Chapter 1

Sum Rules for Jacobi Matrices and Their Applications to Spectral Theory

We show and prove a bound of a Jacobi matrix. And we give complete description for the point and absolutely continuous spectrum, while for the singular continuous spectrum additional assumptions are needed, we prove a characterization of a characteristic function of a row contraction operator and verify its defect operator. We also prove a commutability of an operator of this row contraction**.**

Section (1-1): Spectral Form for Jacobi Matrices:

The case of some rules and were efficiently used to relate properties of elements of a Jacobi matrix of certain class with its special properties. For instance spectral data of Jacobi matrices being a Hilbert space-Schmeidt perturbation of the free Jacobi matrix were characterization [42,101,135] and

we suggest a modification of the method that permits us to work with higher order sum rules. We obtain sufficient conditions for a Jacobi matrix to satisfy certain constraints on its spectral measure. We consider a Jacobi matrix [129,124].

$$J : J(a,b) : \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Where $a = \{a_k\}, a_{k>0}$ and $b = \{b_k\}, b_k \in$ i , We assume that J is a compact perturbation of the free Jacobi matrix J_0

$$J_0 = \begin{pmatrix} 0 & 1 & 0 & \\ 1 & 0 & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix} \qquad (1)$$

A scalar spectral measure $\delta = \delta(J)$ is defined by the formula $((J - z)e_0, e_0) = \int_{\mathbb{R}} \frac{d\delta(x)}{x - z}$ with $z \in \mathbb{C} \setminus \mathbb{R}$, the absolutely continuous spectrum $\delta_{ac}(J)$ of fills in [-2, 2]and the discrete spectrum consist of two sequences $\{x_j^{\pm}\}$ with properties $\bar{x}_j < -2$, $\bar{x}_j \to 2$ and $x_j^+ > 2,\ x_j^+ \to 2$

Let $\partial_a = \{a_k - a_{k-1}\}$ for a given and $k \in N$ we construct a sequence $\gamma_k(\alpha)$ by formula $\gamma_k(a)_j = \alpha_j^k - \alpha_j \ldots \alpha_{j+k-1}$ where $\alpha = a - 1$ and 1 is a sequence of units

Theorem (1-1-1) [87]:

Let $J = J(a, b)$ be a Jacobi matrix described above. If

(i) $a - 1, b \in L^{m+1}, \partial_a, \partial_b \in L^2$

(ii) $\gamma_k(a) \in L', k = 3, [(m+1)/2]$ (2)

Then

$$(i')\ \int_{-2}^{2} \log \delta'(x).\left(4 - x^2\right)^{m - \frac{1}{2}} dx > -\infty$$

$$(ii')\ \sum_j \left(x^{\pm 2} - 4\right)^{m + 1/2} < \infty \qquad (3)$$

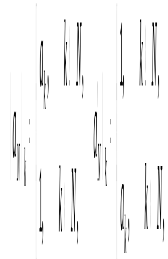
When $m = 1$ the theorem gives the fact of theorem (1-1-1)

Proof:

Define $\phi_m(J)$ as $\varphi_m(J) = \varphi_m(\delta) = \varphi_{m,1}(\delta) + \varphi_{m,2}(\delta)$

$$= \frac{1}{2\pi} \int_{-2}^{2} \log \frac{1}{\delta'(x)} . \left(4 - x^2\right)^{m - \frac{1}{2}} dx + \sum_j G_m\left(x_j^{\pm}\right) .$$

We have to show that $\phi_m(J) < \infty$. We put $a_N = \{(a_N)_k\}$ and $a'_N = \{(a'_N)_k\}$, where



Define sequences b_N, b'_N in the same way (of course, with replaced by $0's$).

Let $J_N = J(a_N, b_N)$. As we readily see, $\dot{a}_N - 1, b_N \to 0, \ \partial a'_N, \ \partial b'_N \to 0$, and $\gamma_k(a'_N) \to 0$ in corresponding norms, as $N \to \infty$ by the Lemma (1-1-4) below, we have for $N' = N - m$

$$\left|\psi_m(J) - \psi_m(J_N)\right| \le \psi_m(a'_{N'}, b_{N'}) \le C_1(\|a'_{N'} - 1\|_{m+1} + \|b_{N'}\|_{m+1}$$
$$+ \|\delta a_{N'}\|_2 + \|\delta b_{N'}\|_2 + \sum_k \|\gamma_k(d_{N'})\|_1)$$

or $\psi_m(J_N) \to \psi_m(J), as\ N \to \infty$

on the other hand $(J_N - z)^{-1} \to (J - z)^{-1}\ for\ z \in \mathbb{C}$ \R, and consequently $\sigma_N \to \sigma$ weakly $\phi_{m,1}(\sigma) \le \lim_N \inf \phi_{m,1}(\sigma_N)$ and $\lim \phi_{m,2}(\sigma_N) = \phi_{m,2}(\sigma)$ we bound the latter quantity $|\psi_{m,2}(J)| = \sum_j |G_m(x_j^{\pm})| \le C_2(\|a-1\|_{m-1}^{m-1} + \|b\|_{m-1}^{m-1})$ with some constant C_2 . Summing up we obtain

$$\phi(\sigma) \le \lim_N \sup \phi(\sigma_N) = \lim_N \sup \psi(J_N) = \lim_{N \to \infty} \psi(J_N) = \psi(J)$$

The proof is complete. It is easy to give simple conditions sufficient for $\gamma_k(a) \in L'$ for the instance put $(A_k(a))_j = \alpha_{j+1} + \ldots + \alpha_{j+k-1} - (k-1)\alpha_j$,then relations $a - 1\ e L^{m-1}, \partial_a \in L^2$ and $A_k(a) \in L^{2(k,m)}$ $2(k,m) = (m+1)!(m+2-k)$ imply that $\gamma_k(a) \in L'$. In particular we have the following corollary.

Corollary (1-1-2) [87]:

Theorem (1-1-1) holds if conditions (i), (ii) are replaced with

$A_k(a) \in L^{2(k,m)}$, $2(k,m) = (m+1)/(m+2-k)$, where $k = \delta,\left[\frac{m+1}{2}\right]$ we observe that relations (i) and (ii) are trivially true in the case of discrete Schrödinger operator i.e., when $J = J(1,b)$.

Corollary (1-1-3) [87]:

Then inequalities $(i')and(ii')$ let hold $J = J(1,b)$. If $b \in L^{m+1}$, $\partial b \in L^2$, the corollary is still true if $b \in L^{m+2}$, m being even. The proof is a sum rule of a special type. First we obtain it assuming rank $(J - J_0) < \infty$. Applying methods we see that

$$\frac{1}{2\pi}\int_{-2}^{2}\log\frac{1}{\delta'(x)}.(4-x^2)^{m-\frac{1}{2}}dx + \sum_j G_m(x_j^{\pm}) = \psi_m(J)$$

Where $\psi_m(J) = \psi_m(a,b)$ and $G_m(x) = (-1)^{m+1}C_0(x^2-4)^{m+\frac{1}{2}} + o(x^2-4)^{m+\frac{3}{2}}$ with $x \in R\setminus[-2,2]$,

C_0 being a positive constant. where

$$\psi_m(J) = tr\left\{\sum_{k=1}^{m}\frac{(-1)^{k+1}}{2^{2k+1}k}(J^{2k} - J_0^{2k}) - \frac{(2m-1)!!}{(2m)!!}\log A\right\} \tag{4}$$

Where $A = diag\{a_k\}$ and $\tilde{C}_m^k = \frac{m!!}{(m-k)!!k!!}$. Notation $k!!$ is used for "even" or "odd" factorials.

Lemma (1-1-4) [5]:

Let $J = J(a,b)$ we have

$$|\psi_m(J)| \le C_1\left(\|a-1\|_{m+1} + \|b\|_{m+1} + \|\partial_a\|_2 + \|\partial b\|_2 + \sum_{k=3}^{[(m+1)/2]}\|\gamma_k(a)\|_1\right) \tag{5}$$

Where C_1 depends on m only. Above, norms $\|\ \|$ refer to the standard L^p- space norms. We begin with considering expressions tr $(J^{2k} - J_0^{2k})$ a rising in (4). Defining $V = J - J_0 = J(a-1,b)$ we have

$$tr(J^{2k} - J_0^{2k}) = tr\sum_{p=1}^{2k}\sum_{i_1+\dots+i_p=2k-p} VJ_0^{i_1}\dots VJ_0^{i_p}$$

we prove the lemma in steps.

Proof:

First we bounded summands corresponding to $P=\frac{m+1}{2},m$ in [87]. We get $|tr(V^p F_p(J_0))| \leq \|V^p F_p(J_0)\|_{s_1} \leq \|F_p(J_0)\|, \|V^p\|_{s_1}$ and for these $p's$

$$\|V^p\|_{s_1} \leq C_{10}\|V^{m-1}\|_{s_1} \leq C_{10}\left(\|a-1\|_{m-1}^{m-1}+\|b\|_{m-1}^{m-1}\right)$$

(6)

With the constant depending on $\|V\|$. Similarly $|tr\,\alpha^p| \leq C_{11}\|a-1\|_{m+1}^{m+1}$, let $p=3,m$ now. As we already mentioned in [134]

$$V^p = \sum_{j=0}^{p}\left(S^i p_{p,j}(a,b)+p_{p,j}(a,b)\bar{S}^j\right) .$$

It is easy to show by induction that the polynomials $P_{p,p}(a,b)$ are particularly simple. Namely $P_{p,p}(a,b)=\alpha\alpha_{(1)}\cdots\alpha_{(p-1)}$ yields that

$$trV^p F_p(J_0) = (-1)^p \frac{(2m-1)!!}{2p(2m)!!} trV^p J_{0,p}$$

$$= (-1)^p \frac{(2m-1)!!}{2p(2m)!!} tr\left(P_{p,p}(a,p)+P_{p,p}(a,b)_p\right)$$

$$= (-1)^p \frac{(2m-1)!!}{2p(2m)!!} \sum_j \alpha_j \alpha_{j+1}\cdots\alpha_{j+(p-1)}$$

Since $trV^p J_{0,s}=0$ for $s \geq p+1$. Hence $tr\left(V^p F_p(J_0)+(-1)^{p+1}\frac{(2m+1)!!}{p(2m)!!}\alpha^p\right)$

$$= (-1)^{p+1}\frac{(2m-1)!!}{2p(2m)!!}\sum_j \alpha_j^p - \alpha_j\alpha_{j+1}\cdots\alpha_{j+(p-1)}$$ and we obtain that

$$\left|trV^p F_p(J_0)+(-1)^{p+1}\frac{(2m-1)!!}{2p(2m)!!}\alpha^p\right| \leq C_{12}\|\gamma_p(a)\|_1 \quad (7)$$

Where C_{12} depends on p,m and sequences $\gamma_k(a)$ are defined in [134] Observe that $\gamma_p(a)=0$ when $p=1$. Furthermore we have for $p=2$ that

$$\sum_j(\alpha_j^2 - \alpha_j\alpha_{j+1}) = \frac{1}{2}\sum_j(\alpha_j^2 - 2\alpha_j\alpha_{j+1}+\alpha_{j+1}^2)$$

$$= \frac{1}{2}\sum_j(\alpha_j - \alpha_{j+1})^2 = \frac{1}{2}\|\partial a\|_2^2$$

So the left hand-side of (7) for $p=2$ can be estimated by $C_{13}\|\partial a\|_2^2$. It is also clear that inclusion and $\partial a \in L^2$ give that $\gamma_p(a) \in L'$ for $p = m/2+1$. Indeed

we have $\alpha^p - \alpha\alpha_{(1)}\ldots\alpha_{p-1} = \sum_{k=1}^{p}\alpha^{p-k}(\alpha-\alpha_{p-k})\alpha_{(p-(k-1))}\ldots\alpha_{p-1}$

The terms in the latter sum look like $\alpha(i_1)\ldots\alpha_{(2p-1)}(\alpha-\alpha_{(i_p)})$ for some $i=(i_1,\ldots,i_p)$. Obviously $\alpha-\alpha_{i_p} = a - a_{ip} = \partial a \in L^2$. Applying the Holder

inequality $\sum_k a_k \ldots a_{p+k} \le \sum_k \left(\sum_{j=1}^{p}\frac{1}{q_j} a_{j,k}^{q_j}\right)$ with $a_{j,k} = |(\alpha_{i_j})_k|,\ q_j = 2(p-1)$ for

