\delta)$ for $\delta\in B((0,\infty))$. Hence

$$\sum_{B,h}((0,\infty))=\frac{1}{\pi}\int_0^x ds\int_R \frac{\sqrt{s}\,d\left(E_B(t)\,h,h\right)}{(t\theta-1)^2+\delta t^2}$$

$$=\int_R \Phi_\theta(x,t)\,d\left(E_B(t)\,h,h\right) \qquad x>0,\ h\in H \tag{65}$$

Where $\Phi_\theta(x,t)=\frac{2}{\pi t^2}\left(\sqrt{x}-\frac{|t\theta-1|}{t}\text{are tan}\left(\frac{t\sqrt{x}}{t\theta-1}\right)\right)\ x>0$. Which yields

$$\sum_B((0,x))h=\int_R \Phi_\theta(x,t)\,dE_B(t)\,h,\ x>0,\ h\in H \tag{66}$$

Thus, formulas (62) and (65) together give the explicit form for the unbounded non-orthogonal spectral measure $\sum_B(.)$ of the extension .

Section (5-3): Inverse Spectral Problem for Direct Sum of Symmetric Operator

We have [30,89,91,92]

Lemma (5-3-1) [133]:

Let be a symmetric operator in the Hilbert space . Suppose that has a gap . Let be a closed subspace of and a self-adjoint operator in such that is a restriction of the adjoint of and $\delta(M)\subset\bar{J}$. Then

$$H_M=H^{*}_{ID(H)+D(M)}=M\oplus G_0 \tag{67}$$

For some symmetric operator with gap in the Hilbert space . In particular has a self-adjoint extension such that

$$\tilde{H}_J=M_J$$

Proof:

We have only to show that the there exists a symmetric operator with gap such that (66) holds. In fact since is a gap of there exists a self-adjoint operator in such that $\delta(G) \cap J = \Theta$ and is an extension of . Then obviously $\tilde{H} = M \oplus G$ is a self-adjoint extension of and satisfies $\tilde{H}_J = M_J$.

Lemma (5-3-2) [133]:

Let be a symmetric operator in the Hilbert space . Let be closed subspace. and a self-adjoint operator in such that is a restriction of . Then $H_M = H^*_{\mid D(H) + D(M)} = M \oplus G_0$

For some symmetric operator in

Proof:

Let $f, \tilde{f} \in D(H)$ and $g, \tilde{g} \in D(M)$. We have

$$\begin{aligned}\left(H_M(f+g), \tilde{f}+\tilde{g}\right) &= \left(Hf, \tilde{f}+\tilde{g}\right) + \left(H^* g, \tilde{f}\right) + \left(Mg, \tilde{g}\right) \\ &= \left(f, H^*\left(\tilde{f}+\tilde{g}\right)\right) + \left(g, H\tilde{f}\right) + \left(g, M\tilde{g}\right) \\ &= \left(f+g, H_M\left(\tilde{f}+\tilde{g}\right)\right)\end{aligned}$$

This H_M is a symmetric operator in the Hilbert space $\mathcal{H}$. Let $f \in D(H_M)$. For every $n \in \mathbb{Z}$ let $P_n = I_{[n,n+1)}(M) P_{\mathcal{H}_0}$.

Since M is a self-adjoint operator in the Hilbert space $\mathcal{H}$ it follows from the spectral theorem that every $n \in \mathbb{Z}$ the operator P_n is an orthogonal projection in $\mathcal{H}$ onto the closed subspace $R(P_n)$ of $\mathcal{H}$

$$R(P_n) \subset D(M), \quad n \in \mathbb{Z} \tag{68}$$

$$R(P_n) \perp R(P_m), \quad n \neq \mathbb{Z} \tag{69}$$

$$\sum_{n \in \mathbb{Z}} P_n = P_{\mathcal{H}_0} \tag{70}$$

$$P_n M g = M P_n g, \; g \in DM, \; n \in \mathbb{Z} \tag{71}$$

Thus we have

$$\begin{aligned}&(P_n H_M f, g) = (H_M f, g) = (f, Mg) \\ &= (P_n f, Mg) = (M P_n f, g)\end{aligned}$$

For every $g \in R(P_n)$. In the second step we have used (67) and the facts that H_M is symmetric and M a restriction of H_M. In the third step we have used (70) and in the last step a gain (67).

