Chapter 3

Homogenous Shifts Operators and Holomorphic Functions

We prove that a homogenous operator is unitary and a reducible homogenous weighted shift is unweighted bilateral shift, also a projective representation is irreducible, if the unitary operator have non common non-trivial reducing subspace. We prove an irreducible projective representation with multiplier and especial multipliers of Mobius. We prove that a group law is determined by a group homomorphism. We give a computation of a normalized representation, that a unitary representation is equivalent to its quasi-invariant is proved and an image under the operator is a multiplier representation is considered and proved [16,18,19,20].

Section (3-1): Homogeneous Operators and Weighted Shift with Multipliers

Lemma (3-1-1) [5]:

If T is a homogenous operator such that T^k is unitary for some positive integer k then T is unitary [120, 122].

Proof:

Let $\phi \in$ Mobs since $\phi(T)$ is unitary, it follow that $(\phi(T))^k$ is unitary equivalent to T^k and hence is unitary I particular by taking $\varphi = \varphi_\beta$ we find that the inverse and the adjoint of $(T-\beta I)^k (I-\bar{\beta}T)^{-k}$ are equal $(T-\beta I)^{-k}(I-\bar{\beta}T)^k$. Since T^k is unitary implies that $(T-\beta I)^{-k}(I-\bar{\beta}T)^k = (T^*-\bar{\beta}I)^k (I-\bar{\beta}T)^k$ and we get $(T^*-\bar{\beta}I)^k (I-\beta T^*)^{-k}$ and hence $T^*T = I$ we have

$$(I-\bar{\beta}T)^k (I-\beta T^*)^k = (T-\beta I)^k \quad (T^*-\bar{\beta}I)^k .$$

For all $\beta \in D$ the two side of this equation is expanding binomially and the binomial rule is $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r}$. By applying this rule we get

$$\left(\sum_{m=0}^{k}(-1)^{m}\binom{k}{m}\beta^{-m}T^{m}\right)\left(\sum_{n=0}^{k}(-1)^{n}\binom{k}{n}\beta^{n}T^{*n}\right)=\sum_{m=0}^{k}\sum_{n=0}^{k}(-1)^{m}(-1)^{n}\binom{k}{m}\binom{k}{n}\beta^{-m}\beta^{n}T^{m}T^{*n}$$

by equaling the coefficients of powers weight $T^{*n}T^{m} = T^{k-n}T^{*k-n}$ for $0 \le m, n \le k$. Noting that our hypothesis on T implies that T is invertible, we find $\frac{T^{m}}{T^{k-m}} = \frac{T^{*k-m}}{T^{*n}}$ implies $T^{m-n-k} = T^{*k-m-n}$ for all m, n in this range, in particular taking $m-n = k-1$ we have $T^{-1} = T^{*}$ a T is unitary.

Theorem (3-1-2) [5]:

Up to unitary equivalence, the only reducible homogenous weighted shift (with non-zero weights) is the un weighted bilateral shift B.

Proof:

Any such operator T is a bilateral shifts and its weight sequence $w_n, \; n \in z$ is periodic say with period, we may assume $w_n > 0$ for all n in z

The spectral radius $r(T)$ of T is given by the following

$$r^{+} = \lim_{n\to\infty}\left[\sup_{j=0}\left(\omega_j \omega_{j+1} \cdots \omega_{n+j-1}\right)\right]^{\frac{1}{n}} \quad , \quad r(T)\max(r^{-}, r^{+})$$ where

$$r^{+} = \lim_{n\to\infty}\left[\underset{j\geq 0}{Sup}\left(\omega_j\omega_{j+1}\ldots\omega_{n+j-1}\right)_{j=0}\right]^{\frac{1}{n}} \quad \text{and} \quad \bar{r} = \lim_{n\to\infty}\left[\underset{j<0}{Sup}\left(\omega_{j-1}\omega_{j-2}\ldots\omega_{j-n}\right)_{j=0}\right]^{\frac{1}{n}}$$

In our case since the weight sequence ω_n is periodic with period k this formula for the spectral radius reduces to

$$r(T) = (\omega_0\omega_1\ldots\omega_{k-1})^{\frac{1}{k}}$$

Now assume that T is also homogenous, then $r(T)=1$. Thus $\omega_0\omega_1\ldots\omega_{k-1}=1$ by the periodicity of the weight sequence, it then follows that $\omega_n\omega_{n+1}\ldots\omega_{n+k-1}=1$ $\forall_n \in Z$ therefor it $x_n, n\in z$ is the orthogonal basis such that $Tx_n = x_{n+k} = B^k x_n$ for all n and hence $T^k = B^k$, since B is unitary show that T^k is unitary therefore T is unitary. Hence $\omega_n = \|Tx_n\| = \|T\|\,\|x_n\|$ since $\|T\| = 1$ implies $\|x_n\| = 1$ for all n. Thus $T = B$.

Definitions (3-1-3) [17]:

If T is an operator on a Hilbert space H then a projective representation π of Mob on H is said to be associated with T if the spectrum of T is contained in D and

$$\phi(T) = \pi(\phi)^{*}\, T\, \pi(\phi) \qquad (1)$$

For all elements ϕ of Mob

Theorem (3-1-4) [17]:

If T is an irreducible homogenous operator, then T has a projective representation of Mob associated with it- Further this representation is uniquely determined by T.

For any projective representation π of Mobs let $\pi^{\#}$ denote the projective representation of Mobs obtained by composing with the automorphism * of Mobs so [73,48]

$$\pi^{\#}(\phi) = \pi(\phi^{*}) \qquad (2)$$

We note.

Proposition (3-1-5) [17]:

If the projective representation is associated with a homogenous operator then is associated with the adjoint of . Further is invertible then is associated with T^{-1} also it is follows that and T^{*-1} have the same associated representation.

Theorem (3-1-6) [17]:

Let be a Hilbert space of function on such that the operator on giver by $(Tf)(x) = xf(x)$, $x \in \mathbb{C}$, $f \in H$ is bounded. Suppose this is a multiplier representation π of Mob on . Then is homogenous and is associated with .

Definition (3-1-7) [17]:

Let be a bounded operator on a Hilbert space then T is called a block shift is an orthogonal decomposition $H = \oplus_{n \in I} w_n$ of in to non-trivial subspace $w_n,\ n \in I$ such that $T(w_n) \subseteq w_{n+1}$ the following is due to Mark Ordower.

Lemma (3-1-8) [17]:

If is an irreducible block shift then the blocks of are uniquely determined by .

Proof:

Fix an element $\alpha \in T$ of infinite order and let V_n , $n \in I$ be blocks of then define a unitary operator by $Sx = \alpha^n x$ for $x \in V_n$, $n \in I$. Notice that by our assumption on the eigenvalue $\alpha^n, n \in I$ of are distinct and the blocks V_n of are precisely the eigenspaces of . If $w_n, n \in J$ are also blocks of then define unitary replacing the blocks V_n , by the definition of .

A simple computation shows that we have $STS^* = \alpha T = S_1 T S_1^*$ hence $S_1^* S$ commutes with since $S_1^* S$ is unitary and is irreducible and $s_1^* s$ is a scalar. That is $s_1 = \beta s$ for $\beta \in T$ therefore has same eigenspaces as thus the blocks of are uniquely determined as eigenspaces of .

To define the projective representation and multipliers, let G to be a locally compact second countable to topological group then a measurable function.

π : $G \to u(H)$

is called a projective representation of on the Hilbert space if there is function $m : G \times G \to T$ such that

$$\pi(1) = 1 \quad , \quad \pi(g_1 g_2) = m(g_1 g_2)\,\pi(g_1)\,\pi(g_2) \tag{3}$$

for all (g_1, g_2) . Two projective representation , in the Hilbert spaces H_1 , H_2 will be called the equivalent if there exists a unitary operator $u : H_1 \to H_2$, and function $\gamma : G \to T$. Such that $\pi_2(g)\,\alpha(g)\,U\,\pi_1(g)$. For all $(g) \in G$

We shall identify two projective representation they are equivalent.