$j=1,\ldots,p-1$ and $a_{p,k} = |(\alpha-\alpha_{i_p})_k|,\ q_p = \frac{1}{2}$ we get that

$$\|\alpha^p - \alpha\alpha_{(1)}\ldots\alpha_{(p-1)}\|_1 \le C_{14}\|\partial a\|_2^2 + \|\alpha\|_{2(p-1)}^{2(p-1)}$$

Which is finite for $p = m2+1$. Thus gathering the above argument which is complete(see [134])we complete the proof of the lemma

Lemma (1-1-5) [87]:

Let $i=(i_1,\ldots,i_p)$ and $\sum_s i_s = n$ then

$$VJ_0^{i_1}\ldots VJ_0^{i_p} : V^p J_0^n + \sum_{\substack{L_1+L_2+L_3=p \\ p_1+p_2+p_3=n}} C_{1,p} J_0^{p_1} V^{L_1} [V^{L_2}, J_0^{p_2}] V^{L_3} J_0^{p_3}$$

$$+\sum_{i}^{m,p_i} A_k[V, J_0] B_k [V, J_0] C_k$$

Where $p=(p_1,p_2,p_3),\ 1=(L_1,L_2,L_3)$ and A_k, B_k, C_k are some bounded operators

Lemma (1-1-6) [87]

Let $\sum_s j = 2k-p$ we have $|tr(VJ_0^{i_1}\ldots VJ_0^{i_p} - V^p J_0^{k-p})| \le C_3(\|\partial a\|_2 + \|\partial b\|_2)$

With C_3 depending on $\|V\|$ only. The lemma exactly bounded ,we may assume that operators and to commute we estimating $\psi_m(J)$

$$\psi_m(J) = tr\left\{\sum_{p=1}^{2m} V^p F_p(J_0) - \frac{(2m-1)!!}{(2m)!!}\log(I+\tilde{\alpha})\right\} \quad (8)$$

Where $\tilde{\alpha} = diag\{a_k\} = A - I$ and

$$F^p(J_0) = \sum_{k=[(p+1)/2]}^{m} \frac{(-1)^{k+1}}{2^{2k+1}} \tilde{C}_{2m-1}^{2k-1} C_{2k}^{p} J_0^{2k-p}$$

Here C_l^j is a usual binomial coefficient, observe that for $p \geq m+1$ we have

$$\left|tr\left(V^p F_p(J_0)\right)\right| \leq \|F_p(J_0)\| \, \|V^p\|_{\mathfrak{S}_1} \leq C_4\left(\|a-1\|_{m+1}^{m+1} + \|b\|_{m+1}^{m+1}\right)$$

Where $\|\cdot\|_{\mathfrak{S}_1}$ is the norm in the class of nuclear operators, hence it remains to bound the first m terms in (8) we have

$$\log(1+\tilde{\alpha}) = \sum_{p=1}^{2m} \frac{(-1)^{p+1}}{p}\tilde{\alpha} + o\left(\tilde{\alpha}^{2m+1}\right)$$

Set $J_{0,p}$ to be a symmetric matrix with 1's on p-th auxiliary diagonals and o's elsewhere the following lemma holds.

Lemma (1-1-7) [87]:

We have $F_p(J_0) = (-1)^{p+1} \dfrac{(2m-1)!!}{2p(2m)!!} J_{0,p}$

Combining this with explicit form of V^p and the series expansion for $\log(I+\tilde{\alpha})$ we get the required bound (7).

Section (1-2): Spectral Properties of Self-adjoint Extensions

Let A be a closed symmetric operator on a separable Hilbert space $\mathfrak{H}$.

If A has equal deficiency indices $n_{\pm}(A)$ =dim $(h \in ran(A \pm iI))$, then A has a lot of self-adjoint extensions. These self-adjoint extensions can be labeled by the so-called Weyl function $M(\cdot)$ [82, 83, 84]. The generalization is based on concept of a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* being an abstract generalization of the Green's identity. Here $\mathcal{H}$ is a separable Hilbert space with dim $(\mathcal{H}) = n_{\pm}(A)$ and Γ_0 and Γ_1 are linear mapping from dom (A^*) to $\mathcal{H}$ so that Green's identity is satisfied [108,119].

The problem is the following. Let $M(.)$ be the Weyl function of a certain self-adjoint extensions A_h of A, introducing the associated scalar Weyl function $M_h(.) = (M(.)h, h)$, $h \in H$ is it possible to localize the different spectral subsets of A_h knowing the boundary values $M_h(x + i0)$, $x \in$ i of the associated scalar Weyl function. Let H be separable Hilbert space. Recall that an operator function $F(.)$ with values in $[H]$ is said to be a Hirglotz or Nevanlina function or R-function if holomorphic in and for every $z \in £_+$

the operator $F(z)$ in is dissipative i.e., $Sm\left(F(z)\right) = \frac{\left(F(z) - F(z)^*\right)}{2i} \geq 0$. In the following we prefer the notion R-function. The class of R–functions with values in $[H]$ is denoted by R_H .If $F(.) \in (R_H)$ then there exist bounded self-adjoint operator L in K, a bounded non-negative operator $R \geq 0$ with $R|K \ominus H = 0$ such that

$$F(z) = C_0 + C_1 z + R^{\frac{1}{2}} (I_k + zL)(L - z)^{-1} R^{\frac{1}{2}} |H, \quad z \in £_+ \tag{9}$$

Denoting by $E_L(.)$ the spectral measure of the self-adjoint operator one immediately obtains from (9) the representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sum_F(t), \; z \in £_+ \tag{10}$$

Where $\sum_F(.)$ is an operator valued Borel measure on given by

$$d\sum_F(t) = \left(1 - t^2\right) R^{\frac{1}{2}} dE_L(.) R^{\frac{1}{2}}, \; t \in \text{i} \tag{11}$$

the measure $\sum_F(.)$ is self-adjoint and obeys

$$\int_{-\infty}^{+\infty} \frac{1}{1 + t^2} d\sum_F(t) \in [H]$$

(12)

In contrast to spectral measures of self-adjoint operators it is not necessary true that ran $\Sigma\delta_1$ is orthogonal to ran $(\Sigma\delta_2)$ for adjoint Borel sets δ_1 and δ_2.

However the measure $\sum_F(.)$ is uniquely determined by the R-function $F(.)$.

The integral in (10) is understood in the strong sense in the following $\sum_F(.)$ is called the spectral measure of $F(.)$ defined by

$$\sum_F(t) := \begin{cases} \sum_F [0,t) & : t > 0 \\ 0 & : t = 0 \\ -\sum_F [t,0) & : t < 0 \end{cases} \tag{13}$$

The distribution function $\sum_F(.)$ is strongly left continuous and satisfies the condition

$$\sum_F(t) = \sum_F(t)^*,\ \sum_F(s) \le \sum_F(t),\ -\infty < s < t < \infty$$

The distribution function $\sum_F(.)$ is called the spectral function of $F(.)$.

We note that the spectral function $\sum_F(.)$ can be obtained by the Stieltjes transformation:

$$\frac{1}{2}\sum_F(t+0) + \sum_F(t) - \frac{1}{2}\sum_F(s+0) + \sum(s) \;=\; w - \lim_{y \to 0} \frac{1}{\pi}\int_s^t Sm\left(F\left(x+iy\right)\right)dx\,,\quad t.s \in \mathbb{R} \tag{14}$$

Where it is used that the spectral function is strongly left continuous.

A will always denote a closed symmetric operator with equal deficiency indices $n_+(A) = n_-(A)$ [97,140,147,148].

We can assume that A is simple. This means that A has no self-adjoint parts.

Definition (1-2-1) [96]:

A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mapping $\Gamma_i : dom(A^*) \to \mathcal{H},\ i = 0,1$ is called a boundary triple for the adjoint operator $A^* \to \mathcal{H}$, $i = 0,1$ is called a boundary triple for the adjoint operator A^* of A if

(i) The second Green's formula takes place

$$(A^* , f) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in dom(A^*) \tag{15}$$

(ii) The mapping $\Gamma = \{\Gamma_0, \Gamma_1\} : dom(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective

Definition (1-2-2) [96]:

(i) A closed linear relation Θ in $\mathcal{H}$ is closed subspace Θ of $\mathcal{H} \oplus \mathcal{H}$.

(ii) The closed linear relation Θ is symmetric if $(g_1, f_2) - (f_1, g_2) = 0$ for all $\{f_1, g_1\}, \{f_2, g_2\} \in \Theta$

(iii) The closed linear relation Θ is self-adjoint if it is maximal symmetric.

Definition (1-2-3) [96]:

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^*

(i) for every self-adjoint relation Θ in $\mathcal{H}$ we put

$$D^\Theta \{f \in dom(A^*) : \Gamma_0 f, \Gamma_1 f \in \Theta\}, A^\Theta = A^* | D^\Theta \tag{16}$$

(ii) In particular we set $A_i = A^{\Theta_i}, i = 0,1, if \Theta_i, i = 0,1$

(iii) If $\Theta = G(B)$ where B is an operator on $\mathcal{H}$,then we set $A^B A^\Theta$

Proposition (1-2-4) [96]:

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* then for every self-adjoint relation Θ in $\mathcal{H}$ the operator A^Θ given by definition (1-2-3) is self-adjoint extension of A the mapping $\Theta \mapsto A^\Theta$ from the set of self-adjoint extensions in $\mathcal{H}$ onto the set Ext_A of self-adjoint extensions of A is bijective. It is well known that Weyl function are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators.

Definition (1-2-5) [96]:

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the operator A^* . The Weyl function of A corresponding to the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is the unique mapping

$M(.):\rho(A_0)\to H$ satisfying

$$\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad f_z \in N_z, \quad z \in \rho(A_0) \tag{17}$$

Where $N_z = \ker(A^* - zI)$ above implicit definition of the Weyl function is correct and the Weyl function $M(.)$ is a R-function obeying

$$o \in \rho(Sm(M(i)))$$

Definition (1-2-6)[96]:

A closed linear relation in is called boundedly invertible if the inverse relation $\Theta = \{g, f\} \in H \times H : \{f, g\} \in \Theta$ is the graph of a bounded operator defined on . we say $\lambda \in £$ belong to the resolvent set $\rho(\theta)$ if the closed linear relation $\Theta - \lambda I = \{\{f, g - \lambda\} : \{f, g\} \in \Theta\}$ is boundedly invertible.

Proposition (1-2-7) [96]:

Let A be a simple closed densely defined symmetric operator in h. Suppose that $\{H, \Gamma_0, \Gamma_1\}$ is a boundary triple for A^* $M(.)$ is the corresponding Weyl function , a self-adjoint relation in and $\lambda \in \rho(A_0)$. Then the following holds.

(i) $\lambda \in \rho(A^\theta)$ if and only if $0 \in \rho(\Theta - M(\lambda))$.

(ii) $\lambda \in \delta_\tau(A^\theta)$ if and only if $0 \in \delta_\tau(\Theta - M(\lambda)), \tau = p, c$

If A is a simple symmetric operator then the Weyl function $M(.)$ determines the pair $\{A, A_0\}$ up to unitary equivalence. We shall often say that $M(.)$ is the Weyl function of the pair $\{A, A_0\}$ or simply of A_0 . We can prove $M_1(.)$ and $M_2(.)$ with values in $[H_1]$ and $[H_2]$ are connected via

$$M_2(z) = K^* M_1(z) K + D \tag{18}$$

Where $D = D^* \in [H_2]$ and $K \in [H_2, H_1]$ is boundedly invertible. With each boundary triple we can associate a so-called -field (.) corresponding to is defined by

$$\gamma(z) = (\Gamma_0 | N_z)^{-1} : H \to N_2, z \in \rho(A_0) \tag{19}$$

One can easily check that

$$\gamma(z) = (A_0 - z_0)(A_0 - z)^{-1}\gamma(z_0),\ z.z_0 \in \rho(A_0) \tag{20}$$

And consequently $\gamma(\cdot)$ is a -field. The -field and the Weyl function $M(\cdot)$ are related by

$$M(z) - M(z_0)^* = (z - \bar{z}_0)\,\gamma(z_0)^*\gamma(z), \qquad z, z_0 \in \rho(A_0) \tag{21}$$

The relation (21) means the $M(\cdot)$ is a -function of a pair $\{A, A_0\}$ .Further we note that if A is simple then $N_z, z \in \rho(A_0)$ is generating with respect to A_0 too .