Thus we have

$$P_n H_M f = M P_n f, \quad n \in \mathbb{Z} \tag{72}$$

Since by (71), (68) and (69)

$$(M P_n f, M P_k f) = (P_n H_M f, P_k H_M f) = 0 \quad k \neq n$$

and $\sum_{n \in \mathbb{Z}} \|M P_n f\|^2 = \sum_{n \in \mathbb{Z}} \|P_n H_M f\|^2 = \left\|P_{\mathcal{H}_0} H_M f\right\|^2 < \infty$

The sequence $\left\{M\sum_{n-N}^{N}P_n f\right\}_{N\in ¢}$ converges in . Since is closed and by (69),

$\lim_{N\to\infty}\sum_{n=-N}^{N}P_n f = P_{H_0} f$ it follows that

$$P_{H_0} f \in D(M) \tag{73}$$

and

$$MP_{H_0} f = \lim_{N\to\infty}\sum_{n=-N}^{N} MP_n f = \sum_{n\in z} P_n H_M f = P_{H_0} H_M f \tag{74}$$

Hare again we have used (71), (69). By (72) and (73)

$$H_M = M \oplus G_0$$

$$G_0 = H_{m/D(H_M) \cap H_0^{\perp}}$$

is a symmetric operator in the Hilbert since H_M is asymmetric operator

.

Proposition (5-3-3) [133]:

Let be a symmetric operator in the separable Hilbert space

suppose that has a gap . Let $\{\lambda\}_{n=1}^{N}$ be a (finite or infinite) sequence in

[54]. Then there exists a self-adjoint extension of such that

$$\delta_p(H') \cap J = \{\lambda_n : n\in N, 1\le n\le N\}$$

and for every eigenvalue of in the multiplicity of equals the number of times it occurs in the sequence $\{\lambda\}_{n=1}^{N}$ if and only if is less than or equal to the deficiency number of $\{\lambda\}_{n=1}^{N}$. In this case can be chosen such that it has a pure point spectrum inside the gap .

Proof:

First we shall do the "only-if-part" . Trivially the assertion of this part is true provided the deficiency number of is finite. But then the "only-if-part" follows from kreiu's theorem suppose now that is less that or equal to the deficiency number of . Then we can choose by induction an orthonarmal system $\{e_n\}_{n=1}^{N}$ such $e_n \in N(H^* - \lambda_n), n \in ¥, 1 \le n \le N$.

Due to the well known fact that the dimension of $N(H^* - \lambda)$ equal the deficiency number of for every regular point of and in particular for every $\lambda \in J$ [107,126].

Let $H_{00} = Span\{e_n : n \in ¥, 1 \le n \le N\}$ and $H_0 = \overline{H_{00}}$, $M_0 = H^*_{1H_{00}}$

By construction $\{e_n\}_{n=1}^{N}$ is an orthonormal base of the Hilbert space and for every $n \in ¥, 1 \le n \le N$ is an eigenvector of corresponding to the real eigenvalue . Thus can be and will be regarded as an operator in the Hilbert space its closure is a self-adjoint operator in has a pure point spectrum $\sigma_p(M) = \{\lambda_n : n \in N, 1 \le n \le N\}$ and for every eigenvalue of the multiplicity of equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^{N}$. is a restriction of since the adjoint of any operator closed. Thus has a self-adjoint extension such that $\tilde{H}_J = M$. i.e., such that has the required properties.

Definition (5-3-4) [133]:

A symmetric operator is significantly deficient if and only if it has a real regular point and $P_{N(H^*-z)} D(\tilde{H}) \ne N(H^* - z)$. For every regular point of

.

Proposition (5-3-5) [133]:

(i) Let be a closed symmetric operator in the Hilbert space . Let be a gap of and $0 \in J$. Let $A = P_{R(H)} H^{-1}, B = P_{R(H)^\perp} H^{-1}$, and an orthogonal projection in $R(H)^\perp$ such that the operator PB belongs to the trace class and let be the zero-operator in the Hilbert space

$R(1-P)$, then for every invertible self-adjoint operator in the

Hilbert space $R(P)$ the operator $\tilde{L} = \begin{pmatrix} A & (PB^*) \\ PB & Q \end{pmatrix} : \begin{matrix} R(H) \\ \oplus \\ D(Q) \end{matrix} \rightarrow \begin{matrix} R(H) \\ \oplus \\ R(P) \end{matrix}$, is

invertible and the operator $\tilde{H} = 0 \oplus \tilde{L}$ is a self-adjoint extension of

such that $\tilde{H}_{ac\,J} ; Q^{-1}_{ac\,J}$.

