Chapter 4

Spectral Applications of Jacobic Matrices

We prove that a trace class of operators is continuous in the unit disc and give a prove of a result concerned to the commutator of two operators, with a prove of a result of a gap and a trace of a gab [71,72,81,113,117].

Section (4-1): Determinates and Jost Function Perturbation

We shall look at the spectral theory of Jacobi matrices that is infinite tridiagonal matrices,

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

With $a_j > 0$ and $b_j \in R$ we suppose that the entries of J are bounded that is $\sup_n |a_n| + \sup |b_n| < \infty$ so that J defines a bounded self-adjoint operator on $L^2(Z_+) = L^2(\{1,2,\ldots\})$. Let δ_j be the obvious vector in $L^2(Z_+)$, that is with components δ_{jn} which are 1 if $n = j$ and 0 if $n \neq j$.

The spectral measure was associated to J is one given by the spectral theorem for the vector δ_1. That is the measure μ defined by

$$m_\mu(E) \equiv \langle \delta_1, (J - E)^{-1} \delta_1 \rangle = \int \frac{d\mu(x)}{x - E}$$

(2)

There is a one-to-one correspondence between bounded Jacobi matrices and unit measures. Applying the Gram-Schmidt process to $\{x^n\}_{n=0}^{\infty}$ one gets orthonormal polynomials $p_n(x) = k_n x^n + \ldots$ with $k_n > 0$ and

$$\int p_n(x) p_m d\mu(x) = \delta_{nm}$$

(3)

These polynomials obey three-terms recurrence

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_{n+1} p_n(x) + a_n p_{n-1}(x)$$

(4)

Where a_n, b_n are the Jacobi matrix coefficients of the Jacobi matrix with spectral measure (and $p_{-1} = 0$) [23,43,85].

We made our choice to start numbering of at $n = 1$ so that we could have for the free Jost function , (well known with $z = e^{ik}$) and arrange for the Jost function to be regular at $z = 0$.

An alternate way of recovering from is the continued fraction expansion for the function $m_{\mu}(z)$ near infinity,

$$m_{\mu}(E) = \frac{1}{-E + b_1 - \left(a_1^2 / (-E + b + \ldots.)\right)} \quad (5)$$

One is especially interested in J's "close" to the free matrix with $a_n = 1$ and $b_n = 0$ that is

$$J_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \end{pmatrix}$$

(6)

Lemma (4.1.1) [128]:

Let be a Jacobic matrix and the corresponding spectral measure. The operator $J - J_0$ is Hilbert-Schmidt that is

$$2\sum (a_n - 1)^2 + \sum b_n^2 < \infty \quad (7)$$

If and only if has the following four properties

(i) (Blumenthal-Weyl Griterion) the support of is $[-2,2] \cup \{E_j^+\}_{j=1}^{N+} \cup \{E_j^-\}_{j=1}^{N-}$ where are each zero finite or infinite and $E_1^+ > E_{12}^+ > \ldots > 2$ and $E_1^- < E_{12}^- < \ldots < -2$ and if is infinite, then $Lim_{j \to \infty} E_j^{\pm} = \pm 2$.

(ii) (Quasi-Szego Condition) ,Let $\mu_{ac}(E) = f(E) dE$ where μ_{ac} is the Lebesque absolutely continuous component of . Then

$$\int_{-2}^{2} Log[f(E)]\sqrt{4-E^2}\,dE > -\infty$$

(8)

(iii) (Lieb-Thirring Bound) .

$$\sum_{j=1}^{N+} \left|E_j^{+} - 2\right|^{\frac{3}{2}} + \sum_{j=1}^{N-} \left|E_j^{-} + 2\right|^{\frac{3}{2}} < \infty \qquad (9)$$

(iv) (Normalization) $\int d\mu(E) = 1$

It is natural to approximate the true perturbation by one of finite rank. We define J_n as the semi-infinite matrix

$$J_n = \begin{bmatrix} b_1 & & a_1 & 0 & & & \\ a_1 & & b_2 & a_2 & & & \\ & \cdots & \cdots & & & & \\ & \cdots & b_{n-1} & a_{n-1} & & & \\ & & a_{n-1} & b_n & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & \cdots \end{bmatrix}$$

(10)

That is J_n has $b_m = 0$ for $m > n$ and $a_m = 1$ for $m \geq n+1$.

Notice that $J_n - J_0$ has rank at most n . We write the $n \times n$ matrix obtained by taking the first n rows and columns of (J or J_n) as $J_{n;F}$. The $n \times n$ matrix formed from J_0 will be called $J_{o;n;F}$. The semi-infinite matrices $J^{(n)}$ obtained from J by dropping the first n rows and columns of J , that is

$$J^{(n)} = \begin{bmatrix} b_{n+1} & a_{n+1} & 0 & \cdots \\ a_{n+1} & b_{n+2} & a_{n+2} & \cdots \\ 0 & a_{n+2} & b_{n+3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(11)

We need some facts about J_0 the free Jacobi matrix. Fax z with $|z| < 1$. Look for solutions of

$$u_{n+1} + u_{n-1} = (z + z^{-1})\, u_n, \; n \geq 2 \qquad (12)$$

As sequences with out any conditions at infinite or $n = 1$. The solutions of (12) a linear combination of the two “obvious” solutions given by

$$u_n^{\pm}(z) = z^{\pm n}$$

u^{+} is ℓ^2 at infinity since $|z| < 1$. The linear combination that obeys $u_u = (z + z^{-1})u_1$ as required by matrix ending at zero is

$$u_n^{(0)}(z) = z^{-n} - z^n \tag{13}$$

The Wronskion of $u^{(0)}$ and u^{+} is $z^{-1} - z$, we see that $(J_0 E(z))^{-1}$ has the matrix elements.

$-(z^{-1} - z)^{-1} u^{(0)}_{\min(n,m)}(z) u^{+}_{\max(n,m)}(z)$ either by a direct calculation or standard Green’s function formula. We have thus proven that

$$(J_0 - E(z))^{-1}_{nm} = -(z^{-1} - z)^{-1}\left[z^{|m-n|} - z^{m+n}\right]$$

(14)

$$= -\sum_{j=0}^{\min(m,n)-1} z^{1+|m-n|+2j}$$

(15)

Where the second comes from

$(z^{-1} - z)(z^{1-n} + z^{3-n} + \ldots + z^{n-1}) = z^{-n} - z^{n}$ by

$$|z| \le 1 \Rightarrow \left|(J_0 - E(z))^{-1}_{nm}\right| \le \min(n,m)|z|^{1+|m-n|} \tag{16}$$

And that while the operator $(J_0 - E(z))^{-1}$ becomes singular as $|z| \to 1$, the matrix elements do not; indeed they are polynomials in z [80,25]. We need an additional fact about J_0 .

Proposition (4.1.2) [128]:

The characteristic polynomial of $J_{0;n;F}$ is

$$\det(E(z) - J_{0;n;F}) = (z^{-n-1} - z^{n+1})/(z^{-1} - z) = U_n\left(\frac{1}{2}E(z)\right)$$

(17)

Where $U_n(\cos\theta)=\sin[(n+1)\theta]/\sin(\theta)$ is the Chebyshev polynomial of the second kind. In particular

$$\lim_{n\to\infty}\frac{\det\left[E(z)-J_{0;n;F}\right]}{\det\left[E(z)-J_{0;n;F}\right]}=z^{-j}$$

(18)

Proposition (4-1-3) [128]:

Let T_m be Chebushev polynomial (of the first kind):

$$T_m(\cos\theta)=\cos(m\theta) \tag{19}$$

Then $$T_r\left[T_m\left(\frac{1}{2}J_{0;n;F}\right)\right]:\begin{cases} n & m:2L(n+1);L\in Z \\ -\frac{1}{2}-\frac{1}{2}(-1)^m & \text{other wise}\end{cases} \tag{20}$$

In particular for fixed, once $n>\frac{1}{2}m-1$ the trace is independent of .

Proof:

As noted above the characteristic polynomial of $J_{0;n;F}$ is $\det\left[2\cos(\theta)-J_{0;n;F}\right]=\sin\left[(n+1)\theta\right]/\sin[\theta]$. This implies that the eigenvalues of $J_{0;n;F}$ are given by

$$E_n^{(k)}=2\cos\left(\frac{k\pi}{n+1}\right),k=1,\dots,n \tag{21}$$

So by (19), $T_m\left(\frac{1}{2}E_n^{(k)}\right)=\cos\left(\frac{km\pi}{n+1}\right)$. Thus

$$\begin{aligned}T_r\left[T_m\left(\frac{1}{2}J_{0;n;F}\right)\right]&=\sum_{k=1}^{n}\cos\left(\frac{km\pi}{n+1}\right)\\&=-\frac{1}{2}-\frac{1}{2}(-1)^m+\frac{1}{2}\sum_{k=-m}^{n+1}\exp\left(\frac{ikm\pi}{n+1}\right)\end{aligned}$$

The final sum is $2n+2$ if a multiple of is $2(n+1)$ and 0 if it is not. Let denote the Schatten classes of operator with norm $\|A\|_p=T_r\left(|A|^p\right)$. In particular

T_1 and T_2 are the trace class and Hilbert-Schmidt operators respectively. For each $A \in T_1$ one can define a complex-valued function $\det(1+A)$, so that

$$|\det(1+A)| \le \exp(\|A\|_1)$$

(22)

And $A \mapsto \det(1+a)$ is continuous.

$$|\det(1+A) - \det(1+B)| \le \|A-B\|_1 \exp(\|A\|_1 + \|B\|_1 + 1)$$

(23)

We will also use the following properties:

$$A, B \in T_1 \Rightarrow \det(1+A)\det(1+B) = \det(1+A+AB)$$

(24)

$$AB, BA \in T_1 \Rightarrow \det(1+AB) = \det(1+BA) \qquad (25)$$

$(1+A)$ is invertible if and only if $\det(1+A) \ne 0$

(26)

$z \mapsto A(z)$ analytic $\Rightarrow \det(1+A(z))$ analytic

(27)

If A is finite rank and P is a finite-dimensional self-adjoint projection

$$PAP = A \Rightarrow \det(1+A) = \det_{pH}(1_{pH} + PAP) \qquad (28)$$

Where $\det_{pH}$ is the standard finite-dimensional determinate.

