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Perfect Measurable and Interpolation Theorems for Variable Exponent Lebesgue Spaces with Relatively Compact Sets

المقيسية التامة ومبرهنات الإستكمال لفضاءات لبيق أسية المتغير مع فئات التراص النسبي

A thesis Sumbittd for Partial Fulfilment of Degree of MSc. In Mathematics

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Dedication

To :

My Parents

Sister

and Brother .

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First and foremost, we would like to thank with all sincerity to the almighty God Allah (Glory to Him) who gave us the strength to do this research and made all difficult tasks very easy.

To my Supervisor, **Prof.Dr. Shawgy Hussein AbdAlla** who inspired our thoughts, encourage our mind to produce and supported us whenever it is needed.

Abstract

We give a general treatment of the union problem for not necessarily perfect or weakly perfect ,measurable spaces. We introduce variable exponent Lebesgue spaces on metric measure spaces and consider a central tool in geometric analysis and the Hardy –Littlewood maximal operator. We study relatively compact sets in variable Lebesgue spaces. The full characterization of such sets is given in the case of variable Lebesgue space on metric measure spaces .

الخلاصة

أعطينا معالجه عامة لمسألة الإتحاد ليست التامة الضرورية أوالتامة الضعيفه للفضاءات المقيسه. تم إدخال فضاءات لبيق أسية المتغير على فضاءات القياس المتري وأعتبرنا الاداء المركزيه في التحليل الهندسي والمؤثر الاعظمي لهاردي- لتلوود . درسنا فئات التراص النسبيه في فضاءات لبيق المتغيره . تم اعطاء التشخيص الكامل لمثل هذه الفئات في حالة فضاء لبيق المتغير على فضاءات القياس المتري

The contents

Chapter1

Perfect measurable spaces

We focused on the following (measurable) union problem: let $\mathcal{A}$ be a σ-algebra of subsets of a space X and$X_i \in \mathcal{A}(i \in I)$. What conditions on $\mathcal{A}$ and $X_i(i \in I)$. Guarantee the existence of a subfamily $X_i(i \in I)$ whose union is not measurable with respect to$\mathcal{A}$, i.e. does not belong to$\mathcal{A}$? Usually it is assumed that the family $X_i(i \in I)$is point-finite, i.e. $\{i \in I: x \in X_i\}$ is finite for all $x \in X$, and that the X_i are small, i.e. belong to some σ-ideal .

Section(1.1) Perfect and Weekly Perfect Measurable Space with Example :

During the recent years a good deal of attention was focused on the following (measurable) union problem: let $\mathcal{A}$ be a σ-algebra of subsets of a space X and $X_i \in \mathcal{A}(i \in I)$. What conditions on $\mathcal{A}$ and $X_i(i \in I)$. guarantee the existence of a subfamily $X_i(i \in I)$ whose union is not measurable with respect to $\mathcal{A}$, i.e. does not belong to$\mathcal{A}$? Usually it is assumed that the family $X_i(i \in I)$is point-finite, i.e.$\{i \in I: x \in X_i\}$ is finite for all $x \in X$, and that the X_i are small, i.e. belong to some a-ideal. The main goal of the present chapter is to give a general treatment of the union problem which yields most of the results in the literature as special cases.

Our own work received impetus from an unpublished result of R. Solovay, which is as follows:

Theorem (1.1.1) [1] : Let $X_i(i \in I)$be a disjoint partition of the interval [0, I] where each X_i has Lebesgue measure zero. Then there is some $J \subseteq I$such that$\bigcup\{Xi: (i \in I\}$ is not Lebesgue measurable.

Theorem(1.1.2) [1] : Let $f: [0,1] \longrightarrow X$ be Lebesgue measurable, where X iS a metric space . Then there is a closed separable $Y \subseteq X$ such that $f^{-1}(Y)$ has Lebesgue measure one.

Theorem(1.1.2) can also be restated as follows. Letting μ denote the Lebesgue measure on $[0,1]$, we can define a Borel measure ν on$X$ by setting

$$\nu(B) = \mu(f^{-1}(B))$$

For$B \subseteq X$, B Borel. theorem(1.1.2) is then equivalent to:there is a closed separable Y such that $\nu(Y) = 1$.

Another equivalent result is the following generalised Lusin theorem (In the mathematical field of real analysis , Lusin's theorem(or Luzin's theorem named for Nikolai Luzin), states that every measurable function is a continuous function on nearly all it's domain . In the informal formulation of J.E.littlewood, "every measurable function is nearly continuous")[5].

Theorem (1.1.3) [1]: Under the assumptions of theorem(1.1.2), there is a compact $K \subseteq [0,1]$such that $\mu(K) > 0$and $f \setminus K$ is continuous. Theorem(1.1.3) is well known when $X$is separable,being only a slight extension of the classical Lusin's theorem. To pinpoint further the nature of the above results it is necessary to comment on their relationship to the problem of the existence of real-valued measurable r.v.m cardinals. This was pointed out by Kuratowski. In outline, supposing that theorem(1.1.1) is false, one defines a measure ν on $\mathcal{P}(I)$by setting

$$\nu(J) = \mu(\cup\{Xi: i \in I\}) \quad (J \subseteq I).$$

Then ν is a countably additive diffuse probability measure on $\mathcal{P}(I)$,and the existence of a real-valued measurable cardinal follows.

Thus the importance of Theorem (1.1.1.) lies in eliminating the assumption that there is no r.v.m cardinal. This situation is fairly typical. the following result of Marczewski and Sikorski.

Theorem(1.1.4) [1] : Suppose that there is no r.v.m cardinal. Let X be a metric space and μ a Borel probability measure on X. Then there is a closed separable subspace Ysuch that $\mu(Y) = 1$. theorem(1.1.4) also easily implies Theorems(1.1.2) and (.1.3). (in the absence of r.v.m. cardinals),The main part of the proof of Theorem (1.1.4) is the construction of a covering of a set of positive measure by disjoint measure zero sets, there by creating the situation described at the outset.

Theorem(1.1.5) [1]: Let X be a metacompact space and , μ a regular Borel probability measure on X. Then either there is a closed Lindel of subspace Ysuch that $\mu(Y) = 1$, or X has a discrete subspace whose cardinality is real-valued measurable.

Once again, in the proof one considers a covering of a set of positive measure by open null sets. Exploiting the metacompactness one refines this covering to a point-finite one. Then, after further work, one succeeds in disjointizing the sets.

Thus, while studying measures on topological spaces it usually suffices to work with disjoint families. However, point-finite coverings enter naturally and it is not always obvious how to accomplish an adequate disjointization. In an abstract situation, this constitutes a much more difficult problem than the disjoint case. Another interesting point is that if the disjointization is not possible, the connection with r.v.m. cardinals in no longer clear. Unlike Theorem(1.1.1),it is clear that the assumption that there is no r.v.m. cardinal cannot be omitted in

Theorems(1.1.4) and (1.1.5) , (just consider a discrete space X of r.v.m. cardinality). A moment of reflection, however, shows that Theorem(1.1.2) partially restores the validity of Theorem(1.1.4), without assuming the non-existence of r.v.m. A more general result is due to Koumoullis and Pachl.

Theorem(1.1.6) [1] : Let X be a metric space and μ a perfect probability Borel measure on X. Then either μ is a Radon measure (that is, for every $\varepsilon > 0$ there is a compact subset K with $\mu(K) > 1 - \varepsilon$), or there is a closed discrete subset Y whose cardinal $|Y| is \{0,1\}$-measurable. Perfect measures were introduced by Gnedenko and Kolmogorov. Theorem(1.1.6) implies Lusin's Theorem for perfect measures on metric spaces and mappings into metric spaces with cardinality < the least measurable cardinal.For Radon measure spaces and mappings into any metric space, Lusin's Theorem was obtained by Fremlin. To generalize theorem(1.1.6) to developable spaces. The key to theorem(1.1.6) is the realization that Theorem(1.1.1) above remains valid for any perfect measure if $|I| < the\, least(\{0,1\})\_$ measurable cardinal. An important advantage associated with the setting of perfect measures is that the cardinality restrictions are much less severe (the least measurable cardinal as opposed to 2^{x0} which is at least implicit in Theorems (1.1.1),(1.1.2),(1.1.3). Having realized the significance of the union problem in measure theory it is natural to inquire what happens for other σ-algebras and ideals.For example, what happens when $\mathcal{A}$ is the σ-algebra of sets with the property of Baire and X_i $(i \in I)$ are meager? Recall that a subset S of a topological space has the property of Baire if Scan be expressed as $S = S_0 \, \Delta \, S_1$where$S_0$is open and S_1, is meager. Bukovsky proved:

Theorem (1.1.7) [1] : Let X_i $(i \in I)$ be a disjoint partition of the interval [0, l] into meager sets. Then there is some $J \subseteq I$ such that $\cup \{X_i \;: i \in J\}$ doesnot have the property of Baire. Another interesting result of this type was recently obtained by Louveau and Simpson .To describe their result, let $\mathbb{N}$ denote the space of natural numbers with the discrete topology. Then $2^{\mathbb{N}}$ traditionally denotes the space of non-empty closed subsets of $\mathbb{N}$ with the Vietoris topology. Louveau and Simpson consider the subspace $\Omega\, of\; 2^{\mathbb{N}}$ consisting of the infinite subsets of $\mathbb{N}$ and obtain the analogue of Theorem (1.1.7) for Ω. Actually, in their result X_i $(i \in I)$is point-finite. But Fremlin shows that in general if Theorem (1.1.7)holds for somespace X, then it also holds when X_i $(i \in I)$ is a point-finite covering of X.The space Ωhas recently come into prominence due to the work of Ellentuck who showed that Ω plays an important role ingeneralizing Ramsey's Theorem. Such generalizations of Ramsey's Theorem have ecently found applications in Banach space theory. Our results on measurable spaces generalize both theorem(1.1.7). aswellas theLouveau-Simpson result. The analogue of theorem(1.1.7) holds for other spaces (developable, Tech-complete and K-analytic). All results discussed so far were connected either with measure or category. A general theorem which includes both of these as special cases is due to Brzuchowskietal. In order to be able to state their result we need adefinition. An ideal $\mathcal{P} \subseteq \mathcal{P}(x)$, where $X$is a topological space, is said to have a Borel base if every set in $\mathcal{P}$ is a subset of a Borel set belonging to $\mathcal{P}$.

Theorem(1.1.8)[1] : Let X be a Polish space (that is,a complete separable metric space) and $\mathcal{P}$ be a σ-ideal with a Borel base. Let X_i $(i \in I)$be a point-finite covering of X by sets from $\mathcal{P}$. Then there is some$J \subseteq I$ $such\ that$ $\cup\{X_i : i \in I\}$does not differ from a Borel set by a set from $\mathcal{P}$. theorem(1.1.8) was obtained by Prikry for the case of measure and category for all separable (not necessarily complete) metric spaces. These results are presented here. They are not covered by the general theory of perfect measurable spaces whereas Theorem (1.1.8) above is.

For a set I, $\mathcal{P}(1)$denotes the power set of I and denotes the cardinal of I is the cardinal of the continuum.

All topological spaces X are assumed to be at least Hausdorff. $\mathfrak{B}(X)$denotes the Borel σ-algebra on X, i.e., the σ-algebra generated by the closed sets in X. we also consider the Baire σ-algebra $\mathfrak{B}_a$ (X) which is generated by the zero sets of continuous functions.

Asusual, if $(X, \mathcal{A})$is a measurable space and $Y$is a topological space, $f: X \to Y$ is $\mathcal{A}$-measurable if for all open $\cup \subseteq Y$ (equivalently, for all Boreal $\cup \subseteq Y$), f^{-1} $(A) \in \mathcal{A}$. Most of the subsequent discussion evolves around a triple $(X, \mathcal{A}, \wp)$ $where$ $(x, \mathcal{A})$is a measurable space and $\wp$is an ideal of subsets of X(i.e. $\wp$ is closed under taking subsets and finite unions). Usually, we assume that $\wp$ is a σ-ideal (i.e. $\wp$is closed under enumerable unions). In this case, if we set

$$\mathcal{A}_{\wp} = \{A \,\Delta\, Y : A \in \mathcal{A}, Y \in \wp\}$$

then $(X, \mathcal{A}_{\wp})$is a measurable space. We say that a family $\mathcal{Y} \subseteq \wp(X)$is a base for $\wp$ if every set in $\wp$ is contained in a set in $\mathcal{Y} \cap \wp$.If there is a base $\mathcal{Y} \subseteq \wp$ we say that $\wp$; has an $\mathcal{A}$-base.

Definition(1.1.9)[1]: $(X, \mathcal{A}, \wp)$is a perfect (resp. weaklyperfect) measurable space if whenever $f: X \to R$ is $\mathcal{A}$-measurable there is an analytic $A \subseteq f(X)$$such\ that$ $X - f^{-1}$ $(A) \in \wp(resp. f^{-1}$(A) $\notin \wp$). In addition,$(X, \mathcal{A}, \wp)$ is weakly perfect if $X \in \wp$.

Essentially the same concept of perfect measurable spaces was introduced independently by Fremlin who used the term ‘semi-perfect’. Our results on these spaces were obtained in 1981 while the first author was visiting the University of Minnesota. Clearly, if $(X, \mathcal{A}, \wp)$is perfect, then it is weakly perfect. Note that in the above definition $f^{-1}(A)$does not necessarily belong to $\mathcal{A}$(i.e.might benon- measurable). However,in most applications, f^{-1} (A)belongs to $\mathcal{A}$ or at least to $\mathcal{A}_{\wp}$. This is true if $\mathcal{A}$ is closed under the Suslin operation, or if $\wp$ is a σ-ideal with an $\mathcal{A}$-base and $(X, \mathcal{A}, \wp)$ is perfect.

The seminal instance of perfect measurable spaces is connected with the concept of perfect measures. Here we shall give a well known equivalent definitions : a probability measure space

$(X, \mathcal{A}, \mu)$is perfect if for every $\mathcal{A}$-measurable $f: X \to R$ there is a Borel subset of B of $\mathbb{R}$ such that $B \subseteq f(X) and\ \mu(f^{-1}(B)) = 1.$ The class of perfect measures includes Borel probability measures on separable complete metric spaces, or more generally Radon probability measures. This follows easily from Lusin's Theorem. Moreover, the completion of a perfect measure space is perfect, as is every $\{0,1\}$-valued probability measure. It is easy to see that iff $(X, \mathcal{A}, \mu)$is a probability measure space, it is perfect iff $(X, \mathcal{A}, \wp)$is perfect where $\wp$ is the ideal of μ-measure zero sets. One direction follows from every Borel set being analytic and the other direction from every analytic set universally measurable.

We can define $(X, \mathcal{A}, \mu)$ to be weakly perfect if the corresponding $(X, \mathcal{A}, \wp)$ is weakly perfect. For a probability Borel measure μ on a separable metric space we have that μ is perfect iff μ is Radon and thatμ is weakly perfect iff there is a compact set of positive measure. Both of these results follow from the proof of Lemma (1.1.19).

The σ-algebra of sets with the property of Baire and the corresponding ideal $-$ that of meager sets$-$ also provide examples of perfect (resp. weakly perfect) measurable spaces, which we shall call perfect (resp. weakly perfect) category spaces. If the underlying space X is a complete separable metric space, then we have an example of a perfect category space. This can be seen by usingthe theorem that if $f: X \to \mathbb{R}$ has the property of Baire, then there is a dense $G_\delta - set\ G$ such that $f \setminus G$is continuous.

IF X is an analytic separable metric space, then$(X, \mathfrak{B}(X), \{\emptyset\})$is perfect, and thus $(X, \mathfrak{B}(X), \wp)$,for an arbitrary $\wp$ is likewise perfect. More generally, $(X, \mathcal{A}, \{\emptyset\})$clearly is perfect iff $f(X)$is analytic for every $\mathcal{A}$-measurable If $(X, \mathcal{A}, \wp)$These are exactly the smooth spaces of Falkner. If $(X, \mathcal{A}, \wp)$ is perfect and$\wp$ is a σ-ideal with an $\mathcal{A}$-base, then $(X, \mathcal{A}, \wp)$is perfect, and similarly for 'weakly perfect'. Hence in particular, if$X$ is an analytic separable metric space and $\wp$ is a σ-ideal with a Borel base, then$(X, \mathfrak{B}(X), \wp)$is perfect.

Theorem(1.1.10) [1] : Let $(X, \mathcal{A}, \wp)$be a perfect (resp. weakly perfect) maesrable space and $X_i(i \in I)\ be\ a$ point-finite covering of X.set

$\mathcal{A}' = \{J \subseteq I: \{X_i : i \in J\} \in \mathcal{A}\},$

$\wp' = \{J \subseteq I: \{X_i : i \in J\} \in \wp\}.$

This theorem is relatively easy to prove if $X_i : i \in I$are disjoint. Then $\mathcal{A}'$ is automatically a σ-algebra. In general $\mathcal{A}'$ is not a σ-algebra; but it is always closed under enumerable unions. Our proof shows that in general the 'paved' space $(I, \mathcal{A}', \wp')$ is perfect (resp. weakly perfect).

Proof : Let $f: I \to (0,1)\ be\ \mathcal{A}'$-measurable. We shall show that there is an analytic $A' \subseteq f(I) such\ that\ I - f^{-1}(A') \in \wp (resp. f^{-1}(A') \notin \wp).$

Let $g: X \to [0,1]^{\mathbb{N}}$ be the associated many-valued function. Then g is $\mathcal{A}$- measurable by Lemma(1.1.17). Since $(X, \mathcal{A}, p)$ is perfect (resp. weakly perfect), there is, by Lemma(1.1.18), an analytic $A \subseteq G(X) such\ that\ X - g^{-1}(A) \in p (resp. g^{-1}(A) \notin p)$. Hence all $\pi_k(A)$ are analytic subsets of $[0,1]$and so is $\bigcup_{k\in\mathbb{N}} \pi_k(A)$.

Moreover, setting

$$A' = \bigcup_{k\in\mathbb{N}} \pi_k(A) - \{1\},$$

We have that A' is analytic and $A' \subseteq f(I)$.

We claim that $X_i \cap g^{-1}(A) = \emptyset\ for\ all\ i \in I - f^{-1}(A')$.Indeed, $if\ x \in X_i \cap g^{-1}(A)$ then, by the definition of g, we have $f(i) = \pi_k\big(g(x)\big)$for some$k\ and\ f(i) \neq 1$.Since $g(x) \in A, f(i) \in \pi_k(A) - \{1\} \subseteq A'$.

$If\ X - g^{-1}(A) \in p$,then by the claim

$$\cup \{X_i : i \in I - f^{-1}(A')\} \in p,$$

that is,$I - f^{-1}(A') \in p'$.Similarly $g'(A) \notin p'$. This completes the proof. For the proof of Theorem(1.1.11) we need to analyze perfect and weakly perfect spaces of the form$(X, p(X), p)$.

Theorem(1.1.11) [1] : Let $(X, \mathcal{A})$be a measurable space and $p\ a\ \sigma$-ideal on X. Let$X_i : i \in I$ be a point-finite covering of X. Also suppose that one of the conditions (a), (b) below holds:

(a) $(X, \mathcal{A}, p)$ is perfect, $|I|$ <the least measurable cardinal and there is no $J \subseteq I\ \ such\ that\ |J| \leq x_0\ and\ X - \cup \{X_i : i \in J\} \in p$.

(b)$(X, \mathcal{A}, p)$isweakly perfect,$|I| \leq c\ and\ X_i \in p\ \ for\ all\ i \in I\ \ and\ x \notin p$. then there is some $J \subseteq I\ such\ that\ \cup \{X_i : i \in J\} \notin \mathcal{A}$.Moreover, if $\mathcal{A}$-base, then there is some $J \subseteq I\ such\ that\ \cup \{X_i : i \in J\} \notin \mathcal{A}_p$.

Proof : We start by proving the first conclusion of theorem(1.1.11). We define a σ-ideal $p'\ on\ I$ as in the statement of theorem (1.1.10). Supposing that case (a) is false it follows from theorem(1.1.10)that $(I, p(1), p')$is perfect. Applying Lemma(1.1.20) for the partition of Iinto singletons, we conclude that $1 - J \in p'$for some at most enumerable J.Hence it easily follows that $X - \cup \{X_i : i \in J\} \in p$, a contradiction.The proof of case (b)follows similarly by contradiction applying Lemma(1.1.21).

The second conclusion follows from the first and Lemma(1.1.22). concluding the proof of Theorem(1.1.11).

Theorem(1.1.12) [1] : Let $(Y_\alpha, \mathcal{A}_\alpha), (\alpha \in K)$, be measurable spaces, where K is arbitrary, and $|Y_\alpha|$ < the least measurable cardinal. Set

$$X = \prod_{\alpha \in K} Y_\alpha$$

And let $\mathcal{A}$ be the usual product σ-algebra on X. Suppose that $\mathscr{p}$ is any σ-ideal such that $(X, \mathcal{A}, \mathscr{p})$ is perfect and $X_i : i \in I$ a point-finite covering of X such that there is no $J \subseteq I \; with \; |J| \leq x_0 \;\; and \; X - \bigcup\{X_i : i \in J\} \in \mathscr{p}$. Then there is some $J \subseteq I \; such \; that \; \cup \{X_i : i \in J\} \notin \mathcal{A}$. Moreover, if $\mathscr{p}$ has an $\mathcal{A}$-base, then there is some $J \subseteq I \; suth \; that \; \cup \{X_i : i \in J\} \notin \mathcal{A}_{\mathscr{p}}$.

Proof: As in Theorem (1.1.11)the second conclusion follows from the first and Lemma(1.1.22) So we shall prove only the first conclusion. Suppose that this is false. We proceed as in the proof of Lemma(1.1.20)(a)→ (c).So we can assume that $X_i \in \mathcal{A} \; for \; all \; i \in I \; and \; that \; X \notin \mathscr{p}$.Then we define

$$\mathscr{p}' = \{J \subseteq I : \cup \{X_i : i \in J\} \in \mathscr{p}\}$$

So that $(1, \mathscr{p}(1), \mathscr{p}')$is perfect and $\mathscr{p}' is \; \omega_1$-saturate Theorem(1.1.10) and Lemma(1.1.19)). Again by Ulam's Theorem, either there exists some $J \subseteq I \; such \; that \; \mathscr{p}' \cap \mathscr{p}(1)$is a proper maximal ideal on J,or there exists a partition of I into $\leq c$sets in $\mathscr{p}'$. By Theorem (1.1.11)the second alternative in Ulam's Theorem cannot occur.

Now if J is as in the first alternative, let λ be the additivity degree of $\mathscr{p}' \cap \mathscr{p}(J). \, then \; \lambda$ is a measurable cardinal and there are disjoint sets $I_\alpha \in \mathscr{p}' \cap \mathscr{p}(J)(\alpha \in \lambda) such \; that$

$$J = \cup \{I_\alpha : \alpha \in \lambda\}.$$

We consider the family

$$X'_\alpha = \cup \{X_i : i \in I_\alpha\}(\alpha \in \lambda).$$

This is a point-finite family of sets from $\mathscr{p}$,the union of every subfamily of X_α belongs to $\mathcal{A}$and

$$(X_1 = \cup \{X'_\alpha : \alpha \in \lambda\} \notin \mathscr{p}.$$

Now we can extend the perfect space

$$(X_1, \wp(X_1) \cap \mathcal{A}, \wp \cap \wp(X_1))$$

Back to X,putting $X - X_1$to a new ideal, the resulting space being perfect. This means that we can assume that $I = \lambda \ and \ \wp' is \ \lambda$complete.

