Sudan University of Science and Technology

College of Graduate Studies

Discontinuous Dynamical Systems and Stabilization

الانظمه الحركيه المتقطعه والاستقراريه

A Thesis Submitted in partial Fulfillment for the Requirements of the M.Sc Degree in Mathematics

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Dedication

To my mother,lovely father;

brothers,specially brother Adam yahia,

sister;

and aunts.

Acknowledgements

Faithfully I would like to thank almighty Allah for giving the strength and stamina to accomplish this work.

I would like to express my gratitude and appreciation to my supervisor Dr.Emad AL-Deen who has given me much of this time to supervision,guidance and correction of work.

To all who have lent me a hand to make the accomplishment of the workpossible,spicily very important person in my life(Mamy:Mona Adam) and friends(Nadia hamad Al-seed and AL-Fatih Babekir)

I sincerely appreciate the endless help of (my family)

Abstract

In this research we consider discontinuous dynamical systems. We discuss the uniqueness of solutions and the stability analysis. We also report a number of sufficient conditions for uniqueness. We also present specific results to piecewise continuous vector fields and differential inclusions, with some examples and applications.

الخلاصه

في هذا البحث اعتبرنا الانظمه الديناميكيه غير المستمره.ناقشنا وحدانيه الحلول وتحليل الاستقرار.ايضا قررنا عدد الشروط الكافيه للوحدانيه.ايضا عرضنا النتائج المحدده لحقول المتجه المستمره القطعيه والتضمينات التفاضليه مع بعض الامثله والتطبيقات.

The contents

chapter (1)

Discontinuous Dynamic Systems,Caratheodary and Filippov Solutions

Section (1.1):Discontinuous Dynamical Systems and Caratheodary:

To discuss the existence of solutions:

Let $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a (non-autonomous) vector field, and consider the differential equation on $\mathbb{R}^d$

$$\dot{x}\,(\mathrm{t}) = X(t, x(t)). \qquad (1.1)$$

A point $x_* \in \mathbb{R}^d$ is an equilibrium of the differential equation if $0 = X(t, x_*)$ for all $t \in \mathbb{R}$. A solution of (1.1) on $[t_0\ , t_1]$ is a continuously differentiable map $\gamma : [t_0\ , t_1] \to \mathbb{R}^d$ such that $\gamma(\mathrm{t}) = X(t, \gamma(t))$. Usually, we refer to γ as a solution with initial condition $\gamma(t_0) = x_0$. If the vector field is autonomous, that is, does not depend explicitly on time, then without loss of generality,we take $t_0 = 0$. A solution is maximal if it cannot be extended, that is, if it is not the result of the truncation of another solution with a larger interval of definition. Note that the interval of definition of a maximal solution might be right half-open.

Essentially, continuity of the vector field suffices to guarantee the existence of solutions, as the following result states.

Proposition(1.1.1):

Let $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Assume that (i) for each $t \in \mathbb{R}$, the map $x \mapsto X(t, x)$ is continuous, (ii) for each $x \in \mathbb{R}^d$, the map $t \mapsto X(t, x)$ is measurable, and (iii) X is locally bounded, that is, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $\varepsilon \in (0, \infty)$ and an integrable function $m : [t, t + \delta] \to (0, \infty)$ such that $\|X(s, y)\|_2 \leq m(s)$ for all $s \in [t, t + \delta]$ and all $y \in B(x, \varepsilon)$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a solution of (1.1) with initial condition $x(x_0) = t_0$.

For autonomous vector fields, Proposition (1.1.1) takes a simpler form $X : \mathbb{R}^d \to \mathbb{R}^d$ must simply be continuous in order to have at least a solution starting from any given initial condition. As the following example shows, if the vector field is discountinuous, then solutions of (1.1) might not exist.

Now we discuss discontinuous vector field with non-existence of solutions:

Consider the autonomous vector field $X : \mathbb{R} \to \mathbb{R}$,

$$X(x) = \begin{cases} -1, & x > 0, \\ 1, & x \le 0. \end{cases} \tag{1.2}$$

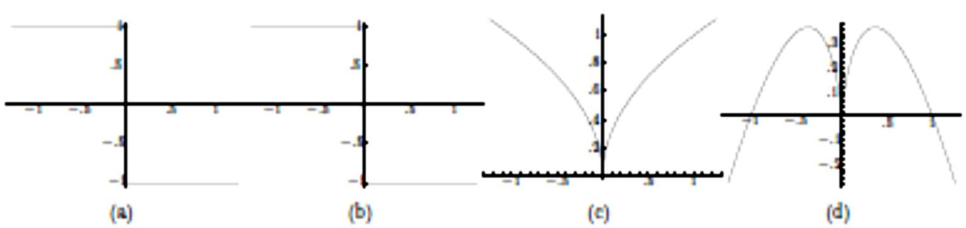
This vector field is discontinuous at 0 (see Figure (1.1)(a)). The associated dynamical system $\dot{x}(t) = X(x(t))$ does not have a solution starting from 0. That is, there does not exist a continuously differentiable map $\gamma : [0, t_0] \to \mathbb{R}$ such that $\dot{\gamma}(t) = X(\gamma(t))$ and $\gamma(0) = 0$. Otherwise, if such a solution exists, then $\dot{\gamma}(0) = 1$, and $\dot{\gamma}(t) = -1$ for any positive t sufficiently small, which contradicts the fact that $\dot{\gamma}$ is continuous.

However, the following example shows that the lack of continuity of the vector field does not necessarily preclude the existence of solutions.

Now we discuss discontinuous vector field with existence of solutions:

Consider the autonomous vector field $X : \mathbb{R} \to \mathbb{R}$,

$$X(x) = -\operatorname{sign}(x) = \begin{cases} -1, & x > 0, \\ 0, & x = 0, \\ 1, & x < 0. \end{cases} \tag{1.3}$$



Figure(1.1). Discontinuous –(a) and (b)– and not-locally Lipschitz – (c) and (d)– vector fields. The vector fields in (a) and (b) do not verify the hypotheses of Proposition(1.1.1) and therefore the existence of solutions is not guaranteed. The vector field in (a) has no solution starting from 0. However, the vector field in (b) has a solution starting from any initial condition. The vector fields in (c) and (d) do not verify the hypotheses of Proposition(1.1.3) and therefore the uniqueness of solutions is not guaranteed. The vector field in (c) has two solutions starting from 0.

However, the vector field in (d) has a unique solution starting from any initial condition.

This vector field is discontinuous at 0 (see Figure(1.1)(b)). However, the associated dynamical system $\dot{x}(t) = X(x(t))$ has a solution starting from each initial condition. Specifically, the maximal solutions are

For $x(0) > 0$, $\gamma : [0, x(0)) \to \mathbb{R}$, $\gamma(t) = x(0) - t$,

For $x(0) = 0$, $\gamma : [0, \infty) \to \mathbb{R}$, $\gamma(t) = 0$,

For $x(0) < 0$, $\gamma : [0, -x(0)) \to \mathbb{R}$, $\gamma(t) = x(0) + t$.

The difference between the vector fields (1.2) and (1.3) is minimal (they are equal up to the value at 0), and yet the question of the existence of solutions has a different answer for each of them. We see later how considering a different notion of solution can reconcile the answers given to the existence question for these vector fields.

Now we discuss uniqueness of solutions:

Next, let us turn our attention to the issue of uniqueness of solutions. Here and in what follows, (right) uniqueness means that, if there exist two solutions with the same initial condition, then they coincide on the intersection of their intervals of existence. Formally, if $\gamma_1: [t_0, t_1] \to \mathbb{R}^d$ and $\gamma_2:[t_0, t_2] \to \mathbb{R}^d$ are solutions of (1.1) with $\gamma_1(t_0) = \gamma_2(t_0)$, then uniqueness means that $\gamma_1(t) = \gamma_2(t)$ for all $t \in [t_0, t_1] \cap [t_0, t_2] = [\mathrm{t}_0 \min [t_1, t_2]]$. The following result provides a sufficient condition for uniqueness.

Definition(1.1.2):(Locally Lipschitz Functions)
A function $f : \mathbb{R}^d \to \mathbb{R}^m$ is locally Lipschitz at $x \in \mathbb{R}^d$ if if there exist $L_x, \epsilon \in (0,\infty)$ such that

$$\|f(\mathcal{Y}) - f(\mathcal{Y}')\|_2 \leq L_x \|\mathcal{Y} - \mathcal{Y}'\|_2$$

for all $\mathcal{Y}, \mathcal{Y}' \in B(x, \epsilon)$. A locally Lipschitz function at x is continuous at x, but the converse is not true ($f: \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{|x|}$), is continuous at 0, but not locally Lipschitz at 0). A function is locally Lipschitz on S □ $\mathbb{R}^d$ if it is locally Lipschitz at x, for all x □ S. We abbreviate "f is locally Lipschitz on $\mathbb{R}^d$" by simply saying "f is locally Lipschitz." Note that continuously differentiable functions at x are locally Lipschitz at x, but the converse is not true ($f: \mathbb{R} \to \mathbb{R}, f(x) = |x|$,is locally Lipschitz at 0, but not differentiable at 0). Here, functions like

$f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^m$,that depend explicitly on time, are locally Lipschitz at $x \in \mathbb{R}^d$ if there exists $\epsilon \in (0,\infty)$ and $L_x:\mathbb{R} \rightarrow (0,\infty)$ such that

$$\|f(t,\mathcal{Y}) - f(t,\mathcal{Y}')\|_2 \leq L_x(t)\|\mathcal{Y} - \mathcal{Y}'\|_2,$$

for all $t \in \mathbb{R}$ and $\mathcal{Y}, \mathcal{Y}' \in B(x,\epsilon)$.

Proposition (1.1.3):

Under the hypotheses of Proposition(1.1.1), further assume that for all $x \in \mathbb{R}^d$, there exist $\varepsilon \in (0, \infty)$ and an integrable function $L_X : \mathbb{R} \rightarrow (0, \infty)$ such that

$$(X(t,y) - X(t,y'))^T(y - y') \leq L_X(t)\|y - y'\|_2^2, \quad (1.4)$$

for all $y, y' \in B(x, \varepsilon)$ and all $t \in \mathbb{R}$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique solution of (1.1) with initial condition $x(t_0) = x_0$.

Equation(1.4) is usually referred to as a one-sided Lipschitz condition. In particular,it is not difficult to see that locally Lipschitz vector fields (see the Definition(1.1.2))verify this condition. The opposite is not true (as an example, consider the vector field $X : \mathbb{R} \rightarrow \mathbb{R}$ defined by $X(x) = x \log(|x|)$ for $x \neq 0$ and $X(0) = 0$, which verifies the one-sided Lipschitz condition (1.4) around 0, but is not locally Lipschitz at 0). Locally Lipschitzness is the most common requirement invoked to guarantee uniqueness of solution.As Proposition(1.1.3) shows, uniqueness is indeed guaranteed under slightly more general conditions.

The following example shows that, if the hypotheses of Proposition(1.1.3) are not verified, then solutions might not be unique.

Now we can discuss the Continuous, not locally Lipschitz vector field with non-uniqueness of solutions:

Consider the autonomous vector field $X: \mathbb{R} \rightarrow \mathbb{R}$,

$$X(x) = \sqrt{|x|} \quad (1.5)$$

This vector field is continuous everywhere, and locally Lipschitz on $\mathbb{R} \setminus \{0\}$ (see Figure(1.1)(c)). Even more, X does not verify equation (1.4) in any neighborhood of 0. The associated dynamical system $\dot{x}(t) = X(x(t))$ has two maximal solutions starting from 0, namely.

$$\gamma_1 : [0,\infty) \to \mathbb{R}, \quad \gamma_1(t) = 0,$$

$$\gamma_2 : [0,\infty) \to \mathbb{R}, \ \gamma_2(t) = {t^2}/{4}.$$

However, there are cases where the hypotheses of Proposition(1.1.3) are not verified, and the differential equation still enjoys uniqueness of solution, as the following example shows.

Now we can discuss the continuous, not locally Lipschitz vector field with uniqueness of solutions:

Consider the autonomous vector field $X: \mathbb{R} \to \mathbb{R}$,

$$X(x) = \begin{cases} -x \log x & x > 0, \\ 0, & x = 0, \\ x \log(-x) & x < 0. \end{cases} \qquad (1.6)$$

This vector field is continuous everywhere, and locally Lipschitz on $\mathbb{R} \setminus \{0\}$ (see Figure (1.1)(d)). Even more, X does not verify equation (1.4) in any neighborhood of 0. However, the associated dynamical system $\dot{x}(t) = X(x(t))$ has a unique solution starting from each initial condition. Specifically, the maximal solution is

For $x(0) > 0$, $\quad \gamma : [0,\infty) \to \mathbb{R}, \quad \gamma(t) = \exp(\log x(0) \exp(-t))$,

For $x(0) = 0$, $\quad \gamma: [0,\infty) \to \mathbb{R}, \quad \gamma(t) = 0$,

For $x(0) < 0$, $\quad \gamma : [0,\infty) \to \mathbb{R}, \quad \gamma(t) = -\exp(\log(-x(0)) \exp(t))$.

Note that the statement of Proposition (1.1.3) prevents us from applying it to discontinuous vector fields, since solutions are not even guaranteed to exist. However, the discontinuous vector field (1.3) verifies the one-sided Lipschitz condition around any point, and indeed, the associated dynamical system enjoys uniqueness of solutions. A natural question is then to ask under what conditions discontinuous vector fields have a unique solution starting from each initial condition. Of course, the answer to this question relies on the notion of solution itself. We explore in detail these questions in the section entitled “Notions of Solution for Discontinuous Dynamical Systems”.

Now we present three more examples of discontinuous dynamical systems. These examples, together with the ones discussed in above, motivate the extension of the classical notion of (continuously differentiable) solution for ordinary differential equations, which is the subject of the next section.

We can now define Brick on a frictional ramp:

Consider a brick sliding on a ramp, an example taken from [34]. As the brick moves down, it experiments a friction force in the opposite direction as a result of the contact with the ramp (see Figure (1.2)(a)). Coulomb's friction law is the most accepted model of friction available.

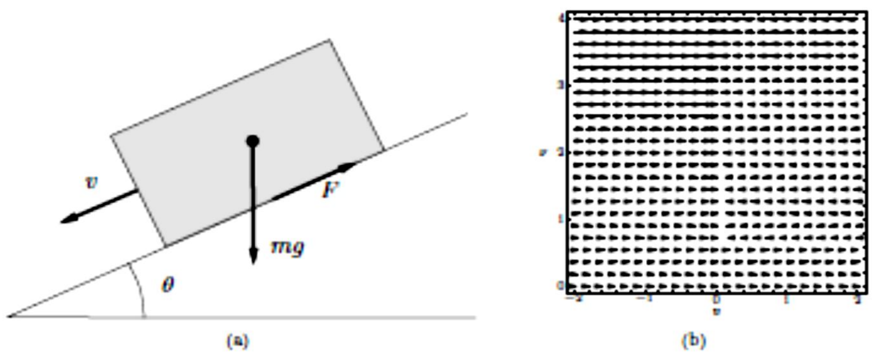


Figure (1.2) Brick sliding on a frictional ramp. The plot in (a) shows the physical quantities used to describe the example. The plot in (b) shows the one-dimensional phase portraits of (1.7) corresponding to values of the friction coefficient between 0 and 4, with a constant ramp incline of π/6.

In its simplest form, it says that the friction force is bounded in magnitude by the normal contact force tismes the coefficient of friction.

The application of Coulomb's law to the brick example gives rise to the equation of motion.

$$m\frac{dv}{dt} = mg\sin\theta - \nu mg\cos\theta \operatorname{sign}(v) \qquad (1.7)$$

where m and v are the mass and velocity of the brick, respectively, g is the constant of gravity, θ is the incline of the ramp, and ν is the coefficient of friction. The right-hand side of this equation is clearly a discontinuous function of v. Figure (1.2)(b) shows the phase plot of this system for different values of the friction coefficient.

Depending on the magnitude of the friction force, one may observe in real experiments that the brick stops and stays stopped. In other words, the brick attains v=0 in finite time, and stays with $v = 0$ for certain period of time. The classical solutions of this differential equation do not exhibit this type of behavior. To see this, note the similarity of (1.7) and (1.3). In order to explain this type of physical evolutions, we need then to understand the discontinuity of the equation, and expand our notion of solution beyond the classical one.

Now we can define the Nonsmooth harmonic oscillator:

Arguably, the harmonic oscillator is one of the most encountered examples in text-books of periodic behavior in physical systems. Here, we introduce a nonsmooth version of it, following [1]. Consider a mechanical system with two degrees of freedom, evolving according to

$$\dot{x}_1(t) = \operatorname{sign}(x_2(t))$$

$$\dot{x}_2(t) = -\operatorname{sign}(x_1(t))$$

The phase portrait of this system is plotted in Figure (1.3)(a).

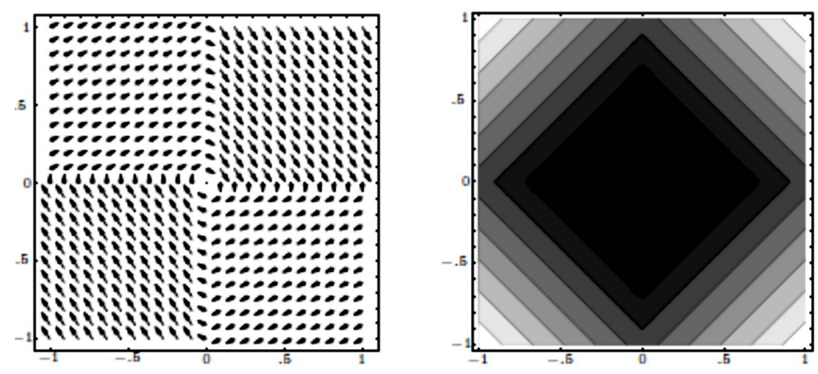


Figure (1.3). Nonsmooth harmonic oscillator. The plot in (a) shows the phase portrait on $[-1, 1]^2$ of the vector field $(x_1, x_2) \longrightarrow (\operatorname{sign}(x_2), -\operatorname{sign}(x_2)$,and the plot in (b) shows the contour plot on $[-1, 1]^2$ of the function$(x_1, x_2) \longmapsto |x_1| + |x_2|$

By looking at the equations of motion, (0, 0) is the unique equilibrium point of the system. Regarding other initial conditions, it seems clear how the system evolves while not in any of the coordinate axes. However, things are not so clear on the axes. If we perform a discretization of the equations of motion, and make the time stepsize smaller and smaller, we find that the trajectories look closer and closer to the set of diamonds plotted in Figure(1.3).These diamonds correspond to the level sets of the function$(x_1, x_2) \longmapsto |x_1| + |x_2|$. This observation is analogous to the fact that the level sets of the function $(x_1, x_2) \longmapsto x_1^2 + x_2^2$ correspond to the trajectories of the classical harmonic oscillator. However, the diamond trajectories are clearly not continuously differentiable, so to consider them as valid solutions we need a different notion of solution than the classical one.

We can define the“Move-away-from-closest-neighbor” interaction law:

Consider n nodes $p_1, . . . , p_n$ evolving in a convex polygon Q according to the interaction rule “move-away-from-closest-neighbor.” Formally, let $S = \{(p_1, . . . , p_n) \in Q^n \mid p_n = p_j$ for some $i \neq j\}$, and consider the nearest-neighbor map $N : Q^n \setminus S \rightarrow Q^n$ defined by

$$N_i(p_1, . . . , p_n) \in \text{argmin}\{\|p_i - q\|_2 \mid q \in \partial Q \cup \{p_1, . . . , p_n\} \setminus \{p_i\}\},$$

where ∂Q denotes the boundary of Q. Note that $N_i(p_1, . . . , p_n)$ is one of the closest nodes to p_i, and that the same point can be the closest neighbor to more than one node. Now, consider the

$$\dot{p}_i = \frac{p_i - N_i(p_1, . . . , p_n)}{\|p_i - N_i(p_1, . . . , p_n)\|_2} \quad i \in \{1, \dots, n\} \tag{1.8}$$

Clearly, changes in the nearest-neighbor map induce discontinuities in the dynamical system. Figure(1.4) shows two instances where these discontinuities occur.

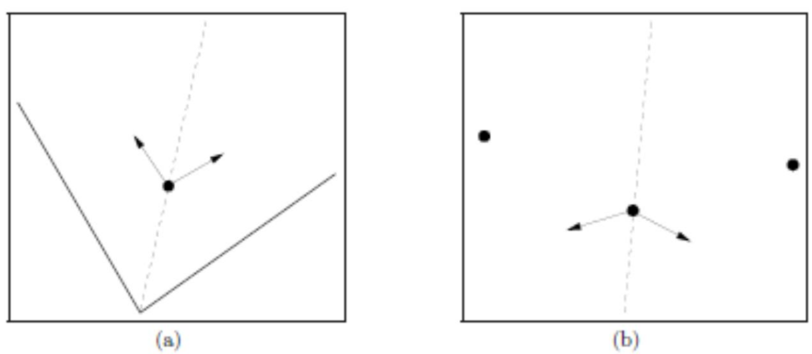


Figure (1.4). “Move-away-from-closest-neighbor” interaction law. The plots in (a) and (b) show two examples of how infinitesimal changes in a node location give rise to different closest neighbors (either polygonal boundaries or other nodes) and hence completely different directions of motion.

To analyze this dynamical system, we need to understand how the discontinuities affect its evolution. It seems reasonable to postulate that the set $Q^n \setminus S$ remains invariant under this flow, that is, that nodes never run into each other, but we need to extend our notion of solution –and redefine our notion of invariance accordingly– in order to ensure it.

Now we investigate the notions of solution for discontinuous dynamical systems:

In the previous illusteration, we have seen that the usual notion of solution for ordinary differential equations is too restrictive when considering discontinuous vector fields. Here, we explore other notions of solution to reconcile the mismatch. In general, one may think that a good way of taking care of the discontinuities of the differential equation (1.1) is by allowing solutions to violate it (that is, do not follow the direction specified by X) at a few time instants. The precise mathematical notion corresponding to this idea is that of Caratheodory solution, which we introduce next.

Definition(1.1.4):(Absolutely continuous functions)

A function γ: $[a, b] \to$ R is absolutely continuous if for all $\epsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ such that any finite collection $(a_1, b_1), \ldots (a_n, b_n)$,) of disjoint open intervals contained in $[a, b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta$ verifies

$$\sum_{i=1}^{n}|\gamma(b_i) - \gamma(a_i)| < \varepsilon.$$

Locally Lipschitz functions are absolutely continuous. The function : $[0, 1] \to \mathbb{R}$, $\gamma(x) = \sqrt{x}$, is absolutely continuous but not locally Lipschitz at 0.Absolutely continuous functions are (uniformly) continuous.The function γ:$[-1, 1] \to \mathbb{R}$ defined by $\gamma(t)$=$t \sin(\frac{1}{t})$ for $t \neq 0$ and $\gamma(0) = 0$ is continuous, but not absolutely continuous. Finally, absolutely continuous functions are differentiable almost everywhere.

We can now define the Caratheodory solutions:

A Caratheodory solution of (1.1) defined on $[t_0, t_1] \subset \mathbb{R}$ is an absolutely continuous map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}(t) = X(t, \gamma(t))$ for almost every $t \in [t_0, t_1]$. The sidebar “Absolutely continuous functions” reviews the notion of absolutely continuous function, and examines some of their properties. Arguably, this notion of solution is the most natural candidate for a discontinuous system (indeed, Caratheodory solutions are also called classical solutions).

Consider, for instance, the vector field $X : \mathbb{R} \to \mathbb{R}$ defined by

$$X(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ -1, & x < 0. \end{cases}$$

This vector field is discontinuous at 0.The associated dynamical system $\dot{x}(t) = X(x(t))$ does not have a (continuously differentiable) solution starting from 0. However, it has two Caratheodory solutions starting from 0, namely,

$\gamma_1 : [0, \infty) \to \mathbb{R}$, $\gamma_1(t) = t$, and $\gamma_2 : [0, \infty) \to \mathbb{R}$, $\gamma_2(t) = -t$. Note that both γ_1 and γ_2 violate the differential equation only at $t = 0$, that is, $\dot{\gamma}_i(0) \neq X(\gamma_i(0))$, for $i = 1,2$.

However, the good news are over soon. The physical motions observed in the brick sliding on a frictional ramp example, where the brick slides for a while and then stays stopped, are not Caratheodory solutions. The discontinuous vector field (1.2) does not admit any Caratheodory solution starting from 0. The "move-away-from-closest-neighbor" interaction law is yet another example where Caratheodory solutions do not exist either, as we show next.

We can discuss the"Move-away-from-closest-neighbor" interaction law for one agent moving in a square:

For the "move-away-from-closest-neighbor" interaction law, consider one agent moving in the square environment $[-1,1]^2 \subset R^2$. Since no other agent is present in the square, the agent just moves away from the closest polygonal boundary, according to the vector field

$$X(x_1, x_2) = \begin{cases} (-1,0), & -x_1 < x_2 \leq x_1, \\ (0,1), & x_2 < x_1 \leq -x_2, \\ (1,0), & x_1 \leq x_2 < x_1 \\ (0,-1), & -x_2 \leq x_1 < x_2 \end{cases} \qquad (1.9)$$

Since on the diagonals of the square,
$\{(a, a) \in [-1, 1]^2 \mid a \in [-1, 1]\} \cup \{(a, -a) \in [-1, 1]^2 \mid a \in [-1, 1]\}$, the "move-away-from-closest-neighbor" interaction law takes multiple values, we have chosen one of them in the definition of X. Figure(1.5) shows the phase portrait. The vector field

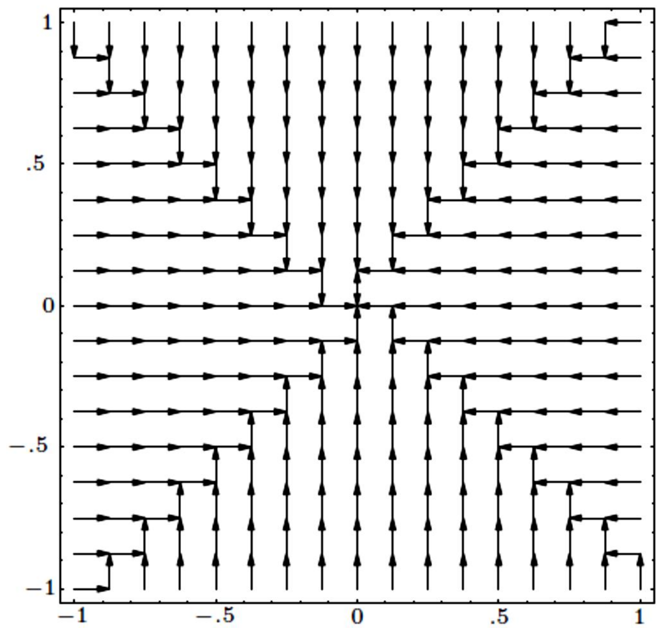


Figure (1.5). Phase portrait of the "move-away-from-closest-neighbor" interaction law for one agent moving in the square $[-1,1]^2 \subset \mathbb{R}^2$. Note that there is no Caratheodory solution starting from an initial condition in the diagonals of the square.

X is discontinuous on the diagonals. It is precisely when the initial condition belongs to these diagonals that the dynamical system $\dot{x}(t)= X(x(t))$ does not admit any Caratheodory solution.

We can now discuss Sufficient conditions for the existence of Caratheodory solutions:

Specific conditions under which Caratheodory solutions exist , and are known as Caratheodory conditions. Actually, they turn out to be a slight generalization of the conditions stated in Proposition (1.1.1), as the following result shows.

Proposition (1.1.5):

Let $X:\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$.Assume that (i) for almost all $t \in \mathbb{R}$, the map $x \longmapsto X(t,x)$ is continuous, (ii) for each $x \in \mathbb{R}^d$, the map $t \longmapsto X(t,x)$ is measurable, and (iii) X is locally essentially bounded, that is, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $\varepsilon \in (0, \infty)$ and an integrable function $m : [t, t + \delta] \to (0, \infty)$ such that $\|X(s,y)\|_2 \leq m(s)$ for all $s \in [t, t + \delta]$ and almost all $y \in B(x, \varepsilon)$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a Caratheodory solution of (1.1) with initial condition $x(t_0) = x_0$.

Note that in the autonomous case, the assumptions of this result amount to ask the vector field to be continuous. This requirement is no improvement with respect to Proposition(1.1.1), since we already know that in the continuous case, continuously differentiable solutions exist. Because of this reason, various authors have explored conditions for the existence of Caratheodory solutions specifically tailored to autonomous vector fields. For reasons of space, we do not go into details here.see[23] to find that directional continuous vector fields admit Caratheodory solutions, and [50] to learn about patchy vector fields, a special family of autonomous, discontinuous vector fields that also admit Caratheodory solutions.

Caratheodory solutions can also be defined for differential inclusions, instead of differential equations. The sidebars "Set-valued Maps" and "Differential Inclusions and Caratheodory Solutions" explain how in detail.

Given the limitations of the notion of Caratheodory solution, an important research thrust in the theory of differential equations has been the identification of more flexible notions of solution for discontinuous vector fields. Let us discuss various alternatives, and illustrate their advantages and disadvantages.

Section(1.2):Relationship between Caratheodary and Filippov solutions

Now we discuss Filippov solutions:

As we have seen when considering the existence of Caratheodory solutions starting from a desired initial condition, focusing on the specific value of the vector field at the initial condition might be too shortsighted. Due to the discontinuities of the vector field, things can be very different arbitrarily close to the initial condition, and this mismatch might indeed make impossible to construct a solution. The vector field in(1.2) and the "move-away-from-closest-neighbor" interaction law are instances of this situation.

What if, instead of focusing on the value of the vector field at each point, we somehow consider how the vector field looks like around each point? The idea of looking at a neighborhood of each point is at the core of the notion of Filippov solution [18]. A closely related notion is that of Krasovskii solution (to ease the exposition, we do not deal with the latter here.

The mathematical treatment to formalize this "neighborhood" idea uses set-valued maps.Let us discuss it informally for an autonomous vector field $X: \mathbb{R}^d \rightarrow \mathbb{R}^d$. Filippov's idea is to associate a set-valued map to X by looking at the neighboring values of X around each point. Specifically, for $x \in \mathbb{R}^d$, one evaluates the vector field X on the points belonging to $B(x, \delta)$, the open ball centered at x of radius $\delta > 0$.

We examine the result when δ gets closer to 0 by performing this operation for increasingly smaller δ. For further flexibility, we may exclude any set of measure zero in the ball $B(x, \delta)$ when evaluating X, so that the outcome is the same for two vector fields that only differ by a set of measure zero.

Mathematically,the previous paragraph can be summarized as follows.For $X: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, define the Filippov set-valued map $F[X]: \mathbb{R} \times \mathbb{R}^d \to B(\mathbb{R}^d)$ by

$$F[X](t, x) = \bigcap_{\delta>0} \bigcap_{\mu(s)=0} \operatorname{co}\{X(t, B) \setminus S)\}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d$$

In this formula, co denotes convex closure, and μ denotes the Lebesgue measure. Because of the way the Filippov set-valued map is defined, its value at a point is actually independent of the value of the vector field at that specific point. Note that this definition also works for maps of the form $X: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$, where d and m are not necessarily equal.

Let us compute this set-valued map for the vector fields (1.2) and (1.3). First of all, note that since both vector fields only differ at 0 (that is, at a set of measure zero), their associated Filippov set-valued maps are identical. Specifically, $F[X] : \mathbb{R} \to B(\mathbb{R})$ with

$$F[X](x) = \begin{cases} -1, & x > 0, \\ [-1,1], & x = 0, \\ 1, & x < 0. \end{cases}$$

Now we are ready to handle the discontinuities in the vector field X. We do so substituting the differential equation $\dot{x}(t) = X(t, x(t))$ by the differential inclusion

$$\dot{x}(t) \in F[X]\big(t, x(t)\big), \tag{1.10}$$

Definition(1.2.1):

Differential inclusions are a generalization of differential equations:at each state, they specify a range of possible evolutions, rather than a single one. These objects are defined by means of set-valued maps, The differential inclusion associated to $\mathcal{F} : \mathbb{R} \times \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ is an equation of the form

$$\dot{x}(t) \in \mathcal{F}\big(t, x(t)\big) \tag{1.11}$$

A point $x_* \in \mathbb{R}^d$ is an equilibrium of the differential inclusion if $0 \in \mathcal{F}(t, x_*)$ for all $t \in \mathbb{R}$.We define the notion of solution of a differential inclusion a *la* Caratheodory. The flexibility providedby the differential inclusion makes things work under fairly general conditions.

A Caratheodory solution of (1.11)defined on $[t_0, t_1] \subset \mathbb{R}$ is an absolutely continuous map γ:$[t_0,t_1] \rightarrow \mathbb{R}^d$such that $\dot{\gamma} \in \mathcal{F}(t, x(t))$for almost every $t \in [t_0, t_1]$.The existence of at least a solution starting from each initial condition is guaranteed by the following result (see, for instance, [6, 80]).

Proposition(1.2.2):

Let $\mathcal{F}$:$\mathbb{R} \times \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ be locally bounded and take nonempty compact and convex values. Assume that, for each $t \in \mathbb{R}$, the set-valued map $x \rightarrow \mathcal{F}(t, x)$ is upper semicontinuous, and, for each $x \in \mathbb{R}^d$, the set-valued map $t \rightarrow \mathcal{F}(t, x)$ is measurable.

Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a solution of (1.11) with initial condition $x(t_0) = x_0$.

This result is sufficient for our purposes. The reader is invited to find in the literature other existence results that work under different assumptions, see for instance [6, 53]. Uniqueness of solutions of differential inclusions is guaranteed by the following result.

Proposition(1.2.3):

Under the hypothesis of Proposition(1.2.2) ,further assume that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $L_x(t)$,$\varepsilon \in (0,\infty)$ such that for almost every $\mathcal{Y}, \mathcal{Y}' \in B(x, \varepsilon)$, one has

$$(v - w)^T(\mathcal{Y} - \mathcal{Y}') \leq L_x(t)\|\mathcal{Y} - \mathcal{Y}'\|_2^2 \qquad (1.12)$$

for all $v \in \mathcal{F}(t, \mathcal{Y})$ and $w \in \mathcal{F}(t, \mathcal{Y}')$ Assume that the function $t \rightarrow L_x(t)$ is integrable. Then,for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique solution of (1.11) with initial condition $x(t_0) = x_0$.Let us present an example of the application of Propositions(1.2.2) and(1.2.3). Following [74], considerthe set-valued map $\mathcal{F}$:$\mathbb{R} \times \rightarrow \mathfrak{B}(\mathbb{R}^d)$ defined by

$$\mathcal{F}(x) = \begin{cases} 0, & x \neq 0, \\ [-1,1], & x = 0. \end{cases}$$

Note that $\mathcal{F}$ is upper semicontinuous, but not lower semicontinuous (and hence, it is not continuous).This set-valued map verifies all the hypotheses in Proposition (1.2.2), and therefore solutionsexist starting from any initial condition. In addition, $\mathcal{F}$ satisfies equation(1.11)as long as $\mathcal{Y}$ and $\mathcal{Y}'$ are different from 0. Therefore, Proposition(1.1.3) guarantees uniqueness. Actually, the solution $\dot{x}(t) \in \mathcal{F}(x(t))$ starting from any initial condition is just the equilibrium solution,and considering the solutions of the latter, as defined in the(Definition(1.2.1).A Filippov solution of (1.1) defined on $[t_0, t_1] \subset \mathbb{R}$ is a solution of the differential inclusion (1.10), that is, an absolutely continuous map $\gamma : [t_0, t_1] \rightarrow \mathbb{R}^d$ such that $\dot{\gamma}(t) \in F\,[X](t, \gamma(t))$ for almost every $t \in [t_0, t_1]$, see the Definition(1.1.4).Because of the way the Filippov set-valued map is defined, note that any vector field that differs from X in a set of measure zero has the same set-valued map, and hence the same set of solutions. The next result establishes mild conditions under which Filippov solutions exist.