Recall that a projective representation of G is called irreducible if the unitary operator $\pi(g)$, $g \in G$ have no common non-trivial reducing subspace. Clearly $m : G \times G \to T$ is a Borel map. In view of equation (3) m satisfies $m(g,1) = 1 = m(1,g)$

$$m(g_1 g_2)\, m(g_1, g_2, g_3) = m(g_1, g_2, g_3)\, m(g_2, g_3) \tag{4}$$

Proof equation (4) **[5]:**

From equation (3) $\pi(g_1, g_2) = m(g_1, g_2)\,\pi(g_1)\,\pi(g_2)$ which implies that

$$m(g_1, g_2) = \pi(g_1, g_2)\,\pi(g_1)\,\pi(g_2)$$

then $m(g,1) = \pi(g) / \pi(g)\,\pi(1) = 1$, $m(1,g) = \pi(g) / \pi(1)\,\pi(g) = 1$, and

$$m(g_1, g_2, g_3) = \pi(g_1, g_2, g_3) / \pi(g_1, g_2)\,\pi(g_3)$$

the left hand side of equation. (4)

$$m(g_1, g_2)\, m(g_1, g_2, g_3) = \frac{\pi(g_1 g_2)}{\pi(g_1)\,\pi(g_2)} \cdot \frac{\pi(g_1 g_2 g_3)}{\pi(g_1 g_2)\,\pi(g_3)} = \frac{\pi(g_1 g_2 g_3)}{\pi(g_1)\,(g_2)\,\pi(g_3)}$$

and the right hand side

$$m(g_1,g_2g_3)\,m(g_2,g_3)=\frac{\pi(g_1g_2g_3)}{\pi(g_1)\,\pi(g_2g_3)}\cdot\frac{\pi(g_2g_3)}{\pi(g_2)\,\pi(g_3)}=\frac{\pi(g_1g_2g_3)}{\pi(g_1)\,\pi(g_2)\,\pi(g_3)}$$

$$m(g_1,g_2)\,m(g_1g_2,g_3)=\pi(g_1,g_2g_3)\,\pi(g_2,g_3)$$

for all group of elements g,g_1,g_2,g_3 any Borel function m into T satisfying (4) is called a multiplier in the group.

Definition (3-1-9) [17]:

Two multipliers m and on the group G are called equivalent if there is Borel function $\gamma: G\to T$ such that $\gamma(g_1,g_2)\,m'(g_1,g_2)=\gamma(g_1)\,\gamma(g_2)\,m(g_1,g_2)$ for all $g_1,g_2\in G$ and clearly equivalent projective reorientation have multipliers, the multipliers equivalent to the trivial multiplier are called exact. The exact multipliers form a subgroup of the multiplier group, the quotient is called the second co homology group $H^2(G,T)$ we shall need.

Theorem (3-1-10) [17]:

Let G be a connected semi-simple Lie group then every projective representation of G is a direct. Integral of irreducible projective representations of G .

Proof:

Let π be a projective representation of G let $\tilde{G}$ be the universal cover of G and let $P:\tilde{G}\to G$ be the covering homomorphism. Define projective representation π_0 of $\tilde{G}$ by $\pi_0(\tilde{x})=\pi(x)$ where $x=P(\tilde{x})$ a trivial computation of $\tilde{G}$ and its multiplier m_0 is given by $m_0(\tilde{x},\tilde{y})=m(x,y)$ where $x=P(\tilde{x})$, $y=P(\tilde{y})$.

However since $\tilde{G}$ is a connected Lie group $H^2(\tilde{G},T)$ is trivial therefore m_0 is exact that is a Borel function $\gamma:\tilde{G}\to T$ such that

$$m(x,y)=m_0(\tilde{x},\tilde{y})=\gamma(\tilde{x})\,\gamma(\tilde{y})/\gamma(\tilde{x}\tilde{y}) \qquad (5)$$

For all $\tilde{x}\tilde{y}\in\tilde{G}$, and $x=P(\tilde{x})$, $y=P(\tilde{y})$

Now we define the ordinary representation of by $\pi(\tilde{x}) = \alpha(\tilde{x})\pi_0(\tilde{x})$ for $\tilde{x} \in \tilde{G}$ the ordinary representation of $\tilde{G}$: $\tilde{\pi}(\tilde{x}) = \int^{\oplus} P_t(\tilde{x})\,dp(t)$, $\tilde{x} \in \tilde{G}$ replacing its definition in term of , we get that for each $x \in G$, $\pi(x) = \int^{\oplus} \gamma(\tilde{x})^{-1} \tilde{\pi}_t\,dp(t)$ for any such that $x = P(\tilde{x})$. So we would like to define $\pi_t : G \to u(\mathrm{H})$ by $\pi_t(x) = \gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ for any as above and verify that thus defined is an irreducible projective representation of with multiplier m. But first we must show that is well defined, that is if $\tilde{x}, \tilde{y}$ are elements of mapping in the same element of under then we need to show

$$\gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x}) = \gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}) \tag{6}$$

Let be the kernel of the covering map . Since is a discrete normal subgroup of the connected topological group $\tilde{G}, \tilde{Z}$ is a central subgroup of . Since for each $t, \tilde{\pi}_t$ is irreducible it follows that there is a Borel function [44]. $\gamma_t : \tilde{Z} \to T$. Such that $\tilde{\pi}_t(\tilde{z}) = \gamma_t(\tilde{z})I$ for all $\tilde{z} \in \tilde{Z}$ we have $\tilde{\pi}(\tilde{z}) = Z(\tilde{z})\pi_0(\tilde{z}) = \gamma(\tilde{z})\pi(1) = \gamma(\tilde{z})I$ for all $\tilde{z} \in \tilde{Z}$.

Therefore evaluating $\tilde{\pi}(\tilde{z})$ using its all in a set of full P measure and all $\tilde{z} \in \tilde{Z}$. Replacing the domain of integration by this subset if need be we may assume that $\gamma_t = \gamma$ for all . Thus

$$\tilde{\pi}_t(\tilde{z}) = \gamma(\tilde{z})I \tag{7}$$

for all $\tilde{z} \in \tilde{Z}$ and for all . Also for $\tilde{x} \in \tilde{G}$ and $\tilde{z} \in \tilde{Z}$ we have

$$\gamma(\tilde{x})r(\tilde{z}) / r(\tilde{x}\tilde{z}) = m(\tilde{x}, \tilde{z}) = m(x, 1) = 1$$

where $x = P(\tilde{x})$ and hence

$$\gamma(\tilde{x}\tilde{z}) = \gamma(\tilde{x})\gamma(\tilde{z}) \tag{8}$$

Now we come back to prove equation (6) **[5]:**

Since $P(\tilde{x}) = P(\tilde{y})$, there is $\tilde{z} \in \tilde{Z}$ such that $\tilde{y} = \tilde{x}\tilde{z}$ using equation (6) we get $\gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}) = \gamma(\tilde{x})^{-1}\gamma(\tilde{z})^{-1}\tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{z})$ from equation (8) we have $\gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{y}) =$

$\gamma(\tilde{x})^{-1}\tilde{\pi}_t(\tilde{x})$ this proves equation (6) and hence π_t is well defined. Now for