Now we set $Z_\xi = \bigcup_{\eta \leq \xi} X_\eta (\xi \in \lambda)$. Since $\wp' is \ \lambda$complete $Z_\xi \notin \wp$. Since $\mathcal{A}$ is a product σ-algebra and $Z_\xi \in \mathcal{A}$, we can pick an enumerable $E_\xi \subseteq \lambda$such that

$$Z_\xi = C_\xi \times \prod \{Y_\alpha : \alpha \in \lambda - E_\xi\}$$

And

$$C_\xi = \prod \{Y_\alpha : \alpha \in E_\xi\}$$

for every $\xi \in \lambda$.

By the Δ-system of Erdos and Rado.There is some $E \subseteq \lambda \ and \ J \subseteq \lambda \ such \ that \ |J| = \lambda \ and \ for \ all \ \xi \ and \ \eta \in \lambda$, if $\xi \neq \eta \ then \ E_\xi \cap E_\eta = E$. Pick $t_\xi \in C_\xi \ for \ \xi \in J$. Since $|\prod\{Y_\alpha : \alpha \in E\}|$ <the least measurable cardinal, there is some k$\subseteq J \ and \ t \in \prod\{Y_\alpha : \alpha \in E\} \ such \ that \ |K| = \lambda \ and \ t_\xi \setminus E = t \ for \xi \in K$. It is now easy to find an $x \in X \ such \ that \ x \setminus E_\xi = t_\xi \ for \ all \ \xi \in K$.Hence$x \in Z_\xi \ for \ all \ \xi \in K$,contradicting the point-finiteness and the proof is complete

Theorem(1.1.13) [1] : If $(X, \mathcal{A}, \wp)$is a perfect measurable space and $X_i : (i \in I)$is a point-finite covering of$X \ such \ that X_i \notin \wp \ and \cup \{X_i : i \in J\} \in \mathcal{A} \ for \ all \ J \subseteq I$, then$I$at most enumerable. We also have useful characterizations of perfect and weakly perfect measurable spaces (Theorems (1.1.14) and (1.1.15).

Proof: Define

$$\wp' = \{J \subseteq I : \cup \{X_i : i \in I\} \in \wp\}.$$

By Theorem(1.1.10), $(1, \wp(X), \wp')$is perfect and Lemma(1.1.19) implies that $\wp' is \ \omega_1$-saturated. Since$\{i\} \notin \wp' for \ every \ i \in I$, it follows that I is at most enumerable.

Theorem(1.1.14)[1] : For a triple $(X, \mathcal{A}, \wp)$the following are equivalent:

(a) $(X, \mathcal{A}, \wp)$ is perfect.

(b) For every countably generated σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ there is some $f: X \to \mathbb{R}$ and an analytic set $A \subseteq f(X)$ such that

(i)$\mathcal{A}' \subseteq f^{-1}\left(\mathfrak{B}(\mathbb{R})\right)$, and (ii) $X - f^{-1}(A) \in \wp$.

(c) For every Suslin scheme $A(s) \in \mathcal{A}$where s ranges over finite sequences from $\mathbb{N}$, there is an analytic $C \subseteq \mathbb{N}^{\mathbb{N}}$such that

(i) $\bigcap_{n\in\mathbb{N}} A(\sigma \setminus n) \neq 0 \; for \; \sigma \in C$, and (ii) $\bigcup_{\sigma \notin C} \bigcap_{n\in\mathbb{N}} A(\sigma \setminus n) \in \wp$.

(d) This is the same as (c), but in addition,$A(s)$are decreasing and disjoint in the sense that if$\sigma, \tau \in \mathbb{N}^{\mathbb{N}}$ $and \; \sigma \neq \tau, then$

$$\bigcap_{n\in\mathbb{N}} A(\sigma \setminus n) \cap \bigcap_{n\in\mathbb{N}} A(\sigma \setminus n) \neq \emptyset.$$

Proof : $(a) \to (b). Let \; \mathcal{A}' \subseteq \mathcal{A}$ be a countably generated σ- algebra. Then there is some$f: X \to \mathbb{R} \; such \; that \; \mathcal{A}' = f^{-1}\left(\mathfrak{B}(\mathbb{R})\right)$. Hence $f \; is \; \mathcal{A} -$ measurable and there is an analytic $A \subseteq f(X) such \; that \; X - f^{-1}(A) \in \wp.$

$(b) \to$ (c). Let$\mathcal{A}'$ be the σ-algebra generated all the $A(s)$. Let $f \; and \; A$ now be as in (b). It suffices to show that

$$C = \{\sigma \in \mathbb{N}^{\mathbb{N}}: \bigcap_{n} A(\sigma \setminus n) \cap f^{-1}(A) \neq \phi\}$$

is analytic.

Let $g: \mathbb{N}^{\mathbb{N}} \to A$ becontinuousandonto.

Pick$B(s) \in \mathfrak{B}(\mathbb{R}) such \; that \; A(s) = f^{-1}\left(B(s)\right)$. We show that

$$C = \left\{\sigma \in \mathbb{N}^{\mathbb{N}}: \bigcap_{n} g^{-1}\left(B(\sigma \setminus n)\right) \neq \phi\right\}.$$

First, if $\tau \in \bigcap_n g^{-1}\left(B(\sigma \setminus n), then \; g(\tau) \in \bigcap_n B(\sigma \setminus n). Sinec \; g(\tau) \in A \subset f(X)\right.$, $there \; is \; an \; x \in X \; such \; that g(\tau) = f(x)$. Hence

(*)$x \in \bigcap_n A(\sigma \setminus n) \cap f^{-1}(A)$

Secondly, if (*) holds, then $f(x) \in A = g(\mathbb{N}^{\mathbb{N}})$. Hence $f(x) = g(\tau)$for some τ.

$Since\ x \in \bigcap_n f^{-1}(B(\sigma \setminus n))$, it follows that $g(\tau) = f(x) \bigcap_n B(\sigma \setminus n)$, so

$$\tau \in \bigcap_n g^{-1}\big(B(\sigma \setminus n)\big).$$

Hence it now follows that C is the projection onto the second coordinate of the Borel set

$$\bigcap_n \bigcup_{|s|=n} g^{-1}\left(B(s)\right) \times I(s),$$

Where $I(s) = \{\sigma \in \mathbb{N}^{\mathbb{N}} \sigma\ extends\ s\}$.

$(c) \rightarrow (d)$ is trivial.

$(d) \rightarrow (a)$. Let $f: X \rightarrow \mathbb{N}^{\mathbb{N}}\ be\ \mathcal{A}$-measurable. We shall show that there \is an analytic $C \subseteq f(X) such\ that\ X - f^{-1}(C) \in \wp$. This suffices because is $\mathbb{N}^{\mathbb{N}}$Borel isomorphic to $\mathbb{R}$.

Let $A(s = f^{-1}\big(I(S)\big)\ where\ I(s)$is as above. The Suslin scheme A(s) is decreasing and disjoint. Let C be as guaranteed by (d). Then C works.

The proof of Theorem(1.1.15)is similar tothat of Theorem(1.1.14).

Theorem(1.1.15)[1] : For a triple$(X, \mathcal{A}, \wp)$, where $\wp$is a proper ideal on $X$the following are equivalent:

(a)$(X, \mathcal{A}, \wp)$ is weakly perfect.

(b) This is the same as in Theorem (1.1.14)(b), except that (ii) becomes$f^{-1}\ (A) \notin \wp$.

(c) The same as in Theorem(1.1.14)(c),except that (ii) becomes $\bigcup_{\sigma \notin C} \bigcap_{n \in \mathbb{N}} A(\sigma \setminus n) \notin \wp$.

(d) The same as (c) with $A(s)$decreasing and disjoint.

We shall now prove Theorems (1.1.10)−(1.1.15). For the proof of Theorem(1.1.10),we need the following definition and Lemmas(1.1.17) and (1.1.18); we also assume that $\wp$is a proper ideal $(i.e. X \notin \wp)$, the other case being trivial.

Definition(1.1.16) [1] : Let $X = \cup \{X_i : i \in I\}$, where the family $\{X_i\}$ is point-finite. Also let $f: I \to \{0,1)$. Then the associated many- valued function g is defined as follows:

For each $x \in X, let\, F_x = \{i \in I: x \in X_i\}$ (hence enumerate F_x is finit).Let $\langle r_0 \ , r_1 \ , \ldots\ldots, r_k \rangle$ enumerate $\{f(i): i \in F_x\}$ in increasing order. We set g(X)=$\langle r_0 \ , r_1 \ , \ldots\ldots, r_k ,1,1, \ldots\ldots \rangle \in [0,1]^{\mathbb{N}}$. thus g:$X \to [0,1]^{\mathbb{N}}$

Lemma(1.1.17) [1] : Let $(X, \mathcal{A})$be a measurable space and X_i $(i \in I)$a point-finite covering of X set

$$\mathcal{A}' = \{J \subseteq I: \cup\{X_i : i \in J\} \in \mathcal{A}\} .$$

Let$f: I \to \{0,1)$be $\mathcal{A}'$ -measurable and g:$X \to [0,1]^{\mathbb{N}}$ be the associated many- valud function. Then g is $\mathcal{A}$ $-$measurable.

Proof: Π_k : $[0,1]^{\mathbb{N}} \to [0,1]$ be the K-th projection. It clearly suffices to show that for all $K, g_k = \Pi_k \circ g\ is\ \mathcal{A}$-measurable, that is, $gk^{-1}\,([0,r)) \in \mathcal{A}\ \ for\, every\, r \in [0,1]$. We prove this by induction on k.

We have g_0(x) < r iff $x \in X_i$for some $i \in I$ such that $f(i) < r$ iff $x \in \cup \{X_i \ : f(i) < r\}$. Since$f\ is \mathcal{A}'$-measurable, $f^{-1}\,([0,1)) \in \mathcal{A}'$ .Hence $\cup \{X_i : f(i) < r\} \in \mathcal{A}$ and g_0 is$\mathcal{A}$-measurable. For the induction step suppose that g_i $(j \leq k)$ are $\mathcal{A}$-measurable.For every open $\cup \subseteq [0,1]$set

$$C_U = \cup \{X_i : f(i) \in U\} .$$

As in the preceding step we conclude that $C_U \in \mathcal{A}$,In particular, for open intervals$(s,r), C_{(s,r)} \in \mathcal{A}$.We now have $g_{k+1}(x) < r$ iff

$$\exists s < r\big(s \in Q \wedge g_k(x) \in C_{(s,r)}\big),$$

Where Q is the set of rational numbers. Hence

$$g_{k+1}^{-1}([\,0,r) = \bigcup_{s<r}(\,g_k^{-1}\,(\big[0,s)) \cap C_{(s,r)})$$

And we are done.

Lemma(1.1.18)[1] : The following are equivalent:

(a) $(X, \mathcal{A}, p)$ is perfect (resp. weakly perfect).

(b) $If\ Y$ is a separable metric space and $f: X \to Y\ is \mathcal{A}$-measurable, then theris an analytic $A \subseteq f(X) such\ that\ X - f^{-1}(A) \in \wp (resp. f^{-1}(A) \notin \wp)$.

Proof: (b) implies (a) is trival. The other direction follows easily from the following well known fact .

If Y is a separable metric space, then there is a one-to-one function $f: X \to \mathbb{R}\ such\ that\ f^{-1}: f(Y) \to Y$ is continuous and for every open $U \subseteq \mathbb{R}, f^{-1}(U)\ isF_0\ inY$.

Lemma(1.1.19) [1] : $(X, \wp(X), \wp)$is perfect, then $\wp\ is\ \omega_1$-saturated (i.e. there is no uncountable disjoint family of sets not belonging to$\wp$).

Proof: Let $X_i\ (i \in I)$be a disjoint family of sets not belonging to $\wp$ and assume that $|I| \leq c$. It suffices to show that$|I| \leq x_0$.Let Z be a totally imperfect (i.e. containing no nonempty perfect set) subset of $\mathbb{R}$ with$|Z| = c$. By a well known theorem, every analytic subset of Z is at most enumerable. Now consider a function $f: X \to \mathbb{R}\ with\ f(X) \subseteq Z, f \setminus X_i$ constant and $f(X_i) \neq f(X_j)$for every $i\ and\ j \in I, i \neq j$. Since $(X, \wp(X), \wp)$is perfect and f is trivially $\wp(X)$-measurable, there is an analytic $A \subseteq f(X) such\ that\ X - f^{-1}(A) \in \wp. Thus\ |A| \leq x_0\ andX_i \subseteq f^{-1}(A). Thus\ |I| \leq x_0$ as required.

Lemma(1.1.20) [1] : The following are equivalent if $\wp$ is a σ-ideal on X:

(a) $(X, \wp(X), \wp)$is perfect.

(b) For every partition of $X, X_i(i \in I) where\ |I| \leq c$, there is an at most enumerable $J \subseteq I\ such\ that\ X - \cup \{X_i: i \in J\} \in \wp$.

(c) The same as (b) except that $|I| <$ the least measurable cardinal.

Proof:$(a) \to (b)$. As in the previous lemma we consider a totally imperfect set Z of real and a function$f: X \to \mathbb{R}\ with\ f(X) \subseteq Z, f \setminus X_i$constantand$f(X_i) \neq f(X_j) for\ every\ i\ and\ j \in I, i \neq j. Since (X, \wp(X), \wp)$is perfect ,there is analytic $A \subseteq f(X)\ such\ that\ X - f^{-1}(A) \in \wp. Then\ |A| \leq x_0$ and setting

$$J = \{i \in I: f(X_i \subseteq A\}$$

We have $|J| \leq x_0\ and\ X - \cup \{X_i: i \in J\} \in \wp$.

$(b) \rightarrow (a)$. For every function $f: X \rightarrow \mathbb{R}$, we apply (b) for the partition $\{f^{-1}(\{y\}): y \in f(X)\}$. Thus there is an at most enumerable, hence also analytic, $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \wp$.

$(c) \rightarrow (a)$ is trivial. So it remains to show $(a) \rightarrow (c)$. Assume that $(X, \wp(X), \wp)$ is perfect and let $X_i (i \in I)$ be a partition of X, where $|I| <$ the least measurable cardinal. By Lemma (1.1.2.11), $\wp$ is ω_1 –saturated, so $J = \{i \in I: X_i \notin \wp\}$ is at most enumerable.

We claim that

$$X_1 = X - \cup \{X_i: i \in J\} \in \wp.$$

Suppose that the claim is false and consider the perfect space

$$\left(X_1, \wp(X_1), \wp \cap \wp(X_1)\right),$$

where $\wp \cap \wp(X_1)$ is a proper ideal on X_1. This space can be 'extended' back to X by putting $X - X_1$ into the new proper ideal on X, the resulting space being perfect. So we can assume that $J = \emptyset$, i.e., $X_i \in \wp$ for all $i \in I$, and that $X \notin \wp$.

Define

$$\wp' = \{J \subseteq I: \cup \{X_{i:} i \in J\} \in \wp\}.$$

By Theorem (1.1.10), $(I, \wp(1), \wp')$ is perfect and Lemma (1.1.19) implies that $\wp'$ is w_1-saturated. Moreover, we have $\{i\} \in \wp'$ for all $i \in I$ and $I \notin \wp'$. either there exists some $J \subseteq I$ such that $\wp' \cap \wp(J)$ is a proper maximal σ- ideal on J, or there exists a partition of I into $\leq c$ sets in $\wp'$. If J is as in the first lternative of Ulam's Theorem, let λ be the additivity degree of the ideal $\wp' \cap \wp(J)$, i.e., the least cardinal such that there are sets $I_\alpha \in \wp' \cap \wp(J) (\alpha \in \lambda)$ with $J = \cup \{I_\alpha: \alpha \in \lambda\}$. Itis known that λ is a measurable cardinal, which is a contradiction since $\lambda \leq |J| \leq |I|$. If the second alternative occurs we come again to a contradiction because $(I, \wp(1), \wp')$ is perfect and we already know that $(a) \rightarrow (b)$. This completes the proof of Lemma(1.1.20).

Lemma(1.1.21) [1] : The following are equivalent if $\wp$ is a proper σ-ideal on X:

(a) $(X, \wp(X), \wp)$ is weakly perfect.

(b) For every partition of $X, X_i (i \in I)$ where $|I| \leq c$, there is an at most enumerable $J \subseteq I$ such that $\cup \{X_i: i \in J) \notin \wp$.

The proof is similar as that of Lemma(1.1.20), $(a) \leftrightarrow (b)$.

Lemma(1.1.22)[1] : Let $(X, \mathcal{A}, \wp)$be perfect (resp.weakly perfect), where $\wp$ $is\ a\ \sigma$-ideal with an $\mathcal{A}$-base. Then $(X, \mathcal{A}, \wp)$is perfect (resp. weakly perfect).

Proof: Let $f: X \to \mathbb{R}\ be\ \mathcal{A}_{\wp}$-measurable.Lemma(1.1.2.14)clearly follows if we can find an$\mathcal{A}$-measurable $g: X \to \mathbb{R}\ such\ that\ \{x: f(x) \neq g(x)\} \in \wp$.To find such a g we proceed as follows. Let$U_n (n \in \mathbb{N})$be an open base for $\mathbb{R}$. Sinc$f\ is\ \mathcal{A}_{\wp}$--measurable and $\wp$ has an $\mathcal{A}$-base we can pick an$S_n \in \mathcal{A} \cap \wp\ such\ that\ f^{-1}(U_n) \cap (X - S_n) \in \mathcal{A}.$ It now suffices to set

$$g(x) = \begin{cases} f(x), & if\ x \notin S \\ 0, & if\ x \in S \end{cases}$$

Theorem(1.1.23) [1] : Let $(X, \mathcal{A})$be a measurable space, $\wp$ $a\sigma$-ideal on X with an$\mathcal{A}$-base and a Y metric space. Assume that $(X, \mathcal{A}, \wp)$is perfect and $|Y|$ <the least measurable cardinal. Then if $F: X \to \wp(Y) is \mathcal{A}_{\wp}$-measurable, there is a separable $Y^{\sim} \subseteq Y\ such\ that\ X - \{x: F(x) \subseteq Y^{\sim}\} \in \wp\ and\ F$ admits an $\mathcal{A}_{\wp}$-measurable selection.

Some Example (1.1.24)[1] :

The results of the previous section deal with arbitrary perfect and weakly perfect measurable spaces. Here we study such space when the underlying space $X$is a topogical space; the σ-algebra is usually the family $\mathfrak{B}(X)$of Borel sets. We find conditions on X so that for an ideal $\wp$on X, $(X, \mathfrak{B}(X), \wp)$ is perfect or weakly perfect. When this is done, the results of the previous section apply and yield more concrete results, where the notions of perfect and weakly perfect measurable spaces are at least implicit.

Theorem(1.1.25)[1] : Let X be a metric, or more generally a developable space with $|X|$ <the least measurable cardinal and let $\wp\ a\ \sigma$-ideal on X.Then $(X, \mathfrak{B}(X), \wp)$is perfect iff there is an analytic $A \subseteq X$ such that $X - A \in \wp$.

Proof: First suppose that there is an analytic$A\ such\ that\ X - A \in \wp$.. Then if $f: X \to \mathbb{R}\ is\ \mathfrak{B}(X)$ −measurable it is easily seen that $f(A)$is analytic and the resultfollows.

To prove the opposite direction, we shall consider the metric case separately since this is the most interesting case and the proof is simpler. Thus suppose that X is metric and also separable at first. Let $f: X \to \mathbb{R}$ be as in the proof of Lemma(1.1.18). Hence there is an analytic $A' \subseteq f(X) suchthat\ X - f^{-1}(A') \in \wp$. It suffices to set $A = f^{-1}(A')$.

We shall now reduce the general metric case to the separable one. We set

$$u = \{U \subseteq X: U\ is\ open\ and\ U \in \wp\}.$$

Set $Y \bigcup \mathcal{u}$. we shall show that $Y \in \wp$. Suppose not. By Stone's theorem, wecan pick a σ-disjoint open refinement $v = \cup \{v_n : n \in \mathbb{N}\}$ of $\mathcal{u}$,where each family $v_n$is disjoint. We can now pick $n \in \mathbb{N}$ $such$ $that$ $Y_n = \cup v_n \notin \wp. Let$ $v_n = \{U_i = i \in I\}$be a one-to-one indexing of sets from As v_n usual, we set

$$\wp' = \{J \subseteq I : \{U_i : i \in J\} \in \wp\}.$$

Clearly $(1, \wp(1), \wp')$is perfect and all singletons belong to $\wp'$while $I \notin \wp'$ and $|I|$ <the least measurable cardinal. This contradicts Lemma (1.1.20). Hence $Y \in \wp. set$ $F = X - Y. F$is closed, hence in $\mathfrak{B}(X)$and thus we can suppose without loss of generality that $X = F$. Hence if $\emptyset \neq U \subseteq X$ and U is $open$ $then$ $U \notin \wp$.

Let $U_i (i \in I)$be a family of non-empty disjoint open subsets. Then if $\wp'$isdefined as above, we have $\wp' = \{\emptyset\}$. Hence $(1, \wp(1), \{\emptyset\})$is perfect, and thus by Lemma(1.1.19),$|I| \leq x_0$.

This completes the proof of Theorem(1.1.26) when X is metric.

Consider a Hausdorff space X is developable if there is a sequence $\mathcal{U}_n (n \in \mathbb{N})$of open covers such that fore very $x \in X$ the family

$$\cup = \{U \in \mathcal{U}_n : x \in U\} (n \in \mathbb{N}),$$

Is neighbourhood basis at x.

Fix a sequence $\mathcal{U}_n (n \in \mathbb{N})$as above. we can pick $\mathcal{F}_n$a closed σ-discrete refinement of $\mathcal{U}_n$ The union of an arbitrary subfamily of $\mathcal{F}_n$ a is an F_σ , hence Borel. By using Lemma (1.1.20) similarly as in the proof of the metric case we can find $\mathscr{b}_n \subseteq \mathcal{F}_n$ $such$ $that$ $|\mathscr{b}_n| \leq x_0$ and $X - \cup \mathscr{b}_n \in \wp. Let$ $\mathcal{V}_n \subseteq \mathcal{U}_n$be at most enumerable and such that for every $E \in \mathscr{b}_n$there is some $U \in \mathcal{V}_n$ containing E. Set

$$Y = \cap (\cup \mathcal{V}_n).$$

Clearly $X - Y \in \wp$. Moreover , $Y$is second countable since

$$\bigcup_n \{U \cap Y : U \in \mathcal{V}_n\}$$

Is an open base for Y,by the definition of developability. Since $Y \in \mathfrak{B}(X)$$and$ $X - Y \in \wp$,we have a reduction of the general case to the second countable case. This is handled similarly as the separable metric case. This is because every second countable Hausdorff space is Borel isomorphic to a subset of $\mathbb{R}$ by an open mapping onto its range.

Theorem(1.1.26) [1] : Let X be a developable space with $|X| \leq c$ and $\wp$ be a proper σ-ideal on X. Then $(X, \mathfrak{B}(X), \wp)$is weakly perfect iff there is an analytic $A \subseteq X$ $such$ $that$ $A \notin \wp$.

Proof: If there is an analytic $A \subseteq X$ $such$ $that$ $A \notin \wp$, the result follows as in the previous theorem. The opposite direction follows from Lemma (1.1.18)and Lemma(1.1.27),concluding the poof of Theorem (1.1.26).

Lemma(1.1.27) [1] : Let X be a developable space with $|X| \leq c$ Then there is a Borel measurable function $f: X \to \mathbb{R}^{\mathbb{N}}$ such that:

(a) for every second countab l$Y \subseteq X, f \setminus Y: Y \to f(Y)$a Borel isomorphism; and

(b) for every analytic $A \subseteq f(X), f^{-1}(A)$is analytic.