Let be a Borel measure on . A support of is a set such that $\mu(\mathbb{R} \setminus S) = 0$ we note that $S \subseteq \tilde{S}$ implies that is a support too. Measures and on are called orthogonal if some of their supports are disjoint. The topological support $S(\mu)$ of is the smallest closed set which is a support of .According to the Lebesgue-Jordan\decomposition $\mu = \mu_s + \mu_{ac}, \mu_s = \mu_{pp} + \mu_{sc}$. Where $\mu_s, \mu_{pp}, \mu_{sc}$ and are the corresponding singular pure point, singular continuous and absolutely continuous measures of respectively. We set $S_T(\mu) = S(\mu_T),\ T = s, pp, sc, ac$ the set $S_s(\mu), S_{pp}(\mu), S_{sc}(\mu), S_{ac}(\mu)$ are closed and called singular, pure point, singular continuous and absolutely continuous supports of , we denote that the closed supports $S_s(\mu)$, $S_{pp}(\mu), S_{ac}(\mu)$ and $S_{sc}(\mu)$ are not generally mutually disjoint to obtain mutually disjoint supports we introduce the following sets .

$$S'_0(\mu) = \{t \in \mathbb{R} : d\mu(t)\,dt\ exists and\ d\mu(t) = \infty\} \tag{22}$$

$$S'_{pp}(\mu) = \{t \in \mathbb{R} : \mu(\{t\}) \neq 0\} \tag{23}$$

$$S'_{sc}(\mu) = \left\{t \in \mathbb{R} : d\mu(t)\,d(t)\ exists\ \frac{d\mu(t)}{dt} = \infty\ and\ \mu(t) = 0\right\} \tag{24}$$

$$S'_{ac}(\mu) = \left\{t \in \mathbb{R} : \frac{d\mu(t)}{dt}\ exists\ \ and\ 0 < d\mu(t)/dt < \infty\right\} \tag{25}$$

Where the distribution function $\mu(.)$ is similar to (13) defined by it turns out that. Since the sets $S'_T(\mu)$, $T=s, pp, sc$ are of Lebesgue measure zero and mutually disjoint we find that for any Borel set $\chi\subset\mathbb{R}$ one has

$$\mu(\chi\cap S'_T(\mu)) = \mu_T(\chi), T = s, pp, sc, ac \qquad (26)$$

The sets $S'_s(\mu)$, $S'_{pp}(\mu)$, $S'_{sc}(\mu)$, and $S'_{ac}(\mu)$ singular pure point ,singular continuous and absolutely continuous supports of respectively. We note that

$$S_{pp}(\mu) = \overline{S'_{pp}(\mu)} \text{ and } S_\tau(\mu) \subseteq \overline{S'_\tau(\mu)} \subseteq S(\mu) \ , \ \tau = s, sc, ac \qquad (27)$$

In general it is not possible to replace inclusion by equalities, let now $\sum(.)$ be a measure with values in (H) the measure $\sum(.)$ admit a Lebesque- Jordan decomposition $\sum = \sum_{ac}, \sum_s, \sum_{pp} + \sum_{sc}$ .As above the notation

$$S_s \sum = S\sum_s, S_{pp}\sum = S\sum_{pp}, S_{sc}\sum = S\left(\sum_{sc}\right) \text{ and } S_{ac}\left(\sum\right) = S\left(\sum_{ac}\right)$$

stand for the singular pure point , singular continuous and absolutely. We get

$$S_p\left(\sum\right) = \{\tau\in\mathbb{R} \} : \sum(\{\tau\}) \neq 0 \qquad (28)$$

we have $S_p\left(\sum\right) = S_{pp}\left(\sum\right)$ and $\overline{S_p(\sum)} = S_{pp}(\sum)$ with each operator-valued measure $\sum(.)$ we can associate a scalar measure $\sum_h(.) = \left(\sum(.)h, h\right), \ h\in\mathrm{H}$. In the following we are interested in the problem whether the spectral properties of the operator valued measure $\sum(.)$ can be characterized by a family of scalar measures. To this end let $\tau = \{h\}_{k=1}^{N}, \ 1\le N \le +\infty$ be a total set in H with we associate the family $\sum_{h_k}\{.\}_{k=1}^{N},$ of scalar measures. Let us introduce the following sets.

$$S'_s\left(\sum,\tau\right) = \cup_{k=1}^{N} S'_s\left(\sum_{h_k}\right) \qquad (29)$$

$$S'_{pp}\left(\sum,\tau\right) = \cup_{k=1}^{N} S'_{pp}\left(\sum_{h_k}\right) \qquad (30)$$

$$S'_{sc}\left(\sum,\tau\right) = \cup_{k=1}^{N} S'_{sc}\left(\sum_{h_k}\right) \mid S'_{pp}\left(\sum\right) \qquad (31)$$

$$S'_{ac}\left(\sum, \tau\right) = \cup_{k=1}^{N} S'_{ac}\left(\sum_{h_k}\right) \mid S'_s\left(\sum\right) \tag{32}$$

Lemma (1-2-8) [96]:

Let be a separable Hilbert space and $T = \{h_k\}_{k=1}^{N}, 1 \le N \le +\infty$ be a total set in .Then the sets $S'_s\left(\sum, \tau\right), S'_{pp}\left(\sum, \tau\right), S'_{sc}\left(\sum, \tau\right)$ and $S'_{ac}\left(\sum, \tau\right)$ are singular ,pure point ,singular continuous and absolutely continuous supports of $\Sigma(.)$ respectively i.e.,

$$\sum\left(\chi \cap S'_{\tau}\left(\sum, \tau\right)\right) = \sum^{\tau}(\chi), \tau = s, pp, sc, ac \tag{33}$$

For any Borel set $\chi \subseteq R$ **.** In particular the following relations hold.

$S'_p \sum = S'_{pp}\left(\sum, \tau\right)$ and

$$S_p\left(\sum\right) \subseteq \overline{S'_p\left(\sum, \tau\right)} \subseteq S\left(\sum\right), \tau = s, sc, ac \tag{34}$$

Proof:

By the Lebesgue-Jordan decomposition one easily gets that for each $h \in H$ We have

$$\left(\sum^{\tau}(\chi)h, h\right) = \sum_{h,\tau}(\chi), \tau = s, pp, sc, ac \tag{35}$$

For any Borel set $\chi \subseteq R$ where $\sum_{h,T}(.)$ arises from the Lebesgue-Jordan decomposition of the scalar measure $\Sigma_h(.)$. Let . Since mes $S'_s\left(\sum, \tau\right) = 0$

We get

$$\left(\sum\left(\chi \cap S'_s\left(\sum, \tau\right)\right) h_k, h_k\right) = \sum_{h_k}\left(\chi \cap S'_s\left(\sum, \tau\right)\right) = \sum_{h_{k,s}}\left(\chi \cap S'_s\left(\sum, \tau\right)\right) \tag{36}$$

For any $h_k \in \tau$ using (35),(36) and

$$\sum_{h_{k,s}}\left(\chi \cap S'_s\left(\sum, \tau\right)\right) = \sum_{h_{k,s}}\left(\chi \cap S'_s\left(\sum, \tau\right) \cap S'_s \sum_{h_k}\right)$$

$$= \sum_{h_{k,s}}\left(\chi \cap S'_s\left(\sum, \tau\right)\right) = \sum_{h_{k,s}}(\chi) \tag{37}$$

We find $\left(\sum\left(\chi\cap S_s'\left(\sum_{;}\tau\right)\right)h_k,h_k\right)=\left(\sum^{s}(\chi)\,h_k,h_k\right)$ for any $h_k\in\tau$. Since is total we finally obtain $\sum\left(\chi\cap S_s'\left(\sum_{;}\tau\right)\right)=\sum^{s}(\chi)$ for any Borel set $\chi\in$ i . Similarly we prove the statements for $\tau=pp,sc,ac.$

Let $x\in S_{pp}'\sum_{;}\tau$. Then there is $h_k\in\tau$ such that $x\in S_{pp}'\left(\sum_{h_k}\right)$. Hence $\sum_{h_k}(\{x\})=\left(\sum(\{x\})h_k,h_k\right)\neq 0$ which yields $\sum(\{x\})\neq 0$ or $x\in S_p(\sum)$, i.e. $S_{pp}'\left(\sum_{;}\tau\right)\subseteq S_p\left(\sum\right)$ conversely if $x\in S_p(\sum)$ then there is a $h\in$ H such that $\sum_h(\{x\})\neq 0$. If this is not the case then for each $h_k\in T$ one has $\sum_{h_k}(\{x\})=\left(\sum(\{x\})h_k,h_k\right)=0$. Since is total this yields $\left(\sum(\{x\})h_k,h_k\right)=\sum_{h_k}(\{x\})=0$

For each $h\in$ H Contrary to the assumption. however, if there is a $h_k\in T$ such that ` $\sum_{h_k}(\{x\})\neq 0$ then $x\in S_{pp}'\left(\sum_{;}\tau\right),\ i,e.S_p\left(\sum\right)\subseteq S_{pp}'\sum_{;}\tau$ hence $S_p\left(\sum\right)=S_{pp}'\sum_{;}\tau$. Further from (33) we get

$$S_T\left(\sum\right)\subseteq\overline{S_T'\left(\sum_{;}\tau\right)}\ ,\ \tau=s,pp,sc,ac$$

Taking (27) into account we get $S_T'\left(\sum_{h_k}\right)\subseteq S\left(\sum_{h_k}\right),\ \tau=s,sc,ac,sc$ for each $h_k\in\tau$. Since $S\left(\sum_{h_k}\right)\subseteq S\left(\sum\right)$ for each $h_k\in T$ we get $S_T'\left(\sum_{;}\tau\right)$ which immediately proves (34). Taking (20) and (21) into account we obtain that $C_1=0$ which leads to the representation.

$$M(z)=C_0+\int_{\mathbb{i}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sum(t)\ ,\ z\in\pounds \tag{38}$$

Lemma (1-2-9) [96]:

Let A be a simple densely defined closed symmetric operator on the a separable Hilbert space with $n_+(A)=n_-(A)$. Further, let $\Pi=[\mathrm{H},\Gamma_0,\Gamma_1]$ be a boundary triple of A^* with Weyl function $\Sigma(\cdot)$. If $E_{A_0}(.)$ is the spectral measure of $A_0 = A^*|\ker(\Gamma_0)(\in Ext_A)$ and $\Sigma(\cdot)$ that of the integral representation (38) of the Weyl function $\Sigma(\cdot)$. Then the measure $E_{A_0}(.)$ and $\Sigma(\cdot)$ are equivalent. In particular one has $\delta_\tau(A_0) = \delta_\tau(\sum)$.

Theorem (1-2-10) [96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space h with $n_+(A) = n_-(A)$. Further let $\Pi = [\mathrm{H}, \Gamma_0, \Gamma_1]$ be a boundary triple of A^* with Weyl function $M(.)$.

If $E_{A_0}(.)$ is the spectral measure of $A_0 = A^* \ker \Gamma_0 (\in Ext_A)$ and $\Sigma(\cdot)$ that of the integral representation (38) of the Weyl function $M(.)$, then for each total set

$\tau = \{h_k\}_{k=1}^{N}, 1 \leq N \leq +\infty$ in h the sets $S'_s(\sum, \tau), S'_{pp}(\sum, \tau), S'_{sc}(\sum, \tau)$ and $S'_{ac}(\sum, \tau)$.

Singular pure point, singular continuous, and absolutely continuous supports of $E_{A_0}(.)$ respectively, i.e. we have

$$E_{A_0}\left(\chi \cap S'_T\left(\sum, \tau\right)\right) = E^T_{A_0}(\chi) \tag{39}$$

For each Borel set $\chi \in \mathrm{R}$. In particular the relations $\delta_p(A_0) = S'_{pp} \sum, \tau$ and

$\delta_\tau(A_0) \subseteq S'_\tau \sum, \tau \subseteq \delta(A_0), \tau = s, sc, ac$ hold.

