Chapter 2
The Representation of Bitriangular and Quasitriangular Operators

We deal with a striking matrix representation for biquasitriangular operators and deduce some consequences of this representation for the structure of biquasitriangular operators. The canonical Jordan model of a Jordan operator is determined by the numerical data. We prove the complementary invariant subspaces for the triangular operator

Section (2-1): Representation of Biquasitriangular Operator

Let H be a complex separable infinite-dimensional Hilbert space and let $y(H)$ denote the algebra of all bounded linear operators on H, we introduced the remarkable class of quasitirangular, operators on H which we shall denote by (QT) [31, 32, 33, 123]. One consequence of the subsequence study of this class was the spectral characterization of non-quasitriangular operators. In particular this theorem implies that every non- quasitriangular operator on H has non trivial hyperinvariant subspace, and thus attention now naturally focuses on the class

$$(BQT)=(QT)\cap(QT)^{*} .$$

Of biquasitriangular operators on H. It was shown that (BQT) is the norm closure of the class (A) of all algebraic operators on H and the norm-closure of the class of all nilpotent operators on H was also determined .

We present a striking matrix representation for biquasitriangular operators and to deduce some consequence of the existence of this representation for the structure theory of biquasitriangular operators. If $T\in y(H)$ we shall denote the spectrum of T by $\delta_e(T)$ and the [left, right] Clakin spectrum of T by $\lfloor\delta_{L_e}(T),\delta_{re}(T)\rfloor$ $\delta_e(T)$. If T is a Fredholm operator we write $j(T)$ for the Fredholm index of T. Moreover if H is a Hilbert space and T is bounded operator mapping H into H such that

$\ker(T)=ket(T^{*})=\{0\}$.

We say that T is a quasiaffinity. We shall say that an operator T in $y(H)$ has a staircase-matrix representation if there exists a orthogonal decomposition of H of the form

$$H=\sum_{n=1}^{\infty}\oplus H_n \qquad (1)$$

Where the subspaces $H_n\ (1\le n<\infty)$ are finite-dimensional such that the matrix of T with respect to this decomposition has the form

$$\begin{bmatrix} A_1 & B_1 & & & & & & & \\ & C_1 & & & & & & & \\ & & A_2 & B_2 & & & & & \\ & D_1 & & C_1 & & B_n & & & \\ & & & & \ddots & & & & \\ & & & D_2 & & A_n & & & \\ & & & & & D_n & C_n & & \\ & & & & & & & A_{n+1} & \\ & & & & & & & & \ddots \end{bmatrix}$$

(2)

Where all the entries except the A_n , B_n , C_n and D_n are understood to be 0 .

Theorem (2-1-1) [37]:

An operator T in $y(H)$ is biquasitriangular if and only if for every $\varepsilon>0$ there exists a compact operator K in $y(H)$ such that $\|K_\varepsilon\|<\varepsilon$ and such that $T-K_\varepsilon$ has a staircase-matrix representation.

Proof:

Suppose first that an operator T in $y(H)$ can be written as a sum $T=S+K$, where K is a compact and S has a staircase-matrix representation of the form (2) with respect to a decomposition of the form (1). To show that is bitriangular, it suffices in view of the fact that (BQT) is invariant under compact perturbations to show that (BQT) is biquasitriangular. Since the finite-dimensional subspaces

$$\mathrm{H}_1, \mathrm{H}_1 \oplus \mathrm{H}_2 \oplus \mathrm{H}_3, \ldots, \mathrm{H}_1 \oplus \cdots \oplus \mathrm{H}_{2n+1}, \ldots$$

are all invariant under , it follows easily from the definition .

That $S \in (QT)$. That $S^* \in (QT)$ is just as obvious, since each of the finite-dimensional subspace

$$\mathrm{H}_1 \oplus \mathrm{H}_2, \mathrm{H}_1 \oplus \cdots \oplus \mathrm{H}_4, \ldots, \mathrm{H}_1 \oplus \cdots \oplus \mathrm{H}_{2n}, \ldots$$

Is invariant under . To prove the other half of the $T \in (BQT)$ and let be any positive number. Then by virtue of the equivalent definitions of quasitriangularity, it follows easily that there exist increasing sequences $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ of finite-rank projections converging strongly to $1 = 1_{\mathrm{H}}$ and satisfying the future conditions

$$P_n\mathrm{H} + T^* P_n \mathrm{H} \subset Q_n \mathrm{H} \qquad (n = 1,2,\ldots)$$
$$Q_n\mathrm{H} + T Q_n \mathrm{H} \subset P_{n+1} \mathrm{H} \qquad (n = 1,2,\ldots)$$

(3)

and

$$\|(1 - P_n) T P_n\| \le \varepsilon / 2^{n+2} \qquad (n = 1,2,\ldots)$$
$$\|(1 - P_n) T^* P_n\| \le \varepsilon / 2^{n+2} \qquad (n = 1,2,\ldots)$$

(4)

It follows from (3) that

$$(1 - P_{n+1}) T P_n = 0 = (1 - Q_{n+1}) T^* Qn$$

(5) and

$$(1 - P_{n+1}) Q_j = 0 = (1 - Q_n) P_j \qquad (1 \le j \le n) \qquad (6)$$

Moreover the inequalities (4) imply that if is defined by the equation

$$k_\varepsilon = \sum_{j=1}^{\infty} [(1 - P_j) T P_j + Q_j T (1 - Q_j)]$$.

Then is a compact operator of norm less than . We define $T_0 = T - k_\varepsilon$.

Then by virtue of (5) and (6) we have the equations **[5]:**

$$(1-P_n)T_0P_n=(1-P_n)TP_n-(1-P_n)\left[\sum_{j=1}^{\infty}(1-P_j)TP_j+Q_jT(1-Q_j)\right]P_n$$
$$=(1-P_n)TP_n-(1-P_n)\left[\sum_{j=1}^{\infty}(1-P_j)TP_j\right]P_n$$
$$=(1-P_n)TP_n-(1-P_n)\left[\sum_{j=1}^{\infty}(1-P_j)TP_j\right]P_n$$
$$=(1-P_n)TP_nTP_n=0 \qquad (n=1,2,\ldots)$$

(7)

By an analogous argument we conclude that

$$Q_nT_0(1-Q_n)=0$$

(8)

We define $H_1=P_1H_1$ and for every positive integer n we get

$$H_{2n}=(Q_n-P_n)H,\ H_{2n+1}=(P_{n+1}-Q_n)H$$

(9)

It follows easily from (7) and (8) that the matrix of $T_0=T-k_\varepsilon$ with respect to the decomposition (1) has the form (2). Thus the theorem is proved.

Corollary (2-1-2) [37]:

Let T be any biquasitriangular operator in $y(H)$ and let ε by any positive number. Then there exists a compact operator K_ε of norm less than ε such that the operator $T-k_\varepsilon$ has a staircase-matrix representation of the form (2) where

(a) for $1\le n<\infty$, each eigenvalue of A_n [respectively, C_n] has algebraic multiplicity one,

(b) for $1\le i,j<\infty$ and $i\ne j$ $\delta(A_i)\cap\delta(A_j)=\phi$, and $\delta(C_i) \cap \delta(C_j)=\varphi$.

(c) for $1\le i,j<\infty,\ \delta(A_i)\cap\delta(C_j)=\phi$

We shall now deduce some consequences of theorem (2-1-1) and corollary (2-1-2).

Recall that two operators A and B acting on Hilbert spaces H and K respectively are called quasisimilar if there exist bounded operators $X:H\to K$

and $Y : H \to K$ with trivial kernels and trivial co-kernels such that $XA = BX$ and $AY = YB$.

Theorem (2-1-3) [37]:

Let $T \in \mathcal{L}(H)$. Then the following statements are equivalent:

(i) $T \in (BQT)$.

(ii) $T = T_0 + K$, where K is compact and T_0 is quasisimilar to a normal operator,

(iii) For every $\varepsilon > 0$ there exists a compact operator K_ε such that $\|K_\varepsilon\| < \varepsilon$ and such $T - K_\varepsilon$ is a quasisimilar to a diagonable normal operator.

We show that the property of being biquasitriangular is not preserved under quasisimilarity.

Proposition (2-1-4) [37]:

There exists a biquasitriangular operator that is quasisimilar to anon-quasisimilar to a unitary operator.

Proof:

A contraction T_0 was constructed that is a quasisimilar to a unitary operator V and has the further property that

$$\sigma(T_0) = \sigma_e(T_0) = \sigma_{le}(T_0) = \{\lambda \in C : |\lambda| \le 1\}$$

Let $T = T_0 \oplus S$, where S is a unilateral shift operator multiplicity one. Then the spectrum of T and the left essential spectrum of T are again the closed unit disc, that T is biquasisimilar. On the other hand T is obviously quasisimilar to $V \oplus S$, which fails to be quasisimilar since the Fredholm index of $V \oplus S$ at the origin is -1. The following proposition is known and extremely useful.

Proposition (2-1-5) [37]:

Suppose that for every positive integer n, A_n and B_n are similar operator. Then $\sum_{n=1}^{\infty} \oplus A_n$ is quasisimilar to $\sum_{n=1}^{\infty} \oplus B_n$.

Proof:

Suppose that $S_n A_n = B_n S_n$, where for every n , S_n is an invertible operator. Then

$$\left(\sum_{n=1}^{\infty}\oplus\alpha_n S_n\right)\left(\sum_{n=1}^{\infty}\oplus A_n\right)=\left(\sum_{n=1}^{\infty}\oplus B_n\right)\left(\sum_{n=1}^{\infty}\oplus\alpha_n S_n\right)$$

and

$$\left(\sum_{n=1}^{\infty}\oplus A_n\right)\left(\sum_{n=1}^{\infty}\oplus B_n S_n^{-1}\right)=\left(\sum_{n=1}^{\infty}\oplus B_n S_n^{-1}\right)\left(\sum_{n=1}^{\infty}\oplus B_n\right)$$

Where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers chosen to make the quasiaffinity $\sum\oplus\alpha_n S_n$ and $\sum\oplus\beta_n S_n^{-1}$ bounded. The result follows.