Proof: We claim that there are partitions $\mathscr{b}_n, (n \in \mathbb{N})$,such that:

(i) the union of every subfamily of $\mathscr{b}_n$ is a Borel set (actually an F_σ-set);

(ii) every open set in X is the union of some members of $\bigcup\{\mathscr{b}_n: n \in \mathbb{N}\}$;

(iii) for every $Y \subseteq X, X$ is second countable iff

$$|\{E \in \mathscr{b}_n: E \cap Y \neq \emptyset\}| \leq x_0$$

for every $n \in \mathbb{N}$.

To see this, let $\mathcal{F}_n(n \in \mathbb{N})$ be as in proof of Theorem(1.1.25)Then

$$\mathcal{F}_n = \cup \left\{\mathcal{F}_{n,m}: m \in \mathbb{N}\right\},$$

Where each $\mathcal{F}_{n,m}$ is a discrete family of closed sets. Now we set

$$\mathscr{b}_{n,m} = \mathcal{F}_{n,m} \cup \{X - \cup \mathcal{F}_{n,m}\}$$

and rearrange$\mathscr{b}_{n,m}(n \in \mathbb{N}, m \in \mathbb{N}) as \mathscr{b}_n(n \in \mathbb{N})$. It is easy to see that (i)-(iii) hold. The metric case is simplef_nr: we consider a σ-discrete base $\mathcal{V} = \cup \{\mathcal{V}_n: n \in \mathbb{N}\}$and set

$$\mathscr{b}_n = \mathcal{V}_n \cup \{X - \bigcup \mathcal{V}_n\}.$$

Now we proceed as follows. We choose a totally imperfect set of reals $Z\ with\ |Z| = c.\ Since\ |\mathscr{b}_n| \leq c, there\ is\ f_n: X \to \mathbb{R}\ such\ that\ f_n(X) \subseteq Z, f_n \setminus E$ is constant and $f_n(E) \neq f_n(E') forevery\ E\ and\ E' \in \mathscr{b}_n, E \neq E'$.Define $f: X \to \mathbb{R}^{\mathbb{N}}$ by

$$f(x) = \langle f_n(x): n \in \mathbb{N} \rangle.$$

Then $f$is Borel measurable by (i). Since X is Hausdorff, (ii) implies that $\bigcup\{\mathscr{b}_n: n \in \mathbb{N}\}$ separates points and so $f$is one-to-one.

By the definition off, it is easy to see that for every $E \in \bigcup\{\mathscr{b}_n: n \in \mathbb{N}\}$ there is a closed $F \subseteq \mathbb{R}^{\mathbb{N}}$such that $E = f^{-1}(F)$.Now, if $Y \subseteq X$ is second countable, then by(iii)

$$|\{E \in \mathscr{b}_n: E \cap Y \neq \emptyset\}| \leq x_0$$

For every $n \in \mathbb{N}$ so by (ii) every relatively open set in Yhas the form $Y \cap f^{-1}(C)$for some $F_\sigma\ set\ C\ in \mathbb{R}^{\mathbb{N}}$This shows that $f \setminus Y$is a Borel isomorphism, concluding the proof of (a).

To prove (b) Let Abe an analytic subset of $f(X)$and let $\pi_n: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$denote the n-th projection. Then $\pi_n(A) \subseteq Z$ is analytic and by the choice of Zit follows that $\pi_n(A)$ is at most enumerable. So we have

$$|\{E \in \mathscr{b}_n: E \cap f^{-1}(A) \neq \emptyset\}| \leq x_0$$

For every n and, by (iii), is $f^{-1}(A)$ second countable. Now (a) implies that$f^{-1}(A)$ is Borel isomorphic to A and so (b) follows.

Remarks(1.1.28) [1] : The proof of Lemma(1.1.25) actually shows that for every second countable $Y \subseteq X, f \setminus Y$ is a Borel isomorphism. in the sense that both functions $f \setminus Y\ and\ (f \setminus Y)^{-1}$map open sets to$F_\sigma$sets.

The cardinality restrictions in Theorems(1.1.25) and (1.1.26) are necessary. Counter-examples are those mentioned after the statement of Theorem(1.1.11) where X is considered as a discrete metric space. However, for the special case of Theorem (1.1.25) when $\mathscr{p} = \{\emptyset\}$, or more generally, if the union of any discrete (equivalently, σ-discrete) family of closed sets from $\mathscr{p}$ belongs to $\mathscr{p}$, no restriction on $X$is necessary. We indicate the necessary modification in the proof of Theorem(1.1.26) for developable X Since $\mathcal{F}_n\ is\ \sigma$-discrete,

$$\bigcup\{F \in \mathcal{F}_n: F \in \mathscr{p}\} \in \mathscr{p}.$$

Set $\mathscr{b}_n = \{F \in \mathcal{F}_n: F \notin \mathscr{p}\}$. Then

$$X - \bigcup \mathscr{b}_n \in \mathscr{p},$$

And using Lemma(1.1.19) $|\mathscr{b}_n| \leq x_0$.The rest is as above.

Corollary(1.1.29)[1]: Let X be a developable space. Then $(X, \mathfrak{B}(X), \{\emptyset\})$ is perfect($(X, \mathfrak{B}(X))$is smooth"in Falkner's terminology) iff $X$is analytic.

Corollary(1.1.30) [1]: Let X be perfect (resp. weakly perfect) and $f: X \to Y$ be $\mathcal{A}$-measurable,where Y isdevelopable and $|Y|$ <the least measurable cardinal (resp.$|Y| \leq c$). Then there is an analytic $A \subseteq Y$ $such\ that\ X - f^{-1}(A) \in \mathscr{p}(\ resp. f^{-1}(A) \notin \mathscr{p}\ if\ \mathscr{p}$ is a proper ideal)

Proof : Set $Z \in \mathscr{p}'\ iff\ f^{-1}(Z) \in \mathscr{p}$.Then $(Y, \mathfrak{B}(Y), \mathscr{p}')$is perfect(resp. weakly perfect). Hence the result follows from Theorems (1.1.25) and (1.1.26)

For measure and category spaces we have:

Corollary(1.1.31) [1] : Let X be a developable space with $|X|$ <theleast measurable cardinal (resp.$|X| \leq c$). A probability Borel measure $\mu\ on\ X$ is perfect (resp. weakly perfect) iff μ is a Radon measure (resp. there is a compact subset of $X$with posiive μ-measure).

Proof: If $A \subseteq X$ is analytic, then A is measurable with respect to any Borel measure on $X$and every Borel measure on A is a Radon measure.

Hence the result follows from Theorems (1.1.25) and (1.1.26).

Corollary (1.1.32)[1] : Let X be a complete metric space with $|X|$ <the least measurable cardinal (resp.$|X| \leq c$) and let $\mathscr{p}$ denote the ideal of meager sets in X. Then $(X, \mathfrak{B}(X), \mathscr{p})$is perfect (resp. weakly perfect) iff X is separable(resp. there exists a nonempty open separable subset of X).

Proof: Assume that $(X, \mathfrak{B}(X), \mathscr{p})$is perfect and$|X| < 1$ the leas measurable cardinal. By Theorem (1.1.25) there is an analytic$A \subseteq X$ such that $X - A$is meager. Then $X - \bar{A}$is open meager, hence $\bar{A}$=X. Since A is separable, so is X. The converse is obvious since $X$is now assumed to be a Polish space(In the mathematical discipline of general topology , a polish space is separable completely metrizable topological space; that is , a space homeomorphic to a complete metrie space that has a countable dense sub set . polish space are mostly studied today because they are the primary setting for descriptive set theory , in cluding the study of Borel equivalence relation . polish space are also a convenient setting for more a dvanced measure theory, in particular in prolability) theory)[6].

Now assume that $(X, \mathfrak{B}(X), \mathscr{p})$is weakly perfect and $|X| \leq c$. By Theorem(1.1.26)there is an analytic set $A \subseteq X$ of the second category. Since A is separable, the interior of A is a nonempty open separable set. Again the converse is obvious since any nonempty open separable set in X is a Polish space. We shall now present several results concerning perfect measurable spaces when the underlying space X is K-analytic.

Lemma(1.1.33) [1] : Let X be a topological space and $\mathscr{p}$ a proper ideal on X. Then $(X, \mathfrak{B}_a(X), \mathscr{p})$ is perfect (resp. weakly perfect) iff for every continuous $f: X \to Y$,where Y is separable metric, there is an analytic $A \subseteq f(X)\ such\ that\ X - f^{-1}(A) \in \mathscr{p}(resp. f^{-1}(A) \notin \mathscr{p})$.

Proof: The implication from left to right follows by Lemma(1.1.18). In the opposite direction it suffices to show that (b) of Theorem(1.1.14) (resp. (1.1.15)) holds. Let $A_n \in \mathfrak{B}_n(X)(n \in \mathbb{N}) and\ \mathcal{A}'$be the σ-algebra generated by the sets A_n. For every n,let$f_n: X \to \mathbb{R}^{\mathbb{N}}$ be a continuous function such that $A_n = f_n^{-1}(B_n)\ for\ some\ B_n \in \mathfrak{B}(\mathbb{R}^{\mathbb{N}})$. Define

$$f: X \to (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$$

By

$$f(x) = \langle f_n(x): n \in \mathbb{N} \rangle$$

Then f is continuous and by our assumption there is an analytic $A \subseteq f(X) such\ that\ X - f^{-1}(A) \in \mathscr{p}, (resp. f^{-1}(A) \notin \mathscr{p})$Moreover, we have

$$\mathcal{A}' \subseteq f^{-1}(\mathfrak{B}((\mathbb{R}^{\mathbb{N}})^{\mathbb{N}})).$$

Since $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$is Borel isomorphic to $\mathbb{R}$,we have (b) of Theorem(1.1.14) (resp. (1.1.15). We recall that a Hausdorff space $X\ is\ K$-analytic, if there is an upper semicontinuous f from $\mathbb{N}^{\mathbb{N}}$into the family of compact subsets of X such that

$$X = \bigcup \{f(\sigma): \sigma \in \mathbb{N}^{\mathbb{N}}\}.$$

Lemma(1.1.34)[1] : $If\ X\ is\ K$-analytic, then $\big(X, \mathfrak{B}_a(X)\big)$is smooth, hence $(X, \mathfrak{B}_a(X), \mathscr{p})$is perfect for every ideal $\mathscr{p}$.

Theorem(1.1.35)]1] : Let Xbe a K-analytic regular space and $\mathscr{p}$ the ideal of meager sets. Then $(X, \mathfrak{B}(X), \mathscr{p}) is$ perfect $iff\ X$ has a closed comeager c.c.c. subset.

Recall that a space $Y\,is$ c.c.c. if every disjoint family of open sets in Y is at most enumerable.

Proof: Suppose$(X, \mathfrak{B}(X), \wp)is$isperfect. Set

$$\mathcal{U} = \{U \subseteq X: U\ open\ and\ meager\}, \qquad V = \bigcup \mathcal{U}$$

By Banach's category theorem , V is meager. Set $Y = X - V$. Then if $\emptyset \neq U \subseteq Y, and\ U$ is relative open, $U \in \wp$. Hence by Theorem(1.1.13) (or an easy argument using Lemma(1.1.19)), $Y$is c.c.c.

Now for the converse let Ybe closed, c.c.c. and corneager. It suffices to show that $(X, \mathfrak{B}(X), \wp')$,where $\wp' = \wp \cap \wp(Y)$,isperfect. $Y\ isK$-analytic as a closed subspace of a K-analytic space. Hence by Lemma (1.1.35),$(Y, \mathfrak{B}_a(Y), \wp)is$perfect. Thus by Lemma(1.1.22) it suffices to show that$\mathfrak{B}(Y)$is included in the $\wp'$- completion of $\mathfrak{B}_a(Y)$.

Let G be an open subset of Y. Let$\mathcal{V}$be a maximal family of non-empty pair-wise disjoint cozero subsets of G . Then $\mathcal{V}$ is at most enumerable and thus $V = \bigcup \mathcal{V} \in \mathfrak{B}_a(X)$and is dense in by the$G$complete regularity. Hence $G - V \in \wp'$and the proof is complete.

Corollary(1.1.36) [1] : Let X be either (i) a regular K-analytic Baire space or(ii) a Cech-complete space. Then $(X, \mathfrak{B}(X), \wp)$where $\wp$is the ideal of meager sets, is perfect $iff\ X$ is c.c.c.

Proof: First note that a Cech-complete space is also a Baire space. As in Theorem(1.1.42) this fact alone suffices to show that if $(X, \mathfrak{B}(X), \wp)is$perfect, then X is c.c.c.

For the opposite direction first suppose (i). Then $X$is a closed comeager c.c.c.subset of itself. Hence the result follows from Theorem(1.1.36).

Now suppose (ii),i.e., X isa G_δ-subset of βX, theStone-Tech compactification of X. Let $\wp'$ be the ideal of meager subsets of βX,. By Theorem(1.1.36),

$$(\beta X, \mathfrak{B}(\beta X), \wp')$$

is perfect. Hence since $X \in \mathfrak{B}(\beta X), (X, \mathfrak{B}(X), \wp' \cap \wp(X))$ is perfect. But since X is comeager $\beta X, \wp' \cap \wp(X)) = \wp$.

The next two corollaries follow directly from Theorem(1.1.11) (resp. Theorem(1.1.13)and Lemma(1.1.34) .

Corollary(1.1.37) [1] : Let X be a K-analytic space and $\wp$ a proper σ-ideal on$X$ with a Baire base. Let$X_i(i \in I)$be a point-finite covering of X by sets from $\wp$, where $|I|$ <the least measurable cardinal. Then there is some $J \subseteq I\ such\ that$ $\bigcup\{X_i: i \in J\}$ does not differ from a Baire set by a set from $\wp$.

Corollary(1.1.38) [1] : Let X be a K-analytic space and $X_i(i \in I)$be a point-finite covering of X by nonempty sets such that $\bigcup\{X_i: i \in J\} \in \mathfrak{B}_a(X)\ for\ all\ J \subseteq I$. Then I is at most enumerable.

Lemma(1.1.39) [1] : Ellentuck$A\ set\ P \subseteq \Omega$is completely Ramsey $iff\ P$ has the property of Baire.

Ellentuck also proves that every meager set inΩ is nowhere dense. Thus $\Omega$is a Baire space.

Theorem(1.1.40)[1]: If $\wp$ is the ideal of meager sets in Ω, then $(\Omega, \mathfrak{B}(\Omega), \wp)$ is weakly perfect.

Proof: Let $f: \Omega \to (0,1)$ be a Borel measurable function. Let $n_0 = 0$ and $A_0 = \mathbb{N}$. Suppos we have defined $n_0, n_1, \dots, n_{k-1}\ in\ \mathbb{N}\ and\ A_0, A_1, \dots, A_{K-1}$infinite subsets of $\mathbb{N}$. Then we choose $A_k \in \Omega(A_{k-1})\ suth\ that\ \{n_{k-1}\} < A_k\ and\ for\ all\ s \subseteq \{n_0, n_1, \dots, n_k\}$

(*) diameter of $f(\Omega(s, A_k)) < 1/2^k$.

We can do this by considering a finite partition $\{T_1, \dots \dots . T_m\}$ of $(0,1)$ to intervals of length $< 1/2^k$. Then Lemma(1.1.39) implies that each $f^{-1}(T_i)(i = 1 \dots \dots m)$is completely Ramsey and $A_k$is found easily by applying the Ramsey property finitely many times. Now we set n_k =min A_k.

By the above construction

$$n_0 < n_1 < \cdots < n_k < \cdots,$$

$$A_0 \supset A_1 \supset \cdots \supset A_k \supset \cdots$$

And $n_j \in A_k\ for\ all\ j \geq k$. Let $A = \{n_0, n_1, \dots\}$Since $\Omega(A) \notin \wp$ it suffices to show that $f(\Omega(A))$ is analytic.

Let $Z = \Omega(A)$endowed with the relative topology from $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}}$has the usual product topology. Observe that Z isG_δ in $\{0,1\}^{\mathbb{N}}$, so $Z \times (0,1)$is a Polish space. If we show that the graph of) $f \setminus Z, G_r(f \setminus Z)$,is closed in $Z \times (0,1)$,then $f\big(\Omega(A)\big)$.being the projection

$G_r(f \setminus Z)\ to\ (0,1)$, of must be analytic. To do this, let $\{X_i\}$be a sequence in $Z\ with\ X_l \to X \in Z\ and\ f(X_1) \to t \in (0,1)$.It suffices to show that $t = f(X). X$ is of the form

$$X = \{n_{k0} < n_{k1} < \cdots\} \subseteq A.$$

Let $\varepsilon > 0$ and choose $l_0 \in \mathbb{N}$ such that

$$1/2^{k_{l_0}} < \varepsilon/2 \quad and\ |f(X_l) - t| < \varepsilon/2 \quad for\ all\ \ l \geq l_0.$$

Since $X_l \to X$there is some $l \geq l_0$ such that

$$s = \{n_{k0}, n_{kl}, \dots, n_{kl_0-1}\}$$

is the set of the first l_0 elements of X_1.Then X andX_l,belonto $\Omega(s, A_{kl_0})$ so (*) implies that $|f(X_l) - f(X)| < 1/2^{k_{l_0}}$. Thus we have

$$|f(X) - t| \leq |f(X\ ) - f(X_l)| + |f(X_l) - t| < \varepsilon.$$

There fore $t = f(X)$.

Corollary(1.1.41) [1] : If $X_i(i \in I)$ is a point-fnite covering of Ω by meager sets, then there is some $J \subseteq I\ such\ that\ \bigcup\{X_i : i \in J\}$ does not have the property of Baire , hence is not Ramsey.

Section (1.2) Related Results :

In this section we prove a result on the union problem for not necessarily perfect, or weakly perfect, measurable spaces(Theorem 1.2.1). We assume instead that $(X, \mathcal{A}, \wp)$satisfies a countability condition and has a Fubini-type property defined below. We also relate the union problem to the rectangle problem Theorem (1.2.5).

First we need some notation. If $\mathscr{b}$ is a family of subsets of X,define $\mathscr{b}_\alpha(\alpha < \omega_1)$ as follows:$\mathscr{b}_0 = \mathscr{b}; for\ \alpha$ odd (resp. even) let $\mathscr{b}_\alpha$be the family of intersections (resp.unions)of at most enumerable subfamilies of $\bigcup_{\beta<\alpha} \mathscr{b}_\beta\ \ for\ each\ Z \subseteq X \times X\ and\ x \in X,\ let\ Z_x$(resp. Z^x) be the vertical (resp. horizontal) section at $x\ of\ Z$. As usual, $\pi_i : X \times X \to X(i = 1,2)$denote the first and second projection.

Definition(1.2.1) [1] : A triple $(X, \mathcal{A}, \wp)$has the weak Fubini property if for every Z in the product σ-algebra $\mathcal{A} \otimes \mathcal{A}\ with\ Z_x \in \wp\ for\ all\ x \in X\ and\ \pi_2(Z) \notin \wp,$ there exists some $y \in \pi_2(Z)\ such\ that\ Z^y \in \wp$.

Theorem(1.2.2.) [1] : Let $(X, \mathcal{A})$be a measurable space and $\mathcal{p}$ a proper σ-ideal on X. Assume that :

(a) there exists $\mathcal{b} \subseteq \mathcal{A}\ with\ |\mathcal{b}| \leq x_0\ and\ some\ 0 < \alpha < \omega_1$ such that $\mathcal{b}_\alpha$ is a base for$\mathcal{p}$; and

(b) $(X, \mathcal{A}, \mathcal{p})$ satisfies the weak Fubini property.

Then for every point-finite covering $X_i(i \in I) of\ X\ with\ X_i \in \mathcal{p}$ there is some $J \subseteq I\ such\ that\ \bigcup\{X_i : i \in J\} \notin \mathcal{A}_\mathcal{p}.$

Proof: Suppose that the theorem is false and let $X_\xi(\xi \in K)$be a counter example to the theorem. Let ρ be minimal such that

$$\tilde{X} = \bigcup\{X_\xi : \xi \in \rho\} \notin \mathcal{p}.$$

Then

$$(\tilde{X}, \mathcal{A}_\mathcal{p} \cap \tilde{X}, \mathcal{p} \cap \tilde{X})$$

Satisfies (a) and (b) and

$$\bigcup\{X_\xi : \xi \in T\} \in \mathcal{A}_\mathcal{p} \cap \tilde{X}$$

For all $T \subseteq \rho$. Thus without loss of generality we assume that $\rho = K$, so that$\tilde{X} = X$and

$$\bigcup\{X_\xi : \xi \in \tau\} \in \mathcal{p}$$

For all $\tau < K$. Obviously, we can also assume that $X \in \mathcal{b}\ and\ \mathcal{b}$ is closed under finite unions and finite intersections; so each $\mathcal{b}_\beta(\beta < \omega_1)$ has the same properties.

For every $\xi \in K$, let

$$Y_\xi = \bigcup_{\eta<\xi} X_\eta$$

And , by (b), choose $C_\xi \in \mathcal{b}_\alpha \cap \mathcal{p}\ such\ that\ Y_\xi \subseteq C_\xi$. We set

$$Z = \bigcup_{\xi \in K} (X_\xi \times C_\xi)$$

And claim that

$$Z \in (\mathcal{A}_{\wp} \times \mathscr{b})_\alpha$$

Where

$$\mathcal{A}_{\wp} \times \mathscr{b} = \{A \times C : A \in \mathcal{A}_{\wp}, C \in \mathscr{b} \}.$$

The claim is proved by induction on α.

If $\alpha = 1$ we set

$$A_C = \bigcup \{ X_\xi : C \subseteq C_\xi \} \in \mathcal{A}_{\wp}$$

for every $C \in \mathscr{b}$. Then we have

$$Z = \bigcup \{ A_C \times C : C \in \mathscr{b}\} \in \left(\mathcal{A}_{\wp} \times \mathscr{b} \right)_\alpha .$$

Now assume that $1 < \alpha < \omega_1$and the claim is true for every $\beta, 0 < \beta < \alpha$.

Case 1: α is even. We fix a sequence $\gamma_n (n \in \mathbb{N})$of ordinals less than$\alpha$ as follows: if α is limit, then $\sup\{\gamma_n : n \in \mathbb{N}\} = \alpha$andif αisnonlimit, say$\alpha = \alpha_0 + 1$,then$\gamma_n = \alpha_0$for all $n \in \mathbb{N}$.

By our assumption on $\mathscr{b}$, for every $\xi \in K$ there is a decreasing sequence $C_{\xi,n} (n \in \mathbb{N})$ *such that* $C_{\xi,n} \in \mathscr{b}_{\gamma_n}$ *and* ,

$$C_\xi = \bigcap_{n=0}^{\infty} C_{\xi,n}.$$

Then

$$Z = \bigcup_{\xi \in K} (X_\xi \times \bigcap_{n=0}^{\infty} C_{\xi,n}) = \bigcap_{n=0}^{\infty} \bigcup_{\xi \in K} (X_\xi \times C_{\xi,n}),$$

Where the last equality is proved as follows: if (x, y) belongs to the right side, then there are $\xi_n \in K(n \in \mathbb{N})$ *such that* $x \in X_{\xi n}$ *and* $y \in C_{\xi,n}$ *for all* $n \in \mathbb{N}$. *Since* $X_\xi(\xi \in K)$ is point-finte $M \subseteq \mathbb{N}$ *and some* $\xi \in K$ *such that* $\xi_n = \xi$ *for all* $n \in M$. Hence$(x, y) \in X_\xi \times C_{\xi,n}$ *for* $n \in M$. But $C_{\xi,n}$ is decreasing$(x, y) \in X_\xi \times C_{\xi,n}$ *for* n. The other direction is obvious. By the induction hypothesis, it follows that $Z \in (\mathcal{A}_{\wp} \times \mathscr{b})_\alpha$.