$x, y \in G$ $\pi_t(xy) = \gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(\tilde{x}\tilde{y})$

we apply $\tilde{\pi}_t(\tilde{x}\tilde{y}) = \tilde{\pi}_t(\tilde{x})\tilde{\pi}_t(\tilde{y})$

we get $\pi_t(xy) = \gamma(\tilde{x}\tilde{y})\tilde{\pi}_t(x)\tilde{\pi}_t(y)$

we use $\pi_t(x) = \gamma(x)^{-1}\tilde{\pi}_t(\tilde{x})$

and $\pi_t(x) = \gamma(\tilde{y})^{-1}\tilde{\pi}_t(\tilde{x})$

this implies $\tilde{\pi}_t(\tilde{x}) = \pi_t(x) / \gamma(\tilde{x})^{-1}$

$\tilde{\pi}_t(\tilde{y}) = \pi_t(y) / \gamma(\tilde{y})^{-1}$

by applying eq. (8) we get

$$\pi_t(xy) = \gamma(\tilde{x}\tilde{y})\frac{\pi_t(xy)}{\gamma(\tilde{x})} \cdot \frac{\pi_t(xy)}{\gamma(\tilde{y})} = \frac{\gamma(\tilde{y})\gamma(\tilde{y})\pi_t(x)\pi_t(y)}{\gamma(\tilde{x})^{-1}\gamma(\tilde{y})^{-1}} = \frac{\gamma(\tilde{x})\gamma(\tilde{y})}{\gamma(\tilde{x}\tilde{y})}\pi_t(x)\pi_t(y)$$

form eq. (8) we get

$$\frac{\gamma(\tilde{x})\gamma(\tilde{y})}{\gamma(\tilde{x}\tilde{y})}\pi_t(x)\pi_t(y) = m_0(\tilde{x}, \tilde{y})\pi_t(x)\pi_t(y)$$

Since $m_0(\tilde{x}, \tilde{y}) = m(x, y)$ then $\pi_t(xy) = m(x, y)\pi_t(x)\pi_t(y)$ where $\tilde{x}, \tilde{y} \in \tilde{G}$ are such that $x = P(\tilde{x}), y = P(\tilde{y})$ this shows that π_t is indeed projective representation of G with multiplier m . Since from the definition of π_t it is clear that π_t and $\tilde{\pi}_t$ have the same invariant subspaces and since the latter is irreducible it follows that each π_t is irreducible. Thus we have the required decomposition of π as a direct integral of irreducible projective representation π_t with the same multiplier as $\pi : \pi = \int^{\oplus} \pi_t dp(t)$. As a consequence of theorem (3-1-10) we have the following corollary, here as above $\tilde{G}$ in the universal cover of $G \cdot P : \tilde{G} \to G$ is the covering map. Fix a Borel section $S : G \to \tilde{G}$ for P such that $S(1) = 1$. Notice that the kernel Z of P is naturally identified with the fundamental group $\pi^1(G)$ of G . Define the map.

$$\alpha : G \times G \to Z \quad \text{by} \quad \alpha(x, y) = S(xy) S(y)^{-1} S(x)^{-1}, \quad x, y \in G \tag{9}$$

For any character (i.e., continuous homomorphism into the circle group T) of $\pi^1(G)$ define $m_x : G \times G \to T$ $m_x(x, y) = x(\alpha(x, y)), \quad x, y \in G$. Since Z is a central subgroup of $\tilde{G}$ it is easy to verity that m_x satisfies the multiplier identity (7). Hence m_x is a multiplier on G for each character x of Z .

Corollary (3-1-11) [17]:

Let G be a connected semi-simple Lie group, then the multipliers m_x are mutually in equivalent and every multiplier on G is equivalent to m_x for a unique character x . In other words $x \to [m_x]$ defines a group isomorphism $H^2(G, T) = Hom H, (G, T)$, for $\varphi \in Möb$, φ' is non-vanishing analytic on D . Hence there is an analytic branch of log φ' on D Fix such a branch for each φ such that

(a) For $\varphi = 1$, log $\varphi' = 0$

(b) The map $(\varphi, z) \to \log \varphi'(z)$ from $Möb \times D$ into C is a Borel function

with such a determination of the logarithm we define the function $(\varphi')^{\frac{N}{2}}$ and

$N > 0$ and $\arg \varphi'$ on D by $\varphi'(\varphi')^{\frac{N}{2}} = \exp\left(\frac{N}{2} \log \varphi'(z)\right)$, and $\arg \varphi'(z) = \operatorname{Im} \log \varphi'(z)$

for $n \in z$ let $f_n : T \to T$ defined by $f_n(z) = Z^n$ in the following all the Hilbert space H is spanned by orthogonal set $\{f_n : n \in I\}$. Where is s subsets of I thus the

Hilbert space of functions specified by the set I and $\{\|f_n\|, n \in I\}$ for $\varphi \in$ MÓb and complex parameters and define the operator $R_{\lambda\mu}(\varphi^{-1})$ on by

$$R_{\lambda\mu}(\varphi^{-1}) f(Z) = \varphi^{-1}(Z)^{\frac{N}{2}} |\varphi'(z)|^{\mu} (f(\varphi)(z)) \qquad z \in T,\ f \in \mathrm{H}, \varphi \in \text{MÓb}$$

We obtain a complete result of the irreducible projective representations of Mob is follows that [10,11] Holomorphic discrete series representations D_{λ}^{+}

here $\lambda > 0, \mu = 0, I = Z^{+}$ and $\|f_n\|^2 = \dfrac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ if $n = 0$ we get $\|f_n\|^2 = 0$ for $n \geq 0$ for each in the representation space there is an [illegible] analytic in such that is the non-tangential bounding value of , by the identification the representation space may be identified with the function Hilbert space $(\mathrm{H})^{(N)}$ of analytic functions on with reproducing kernel

$(1 - 2\overline{w})^{-N}$, $z, w \in D$.

Principal series representation $C_{\lambda\delta}\ -1 < \lambda \leq 1, s$ purely imaginary.

equation **[5]:**

$$\|f_n\|^2 = \frac{\Gamma(n+1)\,\Gamma(\lambda)}{\Gamma(n+\lambda)} = \frac{n\Gamma(n)\,\Gamma(\lambda)}{n\Gamma(n)} = \Gamma(\lambda) = 1$$ where $\lambda \leq 1$ so $\|f_n\|^2 = 1$, here

$\lambda = \lambda, \mu = \dfrac{1-\lambda}{2} + s$, $I = Z$, $\|f_n\| = 1$ for all and the complementary series

representation $C_{\lambda,\delta}, -1 < \lambda < 1, 0 < \delta < \frac{1}{2}(1-|\lambda|)$, here $\lambda = \lambda, \mu = \frac{1}{2}\left(1 - \frac{\lambda}{2}\right) + \delta, I = Z$

and

$$\|f_n\|^2 = \mathrm{C}_{k=0}^{|n|-1} \frac{k \pm \frac{\lambda}{2} + \frac{1}{2} - \delta}{k \pm \frac{\lambda}{2} + \frac{1}{2} + \delta} \ , n \in Z$$

where one takes the upper or lower sign according as n is positive or negative.

Theorem (3-1-12) [17]:

(i) m_ω is a multiplier of Mobs for each $\omega \in T$ up to equivalent m_ω , $\omega \in T$ are all the multipliers in other words, H^2 (Mob) is naturally isomorphic to T via the map $\omega \mapsto m_\omega$.

(ii) For each of the representations of Mob result above.