Proof:

Since by lemma (1-2-8) the sets $S'_T(\delta, \tau), \tau = s, pp, sc, ac$ are supports of $\Sigma(\cdot)$, one immediately gets from lemma (1-2-9) that the same sets are supports of $E_{A_0}(.)$ of the same type, i.e., (39) holds . If $x \in S'_{pp}(\sum, \tau)$ then there is at least one $k = 1, 2, \ldots, N$ such that

$$\left(E_{A_0}(\{x\})\, \gamma(i)\, h_k, \gamma(i)\, h_k\right) \neq 0$$

Hence $x \in \delta_p(A_0)$ conversely, if $x \in \delta_p(A_0)$ then due to the fact that $\gamma(i)\tau$ is

Generating for $E_{A_0}(.)$ then is at least one $k = 1, 2, \ldots, N$ such that

$$\left(E_{A_0}(\{x\})\,\gamma(i)\,h_k,\gamma(i)\,h_k\right) \neq 0$$

Hence $x \in S'_{pp}\sum_{;}\tau$ which proves $\delta_p(A_0) = S'_{pp}\left(\sum \tau\right)$ the relations $\delta_s(A_0) \subseteq \overline{S'_\tau\left(\sum_{;}\tau\right)} \subseteq$

$\delta(A_0)$, $\tau = s, sc, ac$ are consequences of lemma (1-2-8) and lemma (1-2-9) .we characterize the spectral properties of the operator-valued measure $\Sigma(.)$ using the boundary behavior of the Weyl-function $M(.)$. A first step is to develop a corresponding theory for scalar measure which satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < +\infty$$

(40)

Let us associate with the Poisson integral

$$V(z) = \int_{\mathbb{R}} \frac{y\,d\mu(t)}{(t-x)^2 + y^2}, z = x + iy \in £_+$$

(41)

Which defines a positive harmonic function in . Conversely it is well known that each positive harmonic function $V_1(z)$ in admits the representation $V_1(z) = ay + V(z)$ with $a \geq 0$ and $V(z)$ of the form (40) and (41). Below we summarize some well-known facts on positive harmonic function

Proposition (1-2-11) [96]:

Let be a positive Radan measure obeying (40) and let $V(z)$ be a positive harmonic function in $z = x + iy \in £_+$ defined by (41). Then one has.

(i) for any $x \in$ the $\lim V(x+io) = \lim V(x+iy)$ exists and is finite , if and only if symmetric derivative $D_\mu(x)$

$$D_\mu(x) = \lim_{\varepsilon \to 0} \frac{\mu(x+\varepsilon) - \mu(x-\varepsilon)}{2\varepsilon} \quad (42)$$

Exists and is finite. In this case one has

$$V(x+io) = \pi D_\mu(x) \quad (43)$$

(ii) if the symmetric derivative $D_\mu(x)$ exists and is infinite the $V(z) \longrightarrow$ as $z \to x$

(iii) for each $x \in R$ one has $Sm(z-x)V(z) \longrightarrow \mu(\{x\})$ as $z \to x$

(iv) $V(z)$ converges to a finite constant as $z \to x$, if and only if the derivative $d\mu(t)\, dt$ exists at $t = x$ and is finite.

The symbol $\to\rangle$ means that the limit $\lim_{r\to 0} V(x + re^{i\theta}), x \in R$ exists uniformly in $\theta \in [\varepsilon, \pi - \varepsilon]$ for each $\varepsilon \in (0, \pi/2)$. Proposition (1-2-11) allows us to introduce measures satisfying (40) the following sets $z = (x + iy)$

$$S_s''(\mu) = \{x \in i : V(z) \to \infty\} \text{ as } z \to x \quad (44)$$

$$S_{pp}''(\mu) = x \in i : Sm \lim_{z\to x} (z - x)V(z) > 0 \quad (45)$$

$$S_{sc}''(\mu) = \{x \in i : V(z) \to \infty \, and \, (z - x)V(z) \to 0 \, as \, z \to x\} \quad (46)$$

$$S_{sc}''(\mu) = \{x \in i : V(x + i0) \, exists \, and \, 0 < V(x + i0) < \infty\} \quad (47)$$

Obviously the sets $S_s''(\mu)$ and $S_{ac}''(\mu)$ as well as $S_{pp}''(\mu)$, $S_{sc}''(\mu)$, and $S_{ac}''(\mu)$ are mutually disjoint. By proposition (1-2-15) one immediately gets that $S_{pp}'(\mu) = S_{pp}''(\mu)$ and

$$S_\tau'(\mu) \subseteq S_\tau''(\mu) \subseteq S(\mu) \quad (48)$$

Indeed the relation $S_{pp}'(\mu) = S_{pp}''(\mu)$ is a consequence of (iii).By (ii) we get

$S_s'(\mu) \subseteq S_s''(\mu)$

Similarly we prove $S_{sc}'(\mu) \subseteq S_{sc}''(\mu)$ using (ii) and(iii).Finally the relation $S_{ac}'(\mu) \subseteq S_{ac}''(\mu)$ follows from (i). We note that it can happen that $S_{sc}'(\mu) \neq 0$ and the inclusion $S_{sc}'(\mu) \subseteq S_{sc}''(\mu)$ is strict even if $\mu_{sc} = 0$. Furthermore we note that from (26) and the inclusion $S_T'(\mu) \subseteq S_T''(\mu)$, $\tau = s, pp, sc, ac$ we find that

$$\mu(\chi \cap S_\tau''(\mu)) = \mu_\tau(x) \quad (49)$$

For any Borel set $\chi \in i$. Now we are going to characterize the spectral parts of the extension A_0 by means of boundary values of the Weyl function $M(.)$.

Using the integral representation (38) of the Weyl function we easily get that

$$V_h(z) = \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} \, d\Sigma_h(t) = Sm(M_h(z)), \; z \in \mathbb{C}, \quad h \in \mathcal{H} \tag{50}$$

Where $M_h(z) = (M(z)h, h), \; z \in \mathbb{C}, \quad h \in \mathcal{H}$ (51)

The function $M_h(.)$ is a scalar R-function. Since $M_h(.)$ arises from the Weyl function we call it the associated scalar Weyl function $V_h(.)$ is imaginary part of the associated scalar Weyl function $M_h(.)$ and the theory developed we can relate the boundary behavior at the real axis the imaginary part of associated scalar Weyl functions with the spectral properties of the self-adjoint extension A_0. To this end in addition to (29) and (32) we introduce.

$$S_s''(\Sigma_{;}\tau) = \cup_{k=1}^{N} S_s''(\Sigma_{h_k}) \tag{52}$$

$$S_{pp}''(\Sigma_{;}\tau) = \cup_{k=1}^{N} S_{pp}''(\Sigma_{h_k}) \tag{53}$$

$$S_{sc}''(\Sigma_{;}\tau) = \cup_{k=1}^{N} S_{sc}''(\Sigma_{h_k}) \setminus S_{pp}''(\Sigma) \tag{54}$$

$$S_{ac}''(\Sigma_{;}\tau) = \cup_{k=1}^{N} S_{ac}''(\Sigma_{h_k}) \setminus S_s''(\Sigma) \tag{55}$$

By definition the sets $S_s''(\Sigma_{;}\tau)$ are disjoint. They holds for $S_{pp}''(\Sigma_{;}\tau)$.

Furthermore we denote that the sets $S_T''(\Sigma_{;}\tau)$ have Lebesgue zero, i.e., mes $S_T''(\Sigma_{;}\tau) = 0, \; \tau = s, pp, sc.$, it turns out that the sets $S_T''(\Sigma_{;}\tau)$ in theorem (1-2-14) can be replaced by the sets $S_T''(\Sigma_{;}\tau)$

Theorem (1-2-12) [96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space H with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$. If $E_{A_0}(.)$ is the spectral measure of $A_0 = A^* | \ker \Gamma_0 (\in Ext_A)$ and total set $T = \{h_k\}_{k=1}^{N}, \; 1 \le N \le +\infty$ in $\mathcal{H}$ the sets $S_s^{\prime\prime}(\sum;\tau)$, $S_{pp}^{\prime\prime}(\sum,\tau)$, $S_{sc}^{\prime\prime}(\sum,\tau)$ and $S_{as}^{\prime\prime}(\sum;\tau)$ are singular ,pure point , singular continuous and absolutely continuous supports of $E_{A_0}(.)$ respectively , i.e., we have

$$E_{A_0}\left(\chi\cap S_\tau^{\prime\prime}\left(\sum,\tau\right)\right)=E_{A_0}^{\tau}(\chi),\ \tau=s,pp,sc,ac \qquad (56)$$

For each Borel set $\chi\in$ i . In particular it hold $\delta_p(A_0)=S_{pp}^{\prime\prime}(\sum,\tau)$ and

$$\delta_\tau(A_0)\subseteq S_\tau^{\prime\prime}\left(\sum,\tau\right)\subseteq\delta(A_0),\ \tau=s,sc,ac \ .$$

Proposition (1-2-13) [96]:

Let $\varphi(.)$ be a scalar R-function. Then for almost all $x\in$ i the limit $\varphi(x+io)=\lim_{y\to 0}\varphi(x+io)$ exists and moreover in this case one has $\varphi(x+io)=\lim_{z\to x}\varphi(z)$.

Theorem (1-2-14) [96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space h with $n_+(A)=n_-(A)$ Further let $\Pi=\{H\Gamma_0,\Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$ and let $E_{A_0}(0)$ be the spectral measure of the self- adjoint extension A of i . If $\tau=\{h_k\}_{k=1}^{N}$, $1\le N\le\infty$ is a total set in H then sets $\Omega_s(M;\tau)$, $\Omega_{pp}(M;\tau)$, $\Omega_{sc}(M;\tau)$ and $\Omega_{ac}(M;\tau)$ are supports of $E_{A_0}(.)$ respectively. i.e., we have

$$E_{A_0}(\chi\cap\Omega_\tau(M;\tau))=E_{A_0}^{\tau}(\chi),\ \tau=s,pp,,sc.ac \qquad (57)$$

For each Borel set $\chi\in R$ In particular it holds $\delta_p(A_0)=\Omega_{pp}(M;\tau)$ and $\delta_\tau(A_0)\subseteq\overline{\Omega_\tau(M;\tau)}\subseteq\delta_\tau(A_0)$ for $\tau=s,sc,ac$. We note that the inclusions $\delta_s(A_0)\subseteq\overline{\Omega_s(M;\tau)}$ and $\delta_{sc}(A_0)\subseteq\overline{\Omega_{sc}(M;\tau)}$ of theorem (1-2-14) may be strict even if $\delta_{sc}(A_0)$ is empty.

Let $\mu(.)$ be a Borel measure on $\mathbb{R}$ and let $x \subseteq \mathbb{R}$ be a Borel set the set

$$CL_{ac}(x) = x \subseteq \mathbb{R} : mes((x-\varepsilon, x+\varepsilon) \cap x) > 0 \ \forall \varepsilon > \delta \tag{58}$$

is called the absolutely continuous closure of set $\mathbb{R}$ obviously the set $CL_{ac}(x) \in \bar{x}$ is always closed and one has

Lemma (1-2-15) [96]:

Let $\phi(.)$ be a scalar R-function which has the representation (10) then $S_{ac}(\mu) = CL_{ac}(\Omega_{ac}(\phi))$

Proof:

If $x \notin CL_{ac}(\Omega_{ac}(\phi))$ then there is an $\varepsilon > 0$ such that $mes(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi) = 0$

$$\mu_{ac}(x-\varepsilon, x+\varepsilon) = \mu_{ac}(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi) = 0 \tag{59}$$

Hence $x \notin S(\mu_{ac}) = S_{ac}(\mu)$ which yields $S_{ac}(\mu) \subseteq CL_{ac}(\Omega_{ac}(\phi))$ conversely if $x \notin S_{ac}(\mu)$ then there is an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\mu) = 0$ then $\mu_{ac}(x-\varepsilon, x+\varepsilon) = 0$ using

$$\mu_{ac}(x-\varepsilon, x+\varepsilon) = \quad \mu_{ac}(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi) \int_{(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi)} \frac{d\mu(t)}{dt} dt = 0 \tag{60}$$

and proposition (1-2-11) (i) and (vi) one gets

$$\mu_{ac}(x-\varepsilon, x+\varepsilon) = \quad \frac{1}{\pi} \int_{(x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi)} Sm(\phi(\tau + i0))\, d\tau = 0 \tag{61}$$

Hence $Sm(\phi(t+i0))dt = 0$ for $a.e.t \in (x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi)$. However by definition of the set $\Omega_{ac}(\phi)$ one has $Sm(\phi(\tau + i0))\, dt > 0$ for all $\tau \in \Omega_{ac}(\phi)$ which implies $mes((x-\varepsilon, x+\varepsilon) \cap \Omega_{ac}(\phi)) = 0$

Hence $x \notin CL_{ac}(\Omega_{ac}(\phi))$ or equivalent $CL_{ac}(\Omega_{ac}(\phi)) \subseteq S_{sc}(\mu)$.