Proposition (2-1-6) [37]:

There exists an operator in $y(H)$ of the form $N+K$, where N is normal and K is compact that is quasisimilar operator.

Theorem (2-1-7) [37]:

Let K and T be nonzero operators in $y(H)$ such that K is a compact quasiaffinity. If T has property that there exists at least one scalar λ_0 such that $T-\lambda_0$ is a Fredholm operator of nonzero (necessarily finite) index then K does not commute with T .

Proof:

We may suppose, without loss of generality that $j(T-\lambda_0)>0$. We can apply the argument to T and K . By the Fredholm theory, there exists a neighborhood N of the point λ_0 such that for $\lambda\in N, \mu_\lambda=\ker(T-\lambda)$ is a nonzero finite-dimensional subspace of H . Suppose now that contrary to the theorem $Tk=kT$. Then $(T-\lambda)k=(T-\lambda)K$ for every scalar λ , and it follows that all of the subspaces $\mu_\lambda(\lambda\in N)$ are invariant under K .

Since μ_λ is finite-dimensional k/μ_λ must have a nonzero eigenvalue —— and an associated eigenspace $\varepsilon_\lambda\subset\mu_\lambda$. Since K is compact, the collection $\{\mu_\lambda\}_{\lambda\in N}$ must be at most countable and thus there exists an uncountable subset $y \subset N$ such that $\mu_{\lambda_1} = \mu_{\lambda_2}$ for all λ_1, λ_2 in . If for each in we choose a unit vector in , then the space $\bigvee_{\lambda \in y}\{f_\lambda\}$ must be finite-dimensional (because each is an eigenvector of corresponding to the eigenvalue). This contradicts the compactness of and the proof is complete.

The preceding theorem and the spectral characterization of non quasisimilar operators yield the following corollary.

Corollary (2-1-8) [37]:

If is a compact quasiaffinity on and commutes with a non-bi quasisimilar operator , then for every scalar such that $T-\lambda$ is a semi-Fredholm operator [illegible] .

We observe that this phenomenon can actually occur.

Proposition (2-1-9) [37]:

There exist a compact quasiaffinity on and a non- quasisimilar operator on such that $KT = TK$.

Proof:

Let be the classical Voltera operator that is, let

$$(Vf)(x) = \int_0^x f(t)dt \qquad (f \in L_2[0,1])$$

Then is similar to $V/2$. In other words, there exists an invertible operator on $L_2[0,1]$ such that $V/2 = XVX^{-1}$. We set

$H = L_2[0,1] \oplus L_2[0,1] \oplus \cdots$ and define and by the matrices

$$\begin{pmatrix} V & & 0 & \\ & V/2 & & \\ O & & V/4 & \\ & & & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & 0 & \\ x & 0 & & 0 \\ O & x & 0 & \\ & & \ddots & \ddots \end{pmatrix}$$

Respectively. Then it is clear that $TK = KT$, and T is not quasisimilar, since T is a semi-Fredholm operator with $j(T) = -\infty$. Since K is obviously a compact quasiaffinity, the proof is complete.

Bitriangular Operators and Jordan Forms with Quasisimilarly Orbit

Section (2-2):

A Hilbert space operator T is called triangular if it has an upper triangular matrix with respect to some orthonarmal basis $\{e_n, n \geq 1\}$ of the underlying space. When both T and T^* are triangular (in general, with respect to different orthogonal bases), T is called bitriangular (class $(B\Delta)$). This is a rich class containing all algebraic operators, diagonal normal operators, block diagonal operators, and all operators, and all operators with a staircase representation. When the Hilbert space is finite dimensional of course every operator is bitriangular [21, 26, 112, 115].

Every operator on a finite dimensional space is similar to a unique Jordan form. In infinite dimensions, operators similar to Jordan forms (direct sums of Jordan blocks) form quite a small class.

It will be shown that every bitriangular operator T is quasisimilar to a canonical Jordan form, called the Jordan model of T.

The bitriangular operators form the largest class of operators which have Jordan models. We have obtained the best possible result concerning the extension of Jordan forms to infinite dimensions.

In particular the results subsume those of A postol Douglas and Foias on models for algebraic operators. Let $nul(T - \lambda, k)$ be the dimensional of $\ker(T - \lambda)^k$ e $\ker(T - \lambda)^{k-1}$.

Infinite dimensions this counts the number of Jordan blocks for of size at least . So we set $\alpha(T-\lambda,k) = nul(T-\lambda)^k - nul(T-\lambda)^{k-1}$, where $\infty - \infty$ is designed to be . Now the Jordan form of is

$$J(T) = \sum_{\lambda \in \delta_p(T)} \oplus \sum_{k \geq 1} (\lambda_k I + J_k)^{(\alpha(T-\lambda,k))} \qquad (10)$$

The bitriangular operators are $T \sim_{qs} J(T)$ the main result yields many consequences. In particular we obtain a complete description of the quasisimilarity orbit

$$2y(T) = \left\{ A \in y(\mathrm{H}) : A \underset{qs}{\sim} T \right\} \qquad (11)$$

of a bitriangular operator .

We also consider the relationship between $2y(T)$ and the closure of similarity orbit .

$$2y(T) = \{ WTW^{-1} : W \in y(\mathrm{H}) \quad is\ invertible \} \qquad (12)$$

Let denote a separable Hilbert space of infinite dimension let $y(\mathrm{H})$ denote the space of bounded linear operators and let or $k(\mathrm{H})$ denote the ideal of compact operators.

In particular, $\delta(T)$, $\delta_l(T)$, $\delta_r(T)$ and $\delta_p(T)$ denote the spectrum, left and right spectrum, and point spectrum respectively, the sets $\delta_e(T)$, $\delta_{Le}(T)$, $\delta_{re}(T)$ are the corresponding parts of the essential spectrum. Also

$\delta_{Lre}(T) = \delta_{le}(T) \cap \delta_{re}(T)$ is the complement of $\rho_{sF}(T)$ the set of points in such that $T - \lambda$ is the semi-Fredholm. The set $\delta_0(T)$ consists of the isolated eigenvalues of finite multiplicity known as normal eigenvalues.

If is a (closed and open) sub set of $\delta(T)$ then $\mathrm{H}(T, \delta)$ denote the corresponding subspace. The range of in $y(\mathrm{H})$ is denoted by $ranA = A\mathrm{H}$, and $\ker A$ denotes its kernel. Also $nul(A) = \dim \ker A$. By $\ker A^w$, we mean $V_{n \geq 1} \ker A^n$

and $nul A^{*}$ is the dimensional of this subspace. When $ran\ A$ is closed and one of $nul(A)$ or $nul(A^{*})$ is finite then is semi-Fredholm and

$$ind(A) = nul(A) - nul(A^{*}) \quad (13)$$

Let $\rho_{sF}^{\pm}(A)$ denote the parts of positive and negative indices respectively.
An operator X in $\mathcal{B}(H)$ is quasiaffinity it is injective and has dense range. An operator S is a quasiaffine transform of an operator T (written $S \prec T$) if there exists a quasiaffinity X such that $TX = XS$.

Two operators S and T are quasilinear (written $S \sim_{qs} T$) if $S \prec T$ and $T \prec S$. Since the product of quasiaffinities is a quasiaffinity, $\prec$ is a partial order and $\sim_{qs}$ is an equivalent relation.
We see that an operator T is triangular if and only if

$$V\left\{\ker(T-\lambda)^{k} : \lambda \in \mathbb{C}, k \geq 1\right\} = H \quad (14)$$

Definitions (2-2-1) [102]:

Let $\ker(A;k)$ denote $A^{k} \ominus \ker A^{k-1}$ and $nul(A;k) = \dim \ker(A;k)\ k \geq 1$. Let

$$ord(A;\lambda) = \begin{cases} 0 & if \quad \ker A-\lambda = 0 \\ n & if \quad nul(A-\lambda;n) \neq 0 = nul(A-\lambda;n+1) \\ \infty & if \quad nul(A-\lambda;n) \neq 0 \ for\ all\ n \geq 1 \end{cases} \quad (15)$$

Lemma (2-2-2) [102]:

Let T be a triangular operator with diagonal $d(T)$. Then $\delta_{p}(T^{*})$ is contained in $\{\bar{\lambda} : \lambda \in d(T)\}$ and

$$nul((T-\lambda)^{*};k) \leq nul(T-\lambda;k) \quad (16)$$

Proof:

Without loss of generality, let $\lambda = 0$. Let $\{e_{j}, j \geq 1\}$ be the orthonormal basis that triangularizes T; and let P_{n} be the orthogonal projection onto $\mathcal{M}_{n} = span\{e_{1},\ldots,e_{n}\}$. Since $\mathcal{M}_{n}$ is invariant for $T, \ker(T^{k}/\mathcal{M}_{n})$ is contained in

$\ker(T^k)$ for each $k \geq 1$. Thus the projection of $\ker(T/\mu_n;k)$ onto $\ker(T;k)$ is injective, and

$$ \tag{17} $$

On the other hand, for any vector

$$(P_n T^{*k}/\mu_n)P_n x = P_n T^{*k} x \tag{18}$$

So $P_n(\ker T^{*k})$ is obtained in $\ker(P_n T^{*k}/\mu_n)$. Moreover for any nonzero vector in $\ker(T^*;k)$, $P_n x$, will not lie in $T\ker\left(T^{k-1}/\mu_n\right)$ for sufficiently large. So

$$nul(T^*;k) \leq \limsup_{n\to\infty} nul(P_n T^*/\mu_n;k)$$

From the linear algebra we obtain $nul(P_n T^*/\mu_n) = nul(T^k/\mu_n)$. For all $k \geq 1$ and $n \geq 1$. Hence for $k \geq 1$ and $n \geq 1$ we have $nul(P_n T^*/\mu_n;k) = nul(T/\mu_n;k)$. Putting these inequalities together yields

$$\begin{aligned} nul(T^*;k) &\leq \limsup_{n\to\infty} nul(P_n T^*/\mu_n;k) \\ &= \limsup_{n\to\infty} nul(T \setminus \mu_n;k) \leq nul(T;k) \end{aligned} \tag{19}$$

