# Chapter One

# The Variational Iteration Method

The main objective of this chapter is a study Variational Iteration Method for partial differential equation (PDE). And we have given examples.

As we discussed the convergence of He's variational iteration method.

## Sec (1.1): Definition of Variational Iteration Method

It was expressed before that Adomian decomposition method, with its changed structure and the clamor term wonder and a portion of the conventional strategies will be utilized as a part of this content. The other understood routines, for example, the converse dispersing strategy, the pseudo strategies won't be utilized here on the grounds that it can be found in numerous different writings.

Furthermore to adomian decay method, those recently created a variational iteration method will make connected. The variational iteration method (VIM) created toward Ji-Huan, he will be completely utilized toward mathematicians to handle a totally assortment for exploratory Also building applications: straight and non -linear, and homogenous and inhomogeneous too. It might have been indicated that this strategy is successful Furthermore dependable for systematic what's more numerical purposes. Those strategies provide for quickly focalized progressive close estimation of the correct result on such an answer exists. The VIM doesn't require particular medicines for non-linear issues Likewise clinched alongside adomian method, bother techniques, and so forth throughout this way, observing and stock arrangement of all instrumentation may be anyhow. Done the thing that follows; we introduce the fundamental steps of the technique..

Consider the differential equation [1]:

$$Lu + Nu = g(t) \tag{1}$$

with the initial conditions depends on the order of operator $L$.

Where $L$ and $N$ are linear and nonlinear operators respectively, and

$g(t)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional for equation (1) in the form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)\left(Lu(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right)d\xi\,, \tag{2}$$

where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. It is obvious now that the main steps of the He variational iteration method require first the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally. Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words we can use:

$$\int \lambda(\xi)u_n'(\xi)d\xi = \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi,$$

$$\int \lambda(\xi)u_n''(\xi)d\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda'(\xi)u_n(\xi)d\xi, \tag{3}$$

and so on. The last two identities can be obtained by integration by parts.

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations function. Consequently, the solution

$$u = \lim_{n\to\infty} u_n \tag{4}$$

In other words, the correction functional (2) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers,and its correction functional respectively:

$$(i)\begin{cases} u' + f\left(u(\xi), u'(\xi)\right) = 0, \lambda = -1, \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]d\xi, \end{cases}$$

$$(ii)\begin{cases} u'' + f\left(u(\xi), u'(\xi), u''(\xi)\right) = 0, \lambda = (\xi - x), \\ u_{n+1} = u_n - \int_0^x (\xi - x)[u_n'' + f(u_n, u_n', u_n'')]d\xi, \end{cases}$$

$$(iii)\begin{cases} u'' + f\left(u(\xi),u'(\xi),u''(\xi),u'''(\xi)\right) = 0, \lambda = -\frac{1}{2}(\xi - x)^2, \\ u_{n+1} = u_n + \int_0^x \frac{1}{2}(\xi - x)^2 \left[u_n'' + f\left(u_n, u_n', u_n'', u_n'''\right)\right] d\xi, \end{cases}$$

$$(iv)\begin{cases} u^{(iv)} + f\left(u(\xi),u'(\xi).u'''(\xi),u^{(iv)}\right) = 0, \lambda = \frac{1}{3}(\xi - x)^3, \\ u_{n+1} = u_n + \int_0^x \frac{1}{3}(\xi - x)^3 \left[u^{(iv)}{}_n + f\left(u_n, u_n', u_n'', u_n''', u_n{}^{(iv)}\right)\right] d\xi, \end{cases}$$

$$(v)\begin{cases} u^{(n)} + f\left(u(\xi),u'(\xi).u'''(\xi),u^{(n)}\right) = 0, \lambda = \frac{(-1)^n}{(n-1)!}(\xi - x)^{(n-1)}, \\ u_{n+1} = u_n + (-1)^n \int_0^x \frac{1}{(n-1)!}(\xi - x)^{(n-1)} \left[u^{(n)}{}_n + f\left(u_n, \ldots, u_n{}^{(n)}\right)\right] d\xi, \end{cases}$$

for $n \geq 1$.

**Example(1.1.1)[1]:**
Consider the following linear partial differential equation

$$u_x + u_y = x + y \quad , \quad u(0, y) = 0 \tag{5}$$

**Solution:**

The correction functional for equation (5) is

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^x \lambda(\xi)\left(\frac{\partial u_n}{\partial \xi}(\xi, y) + \frac{\partial \tilde{u}_n}{\partial y}(\xi, y) - \xi - y\right) d\xi \tag{6}$$

Using (3), the stationary conditions

$$1 + \lambda|_{\xi = x} = 0 \tag{7}$$

$$\lambda'|_{\xi = x} = 0$$

Follow immediately. This in turn gives

$$\lambda = -1 \tag{8}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (6) gives the iteration formula

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^x \lambda(\xi)\left(\frac{\partial u_n}{\partial \xi}(\xi,y)+\frac{\partial u_n}{\partial y}(\xi,y)-\xi-y\right)d\xi, n\geq 0 \tag{9}$$

As state before, we can select $u_0(x,y)=u(0,y)=0$ from the given conditions.