Proposition (1-2-16) [96]:

Let A be a simple densely defined closed symmetric operator a separable Hilbert space with $n_+(A) = n_-(A)$. Further let $\Pi = \{H, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$ If $\tau = \{h_k\}_{k=1}^{N}$, $1 \leq N \leq \infty$ is a total set in $\mathfrak{H}$ then the absolutely continuous spectrum of the self-adjoint extension A_0 of A is given by.

$$\delta_{ac}(A_0) = \overline{\cup_{k=1}^{N} CL_{ac}\left(\Omega_{ac}\left(M_{h_k}\right)\right)} \quad (62)$$

Theorem (1-2-17)[96]:

Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(.)$.

If $\tau = \{h_k\}_{k=1}^{N}$, $1 \le N \le \infty$ is a total set in $\mathcal{H}$, then for the self-adjoint extension A_0 of A the following conclusions are valid :

(i) The self-adjoint extension A_0 of A has no point spectrum within the interval (a,b). i.e., $\delta_{pp}(A_0) \cap (a,b) = \emptyset$ if and only if for each $k = 1,2,\ldots N$ one has

$$\lim_{y \to 0} yM_{hk}(x+iy) = 0$$

(63)

for all $x \in (a,b)$. In this case the following relation holds

$$\delta(A_0) \cap (a,b) = \frac{\delta_c(A_0) \cap (a,b)}{\cup_{k=1}^{N} \Omega_{ac}(M_{hk}) \cup} = \overline{\cup_{k=1}^{N} \Omega_{ac}(M_{hk})} \cap (a,b) \quad (64)$$

(ii) The self-adjoint extension A_0 of A has no singular continuous spectrum within the interval (a,b), i.e. $\delta_{ac}(A_0) \cap (a,b) = \emptyset$ if for each $k = 1,2,\ldots N$ the set $\Omega_{ac}(M_{hk}) \cap (a,b)$ is countable in particular, if $(a,b) | \Omega_{ac}(M_{hk})$ is countable.

(iii) The self-adjoint extension A_0 of A has no absolutely continuous spectrum within the interval (a,b) .i.e., $\delta_{ac}(A_0) \cap (a,b) = \emptyset$ if and only if for each $k = 1,2,\ldots N$ the condition

$$Sm(M_{hk}(x+i0)) = 0 \quad (65)$$

holds for a.e. $x \in (a,b)$. in this case we have

$$\delta_s(A_0)\cap(a,b)=\overline{\Omega_s(M;\tau)}\cap(a,b)$$

Proof:

(i) If condition (65) is satisfied for all $x\in(a,b)$ and all $k=1,2,\dots N$, then a simple computation shows that $\lim_{z\to x}(z-x)M_{hk}=0$ holds for all $x\in(a,b)$ and each $k=1,2,\dots N$ too. Therefore $\Omega_{pp}(M_{hk})\cap(a,b)=0$ for $k=1,2,\dots N$ whichyields $\Omega_{pp}(M;T)\cap(a,b)=0$ theorem (1-2-14). Implies $\delta_p(A_0)\cap$ $(a,b)=0$ which yields $\delta_{pp}(A_0)\cap(a,b)=0$.

(ii) Conversely if $\delta_{pp}(A_0)\cap(a,b)=0$ then $\delta_p(A_0)\cap(a,b)=0$ again by theorem (1-2-14) we find $\delta_{pp}(A_0)\cap(a,b)=0$ therefore $\delta_{pp}(A_0)\cap(a,b)=0$ for each $k=1,2,\dots N$. However this implies that $\lim_{z\to x}(z-x)M_{hk}(z)=0$ which yields $\lim_{y\to 0} yM_{hk}(x+iy)=0$ for all $x\in[a,b]$ and each $k=1,2,\dots N$. The first of relation (64) is consequence of $\delta(A_0)=\delta_{pp}(A_0)U\delta_c(A_0)$ and $\delta_{pp}(A_0)\cap$ $(a,b)=0$. The second part of relation (64) is a consequence of theorem (1-2-18) which shows that

$$\delta_\tau(A_0)\subseteq\overline{\Omega_\tau(M;\tau)}=\overline{\cup_{k=1}^{N}\Omega_\tau(M;\tau)}\subseteq\delta(A_0)\ ,\ \tau=sc,ac \tag{67}$$

and $\delta_c(A_0)=\delta_{sc}(A_0)U\delta_{ac}(A_0)$ .Both facts imply that $\delta_c(A_0)\cap(a,b)\subseteq$

$$\overline{\cup_{k=1}^{N}\Omega_{ac}(M_{hk})}\cup\overline{\cup_{k=1}^{N}\Omega_{ac}(M_{hk})}\cap(a,b)\subseteq\delta(A_0)\cap(a,b)=\delta_c(A_0)\cap(a,b) \tag{68}$$

Which proves (64)

(ii) By (53) we gets that $S_{ac}^{\prime}(\sum_{hk})=S^{\prime}(\sum_{hk,sc})\subseteq S_{sc}^{\prime}(\sum_{hk})\Omega c(M_{hk})$. Therefore if $\Omega_{ac}(M_{hk})\cap(a,b)$ is countable, then so is $S_{ac}^{\prime\prime}(\sum_{hk})\cap(a,b)$ this yields that the singular continuous measure $\sum_{hk,sc}(.)$ is supported within the interval (a,b) on a countable set. However this implies that $\sum_{hk,sc}(a,b)=0$ for each

$k=1,2,\dots,N$ and every $h\in H$ one has $\sum_{hK,sc}(a,b)=0$ which yields $\sum^{c}(a,b)=0$. Therefore by lemma(1-2-9) one gets $E^{sc}_{A_0}(a,b)=0$ which proves $\delta_{sc}(A_0)\cap(a,b)=0$. If $(a,b)\setminus\Omega_{ac}M_{hk}$ is countable, then by $\Omega_{sc}(M_{hk})\subseteq(a,b)\setminus\Omega_{ac}(M_{hk})$ the set $\Omega_{sc}(M_{hk})$ is countable too which completes the proof (ii).

(iii) If for each $k=1,2,\dots,N$ the condition (65) holds for a.e. $x\in(a,b)$ each $\varepsilon>0$ one has $mes(x-\varepsilon,x+\varepsilon)\cap\Omega_{ac}(M_{hk})\cap(a,b)=\theta$ hence $CL_{ac}(\Omega_{ac}(M_{hk}))\cap(a,b)=0$ taking proposition (1-2-16) into account we find $\delta_{ac}(A_0)\cap(a,b)=0$. Conversely if $\delta_{ac}(A_0)\cap(a,b)=0$ then proposition (1-2-16) for each $k=1,2,\dots,N$ we have

$$CL_{ac}(\Omega_{ac}(M_{hk}))\cap(a,b)=CL_{ac}(\Omega_{ac}(M_{hk}))\cap(a,b)=0$$

Which verifies condition (65) for a.e $x\in(a,b)$. Using $\delta(A_0)\cap(a,b)=\delta_s(A_0)$ and $\delta_s(A_0)\subseteq\overline{\Omega_s(M;\tau)}\subseteq\delta(A_0)$ which was proved in theorem (1-2-14)

Section (1-3): Characteristic Function for Row Contraction and Factorization

We give new result for the characteristic function associated with an arbitrary row contraction and show that

$$I - \Theta_T \Theta_T^* = k_T k_T^*$$

Where k_T is the Poisson kernel of T. Consequently we will show that the curvature invariant and characteristic associated with a Hilbert model over $\mathcal{F}_n^+$ generated by an arbitrary row contraction T can be expressed only in terms of the characteristic function of T [62,63].

The characteristic function associated with an arbitrary row condition $T = [T_1, \dots, T_n],\ T_i \in B(\mathcal{H})$ was introduced for the classical case $n = 1$ and it was proved to be a complete unitary invariant for completely non-coisometric (c.n.c) row contraction. Using the characteristic of multi-analytic operator on Fock spaces, one can easily see that the characteristic function of T is multi-analytic operator [64, 65].

$$\Theta_T(R_1, \dots, R_n): \quad F^2(H_n) \otimes \mathcal{D}_{T^*} \longrightarrow F^2(H_n) \otimes \mathcal{D}_T$$

With the formal Fourier representation

$$-I_{F^2(H_n)} \otimes T + \left(I_{F^2(H_n)} \otimes \Delta_T\right)\left(I_{F^2(H_n) \otimes \mathcal{H}} - \sum_{i=1}^{n} R_i \otimes T_i^*\right)^{-1} [R_1 \otimes I_{\mathcal{H}}, \dots, R_n \otimes I_{\mathcal{H}}]\left(I_{F^2(H_n)} \otimes \Delta_*\right)$$

where $R_1, \dots, R_n$ are the right creation operator on the full Fock space $F^2(H_n)$

are $\Delta_T = \left(I_{\mathcal{H}} - \sum_{i=1}^{n} T_i T_i^*\right)^{\frac{1}{2}} \in B(\mathcal{H})$ and $\Delta_{T^*} = (I_{\mathcal{H}} - T^*T) \in B(\mathcal{H}^{(n)})$ while the defect spaces are $D_T = \overline{\Delta_T \mathcal{H}}$ and $D_{T^*} = \overline{\Delta_{T^*} \mathcal{H}^n}$ where $\mathcal{H}^n$ denotes the direct sum of n copies of $\mathcal{H}$, we need the following result .

Lemma (1-3-1) [61]:

If $\Theta(R_1,\dots,R_n)\in R_n^\infty\bar{\otimes}B(\mathcal{H},k)$. Then $SOT-\lim_{r\to 1}\Theta(rR_1,\dots,rR_n)^*=\Theta(R_1,\dots,R_n)^*$

Proof:

We know that any multi-analytic operator $\Theta(R_1,\dots,R_n)^*$ with formal Fourier representation $\Theta(R_1,\dots,R_n)\ \sim\ \sum_{k=0}^{\infty}\sum_{|\alpha|=k}R_\alpha\otimes\theta_\alpha$ $\theta_\alpha\in B(\mathcal{H},k)$ has the property that $\theta(R_1,\dots,R_n)=SOT-\lim_{T\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k}r^{|\alpha|}R_\alpha\otimes\theta_\alpha$ where the series converge in the uniform norm for each $r\in(0,1)$ now note that for every $\beta\in F_n^+, h\in\mathcal{H},$ and $g\in F^2(H_n)\otimes k$ we have

$$\left\langle\Theta(R_1,\dots,R_n)^*(e_\beta\otimes h),g\right\rangle=\left\langle e_\beta\otimes h,\Theta(R_1,\dots,R_n)g\right\rangle$$

$$=\left\langle e_\beta\otimes h\left(\sum_{\alpha\in F_n^+,|\alpha|\le|\beta|}R_\alpha\otimes\theta_\alpha\right)g\right\rangle$$

$$=\left\langle\left(\sum_{\alpha\in F_n^+,|\alpha|\le|\beta|}R_\alpha^*\otimes\theta_\alpha^*\right)(e_\beta\otimes h)\,g\right\rangle$$

therefore

$$\Theta(R_1,\dots,R_n)^*(e_\beta\otimes h)=\left(\sum_{\alpha\in F_n^+,|\alpha|\le|\beta|}R_\alpha^*\otimes\theta_\alpha^*\right)(e_\beta\otimes h)$$

similarly we have

$$\Theta(rR_1,\dots,rR_n)^*(e_\beta\otimes h)=\left(\sum_{\alpha\in F_n^+,|\alpha|\le|\beta|}r^{|\alpha|}R_\alpha^*\otimes\theta_\alpha^*\right)(e_\beta\otimes h)$$

Using the last two equalities we obtain

$$\lim_{r\to 1}\Theta(rR_1,\dots,rR_n)^*(e_\beta\otimes h)=\Theta(R_1,\dots,R_n)^*(e_\beta\otimes h)$$

for any $\beta\in F_n^+$ and h on the other hand according to non commutative von Neumann inequality $\|\Theta(rR_1,\dots,rR_n)^*\|\le\|\Theta(R_1,\dots,R_n)^*\|$ for any $r\in(0,1)$. Hence and due to the fact that the closed span of all vectors $e_\alpha\otimes h$ with $\beta\in F_n^+, h\in\mathcal{H},$ coincides with $F^2(H_n)\otimes\mathcal{H}$ we deduce that

$$SOT-\lim_{r\to 1}\Theta(rR_1,\dots,rR_n)^*=\Theta(R_1,\dots,R_n)^*$$

The proof is complete. The following factorization play an important role

Theorem (1-3-2) [5]:

Let $T = [T_1, \ldots, T_n]$, $T_i \in B(H)$ be a row contraction, then

$$I - \Theta_T \Theta_T^* = k_T k_T^*$$

(69)

where Θ_T is the characteristic function of T and k_T is the corresponding Poisson kernel.