The associated multiplier is m_ω where $we = e^{i\pi N}$ in each case except for the auti-holomorphic discrete series, from the definition of $R_{\lambda,\mu}$ one calculates that the associated multiplier m is given by

$$m\left(\phi_1^{-1},\phi_2^{-1}\right) = \frac{\left(\left(\phi_2\phi_1\right)'(z)\div\right)^{\frac{\lambda}{2}}}{\left(\phi_1'(z)^{\frac{\lambda}{2}}\div\right)\left(\phi_1'\left(\kappa_1(z)\right)\right)^{\frac{\lambda}{2}}}, z \in T$$

For any two elements φ_1, φ_2 of Mob to show this we have **[5]:**

$\pi(e)=1$ from equation (3) $\pi(g_1,g_2) = m(g_1,g_2)\,\pi(g_1)\pi(g_2)$ by applying (3) if $R_{\lambda,\mu} = \pi$ then $\left(\pi\left(\varphi_1^{-1},\varphi_2^{-1}\right)f\right)z = m\left(\varphi_1^{-1},\varphi_2^{-1}\right)\pi\left(\varphi_1^{-1}\right),\left(\varphi_2^{-1}\right)$ implies that

$$m\left(\varphi_1^{-1},\varphi_2^{-1}\right) = \frac{\left(\pi\left(\varphi_1^{-1},\varphi_2^{-1}\right)f\right)z}{\pi\left(\varphi_1^{-1}\right),\left(\varphi_2^{-1}\right)}$$

substituted

$$R_{\lambda,\mu} = \pi \quad , \quad m\left(\varphi_1^{-1},\varphi_2^{-1}\right) = \frac{\left(R_{\lambda,\mu}\left(\varphi_1^{-1},\varphi_2^{-1}\right)f\right)z}{R_{\lambda,\mu}\left(\varphi_1^{-1}\right),\left(\varphi_2^{-1}\right)}$$

but since

$$\left(R_{\lambda,\mu}\left(\phi^{-1}\right)f\right)z = \phi'(z)^{\frac{\lambda}{2}}\left|\phi'(z)\right|^{\lambda}\left(f\phi(z)\right)$$

Implies

$$m\left(\phi_1^{-1}\phi_2^{-1}\right) = \frac{\phi_1^{-1}(z)^{\frac{\lambda}{2}}\phi_2^{-1}(z)^{\frac{\lambda}{2}}\left|\left(\phi_1\phi_2\right)(z)\right|^{\mu} f\left(\phi_2\left(\phi_2\right)(z)\right)}{R_{\lambda,\mu}\phi_1^{-1}R_{\lambda,\mu}\phi_2^{-1}}$$

$$= \frac{\phi_1^{1}(z)^{\frac{\lambda}{2}}\phi_2^{1}(z)^{\frac{\lambda}{2}}\left|\phi_1\phi\left(_2 z\right)\right|^{\mu} f\left(\phi_2\left(\phi_2\right)(z)\right)}{R_{\lambda,\mu}\left(\left(\phi_1^{1}\phi_2^{1}\right)f\right)(z)}$$

Then

$$m\left(\phi_1^{-1}\phi_2^{-1}\right)=\frac{\phi_1(z)^{\frac{\lambda}{2}}-\phi_2(z)^{\frac{\lambda}{2}}\left|\phi_1\phi_2(z)\right|^{\mu}f\left(\phi_2(\phi_1 z)\right)}{\phi_1'(z)^{\frac{\lambda}{2}}\left(\phi_2'(\phi_1)(z)\right)^{\frac{\lambda}{2}}\left|\phi_1\phi_2(z)\right|^{\mu}f\left(\phi_2(\phi_1 z)\right)}=\frac{\left(\phi_1\phi_2\right)'(z)^{\frac{\lambda}{2}}}{\phi_1^1(z)^{\frac{\lambda}{2}}\left(\phi_2^1(\phi_1)(z)\right)^{\frac{\lambda}{2}}}$$

Notice that the right hand side of this equation is an analytic function of z in and it is of constant modulus1 in view of the chain rule for differentiation there fore by the maximum modulus principle, this formula is independent of z for $z \ in \ \bar{D}$. Hence we may take z = 0 in this formula and thus $m = m_\omega$ with $w = e^{i\pi N}$ so m is the multiplier associated with is since $\bar{D}_\lambda = D_N^{+\#}$ it follows that if $\pi = \bar{D}_\lambda$ is the anti-holomorphic discrete series, then multiplier is m_ω where $we = e^{i\pi N}$. The multiplier m_ω , $w \in T$ are naturally equivalent (since $w \rightarrow [m_\omega]$) is clearly a group homomorphism from onto $H^2(\text{Mób}, \text{T})$ this amounts to verifying that m_ω is never exact for $w \neq 1$ this fact may be deduced from corollary (3-1-11) as follows [40,132]. Identify Mob with $T \times D \ via \ \varphi_{\alpha,\beta} \ (\alpha,\beta)$ the group law on $T \times D$ is given by

$$(\alpha_1,\beta_1)(\alpha_2,\beta_2)=\left(\alpha_1\alpha_2.\frac{1+\bar{\alpha}_2\beta_1\bar{\beta}_2}{1+\alpha_2\bar{\beta}_1\beta_2},\frac{\beta_1+\alpha_2\beta_2}{\alpha_2+\beta_1\beta_2}\right)$$, the identity in $T \times D$ is (1,0) and inverse map is $(\alpha,\beta)^{-1} = (\bar{\alpha}, -\alpha\beta)$ then the universal cover is naturally identified with $R \times D$ taking covering map. $R \times D \longrightarrow T \times D$ to be $P(t,\beta) = (e^{2\pi it},\beta)$, the group low on $R \times D$ is determined by the requirement that be a group homomorphism as follows

$$(t_1,\beta_1)(t_2,\beta_2)=t+_1 t_2+\frac{1}{\pi}Im\log\left(1+e^{-2\pi it}\beta_1\bar{\beta}_2 h\right)\frac{\beta_1+e^{2\pi it_2}\beta_2}{e^{2\pi it_2}+\beta_1\bar{\beta}_2}$$

to show this we have **[5]:**

Let $\alpha_1 = e^{2\pi it_1}$, $\alpha_2 = e^{2\pi it_2}$. Substitute t_1 and t_2 in the following equation

$$(\alpha_1,\beta_1)(\alpha_2,\beta_2)=\left(\alpha_1\alpha_2.\frac{1+\alpha_2'\beta_1\bar{\beta}_2}{1+\alpha_2\beta'_1\beta_2},\frac{\beta_1+\alpha_2\beta_2}{\alpha_2+\beta_1\beta_2'}\right)$$

we get

$$(t_1,\beta_1)(t_2,\beta_2) = \left(e^{2\pi i t_1} e^{2\pi i t_2} \frac{1+e^{-2\pi i t_2}\beta_1\bar\beta_2}{1+e^{2\pi i t_2}\beta_1\bar\beta_2}, \frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\bar\beta_2} \right)$$

$$= \left(e^{2\pi i (t_1+t_2)} \cdot \left(1+e^{-2\pi i t_2}\beta_1\bar\beta_2\right)\left(1+e^{2\pi i t_2}\beta_1\bar\beta_2\right)^{-1}, \frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\bar\beta_2} \right)$$

$$= \left(e^{2\pi i (t_1+t_2)} \cdot \left(1+e^{-2\pi i t_2}\beta_1\bar\beta_2\right)\left(1+e^{-2\pi i t_2}\beta_1\bar\beta_2\right), \frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\bar\beta_2} \right)$$

$$= \left(e^{2\pi i (t_1+t_2)} \cdot \left(1+e^{-2\pi i t_2}\beta_1\beta_2'\right)^2, \frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\bar\beta_2} \right),$$

and this gives

$$(t_1,\beta_2)(t_2,\beta_2) = T_1 + T_2 + \frac{1}{\pi}\operatorname{Im}\log\left(1-e^{-2\pi i t_2}\beta_1\beta_2, \frac{\beta_1+e^{2\pi i t_2}\beta_2}{e^{2\pi i t_2}+\beta_1\bar\beta_2}\right)$$

where (log) denote the principle branch of the logarithm on right halt plane.