Case 2: α is odd.

We have $\alpha = \alpha_0 + 1$. *Let*$C_{\xi,n}(n \in \mathbb{N})$ be a sequence in $\mathscr{b}_{\alpha_0}$ such that

$$C_\xi = \bigcup_{n=0}^{\infty} C_{\xi,n}$$

Then

$Z = \bigcup_{\xi \in K}(X_\xi \times \bigcup_{n=0}^{\infty} C_{\xi,n}) = \bigcup_{n=0}^{\infty} \bigcup_{\xi \in K}(X_\xi \times C_{\xi,n}) \in (\mathcal{A}_{\wp} \times \mathscr{b})_\alpha$.

Thus the proof of the claim is complete.

Now if $x \in X$and

$$\{\xi \in K: x \in X_\xi\} = \{\xi_1, \xi_2, \dots, \xi_l\},$$

Then it is easy to see that

$$Z_x = \bigcup_{j=0}^{l} C_{\xi j}\,.$$

Thus $Z_x \in \wp$ for every $x \in X$.

Since $\pi_2(Z) = X \notin \wp$,by (b) there is some $y \in X$ *such that* $Z^y \in \wp$. Let $\eta \in K$ *with* $y \in X_\eta$.For every $\xi > \eta$ we have $y \in Y_\xi \subseteq C_\xi$, hence $X_\xi \times \{y\} \subseteq X_\xi \times C_\xi$.

Therefore

$$\bigcup_{\xi > \eta} X_\xi \times \{y\} \subseteq Z,$$

i.e., $\bigcup_{\xi>\eta} X_\xi \subseteq Z^y$, Since $\bigcup_{\xi\leq\eta} X_\xi \in \mathscr{p}$, this contradicts the fact that $Z^y \in \mathscr{p}$, and completes the proof of the theorem.

Corollary(1.2.3) [1] : Let $(X, \mathcal{A}, \mu)$ be a probability measure space, where $\mathcal{A}$ is countably generated. Then for every point-finite covering $X_i (i \in I)\ of\ X\ with\ \mu^*(X_i) = 0$, there is some $J \subseteq I$ such that $\bigcup\{X_i : i \in J\}$ is not μ-measurable.

Proof: We apply Theorem (1.2.2.), when$\mathscr{p} = \{A \subseteq X : \mu^*(A) = 0\}$and $\mathcal{A}$ is the σ-algebra of μ–measurable sets. Since condition (b) of Theorem(1.2.2.) is trivially a special case of Fubini's Theorem, it suffices to verify condition (a).

Since $\mathcal{A}$is countably generated, there is $f: X \to \mathbb{R}\ such\ that\ \mathcal{A} = f^{-1}(\mathfrak{B}(\mathbb{R}))$. Let $f(\mu)$be the image measure on $\mathbb{R}$defined by

$$f(\mu)(B) = \mu\,(f^{-1}(B)\,)$$

for every $B \in \mathfrak{B}(\mathbb{R})$and let $\mathcal{V}$ be a countable base for the topology of $\mathbb{R}$. Since $f(\mu)$ is regular $\mathcal{V}_2$ is a base for the ideal of $f(\mu)$-measure zero sets. Thus, if we set $\mathscr{b} = f^{-1}(\mathcal{V})$,then $\mathscr{b}_2$is a base for $\mathscr{p}$.

Corollary(1.2.4) [1] : Let X be a separable metric space, which is not meager in itself. Then for every point-finite covering$X_i (i \in I)\ of\ X$ with each $X_i$meager, there is some $J \subseteq I$ such that $\bigcup\{X_i : i \in J\}$ does not have the property of Baire.

Proof: Let $\mathcal{A} == \mathfrak{B}(X)\ and\ \mathscr{p}$be the ideal of meager sets in X, so that $\mathcal{A}_{\mathscr{p}}$ is the σ-algebra of sets with the property of Baire. Condition (b) of Theorem(1.2.2)follows from Kuratowski-UlamTheorem, the category analogue of Fubini's Theorem. Moreover, using the fact that every meager set is included in an F_σ meager set and that X is second countable, we conclude that condition (a) holds. Thus the result follows from Theorem(1.2.2).

Theorem(1.2.5) [1] : Let$(X, \mathcal{A})$be a measurable space and $\mathscr{b}$ a point-finite covering of$X$such that $|\mathscr{b}|$ has the rectangle property. Then the following are equivalent:

(a) There exists$\mathfrak{B} \subseteq \mathscr{b}\ such\ that\ \bigcup \mathfrak{B} \notin \mathcal{A}.$

(b) There exists $\mathfrak{B} \subseteq \mathscr{b}_d\ such\ that\ \bigcup \mathfrak{B} \notin \mathcal{A}.$

Moreover, if $|\mathscr{b}|$ does not have the rectangle property the result fails.

For a family of sets $\mathscr{b}$ let $\mathscr{b}_u$ be the family of all unions of elements from $\mathscr{b}$ and $\sigma(\mathscr{b})$ be the σ-algebra of sets generated by $\mathscr{b}$. Then we have the following lemma which immediately implies Theorem(1.2.5) .

Lemma(1.2.6) [1] : (a) Let $\mathscr{b}$ be a point-finite covering of a set X such that $K = |\mathscr{b}|$has the rectangle property. Then

(*) $\sigma(\mathscr{b}_{du}) = \sigma(\mathscr{b}_d)$.

(b) $if\ K$is a cardinal without the rectangle property, then there is a point-finite covering$\mathscr{b}$of a set X such that $|X| = |\mathscr{b}| = K$and(*) fails.

Proof: (a) Clearly, $\sigma(\mathscr{b}_{du}) \supseteq \sigma(\mathscr{b}_d)$,so it suffices to show that

$\sigma(\mathscr{b}_{du}) \supseteq \sigma(\mathscr{b}_d)$Let

$$\mathscr{b} = \{A_\xi : \xi \in K\}$$

And define

$$\mathcal{Y} = \{S \in \mathcal{P}(K \times K): \bigcup_{(\xi,\eta)\in S} (A_\xi \cap A_\eta) \in \sigma(\mathscr{b}_{du})\}.$$

We claim that $\mathcal{Y}$ has the following properties:

(i) If S_1 and $S_2 \in \mathcal{P}(K)$, then $S_1 \times S_2 = \mathcal{Y}$.

(ii) If $S_i \in \mathcal{Y}(i \in \mathbb{N})$, then $\bigcup_i S_i \in \mathcal{Y}$.

(iii) If $S_i \in \mathcal{Y}$ and $S_i \supseteq S_{i+1}(i \in \mathbb{N})$, then $\bigcap_i S_i \in \mathcal{Y}$.

To prove (i)-(iii), it is enough to observe that

$$\bigcup_{(\xi,\eta)\in S_1\times S_2} (A_\xi \cap A_\eta) = \left(\bigcup_{\xi\in S_1} A_\xi\right) \cap \left(\bigcup_{\eta\in S_2} A_\eta\right),$$

$$\bigcup_{(\xi,\eta)\in \bigcup_i S_i} (A_\xi \cap A_\eta) = \bigcup_i \bigcup_{(\xi,\eta)\in S_i} (A_\xi \cap A_\eta),$$

And

$$\bigcup_{(\xi,\eta)\in\bigcap_i S_i} (A_\xi \cap A_\eta) = \bigcap_i \bigcap_{(\xi,\eta)\in S_i} (A_\xi \cap A_\eta)$$

Where S_i are as in (i), (ii) and (iii), respectively. We prove only the inclusion '⊇' in the last equation; the rest is verified directly without using the point-finiteness.

Let

$$x = \bigcap_i \bigcup_{(\xi,\eta)\in S_i} (A_\xi \cap A_\eta).$$

Choose $(\xi_i, \eta_i) \in S_i$ *such that* $x \in A_{\xi i} \cap A_{\eta i}$ .By the point-finiteness of $\mathscr{b}$ the same(ξ_i, η_i) occurs infinitely often and since S_i is decreasing, there is $(\xi,\eta) \in \bigcap_i S_i$such that $x \in A_\xi \cap A_\eta$.Hence

$$x \in \bigcup_{(\xi,\eta)\in\bigcap_i S_i} (A_\xi \cap A_\eta).$$

Now, since the family of finite unions of rectangles is an algebra, it follows (i)-(iii) that $\mathcal{Y}$contains the σ-algebra generated by rectangles. But K has the rectangle property, so $\mathcal{Y} = \mathcal{P}(K \times K)$. Thus we have proved that

$$\bigcup_{(\xi_1,\dots,\xi_n)} (A_{\xi_1} \cap A_{\xi_2} \cap \dots \cap A_{\xi_n}) \in \sigma(\mathscr{b}_u)$$

Holds for every $S \subseteq K^n$when $n = 2$. By the comments before Theorem(1.2.5) thisis proved similarly for every finite $n > 2$, while the case $n < 2$ is trivial.

Finally, observe that an arbitrary member of $\mathscr{b}_{du}$has the form

$$\bigcup_{n=0}^{\infty} \bigcup_{(\xi_1,\dots,\xi_n)\in S_n} (A_{\xi_1} \cap A_{\xi_2} \cap \dots \cap A_{\xi_n}),$$

Where $S_n \subseteq K^n$. Hence, by the above, it belongs to $\sigma(\mathscr{b}_u)$.

(b) Let $X = K \times K$ and

$$\mathscr{b} = \big\{\{\xi\} \times K: \xi \in K\big\} \cup \{K \times \{\eta\}: \eta \in K\}$$

Then, $|X| = |\mathscr{b}| = K$ and $\mathscr{b}$ is point-finite. (In fact, every subfamily of with nonempty intersection has cardinal ≤ 2.) Moreover, $\sigma(\mathscr{b}_{du}) = \mathcal{P}(K \times K)$ and $\sigma(\mathscr{b}_u)$.is the σ-algebra generated by the rectangles in $K \times K$. Since K does not have the rectangle property, $\sigma(\mathscr{b}_u) \neq \sigma(\mathscr{b}_{du})$.

Chapter 2

Metric Space and Variable Exponent Lebesgue SpaceS

We show that the maximal operator is bounded provided the variable exponent satisfies a log H¨older type estimate. This condition is known to be essentially sharp in real Euclidean space, however, we show that this is not so in metric spaces .

Section(2.1) Metric Measure and Variable Exponent Lebesgue Spaces:

Lebesgue and Sobolev spaces defined on metric spaces are to-date well understood. These are function spaces in which we replace the Euclidean space from classical analysis by a metric space equipped with a measure (satisfying some conditions). Such spaces are natural abstractions in many situations - consider for instance a plate joined to a block, or a rod joined to a plate (Figure 1). Neither of these domains can be adequately analyzed in Euclidean space, since the joined plate or rod has measure zero.

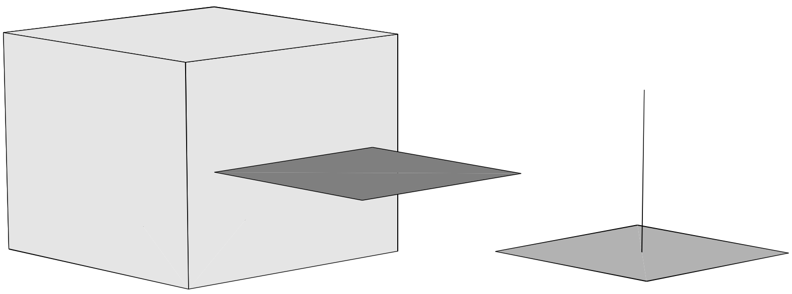

Figure 1. Some metric spaces

In classical Lebesgue and Sobolev theory many properties depend crucially on the dimension of the underlying Euclidean space.Examples include the boundedness of convolution with a Riesz kernel and the Sobolev embedding. In general, metric measure spaces do not have a dimension which could reasonably take the place of the Euclidean dimension in these cases. This problem has been "dealt with" by considering spaces with a doubling measure. However, this really gives only a lower dimension; i.e., $\mu(B(x,r)) \geq Cr^Q$. Moreover, this dimension is the same in every part of the space, which does not correspond to the geometry of the domain. To tackle this problem we propose that analysis on metric measure spaces should be done in Lebesgue spaces where the exponent is allowed to vary, to reflect the non-homogeneity of the underlying space. Then perhaps a local uniformness condition of the kind $\mu(B(x,r)) \approx r^{q(x)}$ will suffice for many properties. Note that this condition is satisfied in the spaces shown in Figure 1.

Variable exponent Lebesgue spaces on Euclidean spaces have attracted a steadily increasing interest over the last couple of years, but the variable exponent framework has not yet been applied to the metric measure space setting (after the completion of this chapter T. Futamura, Y. Mizuta and T. Shimomura have written a couple of papers o(related questions. Variable exponent spaces have been independently discovered by several investigators .These investigations were motivated by differential equations with non-standard coercivity conditions, arising for instance from modeling certain fluids .For some of the latest advances .

In this article we initiate the investigation of variable exponent Lebesgue spaces on metric spaces by considering the Hardy–Littlewood maximal operator M (In mathematics, the Hard-Littlewood maximal operator M is a significant non-linear operator used in real analysis and harmonic analysis. It takes a locally integerable function $f: R^d \to C$ and returns another function Mf that , at each point $x \in R^d$. gives the maximum centered at that point .more precisely ,

$$Mf(x) = sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

Where $B(x,r)$ of ball of radius r centered at x , and $|E|$ denotes the d-dimensional Lebesgue measure of $E \subset R^d$.

The averages are jointly continuous in the x and r , therefore the maximal function Mf , being the supremum over > 0 , is measurable . it is not obvious that Mf is finite almost everywhere)[7]. It has proved to be a very useful tool in geometric analysis both in $\mathbb{R}^n$ and in metric measure spaces. One central question is whether M maps L^p to L^p. In metric spaces with a doubling measure everything works as in the classical case; i.e., L^p maps to itself provided p >1. In variable exponent Lebesgue spaces on $\mathbb{R}^n$ the situation is a bit more precarious: $L^{p(\cdot)}$ maps to $L^{p(\cdot)}$ only when $p(\cdot)$ is sufficiently regular. Due to the efforts of L. Pick & M. R˚uˇziˇcka, L. Diening, A. Nekvinda, and D. Cruz-Uribe, A. Fiorenze& C. Neugebauer, the essentially sharp condition on $p(\cdot)$ is known.

We will show in this section that the condition from the Euclidean setting is sufficient but not necessary in metric measure spaces. For the first of these results we adapt a method of L. we show that a so-called weak type estimate of the maximal operator holds irrespective of the variation of the exponent. This result and its proof are similar to their analogues in Euclidean spaces. D. Cruz-Uribe, A. Fiorenze and C. Neugebauer and A. Nekvinda have given a condition for when the maximal operator is bounded on Lebesgue spaces defined on the whole of $\mathbb{R}^n$. Unfortunately, we are not able to generalize these global results.

A *metric measure space* we mean a triple (X,d,μ), where X is a set, d is a metric on X and μ is a non-negative Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X,d,μ). For $x \in X$ and $r \geq 0$ we denote by $B(x,r)$ the open ball centered at x with radius r. We use the convention that C denotes a constant whose value can change even between different occurrences in a chain of inequalities.

A metric measure space X or a measure μ is said to be *doubling* if there is a constant $C \geq 1$ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \qquad (1)$$

For every open ball $B(x,r) \subset X$. The constant C in (1) is called the *doubling constant* of μ. By the doubling property, if $B(y,R)$ is an open ball in X, $x \in B(y,R)$ and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq C_Q \left(\frac{r}{R}\right)^Q \qquad (2)$$

For some C_Q and Q depending only on the doubling constant. For example, in $\mathbb{R}^n$ with the Lebesgue measure (2) holds with Q equal to the dimension n.

We say that the measure μ is *lower Ahlfors Q-regular* if there exists a constant $C > 0$ such that $\mu(B) \geq C \operatorname{diam}(B)^Q$ for every ball $B \subset X$ with diam $B \leq 2$ diam X . We say that μ is *upper Ahlfors q–regular* if there exists a constant $C > 0$ such that $\mu(B) \leq C \operatorname{diam}(B)^q$ for every ball $B \subset X$ with diam $B \leq$ 2diam X. The measure μ is *Ahlfors Q–regular* if it is upper and lower Ahlfors Q–regular; i.e., if $\mu(B) \approx \operatorname{diam}(B)^Q$ for every ball $B \subset X$ with diam $B \leq$ 2diam X.

Lemma(2.1.1) [2] : *If X is a bounded doubling metric measure space, so that* $\mu(X) < \infty$ *and* $\operatorname{diam}(X) < \infty$, *then it is lower Ahlfors Q-regular.*

Proof: By property (2) we obtain for every $x \in X$ and $0 < r < \operatorname{diam}(X)$

$$\mu\big(B(x,r)\big) \geq C_Q \frac{\mu(X)}{diam(X)^Q} r^Q = C_Q x^{r^Q} .$$

We call a measurable function $p: X \rightarrow [1,\infty)$ a *variable exponent*. For $A \subset X$

We define $p_A^+ = ess\ sup_{\ x \in A}\ p(x)$ and $p_A^- = ess\ inf_{\ x \in A}\ p(x)$; we use the abbreviations $p^+ = p_X^+$ and $p^- = p_X^-$. For a μ–measurable function u: $X \to \mathbb{R}$ we define the *modular*

$$\varrho_{p(\cdot)}(u) = \int_X u(y)|^{p(y)} d\mu(y) \qquad .$$

and the *norm*

$\|u\|_{p(\cdot)} = \inf\{\lambda > 0: \varrho_{p(\cdot)}(u/\lambda) \leq 1\}$.

Sometimes we use the notation $\|u\|_{p(\cdot),X}$ when we also want to indicate in what metric space the norm is taken. The *variable exponent Lebesgue spaces on X*, $L^{p(\cdot)}(X,d,\mu)$, consists of those μ-measurable functions u: $X \to$ R for which there exists $\lambda > 0$ such that $\varrho_{p(\cdot)}(\lambda u) < \infty$. This space is an Orlicz-Museliak space.

It is easy to see that $\|\cdot\|_{p(\cdot)}$ is a norm. if $\|f\|_{p(\cdot)} \leq 1$, then $\varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$. Moreover, if $p^+ < \infty$, then $\varrho_{p(\cdot)}(f_i) \to 0$ if and only if $\|f_n\|_{p(\cdot)} \to 0$. H¨older's inequality (i.e., $\|fg\|_1 \leq C\|f\|_{p(\cdot)} \|g\|_{p'(.)}$) holds also in variable exponent Lebesgue spaces, the proof being as in the Euclidean setting.

Lemma(2.1.2) [2] : *The space $L^{p(\cdot)}(X)$ is a Banach space.*

Proof: Let (f_i) be a Cauchy sequence in $L^{p(\cdot)}(X)$. Then there is a subsequence, denoted again by (f_i), such that

$$\|f_{i+1} - f_i\|_{p(x)} < 2^{-i} \qquad (3)$$

We set

$$gk \sum_{i=1}^{k} |f_{i+1} - f_i \ | \ and\ g = \sum_{i=1}^{\infty} |f_{i+1} - f_i\,|.$$

By (3) and the triangle inequality, we obtain $\|g_k\|_{p(\cdot)} < 1$ for every k. Fatou's Lemma applied to $(g_k^{p(.)})$ gives

$$\int_x \liminf gk\ (x)\ d\mu(x) \leq \liminf \int_x gk\ (x)^{\ p(x)} d\mu(x) \leq \ \|gk\|\ \ p(.) < 1.$$

In particular $g(x) < \infty$ almost everywhere, so that

$$f = f_1 + \sum_{i=1}^{\infty} (f_{i+1} - f_i)$$

exists almost everywhere, and $f(x) = \lim f_i(x)$ almost everywhere. Now we need to prove that f is the $L^{p(\cdot)}$-limit of (f_i). Let $\varepsilon > 0$. There exists N such that $\|f_i - f_j\|_{p(\cdot)} < \varepsilon$ for $i,j > N$. For every $m > N$, Fatou's Lemma implies that

$$\int_X |f - f_m|^{p(\cdot)} d\mu \leq \liminf_{i\to\infty} \int_X |f_i - f_m|^{p(\cdot)} d\mu \leq \liminf_{i\to\infty} \|f_i - f_m\|_{p(\cdot)} \leq \varepsilon$$

and hence $f \in L^{p(\cdot)}(X)$ and $\|f - f_m\|_{p(\cdot)} \to 0$ as $m \to \infty$.

In the next theorem we use the method of Kov´aˇcik and R´akosn´ık to show that continuous functions are dense in variable exponent Lebesgue space.

Theorem(2.1.3)[2]: *Let X be a locally compact doubling space and let $p^+ < \infty$. Then continuous functions with compact support are dense in $L^{p(\cdot)}(X)$.*

Proof : Let $f \in L^{p(\cdot)}(X)$ and define

$$f_n(x) = \begin{cases} f(x); & if\ |f(x)| \leq n\ and\ x \in B(0,n), \\ n\ sign\ f(x), & if\ |f(x)| \leq n\ and\ x \in B(0,n), \\ 0, & else\ where\ . \end{cases}$$

Then each f_n is bounded and has a bounded support. By the Lebesgue Dominated Convergence Theorem $\rho_{p(\cdot)}(f - f_n) \to 0$ as $n \to \infty$, and therefore also $\|f_n - f\|_{p(\cdot)} \to 0$. This shows that the set of bounded functions with bounded support is dense in $L^{p(\cdot)}$.

Let $\epsilon > 0$ and choose a bounded function $g \in L^{p(\cdot)}(X)$ with bounded support such that

$$\|f - g\|_{p(.)} < \epsilon \qquad (4)$$

By Luzin's theorem there exists a continuous function h with compact support in X and an open set U such that

$$\mu(U) < \min\left\{1, \left(\frac{\epsilon}{2\|g\|_\infty}\right)^{p^+}\right\},$$

$g(x) = h(x)$ in $X \setminus U$, and $\sup |h(x)| = \sup_{X\setminus U} |g(x)| \leq \|g\|_\infty$. Hence,

$$\rho_{p(\cdot)}((g-h)/\epsilon) \leq \max\left\{1, (2\|g\|_\infty/\epsilon)^{p^+}\right\}\mu(U) \leq 1$$

which implies $\|g-h\|_{p(\cdot)} \leq \epsilon$. Together with (4) this gives

$$\|f-h\|_{p(\cdot)} \leq 2\epsilon.$$

The following condition has emerged as the right one to guarantee regularity of variable exponent Lebesgue spaces in the Euclidean setting. We say that p is log-*H¨older continuous* if

$$|p(x)-p(y)| \leq \frac{C}{-\log d(x,y)}, \qquad (5)$$

When $d(x,y) \leq 1/2$. This condition has also been called Dini-Lipschitz, weak Lipschitz and 0-H¨older. Since it is the limiting case of α-H¨older continuity, we think that the term log-H¨older is the most descriptive one. The following lemma illustrates the geometrical significance of log-H¨older continuous exponents.

Lemma(2.1.4) [2] : *Assume that $p^+ < \infty$ and define two conditions:*

(i) *p is log-H¨older continuous;*

(ii) *for all balls $B \subset X$ we have $\mu(B)^{p_B^- - p_B^+} \leq C$.*

If μ is lower Ahlfors Q-regular, then (i) *implies* (ii). *If μ is upper Ahlfors qregular, then* (ii) *implies* (i).

Proof : Suppose first that μ is lower Ahlfors Q-regular and that (i) holds.