Using this selection into (9) we obtain the following successive approximations

$$u_0(x,y)=0$$

$$u_1(x,y)=u_0(x,y)-\int_0^x\left(\frac{\partial u_0(\xi,y)}{\partial \xi}+\frac{\partial u_0(\xi,y)}{\partial y}-\xi-y\right)d\xi=\frac{1}{2}x^2+xy,$$

$$u_2(x,y)=u_1(x,y)-\int_0^x\left(\frac{\partial u_1(\xi,y)}{\partial \xi}+\frac{\partial u_1(\xi,y)}{\partial y}-\xi-y\right)d\xi=xy,$$

$$u_3(x,y)=u_2(x,y)-\int_0^x\left(\frac{\partial u_2(\xi,y)}{\partial \xi}+\frac{\partial u_2(\xi,y)}{\partial y}-\xi-y\right)d\xi=xy,$$

:

$$u_n(x,y)=xy.$$

The VIM admits the use of

$$u(x,y)=\lim_{n\to\infty}u_n(x,y),$$

That gives the exact solution by

$$u(x,y)=xy.$$

**Example (1.1.2)[1]:**

Consider the following homogeneous partial differential equation:

$$u_y+xu_x=3u, u(x,0)=x^2 \tag{10}$$

**Solution:**

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^y \lambda(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi}+x\,\frac{\partial \tilde{u}_n(x,\xi)}{\partial x}-3\tilde{u}_n(x,\xi\right]d\xi \tag{11}$$

As presented before, the stationary condition are:

$$1+\lambda\,|_{\xi=x}=0$$

$$\lambda'\,|_{\xi=x}=0$$

and this gives

$$\lambda=-1$$

Substituting this value of the Lagrange multiplier $\lambda=-1$ into the functional (11) gives the iteration formula

$$u_{n+1}(x,y)=u_n(x,y)-\int_0^y\left[\frac{\partial u_n(x,\xi)}{\partial\xi}+x\,\frac{\partial u_n(x,\xi)}{\partial x}-3u_n(x,\xi)\right]d\xi \qquad (12)$$

As state before, we can select $u_0(x,y)=u(x,0)=x^2$ from the given conditions. Using this selection into (12) we obtain the following successive approximations

$$u_0(x,y)=x^2$$

$$u_1(x,y)=x^2-\int_0^y\left[\frac{\partial u_0(x,\xi)}{\partial\xi}+x\,\frac{\partial u_0(x,\xi)}{\partial x}-3u_0(x,\xi\right]d\xi=x^2+x^2y,$$

$$u_2(x,y)=x^2+x^2y-\int_0^y\left[\frac{\partial u_1(x,\xi)}{\partial\xi}+x\,\frac{\partial u_1(x,\xi)}{\partial x}-3u_1(x,\xi\right]d\xi$$
$$=x^2+x^2y+\frac{1}{2!}x^2y^2,$$

$$u_3(x,y)=x^2+x^2y+\frac{1}{2!}x^2y^2-\int_0^y\left[\frac{\partial u_2(x,\xi)}{\partial\xi}+x\,\frac{\partial u_2(x,\xi)}{\partial x}-3u_2(x,\xi\right]d\xi$$
$$=x^2+x^2y+\frac{1}{2!}x^2y^2+\frac{1}{3!}x^3y^3,$$

$$\vdots$$

$$u_n(x,y)=x^2\left(1+y+y^2+y^3+y^4+\ldots\right)$$

The VIM admits the use of

$$u(x,y)=\lim_{n\to\infty}u_n(x,y)$$

that give exact solution by

$$u(x,y)=x^2e^y$$

## Example (1.1.3) [1]:

$$u_x + u_y + u_z = 3, u(0, y, z) = y + z \qquad (13)$$

**Solution:**

The correction functionalfor Eq. (13) is

$$u_{n+1}(x, y, z) = u_n(x, y, z) + \int_0^x \lambda(\xi)\left[\frac{\partial u_n(\xi, y, z)}{\partial \xi} + \frac{\partial \tilde{u}_n(\xi, y, z)}{\partial y} + \frac{\partial \tilde{u}_n(\xi, y, z)}{\partial z} - 3\right] d\xi, n \geq 0$$

This also gives

$$\lambda = -1$$

Consequently, we obtain the iteration formula

$$u_{n+1}(x, y, z) = u_n(x, y, z) - \int_0^x \left[\frac{\partial u_n(\xi, y, z)}{\partial \xi} + \frac{\partial u_n(\xi, y, z)}{\partial y} + \frac{\partial u_n(\xi, y, z)}{\partial z} - 3\right] d\xi, \qquad (14)$$

We select$u_0(x, y, z) = y + z$ from the given conditions. Using this selection into (12) we obtain the following successive approximations

$$u_0(x, y, z) = y + z$$

$$u_1(x, y, z) = y + z - \int_0^x \left[\frac{\partial u_0(\xi, y, z)}{\partial \xi} + \frac{\partial u_0(\xi, y, z)}{\partial y} + \frac{\partial u_0(\xi, y, z)}{\partial z} - 3\right] d\xi = x + y + z,$$

$$u_1(x, y, z) = x + y + z - \int_0^x \left[\frac{\partial u_1(\xi, y, z)}{\partial \xi} + \frac{\partial u_1(\xi, y, z)}{\partial y} + \frac{\partial u_1(\xi, y, z)}{\partial z} - 3\right] d\xi = x + y + z,$$

$$\vdots$$

$$u_n(x, y, z) = x + y + z$$

## Sec (1.2): Nonlinear PDEs by VIM

Similarly, as state before, those variational iteration method handles nonlinear issues in a parallel way to that utilized for straight issues. Those primary step is to figure out the Lagrange multiplier $\lambda(\xi)$ , afterward those progressive approximations could be acquired clinched alongside a recursive way. In the following, we will analyze the same samples examined in front of to delineate those forces of those VIM.