Proof:

Denoting $\tilde{T} = [I_{F^2(H_n)} \otimes T_1, \ldots, I_{F^2(H_n)} \otimes T_n]$ and

$$\tilde{R} = [R_1 \otimes I_H, \ldots, R_n \otimes I_H]$$

the characteristic function of T has representation

$$\Theta(R_1, \ldots, R_n) = SOT - \lim_{r \to 1} \left[-\tilde{T} + \Delta_{\tilde{T}} \left(I_{F^2(H_n)} - r\hat{R}\tilde{T}^* \right)^{-1} r\hat{R}\Delta_{\tilde{T}} \right] \tag{70}$$

define the operator

$A = \tilde{T}^*$, $B = \Delta_{T^*}$, $C = \Delta_{T^*}$, $D = -\tilde{T}$, and $z = r\hat{R}$, $0 < r < 1$

and note that

$$\begin{pmatrix} A & B \\ c & D \end{pmatrix} = \begin{pmatrix} \tilde{T}^* & \Delta_{T^*} \\ \Delta_{\tilde{T}} & -\tilde{T} \end{pmatrix}$$ is a unitary operator therefore

$$AA^* + BB^* = I, CC^* + DD^* = I \quad \text{and} \quad AC^* + BD^* = 0 \tag{71}$$

define $\phi(z) = D + C(I - ZA)^{-1} ZB$

And notice that using relation (71) we have **[5]:**

$$I - \phi(Z)\phi(Z)^* = I - DD^* - C(I - ZA)^{-1} ZBD^* - DB^* Z^* (I - A^* Z^*)^{-1} C^*$$

$$- C(I - ZA)^{-1} ZBB^* Z^* (I - A^* Z^*)^{-1} C^*$$

$$= CC^* + C(I - ZA)^{-1} ZAc^* + CA^* Z^* (I - A^* Z^*)^{-1} C^*$$

$$- C(I - ZA)^{-1} ZZ^* (I - A^* Z^*)^{-1} C^*$$

$$+ C(I - ZA)^{-1} (ZAA^* Z^*)(I - A^* Z^*)^{-1} C^*$$

$$= C(I - ZA)^{-1} ((I - ZA)(I - A^* Z^*) + ZA(I - A^* Z^*))$$

$$-(I-ZA)^{-1}A^*Z^* - Z^*Z^* + ZAA^*Z^*(I-A^*Z^*)^{-1}C^*$$

$$= \in (I-ZA)^{-1}(I-ZZ^*)(I-A^*Z^*)^{-1}C^*$$

Therefore

$$I - \Theta(Z)\Theta(Z)^* = \in (I-ZA)^{-1}(I-ZZ^*)(I-A^*Z^*)^{-1}C^*$$

(72)

Therefore according to our notation for any $r \in (0,1)$ the defect operator $I - \Theta_T(rR_1,\ldots,rR_n)\Theta_T(rR_1,\ldots,rR_n)^*$ is equal to the product **[5]:**

$$\Delta_T(I - r\hat{R}\tilde{T}^*)^{-1}(I - r^2\hat{R}^*\hat{R}^*)(I - r\tilde{T}\hat{R}^*)^{-1}\Delta_T$$

$$=(I \otimes \Delta_T)\left(I - r\sum_{i=1}^{n} R_i \otimes T_i^*\right)^{-1}\left[\left(1 - r^2\sum_{i=1}^{n}\tilde{R}_i R_i^*\right)\otimes I\right]\left(I - r\sum_{i=1}^{n} R_i^* \otimes T_i\right)^{-1}(I \mp \otimes \Delta_T)$$

$$=\sum_{k=0}^{\infty}\sum_{|\alpha|=k} r^{|r|} R_r \otimes \Delta_T T_r^* \left[\left(I - r^2\sum_{i=1}^{n} R_i R_i^*\right)\otimes I\right]_i \left(\sum_{p=0}^{\infty}\sum_{|\beta|=p} r^{|\beta|} R_\beta^* \otimes T_{\tilde{\beta}}\Delta_T\right)$$

$$=\sum_{k,p=0}^{\infty}\sum_{|r|=k,|\beta|=p} r^{|r|+|\beta|} R_r\left(I - r^2\sum_{i=1}^{n} R_i R_i^*\right) R_\beta^* \otimes \Delta_T T_{\tilde{r}}^* T_{\tilde{\beta}}\Delta_T$$

Now for every $\alpha, \omega \in F_n^+,\ h \in D_T$ we have

$$\left\langle [I - \Theta_T(rR_1,\ldots,rR_n)\Theta_T(rR_1,\ldots,rR_n)^*](e_\alpha \otimes h), e_\omega \otimes k\right\rangle$$

$$=\sum_{r \in F_n^+, |r| \le |\omega|, \beta \in F_n^+, |\beta| \le |\alpha|} \sum \left\langle r^{|r|+|\beta|}\left(I - r^2\sum_{i=1}^{n} R_i R_i^*\right) R_\beta^* e_\alpha, e_\omega\right\rangle \left\langle \Delta_T T_{\tilde{r}}^* T_\beta \Delta_T h, k\right\rangle$$

using lemma (1-3-1) we have

$$SOT - \lim_{r \to 1}\Theta_T(rR_1,\ldots,rR_n)^* = \Theta_T(R_1,\ldots,R_n)^*$$

therefore the above computations imply that

$$\left\langle [I - \Theta_T(R_1,\ldots,R_n)\Theta_T(R_1,\ldots,R_n)^*](e_\alpha \otimes h), e_\omega \otimes k\right\rangle$$

$$=\sum_{r \in F_n^+, |r| \le |\omega|, \beta \in F_n^+, |\beta| \le |\alpha|} \sum \left\langle R_T P_{\mathbb{C}} R_\beta^* e_\alpha, e_\omega\right\rangle \left\langle \Delta_T T_{\tilde{r}}^* T_{\tilde{\beta}} \Delta_T h, k\right\rangle$$

$$=\sum_{r \in F_n^+, |r| \le |\omega|} \left\langle R_r(1), e_\omega\right\rangle \left\langle \Delta_T T_{\tilde{r}}^* T_{\tilde{\alpha}} \Delta_T h, k\right\rangle = \left\langle \Delta_T T_{\tilde{\omega}}^* T_{\tilde{\alpha}} \Delta_T h, k\right\rangle$$

For any $h,k \in D_T$ and $\alpha, \omega \in F_n^+$. Summing up the above computation we deduce that $I - \Theta_T(R_1,\ldots,R_n)\Theta_T(R_1,\ldots,R_n)^* = k_T k_T^*$ which complete the proof.

We recall that the spectral radius of an n-tuple of operators $X = [X_1, \ldots, X_n]$ is defined by

$$r(X) = \lim_{k \to \infty} \left\| \sum_{|\alpha| = k} X_\alpha X_\alpha^* \right\|^{1/2k}$$

A closer look at the proof of theorem (1-3-2) reveals the following factorization result. We should add that operator $I - \hat{X}\tilde{T}^*$ invertible because $r(X) < 1$.

Corollary (1-3-3) [61]:

Let $T = [T_1, \ldots, T_n], T_i \in B(\mathcal{H})$ be a row contraction and let be its characteristic function. If $X = [X_1, \ldots, X_n], X_i \in B(\mathcal{K})$ is a row contraction with spectral radius $r(X) < 1$, then $I_{k \otimes D_T} - \Theta_T(X_1, \ldots, X_n)\Theta_T(X_1, \ldots, X_n)^* = \Delta_{\tilde{T}}\left(I - \hat{X}\tilde{T}\right)^{-1}$

$\left(I - \hat{X}\hat{X}^*\right)\left(I - \tilde{T}\hat{X}^*\right)^{-1}\Delta_{\tilde{T}}$. Where $\hat{X} = [X_1 \otimes I_{\mathcal{H}}, \ldots, X_n \otimes I_{\mathcal{H}}]$ notation as in proof of theorem (1-3-2).

Theorem (1.3.3) [5]:

Let $T \in B(H)$ be arrow counteraction, if $\Theta_T \Theta_T^* + k_T . k_T^* = I$, where is the characteristic function of and is the corresponding Poisson kernel, then for $A = \tilde{T}, B = \nabla_{\tilde{T}}$ and $C = 0$ the operator $S = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$ commute with if and only if commute with and A, B are self-adjoint.

Proof:

If $S^*S = \begin{bmatrix} A^*A & A^*B + B^*A \\ 0 & A^*A \end{bmatrix}$ and $SS^* = \begin{bmatrix} AA^* & AB^* + BA^* \\ 0 & AA^* \end{bmatrix}$

Equating, then $S^*S = SS^*$ if and only if $A^*A = AA^*$ is normal and $A^*B = BA^*$.

Theorem (1-3-4) [61]:

Let $\pounds F_n^+$ be the complete free semigroup algebra generated by the free semigroup F_n^+ with generators $g_1,\ldots,g_n$ and natural element g_0 any n-tuple $T_1,\ldots,T_n$ of bounded operators on a Hilbert space $\mathcal{H}$ gives rise to a Hilbert model over $\pounds F_n^+$ in the natural way $f.h = f(T_1,\ldots,T_n)h, \quad f \in \pounds F_n^+ \quad h \in \mathcal{H}$ we say that $\mathcal{H}$ is a contractive F_n^+ model if $T = (T_1,\ldots,T_n)$ is a row contraction which is equivalent to $\|g_1.h_1 + \ldots + g_n h_n\|^2 \leq \|h_1\|^{2} + \ldots + \|h_n\|^2, h_1,\ldots,h_n \in \mathcal{H}$ we say that $\mathcal{H}$ is of finite $rank \ if \ rank(\mathcal{H}) = rank \ \Delta_T < \infty$. The curvature invariant and Euler characteristic associated with an arbitrary row contraction T were introduced

and studied we recall $Cuv(T) = \lim\limits_{m\to\infty} \dfrac{trace\left[I - \phi_T^m(I)\right]}{I + n + \ldots + n^{m-1}}$ and

$\chi(T) = \lim\limits_{m\to\infty} \dfrac{rank\left[I - \phi_T^m(I)\right]}{1 + n + \ldots + n^{m-1}}$ where ϕ_T is the completely positive map associated

with T i.e., $\phi_T(X) = \sum\limits_{i=1}^{n} T_i \times T_i^*$ using theorem (1-3-2) and some results we can show that the curvature and the Euler characteristic of arrow contraction T can be expressed only in terms of the standard characteristic functions θ_T.

Theorem (1-3-5) [61]:

Let $T = (T_1,\ldots,T_n)$, $T_i \in B(\mathcal{H})$ contraction with rank $\Delta_T < \infty$ and $Cur(T)$ and $\chi(T)$ denote its curvature and Euler characteristic respectively then

$$Curv(T) = rank\ \Delta_T - \lim_{m\to\infty} \frac{trace\left[\theta_T\theta_T^*(p_m \otimes I)\right]}{n^m} \quad \text{and} \quad \chi(T) = \lim_{m\to\infty} \frac{rank\left[\left(I - \theta_T\theta_T^*\right)P_{\leq m} \otimes I\right]}{1 + n + \ldots + n^{m-1}}$$

Where $P_m (resp. P_{\leq m})$ in the orthogonal projection of full Fock space $F^2(H_n)$ onto the subspace of all homogenous polynomials of degree (resp-polynomials of degree $\leq m$).

Proof:

Since $Curv(T) = \lim_{m\to\infty} \frac{trace\left[K_T K_T^* \left(p_m \otimes I\right)\right]}{n^m}$. Using the factorization results of theorem(1-3-2) the first result follows

$$\chi(T) = \lim_{m\to\infty} \frac{rank\left[K_T^* \left(P_{\le m} \otimes I\right) K_T\right]}{1+n+\ldots+n^{m-1}} \tag{73}$$

Since $K_T^*(P_{\le m} \otimes I)$ has finite rank we have $rank K_T^*(P_{\le m} \otimes I) = rank\left[ran\ k_T^*(P_{\le} \otimes I)\right]$ on the other hand ,since K_T is one- to-one on the range of $K_T^*(P_{\le m} \otimes I)$ we also have $rank\left[K_T^*(P_{\le m} \otimes I)\right] = rank\left[K_T\ k_T^*(P_{\le} \otimes I)\right]$. Hence using relation (73) and theorem (1-3-2) we complete the proof.