Proposition(1.2.4):

For X: $\mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ measurable and locally essentially bounded, there exists at least a Filippov solution of (1.1) starting from any initial condition.

The hypotheses of this proposition on the vector field guarantee that the associated Filippov set-valued map verifies all hypothesis of Proposition(1.2.2), and hence the existence of solutions follows. As an application of this result, since the autonomous vector fields in (1.2) and (1.3) are bounded, Filippov solutions exist starting from any initial condition. Both vector fields have the same (maximal) solutions.Specifically, For $x(0) > 0$, $\gamma : [0, \infty) \rightarrow \mathbb{R}$, $\gamma(t) = |x(0) - t|_+$

For $x(0) = 0$, $\gamma : [0, \infty) \rightarrow \mathbb{R}$, $\gamma(t) = 0$,

For $x(0) < 0$, $\gamma : [0, \infty) \rightarrow \mathbb{R}$, $\gamma(t) = |x(0) + t|_-$

Following a similar line of reasoning, one can show that the physical motions observed in the brick sliding on a frictional ramp example, where the brick slides for a while and then stays stopped, are indeed Filippov solutions. Similar computations can be made for the "move-away- from-closest-neighbor" interaction law to show that Filippov solutions exist starting from any initial condition, as we show next.

Now we can discuss the "Move-away-from-closest-neighbor" interaction law for one agent in a square –revisited:

Consider again the discontinuous vector field for one agent moving in a square under the "move-away-from-closest-neighbor" interaction law. The corresponding set-valued map $F[X] : |-1,1|^2 \rightarrow B(\mathbb{R}^2)$ is given by

$$F[X](x_1, x_2)=\begin{cases}\{(\mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{R}^2 \mid |\mathcal{Y}_1 + \mathcal{Y}_2| \leq 1, |\mathcal{Y}_1 - \mathcal{Y}_2| \leq 1\} & (x_1, x_2) = (0,0), \\ \{(-1,0)\}, & -x_1 < x_2 < x_1, \\ \{(\mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{R}^2 \mid \mathcal{Y}_1 + \mathcal{Y}_2 = -1, \mathcal{Y}_1 \in [-1,0]\},\} & 0 < x_2 = x_1, \\ \{(0,1)\} & x_2 < x_1 < -x_2, \\ \{(\mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{R}^2 \mid \mathcal{Y}_1 - \mathcal{Y}_2 = -1, \mathcal{Y}_1 \in [-1,0]\}, & 0 < -x_1 = x_2, \\ \{(1,0)\}, & x_1 < x_2 < -x_1, \\ \{(\mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{R}^2 \mid \mathcal{Y}_1 + \mathcal{Y}_2 = 1, \mathcal{Y}_1 \in [0,1]\}, & x_2 = x_1 < 0, \\ \{(0,-1)\}, & -x_2 < x_1 < x_2, \\ \{(\mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{R}^2 \mid \mathcal{Y}_1 - \mathcal{Y}_2 = 1, \mathcal{Y}_1 \in [0,1]\}, & 0 < x_1 = -x_2\end{cases}$$

According to Proposition (1.2.4), since X is bounded, Filippov solutions exist. In particular, the solutions starting from any point in a diagonal are nice straight lines flowing along the diagonal itself and reaching (0,0). For example, the maximal solution $\gamma : [0, \infty) \rightarrow \mathbb{R}^2$ starting from $(a, a) \in \mathbb{R}^2$ is

$$t \mapsto \gamma(t) = \begin{cases}(a - \text{sign}(a)t, a - sign(a)t), & t \leq |a|, \\ (0,0), & t \geq |a|.\end{cases}$$

Note that the behavior of this solution is quite different from what one might expect by looking at the vector field at the points of continuity. Indeed, the solution slides along the diagonals, following a convex combination of the limiting values of X around them, rather the direction specified by X itself. We study in more detail this type of behavior in the section entitled "Piecewise continuous vector fields and sliding motions."

We can now define the Relationship between Caratheodory and Filippov solutions:

One may pose the question: how are Caratheodory and Filippov solutions related? The answer is that not much. An example of a vector field for which both notions of solution exist but Filippov solutions are not Caratheodory solutions is given in [50]. The converse is not true either. For instance, the vector field

$$X(x) = \begin{cases}1, & x \neq 0, \\ 0, & x = 0.\end{cases}$$

has $t \mapsto 0$ as a Caratheodory solution starting from 0. However, the associated Filippov set-valued map is $F[X] : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{R})$, $F[X](x) = \{1\}$, and hence the unique Filippov solution starting from 0 is $t \rightarrow t$. On a related note, Caratheodory solutions are always Krasovskii solutions (but the converse is not true, see [52]).

We can now discuss the Computing the Filippov set-valued map:

In general, computing the Filippov set-valued map can be a daunting task.The work [28] develops a calculus that simplifies its calculation. We summarize here some useful facts.

Definition(1.2.5):(Consistency)

For $X: \mathbb{R}^d \rightarrow \mathbb{R}^d$ continuous at $x \in \mathbb{R}^d$,

$$F\,[X](x) = \{X\,(x)\}. \quad (1.13)$$

Definition(1.2.6)(Sum rule).

For $X_1,X_2:\mathbb{R}^d \longrightarrow \mathbb{R}^m$ locally bounded at $x \in \mathbb{R}^d$,

$$F[X_1 + X_2](x) \subset F[X_1](x) + F[X_2](x) \quad (1.14)$$

Definition(1.2.7)(Product rule).

For $X_1:\mathbb{R}^d \longrightarrow \mathbb{R}^m$ and $X_2:\mathbb{R}^d \longrightarrow \mathbb{R}^n$ locally bounded at $x \in \mathbb{R}^d$,

$$F[(X_1 + X_2)](x) \subset F[X_1](x) \times F[X_2](x) \quad (1.15)$$

Moreover, if one of the vector fields is continuous at x, then equality holds;

Definition(1.2.8)(Chain rule).

For $: \mathbb{R}^d \rightarrow \mathbb{R}^n$ continuously differentiable at $x \in \mathbb{R}^d$ with rank n, and,

$X : \mathbb{R}^n \rightarrow \mathbb{R}^m$ locally bounded at $Y(x) \in \mathbb{R}^n$

$$F\,[Xo\;Y\,](x) = F\,[X]\big(Y(x)\big). \quad (1.16)$$

Definition(1-2-9):(Matrix transformation rule).

For $: \mathbb{R}^d \longrightarrow \mathbb{R}^m$ locally bounded at $x \in \mathbb{R}^d$ and $Z : \mathbb{R}^d \rightarrow \mathbb{R}^{d\times m}$ continuous at $x \in \mathbb{R}^d$,

$$F\,[Z\,X](x) = Z(x)F\,[X](x) \quad (1.17)$$

Similar expressions can be developed for non-autonomous vector fields.

We conclude this section with an alternative description of the Filippov set-valued map. For $X: \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ measurable and locally essentially bounded, one can show that, for each $t \in \mathbb{R}$, there exists $S_t \subset \mathbb{R}^d$ of measure zero such that

$$F[X](t,x) = \text{co}\{ \lim_{i\to\infty} X(t,x_i) \,|x_i \longrightarrow x \,, x_i \notin S \cup S_t\},$$

where S is any set of measure zero. As we see later when discussing nonsmooth functions, this description has a remarkable parallelism with the notion of generalized gradient of a locally Lipschitz function.

We can discuss the Piecewise continuous vector fields and sliding motions:

When dealing with discontinuous dynamics, one often encounters vector fields that are continuous everywhere except at a surface of the state space. Indeed, the examples of discontinuous vector fields that we have introduced so far all fall into this situation. This problem can be naturally interpreted by considering two continuous dynamical systems, each one defined on one side of the surface, glued together to give rise to a discontinuous dynamical system on the overall state space. Here, we analyze the properties of the Filippov solutions in this sort of (quite common) situations.

Let us consider a piecewise continuous vector field X: $\mathbb{R}^d \to \mathbb{R}^d$. Here, by piecewise continuous we mean that there exists a finite collection of disjoint domains $\mathcal{D}_1, \dots, \mathcal{D}_m \subset \mathbb{R}^d$ (that is, open and connected sets) that partition $\mathbb{R}^d$ (that is, $\mathbb{R}^d = \cup_{k=1}^{m} \overline{\mathcal{D}}_k$) such that the vector field X is continuous on each $\overline{\mathcal{D}}_k$, for $k \in \{1, \dots, m\}$. More general definitions are also possible (by considering, for instance, non-autonomous vector fields), but we restrict our attention to this one for simplicity. Clearly, a point of discontinuity of X must belong to one of the boundaries of the sets $\mathcal{D}_1, \dots, \mathcal{D}_m$. Let us denote by $S_x \subset \partial\mathcal{D}_1 \cup . . . \cup \partial\mathcal{D}_m$ the set of points where X is discontinuous. Note that $S_X x$ has measure zero.

The Filippov set-valued map associated with X takes a particularly simple expression for piecewise continuous vector fields, namely,

$$F[X](t,x) = \text{co}\{ \lim_{i\to\infty} X(x_i) \,|x_i \longrightarrow x \,, x_i \notin S \cup S_X \}.$$

This set-valued map can be easily computed as follows. At points of continuity of X, that is, for $x \notin S_X$, we deduce $F\,[X](x) = \{X(x)\}$, using the consistency property (1.13). At points of discontinuity of X, that is, for $x \in S_X$, one can prove that $F\,[X](x)$ is a convex polyhedron in $\mathbb{R}^d$ with vertices of the form

$$X_{|\overline{\mathcal{D}_k}}(x) = \lim_{i\to\infty} X(x), with\ |x_i \longrightarrow x \,, x_i \in \mathcal{D}_k, x_i \notin S_X,$$

for some $k \in \{1, \dots, m\}$.

As an illustration, let us consider the systems in the section "Examples of discontinuous dynamical systems."

The vector field in the brick sliding on a frictional ramp example is piecewise continuous, with $\mathcal{D}_1 = \{v \in \mathbb{R} \mid v > 0\}$ and $\mathcal{D}_2 = \{v \in \mathbb{R} \mid v < 0\}$. Its associated Filippov set-valued map $F[X] : \mathbb{R} \rightarrow \mathbb{R}$,

$$F[X](v)\begin{cases}\{g(\sin\theta - v\cos\theta)\}, & v > 0,\\ \{g(\sin\theta - dv\cos\theta) | d \in [-1,1]\}, & v = 0,\\ \{g(\sin\theta + v\cos\theta)\}, & v < 0.\end{cases}$$

is singleton-valued outside $S_X = \{0\}$, and a closed segment at 0.

The discontinuous "move-away-from-closest-neighbor" vector field for one agent moving in the square $X : [-1,1]^2 \rightarrow \mathbb{R}^2$ is piecewise continuous, with $\mathcal{D}_1 = \{(x_1, x_2) \in [-1,1]^2 - x_1 < x_2 < x_1\}$,

$\mathcal{D}_2 = \{(x_1, x_2) \in [-1,1]^2\ x_2 < x_1 < -x_2\}$,

$\mathcal{D}_3 = \{(x_1, x_2) \in [-1,1]^2\ x_1 < x_2 < -x_1\}$,

$\mathcal{D}_4 = \{(x_1, x_2) \in [-1,1]^2\ \ -x_2 < x_1 < x_2\}$,

Its Filippov set-valued map, described in the section "Move-away-from-closest-neighbor interaction law for one agent in a square –revisited," maps points outside $S_X = \{(a, a) \in [-1, 1]^2 \mid a \in [-1, 1]\} \cup \{(a, -a) \in [-1, 1]^2 \mid a \in [-1, 1]\}$ to singletons, points in $S_X \setminus \{(0, 0)\}$ to closed segments, and (0, 0) to a square polygon.

The nonsmooth harmonic oscillator also falls into this category. The vector field $X : \mathrm{R}^2 \rightarrow \mathrm{R}^2$, $X(x_1, x_2) = (\mathrm{sign}(x_2), -\mathrm{sign}(x_1))$, is continuous on each one of the quadrants$\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\}$, with

$$\mathcal{D}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\},\ \mathcal{D}_2 = \{(x_1, x_2) \in \mathbb{R}^d \mid x_1 > 0, x_2 < 0\}$$

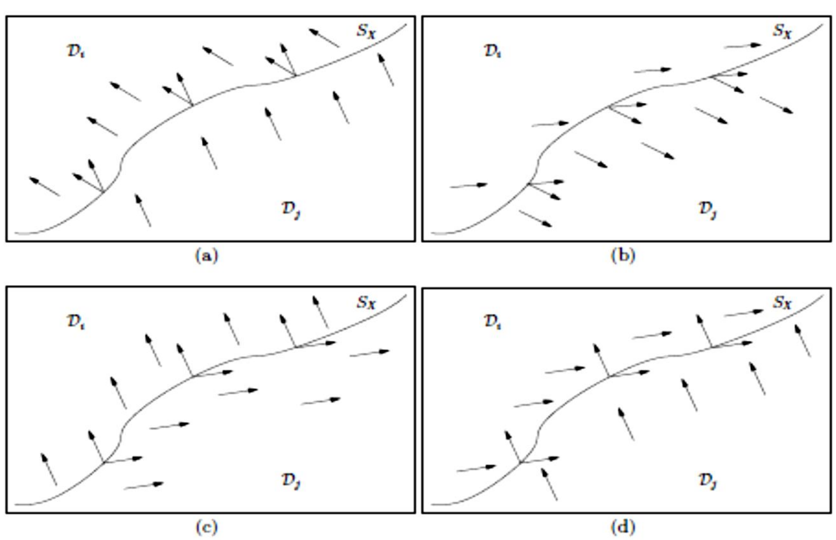
$$\mathcal{D}_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0, x_2 < 0\}, \mathcal{D}_4 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0, x_2 > 0\}$$

and discontinuous on $S_X = \{(x_1, 0) \mid x_1 \in \mathbb{R}\} \cup \{(0, x_2) \mid x_2 \in \mathbb{R}\}$. Therefore, X is piecewise continuous. Its Filippov set-valued map $F[X] : \mathbb{R}^2 \rightarrow B(\mathbb{R}^2)$ is given by

$$F[X](x_1, x_2) = \begin{cases} \{\text{sign}(x_2), -\text{sign}(x_1))\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ [-1,1] \times \{-\text{sign}(x_1)\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \{\text{sign}(x_2)\} \times [-1,1], & x_1 = 0 \text{and } x_2 \neq 0, \\ [-1,1] \times [-1,1], & x_1 = 0 \text{and } x_2 = 0. \end{cases}$$

Let us now discuss what happens on the points of discontinuity of the vector field $X: \mathbb{R}^d \to \mathbb{R}^d$. Let $x \in S_X$ belong to just two boundary sets,$x \in \partial\mathcal{D}_i \cap \partial\mathcal{D}_j$ for some $i, j \in \{1, \dots, m\}$. In this case, one can see that $F[X](x)$ = co$\{X_{|\overline{\mathcal{D}_i}}(x)$, $X_{|\overline{\mathcal{D}_j}}$ $(x)\}$. We consider the following possibilities:(i) if all the vectors belonging to $F[X](x)$ point in the direction of $\mathcal{D}_i$,then any Filippov solution that reaches S_X at x continues its motion in $\mathcal{D}_i$, (see Figure (1.6)(a));(ii) likewise, if all the vectors belonging to $F[X](x)$ point in the direction of $\mathcal{D}_j$,then any Filippov solution that reaches S_X at x continues its motion in $\mathcal{D}_j$, (see Figure (1.6)(b)); and (iii) however, if a vector belonging to $F[X](x)$ is tangent to S_X, then either Filippov solutions start at x and leave S_X immediately (see Figure (1.6)(c)), or there exists Filippov solutions that reach the set S_X at x, and stay in S_X afterward (see Figure(1.6)(d)).

The latter kind of trajectories are called sliding motions, since they slide along the boundaries of the sets where the vector field is continuous. This is the type of behavior that we saw in the example of the "move-away-from-closest-neighbor" interaction law. Sliding motions can also occur along points belonging to the intersection of more than two sets in $\mathcal{D}_1, \dots, \mathcal{D}_m$.The theory



Figure(1.6). Piecewise continuous vector fields. The dynamical systems are continuous on $\mathcal{D}_1$ and $\mathcal{D}_2$, and discontinuous at S_X . In cases (a) and (b), Filippov solutions cross the set of discontinuity. In case (c), there are two Filippov solutions starting from points in S_X.Finally, in case (d),Filippov solutions that reach S_X continue its motion sliding along it.

of sliding mode control builds on the existence of this type of trajectories to design stabilizing feedback controllers. These controllers induce sliding surfaces with the right properties in the state space so that the closed-loop system is stable.

The solutions of piecewise continuous vector fields in (i) and (ii) above occur frequently in state-dependent switching dynamical systems. Consider, for instance, the case of two unstable dynamical systems defined on the whole state space. It is conceivable that, by identifying an appropriate switching surface, one can synthesize a stable discontinuous dynamical system on the overall state space.

We can now discuss the Uniqueness of Filippov solutions:

In general, discontinuous dynamical systems do not have unique Filippov solutions.As an example, consider the vector field $X: \mathbb{R} \rightarrow \mathbb{R}$, $X(x)$ = sign(x). For any $x_0 \in \mathbb{R} \setminus \{0\}$, there is a unique Filippov solution starting from x_0. Instead, there are three (maximal) solutions $\gamma_1, \gamma_2, \gamma_3$: $[0, \infty) \rightarrow \mathbb{R}$ starting from $x(0) = 0$, specifically

$$t \longmapsto \gamma_1(t) = -t, \qquad t \longmapsto \gamma_2(t) = 0, \qquad t \longmapsto \gamma_3(t) = t.$$

The situation depicted in Figure(1.6)(c) is yet another qualitative example where multiple Filippov solutions exist starting from the same initial condition.

In this section, we provide two complementary uniqueness results for Filippov solutions.The first result considers the Filippov set-valued map associated with the discontinuous vector field, and identifies conditions that allow to apply Proposition (1.2.3)to the resulting differential inclusion.

Proposition (1.2.10):

Let $X : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable and locally essentially bounded. Assume that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $L_X(t)$, $\varepsilon \in (0, \infty)$ such that for almost every $\mathcal{Y}, \mathcal{Y}' \in B(x, \varepsilon)$, one has

$$\left(X(t, \mathcal{Y}) - X(t, \mathcal{Y}')\right)^T (\mathcal{Y} - \mathcal{Y}') \leq L_x(t) \, \|\mathcal{Y} - \mathcal{Y}'\|_2^2, \qquad (1.18).$$

Assume that the resulting function $t \longmapsto L_X(t)$ is integrable. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique Filippov solution of (1.1) with initial condition $x(t_0) = x_0$.

Let us apply this result to an example. Consider the vector field $X: \mathbb{R} \to \mathbb{R}$ defined by

$$X(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ -1, & x \notin \mathbb{Q}. \end{cases}$$

Note that this vector field is discontinuous everywhere in $\mathbb{R}$.The associated Filippov set-valued map $[X] : \mathbb{R} \to \mathbb{R}$ is $F[X](x) = \{-1\}$ (since $\mathbb{Q}$ has measure zero in $\mathbb{R}$,the value of the vector field at rational points does not play any role in the computation of $F[X]$).Clearly,equation(1.18) is verified for all $\mathcal{Y}, \mathcal{Y}' \notin \mathbb{Q}$. Therefore, there exists a unique solution starting from each initial condition (more precisely, the curve $\gamma: [0, \infty) \to \mathbb{R}$, $t \longmapsto \gamma(t) = x(0) - t$).

In general, the Lipschitz-type condition(1.18) is somewhat restrictive. This assertion is justified by the observation that, in dimension higher than one, piecewise continuous vector fields (arguably, the simpler class of discontinuous vector fields) do not verify the hypotheses of Proposition(1.2.10).

Next, the following result identifies sufficient conditions for uniqueness specifically tailored for piecewise continuous vector fields.

Proposition (1.2.12):

Let $X: \mathbb{R}^d \to \mathbb{R}^d$ be a piecewise continuous vector field, with $\mathbb{R}^d = \mathcal{D}_1 \cup \mathcal{D}_2$ Let $S_X = \partial\mathcal{D}_1 = \partial\mathcal{D}_2$ be the point set where X is discontinuous, and assume S_X is C^2 (that is, around a neighborhood of any of its points, the set can be expressed as the zero level set of twice continuously differentiable functions). Further assume that $X_{|\bar{\mathcal{D}}_i}$ is continuously differentiable on $\mathcal{D}_i$, for $i \in \{1, 2\}$, and that $X_{|\bar{\mathcal{D}}_1} - X_{|\bar{\mathcal{D}}_2}$ is continuously differentiable on S_X. If for each $x \in S_X$, either $X_{|\bar{\mathcal{D}}_1}(x)$ points in the direction of $\mathcal{D}_2$, or $X_{|\bar{\mathcal{D}}_2}(x)$ points in the direction of $\mathcal{D}_1$, then there exists a unique Filippov solution of (1.1) starting from each initial condition.

Note that the hypothesis on X already guarantees uniqueness of solution on each of the domains $\mathcal{D}_1$ and $\mathcal{D}_2$. Roughly speaking, the additional assumptions on X along S_X take care of guaranteeing that uniqueness is not disrupted by the discontinuities. Under the stated assumptions, when reaching S_X , Filippov solutions might cross it or slide along it. Situations like the one depicted in Figure(1.6)(c) are ruled out.

For the brick sliding on a frictional ramp example, at $v = 0$, the vector $X_{|\bar{\mathcal{D}}_1}(0)$ points in the direction of $\mathcal{D}_2$, and the vector $X_{|\bar{\mathcal{D}}_2}$ (0) points in the direction of $\mathcal{D}_1$. Proposition(1.2.12) then ensures that there exists a unique solution starting from each initial condition;

For the discontinuous vector field for one agent moving in the square $[-1, 1]^2$ under the "move-away-from-closest-neighbor" interaction law, it is convenient to define $\mathcal{D}_5 = \mathcal{D}_i$. Then, at any $(x_1, x_2) \in \partial\mathcal{D}_i \cap \partial\mathcal{D}_{i+1} \setminus \{(0, 0)\}$, with $i \in \{1, . . . , 4\}$, the vector $X_{|\bar{\mathcal{D}}_i}$ (x_1, x_2) points in the direction of $\mathcal{D}_{i+1}$, and the vector $X_{|\bar{\mathcal{D}}_{i+1}}$ (x_1, x_2) points in the direction of $\mathcal{D}_i$, see Figure(1.5).

Moreover, there is only one solution (the equilibrium one) starting from (0, 0). Therefore, using Proposition(1.2.12), we conclude that uniqueness of solutions holds;

For the nonsmooth harmonic oscillator, it is also convenient to define $\mathcal{D}_5 = \mathcal{D}_1$. Then, we can write that, for any $(x_1, x_2) \in \partial\mathcal{D}_i \cap \partial\mathcal{D}_{i+1} \setminus \{(0, 0)\}$, with $i \in \{1, . . . , 4\}$, the vector $X_{|\bar{\mathcal{D}}_i}$ (x_1, x_2) points in the direction of $\mathcal{D}_{i+1}$, see Figure(1.3)(a). Moreover, there is only one solution (the equilibrium one) starting from (0,0).Therefore, using Proposition(1.2.12), we conclude that uniqueness of solutions holds.

Proposition(1.2.12)can be also applied to piecewise continuous vector fields with an arbitrary number of partitioning domains, provided that set where the vector field is discontinuous is composed of a disjoint union of surfaces resulting from the pairwise intersection of the boundaries of two domains. Other versions of this result can also be stated for non-autonomous piecewise continuous vector fields, and for situations when more than two domains intersect at points of discontinuity.

We can investigate the Solutions of control systems with discontinuous input functions:

Let $X: \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d$, with $\mathcal{U} \subset \mathbb{R}^m$ the space of admissible controls, and consider the control equation on $\mathbb{R}^d$,

$$\dot{x}(t) = X\big(t, x(t), u(t)\big). \tag{1.20}$$

At first sight, the most natural way of identifying a notion of solution of this equation would seem to be as follows: select a control input, either open-loop $u: \mathbb{R} \to \mathcal{U}$, closed-loop $u : \mathbb{R}^d \to \mathcal{U}$, or a combination of both $u : \mathbb{R} \times \mathbb{R}^d \to \mathcal{U}$, and then consider the resulting non-autonomous differential equation. In this way, one is back to confronting the question posed above, that is, suitable notions of solution fo a discontinuous differential equation.

There are at least a couple of important alternatives to this approach that have been considered. We discuss them next.

We can discuss the Solutions via differential inclusions:

In a similar way as we have done so far, one may associate to the original control system (1.20) a differential inclusion, and build on it to define the notion of solution. This approach goes as follows: define the set-valued map

$G[X] : \mathbb{R} \times \mathbb{R}^d \to B(\mathbb{R}^d)$ by

$$G[X](t, x) = \{X(t, x, u) \mid u \in \mathcal{U}\}.$$

In other words, the set-valued map captures all the directions in $\mathbb{R}^d$ that can be generated with controls belonging to U. Consider now the differential inclusion

$$\dot{x}\ (t) \in\ G[X]\big(t, x(t)\big). \tag{1.20}$$

A solution of (1.20) defined on $[t_0,\ t_1] \subset \mathbb{R}$ is a solution of the differential inclusion(1.20),that is, an absolutely continuous map $\gamma : [t_0,\ t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}\ (t) \in G[X](t, \gamma(t))$ for almost every $t \in [t_0,\ t_1]$.

Clearly, given u: $\mathbb{R} \to \mathcal{U}$, any Caratheodory solution of the control system is also a solution of the associated differential inclusion. Alternatively, one can show [18] that, if X is continuous and $\mathcal{U}$ is compact, the converse is also true. Considering the differential inclusion has the advantage of not focusing the attention on any particular control input, and therefore allows to comprehensively study and understand the properties of the control system as a whole.

We can define the Sample-and-hold solutions:

Here we introduce the notion of sample-and-hold solution for control systems [57]. As we see later, this notion plays a key role in the stabilization question for asymptotically controllable systems.

A partition of the interval $[t_0,\ t_1]$ is a strictly increasing sequence

$\pi = \{s_0 = t_0 < s_1 < \cdot\cdot\cdot < s_N = t_1\}$. Note that the partition does not need to be finite, and that one can define the notion of partition of $[t_0,\ \infty)$ similarly. The diameter of π is diam(π) = $\sup\{s_i - s_{i-1} \mid i \in \{1, \ldots, N\ \}\}$. Given a feedback law$u$: $\mathbb{R} \times \mathbb{R}^d \to \mathcal{U}$ and a partition π of $[t_0,\ t_1]$, a π-solution of (1.20) defined on $[t_0,\ t_1] \subset \mathbb{R}$ is the map $\gamma : [t_0,\ t_1] \to \mathbb{R}^d$recursively defined as follows: for $i \in \{1, \ldots, N - 1\}$, the curve $[t_{i-1},\ t_i] \ni$ t $\longmapsto \gamma(t)$ is a Caratheodory solution of the differential equation

$$\dot{x}(t) = X(t, x(t), u(t_{i-1}, x(t_{i-1}))).$$

Roughly speaking, the control is held fixed throughout each interval of the partition at the value corresponding to the state at the beginning of the interval, and then the corresponding differential equation is solved, which explains why π-solutions are also referred to as sample-and-hold solutions. From our previous discussion on Caratheodory solutions, it is not difficult to derive conditions on the control system for the existence of π-solutions. Indeed, existence of π-solutions is guaranteed if (i) for all $u \in \mathcal{U} \subset \mathbb{R}^m$ and almost all $t \in \mathbb{R}$, the map $x \longmapsto X(t, x, u)$ is continuous, (ii) for all $u \in \mathcal{U} \subset \mathbb{R}^m$ and all $x \in \mathbb{R}^d$, the map $t \longmapsto X(t, x, u)$ is measurable, and (iii) for all $u \in \mathcal{U} \subset \mathbb{R}^m$, $(t, x) \rightarrow X(t, x, u)$ is locally essentially bounded.

Generalized sample-and-hold solutions of (1.20) are defined in [9] as the uniform limit of a sequence of π-solutions of (1.20) as diam(π) → 0. Interestingly, in general, generalized sample-and- hold solutions are not Caratheodory solutions, although conditions exist under which the inclusion holds, see [52].

Chapter (2)

Nonsmooth Analysis and stability

Secation(2-1):Nonsmooth analysis

If we have "gone discontinuous" with differential equations, we now "go nonsmooth" with the candidate Lyapunov functions. When examining the stability properties of discontinuous differential equations and differential inclusions,there are additional reasons to consider nonsmooth Lyapunov functions. The nonsmooth harmonic oscillator is a good example of what we mean, because it does not admit any smooth Lyapunovfunction. To see why, recall that all the Filippov solutions of the discontinuous system are periodic (see Figure(1.3). If such a smooth function exists, it necessarily has to be constant on each diamond. Therefore, since the level sets of the function are necessarily one-dimensional, each diamond would be a level set, which contradicts the fact that the function is smooth. This observation, taken from [9], is a simple illustration that our efforts to consider nonsmooth Lyapunov functions when considering discontinuous dynamics are not gratuitous.

In this section we discuss two tools from nonsmooth analysis: generalized gradients and proximal subdifferentials, see for instance [53,55]. As with the concept of solution of discontinuous differential equations, the literature is full of generalized derivative notions for the case when a function fails to be differentiable. These notions include, in addition to the two considered in this section, generalized (super or sub) differentials, (upper or lower, right or left) Dini derivatives, and contingent derivatives. Here, we have chosen to focus on the notions of generalized gradients and proximal subdifferentials because of their important role on providing applicable stability tools for discontinuous differential equations. The functions considered here are always defined on a finite dimensional Euclidean space, but we note that these objects are actually well-defined in Banach and Hilbert spaces.

Now we can define the generalized gradients of locally Lipschitz function:

From Rademacher's theorem[55],locally Lipschitz functions are differentiable almost everywhere (in the sense of Lebesgue measure). When considering a locally Lipschitz function as a candidate Lyapunov function, this statement may rise the following question: if the gradient of a locally Lipschitz function exists almost everywhere, should we really care for those points where it does not exist? Conceivably, the solutions of the dynamical systems under study stay almost everywhere away from the "bad" points where we do not have any gradient of the function.

However, such assumption turns out not to be true in general. As we show later, there are cases when the solutions of the dynamical system insist on staying on the "bad" points forever. In that case, having some sort of gradient information is helpful.

Let $f: \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. If Ω_f denotes the set of points in $\mathbb{R}^d$ at which f fails to be differentiable, and S denotes any set of measure zero, the generalized gradient $f: \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ is defined by

$$\partial f(x) = \text{co}\left\{\lim_{i\to\infty} \nabla f(x_i) \middle| x_i \to x, x_i \notin S \cup \Omega_f\right\}.$$

From the definition, the generalized gradient at a point provides convex combinations of all thepossible limits of the gradient at neighboring points (where the function is in fact differentiable).

Note that this definition coincides with $\nabla f(x)$ when f is continuously differentiable at x. Other equivalent definitions of the generalized gradient can be found in [55].

Let us compute the generalized gradient in a particular case. Consider the locallyLipschitz function $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$. The function is differentiable everywhere except for 0.Actually, $\nabla f(x) = 1$ for $x > 0$ and $\nabla f(x) = -1$ for $x < 0$. Therefore, we deduce

$$\partial f(0) = \text{co}\{1, -1\} = [-1, 1].$$

Now we can discuss the computing the generalized gradient:

As one might imagine, the computation of the generalized gradient of a locally Lipschitz function is not an easy task in general. In addition to the "brute force" approach, there are a number of results that can help us compute it. Many of the standard results that are valid for usual derivatives have their counterpart in this setting. We summarize some of them here.

Definition(2.1.1):(Dilation rule)

For $f: \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$, the function sf is locally Lipschitz at x, and

$$\partial(sf)(x) = s\,\partial f(x). \qquad (2.1)$$

Definition(2.1.2):(Sum rule)

For $f_1, f_1 : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$, and any scalars $s_1, s_2 \in \mathbb{R}$, the function $s_1 f_1 + s_2 f_2$ is locally Lipschitz at x, and

$$\partial(s_1 f_1 + s_2 f_2)(x) \subset s_1 \partial f_1 + s_2 \partial f_2 \qquad (2.2)$$

Moreover,if $f_1$and f_2 are regula at x and $s_1, s_2 \in [0, \infty)$,then equality holds and $s_1 f_1 + s_2 f_2$ is regular at x.

Definition(2.1.3):(Product rule)

For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$, the function $f_1 f_2$ is locally Lipschitz at x, and

$$\partial(f_1 f_2)(x) \subset f_2(x)\partial f_1(x) + f_1(x)\partial f_2(x). \qquad (2.3)$$

Moreover, if f_1 and f_1 are regular at x, and $f_1(x), f_2(x) \geq 0$, then equality holds and $f_1 f_2$is regular at x.

Definition(2.1.4): (Quotient rule)

For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$,with $f_2(x) \neq 0$, the function $f_1 f_2$ is locally Lipschitz at x, and

$$\partial\left(\frac{f_1}{f_2}\right)(x) \subset \frac{f_2(x)\partial f_1(x) - f_1(x)\partial f_2(x)}{f_2^2(x)}\,. \qquad (2.4)$$

Moreover, if f_1 and $-f_1$ are regular at x, and $f_1(x) \geq 0, f_2(x) > 0$, then equality holds and $f_1 / f_2$is regular at x.

Definition(2.1.5):(Chain rule)

For $h : \mathbb{R}^d \to \mathbb{R}^m$, with each component locally Lipschitz at $x \in \mathbb{R}^d$, and $g : \mathbb{R}^m \to \mathbb{R}$ at $h(x) \in \mathbb{R}^m$ the function $g \circ h$ is locally Lipschitz at x, and

$$\partial(g \circ h)(x) \subset co\left\{\sum_{k=1}^{m} \alpha_k \zeta_k \middle| (\alpha_1, \dots, \alpha_m) \in \partial g\big(h(x)\big), (\zeta_1, \dots, \zeta_m) \in \partial h_1(x) \times \dots \times \partial h_m(x)\right\}. \quad (2.5)$$

Moreover, if g is regular at $h(x)$, each component of h is regular at x,and every element of $\partial g(h(x))$belongs to $[0, \infty)^d$, then equality holds and goh is regular at x.

Let us highlight here a particularly useful result from [55, Proposition 2.3.12] concerning the generalized gradient of max and min functions.

Proposition(2.1.6):

$f_k: \mathbb{R}^d \to \mathbb{R}, k \in \{1, \dots m\}$ be locally Lipschitz functions at $x \in \mathbb{R}^d$ and consider the functions

$$f_{\max}(x') = \max\{f_k(x') | k \in \{1, \dots, m\}\}, \quad f_{\min}(x') = \min\{f_k(x') | k \in \{1, \dots, m\}\}.$$

Then,

(i) $f_{\max}$ and $f_{\min}$ are locally Lipschitz at x,

(ii) f $I_{\max}(x')$denotes the set of indexes k for which $f_k(x') = f_{\max}(x')$, we have

$$\partial f_{\max}(x) \subset \text{co}\{\partial f_i(x) | i \in I_{\max}(x)\}, \quad (2.6)$$

and if $f_i, i \in I_{\max}(x)$, is regular at x, then equality holds and $f_{\max}$ is regular at x,

(iii) if $I_{\min}(x')$ denotes the set of indexes k for which $f_k(x') = f_{min}(x')$, we have

$$\partial f_{\min}(x) \subset \text{co}\{\partial f_i(x) | i \in I_{\min}(x)\}, \quad (2.7)$$

And $if\ -f_i, i \in I_{\min}(x)$,is regular at x, then equality holds and $-f_{\min}$ is regular at x.

As a consequence of this result, the maximum of a finite set of continuously differentiable functions is a locally Lipschitz and regular function, and its generalized gradient is easily computable at each point as the convex closure of the gradients of the functions that attain the maximum at that particular point. As an example, the function $f_1(x) = |x|$ can be re-written as $f_1(x) = max\{x, -x\}$. Both $x \longmapsto x$ and $x \longmapsto -x$ are continuously differentiable, and hence locally Lipschitz and regular.