Since $p_B^- - p_B^+ \leq 0$, it suffices to check (ii) for balls B with radius r less than $\frac{1}{2}$ By (5) we obtain

$$\mu(B)^{p_B^- - p_B^+} \leq (Cr^Q)^{p_B^- - p_B^+} \leq (Cr)^{\frac{-QC}{\log(1/2r)}} \leq C.$$

Suppose then that (ii) holds and that μ is upper Ahlfors$q$-regular. Let $x,y \in X$ be points with $d(x,y) \leq 1/2$ and denote $B = B(x,2d(x,y))$. Using the Ahlfors regularity for the first inequality and condition (ii) for the third inequality we find that

$$\begin{aligned} d(x,y)^{-|p(x)-p(y)|} &= \left(d(x,y)^q\right)^{\frac{-|p(x)-p(y)|}{q}} \leq \left(C\mu(B)\right)^{\frac{-|p(x)-p(y)|}{q}} \\ &\leq C \max\left\{1, \mu(B)^{p_B^- - p_B^-}\right\}^{1/q} \leq C, \end{aligned}$$

so$p$is log-H¨older continuous.

Section(2.2) Strong and Weak Types Estimate with Lebesgue Points :

In this section we derive a weak type inequality of the norm of the maximal operator. This result is obtained by adapting the method used by Cruz-Uribe, Fiorenze & Neugebauer. Their result is stated assuming that the inverse of the variable exponent belongs to a reverse H¨older class. Instead of this assumption we will assume that $p^+ < \infty$. This stronger assumption is motivated by the fact that almost nothing is known of the properties of variable exponent Lebesgue or Sobolev spaces when $p^+ = \infty$. Also our assumption allows us to simplify and shorten their proof.

For every locally integrable function on X we set

$$\mathcal{M}_{B(x,r)} f = \fint_{B(x,r)} |f(y)|\, d\mu(y)$$

and

$$\mathrm{M}f(x) = \sup_{r>0} \mathrm{M}_{B(x,r)} f.$$

The operator M is called the *Hardy–Littlewood maximal operator*. If X is doubling, then for every $t > 0$ and $f \in L^1(X)$ we have the *weak type estimate*

$$\mu\big(\{x \in X : \mathcal{M}f(x) > t\}\big) \le \frac{C_1}{t} \int_X |f|\, d\mu, \qquad (6)$$

and for every $f \in L^q(X)$, $1 < q \le \infty$, the *strong type estimate*

$$\|Mf\|_{qX} \le C_q \|f\|_{qX} .$$

The constants C_1 and C_q depend only on q and the doubling constant.

Our first goal is to generalize the latter result to variable exponent Lebesgue spaces, adapting the method of L. The following lemma is a streamlined version .

Lemma(2.2.1)[2] : *Suppose that X is a bounded doubling space and that p is* log*H¨older continuous with $p^+ < \infty$. Let $f \in L^{p(\cdot)}(X,\mu)$ be such that*

$$(1 + \mu(X))\, \|f\|_{p(\cdot)} \le 1.$$

Then for every $x \in X$ we have

$$\left[\mathcal{M}f(x)\right]^{p(x)} \le C(\mathcal{M}(f^p)(x)+1).$$

Proof : Fix $x \in X$ and let B be a ball centered at x. we have $\|f\|_{p^-_B} \le (1 + \mu(X))\, \|f\|_{p(\cdot)} \le 1$. Using H¨older's inequality for the fixed exponent p^-_B, $\|f\|_{p^-_B} \le 1$. and lemma (2.1.1) and (2.13) we find that

$$\begin{aligned}(\mathcal{M}_B f)^{p(x)} &\le \left(\fint_B |f(y)|^{p_B^-}\, d\mu(y)\right)^{p(x)/p_B^-} = \mu(B)^{-p(x)/p_B^-} \|f\|_{p_B^-}^{p(x)} \\ &\le \mu(B)^{-p(x)/p_B^-} \|f(y)\|_{p_B^-}^{p_B^-} = \mu(B)^{(p_B^- - p(x))/p_B^-} \fint_B |f(y)|^{p_B^-}\, d\mu(y) \\ &\le C \fint_B |f(y)|^{p(y)} + 1\, d\mu(y) \le C(\mathcal{M}_B(f^p)+1).\end{aligned}$$

Since the constant does not depend on the ball B, the result follows by taking the supremum over B.

Theorem(2.2.2) [2] : *Let X be a bounded doubling space. Suppose that p is* log-*H¨older continuous with $1 < p^- \le p^+ < \infty$. Then*

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}. \qquad (7)$$

Proof : Define $q(x) = p(x)/p^-$ for $x \in X$. Since (7) is homogeneous, it suffices to assume that $\|f\|_{p(\cdot)} \le 1/(1+\mu(X))^2$ and prove that $\|Mf\|_{p(\cdot)} \le C$. Since q is smaller than p, we find that

$$(1 + \mu(X))\, \|f\|_{q(\cdot)} \le (1 + \mu(X))^2 \|f\|_{p(\cdot)} \le 1.$$

Thus f satisfies the assumptions of Lemma (2.2.1.)(with p replaced by q), and so

$$\left[\mathcal{M}f(x)\right]^{q(x)} \le C(\mathcal{M}(f^q)(x)+1)$$

for every $x \in X$. Since $p^- > 1$, M: $L^{p^-} \to L^{p^-}$ is a bounded operator, and we get

$$\begin{aligned}\rho_{p(\cdot)}(\mathcal{M}f) = \|(\mathcal{M}f)^{q(\cdot)}\|_{p^-}^{p} &\le C\Big(\|\mathcal{M}(f^q)\|_{p^-} + \|1\|_{p^-}\Big)^{p^-} \\ &\le C\Big(\|f^q\|_{p^-} + \mu(X)^{1/p^-}\Big)^{p^-} \le C.\end{aligned}$$

Since $p^+ < \infty$, this implies that $\|Mf\|_{p(\cdot)} \le C$, which proves the claim.

An example by L. Pick and M. R˚uˇziˇcka showed that log-H¨older continuity of the exponent is essentially the optimal condition for when the maximal operator is bounded on variable exponent Lebesgue spaces defined on Euclidean spaces. The next example features a metric space of the kind considered in the introduction. It shows that in metric spaces the maximal function can be bounded even though the variable exponent is not log-H¨older continuous. In particular, this example implies that there is still work to be done on this question in metric spaces.

Example(2.2.3)[2]: Let $X_1 = \{(x,0) \in \mathbb{R}^2 : 0 \le x < 1/4\}$ and $X_2 = \{(x,y) \in B(0,1/2) : x < 0\}$ and define $(X,\mu) = (X_1,m_1) \cup (X_2,m_2)$, where m_i denotes the i-dimension Lebesgue measure. The space X is shown left most in Figure 2.

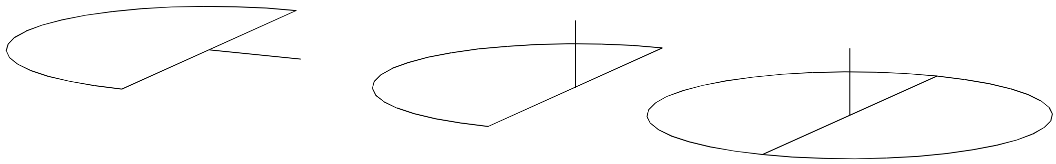

Figure 2. The space X and some variants

Remark(2.2.4) [2] : The space X in the previous example is essentially the same as the right-hand-side space in Figure 1, since X is mapped by a bilipshcitz mapping to the space in the middle of Figure 2 and can then be extended to the third space in Figure 2 by reflection. In particular, the maximal operator is bounded for exactly the same exponents in all these cases.

Theorem(2.2.5)[2] :*Let X and μ be as in Example (2.2.3). We set the exponent p equal to s on X_1 and to t on X_2* $(s,t>1)$. *Consider the boundedness of the maximal operator from $L^{p(\cdot)}(X)$ to itself:*

(i) *if $t \ge 2$, then* M *is bounded if and only if $s \ge t$;*

(ii) *if $t < 2$ and $s < t/(2-t)$, then* M *is bounded;*

(iii) *if $t < 2$ and $s > t/(2-t)$, then* M *is not bounded.*

Proof : Take $f \in L^{p(\cdot)}(X,\mu)$. For the ball $B \subset X$ we find that $M_B f$ equals

$$\frac{1}{m_1(B\cap X_1)+m_2(B\cap X_2)}\left(\int_{B\cap X_1}|f(y)|\,dm_1(y)+\int_{B\cap X_2}|f(y)|\,dm_2(y)\right).$$

We suppose first that $s \ge t$ and either $t \ge 2$ or $s < t/(2-t)$ and show that the maximal operator is bounded. If we denote $f_1 = f\chi_{X1}$ and $f_2 = f\chi_{X2}$ (χ_S denotes the characteristic function of the set S), then we see that

$$\|Mf\|_{p(.)} \leq \|Mf_1\|_{s,X_1} + \|Mf_2\|_{s,X_1} + \|Mf\|_{s,X_2} \quad .$$

Since $f \in L^{p(\cdot)}(X) \subset L^t(X)$, classical Lebesgue theory in metric spaces tells us that the third term in the last expression is bounded by $C\|f\|_{p(\cdot),X}$. Since the first term lives only in X_1 and $f_1 \in L^s(X)$, we get an upper bound for it in terms of $\|f_1\|_{s,X1}$. Therefore we have shown that

$$\|\mathcal{M}f\|_{p(\cdot)} \leq C\left(\|f_1\|_{s,X_1} + \|f\|_{p(\cdot),X}\right) + \|\mathcal{M}f_2\|_{s,X_1}$$

And all that remains is to show is that $\|Mf_2\|_{s,X1} \leq C\|f_2\|_{t,X2}$.

The idea of the proof is that the only really important part of the norm is the origin and its surroundings. Therefore we can make estimates when calculating the norms that have an effect only far from the origin, without this having a large effect on the result.

In what follows we will denote by $B(r)$ a ball centered at the origin with radius $r > 0$. For a ball B centered on X_1 and intersecting X_2, let B' be the ball such that $B \cap \{(0,y): y \in \mathrm{R}\}$ is a diametrical chord. Then for $x \in X_1$ and a ball B centered at x we have

$$\begin{aligned}\mathcal{M}_B f_2 = \fint_B |f_2|\, d\mu &\leq \frac{1}{\mu(B)} \int_{B'} |f_2|\, d\mu \leq \frac{1}{\mu(B \cap X_1)} \int_{B' \cap X_2} |f_2|\, dm_2 \\ &\leq \frac{1}{\min\{|x| + r, 1/4\}} \int_{B' \cap X_2} |f_2|\, dm_2,\end{aligned}$$

where r is the radius of B'. Therefore:

$$\mathcal{M}f_2(x) \leq \frac{1}{\min\{|x| + r(x), 1/4\}} \int_{B(r(x)) \cap X_2} |f_2|\, dm_2$$

For some suitable function $r: X_1 \to [0,1/2)$. For those $x \in X_1$ with $|x|+r(x) > 1/4$ this means that

$$Mf_2(x) \leq 4 \quad \int |f_2| dm_2 \leq C\|f_2\|_{t,X2},$$

So we need not worry about these points. For the other points we calculate

$$\|\mathcal{M}f_2\|_{s,X_1} \leq \left(\int_0^{1/4} \left(\frac{1}{|x| + r(x)} \int_{B(r(x)) \cap X_2} |f_2|\, dm_2\right)^s dx\right)^{1/s} \qquad (8)$$

By H¨older's inequality we have

$\int_{B(r)\cap X2} \int|f_2|dm_2 \leq \|f_2\|_{t,B(r)\cap X_2}\|1\|_{t',B(r)\cap X2} \leq cr^{2(1-1/t)}\| f_2\|_{t,X_2}$.

Using this in (8) gives

$$\|\mathcal{M}f_2\|_{s,X_1} \leq \left(\int_0^{1/4}\left(\frac{Cr(x)^{2(1-1/t)}}{|x|+r(x)}\|f_2\|_{t,X_2}\right)^s dx\right)^{1/s}$$
$$-C\left(\int_0^{1/4}\left(\frac{r(x)^{2(1-1/t)}}{|x|+r(x)}\right)^s dx\right)^{1/s}\|f_2\|_{t,X_2}.$$

By considering the partial derivative with respect to r, we see that

$$\frac{r^a}{|x|+r} \leq \begin{cases} \frac{a^a}{(1-a)^{a-1}} x^{1-2/t} & if\ a < 1 \\ \frac{1}{1+x} & if\ a \geq 1 \end{cases},$$

Where $a= 2(1 - 1/t)$. Using this in our previous estimate gives $\|Mf_2\|_{s,X1} \leq \|f_2\|_{t,X2}$ if either $a \geq 1$ (that is $t \geq 2$), or $a <1$ and

$$\int_0^{1/4} x^{(1-2/t)s}\, dx < \infty$$

This integral is finite provided $(1 - 2/t)s>-1$; i.e., $s < t/(2 - t)$, which concludes the proof of (ii) and sufficiency in (i).

If $t > s$, it is easy to see that the maximal operator is not bounded (consider the function defined as $|x|^{-2/(s+t)}$ on X_1 and 0 on X_2). This completes the proof of (i).

To prove (iii) we will construct a function $g \in L^{p(\cdot)}(X)$ for which $Mg \notin L^s(X_1)$ for fixed $t <2$ and $s > t/(2 - t)$ It turns out that

$$g(x) = \left(|x| \log(1/|x|)\right)^{-2/t}\chi_{X_2}$$

does the trick. We show that $g \in L^t(X_2)$:

$$|g\|_{t,X_2}^t = \int_0^{1/2} r\left(r\log(1/r)\right)^{-2} dr = \int_0^{1/2} r^{-1}\log^{-2}(1/r)\, dr = \frac{1}{\log 2}.$$

For $x \in X_1$we find that

$$\mathcal{M}g(x) \geq \fint_{B(x,2|x|)} |g(y)|\, d\mu(y) \geq \frac{1}{3|x| + \pi(2|x|)^2} \int_{B(0,|x|)} g(y)\, dm_2(y)$$

$$= \frac{1}{3+4\pi|x|} \fint_0^{|x|} rg(r)\, dr \geq \frac{1}{3+\pi} \fint_0^{|x|} r^{1-2/t} \log^{-2/t}\left(\tfrac{1}{r}\right) dr.$$

Since $r \to r^{1-2/t} \log^{-2/t}(1/r)$, is convex, we find, by Jensen's inequality, that

$$\fint_0^{2y} r^{1-2/t} \log^{-2/t}\left(\tfrac{1}{r}\right) dr \geq y^{1-2/t} \log^{-2/t}\left(\tfrac{1}{y}\right)$$

Using this in the previous estimate gives

$$\|\mathcal{M}g\|_{s,X_1}^s = \int_0^{1/4} (\mathcal{M}g(x))^s\, dx$$

$$\geq \frac{1}{(3+\pi)^s} \int_0^{1/4} (x/2)^{-(2-t)s/t} \log^{-2s/t}\left(\tfrac{2}{x}\right) dx$$

Since $(2-t)s/t > 1$, this integral is divergent, $\mathrm{M}g \notin L^s(X_1)$, which was to be shown.

Lemma(2.2.6) [2] : *Let X be a doubling space and assume that $p^+ < \infty$. Let also $f \in L^{p(\cdot)}(X)$ and $t > 0$. Suppose that B is a ball such that*

$$\int_B \frac{|f(y)|}{t}\, d\mu(y) > \mu(B)$$

Then there exists a constant C depending only on p such that

$$\mu(B) \leq C \int_B \left(\frac{|f(y)|}{t}\right)^{p(y)} d\mu(y)$$

Proof : We choose a sequence of simple functions (s_n) with $s_n \geq p^-_B$ such that the sequence $(s_n(x))$ increases monotonically to $p(x)$ on B. Then for each n,

$$s_n(x) \sum_{j=1}^{k_n} \propto n, jx\, A_{n,j}\,(x),$$

where the sets $A_{n,1},\ldots,A_{n,kn}$ are disjoint and $\cup_j A_{n,j}= B$.

Using H¨older's and Young's inequalities we have

$$\begin{aligned}\int_{A_{n,j}} \frac{|f(y)|}{t}\,d\mu(y) &\le \left(\int_{A_{n,j}} \left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} d\mu(y)\right)^{1/\alpha_{n,j}} \mu(A_{n,j})^{1/\alpha'_{n,j}} \\ &\le \frac{1}{\alpha_{n,j}} \int_{A_{n,j}} \left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} d\mu(y) + \frac{\mu(A_{n,j})}{\alpha'_{n,j}} \\ &\le \frac{1}{p_B^-} \int_{A_{n,j}} \left(\frac{|f(y)|}{t}\right)^{s_n(y)} d\mu(y) + \frac{\mu(A_{n,j})}{(p_B^+)'}\end{aligned}$$

Adding the inequalities from 1 to k_n gives

$$\int_B \frac{|f(y)|}{t}\,d\mu(y) \le \frac{1}{p_B^-} \int_B \left(\frac{|f(y)|}{t}\right)^{s_n(y)} d\mu(y) + \frac{\mu(B)}{(p_B^+)'}$$

This inequality holds for all n, hence the monotone convergence theorem implies that

$$\mu(B) < \int_B \frac{|f(y)|}{t}\,d\mu(y) \le \frac{1}{p_B^-} \int_B \left(\frac{|f(y)|}{t}\right)^{p(y)} d\mu(y) + \frac{\mu(B)}{(p_B^+)'}$$

Since $p^+ < \infty$, it follows that $(p_B^+)' > 1$, so we are done.

Theorem(2.2.7) [2] : *Let X be a doubling space and assume that $p^+ < \infty$. Then for all $f \in L^{p(\cdot)}(X)$ and $t > 0$*

$$\mu\big(\{x \in X : \mathcal{M}f(x) > t\}\big) \le C \int_X \left(\frac{|f(y)|}{t}\right)^{p(y)} d\mu(y) \qquad (9)$$

Proof : Let $n > 0$ and define

$$\mathcal{M}_n f(x) = \sup_{r \le n} \fint_{B(x,r)} |f(y)|\,d\mu(y) .$$

Then the sequence $(\mathcal{M}_n f(x))$ is increasing and converges to M$f(x)$ for every $x \in X$. By the monotone convergence theorem, we find for each $t > 0$ that

$$\mu(\{x \in X : \mathcal{M}f(x) > t\}) = \lim_{n\to\infty} \mu(\{x \in X : \mathcal{M}_n f(x) > t\}),$$

and therefore it suffices to prove (9) for M_n with constant independent of n.

Let $t > 0$ and denote $E_n = \{x \in X: M_n f(x) > t\}$. Then for each $x \in E_n$ there is a ball $B_x = B(x, r_x)$ such that

$$\fint_{B_x} |f(y)|\, d\mu(y) > t.$$

By the standard covering theorem we find a disjoint family $\mathcal{F}$ of balls B_x with

$$E_n \;=\subset \textstyle\sum_{B\in\mathcal{F}} \mu(B)$$

Using the doubling property and Lemma (9)this implies that

$$\mu(E_n) \le \sum_{B\in\mathcal{F}} \mu(5B) \le C \sum_{B\in\mathcal{F}} \mu(B)$$
$$\le C \sum_{B\in\mathcal{F}} \int_B \left(\frac{|f(y)|}{t}\right)^{p(y)} d\mu(y) \le C \int_X \left(\frac{|f(y)|}{t}\right)^{p(y)} d\mu(y),$$

where the constant C depends on p^+ and the doubling constant.

A standard application of the weak type estimate is the Lebesgue point theorem. We show that this method also work in our case, provided we know that continuous functions are dense in $L^{p(\cdot)}(X)$. The theorem generalizes to metric measure spaces.

Theorem(2.2.8) [2] : *Let X be a locally compact doubling space and assume that $p^+ < \infty$. Then for every $u \in L^{p(.)}_{loc}(X)$ and almost every $x \in X$*

$$\limsup_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)|^{p(y)}\, d\mu(y) = 0$$

Proof: We may assume that $u \in L^{p(\cdot)}(X)$. Otherwise we can study a function $u\chi_{Br}$, where χ_{Br} is the characteristic function of a ball with radius $r > 0$, and the theorem follows by the subadditivity of the measure. We define

$$Lu(x) = \limsup_{r \to 0} \fint_{B(x,r)} |u(y) - u(x)|^{p(y)} \, d\mu(y)$$

For $x \in X$ we obtain

$$Lu(x) \leq 2^{P^+} \limsup \fint_{B(x.r)} |u(y)|^{p(y)} d\mu(y).$$

$$+ \; 2^{P^+} \limsup \fint_{B(x,r)} |u(x)|^{p(y)} \, d\mu(y)$$

$$\leq 2^{P^+} \, Mu^{p(\cdot)}(x) + 2^{P^+} \max\{|u(x)|, |u(x)|^{p^+}\}$$

We easily see that $L(u_1+u_2)(x) \leq 2^{p^+}Lu_1(x)+2^{p^+}Lu_2(x)$. Let v be continuous and note that $Lv$is identically zero. This yields

$$\begin{aligned}
\mu\big(\{x \in X : Lu(x) > t\}\big) & \\
&= \mu\big(\{x \in X : L(u-v)(x) > t\}\big) \\
&\leq \mu\left(\left\{x \in X : \mathcal{M}(u-v)^{p(\cdot)}(x) > \tfrac{t}{2^{p^+ + 1}}\right\}\right) \\
&\quad + \mu\left(\left\{x \in X : |u(x) - v(x)| > \min\left\{\tfrac{t}{2^{p^+ + 1}}, \tfrac{t^{1/p^+}}{2^{1+1/p^+}}\right\}\right\}\right)
\end{aligned}$$

We use the weak type estimate (6) for $(u - v)^{p(\cdot)} \in L^1(X)$ or Theorem(2.2.7) to get

$$\begin{aligned}
\mu\big(\{x \in X : Lu(x) > t\}\big) \leq C(t) \int_X |u(y) - v(y)|^{p(y)} d\mu(y) & \\
+ \mu\big(\{x \in X : |u(x) - v(x)| > C(t)\}\big).
\end{aligned}$$

Since continuous functions are dense in $L^{p(\cdot)}(X)$ by theorem (2.1.3) and since convergence in norm implies convergence in measure, we can choose $v \in L^{p(\cdot)}(X)$ to make the right-hand-side of the previous inequality arbitrarily small. Since the left-hand-side does not depend on v, it follows that it has to equal zero for every t >0. Therefore, by the subadditivity of μ,

$$\mu\big(\{x \in X : Lu(x) > 0\}\big) \leq \sum_{i=1}^{\infty} \mu\big(\{x \in X : Lu(x) > 1/i\}\big) = 0$$

So $Lu(x) = 0$ almost everywhere.

1. # Chapter 3

Variable exponent Lebesgue space and Interpolation theorem

When Hardy–Littlewood maximal operator is bounded on $L^{p(.)}(\mathbb{R}^n)$ space we prove $\left[L^{p(.)}(\mathbb{R}^n), BMO(\mathbb{R}^n)\right]_\theta = L^{q(.)}(\mathbb{R}^n)$ where $q(\cdot) = p(\cdot)/(1-\theta)$ and $\left[L^{p(.)}(\mathbb{R}^n), H^1(\mathbb{R}^n)\right]_\theta = L^{q(.)}(\mathbb{R}^n)$ where $1/q(\cdot) = \theta + (1-\theta)/p(\cdot)$.

2. **Section(3.1) The Characterizations of variable exponent Lebesgue spaces :**

3. The Lebesgue spaces $\boldsymbol{L^{p(.)}}(\mathbb{R}^{\boldsymbol{n}})$ with variable exponent and the corresponding variable Sobolev spaces $\boldsymbol{w^{k,p(.)}}(\mathbb{R}^{\boldsymbol{n}})$ are of interest for their applications to modelling problems in physics, and to the study of variational integrals and partial differential equations with nonstandard growth condition.