In particular if is not in , then $\ker(T) \cap \mu_n = 0$ for all $n \geq 1$ and thus $\ker T^* = \{0\}$. Consider an operator in $\mathcal{L}(H \oplus H)$ of the form

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{20}$$

Where A, B, C belong to $\mathcal{L}(H)$. If and are triangular. For example, if is the compact backward weighted shift defined by

$$Ae_1 = 0, Ae_1 = (j-1)e_{j-1} \quad \text{for } j \geq 2$$

(with respect to the orthormal basis $\{e_j\}_{j=1}^{\infty}$ of H_1), $B = 0$ and is any operator mapping injectively onto a linear manifold of such that $ranC \cap ranA = R \cap AH$, then a straightforward computation shows that and

$$\vee\{\ker T^{k} : k = 1,2,3,\ldots\} = H \ominus \{0\} = H \ominus H \qquad (21)$$

If $R \in y(H)$ is a strict contraction (i.e., $\|R\| < 1$), and S is the backward shift of multiplicity one then $S^{*(\infty)}$ is unitarily equivalent to

$$\begin{pmatrix} R & * \\ 0 & S^{*(\infty)} \end{pmatrix}$$

The operator $S^{*(\infty)}$ is triangular, but the $(2,2)$-entry of the above matrix is not in general.

Nevertheless, the (2, 2)-entry is always triangular if the 2×2 operator matrix is triangular.

Lemma (2-2-3) [102]:

Let T be triangular operator with diagonal $d(T) = \{\lambda_j, j \geq 1\}$. Suppose that $\mathcal{M}$ is an invariant subspace for T and $B = (T^*/\mathcal{M})^* = P_{\mathcal{M}^\perp} T/\mathcal{M}^\perp$. Then B is triangular, and basis can be chosen so that $d(B) \subseteq d(T)$. In particular, $\sigma((T^*/\mathcal{M})^*) \cap d(T)^*$ is not empty.

Proof:

By our previous remarks the triangular of T implies that $H = \vee \ker(T - \lambda_j)^{w}$. With respect to $H = \mathcal{M} \oplus \mathcal{M}^\perp$, we can write

$$T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \text{and} \quad (T-\lambda)^{k} = \begin{bmatrix} (A-\lambda)^{k} & * \\ 0 & (B-\lambda)^{k} \end{bmatrix} \qquad (22)$$

So any vector $x \oplus y$ in $\ker(T - \lambda_j)^{k}$ yields the vector y in $\ker(B - \lambda)^{k}$. It follows immediately that $\mathcal{M}^\perp = \vee \ker(B - \lambda)^{w}$. Hence B is triangular with $d(B) \subseteq d(T)$. In particular, $\delta_p(B)$ meets $d(T)$. So $\sigma((T^*/\mathcal{M})^*) \cap d(T)^*$ is not empty.

Corollary (2-2-4) [102]:

If T is triangular, $d(T) = \{\lambda_j\}_{j=1}^{\infty}$ (with respect to some orthogonal basis) and

$$T=\begin{pmatrix} T_1 & & & * \\ & T_2 & & \\ & & T_3 & \\ O & & \text{o} & \end{pmatrix} \begin{matrix} \ker(T-\lambda_1)^w \\ \ker\left[(T-\lambda_1)(T-\lambda_2)\right]^w \Theta \ker(T-\lambda_1)^w \\ \ker\left[(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)\right]^w \\ \text{e}\ \ker\left[(T-\lambda_1)(T-\lambda_2)\right]^w \end{matrix} \qquad (23)$$

Then $T_1, T_2, \ldots$ are triangular operators. $T_1, T_2, \ldots$ may act on finite dimensional spaces; if $\lambda_k=\lambda_j$ for some $j<k$ then T_k acts on the trivial space $\{0\}$.

Proof:

It is obvious that $T_1=T/\ker(T-\lambda_1)^w$. Is triangular. By lemma (2-1-3)

$$B_2=\begin{pmatrix} T_2 & & * \\ & T_3 & \\ 0 & & \ddots \end{pmatrix} \qquad (24)$$

Is also triangular and a straight forward computational shows that $T_2=B_2/\ker(B_2-\lambda_2)^w$ is triangular. The result follows by induction. Our next result can be applied to a wide class of operators, not necessarily triangular. Observe that if

$$A=\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & A_3 & \\ 0 & & & \ddots \end{pmatrix} \begin{matrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \mathrm{H}_3 \\ \vdots \end{matrix} \in y\left(\sum_{j=1}^{\infty}\oplus \mathrm{H}_j\right)$$

(25)

where $A_j \in y(\mathrm{H}_j)$ for $j \geq 1$, and interior $\delta(A_j)=\phi$ for all j , then

$$\delta(A)=\left[\bigcup_{j=1}^{\infty}\delta(A_j)\right]^-$$

(and each component of $\delta(A)$ meets $\left[\bigcup_{j=1}^{\infty}\delta(A_j)\right]^-$ but in general this inclusion is proper. For instance it can happen that $\delta(A)=\left[\bigcup_{j=1}^{\infty}\delta(A_j)\right]^-$ is a totally disconnected set but $\delta(A)$ is connected.

Proposition (2-2-5) [102]:

Let $$A=\begin{pmatrix} A_1 & A_{12} & A_{13} & \cdots \\ & A_2 & A_{23} & \cdots \\ & A_3 & A_{33} & \cdots \\ O & & & \ddots \end{pmatrix}\begin{matrix} H_1 \\ H_2 \\ H_3 \\ \vdots \end{matrix} \tag{26}$$

and assume that $\delta(A_j)\cap\delta(A_k)=\phi$ if $j\neq k$. Then $\sum_{j=1}^{\infty}\oplus A_j < A$.

Proof:

It will be shown that there exists an upper triangular operator matrix of the form

$$X=\begin{pmatrix} b_1 & b_1X_{12} & b_3X_{13} & \cdots \\ & b_2 & b_3X_{23} & \cdots \\ & & b_3 & \cdots \\ O & & & \ddots \end{pmatrix}\begin{matrix} H_1 \\ H_2 \\ H_3 \\ \vdots \end{matrix} \tag{27}$$

Such that is a quasiaffinity and

$$AX=X\left(\sum_{j=1}^{\infty}\oplus A_j\right) \tag{28}$$

Here $\{b_j\}_{j=1}^{\infty}$ is a strictly decreasing sequence of positive real's converging to , $(b_1=1)$. The $X_{ik's}$ are inductively defined as follows: if we formally write

$AX=X\left(\sum_{j=1}^{\infty}\oplus A_j\right)$ (or, equivalently, $AX-X\left(\sum_{j=1}^{\infty}\oplus A_j\right)=0$), we obtain

$$0=\left(AX-X\left(\sum_{j=1}^{\infty}\oplus A_j\right)\right)_{ik}=b_k\left\{A_iX_{ik}+\sum_{r=i+1}^{k-1}A_{ir}X_{ir}+A_{ik}-X_{ik}A_k\right\}$$

$$=b_k\left\{(A_iX_{ik}-X_{ik}A_k)+A_{ik}+\sum_{r=i+1}^{k-1}A_{ir}X_{ir}\right\}$$

For $1\le i<k<\infty$, and the remaining entries of the matrix vanish identically. The (1,2)-entry shows that $A_1X_{12}-X_{12}A_2=-A_{12}$. Since $\delta(A_1)\cap\delta(A_2)=\phi$,

Rosenblum's theorem establishes the invertibility of $\tau_{A_1,A_2}\in y[y(H_2,H_1)]$,

where $\tau_{A_1,A_2}(X)=A_1X-XA_2$. Whence we readily obtain the unique solution

$$X_{12}=-\tau_{A_1,A_2}^{-1}(A_{12})\in y(H_1,H_2)$$

We proceed by induction; suppose the $k-1$ columns have been determined, and consider the 10th column. The above equation show that

$$\tau_{A_i,A_k}(X_{ik}) = -A_{ik} - \sum_{r=i+1}^{r-1} A_{ir} X_{rk}, \quad i = 1,2,\ldots,k-1$$

Which define the kth column. It is easily seen that the matrix defining X represents, indeed, a boundary linear mapping provided $b_j \to 0$ fast enough. and

$$AX = X\left(\sum_{j=1}^{\infty} \oplus A_j\right) .$$

For all possible choices of the b_j 's . Moreover

$$X\left(\sum_{j=1}^{k} \oplus H_j\right) = \sum_{j=1}^{k} \oplus H_j \qquad \text{(for all } k = 1,2,\ldots \text{)}$$

So that $(ranX)^{-} \supset V\left\{\sum_{j=1}^{k} \oplus H_j\right\} = \sum_{j=1}^{k} \oplus H_j = H$. Thus it only remains to show that the b_j's can be chosen so that X is injective. To this end, choose the b_j decreasing to 0 so fast that

$$\sum_{j=i+1}^{\infty} b_j \|X_{ij}\| < 2^{-i} b_i \qquad \text{for each } i \geq 1$$

(For example, one could recursively define $b_j = 2^{-j} b_{j-1} \min_{1 \le i < j} \|X_{ij}\|^{-1}$). Let

$x = \sum_{j=1}^{\infty} \oplus x_j$, for x_j in H_j be any vector in $\ker X$. The ith coordinate of X_x

is, $0 = b_i x_i + \sum_{j=i+1}^{\infty} b_j X_{ij} x_j$

Hence

$$\|x_i\| \le b_i^{-1} \sum_{j=i+1}^{\infty} b_j \|X_{ij}\| \, \|x\| \le 2^{-i} \|x\|; \quad \text{Whence} \quad \|x\|^2 = \sum \|x_i\|^2 \le \|x\|^2 / 3$$

(29)

Which implies $x = 0$.