Therefore, according to Proposition(2.1.6)(i) and (ii), so is f_1, and its generalized gradient is

$$\partial f_1(x) = \begin{cases} \{1\}, & x > 0, \\ [-1,1], & x = 0, \\ \{-1\}, & x < 0, \end{cases} \qquad (2.8)$$

Which is the same result that we obtained warlier by direct computation.

Definition(2.1.7):(Regular Functions)
Let us recall here the notion of regular function. To introduce it, we need to first define what right directional derivatives and generalized right directional derivatives are. Given $f : \mathbb{R}^d \to \mathbb{R}$, the right directional derivative of f at x in the direction of $v \in \mathbb{R}^d$ is defined as

$$f'(x,v) = \lim_{h\to 0^+} \frac{f(x+hv)-f(x)}{h},$$

when this limits exists. On the other hand, the generalized directional derivative of f at x in the direction of $v \in \mathbb{R}^d$ is defined as

$$f^0(x,v) = \lim_{\substack{y\to x \\ h\to 0^+}} sup \frac{f(y+hv)-f(y)}{h} = \lim_{\substack{\delta\to 0^+ \\ \varepsilon\to 0^+}} sup_{\substack{y\in B(x,\delta) \\ h\in[0,\varepsilon)}} \frac{f(y+hv)-f(y)}{h}.$$

This latter notion has the advantage of always being well-defined. In general, these directional derivatives may not be equal. When they are, we call the function regular. More formally, afunction $f : \mathbb{R}^d \to \mathbb{R}$ is regular at $x \in \mathbb{R}^d$ if for all $x \in \mathbb{R}^d$, the right directional derivative of f at x in the direction of v exists, and $f'(x;v) = f^0(x;v)$. A continuously differentiable function at x is regular at x. Also, a convex and locally Lipschitz function at x is regular (of. [55,Proposition 2.3.6]). An example of a non-regular function is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -|x|$. The function is continuously differentiable everywhere except for zero, so it is regular on $\mathbb{R} \setminus \{0\}$.
However, its directional derivatives

$$f'(0;v) = \begin{cases} -v, & v > 0, \\ v & v < 0. \end{cases} \quad f^0(0,v) = \begin{cases} v, & v > 0, \\ -v, & v < 0. \end{cases}$$

do not coincide. Hence, the function is not regular at 0.

Note that the minimum of a finite set of regular functions is in general not regular.Asimple example is given by $f_2(x)$=min$\{x,-x\}$=$-|x|$,which is not regular at 0,as we showed in the definition(2.1.7).However,according to Proposition(2.1.6)(i)and(iii),this fact dose not mean that its generalized gradient cannot be computed.Indeed,one has

$$\partial f_2(x) = \begin{cases} \{-1\}, & x > 0, \\ [-1,1], & x = 0, \\ \{1\}, & x < 0, \end{cases} \tag{2.9}$$

We can now define the critical points and directions of descent:

Acritical point of $f\colon \mathbb{R}^d \to \mathbb{R}$ is a point $x \in \mathbb{R}^d$ such that $0 \in \partial f(x)$.The maxima and minima of locally Lipschitz functions are critical points according to this definition. As an example, $x = 0$ is a minimum of $f(x) = |x|$, and indeed one verifies that $0 \in \partial f(0)$.

When the function f is continuously differentiable, the gradient ∇f provides the direction of maximum ascent (respectively, $-\nabla f$ provides the direction of maximum descent).When considering locally Lipschitz functions, however, one faces the following question: given that the generalized gradient is a set of directions, rather than a single one, which one are the right ones to choose? Without loss of generality, we restrict our discussion to directions of descent, since a direction of descent of $-f$ corresponds to a direction of ascent of f , and f is locally Lipschitz if and only if $-f$ is locally Lipschitz.

Let Ln : $\mathfrak{B}(\mathbb{R}^d) \to \mathfrak{B}(\mathbb{R}^d)$ be the set-valued map that associates to each subset S of $\mathbb{R}^d$the set of least-norm elements of its closure $\bar{S}$. If the set S is convex, then the set $\mathrm{Ln}(S)$ reduces to a singleton and we note the equivalence $\mathrm{Ln}S) = \mathrm{proj}_S(0)$. For a locally Lipschitz function f, consider the generalized gradient vector field $\mathrm{Ln}(\partial f)\colon \mathbb{R}^d \to \mathbb{R}^d$

$$x \to \mathrm{Ln}(\partial f)(x) = \mathrm{Ln}\big(\partial f(x)\big).$$

It turns out that $\mathrm{Ln}(\partial f)(x)$ is a direction of descent at $x \in \mathbb{R}^d$. More precisely, following [55], one finds that if $0 \notin \partial f(x)$, then there exists $T > 0$ such that

$$f\big(x - t\mathrm{Ln}(\partial f)(x)\big) \le f(x) - \frac{t}{2}\|\mathrm{Ln}(\partial f)(x)\|_2^2, \quad 0 < t < T \tag{2.10}$$

We can now discuss the minimum distance to polygonal boundary:

Let $Q \subset \mathbb{R}^2$ be a convex polygon. Consider the minimum distance function $sm_Q : Q \to \mathbb{R}$ from any point within the polygon to its boundary defined by

$$\mathrm{sm}_Q(p) = \min\{\|p - q\|_2 \mid q \in \partial Q\}.$$

Note that the value of sm_Q corresponds to the radius of the largest disk contained in the polygon with center p. Moreover, this function is locally Lipschitz on Q. To show this, simply rewrite the function as

$$\mathrm{sm}_Q(p) = \min\{\mathrm{dist}(p, e) \mid e \text{ edge of } Q\},$$

where $\mathrm{dist}(p, e)$ denotes the Euclidean distance from the point p to the edge e. Let us consider the generalized gradient vector field corresponding to this function (if one prefers to have a function defined on the whole space, as we have been using in this section, one can easily extend the definition of sm_Q outside Q by setting

$\mathrm{sm}_Q(p) = -\min\{\|p - q\|_2 \mid q \in \partial Q\}$for $p \notin Q$, and proceed with the discussion). Applying Proposition(2.1.6)(iii), we deduce that $-\mathrm{sm}_Q$ is regular on Q and its generalized gradient is

$$\partial \mathrm{sm}_Q(p) = \mathrm{co}\{\mathrm{n_e} \mid \mathrm{e} \text{ edge of } Q \text{ such that } \mathrm{sm}_Q(p) = \mathrm{dist}(p, e)\}, p \in Q,$$

where n_e denotes the unit normal to the edge e pointing toward the interior of Q. Therefore, at points p in Q where there is a unique edge e of Q which is closest to p, the function sm_Q is differentiable, and its generalized gradient vector field is given by $\mathrm{Ln}(\mathrm{sm}_Q)(p) = n_e$. Note that this vector field corresponds to the “move-away-from-closest-neighbor” interaction law for one agent moving in the polygon!At points p of Q where various edges $\{e_1, \ldots, e_m\}$ are at the same minimum distance to p, the function sm_Q is not differentiable, and its generalized gradient vector field is given by the least-norm element in $co\{n_{e_1}, \ldots, n_{e_m}\}$. If p is not a critical point, 0 does not belong to the latter set, and the least-norm element points in the direction of the bisector line between two of the edges in $\{e_1, \ldots, e_m\}$. Figure(2.1)shows a plot of the generalized gradient vector field of sm_Q on the square $Q = [-1, 1]^2$. Note the similarity with the plot in Figure(1.5).

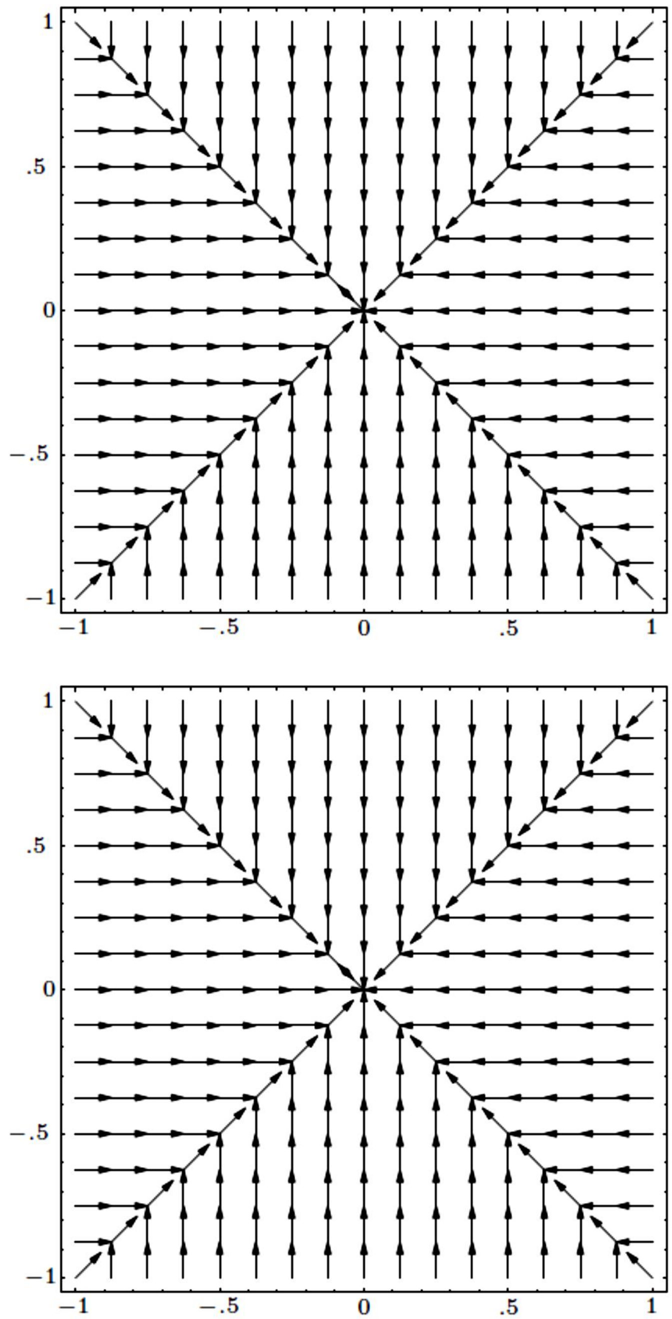


Figure (2.1). Generalized gradient vector field. The plot shows the generalized gradient vector field of the minimum distance to polygonal boundary function $sm_Q : Q \to \mathbb{R}$ on the square $[-1, 1]^2$. The vector field is discontinuous on the diagonals of the square.

Indeed, one can characterize [69] the critical points of sm_Q as $0 \in \partial sm_Q(p)$ if and only if p belongs to the incenter set of Q.

The incenter set of Q is composed of the centers of the largest-radius disks contained in Q. In general, the incenter set is not a singleton (think, for instance, of a rectangle), but a segment. However, one can also show that is not $0 \in \text{interior}(\partial sm_Q(p))$, then the incenter set of Q is the singleton $\{p\}$.

We can now discuss the nonsmooth gradient flows:

Given a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, one can define the nonsmooth analog of the classical gradient flow of a differentiable function as

$$\dot{x}(t) = -\text{Ln}(\partial f)(x(t)). \tag{2.11}$$

According to (2.10), unless the flow is already at a critical point, $-\text{Ln}(\partial f)(x)$ is always a direction of descent at x. Note that this nonsmooth gradient vector field is discontinuous, and therefore we have to specify the notion of solution that we consider. In this case, we select the Filippov notion. Since f is locally Lipschitz, $\text{Ln}(\partial f) = \nabla f$ almost everywhere. A remarkable fact [28] is that the Filippov set-valued map associated with the nonsmooth gradient flow of f is precisely the generalized gradient of the function, that is,Filippov set-valued map of nonsmooth gradient.For $f : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz, the Filippov set-valued map $F[\text{Ln}(\partial f)] : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ of the nonsmooth gradient of f is equal to the generalized gradient $\partial f : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ of f,

$$F[\text{Ln}(\partial f)](x) = \partial f(x), x \in \mathbb{R}^d.$$

As a consequence of this result, note that the discontinuous system (2.11) is equivalent to the differential inclusion

$$\dot{x}(t) \in -\partial f(x(t)).$$

How can we analyze the asymptotic behavior of the trajectories of this system? When the function f is differentiable, the LaSalle Invariance Principle allows us to deduce that, for functions with bounded level sets, the trajectories of the gradient flow asymptotically converge to the set of critical points. The key tool behind this result is being able to establish that the value of the function decreases along the trajectories of the system. This behavior is formally expressed through the notion of Lie derivative. We discuss later suitable generalizations of the notion of Lie derivative to the nonsmooth case. These notions allow us, among other things, to study the asymptotic convergence properties of the trajectories of nonsmooth gradient flows.

Now we can define the proximal subdifferentials of lower semicontinuous functions:

A complementary set of nonsmooth analysis tools to deal with Lyapunov functions is given by proximal subdifferentials. This concept has the advantage of being defined for a larger class of functions, namely, lower semicontinuous (instead of locally Lipschitz) functions. Generalized gradients provide us with directional descent information, that is, directions along which the function decreases. The price we pay by using proximal subdifferentials is that explicit descent directions are in generally not known to us. Proximal subdifferentials, however, still allow us to reason about the monotonic properties of the function, which as we show later, turns out to be sufficient to provide stability results.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous at $x \in \mathbb{R}^d$ if for all $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that $f(y) \geq f(x) - \varepsilon$, for $y \in B(x, \delta)$. We restrict our attention to real-valued lower semicontinuous functions. Lower semicontinuous functions with extended real values are considered in [53]. A function $f : \mathbb{R}^d \to \mathbb{R}$ is upper semicontinuous at $x \in \mathbb{R}^d$ if $-f$ is lower semicontinuous at x. Note that f is continuous at x if and only if f is both upper and lower semicontinuous at x. For a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$, a vector $\zeta \in \mathbb{R}^d$ is a proximal subgradient of f at $x \in \mathbb{R}^d$ if there exist $\sigma, \delta \in (0, \infty)$ such that for all $y \in B(x, \delta)$,

$$f(y) \geq f(x) + \zeta(y - x) - \sigma^2 \|y - x\|_2^2. \qquad (2.12)$$

The set of all proximal subgradients of f at x is the proximal subdifferential of f at x, and denoted $\partial_P f(x)$.The proximal subdifferential at x is always convex. However, it is not necessarily open, closed, bounded or nonempty. Geometrically, the definition of proximal subgradient can be interpreted as follows. Equation (2.1) is equivalent to saying that, around x, the function $y \longmapsto f(y)$ majorizes the quadratic function $y \longmapsto f(x) + \zeta(y - x) - \sigma^2 \|y - x\|_2^2$. In other words, there exists a parabola that locally fits under the graph of f at $(x, f(x))$. This geometric interpretation is indeed very useful for the explicit computation of the proximal subdifferential.

Let us compute the proximal subdifferential in two particular cases. Consider the locally Lipschitz functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}, f_1(x) = |x|$ and $f_2(x) = -|x|$. Using the geometric interpretation of (2.12), it is not difficult to see that

$$\partial_p f_1(x) = \begin{cases} \{1\}, & x < 0, \\ [-1,1], & x = 0, \\ \{-1\}, & x > 0, \end{cases}$$

$$\partial_p f_2(x) = \begin{cases} \{-1\}, & x < 0, \\ \emptyset, & x = 0, \\ \{1\}, & x > 0, \end{cases}$$

Compare this result with the generalized gradients of f_1 in (2.8) and of f_2 in (2.9).

Unlike the case of generalized gradients, the proximal subdifferential may not coincide with $\nabla f(x)$ when f is continuously differentiable. The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto -|x|^{3/2}$, is continuously differentiable, but $\partial_P f(0) = \emptyset$. In fact, [37] provides an example of a continuously differentiable function on $\mathbb{R}$ which has an empty proximal subdifferential almost everywhere. However, it should be noted that the density theorem (of. [53, Theorem 3.1]) states that the proximal subdifferential of a lower semicontinuous function is always nonempty in a dense set of its domain of definition.

On the other hand, the function $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{|x|}$ provides an example where proximal subdifferentials are more useful than generalized gradients. The function is continuous at 0, but notlocally Lipschitz at 0, which precisely corresponds to its global minimum. Hence the generalized gradient does not help us here. The function is lower semicontinuous, and has a well-defined proximal subdifferential,

$$\partial_P f(x) = \begin{cases} \left\{\frac{1}{2}\frac{1}{\sqrt{x}}\right\}, & x > 0, \\ \mathbb{R}, & x = 0, \\ \left\{-\frac{1}{2}\frac{1}{\sqrt{-x}}\right\}, & x < 0, \end{cases}$$

If $f : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^d$, then the proximal subdifferential of f at x is bounded. In general, the relationship between the generalized gradient and the proximal subdifferential of a function f locally Lipschitz at $x \in \mathbb{R}^d$ is expressed by

$$\partial f(x) = \mathrm{co}\left\{ \lim_{n\to\infty} \zeta_n \in \mathbb{R}^d \,\middle|\, \zeta_n \in \partial_P f(x_n) \; and \; \lim_{n\to\infty} x_n = x \right\}.$$

We can now discuss the Computing of the proximal subdifferential:

As with the generalized gradient, the computation of the proximal subdifferential gradient of a lower semicontinuous function is not straightforward in general. Here we provide some useful results following the exposition in [53].

Definition(2.1.8):(Dilation rule)

For $f : \mathbb{R}^d \to \mathbb{R}$ lower semicontinuous at $x \in \mathbb{R}^d$ and $s \in (0, \infty)$, the function sf is lower semicontinuous at x, and

$$\partial P\ (sf)(x) = s\, \partial_P f(x). \tag{2.13}$$

Definition:(2.1.9):(Sum rule)

For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ lower semicontinuous at $x \in \mathbb{R}^d$, the function $f_1 + f_2$ is lower semicontinuous at x, and

$$\partial_P f_1(x) + \partial_P f_2(x) \subset \partial_P (f_1 + f_2)(x). \tag{2.14}$$

Moreover, if either f_1 or f_2 are twice continuously differentiable, then equality holds.

Definition(2.1.10):(Chain rule)

For either $h : \mathbb{R}^d \to \mathbb{R}^m$ linear and $g : \mathbb{R}^m \to \mathbb{R}$ lower semicontinuous at $h(x) \in \mathbb{R}^m$, or $h : \mathbb{R}^d \to \mathbb{R}^m$ locally Lipschitz at $x \in \mathbb{R}^d$ and $g : \mathbb{R}^m \to \mathbb{R}$ locally Lipschitz at $h(x) \in \mathbb{R}^m$, the following holds: for $\zeta \in \partial_P (g \circ h)(x)$ and any $\varepsilon \in (0, \infty)$, there exist $\tilde{x} \in \mathbb{R}^d, \tilde{y} \in \mathbb{R}^m$, and $\gamma \in \partial_P\, g(\tilde{y})$ with $\max\left\{\left\|\tilde{x} - x\right\|_2, \|\tilde{y} - h(x)\|_2\right\} < \varepsilon$ such that $\|h(\tilde{x}) - h(x)\|_2 < \varepsilon$ and

$$\zeta \in \partial_P\ (\langle \gamma, h(\cdot)\rangle)(\tilde{x}) + \varepsilon B(0, 1). \tag{2.15}$$

The statement of the chain rule above shows one of the characteristic features when dealing with proximal subdifferentials: in many occasions, arguments and results are expressed with objects evaluated at points in a neighborhood of the specific point of interest, rather than at the point itself.

The computation of the proximal subdifferential of twice continuously differentiable functions is particularly simple. For $f : \mathbb{R}^d \to \mathbb{R}$ twice continuously differentiable on $U \subset \mathbb{R}^d$ open, one has

$$\partial_P f(x) = \{\nabla f(x)\}, \text{ for all } x \in U. \tag{2.16}$$

This simplicity also works for continuously differentiable convex functions, as the following result states.

Proposition(2.1.11):

Let $f : \mathbb{R}^d \to \mathbb{R}$ be lower semicontinuous and convex, and let $x \in \mathbb{R}^d$. Then,

(i) $\zeta \in \partial_P f(x)$ if and only if $f(y) \geq f(x) + \zeta(y - x)$, for all $y \in \mathbb{R}^d$;

(ii) the map $x \longmapsto \partial_P f(x)$ takes nonempty, compact and convex values, and is upper semicontinuous and locally bounded;

(iii) if, in addition, f is continuously differentiable, then $\partial_P f(x) = \{\nabla f(x)\}$, for all $x \in \mathbb{R}^d$;

Regarding critical points, if x is a local minimum of a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$, then $0 \in \partial_P f(x)$. Conversely, if f is lower semicontinuous and convex, and $0 \in \partial_P f(x)$, then x is a global minimum of f. If one is interested in maxima, then instead of the notions of lower semicontinuous functions, convex functions and proximal subdifferentials, one needs to consider upper semicontinuous functions, concave functions and proximal superdifferentials, respectively (see [53]).

We can now discuss the Gradient differential inclusions:

In general, one cannot define a nonsmooth gradient flow associated to a lower semicontinuous function, because, as we have observed above, the proximal subdifferential might be empty almost everywhere. However, following Proposition (2.1.11)(ii), we can associate a nonsmooth gradient flow to functions that are lower semicontinuous and convex, as we briefly discuss next following [104].

Let $f : \mathbb{R}^d \to \mathbb{R}$ be lower semicontinuous and convex. Consider the gradient differential inclusion

$$\dot{x}(t) \in -\partial_P f\big(x(t)\big). \tag{2.17}$$

Using the properties of the proximal subdifferential stated in Proposition(2.1.11)(ii), existence of solutions of this differential inclusion is guaranteed by Proposition(1.2.2).Moreover, uniqueness of solutions can also be established.To show this, let $x, y \in \mathbb{R}^d$,and take $\zeta_1 \in -\partial_P f(x)$ and $\zeta_2 \in -\partial_P f(y)$. Using Proposition(2.1.11)(i), we have

$$f(y) \geq f(x) - \zeta_1(y - x), \qquad f(x) \geq f(y) - \zeta_2(x - y).$$

From here,we deduce $-\zeta_1(y - x) \leq f(y) - f(x) \leq -\zeta_2(y - x)$,and therefore $(\zeta_2 - \zeta_1)(y - x) \leq 0$,which, in particular, implies that the set-valued map $x \longmapsto -\partial_P f(x)$ verifies the one-sided Lipschitz condition(1.12)). Proposition(1.2.3) guarantees then uniqueness of solutions.

Once we know that solutions exist and are unique, the next natural question is to understand their asymptotic behavior.To analyze it, we need to introduce tools specifically tailored for this nonsmooth setting that allow us to establish, among other things, the monotonic behavior of the function f along the solutions.

Section(2-2):Nonsmooth Stability Analysis:

In this section, we present tools to study the stability properties of discontinuous dynamical systems.Unless explicitly mentioned otherwise, the stability notions employed here correspond to the usual ones for differential equations,see,for instance [66]. The presentation of the results focuses on the setup of autonomous differential inclusions,

$$\dot{x}(t) \in \mathcal{F}(x(t)), \tag{2.18}$$

Where $\mathcal{F}: \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$. Throughout the section, we assume that the set-valued map $\mathcal{F}$ verifies the hypothesis of Proposition(1.2.2),so that the existence of solutions of the differential inclusion is guaranteed. From our previous discussion, it should be clear that the setup of differential inclusions has a direct application to the scenario of discontinuous differential equations and control systems. The results presented here can be easily made explicit for the notions of solution introduced earlier.

Before proceeding with our exposition, let us make a couple of remarks. The first one concerns the wordings “strong” and “weak.” As we already observed, solutions of discontinuous systems are generally not unique. Therefore, when considering properties such as Lyapunov stability or invariance, one needs to specify if one is paying attention to a particular solution starting from an initial condition (“weak”) or to all the solutions starting from an initial condition (“strong”). As an example, a set $M \subset \mathbb{R}^d$ is weakly invariant for (2.18) if for each $x_0 \in M, M$ contains a maximal solution of (2.18) with initial condition x_0.

The second remark concerns the notion of limit point of solutions of the differential inclusion. A point $x \in \mathbb{R}^d$ is a limit point of a solution γ of (2.18) if there exists a sequence $\{t^n\}_{n \in N}$ such that $\gamma(t_n) \rightarrow x$ as $n \rightarrow \infty$. We denote by $\Omega(\gamma)$ the set of limit points of γ. Under the hypothesis of Proposition(1.2.2), $\Omega(\gamma)$ is a weakly invariant set. Moreover, if the solution γ lies in a bounded domain, then $\Omega(\gamma)$is nonempty, bounded, connected, and $\gamma(t) \rightarrow \Omega(\gamma)$as $\rightarrow \infty$, see [18].

We can now investigate the stability analysis via generalized gradients of nonsmooth Lyapunov functions:

In the following, we discuss nonsmooth stability analysis results that invoke locally Lipschitz functions and their generalized gradients. We have chosen a number of important results taken from different sources in the literature. The discussion presented here does not intend to be a comprehensive account of such a vast topic, but rather serve as a motivation to further explore it.

Now we can define the Lie derivatives and monotonicity:

A common theme in stability analysis is the possibility of establishing the monotonic evolution along the trajectories of the system of a candidate Lyapunov function. Mathematically, the evolution of a function along trajectories is captured by the notion of Lie derivative. Our first task here is then to generalize this notion to the setup of discontinuous systems following [1], see also [36, 46].

Given a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and a set-valued map

$\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$, the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}} f : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R})$ of $\mathcal{F}$with respect to $\mathcal{F}$ at x is defined as

$$\tilde{\mathcal{L}}_{\mathcal{F}} f(x) = \{a =\in \mathbb{R} | \text{ there exists } v \in \mathcal{F}(x) \text{ such that } \zeta^T v = a, for\ all\ \zeta \in \partial f(x)\}. \quad (2.19)$$

If $\mathcal{F}$ takes convex and compact values, for each $x \in \mathbb{R}^d$, $\tilde{\mathcal{L}}_{\mathcal{F}} f(x)$ is a closed and bounded interval in $\mathbb{R}$, possibly empty. If $f$is continuously differentiable at x, then $\tilde{\mathcal{L}}_{\mathcal{F}} f(x) = \{(\nabla f)^T v \mid v \in \mathcal{F}(x)\}$.The importance of the set-valued Lie derivative stems from the fact that it allows us to study how the function f evolves along the solutions of a differential inclusion without having to obtain them in closed form. Specifically, we have the following result.

Proposition(2.2.1):

Let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be a solution of the differential inclusion (2.18), and let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function. Then

(i) the composition $t \mapsto f(\gamma(t))$ is differentiable at almost every $t \in [t_0, t_1]$, and

(ii) the derivative of $t \mapsto f(\gamma(t))$ verifies

$$\frac{d}{dt}\Big(f\big(\gamma(t)\big)\Big) \in \tilde{\mathcal{L}}_{\mathcal{F}} f\big(\gamma(t)\big) \text{ for almost every } t \in [t_0, t_1] \quad (2.20)$$

Given a discontinuous vector field $X : \mathbb{R}^d \to \mathbb{R}^d$, consider the solutions of (1.1) in the Filippov sense. In that case, with a little abuse of notation, we denote $\tilde{\mathcal{L}}_X f = \tilde{\mathcal{L}}_{\mathcal{F}[X]} f$. Note that if X is continuous at x, then $F[X](x) = \{X(x)\}$, and therefore, $\tilde{\mathcal{L}}_X f(x)$ corresponds to the singleton $\{\mathcal{L}_{\mathcal{F}} f(x)\}$, the usual Lie derivative of f in the direction of X at x.

Let us illustrate the importance of this result in an example.

We can now discuss the Monotonicity in the nonsmooth harmonic oscillator:

For the nonsmooth harmonic oscillator, consider the locally Lipschitz and regular map $f : \mathbb{R}^d \to \mathbb{R}, f(x_1, x_2) = |x_1| + |x_2|$ (recall that Figure(1.3)(b) shows the contour plot of f). Let us determine how the function evolves along the solutions of the dynamical system by looking at the set-valued Lie derivative. First, we compute the generalized gradient of f. To do so, we rewrite the function as $f(x_1, x_2) = max\{x_1, -x_1\} + \max\{x_2, -x_2\}$, and apply Proposition(2.1.6)(ii) and the sum rule to find

$$\partial f(x_1, x_2) = \begin{cases} \{(\text{sign}(x_1), \text{sign}(x_2))\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \{\text{sign}(x_1)\} \times [-1,1], & x_1 \neq 0 \text{ and } x_2 = 0, \\ [-1,1] \times \{\text{sign}(x_2)\}, & x_1 = 0 \text{ and } x_2 \neq 0, \\ [-1,1] \times [-1,1], & x_1 = 0 \text{ and } x_2 = 0, \end{cases}$$

With this information, we are ready to compute the set-valued Lie derivative

$$\tilde{\mathcal{L}}_X f(x_1, x_2) = \begin{cases} \{0\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \emptyset, & x_1 \neq 0 \text{ and } x_2 = 0, \\ \emptyset, & x_1 = 0 \text{ and } x_2 \neq 0, \\ \{0\}, & x_1 = 0 \text{ and } x_2 = 0, \end{cases}$$

From this equation and (2.20), we conclude that the function f is constant along the solutions of the discontinuous dynamical system. Indeed, the level sets of the function f are exactly the diamond figures described by the solutions of the system.

We can now explain the stability results:

The above discussion on monotonicity is the stepping stone to provide stability results using locally Lipschitz functions and generalized gradient information. Proposition (2.2.1)provides a criterion to determine the monotonic behavior of the solutions of discontinuous dynamics along locally Lipschitz functions. This result, together with the right "positive definite" assumptions on the candidate Lyapunov function allows us to synthesize checkable stability tests. We start by formulating the natural extension of Lyapunov stability theorem for ODEs. In this and in forthcoming statements, it is convenient to adopt the convention $\max \emptyset = -\infty$.

Theorem(2.2.2):

Let $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$be a set-valued map satisfying the hypothesis of Proposition(1.2.2) Let x_* be an equilibrium of the differential inclusion (2.18), and let $\mathcal{D} \subset \mathbb{R}^d$ be a domain with $x_* \in \mathcal{D}. Let\ f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

(i) f is locally Lipschitz and regular on $\mathcal{D}$;

(ii) $f(x_*) = 0$, and $f(x) > 0$ for $x \in \mathcal{D} \setminus \{x_*\}$

(iii) max $\tilde{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$ for all $x \in \mathcal{D}$.

Then, x_* is a strongly stable equilibrium of (2.18). In addition, if (iii) above is substituted by

(iii)' max $\tilde{\mathcal{L}}_X f(x) < 0$ for all $x \in \mathcal{D} \setminus \{x_*\}$,

then x_* is a strongly asymptotically stable equilibrium of (2.18).

Let us apply this result to the nonsmooth harmonic oscillator. The function $(x_1, x_2) \rightarrow |x_1| + |x_2|$ verifies hypothesis(i)-(iii) of Theorem(2.2.2)on $\mathcal{D} = \mathbb{R}^d$. Therefore, we conclude that 0 is a strongly stable equilibrium. From the phase portrait in Figure(1.3)(a), it is clear that 0 is not strongly asymptotically stable.We are invited to use Theorem(2.2.2)to deduce that the nonsmooth harmonic oscillator under dissipation, with vector field $(x_1, x_2) \mapsto (\text{sign}(x_2), -\text{sign}(x_1) - \frac{1}{2} sign(x_2))$, has 0 as a strongly asymptotically stable equilibrium.

Another important result in the theory of differential equations is the LaSalle Invariance Principle. In many situations, this principle allows us to figure out the asymptotic convergence properties of the solutions of a differential equation. Here, we build on our previous discussion to present a generalization to differential inclusions(2.18) and nonsmooth Lyapunov functions. Needless to say, this principle is also suitable for discontinuous differential equations. The formulation is taken from [1], and slightly generalizes the one presented in [36].

Theorem(2-2-3):

Let $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition (1.2.2), and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally Lipschitz and regular function. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for (2.18), and assume that max $\tilde{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$ for all $x \in S$. Then, any solution $\gamma : [t_0, \infty) \rightarrow \mathbb{R}^d$ of (2.18) starting at S converges to the largest weakly invariant set M contained in

$$S \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}} f(x)\}}.$$

Moreover, if the set M is a finite collection of points, then the limit of all solutions starting at S exists and equals one of them.

Let us show next an application of this result to nonsmooth gradient flows.

We can now discuss the nonsmooth gradient flows revisited:

Consider the nonsmooth gradient flow (2.11) of a locally Lipschitz function f. Assume further that the function f is regular. Let us examine how the function evolves along the solutions of the flow using the set-valued Lie derivative. Given $x \in \mathbb{R}^d$, let $a \in \mathcal{L}_{-Ln(\partial f)} f(x)$. By definition, there exists $v \in F[-Ln(\partial f)](x) = -\partial f(x)$ such that

$$a = \zeta^T v, \text{ for all } \zeta \in \partial f(x).$$

Since the equality holds for any element in the generalized gradient of f at x, we may choose in particular $\zeta = -v \in \partial f(x)$. Therefore,

$$a = (-v)^T v = -\|v\|_2^2 \leq 0.$$

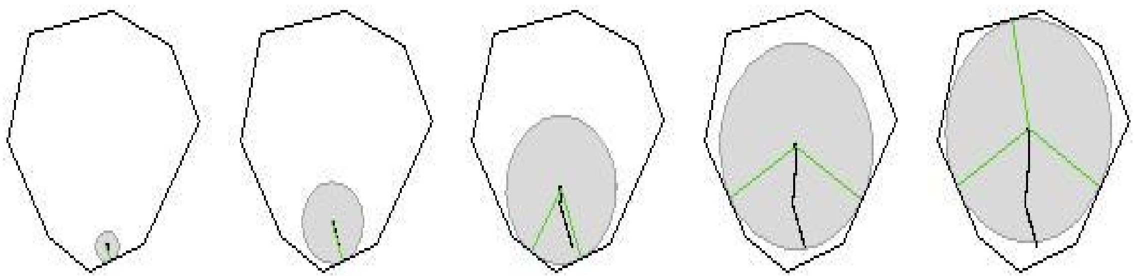
From this equation, we conclude that the elements of $\mathcal{L}_{-Ln(\partial f)} f$ all belong to $(-\infty, 0]$, and therefore,from equation (2.20), the function f monotonically decreases along the solutions of its nonsmooth gradient flow.

The application of the Lyapunov stability theorem and the LaSalle Invariance Principle above gives now rise to the following nice nonsmooth counterpart of the classical smooth results [120] for gradient flows.

We can now define the stability of nonsmooth gradient flows:

Let f be a locally Lipschitz and regular function. Then, the strict minima of f are strongly stable equilibria of the nonsmooth gradient flow of f. Furthermore, if the level sets of f are bounded, then the solutions of the nonsmooth gradient flow asymptotically converge to the set of critical points of f.

As an illustration, consider the nonsmooth gradient flow of $-sm_Q$ (the minimum distance to polygonal boundary function). Uniqueness of solutions for this flow can be guaranteed via Proposition(1.2.12).Regarding convergence, the application of the above result on the stability of nonsmooth gradient flows guarantee that solutions converge asymptotically to the incenter set. Indeed, one can show [69] that the incenter set is attained in finite time, and hence convergence occurs to individual points. In all, one can interpret the nonsmooth gradient flow as a "sphere-packing algorithm," in the sense that, starting from any initial point, it monotonically maximizes the radius of the largest disk contained in the polygon (that is, sm_Q!) until it reaches an incenter point. An illustration of this fact is shown in Figure(2.2).



Figure(2.2)From left to right, evolution of the nonsmooth gradient flow of the function $-sm_Q$ in a convex polygon. At each snapshot, the value of sm_Q is the radius of the largest disk (plotted in light gray) contained in the polygon with center at the current location. The flow converges in finite time to the incenter set, that for this polygon, is a singleton.