For $A \in T_2, (1+A)e^{-A} - 1 \in T_1$, so one define

$$\det_2(1+A) = \det((1+A)e^{-A})$$

(29)

Then

$$|\det_2(1+A)| \le \exp(\|A\|_2^2)$$

(30)

$$|\det_2(1+A) - \det_2(1+B)| \le \|A-B\|_2 \exp((\|A\|_2 + \|B\|_2 + 1)^2)$$

(31)

and, if $A \in T_1$

$$\det{}_2(1+A) = \det(1+A)e^{-Tr(A)}$$

(32)

or

$$\det(1+A) = \det{}_2(1+A)e^{Tr(A)}$$

(33)

To estimate the T_p norms we use

Lemma (4-1-4) [128]:

If A is a matrix and $\|\cdot\|_p$ the Schatten T_p norm, then

(i) $\|A\|_2^2 = \sum_{n,m} |a_{nm}|^2$

(34)

(ii) $\|A\|_1 = \sum_{n,m} |a_{nm}|$

(35)

(iii) For any j and p $\sum_n |a_{n,n+j}|^p \le \|A\|_p^p$

(36)

Theorem (4-1-5) [128]:

Let $C_n = \max(|a_{n-1} - 1|, |b_n|, |a_n - 1|)$. For any $p \in [1,\infty)$.

$$\frac{1}{3}\left(\sum_n |C_n|^p\right)^{1/p} \le \|\delta J\|_p \le 3\left(\sum_n |C_n|^p\right)^{1/p} \qquad (37)$$

Proof:

The right side is immediate Holder's inequality , for trace ideals. The left most inequality follows from $\delta J = C^{\frac{1}{2}} U C^{\frac{1}{2}}$ and

$$\left(\sum_n |C_n|^p\right)^{1/p} \le \left(\sum_n |b_n|^p\right)^{1/p} + 2\left(\sum_n |a_n - 1|^p\right)^{1/p} .$$

With these preliminaries out of the way we can begin discussing the perterbution determinate L . For any J with $\delta J \in T_1$ (By (37) this is equivalent to $\sum |a_n - 1| + \sum |b_n| < \infty$).

$$L(z;J) = \det\left[J - E(z)(J_0 - E(z))^{-1}\right] \qquad (38)$$

For all $|z| < 1$. Since

$$(J - E)(J_0 - E)^{-1} = 1 + \delta J (J_0 - E(z))^{-1} \qquad (39)$$

The determinant in (39) is of the form $1 + A$ with $A \in T_1$.

Theorem (4-1-6) [128]:

Suppose $\delta J \in T_1$.

(i) $L(z; J)$ is analytic in $D = \{z / |z| < 1\}$.

(ii) $L(z; J)$ has a zero in D only at points z_j where $E(z_j)$ is an eigenvalue of J , and it has zero at all such points. All zero are simple.

(iii) If δJ is finite range then $L(z; J)$ is a polynomial and so has an analytic continuation to all of $\mathbb{C}$.

Proof:

(i) follows from (28)

(ii) If $E_0 = E(z_0)$ is not an eigenvalue of J , then $E_0 \notin \sigma(J)$ since $E : D \to \mathbb{C} \setminus [-2, 2]$ and $\sigma_{ess}(J) = [-2, 2]$. Thus $(J - E_0)/(J_0 - E_0)$ has an inverse (namely $(J_0 - E_0)/(J - E_0)$, and so by (27), $L(z; J) \neq 0$. Finally, if $E(z_0)$ is an eigenvalue of J are simple by a Wronskian a requirement, Simon. That L has a simple zero under these circumstances comes from the following [103.104]. If P is the project on to the eigenvector at $E_0 = E(z_0)$, then $(J - E)^{-1}(1 - P)$ has a removable singularity at E_0 . Define

$$C(E) = (J - E)^{-1}(1 - P) + P \qquad (40)$$

So

$$(J - E)C(E) = 1 - P + (E_0 - E)P \qquad (41)$$

Define

$$
\begin{aligned}
D(E) &\equiv (J_0 - E)C(E) \\
&\equiv -\delta J C(E) + (J - E)C(E) \\
&\equiv 1 - P + (E_0 - E)P - \delta J C(E) \\
&\equiv 1 + trace\ class
\end{aligned} \quad (42)
$$

Moreover

$$
D(E)[(J - E)/(J_0 - E)] = (J_0 - E)[1 - P + (E_0 - E)P](J_0 - E)^{-1} =
$$
$$
1 + (J_0 - E)[-P + (E_0 - E)P] \quad (J_0 - E)^{-1}
$$

Thus by (25) first and then (26),

$$
\begin{aligned}
\det(D(E(z))) &= \det\left(1 + (J_0 - E)[-P + (E_0 - E)P](J_0 - E)^{-1}\right) \\
&= \det(1 - p + (E_0 - E)P) \\
&= E_0 - E(z)
\end{aligned}
$$

Where we used (29) in the last step. Since $L(z;J)$ has a zero at and

$E_0 - E(z) = (z - z_0)\left[1 - \frac{1}{zz_0}\right]$ has a simple zero, $L(z;J)$ has a simple zero.

(iii) Suppose has range that is $N = \max\{n / |b_n| + |a_{n-1} - 1| > 0\}$ and let $P^{(N)}$ be the projection on to the span of $\{\delta_j\}_{j=1}^{N}$. As $P^{(N)}\delta J = \delta J$,

$\delta J (J_0 - E)^{-1} = P^{(N)} P^{(N)} \delta J (J_0 - E)^{-1}$. By (26)

$$
L(z;J) = \det\left(1 + P^{(N)} \delta J (J_0 - E)^{-1} P^{(N)}\right)
$$

Thus by (16), $L(z,J)$ is a polynomial if is finite range.

Lemma (4-1-7) [128]:

Let be diagonal positive trace class matrix. For $|z| < 1$, define

$$
A(z) = C^{\frac{1}{2}} (J_0 - E(z))^{-1} C^{\frac{1}{2}} \quad (43)
$$

Then, as Hilbert-Schmidt operator-valued function $A(z)$ extends continuously

to $\bar{D} \setminus [-1,1]$.

If $\sum_n nC_n < \infty$ (44)

It has a Hilbert-Schmidt continuation to .

Proof:

Let $A_{nm}(z)$ be the matrix elements of $A(z)$. It follows from $|z|\to 1$ and (14), (15) that

$$|A_{nm}(z)| \le 2C_n^{\frac{1}{2}}C_m^{\frac{1}{2}}|z-1|^{-1}|z+1|^{-1}$$

(45)

$$|A_{nm}(z)| \le \min(m,n)C_n^{\frac{1}{2}}C_m^{\frac{1}{2}}$$

(46)

and each $A_{n,m}(z)$ has a continuous extension to . It follows from (46) the dominated convergence theorem and

$$\sum_{n,m}(C_n^{1/2}C_m^{1/2}) = \left(\sum_n C_n\right)^2$$

That so long as stays a way from $\pm 1, \{A_{n,m}(z)\}_{n,m}$, is continuous in the

space $L^2((1,\infty)\times(1,\infty))$ so $A(z)$ is Hilbert-Schmidt and continuous on $\bar{D}\setminus[-1,1]$.

Moreover $\sum n^{\alpha}C_n < \infty$, and $\sum_{n,m}\left[\min(m,n)C_n^{\frac{1}{2}}C_m^{\frac{1}{2}}\right] \le \sum_{mn} mnC_nC_m = \left(\sum_n nC_n\right)^2$ imply

that $A(z)$ is Hilbert-Schmidt on if (45) holds.

Lemma (4-1-8) [5]:

Let be trace class. Then

$$t(z) = Tr\left((\delta J)(J_0 - E(z))^{-1}\right) \tag{47}$$

Has a continuous in $\bar{D}\setminus[-1,1]$. If

$$\sum_{n=1}^{\infty} n\left[|a_n - 1| + |b_n|\right] < \infty$$

holds, $t(z)$ can be continued to .

Proof:

$t(z)=t_1(z)+t_2(z)+t_3(z)$ where $t_1(z)=\sum b_n(J_0-E(z))^{-1}_{nn}$, $t_2(z)=\sum(a_n-1)$ $(J_0-E(z))^{-1}_{n+1,n}$

$t_3(z)=\sum(a_n-1)(J_0-E(z))^{-1}_{n,n+1}$

Since by (15), (17) $|(J_0-E(z))^{-1}_{nm}|\leq 2|z-1|^{-1}|z+1|^{-1}$ $\leq \min(n,m)$. The result is imm-ediate.

Theorem (4-1-9) [128]:

If δJ is trace class. $L(z;J)$ can be extended to a continuous function on

$\bar{D}\backslash[-1,1]$ with

$$|L(z;J)|\leq \exp\{C[\|\delta J\|_1+\|\delta J\|_1^2]|z-1|^{-2}|z+1|^{-2}\}$$

(48)

For universal constant C . If $\sum_{n=1}^{\infty} n[|a_n-1|+|b_n|]<\infty$ holds, and L can be extended to all of $\bar{D}$ with

$$|L(z;J)|\leq \exp\left\{\tilde{C}\left[1+\sum_{n=1}^{\infty} n[|a_n-1|+|b_n|]\right]^2\right\}$$

(49)

For a universal constant $\tilde{C}$.

Lemma (4-1-10) [128]:

Let C be a positive diagonal trace class operator.

Then $$\lim_{|x|\to 1, x\ real}(1-|x|)\left\|C^{\frac{1}{2}}(J_0-E(x))^{-1}C^{\frac{1}{2}}\right\|_1=0$$

(50)

Proof:

For $x<0, E(x)<-2$ and $J_0-E(x)>0$, while for $x>0, E(x)>2$ so $J_0-E(x)<0$. It follows that

$$\left\| C^{\frac{1}{2}}(J_0 - E(x))^{-1} C^{\frac{1}{2}} \right\|_1 = \left| Tr\left(C^{\frac{1}{2}}(J_0 - E(x))^{-1} C^{\frac{1}{2}} \right) \right| \leq \sum_n C_n \left| (J_0 - E(x))^{-1}_{nn} \right| \quad (51)$$

By (15) $(1+|x|)|(J_0 - E(x))^{-1}_{nn}| = 0$ and by (16) for each fixed ,

$$\lim_{|x| \to 1, x\ real} (1+|x|)|(J_0 - E(x))^{-1}_{nn}| = 0$$

The dominated convergence theorem proves (51).