Remark (2-2-6) [102]:

Under certain circumstances these two result can be applie to a triangular operator [illegible] to obtain triangular operators T_k with $\delta_p(T_k)=\{\mu_k\}$, and $\sum_{k\geq 1}\oplus T_k < T$. However to do this we require $\delta(T_k)$ to be pair wise disjoint.

Proposition (2-2-7) [102]:

Suppose [illegible] is a triangular operator and for some [illegible] in [illegible] and some integral [illegible], $nul((T-\lambda)^{*};k)<\infty$. Then $ord(T^{*};\lambda)<\infty$. That is there is an integer [illegible] so that $\ker(T-\lambda)^{*w}=\ker(T-\lambda)^{*m}$.

Lemma (2-2-8) [102]:

Let and be operators on . Assume that has dense range and $AX = XB$. Then $X\ker(B-\lambda)^k$ is contained in $\ker(A-\lambda)^k$.

Thus if triangular so is .

Lemma (2-2-9) [102]:

If is a quasiaffine transform of then $nul(B-\lambda k) \le nul(A-\lambda k)$ for all $\lambda \in \mathbb{C}, k \ge 1$. In particular if $A \sim_{qs} B$, the $nul(B-\lambda k) = nul(A-\lambda k)$ for all $\lambda \in C$ and $k \ge 1$.

Theorem (2-2-10) [102]:

Let be a bitriangular operator with diagonal $d(T) = \{\lambda_n, n \ge 1\}$ with respect to the triangularizing basis. Then

(i) $d(T) = d(T^*)^{\sim} = \sigma_p(T) = \sigma_p(T^*)^{\sim}$. Moreover

$nul(T-\lambda k) = nul((T-\lambda)^*, k)$

For all . Thus each in $\sigma_p(T)$ occur in $d(T)$ exactly $\dim\ker(T-\lambda)^w$ times. If $nul(T-\lambda k) < \infty$ for some , then $ord(T;\lambda) < \infty$.

(ii) If $\lambda \notin \sigma_p(T)$ and $T-\lambda$ is semi Fredholm, then $T-\lambda$ is invertible thus $\sigma(T) = \sigma_{Lre}(T) \cup \sigma_p(T)$.

(iii) Every nonempty clouse open subset of $\sigma(T)$ meets $\sigma_p(T)$ and card $\{j : \lambda_j \in \delta\} = \dim H(T, \delta)$. Hence each component of $\sigma(T)$ meets $\sigma_p(T)^{-}$.

Proof:

Note that we use the notation $\tilde{\Sigma} = \{\bar{\lambda} : \lambda \in \Sigma\}$ to avoid confusion with the notation the closure of a set. By lemma (2-2-2) $\sigma_p(T^*)^{\sim}$ is contained in $d(T)$, which is a subset of $\sigma_p(T)$. And a $\sigma_p(T)$ is contained in $d(T^*)^{\sim}$ which is a subset of $\sigma_p(T^*)^*$. So equality of these four sets in assured. Moreover, by the same lemma,

$$nul(T-\lambda;k)=nul\left((T-\lambda)^*,k\right)$$

Form the proof of this lemma, one sees that

$$nul\left((T-\lambda)^{*k}\right)\leq\lim_{n\to\infty}nul\left((T-\lambda/\mu_n)^k\right)\leq nul(T-\lambda)^k=nul\left((T-\lambda)^{*k}\right) \quad (30)$$

Since the number of occurrences of in $d(T)$ is easily seen to be

$\lim_{n\to\infty}nul(T-\lambda/\mu_n)^k$, it follows that this equals $nul(T-\lambda)^w$. The last statement of (i) is a consequences of proposition (2-2-7).

A semi-Fredholm triangular operator has index . Thus such an operator is either invertible, or belongs $\sigma_p(T)-\sigma_p(T^*)^*$. Hence $\sigma(T)=\sigma_{re}(T)\cup\sigma_p(T)$.

If is a clouse open subset of $\sigma(T)$ then by the Riesz functional calculus, is similar to an operator $T_\delta\oplus T_{\delta'}$. Such that $\delta(T_\delta)=\delta$ and $\delta(T_{\delta'})$ is disjoint from . By lemma (2-2-3), T_δ is bitriangular. Hence $\delta_p(T_\delta)$ is a non-empty subset of $\sigma_p(T)=d(T)$. Indeed, $d(T_\delta)$ is necessarily a subset of $d(T)$.

Of cardinality $\dim H(T,\delta)$. That each component of $\delta(T)$ meets $\delta_p(T)^-$ is a simple topological consequence.

Proposition (2-2-11) [102]:

An operator is bitriangular if and only if

$$V\left\{\ker(T-\lambda)^k:\lambda\in C,k\geq1\right\}=H=V\left\{\ker(T-\lambda)^{*k}\ \lambda:C,\quad k\geq1\right\} \quad (31)$$

Hence the class $(B\Delta)$ is closed under quasisimilarities. We were able to apply Remark (2-2-6) to both and . We would obtain triangular operators T_k and T_k' for $k\geq1$, such that $\delta_p(T_k)=\delta_p(T_k')=\{\mu_k\}$, Where $\{u_k,k\geq1\}=\sigma_p(T)$ so that

$\sum_{k\geq1}\oplus T_k \prec T$ and $\sum_{k\geq1}\oplus T_k'^* \prec T^*$. Thus

$$\sum_{k\geq1}\oplus T_k \prec T \prec \sum_{k\geq1}\oplus T_k' \quad (32)$$

If we know that $T_k \sim_{qs} T_k'$ for each k, this would reduce to consideration of the case $\sigma_p(T)=\{\mu\}$. When $null(T-\mu;m)=0$ for some $m, T-\mu$, is nilpotent and the Apostol Dauglas theorem applies. So the case $nul(T-\mu;k)=\infty$ for all k remains.

An operator is quasitriangular if it has a compact perturbation which is triangular, and an operator T is biquasitriangular if both T and T^* are quasi triangular. If $\mu_1 \subset N_1 \subset \mu_2 \subset N_2 \subset \ldots$ such that μ_k is invariant for T and N_k is invariant for T^* for all $k \geq 1$. With respect to the decomposition

$$H = \mu_1 \oplus (N_1 \ominus \mu_1) \oplus (\mu_2 \ominus N_1) \oplus (N_2 \ominus \mu_2) \oplus \ldots$$

The operator T has the matrix form

$$T = \begin{bmatrix} A_1 & B_1 & & & & & \\ & C_1 & & & & O & \\ & D_1 & A_2 & B_2 & & & \\ & & & C_2 & & & \\ & & & D_2 & A_3 & B_3 & \\ & O & & & & D_3 & \\ & & & & & & \ddots \end{bmatrix}$$

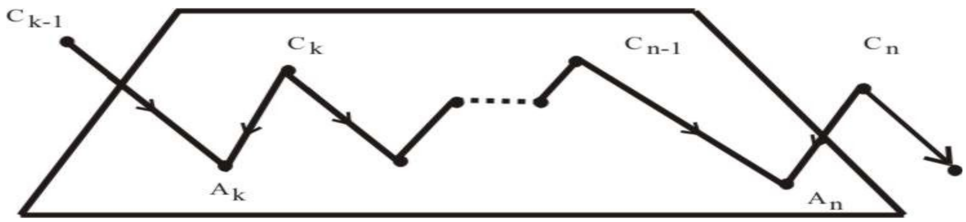
(33)

From this form it is clear that T is both block upper triangular and block lower triangular, and so is (BA). We introduce the following diagrammatic device to represent the matrix

$$\begin{array}{l} \ker(T-\lambda_1)^w \\ \ker\left[(T-\lambda_1)(T-\lambda_2)\right]^r \ominus \ker(T-\lambda_1)^w \\ \ker\left[(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)\right]^r \\ \ominus \ker\left[(T-\lambda_1)(T-\lambda_2)\right]^r \end{array} \tag{34}$$

Note that A_k represents the operator mapping the invariant subspace $\mu_k \ominus N_{k-1}$ into itself. Similarly C_k maps $N_k \ominus \mu_k$ into itself and this is T^* invariant.

Diagrammatically



(35)

Similarly $N_n \oplus u_k (k \le n)$ is T invariant and can be collapsed into one.

Example (2-2-12) [102]

Note every bitriangular operator admits a fair case representation. Let

$$R = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ & 1/2 & & & O \\ & & 1/3 & & \\ & O & & 1/4 & \\ & & & & \ddots \end{bmatrix}$$

With respect to the given basis $e_n, n \ge 1$, R , can be written

$$R = \sum_{n \ge 1} n^{-1} e_n \otimes e_n^{e} + e_1 \otimes \left(\sum_{n \ge 2} n^{-1} e_n \right)^* \qquad (36)$$

This is a compact triangular operator. Let

$$S = \begin{bmatrix} 1 & 1 & 1/2 & 1/3 & \cdots \\ & 1 & & & \\ & & 1 & & 0 \\ & 0 & & 1 & \\ & & & & \ddots \end{bmatrix} = I + e_1 \otimes \left(\sum_{n \ge 1} n^{-1} e_{n+1} \right)^*$$

A simple computation shows that S in invertible and

$$SRS^{-1} = \sum_{n \ge 1} n^{-1} e_n \otimes e_n^* = diag\left(n^{-1}\right) \qquad (37)$$

This is diagonal, and thus R is bitriangular. The eigenvalues of R are $\{n^{-1}, n \ge 1\}$ and the corresponding eigenvector are $f_1 = e_1$ and $f_n = S^{-1} e_n - (n-1)^{-1}$ for $n \ge 2$. The corresponding eigenvectors for R^* are

$g_1 = S^{*}e_1 = e_1 + \sum_{n\ge 2}(n-1)^{-1}e_n$ and $g_n = e_n$ for $n \ge 2$. Since $\{g_n, n \ge 1\}$ forms a basis for any staircase form for must contain in some N_k.