What if, instead, one is interested in packing more than one sphere within the polygon, say for example n spheres? It turns out that the "move-away-from-closest-neighbor" interaction law is a discontinuous dynamical system that solves this problem, where the solutions are understood in the Filippov sense.

We can now discuss the finite-time convergent gradient flows of smooth functions:

General results on finite-time convergence for discontinuous dynamical systems can be found in [28, 70]. Here, we briefly discuss the finite-convergence properties of a class of nonsmooth gradient flows.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function, with bounded level sets. As we have mentioned before, the solutions of the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ converge asymptotically toward the set of critical points of f. However, they cannot reach them in finite time. Here, we slightly modify the gradient flow to turn it into two different nonsmooth flows that achieve finitetime convergence.

Consider the discontinuous differential equations

$$\dot{x}(t) = -\frac{\nabla f(x(t))}{\|\nabla f(x(t))\|_2}, \tag{2.21}$$

$$\dot{x}(t) = -\text{sign}\left(\nabla f(x(t))\right), \tag{2.22}$$

where $\|\cdot\|_2$ denotes the Euclidean distance and $\text{sign}(x) = (\text{sign}(x_1), \ldots, \text{sign}(x_d)) \in \mathbb{R}^d$. We understand the solutions of these systems in the Filippov sense. The nonsmooth vector field (2.21) always moves in the direction of the gradient with unit speed. The nonsmooth vector field (2.22), instead, specifies the direction of motion via a binary quantization of the direction of the gradient. For these discontinuous systems, one can establish the following result.

Finite-time convergence of nonsmooth gradient flows. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for (2.21) (resp., for (2.22)). If the Hessian of f is positive definite at each critical point of f in S, then each solution of (2.21) (resp. (2.22)) starting from S converges in finite time to a minimum of f.

The proof of this result builds on the stability tools presented in this section. Specifically, the LaSalle Invariance Principle can be used to establish convergence toward the set of critical points of the function. To establish finite-time convergence, one derives bounds on the evolution of the function along the solutions of the discontinuous dynamics using the set-valued Lie derivative. This analysis also allows to provide upper bounds on the convergence time.

Now we can define the finite-time consensus:

Arguably, the ability to reach consensus, or agreement, upon some (a priori unknown) quantity is critical for any multiagent system. Network coordination problems require individual agents to agree on the identity of a leader, jointly synchronize their operation, decide which specific pattern to form, balance the computational load or fuse consistently the information gathered on some spatial process. Here, we briefly comment on two discontinuous algorithms that achieve consensus in finite time, following [70].

Consider a network of n agents with states $p_1, \ldots, p_n \in R$. Let $G = (\{1, \ldots, n\}, E)$ be an undirected graph with n vertices, describing the topology of the network.Two agents p_i and p_j agree if and only if $p_i = p_j$ .The disagreement function $\Phi_G : \mathbb{R}^n \to [0, \infty)$ quantifies the group disagreement

$$\Phi_G(p_1, \dots, p_2) = \frac{1}{2} \sum_{(i,j)\in E} \left(p_j - p_i\right)^2.$$

We can now discuss the Stability analysis via proximal subdifferentials of nonsmooth Lyapunov functions**:**

This section presents stability tools for differential inclusions using lower semicontinuous functions as candidate Lyapunov functions. We make use of proximal subdifferentials to study the monotonic evolution of the candidate Lyapunov functions along the solutions of the differential inclusions. As in the previous section, we have chosen to present a few representative and useful results.

We can now define the Lie derivatives and monotonicity:

Let $\mathcal{D} \subset \mathbb{R}^d$ be a domain. A lower semicontinuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is weakly nonincreasing on $\mathcal{D}$ for a set-valued map $\mathcal{D} : \mathbb{R}^d \rightarrow B(\mathbb{R}^d)$ if for any $x \in \mathcal{D}$, there exists a solution $\gamma: [t_0, t_1] \rightarrow \mathbb{R}^d$of the differential inclusion (2.18) starting at x and lying in $\mathcal{D}$ that satisfies

$$f(\gamma(t)) \leq f((0)) = f(x)\, for\ all\ t \in [t_0, t_1].$$

If in addition, f is continuous, then being weakly nonincreasing is equivalent to the property of having a solution starting at x such that $t \mapsto f(\gamma(t))$ is monotonically nonincreasing on $[t_0, t_1]$.

Similarly, a lower semicontinuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is strongly nonincreasing on $\mathcal{D}$ for a set-valued map $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ if for any $x \in \mathcal{D}$, all solutions $\gamma: [t_0, t_1] \rightarrow \mathbb{R}^d$ of the differential inclusion (2.18) starting at x and lying in $\mathcal{D}$ satisfy

$$f((t)) \leq f((0)) = f(x)\, for\ all\ t \in [t_0, t_1].$$

Note that being strongly nonincreasing is equivalent to the property of having $t \mapsto f((t))$ be monotonically nonincreasing on $[t_0, t_1]$ for all solutions of the differential inclusion.

Given a set-valued map $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$taking nonempty, compact values, and a lower semicontinuous function $\mathcal{F} : \mathbb{R}^d \rightarrow \mathbb{R}$, the lower and upper set-valued Lie derivatives $\underline{\mathcal{L}}_{\mathcal{F}} f, \overline{\mathcal{L}}_{\mathcal{F}} f : \mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ of f with respect to $\mathcal{F}$ at x are defined by, respectively

$\underline{\mathcal{L}}_{\mathcal{F}} f(x) = \{a \in \mathbb{R} \mid$ there exists $\zeta \in \partial_P f(x)$

such that $a = \min\{\zeta^T v \mid v \in \mathcal{F}(x)\}\}$,

$\overline{\mathcal{L}}_{\mathcal{F}} f(x) = \{a \in \mathbb{R} \mid$ there exists $\zeta \in \partial_P f(x)$

such that $a = \min\{\zeta^T v \mid v \in \mathcal{F}(x)\}$,If, in addition, $\mathcal{F}$ takes convex values, then for each $\zeta \in \partial_P f(x)$, the set $\{\zeta^T v \mid v \in \mathcal{F}(x)\}$ is a closed interval of the form $[\min\{\zeta^T v \mid v \in \mathcal{F}(x)\}, \max\{\zeta^T v \mid v \in \mathcal{F}(x)\}]$. Note that the lower and upper set-valued Lie derivatives at a point x might be empty.

The lower and upper set-valued Lie derivatives play a similar role for lower semicontinuous functions to the one played by the set-valued Lie derivative e $\tilde{\mathcal{L}}_{\mathcal{F}} f$ for locally Lipschitz functions. These objects allow us to study how the function f evolves along the solutions of a differential inclusion without having to obtain them in closed form. Specifically, we have the following result. In this and in forthcoming statements, it is convenient to adopt the convention sup $\emptyset = -\infty$.

Proposition(2.2.4):

Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$be a set-valued map satisfying the hypothesis of Proposition(1.2.2), and consider the associated differential inclusion (2.18). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function, and $D \subset \mathbb{R}^d$ open. Then,

(i) The function f is weakly nonincreasing on $\mathcal{D}$ if and only if

$$\sup \underline{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0, \text{ for all } x \in \mathcal{D};$$

(ii) If, in addition, either $\mathcal{F}$ is locally Lipschitz on $\mathcal{D}$, or $\mathcal{F}$ is continuous on $\mathcal{D}$ and f is locally Lipschitz on $\mathcal{D}$, then f is strongly nonincreasing on $\mathcal{D}$ if and only if

$\sup \overline{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$, for all $x \in \mathcal{D}$.

Let us illustrate this result in a particular example.

We can now discuss the cart on a circle:

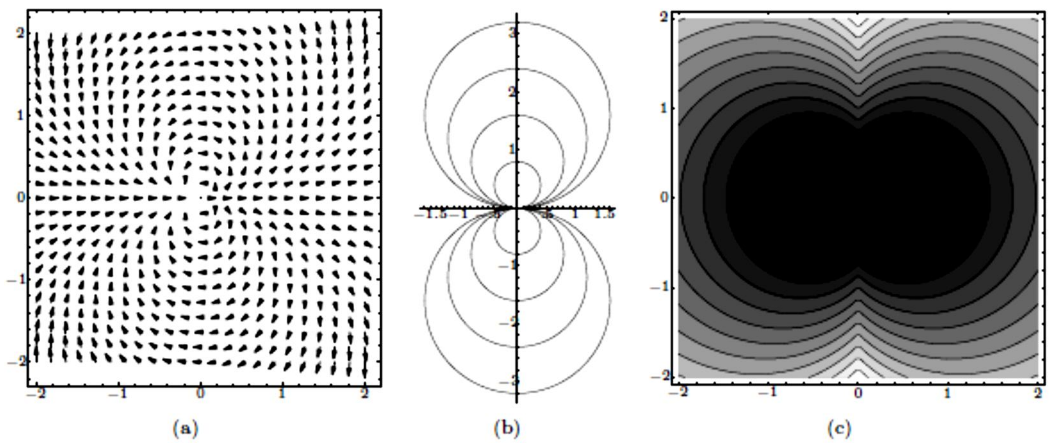
Consider, following [45, 46], the driftless control system on $\mathbb{R}^2$

$$\dot{x}_1 = (x_1^2 - x_2^2)u,$$

$$\dot{x}_2 = 2x_1 x_2 u,$$

with $u \in \mathbb{R}$. The phase portrait of the vector field

$(x_1, x_2) \longmapsto g(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$ is plotted in Figure(2.3)(a).



Figure(2.3). Cart on a circle. The plot in (a) shows the phase portrait of the vector field $(x_1, x_2) \longmapsto (x_1^2 - x_2^2, 2x_1x_2)$the plot in (b) shows its integral curves, and the plot in (c) shows the contour plot of the function

$$0 \neq (x_1, x_2) \longmapsto \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2} + |x_1|}, (0,0) \mapsto 0.$$

Alternatively, consider the associated set-valued map $\mathcal{F} : \mathbb{R}^2 \to \mathfrak{B}(\mathbb{R}^2)$ defined by $\mathcal{F}(x_1, x_2) = \{g(x_1, x_2)\, u \mid u \in \mathbb{R}\}$. Note that F does not take compact values. Therefore, instead of considering $\mathcal{F}$, we take any nondecreasing map $\sigma : [0, \infty) \to [0, \infty)$, and define the set-valued map $\mathcal{F}_\sigma : \mathbb{R}^2 \to \mathfrak{B}(\mathbb{R}^2)$ given by $\mathcal{F}_\sigma(x_1, x_2) = \{g(x_1, x_2)\, u \in \mathbb{R}^2 \mid |u| \leq \sigma(\|(x_1, x_2)\|)\}$.

Consider the locally Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2} + |x_1|}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

The level set curves of this function are depicted in Figure (2.3)(b). Let us determine how f evolves along the solutions of the control system by using the lower and upper set-valued Lie derivatives.

First, let us compute the proximal subdifferential of f. Using the fact that f is twice continuously differentiable on the open right and left half-planes, together with the geometric interpretation of proximal subgradients, we obtain

$$\partial_P f(x_1, x_2) = \begin{cases} \left\{ -\dfrac{x_1^2 + x_2^2 - 2x_1\sqrt{x_1^2 + x_2^2}}{x_1^2 + x_2^2 + x_1\sqrt{x_1^2 + x_2^2}}, \dfrac{x_2\left(2x_1 + \sqrt{x_1^2 + x_2^2}\right)}{\left(x_1 + \sqrt{x_1^2 + x_2^2}\right)^2} \right\}, & x_1 > 0, \\ \emptyset, & x_1 = 0, \\ \left\{ \dfrac{x_1^2 + x_2^2 - 2x_1\sqrt{x_1^2 + x_2^2}}{x_1^2 + x_2^2 - x_1\sqrt{x_1^2 + x_2^2}}, \dfrac{x_2\left(-2x_1 + \sqrt{x_1^2 + x_2^2}\right)}{\left(x_1 - \sqrt{x_1^2 + x_2^2}\right)^2} \right\} & x_1 < 0. \end{cases}$$

With this information, we compute the set

$$\{\zeta^T v | \zeta \in \partial_P f(x_1, x_2), v \in \mathcal{F}_\sigma(x_1, x_2)\} = \begin{cases} \left\{ u \dfrac{(x_1^2 + x_2^2)^2}{x_1^2 + x_2^2 + x_1\sqrt{x_1^2 + x_2^2}} \middle| |u| \le \sigma(\|(x_1, x_2)\|_2) \right\}, & x_1 > 0, \\ \emptyset, & x_1 = 0, \\ \left\{ -u \dfrac{(x_1^2 + x_2^2)^2}{x_1^2 + x_2^2 - x_1\sqrt{x_1^2 + x_2^2}} \middle| |u| \le \sigma(\|(x_1, x_2)\|_2) \right\} & x_1 < 0. \end{cases}$$

We are now ready to compute the lower and upper set-valued Lie derivatives as

$$\underline{\mathcal{L}}_{\mathcal{F}} f(x_1, x_2) = \begin{cases} -\sigma(\|(x_1, x_2)\|_2) \dfrac{(x_1^2 + x_2^2)^{3/2}}{\sqrt{x_1^2 + x_2^2} + |x_1|}, & x_1 \neq 0, \\ -\infty, & x_1 = 0 \end{cases}$$

$$\overline{\mathcal{L}}_{\mathcal{F}} f(x_1, x_2) = \begin{cases} \sigma(\|(x_1, x_2)\|_2) \dfrac{(x_1^2 + x_2^2)^{3/2}}{\sqrt{x_1^2 + x_2^2} + |x_1|}, & x_1 \neq 0, \\ -\infty, & x_1 = 0 \end{cases}$$

Therefore $\sup \underline{\mathcal{L}}_{\mathcal{F}} f(x_1, x_2) \le 0$, for all $(x_1, x_2) \in \mathbb{R}^2$. Using now Proposition(2.2.4)(i), we deduce that the function f is weakly nonincreasing on $\mathbb{R}^2$. Since f is continuous, this fact is equivalent to saying that there exists a choice of control input u such that the solution γ of the resulting dynamical system satisfies that $t \mapsto f(\gamma(t))$ is monotonically nonincreasing.

We can now define the stability results:

The results presented in the previous section establishing the monotonic behavior of lower semicontinuous functions allow us to provide tools for stability analysis. We present here an exposition parallel to the one for locally Lipschitz functions and generalized gradients. We start by presenting a result on Lyapunov stability.

Theorem(2.2.5):

Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$be a set-valued map satisfying the hypothesis of Proposition(1.2.2).Let x_* be an equilibrium of the differential inclusion (2.18), and let $\mathcal{D} \subset \mathbb{R}^d$ be a domain with $x_* \in \mathcal{D}$. Let $f : \mathbb{R}^d \to \mathbb{R}$ and assume that

(i) $\mathcal{F}$ is continuous on $\mathcal{D}$ and f is locally Lipschitz on $\mathcal{D}$, or $\mathcal{F}$is locally Lipschitz on $\mathcal{D}$ and f is lower semicontinuous on $\mathcal{D}$, and f is continuous at x_* ;

(ii) $f(x_*) = 0, and\ f(x) > 0\ for\ x \in \mathcal{D} \setminus \{x_*\}$;

(iii) $sup\bar{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0\ for\ all\ x \in \mathcal{D}$.

Then, x_* is a strongly stable equilibrium of (2.18). In addition, if (iii) above is substituted by

(iii)' $\sup\bar{\mathcal{L}}_{\mathcal{F}} f(x) < 0$ for all $x \in \mathcal{D} \setminus \{x_*\}$,

Then x_* is a strongly asymptotically equilibrium of (2.18)

A similar result can be stated for weakly stable equilibria substituting (i) by "(i') f is continuous on $\mathcal{D}$," and the upper set-valued Lie derivative by the lower set-valued Lie derivative in (iii) and (iii'). Note that, if the differential inclusion (2.18) has unique solutions starting from any initial condition, then the notions of strong and weak stability coincide, and it is sufficient to verify the simpler requirements of the result for weak stability.

In a similar way to the case of continuous differential equations, global asymptotic stability can be established by requiring the Lyapunov function f to be continuous and radially unbounded. Indeed, this type of global results are commonly invoked when dealing with the stabilization of control systems by referring to control Lyapunov functions [42] or Lyapunov pairs [53]. Two lower semicontinuous functions , $g:\mathbb{R}^d \to \mathbb{R}$ are a Lyapunov pair for an equilibrium $x_* \in \mathbb{R}^d$ if they satisfy that $f(x)$, $g(x) \geq 0$ for $x \in \mathbb{R}^d$, and $g(x) = 0$ if and only if $x = x_*$; f is radially unbounded, and moreover,

$$\sup \underline{\mathcal{L}}_{\mathcal{F}} f(x) \leq -g(x), \text{ for all } x \in \mathbb{R}^d.$$

If an equilibrium x_* of (2.18) admits a Lyapunov pair, then one can show that there exists at least one solution starting from any initial condition that asymptotically converges to the equilibrium, see [53].

As an application of this discussion and the version of Theorem(2.2.5)for weak stability, consider the cart on a circle example. Setting x_*= (0, 0) and $\mathcal{D} = \mathbb{R}^2$, and taking into account our previous computation of the lower set-valued Lie derivative, we conclude that (0, 0) is a (globally) weakly asymptotically stable equilibrium.

We now turn our attention to the extension of LaSalle Invariance Principle for differential inclusions using lower semicontinuous functions and proximal subdifferentials.

Theorem(2.2.6):

Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition(1.2.2), and let $f : \mathbb{R}^d \to \mathbb{R}$. Assume either $\mathcal{F}$ is continuous and f is locally Lipschitz, or $\mathcal{F}$ is locally Lipschitz and f is continuous. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for (2.18),and assume that $\sup \bar{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$ for all $x \in S$. Then, any solution : $[t_0, \infty) \to \mathbb{R}^d$ of (22.18) starting at S converges to the largest weakly invariant set M contained in

$$S \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \bar{\mathcal{L}}_{\mathcal{F}} f(x)\}}$$

Moreover, if the set M is a finite collection of points, then the limit of all solutions starting at S exists and equals one of them.

Let us apply this result to gradient differential inclusions.

We can discuss the gradient differential inclusions revisited:

Consider the gradient differential inclusion (2.17) associated to a continuous and convex function f: $\mathbb{R}^d \to \mathbb{R}$.Let us study here the asymptotic behavior of the solutions. From our previous discussion, we know that solutions exist and are unique. In particular, this fact means that in this case the notions of weakly nonincreasing and strongly nonincreasing function coincide. Therefore,let us simply show that the function f is weakly nonincreasing on $\mathbb{R}^d$ for the gradient differential inclusion.

For any $\zeta \in \partial_P f(x)$, there is $v = -\zeta \in -\partial_P f(x)$ such that $\zeta^T v = -\|\zeta\|_2^2 \leq 0$. In particular, this implies

$$\underline{\mathcal{L}}_{-\partial_P} f(x) \leq 0, \text{ for all } x \in \mathbb{R}^d.$$

Proposition(2.2.4)(i) now guarantees that f is weakly nonincreasing on $\mathbb{R}^d$. Since the solutions of the gradient differential inclusion are unique, f is monotonically nonincreasing.

The application of the Lyapunov stability theorem and the LaSalle Invariance Principle above gives now rise to the following nice nonsmooth counterpart of the classical smooth results for gradient flows.

Stability of gradient differential inclusions. Let f be a continuous and convex function. Then, the strict minima of f are strongly stable equilibria of the gradient differential inclusion associated to f. Furthermore, if the level sets of f are bounded, then the solutions of the gradient differential inclusion asymptotically converge to the set of minima of f.

We can now investigate the Stabilization of control systems**:**

Consider an autonomous control system on $\mathbb{R}^d$ of the form

$$\dot{x} = X(x, u), \qquad (2.23)$$

where $X: \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ (note that the space of admissible controls is $\mathcal{U} \subset \mathbb{R}^m$). The system is locally (respectively globally) continuously stabilizable if there exists a continuous map $k : \mathbb{R}^d \to \mathbb{R}^m$ such that the closed-loop system

$$\dot{x} = X\big(x, k(x)\big)$$

is locally (respectively globally) asymptotically stable at the origin. The celebrated result by Brockett [113], see also [44, 6], states that many control systems are not continuously stabilizable.

Theorem(2.2.7):

Let X: $\mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be continuous and $X(0, 0) = 0$. A necessary condition for the existence of a continuous stabilizer of the control system (2.23) is that X maps any neighborhood of the origin in $\mathbb{R}^d \times \mathbb{R}^m$ onto some neighborhood of the origin in $\mathbb{R}^d$.

In particular, Theorem(2.2.7) implies that driftless control systems of the form

$$\dot{x} = u_1 X_1(x) + \ldots + u_m X_m(x), \tag{2.24}$$

with $m < n$, and $X_i : \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{1, \ldots, m\}$ continuous, cannot be stabilized by a continuous feedback.

The condition in Theorem (2.2.7), is only necessary. There exist control systems that satisfy it, and still cannot be stabilized by means of acontinuous stabilizer. The cart on a circle example is one of them. The map $((x_1, x_2), u) \to g(x)u$ is onto any neighborhood of (0, 0).However, it cannot be stabilized with a continuous $k : \mathbb{R}^2 \to \mathbb{R}$, see [42] for various ways to justify it.

The obstruction to the existence of continuous stabilizers has motivated the search for time-varying and discontinuous feedback stabilizers. Regarding the latter, an immediate question pops up: if one uses a discontinuous map $k : \mathbb{R}^m \to \mathbb{R}^d$, how should the solutions of the resulting discontinuous dynamical system $\dot{x} = X(x, k(x))$ be understood? From the previous discussion, we know that Caratheodory solutions are not a good candidate, since in many situations they fail to exist. The following result [47,76], shows that Filippov solutions are not a good candidate either.

Theorem(2.2.8):

Let X: $\mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be continuous and $X(0, 0) = 0$. Assume that for each $U \subset \mathbb{R}^m$ and each $x \in \mathbb{R}^d$, one has $X(x, \mathrm{co}U) = \mathrm{co}X(x, U)$. Then, a necessary condition for the existence of a measurable, locally bounded stabilizer of the control system (2.23) (where solutions are understood in the Filippov sense) is that X maps any neighborhood of the origin in $\mathbb{R}^d \times \mathbb{R}^m$ onto some neighborhood of the origin in $\mathbb{R}^d$.

In particular, driftless control systems of the form (2.24) cannot be stabilized by means of a discontinuous feedback if solutions are understood in the Filippov sense. This impossibility result, however, can be overcome if solutions are understood in the sample-and-hold sense, as shown in [57]. This work used this notion to solve the open question concerning the relationship between asymptotic controllability and feedback stabilization.

Let us briefly discuss this result in the light of our previous exposition. Consider the differential inclusion (1.20) associated with the control system (2.23). The system (2.23) is (open loop) globally asymptotically controllable (to the origin) if 0 is a Lyapunov stable equilibrium of (1.20), and every point $x \in \mathbb{R}^d$ has the property that there exists a solution of (1.20) satisfying $x(0) = x$ and

$\lim_{t\to\infty} x(t) = 0$. On the other hand, a feedback $k : \mathbb{R}^d \to \mathbb{R}^m$ stabilizes the system (2.23) in the sample-and-hold sense if, for all $x_0 \in \mathbb{R}^d$ and all $\varepsilon \in (0, \infty)$, there exist $\delta, T \in (0, \infty)$ such that, for any partition π of $[0, t_1]$ with $\text{diam}(\pi) < \delta$, the corresponding π-solution of (2.23) starting at x_0 satisfies $\|\gamma(t)\| \le \varepsilon$ for all $t \ge T$.

The following result states that both notions, global asymptotic controllability and the existence of a feedback stabilizer, are equivalent.

Theorem(2.2.9):

Let $X: \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be continuous and $X(0, 0) = 0$. Then, the control system (2.23) is globally asymptotically controllable if and only if it admits a measurable, locally bounded stabilizer in the sample-and-hold sense.

The implication from right to left is clear. The converse implication is proved by explicit construction of the stabilizer, and is based on the fact that the control system (2.23) is globally asymptotically controllable if and only if it admits a continuous Lyapunov pair, see [41].

Using the continuous Lyapunov function provided by this character rization,one constructs explicitly the discontinuous feedback for the control system (2.23), see [57, 42]. The existence of a Lyapunov pair “in the sense of generalized gradients” (that is, when instead of using the lower set-valued Lie derivative involving proximal subdifferential, one uses the set-valued Lie derivative involving the generalized gradient) turns out to be equivalent to the existence of a stabilizing feedback in the sense of Filippov, see [89].

As an illustration, consider the cart on a circle example. We have already shown that (0, 0) is a globally weakly asymptotically stable equilibrium of the differential inclusion associated with the control system. Therefore, the control system is globally asymptotically controllable,and can be stabilized in the sample-and-hold sense by means of a discontinuous feedback. The stabilizing feedback that results from the proof of Theorem(2.2.9) is the following, see [45, 51]: if to the left of the x_2 axis, move in the direction of the vector field g, if to the right of the x_2 axis, move in the opposite direction of the vector field g, and make an arbitrary decision on the x_2-axis. The stabilizing nature of this feedback can be graphically checked in Figure (2.3)(a) and (b).

Remarkably, for systems affine in the control, there exist [91] stabilizing feedbacks whose discontinuities form a set of measure zero, and, moreover, the discontinuity set is repulsive for the solutions of the closed-loop system. In particular, this fact means that in applying the feedback,the solutions can be understood in the Caratheodory sense. This situation is exactly what we see in the cart on a circle example.

Chapter (3)

Discontinuous Differential Equations,non smooth Analysis and stability

Section(3-1):Discontinuous Differential Equations

The present chapter is motivated both by the intrinsic mathematical interest in the study of differential equations with discontinuous righthand sides and by the fact that such equations often occur in control theory because of the application of discontinuous feedback laws in control systems. The survey on different possible concepts of solution is then strongly influenced by the requirements of applications in control theory.

We consider both time-dependent and autonomous Cauchy problems:

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases} \qquad (3.1)$$

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases} \qquad (3.2)$$

Where $x \in \mathbb{R}^n$.

Let us recall that a classical solution of one of the previous Cauchy problems on an interval $I \subset \mathbb{R}$ is an everywhere differentiable function

which satisfies (3.1) (or (1.2)) at every $t \in I$.

If the function f is continuous then the Cauchy problem (3.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)ds \qquad (3.3)$$

The so called Carath´eodory solutions of (3.1) are solutions of (3.3), which can exist even if f is not continuous. The classical conditions on f which guarantee the existence of these solutions are Carath´eodory conditions, that we specify in this section.

If the study of equations whose righ hand sides do not satisfy Carath´eodory conditions is needed, two alternative ways can be pursued. The first one, consists in looking for the weakest hypothesis on the vector field f which guarantee the existence of Carath´eodory solutions (see [51, 73, 24, 25]). The second one, followed by most authors who work in control theory, consists in introducing generalized solutions.

Besides Carath´eodory solutions, we focus on Euler, generalized sampling, Krasovskii and Filippov solutions.

Among the authors who have introduced other kinds of generalized solutions, let us mention Hermes ([101]), Ambrosio ([86, 85]) and Sentis ([111, 108]).

Concerning bibliography, let us stress that a very important article is

the one by H´ajek ([101]), dated 1979, where Hermes, Krasovskii and Filippov solutions are compared. To the best knowledge of the author there is no recent work in which something similar has been done.

Now we begin by studing Caratheodory Solutions.

Definition(3.1.1):

An absolutely continuous function φ:$[[t_0, t_0 + a] \to \mathbb{R}^n$ is said to be a Carath´eodory solution of (3.1) if it satisfies (3.3) for all $t \in [t_0, t_0 + a]$ or, equivalently, if it satisfies (3.1) for almost every $t \in [t_0, t_0 + a]$**.**

We denote the set of Carath´eodory solutions of (3-1) by $\mathcal{C}$.

We say that there exists a local solution of (3.1) if there exists $\delta > 0$ such that there exists a Carath´eodory solution of (3.1) on $[t_0, t_0 + a]$.

Definition (3.1.2):

Let I be any interval of $\mathbb{R}, D$ any subset of $\mathbb{R}^n, f : I \times D \to \mathbb{R}^n$. The function $f : I \times D \to \mathbb{R}^n$ is said to satisfy the Carath´eodory conditions on $I \times D$ if:

(i) f is defined and continuous with respect to x for a.e. $t \in I$,

(ii) f is measurable with respect to t for each $x \in D$,

(iii) there exists a nonnegative summable function $m : I \to \mathbb{R}$ such that $\|f(t,x)\| \le m(t)$ for all $t \in I$.

Theorem(3.1.3):

Let f be defined on $\mathbb{R} = \{(t,x) : t_0 \le t \le t_0 + a, \|x - x_0\| \le b\}$.If f satisfies the Carath´eodory conditions on R, then there exists a Carath´eodory solution of (3.1) on $[t_0, t_0 + \delta]$, where δ is such that $\int_{t_0}^{t_0+\delta} m(t)\, dt \le b$.

Moreover if there exists a summable function $l : [t_0, t_0 + \delta] \to \mathbb{R}$ such that for all $t \in [t_0, t_0 + \delta]$and for all $x, \mathcal{Y}$ such that$\|x - x_0\| \le b, \|\mathcal{Y} - \mathcal{Y}_0\| \le b$ one has

$$\|f(t,x) - f(t,\mathcal{Y})\| \le l(t)\|x - \mathcal{Y}\| \qquad (3.4)$$

Then the solution on $[t_0, t_0 + \delta]$ is unique.

In [15, 51, 24, 25], the authors consider the autonomous Cauchy problem (3.1.2) in order to get weaker conditions which guarantee the existence of Carath´eodory solutions. Let us introduce these conditions.

Definition(3.1.4):

The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be directionally continuous if there exists $\delta > 0$ such that, for every $x \in \mathbb{R}^n$, if $f(x) \neq 0$ and $x_n \to x$ with

$$\left| \frac{x_n - x}{|x_n - x|} - \frac{f(x)}{|f(x)|} \right| < \delta \qquad \forall n \ge 1,$$

then $f(x_n) \to f(x)$.

Directional continuity asks $f(x_n) \to f(x)$ only for sequences converging to x contained inside a cone with vertex at x and opening δ around an axis having the direction of $f(x)$.

Definition (3.1.5):

The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$is said to be self tangent if, for every $x \in \mathbb{R}^n$, there exist two sequences $x_n \to x$ and $t_n > 0$ such that

$$\lim_{n\to\infty}\frac{x_n - x}{t_n} = \lim_{n\to\infty} f(x_n) = f(x).$$

Let us remark that directional continuity implies self tangency. In [23,24] it is proved that directional continuity of f implies the existence of local Carath´eodory solution of (3.2). The same is proved in [15] under the assumption that f is self tangent and has locally closed graph.

We now define patchy vector fields, which have been introduced in [51]. They are the superposition of inward-pointing vector fields. They also guarantee the existence of local Carath´eodory solutions of (3.2).

Definition (3.1.6):

Let $\Omega \subset \mathbb{R}^n$ be an open domain with smooth boundary $\partial\Omega$. A smooth vector field f defined on a neighbourhood of $\overline{\Omega}$ is said to be an inward pointing vector field on Ω if at every boundary point $x \in \partial\Omega$ the inner product of f with the outer normal n satisfies $f(x) \cdot n < 0$. The pair (Ω, f) is said to be a patch.

Definition(3.1.7):

$f : \Omega \to \mathbb{R}^n$ is said to be a patchy vector field if there exists a family of patches $\{(\Omega_\alpha, f_\alpha) : \alpha \in \mathcal{A}\}$ such that:

- $\mathcal{A}$ is a totally ordered index set,

- the open sets Ω_α form a locally finite covering of Ω,

- the vector field f can be written in the form

$$f(x) = f_\alpha(x) \qquad if \qquad x \in \Omega \backslash \bigcup_{\beta>\alpha} \Omega_\beta .$$

We can now define Euler Solution:

In order to define generalized solutions mainly two approaches can be pursued. The first one consists in defining aroximate solutions by means of an algorithm and taking as generalized solutions the uniform limits of such approximate solutions. Euler and generalized sampling solutions are constructed in this way. In the second approach one associates a differential inclusion to the differential equation and defines generalized solutions as solutions of the associated differential inclusion. We discuss this approach in this section.

Definition(3.1.8):

An Euler polygonal ϵ-approximate solution associated to the Cauchy problem (3.1) and to the partition $\pi = \{t_0, t_1, \dots, t_N\}$ of $[t_0, t_0 + a]$,with $t_N = t_0 + a$ and $\mu_\pi = \max\{t_{i+1} - t_i, 0 \le i \le N - 1\} < \epsilon$, is the piecewise affine function defined by

$$\begin{cases} \varphi_\pi(t_0) = \bar{x}_0 & t \in [t_0, t_1] \\ \varphi_\pi(t) = \varphi_\pi(t_i) + \left(e_i + f(t_i, x_i(t_i) + e'_i)\right)(t - t_i) & t \in [t_i, t_{i+1}] \end{cases}$$

Where $i = 1, \dots, N$, $\|x_0 - \bar{x}_0\| < \epsilon, e_i, e'_i \in \mathbb{R}^n$ with$|e_i| < \epsilon, |e'_i| < \epsilon$.

e_i and e'_i can be seen as respectively inner and outer perturbations.

Let us remark that Euler polygonal ϵ-approximate solutions are absolutely continuous functions.

Definition(3.1.9):

A function φ: $[t_0, t_0 + a] \to \mathbb{R}^n$ is said to be

(1) an Euler solution of (3.1) if it is the uniform limit as $\epsilon \to 0$ of a sequence of Euler polygonal ϵ-approximate solutions with $\bar{x}_0 = x_0, e_i \equiv e'_i \equiv 0$

for all i;

(2) an Euler externally disturbed solution of (3.1) if it is the uniform limit as $\epsilon \to 0$ of a sequence of Euler polygonal ϵ-approximate solutions with $e'_i \equiv 0$ for all i;

(3) an Euler disturbed solution of (3.1) if it is the uniform limit as $\epsilon \to 0$ of a sequence of Euler polygonal ϵ-approximate solutions.

We denote the set of Euler solutions of (3.1) by $\mathcal{E}$, the set of Euler externally disturbed solutions by $\mathcal{E}_E$ and the set of Euler disturbed solutions with $\mathcal{E}_D$. Note that Euler disturbed solutions are sometimes addressed as weak generalized solutions (see [30]).

Obviously we have that $\mathcal{E} \subseteq \mathcal{E}_E \subseteq \mathcal{E}_D$. We mainly focus on Euler solutions.let us remark that Euler solution are interesting from mathemathical point of view, while they don't seem to have a physical meaning. Then let us point out their mathematical interest. One possible proof of Peano existence theorem of classical solutions of (3.1) is based on the construction of a sequence of Euler ϵ-approximate solutions. By means of Ascoli and Arzel`a theorem, this sequence is proved to admit a subsequence convergent to a continuous function, which is a solution of the Cauchy problem (see [64], page 36). If f is not continuous, the limit function does not necessarly verifies the equation, but it is taken as a (Euler) solution by definition. Going on this way, we prove the following theorem.

Theorem (3.1.10):

If f is bounded on the set $\mathbb{R} = \{(t, x) : t_0 \leq t \leq t_0 + a, \|x - x_0\| \leq b\}$, then a local Euler solution of (3.1) exists. Moreover it is absolutely continuous.

Proof:

Let M be such that $\|f(t, x)\| \leq M$ for all $(t, x) \in R, \delta = \min\{a, \frac{b}{M}\}$. Let π_j be a sequence of partitions of $[t_0, t_0 + \delta]$ with $\mu_{\pi_j} < \epsilon_j \to 0$, and φ_j the corresponding Euler polygonal ϵ_j-approximate solution.

Let us first remark that $(t, \varphi_j(t)) \in R$ for every j and for every $t \in [t_0, t_0 + \delta]$. In fact $\|\varphi_j(t_1) - x_0\| \leq \delta M \leq b$. From this inequality it also follows that the set of continuous functions $\{\varphi_j\}$ is equi-bounded. Let us show that it is also equi-continuous. For all δ > 0 we have that, for all $t_1, t_2 \in [t_0, t_0 + \delta]$, if $|t_1 - t_2| < \frac{\epsilon}{M}$,then $\|\varphi_j(t_1) - \varphi_j(t_2)\| \leq \delta M \leq \epsilon$ for all j. By the Ascoli and Arzel`a theorem it follows that there exists a subsequence of 'j uniformly converging to a continuous function ' defined on $[t_0, t_0 + \delta]$. Such a function is an Euler solution of (3.1) by definition.