**Example (1.2.1)[1]:**

$$u_t + uu_x = 1 + x + t, u(x,0) = x, t > 0 \tag{15}$$

where $u = u(x,t)$

**Solution:**

Note that the equation in homogeneous. The correction, functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - 1 - x - \xi\right] d\xi \tag{16}$$

The stationary conditions give $\lambda = -1$ . Based on this, we obtain the iteration formula.

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n \frac{\partial u_n(x,\xi)}{\partial x} - 1 - x - \xi\right] d\xi$$

Selecting $u_0(x,t) = x$ from the given initial condition yields the successive approximations.

$$u_0(x,t) = x$$

$$u_1(x,t)=x-\int_0^t\left[\frac{\partial u_0(x,\xi)}{\partial\xi}+u_0\frac{\partial u_0(x,\xi)}{\partial x}-1-x-\xi\right]d\xi$$

$$=x+t-\frac{t^2}{2!}$$

$$u_2(x,t)=x+t-\frac{t^2}{2!}-\int_0^t\left[\frac{\partial u_1(x,\xi)}{\partial\xi}+u_1\frac{\partial u_1(x,\xi)}{\partial x}-1-x-\xi\right]d\xi$$

$$=x+t-\frac{t^3}{3!}$$

$$u_3(x,t)=x+t-\frac{t^3}{3!}-\int_0^t\left[\frac{\partial u_2(x,\xi)}{\partial\xi}+u_2\frac{\partial u_2(x,\xi)}{\partial x}-1-x-\xi\right]d\xi$$

$$=x+t-\frac{t^4}{4!}$$

$$u_4(x,t)=x+t-\frac{t^4}{4!}-\int_0^t\left[\frac{\partial u_3(x,\xi)}{\partial\xi}+u_3\frac{\partial u_3(x,\xi)}{\partial x}-1-x-\xi\right]d\xi$$

$$=x+t-\frac{t^5}{5!}$$

$$\vdots$$

$$u(x,t)=x+t-\frac{t^2}{2!}-\frac{t^3}{3!}-\frac{t^4}{4!}+\ldots$$

The VIM admits the use of

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$

Then

$$u(x,t)=x+t$$

**Example (1.2.2) [1]:**

$$u_t=2x^2-\frac{1}{8}u_x{}^2,u(x,0)=0 \qquad (17)$$

**Solution:**

Proceeding as befor we obtain the iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\xi)}{\partial\xi}-2x^2-\frac{1}{8}u^2{}_{nx}(x,\xi)\right)d\xi,n\geq 0 \qquad (18)$$

Selecting $u_0(x,t)=0$ from the given initial condition yields the successive approximations:

$$u_0(x,t)=0$$

$$u_1(x,t)=0-\int_0^t\left(\frac{\partial u_0(x,\xi)}{\partial \xi}-2x^2-\frac{1}{8}u^2{}_{0x}(x,\xi)\right)d\xi=2x^2t$$

$$u_2(x,t)=2x^2t-\int_0^t\left(\frac{\partial u_1(x,\xi)}{\partial \xi}-2x^2-\frac{1}{8}u^2{}_{1x}(x,\xi)\right)d\xi$$
$$=2x^2t+\frac{2}{3}x^2t^3$$

$$u_3(x,t)=2x^2t+\frac{2}{3}x^2t^3-\int_0^t\left(\frac{\partial u_2(x,\xi)}{\partial \xi}-2x^2-\frac{1}{8}u^2{}_{2x}(x,\xi)\right)d\xi$$
$$=2x^2t+\frac{2}{3}x^2t^3+\frac{4}{15}x^2t^5+\frac{2}{63}x^2t^7$$

:

$$u_n(x,t)=2x^2\left(t+\frac{t^3}{3}+\frac{2}{15}t^5+\frac{1}{63}t^7+\ldots\right)$$

As a result, the exact solutions are given by

$$u(x,t)=2x^2\tanh t$$

## Sec(1.3): Systems of Linear and Nonlinear PDFs by Variational Iteration Method:

We will apply the variation iteration method for solving systems of linear partial differential equation. We write a system in an operator form by:

$$\begin{aligned} L_1 u + R_1(u,v) &= g_1 \\ L_2 v + R_2(u,v) &= g_2 \end{aligned} \tag{19}$$

where $u = u(x,t)$, with initial data

$$\begin{aligned} u(x,0) &= f_1(x) \\ v(x,0) &= f_2(x) \end{aligned}$$

where $L_1$ is considered a first order partial differential operator, and $R_j, 1 \le j \le 3,$ are linear operators and $R_1$, and $R_2$ are source terms.

Following the discussion presented above for a variational iteration method, the following correction, functional for the system (19) can be set in the form:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1 \left[ Lu_n(\xi) + R_1(\tilde{u}_n, \tilde{v}_n) - g_1(\xi) \right] d\xi \\ v_{n+1}(x,t) &= v_{n+1}(x,t) + \int_0^t \lambda_2 \left[ Lu_n(\xi) + R_2(\tilde{u}_n, \tilde{v}_n) - g_2(\xi) \right] d\xi \end{aligned} \tag{20}$$

where $\lambda_j, j = 1,2$ is the general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ and $\tilde{v}_n$ as restricted variations which means $\delta\tilde{u}_n = 0$ , and $\delta\tilde{v}_n = 0$ . The Lagrange multipliers $\lambda_j, j = 1,2$ will be identified optimally via integration by parts as introduced before. The successive approximation $u_{n+1}(x,t)$ and $v_{n+1}(x,t), n \ge 0,$ of the solution $u(x,t)$ and $v(x,t)$ will follow immediately uponusing the obtained Lagrange multipliers and by using selective functions $u_0$ and $v_0$.