But then μ_{k+1} must both contain and be spanned by a finite subset of $\{f_n, n \ge 1\}$. This is clearly impossible. This example is similar to δRS^{-1} which being diagonal has a staircase model.

Lemma (2-2-13) [102]:

Let and be finite dimensional subspaces, and let and be the corresponding projections. Suppose that $\|M - M^{\perp}NM\| = \delta < (\varepsilon/\dim \mu)^2 \le \frac{1}{2}$.

Then there is an operator $S = I + X$ such that $X = M^{\perp}XM$ and $\|X\|_1 \le 2\varepsilon$, so that SN contains . Similarly there is an operator $T = I + Y$ such that $Y = N^{\perp}YN$ and $\|Y\|_1 < 2\varepsilon$, so that $T\mu$ is contained in .

Proof:

Note that $M^{\perp} + MNM \ge (1-\delta)I$, so $A = NM(M^{\perp} + MNM)^{-1}$, has norm at most $(1-\delta)^{-1}$. With respect to $H = \mu \oplus \mu^{\perp}$, has the matrix $\begin{pmatrix} 1 & 0 \\ H & 0 \end{pmatrix}$. Thus

$\|H\| \le \left((1-\delta)^{-2} - 1\right)^{\frac{1}{2}} < 2\delta^{\frac{1}{2}}$. Hence $\|H\|_1 \le 2\delta^{\frac{1}{2}}(\dim \mu) < 2\varepsilon$. The range of is a subspace of . The operator $S = \begin{pmatrix} 1 & 0 \\ -H & 1 \end{pmatrix}$ is of the desired form, and it maps onto . Hence SN contains . For the second statement let be the projection onto $N\mu$. Decompose $H = N \oplus N^{\perp}$. As above there is an operator $B = \begin{pmatrix} P/N & 0 \\ k & 0 \end{pmatrix}$ with range equal to and $\ker B = P^{\perp}H$. Let $T = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$. Then $kP = k$. T maps onto $N\mu$. As before

$$\|k\|_1 \le \|k\| \dim \mu < 2\varepsilon \tag{38}$$

Proposition (2-2-14) [102]:

Let T be a bitriangular operator. Then given $0<\varepsilon<1$, there exists a trace class operator K with $\|X\|_1<\varepsilon$ such that $(I+X)T(1+X)^{-1}$ admits a staircase representation.

By a Jordan operator we mean a direct sum of dimensional operators λI_k+J_k, where $\lambda\in C, I_k$ is the identity on C^k and J_k is the standard Jordan nilpotent operator of order k given by $J_k e_1=0$, $J_k e_i=e_{i-1}$ for $2\le i\le k$.

Theorem (2-2-15) [102]:

Let T be a bitriangular operator with $\delta_p(T)=\{\lambda_n, n\in I\}$. Suppose that for all $n\in I$ and all $k\ge 1$, $nul(T-\lambda_n;k)$ is 0 or ∞. Then there are a Jordan operators H_1 and H_2 such that $H_1\prec T\prec H_2$, and

$$nul(H_1-\lambda_n;k)=nul(H_2-\lambda_n;k)=nul(T-\lambda_n;k)$$ for all $n\in I$ and $k\ge 1$.

Lemma (2-2-16) [5]:

Let $m_1\le n_1\le m_2\le n_2\le \ldots$ be a monotone increasing sequence of positive integers. Let $A=\sum_{j=1}^{\infty}\oplus J_{mj}$ and $B=\sum_{j=1}^{\infty}\oplus J_{nj}$. Then $A\sim_{qs} B$.

Proof

Let $\{e_i^{(j)}:1\le i\le m_j\}$ and $\{F_i^{(j)}:1\le i\le n_j\}$ be the canonical basis for the spaces H_j and K_j on which J_{mj} and J_{nj} act respectively. Define $d_j=n_j-m_j$. Then define a linear operator X by

$$Xf_i^{(j)}=(J!)^{-1}e_{i-dj}^{(j)}-((j+1)!)^{-1}e_i^{(j+1)} \tag{39}$$

Where $e_s^{(j)}\equiv 0$ for $s\le 0$. Note that X extends to compact operator from $k=\sum_{j=1}^{\infty}\oplus k_j$ into $H=\sum_{j=1}^{\infty}\oplus H_j$. A routine computation on these basis vector shows that $AX=XB$. We show that X is a quasiaffinity. Suppose that $x=\sum_{j=1}\sum_{i=1}^{nj}a_i^{(j)}f_i^{(j)}$ and $0=Xx$. Then $0=\sum_{j\ge 1}\sum_{i=1}^{n_j}a_i^{(j)}\left(\frac{1}{J!}e_{i-dj}^{(j)}-\frac{1}{(j+1)!}e_i^{(j+1)}\right)$ from equation (57) and

$$0=\sum_{j\ge 1}\sum_{i=1}^{m_j}\frac{1}{j!}a_{i+dj}^{(j)}e_i^{(j)}-\sum_{j=2}\sum_{i=1}^{n_j-1}\frac{1}{j!}a_i^{(j-1)}e_i^{(j)}$$
$$=\sum_{j\ge 1}\sum_{i=1}^{m_j}\frac{1}{j!}\left(a_{i+dj}^{(j)}-a_i^{(j-1)}\right)e_i^{(j)}$$

(40)

Where we drop the convention $a_i^{(0)}\equiv 0$. Examination of each coefficient yields

$$a_i^{(j)}=a_{i+dj+1}^{(j+1)}=a_{i+dj+1+dj+2}^{(j+2)}=\ldots$$

Since $\|x\|\to\infty$ all coefficients must be 0 , so X is injective to see that X has dense range, note that

$$X\left(j!\,f_{i+dj}^{(j)}\right)=e_i^{(j)}-(j+1)^{-1}e_{i+dj}^{(j+1)}$$

$$X\left(j!\,f_{i+dj+\ldots+dj+s}^{(j+s)}\right)=\frac{J!}{(j+s)!}e_{i+dj+\ldots+dj+s+1}^{(j+s)}-\frac{J!}{(j+s+1)!}e_{i+dj+\ldots+dj+s}^{(j+s+1)}$$

Summing these terms for $0\le s\le p-1$ yields

$$e_i^{(j)}-(j!/(j+p)!)e_{i+dj+\ldots+dj+p-1}^{(j+p)}$$

(41)

In range (X) . Consequently, $e_i^{(j)}$ belongs to $ran(X)^-$ for all i and j . So X has dense range. It follows that $B \prec A$. But then $A\cong A^* \prec B^*\cong B$ So $A\sim_{qs}B$.

Theorem (2-2-17) [102]:

Let $\{m_k, k\ge 1\}$ and $\{n_k, k\ge 1\}$ be sequences of positive integers. Set

$A=\sum_{k=1}^{\infty}\oplus J_{mk}$ and $B=\sum_{k=1}^{\infty}\oplus J_{nk}$. Then $A\sim_{qs}B$ if and only if $nul(A;k)=nul(B;k)$ for all $k\ge 1$.

Proof:

The necessity follows from lemma (2-2-9). So we suppose that $nul(A;k)=nul(B;k)$ for $k\ge 1$. It is easy to see that $nul(A;k)-nul(A;k-1)$, Jordan blocks of size k in A for $k_0 < l \le l_0$. The same holds for B and the sum of these blocks for A and B are thus unitarily equivalent. By restricting our attention to remaining summands, we can assume that $m_k\le k_0$ and $n_k\le k_0$ for all $k\ge 1$ with equality holding infinitely often in both cases.

In either case: (i) $Sup\, m_j = \infty = \sup n_j$ or (ii) $\sup m_j = \lim\sup m_j = \lim\sup n_j = k_0 < \infty$; it is routine exercise to split $\{m_j\}$ and $\{n_j\}$ into at most countably many infinite subsets, which we denote by $\{m_{ij}, j \geq 1\}$ and $\{n_{ij}, j \geq 1\}$ so that for each i , either $m_{i1} \leq n_{i1} \leq m_{i2} \leq n_{i2} \leq \ldots$ or $n_{i1} \leq m_{i1} \leq n_{i2} \leq m_{i2} \leq \ldots$ let $A_i = \sum_{j\geq 1} \oplus J_{m_{ij}}$ and $B_i = \sum_{j\geq 1} \oplus J_{n_{ij}}$. By lemma (2-2-16), $A_i \sim_{qs} B_i$. Hence $A \sim_{qs} B$ as desired.

It now apparent by comparison of theorems (2-2-15) and (2-2-17) how to obtain a Jordan operator quasisimilar to given bitriangular operator T . We wish to define a canonical Jordan model for T , which we denote by $J(T)$. Define

$$x(T-\lambda;k) = nul(T-\lambda;k) - nul(T-\lambda;k+1)$$

Where $\infty - \infty$ is deigned to be 0 . By analogy with the finite dimensional case, let

$$J(T) = \sum_{\lambda\in\delta_p(T)} \oplus J(T;\lambda) = \sum_{\lambda\in\delta_p(T)} \sum_{k\geq 1} \oplus(\lambda I_k + J_k)^{(x(T-\lambda;k))}$$

Note that there are three cases:

(i) When $nul(T-\lambda)^w < \infty$, $J(T;\lambda)$ is the Jordan form of $T \setminus \ker(T-k)^w$.