The identity in $R \times D$ is (0,0) and the inverse map is $(t,\beta)^{-1} = (-t - e^{2\pi i t})$ and the kernel of the covering map is identified with additive group via $n \to (n,0)$ so we choose a Borel branch of the argument function satisfying $\arg(\bar Z) = \arg(Z),\ z \in T$ we make an explicit choice of the Borel function $(\varphi z) \to \arg(\phi(z))$ as follows $\arg \phi_{\alpha,\beta}(z) = \arg(\alpha) - 2\operatorname{Im}\log(1-\beta)$ let's also choose function $s : T \times D \to R \times D$ as follows $S(\alpha,\beta) = \left(\frac{1}{2\pi}(\alpha),\beta\right)$ and easy computation shows that for these choices we have $S(\phi_1\phi_2)S(\phi_2^{-1})S(\phi_1^{-1}) = -n(\phi_1\phi_2)$ for φ_1,φ_2 in Mob. Hence we get that for $w \in T,\ m_w = m_\chi$ where $\chi = \chi_w$ is the character maps to w^{-n} of . Thus the map $w \to [m_w]$ is but a special case of the isomorphism $\chi \to m_\chi$ of corollary (3-1-11) to show the simple representation of the Moby's group let be the maximal compact subgroup of

Mob given by $\{\varphi_{\alpha 0} : \alpha \in T\}$ of course is isomorphic to the circle group via $via\ \alpha \longrightarrow \varphi_{\alpha 0}$.

Definition (3-1-13) [17]:

Let be a projective representation of Mob and we shall say is normalized if π / k is an ordinary representation of .

Lemma (3-1-14) [17]:

For any projective representation of Mob then δ/k is projective repress-entation of say with multiplier m. But $H^2(k)$ so there exists a Borel function $f : k \to T$ such that $m(x, y) = \frac{f(x)f(y)}{f(xy)}$, $x, y \in k$. Extend to a Borel function $g : \text{Möb} \to T$. Define by $\pi(x) = g(x)\delta(x)$, $x \in \text{Möb}$ then is normalized and equivalent to for $n \in Z$, let be the character of given by $\chi_n(x) = x^{-n}$, $x \in T$ for any normalized projective representation of Mob and $n \in Z$ let $V_n \pi = \{v \in H : \pi(x)v = \chi_n(x)v_1, \forall x \in T\}$ then $H = \oplus_{n \in Z} V_n \pi$. The subspace $V_n(\pi)$ are usually called the -isotopic subspaces of put $d_n(\pi) = \dim V_n \pi$ and $T(\pi) = \{n \in Z : d_n(\pi) \neq 0\}$.

Definition (3-1-15) [17]:

(a) A subset of is said to be connected if for any three elements $a < b < c$ in $Z . a . C \in A$ implies $b \in A$. If is any subset of . a connected subset of . Since the union of two intersecting connected sets is clearly connected, the connected components of a set partition the set.

(b) let be a normalized projective representations of Mob we shall say that is connected $T(\pi)$ is connected will be called simple if is connected and further $d_n(\pi) \leq 1$ for all $n \in Z$. More generally projective representation is connected simple if it's equivalent to a connected simple (normalized) representation.

Remark (3-1-16) [17]:

If and are equivalent normalized representations then there is an integer such that $V_n(\delta) = V_{n+h}(\pi)$ for all $n \in Z$ consequently $T(\delta)$ is an additive translate of $T(\pi)$. Hence is connected simple if and only if is thus the definitions given above are constant and we need this

Lemma (3-1-17) [17]:

Let be normalized projective representation of Mob then each connected component of $T(\pi)$ is unbounded.

Theorem (3-1-18) [17]:

Up to equivalent the only simple projective representations of Mob are the irreducible projective representations of Mob and the representations

$$D_{\lambda}^{+} \oplus D_{2-\lambda}^{-}, \quad 0 < \lambda < 2$$

Theorem (3-1-19) [17]:

If is an irreducible homogenous operator the is a block shift. If is a normalized representation associated with then the blocks of are precisely the -isotopic subspaces.

$$V_n(\pi), \quad n \in T(\pi) \; .$$

Proof:

If is an irreducible block shift then the blocks of are uniquely determined by . Then

$$T(V_n(\pi)) \subseteq V_{n+1}(\pi) \quad \text{for} \quad n \in T(\pi) \qquad (10)$$

indeed since is irreducible then equation (10) show that is connected and $b \notin T(\pi)$ then (10) would imply that $\oplus_{n \geq b} V_n(\pi)$ is a non-trivial. Since unbounded by theorem (3-1-21) it follows that be re-indexing, the index can be taken to be either all integer or the non-positive integers, therefore is a block shift. So it only remains to prove (10). To do this, fix $n \in T(\pi)$ and $v \in_n (\pi)$ for $x \in k$ we have $\pi(x)v = \chi_n(x)v$. Consequently

$$\pi(x)Tv = \pi(x^{-1})^{*} Tv$$

$$= \pi(x^{-1})^{*} T(x^{-1})(\pi(x)v)$$

$$= (x^{-1}T)^{*} T(x^{-N}v) = x^{-((n+1))}Tv$$

So $Tv \in V_{n+1}(\pi)$, this proves (10)

Lemma (3-1-20) [17]:

Let T be any homogenous weighted shift, let be the projective Representation of associated with T. Then up to equivalent is one of the representations further

(a) If T is a forward shift then the associated representation is holomorphic discrete series.

(b) If T is a back word shift then the associated representation is auti-holomorphic discrete series.

(c) If T is a bilateral shift then the associated representation is either principle series or complementary series.

Theorem (3-1-21) [17]:

Up to unitary equivalence the only homogenous weighted shifts are reducible.

Proof:

Let T be homogenous weighted shift. If T is reducible we are done by theorem (3-1-2). So assume T is irreducible then by theorem (3-1-4) there is a projective, representation π of Mob associated with T. By lemma (3-1-3) π is one of the representation. Further replacing T by T^* if necessary, we may assume that T is either a foreword or bi-lateral shift.

According π is either a homomorphic discrete series representation or a principal complementary series representation. Hence $\pi = R_{\lambda,\mu}$ for some parameters λ,μ recall that the representation space H_π is the closed span of the function $f_n, \; n \in I$ where $f_n(z) = Z^n, n \in I$ and $I \in Z^+$ in the former case and $I = Z$ in the a case the element's $f_n, \; n \in I$ form a complete orthogonal set of vectors in H_π , but these vectors are not unit vectors . Their norms are as given before .Since T is a weighted shift with respect to the orthogonal basis of obtained H_π by normalizing $f_n s$ where are scalar $a_n > 0, \; n \in I$ such that

$$Tf_n = a_n f_{n+1}, \quad n \in I$$

Notice that since the $f_n s$ are not normalized the numbers an are not the weights of the weighted shift T. These weights are given by follows there the adjoint T acts by $w_n = a \| f_{n+1} \| / \| f \|$, $n \in I$

Its follows that the adjoint act by $T^* f_n = \frac{\| f_n \|^2}{\| f_{n-1} \|^2} an - 1 f_{n-1}$, $n \in I$ where one puts

$a_{-1} = 0$ in case $I = Z^+$ let M be multiplication operator on H_π define by $M f_n = f_{n+1}$, $n \in I$.