The initial value may be used for the selective zeroth approximations. With the Lagrange multipliers $\lambda_j$ determined, several approximations

$u_j(x,t), v_j(x,t), j \geq 0,$ can be computed. Consequently, the solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t) \end{aligned} \tag{21}$$

To give a clear overview of the analysis introduced above, the two examples that were studied before will be used to explain the technique that we summarized before, therefore we will keep the same numbers.

**Example(1.3.1) [1]:**

We first consider the linear system:

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + u_x &= 0 \end{aligned} \tag{22}$$

with the initial data

$$u(x,0) = e^{x}, v(x,0) = e^{-x} \tag{23}$$

where $u = u(x,t)$ and $v = v(x,t)$

**Solution:**

The correction, functional for (22) are

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{v}_n(x,\xi)}{\partial x}\right] d\xi \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi)\left[\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x}\right] d\xi \end{aligned} \tag{24}$$

This gives the stationary conditions

$$\begin{aligned} 1 + \lambda_1 |_{\xi = t} &= 0 \\ \lambda_1'(\xi = t) &= 0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} 1 + \lambda_2 |_{\xi = t} &= 0 \\ \lambda_2'(\xi = t) &= 0 \end{aligned}$$

As a result, we find

$$\lambda_1 = \lambda_2 = -1 \tag{26}$$

Substituting these values of the Lagrange multipliers into the functional (24) gives the iteration formulas

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left[ \frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial v_n(x,\xi)}{\partial x} \right] d\xi \\ v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \left[ \frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} \right] d\xi, n \geq 0 \end{aligned} \tag{27}$$

We can select $u_0(x,0) = e^x, v_0(x,0) = e^{-x}$ by using the given initial values. Accordingly, we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= e^x, \\ v_0(x,t) &= e^{-x} \\ u_1(x,t) &= e^x + te^{-x} \\ v_1(x,t) &= e^{-x} - te^x \\ u_2(x,t) &= e^x + te^{-x} + \frac{1}{2!}t^2 e^x \\ v_2(x,t) &= e^{-x} - te^x + \frac{1}{2!}t^2 e^{-x} \\ u_3(x,t) &= e^x + te^{-x} + \frac{1}{2!}t^2 e^x + \frac{1}{3!}t^3 e^{-x} \\ v_3(x,t) &= e^{-x} - te^x + \frac{1}{2!}t^2 e^{-x} - \frac{1}{3!}t^3 e^x, \end{aligned} \tag{28}$$

$$\vdots$$

$$\begin{aligned} u_n(x,t) &= e^x \left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \ldots\right) + e^{-x}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \ldots\right) \\ v_n(x,t) &= e^{-x} \left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \ldots\right) - e^{-x}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \ldots\right) \end{aligned}$$

Since the

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t) \end{aligned} \tag{29}$$

Consequently, the exact analytical solutions are of the form

$$(u,v)=\left(e^{x}\cosh t+e^{-x}\sinh t, e^{-x}\cosh t-e^{x}\sinh t\right) \qquad (30)$$

**Example(1.3.2) [1]:**

Consider the linear system of partial differential equations

$$\begin{aligned} u_t+u_x-2v&=0 \\ v_t+v_x+2u&=0 \end{aligned} \qquad (31)$$

with the initial data

$$u(x,0)=\sin x \qquad , \qquad v(x,0)=\cos x$$

**Solution:**

The correction functional for (31) read:

$$\begin{aligned} u_{n+1}(x,t)&=u_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi}+\frac{\partial \tilde{u}_n(x,\xi)}{\partial x}-2\tilde{v}(x,\xi)\right]d\xi \\ v_{n+1}(x,t)&=v_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial v_n(x,\xi)}{\partial \xi}+\frac{\partial \tilde{v}_n(x,\xi)}{\partial x}+2\tilde{u}(x,\xi)\right]d\xi, n\geq 0 \end{aligned} \qquad (32)$$

Proceeding as before, we find

$$\lambda_1=\lambda_2=-1$$

Substituting these values of the Lagrange multipliers into the functional (32) gives the iteration formals

$$\begin{aligned} u_{n+1}(x,t)&=u_n(x,t)-\int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi}+\frac{\partial u_n(x,\xi)}{\partial x}-2v(x,\xi)\right]d\xi \\ v_{n+1}(x,t)&=v_n(x,t)-\int_0^t \left[\frac{\partial v_n(x,\xi)}{\partial \xi}+\frac{\partial v_n(x,\xi)}{\partial x}+2u(x,\xi)\right]d\xi \end{aligned} \qquad (33)$$

We can select $u_0(x,0)=\sin x$, $v_0(x,0)=\cos x$, by using the gives initial values. Accordingly, we obtain the following successive approximations.