(ii) When $nul(T-\lambda;k_0) = \infty > nul(T-\lambda;k) > 0$ for $k_0 < k < I_0$ and $nul(T-\lambda;I_0) = 0$, then $J(T;\lambda)$ equals $\sum_{k=1}^{k_0} \oplus(\lambda I_k + J_k)^{(\infty)} \oplus \bigoplus_{k=k_0+1}^{I_0-1} \oplus(\lambda I_k + J_k)^{x(T-\lambda;k)}$

(iii) When $nul(T-\lambda;k) = \infty$ for all $k \geq 1$, then $J(T;\lambda)$ equal $\sum_{k\geq 1} \oplus(\lambda I_k + J_k)^{(\infty)}$

By proposition (2-2-7), these cases are exhaustive. We can obtain two special cases of the main theorem as corollaries of theorems (2-2-15) and (2-2-17).

Section (2-3): Complementary Invariant Subspaces and the Relative boundedness of Triangular and Bitriangular Operators

Corollary (2-3-1) [102]:

Let T be a bitriangular operator such that $nul(T-\lambda)^k = 0$ or for each $\lambda \in \mathbb{C}$ and $k \geq 1$. The $T \underset{qs}{\sim} J(T)$. [34, 41, 105, 38].

Corollary (2-3-2) [102]:

Let be an algebric operator. Then $T \underset{qs}{\sim} J(T)$. Recall that two subspa-ces and are quasicomplementary if $M \cap N = 0$ and $M \vee N = H$. We require a technical lemma.

Lemma (2-3-3) [102]:

Let $A = \sum_{k \geq 1} \oplus A_k$ be an operator in $\mathcal{Y}(H \oplus H)$, where

$\delta_p(A_k) = \{\lambda_k\}$ are distend complex numbers and $J = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ for $k \geq 1$. Suppose

is an operator in such that $B \underset{qs}{\sim} A$. For each subset $\Gamma \subset £$, set

$H(B, \Gamma) = V\left\{\ker(B - \lambda)^w : \lambda \in \Gamma\right\}$.

Then $H(B, \Gamma)$ and $H(B, £ \setminus \Gamma)$ are quasicomplementary hyperinvariant subspaces . If is a quasiaffinity such that $AY = YB$ then $YH(B, \Gamma)^{-} = H(A, \Gamma)$

. If $\{\Gamma_x\}$ is a collection of subsets of , then $H\left(B, \underset{x}{U} \Gamma_x\right) = \underset{x}{V} H(B, \Gamma_x)$ If

$\cap_x \Gamma_x = \Phi$, then $\underset{x}{\cap} H(B, \Gamma_x) = \{0\}$. We prove the following result.

Theorem (2-3-4) [102]:

Let be a bitriangular operator. Then $T \underset{qs}{\sim} J(T)$.

Proof:

Let $\delta_p(T)$ be a bitriangular operator. Then $T \underset{qs}{\sim} J(T)$. Where

$ord(T, \mu_i) = \infty$ so that $nul(T - \mu_i; k) = \infty$ for all $k > m_j, i \geq 1$ and $ord(T, v_j) = m_j < \infty$ so that

$$nul(T - v_j, k) = 0 \quad \text{for all} \quad k > m_j, j \geq 1$$

Define $R = \sum_{j \geq 1} \oplus v_j I + J_{m_j}^{(\infty)} \in \mathcal{Y}(R)$. Then $T \oplus R$ is a bitriangular operator in

$\mathcal{Y}(H \oplus R)$. Moreover, $\delta_p(T \oplus R) = \delta_p(T)$,

$$nul\left(T \oplus R - v_j I; k\right) = \begin{cases} \infty & , k \le m_j \\ 0 & , k > m_j \end{cases} , \quad nul(T \oplus R - \mu_i I; k) = \infty \quad \text{for} \quad k \ge 1$$

and $\ker(T \oplus R - \mu_i I)^w = \ker(T - \mu_i I)^w \oplus \{0\}$ for $i \ge 1$. Let $J = J(T \oplus R)$.

By corollary (2-3-1), $J \underset{qs}{\sim} T \oplus R$. Let X be a quasiaffinity such that $JX =$ $X(T \oplus R)$. Let $H_0 = X(\mathrm{H} \oplus \{0\})^-$ and set $X_0 = X / \mathrm{H} \oplus \{0\}$, considered as an element of $\mathcal{Y}(\mathrm{H}, \mathrm{H}_0)$. Let $J_0 = J / \mathrm{H}_0$. Clearly, $J_0 X_0 = X_0 T$ so that $T \prec J_0$. We wish to show that J_0 is quasisimalar to a Jordan operator. By lemma (2-3-3),

$$X_0 \mathrm{H}(T, \Gamma_1)^- = X\mathrm{H}(T \oplus R, \Gamma_1)^- = \mathrm{H}(J, \Gamma_1) = \mathrm{H}(J_0, \Gamma_1)$$

and

$$X\mathrm{H}(T \oplus R, \Gamma_2)^- = \mathrm{H}(J, \Gamma_2) = \mathrm{H}(J, \Gamma_1)^{\perp}$$

also

$$\mathrm{H}_0 = XV\left\{\ker\left(T - \lambda\right)^w : \lambda \in \delta_\rho(T)\right\} \subseteq V\left\{\ker\left(J_0 - \lambda \in \delta_\rho(T)\right)\right\} \subseteq \mathrm{H}_0$$

Hence

$$\mathrm{H}_0 = V\left\{\ker(J_0 - \lambda)^w : \lambda \in \delta_\rho(T)\right\} = \sum \oplus \left\{\mathrm{H}_0 \cap \ker(J - \lambda)^w : \lambda \in \delta_\rho(T)\right\}$$

When $\lambda = \mu_i \in \Gamma_1$, then $\mathrm{H}_0 \cap \ker(J - \mu_i)^w = \ker(J - \mu_i)^w$ and $J_0 / \ker(J_0 - \mu_i)^w$ is a Jordan operator. When $\lambda = v_j$, $\ker\left(J - v_j\right)^w = \ker\left(J - v_j\right)^{m_j}$, so $J_0 / \ker\left(J_0 - v_j\right)^{m_j}$ is algebric and thus by corollary (2-3-2), $J_0 / \ker\left(J_0 - v_j\right)^{m_j}$ is quasisimilar to Jordan operator. Thus J_1 is thus $J_1^* \prec T^*$. By lemma (2-2-9), and theorem (2-2-10)

$$\begin{aligned} nul\left(J_1 - \lambda; k\right) &= nul\left(\left(J_1 - \lambda\right)^*; k\right) \\ &\le nul\left(\left(J_1 - \lambda\right)^*; k\right) = nul\left(T - \lambda; k\right) \\ &\le nul\left(J_1 - \lambda; k\right) \end{aligned}$$

Consequently, $nul(J_1 - \lambda; k) = nul(T - \lambda; k)$ for all λ in $\mathbb{C}$ and $k \ge 1$.

Treating T^* similarly one finds a Jordan operator J_2 so that $T^* \prec J_2^*$, whence $J_2 \prec T$. As above

$$nul(J_2-\lambda;k)=nul((J_2-\lambda)^*;k)=nul((T-\lambda)^*;k)=nul((T-\lambda);k)$$

By theorem (2-2-17), $J_1 \underset{qs}{\sim} J_2 \underset{qs}{\sim} J(T)$. Thus $J_1 \sim_{qs} J(T)$.

Corollary (2-3-5) [102]:

Let S and T be bitriangular operators. Then the following are equivalent:

(i) $S \sim_{qs} T$

(ii) $nul(S-\lambda)^k = nul(T-\lambda)^k$ for all $\lambda \in \mathbb{C}, k \geq 1$

(iii) $J(S) \cong J(T)$

Corollary (2-3-6) [102]:

Let T be a bitriangular operator such that $\sigma_p(T)$ is real. The $T \sim_{qs} T^*$.

Corollary (2-3-7) [102]:

Let S and T be bitriangular operators such that $S \prec T$. Then $S \sim_{qs} T$.

Corollary (2-3-8) [102]:

Suppose S and T are bitriangular operators such that $S \prec_i T$ and $T \prec_i S$. Then $T \sim_{qs} S$.

Lemma (2-3-9) [102]:

Let T be a bitriangular operator and let $\varepsilon > 0$. Then T is quasisimilar to an operator of the form $N + K$, where N is a diagonal normal, K is quasi-nilpotent trace class operator which commutes with N, $\|K\|_1 < \varepsilon, \sigma_p(N) = \sigma_p(T)$ and $\sigma(N) = \sigma(N+K) = \sigma_p(T)^-$.

Proposition (2-3-10) [102]:

Let T be a bitriangular operator. A compact subset A of $\mathbb{C}$ is the spectr-um of an operator $S \sim_{qs} T$ if and only if (i) A contains $\sigma_p(T)$ (ii) each compo-nent Ω of A meets $\sigma_p(T)^-$, and (ii) each component Ω of A which is not a sin-gleton meets $\sigma_p(T)_u$.

Corollary (2-3-11) [102]:

Let T be a bitriangular operator. Suppose that $\lambda \in \delta_p(T)$ is an isolated point of $\delta_p(T)^-$; $nul(T-\lambda)=\infty$, and $nul(T-\lambda)=\infty$ and $2 \le ord(T,\lambda) < \infty$. Then $2y(T)$ is not contained in $y(S)^-$ for any $S \sim_{qs} T$.