(ii) There exists an orthogonal projection in the Hilbert space $R(H)^{\perp}$ such that the operator belongs to the trace class and $R(P)$ is infinite dimensional if and only if the operator is significantly deficient in the sense of the definition (5-3-4).

Corollary (5-3-6) [133]:

Let be a significantly deficienct symmetric operator and let be a gap of . Then for every self-adjoint operator in a separable Hilbert space there exists a self-adjoint extension of such that

$\tilde{H}_{acJ} ; M'_{acJ}$.

Definition (5-3-7) [133]:

A symmetric operator is weakly significantly deficient if there exists a real regular point of and a real number which is not an eigenvalue of the operator $A = P_{R(\tilde{H}-\nu)}(\tilde{H}-\nu)^{-1}$. Such that $R(B(A-\lambda)) \neq N(H^* - \nu)$. Where

$B = P_{N(H^*-\nu)}(\tilde{H}-\nu)^{-1}$. We may assume that the operator is closed and $\nu = 0$.

$R(A-\lambda)$ is dense in the Hilbert space $R(H)$ since is self-adjoint and is not an eigenvalue of . Since is bounded and $R(B)$ is dense in $R(H)^{\perp}$ this implies that $R(B(A-\lambda))$ is dense in $R(H)^{-\perp}$. Thus we can replace by $B(A-\lambda)$ in the consideration at the beginning and get that there exists an orthogonal projection in the Hilbert space $R(H)^{\perp}$ such that $\dim R(P)=\infty$. And the operator $PB(A-\lambda)$ belongs to the trace class. Let be the zero operator in $R(1-P)$. We have shown that $H_0 = H^*_{|D(H)+R(1-P)}$ can be decomposed as

$$H_0 = 0 \oplus G_0 \tag{75}$$

for some continuously invertible operator in $R(H) \oplus R(P)$ such

$$G_0^{-1} = \begin{pmatrix} A \\ PB \end{pmatrix} : R(H) \to \begin{matrix} R(H) \\ \oplus \\ R(P) \end{matrix}$$

Let be any invertible self-adjoint operator in the Hilbert space $R(P)$. By the given considerations has an invertible self-adjoint extension such that

$$G^{-1} - \lambda = \begin{pmatrix} A-\lambda & B^*P \\ PB & Q-\lambda \end{pmatrix} : \begin{matrix} R(H) \\ \oplus \\ D(Q) \end{matrix} \to \begin{matrix} R(H) \\ \oplus \\ R(P) \end{matrix} \tag{76}$$

The following simple lemma will play role in the investigation of the absolutely continuous spectrum of the operator .

Lemma (5-3-8) [133]:

Let be a bounded self-adjoint operator and a self-adjoint operator such that $k_2 k_1 I_{\Delta}(k_1)$ belongs to the trace class for every bounded interval . Then

$$(k_1 + k_2)_{ac} = k_1' \oplus R$$

For some self-adjoint operator and such that $k_1' ; k_{1ac}$.

Corollary (5-3-9) [133]:

Let be a symmetric operator with gap in the complex Hilbert space suppose that $H = \overset{\infty}{\underset{n=1}{\oplus}} H_n$

For some symmetric operator $H_n, n \in ¥$ with strictly positive deficiency numbers.

Then for every self-adjoint operator in a separable Hilbert space there exists a self-adjoint extension of such that

$$\tilde{H}_J \simeq M'_{acJ}$$

Proof:

It easily follows from the spectral theorem that $M'_{ac\,J} \simeq \bigoplus_{n=1}^{\infty} Q_{\rho n}$

for suitabley chosen $\rho_n \in L^{1,+}_{loc}, \rho_n = 0 \quad dx - a.e$. On $\mathbb{R} \setminus J, n \in \mathbb{N}$. Since can be decomposed into infinitely sets the operator can be decomposed as

$$H = \bigoplus_{n=1}^{\infty} H^{(n)} \quad ,$$

where for every $n \in \mathbb{N}$ operator $H^{(n)}$ is the orthogonal sum of two operators with infinite deficiency numbers. For every $n \in \mathbb{N}$ there exists a self-adjoint extension of $H^{(n)}$ such that $\tilde{H}_{nJ} \simeq Q_{\rho n}$

Then $\tilde{H} \simeq \bigoplus_{n=1}^{\infty} \tilde{H}_n$ is a self-adjoint extension of and $\tilde{H}_J = \bigoplus_{n=1}^{\infty} \tilde{H}_{nJ} \simeq \bigoplus_{n=1}^{\infty} Q_{\rho n} \simeq M'_{acJ}$