Notice that for each representation is corresponding operator M. Also in case I is invertible M^{-1} is also exist. Let β be a fixed but arbitrary element of D and let $\varphi_\beta = \varphi_{-1,\beta} \in$ Mob. Notice that φ_β is an involution and this simplifies the following computation of $\pi(\varphi_\beta)$ a little bit indeed a straight foreword calculation shows that for $\pi = R_{\lambda,\mu}$ we have

$$\langle \pi(\varphi_\beta) f_m , f_n \rangle = C(-1)^n \bar{B}^{n-m} \| f_n \|^2 \sum_{k \geq (m-n)^+} C_k (m,n) r^k \ , 0 \leq r \leq 1 \qquad (11)$$

where we have put $r = |\beta|^2$, $C = \varphi_\beta^1(0)^{\frac{N}{2+m}}$ and [illegible] since is associated with from the following equation (4) we have $T\pi(\varphi_\beta)(I-\bar{\beta}T) = \pi(\varphi_\beta)(\beta - T)$ to show this **[5]:**

we analysis the two sides of the above equation we get

$$T(\pi(\varphi_\beta) - T\pi(\varphi_\beta)\bar{\beta}T = \pi(\varphi_\beta)\beta - \pi(\varphi_\beta)T$$

implies $T\pi(\varphi_\beta) + \pi(\varphi_\beta)T = \pi(\varphi_\beta)\beta + \bar{\beta}T\pi(\varphi_\beta)T$ and

$$\bar{\beta}T\pi(\varphi_\beta)T + \pi(\varphi_\beta)T = T\pi(\varphi_\beta)T + \pi(\varphi_\beta)T$$

where m, n fix in I, we evaluate each side of the above equation at and take the inner product of the resulting vectors with we have for the instance

$$\langle T\pi(\varphi_\beta)Tf_m, f_n\rangle = \langle \pi(\varphi_\beta)Tf_m, T^* f_n\rangle \quad = a_m \bar{a}_{n-1} \frac{\| f_n \|^2}{\| f_{n-1} \|^2} \langle \pi(\varphi_\beta) f_{m+1}, f_{n-1} \rangle$$

and similarly for the other three terms . Now substituting from equation (11)

we get $\pi(\varphi_\beta) f_{m+1}, f_{n+1} = C(-1)^n \bar{B}^{n-m} \| f_n \|^2 \sum_{k \ge (m-n+2)} C_k (m+1, n-1) r^k$, by applying equation (11) in the main equation we have

$$\langle \pi(\varphi_\beta) T f_m, T^* f_n \rangle = a_m \bar{a}_{n-1} \frac{\| f_n \|^2}{\| f_{n-1} \|^2} C(-1)^n \bar{B}^{n-m} \| h_{n-1} \|^2 \sum_{k \ge (m-n+2)} C_k (m+1, n-1) r^k$$

by comparing with the equation (11) we get

$$a_m \bar{a}_{n-1} C(-1) \bar{B}^{n-m\,n} \| f_n \|^2 \sum_{k \ge (m-n+2)} C_k (m+1, n-1) r^k =$$

$$C(-1)^n \bar{B}^{n-m} \| f_n \|^2 \sum_{k \ge (m-n+2)} C_k (m, n) r^k$$

where $0 \le r \le 1$,

$$a_m \bar{a}_{n-1} \sum_{k \ge (m-n+2)} C_k (m+1, n-1) r^k = \sum_{k \ge (m-n+2)} C_k (m, n) r^k$$

we canceling the common factor $C(-1)^{n-1} \| f_n \|^2 \bar{B}^{n-m}$ we have the following identity in the indeterminate which is obtained from the above

$$\bar{a}_{n-1} \sum_{k \ge (m-n+2)} C_k (m, n-1) r^k = a_m \sum_{k \ge (m-n+2)} C_k (m+1, n) r^k$$

(12)

taking $m = n$ in equation (12) and equating the coefficients of we obtain

$$(n+1-\mu)a_n = (n-\mu)\overline{a}_{n-1}+1 \qquad n\in I \tag{13}$$

Homogeneous Operators and Mobius Group

Section (3-2):

Definition (3-2-1) [15]:

An operator T is called homogenous if $\varphi(T)$ is unitary equivalent to T for all φ in MObs which are analytic on the spectrum of T.

Lemma (3-2-2) [15]:

Let $\Theta: \mathbb{D} \to B(k, L)$ be a pure contraction valued inner analytic function. Let denote the invariant subspace $H^2(\mathbb{D}) \otimes$ corresponding to in the sense of Beur's theorem. That is $\mu = \{z \mapsto \theta(z) f(z) : f \in H^2(\mathbb{D}) \otimes k\}$ then coincides with the characteristic function of the compression of multiplication by to the subspace [14,45].

Lemma (3-2-3) [15]:

Let be contraction in the class $C_{\cdot 0}$ with characteristic function let be a scalar in the range $0 < \mu < 1$ and put $\delta = \sqrt{1 - \mu^2}$. Then with respect to the decomposion $M^{\perp} \oplus M \oplus H_k^2$ of its domain the operator $T[\mu]$

$$H_L^2 \oplus H_L^2 \oplus H_L^2 \to H_L^2$$

has the block matrix representation.

$$T[\mu] : \begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & ME \\ 0 & 0 & N^* \end{bmatrix} .$$

Theorem (3-2-4) [15]:

Let be the characteristic function of a homogenous $C_{\cdot 0}$ contraction. If is a compact operator then must be constant function.

Proof

Let $\Theta: \mathbb{D} \to B(k, L)$ be the characteristic function of a homogeneous operator. Assume $C = \Theta(0)$ compact. Replacing by a coincident analytic function if necessary we may assume without loss of generality that $k = L$ and $C \geq 0$. By lemma(3-2-3) there exists unitaries U_z, V_z such that $\Theta(z) = U_z C V_z,\ z \in \mathbb{D}$. let $\lambda_1 > \lambda_2 > \dots$ be the non-zero eigenvalues of compact positive operator at this point shows the eigenspace corresponding to the eigenvalue a common reducing subspace for $U_z, V_z, z \in \mathbb{D}$ and hence for $\theta(z), z \in \mathbb{D}$ we can write $\theta(z) = \theta_1(z) \oplus \theta_2(z)$ where is an analytic function into $B(k_1)$. Since must be a constant Replacing the same argument with one concludes by induction on that the eigenspace corresponding to the eigenvalue is reducing for $\theta(z), z \in \mathbb{D}$ and the projection of to each is a constant function. Since the same is obviously true of the zero eigenvalue we are done.

Definition (3-2-5) [15]:

Two projective representation , of on the Hilbert spaces H_1, H_2 (respectively) will be called equivalent it there exists a unitary operator $U : H_1 \to H_2$ and a function (necessarily Borel), $f : G \to \mathbb{T}$ such that $\pi_2(\varphi)U = f(\varphi)U\pi_1(\varphi)$ for all $\varphi \in G$.

If is an operator on a Hilbert space then the projective representation of Möb on is said to be associated with if the spectrum of is contained in and

$$\varphi(T) = \pi(\varphi)^* T \pi(\varphi) \tag{15}$$

for all elements of Möb clearly if has an associated representation then is homogeneous. In the converse direct we have.

Theorem (3-2-6) [15]:

If is an irreducible homogeneous operator then has a projective representation of Möb associated with it. This projective representation is unique up to equivalence.

Theorem (3-2-7) [15]:

If is an irreducible homogeneous contraction then its characteristic function $\theta : \mathbb{D} \to B(k, l)$ is given by $\theta(z) = \pi(\varphi_z)^* C \sigma(\varphi_z),\ z \in \mathbb{D}$ where and are two projective representation of Möb with a common multiplier. Further $C : k \to l$ is a pure contraction which intertwines σ/k and π/l conversely whenever π, σ are projective representation of Möb with a common multiplier and is a purely contractive intertwine between σ/k and π/k such that the function defined by

$$\theta(z) = \pi(\varphi_z)^* C \sigma(\varphi_z)$$

is analytic on then is the characteristic function of a homogeneous contraction. Here φ_z is the involution in Möb which interchanges and also $k = \{\varphi \in \text{Möb} : \varphi(0) = 0\}$ is the standard maximal compact subgroup of Möb).

Lemma (3-2-8) [15]:

The only $C_{\cdot 0}$ contractions with a constant characteristic function are the direct integrals of the operators $M^{(1)*}$ and $B_\lambda, \lambda > 0$. The examples of homogeneous operator given so a are all weighted shifts.