$$u_0(x,0)=\sin x$$

$$v_0(x,0)=\cos x$$

$$u_1(x,t)=\sin x+t\cos x$$
$$v_1(x,t)=\cos x-t\sin x$$
$$u_2(x,t)=\sin x+t\cos x-\frac{t^2}{2!}\sin x$$
$$v_2(x,t)=\cos x-t\sin x-\frac{t^2}{2!}\cos x$$
$$u_3(x,t)=\sin x+t\cos x-\frac{t^2}{2!}\sin x+\frac{t^3}{3!}\cos x$$
$$v_3(x,t)=\cos x-t\sin x-\frac{t^2}{2!}\cos x-\frac{t^3}{3!}\sin x$$

$$\vdots$$

$$u_n(x,t)=\sin x\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right)+\cos x\left(t+\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots\right)$$
$$v_n(x,t)=\cos x\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right)+\sin x\left(t+\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots\right)$$

Since the

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$
$$v(x,t)=\lim_{n\to\infty}v_n(x,t)$$

This gives the pair of solutions $(u,v)$ in a closed form by

$$(u,v)=\left(\sin(x+t),\cos(x+t)\right)$$

Frameworks from claiming nonlinear halfway differential equations emerge to asignificant number experimental models for example, such that the proliferation about shallow water waves and the Brusselator model from claiming concoction reaction-diffusion model. To utilize those VIM, we compose an arrangement done an driver structure toward.

$$\begin{aligned}L_1u+R_1(u,v,w)+N_1(u,v,w)&=g_1\\L_2u+R_2(u,v,w)+N_2(u,v,w)&=g_2\\L_3u+R_3(u,v,w)+N_3(u,v,w)&=g_3\end{aligned}\qquad(34)$$

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x), \\ w(x,0) &= f_3(x), \end{aligned} \tag{35}$$

where $L_t$ is considered a first order partial differential operator $R_j, 1 \le j \le 3$ and $N_j, 1 \le j \le 3$ are linear and nonlinear operators respectively, and $g_1, g_2$ and $g_3$ are source terms. The correction, functional for equations of the system (35) can be

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda(\xi)\left[Lu_n(x,\xi) + R_1(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) + N_1(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) - g_1(\xi)\right] d\xi, \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda(\xi)\left[Lv_n(x,\xi) + R_2(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) + N_2(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) - g_1(\xi)\right] d\xi, \\ w_{n+1}(x,t) &= w_n(x,t) + \int_0^t \lambda(\xi)\left[Lu_n(x,\xi) + R_3(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) + N_3(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) - g_1(\xi)\right] d\xi, \end{aligned} \tag{36}$$

Where $\lambda_j, 1 \le j \le 3$ are general Lagrange's multipliers, which can be identified optimally via the variational theory, and $\tilde{u}_n, \tilde{v}_n$, and $\tilde{w}_n$ as restricted variations which means $\delta\tilde{u}_n = 0, \delta\tilde{v}_n = 0$ and $\delta\tilde{w}_n = 0$. It is required first to determine the Lagrange multipliers $\lambda_j$ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), v_{n+1}(x,t), w_{n+1}(x,t), n \ge 0$, of the solutions $u(x,t), v(x,t)$ and $w(x,t)$ will follow immediately upon using the obtained Lagrange multipliers and by using selective functions $u_0, v_0$ and $w_0$ . The initial values are usually used for the selective zeroth approximations. With the Lagrange multipliers $\lambda_j$ determined, then several approximations $u_j(x,t), v_j(x,t), w_j(x,t), n \ge 0$ can be determined. Consequently, the solutions are given by:

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t), \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t), \\ w(x,t) &= \lim_{n\to\infty} w_n(x,t) \end{aligned} \tag{37}$$

To give a clear overview of the analysis introduced above, we will apply the VIM to the same two illustrative systems of partial differential equations that were studied in the previous section.

**Example(1.3.3) [1]:**

Consider the inhomogeneous nonlinear system

PDE

$$u_t + vu_x + u = 1$$
$$v_t - uv_x - v = 1 \tag{38}$$

IC $\quad u(x,0) = e^x, v(x,0) = e^{-x}$

**Solution:**

The correction functionals for (38) read

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{v}_n(x,\xi)\frac{\partial \tilde{u}_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) - 1\right)d\xi \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi)\left(\frac{\partial v_n(x,\xi)}{\partial \xi} - \tilde{u}_n(x,\xi)\frac{\partial \tilde{v}_n(x,\xi)}{\partial x} + \tilde{v}_n(x,\xi) - 1\right)d\xi \end{aligned} \tag{39}$$

The stationary conditions are given by:

$$1 + \lambda_1 = 0,\ \lambda_1'(\xi = t) = 0$$

$$1 + \lambda_2 = 0,\ \lambda_2'(\xi = t) = 0$$

So that

$$\lambda_1 = \lambda_2 = -1 \tag{40}$$

Substituting these values of the Lagrange multipliers into the functionals (39)gives the iteration formulas

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + v_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + u_n(x,\xi) - 1\right)d\xi \\ v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - u_n(x,\xi)\frac{\partial v_n(x,\xi)}{\partial x} + v_n(x,\xi) - 1\right)d\xi \end{aligned} \tag{41}$$

The zeroth approximations $u_0(x,t) = e^x$ and $v_0(x,t) = e^{-x}$ are selected by using the given initial conditions. Therefore, we obtain the following successive approximations.

$$u_0(x,t) = e^x, \qquad v_0(x,t) = e^{-x}$$
$$u_1(x,t) = e^x - te^x, \qquad v_1(x,t) = e^{-x} + te^{-x}$$
$$u_1(x,t) = e^x - te^x + \frac{t^2}{2!}e^x + noiseterms, \ ,$$
$$v_2(x,t) = e^{-x} + te^{-x} + \frac{t^2}{2!}e^{-x} + noiseterms,$$

By canceling the noise terms between $u_2, u_3, \ldots$ and between $v_2, v_3, \ldots$, we find

$$u_n(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right), v_n(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)$$

And as a result, the exact solution s are given by:

$$u(x,t) = e^{x-t}, v(x,t) = e^{-x+t}$$

Obtained upon using the Taylor expansion for $e^{-t}$ and $e^t$. It is obvious that we did not use any transformation formulas or linearization assumptions for handling then onlinear terms.