Let us now show that φ is absolutely continuous. We have already remarked that the sequence $\{\varphi_j\}$ is equibounded, then for each $t \in [t_0, t_0 + \delta]$ the set $\{\varphi_j(t)\}$ is reletively compact. Moreover$\|\varphi_j(t)\| \leq M$ for all $t \in [t_0, t_0 + \delta]$. It follows that there exists a subsequence of $\{\varphi_j\}$ uniformly converging to an absolutely continuous function. Since we already know that the whole sequence $\{\varphi_j\}$ uniformly converges to φ, it follows that φ is absolutely continuous.

Let us now compare Euler and Carath´eodory solutions.The following examples respectively show that $\mathcal{C} \nsubseteq \mathcal{E}$and $\mathcal{E} \nsubseteq \mathcal{C}$.

Example (3.1.11):

Let us consider the Cauchy problem (3.2) with $f(x) = \frac{3}{2}x^{\frac{1}{3}}, t_0 = 0, a = 1$ and $x_0 = 0$ Carath´eodory solutions are $t^{\frac{3}{2}}, t^{-\frac{3}{2}}$ and 0,while the only Euler solution is 0.

Example (3.1.12):

Let us consider the Cauchy problem (3.2) with

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}\backslash Q \\ -1 & \text{if } x \in Q \end{cases}$$

$t_0 = 0, a = 1, x_0 = 0$ and the sequence of partitions of the interval $[0,1]$ given by $\pi_N = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}$. The corresponding Euler solution is $-t$, while the unique Carath´eodory solution is t.

Note that in the previous example the Euler solution −t does not satisfy (3.1) at a.e. $t \in [0,1]$, but, at least, it satisfies it on a dense subset of $[0,1]$.Exercise(1.6) (b) in [61] is an example of an Euler solution of a Cauchy problem of the form (3.2) on an interval I that does not satisfy the equation at any $t \in I$.

In Theorem(4.1.7) in [61], it is proved that, under the assumptions that f is continuous with respect to (t, x), $\mathcal{E} \subseteq \mathcal{C}$. Example (3.1.11) shows that $\mathcal{E} \nsubseteq \mathcal{C}$ even if f is continuous.

In order to get $\mathcal{E} \equiv \mathcal{C}$, the easiest possibility is to ask f to be continuous with respect to (t, x) and Lipschitz continuous with respect to x, in the sense of condition (3.4).In fact, in this case, there exist both an Euler and a Carath´eodory solution, which is unique. Since the Euler solution is also a Carath´eodory solution, the two must coincide. In [51] it is proved that if the system is autonomous and f is patchy, then $\mathcal{C} \equiv \mathcal{E}_E$. From this fact it follows in particular that, if f is patchy, $\mathcal{E} \subseteq \mathcal{C}$.

Now we discuss Generalized Sampling Solution:

Sampling solutions have been introduced by Krasovskii and Subbotin (see[95]) in the contest of differential games, and then used in [58] in order to prove that asymptotic controllability implies feedback stabilization. Here we consider generalized sampling solutions which are uniform limits of (not generalized) sampling solutions. Roughly speaking, generalized sampling solutions are obtained as limits of solutions of a sequence of systems in which the control is piecewise constant. The aim of the present section is to see to what extent they have sense in the general contest of discontinuous differential equations. We introduce them for systems of the form

$$\begin{cases} \dot{x} = f(t, x(t), k(t, x(t))) \\ x(t_0) = x_0 \end{cases} \qquad (3.5)$$

where $k : \mathbb{R}^{n+1} \to \mathbb{R}^m$ is, in general, a discontinuous function, not necessary to be thought as a control.

Definition(3.1.13): An ϵ-trajectory associated to the Cauchy problem (3.5) and to the partition $\pi = \{t_0, t_1, \ldots, t_N\}$ of $[t_0, t_0 + a]$ with $t_N = t_0 + a$ and $\mu_\pi < \epsilon$, is a function obtained by iteratively solving the following integral equations

$$\varphi_\pi(t) = x_0 + \int_{t_0}^{t_1} f(\tau, \varphi_\pi(\tau), k(t_0, x_0))\, d\tau \qquad t \in [t_0, t_1]$$

$$\varphi_\pi(t) = \varphi_\pi(t_i) + \int_{t_i}^{t} f\left(\tau, \varphi_\pi(\tau), k\big(t_i, \varphi_\pi(t_i)\big)\right) d\tau \quad t \in [t_i, t_{i+1}], i = 1, \ldots N-1$$

Let us remark that ϵ-trajectories do not necessarly exist, nor are unique. Nevertheless, if an ϵ-trajectory exists, then it is absolutely continuous.

Definition (3.1.14):

A function $\varphi : [t_0, t_0 + a] \to \mathbb{R}^n$ is said to be a generalized sampling solution of (3.1) if it is the uniform limit of a sequence of ϵ-trajectories as $\epsilon \to 0$.

It is important to emphasize that, in the definition of generalized sampling solution on $[t_0, t_0 + a]$, it is implicit that a sequence of ϵ-trajectories of (3.5) on $[t_0, t_0 + a]$ does exist.

We denote the set of generalized sampling solutions of (3.5) by $\mathcal{S}$. As for Euler solutions it would be possible to define also externally disturbed generalized sampling solutions and disturbed generalized sampling solutions.

Let us now state a local existence theorem for generalized sampling solutions.

Theorem(3.1.15):

Let f be defined on the set $Q = \{(t, x, u) : t_0 \le t \le t_0 + a, \|x - x_0\| \le b, u \in \mathbb{R}^m\}. If\ f$ is such that

(i) for all fixed $u \in \mathbb{R}^m$, f is measurable in t for all x and continuous in

x for a.e. t

(ii) there exists a positive summable function $m : [t_0, t_0 + a] \to \mathbb{R}$ such that $\|f(t, x, u)\| \le m(t)$ for all $(t, x, u) \in Q$

then there exists at least one local generalized sampling solution of (3.5).Moreover it is absolutely continuous.

Proof:

Let $\{\pi_j\}$ be a sequence of partitions of $[t_0, t_0 + a]$ with $\mu_{\pi_j} < \epsilon_j \to 0$. By Theorem (3.1.3), there exists $\delta > 0$ such that for each partition π_j acorresponding ϵ_j-trajectory φ_j exists (it is sufficient that $\int_{t_0}^{t_0+\delta} m(t)dt \leq b$).

Let us remark that for every $t \leq [t_0, t_0 + \delta]$ and for all u, $(t, \varphi_j(t), u) \in Q$.In fact$\left\|\varphi_j(t) - x_0\right\| \leq \left\|\int_{t_0}^{t} m(\tau)d\tau\right\| \leq \int_{t_0}^{t_0+\delta} m(\tau)d\tau \leq b$. From this inequality it also follows that the sequence of functions $\{\varphi_j\}$ is equibounded. Let us show that it is equi-continuous. Let $\epsilon > 0$ be arbitrarily fixed and let $\gamma > 0$ be such that for any $t_1, t_2 \in [t_0, t_0 + a]$, if $| t_1 - t_2| < \gamma$ then $\int_{t_1}^{t_2} m(\tau)d\tau \leq \epsilon$ (such a γ exists because of the absolute continuity of Lebesgue integral). Let us then consider $t_1, t_2 \in [t_0, t_0 + a]$ such that$| t_1 - t_2| < \gamma$. We get that $\left\|\varphi_j(t_1) - \varphi_j(t_2)\right\| \leq \int_{t_1}^{t_2} m(\tau)d\tau \leq \epsilon$.

By the Ascoli and Arzel`a theorem it follows that the sequence $\{\varphi_j\}$ admits a subsequence uniformly converging to a continuous function φ, that is a generalized sampling solution of (3.5) by definition. As in Theorem (3.1.10), the absolute continuity of φ follows by Theorem (4)in [74].

Remark (3.1.16):

Analogous theorems could be stated if f verifies some conditions which guarantee the existence of ϵ-trajectories for any sequence of partitions of an interval $[t_0, t_0 + \delta] \subseteq I$ for some φ and for every fixed u.

Remark(3.1.17):

If we assume that the feedback law k is locally bounded and $M > 0$ is such that $\|k(t, x) - k(t_0, x_0)\| \leq M$ for all $t \in [t_0, t_0 + a]$ and for all x such that $\|x - x_0\| \leq b$, then hypothesis (ii) in the previous theorem can be weakened to the following:

(iibis) there exists a summable function $m : [t_0, t_0 + a] \leq \mathbb{R}$ such that

$\|f(t, x, u)\| \leq m(t)$ for all $(t, x, u) \in Q$ such that $\|u - k(t_0, x_0)\| \leq M$

As for Euler solutions, the existence a.e. of the derivative does not imply that a generalized sampling solution satisfies(3.5) a.e..We can reinterpret Example (3.1.12) in terms of generalized sampling solutions by posing $f(x) = k(x)$.

We get that $-t$ is a generalized sampling solution, but not a Carath´eodory solution, then $\mathcal{S} \nsubseteq \mathcal{C}$. Analogously, by reinterpreting Example(3.1.11), we get $\mathcal{C} \nsubseteq \mathcal{S}$. This can be also seen by means of the example in [51].

It is then natural to look for conditions which guarantee $\mathcal{S} \nsubseteq \mathcal{C}$. The following theorem is analogous to Theorem (4.1.7) in [61], page 183, for Euler solutions.

Theorem(3.1.18):

If f is continuous with respect to (t, x, u) on Q,there exists a positive and summable function $m: [t_0, t_0 + a]$ such that $\|f(t, x, u)\| \leq m(t)$ for all $(t, x, u) \in Q$ and k is continuous with respect to (t, x), then every generalized sampling solution of (3.5) on $[t_0, t_0 + a]$ is also a Carath´eodory solution.

Proof:

Let φ be a generalized sampling solution of (3.5),$\{\varphi_j\}$ a sequence of φ_j-trajectories corresponding to the sequence $\{\varphi_j\}$ of partitions of the interval $[t_0, t_0 + a]$ with $\mu_{\pi_j} < \epsilon_j \to 0$, such that $\varphi_j \to \varphi$ uniformly.

Let us posit $k_j(t) = k(t_i , \varphi_j(t_i)), t \in [t_i, t_{i+1}]$. Since k is continuous in (t, x) and φ_j is continuous in t for all j, we have that $k_j(t) \to k(t, \varphi(t))$ and also $f(t, \varphi_j(t), k_j(t)) \to f(t, \varphi(t), k(t, \varphi(t)))$ f every Moreover $\|f(t, \varphi_j(t), k_j(t))\| \leq m(t), then\ \varphi_j(t) \to x_0 + \int_{t_0}^{t} f\left(\tau, \varphi_j(\tau), k_j(\tau)\right) d\tau.$ On the other hand we know that $\varphi_j \to \varphi$ uniformly, then $\varphi(t) = x_0 + \int_{t_0}^{t} f(\tau, \varphi(\tau), k(t, \varphi(t)))d\tau$ for all t, i.e.φ is a Carath´eodory solution of (3.5).

Let us end this section by comparing generalized sampling solutions with Euler solutions. In general, Euler solutions can be seen as a particular case of generalized sampling solutions, when the equation in the Cauchy problemis given by $\dot{x} = k(t, x)$. Nevertheless these two kinds of generalized solutions are not really tightly connected. The example in [51], if reinterpreted in terms of equation (3.5), shows that $\mathcal{E} \nsubseteq \mathcal{S}$. Moreover the following example shows that $\mathcal{S} \nsubseteq \mathcal{E}$.

Example(3.1.19):

Let us consider the Cauchy problem (3.5) with $f\left(t, x, k(t, x)\right) = \frac{3}{2} k(x) x^{\frac{1}{3}}$, where k is defined by

$$k(x) = \begin{cases} 1 & \text{if } x \in \{0\} \cup \{(\mathbb{R} \backslash Q) \cap [0,1]\} \\ 0 & \text{if } x \in Q \cap (0,1] \end{cases}$$

$t_0 = 0, a = \pi$ and $x_0 = 0$. The unique Euler solution is 0.

On the other hand, by considering the sequence of partitions $\pi_j = \{0, \frac{\pi}{j}, \frac{2\pi}{j}, \dots, (j-1)\frac{\pi}{j}, \pi\}$, we get that, besides 0, also $t^{\frac{3}{2}}$ and $t^{-\frac{3}{2}}$ are generalized sampling solutions.

Now we will discuss the Krasovskii and Filippov Solutions:

The idea behind the concepts of Krasovskii and Filippov solutions is that the value of a solution at a certain point should be determined by the behavior of its derivative in the nearby points. Moreover the definition of Filippov solution suggests that possible misbehaviour of the derivative on null measure sets could be ignored.

More precisely, if we denote by $\overline{co}$ the convex closure and by μ the usual Lebesgue measure in $\mathbb{R}^n$, we have the following definitions.

Definition(3.1.20):

An absolutely continuous function $\varphi : [t_0, t_0 + a] \to \mathbb{R}^n$ is said to be

- a Krasovskii solution of (3.1) if it is a solution of the differential inclusion

$$\dot{x} \in Kf(t, x) = \cap_{\delta>0} \overline{co}\, f\big(t, B(x, \delta)\big) \tag{3.6}$$

i.e. φ satisfies (3.6)for $t \in [t_0, t_0 + a]$,

- a Filippov solution to (3.1) if it is a solution of the differential inclusion

$$\dot{x} \in Ff(t, x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}\, f(t, B(x, \delta)\backslash N) \tag{3.7}$$

i.e. φ satisfies (3.7)for a.e $t \in [t_0, t_0 + a]$.

We denote the sets of Krasovskii and Filippov solutions to (3.1) respectively by $\mathcal{K}$ and $\mathcal{F}$.

Let us consider the set-valued functions $f, Ff : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. If f is locally bounded,then Kf and Ff are both upper semicontinuous and have nonempty, compact and convex values. The same is still true for Ff if f is just locally essentially bounded. From this remark and a classical existence theorem for differential inclusion (see [74], page 97), the local existence theorem for Krasovskii and Filippov solutions follows.

Theorem(3.1.21):

If $f : t \in [t_0, t_0 + a] \times \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded (locally essentially bounded), then a local Krasovskii (Filippov) solution of (3.1) exists.

Obviously, if f is locally bounded, $\mathcal{F} \subseteq \mathcal{K}$. An interesting condition in order to get $\mathcal{F} = \mathcal{K}$ is given in [101] (Lemma 2.8).We report it for autonomous systems, but it can be generalized to nonautonomous ones.

Proposition(3.1.22):

If there exists a disjoint decomposition $\mathbb{R}^n = \cup\Omega_i$ with $\Omega_i \subset \overline{\text{Int}\Omega_i}$ and continuous functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $f = f_i$ on Ω_i, then each Krasovskii solution of (3.2) is a Filippov solution, i.e. , $\mathcal{K} \subseteq \mathcal{F}$.

If f is continuous with respect to x, then $Kf(t,x) = Ff(t,x) = \{f(t,x)\}$,so that $\mathcal{K} \equiv \mathcal{F} \equiv \mathcal{C}$. Let us remark that this does not always occur when Carath´eodory solutions exist. The example in [AB] shows that $\mathcal{K} \nsubseteq \mathcal{C}$ and $\mathcal{F} \nsubseteq \mathcal{C}$ (note that in this example, thanks to Proposition (1.1.22), $\mathcal{K} = \mathcal{F}$). As far as the opposite inclusions are concerned, we have that $\mathcal{C} \subseteq \mathcal{K}$, while $\mathcal{C} \nsubseteq \mathcal{F}$. The inclusion $\mathcal{C} \subseteq \mathcal{K}$ is due to the fact that $f(t,x) \in f(t, B(x,\delta))$ for every (t,x) and $\delta > 0$ and then $f(t,x) \in Kf(t,x)$. On the other hand $\mathcal{C} \nsubseteq \mathcal{F}$ is shown by the following example.

Example(3.1.23):

Let us consider the Cauchy problem (3.2) with

$$f(x_1, x_2) = \begin{cases} (0,0) & \text{if } (x_1, x_2) = (0,0) \\ (1,0) & \text{if } (x_1, x_2) \neq (0,0) \end{cases}$$

$t_0 = 0, (x_{1_0}, x_{2_0}) = (0,0)$. The function $x_1(t) = 0, x_2(t) = 0$ is a Carath´eodory solution of the Cauchy problem(3.5).On the other hand, since $F(x_1, x_2) = (1,0)$ for each $(x_1, x_2) \in \mathbb{R}^2$, the unique Filippov solution is $x_1(t) = t, x_2(t) = 0$.

Let us now compare Krasovskii and Filippov solutions with Euler solutions. The example in [2] shows that $\mathcal{F} \nsubseteq \mathcal{E}$. In the same example $\mathcal{K} = \mathcal{F}$, then also $\mathcal{K} \nsubseteq \mathcal{E}$. The opposite inclusions don't hold too. This can be seen by means of the following example.

Example(3.1.24):

Let us consider the Cauchy problem (3.1) with

$$f(t,x) = \begin{cases} 1 & \text{if } t \in \mathbb{R}\backslash Q \\ -1 & \text{if } t \in Q \end{cases}$$

$t_0 = 0, a = 1, x_0 = 0$ and the sequence of partitions of the interval $[0,1]$ given by $\pi_N = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}$. The corresponding Euler solution is $-t$, while the unique both Krasovskii and Filippov solution is t.

Nevertheless, as mentioned in [16], if the system is autonomous it may be proved that $\mathcal{E}_D \equiv \mathcal{K}$, then also $\mathcal{E} \subseteq \mathcal{K}$. On the other hand, even for the autonomous case, $\mathcal{E} \nsubseteq \mathcal{F}$, as it can be seen by means of Example(3.1.12): $-t$ is an Euler solution, while the only Filippov solution is t.

We now compare Krasovskii and Filippov solutions with generalized sampling solutions.The example in [2], if reinterpreted in terms of the Cauchy problem(3.5), shows that $\mathcal{K} = \mathcal{F} \nsubseteq \mathcal{S}$. Moreover,if in Example(3.1.24) we pose $f(t,x) = k(t,x)$, we also get that $\mathcal{S} \nsubseteq \mathcal{K}$ and $\mathcal{S} \nsubseteq \mathcal{F}$. Finally, in the autonomous case, we have that $\mathcal{S} \subseteq \mathcal{K}$ (see [121], Lemma 2.9).

Secation(3.2): Elements in nonsmooth analysis:

In our study of stability and stabilization we need to deal with nonsmooth Lyapunov functions. The central problem is then to have conditions which guarantee the decrease of functions which, in general, are not differentiable.Such conditions can be based on different kinds of generalized derivatives and gradients. We only introduce generalized derivatives and gradients that are needed in the following chapters.

We can now define functions of one variable:

.Let I be any interval of $\mathbb{R}$ and $V : I \to \mathbb{R}$. We recall some basic results (see[45], page 207).

Proposition (3.2.1):

Let V be absolutely continuous on each compact subinterval of I. V is non-increasing on I if and only if $\dot{V}(t) \leq 0$ for a.e. $t \in I$.

If V is just continuous its decrease can be characterized by means of Dini derivatives. Let us then recall some definitions. We denote $r(h,t) = \frac{V(h+t)-V(t)}{h}$.

upper right Dini derivative: $\overline{D+V}(t) = \limsup_{h\downarrow 0} r(h,t)$

upper left Dini derivative: $\overline{D-V}(t) = \limsup_{h\uparrow 0} r(h,t)$

lower right Dini derivative: $\underline{D+V}(t) = \lim\inf_{h\downarrow 0} r(h,t)$

lower left Dini derivative: $\underline{D-V}(t) = \lim\inf f_{h\uparrow 0}\, r(h,t)$.

Proposition (3.2.2):

Let V be continuous. Then the following statements are equivalent:

(i) V is non − increasing on I

(ii) $\overline{D+V}(t)$ _ 0 for all $t \in I$

(iii) $\overline{D-V}(t)$ _ 0 for all $t \in I$

(iv) $\underline{D+V}(t)$ _ 0 for all $t \in I$

(v) $\underline{D-V}(t)$ _ 0 for all $t \in I$.

Also in the case V is just either lower semi-continuous or upper semicontinuous, criteria for the decrease of V based on Dini derivatives can be formulated.

Now we investigate the functions of several variables:generalized directional derivatives and gradients:

A fundamental notion associated to functions of several variables is that of directional derivative. Many different definitions can be given for nonsmooth functions and, in connection with them, also different notions of generalized gradients come out.

Let $V : \mathbb{R}^n \to \mathbb{R}$ and $(h,x,v) = \frac{V(x+hv)-V(x)}{h}$. Dini generalized directional derivatives are

upper right Dini directional derivative: $\overline{D+V}(x,v) = \lim\sup_{h\downarrow 0} R(h,x,v)$

upper left Dini directional derivative: $\overline{D-V}(x,v) = \lim\sup_{h\uparrow 0} R(h,x,v)$

lower right Dini directional derivative: $\underline{D+V}(x,v) = \lim\inf_{h\downarrow 0} R(h,x,v)$

lower left Dini directional derivative: $\underline{D-V}(x,v) = \lim\inf f_{h\uparrow 0}\, R(h,x,v)$.

Other generalized directional derivatives are contingent directional derivatives. The upper right contingent derivative is defined as

$$\overline{D^+}_K V(x,v) = \lim_{h\downarrow 0} \sup_{w\to v} R(h,x,w)$$

The other contingent derivatives can be defined analogously. If V is locally Lipschitz continuous contingent derivatives and Dini derivatives coincide.

We mainly focus on Clarke generalized derivatives and gradient (see [54]):

upper Clarke directional derivative: $\overline{D}_C \mathrm{V}\ (x, v) = \lim\ \sup_{h \to 0\ y \to x} R(h, y, v)$

lower Clarke directional derivative: $\underline{D}_C V\ (x, v) = \lim\ \inf_{h \to 0\ y \to x} R(h, y, v)$

From the definitions it follows that

$$\underline{D}CV\ (x, v) \leq \underline{D^+}V\ (x, v) \leq \overline{D^+}V\ (x, v) \leq \overline{D}_C V\ (x, v)$$

Dini,contingent and also Clarke directional derivatives are positively homogeneous with respect to v. Moreover Clarke directional derivative is subadditive (and hence convex) as a function of v.

Clarke generalized gradient is defined by means of Clarke directional derivatives as

$$\partial_C V\ (x) = \left\{p \in \mathbb{R}^n : \underline{D}_C V\ (x, v) \leq p \cdot v \leq \overline{D}_C V\ (x, v) \forall v \in \mathbb{R}^n\right\}$$

In a similar way, by means of other generalized directional derivatives, other generalized gradients can be defined.

For each x, the set $\partial_C V\ (x)$ is convex and closed.

The connection between Clarke generalized derivatives and gradient can also be seen by means of the following equalities:

$\overline{D}_C V\ (x, v) = \sup\{p \cdot v, p \in \partial_C V\ (x)\}$ $\quad$ $\underline{D}_C V\ (x, v) = \inf\{p \cdot v, p \in \partial_C V\ (x)\}$

From these it follows that $\underline{D}_C V\ (x, v) = -\overline{D}_C(x, -v)$.

Very important properties of Clarke generalized directional derivatives and gradient arise if V is locally Lipschitz continuous. Let us recall that in this case, by the well known Rademacher theorem, the gradient ∇V of V exists a.e.. Moreover for every $v \in \mathbb{R}^n \nabla V(\cdot) \cdot v$ is a measurable function (see [92], page 83). Let us then denote by N the set of zero measure where the gradient of V does not exist and let S be any subset of $\mathbb{R}^n$ of zero measure. We have that

$$\partial_C V(x) = co\left\{\lim_{i \to +\infty} \nabla V\ (x_i) : x_i \to x, x_i \notin S\ x_i \notin N\right\} \qquad (3.8)$$

Still if V is Lipschitz continuous, its gradient is bounded, then $\partial_C V\ (t, x)$ is not just closed but also bounded, and then compact. Thanks to this characterization of $\partial_C V\ (x)$ it can be proved that, if V is C^1, then $\partial_C V$ (x) $= \{\nabla V\ (x)\}$. Still in the case V is locally Lipschitz continuous, if U_x is a compact neighbourhood of x and L_x is the Lipschitz constant of V on U_x, we have that

$$-L_x\|v\| \leq \underline{D}_C V\ (y, v) \leq \overline{D}_C V\ (y, v) \leq L_x\|v\| \quad \forall y \in U_x \ \forall v \in \mathbb{R}^n.$$

Concluding this paragraph let us recall how Clarke upper directional derivative is used in the definition of C-regular functions.

Definition(3.2.3):

$V : \mathbb{R}^n \to \mathbb{R}$ is said to be C-regular at x if for every $v \in \mathbb{R}^n$

(i) there exists the usual right directional derivative

$$D^+V\ (x, v) = \lim_{h\downarrow 0} R(h, x, v)$$

(ii) $\overline{D}_C V\ (x, v) = D^+V\ (x, v)$.

V is said to be C-regular if it is regular at each $x \in \mathbb{R}^n$.

Let us remark that a convex function is not only Lipschitz continuous, but it is also regular.

We can now define A chain rule:

We now restrict our attention to functions which are the composition of a locally Lipschitz continuous funtion $V : \mathbb{R}^n \to \mathbb{R}$ and an absolute continuous function $\psi: \mathbb{R} \to \mathbb{R}^n$ First of all, let us remark that, in these hypothesis, $V \circ \psi : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function and then its derivative exists a.e.. In [75] the authors prove a chain rule, that we now state in the particular case that is of interest for us.

Proposition(3.2.4):

If $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and $\psi: \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then for a.e. t there exists $p_0 \in \partial_C V\left(\psi(t)\right)$ such that
$\frac{d}{dx} V\ (\psi\ (t)) = p_0 \cdot \dot{\psi}\ (t)$.

In this context it becomes very interesting the notion of healthy function introduced in [94], and that we slightly modify for functions V having an explicit dependence on time.

Definition(3.2.5):

We say that a locally Lipschitz function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is healthy if for every absolute continuous function $\varphi: \mathbb{R} \to \mathbb{R}^n$and for a.e. t the set $\partial_C V\left(t, \varphi(t)\right)$ is a subset of an affine subspace orthogonal to $(1, \dot{\varphi}\ (t))$.

Let us remark that C-regular functions (and then also convex functions) are healthy. The interest in healthy functions is motivated by the following proposition, that can be seen as a chain rule for healthy functions and is easily proved by means of Proposition(3.2.4).

Definition(3.2.6):

If $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is healthy and $\psi : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then the set $\{ p \cdot (1, \dot{\varphi}\ (t)), p \in \partial_C V\left(t, \varphi(t)\right)\}$ is reduced to the singleton $\left\{\frac{d}{dt} V\left(t, \varphi(t)\right)\right\}$ $for\ a.e.\ t.$

Proof :

Let $\psi : \mathbb{R} \to \mathbb{R}^{n+1}$ be defined by $\psi\ (t) = (t, \varphi(t). \psi$ is absolutely continuous and the same is true for $V \circ \psi(t) = V\left(t, \varphi(t)\right)$.Then $\frac{d}{dt} V \circ \psi(t) = \frac{d}{dt} V\left(t, \varphi(t)\right)$exists a.e.. By Proposition(3.2.4) we have that for a.e. t there exists $p_0 \in \partial_C V\left(t, \varphi(t)\right)$ such that $\frac{d}{dt} V \circ \psi(t) = p_0 \cdot \dot{\psi}\ (t) = p_0 \cdot (1, \dot{\varphi}\ (t))$ and, by the definition of healthy function, it follows that $\frac{d}{dt} V\left(t, \varphi(t)\right) = p \cdot (1, \dot{\varphi}\ (t)$ for all $p \in \partial_C V\left(t, \varphi(t)\right)$.

Now we discuss the monotonicity along solutions of differential inclusions:

In the following we still consider functions which are the composition of a healthy function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ and an absolutely continuous function

$\varphi : \mathbb{R} \to \mathbb{R}^n$, but in the particular case φ is a solution of a differential

inclusion of the form

$$\dot{x} \in F(t, x) \tag{3.9}$$

with the initial condition $x(t_0) = x_0$, where $F : \mathbb{R}^{n+1} \to 2^{\mathbb{R}^n} \backslash \emptyset$ is an upper semi-continuous set-valued function with compact and convex values.

Definition (3.2.7):

A function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is said to be decreasing along F if for each solution $\varphi(t)$of (3.9) on I and all $t_1, t_2 \in I$ one has

$$t_1 \leq t_2 \Rightarrow V(t_2, \varphi(t_2)) \leq V(t_1, \varphi(t_1)) \qquad (3.10)$$

Note that a function V with the property (3.10) is said to be a Lyapunov function for (3.9). We don't emphazise the point of view of Lyapunov functions now because we will focus on them in this Chapter, in connection with the problem of stability of differential inclusions.

Let us remark that the definition of monotonicity we have given is "strong", in the sense that it refers to all solutions of (3.9).One could analogously define "weak" monotonicity, by refering only to some solutions of (3.9).

In the case V is C^1 the classical condition which guarantees its decrease along F is that for all t and all x

$$D^+V((t,x),(1,v)) = \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot v \leq 0 \, \forall v \in F(t,x)$$

It is from this condition that one gets inspiration for the nonsmooth case If V locally Lipschitz continuous, a condition based on Dini derivatives which implies (3.10) is that for a.e.t and all x (see [12]):

$$D^+V((t,x)(1,v)) \leq 0 \qquad \forall v \in F(t,x) \qquad (3.11)$$

Note that (3.10) does not imply (3.11). This can be seen by means of

Example(3.3.19) in this Chapter.

Since $D^+V\big((t,x)(1,v)\big) \leq \overline{D}_C V\big((t,x)(1,v)\big) = \max\{p \cdot (1,v), p \in \partial_C V((t,x)\}$ for all (t,x) and for all $v \in F(t,x)$, also the condition

$$\text{for a.e. } t \; \forall x \; \forall v \in F(t,x) \max\{p \cdot (1,v), p \in \partial_C V(t,x)\} \leq 0 \qquad (3.12)$$

guarantees the monotonicity of V along F. The advantage of this last condition is that, thanks to the characterization of Clarke generalized gradient for locally Lipschitz continuous functions, it can be relatively easily computed. On the other hand it is not very sharp, except for the case $F(t,x) = \{F(x)\}$, where F is continuous. In this case we have the following proposition (see[10]).

Proposition(3.2.8):

$F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \backslash \emptyset$ be continuous and let $V : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. V decreases along all solutions of

$$\dot{x} \in F(x) \tag{3.13}$$

if and anly if

$$\forall x \in \mathbb{R}^n \, \forall v \in F(x) \ \max\{p \cdot v, p \in \partial_C V(x)\} \leq 0$$

Proof :

The proof that if for all $x \in \mathbb{R}^n$ and for all $v \in F(x)$ one has $\max\{p \cdot v, p \in \partial_C V(x)\} \leq 0$ then V decreases along all solutions of (3.13) trivially follows from the previous discussion.

Let us prove that if V decreases along all solutions of (3.13) then for all x and for all $p \in \partial_C V(x)$ one has $p \cdot v \leq 0$. Let us suppose by contradiction that there exists a $x_0 \in \mathbb{R}^n, v_0 \in F(x_0)$ and $p_0 \in \partial_C V(x_0)$ such that $p_0 \cdot v_0 > 0$.By (3.8) there exist

a) $\lambda_1, \dots \lambda_m > 0$ such that $\sum_{i=1}^{m} \lambda_i = 1$

b) $p^{(1)}, \dots p^{(m)} \in \mathbb{R}^nand\left\{x_k^{(1)}\right\} \subset \mathbb{R}^n, \dots, \left\{x_k^{(m)}\right\} \subset \mathbb{R}^n$ such that for all $i \in \{1, \dots, m\}$ and for all k there exists $\nabla V(x_k^{(i)})$,

$$p^{(i)} = \lim_{k \to +\infty} \nabla V\left(x_k^{(i)}\right) \text{ and } \lim_{k \to +\infty} x_k^{(i)} = x_0$$

Such that $p_0 = \lambda_1 p^{(1)} + \cdots + \lambda_m p^{(m)}$. Since $p_0 . v_0 = \lambda_1 p^{(1)} . v_0 + \cdots + \lambda_m p^{(m)} . v_0 > 0$, there exists $j \in \{1, \dots m\}$ such that $p^{(j)} . v_0 > 0$. Let $\left\{x_k^{(j)}\right\}$ be a sequence as in (b).

Let us fix $\epsilon < \min\left\{1, \frac{p^{(j)} . v_0}{2(\|v_0\| + \| p^{(i)}\| + 1)}\right\}$.

Since $p^{(j)} = \lim_{k \to +\infty} \nabla V\left(x_k^{(j)}\right)$, there exists $w_k \in B(0,1)$ such that $\nabla V\left(x_k^{(j)}\right) = p^{(j)} + \epsilon w_k$where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ centered at the origin.

Moreover $x_k^{(j)} \to x_0$ and F is continuous, then there exists $\tilde{k}$ such that for all $k > \tilde{k}$ there exist $v_k \in F\left(x_k^{(j)}\right)$ and $z_k \in B(0,1)$, such that $v_k = v_0 + \epsilon z_k$. Then for

all $k > \max\{\bar{k}, \tilde{k}\}$ there exist $v_k \in F\left(x_k^{(j)}\right)$ and $w_k, z_k \in B(0,1)$ such that $\nabla V\left(x_k^{(j)}\right) \cdot v_k = \left(p^{(j)} + \epsilon w_k\right) \cdot (v_0 + \epsilon z_k) = p^{(j)} \cdot v_0 - \epsilon|w_k . v_0 + z_k .$
$p^{(j)} + \epsilon w_k \cdot z_k| \geq p^{(j)} \cdot v_0 - \epsilon\left(\|v_0\| + \| p^{(i)}\| + 1\right) > \frac{p^{(j)} \cdot v_0}{2} > 0.$

Let us fix $K > \max\{\bar{k}, \tilde{k}\}$ and let us consider the solution $\varphi(t)$ of (3.2) with the initial conditions $\varphi(t_0) = x_K^{(j)}$ and $\dot{\varphi}(t_0) = v_K^{(j)}$. The existence of such a solution is guaranteed by Theorem(2.3)in [105].

Since V decreases along solutions of (3.13), then $\frac{d}{dt}V\,(\varphi(t)) \leq 0$ a.e., i.e. on the set where $V \circ \varphi$ is differentiable. In particular we have that

$\frac{d}{dt}V\,(\varphi(t_0)) \leq 0$. On the other hand
$\frac{d}{dt}V\,(\varphi(t_0) = \nabla V\,(\varphi(t_0) \cdot v_K = \nabla V\left(x_K^{(j)}\right) \cdot v_K > 0$, that is a contradiction.

In the general case, in order to get a condition sharper than (3.12), we need to define the set-valued derivative of V with respect to (3.9):

$$\bar{V}^{(3.9)}(t,x) = \{a \in \mathbb{R} : \exists v \in F(t,x) \text{such that } p \cdot (1,v) = a \quad \forall p \in \partial_C V(t,x)\}.$$

Remark(3.2.9):

By Proposition(3.2.6), if $\varphi(t)$ is any solution of (3.9) we have that
$\frac{d}{dt}V\left(t,\varphi(t)\right) \in \bar{V}^{(3.9)}\left(t,\varphi(t)\right)$ for a.e.t.

Lemma(3.2.10):

Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be a Lipschitz continuous function. For each fixed $(t,x) \in \mathbb{R}^{n+1}$the set $\bar{V}^{(3.9)}(t,x)$ is a closed and bounded interval, possibly empty.