Lemma (3-2-9) [15]:

Up to unitary equivalence the only irreducible homogeneous operators in the $Cowen - Douglas\ B_2(\mathbb{D})$ are the adjoin of the operators $W^{(\lambda_1,\lambda_2)}$, $\lambda_1 > 0, \lambda_2 > 0$ Wilk's operator $T^*_{\lambda,\rho}$ is unitary equivalent to the operator $W_2^{(\lambda_1,\lambda_2)}$

Theorem (3-2-10) [15]:

For $k = 1, 2, \ldots$ and real number $\lambda > k$ the characteristic function of the operator $W_k^{(1,\lambda-k)}$ coincides with the inner analytic faction $\theta_k^{(\lambda)} : \mathbb{D} \to B(H^{\lambda+k}), H^{(\lambda-k)}$ given by $\theta_k^{(\lambda)}(z) = C_{\lambda,k} D_\lambda^+(\varphi_z), z \in \mathbb{D}$. Here ∂^{k*} is the adjoin of the k-times differentiation operator $\partial^k : H^{(\lambda-k)} \to H^{(\lambda+k)}$ and

$$C_{\lambda,k} = \pi^k_{e=-(k-1)}(\lambda - e)^{-\frac{1}{2}} .$$

Proof:

It is easy to check that $C = c_{\lambda,k}\partial^{k*}$ is a pure contraction intertwining the restrictions to k of $D^+_{\lambda+k}$ and $D^+_{\lambda-k}$ since we already know that is an inner analytic function $k = 1$ the recurrence formula $\theta_{k+1}^{(\lambda)} = \theta_1^{(\lambda-k)} \theta_{k-1}^{(\lambda)} \theta_1^{(\lambda+k)}$ for $k \geq 1, \lambda > k + 1$ with the interpretation that $\theta_0^{(\lambda)}$ denotes the constant function) shows that $\theta_k^{(\lambda)}$ is an inner analytic function on $\mathbb{D}$ for $\lambda > k, k = 1,2,\ldots$ a $C_{.nu}$ contraction T in the class C and T is the compression to $M^{\perp}$ of the multiplication operator $H^{(1)} \otimes H^{\lambda k}$ where M is the invariant subspace corresponding to this inner function. But one can verity that M is the subspace consisting of the functions vanishing to order k on the diagonal therefore $T = W_k^{(1, \lambda - k)}$.

Lemma (3-2-11) [15]:

Every normal homogeneous operator is direct sum (countable many) copies of T and T .

Let us define an atomic homogeneous operator to be a homogeneous operator which can not be written as the direct sum of two homogeneous operators. We have

Corollary (3-2-12) [15]:

T and T are atomic (but reducible) homogeneous operators T is a C_{nu} contraction.

Lemma (3-2-13) [15]:

The characteristic function $\theta_N : \mathrm{D} \to \mathrm{B}(\mathrm{L}^2(\mathrm{D}))$ of the operator N is given by the formula.

Lemma (3-2-14) [15]:

Up to unitary equivalence we have $N = B \otimes C$ where the positive contraction C is given on a Hilbert space with orthogonal basis $\{f_n : n \geq 0\}$ the formula $Cf_n = a_n f_{n-1} + b_n f_n + a_{n+1}, \quad n = 0,1,2,\ldots$ where $(f_{-1} = 0)$ and the constants a_n, b_n are given by

$$a_n = \frac{\sqrt{n(n+1)}}{4n+2} \quad , \quad bn = \frac{2(n-1)^2}{(2n+1)(2n+3)} \, , \, n \geq 0$$

Theorem (3-2-15) [15]:

If T is a bounded operator then $\|T^m\| = 0(m) \, as \, m \to \infty$

Proof:

Say $\|\varphi(T)\|\le C$ for $\emptyset \in$ Mob for any $\varphi \in$ Möb we have an expansion

$\varphi(z)=\sum_{m=0}^{\infty} a_m z^m$ valid in the closed unit disc. Hence $a_m T^m = \int_T \varphi(\alpha T)\alpha^{-1} d\alpha$

where the integral is with respect to the normalized Haar measure on therefore we get the estimate $|a_m|\|T^m\| \le C$ for all choosing $\varphi = \varphi_{\alpha,\beta}$ we see that for $m \ge 1$, $|a_m| = (1-r^2)r^{m-1}$ where $r = |\beta|$. The choice $r = \sqrt{(m-1)/(m+1)}$ gives $|a_m| = O(1/m)$ and hence $\|T^m\| = O(m)$.

Section (3-3): Homogeneous operators of Holomerphic Function

Proposition (3-3-1) [5]:

Suppose has a reproducing kernel then is unitary representation if and only if is quasi-invariant [67].

Proof:

Assume that is quasi-invariant, we have to show that the linear transformation is unitary. We note writing $\tilde{w} = g^{-1}(w)$ and

$$\tilde{w}^1 = g^{-1}(w^1), \langle U(g^{-1})k(.,w)\zeta, U(g^{-1})k(.,w^1)\eta\rangle$$

$$= \langle J(g,.)k(g(.),w)\zeta, J(g,.)k(g(.),w)\eta\rangle$$

$$= \left\langle k(.,\tilde{w})J(g,\tilde{w})^{*-1}\zeta, k(.,\tilde{w}^1) J(g,\tilde{w}^1)^{*-1}\right\rangle$$

$$= \left\langle k(\tilde{w}^1,\tilde{w})J(g,\tilde{w})^{*-1}\zeta, J(g,\tilde{w}^{1})^{*-1}\zeta\right\rangle$$

$$= \left\langle J(g,\tilde{w}^1)^{-1}k(\tilde{w}^1,\tilde{w})J(g,\tilde{w})^{*-1},\zeta,\eta\right\rangle$$

Since $U(g^{-1})k(.,w)\zeta, U(g^{-1})\langle k(.,w^1)\eta\rangle = \langle k(w^1,w)\zeta,\eta\rangle$. and it follows that $U(g^{-1})$ is isomeric. On the other hand is unitary then the reproducing kernel of the Hilbert space satisfies the transformation rule . It follows from uniqueness of the reproducing kernel that the expansion is independent of the choice of the orthogonal basis. Consequently we also have $k(z,w)=\sum_{L=0}(U_{g^{-1}}e_L)(w)^*$.

We need a relation between $g''(z)$ and $g'(z)$ the elements of G_0 are the

matrices acting on the inequalities $|a-1|<\frac{1}{2}|b|<\frac{1}{2}$ determine a

simply connected neighborhood U_0 o in G_0 under the natural projections it is dimorphic with a neighborhood of of in , so we may use a,b satisfying inequalities parameterize for we have

$g'(z)=(b'z+a')^{-2}$ by taking $\frac{d}{dz}g'(z)$ we get $g''(z)=-2\bar{b}(\bar{b}z+\bar{a})^{-3}$

which gives a relation

$$g''(z)=-2cg'(z)^{\frac{3}{2}}$$

(16)

to proof (16) **[5]:**

We use the relation $g'(z)=(\bar{b}z+\bar{a})^{-2}$ where $g'(z)^2=\bar{b}z+\bar{a}$, $\bar{b}=g'(z)-\hat{a}/z$ and

$$\bar{b}=g''(z)=-2(g'(z))^2-\bar{a}/z\,(g'(z))^3=-2(g''(z))^{\frac{2}{3}}\left(\frac{-a^2}{2(g'(z))^3}\right)=-2cg'(z)^{\frac{3}{2}}$$

where $C=\frac{-a^2}{z(g'(z))^3}$ the prove is complete. Where depends $C=C_g$ on real

analytically and is independent of the meaning of $g|z|^{\frac{3}{2}}$ is as defined earlier since both sides are real analytic (16) remains true on all of $\tilde{G}>0$.

Definition (3-3-2) [4]:

Let $J : \tilde{G}\times D \longrightarrow \mathbb{C}^{m+1\times m+1}$ be the function given by the formula for here is the constant depending on the following lemma is used for showing that is a multiplier representation.