Proposition (5-3-10) [133]:

Let be a symmetric operator in some Hilbert space . Suppose that the operator has some gap and its deficiency number is infinite. Let be an open subset of . Then has a self-adjoint extension such that

$$\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J$$

Corollary (5-3-11) [133]:

Let be a symmetric operator in the Hilbert space . Suppose has a gap and the deficiency number of is infinite. Let be anon-empty open subset of . Then there exists self-adjoint extensions and and a non-empty compact subset of with lebesque measure zero such that and have a purely singular continuous spectrum in the gap of

$$\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J \quad \text{and} \quad \sigma_{sc}(\tilde{H}) \cap J = C_0$$

We shall combine studying the point singular continuous and absolutely continuous spectra.

Theorem (5-3-12) [133]:

Let be a symmetric operator in the Hilbert space . Suppose that has a gap and its deficiency number is infinite. Then for every open subset of and every finite or infinite sequence $\{\lambda_n\}_{n=1}^{N}$ in there exists a self-adjoint extension of with the following properties.

(i) $\sigma_p(\tilde{H}) \cap J = \{\lambda_n : n \in \mathbb{N}, n \le N\}$ and for every eigenvalue of in the multiplicity of equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^{N}$.

(ii) $\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J$

(iii) $\sigma_{ac}(\tilde{H}) \cap J = 0$

Proof:

Let be any point in and if $N < \infty$, let $\lambda_n = \lambda_0$ for all $n \in N, n \in 2N$. Since for every regular point of and in particular for $\lambda = \lambda_0$, the dimension of $N(H^* - \lambda)$ equals the deficiency number of , we can choose by induction an orthogonal system $\{e_n\}_{n \in N}$ such that $e_n \in N(H^* - \lambda_n), n \in N$ and $e_j \in N(H^* - \lambda_0)$, $J = 2n - 1, n \in N$ as in the proof of proposition (5-3-3) we can show that there exist self-adjoint operators and in the Hilbert space

$H_0 = [e_{2n} : n \in \mathbb{N}, n \le N]$, $H'_0 = [e_n : n \in \mathbb{N}]$

Respectively such that

$e_{2n} \in N(M - \lambda_n), n \in \mathbb{N}, n \le N$, $e_{2n} \in N(M' - \lambda_n), n \in N$

$e_j \in N(M' - \lambda_0), J = 2n - 1, n \in \mathbb{N}$

and and are restrictions of the adjoint of . Obviously is also a restriction of the operator and therefore, $MH'_M, H_M \subset H'_M$.

Where the operators H_M and H_M' are defined by (66). Moreover the self-adjoint operator has a pure point spectrum $\delta_p(M) = \{\lambda_n : n \in \mathbb{N}, n \le N\}$ and for every eigenvalue of the multiplicity of equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^{N}$. By lemma (5-3-11) the operator H_M can be decomposed as $H_M = M \oplus G_0$. Where is a symmetric operator in the Hilbert space and is also a gap of . We have only to show that the deficiency number of is infinite. In fact, then corollary (5-3-18) yields that there exists a self-adjoint operator in such that is an extension of and

(i) $\delta_{sc}(G) \cap J = \overline{J_0} \cap J$, $\delta_{ac}(G) \cap J = \delta_{pp}(G) \cap J = 0$

Then obviously the operator $\tilde{H} = M \oplus G$ is a self-adjoint extension of with the required spectral properties. By lemma (5-3-1) the operator H_M' can be decomposed as $H_M' = M' \oplus G_0'$. Where is a symmetric operator in the Hilbert space and is also a gap of . Since the symmetric operator has a gap it has a self-adjoint extension . Let $\tilde{H} = M \oplus G'$. Since $M \subset H' = H'^*$ we have $H_M' = H'$. Since is a self-adjoint operator in and $\delta(M) \subset \bar{J}$ it follows from lemma (5-3-1) that

$$H' = M \oplus G'$$

for some self-adjoint operator in the Hilbert space . Since

$H'_{ej} = M'_{ej} = G'_{ej} = \lambda_0 e_j$, $j = 2n - 1, n \in \mathbb{N}$ the point in the gap of is an eigenvalue of with infinite multiplicity. Since obviously is a self-adjoint extension of this implies, by Krein's theorem, i.e., that the deficiency number of is infinite.