Moreover, if V is differentiable, then

$$\bar{V}^{(3,9)}(t,x) = \left\{\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot v, v \in F(t,x)\right\}.$$

Proof:

We first prove that $\bar{V}^{(3.9)}(t,x)$ is closed. Let $\{a_n\} \subset \bar{V}^{(3.9)}(t,x)$,$a_n \to a$. For each n there exists $v_n \in F(t,x)$ such that $p \cdot (1, v_n) = a_n$ for all $p \in \partial_C V\,(t,x)$. Since $F(t,x)$ is compact there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ converging to

$v \in F(t,x)$. We get that $a_{n_j} = p \cdot (1, v_{n_j}) \to p \cdot v$.By the uniqueness of the limit we have $p \cdot (1, v) = a \in \bar{V}^{(3.9)}(t,x)$.

Let us now prove that $\bar{V}^{(3.9)}(t,x)$ is bounded. Let $U_{t,x}$ be a compact neighbourhood of (t,x), $L_{t,x}$ be the Lipschitz constant of V on $U_{t,x}$ and $M_{t,x}$be such that $\|(1,v)\| \le M_{t,x}$for all $v \in F(t,x)$. Let $a \in \bar{V}^{(3.9)}(t,x)$.Since there exists $v \in F(t,x)$ such that $a = p \cdot (1,v)$ for all $p \in \partial_C V\,(t,x)$, we get that $|a| \le \|p\|\|(1,v)\| \le L_{t,x}M_{t,x}$.

Let us show that $\bar{V}^{(3.9)}(t,x)$ is convex. Let $a_1, a_2 \in \bar{V}^{(3.9)}(t,x))$. There exist $v_1, v_2 \in F(t,x)$ such that $a_1, = p \cdot (1, v_1)$ and $a_2 = p \cdot (1, v_2)$ for all $p \in \partial_C V\,(t,x)$.Let $\tau \in [0,1]$ and let us consider $v = \tau v_1 + (1 - \tau)v_2$.Since $F(t,x)$ is convex, $v \in F(t,x)$. For all $p \in \partial_C V\,(t,x)$ we have that $p \cdot v = \tau a_1 + (1 - \tau)a_2$, so that $\tau a_1 + (1 - \tau)a_2 \in \bar{V}^{(3.9)}(t,x)$.

Finally $\bar{V}^{(3.9)}(t,x) = \left\{\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot v, v \in F(t,x)\right\}$ is an immediate consequence of the fact that, if V is differentiable, then $\partial_C V\,(t,x) = \left\{\frac{\partial V}{\partial t}(t,x), \nabla V(t,x)\right\}$.

Proposition(3.2.11):

If $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is locally Lipschitz continuous then for all $(t,x) \in \mathbb{R}^{n+1}$

$$\max \bar{V}^{(2,2)}(t,x) \le \max_{v \in F(t,x)} \underline{D^+}\, V\big((t,x)(1,v)\big) \tag{3.14}$$

Proof:

Let $\bar{a} = \max \bar{V}^{(3.9)}(t,x)$ There exists $\bar{v} \in F(t,x)$ such that $p \cdot \bar{v} = \bar{a}$ for all $p \in \partial_C V\,(t,x)$,so that

$$\bar{a} = \underline{D_C} V\big((t,x),(1,v)\big) \le \underline{D^+}\, V\big((t,x),(1,v)\big) \le \max_{v \in F(t,x)} \underline{D^+}\, V\big((t,x),(1,v)\big).$$

Proposition(3.2.12):

If $V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a healthy function and $\max \bar{V}^{(3.9)}(t,x) \le 0$ for a. e. t and all x, then V decreases along F .

Proof:

Let $\varphi(t)$ be any solution of (3.9) and let us consider the absolute continuous function $V\ (t,\varphi(t))$. By Proposition (3.2.6) the set $\{p\ \cdot\ (1,\dot{\varphi}(t)), p\ \in \partial_C V\ (t,\varphi(t))\}$ reduces to the singleton $\left\{\frac{dV}{dt}(t,x)\right\}$ for a.e. t.

Since $\dot{\varphi}(t) \in F\big(t,\varphi(t)\big)$ for a.e. t, we have .
$\left\{\frac{dV}{dt}(t,x)\right\}\{p\ \cdot\ \big(1,\dot{\varphi}(t)\big), p\ \in \partial_C V\ \big(t,\varphi(t)\big)\}\ \{a \in \mathbb{R} :$ $\exists v F\big(t,\varphi(t)\big)$ such that $p.v = a \forall p \in F\big(t,\varphi(t)\big)\} = \bar{V}^{(2,2)}(t,x)$.Then, if $\max \bar{V}^{(3.9)}(t,x) \leq 0$ for a.e. t and for all x,we get the $\frac{dV}{dt}\big(t,\varphi(t)\big) \leq 0$ for a.e t. Finally Proposition(3.2.1), implies that V decreases along $\varphi(t)$.

Remark(3.2.13):

Let $\varphi(t)$ be any solution of (1.9). $\bar{V}^{(3.9)}\big(t,\varphi(t)\big) = \emptyset$ can occur only on a zero measure set. In fact for a.e. t there exists

$\frac{dV}{dt}\big(t,\varphi(t)\big) \in \bar{V}^{(3.9)}\big(t,\varphi(t)\big)$ so that $\bar{V}^{(3.9)}\big(t,\varphi(t)\big) \neq \emptyset$ for a.e. t. We can then extend theconclusion of Proposition (1.2.12) to the case $\bar{V}^{(3.9)}(t,x) = \emptyset$ by posing $\max \bar{V}^{(3.9)}(t,x) = -\infty$ if $\bar{V}^{(3.9)}(t,x) = \emptyset$.

Remark(3.2.14):

An analogous version of Proposition(3.2.12)was given in[37] for Cregularfunctions. The set-valued derivative used is slightly different and, in general, it is a set larger than $\bar{V}^{(3.9)}$. We show this by means of Example(3.3.8)in this Chapter.

Remark(3.2.15):

The converse of inequality(3.14) does not hold (see Example(3.3.8) in this chapter). This means that, if V is healthy, the stability criterion based on $\bar{V}^{(3.9)}$ works better than the criterion based on Dini lower right derivative.

Remark(3.2.16):

It is important to emphasize that, if instead of the differential inclusion(3.9) we consider the autonomous differential inclusion(3.13) and Lyapunov functions $V : \mathbb{R}^n \rightarrow \mathbb{R}$ not depending on time, we obtain results perfectly analogous to those described in this section (see [2])

Section(3-3):Stability of differential inclusions

The problem of stability of differential inclusions is of primary interest for us.In fact, as we have seen in this Chapter, discontinuous differential equations are often interpreted in terms of differential inclusions, so that the study of stability of discontinuous differential equations can coincide with the study of stability of differential inclusions. Since in general a differential inclusion has not a unique solution, the stability property is usually said to be strong or weak according to the fact that it refers to all or just some of its solutions.

We only consider strong stability, so we omit to mention the adjective strong in the following. But let us define stability precisely.

Definition(3.3.1):

The differential inclusion (3.9) is said to be stable at $x = 0$ if for all $\epsilon > 0$ there exists$\delta > 0$ such that for each initial condition $(t_0, x_0)$and each solution $\varphi(t)$of (3.9) such that $\varphi(t_0)= x_0$,

$$\|x_0\| < \delta \Rightarrow \|\varphi(t)\| < \epsilon \forall t \leq t_0.$$

More precisely this concept of stability is uniform, in the sense that δ does not depend on t_0.

Note that if the differential inclusion (3.9) is stable at $x = 0$, then $x = 0$ is an equilibrium point for it, i.e. $0 \in F(t, 0)$ for all $t \geq t_0$. In fact, if $x_0 = 0$,then $\|x_0\| < \delta$ for all $\delta$and , if $\varphi(t)$ is a solution of (3.9) with $\varphi(t_0)$= 0,then for all ϵ one has $\|\varphi(t)\| < \epsilon$ for all $t \geq t_0$, i.e. $\varphi(t) = 0$. Since $\varphi(t) = 0$ is a solution of (3.9) and $\dot{\varphi}(t) \equiv 0$, we get that $0 \in F(t, 0)$ for all t. In this way, we have also proved that$\varphi(t) \equiv 0$ is the unique solution of (3.9) such that $\varphi(t_0)$= 0.

we can now define the Lyapunov's direct method:

Lyapunov's direct method (also called Lyapunov's second method), originated in order to study stability of differential equations, but it can be also successfully applied to differential inclusions. It makes it possible to investigate the stability of (3.9) without knowing the explicit form of its solutions, but just using the differential inclusion itself. The method is based on the knowledge of Lyapunov functions, which can be seen as a generalization of the concept of energy.

Definition(3.3.2):

$V: \mathbb{R}^n \to \mathbb{R}$ is said to be a Lyapunov function for (3.9) $\varphi(t)$ if for each solution of (3.9) on $I \subseteq \mathbb{R}$ and for all $t_1, t_2 \in I$ condition (3.10) holds.

Let us emphasize that we always consider differential inclusions of the form(3.9) where F is an upper semi-continuous set-valued map with nonempty,compact and convex values. In these hypothesis the existence of at least one solution of (3.9) is ensured (see, e.g., [74], page 37).

A well known version of first Lyapunov theorem for differential inclusions is the following. Because of its importance we also prove it.

Theorem(3.3.3):

If there exist a Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}$ for (3.9) and two continuous, strictly increasing functions $a, b: \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) $a(0) = b(0) = 0$ and $a(r) > 0$ for $r > 0$

(ii) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$ for all (t, x) then (1-9) is stable at $x = 0$.

Remark(3.3.4):

Note that hypothesis (i) and (ii) imply that

1) $V(t, x) \geq 0$ for all (t, x)

2) $V(t, 0) = 0$ for all t

3) $V(t, x)$ is continuous with respect to x at $(t, 0)$ for all t.

Proof:

We want to prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all t_0 and all solutions $\varphi(t)$ of (1.9) with $\varphi(t_0) = x_0$, $\|x_0\| < \delta$ implies$\|\varphi(t)\| < \epsilon$for all $t \geq t_0$.

Let $\epsilon > 0$ and t_0 be given, and let us consider $a(\epsilon)$. By the continuity of b there exists $\delta > 0$ such that if $\|x_0\| \leq \delta$then $V(t_0, x_0) \leq b(\|x_0\|) < a(\epsilon)$.

Since V is a Lyapunov function, for every solution $\varphi(t$ of (3.9) with$\varphi(t_0) = x_0$

$$V(t, \varphi(t) \leq V(t_0, x_0) < a(\epsilon) \quad \forall t \geq t_0 \tag{3.15}$$

From this inequality it follows that $\|\varphi(t) < \epsilon\|$for all $t \geq t_0$.

In fact otherwise there would exist t such that $\|\varphi(\bar{t})\| \geq \epsilon$ and, by (i) and

(ii), $V\ (\bar{t}, \varphi(\bar{t})) \geq a(\|\varphi(\bar{t})\|) \geq a(\epsilon)$, that is a contradiction to (3.15).

In order to apply Lyapunov's second method and prove the stability of a differential inclusion, the fundamental tool is to find a Lyapunov function and, in particular,to verify condition(3.10),without knowing the explicit form of its solutions. We are then led back to the problem of monotonicity along solutions of a differential inclusion that we discussed in this Section.

The following result is a corollary of the previous theorem and Proposition(3.2.12) in this chapter.

Corollary(3.3.5):

If there exists a function $V : \mathbb{R}^{n+1} \to \mathbb{R}$ such that max $\dot{V}^{(3.9)}(t, x) \leq 0$ for a.e.t and for all x and two continuous strictly increasing functions $a, b : \mathbb{R}^{+} \to \mathbb{R}^{+}$ such that hypothesis (i) and (ii) of Theorem(3.2)hold, then the differential inclusion (1.9) is stable at $x = 0$.

Remark(3.3.6):

If instead of (3.9) we consider the autonomous differential inclusion (3.13), it makes more sense to consider Lyapunov functions not depending on t. Hypothesis (i) and (ii) in Theorem(3.3.3)can then be changed into the following:

$V : \mathbb{R}^{n} \to \mathbb{R}$ is positive definite and continuous at $x = 0$.

The conclusion of the previous corollary still holds with $\dot{V}^{(3.9)}$replaced by $\dot{V}^{(2.6)}(x) = \{a \in \mathbb{R} : \exists v \in F(x) \text{ s.t. } p \cdot v = a\ \forall p \in \partial_C V\ (x)\}$ (see [2]).

Remark(3.3.7):

Nonsmooth Lyapunov functions and generalized derivatives have been previously used in the literature on stability mainly in connection with the problem of asymptotic stability and stabilization: see for instance [37,39, 58, 48,114].

Example(3.3.8):

We consider a system of the form (3.2) in $\mathbb{R}^2$ where $f(x_1, x_2) = (-sgn\ x_2, sgn\ x_1)^T$ (Fig(3.1). According to the Filippov's approach, this leads to the differential inclusion (3.13), where

$$F(x_1, x_2) = Ff(x_1, x_2) =$$

$$\begin{cases} \{-\text{sgn } x_2\} \times \{\text{sgn } x_1\} & \text{at}(x_1, x_2), x_1 \neq 0 \text{ and} x_2 \neq 0 \\ [-1,1] \times \{\text{sgn } x_1\} & \text{at}(x_1, 0), x_1 \neq 0 \\ \{-\text{sgn } x_2\} \times [-1,1] & \text{at}(0, x_2), x_2 \neq 0 \\ \overline{co}\{(1,1), (-1,1), (-1,-1), (1,-1)\} & \text{at}(0,0) \end{cases}$$

Let us now consider $V\ (x_1, x_2) = |x_1| + |x_2|$. We have

$$\partial_C V\ (x_1, x_2) = \begin{cases} \{-\text{sgn } x_2\} \times \{\text{sgn } x_1\} & \text{at}(x_1, x_2), x_1 \neq 0 \text{ and} x_2 \neq 0 \\ [-1,1] \times \{\text{sgn } x_1\} & \text{at}(x_1, 0), x_1 \neq 0 \\ \{-\text{sgn } x_2\} \times [-1,1] & \text{at}(0, x_2), x_2 \neq 0 \\ \overline{co}\{(1,1), (-1,1), (-1,-1), (1,-1)\} & \text{at}(0,0) \end{cases}$$

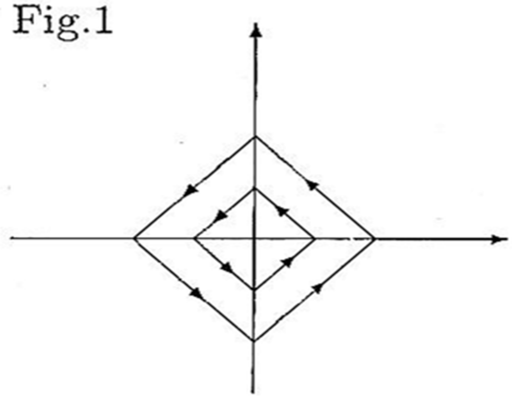
so that

$$\dot{\tilde{V}}^{(2.6)}(x_1, x_2) = \begin{cases} \{0\} & \text{at}(x_1, x_2), x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at}(x_1, 0), x_1 \neq 0 \\ \emptyset & \text{at}(0, x_2), x_2 \neq 0 \\ \{0\} & \text{at } (0,0) \end{cases}$$

Since for all $(x_1, x_2) \in \mathbb{R}^2$ one has max $\dot{\tilde{V}}^{(3.13)}(x_1, x_2) \leq 0$, by Corollary(3.3.5), the system is stable at $x = 0$. Let us remark that max $\dot{\tilde{V}}^{(3.13)}(0, x_2)$ $= -\infty <$

$\underline{D+}V\ ((0, x_2), (-1, 1)) = 2 \leq \max_{v \in F(0, x_2)} \underline{D+}V\ ((0, x_2), v)$. This means that a test based on Dini derivative is inconclusive. Moreover $\dot{\tilde{V}}^{(3.13)}(0, x_2) \neq \emptyset$ is a strict subset of the set $\dot{\tilde{V}}^{(3.13)}(0, x_2) = \{0\}$ considered in [37].

Fig.1



Remark(3.3.9):

The previous example clearly shows that, in general, there is no hope to find smooth Lyapunov functions for discontinuous equations.

Let us assume by contradiction that a smooth Lyapunov function V does exist. V is constant on cycles and any cycle is a level set for V . Then V has nonsmooth level sets, and this contradicts the fact that V is smooth.

Example(3.3.10):

Nonsmooth harmonic oscillator. Let us consider the scalar differential equation

$$\ddot{x} = -\mathrm{sgn} x \qquad (3.16)$$

We can associate to this equation a system of the form (3.2) in $\mathbb{R}^2$ where

$f(x_1, x_2) = (x_2, -sgn\ x_1)^T$. The Filippov multivalued map associated to the system is

$$F(x_1, x_2) = Ff(x_1, x_2) == \begin{cases} \{x_2\} \times \{-\mathrm{sgn} x_1\} & \mathrm{at}(x_1, x_2), x_1 \neq 0 \\ \{x_2\} \times [-1,1] & \mathrm{at}(0, x_2) \end{cases}$$

Let us now consider $V(x_1, x_2) = |x_1| + \frac{x_2^2}{2}$. We have

$$\partial_C V(x_1, x_2) = \begin{cases} \{\mathrm{sgn}\ x_1\} \times \{x_2\} & \mathrm{at}(x_1, x_2), x_1 \neq 0 \\ [-1,1] \times \{x_2\} & \mathrm{at}(0, x_2) \end{cases}$$

so that

$$\dot{\bar{V}}^{((2.6))}(x_1,x_2) = \begin{cases} \{0\} & \text{at}(x_1,x_2), x_1 \neq 0 \\ \{0\} & \text{at}(x_1,x_2), x_1 = 0, x_2 = 0 \\ \emptyset & \text{at } (0,x_2), x_2 \neq 0 \end{cases}$$

Since for all $(x_1, x_2) \in \mathbb{R}^2$ one has max $\dot{\bar{V}}^{(3.13)}(x_1, x_2) \leq 0$, by Corollary(3.3.5),

the system is stable at $x = 0$.

Example(3.3.11)

Gradient vector fields. It is well known that if $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite smooth function then the equation

$$\dot{x} = -\nabla V\ (x)$$

has an asymptotically stable equlibrium at the origin. For a locally Lipschitz, healthy positive definite function , a natural substitute of the previous equation is the differential inclusion

$$\dot{x} \in -\partial_C V\ (x).$$

Let $a \in \dot{\bar{V}}^{(3.13)}(x)$, where $F(x) = -\partial_C V\ (x)$. Then there exists $v \in -\partial_C V\ (x)$ such that $p \cdot v = a$ for each $p \in \partial_C V\ (x)$. In particular the equality must be true for $p = v$. But then $a = -|v|^2 \leq 0$. According to Corollary(3.3.5), we conclude that any differential inclusion of the form $\dot{x} \in -\partial_C V\ (x)$ is stable at the origin.

Now we investigate the Inverse Lyapunov's theorems:

We devote just a short paragraph to the problem of inverting Theorem(3.3.3).The cases of a continuous or smooth, time dependent or autonomous single valued righ thand side of (3.9) have been widely treated in the literature: see [99, 98, 72,116, 71, 68] and [5] for an overview on the problem. It is important to emphasize that really the most important issue is not the existence of a Lyapunov function, that is relatively easily obtained, but its regularity. Actually, regularity of Lyapunov functions plays an important role in the applications, for example in connection with the problem of asymptotic stabilization of a control system, as we will see in Chapter 4. We have already remarked that, in general, for discontinuous systems, there is no hope to find a smooth Lyapunov function. In [71] the authors give an example of a system of the form (3.1) with f continuous such that a continuous Lyapunov function does not exist. Moreover they prove that the existence of a continuous Lyapunov function becomes a necessary and sufficient condition if the notion of stability is conveniently strengthened(see also [68] for the autonomous case).

From our point of view, it is interesting to know a notion of stability equivalent to the existence of a Lipschitz continuous (or regular or healthy) Lyapunov function, both for autonomous and time dependent systems. For time dependent systems the problem has been solved by Kurzweil and Vrkoˇc([71]) in the case f is continuous, by means of the notion of robust stability.For autonomous systems the problem has been very recently solved in the scalar case (see [6]), but it is still open in $\mathbb{R}^n$.

Going back to discontinuous equations and differential inclusions, let us mention just two results. The first one can be found in [83]. It essentially is the inverse of Theorem (3.3.3).

In the second one ([5]), the authors generalize the mentioned result in [71] to differential inclusions of the form (3.9).

We can now define the Asymptotic Stability and Invariance Principle:

As shown in the previous section, if a Lyapunov function is known, one can get some conclusions about stability, but nothing can be said in general about asymptotic stability, whose definition is the following.

Definition(3.3.12):

The differential inclusion (3.9) is said to be asymptotically stable at $x = 0$ if

(i) (3.9) is stable at $x = 0$

(ii) there exists $\eta > 0$ such that if $\|x_0\| \leq \eta$then for all t_0 and for all solutions $\varphi(t)$ of (3.9) with initial condition $\varphi(t_0) = x_0$ one has $\lim_{t\to\infty} \varphi(t) = 0$.

Note that the given concept of asymptotic stability is strong, in the sense that it refers to all solutions of (3.9), and uniform, because η does not depend on t_0.

Both for autonomous and time dependent differential equations with continuous right handside there are classical results which give asymptotic stability.They are based on the knowledge of a smooth Lyapunov functions whose derivative with respect to the system is negative definite.

In the autonomous case some asymptotic stability results can be achieved by means of LaSalle principle (also called invariance principle) even if the derivative of the Lyapunov function with respect to the system is not known to be negative definite. Beside its application in the study of asymptotic stability,LaSalle principle is interesting by itself, because it provides information on the behaviour of solutions. Even if a “plain” version of LaSalle principle for time-dependent systems does not exist, many results in this direction can be mentioned: [100, 33, 32] and references there in.

Going back to differential inclusions and nonsmooth Lyapunov functions,we now limit ourselves to consider the autonomous case. We give a version of the invariance principle based on the notion of set-valued derivative with respect to (3.13) and compare it with similar early results.

The following definitions (see [20], page 129) are useful to formulate and prove such an invariance theorem.

Definition(3.3.13):

A point $q \in \mathbb{R}^n$ is said to be a limit point for a solution $\varphi(t)$ of (3.13) if there exists a sequence $\{t_i\}, t_i \to +\infty$ as $i \to +\infty$, such that $\varphi(t_i) \to q$ as $i \to +\infty$.

The set of the limit points of $\varphi(t)$ is said to be the limit set of $\varphi(t)$and is denoted by $\Omega(\varphi)$.

Definition(3.3.14):

A set Ω is said to be a weakly invariant set for (3.13) if through each point $x_0 \in \Omega$ there exists a maximal solution of (3.13) lying in Ω .

We recall that under the assumption that F is an upper semi-continuous multivalued map with compact, convex values, if $\varphi(t)$is a solution of the autonomous differential inclusion(3.13) and $\Omega(\varphi)$ is its limit set, then $\Omega(\varphi)$ is weakly invariant and if $\varphi(t), t \in \mathbb{R}_+$, lies in a bounded domain, then $\Omega(\varphi)$ is nonempty, bounded, connected and $\mathrm{dist}(\varphi(t), \Omega(\varphi)) \to 0$ as $t \to +\infty$ (see[20], page 129).

Theorem(3.3.15):

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and healthy

Lyapunov function for (3.13). Let us assume that for some $l > 0$, the connected component L_l of the level set $\{x \in \mathbb{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded. Let $x_0 \in L_l$ and $\varphi(t)$ be any solution of (3.13) such that $\varphi(t_0) = x_0$.

Let

$$Z_V^{(3.13)} = \{x \in \mathbb{R}^n : 0 \in \dot{\bar{V}}^{(3.13)}(x)\}$$

and let M be the largest weakly invariant subset of $\overline{Z_V^{(3.13)}} \cap L_l$.
Then dist $(\varphi(t), M) \to 0$ as $t \to +\infty$.

Proof:

Let $\Omega(\varphi)$be the limit set of $\varphi(t)$. Let us remark that $\varphi(t)$is bounded. In fact otherwise there would exist $t_1 > 0$ such that $\varphi(t_1) \notin L_l$ and, since $\varphi(t)$ is continuous,$\varphi(t_1)$ is not in any other connected component of $\{x \in \mathbb{R}^n : V(x) \le l\}$. Then $V(\varphi(t_1)) > l \ge V(x_0)$, that is impossible since $Vo\varphi$ is decreasing.

Let us prove that $\Omega(\varphi) \subseteq \overline{Z_V^{(3.13)}} \cap L_l$.Because of the definition of L_l, $\Omega(\varphi) \subseteq L_l$.

We now prove that $\Omega(\varphi) \subseteq \overline{Z_V^{(3.13)}}$.

Let us remark that V is constant on $\Omega(\varphi)$.Indeed, since $+\infty Vo\varphi$ is decreasing and bounded from below, there exists $\lim_{t\to+\infty} V(\varphi(t)) = c \ge 0$. Let$\mathcal{Y} \in \Omega(\varphi)$. There exists a sequence $\{t_n\}, t_n \to +\infty$, such that $\lim_{t\to+\infty} \varphi(t_n) = \mathcal{Y}$ and, by the continuity of V ,$V(\mathcal{Y}) = c$.

Let $\mathcal{Y} \in \Omega(\varphi)$ and $\psi(t)$ be a solution of (3.9) lying in $\Omega(\varphi)$ such that $\psi(t) =\mathcal{Y}$. Since $V(\psi(t)) = c$ for all t, we have $\frac{d}{dt} V(\psi(t)) = 0$ for all t.

Therefore $0 \in \dot{\bar{V}}^{(3.13)}(\psi(t))$ almost every where, namely $\psi(t) \in Z_V^{(3.13)}$almost everywhere.

Let $\{t_i\}, t_i \to 0$, be a sequence such that $\psi(t_i) \in Z_V^{(3.13)}$ for all i. Since is continuous $\lim_{t\to+\infty} \psi(t_i) = \psi(0) = \mathcal{Y} \in \overline{Z_V^{(2.6)}}$.

From the fact that $\Omega(\varphi)$is weakly invariant it follows that $\Omega(\varphi) \subseteq M$ and from the fact that dist$\varphi(t)\Omega(\varphi) \to 0$ as $t \to +\infty$ it follows that dist$((\varphi t), M) \to 0$ as $t \to +\infty$.

Remark(3.3.16):

Early versions of the invariance principle for differential inclusions can be found in [37] and [48]. Although the result presented here has been largely inspired by both of them, certain differences should be pointed out. First of all, we emphasize that Theorem(3.3.15) is more general than Theorem(3.2)of [37] since no assumption about uniqueness of solutionsis required. As far as Ryan's invariance principle is concerned, essentially two remarks have to be done. On one hand Ryan's result refers to merely locally Lipschitz continuous Lyapunov functions, while we deal with locally Lipschitz continuous and also healthy Lyapunov functions. On the other hand our identification of the "bad" set $Z_V^{(3.13)}$ is sharper than Ryan's one.Finally, Example(3.3.20) shows a case in which Theorem(3.3.15) can be used in order to compute the limit set, while Ryan's invariance principle doesn't help.

Remark(3.3.17):

Example(3.3.8) of the this Section shows that,in the conclusion of Theorem(3.3.15), we cannot avoid to take, in general,the closure of $Z_V^{(3.13)}$ .Indeed, in Example(3.3.8) each trajectory is a closed path that coincides with its limit set and crosses the coordinates axis.

Remark(3.3.18):

As a consequence of the invariance principle we get asymptotic stability in the case that a Lyapunov function for (3.13) is known and the set $Z_V^{(3.13)}$ reduces to the origin.

Example(3.3.19):

Smooth oscillator with nonsmooth friction and uncertain coefficients.Let us consider a differential inclusion of the form (3.13) in $\mathbb{R}^2$,where

$$F(x_1,x_2) = \begin{cases} [-2x_2-1,-x_2-1]\times\{x_1\} & \text{at } (x_1,x_2), x_1>0 \text{ and } x_2>0 \\ \{-x_2-\operatorname{sgn}x_1\}\times\{x_1\} \text{ at } (x_1,x_2)\in\mathbb{R}^2\setminus(\{(0,x_2)\}\cup\{(x_1,x_2),x_1>0 \text{ and } x_2>0\}) & \\ [-2x_2-1,-x_2+1]\times\{0\} & \text{at } (0,x_2), x_2>0 \\ [-x_2-1,-x_2+1]\times\{0\} & \text{at } (0,x_2), x_2<0 \\ [-1,1]\times\{0\} & \text{at } (0,0). \end{cases}$$

Let us now consider the smooth function $V\ (x_1,x_2) = \frac{x_1^2+x_2^2}{2}$. In this case

$\dot{V}^{(2.6)}(x)$

$$= \begin{cases} \{[-1,0]x_1x_2 - x_1\} & \text{at } (x_1,x_2), x_1 > 0 \text{ and } x_2 > 0 \\ \{-|x_1|\}\ at\ (x_1,x_2) \in \mathbb{R}^2 \backslash (\{(0,x_2)\mathbb{R}\} \cup \{(x_1,x_2), x_1 > 0 & \text{and } x_2 > 0\}) \\ \{0\} & \text{at } (0,x_2), x_2 \neq 0 \\ \{0\} & \text{at } (0,0), \end{cases}$$

then $Z_V^{(3.13)} = \{(0,x_2), x_2 \in \mathbb{R}\}$.

Let us now determine the largest weakly invariant subset M of $Z_V^{(3.13)}$.

Let us remark that , if $|x_2| \leq 1$, then $(0,0) \in F(x_1,x_2)$, hence the segment $\overline{P_1P_2}$, where $P_1 = (0,1)$ and $P_2 = (0,-1)$, is a weakly invariant subset of $Z_V^{(3.13)}$.

Moreover, if $|x_2| > 1$, all the vectors $v \in F(0,x_2)$ point in the same direction, hence each trajectory, starting in $(0,x_2)$, with $|x_2| > 1$, leaves thex_2-axis.

We conclude that $M = \overline{P_1P_2}$, i.e. all trajectories of the differential inclusion(3.13) tend to the segment $\overline{P_1P_2}$ as $t \to +\infty$. In fact each solution is attracted by a single point of the segment $\overline{P_1P_2}$. This follows by the proof of Theorem(3.3.15).Indeed each solution is attracted by the set $Z_V^{(1-13)} \cap L_l \cap V^{-1}(c)$for some c.

Example(3.3.20):

Nonsmooth harmonic oscillator with nonsmooth friction (Fig(3.2). Let us consider a system of the form (3.2) in $\mathbb{R}^2$ where $f(x_1,x_2) = \left(-\text{sgn}\ x_2 - \frac{1}{2}\text{sgn}\ x_1, \text{sgn}\ x_1\right)^T$. Filippov solutions of (3.2) are solutions of the differential inclusion (3.13), where

$$F(x_1,x_2) = Ff(x_1,x_2)$$

$$= \begin{cases} \left\{-\text{sgn}\ x_2 - \frac{1}{2}\text{sgn}\ x_1\right\} \times \{\text{sgn}\ x_1\} & \text{at } (x_1,x_2), \quad x_1 \neq 0, \quad x_2 \neq 0 \\ \overline{co}\left\{\left(-\frac{3}{2},1\right),\left(\frac{1}{2},1\right)\right\} & \text{at } (x_1,0), \quad x_1 > 0 \\ \overline{co}\left\{\left(-\frac{1}{2},-1\right),\left(\frac{3}{2},-1\right)\right\} & \text{at } (x_1,0), \quad x_1 < 0 \\ \overline{co}\left\{\left(-\frac{3}{2},1\right),\left(-\frac{1}{2},-1\right)\right\} & \text{at } (0,x_2), \quad x_2 > 0 \\ \overline{co}\left\{\left(\frac{3}{2},-1\right),\left(\frac{1}{2},1\right)\right\} & \text{at } (0,x_2), \quad x_2 < 0 \\ \overline{co}\left\{\left(-\frac{1}{2},-1\right),\left(\frac{1}{2},1\right)\left(-\frac{3}{2},1\right),\left(\frac{3}{2},-1\right)\right\} & \text{at } (0,0), \end{cases}$$

Let us now consider $V\ (x_1, x_2)\ =\ |x_1|\ +\ |x_2|$. In this case

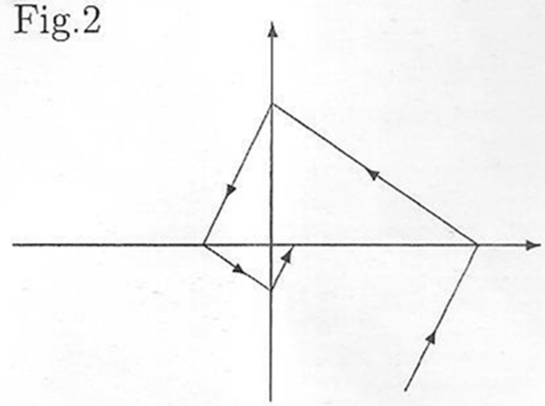
$$\dot{\bar{V}}^{(3.13)}(x_1, x_2) = \begin{cases} \left\{-\frac{1}{2}\right\} & \text{at } (x_1, x_2), x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at } (x_1, 0), x_1 \neq 0 \\ \emptyset & \text{at } (0, x_2), x_2 \neq 0 \\ \{0\} & \text{at } (0,0) \end{cases}$$

then V is a Lyapunov function for the system, that is stable at $x\ =\ 0$.

Moreover $Z_V^{(3.13)}\ =\ \{(0,0)\}$, hence the solutions tend to $(0,0)$ as $t\ \rightarrow +\infty$ (see Fig(3.2). Let us remark that in this example Ryan's invariance principle doesn't help if we want to compute the limit set of the differential inclusion.

In fact, if $x_2\ >\ 0$, we have that $\max\{V^{\circ}((0,x_2),v), v\ \in\ F(0,x_2)\}\ =\ \frac{5}{2}>0$

Fig.2



Chapter (4)

Asympototic and External Stabilization

Section(4.1)Asymptotic stabilization of control systems

We now turn our attention to control ssystems of the form

$$\begin{cases} \dot{x} = f(x, u) \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

Where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, f: \mathbb{R}^{n+m} \to \mathbb{R}^n$is locally essentially bounded and continuous with respect to u and $f(0,0) = 0$. The parameter u is said to be the control.

Definition(4.1.1):

System (4.1) is said to be (locally) asymptotically stabilizableat $x = 0$ if there exists $\delta > 0$ and a measurable function $u: \mathbb{R}^n \to \mathbb{R}^m$ (called feedback law) such that for any initial condition x_0 such that $\|x_0\| < \delta$the system

$$\begin{cases} \dot{x} = f\big(x, u(x)\big) \\ x(t_0) = x_0 \end{cases} \tag{4.2}$$

is asymptotically stable at $x = 0$.

Note that, from now on, if a system is discontinuous, solutions are intended in some generalized sense (see Chapter 3).

Now we discuss the Asymptotic stabilizability and asymptotic controllability:

The problem of asymptotic stabilizability is historically tied to the problem of asymptotic controllability to zero. In fact, for linear systems, these two concepts are equivalent.

Definition(4.1.2):

System (4.1) is said to be (locally) asymptotically controllable to zero if

1) there exists $\eta > 0$ such that for allx_0 with $\|x_0\| < \eta$ there exists a control $u: \mathbb{R} \to \mathbb{R}^m$ such that for every solution $\varphi(t)$ of the system

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) \\ x(t_0) = x_0 \end{cases} \tag{4.3}$$

$\varphi(t) \to 0 \; as \; t \to +\infty$

2) for all $\epsilon > 0$ there exists $\delta > 0$, $(\delta \leq \eta)$, such that for all x_0 with $\|x_0\| < \delta$ there exists a control u as in (4.1) such that for every solution $\varphi(t)$ of (4.3) one has $\|\varphi(t)\| < \epsilon$for all $t \geq t_0$.

It is evident that an asymptotically stabilizable system is also asymptotically controllable, but the converse is not obvious at all.