Given a measurable function $(.) : \mathbb{R}^{\boldsymbol{n}} \to [1, \infty)$, $\boldsymbol{L^{p(.)}}(\mathbb{R}^{\boldsymbol{n}})$ denotes the set of measurable functions f on $\mathbb{R}^{\boldsymbol{n}}$ that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty$$

This set becomes a Banach function space when equipped with the norm

$$||f||_{p(\cdot)} = \inf\left\{\lambda > 0: \int \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}$$

There is interesting whether it is possible to transfer complex and real interpolation results to variable exponent Lebesgue spaces. It was shown that Riesz–Thorin theorem (In mathematics , Riesz–Thorin theorem , ofent referred to as the Riesz–Thorin interpolation theorem or the Riesz–Thorin convexity theorem is a result a bout interpolation of operators. it is named after marcel Riesz and his student G.O Lof Thorin.

The theorem boundeds the norms of linear maps a cting between L^P spaces. It is usefulness stems from the fact that is some of thess spaces have rather simpler structure than others. Usually that refers to L^2 which is a Hilbert space , or to L^1 and L^∞ . therefore one may prove theorems about the more complicated cases by proving theorem in two simple cases and then using the Riesz–Thorin theorem to pass from the simple cases to the complicated cases)[8].

is valid on $L^{p(.)}$ (Ω) spaces, i.e a linear operator T which is bounded from $L^{p_j(.)}(\Omega)$ to $L^{p_j(.)}(\Omega)$, j = 0,1, is also bounded from $L^{p_\theta(.)}$ (Ω) to $L^{p_\theta(.)}$ (Ω). Here $p_\theta(.)$ is de- finedusual way, $1/p\theta(\cdot) = (1-\theta)/p0(\cdot)$. $[\boldsymbol{L^{p_0(.)}}(\Omega), \boldsymbol{L^{p_1(.)}}(\Omega)]_{\boldsymbol{\theta}} \approx L^{p_\theta(.)}$ (Ω), $1 < (pj)_- \leq (pj)_+ < \infty$ (below we will denote p_- =ess inf $p(\cdot)$ andp_+ = ess sup $p(\cdot)$), where $[A_1,A_2]_\theta$ denotes the complex interpolation space of the Banach spaces A_1 and A_2 .

Recall that $BMO(\mathbb{R}^{\boldsymbol{n}})$ denotes the space of functions of bounded mean oscillation with the seminorm

$$\|f\|_{BMO} = \sup \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

Where $f_Q = \frac{1}{|Q|} \int_Q f(x)\,dx$ and supremum is taken over all cubes in $\mathbb{R}^{\boldsymbol{n}}$.

Among the various equivalent characterizations of the Hardy space $H^1(\mathbb{R}^{\boldsymbol{n}})$ one of the simplest is using a maximal function fix a function $\phi \in S(\mathbb{R}^n)$ with $\int \phi(x)\,dx = 1$ and define the maximal function of a distribution $f \in S'(\mathbb{R}^n)$ by

$$M_\phi f(x) = \sup_{t>0} \left| f * \phi_t(x) \right|, \quad x \in \mathbb{R}^n$$

Then $f \in (\mathbb{R}^{\boldsymbol{n}})$ if $M_\emptyset f \in L^1(\mathbb{R}^{\boldsymbol{n}})$ and one can set

$$\|f\|_{H^1} = \|M_\emptyset f\|_1 .$$

In this chapter we study complex interpolation spaces $[\boldsymbol{L^{p(.)}}(\mathbb{R}^{\boldsymbol{n}}), \boldsymbol{BMO}(\mathbb{R}^{\boldsymbol{n}})]_{\boldsymbol{\theta}}$ and $[\boldsymbol{L^{p(.)}}(\mathbb{R}^{\boldsymbol{n}}), \boldsymbol{H^1}(\mathbb{R}^{\boldsymbol{n}})]_{\boldsymbol{\theta}}$ when Hardy–Littlewood maximal operator is bounded on $\boldsymbol{L^{p(.)}}(\mathbb{R}^{\boldsymbol{n}})$. We follow the approach of Frazier and Jawerth which takes advantage of the fact that, unlike

distribution spaces $F^{\cdot\alpha}{}_{p,q}$ the sequence spaces $\dot{f}^{\alpha}_{p,q}$ are quasi-Banach lattices. Hence, by computing Calderon spaces we can deduce complex interpolation results for $F^{\cdot\alpha}{}_{p,q}$ spaces.

Let $B(x,r)$ denote the open ball in $\mathbb{R}^{\boldsymbol{n}}$ of radius r and center x. By $|B(x,r)|$ we denote dimensional Lebesgue measure of $B(x,r)$. The Hardy–Littlewood maximal operator M is defined on locally integrable function f on $\mathbb{R}^n$ by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy.$$

By $p(\mathbb{R}^n)$ denote the class of all exponents $p(.)$ with property $1 < a \leq p(t) \leq b < \infty; t \in \mathbb{R}^n$ and define $B(\mathbb{R}^n)$ to be the set of exponents $P(.) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $\boldsymbol{L^{p(.)}}(\mathbb{R}^n)$.

Below by $p'(.)$ we denote the conjugate exponent of $p(.)(1/p(t) + 1/p'(t) = 1, t \in \mathbb{R}^n)$.

In harmonic analysis a fundamental operator is the Hardy–Littlewood maximal operator M. In many applications a crucial step has been to show that operator M is bounded on a variable L^Pspace. Note that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $\boldsymbol{L^{p(.)}}$ whenever the Hardy–Littlewood maximal operator is bounded on $\boldsymbol{L^{p(.)}}(\mathbb{R}^n)$. Conditions for the boundedness of the maximal and singular operators on spaces $\boldsymbol{L^{p(.)}}(\mathbb{R}^n)$.

Theorem(3.1.1) [3]: *Let* $\boldsymbol{p}(.) \in \boldsymbol{B}(\mathbb{R}^n)$ *and* $0 < \theta < 1$. *Then*

$$\left[\boldsymbol{L^{p(.)}}(\mathbb{R}^n), \boldsymbol{BMO}(\mathbb{R}^n)\right]_{\boldsymbol{\theta}} = \boldsymbol{L^{q(.)}}(\mathbb{R}^n), \boldsymbol{where}\ \boldsymbol{q}(.) = \frac{\boldsymbol{p}(.)}{\boldsymbol{1-\theta}}, \qquad \textbf{(1)}$$

$$\left[\boldsymbol{L^{p(.)}}(\mathbb{R}^n), \boldsymbol{H^1}(\mathbb{R}^n)\right]_{\boldsymbol{\theta}} = \boldsymbol{L^{q(.)}}(\mathbb{R}^n), \boldsymbol{where}\ \boldsymbol{1}/\boldsymbol{q}(.) = \boldsymbol{\theta} + (\boldsymbol{1}-\boldsymbol{\theta})/\boldsymbol{p}(.). \qquad \textbf{(2)}$$

For the case $p(.)$ = const, this is the classical result of Fefferman and Stein. Later several authors have extended Fefferman–Stein's complex interpolation result in the sense of replacing L^p on left side by L^∞or *BMO* for different approaches to this result).

Proof : We can obtain results regarding complex interpolation from Theorem (3.2.1) Let $[X_0,X_1]_\theta$ denote the space obtained from X_0 and X_1 by the complex interpolation method. Suppose X_0 and X_1 are Banach lattices on a measure space Ω, and let $X = X_0^{1-\theta} X_1^{\theta}$ for some

$\theta \in (0,1)$ show that $\left[X_0, X_1 = X_0^{1-\theta} X_1^{\theta}\right]_\theta$ under the hypothesis that X has Fatou property. Note that $L^{p(.)}$, if $p(.) \in L^\infty$ satisfies the above property by the Lebesgue dominated convergence theorem . Hence, we obtain

$$[\dot{f}^0_{p(\cdot),2}, \dot{f}^0_{\infty,2}]_\theta = \dot{f}^0_{q(\cdot),2}.$$

Applying Theorem (3.1. 3) we obtain (31.1).

Corollary(3.1.2)[3]: *Let* $p(.) \in B(\mathbb{R}^n)$, $\mathbf{0} < \boldsymbol{\theta} < \mathbf{1}$ ***and*** $\boldsymbol{q}(.) = \frac{p(.)}{1-\theta}$. *Then there exists a constant* C *such that*

$$\|f\|_{q(.)} \leq C\|f\|_{p(.)}^{1-\theta} \; \|f\|_{BMO}^{\theta} \qquad (3)$$

Holds for all $f \in L^{p(.)}(\mathbb{R}^n) \cap \boldsymbol{BMO}(\mathbb{R}^n)$.

Note that in case $P(.) \equiv$ const (3) was shown by Chen and Zhu by using the John–Nirenberg type inequality .

To prove (3.1.2), we first observe that $(L^{p(\cdot)}(\mathbb{R}^n))^* = (L^{p(\cdot)}(\mathbb{R}^n))' = L^{p'(\cdot)}(\mathbb{R}^n)$ and consequently

$$(\dot{f}^{0}_{p(\cdot),2})^* = (\dot{f}^{0}_{p(\cdot),2})' = \dot{f}_{p'(\cdot),2}$$

Note also that $\left(f^{\cdot 0}_{1,2}\right)^* = \left(f^{\cdot 0}_{1,2}\right)' = f^{\cdot 0}_{\infty,2}$ and $f^{\cdot 0}_{\infty,2} = \left(f^{\cdot 0}_{\infty,2}\right)'(f^{\cdot 0}_{1,2})$ is order continuous Banach lattices) .

According to (13) we can write

$$\begin{aligned}((\dot{f}^{0}_{p'(\cdot),2})^{1-\theta}(\dot{f}^{0}_{\infty,2})^{\theta})' &= ((\dot{f}^{0}_{p'(\cdot),2})')^{1-\theta}((\dot{f}^{0}_{\infty,2})')^{\theta} \\ &= (\dot{f}^{0}_{p(\cdot),2})^{1-\theta}(\dot{f}^{0}_{1,2})^{\theta} = [\dot{f}^{0}_{p(\cdot),2}, \dot{f}^{0}_{1,2}]_{\theta}\end{aligned}$$

Applying Theorems (3.2.1), (3.1.3) and (3.1.1) we can write $L^{q'(.)}(\mathbb{R}^n)' = \left[L^{p(.)}(\mathbb{R}^n), H^1(\mathbb{R}^n)\right]_\theta$

Where $1/q'(.) = (1-\theta)/p'(.)$ and consequently $L^{q(.)}(\mathbb{R}^n) = \left[L^{p(.)}(\mathbb{R}^n), H^1(\mathbb{R}^n)\right]_\theta$ where $1/q(.) = (1-\theta)/p(.)$.

Let $f \in L^{p(.)}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ *and* $s = \left\{(S_\varphi)_Q\right\}_{Q\in\mathcal{D}}$. From (12) and (14) we obtain

$$\|s\|_{f^{\cdot 0}_{q(.).2}} = \|s^{1-\theta}\, s^{\theta}\|_{f^{\cdot 0}_{q(.).2}} \leq C\|s\|^{1-\theta}_{f^{\cdot 0}_{p(.).2}} \|s\|^{\theta}_{f^{\cdot 0}_{\infty.2}}$$

This completes the proof of (3) .

Many function spaces arising in harmonic analysis admit decompositions into simpler building blocks, often called atoms or molecules,that have some additional desirable properties.

One of the possible directions, where decomposition techniques are very useful, is the study of a large class of general Triebel–Lizorkin spaces $F^{\cdot\alpha}{}_{p,q}$ (homogeneous) and $F^{\cdot\alpha}{}_{p,q}$ (inhomogeneous), $\alpha \in \mathbb{R}, 0 < p, q \leqslant \infty$ which includes many well-known classical function spaces. In particular, $L^p \approx F^{\cdot\alpha}{}_{p,2} \approx F^{\cdot\alpha}{}_{p,2}$ when 1 < p <∞ (here ≈ means that the (quasi)-norms are equivalent) and $F^{\cdot\alpha}{}_{p,2} \approx H^p$ when $0 < p \leqslant 1$The atomic and molecular decomposition results for homogeneous Triebel–Lizorkin spaces were first obtained by Frazier and Jawerth.

For $v \in \mathbb{Z}$ and $\in \mathbb{Z}^n$, let Q_{vk} be the dyadic cube 2^{-v} $[0,1]^n + k$. Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$. For a cube Q let x_Q denote the "lower left corner".

The set $S(\mathbb{R}^n)$ denotes the usual Schwartz space of rapidly decreasing complex-valued functions and $S'(\mathbb{R}^n)$ denotes the dual space of tempered distributions. We denote the Fourier transform of $\varphi\ by\ \hat{\psi}$Let functions $\varphi, \psi \in S(\mathbb{R}^n)$ satisfy the following conditions:

$$\operatorname{supp}\hat{\varphi}, \hat{\psi} \subset \left\{\xi \in \mathbb{R}^n: \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}, \tag{4}$$

and $$\left|\hat{\psi}(\xi)\right|, \left|\hat{\psi}(\xi)\right| \geq c > 0 \quad \text{when} \quad \frac{3}{5} \leqslant |\xi| \leqslant \frac{5}{3} \quad , \tag{5}$$

$$\sum_{v\in\mathbb{Z}} \widehat{\varphi}\,(2^{-v}\xi)\hat{\psi}(2^{-v}\xi) = 1 \quad if\ \xi \neq 0. \tag{6}$$

Here $\tilde{\varphi}$ (x) = $\overline{\varphi(-x)}$. we set φ_V(x) = $2^{V\,n}\,\varphi(2^V\ x)\ and\ \psi_{\,V}(x) = 2^{V\,n}\,\psi\,(2^V\ x), \nu \in \mathbb{Z}$.

$$\varphi_Q\,(x) = |Q\,|^{\frac{1}{2}}\,\varphi_\nu\,\left(x - x_Q\right) if\ Q = Q_{\,\nu k,} and\ similarly\ \psi_{\,Q} .$$

For φ and ψ satisfying (4) - (6) the φ - transform S_φ *is the map taking each* $f \in S(\mathbb{R}^n)$ / P (the space of tempered distributions modulo polynomials) to the sequence $S\varphi f$ = $\{(S\varphi f)_Q\}_{Q \in D}$ defined by $(S\varphi f)_Q = \langle f, \varphi_Q \rangle$ for Q dyadic. Here ‹ ·,·›denotes the usual inner product on $L^2(\mathbb{R}^n; \mathbb{C})$. The inverse of φ-transform T_ψ is the map taking a sequence $s = \{s_Q\}_{Q \in D}$ *to* $T_\psi s = \sum Q \in D^s\, Q\psi Q$.

The starting point in the theory of discrete φ -transforms of Frazier and Jawerth is the representation formula for tempered distributions: If $f \in S'(\mathbb{R}^n)/\mathcal{P}$, then

$$f(x) = \sum_{Q \in Q} \langle f, \varphi_Q \rangle\, \varphi_Q(\mathrm{x}) ,$$

Where the convergence of the above series, as well as the equality, is in $S'(\mathbb{R}^n)/\mathcal{P}$.

Motivated by the classical definition of homogeneous Triebel–Lizorkin spaces, Frazier, Jawerth and Weiss we define variable discrete homogeneous Triebel–Lizorkin spaces $f^0_{p(\cdot),2}$ as follows: for $P(.) \in p(\mathbb{R}^n)$ let $f^0_{p(\cdot),2}$ be the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in D}$ such that

$$\|s\|_{f^{.0}_{p(.),2}} = \|\left(\sum_{Q \in D}(|s_Q| \hat{X}_Q)^2\right)^{1/2}\|_{P(.)} < \infty ,$$

Where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of Q.

Theorem (3.1.3) [3]: *Suppose* $P(.) \in B(\mathbb{R}^n)$ *The operators* $S_\varphi : L^{p(.)}(\mathbb{R}^n) \to f^{.0}_{p(\cdot),2}$ and Tψ : $f^{.0}_{p(\cdot),2} \to L^{p(.)}(\mathbb{R}^n)$ are *bounded. Furthermore,* Tψ 0 S_φ *is the identity on.* In particular

$$\|f\|_{p(.)} \approx \| S_\varphi : f\|_{f^{.0}_{p(\cdot),2}}.$$

The above norm estimate is consequence of the extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Perez and the weighted norm inequalities for

$$\mathcal{W}f(x) = \left(\sum_{Q \in \mathcal{D}} \left(|\langle f, \varphi_Q \rangle| \tilde{\chi}_Q \right)^2 \right)^{1/2}$$

function, given by Frazier and Jawerth. We describe these results.

Under a weight we mean a non-negative, locally integrable function w. When $1 < p < \infty$, we say $w \in A_p$ if for every cube Q.

$$\frac{1}{|Q|} \int_Q w(x) dx \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} \leq C < \infty .$$

By $A_{p,w}$ we denote the infimum over the constants on the right-hand side of the last inequality. By $\mathcal{F}$ we will denote a family of ordered pairs of non-negative, measurable functions (f,g) We say that an inequality

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)\, dx \leqslant C \int_{\mathbb{R}^n} g(x)^{p_0} w(x)\, dx \quad (0 < p_0 < \infty) \tag{7}$$

holds for any $(f,g) \in \mathcal{F}$ and $w \in A_q$ (for some q, $1 < q < \infty$) if it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant C depends only on p_0 and on the constant $A_{q,w}$.

Proof : From the assumption $P(.) \in B(\mathbb{R}^n)$ we get that there exists $1 < p_1 < p-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$. An application of Theorem (3.1.4) and (3.1.7) with the pairs $(\mathcal{W}f, |f|)$ gives the norm estimate $\| s_\varphi f \|_{f^0{}_{p(\cdot),2}} \leq C \| f \|_{p(\cdot)}$ provided $f \in C_0^\infty(\mathbb{R}^n)$. Note that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(.)}(\mathbb{R}^n)$ and consequently this inequality is also valid for all $\in L^{p(.)}(\mathbb{R}^n)$. Analogously we obtain opposite direction inequality.

The extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Perez, implies among other things the Fefferman–Stein vector-valued inequality for variable exponent Lebesgue spaces.

Theorem(3.1.4) [3]: *Given a family $\mathcal{F}$, assume that* (7) *holds for some* $1 < p_0 < \infty$ *, for every weight* $w \in A_{p0}$ *and for all* $(f,g) \in \mathcal{F}$ *. Let* $P(.) \in p(\mathbb{R}^n)$ *be such that there exists* $1 < p_1 < p-$, *with* $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$. *Then*

$$\| f\|_{p(\cdot)} \leqslant C\|g\|_{p(\cdot)}$$

for all $(f,g) \in \mathcal{F}$ *such that* $f \in L^{p(.)}(\mathbb{R}^n)$ . .

Let w be a non-negative function satisfying "doubling condition" $w(2Q) \leqslant Cw(Q)$, where, for a measurable set. E $w(E) = \int_E w(x)\,dx$ The discrete Triebel–Lizorkin weighted sequence space $f^{0}_{p\,(.),2}(\mathfrak{w})$ is defined as the collection of all complex-valued sequences s such that

$$s\|_{\dot f^{0}_{p,2}(w)} = \left\| \left(\sum_{Q\in\mathcal{D}} (|s_Q|\tilde{\chi}_Q)^2 \right)^{1/2} \right\|_{L^p(w)} < \infty$$

Where $L^p(\boldsymbol{w})$ is weighted Lebesgue space.

Theorem(3.1.5) [3]: *Suppose* p ∈ (1, ∞), w ∈ A_p. *The operators* S_φ: $L^p(w) \to$ $\dot f^{0}_{p\,(.),2}$ *and* T_ψ: $\dot f^{0}_{p\,(.),2} \to L^p(w)$ *are bounded. Furthermore,* $T_\psi \circ S_\varphi$ *is the identity on* $L^p(w)$. *In*

particular

$$\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \approx \int_{\mathbb{R}^n} \mathcal{W}f(x)^p w(x)\,dx. \qquad (8)$$

Theorem(3.1.6)[3]: *Let* $p(.) \in B(\mathbb{R}^n)$ *and* $1 < q < \infty$. *Then*

$$\left\| \|Mf_i\|_{l^q} \right\|_{p(\cdot)} \leqslant C \left\| \|f_i\|_{l^q} \right\|_{p(\cdot)}.$$

Theorem(3.1.7) [3]: *Let* ε > 0, $p(.) \in B(\mathbb{R}^n)$. *Suppose that for each dyadic cube* $Q, E_Q \subseteq Q$ *is a measurable set with* $|E_Q|/|Q| \geqslant \varepsilon$. *Then*

$$\left\|\{s_Q\}_Q\right\|_{\dot f^{0}_{p(\cdot),2}} \approx \left\| \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\tilde{\chi}_{E_Q}\right)^2 \right)^{1/2} \right\|_{p(\cdot)} , \qquad (9)$$

Where $\tilde{\chi}_{E_Q} = |E_Q|^{-1/2}\chi_{E_Q}$

Proof: Since $\widetilde{x}\, E_Q \leq \varepsilon^{-1/2} \widetilde{x}_Q$, one direction is trivial. For the other , not that for all A > 0, $X_Q \leq \varepsilon^{-1/A} \left(M(X_{E_Q}^A)\right)^{1/A}$, where M denotes the Hardy–Littlewood operator. Select A such that p–/A , 2/A > 1. Note that $\|f\|_{p(\cdot)} = \|f^A\|_{p(\cdot)/A}^{1/A}$ and by theorem(3.1.6) we have

$$\left\|\{s_Q\}_Q\right\|_{\dot{f}^0_{p(\cdot),2}} \leqslant \varepsilon^{-1/A} \left\| \left(\sum_{Q\in\mathcal{D}} \left(M\left(|s_Q|\widetilde{\chi}_{E_Q}\right)^A\right)^{2/A} \right)^{A/2} \right\|_{p(\cdot)/A}^{1/A}$$

$$\leq C\varepsilon^{-1/A} \left\| \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_{E_Q}\right)^2 \right)^{1/2} \right\|_{p(\cdot)}. \qquad \square$$

Corollary(3.1.8) [3] : *Let* , $p(.) \in B(\mathbb{R}^n)$. *Then* s = $\{s_Q\}_{Q\in\mathcal{D}} \in \dot{f}^0_{p\,(.),2}$ if and only if for each Q *dyadic there is a subset* $E_Q \subset Q$ with $|E_Q|/|Q| > 1/2$ (or any other fixed number 0 < s < 1)such that

$$\left\| \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_{E_Q}\right)^2 \right)^{1/2} \right\|_{p(\cdot)} < \infty$$

For space $BMO(\mathbb{R}^n)$ we have the following identification $\|f\|_{BMO} \approx \|S_\varphi f\|_{\dot{f}^0_{\infty,2}}$ where discrete Triebel–Lizorkin sequence space $\dot{f}^0_{\infty,2}$ is defined as the collection of all complex valued sequences such that

$\|s\|$

Note also that $\|s\|_{\dot{f}^0_{\infty,2}} = \sup_{P \text{ dyadic}} \left(\frac{1}{|P|}\int_P \sum_{Q\subset P} (|s_Q|\widetilde{\chi}_Q(t))^2\right)^{1/2}$ $s = \{s_Q\}_{Q\in\mathcal{D}} \in \dot{f}^0_{p(\cdot),2}$ if and only < ∞ for each Q there is a subset $E_Q \subset Q$ with $|E_Q|/|Q| > 1/2$ (or any other fixed number 0 < ε < 1) such that

$$\left\| \left(\sum_{Q\in\mathcal{D}} \left(|s_Q| \widetilde{\chi}_{E_Q} \right)^2 \right)^{1/2} \right\|_{L^\infty} < +\infty \qquad (10)$$

Moreover, the infimum of this expression over all such collections $\{E_Q\}_{Q\in\mathcal{D}}$ is equivalent to $\|s\|_{\dot{f}^0_{\infty,2}}$.