Theorem (2-3-12) [102]:

Let T be a bitriangular operator. If for each isolated point λ of $\delta_p(T)^-$, either (i) $ord(T;\lambda)=1$, or (iii) $ord(T;\lambda)=\infty$, then there is an operator $S \sim_{qs} T$ such that $y(S)^-$ contains $2y(S)^-$. Conversely, if $y(S)^- \supset 2y(T)$ for some $S \sim_{qs} T$, then T satisfies the conditions above. In particular, $2y(T) \supset y(S)^-$ if and only if $\delta(T_\lambda)=\{\lambda\}$, and $(T_\lambda - \lambda I)^k$ is not compact $T_\lambda \oplus T^1$, where $\delta(T)^-$ $=\delta(T)\setminus\{\lambda\}, \delta(T_\lambda)=\{\lambda\}$ and is not compact for all $k \ge 1$.

Proof:

Corollary (2-3-11) shows that condition on isolated points is necessary. Further, if $y(T)^-$ contains $2y(T)$, then lemma (2-3-9) and the upper semi-continuity of the spectrum imply that $\delta(T)=\delta_p(T)^-$. So when λ is an isolated point of $\delta(T)^-$ the Riesz functional calculus implies that $T \sim T_\lambda \oplus T^1$ such that $\delta(T_\lambda)=\{\lambda\}$ and $\delta(T^1)=\delta(T)/\{\lambda\}$. When (iii) holds $T_\lambda - \lambda I$ is a non-nilpotent quasinilpotent. If $(T_\lambda - \lambda I)^k$ is compact that this properly persists for any operator A in $y(T)^-$ such that λ is still an isolated point of $\delta(A)$. But $2y(T)$ contains operators S such that $S \sim S_\lambda \oplus S^-$, $\delta(S_\lambda)=\{\lambda\}$ and $\delta(S') : \delta(T) \setminus \lambda$, and so that $(T_\lambda - \lambda I)^k$ is not compact for any k . This proves necessity. The converse follows from the similarity orbit.

Example (2-3-13) [102]:

Let us look at our Jordan models. In lemma (2-2-3) we showed that the 2-2 corner of a triangular form of a triangular operator is triangular whereas the 1-1 corner need not be triangular. This phenomenon occur even for not be triangular Jordan operators. Let $T = \sum_{n \ge 1} \oplus J_n$ act on $H = \sum_{n \ge 1} \oplus H_n$ is n-dimensional with standard basis $\{e_i^{(n)} : 1 \le i \le n\}$ so that $Te_i^{(n)} = e_{i-1}^{(n)}$ for $2 \le i \le n$ and $Te_i^{(n)} = 0$.

Let $X_k = \sum_{n>k} \alpha_n e_{n-k}^{(n)}$, $\alpha_n = (n(n-1))^{-\frac{1}{2}}$. Then $\{x_k, k \ge 0\}$ are pair wise orthogonal and $Tx_k = x_{k-1}$. Let $\mu = span\{x_k, k \ge 0\}$. Clearly, is a weight shift with weights $\|x_{k+1}\| \|x_k\|^{-1} = (k+1/k+2)^{\frac{1}{2}}$. Hence is a Fredholm operator of index-1, and thus is not triangular.

An even more striking example is obtains by taking $N = span\{x_0, N_0\}$, where $N_0 = span\{e_i^{(n)} : 1 \le i \le n-1, n \ge 2\}$. This is invariant for , and by lemma (2-2-3), is triangular. However $x_k = T^k x_0 \ne 0$ is orthogonal to $T^k N_0$ for all $k \ge 0$. So we need that

$$\ker(T \setminus N)^w = N_0 \le N$$

So is not triangular. Nevertheless, N_0 has co-dimension 1 in and $T \setminus N_0 \cong \sum_{n \ge 2} \oplus J_{n-1} \cong T$. So $T : T \setminus N_0 \; T \setminus N \; T$. But is not quasisimilar to any bitriangular operator. This shows that corollary (2-3-8) cannot be extended much. On the other hand hyper invariant subspaces of Jordan model are very easily described.

Lemma (2-3-14) [102]:

Let $T = \sum_{k \geq 1} \oplus \lambda_k I + J_{(k)}$, where $\lambda_{(k)}$ are distinct and each $J_{(k)}$ is the direct sum of nilpotent Jordan blocks. Then the hyperinvariant subspaces of T are precisely $Lat_h(T) = \left\{ \sum_{k \geq 1} \oplus \mu_k \in Lat_h(J_{(k)}) \right\}$. If $J = \sum_{n \geq 1} \oplus J_k$ is a direct sum of Jordan blocks, then $Lat_h(J)$ consists of all subspaces of the form

$\sum_{n \geq 1} \oplus \mu_{i_n}^{(n)}$ where $\mu_{i_n}^{(n)} = span\{e_j^{(n)} : 1 \leq j \leq i_n\}$ and

$i_n \leq i_m \leq i_n + (k_m - k_n)$ if $k_m \geq k_n$

Corollary (2-3-15) [102]:

If T is a Jordan bitriangular operator and N is hyperinvariant for T , then T/N and $P_\mu^\perp T/\mu^\perp$ are both Jordan bitriangular operators.

Example (2-3-16) [102]:

It is easy to give an example that show that quasisimilarity does not preserve the hyperlattice. Let $A = J_3^{(\infty)}$ and $B = J_2 \oplus J_3^{(\infty)}$. Then $A \sim_{qs} B$. But $Lat_h(A)$ consists of $\{0\}$, $\ker A, \ker A^2$, and H , whereas $Lat_h(B)$ consists of $\{0\}$, $\ker J_2 \oplus J_3^{2(\infty)}$, $\ker B^2$ and H .

Example (2-3-17) [102]:

Lemmas (2-2-7) and (2-2-8) might suggest that if $A \sim_{qs} B$ and if X is a quasiaffinity such that $BX = XH$, then $X(\ker A^k)^- = \ker B^k$. This is far from true. Take $A = J_n^{(\infty)}$ and $B = J_n^{(\infty)} \oplus J_m^{(\infty)}$ with $m < 1$ so that $A \sim_{qs} B$. Represent these operators on $H^{(n)}$ and $H^{(n+m)}$ by

$$A=\begin{pmatrix} 0 & 1 & & & O \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & \ddots & 1 \\ O & & & & 0 \end{pmatrix} \quad \text{and} \quad B=\left[\begin{array}{ccccc|ccccc} 0 & 1 & & & o & & & & & \\ & 0 & 1 & & & & & & & \\ & & \ddots & \ddots & & & & O & & \\ & & & 0 & 1 & & & & & \\ o & & & & 0 & & & & & \\ \hline & & & & & 0 & 1 & & & o \\ & & & & & & 0 & 1 & & \\ & & O & & & & & \ddots & \ddots & \\ & & & & & & & & 0 & 1 \\ & & & & & o & & & & 0 \end{array}\right]$$

A routine computation shows that if and are quasiaffinities such that $RanY^- \cap Ranz^- = \{0\}$, then

$$X=\left[\begin{array}{cccc} Y & & & \\ & Y & & O \\ & & \ddots & \\ & O & & \ddots \\ & & & Y \\ \hline z & & & O \\ & z & & \\ O & & \ddots & \\ & & & \ddots \\ & O & & z \end{array}\right]$$

is a quasiaffinity of $H^{(n)}$ into $H^{(n+m)}$ such that $BX = XA$. Clearly $X(\ker A^k)^-$ is a proper subspace of $\ker B^k$ for $1 \le k \le n-1$.

Example (2-3-18) [102]:

The situation is even worse when is bitriangular but not Jordan. Let be the Jordan model of , and let and be quasiaffinities such that $YT = JY$ and $TX = XJ$. By lemma (2-3-3), $Y\ker(T-\lambda)^w = \ker(J-\lambda)^w$ for all in $\delta_p(T)$. On the other hand, even if $\ker T^w = \ker T$ it may occur that $X(\ker J)^-$ is a proper subspace of $\ker T$. Let be the diagonal normal compact operator ?. Consider the operator in $y(H \oplus H)$ given by

$$T=\begin{bmatrix}0 & D\\ 0 & D^3\end{bmatrix}\cong\sum_{n\geq 1}\oplus\begin{pmatrix}0 & n^{-1}\\ 0 & n^{-3}\end{pmatrix}$$

This is block diagonal and thus is bitriangular. Its Jordan form is easily seen to be $J=\begin{bmatrix}0 & 0\\ 0 & D\end{bmatrix}$. Let $\{e_n, n\geq 1\}$ be the standard basis diagonalizing D, and set $f=\sum_{n\geq 1} e_n/n$. Let W be any isomtery of H onto $\{f\}^{\perp}$. Then $X=\begin{bmatrix}W & D\\ 0 & D^3\end{bmatrix}$ Is an operator satisfies $TX=XJ$. Moreover X is one to one, since $0=X\begin{pmatrix}x\\ y\end{pmatrix}$, implies $D^3y=0$, whence $y=0$, hence $Wx=0$; so $\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$. Next, notice that

$$X\begin{pmatrix}0\\ n^2e_n\end{pmatrix}=\begin{pmatrix}ne_n\\ n^{-1}e_n\end{pmatrix}$$

Let $r=\|f\|^2$. Since $(rne_n, f)=\|f\|^2$, there is a vector x_n such that $Wx_n=f-rne_n$.

So $X\begin{pmatrix}x_n\\ rn^2e_n\end{pmatrix}=\begin{pmatrix}f\\ rn^{-1}e_n\end{pmatrix}$ converges to $\begin{pmatrix}f\\ 0\end{pmatrix}$. It follows that

$$RanX^{-}\geq Ran(W)^{-}VC\begin{pmatrix}f\\ 0\end{pmatrix}=H\oplus 0$$

Whence $RanX^{-1}=H\oplus Ran(D^3)^{-}=H\oplus H$. Thus X is a quasiaffinity.

The final is the observation that $X\ker J=RanW$ is a proper subspace of $\ker T$. The only good thing we can say a bout this situation is that the smallest hyperinvariant subspace of T containing $RanW$ is all of $\ker T$. This is because the rank two projection onto $span\left\{\begin{pmatrix}e_n\\ 0\end{pmatrix},\begin{pmatrix}0\\ e_n\end{pmatrix}\right\}$ commute with T.