Theorem (4-1-11) [128]:

$$\lim_{|x| \to 1, x\ real} Sup(1+|x|)\log|L(x;J)| \leq 0$$

(52)

Proof:

Use $\delta J = C^{\frac{1}{2}} \cup C^{\frac{1}{2}}$ and (25) to write

$L(x;J) = \det\left(1 + UC^{\frac{1}{2}}(J_0 - E(x))^{-1} C^{\frac{1}{2}}\right)$ and then (23) and (37) to obtain

$$\log|L(x;J)| \leq \left\| UC^{\frac{1}{2}}(J_0 - E(x))^{-1} C^{\frac{1}{2}} \right\|_1 \leq 3\left\| C^{\frac{1}{2}}(J_0 - E(x))^{-1} C^{\frac{1}{2}} \right\|_1$$

The result now follows from the lemma (4-1-10). We find the Taylor coefficients for $L(z;J)$ at $z=0$.

Theorem (4-1-12) [128]:

If δJ is trace class, then for each $n, T_n(J/2) - T_n(J_0/2)$ is trace class. Moreover, near $z=0$

$$Log[L(z;J)] = \sum_{n=1}^{\infty} C_n(J) z^n$$

(53)

Where $C_n(J) = -\frac{2}{n} Tr\left[T_n\left(\frac{1}{2}J\right) - T_n\left(\frac{1}{2}J_0\right) \right]$ (54)

In particular $C_1(J) = -Tr(J - J_0) = -\sum_{m=1}^{\infty} b_m$ (55)

$$C_2(J) = -\frac{1}{2} Tr(J^2 - J_0^2) = -\frac{1}{2}\sum_{m=1}^{\infty}[b_m^2 + 2(a_m^2 - 1)] \quad (56)$$

Proof:

To prove $T_n(J/2) - T_n(J_0/2)$ is trace class, we need only show that $J^m - J_0^m = \sum_{j=1}^{m-1} J^j \delta J^{m-1-j}$ is trace class and that's obvious. Let $\tilde{\delta J}_{n;F}$ be $\delta J_{n;F}$ extended to $L^2(Z_+)$ by setting it equal to zero matrix on $L^2(j \geq n)$. Let $\tilde{J}_{0;n}$ be J_0 with a_{n+1} set equal to zero. Then $\tilde{\delta J}_{n;F}(\tilde{J}_{0;n} - E)^{-1} \to \delta J (J_0 - E)^{-1}$. In trace norm, this means that

$$\det\left(\frac{J_{n;F} - E(z)}{J_{0,n;F} - E(z)}\right) \to L(z; J) \qquad (57)$$

The convergence is uniform on a small circle at $z = 0$ so the Taylor series coefficient converges imply (54) and (55).

Section (4-2): Data and Spectral

Theorem (4-2-1) [128]:

Suppose $M(z; J)$ is meromorphic in a neighborhood of $\bar{D}$ [25]. Then J and $J^{(1)}$ have finitely many eigenvalues out $[-2,2]$ and if

$$C_0(J) = \frac{1}{4\pi}\int_0^{2\pi} \log\left(\frac{\sin\theta}{\operatorname{Im} M(e^{i\theta}; J)}\right) d\theta - \sum_{j=1}^{N} \log\left|\beta_j(J)\right|$$

(58)

With J, then

$$C_0(J) = -\sum_{n=1}^{\infty}(a_1) + C_0(J^{(1)})$$

(59)

In particular, if δJ is finite rank, then the C_0 sum rule holds:

$$C_0(J) = -\sum_{n=1}^{\infty} \log(a_n)$$

(60)

Proof:

The eigenvalues, E_j of J out side $[-2,2]$ are precisely the poles of $m(E;J)$ and so the poles of $M(z;J)$ under $E_j = z_j + z_j^{-1}$ By

$$-M(z,J)^{-1} = -\left(z + z^{-1}\right) + b_1 + a_1^2 M\left(z, J^{(1)}\right)$$

the poles of $M(z;J^{(1)})$ are exactly the zeros of $M(z;J)$. Thus $\{\beta_j(J)^{-1}\}$ are the poles of $M(z;J)$ and $\{\beta_j(J^{(1)})^{-1}\}$ are its zeros. Since $g(0;J)=1$ by equation (5),

and $\log\left|f(0)\right| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(e^{i\theta}\right)\right|d\theta + \sum_{j=1}^{m}\log\left|z_j\right|$ and $\sum_{j=1}^{m}\log\left|z_j\right| \to \sum_{j=1}^{m}\log\left|z_j\right| - \sum_{j=1}^{m}\log\left|P_j\right|$

becomes

$$\frac{1}{2\pi}\int\log\left(\left|g\left(e^{i\theta};J\right)\right|\right)d\theta = -\sum_j\log\left(\left|\beta_j(J)\right|\right) + \sum_j\log\left(\left|\beta_j\left(J^{(1)}\right)\right|\right)$$

this formula implies (59). By

$$-M\left(e^{i\theta}, J\right)^{-1} = -\cos\theta + b_1 + a_1^2 M\left(e^{i\theta}, J^{(1)}\right)$$

if $M(z;J)$ is meromorphic in a neighborhood of $\bar{D}$, so is $M(z;J^{(1)})$. So we can iterate (59). The free function is

$$M(z;J_0) = z$$

(61)

By equation (15) with $m = n = 1$, so $C_0(J_0) = 0$ and thus if δJ is finite rank the remainder is zero after finitely many steps.

To get the higher-order sum rules we need to compute the power series for $\log(g(z;J))$ about $z=0$. For low-order, we can do this by hand. Indeed by (1) and (5) for $J^{(1)}$,

$$\begin{aligned} g(z;J) &= \left(z\left[\left(z + z^{-1}\right) - b_1 - a_1^2 z + O\left(z^2\right)\right]\right)^{-1} \\ &= \left(1 - b_1 z - \left(a_1^2 - 1\right)z^2 + O\left(z^3\right)\right)^{-1} \\ &= 1 + b_1 z + \left(\left(a_1^2 - 1\right) + b_1^2\right)z^2 + O\left(z^3\right) \end{aligned}$$

So since $\log(1+w) = w - \frac{1}{2}w^2 + O(w^3)$.

$$\log(g(z;J)) = b_1 z + \left(\frac{1}{2}b_1^2 + a_1^2 - 1\right)z^2 + O(z^3) \tag{62}$$

Therefore by mimicking the proof of theorem (4-2-1) we have

Lemma (4-2-2)[128]:

Suppose $M(z;J)$ is meromorphic in a neighborhood of $\bar{D}$. Let

$$C_n(J) = -\frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{\sin\theta}{InM\left(e^{i\theta}\right)}\right)\cos(n\theta)\,d\theta + \frac{1}{n}\left[\sum_j \beta_j(J)^n - \beta_j(J)^{-n}\right]$$

(63)

Then $C_1(J) = b_1 + C_1\left(J^{(1)}\right)$ (64)

$$C_2(J) = \left[\frac{1}{2}b_1^2 + \left(a_1^2 - 1\right) + C_2\left(J^{(1)}\right)\right] \tag{65}$$

If

$$P_2(J) = \frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{\sin\theta}{ImM\left(e^{i\theta}\right)}\right)\sin^2\theta\, d\theta + \sum_j F\left(e_j(J)\right) \tag{66}$$

With F given by

$$F(e) = \frac{1}{4}\left(\beta^2 - \beta^{-2}\right) - \log|\beta|^4$$

then writing $G(a) = a^2 - 1 - 2\log(a)$.

$$P_2(J) = \frac{1}{4}b_1^2 + \frac{1}{2}G(a_1) + P_2\left(J^{(1)}\right)$$

(67)

In particular, if δJ is finite rank, we have the sum rules C_1 , C_2 , P_2 . To go to order large than two we expand $\log(g(z:J))$ systematically as follows:

We begin by noting that (Cramer's rule)

$$g(z;J) = \lim_{n\to\infty} g_n(z;J)$$

(68)

Where

$$g_n(z;J) = \frac{z^{-1}\det\left(z + z^{-1} - J^{(1)}_{n-1;F}\right)}{\det\left(z + z^{-1} - J_{n;F}\right)}$$

(69)

$$= \frac{1}{1+z^2}\frac{\det\left(1 - E(z)^{-1}J_{n-1;F}\right)}{\det\left(1 - E(z)^{-1}J_{n-1;F}\right)}$$

(70)

Where we used $z(E(z)) = 1 + z^2$ and the fact that because the numerator is a matrix of order one less than the denominator we get an extra factor of $E(z)$.

Writing $F_j(x)$ for $\frac{2}{j}[T_j(0) - T_j(x/2)]$,

$$\log\, g_n(z:J) = -\log\left(1+z^2\right) + \sum_{j=1}^{\infty} z^j \left[Tr\left(F_j\left(J^{(1)}_{n-1;F}\right)\right) - Tr\left(F_j\left(J_{n;F}\right)\right)\right]$$

(71)

$$= -\log\left(1 - z^2\right) - \sum_{j=1}^{\infty}\frac{z^{2j}}{j}(-1)^j + \sum_{j=1}^{\infty}\frac{2z^j}{j}\left[Tr\left(T_j\left(\frac{1}{2}J_{n;F}\right)\right) - Tr\left(T_j\left(\frac{1}{2}J^{(1)}_{n-1;F}\right)\right)\right]$$

(72)

Where we picked in the first sum because $J_{n;F}$ has dimension one greater than $J^{(1)}_{n-1;F}$ so the $T_j(0)$ terms in $F_j(J_{n,F})$ and $J^{(1)}_{n-1;F}$ contribute differently. Notice

$$\sum_{j=1}^{\infty}\frac{z^{2j}}{j}(-1)^j = -\log\left(1+z^2\right)$$

So the first two terms and $g_n(z;J)$ converges to $g(z;J)$ in a neighborhood of $z=0$ it's Taylor coefficients converge. Thus

Proposition (4-2-3) [128]:

$$\alpha_j\left(J, J^{(1)}\right) = \lim_{n\to\infty}\left[Tr\left(Tj\left(\frac{1}{2}J_{n;F}\right)\right) - Tr\left(Tj\left(\frac{1}{2}J^{(1)}_{n-1;F}\right)\right)\right]$$

(73)

exists and for small

$$\log g(z;J) = \sum_{j=1}^{\infty}\frac{2z^j}{j}\alpha_j\left(J, J^{(1)}\right)$$

(74)

Theorem (4-2-4) [128]:

Suppose $M(z;J)$ is meromorphic in a neighborhood of . Let $C_n(J)$ be given

by $G(a) = a^2 - 1 - 2\log(a)$ and by $F(e) = \frac{1}{4}(\beta^2 - \beta^{-2}) - \log|\beta|^4$ .Then

$$C_n(J) = \frac{2}{n}\alpha_n\left(J, J^{(1)}\right) + C_n\left(J^{(1)}\right) \tag{75}$$

In particular, if is finite rank, we have the sum rule of (63)

Proof:

The only remaining point is why if finite rank, we have recovered the same sum rule is in (63). Iterating (75) when has rank in gives

$$C_n(J) = \frac{2}{n}\sum_{j=1}^{m}\alpha_n\left(J^{(J-1)}, J^{(j)}\right) = \lim_{L\to\infty}\frac{2}{n}\left[Tr\left[Tn\left(\frac{1}{2}J_{L;F}\right) - Tn\left(\frac{1}{2}J_{0,L-m;F}\right)\right]\right] \tag{76}$$

While (63) reads

$$C_n(J) = \frac{2}{n}Tr\left[T_u\left(\frac{1}{2}J\right) - T_u\left(\frac{1}{2}J_0\right)\right] = \lim_{L\to\infty}\frac{2}{n}\left[Tr\left[T_u\left(\frac{1}{2}J_{L;F}\right)\right] - Tr\left[T_u\left(\frac{1}{2}J_{0,L-m;F}\right)\right]\right] \tag{77}$$

That (76) and (77) are the same is a consequence of proposition (4-1-3).