For Hardy space $H^1(\mathbb{R}^n)$ we have the following identification $\|f\|_{H^1} \approx \|S_\varphi f\|_{\dot{f}^0_{1,2}}$,

where is defined as the collection of all complex-valued sequences s = $\{s_Q\}_{Q\in\mathcal{D}}$ such that

$$\|s\|_{\dot{f}^0_{1,2}} = \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}} \left(|s_Q| \widetilde{\chi}_Q(x) \right)^2 \right)^{1/2} dx < \infty$$

Section(3.2) Calderon- Product and Interpolation Property :

Let (Ω,μ) be a complete σ-finite measure space. By S we denote the collection of all realvalued measurable functions on Ω. A Banach subspace X in S is said to be Banach lattices on (Ω,μ) if:

(1)the norm $\|f\|_X$ is defined for every measurable function f and $f \in X$ if and only if

(2) $|\|f\||_X < \infty, \|f\|_X = 0$ if and if $f = 0$ a.e;

(3) if $0 \le f \le g$ a.e ,then $\|f\|_E \le \|g\|_E$;

(4) if $0 \le f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$ (Fatou property).

A Banach lattice modelled on the discrete set is called a Banach sequence lattice (below we assume that suppX= Ω).

If X is a Banach lattice on Ω, then the Köthe dual X' of X is a Banachlattice which can beidentified with the space of all functionals possessing an integral representation. That is, X'

$$= \left\{ f \in S;\ \sup_{\|g\|_X \le 1} \int_\Omega |fg|\, d\mu < \infty \right\}$$

The space X' is a Banach lattice on Ω and a closed normed subspace of conjugate space X^* .

Recall that X is order continuous if for any $f \in X$ and $|f_n| \to 0 a.e., \|f_n\|_X \to 0$.

Note that if X is order continuous then $E^* = E'$ and $E = (E^*)'$.

Suppose that X_0 and X_1 are Banach lattices on Ω. If $0 < \theta < 1$, the Calderon product $X_0^{1-\theta} X_1^{\theta}$ of X_0 and X_1 is defined to be the set of μ-measurable functions f on Ω such that there exist

$v \in X_0$ with $\|v\|_{X_0} \leqslant 1, w \in X_1$, with $\|w\|_{X_1} \leq 1$ $and \lambda > 0$ such that

$$|f(x)| \leqslant \lambda |v(x)|^{1-\theta} |w(x)|^{\theta} \quad \text{for } \mu \text{ a.e. } x. \qquad (11)$$

We set

$$\|u\|_{X_0^{1-\theta} X_1^{\theta}} = inf\{\lambda > 0 : (11) hold\ with\ \|V\|_{X_0} \leq 1\ and\ \|w\|_{X_1} \leq 1\ \}$$

In the sequel we will need the following obvious inequality: if $f \in X_0$ and $g \in X_1$ then

$$\|f^{1-\theta} g^{\theta}\|_{X_0^{1-\theta} X_1^{\theta}} \leqslant \|f\|_{X_0}^{1-\theta} \|g\|_{X_1}^{\theta} \qquad (12)$$

Indeed if $\| f \| x_0 \| g \| x_1 \neq 0$ $then$ for u = $f / \| f \| x_0$ and $v = g / \| g \| x_1$ we have $\| u \| x_0 = \| v \| x_1 = 1$ and $| f |^{1-0} {}^{1-0}_{x_0} \| g \|^{0}_{x} | u |^{1-0} | v |^{\theta} =$

We will use further also the well-known fact that

$$(X_0^{1-\theta} X_1^{\theta})' = (X_0')^{1-\theta} (X_1')^{\theta} \qquad (13)$$

4. **Theorem (3.2.1) [3]:** *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n), 0 < \theta < 1$ and $q(\cdot) = \frac{p(\cdot)}{1-\theta}$ *Then*

$$(f^{s0}_{p(\cdot),2})^{1-\theta} (f^{s0}_{\infty,2})^{\theta} = f^{s0}_{q(\cdot),2} \qquad (14)$$

Proof: Let $X_0 = \dot{f}^0_{p(\cdot),2}$ and $X_1 = \dot{f}^0_{\infty,2}$. Suppose $s \in X_0^{1-\theta} X_1^\theta$ and $\| s \|_{X_0^{1-\theta} X_1^\theta} = 1$. Then, there exist sequences $r = \{r_Q\}_{Q \in D}$ and $t = \{t_Q\}_{Q \in D}$ such that

$\| r \|_{X_0} \leq 1, \ \| t \|_{X_1} \leq 1, |s_Q| \, 2 \, |r_Q|^{1-0} \ |t_Q|^{\theta}$ for all $Q \in D$.

Let ε = 3/4, then there exist sets $E_Q \subset Q$ for each dyadic cube Q such that $|E_Q|/|Q| > 3/4$ and

$$\| r \|_{X_0} \approx \left\| \left(\sum_{Q \in \mathcal{D}} (|r_Q| \widetilde{\chi}_{E_Q})^2 \right)^{1/2} \right\|_{p(\cdot)} \tag{15}$$

$$\| t \|_{X_1} \approx \left\| \left(\sum_{Q \in \mathcal{D}} (|t_Q| \widetilde{\chi}_{E_Q})^2 \right)^{1/2} \right\|_{L^\infty} \tag{16}$$

Applying Hölder's inequality with conjugate exponents 1/(1 − θ) and 1/θ we obtain

$$\left(\sum_{Q \in \mathcal{D}} (|s_Q| \widetilde{\chi}_{E_Q})^2 \right)^{1/2} \leqslant 2 \left(\sum_{Q \in \mathcal{D}} (|r_Q| \widetilde{\chi}_{E_Q})^2 \right)^{(1-\theta)/2} \left(\sum_{Q \in \mathcal{D}} (|t_Q| \widetilde{\chi}_{E_Q})^2 \right)^{\theta/2};$$

Eqs. (15) and (16) give

$$\int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{D}} (|s_Q| \widetilde{\chi}_{E_Q}(x))^2 \right)^{q(x)/2} dx \leqslant C \int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{D}} (|r_Q| \widetilde{\chi}_{E_Q}(x))^2 \right)^{(1-\theta)q(x)/2} dx \leqslant C.$$

According to Theorem (3.1.7) we have $\| s \|_{\dot{f}^0_{q(\cdot),2}} \leqslant C$ and consequently

$$\left(\dot{f}^0_{p(\cdot),2}\right)^{1-\theta} \left(\dot{f}^0_{\infty,2}\right)^{\theta} \subset \dot{f}^0_{q(\cdot),2}.$$

Suppose $s = f^{;0}_{q(.),2}$. For $k \in \mathbb{Z}$, define

$$\Omega_k = \left\{ x \in \mathbb{R}^n : \left(\sum_{Q \in \mathcal{D}} \left(|s_Q| \widetilde{\chi}_Q(x) \right)^2 \right)^{1/2} > 2^k \right\},$$

$$\mathcal{Q}_k = \left\{ Q \in \mathcal{D} : |Q \cap \Omega_k| \geqslant |Q|/2 \text{ and } |Q \cap \Omega_{k+1}| < |Q|/2 \right\}.||$$

Note that if $Q \notin \bigcup_{k \in \mathbb{Z}} \mathcal{Q}_k$ then $s_Q = 0$, then we set and in this case set $r_Q = t_Q = 0$. Other wise, if $Q \in Qk$ for some $k \in \mathbb{Z}$ then we set

$r_Q = |s_Q|/A_Q$ and $t_Q = |s_Q|/B_Q$

where $A_Q = 2^{k(1-1/(1-\theta))} = 2^{k\delta}$, $B_Q = 2^k$. A direct calculation shows that $|s_Q| = |r_Q|^{1-\theta} |t_Q|^{\theta}$.

In addition, we claim that $\| r \| x_0, \|x_1\| \leq C$ To prove $\|X\|_{X_0} \leq C$ inequality we use Theorem (3.1.7) with $E_Q = Q \cap \Omega_k$, $Q \in Qk$,

$$\int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{D}} \left(|r_Q| \widetilde{\chi}_{E_Q}(x) \right)^2 \right)^{p(x)/2} dx$$

$$= \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \sum_{Q \subset \mathcal{Q}_k} \left(2^{-k\delta} |s_Q| \widetilde{\chi}_{E_Q}(x) \right)^2 \right)^{p(x)/2} dx$$

$$\leq \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \chi_{\Omega_k} \sum_{Q \subset \mathcal{Q}_k} \left(2^{-k\delta} |s_Q| \widetilde{\chi}_{E_Q}(x) \right)^2 \right)^{p(x)/2} dx$$

$$\int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{D}} \left(|s_Q| \widetilde{\chi}_Q(x) \right)^2 \right)^{p(x)/2(1-\theta)} dx$$

Here, we used that $\delta < 0$ and

$$2^{-k\delta} \chi_{\Omega_k} \leqslant \left(\sum_{Q \in \mathcal{D}} \left(|s_Q| \widetilde{\chi}_Q \right)^2 \right)^{-\delta/2}.$$

To prove $\| t \| x_1 \leq C$ inequality we use similar argument as above by redefining $E_Q = Q \cap (\Omega_{K-1})^c$, $Q \in Qk$,

$$\| t\|_{X_1} \leqslant C \left\| \left(\sum_{k\in\mathbb{Z}} \sum_{Q\subset\mathcal{Q}_k} (2^{-k}|s_Q|\widetilde{\chi}_{E_Q})^2 \right)^{1/2} \right\|_{L^\infty}$$

$$\leq C \left\| \left(\sum_{k\in\mathbb{Z}} \chi_{(\Omega_{k+1})^c} \sum_{Q\in\mathcal{Q}_k} (2^{-k}|s_Q|\widetilde{\chi}_{E_Q})^2 \right)^{1/2} \right\|_{L^\infty}$$

$$\leq C \left\| \left(\sum_{Q\in\mathcal{D}} (|s_Q|\widetilde{\chi}_{E_Q})^2 \right)^{1/2} \right\|_{L^\infty} \leqslant C.$$

Hence, we have $s \in X_0^{1-\theta} X_1^{\theta}$.

Chapter 4

Relatively compact sets

This chapter contains a detailed discussion of relatively compact sets in variable Lebesgue space on the Euclidean space. Moreover, some applications of the theorems are given.

Section (4.1) Precompactness in $L^{p(\cdot)}(X, \varrho, \mu)$:

Variable exponent Lebesgue and Sobolevs paces are natural extension of classical constant exponent L^p-spaces. Such kind of theory finds many applications for example in nonlinear elastic mechanics, electrorheological fluids or image restoration .

During the last decade Lebesgue and Sobolev spaces with variable exponent have been intensively studied ; for instance the following Surveying . In particular, the Sobolev inequalities have been shown for variable exponent spaces on Euclidean spaces and on Riemannian manifolds. Moreover, the other type of spaces with variable exponent has been considered, e.g. Hardy, Campanato, Besov .

Recently , the theory of variable exponent spaces has been extended on metric measure spaces.

In this article we investigate relatively compact (precompact) sets in variable Lebesgue space. In classical L^p-spaces relatively compact sets are characterized by the celebrated Riesz–Kolmogorov theorem .

The aim of this paper is to give achara cterization of precompact sets in variable Lebesgue space on metric measure spaces.

Moreover,we discuss with details the case of relatively compact sets in variable Lebe sgue space on the Euclidean space.

Let us mention some generalizations of the Riesz–Kolmogorov theorem . For instance ,

contain a characterizations of precompact sets in $L^p(X,\varrho,\mu)$,where (X, ϱ,μ) is a metric measure space . Weil showed the compactness theorem in $L^p(G)$, where G is alocally compact group. formulated Kolmogorov theorem for p= 2 I n terms of the Fourier transform.

We introduce the required norms,function spaces and recall standard results from the theory of variable exponent spaces. We also recall basic facts about metric measure spaces there. Characterization of relatively compact sets in variable Lebesgue space on metric measure spaces . Moreover, this chapter contains the characterization of strong convergence in variable Lebesgue spaces. we discuss the case of the Euclidean space, In particular we construct examples and counter examples.

We recall some notation and basic facts about variable exponent Lebesgue spaces. Most of the properties of these spaces can be found in the book of Cruz-Uribe and Fiorenza and in the monograph of Diening, Harjulehto, Hästö and Růˆziˆcka.

Let (Ω,μ) be a σ-finite ,complete measure space. By a variable exponent we shall mean a bounded measurable function p: $\Omega\rightarrow[1,\infty)$. The set of variable exponents on Ω we denote by $P(\Omega)$. For $U\subset \Omega$, we put

$$p^{+}(U) = esssup_{x\in U}P(x)\,, \qquad p^{-}(U) = essinf_{x\in U}P(x) \ .$$

If $U = \Omega$, we shall write p^{+}, p^{-}. In this chapter we assume that variable exponent functions are bounded ,i.e. $p^{+} < \infty$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of those μ-measurable functions $f : \Omega \to R$ for which semimodular

$$\rho_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)}\, d\mu(x)$$

Is finite. This is a Banach space with respect to the following Luxemburg norm

$$\|f\|_{p(.)} = inf\left\{\lambda > 0 : \rho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\} \quad ,$$

Where $f \in L^{p(\cdot)}(\Omega)$. Variable Lebesgue space is a special case of the Musielak–Orlicz spaces. If the variable exponent p is constant, then $L^{p(\cdot)}(\Omega)$ is an ordinary Lebesgue space. Moreover, the Hölder inequality

$$\|fg\|_{L^{r}(\Omega)} \leq 2\|f\|_{L^{p(.)}(\Omega)}\|g\|_{L^{p'(.)}(\Omega)}$$

holds, where as usual, $1 = \frac{1}{p(x)} + \frac{1}{p'(x)}$. If $p^{-} > 1$, then the dual space to $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$. Moreover $L^{p(\cdot)}(\Omega)$ is reflexive.

It is needed to pass between norm and semimodular very often. In general there are no functional relation ship between norm and modular but we do have the following useful result.

Proposition (4.1.1) [4]: let $f \in L^{p(\cdot)}$ .then

(i) $if \|f\|_{L^{p(.)}(\Omega)} \leq 1 \; then \; \rho_{p(.)}(f) \leq \|f\|_{L^{p(.)}}$;

(ii) $if \|f\|_{L^{p(.)}(\Omega)} > 1 \; then \; \rho_{p(.)}(f) \geq \|f\|_{L^{p(.)}}$;

(iii) $\|f\|_{L^{p(.)}(\Omega)} \leq \rho_{p(.)}(f) + 1$.

Let (X,ϱ) be a metric space, $\Omega \subset X$. We say that a function $p: \Omega \to R$ is locally log-Hölder continuous on Ω, if

$$\exists c_1 > 0 \quad \forall_{x,y\in\Omega} \qquad |p(x)-p(y)| \leq \frac{c_1}{log\left(e+\frac{1}{\varrho\,(x,y)}\right)} \quad .$$

Moreover, we say that exponent p satisfies log-Hölder decay condition at infinity with base point $x_0 \in X$ if

$$\exists p_\infty \in \mathbb{R} \quad \exists c_1 > 0 \quad \forall_{x\in\Omega} \qquad |p(x)-p_\infty| \leq \frac{c_2}{log(e+\varrho\,(x,x_0\ \ ))} \quad .$$

We will say that p is globally log-Hölder continuous on Ω if it is locally log-Hölder continuous on Ω and satisfies log-Hölder decay condition at infinity. Then the constant

$\boldsymbol{C}_{\mathbf{log}}\,(p) \coloneqq \max\{C_1\,,\, C_2\}$

Will be called log-Hölder constant related to exponent p. Subsequently, we define the set of log-Hölder continuous exponents

$$\mathcal{P}_{log}(\Omega) = \{p \in \mathcal{P}(\Omega); p\text{is globally log} - \text{Hölder continuous}\}\ .$$

Lemma (4.1.2.) [4]: *If* $\Omega \subset \mathbb{R}^n, p \in \mathcal{P}_{log}(\Omega)$, *then there existsan exponent* $q \in \mathcal{P}_{log}(\mathbb{R}^n)$, *which is an extension of p on* $\mathbb{R}^n$ *and satisfie* $, p_- = p_-$ *and* $p_+ = p_+$. *If* Ω *is unbounded then also* $p_\infty = p_\infty$.

Let (X,ϱ,μ) be a metric measure space equipped with ametric ρ and the Borel regular measure μ. We assume throughout the chapter that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. Additionally, we assume that the measure μ satisfies adoubling condition. It means that, there exists a constant $C_{\boldsymbol{\mu}} > 0$ such that for every ball $B(x,r)$,

$$\mu(B(x,2r)) \leq C_\mu \mu(B(x,r)).$$

It is well known that the doubling condition implies that, there exists a positive constant D satisfying

$$\frac{\mu\,(\,B\,(X_2,r_2)\,)}{\mu\,(\,B\,(X_1,r_1)\,)} \;\leq\; \acute{D}\left(\frac{r_2}{r_1}\right)^s \qquad \text{where} \qquad s = \log_2 C_\mu\ ,$$

For all balls $B(x_2,r_2)$ and $B(x_1,r_1)$ with $r_2 \geq r_1 > 0$ and $x_1 \in B(x_2,r_2)$. It follows from the above inequality that if X is bounded, then there exists $b>0$ such that the following inequality holds for $r \leq \text{diam } X$

$$\mu(B(x,r)) \geq br^s. \qquad (1)$$

On the other hand, if the metric measure space equipped with a doubling measure is not bounded, then inequality (1) does not necessarily hold.

Let $\alpha > 0$. We say that a measure μ is lower Ahlfors α-regular if there exists a constant c such that

$$cr^{\alpha} \leq \mu(B(x,r))$$

for all $x \in X$ and all positive$r$. Ameasure μ is upper Ahlfors α-regular if there exists a constant C such that

$$\mu(B(x,r)) \leq Cr^{\alpha}$$

for all $x \in X$ and all positive$r$. We shall call a measure μ Ahlfors α-regular if it is lower and upper Ahlfors α-regular.

It easily follows that an Ahlfors α-regular measure is doubling, but the opposite does not hold.

We are now in apposition to recall the notion of the Hajłasz–Sobolev space with variable exponent .

Definition(4.1.3) [4] : Let (X, ϱ, μ) be a metric measure space. We define the set of weak gradient of a function $u: X \to \mathbb{R}$ as

$$D(u) := \{g \geq 0 : |u(x) - u(y)| \leq \varrho(x,y)(g(x) + g(y)) \text{ for almost all } , y \in X \}.$$

Let $\mathcal{P} \in \mathcal{P}(X,\mu), 1 \leq \mathcal{P}- \leq \mathcal{P}+ < \infty$. was say that the function u belongs to the hajlasz – sobolev $M^{1,p(.)}(x,\mu)$ if $u \in L^{p(.)}(x,\mu)$ $and \mathcal{D}(u) \cap L^{p(.)}(x,\mu) \neq \emptyset$. $M^{1,P(.)}(X,\mu)$ is a Banach space with thenorm defined as follows

$$\|u\|_{M^{1,P(.)}(X,\mu)} = \|u\|_{L^{P(.)}(X,\mu)} + inf_{g \in \mathcal{D}(u)} \|g\|_{L^{P(.)}(X,\mu)} .$$

If f is locally integr able and A is a measurable set then by $(f)_A$ we denote the integral average of the function f over the set A , that is

$Fd\mu$ $\quad f\,d\mu.$ $\quad (f)_A := \int_A \quad = \frac{1}{\mu(A)} \int$

A

In this chapter we study relatively compact sets in $L^{p(\cdot)}(X,\varrho,\mu)$, where (X,ϱ,μ) is a metric space equipped with doubling measure. We start our discussion with the presentation of a Vitali convergence type theorem.

Theorem(4.1.4) [4]: *Let* (X, μ) be *a measure space and let* $\mathcal{P} \in \mathcal{P}(X,\mu), 1 \le \mathcal{P}- \le \mathcal{P}+ < \infty$. *Assume that* $f_n, f \in L^{p(.)}(x,\mu)$ *for any* $n \in \mathbb{N}$. *Then* $f_n \xrightarrow{n\to\infty} f$ *in* $L^{p(.)}(X,\mu)$

$(i) f_n \xrightarrow{n\to\infty} f$ *in measure ;*

(ii) *family* $\{f_n\}_{n\in\mathbb{N}}$ *is* $p(.) -$ *equi* $-$ *integrable, i.e.* $\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{A\subset X}\ \mu(A) < \delta \Rightarrow sup_{n\in\mathbb{N}} \int_A |f_n(x)|^{p(x)}\, d\mu(x) < \varepsilon;$

(iii) *family* $\{f_n\}_{n\in\mathbb{N}}$ *decays uniformly at infinity , i.e* $\forall_{\varepsilon>0}\ \exists_{c_0\subset X}\ \mu(c_0) < \infty \Rightarrow sup_{n\in\mathbb{N}} \int_{X/c_0} |f_n(x)|^{p(x)}\, d\mu(x) < \varepsilon;$

Proof: We first show sufficiency of conditions (i) – (iii) . By the Riesz Theorem $\{f_n\}$ has a subsequence convergent almost everywhere in (X,μ) to $f \in L^{p(\cdot)}(X,\mu)$ (we will use the notation $\{f_n\}$ for this subsequence). We fix $\varepsilon>0$ and let $C_0 \subset X$, $\delta > 0$ be taken from the assumptions. Then, there exists $C' \subset X$ of finite measure such that

$$\int_{X/C'} |f_n(x)|^{p(x)}\, d\mu(x) < \varepsilon .$$

We put $C = C_0 \cup C'$. Since $\mu(C) < \infty$, then by the Egorov Theorem we can choose $E \subset C$ such that $\mu(C\backslash E) < \delta$ and $f_n \rightrightarrows f$ on E for $n \to \infty$, where by $\rightrightarrows$ we denote Uniform

convergence. We have

$$\int_X |f_n(x) - f(x)|^{p(x)} d\mu(x) = \int_{C\backslash E} |f_n(x) - f(x)|^{p(x)} d\mu(x) +$$

$$\int_E |f_n(x) - f(x)|^{p(x)} d\mu(x) + \int_{X\backslash C} |f_n(x) - f(x)|^{p(x)} d\mu(x) = I_1 + I_2 + I_3.$$

It follows from (ii) and the Fatou Lemma that

$$I_1 = \int_{C \setminus E} |f_n(x) - f(x)|^{p(x)} d\mu(x) \leq$$

$$2^{p+-1} \left(\int_{C \setminus E} |f_n(x)|^{p(x)} d\mu(x) + \int_{C \setminus E} |f(x)|^{p(x)} d\mu(x) \right) \leq$$

$$2^{p+-1} \left(\varepsilon + \liminf_{n\to\infty} + \int_{C \setminus E} |f_n(x)|^{p(x)} d\mu(x) \right) < 2^{p+} \varepsilon. \ {}^{n}\to\infty$$

We conclude from $f_n \rightrightarrows f$ on E for $n \to \infty$ that also $|f_n(\cdot) - f(\cdot)|^{p(\cdot)} \rightrightarrows 0$ on E for $n \to \infty$. Therefore, $I_2 < \varepsilon$ for sufficiently large $n \in \mathbb{N}$. Moreover, we get

$$I_3 = \int_{X \setminus C} |f_n(x) - f(x)|^{p(x)} d\mu(x) \leq$$

$$2^{p+-1} \left(\int_{X \setminus C_0} |f_n(x)|^{p(x)} d\mu(x) + \int_{X \setminus \acute{C}} |f(x)|^{p(x)} d\mu(x) \right) < 2^{p+} \varepsilon.$$

Finally, we get

$$\int_X |f_n(x) - f(x)|^{p(x)} d\mu(x) < 2^{p++1} + 1\varepsilon.$$

Now, let us assume that $f_n \xrightarrow{n\to\infty} f$ in $L^{p(\cdot)}(X,\mu)$. We shall show that conditions (*i*)–(*iii*) are satisfied.