Lemma (3-3-3) [4]:

For any we have the formula

$$J(g,z)_{p,L}=\begin{cases}\binom{p}{L}(-C)^{p-L}(g')^{\lambda-\frac{m}{2}+\frac{p+L}{2}}(z) & if \quad p\geq L\\ 0 & if \quad p<L\end{cases} \tag{17}$$

for $g\in\tilde{G}$. Here is the constant depending on as in (16). The following lemma is used for showing that is a multiplier representation.

Lemma (3-3-4) [4]:

For any $g\in G$ we have the formula

$$g'^{L}\left(f\circ g\right)^{(k)}=\sum_{i=0}^{k}\binom{k}{i}(2L)_{k-i}\,C^{k-i}\,g'^{L-\frac{k-i}{2}}\left(f^{(i)}\circ g\right)$$

Proof:

The proof is by induction using formula (17) for $k=0$ the formula is an identity assume the formula to be valid for some then

$$g''(f^{(i)} o g) + = (g')^{L+\frac{p+L}{2}} \left(f^{(i+1)} o g\right) g'$$

$$[illegible]$$

Now we observe that

$$[illegible]$$

$$[illegible]$$

$$= (2L+i)_{k+1-i}\binom{k+1}{i}.$$

Thus $g^{L}(fog)^{k+1} = (2L+i)_{k+1-i}\binom{k+1}{i} c^{k+1-i} g^{L-\frac{k-i+1}{2}}$ completing the induction step we can prove the following theorem.

Theorem (3-3-5) [5]:

The image of $\oplus_0^m D_\lambda^+$ under Γ is a multiplier representation with the multiplier given by $J(g,z)$ as in (17).

Proof:

It will be enough to show $\Gamma_j\left(D_{\lambda_j}^+(g^{-1})f\right)=J(g,\cdot)((\Gamma_j f)og)$. For each $j, 0 \le j \le m$ we compute p^{th} the component on both sides. For $p<j$ both sides are zero by definition of Γ_j and knowing that $J(g,z)_{p,L}=0$ for $L>0$ this comes for equation (17), for $p \ge j$, we have using the lemma (3-3-4)

$$\left(\left(\Gamma_j D_{\lambda_i}^+\right)\left(g^{-1}\right) f\right)_p = \binom{p}{i} \frac{1}{\left(2\lambda_j\right)_{p-j}} \left(\left(g'\right)^{\lambda_i} fo\left(g\right)^{\lambda_j} fog\right)^{p-j}$$

$$(g')^{\lambda_j} + \frac{p-j+i}{2}\left(f^{(i)}og\right)$$

$$=\sum_{L-j}^{m}\frac{p!}{(p-j)!\,j!}\frac{(p-j)!}{(p-j-L+j)!(L-j)!}\frac{1}{(2\lambda_j)_{L-j}}(-C)^{p-e}$$

$$(g')^{\lambda_j-j+\frac{p+L}{2}}\left(f^{L-j}og\right)$$

$$=\sum_{L=j}^{m}\frac{p!}{j!(L-j)(p-L)!}\frac{1}{(2\lambda_j)_{e-i}}(-C)^{p-L}(g')^{\lambda_i-j+\frac{p+L}{2}}\left(f^{(L-j)}og\right)$$

$$=\quad\sum_{L=0}^{m}J(\varphi.)_{p,L}\left((\Gamma_j f)og\right)_L$$

The orthogonal basis on the operator M the vector $e_n^j(z)=\Gamma_j\left(\sqrt{\frac{(2\lambda_j)n}{n!}}z^n\right)$ clearly form an orthogonal basis Hilbert space $A^{(\lambda_j)}(D)$ we have by definition of Γ_j

$$e_n^j(z)_L:\begin{cases}0 & L<j\ or\ L>n+j\\ \binom{L}{j}\frac{\sqrt{n!}}{(n-L+j)}\frac{\sqrt{(2\lambda_j)n}}{(2\lambda_i)_{L-j}}z^{n-L-j} & L\geq j\ and\ L\leq n+j\end{cases}\qquad(18)$$

We compute the reproducing kernel $B^{\lambda_j}(\mathbb{D})$ for the Hilbert space $A^{(\lambda_j)}(\mathbb{D})$. We have $B^{\lambda_j}(z,w) = \sum_{n=0}^{\infty} (\Gamma_j e_n^j)(z)(\Gamma_j e_n^j(w))^*$

$$= \sum_{n=0}^{\infty} \left(\Gamma_j e_n^j \right)(z) \sum_{n=0}^{\infty} \left(\Gamma_j e_n^j (w) \right)^* = \sum_{n=0}^{\infty} \left(\Gamma_j e_n^j (z) \right) \sum_{n=0}^{\infty} \left(\Gamma_j e_n^j (w) \right)^*$$
$$= \Gamma_j^{(z)} \Gamma_j^{(\bar{w})} B^{\lambda_j} (z,w) \qquad (19)$$

Since the series converges uniformly on compact subsets. Explicitly

$$B^{\lambda_j}(z,w)_{p,L} = \begin{cases} \frac{L!\,p!}{j!\,j!}\frac{1}{(2\lambda_j)_{L-j}}\frac{1}{(2\lambda_j)_{p-j}}\, \partial^{p-j}\bar{\partial}^{L-j} B^{\lambda_j}(z,w) & \text{if } L,P \geq j \\ 0 & \text{otherwise} \end{cases} \qquad (20)$$

In particular it follows that $B^{\lambda_j}(0,0)$ is diagonal and

$$B^{\lambda_j}(0,0)_{L,L} = \begin{cases} 0 & \text{if } L < j \\ \binom{L}{j}^2 \frac{(L-j)!}{(2\lambda_j)_{L-j}} & \text{if } L \geq j \end{cases} \qquad (21)$$

Then $$B^{\lambda_j}(0,0)_{L,L} = \sum_{j=0}^{m} B^{\lambda_j}(0,0)\mu_j^2 \qquad (22)$$

a useful formula for $B^{\lambda,\mu}(z,w)$ and $z \in \mathbb{D}$ we get set $p_z = \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in SU(1,1)$ we also write p_z for the corresponding element of G such that p_z depends continuously on $z \in \mathbb{D}$ and $p_0 = e$ then $p_z(0) = z$, $p_z^{-1} = p_{-z}$. By theorem (3-3-5) holds for $B^{\lambda,\mu}$ and gives

$$J_{p_{-z}}(z) B^{\lambda\mu}(0,0) J_{p_{-z}}(z)^* = B^{\lambda\mu}(z,z)$$

(23)

we have $p'_{-z}(\zeta) = \frac{1-|z|^2}{(1-\bar{z}\zeta)^2}$, $p'_{-z}(z) = (1-|z|^2)^{-1}$ the ξ of (16) corresponding to p_{-z} is $\frac{\bar{z}}{(1-|z|)^2}$ so (10) gives

$$J_{p_{-z}}(z)_{p,L} = \begin{cases} (1-|z|^2)^{-\lambda-\frac{m}{2}+p} \binom{p}{L} \bar{z}^{p-L} & p \geq L \\ 0 & p < L \end{cases}$$

which can be written in matrix forms as

$$J_{p_{-z}}(z) = 1-|z^2|^{-\lambda-\frac{m}{2}} D(|z|^2) \exp(\bar{z} S_m)$$

(24)

where $D(|z|^2)_{p,L} = (1-|z^2|)^{m-L} \delta_{p,L}$ is diagonal and S_m is the foreward shift on $\mathbb{C}^{m+1}$ with weight sequence $\{1,\ldots,m\}$ that is $(S_m)_{L,p} = L\delta_{p+1,L}$, $0 \leq p,L \leq m$

substituting (24) in to (5) and polarizing we obtain $B^{(\lambda,\mu)}(z,w) = (1 - z\bar{w})^{-2\lambda - m}$

$D(z\bar{w})\exp(\bar{w}S_m)$

$$B^{(\lambda,\mu)}(0,0) = \exp(zS_m^*)D(z\bar{w})$$

(25)