A constrained characteristic function is associated with any constrained row contraction. For pure constrained row contractions we show that this characteristic function is a complete unitary invariant and provide a model in terms of it. We also show that Arveson's curvature invariant and Euler characteristic associated with a Hilbert model over $C[Z_1,\ldots,Z_n]$ generated by a commuting row contraction can be expressed only in terms of the constrained characteristic function .

Let be a wot-closed two –sided ideal of the non commutative analytic Toeplitz algebra F_n^∞ generated by a family of polynomials P_J we define the constrained characteristic function associated with a J-constrained row contraction $T = [T_1,\ldots,T_n], T_i \in B(\mathrm{H}),$ to be the multi-analytic operator (with respect to the constrained shift $[B_1,\ldots,B_n]$,

$\Theta_{J,T}(W_1,\ldots,W_n) : N_J \otimes D_{T^*} \to N_J \otimes D_T,$

Defined by the formal Fourier representation

$$-I_{N_J} \otimes T + (N_J \otimes \Delta_T)\left(I_{N_T \otimes \mathrm{H}} - \sum_{i=1}^{n} W_i \otimes T_i^*\right)^{-1} [W_1 \otimes I_{\mathrm{H}},\ldots,W_n \otimes I_{\mathrm{H}}](I_{N_J} \otimes \Delta_{T^*})$$

Taking into account that N_T a co-invariant subspace under $R_1,\ldots,R_n$ can see that $\theta_{J,T}$ is the maximal J-constrained pieced of the standard characteristic function θ_T of the row contraction T. More precisely we have

$$\Theta_T(R_1,\ldots,R_n)^*(N_J\otimes D_T)\subseteq N_J\otimes D_{T^*}\ and$$

$$P_{N_J\otimes D_T}\Theta_T(R_1,\ldots,R_n)|N_J\otimes D_{T^*}=\Theta_{J,T}((W_1,\ldots,W_n))$$

(74)

Let us remark that the above definition of the constrained characteristic function makes sense when J is an arbitrary wot-closed two sided ideal of F_n^∞ and $T=[T_1,\ldots,T_n]$ is an arbitrary c.n.c constrained row contraction.

Theorem (1-3-6) [61]:

Let $J\neq F_n^\infty$ be wot-closed two seded ideal of F_n^∞ generated by a family of polynomials P_J.

$T=[T_1,\ldots,T_n],T_i\in B(H)$, a J-constrained row contraction then

$$I_{NJ\otimes D_T}-\Theta_{J,T}=K_{J,T}K_{J,T}^* \qquad (75)$$

where $\theta_{J,T}$ is the constrained characteristic function of T and $K_{J,T}$ is the corresponding constrained Poisson Kernel

Proof:

The constrained Poisson Kernel associated with T is $K_{J,T}H\to N_J\otimes\overline{\Delta_T H}$ $K_{J,T}:H\to N_J\otimes\overline{\Delta_T H}$ defined

$$K_{J,T}=\left(P_{N_J}\otimes I_{\overline{\Delta_T H}}K_T\right) \qquad (76)$$

Where K_T is standard Poisson Kernel of T. As well as $K_T\subset N_T\otimes\overline{\Delta_T H}$ using theorem (1-3-2) and taking the compression of relation (69) to the subspace $N_J\otimes D_T\subset F^2(H_n)\otimes D_T$ we obtain

$$I_{N_J\otimes D_T}-P_{N_J\otimes D_T}\Theta_T(R_1,\ldots,R_n)\Theta_T(R_1,\ldots,R_n)|N_J\otimes D_T=P_{N_J\otimes D_T}K_TK_{T^*}|N_J\otimes D_T$$

taking into account relation (74) and (76) and that $W_i^*=R_i^*|N_J,\ i=1,\ldots,n$ we refer that $I_{N_J\otimes D_T}-\Theta_{J,T}(W_1,\ldots,W_n)\Theta_{J,T}(W_1,\ldots,W_n)^*=K_{J,T}K_{J,T}^*$ as in the proof of theorem (1-3-5) one can use corollary (1-3-3) to obtain the following constrained version of it.

Corollary (1-3-7) [61]:

Let $J \neq F_n^{\infty}$ be closed sided ideal of F_n^{∞} generated by a family of polynomials P_J and let $T = [T_1, \ldots, T_n], T_i \in B(\mathcal{H})$, be a J-constrained row contraction. If $X = [X_1, \ldots, X_n],\ X_i \in B(\mathcal{K})$, is a J-constrained row contraction with spectral radius $r(X) < 1$, then

$$I_{\mathcal{K} \otimes \mathcal{D}_T} - \Theta_{J,T}(X_1, \ldots, X_n)\Theta_{J,T}(X_1, \ldots, X_n)^* = \Delta_T\left(I - \hat{X}\tilde{T}^*\right)^{-1}\left(I - \tilde{X}\hat{X}^*\right)\left(I - \tilde{T}\hat{X}^*\right)^{-1}\Delta_T$$

where $\hat{X} = [X_1 \otimes I_{\mathcal{H}}, \ldots, X_n \otimes I_{\mathcal{H}}]$ and the other notations are from the proof of theorem (1-3-2). Now we present a model for pure constrained row contraction in terms of characteristic function.

Theorem (1-3-8) [61]:

Let $J \neq F_n^{\infty}$ be a wot-closed two-sided ideal of F_n^{∞} and $T = [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a pure J-constrained row contraction. Then the constrained characteristic function $\Theta_{J,T} \in W(W_1, \ldots, W_n)\bar{\otimes} B(\mathcal{D}_{T^*}, \mathcal{D}_T)$ is a partial isometry and T is unitarily equivalent to the row contraction

$$[P_{H_{J,T}}(B_1 \otimes I_{\mathcal{D}_T})|H_{J,T}, \ldots, P_H(B_n \otimes I_{\mathcal{D}_T})|H_{J,T}]$$

(77)

where $P_{H_{J,T}}$ is the orthogonal projection of $N_J \otimes \mathcal{D}_T$ on the Hilbert space

$$H_{J,T} = (N_J \otimes \mathcal{D}_T) \ominus \Theta_{J,T}(N_J \otimes \mathcal{D}_{T^*})$$

Theorem (1-3-9) [61]:

Let $J \neq F_n^{\infty}$ be a wot-closed two-sided ideal of F_n^{∞} and let $T = [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H}), T' = [T_1', \ldots, T_n'] T_i' \in B(\mathcal{H}')$ be two J-constrained pure contractions then T and T' are unitarily equivalent if and only if their constrained characteristic function $\theta_{J,T}$ and $\theta_{J,T'}$ coincide.

Proof:

Assume that and are unitarily equivalent $U:H\to H'$ let be a unitary operator such that $T_i = U^* T_i' U$ for any $i = 1,\dots,n$ simple computation

reveal that $U\Delta_T = \Delta_{T'}U$ and $\left(\otimes_{i=1}^{n} U\right)\Delta_{T^*} = \Delta_{T'^*}\left(\otimes_{i=1}^{n} U\right)$ define the unitary operator

and by setting $\tau = U|D_T : D_T \to D_{T'}$ and $\tau' = \left(\otimes_{i=1}^{n} U\right)\big|D_{T^*} \to D_{T'^*}$ Taking into account the definition of the constrained characteristic function it is easy to see that

$$(I_{N_J}\otimes T)\theta_{J,T} = \theta_{J,T'}(I_{N_J}\otimes T')$$

Conversely assume that the characteristic function of and coincide. According to the remarks preceding the theorem there exist unitary operators

$\tau : D_T \to D_{T'}$ and $\tau_* : D_{T^*} \to D_{T'^*}$ such that the following diagram

$$\begin{array}{ccc} N_j \otimes D_{T^*} & \xrightarrow{\Phi_{j,T}} & N_j \otimes D_T \\ \downarrow I_{N_J}\otimes \tau^* & & \downarrow I_{N_J}\otimes \tau \\ N_j \otimes D_{T'^*} & \xrightarrow{\Phi_{j,T'}} & N_j \otimes D_{T'} \end{array}$$

Is commutative i.e.,

$$(I_{N_J}\otimes \tau)\phi_{J,T} = \phi_{J,T'} = \phi_{J,T'}(I_{N_J}\otimes \tau_*)$$

(78)

Hence we deduce that $\Gamma = (I_{N_J}\otimes \tau)|H_{j,T} : H_{j,T} \to H_{j,T'}$ is a unitary operator where $H_{j,T}$ and $H_{j,T'}$ are the model spaces for and respectively. Since $(B_i^*\otimes I_{D_T})(I_{N_J}\otimes T^*) = (I_{N_J}\otimes T^*)(B_i^*\otimes I_{D_{T'}}),\ i = 1,\dots,n$ and $H_{j,T}(resp.H_{j,T'})$ is a co-invariant subspace under $B_i\otimes I_{D_T}(respp.B_i\otimes I_{D_{T'}}),\ i=1,\dots,n$ we deduce that

$$(B_i^*\otimes I_{D_T})(H_{J,T})\Gamma^* = \Gamma^*(B_i^*\otimes I_{D_{T'}})(I_{N_J}\otimes T^*)(H_{J,T'}),\ i = 1,\dots,n$$

Hence we obtain

$$[P_{H_{J,T}}(B_i^* \otimes I_{D_T})(H_{J,T}) = P_{H_{J,T'}}(B_i \otimes I_{D_{T'}})(H_{J,T'})], \; i = 1,\dots,n$$

Now using theorem (1-3-7) we conclude that and are unitary equivalent. The proof is complete.

Theorem (1-3-10) [61]:

Let be a WOT-closed two- sided ideal of F_n^∞ such that $1 \in N_J$ and condition $\overline{span}\{\beta_\alpha\beta_\alpha^* : \alpha,\beta \in F_n^+\} = c^*(B_1,\dots,B_n)$ is satisfied. If $M \in N_J$ is an invariant subspace under $B_1,\dots,B_n$ and $T = [T_1,\dots,T_n],\ T_i = P_{M^\perp}B_i|M^\perp,\ i = 1,\dots,n$ then $M = \Theta_{J,T}(N_T \otimes D_{T^*})$ where $\theta_{J,T}$ is the constrained characteristic function of the reproducing Kernel Hilbert space with reproducing Kernel $K_n B_n \times B_n \to £$ defined by

$$K_n(z,w) = \frac{1}{1-\langle z,w \rangle £^n}, \; z,w \in B_n$$

The algebra $W_n^\infty = P_{F_n}|F_n$ was proved to be the -closed algebra generated by the operators $B_i, i = 1,\dots,n$ and the identity. Moreover W_n^∞ can be identified with the algebra of all multiplies of H^2 under this identification the creation operator $B_1,\dots,B_n$ become the multiplication operator $M_{z_1},\dots,M_{z_n}$ by the coordinate functions $z_1,\dots,z_n$ of . Let $T = [T_1,\dots,T_n], T_i \in B(H)$, be a constrained row contraction i.e., $T_iT_j = T_jT_i,\ i,j = 1,\dots,n$ under the above-mentioned identifications the constrained characteristic function of the multiplier (multiplication operator)

$$\Theta_{J_C},T : H^2 \otimes D_{T^*} \to H^2 \otimes D_T$$

defined by the operator. Valued analytic function on the open unit ball

$$B_n = \left\{ z = (z_1,\dots,z_n) \in £_n : |z| = \left(|z|^2 + \dots + |z|^2\right)^{\frac{1}{2}} < 1 \right\}$$

given by

$$\Theta_{J_C T}(z) = -T + \Delta_T (I - z_1T_1^* - \dots - z_nT_n^*)^{-1}[z_1 I_H,\dots,z_n I_H]\Delta_{T^*}, \; z \in B_n$$

We going to use the same notation for the multiplication operator $M_{\Theta_{J_C,T}} \in B(H^2 \otimes D_{T^*}, D_T)$ and its symbol $\theta_{J_C T}$ which is a $B(D_{T^*}, D_T)$ valued bounded analytic function in B_n. All the results of this part can be written in this commutative setting. Using theorem (1-3-5) and corollary (1-3-6) and some result we show that Euler characteristic associated with a commutative row contraction T with rank $\Delta_T < \infty$ can be expressed in terms of the constrained characteristic function $\theta_{J_C T}$.