In the sum rules C_0 and P_2 of most interest to us there appear two terms involving integrals of logarithms:

$$z(J) = \frac{1}{4\pi}\int_0^{2\pi}\log\left(\frac{\sin\theta}{\operatorname{Im}M\left(e^{i\theta}, J\right)}\right)d\theta \tag{78}$$

and

$$Q(J)=\frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{\sin\theta}{\operatorname{Im}M\left(e^{i\theta},J\right)}\right)\sin\theta d\theta \tag{79}$$

One should think of μ as related to the original spectral measure on $\sigma(J)=[-2,2]$ as

$$\operatorname{Im}M\left(e^{i\theta}\right)=\pi\frac{d\mu_{ac}}{dE}(2\cos\theta) \tag{80}$$

In which case (78), (79) can be rewritten

$$z(J)=\frac{1}{2\pi}\int_{-2}^{2}\log\left(\frac{\sqrt{4-E^2}}{2\pi d\mu_{ac}/dE}\right)\frac{dE}{\sqrt{4-E^2}} \tag{81}$$

and

$$Q(J)=\frac{1}{4\pi}\int_{-2}^{2}\log\left(\frac{\sqrt{4-E^2}}{2\pi d\mu_{ac}/dE}\right)\sqrt{4-E^2}\,dE$$

(82)

Our main equation is to view z and Q as function prove if $\mu_n\to\mu$ weakly, then $z(\mu_n)$ (resp. $Q(\mu_n)$) obeys

$$z(\mu)\le\liminf z(\mu_n);Q(\mu)\le\liminf Q(\mu_n) \tag{83}$$

That is, that z and Q are weakly lower semi continuous. This will let us prove sum rule-type inequalities in great generality.

Theorem (4-2-5) [128]:

If $J-J_0$ is compact and

(i) $$\sum_j\left|E_j^+-2\right|^{\frac{1}{2}}+\sum_j\left|E_j^--2\right|^{\frac{1}{2}}<\infty \tag{84}$$

(ii) $\limsup N\to\infty\ a_1,\dots,a_N>0$,

then Szego condition holds.

Is equivalent to the

Theorem (4-2-6) [128]:

Let J be a Jacobic matrix with $\sigma_{ess}(J)\subset[-2,2]$ and

$$\sum_k e_k(J)^{\frac{1}{2}}<\infty \tag{85}$$

$$\lim_{N\to\infty}\sup\sum_{j=1}^{N}\log\left(a_j\right) > -\infty \qquad (86)$$

Then

(i) $\sigma_{ess}(J) = [-2,2]$

(ii) The Szego condition holds that is $Z(J) < \infty$ with .

(iii) $\sigma_{ac}(J) = [-2,2]$; indeed, the essential support of is $[-2,2]$.

Proof:

Pick $N_1, N_2, \ldots$ (tending to) so that

$$\inf\left(\sum_{j=1}^{N_L}\log\left(a_j\right)\right) > -\infty \qquad (87)$$

And let J_{N_L} be given by (11). By theorem (4-1-24)

$$Z\left(J_{N_L}\right) \le -\sum_{j=1}^{N_L}\log\left(a_j\right) + \sum\log\left(\left|\beta_k\left(J_{N_L}\right)\right|\right) \qquad (88)$$

$$\le -\inf_{L}\sum_{j=1}^{N_L}\log + \sum\log\left(\left|\beta_k\left(J_{N_L}\right)\right|\right) + 2\log\left(\left|\beta_1(J)\right| + 2\right)$$

Where in $C_n(A,B) = -\frac{2}{n}Tr\left[T_n\left(\frac{1}{2}A\right) - T_n\left(\frac{1}{2}B\right)\right]$ and the fact at solving $e_1(J)+1 = \tilde{\beta}_0 + \tilde{\beta}_0^{-1}$ $\left(i.e. 1+\beta_1+\beta_1^{-1} = \tilde{\beta}_0 + \tilde{\beta}_0^{-1}\right)$ has $\tilde{\beta}_0 \le \beta_1(J)+2$. For later purposes we note that if $\left|b_n(J)\right| + \left|a_n(J)-1\right| \to 0$. Now use (88) to see that

$$Z(J) \le \liminf Z\left(J_{N_L}\right) < \infty$$

This proves (ii). But (ii) implies $\frac{d\mu_{ac}}{dE} > 0$ a.e . On $E \in [-2,2]$, that is $[-2,2]$ is the essential support of . That proves (iii). (i) is then immediate.

Theorem (4-2-7) [128]:

If $J - J_0$ is in trace class, that is

$$\sum_n\left|a_n - 1\right| + \sum_n\left|b_n - 1\right| < \infty \qquad (89)$$

Then the Szego condition holds.

Proof:

The prove of Szego condition holds under the slightly stronger hypothesis

$$\sum_n (\log n)|a_n - 1| + \sum_n (\log n)|b_n| < \infty \quad .$$

We need only check that $J - J_0$ trace class implies (85) and (86). The finiteness of (85) follows from abound of Hundertmark–Simon,

$$\sum \left[|e_k(J)| |e_k(J) + 4| \right]^{\frac{1}{2}} \le \sum_n |b_n| + 2|a_n - 1|$$

Where $e_k(J) = |E^{\pm}| - 2$ so $|e||e+4| = (E^{\pm})^2 - 4$.

Condition (86) is immediate for as is well-known $a_j > 0$ and $\sum (|a_j| - 1) < \infty$

implies $\prod a_j$ is absolutely convergent that is $\sum |\log(a_j)| < \infty$.

Theorem (4-2-8) [128]:

If J is a Jacobi matrix with $a_n \equiv 1$ and $\sum_n |e_n(J)|^{\frac{1}{2}} < \infty$ then $\delta_{ac}(J) = [-2, 2]$

.

Is equivalent to the

Corollary (4-2-9) [128]:

A discrete half-line Schrodinger operator (i.e., $a_n \equiv 1$) with $\delta_{ess}(J) \subset [-2, 2]$

and $\sum e_n(J)^{\frac{1}{2}} < \infty$ has $\delta_{ac} = [-2, 2]$

Lemma (4-2-10) [128]:

If $\delta(J) \subset [-2, 2]$ and

(i) $\limsup_N \sum_{n=1}^{N} \log(a_n) > -\infty$ then the Szego condition holds. If $\delta(J) \subset [-2, 2]$

and either (i) or the Szego condition holds , then

(ii) $\sum_{n=1}^{\infty} (a_n - 1)^2 + \sum_{n=1}^{\infty} b_n^2 < \infty$,

(iii) $\lim_{N \to \infty} \sum_{n=1}^{N} b_n$ exists (and is finite).

In particular if $\sigma(J)\subset[-2,2]$, then (i) implies (ii)-(iv). Next, we deduce some additional aspects of lemma.

Corollary (4-2-11) [128]:

If $\sigma_{ess}(J)\subset[-2,2]$ and (85), (86) holds, then $J-J_0\in\tau_2$, that is

$$\sum b_n^2+\sum(a_n-1)^2<\infty \tag{90}$$

Proof (4-2-12) [128]:

(90) holds if $\sum_k e_k(J)^{\frac{3}{2}}<\infty$ and $Q(J)$ (given by (24)) is finite. By (85) and $e_k(J)^{\frac{3}{2}}\le e_1(J)e_k(J)^{\frac{1}{2}}$, we have that $\sum e_k(J)^{\frac{3}{2}}<\infty$.

Moreover, $Z(J)<\infty$ (i.e., theorem (4-2-6) implies $Q(J)<\infty$.

For in any event $\int \operatorname{Im} M.d\theta<\infty$ implies

Thus

$$\begin{aligned}&\int_0^{2\pi}\log_-\left(\frac{\sin\theta}{\operatorname{Im} M}\right)\sin^2(\theta)d\theta<\infty\\ &\Rightarrow\int_0^{2\pi}\log_+\left(\frac{\sin\theta}{\operatorname{Im} M}\right)\sin^2\theta d\theta<\infty \quad .\\ &\Rightarrow Q(J)<\infty\end{aligned}$$

We will start with $\sum(a_n-1)$. Because $\sum(a_n-1)^2<\infty$, it is easy to see that $\sum(a_n-1)$ is conditionally convergent it and only if $\sum\log(a_n)$ is conditionally convergent. By (88) and the fact that $J-J_0$ is compact, we have:

Proposition (4-2-13) [128]:

If (86) holds and $\sigma(J)\subset[-2,2]$, that is no eigenvalues out side $[-2,2]$, then

$$Z(J)\le-\limsup\left[\sum_{j=1}^{N}\log\left(a_j\right)\right] \tag{91}$$

We are heading towards a proof that

$$Z(J) \geq -\limsup\left[\sum_{j=1}^{N}\log(a_j)\right] \quad (92)$$

From which it follows that the limit exists and equals $Z(J)$.