Proposition (2-3-19) [102]:

Let S and T be quasisimilar bitriangular operators. Then there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$ and such that

$XH(S;T)^{-} = H(T,\Gamma)$ and $YH(T,\Gamma)^{-} = H(S,\Gamma)$ for every subset Γ of $\mathbb{C}$.

Proof:

It suffices to assume that $S = J$ is the Jordan model of T. By lemma (2-3-3), we need only construct X. Let $T_\lambda = T/\ker(T-\lambda)^{w}$ for each λ in $\delta_p(T)$. Clearly T_λ is triangular. By lemma (2-3-3), T_λ^{*} is also triangular. So T_λ is bitriangular. It is obvious that

$$nul(T_\lambda - \lambda k) = nul(T - \lambda k) \quad \text{for } k \geq 1$$

Thus an application of corollary (2-3-5) yields $T \sim_{qs} \sum_{\lambda \in \delta_p(T)} \oplus T_\lambda$. Let J_λ be the Jordan model of T_λ. So $J = \sum_{\lambda \in \delta_p(T)} \oplus J_\lambda$. Let X_λ be a quasiaffinity such that $T_\lambda X_\lambda = X_\lambda J_\lambda$. Now let P_λ be the orthogonal projection onto the domain of J_λ, and let W_λ be the natural injection of $\ker(T-\lambda)^{w}$ into H. For suitably chosen positive constants C_λ, the operator

$$X = \sum C_\lambda W_\lambda X_\lambda P_\lambda$$

Is a bounded operator $\left(C_{\lambda_n} = 2^{-n} \| X C_{\lambda_n} \|^{-1}\right)$. It clear that $TX = XJ$ and that $XH(J,\lambda)^{-} = H(T,\lambda)$ for $\lambda \in \delta_p(T)$. In particular, X has dense range. For any subset Γ of $\mathbb{C}$

$$XH(J,\Gamma)^{-} = \bigvee_{\lambda \in \Gamma} XH(J,\lambda)^{-} = \bigvee_{\lambda \in \Gamma} H(T,\Gamma) \quad .$$

By lemma (2-3-3) $H(T,\{\lambda\}) \cap H(T, C \setminus \{\lambda\}) = \{0\}$. Thus if $v = \sum \oplus v_\lambda$ lies in $\ker X$, then

$$C_\lambda W_\lambda X_\lambda v_\lambda = -\sum_{\mu \neq \lambda} C_\mu W_\mu X_\mu X_\mu$$

Belongs to this intersection and hence is 0. Since X_λ is injective, $v_\lambda = 0$, So X is quasiaffinity.

Proposition (2-3-20) [102]:

For T in $y(H)$ the following are equivalent:

(i) Every subspace of finite or co-finite dimension has a complement in lat T;

(ii) $H = \vee\{\ker(T-\lambda) : \lambda \in C\} = \vee\{\ker(T-\lambda)^{*} : \lambda \in C\}$?

(iii) T is quasisimilar to a diagonal normal operator.

Lemma (2-3-21) [102]:

Let Δ be the class of injective bitriangular compact operator. Then $\zeta_{qs} =$ $\{T \in (B\Delta) : \sigma_p(T) = \{\lambda_k : k \geq 1\}\}$ is a sequence of non-zero complex numbers converging to 0, and $nul(T-\lambda)^{w} < \infty$ for all λ.

Proposition (2-3-22) [5]:

Let $T \in y(H)$ and let p be a polynomial. Suppose μ is an invariant subspace of T contained in $\ker p(T)$ such that $p_{\mu}/\ker p(T)^{*}$ is inverible in $y\left(\ker p(T)^{*}, \mu\right)$. Then $T \sim (T/\mu) \oplus T_{\mu^{\perp}}$ where $T_{\mu^{\perp}}$ is the compression of T to $\mu^{\perp}$. Moreover $p(T/\mu) = 0$ and $p(T_{\mu^{\perp}})^{*}$ is injective.

Proof:

Let $p(t) = \Pi_{k=1}^{m}(t-\lambda_k)^{nk}$. So $\sigma(T/\mu)$ is contained in $\{\lambda_1, \ldots, \lambda_m\}$ and T/μ

$\sim \sum_{k=1}^{m} \oplus T_k$, where $(T_k - \lambda_k I)^{nk} = 0$. It suffices to prove the result in the case $m = 1$,

$\lambda = 0$, $p(t) = t^{n}$, for the general case will follow by a straight forward induction argument. Split $H = \mu \oplus \mu^{\perp}$ and decompose

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

Then $A = T/\mu$ satisfies $A^{n} = 0$. Compute for $k \geq 1$,

$$T^{k} = \begin{pmatrix} A^{k} & C_k \\ 0 & B^{k} \end{pmatrix}$$

Where $C_k = AC_{k-1} + CB^{k-1}$, $A^{k-1}C + C_{k-1}B$. In particular $T^n = \begin{pmatrix} 0 & C_n \\ 0 & B^n \end{pmatrix}$ and

$T^{*n} = \begin{pmatrix} 0 & 0 \\ C_n^* & B^{*n} \end{pmatrix}$. For each vector $f = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\ker T^{*n}$, we have $0 = C_v^* x + B^{*n} y$.

By hypothesis the map taking to defines an isomorphism of $\ker T^{*n}$ onto . Hence we deduce that $R_{an} C_n^* \subseteq R_{an} B^{*n}$. By a well known results there exists an operator in $y(\mu^{\perp}, \mu)$ such that $C_n^* = -B^{*n} X^*$. Equivalently $C_n = -XB^n$ now notice that

$$(C - Ax + XB)B^n = (CB^{n-1})B - A(XB^n) + (XB^n)B$$

$$= (C_n - AC_{n-1})B + AC_n - C_n B$$

$$= A(C_n - C_{n-1}B) = A(A^{n-1}C)$$

Since B^{*n} is injective B^r has dense range. Thus we obtain $C = AX - XB$

Hence $T \sim \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

We observe that a consequence of this proposition is that

$\ker p(T^*) = \ker p(T)^*(T - \lambda_k)^*$ for $1 \le k \le n$. To see this replace by the similar operator $A^* \oplus B^*$ on $H_1 \oplus H_2$.

Since $p(A)^* = 0$ and $p(B)^*$ is injective it follows that $p(B)^*(B - \lambda_k)^*$ is injective for $1 \le k \le n$ and both kernels above are $H_1 \oplus 0$. The preceding proposition may have rather limited application.

Example (2-3-23) [102]:

Let be the triangular operator given by

$$T = \begin{bmatrix} 0 & -1 & & & & & \\ & 0 & -1/2 & & O & & \\ & & 1 & -1/3 & & & \\ & & & 1/2 & -1/4 & & \\ & & & & 1/3 & -1/5 & \\ & O & & & & \ddots & \\ & & & & & & \end{bmatrix}$$

With respect to a basis $\{e_n, n \geq 1\}$. Then $\ker T = Ce_1$ and $\ker T^2 = span\{e_r, e_n\}$. A routine calculation shows that if $x = \sum_{n \geq 3} a_n e_n$ and $Tx \in \ker T^2$, then

$a_n = \frac{1}{2}(n-1)(n-2)a_3$ for $n \geq 4$. Hence $x = 0$, and $\ker T^n = \ker T^2 \supsetneq \ker T$ is two-dimensional for $n \geq 2$. Another simple calculation yields $\ker T^* = C\zeta$, where $\zeta = \sum_{n \geq 2} (n-1)^{-1} e_n$. If $\ker T^* \neq \ker T^*$ then there is a (unique) vector $x = \sum_{n \geq 1} a_n e_n$ such that $a_2 = 0$ and $T^* x = \zeta$. But this forces $a_n = n/2 - 1$ for $n \geq 3$ which is absurd. So $\ker T^{*n} = \ker T^*$ is one-dimensional for $n \geq 1$. Now $P_{\ker T} \ker T^* = \{0\}$ so our proposition does not apply. Indeed the conclusion is false. For if T is similar to an operator of the form $A = (T/\ker T) \oplus T_1$, then

$$1 = nul(T) = nul(A) = 1 + nul(T_1)$$

So T_1 is injective and $nul(T^n) = nul(A^n) = 1$ for all $n \geq 1$, contrary to fact.

Examples (2-3-24) [102]:

The precise relationship between $\ker(T^k)$ and $\ker(T^{*k})$ for a triangular operator T is rather mysterious, even when T is compact. Consider the following three examples. Let

$$T = \begin{bmatrix} A_1 & -b_1 & & & \\ & a_2 & -b_2 & & \\ & & a_3 & -b_3 & \\ & & & & \ddots \end{bmatrix}$$

Where $a_1 = b_1 = 1$ and $b_j = 1/\log j$, $a_j = j/(j+1)$

Then T is compact and $\sigma(T) = \{0\} \cup \{a_j, j \geq 1\}$. Since 0 is not a diagonal entry of T, T^* is injective (lemma (2-2-2)). However $\ker T^n = C\zeta$ for all $n \geq 1$ where $\zeta = \sum_{n\geq 1} e_n / n$. Let

$$A = \begin{bmatrix} 0 & & 0 & 1 & 1/2 & 1/3 & \cdots \\ & 0 & 1 & & & & \\ & & 0 & 1/2 & & O & \\ & & & 0 & 1/3 & & \\ & & & & 0 & & \\ & O & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

In this case, $d(A) = \{0, 0, \ldots\}$. A simple computation yields that A^* is injective (so $\sigma_p(A)^*$ is empty) and thus $nul(A^{*k}) = 0$ for all $k \geq 1$. However $nul(A^k) = k+1$ for all $k \geq 1$.