Theorem (1-3-11) [61]:

Let $T = [T_1, \dots, T_n], T_i \in B(H)$, be a commutative row contraction with rank $\Delta_T < \infty$ and let $K(T)$ and $\chi(T)$ denote Arveson's curvature and Euler characteristic respectively.

Then

$$K(T) = \int_{\partial B_n} \lim_{r \to 1} trace \left[I - \Theta_{J_C,T}(r\zeta)\, \Theta_{J_C,T}(r\zeta)^* \right] d\partial(\zeta)$$

$$= rank\ \Delta_T - (n-1) \lim_{m \to \infty} \frac{trace\left[\Theta_{J_C,T} \Theta^*_{J_C,T} \left(Q_m \otimes I_{D_T} \right) \right]}{n^m}$$

where Q_m is the projection of H^2 onto the subspace of homogeneous polynomials of degree m and

$$\chi(T) = n! \lim_{m \to \infty} \Delta_T - (n-1) \frac{rank\left[\left(1 - \Theta_{J_C,T} \Theta^*_{J_C,T}\right) \left(Q_{\le m} \otimes I_{D_T} \right) \right]}{m^n}$$

Where Q_m is the projection of H^2 onto the subspace of all polynomials of degree $\le m$.

Proof:

Using the factorization result of corollary (1-3-6) in our particular case we obtain

$$\left(1 - \Theta_{J_C,T}(z) \Theta_{J_C,T}(z)^*\right) = \left(1 - |z|^2\right) \Delta_T \left(I - z_1 T_1^* - \dots - z_n T_n^*\right)^{-1} \left(1 - \bar{z}_1 T_1 - \dots - \bar{z}_n\right)^{-1} \Delta_T$$

for any $z \in B_n$ the first formula follows from the definition of the curvature and the above-mentioned factorization for the constrained characteristic function of . Using (60) and we have.

$$K(T) = (n-1) \lim_{r \to 1} \frac{trace\left[(P_m \otimes I) K_T K_T^*\right]}{m^{n-1}}$$

Where K_T is the Poisson on Kernel of and is the orthogonal projection of $F^2(H_n)$ onto the subspace of all homogeneous polynomials of degree . Since is a commutative row contraction -constrained we have range $K_T C F_s^2 \otimes D_T$ and the constrained Poisson Kernel satisfies the equation $K_{J_C,T} = (P_{F_s^2} \otimes I) K_T$ where is the symmetric Fock space. Using the standard properties. For the trace and above relation we deduce that

$$K(T) = (n-1) \lim_{r \to 1} \frac{trace\left[K_{J_C T} K_{J_C T}^* (Q_m \otimes I)\right]}{m^{n-1}} \tag{79}$$

where $Q_m = P_{F_s^2} P_m | F_s^2$ is the projection of onto the subspace of homogenous polynomials of degree. According to theorem (1-3-5) we have

$$I - \Theta_{J_C T} \Theta_{J_C T}^* = K_{J_C T} K_{J_C T}^* \tag{80}$$

taking into account relation (79),(80) we deduce the second formula for the curvature. Here of course we used Arveson's identification of symmetric Fock space with his space and Arveson's showed that his Euler characteristic satisfies the equation $\chi(T) = n! \lim_{m \to \infty} \frac{rank\left[1 - \varphi_T^{m+1}(1)\right]}{m^n}$ we here is the completely positive map associated with we get

$$\chi(T) = n! \lim_{m \to \infty} \frac{rank\left[K_T^* (P_{\le m} \otimes I) K_T\right]}{m^n} \tag{81}$$

Where $P_{\le m}$ is the orthogonal projection of $F_s^2(H_n)$ on the subspace of all polynomials of degree $\le m$. Using again that range $K_T \subset F_s^2 \otimes D_T$ and the contained Poisson Kernel satisfies the equation $K_{J_C,T} = (P_{F_s^2} \otimes I)K_T$ we deduce that

$$rank\,[K_T^*(P_{\leq m} \oplus I)K_T] = rank[K_T^*(P_{\leq m} \oplus I)(P_{F_s^2} \otimes I)K_T]$$

$$= rank[K_T^*(Q_{\leq m} \oplus I)(P_{F_s^2} \otimes I)K_{J_C,T}]$$

$$= rank[K_{J_C,T}^*(Q_{\leq m} \oplus I)(P_{F_s^2} \otimes I)]$$

$$= rank[K_{J_C,T}K_{J_C,T}^*(Q_{\leq m} \oplus I)]$$

Where $Q_{\leq m}$ is the projection of F_s^2 onto the subspace of all polynomials of degree $\leq m$ the last two equalities hold since the operator $K_{J_C,T}^*(Q_{\leq m} \oplus I)$ has finite rank and $K_{J_C,T}$ is one -to-one on the range of $K_{J_C,T}^*(Q_{\leq m} \oplus I)$. Now using relation (81) we obtain the last formula of the theorem. The proof is complete.

Let $[T_1, \ldots, T_n],\ T_i \in B(\mathcal{H})$ be a pure row contraction and let J be a wot-closed two-sided ideal of F_n^∞ such that $\Phi[T_1, \ldots, T_n] = 0$ for any

$$\Phi[S_1, \ldots, S_n] \in J$$

(82)

where $\Phi[T_1, \ldots, T_n]$ is defined using the F_n^∞ functional calculus for row contraction is unitary equivalent to the compression of $B_1 \otimes I_k, \ldots, B_k \otimes I_k$ to a co-invariant subspace under each operator $B_i \otimes I_k\ ,\ i = 1, \ldots, n$ therefore we have the $T_i = P \in (B_i \otimes I_k)\ ,\ i = 1, \ldots, n$ following result is a commutative lifting theorem for pure constrained row contraction.

Theorem (1-3-12) [61]:

Let $J \neq F_n^\infty$ be a WOT-closed two- sided ideal of the non commutative analytic Toeplitz algebra F_n^∞ , and $[B_1, \ldots, B_n]$ and $[W_1, \ldots, W_n]$ be the corresponding constrained shifts acting on N_J . For each $j = 1, 2,$ let A be a

Hilbert space and $\mathcal{E}_j \subseteq N_j \otimes K_j$ be a co-invariant subspace under each operator $B_i \otimes I_K$, $i = 1,\dots,n$ if $X : \mathcal{E}_1 \to \mathcal{E}_2$ is a bounded such operator such that

$$X_{\mathcal{E}_1}\left[P\left(B_i \otimes I_{k_1}\right)|_{\mathcal{E}_1}\right] = P_{\mathcal{E}_2}\left(B_i \otimes I_{k_2}\right)|_{\mathcal{E}_2} X \ , i = 1,\dots,n$$

then there exists

$$G(W_1,\dots,W_n) \in W(W_1,\dots,W_n)\overline{\otimes} B(k_1,k)$$

(83) such that

$$G(W_1,\dots,W_n)^* \mathcal{E}_2 = X^* \quad \text{and} \quad \|G(W_1,\dots,W_n)\| = \|X\|$$

In particular if $\mathcal{E}_j = G \otimes I_j$ where $\mathcal{E}$ is a co-invariant subspace under each operator $B_i, i = 1,\dots,n$ then the above implication becomes an equivalence.

Corollary (1-3-13) [61]:

Let $J \neq F_n^{\infty}$ be a wot-closed two- sided ideal of the non commutative analytic Toeplitz algebra F_n^{∞} and let $B_1,\dots,B_n$ and $W_1,\dots,W_n$ be the corresponding constrained shift acting on N_J . If $\mathcal{K}$ is a Hilbert space and $G \subseteq N_J$ is an invariant subspace under each operator B_i^* and W_n^* , $i = 1,\dots,n$

then $\{[P_G W(B_1,\dots,B_n)|G] \otimes I_k\}' = [P_G W(W_1,\dots,W_n)|G]\overline{\otimes} B(K)$ we remark that theorem (1-3-10) can be extended to the following more general setting. The proof follows exactly the same lines so we shall omit it. For each $j = 1,2.$ let J_j be a wot-closed two- sided ideal of F_n^{∞} and let $[B_1^{(J)},\dots,B_n^{(j)}]$ be the corresponding constrained shift acting on N_{J_j} . Let $\mathcal{E}_j \subseteq N_{J_j} \otimes k_j$ be an invariant subspace under each operator $B_i^{(j)^*} \otimes I_{k_j}$, $i = 1,\dots,n$ where K_j A Hilbert space. If $X : \mathcal{E}_1 \to \mathcal{E}_2$ is a bounded operator such that $XP_{\mathcal{E}_1}\left(B_i^{(1)} \otimes I_{k_1}\right)|_{\mathcal{E}_1}$ = $XP_{\mathcal{E}_2}\left(B_i^{(2)} \otimes I_{k_2}\right)|_{\mathcal{E}_2}$ X , $i = 1,\dots,n$ then there exists $G \in [P_{N_{J_2}} R_n^{\infty}|N_{J_1}]\overline{\otimes} B(k_1,k_2)$ such that $P_{\mathcal{E}_2} G|_{\mathcal{E}_1} = X$ and $\|G\| = \|X\|$.

Theorem (1-3-14) [61]:

Let J be a wot-closed two- sided ideal of F_n^∞ and let $B_1,\dots,B_n$ be the corresponding constrained shift acting on N_J . Let $\lambda_1,\dots,\lambda_k$ be k distinct point in the zero set $z_J=\{\lambda\in B_n : f(\lambda)=0 \ for\ any\ f\in J\}$ and let $A_1,\dots,A_k\in B(k)$ then there exists $\phi(B_1,\dots,B_k)\in W(B_1,\dots,B_m)\overline{\otimes}B(k)$ such that $\|\phi(B_1,\dots,B_n)\|\le 1$ and $\phi(\lambda_j)=A_j\ ,\ j=1,\dots,k$ if and only if the operator matrix

$$\left[\frac{I_k-A_iA_j^*}{I-\langle\lambda_i,\lambda_j\rangle}\right]_{k\times k}$$

(84)

is positive semi definite

Proof:

Let $\lambda_j=(\lambda_{j_1},\dots,\lambda_{j_n})\in \mathbb{C}^n,\ j=1,\dots,k$ and denote $\lambda_{j\alpha}=\lambda_{ji_1}\lambda_{ji_2}\dots\lambda_{jim}$ if $\alpha=g_{i1}g_{i2}\dots g_{im}\in F_n^+$ and λ_{jg_0} define $Z_{\lambda_j}=\sum_{\alpha\in F_n^+}\overline{\lambda}_{j\alpha}e_\alpha\ ,\ j=1,2,\dots,k$ notice that for any $f\in J,\lambda\in z_J$ and $\alpha,\beta\in F_n^+$ we have $\langle IS_\alpha f(S_1,\dots,S_n)S_\beta(1),z_\lambda\rangle=\lambda_\alpha f(\lambda)\lambda_\beta=0$ which implies $z_\lambda\in N_J$ for any $\lambda\in z_J$. Note also that since $B_i^*=S_i^*|N_J$ for $i=1,\dots,n$ we have $B_{iz\lambda_j}^*=\overline{\lambda}_{jiz\lambda}$ for $i=1,\dots,n$ and $j=1,\dots,k$. Define the subspace $M=Span\{z\lambda_j : j=1,\dots,k\}$, and the operators $X_i\in B(M\otimes k)$ by setting $X=P_MB_i|_M\otimes I_k$, $i=1,\dots,n$. Since $z_{\lambda_1},\dots,z_{\lambda_k}$ are linearly independent we can define an operator $T\in B(M\otimes k)$ by setting $T^*(Z_{\lambda_j}\otimes h)=Z_{\lambda_j}\otimes A_j^*h$ for any $h\in k$ and $(j=1,\dots,k)$ notice that $TX_i=X_iT$ for $i=1,\dots,n$ can apply theorem (1-3-12) and find $\phi(W_1,\dots,W_n)\in$ $W(W_1,\dots,W_n)\overline{\otimes}B(k)$ such that

$$\phi(W_1,\dots,W_n)^*\phi(W_1,\dots,W_n)^*M=T^*$$

(85)

and $\|\phi(W_1,\dots,W_n)\|=\|T\|$ one can prove that $\phi(\lambda_j)=A_j\ ,\ j=1,\dots,k$ if and only if (85) holds. Moreover $\|\phi(W_1,\dots,W_n)\|\le 1$ if and only if $TT^*\le I_M$ which is equivalent to the fact that the operator matrix (84) is positive semi definite. This completes the proof.