Lemma (4-2-14) [128]:

If $\delta(J) \subset [-2,2]$, then $\log[z^{-1}M(z;J)]$ lie in every $H^p(D)$ space $p<\infty$. In particular $Z^{-1}M(z;J)$ is a Nevanlinna function with no singular inner part

Proposition (4-2-15) [128]:

Let $\delta(J) \subset [-2,2]$. Suppose $Z(J)<\infty$. Let (by given by (61) and (66) (where the $\beta(J)$ terms are absent). Sum rules in particular .

$$Z(J) = -\log(a_1) + Z(J)^{(1)} \quad (93)$$

$$C_1(J) = b_1 + C_1(J)^{(1)} \quad (94)$$

Theorem (4-2-16) [128]:

If J is such that $Z(J)<\infty$ and $\delta(J) \subset [-2,2]$, then

(i) $\lim_{N\to\infty}\sum_{j=1}^{N}\log(a_j)$ exists.

(ii) The limit in (i) is $-Z(J)$.

(iii) $$\lim_{n\to\infty} Z(J)^{(1)} = 0 \quad (-Z(J)) \quad (95)$$

Proof:

By (93) $$Z(J) + \sum_{j=1}^{n}\log(a_j) = Z\left(J^{(n)}\right) \quad (96)$$

Since $J - J_0 \in L_2, \mu_{J^{(n)}} \to \mu_{J_0}$ weakly

$\text{limit inf } Z\left(J^{(n)}\right) \geq 0$, or by (96)

$$\text{limit inf}\left[\sum_{j=1}^{n}\log(a_j)\right] \geq -Z(J) \quad (97)$$

But (87) says

$$\text{limit sup}\left[\sum_{j=1}^{n}\log\left(a_j\right)\right]\le -Z\left(J\right)$$

Thus the limit exists and equal $Z(J)$, proving (i) and (ii).

Moreover, by (96) (i) and (ii) imply (iii).

Lemma (4-2-17) [128]:

Let $d\mu$ be a probability measure and suppose $f_n \ge 0$, $\int f_n d\mu \le 1$ and

$$\lim_{n\to\infty}\int \log(f_n)\,d\mu = 0 \tag{98}$$

Then

$$\int\left|\log(f_n)\right|d\mu + \int\left|f_n - 1\right|d\mu \longrightarrow 0 \tag{99}$$

Proof:

Let $H(y) = -\log(y) - 1 + y$ (100)

Then

(i) $H(y) \ge 0$ for all y.

(ii) $\inf_{|y-1|\ge\varepsilon} H(y) > 0$

(iii) $H(y) \ge \frac{1}{2}y$ if $y \ge 8$.

(i) is concavity of $\log(y)$, (ii) is strict concavity, and

(ii) holds because $-\log y - 1 + \frac{1}{2}y$ is monotone on $(2,\infty)$ and >0 at $y=8$ since $\log(8)$ is slightly more than 2.

Since $\int (f_n - 1)\,d\mu \le 0$, (i) implies that

$$\int (f_n - 1)\,d\mu(x) \longrightarrow 1 \tag{101}$$

and

$$\lim_{n\to\infty}\int H\left(f_n(x)\right)d\mu(x) \longrightarrow 0 \tag{102}$$

Since $H \ge 0$ and the above imply $f_n \to 1$ in measure.

$$\mu\left(\left\{x / \left|f_n(x) - 1\right| > \varepsilon\right\}\right) \longrightarrow 0 \tag{103}$$

By (i), (iii), and (102)

$$\int_{f_n(x)>8} |f_n(x)|\, d\mu \to 0 \tag{104}$$

Now (103), (104) imply that

$$\int |f_n(x) - 1|\, d\mu \to 0$$

And this together with (102) implies $\int |\log(f_n)|\, d\mu = 0$.

Proposition (4-2-18) [128]:

Suppose $Z(J) < \infty$ and $\delta(J) \subset [-2, 2]$. Then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \log \left| \left(\frac{\sin\theta}{\operatorname{Im} M\left(e^{i\theta}, J^{(n)}\right)} \right) \right| d\theta = 0 \tag{105}$$

Proof:

By (95) the result is true it $\|$ is dropped. Thus it suffices to show

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \log_{-} \left(\frac{\sin\theta}{\operatorname{Im} M\left(e^{i\theta}, J^{(n)}\right)} \right) d\theta = 0$$

Or equivalently

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \log_{+} \left(\frac{\operatorname{Im} M\left(e^{i\theta}, J^{(n)}\right)}{\sin\theta} \right) d\theta = 0 \tag{106}$$

Now, let $d\mu_0(\theta) = \frac{1}{\pi} \sin^2\theta\, d\theta$ and $f_n(\Theta) = (\sin\Theta)^{-1} \operatorname{Im} M\left(e^{i\Theta}, J^{(n)}\right)$.

By (102)

$$\int_{-\pi}^{\pi} f_n(\theta)\, d\mu_0 \le 1 \tag{107}$$

corollary (4-2-11), which implies $\left\| J^{(n)} - J_0 \right\|_2^2 \to 0$, $\int \log(f_n(\Theta))\, d\mu_0(\Theta) \to 0$, so by lemma (4-2-16), we control $|\log|$ and so $\log_{+}$;

That is

$$\log_+\left(\frac{a}{b}\right)\leq \log_+(a)+\log_-(b)=2\log_+\left(a^{\frac{1}{2}}\right)+\log_-(b)$$
$$\leq 2a^{\frac{1}{2}}+\log_-(b)$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\log_+\left(\frac{\operatorname{Im} M\left(e^{i\theta},J^{(n)}\right)}{\sin\theta}\right)\sin^2\theta d\theta=0 \qquad (108)$$

Thus, to prove (106) we need only prove

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{\substack{|\theta|<\varepsilon \\ or\ |\Pi-\theta|<\varepsilon}}\log_+\left(\frac{\operatorname{Im} M\left(e^{i\theta},J^{(n)}\right)}{\sin\theta}\right)d\theta=0$$

(109)

To do this use

With $a=\sin\theta\operatorname{Im} M\left(e^{i\theta},J^{(n)}\right)$ and $b=\sin^2\theta$. The contribution of $\log_-(b)$ in (108) is integrable and ε-independent, and so goes to zero as $\varepsilon\to 0$. The contribution of the $2a^{\frac{1}{2}}$ term is, by the Schwartz inequality bounded by

$$(4\varepsilon)^{\frac{1}{2}}\left(4\int_{-\pi}^{\pi}f_n(\theta)\,d\mu_0(\theta)\right)^{\frac{1}{2}}$$

Also goes to zero as $\varepsilon\to 0$. Thus (108) is proven

Corollary (4-2-19) [128]:

If δW is finite rank, then $(1-z^2)L(z;W)$ is a polynomial and in particular, $L(z,W)$ is a rational function.

Theorem (4-2-20) [128]:

If δW is finite rank, then

$$C_0:\frac{1}{2\pi}\int_0^{2\pi}\log\left|\left(e^{i\theta};W\right)\right|d\theta=\sum_{j=1}^{N(W)}\log\left|\beta_j(W)\right|$$

(110)

$$C_1:\frac{1}{\pi}\int_0^{2\pi}\log\left|L\left(e^{i\theta};W\right)\right|\cos(n\theta)=\frac{1}{n}\sum_{j=1}^{N(W)}\left[\beta_j^n-\beta_j^{-n}\right]-\frac{2}{n}Tr\left(T_n\left(\frac{1}{2}W\right)-T_n\left(\frac{1}{2}W_0\right)\right)$$

for ≥ 1.

The final element of our proof is an inequality for $L(e^{i\theta};W)$ that depends on what a physicist would call conservation of probability.

Proposition (4-2-21) [128]:

Let δW be trace class. Then for all $\theta = 0$,

$$|L(e^{i\theta};W)| \geq \prod_{j=-\infty}^{\infty} a_j$$

(111)

Proof:

As above we can suppose that δW is finite range. Choose R so that all non zero matrix elements of δW have indices lying with in $(-R, R)$. By (111), is equivalent to

$|\alpha_L| \geq 1$

Where L is given by

$$\lim_{n\to\infty} z^{-n} u_n^{+}(z; w) = 1 \tag{112}$$

Since $u_n^{+}(z; W)$ is real for z real we have

$u_n^{+}(z; W) = \overline{u_n^{+}(z; W)}$.

Thus for $z = e^{i\theta}$, $\theta \neq 0, \pi$ and $n < -R$,

$u_n^{+}(e^{i\theta}; W) = \alpha_L(e^{i\theta}) e^{in\theta} + \beta_L(e^{i\theta}) e^{-in\theta}$

$u_n^{+}(e^{-i\theta}; W) = \overline{\alpha_L(e^{i\theta})} e^{-in\theta} + \overline{\beta_L(e^{i\theta})} e^{in\theta}$

Computing the Wronskian of the left-hand sides for $n > R$, where $u_n^{+} = z^n$ and then the Wronskian of the right-hand sides for $n < -R$, we find

$i(\sin\theta) = i(\sin\theta)\left[|\alpha_L|^2 - |\beta_L|^2\right]$

Or since $\theta \neq 0, \pi$, $|\alpha_L|^2 = 1 + |\beta_L|^2$

(113)

Form which $-M\left(e^{i\theta}, J\right)^{-1} = -2\cos\theta + b_1 + a_1^2 M\left(e^{i\theta}, J^{(1)}\right)$ is obvious.

Theorem (4-2-22) [128]

Let W be a whole-line operator with $a_n = 1$ and $\sigma(W) \subset [-2, 2]$. Then

$W = W_0$, that is, $b_n = 0$. The proof works if $\limsup_{\substack{n\to\infty \\ m\to\infty}} \left[\sum_{j=-n}^{m} \log(a_j)\right] \geq 0$

Proof:

Let

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \log\left|\left(e^{i\theta}; W^{(n)}\right)\right| d\theta = 0$$

(114)

Since $a_n = 1$, (110) implies $\log|L(e^{i\theta}; W^{(n)})| \to 0$, and so (113) implies

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \cos(2\theta)\log\left|L\left(e^{i\theta}, W^{(n)}\right)\right| d\theta = 0$$

(115)

By $\log|L(z;w)| = \sum_{n=1}^{\infty} C_n(w) Z^n$, , $a_m = 1$, we see

$$\lim_{n\to\infty} \sum_{|j|<n} b_j^2 = 0$$

Which implies $b = 0$.