**Example(1.3.4) [1]:**

Consider the following nonlinear system

$$\begin{aligned} u_t - v_x w_y &= 1, \\ v_t - w_x u_y &= 5, \\ w_t - u_x v_y &= 5 \end{aligned} \qquad (42)$$

With the initial conditions:

$$u(x,y,0) = x + 2y,\ v(x,y,0) = x - 2y,\ w(x,y,0) = -x + 2y \qquad (43)$$

**Solution**

Proceeding as before we find

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

Substituting these values of the Lagrange multipliers gives the iteration formulas

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t \left[\frac{\partial u_n(x,y,\xi)}{\partial \xi} - \frac{\partial v_n(x,y,\xi)}{\partial x} \times \frac{\partial w_n(x,y,\xi)}{\partial y} - 1\right] d\xi,$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t \left[\frac{\partial v_n(x,y,\xi)}{\partial \xi} - \frac{\partial w_n(x,y,\xi)}{\partial x} \times \frac{\partial u_n(x,y,\xi)}{\partial y} - 5\right] d\xi,$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) - \int_0^t \left[\frac{\partial w_n(x,y,\xi)}{\partial \xi} - \frac{\partial u_n(x,y,\xi)}{\partial x} \times \frac{\partial v_n(x,y,\xi)}{\partial y} - 5\right] d\xi$$

The zeroth approximations

$$u_0(x,y,t) = x + 2y$$
$$v_0(x,y,t) = x - 2y$$
$$w_0(x,y,t) = -x + 2y$$

are selected by using the given initial conditions. Consequently, the following successive approximations

$$u_0(x,y,t) = x + 2y$$
$$v_0(x,y,t) = x - 2y$$
$$w_0(x,y,t) = -x + 2y$$
$$u_1(x,y,t) = x + 2y + 3t$$
$$v_1(x,y,t) = x - 2y + 3t$$
$$w_1(x,y,t) = -x + 2y + 3t$$
$$\vdots$$
$$u_n(x,y,t) = x + 2y + 3t$$
$$v_n(x,y,t) = x - 2y + 3t$$
$$w_n(x,y,t) = -x + 2y + 3t$$

are readily obtained. Notice that the successive approximations became the same for u after obtaining the first approximation. The same conclusion can be made for$v$ and $w$. Based on this, the exact solutions are given by:

$$u(x,y,t) = x + 2y + 3t$$
$$v(x,y,t) = x - 2y + 3t$$
$$w(x,y,t) = -x + 2y + 3t$$

**Example(1.3.5) [1]:**

Consider the following nonlinear system

$$u_t + u_x v_x = 2,\ v_t + u_x v_x = 0,\ u(x,0) = x,\ v(x,0) = x \tag{44}$$

**Solution**

The correction, functional:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda_1(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} \times \frac{\partial v_n(x,\xi)}{\partial x} - 2\right] d\xi \\ v_{n+1}(x,t) &= v_n(x,t) + \int_0^t \lambda_2(\xi)\left[\frac{\partial v_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} \times \frac{\partial v_n(x,\xi)}{\partial x}\right] d\xi \end{aligned} \tag{45}$$

The stationary conditions are given by:

$$\lambda_1 = \lambda_2 = -1$$

Substituting these values of the Lagrange multiplies into the functionals (45) gives the iteration formulas:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[ \frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} \times \frac{\partial v_n(x,\xi)}{\partial x} - 2 \right] d\xi$$
$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left[ \frac{\partial v_n(x,t)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} \times \frac{\partial v_x(x,\xi)}{\partial x} \right] d\xi$$

The zeroth approximations $u_0(x,t) = x,$ and $v_0(x,t) = x$ are selected by using the given initial conditions. Therefore, we obtain the following successive approximations.

$$u_0(x,t) = x,\ v_0(x,t) = x$$

$$u_1(x,t) = x + t,\ v_1(x,t) = x - t$$

$$u_2(x,t) = x + t,\ v_2(x,t) = x - t$$

$$\vdots$$

$$u(x,t) = x + t,\ v(x,t) = x - t$$

**Example(1.3.6) [1]:**

$$u_t + u_x v_x - w_y = 1,\ v_t + v_x w_x + u_y = 1,\ w_t + w_x u_x - v_y = 1 \qquad (46)$$

$$u(x,y,0) = x + y,\ v(x,y,0) = x - y,\ w(x,y,0) = -x + y$$

**Solution:**

Proceeding as before we find

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

Substituting these values of the Lagrange multipliers gives the iteration formulas:

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t\left[\frac{\partial u_n(x,y,\xi)}{\partial \xi}+\frac{\partial u_n(x,y,\xi)}{\partial x}\times\frac{\partial v_n(x,y,\xi)}{\partial x}-\frac{\partial w_n(x,y,\xi)}{\partial y}-1\right]d\xi,$$
$$v_{n+1}(x,y,t)=v_n(x,y,t)-\int_0^t\left[\frac{\partial v_n(x,y,\xi)}{\partial \xi}+\frac{\partial v_n(x,y,\xi)}{\partial x}\times\frac{\partial w_n(x,y,\xi)}{\partial x}+\frac{\partial u_n(x,y,\xi)}{\partial y}-1\right]d\xi,$$
$$w_{n+1}(x,y,t)=w_n(x,y,t)-\int_0^t\left[\frac{\partial w_n(x,y,\xi)}{\partial \xi}+\frac{\partial w_n(x,y,\xi)}{\partial x}\times\frac{\partial u_n(x,y,\xi)}{\partial x}-\frac{\partial v_n(x,y,\xi)}{\partial y}-1\right]d\xi$$

The zeroth approximations

$$u_0(x,y,0)=x+y$$
$$v_0(x,y,0)=x-y$$
$$w_0(x,y,0)=-x+y$$

are selected by using the given initial conditions. Consequently, the following successive approximations:

$$u_0(x,y,0)=x+y,\ v_0(x,y,0)=x-y,\ w_0(x,y,0)=-x+y$$

$$u_1(x,y,0)=x+y+t,\ v_1(x,y,0)=x-y+t,\ w_1(x,y,0)=-x+y+3t$$

$$u_2(x,y,0)=x+y+t,\ v_2(x,y,0)=x-y+t,\ w_2(x,y,0)=-x+y+3t$$

:

$$u_n(x,y,0)=x+y+t,\ v_n(x,y,0)=x-y+t,\ w_n(x,y,0)=-x+y+3t$$

## Sec (1.4): On the Convergence of He's Variational Iteration Method

In this section we will consider He's a variational iteration method for solving second-order initial value problems.

The variational iteration method changes the differential equation to a recurring sequence of functions. The limit of that sequence is considered as the solution of the partial differential equation. Now consider the second - order partial differential equation in the following form (most general form can be considered without loss of generality):

$$F\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x\,\partial t}\right)=0, \tag{47}$$

with specified initial conditions.

The variational iteration method changes the partial differential equation to a correction, functional in $t$ -direction in the following form:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda F\left(u_n,\frac{\partial u_n}{\partial \xi},\frac{\partial \tilde{u}_n}{\partial x},\frac{\partial^2 \tilde{u}_n}{\partial x^2},\frac{\partial^2 \tilde{u}_n}{\partial \xi^2},\frac{\partial^2 \tilde{u}_n}{\partial x\,\partial \xi}\right)d\,\xi, \tag{48}$$

where $\tilde{u}_n$ is considered as He's monographs i.e. $\delta(\tilde{u}_n)=0.$ To find the optimal value of $\lambda$ , we make correction functional stationary in the following form:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda F\left(u_n,\frac{\partial u_n}{\partial \xi},\frac{\partial \tilde{u}_n}{\partial x},\frac{\partial^2 \tilde{u}_n}{\partial x^2},\frac{\partial^2 \tilde{u}_n}{\partial \xi^2},\frac{\partial^2 \tilde{u}_n}{\partial x\,\partial \xi}\right)d\,\xi=0, \tag{49}$$

Which results in the stationary conditions and consequently the optimal value is obtained. In fact the solution of the differential equation is considered as the fixed point of the following functional under the suitable choice of the initial term $u_0(x,t)$:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda F\left(u_n,\frac{\partial u_n}{\partial \xi},\frac{\partial u_n}{\partial x},\frac{\partial^2 u_n}{\partial x^2},\frac{\partial^2 u_n}{\partial \xi^2},\frac{\partial^2 u_n}{\partial x\,\partial \xi}\right)d\,\xi, \tag{50}$$

**Definition (1.4.1) [6]:** A variable quantity $v$ is a functional dependent on a function $u(x)$ if to each function $u(x)$ of a certain class of functions $u(x)$

there corresponds a value $v$ . The variation of a functional $v\left[u(x)\right]$is defined in the following form:

$$\delta v\left[u(x)\right]=\frac{\partial}{\partial\alpha}v\left[u(x)+\alpha\delta u\right]\Big|_{\alpha=0}. \qquad (51)$$

As a well-known result, we have [7]:

**Theorem (1.4.2) [6]**: If a functional $v\left[u(x)\right]$which has a variation achieves a maximum or a minimum at $u=u_0(x)$, where$u(x)$ is an interior point of the domain of definition of the functional, then at $u=u_0(x)$,

$$\delta v=0. \qquad (52)$$

Also as a very powerful tool we have [8]:

**Theorem (1.4.3) [6]:**(Banach's fixed point theorem). Assume that X is a Banach space and

$$A:X\to X$$

is a nonlinear mapping ,and suppose that

$$\left\|A\left[u\right]-A\left[\bar{u}\right]\right\|\le\gamma\left\|u-\bar{u}\right\|,\, u,\bar{u}\in X \qquad (53)$$

For some constant$\gamma<1$.Then A has a unique fixed point. Furthermore, the sequence

$$u_{n+1}=A\left[u_n\right] \qquad (54)$$

with an arbitrary choice of $u_0\in X$ ,converges to the fixed point of A and

$$\left\|u_k-u_l\right\|\le\left\|u_1-u_0\right\|\sum_{j=l-1}^{k-2}\gamma^j \qquad (55)$$

According to Theorem (1.4.3), for the nonlinear mapping

$$A\left[u\right]=u(x,t)+\int_0^t\lambda F\left(u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial\xi^2},\frac{\partial^2 u}{\partial x\,\partial\xi}\right)\mathrm{d}\xi, \qquad (56)$$

a sufficient condition for convergence of the variational iteration method is strictly contraction of A. Furthermore, the sequence (50) converges to the fixed point of A which also is the solution of the partial differential (47).