In an Sussmann [67] shows an analytic system which is globally asymptotically controllable but not globally asymptotically stabilizable by means of a continuous (static) feedback law. More examples of controllable systems which can not be stabilized by means of continuous static feedback laws can be given by means of the following Brockett's condition ([115]). It is a topological necessary condition for a nonlinear smooth control system to be asymptotically stabilizable by means of a continuous (static) feedback law.

Theorem(4.1.3):

If f is locally Lipschitz continuous and the control system (4.1) can be (locally) asymptotically stabilized by means of a continuous (static) feedback law, then the image of any neighborhood of $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ is a neighborhood of $0 \in \mathbb{R}^n$.

From this result it arises the problem of stabilizing systems which don't admit continuous stabilizing feedback laws. Mainly two alternative ways can be pursued. The first one consists in introducing continuous time-varying feedback laws (see [79] for an overview on this point of view), while the second one makes use of discontinuous feedback laws. We devote our attention to this second point of view.

With the introduction of discontinuous feedback laws two problems arise: one must choose which kind of discontinuities allow and then, according to that choice, an appropriate definition of solution.

In [77], the authors consider an affine input system of the form where

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} u_i g_i(x) \qquad (4.4)$$

where $f, g_1, \ldots, g_m$ are continuous vector fields of $\mathbb{R}^n$ and G is the matrix whose columns are $g_1, \ldots, g_m$. The feedback laws are taken to be such that $u \in L^{\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ and

$$\text{esssup}\,\{\|u(x)\|, \|x\| < \epsilon\} \to 0 \; as \; \epsilon \to 0 \qquad (4.5)$$

This last condition can be seen as a sort of continuity of the feedback at the origin. Admitting this kind of feedback the most natural concept of solution is that of Filippov.In this context Coron and Rosier prove that the existence of a discontinuous feedback law implies the existence of a continuous one. Note that if (4.5) is not satisfied, Coron and Rosier's result does not hold anymore. This can be seen by means of the following example (see [49]).

Example(4.1.4):

Let us consider the scalar differential equation

$$\dot{x} = x + u|x| \quad (4.6)$$

where $x, u \in IR$. The feedback law $u(x) = -2\mathrm{sgn}x$ asymptotically stabilizes it but it doesn't satisfy (4.5). Let us prove that there doesn't exist a continuous asymptotically stabilizing feedback by contradiction. Assume that $\tilde{u}(x)$ is a continuous asymptotically stabilizing feedback. Note that (3.2) is asymptotically stable at $x = 0$ if and only if for all $x \in \mathbb{R}\backslash\{0\}$ one has $xf(x) < 0$. This implies that

(a) if $x > 0$ then $xf(x) = x^2\big(1 + \tilde{u}(x)\big) < 0$ and $\tilde{u}(x) < -1$

(b) if $x > 0$ then $xf(x) = x^2\big(1 - \tilde{u}(x)\big) < 0$ and $\tilde{u}(x) > 1$

that is a contradiction to the continuity of $\tilde{u}$. Finally note that in this example Filippov solutions of the implemented system are simply classical solutions.

In [49] essentially affine systems are considered and feedback laws are assumed to be upper semi-continuous multivalued maps with compact and convex values. Ryan proves that, if solutions are intended in the Filippov's or Krasovskii's sense, then Brockett's topological necessary condition still holds.

The previous Filippov and Krasovskii solutions are not the most adequate in order to prove that, for general non linear systems, asymptotic controllability implies asymptotic stabilizability. Actually this problem has been recently solved by means of different kinds of feedbacks and solutions.

In [58] the authors solve the problem by considering locally bounded feedback laws and (not generalized) sampling solutions. A technique analogous to that used in [58] is used by Rifford ([89]) for Euler solutions.

Finally, a totally different approach has been used by Ancona and Bressan([51]). They introduce a new class of piecewise smooth feedback laws,called patchy feedback, and consider Carath´eodory solutions.

We can now define the Discontinuous feedbacks: two examples:

The previous paragraph should have motivated the use of discontinuous feedback laws, but, actually, there is still the problem of constructing them.In(see, for example, [14, 79, 93]) concrete strategies in order to stabilize class of systems which do not satisfy Brockett's condition are suggested. Moreover the fact that they work well is also proved by means of numerical experiments. Nevertheless it is not always clear in which sense solutions have to be considered. In particular, sometimes they are taken in the Filippov's sense, while some other times they seem to be thought in the Carath´eodory's sense. We now try to illustrate the problem of choosing a good definition of solution by means of two examples. The first one is the classical example of the nonholonomic integrator ([115]). The feedback law we consider has been suggested by Bloch and Drakunov ([14]).

Example(4.1.5):

Non holonomic integrator. Let us consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = v \\ \dot{x}_3 = x_1 v - x_2 u \end{cases} \tag{4.7}$$

This system does not satisfy Brockett's condition, then a continuous stabilizing feedback does not exist. Let $(u_0, v_0) \in \mathbb{R}^2\{(0,0)\}$ be a fixed vector,α, β be positive constants and $\rho = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{\beta}{2\alpha}(x_1^2 + x_2^2) < |x_3|\right\}$.

We consider the feedback law

$$\begin{pmatrix} u(x_1, x_2, x_3) \\ v(x_1, x_2, x_3) \end{pmatrix} = \begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} & \text{if } (x_1, x_2, x_3) \in \rho \\ \begin{pmatrix} -\alpha x_1 + \beta x_2 \text{sgn} x_3 \\ -\alpha x_2 - \beta x_1 \text{sgn} x_3 \end{pmatrix} & \text{if } (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \rho \end{cases} \tag{4.8}$$

Note that on the surfaces $x_3 = 0$ and $\partial\rho$ the feedback is discontinuous.Let us denote $k_1(x_1, x_2, x_3) = (u_0, v_0, v_0 x_1 - u_0 x_2)^T$ and $k_2(x_1, x_2, x_3) = (-\alpha x_1 + \beta x_2 \text{sgn} x_3, -\alpha x_2 - \beta x_1 \text{sgn} x_3, -\beta x_1^2 \text{sgn} x_3 - \beta x_2^2 \text{sgn} x_3)^T$,

i.e. k_1 and k_2 are the values of the implemented system respectively on ρ and on $IR^3 \setminus \rho$.Let us remark that on the set $S = \partial\rho \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (\alpha u_0 + \beta v_0) x_2 = (\alpha v_0 - \beta u_0) x_1, \text{sgn} x_1 = \text{sgn}(\alpha u_0 + \beta v_0)\}$ the vectors k_1 and k_2 are parallel and have opposite directions, then the right-hand side of the implemented system is not

patchy (in the sense of [51]).
We briefly examine the behavior of some of the different kinds of solutions of the implemented system that can be considered. All of Carathéodory and Euler solutions actually tend to the origin, but the same is not true for Krasovskii and Filippov solutions. In fact the points of S are equilibrium points for the associated differential inclusions.

Note that, in this example, the value given to the feedback on the discontinuity surfaces is essential. In particular if, in a natural way, we define either $\left(u(x_1,x_2,0),v(x_1,x_2,0)\right)^T = (-\alpha x_1+\beta x_2,-\alpha x_2-\beta x_1)^T$ or

$\left(u(x_1,x_2,0),v(x_1,x_2,0)\right)^T = (-\alpha x_1-\beta x_2,-\alpha_2+\beta x_1)^T$ Carathéodory solutions of the implemented system do not exist for arbitrary initial conditions anymore.

In the following example the system considered is not stabilizable by means of a continuous feedback law even if Brockett's condition is satisfied (for a proof sees [123] and also [43]). We consider the stabilizing feedback law suggested in [103].

Example(4.1.6):

Let us consider the system

$$\begin{cases}\dot{x}_1 = (x_1^2-x_2^2)u \\ \dot{x}_2 = 2x_1x_2u\end{cases} \tag{4.9}$$

The trajectories of the system when $u = 1$ are shown in Fig(4.1).

Let us introduce the arc length of the circles passing through the points

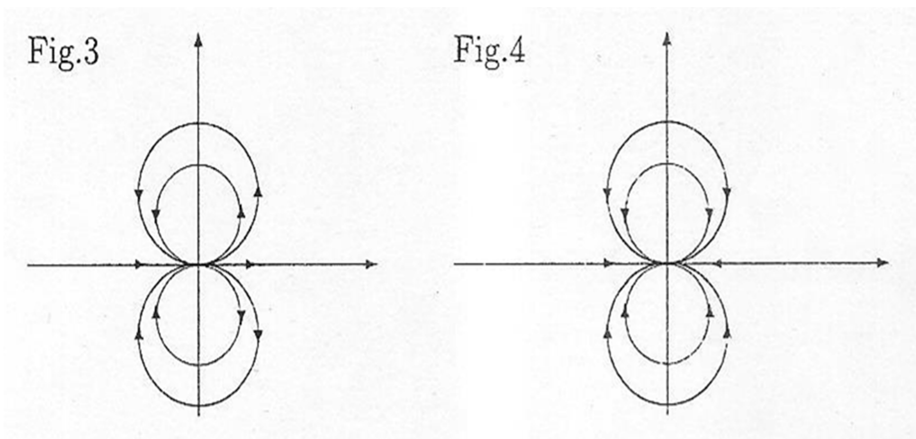
(x_1,x_2) and $(0,0)$ and with the center on the $x_2 - axis$:

$$a(x_1,x_2) = \begin{cases}x_1 & \text{if } x_2 = 0 \\ \frac{x_1^2+x_2^2}{x_2}\operatorname{arcten}\frac{x_2}{x_1} & \text{if } x_2 \neq 0.\end{cases} \tag{4.10}$$

We define the feedback law$u(x_1,x_2) = -ka(x_1,x_2)$, where k is a positive constant. The x_2-axis is a discontinuity line for the feedback, in fact, for $x_2 \neq 0$, we have that $\lim_{x_1\to 0^+} u(x_1,x_2) = k\frac{\pi}{2}x_2$ and $\lim_{x_1\to 0^-} u(x_1,x_2) = -k\frac{\pi}{2}x_2$. From this fact it immediately follows that the points of the x_2-axis are equilibrium points for the associated Krasovskii and Filippov differential inclusions, and then not all of Krasovskii and Filippov solutions of the implemented system converge to the origin.

Note that if we posit either $u(0, x_2) = k\frac{\pi}{2}x_2$ or $u(0, x_2) = -k\frac{\pi}{2}x_2$, the feedback is not patchy. Nevertheless Carathéodory solutions do exist: the trajectories of the system are either the right or the left half circles (see Fig(4.2)). The same happens if, instead of Carathéodory solutions, we consider Euler solutions.

Intuitively, when initial conditions are taken on the x_2-axis, it would be more desirable to have both the right and the left half circles as trajectories of the system. It is possible to get them by considering Euler externally disturbed solutions.



Now we discuss the Discontinuous damping feedback:

In this paragraph we see an example of discontinuous feedback laws which stabilize a wide class of systems.

We study stabilization of autonomous systems affine in the control by means of discontinuous damping feedbacks. Let us first go back to smooth systems for a while.

We devoted to nonlinear feedback stabilization, Jurdjevic and Quinn used the idea introduced in [38] that the stability properties of the affine system (4.4) can be enhanced by setting

$$u = u(x) = -\alpha\big(\nabla V\,(x)G(x)\big)^T \qquad (4.11)$$

where V is a Lyapunov function for the unforced system (3.2), the row vector $\nabla V(x)$ denotes its gradient and a is a positive real parameter (see [119]; see also [11] for subsequent developments and improvements). More precisely, assume that

(A) the origin is Lyapunov stable for (3.2) and a positive definite Lyapunov function $V \in C^1$such that $\dot{V}$ is negative semi-definite is known;
(B) an additional condition, involving Lie brackets of the vector fields $f, g_1, \dots, g_m$, holds.

Then, Jurdjevic and Quinn proved that (4.4) can be asymptotically stabilized by means of the feedback law (4.11).

Although it has been largely and successfully exploited in the literature both from a practical and a theoretical point of view, a weakness of the method related to assumption (A) should be pointed out. Indeed, we have already remarked that, even for smooth f, Lyapunov stability does not imply in general the existence of a (not even) continuous Lyapunov function.

When it is known that the unforced system is stable but the existence of a C^1 Lyapunov function cannot be guaranteed, two alternative ways can be pursued:

1) to introduce time dependent Lyapunov functions. In this case the Jurdjevic and Quinn method can be extended (see [87]) but it gives rise, of course, to a time dependent feedback;
2) to replace the (classical) gradient in (4.11) by some type of generalized gradient. This in general leads to discontinuous feedback, so that we have to choose in which sense solutions of the discontinuous differential equation involved have to be interpreted.

We devote our attention to the second point of view.

Now we investigate the Filippov solutions of the closed loop system:

The main assumption we make in the following is that we know a Lipschitz continuous and healthy Lyapunov function for the unforced system (3.2). In general, this implies that the origin is a stable equilibrium point for system (3.2). Moreover, if G is continuous, the feedback law (4.11) is defined a.e. and it is locally essentially bounded and measurable.

In fact, if L_x is the Lipschitz constant of V in a compact neighbourhood

U_x of x,

$$\|u(x)\| \leq \alpha \|\nabla V(x)\| \|G(x)\| \leq \alpha L_x \|G(x)\| \text{ a.e. in } U_x$$

that is bounded in U_x because G is continuous. As already remarked, for every $v \in \mathbb{R}^n$, $\nabla V(\cdot) \cdot v$ is measurable; hence u is also measurable. On the other hand note that, in general, u does not satisfy (4.5).

From this fact it follows that the right hand-side of the equation

$$\dot{x} = f(x) - \alpha G(x)\big(\nabla V(x)G(x)\big)^T \tag{4.12}$$

is also locally essentially bounded and measurable on $\mathbb{R}^n$.

Among the various solutions introduced in Chapter 1, the most adeguate to this context seem then to be Filippov solutions. We make the following assumptions:

(f0) $f \in L^{\infty}_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, $0 \in Ff(0)$;

(G0) $G \in C(\mathbb{R}^n;\ \mathbb{R}^{n\times m})$;

(V0) $V: \mathbb{R}^n \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous and healthy function.

In particular, note that in general we don't assume f to be continuous. In the smooth case, the proof that (4.11) stabilizes system (4.4) is divided into two steps. First one proves that the stability property of system (4.4) is not affected by the application of the feedback (4.11). After that, by means of LaSalle's principle, it is proved that solutions actually tend to the origin.In the particular case the function f is continuous the first step still holds, as the following proposition shows.

Proposition(4.1.7):

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $f: \mathbb{R}^n \to R$ is a Lyapunov function for the unforced system (3.2) and ($V0$) and ($G0$) hold, then system (4.2) is stable at $x = 0$.

Before proving the proposition let us recall some results that simplify the calculation of Filippov's multivalued maps (see [27]).

Proof :

Let us prove that $\max \dot{\bar{V}}^{(4.12)} \leq 0$. Indeed from this, by Corollary(3.3.5) in Chapter 3, it follows the thesis. Let $a \in \dot{\bar{V}}^{(4.12)}(x)$. By Proposition(4.1.8), we have that $(f - \nabla\alpha G(\nabla VG)^T)(x) = f(x) - \alpha G(x)\big(\partial_C V(x)G(x)\big)^T$, then there exists $q \in \partial_C V(x)$ such that $p \cdot v = p \cdot f(x) - \alpha\big(pG(x)\big) \cdot \big(q \cdot G(x)\big)$ for all $p \in \partial_C V(x)$. In particular, for $p = q$ we get that $a = q \cdot f(x) - \alpha\|qG(x)\|^2$. Since by Proposition(3.2.8) in Chapter 3 $p \cdot f(x) \leq 0$ for all $p \in \partial_C V(x)$, we get that $a \leq 0$.

If f is not continuous the Proposition(4.1.7) fails to be true.This is proved by means of the following example.

Proposition(4.1.8):

(i) If $f \in C(\mathbb{R}^n;\ \mathbb{R}^n)$ then $Ff(x) = \{f(x)\}$ for all $x \in \mathbb{R}^n$.

(ii) If $f, g \in L^{\infty}_{loc}(\mathbb{R}^n;\ \mathbb{R}^n)$ then $F(f + g)(x) \subseteq Ff(x) + Fg(x)$ for all $x \in \mathbb{R}^n$.

Moreover if $f \in C(IR^n;\ IR^n)$ then $F(f + g)(x) = f(x) + Fg(x)$ for all $x \in \mathbb{R}^n$.

(iii) If $G \in G(\mathbb{R}^n; \mathbb{R}^{n\times m}), u \in L^{\infty}_{loc}(\mathbb{R}^n;\ \mathbb{R}^n)$ then $F(Gu)(x) = G(x)Fu(x)$ for all $x \in \mathbb{R}^n$.

(iv) If $V: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, then $F(\nabla V\ )(x) = \partial_C V\ (x)$ for all $x \in \mathbb{R}^n$.

Example(4.1.9):

Let us consider a single-input system of the form (4.4) in $\mathbb{R}^2$, where

$$f(x_1, x_2) = \begin{cases} (\text{sgn}x_1 - 2)^T & \text{at } (x_1, x_2), \quad x_2 \geq 0 \\ (0,0)^T & \text{at } (x_1, x_2), \quad x_2 \geq 0 \end{cases}$$

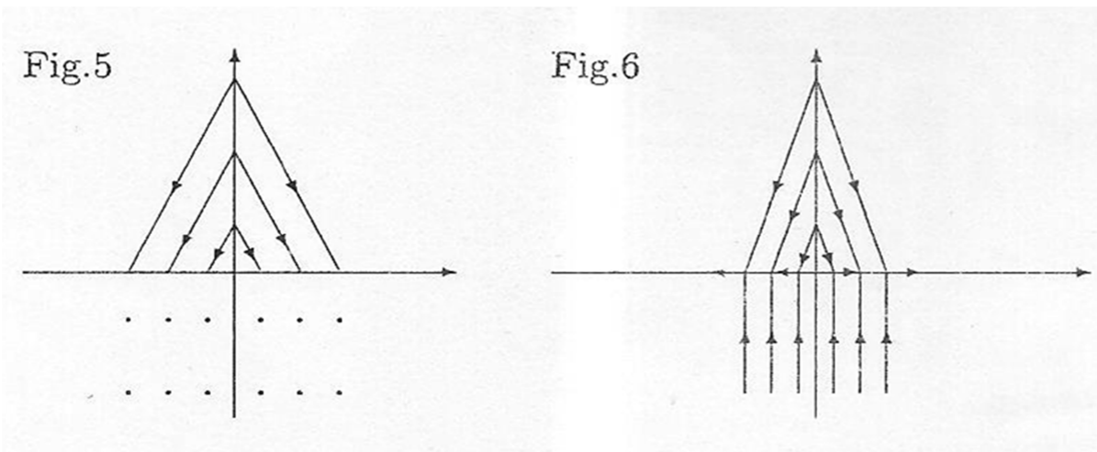
$G(x_1, x_2) = (0, 1)^T$, and the function $V\ (x_1, x_2) = |x_1| + |x_2|$. By computing $\dot{V}^{(1.2)}(x_1, x_2)$, it is easily proved that system (4.3) is stable at $x = 0$ (see Fig(4.3).

Let us now consider system (4.12).

$$F(f - \alpha G(\nabla VG)^T(x_1, x_2)$$

$$= \begin{cases} \{\text{sgn}x_1\} \times \{-2 - \alpha\} & \text{at } (x_1, x_2), x_1 \neq 0, x_2 > 0 \\ \{0\} \times \{\alpha\} & \text{at } (x_1, x_2), x_2 < 0 \\ \{-1,1\} \times \{-2, \alpha\} & \text{at } (0, x_2), x_2 > 0 \\ \overline{\text{co}}.\{(\text{sgn}x_1, -2, -\alpha)^T, (0, \alpha)^T & \text{at } (x_1, 0). x_1 \neq 0 \\ \overline{\text{co}}.\{(1, -2, -\alpha)^T, (0, \alpha)^T, (-1, -2, \alpha)^T & \text{at } (0,0) \end{cases}$$

Let us remark that for all $\alpha < 0$ and for all the points $(x_1, 0)$ with $x_1 \neq 0$, there exists a trajectory starting from $(x_1, 0)$ which lies on the x_1-axis and goes to infinity. This is obtained by considering the vector$\left(\frac{\alpha}{2(1+\alpha)}, 0\right) \in F(f - \alpha G(\nabla VG))(x_1, 0)$ if $x_1 > 0$, and the vector $\left(-\frac{\alpha}{2(1+\alpha)}, 0\right) \in F(f - \alpha G(\nabla VG))(x_1, 0)$ (see Fig(4.4) in the case $\alpha = 1$).



As the previous example shows, in the nonsmooth case, in order to guarantee the conservation of stability for the closed loop system, we need to add some extra assumptions. Actually, we do not present a unique condition, but we list some alternative conditions which, combined together in a convenient way, allow us to get not only the stability of system (4.12), but also the stabilizability of system (4.4). Note that in these conditions the variable x is not yet quantified. Since the role of x will depend on the circumstances, it is convenient to specify it later. The possible conditions are the following:

(f1) $\max \dot{\bar{V}}^{(3.2)}(x) \leq 0$;

(f2) for all $v \in Ff(x)$ there exists $p \in \partial\_C\, V(x)$ such that $p \cdot v \leq 0$;

(f3) for all $v \in Ff(x)$ and for all $p \in \partial\_C\, V(x), p \cdot v \leq 0$;

(G1) there exists $c \in \mathbb{R}$ such that for all $p, q \in \partial\_C\, V(x)$, $(pG(x)) \cdot (qG(x)) = c^2$ (c may depend on x);

(G2) either $(pG(x)) \cdot (qG(x)) > 0$ for all $p, q \in \partial\_C\, V(x)$, or
$(pG(x)) \cdot (qG(x)) = 0$ for all $p, q \in \partial\_C\, V(x)$;

(G3) $(pG(x)) \cdot (qG(x)) \geq 0$ for all $p, q \in \partial\_C\, V(x)$;

(fG1) there exists $\alpha > 0$ such that for all $v \in Ff(x)$ and for all $q \in \partial_C V(x)$ there exist $p_1, p_2 \in \partial_C V(x)$ such that

$$(p_1 - p_2) . (v - \alpha G(x)(qG(x))^T) \neq 0.$$

By definition of $\dot{\bar{V}}^{(3.2)})(x)$, (f1) can be restated by saying that if there exists $v \in Ff(x)$ such that for all $p \in \partial_C V(x)$ one has $p \cdot v = a$, then $a \leq 0$. Conditions (f1),(f2) and (f3) can then be seen as geometric conditions on mutual positions of the sets $Ff(x)$ and $\partial_C V(x)$Moreover we have that (f3)⟹(f2)⟹(f1) and all of these conditions imply that V is a Lyapunov function for the unforced system. Note that if f is continuous, (f3) is equivalent to the fact that V is a Lyapunov function for the unforced system (see Proposition(3.2.8), Chapter 3).

In order to interpretate conditions (G1), (G2) and (G3), let us consider the set $H(x) = \{pG(x), p \in \partial_C V(x)\}$. (G1) implies that $H(x)$ reduces to a single vector, while (G2) and (G3) are conditions on the size of $H(x)$. For these conditions it holds that (G1) ⟹ (G2) ⟹ (G3).

The meaning of condition (fG1) is explained by the following lemma.

Lemma(4.1.10) :

Assume that conditions (f0), (G0) and (V0) hold for some $x \in \mathbb{R}^n$. There exists $\alpha = \alpha(x) > 0$ such that condition (fG1) holds if and only

if $\dot{\bar{V}}^{(4.12)}(x) = \emptyset$

Proof:

We prove the statement by contradiction.

Let us suppose that for all $\alpha > 0$ one has $\dot{\bar{V}}^{(4.12)}(x) \neq \emptyset$.Then there exist a $\alpha \in \mathbb{R}, w \in F(f - \alpha G(\nabla VG)^T)(x)$ such that, for all $p \in \partial_C V(x), p \cdot w = a$. By (ii), (iii) and (iv) in Proposition(4.1.8) it follows that there exist $v \in Ff(x)$ and $q \in \partial_C V(x)$ such that for all $p \in \partial_C V(x)$, $p \cdot (v - \alpha G(x)G(x)^T q) = a$. Let $p_1, p_2 \in \partial_C V(x)$. We have $p_1 \cdot (v - \alpha G(x)G(x)^T q) = = p_2 \cdot (v - \alpha G(x)G(x)^T q) = a$, hence $(p_1 - p_2) \cdot (v - \alpha G(x)G(x)^T q) = 0$, which is a contradiction to (fG1).

The viceversa is easily proved by contradiction.

Now we discuss the Conservation of stability:

From the previous discussion it follows that, in order to prove a stabilization result for system (4.4) by means of the feedback law (4.11), the first step is to give some sufficient conditions for system (4.12) being stable. We do that in the following lemma.

Lemma(4.1.11):

Let us assume that (f0), (G0), (V0) hold and (f1) holds for all

$x \in \mathbb{R}^n \backslash N$, where

$$N = \{x \in \mathbb{R}^n \text{ such that } V \text{ is not differentiable at } x\}.$$

Let us suppose further that for each $x \in N$ one of the following combinations of conditions holds: (i) (f1) and (G1), (ii) (fG1) for some α independent of x, (iii) (f2) and (G3), (iv) (f3).

Then for each $x \in \mathbb{R}^n$ V, max $\dot{\vec{V}}^{(4.12)}(x) \leq 0$.

Moreover, if the use of (fG1) can be avoided, the choice of α can be

arbitrary.

Proof:

Let $a \in \dot{\vec{V}}^{(4.12)}(x)$. Then there exists $w \in F(f - \alpha G(\nabla V\, G)^T)(x)$ such that, for all $p \in \partial_C V(x)$, $p \cdot w = a$. From (ii), (iii) and (iv) in Proposition(4.1.8) it follows that there exist $v \in Ff(x)$ and $q \in \partial_C V(x)$) such tha $w = v - \alpha G(x)\left(qG(x)\right)^T$. In the following we will use this representation for w without mentioning it explicitly.

We distinguish five cases: (o) for $x \in \mathbb{R}^n \backslash N$ and (i), (ii), (iii), (iv) for

$x \in N$.

(o) In this case $\partial_C V(x) = \{\nabla V(x)\}$,then $a = \nabla V(x) \cdot w = \nabla V(x) \cdot \left(v - \alpha G(x)\left(\nabla V(x) G(x)\right)^T\right)$ and $\nabla V(x) \cdot v = a + \alpha \left\|\left(\nabla V(x) G(x)\right)^T\right\|^2 = b$, where $b \in \dot{\vec{V}}^{(3.2)}(x)$. Since by assumption max $\dot{\vec{V}}^{(3.2)} \leq 0$, we also have that $b \leq 0$, hence $a = b - \alpha \left\|\left(\nabla V(x) G(x)\right)^T\right\|^2 \leq 0$.

(i) In this case $a = p \cdot w = p \cdot v - \alpha(pG(x))(qG(x))^T = p \cdot v - \alpha c^2$ for each $p \in \partial_C V(x)$. Hence the proof that max $\dot{\vec{V}}^{(4.12)}(x) \leq 0$ is analogous the one in (o).

(ii) From assumption (fG1) and Lemma(4.1.10) it follows that $\dot{\vec{V}}^{(4.12)}(x) = \emptyset$ for suitable choice of α.

(iii) Since (f2) implies (f1), clearly it is sufficient to prove that for all $w \in F(f + Gu)(x)$ there exists $p \in \partial_C V(x)$ such that $p \cdot w \leq 0$. Let $p \in \partial_C V(x)$ such that $p \cdot v \leq 0$ (such a p exists because of (f2)). By (G3) we get $a = p \cdot w = p \cdot v - \alpha(pG(x))^T \cdot (qG(x))^T \leq 0$, as required.

(iv) For all $p \in \partial_C V(x)$, $a = p \cdot w = p \cdot v - \alpha(pG(x))^T \cdot (qG(x))^T$. In particular, for $p = q$ we get $a = q \cdot w = q \cdot v - \alpha \left\|(qG(x))^T\right\|^2$ that is non positive because of (f3).

From the previous lemma and Corollary((3.3.5) in chapter 3) it follows that system (4.12) is stable at $x = 0$.

Now we study the Improvement of stability:

In order to study asymptotic stabilization of system (4.4) let us introduce the sets

$$Z_V^{4.12} = \{x \in R^n : 0 \in \dot{\vec{V}}^{(4.12)}(x)\}$$

and

$$Z_V^{1.2} = \{x \in R^n : 0 \in \dot{\vec{V}}^{(3.2)}(x)\}.$$

Let us recall that, if the connected component L_l of the level set $\{x \in \mathbb{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded, by the invariance theorem stated in Chapter 3, the solutions of systems (3.2) and (4.12), with initial condition $x_0 \in L_l$, respectively tend to $\overline{Z_V^{(3.2)}} \cap L_l$ and $\overline{Z_V^{(4.12)}} \cap L_l$.

Lemma(4.1.12):

Let us assume that (f0), (G0), (V0) hold and that (f1) holds for all $x \in \mathbb{R}^n \backslash N$. Let us suppose that for each $x \in N$ one of the following pairs of conditions holds: (i) (f1) and (G1), (ii) (f1) and (fG1) for some α independent of x, (iii) (f2) and (G2), (iv) (f3) and (G3).

Then $Z_V^{(4.12)} \subseteq Z_V^{(3.2)}$.

Proof:

$x \in Z_V^{(4.12)}$ means that there exists $w \in F(f - \alpha G(\nabla V\, G)^T)(x)$ such that, for all $p \in \partial_C V\,(x)$, $p \cdot w = 0$. Using the decomposition of w already mentioned in the proof of Lemma(4.1.11), we get that there exist $v \in Ff(x)$ and $q \in \partial_C V\,(x)$ such that $p \cdot w = p \cdot v - \alpha\big(pG(x)\big)^T \cdot \big(qG(x)\big)^T = 0$,i.e. $p \cdot v = \alpha\big(pG(x)\big)^T \cdot \big(qG(x)\big)^T$. Again, we distinguish five cases: (o) $x \in \mathbb{R}^n \backslash N$, (i), (ii), (iii), (iv).

(o) In this case $\partial_C V\,(x) = \{\nabla V\,(x)\}$ then
$\nabla V\,(x) \cdot v = \alpha \left\| \big(\nabla V\,(x) G(x)\big)^T \right\|^2 = b \geq 0$. On the other hand, since
$b \in \dot{\tilde{V}}^{(1.2)}(x)$ and max $\dot{\tilde{V}}^{(1.2)}(x) \leq 0$, $b \leq 0$, hence $b = 0$, i.e. there exists $v \in Ff(x)$suchthat $\nabla V(x) \cdot v = 0$ and $x \in Z_V^{(3.2)}$.

(i) The proof is analogous to the one in (o).

(ii) By Lemma(4.1.11), $\dot{\tilde{V}}^{(4.12)}(x) = \emptyset$, so that $0 \notin \dot{\tilde{V}}^{(4.12)}(x)$ and $x \notin Z_V^{(4.12)}$.

(iii) $p \cdot v = \alpha\big(pG(x)\big)^T \cdot \big(qG(x)\big)^T$ implies that x is such that for all
$p, q \in \partial_C V\,(x)$, $\big(pG(x)\big)^T \cdot \big(qG(x)\big)^T = 0$, otherwise for all $p \in \partial_C V\,(x)$ one has
$p \cdot w > 0$, which contradicts (f2). We conclude that, for all
$p \in \partial_C V\,(x)$, $p \cdot v = 0$, i.e. $x \in Z_V^{(4.12)}$.

(iv) $p \cdot v = \alpha\big(pG(x)\big)^T \cdot \big(qG(x)\big)^T \geq= 0$ because of (G3). On the other hand, by condition (f3), for all $p \in \partial_C V\,(x)$ we have $p \cdot v \leq 0$, hence, for all
$p \in \partial_C V\,(x)$, $p \cdot v = 0$, i.e. $x \in Z^{(3)}\, V$.

We can finally summarize the results of the present section in the following theorem.

Theorem(4.1.13):

Let us assume that (f0), (G0) and (V0) hold and (f1) holds for all $x \in \mathbb{R}^n \backslash N$. If N can be decomposed as a union $N = N_{11} \cup N_{12} \cup N_2 \cup N_3$ such that

(i) for all $x \in N_{11} \cup N_{12}$ (f1) holds; for all $x \in N_{11} \backslash \{0\}$, (G1) holds and for all $x \in N_{12} \backslash \{0\}$, (fG1) holds with α independent of x;

(ii) for all $x \in N_2$, (f2) holds and for all $x \in N_2 \backslash \{0\}$, (G2) holds;

(iii) for all $x \in N_3$, (f3) holds and for all $x \in N_3 \backslash \{0\}$, (G3) holds.

Then, there exists $\alpha > 0$ such that

(A) (4.12) is stable at $x = 0$,

(B) $Z_V^{(4.12)} \subseteq Z_V^{(3.2)}$.

Moreover let us assume that

(V1) there exists $l > 0$ such that the connected component L_l of the level set

$\{x \in \mathbb{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded,

(fG2) the largest weakly invariant subset of $\overline{Z_V^{(4.12)}} \cap L_l$ is $\{0\}$.

Then

(C) (4.4) is asymptotically stabilizable by means of the feedback law (4.11).

Finally, if the use of (fG1) can be avoided, the choice of $\alpha$is arbitrary.

Corollary(4.1.14):

Let us assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and that, if there exists $p \in \partial_C V(x)$ such that $\|pG(x)\| = 0$, then $x = 0$. Then system (4.4) is asymptotically stabilizable by means of the feedback law (4.11).

Remark(4.1.15):

If $V \in C^1$ then $N = \emptyset$, hence we only need to check condition (f1) in order to get the stability of (4.12), and conditions (V1) and (fG2) to get the asymptotic stabilization of (4.4), i.e. we have a classical-like stabilization theorem that can be applied in the case the only assumptions on f are measurability and local boundedness.

We can now explain the examples:

In the present illusteration we illustrate the various situations described in

Theorem (4.1.13) by means of some examples.

Example(4.1.16):

Let us consider a system of the form (4.4) in $\mathbb{R}^2$, where $f(x_1,x_2) = (-\text{sgn}x_2, \text{sgn}x_1)^T$ and $G(x_1,x_2) = (x_1,x_2)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$.

As shown in Example(3.3.8)in chapter 3)), for all $(x_1,x_2) \in \mathbb{R}^2$ we have max $\dot{\bar{V}}^{(3.2)}(x_1,x_2) \leq 0$.

$N = \{(x_1,0), x_1 \in \mathbb{R}\} \cup \{(0,x_2), x_2 \in \mathbb{R}\}$. Let us consider $p = (p_1,p_2)$ and $q = (q_1,q_2) \in \partial_C V(x_1,x_2)$

$$(pG(x_1,x_2)) \cdot (qG(x_1,x_2)) = \begin{cases} x_1^2 & (x_1,0), x_1 \neq 0 \\ x_2^2 & (0,x_2), x_2 \neq 0 \end{cases}$$

i.e. condition (G1) is verified for all $(x_1,x_2) \in N$. Then, by (A) in Theorem(4.1.13), for all $\alpha > 0$ system (4.4) with the feedback (4.11) is stable at $x = 0$.

Moreover let us consider the set

$$\dot{\bar{V}}^{(4.12)}(x_1,x_2) = \{a \in \mathbb{R} : \exists v \in Ff(x_1,x_2)\ \exists q \in \partial_C V(x_1,x_2) \text{ such that}$$

$$\forall p \in \partial_C V(x_1,x_2), p \cdot (v - \alpha G(x_1,x_2)((qG(x_1,x_2))^T) = a\}$$

$$\dot{\bar{\bar{V}}}^{(4.12)}(x_1,x_2) = \begin{cases} -\alpha(|x_1| = |x_2|)^2 & \text{at}(x_1,x_2)\ x_1 \neq 0\ and x_2 \neq 0 \\ \emptyset & \text{at } (x_1,0), x_1 \neq 0 \\ \emptyset & \text{at } (0,x_2)\ \ x_2 \neq 0 \\ \{0\} & \text{at}(0,0) \end{cases}$$

From Proposition(4.1.8)it follows that for all $(x_1,x_2) \in \mathbb{R}^2 \dot{\bar{V}}^{(4.12)}(x_1,x_2) \subseteq \dot{\bar{\bar{V}}}^{(4.12)}(x_1,x_2)$, hence $Z_V^{(4.12)} = \{(0,0)\}$ and, by (C) in Theorem(4.1.13), the system is asymptotically stabilizable by means of the feedback law (4.11).