Commutation of Certain Operators and Pertaining Estimates

Section (4-3):

Let $a = \{a_k\}$, $a_k > 0$, $b_k \in R$ and

$$J = J(1,0) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & . \\ 0 & . & . \end{bmatrix} \tag{116}$$

Be a Jacobi matrix . The free (or chebyshev) Jacobi matrix is given by

$$J(1,0) = J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & . \\ 0 & . & . \end{bmatrix}$$

The scalar spectral measure $\delta = \delta(J)$ of is defined by the relation

$$\left((J - Z)^{-1} e_0, e_0\right) = \int_R \frac{d\delta(x)}{x - z}$$

(117)

Where $z \in C \setminus$. The density of the absolutely continuous component of is denoted by [3,6,99]

we consider which are compact perturbations of . In this case the absolutely continuous spectrum $\delta_{ac}(J)$ coincides with $[-2,2]$, and the discrete spectrum lies on two sequences $\{x_j^{\pm}\}$ with properties $[-2,2]$ or $x_j^- \to -2, x_j^- < -2, x_j^+ \to 2, x_j^+ > 2$.

The results we obtain so called sum rules [127].

Theorem (4-3-1) [134]:

Let $J = J(a,b)$ be a Jacobi matrix. Then $J - J_0$ is Hilbert-Schmidt if and only if

(i) $\int_{-2}^{2} \log \delta'(x)\left(4 - x^2\right)^{\frac{1}{2}} dx > \infty$ (ii) $\sum_j (a_j - 1)^2 + \sum_j b_j^2 < \infty$.

Theorem (4-3-2) [134]:

Let $J = J(a,b)$ is a Jacobi matrix and $J - J_0 \in \delta_3$. Then far a fixed m ,

$\sum_j (\alpha_j + \ldots + \alpha_{j+m-1})^2 + \sum_j (b_j + \ldots + b_{j+m-1})^2 < \infty$.

The space $L^p(Z_+), p \geq 1$ are denoted by ℓ^p . We also set [illegible] and

to be the unit disk $\{\zeta : |\zeta| < 1\}$ and the unit circle $\{\zeta : |\zeta| = 1\}$ correspondingly.

Some fact on one-sided Jacobi matrices, let $J = J(a,b)$ be a Jacobi matrix defined in (116) and acting on $L^2(Z_+)$. Let $\{e_k\}_{k \in Z_+}$ be the standard basis in the space. It is easy to see the so-called Weyl function.

$$M(z) = \left((J - z)^{-1} e_0, e_0\right)$$

associated to J and admit representation (117) with a measure $\delta = \delta(J)$. The measure is called a spectral measure of J and is unique up to normalization. We have

$$\delta = \frac{1}{\pi} weak - \lim_{y \to 0^+} \operatorname{Im}(. + iy)$$

and moreover $\delta'(x) = \frac{1}{\pi} \lim_{y \to 0^+} \operatorname{Im} M(x + iy)$ for almost all $x \in \text{i}$.

Suppose that rank $(J - J_0) < \infty$.

The function M is meromorphic on $\pounds^{-} \backslash [-2,2]$. It is often convenient to uniformize the domain with the help of maps $\zeta(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right)$, $z \in \pounds^{-}[-2,2]$

and $z(\zeta) = \zeta + \frac{1}{\zeta}$, $\zeta \in D$. It is clear that

$\zeta : £^{-} \backslash [-2,2] \to D$, $z : D \to £^{-} \backslash [-2,2]$ and the maps are mutually inverse if and only if

(i) $\int_{-2}^{2} \log \delta(x) w_m(x) dx > -\infty$ (ii) $\sum_j \left(x_j^{\pm 2} - 4\right)^{\frac{3}{2}} < \infty$

Where $w_m(x) = (4-x)^{-\frac{1}{2}}\left(1 - T_m^2(x/2)\right)$ and is the m-th chebyshev polynomial. First it turns out that computations pertaining to sum rules are much simpler on the domain $£^{-} \backslash [-2,2]$ than on the unit disk $[\zeta \mid \leq -1]$.

The second commutations of operators and bounds coming from relations between classes of compact operators [136]. We set $\delta a = \{a_{k+1} - a_k\}$ and $\gamma_k(a) = \{(\gamma_k(a))_j\}$

Where $(\gamma_k(a))_j = \alpha_j^k - \alpha_j \alpha_{j+1} \cdots \alpha_{j+(k-1)}$

(118)

Corollary (4-3-3) [134]:

Let $J = J(a,b)$. Then if $b \in L^{m+1}$ and $\partial a \in L^2$ the relation (i) hold true [132]. And let us consider a generalized eigenvector $u(\zeta) = \{u_j(\zeta)\}$ of (that is, $Ju(\zeta) = \left(\zeta + \frac{1}{\zeta}\right) u(\zeta)$) with the property

$$\lim_{j \to +\infty} \zeta^{-j} u_j(\zeta) = 1$$

The vector and the function are called the Jost solution and Jost function respectively, we have the following lemma.

Lemma (4-3-4) [134]:

Let rank $(J - J_0) < \infty$. Then $u_0(z) = u_0(\zeta(z)) = \frac{1}{A_0'} \det(J_0 - z)^{-1}$, where $A_0' = \prod_j a_j$, and $z \in £^{-} \backslash [-2,2]$. Furthermore $|u_0(x)|^2 = \frac{\sqrt{4 - x^2}}{\delta(x)}$.

Almost every where on $[-2,2]$.

Lemma (4-3-5) [134]:

Let $J = J(a,b)$ and $a, b \in L^{m+1}$. Then

$$\sum_j \left(x_j^{\pm 2} - 4\right)^{m+\frac{1}{2}} \le C_0 \left(\sum_j |a-1|^{m+1} + \sum_j |b|^{m+1} \right)$$

With a constant C_0 depending on m.

Let $a = \{a_k\}_{k \in z}$, $b = \{b_k\}_{k \in z}$, and $J = J(a,b)$ be a Jacobi matrix, acting on $L^2(Z)$. We define a 2×2 - matrix-valued function M with the help of the formula $M(z) = \zeta (J - z)^{-1} \zeta^*$, and consequently, it can be represented as

$M(z) = \int_R \frac{d\sum(x)}{x - z}$, where Σ is a 2×2 -matrix-valued measure Σ. The density of it's absolutely continuous component is denoted by Σ', [100,150]. Let $J_0 = J(1.0)$ where 1 and 0 are two-sided sequences of 1's is and o's Assume that rank $(J - J_0) < \infty$. In this case absolutely continuous spectrum $\delta_{ac}(J)$ of J coincides with $[-2,2]$. The discrete spectrum of J lies on sequences $\{x_j^{\pm}\}$ with properties $x_j^{+} \to 2, x_j^{+} > 2$, and $x_j^{-} \to 2, x^{-} < -2$.

Consider the Jost solution $u_{\pm}$ satisfying the relations

$$J u_{\pm}(\zeta) = \left(\zeta + \frac{1}{\zeta} \right) u_{\pm}(\zeta) \quad , \quad \lim_{J \to \pm\infty} \zeta^{\pm j} u_{\pm}(\zeta) = 1$$

Where $\zeta \in D \setminus [-1,1]$. It is not difficult to see that vectors $u_{\pm}(1/\zeta)$, $\zeta \in T$, are linearly independent and we have for some functions $s, s_{\pm}$ that

$u_{\pm}(\zeta) = s(\zeta) u_{\pm}(1/\zeta) + s_{\pm}(\zeta) u_{\pm}(\zeta)$. Where $\zeta \in \Pi \setminus [-1,1]$.

Lemma (4-3-6) [134]:

Let $J = J(a,b), a_0 = 1$, and s be the transmission coefficient of J. Then

$$\det\left(2\pi \sum{}'(x) \right) = \left| s(\zeta(x)) \right|^2$$

For almost all $x \in [-2,2]$. The theorem suggests that the Jost function u for one-side Jacobi matrices is a right counterpart of transmission coefficient for two-sided Jacobi matrices. Let K be a compact operator on a separable

Hilbert space H . The singular values $\{s_k(A)\}, s_k(A) \to 0$, are defined as $s_k(A) = \lambda_k(A^*A)^{\frac{1}{2}}$, where $\lambda_k(A^*A)$ is the k-th eigen value of operator A^*A .The Schatten-von Neumann classes are given by the relation

$$S_p = \left\{ A - compact : \|A\|_{sp}^{p} = \sum_k \delta_k (A)^{\rho} < \infty \right\}$$

Where $p \geq 1$. In particular S_1 and S_2 describe classes of nuclear and Hilbert-Schmidt operator, respectively. The sets S_p are ideals, that is

$$\|ABC\|_{Sp} \leq \|B\| \|A\|_{sp} \|C\|$$

For any bounded operators B , C , on H and $A \in S_p$. We also have the Holder inequality for $S_p's$, i.e.,

$$\|A_1 \cdots A_n\|_{S_1} \leq \|A_1\|_{Sp_1} \cdots \|A_n\|_{spn}$$

Where $A_j \in S_{p_j}$, $j = 1, \ldots, n$ and $\sum_j 1/P_j = 1$. Suppose now that A, B are some operators on H . We suppose A to be of finite rank. Let, $\{e_j\}$ be a fixed in the space. By TrA we mean $trA = \sum_j (Ae_j, e_j)$, and clearly $|trA| \leq \|A\|_{s_1}$. We define the commutator $[A, B]$ of A and B by

$$[A, b] = AB - BA$$

Lemma (4-3-7) [5]:

Let A, B be some operators. Then

$$[A^k, b] = \sum_{j=0}^{k-1} A^{k-1-j} [A, B] A^j$$

(119)

Proof:

The proof of the lemma immediately follow by induction from the equality

$$[AB, C] = A[B, C] + [A, C]B$$

Or by the induction of the equation (119), we assume the equation (119) true for $j = 1, k, k - 1$ we get

Since $j = 1$ we have in right-hand side

$$\sum_{j=0}^{k-1} A^{k-1-j}[A,B]A^{j} = A^{k-2}[A,B]A = A^{k-1}[A;B]$$

(1)`

And $j = k$ implies that

$$\sum_{j=0}^{k-1} A^{k-1-j}[A,B]A^{j} = A^{k-1-k}[A,B]A^{k} = A^{k-1}[A,B]$$

(2)`

also $j = k - 1$ we get

$$\sum_{j=0}^{k-1} A^{k-1-j}[A,B]A^{j} = A^{-2}[A,B]A^{k+1} = A^{k-1}[A,B]$$

(3)`

From (1)`, (2)`, and (3)` the induction is true so the proof is complete.

Of course, the lemma also implies that

$$[A,B^{k}] = \sum_{j=0}^{k-1} B^{k-1-j}[A,B]B^{j}$$

(120)

We suppose first that rank $(J - J_0) < \infty$. We have the following proposition.