Consider the sequence (50) in the following form:

$$u_{n+1} - u_n = \int_0^t \lambda F\left(u_n, \frac{\partial u_n}{\partial \xi}, \frac{\partial u_n}{\partial x}, \frac{\partial^2 u_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial \xi^2}, \frac{\partial^2 u_n}{\partial x \, \partial \xi}\right) d\xi, \tag{57}$$

It is clear that the optimal value  must be chosen such that extremities the residual functional

$$\int_0^t \lambda F\left(u_n, \frac{\partial u_n}{\partial \xi}, \frac{\partial u_n}{\partial x}, \frac{\partial^2 u_n}{\partial x^2}, \frac{\partial^2 u_n}{\partial \xi^2}, \frac{\partial^2 u_n}{\partial x \, \partial \xi}\right) d\xi, \tag{58}$$

which is equivalent to the extermination of A. But in Theorem (1.4.2) the necessary condition for minimization is given.

As a direct result which shows the relation between He's variational technique and A domian Decomposition method we have:

**Remark (1.4.4) [6]:** For the time-dependent partial differential in the form of

$$u_t + F(u, u_x, u_{xx}) = 0, \tag{59}$$

with the properly given initial condition, then He's a variational method and decomposition procedure of A domian are equivalent.

**Example (1.4.5) [6]:**

As the first example considers the telegraph equation

$$u_{tt} + 2du_t - u_{xx} = 0 \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{60}$$

$$u = g \quad u_t = h \quad (x,t) \in \mathbb{R} \times \{t = 0\}, \tag{61}$$

For $d > 0$, the term $2du_t$ representing a physical damping of wave propagation. This equation is investigated in [8].

Using the well-known He's a variational iteration method we have

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(\frac{\partial^2 u_n}{\partial x^2}(x,\xi)+2\,\mathrm{d}\frac{\partial u_n}{\partial \xi}(x,\xi)-\frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,\xi)\right)d\xi. \qquad (62)$$

To find the optimal value of $\lambda$ we have

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda\left(\frac{\partial^2 u_n}{\partial x^2}(x,\xi)+2\,\mathrm{d}\frac{\partial u_n}{\partial \xi}(x,\xi)-\frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,\xi)\right)d\xi. \qquad (63)$$

or

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(\frac{\partial^2 u_n}{\partial x^2}(x,\xi)+2\,\mathrm{d}\frac{\partial u_n}{\partial \xi}(x,\xi)\right)d\xi. \qquad (64)$$

which results

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)-\delta\lambda' u_n(x,t)\Big|_{\xi=t}+\delta\lambda\frac{\partial u_n}{\partial \xi}(x,t)\Big|_{\xi=t}$$

$$+\int_0^t \delta\lambda'' u_n(x,\xi)d\xi+2d\,\delta\lambda u_n(x,\xi)\Big|_{\xi=t}-2d\int_0^t \delta\lambda' u_n(x,\xi)d\xi=0. \qquad (65)$$

Therefore, the stationary conditions are obtained in the following form:

$$1+2d\,\lambda(\xi)-\lambda'(\xi)=0\Big|_{\xi=t},$$

$$\lambda''(\xi)-2d\,\lambda'(\xi)=0\Big|_{\xi=t},$$

which yields

$$\lambda(\xi)=d\,(t-\xi)\left(t-\xi-\frac{1}{d}\right), \qquad (66)$$

and the desired sequence is found as

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t d\,(t-\xi)\left(t-\xi-\frac{1}{d}\right)\left(\frac{\partial^2 u_n}{\partial \xi^2}(x,\xi)+2\,\mathrm{d}\frac{\partial u_n}{\partial \xi}(x,\xi)-\frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,\xi)\right)d\xi.$$

**Example (1.4.6) [6]:**

As another example, consider the beam equation in the following form

$$\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}=0\ \ (x,t)\in \mathbb{R}\times(0,\infty), \qquad (67)$$

$$u=g\ \ (x,t)\in \mathbb{R}\times\{t=0\}.$$

Using the variational iteration method we have

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left( \frac{\partial u_n}{\partial \xi}(x,\xi) + \frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi) \right) d\xi. \tag{68}$$

Imposing the stationary condition we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda \left( \frac{\partial u_n}{\partial \xi}(x,\xi) + \frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi) \right) d\xi. \tag{69}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda \frac{\partial u_n}{\partial \xi}(x,\xi) d\xi.$$

Thus, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \lambda u_n(x,\xi)\Big|_{\xi=t} - \int_0^t \delta \lambda' u_n(x,\xi) d\xi = 0. \tag{70}$$

Hence we have the following stationary conditions:

$$\lambda'(\xi) = 0\Big|_{\xi=t},$$
$$1 + \lambda(\xi) = 0\Big|_{\xi=t},$$

which yields

$$\lambda = -1$$

Therefore, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left( \frac{\partial u_n}{\partial \xi}(x,\xi) + \frac{\partial^4 \tilde{u}_n}{\partial x^4}(x,\xi) \right) d\xi.$$