Example(4.1.17):

Let us consider a system of the form (4.4) in $\mathbb{R}^2$, where $f(x_1,x_2) = (-\text{sgn}x_2, \text{sgn}x_1)^T$ and $G(x_1,x_2) = (1,0)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$.

As shown in Example(3.3.8) in chapter(3)), for all $(x_1,x_2) \in \mathbb{R}^2$ one has max $\dot{\bar{V}}^{(3.2)}(x_1,x_2) \leq 0$.Also in this case $N = \{(x_1,0), x_1 \in \mathbb{R}\} \cup \{(0,x_2), x_2 \in \mathbb{R}\}$.

Condition (G1) is verified on $\{(x_1, 0), x_1 \in \mathbb{R}\}$ but not on $\{(0, x_2), x_2 \in \mathbb{R}\}$. Nevertheless, for $\alpha \in (0, 1)$, condition (fG1) is verified on $\{(0, x_2), x_2 \in \mathbb{R}\}$, in fact

$$\{v - \alpha G(0, x_2)\big(qG(0, x_2)\big)^T \text{, for } v \in Ff(0, x_2), q \in \partial_C V\,(0, x_2), x_2 \neq 0\}$$
$$= \begin{cases} [-1-\alpha, -1+\alpha] \times [-1,1] & at(0, x_2), x_2 > 0 \\ [1-\alpha, 1+\alpha] \times [-1,1] & at(0, x_2), x_2 < 0 \end{cases}$$

and

$$\{(p_1 - p_2);\ p_1, p_2 \in \partial_C V\,(0, x_2), x_2 \neq 0\} = ([-2, 2], 0)^T .$$

By (A) in Theorem(4.1.14), it follows that system (4.12) is stable at $x = 0$ with $\alpha \in(0, 1)$. Moreover computations analogous to those of the previous example show that $Z_V^{(4.12)} = \{(0, 0)\}$. Hence, by (C) in Theorem(4.1.13), the system is asymptotically stabilizable by means of the feedback law (4.11) with a fixed $\alpha \in (0, 1)$. Example(3.3.20)(is actually a particular case of the present example, with $\alpha = \frac{1}{2}$. Fig.1 and Fig(3.2) show the behaviour of the system before and after the application of the feedback.

Remark(4.1.18):

By direct computation, it is possible to see that the closed loop system considered in the previous example is actually stable for all $\alpha > 0$.However, for $\alpha > 1$, no one of the alternative conditions of Theorem(4.1.13) can be applied. This shows that Theorem(4.1.13) does not cover all the possible cases.

Example(4.1.19):

Let us consider a system of the form (4.12) in $\mathbb{R}^2$, where

$f(x_1, x_2) = (-\text{sgn} x_2, \text{sgn} x_1)^T$and $(x_1, x_2) = \left(x_1 + \frac{1}{2}x_2 , x_2 + \frac{1}{2}x_1\right)^T$, and the function $V\,(x_1, x_2) = |x_1| + |x_2|$.

As shown in Example(3.3.8), for all $(x_1, x_2) \in \mathbb{R}^2$ we have max $\dot{\bar{V}}^{(3.2)}(x_1, x_2) \leq 0$,so that, also in this case,
$N = \{(x_1, 0), x_1 \in \mathbb{R}\} \cup \{(0, x_2), x_2 \in \mathbb{R}\}$. On N condition (f2) is satisfied. Moreover for all $(x_1, x_2) \in N$ and for all $p, q \in \partial_C V\,(x_1, x_2)$ we have $(pG(x)) \cdot (qG(x)) > 0$, i.e. condition (G2) is satisfied in N. By (A) in Theorem(4.1.13), it follows that system (4.12) is stable at $x = 0$ for all $\alpha > 0$. Moreover

$Z_V^{(4.12)} = \{(0,0)\}$, hence, by (C) in Theorem(4.1.13), the system is asymptotically stabilizable by means of the feedback law (4.11) for all $\alpha > 0$.

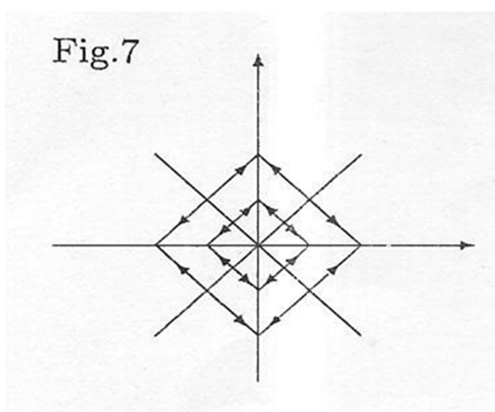
Example(4.1.20):

Let us consider a system of the form (3.2) in $\mathbb{R}^2$, where

$$f(x_1,x_2)\begin{cases}(x_2,-x_2)^T & \text{at}(x_1,x_2), 0 \le x_2 \le x_1\\ (-x_1,x_1)^T & \text{at}(x_1,x_2), 0 \le x_1 \le x_2\\ (-x_1,-x_1)^T & \text{at}(x_1,x_2), 0 \le -x_1 \le x_2\\ (-x_2,-x_2)^T & \text{at}(x_1,x_2), 0 \le x_2 \le -x_1\\ (x_2,-x_2)^T & \text{at}(x_1,x_2), x_1 \le x_2 \le 0\\ (-x_1,x_1)^T & \text{at}(x_1,x_2), x_2 \le x_1 \le 0\\ (-x_1,-x_1)^T & \text{at}(x_1,x_2), x_2 \le -x_1 \le 0\\ (-x_2,-x_2)^T & \text{at}(x_1,x_2), -x_1 \le x_2 \le 0\end{cases}$$

and $(x_1,x_2) = (x_1 + x_2, x_1 + x_2)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$.

By computing $Ff(x_1,x_2)$, it is easy to see that (f3) is verified, then (4.2) is stable at $x = 0$ (see Fig(4.5). Since condition (G3) is satisfied on N (note that (G2) is not satisfied on N), then not only system (4.12) is stable at $x = 0$, but also $Z_V^{4.12} \subseteq Z_V^{1.2}$. Actually in this case it can be shown that the feedback law (4.11) does not stabilize system (4.12) asymptotically.



Now we can discuss Krasovskii solutions of the closed loop system:

In the present section we briefly investigate the effect of the Jurdjevic an Quinn's feedback on the affine system in the case its solutions are intended in the Krasovskii's sense. Note that, thanks to the definition of Filippov solutions, in the previous paragraphs it was not important to specify explicitly the values taken by the righthand side of (4.12) on the subset N of $\mathbb{R}^n$ w here $\nabla V(x)$does not exist. Now, in order to consider Krasovskii solutions,we can't anymore ignore the values of the righthand side of (4.12) on zero measure sets, then we define the feedback law in a slightly different way.

Let

$$\tilde{\nabla} V(x) = \begin{cases} \nabla V(x) & if\ x \in \mathbb{R}^n \setminus N \\ \bar{p}\ where\ \bar{p}\ \ is\ any\ fixed\ vevtor\ in\ \partial_C V(x), & if\ x \in N \end{cases} \qquad (4.13)$$

We define

$$u(x) = -\alpha\left(\tilde{\nabla} V\ (x) G(x)\right)^T \qquad (4.14)$$

The righthand side of (4.12) is locally bounded and it makes sense to consider its Krasovskii solutions.

The essential tool in order to deal with Krasovskii solutions and make explicit computations is the analogous of Proposition(4.1.8).

Proposition(4.1.21):

(i) If $f \in C(\mathbb{R}^n; \mathbb{R}^n)\ then\ Kf(x) = \{f(x)\}$ for all

$x \in \mathbb{R}^n$.

(ii) If f, g are locally bounded then $K(f + g)(x) \subseteq Kf(x) + Kg(x)$ for all

$x \in \mathbb{R}^n$.

Moreover if $f \in C(\mathbb{R}^n; \mathbb{R}^n)$ then $K(f + g)(x) = f(x) + Kg(x)$ for all $x \in \mathbb{R}^n$.

(iii) If $G \in (\mathbb{R}^n; \mathbb{R}^{n\times m})$ and u is locally bounded then
$K(Gu)(x) = G(x)Ku(x)$ for all $x \in \mathbb{R}^n$.

(iv) If $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, then $K(\tilde{\nabla} V)(x) =$

$\partial_C V\ (x)$ for all $x \in \mathbb{R}^n$.

The proof of (i), (ii) and (iii) is perfectly analogous to that of Paden and Sastry (see [PS]), while the proof of (iv) needs some extra remarks.

Proof of (iv):

Let us first remark that from the definition of the multivalued map Kf it follows that, in general, $Kf(x)$ = co$\{\lim_i f(x_i), x_i \to x\}$.

Let us prove that $\partial_C V\ (x) \subseteq K(\widetilde{\nabla} V\ (x))$. Let $p \in \partial_C V\ (x)$. Because of (3.8) and there exist m sequences $\{x_i^{(k)}\}, x_i^{(k)} \to x$ as $i \to \infty 1, x_i^{(k)} \notin N, k = 1, \dots, m$ and m scalars $\lambda_k > 0$ with $\sum_{k=1}^{m} \lambda_k = 1$ _k = 1 such that $p = \sum_{k=1}^{m} \lambda_k \lim_i \nabla V\left(x_i^{(k)}\right)$ Since $\nabla V\ (x) = \widetilde{\nabla} V\ (x)$ on N, we have that $p = \sum_{k=1}^{m} \lambda_k \lim_i \widetilde{\nabla} V\left(x_i^{(k)}\right)$, and then $p \in K(\widetilde{\nabla} V\ (x))$.

Let us now prove that $K(\widetilde{\nabla} V\ (x)) \subseteq \partial_C V\ (x). p$ 2 $K(\tilde{}rV\ (x))$ can be written as $p = \sum_{k=1}^{m} \lambda_k\, p_k$, where, for all $k = 1, \dots, m$,
$p_k = \lim_i \widetilde{\nabla} V\left(x_i^{(k)}\right), = \lambda_k > 0, \sum_{k=1}^{m} \lambda_k = 1$ and $x_i^{(k)} \to x\ as\ i \to \infty$.

Let us emphazise that $\widetilde{\nabla} V\ (x_i^{(k)}) \in \partial_C V\ (x_i^{(k)})$ for all , i. . Since $\partial_C V$, as a multivalued map from $\mathbb{R}^n$ to $2^{\mathbb{R}^n} \setminus \emptyset$, is upper semicontinuous then its graph is closed and $p_k \in \partial_C V\ (x)$ for all k.

From the convexity of $\partial_C V\ (x)$ it finally follows that $p \in \partial_C V\ (x)$.

Thanks to the previous proposition one can get stabilizations results perfectly analogous to those obtained in the context of Filippov solutions, simply by replacing Filippov's multivalued maps with Krasovskii's ones.In particular, also when Krasovskii solutions are considered, the damping feedback may destabilize the system. This can be still proven by means of Example(4.1.10) and slightly different computations.

Section(4.2)External Stabilization:

In the present section we apply the technique we have used for the stabilization of discontinuous systems affine in the control to the problem of external stabilization of the same kind of systems.

We begin by study UBIBS Stability:

Let us consider a time dependent nonlinear system of the form

$$\begin{cases} \dot{x} = f(t,x,u) \\ x(t_0) = x_0 \end{cases} \quad (4.15)$$

where $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is locally essentially bounded and measurable with respect to (t,x) and continuous with respect to u. As in this Chapter, solutions of (4.15) are intended in the Filippov's sense.

We are interested in intrinsic stability properties which take into account the presence of the control in the system. Many different concepts have been recently introduced: ISS (input-to-state stability), iISS (integral input-to-state stability), IOS (input-to-output stability), OSS (output-tostate stability), BIBO (bounded-input bounded-output) stability, UBIBS (uniform bounded input bounded state) stability (see [43] and [BM] for an overview on these problems).

We focus on UBIBS stability.

Definition(4.2.1):

System (4.15) is said to be UBIBS stable if for each $R > 0$ there exists $S > 0$ such that for each $(t_0, x_0) \in \mathbb{R}^{n+1}$, and each input $u \in L^{\infty}_{loc}(\mathbb{R};\ \mathbb{R}^m)$, if $\varphi(t)$ indicates any solution of (4.15), one has

$$\|x_0\| < R, \|u\|_\infty < R \implies \|\varphi(t)\| < S\ \forall t \geq t_0.$$

UBIBS stability is related to Lagrange stability, that, roughly speaking, can be seen as (Lyapunov) stability "in the large".

If we posit $u = 0$ in system (4.15) we get the unforced system

$$\begin{cases} \dot{x} = f(t,x,0) \\ x(t_0) = x_0 \end{cases} \quad (4.16)$$

Definition(4.2.2)

System (4.16) is said to be (uniformly) Lagrange stable if for each $\mathbb{R} > 0$ there exists $S > 0$ such that for each $(t_0, x_0) \in \mathbb{R}^{n+1}$, if $\varphi(t)$ indicates any solution of (4.16), one has

$$\|x_0\| < R\ ) \implies (t)k < \|\varphi(t)\| \geq t_0.$$

It is clear that if (4.15) is UBIBS stable then (4.16) is Lagrange stable, while the converse is not true.

Lagrange stability has been characterized in terms of (smooth) Lyapunov-like functions in [117] in the case of a system with continuous righthand side and then this result has beem generalized in [5] to the case of discontinuous systems.

Definition(4.2.3):

$V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a Lyapunov-like function for

$$\dot{x} = f(t,x) \tag{4.17}$$

if there exists $S > 0$ such that for each solution $\varphi(t)$ of (4.17) and each pair of points t_1, t_2 such that $\|\varphi(t)\| \geq S$ for all $t \in [t_1, t_2]$, condition (3.10) holds.

Lyapunov-like functions differ from Lyapunov functions for the fact that they have to be defined "in the large", while Lyapunov functions could have been defined only on a neighbourhood of the origin.
In [VL, BR1], sufficient conditions for UBIBS stability in terms of (smooth) control Lyapunov-like functions have been given.

Definition(4.2.4):

$V : \mathbb{R}^{n+1} \to \mathbb{R}$ is a control Lyapunov-like function for

$$\dot{x} = f(t,x,u) \tag{4.18}$$

if for all $R > 0$ there exists $S > 0$ such that for each control such that

$\|u\|_\infty < R$, and each solution $\varphi(t)$ of

$$\dot{x}(t) = f(t,x(t),u(t)) \tag{4.19}$$

one has that for each pair of points t_1, t_2 such that $\|\varphi(t)\| \geq S$ for all $t \in [t_1, t_2]$, condition (3.10) holds.

As for Lyapunov functions, in order to verify if a given function V is actually a control Lyapunov-like function, it is important to have sufficient conditions for V to decrease along trajectories of (4.19) that don't involve explicitly neither the control function, nor the solutions of the system.

We give a result analogous to Theorem(1) in [106] and Theorem(6.2) in[5]. It differs from both for the fact that it involves control Lyapunov-like functions which are not of class C^1, but just locally Lipschitz continuous and healthy.

Lemma(4.2.5):

If there exists a control Lyapunov-like function V for (4.15), such that

(V0t) there exist $L > 0$ and two continuous, strictly increasing, positive functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $\lim_{r \to +\infty} a(r) = +\infty$ and for all $t \geq t_0$ and for all x

$$\|x\| > L \,) \, a(\|x\|) \leq V\,(t, x) \leq b(\|x\|)$$

then system (4.15) is UBIBS stable.

Proof:

We prove the statement by contradiction, by assuming that there exists $\bar{R}$ such that for all $S > 0$ there exist $\overline{x_0}$ and $\bar{u}$: $[0, +\infty) \to \mathbb{R}^m$ such that $\|\overline{x_0}\| < \bar{R}$, $\|\bar{u}\|_\infty < \bar{R}$ and there exists a solution $\bar{\varphi}\,(t)$ of (4.19) with $u = \bar{u}$, and $\bar{t} > 0$ such that $\|\bar{\varphi}(\bar{t})\| \geq S$.

Because of (V0t), there exists $S_M > 0$ such that if $\|x\| > S_M$, then

$V\,(t, x) > M = b(\bar{R}) \geq \max\{V\,(t, x), \|x\| = \bar{R}, t \geq 0\}$ for all $t \geq t_0$.

Let us consider $S > \max\{\bar{R}, S_M\}$. By hypothesis there exist $\overline{x_0}$ and $\bar{u} : [0, +\infty) \to \mathbb{R}^m$ such that $\|\overline{x_0}\| < \bar{R}, \|\bar{u}\|_\infty < \bar{R}$ and there exists a solution $\varphi(t)$ of (4.19) with $u = \bar{u}$, and $\overline{t} > 0$ such that $\|\bar{\varphi}(\bar{t})\| \geq S$. Then there also exist $t_1, t_2 > 0$ such that $\bar{t} \in [t_1, t_2]$, $\|\bar{\varphi}(t_1)\| = \bar{R}, \|\bar{\varphi}(t)\| \geq \bar{R}$, for all $t \in [t_1, t_2]$ and $\|\bar{\varphi}(t_2)\| \geq S$.Then

$$V\,(t_2, \bar{\varphi}(t_2)) > M \geq V\,(t_1, \bar{\varphi}(t_1)). \tag{4.20}$$

that contradicts (3.10).

Before stating next lemma, let us introduce some set-valued derivatives.

$\dot{\bar{V}}^{(4.15)}\,(t, x, u) = \{a \in \mathbb{R} : \exists v \in Ff(t, x, u)$ such that

$$\forall p \in \partial_C V\,(t, x) \quad p \cdot (1, v) = a\}.$$

Analogously, if $t > 0, x \in \mathbb{R}^n$ and a measurable and locally essentially bounded $u : \mathbb{R} \to \mathbb{R}^m$ are fixed, we set

$\cdot\dot{\bar{V}}^{(4.19)}_{u(.)}\,(t, x) = \{a \in \mathbb{R} : \exists v \in Ff(t, x, u(t))$ such that

$$\forall p \in \partial_C V\,(t, x) \quad p \cdot (1, v) = a\}.$$

and, if $t > 0$ and $x \in \mathbb{R}^n$are fixed, we define

$\dot{\bar{V}}^{(4.16)}\,(t, x) = \{a \in \mathbb{R} : \exists v \in Ff(t, x, u(t))$ such that

$$\forall p \in \partial_C V\,(t,x) \quad p \cdot (1,v) = a\}.$$

Note that if $\bar{\varphi}$ is any solution of (4.19) with $u(t) = \bar{u}(t)$ we have

$$\dot{\bar{V}}_{\bar{u}(.)}^{(4.19)}\,\big(t,\bar{\varphi}(t)\big) \subseteq \dot{\bar{V}}^{(2-15)}(t,\bar{\varphi}(t),\bar{u}(t)).$$

Lemma (4.2.6):

Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ locally Lipschitz continuous and healthy.If
(V0t) holds and
(fut) for all $R > 0$ there exists $S > \max\{L,R\}$ such that for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$ the following holds:

$$\|x\| > S, \|u\| < R \Longrightarrow \max \dot{\bar{V}}^{(4.15)}\,(t,x,u) \leq 0 \text{ for a.e. } t \geq 0$$

then V is a control Lyapunov-like function for (4.15).

Proof:

Let $R > 0$ be fixed and let us choose S corresponding to R as in (fut). Let us also fix $\bar{u}$ such that $\|\bar{u}\|_\infty < R$. Let $\bar{\varphi}(t)$ be any solution of (4.19) with $u = \bar{u}$ and let t_1, t_2 be such that for all $t \in [t_1,t_2]$ one has $\|\bar{\varphi}(t)\| \geq S$. By Remark(3.2.9)in chapter3), $\frac{d}{dt}V(t,\bar{\varphi}(t)) \in \dot{\bar{V}}_{u(.)}^{(2-15)}\,(t,\bar{\varphi}(t))$a.e.. Moreover, as remarked before stating the lemma,$\dot{\bar{V}}_{\bar{u}(.)}^{(4.19)}\,\big(t,\bar{\varphi}(t)\big) \subseteq \dot{\bar{V}}^{(4.15)}\big(t,\bar{\varphi}(t),\bar{u}(t)\big)$.
Since $\|\bar{u}\| < R$ a.e. and $\|\bar{\varphi}\| > S$ for all $t \in [t_1,t_2]$, by virtue of (fut) we have $\frac{d}{dt}V\big(t,\bar{\varphi}(t)\big) \leq 0$for a.e. $t \in [t_1,t_2]$. We get that $V o\,\bar{\varphi}$ decreases in $[t_1,t_2]$.

The following theorem is now an obvious consequence of the two previous lemmas.

Theorem(4.2.7):

Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be locally Lipschitz continuous and healthy and such that (V0t) and (fut) hold. Then system (4.15) is UBIBS stable.

Remark(4.2.8):

In order to get a sufficient condition for system (4.16) to be uniformly Lagrange stable, one can state Theorem(4.2.7)in the case $u = 0$. In this case the control Lyapunov-like function V simply becomes a Lyapunov like function.

Remark(4.2.9):

If system (4.15) is autonomous it is possible to state a theorem analogous to Theorem(4.2.7)for a control Lyapunov-like function V not depending on time.

Now we discuss the UBIBS Stabilizability:

We now turn our attention to the external stabilizabity property associated to UBIBS stability.

Definition(4.2.10):

System (4.15) is said to be UBIBS stabilizable if there exists a function $k \in L^{\infty}_{loc}(\mathbb{R}^{n+1}; \mathbb{R}^m)$ such that the closed loop system

$$\dot{x} = f(t, x, k(t, x) + v) \qquad (4.21)$$

(with v as input) is UBIBS stable.

We study UBIBS stabilization for systems of the form

$$\dot{x} = f(t, x) + G(t, x)u = f(t, x) + \sum_{i=1}^{m} u_i g_i(t, x) \qquad (4.22)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is measurable and locally essentially bounded, $g_1, \dots, g_m \in C(\mathbb{R}^{n+1}; \mathbb{R}^n)$ for all $i \in \{1, \dots, m\}$ and G is the matrix whose columns are $g_1, \dots, g_m$.

We are interested in finding conditions which guarantee UBIBS stabilizabilty of system (4.22) when the unforced system(3.1) is known to be Lagrange stable. Our result essentially recalls Theorem(6.2)in [5] and Theorem(5) in [91], with the difference that the control Lyapunov-like function involved is not smooth.

We do not give a unique condition for system (4.22) to be UBIBS stabilizable, but some alternative conditions which, combined together, give the external stabilizability of the system. Before stating the theorem we list these conditions. Note that the variable x is not yet quantified. Since its role depends on different situations, it is convenient to specify it later.

(f1t) max $\dot{\bar{V}}^{(3.1)}(t, x) \leq 0$;

(f2t) for all $z \in Ff(t, x)$ there exists $\bar{p} \in \partial_C V\ (t, x)$ such that $\bar{p} \cdot (1, z) \leq 0$;

(f3t) for all $z \in Ff(t, x)$ and for all $p \in \partial_C V\ (t, x), p \cdot (1, z) \leq 0$;

(G1t) for each $i \in \{1, \dots, m\}$ there exists $c_{t,x}^{i} \in \mathbb{R}$ such that for all $p \in \partial_C V(t,x), p \cdot \big(1, g_i(t,x)\big) = c_{(t,x)}^{i}$;

(G2t) for each $i \in \{1, \dots, m\}$ only one of the following mutually exclusive conditions holds:

– for all $p \in \partial_C V(t,x) \quad p \cdot (1, g_i(t,x)) > 0$,

– for all $p \in \partial_C V(t,x) \quad p \cdot (1, g_i(t,x)) < 0$,

– for all $p \in \partial_C V(t,x) \quad p \cdot (1, g_i(t,x)) = 0$;

(G3t) there exists $\in \{1, \dots, m\}$ such that for each $i \in \{1, \dots, m\} \backslash \{\bar{\imath}\}$ only one of the following mutually exclusive conditions holds:

– for all $p \in \partial_C V(t,x) \quad p \cdot (1, g_i(t,x)) > 0$,

– for all $p \in \partial_C V(t,x) \quad p \cdot (1, g_i(t,x)) < 0$,

– for all $p \in \partial_C V(t,x) \quad p \cdot (1, g_i(t,x)) = 0$;

Let us remark that (f3t)⟹(f2t)⟹(f1t) and (G1t)⟹(G2t)⟹(G3t).

Theorem(4.2.11):

Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be locally Lipschitz continuous, healthy and such that there exists $L > 0$ such that (V0t) hold.
If for all $x \in \mathbb{R}^n$ with $\|x\| > L$ one of the following couples of conditions holds for a.e. $t \geq 0$:
(i) (f1t) and (G1t), (ii) (f2t) and (G2t), (iii) (f3t) and (G3t), then system (4.22) is UBIBS stabilizable.

Let us make some remarks.

If for all $x \in \mathbb{R}^n$ with $\|x\| > L$ assumption (f1t) (or (f2t) or (f3t)) holds for a.e. $t \geq t_0$, then, by Theorem(4.2.7), system (3.1) is uniformly Lagrange stable.Actually in [BR1] the authors introduce the concept of robust uniform Lagrange stability and prove that it is equivalent to the existence of a locally Lipschitz continuou Lyapunov-like function. Then assumption (f1t) (or (f2t) or (f3t)) implies more than uniform Lagrange stability of system (3.1).In [Ro], the author has also proved that, under mild additional assumptions on f, robust Lagrange stability implies the existence of a C^∞ Lyapunov-like function, but the proof of this result is not actually constructive. Then we could still have to deal with nonsmooth Lyapunov-like functions even if we know that there exist smooth ones.

Moreover Theorem(4.2.11)can be restated for autonomous systems with the function V not depending on time. In this case the feedback law is autonomous and it is possible to deal with a situation in which the results in [Ro] do not help.

Finally let us remark that if f is locally Lipschitz continuous, then, by [117] (page 105), the Lagrange stability of system (3.1) implies the existence of a time-dependent Lyapunov-like function of class C^{∞}. In this case, in order to get UBIBS stabilizability of system (4.22), the regularity assumption on G can be weakened to $G \in L^{\infty}_{loc}(\mathbb{R}^{n+1};\mathbb{R}^{m})$ (as in [3]).

Now we discuss the Proof of Theorem(2-2-11):

We first state and prove a lemma.

Lemma(4.2.12):

Let $V : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that there exists $L > 0$ such that (V0t) and (V1) hold. If $(\bar{t}, \bar{x})$, with $\|\bar{x}\| > L$, is such that, for all $p \in \partial_C V\ (\bar{t}, \bar{x})\ p \cdot (1, g_i(\bar{t}, \bar{x})) > 0$, then there exists $\delta_{\bar{x}} > 0$ such that, for all $x \in B(\bar{x}, \delta_{\bar{x}})$, for all $p \in \partial_C V\ (\bar{t}, x)\ p \cdot (1, g_i(\bar{t}, x)) > 0$.

Analagously if $(\bar{t}, \bar{x})$, with$\|\bar{x}\| > L$, is such that for all $p \in \partial_C V\ (\bar{t}, \bar{x})$ $p \cdot (1, g_i(\bar{t}, \bar{x})) < 0$, then there exists $\delta_{\bar{x}} > 0$ such that, for all $x \in B(\bar{x}, \delta_{\bar{x}})$,for all $p \in \partial_C V\ (\bar{t}, x)\ p \cdot (1, g_i(\bar{t}, x)) < 0$.

Proof:

Let $\gamma < 0$ be such that $\|\bar{x}\| > L + \gamma$, and let $L_{\bar{x}} > 0$ be the Lipschitz constant of V in the set $\{\bar{t}\} \times B(\bar{x}, \gamma)$. For all $(\bar{t}, x) \in \{\bar{t}\} \times B(\bar{x}, \gamma)$ and for all $p \in \partial_C V\ (\bar{t}, x) \|p\| L_{\bar{x}}$ (see [54], page 27).

Since g_i is continuous there exist η and M such that $\|(1, g_i(\bar{t}, x))\| \leq M$ in $\{\bar{t}\} \times B(\bar{x}, \eta)$.

Let $d = \min\{p \cdot (1, g_i(\bar{t}, \bar{x})), p \in \partial_C V\ (\bar{t}, \bar{x})\}$. By assumption $d > 0$.

Let us consider $\epsilon < \frac{d}{2(L_{\bar{x}} + M)}$.

By the continuity of g_i, there exists δ_i such that, if $\|x - \bar{x}\| < \delta_i$, then

$\|(1, g_i(\bar{t}, x)) - (1, g_i(\bar{t}, \bar{x}))\| < \epsilon$.

By the upper semi-continuity of $\partial_C V$ (see [54], page 29), there exists

$\delta_V > 0$ such that, if $\|x - \bar{x}\| < \delta_V$, then $\partial_C V(\bar{t}, x) \subseteq \partial_C V(\bar{t}, \bar{x}) + \epsilon B(0,1)$, i.e. for all $p \in \partial_C V(\bar{t}, x)$ there exists $\bar{p} \in \partial_C V(\bar{t}, \bar{x})$ such that $\|p - \bar{p}\| < \epsilon$.

Let $\delta_{\bar{x}}$= min$\{\gamma, \eta, \delta_i, \delta_V\}$, x be such that $\|x - \bar{x}\| < \delta_x$ and $p \in \partial_C V(\bar{t}, x), \bar{p} \in \partial_C V(\bar{t}, \bar{x})$ be such that $\|p - \bar{p}\| < \epsilon$.

It is easy to see that $\left|p.\left(1, g_i(\bar{t}, x)\right) - \bar{p}.(1, g_i(\bar{t}, \bar{x}))\right| < \frac{d}{2}$, hence

$p.\left(1, g_i(\bar{t}, x)\right) > \bar{p} - \left(1, g_i(\bar{t}, \bar{x})\right) - \frac{d}{2} = \frac{d}{2} > 0.$

The second part of the lemma can be proved in a perfectly analogous way.

Proof of Theorem(4.2.11):

For each $x \in \mathbb{R}^n$, let N_x be the zero-measure subset of $\mathbb{R}^+$ in which no one of the couples of conditions (i), (ii) and (iii) holds. Let $k : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $k(x) = (k_1(t, x), \ldots, k_m(t, x))$, be defined by

$$k_i(t, x) = \begin{cases} -\|x\| & \text{if } \forall p \in \partial_C V(t, x) \quad p.\left(1, g_i(t, x)\right) > 0 \\ 0 & \text{if } \forall p \in \partial_C V(t, x) \quad p.\left(1, g_i(t, x)\right) = 0 \\ & \text{or (f3)and}(G3) \text{ hold and } \; i = \bar{\imath}, or\ t \in N_x \\ \|x\| & \text{if } \forall p \in \partial_C V(T, X) P.\left(1, g_i(t, x)\right) < 0 \end{cases}$$

It is clear that $k : \mathbb{R} \to \mathbb{R}^m$ is locally essentially bounded.

By Theorem(4.2.7)it sufficient to prove that for all $R > 0$ there exists

$\rho > L, R$ such that for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ the following holds:

$$\|x\| > \rho, \|v\| < R \Longrightarrow \max \dot{\nabla}^{(4.21)}(t, x) \_ 0 \; for\ all\ t \in \mathbb{R}^+ \backslash N_x$$

where $\dot{\bar{V}}^{(4.22)}(t, x) = \{a \in \mathbb{R} : \exists w \in F(f(t, x) + G(t, x)k(t, x) + G(t, x)v)$ such that $\forall p \in \partial_C V(t, x)\, p \cdot (1, w) = a\}$.

Let x be fixed and $t \in \mathbb{R}^+ \backslash N_x$. Let $a \in \dot{\bar{V}}^{(4.22)}(t, x)$, $w \in F(f(t, x) + G(t, x)k(t, x) + G(t, x)v)$ be such that for all $p \in \partial_C V(t, x)\, p \cdot w = a$.

By Proposition(4.2.7) we have that $F(f(t, x) + G(t, x)(k(t, x) + v)(x) \subseteq Ff(t, x) + \sum_{i=1}^{m} g_i(t, x) F(k_i(t, x) +$

$v_i)$, then there exists $z \in Ff(t,x), z_i \in F(k_i(t,x) + v_i), i \in \{1,\ldots,m\}$, such that $w = z + \sum_{i=1}^{m} g_i(t,x) z_i$.

Let us show that $a \leq 0$. We distinguish the three cases (i), (ii), (iii).

(i) $b = p \cdot (1,z) = a - \sum_{i=1}^{m} c_i^{t,x} zi$ does not depend on p, then $b \in \dot{\bar{V}}^{(3.1)}(t,x)$ and, by (f1t), $b \leq 0$.

Let us now show that for each $\in \{1,\ldots,m\}$ $c_{t,x}^i z_i \leq 0$. If i is suchthat $c_{t,x}^i = 0$, obviously $c_{t,x}^i z_i \leq 0$.If i is such that $c_{t,x}^i > 0$. then, by Lemma(4.2.12), there exists δ_x such that $k_i(t,y) = -\|\mathcal{Y}\|\ in\ \{\bar{t}\} \times B(x,\delta_x)$, then k_i is continuous at x with respect to $\mathcal{Y}$. This implies that $F(k_i(t,x) + v_i) = -\|x\| + v_i, i.e. z_i = -\|x\| + v_i$ and $c_{t,x}^i z_i \leq 0$, provided that $\|v\| > \rho \geq \max\{L,R\}$.

The case in which i is such that $c_{t,x}^i < 0$ can be treated analogously. We finally get that $a = b + \sum_{i=1}^{m} c_{t,x}^i z_i \leq 0$.

(ii) By (f2t) there exists $\bar{p} \in \partial_C V\ (t,x)$ such that $\bar{p} \cdot (1,z) \leq 0. a = \bar{p} \cdot (1,z) + \sum_{i=1}^{m} \bar{p} . \big(1, g_i(t,x)\big) z_i$.The fact that for each $i \in \{1,\ldots,m\}$ we have $\bar{p} \cdot (1, g_i(t,x)) zi \leq 0$ can be proved as in (i) we have proved that for each $i \in \{1,\ldots,m\} c_{t,x}^i z_i \leq 0$. We finally get that $a \leq 0$. (iii) Let us remark that if (G2t) is not verified, i.e. we are not in the case (ii), there exists $\bar{p} \in \partial_C V\ (t,x)$ corresponding to $\bar{\imath}$ such that $\bar{p} . \big(1, g_i(t,x)\big) = 0$.Indeed, because of the convexity of $\partial_C V\ (t,x)$, for all $v \in \mathbb{R}^n$, if there exist $p_1, p_2 \in \partial_C V\ (t,x)$ such that $p_1 \cdot v > 0$ and $p_2 \cdot v < 0$, then there also exists $p_3 \in \partial_C V\ (t,x)$ such that $p_3 \cdot v = 0$.

Let $\bar{p} \in \partial_C V\ (t,x)$ be such that $\bar{p} \cdot (1, g_i(t,x)) = 0$. For all $p \in \partial_C V\ (t,x) a = p \cdot (1,w)$. In particular we have $a = \bar{p} \cdot (1,w) = \bar{p} \cdot (1,z) + \sum_{i \neq \bar{\imath}} \bar{p}. \big(1, g_i(t,x)\big) z_i + \bar{p} \cdot \big(1, g_{\bar{\imath}}(t,x)\big) z_{\bar{\imath}}$ .By (f3t), $\bar{p} \cdot (1,z) \leq 0$. If $i \neq \bar{\imath}$ i the proofthat $\bar{p} \cdot \big(1, g_i(t,x)\big) z_i \leq 0$ is the same as in (ii). If $i \neq \bar{\imath}$, because of the choice of $\bar{p}, \bar{p} \cdot (1, g_{\bar{\imath}}(t,x)) = 0$. Also in this case we can then conclude that $a \leq 0$.

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