Proposition (4-3-8) [134]:

Let be the Jost function of and be a real entire function. Then

$$\int_{x_1^-}^{x_1^+} \lambda(x)dx = \operatorname{Re} s_{\infty}\left\{\frac{p(z)}{\sqrt{z^2-4}}\log u_0(z)\right\} \tag{121}$$

Where is a function defined by relations

$$\lambda(x) = \begin{cases} \dfrac{p(x)}{\sqrt{x^2-4}}\lambda_0(x), & x \notin [-2,2] \\ -\dfrac{p(x)}{2\pi\sqrt{4-x^2}}\log\dfrac{\sqrt{4-x^2}}{\delta'(x)}, & x \in [-2,2] \end{cases}$$

and $$\lambda(x) := \begin{cases} \#\{x_j^+ : x_j^+ > x\}, & x > 2 \\ \#\{x_j^+ : x_j^+ < x\}, & x < -2 \\ 0, & x \in [-2,2] \end{cases}$$

Proof:

Let $F(z) = \frac{p(z)}{\sqrt{z^2-4}} \log u_0(z)$.

We choose the branch of $\sqrt{z^2-4}$ with the properties $\sqrt{z^2-4} > 0$, when $z < -2$, $\sqrt{z^2-4} \in i\,\mathbb{R}_+$ when $z \in [-2,2]$, and $\sqrt{z^2-4} < 0$, when $z > 2$ we readily see that the function also has well defined boundary values on the upper and lower edges of $[x_1^-, x_1^+]$. We denote them by $F_\pm$, respectively.

For a sufficiently big $r > 0$, we have by definition of the residue at $z = \infty$ that

$$-\frac{1}{2\pi i} \int_{z:|z|=r} F(z)\,dz = \operatorname{Res}_\infty F(z)$$

We have at the left-hand side of the equality

$$-\frac{1}{2\pi i} \int_{z:|z|=r} F(z)\,dz = \frac{1}{2\pi i}\left(\int_{x_1^-}^{x_1^+} F(x)_+\,dx + \int_{x_1^-}^{x_1^+} F(x)_-\,dx \right)$$

Since $F(x)_- = \overline{F(x)_+}$, $x \in [x_1^-, x_1^+]$ we continue as

$$-\frac{1}{2\pi i} \int_{x_1^-}^{x_1^+} (F(x)_+ F(x)_-)\,dx = \frac{1}{\pi} \int_{x_1^-}^{x_1^+} \operatorname{Im} F(x)_+\,dx$$

(122)

We note that $(\sqrt{z^2-4})_+ = i\sqrt{4-x^2}$ for $x \in [-2,2]$, and by lemma (4-3-4).

$$\operatorname{Re} \log u_0(x)_+ = \log|u_0(x)| = \frac{1}{2} \log \frac{\sqrt{4-x}}{\delta(x)}$$

Furthermore $$\operatorname{Im}(\log u_0(x)_+) := \pi \begin{cases} -\#\{x_j^+ : x_j^+ > x\}, x > 2 \\ \#\{x_j^- : x_j^- < x\}, x < -2 \end{cases}$$

Consequently

$$\operatorname{Im}\left(\frac{p(z)}{\sqrt{z^2-4}}\log u_0(z)\right)_+ = \begin{cases} -\frac{p(x)}{2\sqrt{4-x^2}}\log\frac{\sqrt{4-x^2}}{\delta'(x)}, x\in[-2,2] \\ \frac{\pi p(x)}{\sqrt{x^2-4}}\lambda_0(x) \quad x\notin[-2,2] \end{cases}$$

Plugging this expression in (7), we obtain

$$\operatorname{Re} s_\infty F(z) = -\frac{1}{2\pi}\int_{-2}^{2}\frac{p(x)}{\sqrt{4-x^2}}\log\frac{\sqrt{4-x^2}}{\delta''(x)}dx + \int_{x_1^-}^{x_1^+}\frac{p(x)}{\sqrt{x^2-4}}\lambda_0(x)dx$$

The proposition is proved.

We are particular concerned with the case (A special sum rule),

$P(z)=P_m(z)=(-1)^{m-\frac{1}{2}}(z^2-4)^m$, where $m\in N$. We have

$$\lambda_m(x) = \begin{cases} \frac{1}{2\pi}(4-x^2)^{m-\frac{1}{2}}\log\frac{\sqrt{4-x^2}}{\delta'(x)}, x\in[-2,2] \\ (-1)^{m+1}(x^2-4)^{m-\frac{1}{2}}\lambda_0(x), x\notin[-2,2] \end{cases}$$

We put $\mu_0\pm=\sum\delta_{x_j}\pm\delta_{x_0}$ being Dirac's delta centered at x_j . We notice that

$\lambda_0(x)=\int_2^x d\mu_{0^+}(s)$ for $x>2$, and we get integrating by parts

$$\int_2^{x_1^+}(x^2-4)^{m-\frac{1}{2}}\lambda_0(x)dx = \int_2^{x_1^+}G_m(x)d\mu_0 = \sum_j G_m(x_j^+)$$

Where $G_m(x)=\int_2^x (S^2-4)^{m-\frac{1}{2}}ds$

(123)

We extend G_m to $x<-2$ in even way and carrying out similar computation

for x_j . We see that $\int_{x_1^-}^{-2}\lambda_m(x)dx+\int_2^{x_1^+}\lambda_m(x)dx=(-1)^{m+1}\sum_j G_m(x_j^\pm)$.

Furthermore the inequality $C_1(x\pm 2)^{m-\frac{1}{2}} \le (x^2-4)^{m-\frac{1}{2}} \le C_2(x\pm 2)^{m-\frac{1}{2}}$ for in $[x_1^-,-2]$ or $(2,x_1^+]$ respectively and some constants C_1, C_2 implies that

$$G_m(x) = C_3(x^2-4)^{m-\frac{1}{2}} + O\left((x^2-4)^{m-\frac{3}{2}}\right)$$

(124)

Summing up we obtain that the left-hand side of (6) is given by the formula

$$\Phi_m(\delta) = \Phi_{m,1}(\delta) + \Phi_{m,2}(\delta) = \frac{1}{2\pi}\int_{-2}^{2}(4-x^2)^{m-\frac{1}{2}} \log\frac{\sqrt{4-x^2}}{\delta(x)}dx + (-1)^{m+1}\sum_j G_m\left(x_j^\pm\right)$$

(125)

Observe that $\Phi_{m,2}(\delta) \ge 0$ when m is odd and $\Phi_{m,2}(\delta) \le 0$ when is even.

Let us compute the right-hand side of equality (6) now. In a neighborhood of $z=\infty$, we have

$$\log u_0(z) = tr(\log(z-J) - \log(z-J_0)) - \log A_0'$$

$$= tr(\log(I-J/z) - \log(I-J_0/z)) - \log A_0'$$

$$-\left\{\log A_0' + \sum_{k=1}^{\infty}\alpha_k tr\left(J^k - J_0^k\right)\frac{1}{z^k}\right\}$$

It is convenient to set A_0, and so $\log A_0' = tr\log A_0$. Furthermore

$P_m(z)/\sqrt{z^2-4} = (-1)^{m+1}(z^2-4)^{m-\frac{1}{2}}$ in the Laurrant series centered at $z=\infty$.

That is, we have

$$(1-x)^{m-\frac{1}{2}} = \sum_{k=0}^{m-1}(-1)^k \tilde{C}_{2m-1}^{2k} x^k + (-1)^m \frac{(2m-1)!!}{(2m)!!}x^m + O\left(x^{m+1}\right) \qquad (126)$$

For small $|x|$, $\tilde{C}_{2m-1}^{2k} = \frac{(2m-1)!!}{(2m-1-2k)!!(2k)!!}$, and $k!!$ refers to “even” or “odd” factorials.

Consequently $(z^2-4)^{m-\frac{1}{2}} = z^{2m-1}\left(1-(4/z^2)\right)^{m-\frac{1}{2}}$ and making use of (126) together with $\tilde{C}_{2m-1}^{2m-2k-2} = \tilde{C}_{2m-1}^{2k+2}$ we see that

$$(-1)^{m+1}\left(z^2-4\right)^{m-\frac{1}{2}} = 2^{2m}\left\{\sum_{k=0}^{m-1}\frac{(-1)^k}{2(k-1)}\tilde{C}_{2m-1}^{2k+1}z^{2k+1}-\frac{(2m-1)!!}{(2m)!!}\frac{1}{z}\right\}+O\left(\frac{1}{z^3}\right)$$

For the sake of brevity we put

$$\log u_0(z)=\sum_{k=0}^{\infty}C_k\frac{1}{Z^k},\left(z^2-4\right)^{m-\frac{1}{2}}=\sum_{k=-1}^{m-1}d_{2k+1}z^{2k+1}+O\left(\frac{1}{z^3}\right)$$

Then $\operatorname{Res}_{\infty}\left(\frac{P_m(z)}{\sqrt{z^2-4}}\log u_0(z)\right)=-\sum_{k=0}^{m}d_{2k-1}C_{2k}$

An elementary computation shows that

$$\psi_m(J)=\operatorname{Res}_{\infty}\left(\frac{P_m(z)}{\sqrt{z^2-4}}\log u_0(z)\right)$$

$$=-\left\{\sum_{k=1}^{m}\frac{(-1)^k}{z^{2k+1}k}\tilde{C}_{2m-1}^{2k-1}\,tr\left(J^{2k}-J_0^{2k}\right)+\frac{(2m-1)!!}{(2m)!!}\,tr\,\log A_0\right\}$$

Comparing (102), (125) and the latter relation we obtain

$$\frac{1}{2\pi}\int_{-2}^{2}\left(4-x^2\right)^{m-\frac{1}{2}}.\log\frac{\sqrt{4-x^2}}{\delta(x)}dx+(-1)^{m+1}\sum_j G_m\left(x_j^{\pm}\right)=\psi_m(J)$$

This is precisely the sum rule we are interested in.

Theorem (4-3-9) [134]:

Let $J=J(a,b)$ be a Jacobi matrix if

(i) $a-1, b\in L^{m+1}, (a), (b)\in L^2$

(127)

(ii) $\gamma_k(a)\in L^1, k=3,[3/2+1]$

(128)

then

(i) $\int_{-2}^{2}\log\delta(x).\left(4-x^2\right)^{m-\frac{1}{2}}dx>\infty$.

(ii) $\sum_j\left(x_j^{\pm}-4\right)^{m+\frac{1}{2}}<\infty$.

Corollary (4-3-10)[5]:

Let $J = J(a,b)$ satisfy assumptions of theorem (4-1-9) and l is odd then $\sum_j \left(x_j^{\pm 2} - 4\right)^{m-\frac{1}{2}} < \infty$.

Let $V = J - J_0$ or $V = J(c,b)$, where $c = a - 1$ and 1 is a sequence consisting of units. Obviously

$$J^{2k} = (J_0 + V)^{2k} = \sum_{p=0}^{2k} \sum_{i_0 + \ldots + i_p = 2k-p} J^{i_0} V J_0^{i_1} \ldots V J_0^{i_p}$$

and consequently

$$tr\left(J^{2k} - J_0^{2k}\right) = tr \sum_{p=1}^{2k} \sum_{i_0 + \ldots + i_p = 2k-p} V J_0^{i_1} \ldots V J_0^{i_p}$$

We agree to write $\tilde{O}(A^2)$ instead of $\sum_{k}^{N} B_k A C_k A D_k$ with some bounded operators B_k , C_k , D_k . Set $|k| = \sum_i k_i$. The following lemmas hold.

Lemma (4-3-11) [134]:

$$V J_0^{k_1} \ldots V J_0^{k_p} = V^p J_0^N + \sum_{|1|=p, |p|=N} C_{1,p} J_0^{p_1} V^{L_1} \left[V^{L_2}, J_0^{p_2}\right] V^{L_3} J_0^{p_3} + \tilde{O}\left([V, J_0]^2\right)$$

(129)

Proof:

We prove this lemma by induction on p. The claim of the lemma is trivial when $p=1$. We suppose that the lemma is valid for p and we prove it for $p+1$. We have $k=(k_1,\dots,k_{p+1})$ and $VJ_0^{k_1}\dots VJ_0^{k_p} = VJ_0^{k_1}\left(V^{p}J_0^{k_1} + \right| +$

$\sum_{|1|=p,|p|=N'} C_{1,p} J_0^{p_1} V^{L_1}[V^{L_2}, J_0^{p_2}]V^{L_3} J_0^{p_3} + \tilde{O}([V,J_0]^2)$, where $N' = k_2 + \dots + k_{p+1}$.

Furthermore

$$
\begin{aligned}
VJ_0^{k_1}V^{p}J_0^{N'} &= V\left(V^{p}J_0^{k_1} + \left[J_0^{k_1}, V^{p}\right]\right)J_0^{N'} = \left(VV^{p}J_0^{k_1} + V\left[J_0^{k_1}, V^{p}\right]\right)J_0^{N_1} \\
&= V^{p+1}J_0^{k_1}J_0^{N} + V\left[J_0^{k_1}, V^{p}\right]J_0^{N_1} = V^{p+1}J_0^{k_1+N'} + V\left[J_0^{k_1}, V^{p}\right]J_0^{N_1}
\end{aligned}
$$

Then taking $p_1' = k_1 + p_1$ we get

$$
VJ_0^{k_1}V^{p_1}V^{L_1}[V^{L_2}, J_0^{p_2}]V^{L_3}J_0^{p_3} = VJ_0^{k_1+p_1}V^{L_1}[V^{L_2}, J_0^{p_2}]V^{L_3}J_0^{p_3}
$$

Put $p_1' = k_1 + p_1$ in the expression

$$
\begin{aligned}
VJ_0^{p_1'}V^{L_q}\left[V^{L_2}, J_0^{p_2}\right]V^{L_3}J_0^{p_3} &= V\left(V^{L_1}J_0^{p_1'} + \left[J_0^{p_1'}, V^{L_1}\right]\right)\left[V^{L_2}p_0^{2}\right]V^{L_3}J_0^{p_3} \\
&= V^{L_1+1}J_0^{p_1'}\left[V^{L_2}, J_0^{p_2}\right]V^{L_3}J_0^{p_3} + \tilde{O}\left([V, J_0]^2\right), \\
&= \left(J_0^{p_1'}V^{L_1+1} + \left[V^{L_1+1}, J_0^{p_1}\right]\right)\left[V^{L_2}, J_0^{p_2}\right]V^{L_3}J_0^{p_3} + \tilde{O}\left([V, J_0]^2\right) \\
&= J_0^{p_1'}V^{L_1+1}[V^{L_2}, J_0^{p_2}]V^{L_3}J_0^{p_3} + \tilde{O}([V, J_0]^2)
\end{aligned}
$$

Above we repeated used formulas to show **[5]:**

$[A^k, B] = \sum_{j=0}^{k-1} A^{k-1-j}[A,B]A^j$ and $[A, B^k] = \sum_{j=0}^{k-1} B^{k-1-j}[A,B]B^j$ where A, B are operators with $A = V^k$, $B = J_0$ and $A = V, B = J_0^k$ respectively. Finally it is plain that

$vJ_0^{k_1}\tilde{O}([V, J_0]) = \tilde{O}([V, J_0]^2)$, the proof is complete.

Lemma (4-1-12) [134]:

Let $|k| = N$. Then

$$
trVJ_0^{k_1}\dots VJ_0^{kp} = trV^{p}J_0^{N} + C_5 trV^{p-1}[V, J_0]J_0^{N-1} + tr\tilde{O}([V, J_0]^2)
$$

(130)

Where C_5 is a constant depending on p and l.

Proof:

Employing lemma (4-1-11) we immediately get the first and the last terms on the right-hand side of the latter inequality. As for the second term we see that,

$$trJ_0^{p_1}...V^{L_1}\left[V^{L_2}, J_0^{p_2}\right]V^{L_3}J_0^{p_3} = trV^{L_1}\left[V^{L_2}, J_0^{p_3}\right]V^{L_3}J_0^{P'}$$
$$= trV^{L_1}\left[V_1^{L_2}, J_0^{P_2}\right]V^{L_3}J_0^{P_1+P_3}$$

Put $p_1 + p_3 = p'$ we get

$$trV^{L_1}\left[V^{L_2}, J_0^{p_2}\right]V^{L_3}J_0^{P_1+P_3} = trV^{L_1}\left[V^{L_2}, J_0^{p_2}\right]V^{L_3}J_0^{P'} = trV^{L_1}\left[V_1^{L_2}, J_0^{P_2}\right]V^{L_3}J_0^{P_1'} + Tr\tilde{O}\left([V, J_0]^2\right)$$, and

put $L_1' = L_1 + L_3$, we get $trV^{L_1'}\left[V^{L_2}J_0^{p_2}\right]J_0^{p_1'} + tr\tilde{O}\left([V, J_0]^2\right)$.

Recalling (119) and (120) we obtain that

$$trV^{L_1'}\left[V^{L_2}, J_0^{p_2}\right]J_0^{p_1'} = trV^{L_1'}\left(\sum_j V^j\left[V, J_0^{P_2}\right]V^{L_2-j-1}\right)J_0^{p_0'}$$

$$= \sum_j trV^{L_1'}\left[V, p_0^2\right]\left(J_0^{p_1'}V^{L_2-j-1} + \left[V^{L_2-j-1}, J_0^{p_1'}\right]\right)$$

Where $L_1' + L_2' - 1 = L_1 + L_2 + L_0 - 1 = p - 1$. We obtain that

$$\sum_j V^{L_1'+L_2'-1}\left[V, J_0^{p_1}\right]J_0^{p_1'} + tr\tilde{O}\left([V, J_0]^2\right)$$

Transforming expression $\left[V, J_0^{p_2}\right]$ in the same way we finish the proof of the lemma. We identify a sequence $a = \{a_k\}$ with diagonal operator diag $\{a_k\}$.

Lemma (4-3-13) [134]:

If $p_1 + 2j - 1 < m - p_1$ and $p_1 > 0$, then $\sum_{k=2j}^{m-p_1}(-1)^k C_{m-p_1}^k \frac{(k-p_1-j)!}{(k-2j)!} = 0$ if $p = 0$, the above expression equals $(2p-1)$!

Proof:

Obviously $x^{p_1-1}(1-x)^{mp_1} = \sum_{k=0}^{m-p_1}(-1)^k C_{m-p_1}^k x^{k+p_1-1}$, and consequently

$$\frac{d^{p_1-2j-1}}{dx^{p_1+2j-1}}\left(x^{p_1-1}(1-x)^{m-p_1}\right) = \sum_{k=2j}^{m-p_1}(-1)^k C_{m-p_1}^k \frac{(k+p_1-1)!}{(k-2j)!}x^{k+2j}$$

We set $x=1$ and notice that, since $p_1+2j-1<m-p_1$ the left-hand sided of the equality equals zero. This proves the first claim of the lemma.

Lemma (4-3-14) [134]:

If $\alpha\in L^{m+1}, \partial a\in L^2$, and $A_k(a)\in L^{2(m,k)}$, where $q(m,k)=(m+1)/(m+2-k)$, than $\gamma_k(a)\in L'$ for $k=3$, $[m/2+1]$.

Proof:

Let $\delta^i=\{\delta^i_j\}$ where $\delta_j=\alpha_{j+1}-\alpha_j$ or $\alpha_{j+i}=\alpha_j+\delta^i_j, i=1,k-1$ obviously $\partial a\in L^2$ yields $\delta\in L^2$ we also have $(\delta_k(a))_j=\alpha^k_j-\alpha_j\alpha_{j+1}\cdots\alpha_{j+(k-1)}$ since

$\alpha_{j+1}=\alpha_j+\delta_j$ and $\alpha_{j+(k-1)}=\alpha_j+\delta^{k-1}_j=-\delta^{k-1}_j\left(\sum_{i=1}^{k-1}\delta^i_j\right)+\text{additional terms}$. Furthermore

we have $(A_k(a))_j=\sum_{i=1}^{k-1}\delta^i_j$ and $\|\gamma_k(a)\|_1\le\sum_i|\alpha_j|^{k-1}\left|\sum_{i=1}^{k-1}\delta^i_j\right|+\sum_j O\left((\partial a)^2_j\right)$ using inequality $ab\le(1/p)a^p+(1/p)b^q$ a with $p=p(m,k)=(m+1)/(k-1)$ and $q=q(m,k)=$ $(m+1)/(m+2-k)$

We obtain

$$\|\gamma_k(a)\|_1\le\frac{1}{p(m,k)}\|a-1\|^{m+1}_{m+1}+\frac{1}{q(m,k)}\|A_k(a)\|^{p(m,k)}_{q(m,k)}+G_5\|\partial a\|^2_2 .$$

The quantity on the right hand side of the inequality is finite by the assumptions of the lemma, it is easy to obtain other sufficient conditions providing $\gamma(a)\in L_1$.