(i) However, for the convenience of the reader we give the direct proof. First we show the $L^{p(\cdot)}$ version of the Chebyshev inequality, which will be also used in the proof of the main result of this section .

Lemma (4.1.5) [4]: for any $\lambda > 0$ and $g \in L^{p(.)}(X,\mu)$ we have

$$\mu(\{x \in X: |g(x)| \geq \lambda\}) \leq \begin{cases} \int_X \frac{|g(x)|^{p(x)}}{\lambda^{p+}} d\mu(x), & if\ 0 < \lambda < 1 \\ \int_X \frac{|g(x)|^{p(x)}}{\lambda^{p-}} d\mu(x), & if\ \lambda \geq 1 . \end{cases}$$

Proof : we have

$$\mu(\{x \in X: |g(x)| \geq \lambda\}) = \int_{\left\{x \in X: \frac{|g(x)|}{\lambda} \geq 1\right\}} d\mu(x) \leq \int_X \frac{|g(x)|^{p(x)}}{\lambda^{p(x)}} d\mu(x) .$$

Simple estimates complete the proof .

By Lemma (4.1.5) for any fixed $\varepsilon>0$ we conclude

$$\mu(\{ x\in X : |f_n(x)-f(x)|\geq\varepsilon\})\leq\frac{1}{\varepsilon^{p+}}\int_X |f_n(x)-f(x)|^{p(x)}d\mu(x)\xrightarrow{n\to\infty}0$$

There fore $f_n\to f$ in measure.

(ii) We fix $\varepsilon>0$. Since $f_n\xrightarrow{n\to\infty} f$ in $L^{p(\cdot)}(X,\mu)$, then there exists $N\in\mathbb{N}$ such that

$$\int_X |f_n(x)-f(x)|^{p(x)}d\mu(x)<\frac{\varepsilon}{2p+} \qquad (2)$$

for any $n\geq N$. From now on we shall use the notation $f_0=f$. We fix $\delta_i>0$, $i=0,1,\dots,N-1$ for which

$$\forall\, i=0,1,\dots,N-1 \quad \forall\, A\subset X,\ \mu(A)<\delta i \quad \int_A |f_i(x)|^{p(x)}d\mu(x)<\frac{\varepsilon}{2^{p+}} \qquad (3)$$

Then for $\delta=\min\{\delta_i: i=0,1,\dots,N-1\}, n<N$ and $A\subset X$ satisfying $\mu(A)<\delta$ we have

$$\int_A |f_n(x)|^{p(x)}d\mu(x)<\varepsilon,$$

while for $n\geq N$ it follows from (2) and (3) that

$$\int_A |f_n(x)|^{p(x)}d\mu(x)\leq 2^{p+-1}\int_A |f_n(x)-f(x)|^{p(x)}d\mu(x)+\int_A |f(x)|^{p(x)}d\mu(x)\leq$$

$$2^{P+-1}\left(\int_X |fn(x)-f(x)|^{p(x)}du(x)+\int_A |f(x)|^{P(X)}du(x)\right)<2^{P+-1}\left(\frac{\varepsilon}{2^{P+}}\ \frac{\varepsilon}{2^{P+}}\right)=\varepsilon .$$

Therefore $\{f_n\}_{n\in\mathbb{N}}$ are $p(\cdot)$-equi-integrable .

(iii) We fix $\varepsilon>0$ and $N \in \mathbb{N}$ so that (2) is satisfied for $n\geq N$. Let $C_i \subset X$, $i= 0, 1,..., N-1$, be the sets of finite measure for which

$$\forall_{i=1,2,\dots,N-1}\ , \int_{X/C_i} |f_i(x)|^{p(x)} d\mu(x) < \frac{\varepsilon}{2^{p+}} \quad .$$

For $C = \bigcup_{i=1}^{N-1} C_i$ and any $n< N$ we obtain

$$\int_{X/C} |f_i(x)|^{p(x)} d\mu(x) \leq \int_{X/C_n} |f_n(x)|^{p(x)} d\mu(x) < \varepsilon,$$

while for $n\geq N$ we have

$$\int_{X/C} |f_n(x)|^{p(x)} d\mu(x) \leq 2^{p+\ -1} \left(\int_X |f_n(x) - f(x)|^{p(x)} d\mu(x) + \int_{X/C_0} |f(x)|^{p(x)} d\mu(x) \right)$$
$$< 2^{p+\ -1} \left(\frac{\varepsilon}{2^{p+}} + \frac{\varepsilon}{2^{p+}} \right) = \varepsilon$$

This completes the proof.

Before the presentation of the main result of this section we recall the following maximal function theorem.

Theorem(4.1.6) [4]: *Let* (X,ϱ,μ) *be a metric measure space with doubling measure , and* $p \in \mathcal{P}_{log}(X,\mu)$ *and* $p_- > 1$*.Then*

$$\|M(f)\|_{L^{P(.)}(X,\mu)} \leq \frac{C_{p-}}{p_- -1} \|f\|_{L^{P(.)}(X,\mu)} \ ,$$

Where constant C depends on $\mu\,(B(x_0,1))$*,*C_μ *– doubling constan to f measure* μ *and* $C_{\log}(p)$ *–log-Hölder constant of p.*

The above result will play an important role in the proof of the following characterization of relatively compact sets in $L^{p(\cdot)}(X, \varrho, \mu)$.

Theorem(4.1.7) [4]: *Let* (X,ϱ,μ) *be a metric measure space equipped with doubling measure,* $p \in \mathcal{P}_{log}(X,\mu)$, $1 < p_- \leq p_+ < \infty$. *Assume that*

$$\forall\, r>_0 \quad h(r) := \inf\{\mu(B(x,r)) : x \in X\} > 0.$$

Then the family $\mathcal{F} \subset L^{p(\cdot)}(X, \mu)$ *is precompact in* $L^{p(\cdot)}(X,\mu)$ *if and only if the following conditions are satisfied:*

(i) $\mathcal{F}$ *is bounded in* $L^{p(\cdot)}(X,\mu)$, *i.e* $\exists_{M>0}\, \forall_{f\in\mathcal{F}} \int_X |f(x)|^{p(x)} d\mu(x) \leq M$;

(ii) $\forall_{\varepsilon>0}\, \exists_{r_0>0}\, \forall_{0<r<r_0}\, \forall_{f\in\mathcal{F}} \int_X \left|f(x) - (f)_{B(x,r)}\right|^{p(x)} d\mu(x) < \varepsilon$;

(iii) $\forall_{\varepsilon>0}\, \exists_{R>0}\, \forall_{f\in\mathcal{F}} \int_{X/B(x_0,R)} |f(x)|^{p(x)} d\mu(x) < \varepsilon$ *for some* $x_0 \in X$.

Section(4.2) Precompactness in $L^{p(\cdot)}(\mathbb{R}^n)$*and* $\ell^{\{pn\}}$ *with Applications to Hajłasz–Sobolev spaces :*

In this section we study relatively compact sets in $L^{p(\cdot)}(\mathbb{R}^n)$. We shall give two different characterizations of precompactness. Uptonow, the only known characterization of precompact sets in $L^{p(\cdot)}(\Omega)$ for a bounded set $\Omega \subset \mathbb{R}^n$ was given in the chapter of Rafeiro and Bandaliev. As a simple corollary from Theorem (4.1.7) we get

Theorem(4.2.1) [4]: *Let* $\mathcal{F} \subset L^{p(\cdot)}(\mathbb{R}^n)$ *and assume that* $p \in \mathcal{P}_{log}(\mathbb{R}^n)$ $1 < p_- \leq p_+ < \infty$. *We denote*

$$f_h(x) := (f)_{B(x,h)} = \int_{B(x,k)} f(y)dy,$$

then the family $\mathcal{F}$ *is precompact in* $L^{p(\cdot)}(\mathbb{R}^n)$ *if and only if the following conditions are satisfied*

(i) $\mathcal{F}$ *is bounded;*

(ii) $\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{|h|<\delta}\ \forall_{f\in\mathcal{F}} \quad \int_{\mathbb{R}^n} |f_h(x) - f(x)|^{p(x)} < \varepsilon$;

(iii) $\forall_{\varepsilon>0}\ \exists_{R>0}\ \forall_{f\in\mathcal{F}} \quad \int_{\mathbb{R}^n/B(0,R)} |f(x)|^{p(x)} dx < \varepsilon$.

Now, we generalize the above result on any $\Omega \subset \mathbb{R}^n$.

Corollary(4.2.2.) [4]: *Let* $\Omega \subset \mathbb{R}^{\boldsymbol{n}}$*be any measurable set,* $\mathcal{F} \subset L^{p(.)}(\Omega)$ $p \in \mathcal{P}_{log}(\boldsymbol{\Omega})$ $1 < p_- \leq p_+ < \infty$, $\tilde{\mathcal{F}} = \{\tilde{f} = f\chi\Omega : f \in \mathcal{F}\}$. *Let* $\tilde{p}$ *be the log-Hölder continuous extension of* p *on* $\mathbb{R}^{\boldsymbol{n}}$ *,existing by virtue of* Lemma (4.1.7) *Then the family* $\mathcal{F}$ *is precompact in* $L^{p(.)}(\Omega)$ *if and only if the following conditions are satisfied*

(i) $\mathcal{F}$ *is bounded;*

(ii) $\forall_{\varepsilon>0}$ $\exists_{\delta>0}$ $\forall_{|h|<\delta}$ $\forall_{\tilde{f}\in\tilde{\mathcal{F}}}$ $\int_{\mathbb{R}^{\boldsymbol{n}}} |\tilde{f}_h(x) - \tilde{f}(x)|^{\tilde{p}(x)} dx < \varepsilon$;

(iii) $\forall_{\varepsilon>0}$ $\exists_{R>0}$ $\forall_{f\in\mathcal{F}}$ $\int_{\boldsymbol{\Omega}/B(0,R)} |f(x)|^{p(x)} dx < \varepsilon$.

Proof: First we note that it is obvious that $\tilde{\mathcal{F}} \subset L^{\tilde{p}(.)}(\mathbb{R}^n)$. By theorem (4.2.1) the family $\tilde{\mathcal{F}}$ is precompact in $L^{\tilde{p}(.)}(\mathbb{R}^n)$. We will show that $\mathcal{F}$ is precompact in $L^{p(.)}(\Omega)$. We fix any sequence $\{f_n\} \subset \mathcal{F}$. Then $\{\tilde{f}_n\}$ has a subsequence $\{\tilde{f}_{nk}\}$ convergent in $L^{\tilde{p}(.)}(\mathbb{R}^n)$ to some $\tilde{f} \subset L^{\tilde{p}(.)}(\mathbb{R}^n)$. Hence, we get$\chi\Omega$

$$\int_{\boldsymbol{\Omega}} \left|f_{nk}(x) - \tilde{f}_{\chi\Omega}(x)\right|^{p(x)} dx \leq \int_{\mathbb{R}^{\boldsymbol{n}}} |\tilde{f}_{nk}(x) - \tilde{f}(x)|^{\tilde{p}(x)} dx \xrightarrow{k\to\infty} 0 .$$

Thus , $f_{nk} \xrightarrow{k\to\infty} \tilde{f}_{\chi\Omega}$ *in* $L^{p(.)}(\Omega)$, so the family $\mathcal{F}$ is precompact in $L^{p(.)}(\Omega)$.

Now, let us assume that $\mathcal{F}$ is precompact in $L^{p(.)}(\Omega)$. Then also $\tilde{\mathcal{F}}$ is precompact in $L^{\tilde{p}(.)}(\mathbb{R}^n)$. Indeed, for any $\{\tilde{f}_n\} \subset \tilde{\mathcal{F}}$ and subsequence $\{f_{nk}\} \subset \{f_n\}$ convergent in $L^{p(.)}(\Omega)$ to some $f \in L^{p(.)}(\Omega)$. we obtain

$$\int_{\mathbb{R}^{\boldsymbol{n}}} |\tilde{f}_{nk}(x) - f_{\chi\Omega}(x)|^{\tilde{p}(x)} dx = \int_{\boldsymbol{\Omega}} |f_{nk}(x) - f(x)|^{p(x)} dx \xrightarrow{k\to\infty} 0$$

Hence, it follows from theorem (4.2.1) that $\tilde{\mathcal{F}}$ satisfies conditions (*i*)–(*iii*). We see that $\mathcal{F}$ satisfies these conditions and the proof is finished.

It is well known that in the case of classical Lebesgue spaces the condition (*ii*) from theorem (4.2.1) can be replaced by the following one

$$\int_{\mathbb{R}^{\boldsymbol{n}}} |f(x+h) - f(x)|^{p(x)} dx \xrightarrow{h\to\infty} 0$$

So, it is natural to ask the question about similar conditions in the case of variable Lebesgue spaces. We show the following sufficient conditions for precompactness in $L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$.

Theorem(4.2.3) [4]: *Let* $\mathcal{F} \subset L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$ *and assume that* $p \in \mathcal{P}(\mathbb{R}^{\boldsymbol{n}}), p_+ < \infty$. *If the following conditions are satisfied*

(i) $\mathcal{F}$ *is bounded;*

(ii) $\forall_{\varepsilon>0} \quad \exists_{\delta>0} \quad \forall_{|h|<\delta} \quad \forall_{\tilde{f}\in\widetilde{\mathcal{F}}} \quad \int_{\mathbb{R}^n} |f(x+h) - f(x)|^{p(x)} dx < \varepsilon$

(iii) $\forall_{\varepsilon>0} \quad \exists_{R>0} \quad \forall_{f\in\mathcal{F}} \qquad \int_{\mathbb{R}^n/B(0,R)} |f(x)|^{p(x)} dx < \varepsilon$.

Then the family $\mathcal{F}$ *is precompact in* $L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$.

Proof : Let us fix ε>0 , let δ >0 and R >0 be taken from conditions (*ii*) and(*iii*). We can find a δ-covering of a ball $B(0,R)$ consisting of separated cubes $\{Q_i\}_{i=1,\dots,N}$ Next, let us define an operator

$$P: L^{p(.)}(\mathbb{R}^{\boldsymbol{n}}) \to w \coloneqq span\,\{\chi Q_i : i = 1, \dots, N\} \subset L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$$

as follows:

$$Pf(x) = \sum_{i=1}^{N} \chi Q_i(x) \int_{Q_i} f(y)dy.$$

We easily obtain that

$$\int_{\mathbb{R}^n} |f(x) - P(f)(x)|^{P(x)} dx =$$

$$\int_{Rn \overset{N}{\underset{i-1}{\cup}} Q_I} |f(X) - P(f)|p^{(x)} dx + \sum_{i=1}^{N} \int_{Q_i} |f(x) - p(f)(x)p^{(x)}\; dx \le$$

$$\int_{\mathbb{R}^n \backslash B(0,R).} |f(x)|p^{(x)} dx + \sum_{i=1}^{N} \int_{Q_i} |f(x) - p(x)\;\; |p^{(x)}\, dx = I_1 + I_2$$

It follows from condition (*iii*) that $I_1 < \varepsilon$. Let us consider I_2. We use the notation $\mu_Q = |Q_i| = \delta^n$. By the Jensen inequality we get

$$I_2 = \sum_{i=1}^{N} \int_{Q_i} |f(x)|p^{(x)} dx = \sum_{i=1}^{N} \int_{Q_i} f(x) - \frac{1}{\mu Q} \int_{Q_i} f(y) dy \ dx =$$

$$\sum_{i=1}^{N} \int_{Q_i} \frac{1}{\mu Q} \int_{Q_i} \big(f(x) - f(y)\big) dy^{\,p(x)} dx \le \sum_{i=1}^{N} \frac{1}{\mu Q} \int_{Q_i} \int_{Q_i} |f(x) - f(y)|p^{(x)} \, dy dx.$$

Next, we change the variables of integration by $y = x + z$ Since $x, y \in Q_i$ then z is contained in the cube $2Q$ with side length equals to 2δ and center in the origin of the coordinate system. By the Fubini theorem and using condition (*ii*),we obtain

$$\begin{aligned} I_2 &\le \sum_{i=1}^{N} \frac{1}{\mu Q} \int_{Q_i} \int_{2Q} |f(x) - f(x+z)|P^{(x)} dz dx \\ &= \sum_{i=1}^{N} \frac{1}{\mu Q} \int_{2Q} \int_{Q_i} |f(x) - f(x+z)|P^{(x)} dx dz \\ &\le \frac{1}{\mu Q} \int_{2Q} \int_{\mathbb{R}_n} |f(x) - f(x+z)| \ P^{(x)} \, dx dz \le \frac{1}{\mu Q} \, 2^n \mu Q^{\varepsilon} = 2^n \varepsilon. \end{aligned}$$

Thus, we have established the inequality $\int_{\mathbb{R}^n} |f(x) - P(f)(x)|^{p(x)} dx < (2^n + 1)\varepsilon$.

Lemma(4.2.4) [4]: *Let (X, d_X) be a metric space .Assume that for any $\varepsilon > 0$ there exist $\delta > 0$, metric space (W, d_W) and function $\Phi: X \to W$ satisfying*

(a) $\Phi[X]$ *is totally bounded;*

(b) $x, y \in X$ *and* $d_W(\Phi(x), \Phi(y)) < \delta$ *then* $d_X(x, y) < \varepsilon$. .

Then the space X is totally bounded.

We show that P is the map that satisfies the assumptions of Lemma(4.2.4).

(a) Using the Hölder inequality we get

$$\int_{\mathbb{R}^n} |P(f)(x)|^{P(x)} dx =$$

$$\int_{\mathbb{R}^n} \left| \sum_{i=1}^{N} XQ_i(x) \frac{1}{\mu Q} \int_{Q_i} f(y)dy \right|^{P(x)} dx \leq \int_{\mathbb{R}^n} \left| \sum_{i=1}^{N} XQ_i(x) \frac{2}{\mu Q} \|f\|_{L^{P(.)}(Q_i)} \|1\|_{L^{P'(.)}(Q_i)} \right|^{P(x)} dx$$

$$\leq \int_{\cup_{i=1}^{N} Q_i} \left| \frac{2}{\mu Q} \|f\|_{L^{P(.)}(\mathbb{R}^n)} \sum_{i=1}^{N} (1 + \mu Q) \right|^{P(x)} dx$$

$$\leq \int_{\cup_{i=1}^{N} Q_i} \left| \frac{2N}{\mu Q} \|f\|_{L^{P(.)}(\mathbb{R}^n)} \sum_{i=1}^{N} (1 + \mu Q) + 1 \right|^{P+} dx$$

$$= \left| \frac{2N}{\mu Q} \|f\|_{L^{P(.)}(\mathbb{R}^n)} \sum_{i=1}^{N} (1 + \mu Q) + 1 \right|^{P+} N_{\mu Q}$$

Boundedness of $\mathcal{F}$ implies boundedness of the image $P[\mathcal{F}]$. Moreover, since the space W is finite dimensional, then $P[\mathcal{F}]$ is also totally bounded.

(b) Let $\|P(f) - P(g)\|_{L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})} < \varepsilon$. Then , we get

$$\|f - g\|_{L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})} - \|f - P(f)\|_{L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})} + \|P(f) - P(g)\|_{L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})} + \|P(g) - g\|_{L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})}$$

$$< 2\big((2^n + 1)\varepsilon\big)^{\frac{1}{p+} + \varepsilon}$$

Therefore, it follows from Lemma (4.2.4) that $\mathcal{F}$ is precompact in $L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$.

Corollary(4.2.5) [4]: *Let $\Omega \subset \mathbb{R}^{\boldsymbol{n}}$ be a measurable set $\mathcal{F} \subset L^{p(.)}(\Omega)$ and Let $\widetilde{\mathcal{F}}$ be family of functions from $\mathcal{F}$ extended by 0 outside Ω , i.e. $\widetilde{\mathcal{F}} = \{\tilde{f} = f_{\chi\Omega} : f \in \widetilde{\mathcal{F}}\}$ Assume that $p \in \mathcal{P}(\boldsymbol{\Omega}), p_+ < \infty$ and put $\tilde{p}(x) = p(x) + p_{-\chi\mathbb{R}^{\boldsymbol{n}}/\Omega}(x)$. If the following conditions are satisfied*

(i) *$\mathcal{F}$ is bounded;*

(ii) $\forall_{\varepsilon>0} \quad \exists_{\delta>0} \quad \forall_{|h|<\delta} \quad \forall_{\tilde{f}\in\widetilde{\mathcal{F}}} \quad \int_{\mathbb{R}^{\boldsymbol{n}}} \left|\tilde{f}(x+h) - \tilde{f}(x)\right|^{\tilde{p}(x)} dx < \varepsilon$

(iii) $\forall_{\varepsilon>0} \quad \exists_{R>0} \quad \forall_{f\in\mathcal{F}} \qquad \int_{\boldsymbol{\Omega}/B(0,R)} |f(x)|^{p(x)} dx < \varepsilon$.

Then the family $\mathcal{F}$ is precompact in $L^{p(.)}(\Omega)$.

Proof: It is obvious that $\widetilde{\mathcal{F}} \subset L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$. Moreover, the family $\widetilde{\mathcal{F}}$ is bounded in $L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$. Condition (*ii*) for F˜ is straight forward. It follows from (*iii*) that

$$\int_{\mathbb{R}^n \setminus B(O,R)} |\widetilde{f}\ (x)|\mathrm{p}^{(\mathrm{x})}\mathrm{dx} = \int_{\mathbb{R}^n \setminus B(O,R)} |\widetilde{f}\ (\mathrm{x})\mathrm{p}^{(\mathrm{x})}\mathrm{dx} < \varepsilon$$

For sufficiently large R >0 and any $\tilde{f} \in \tilde{\mathcal{F}}$. By virtue of Theorem (4.2. 3) the family $\widetilde{\mathcal{F}}$ is precompact in $L^{p(.)}(\mathbb{R}^{\boldsymbol{n}})$. We shall show that also $\mathcal{F}$ is precompact in $L^{p(.)}(\boldsymbol{\Omega})$. For this purpose let us fix $\{f_n\} \subset \mathcal{F}$ and let $\{\tilde{f}_{nk}\} \subset \{\tilde{f}_n\}$ be a subsequence convergent to some $\tilde{f} \in L^{\tilde{p}(.)}(\mathbb{R}^{\boldsymbol{n}})\ in\ L^{\tilde{p}(.)}(\mathbb{R}^{\boldsymbol{n}})$. The subsequence $\tilde{f}_{nk}$ can be chosen to converge point wise almost everywhere, and so $\tilde{f}$ is equal to 0 a.e. on $\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{\Omega}$. Therefore, we easily note that

$$\int_{\boldsymbol{\Omega}} \left|f_{nk}(x) - \tilde{f}(x)_{\chi\Omega}(x)\right|^{p(x)} dx = \int_{\mathbb{R}^n} \left|\tilde{f}_{nk}(x+h) - \tilde{f}(x)\right|^{\tilde{p}(x)} dx \xrightarrow{k\to\infty} 0\ .$$

Hence, $f_{nk} \xrightarrow{k\to\infty} 0\ \tilde{f}_{\chi\Omega}$ in $L^{p(.)}(\boldsymbol{\Omega})$ and the proof is complete.

Example (4.2.6) [4]: Let $\Omega = \mathbb{R}_+, 0 < A < B < \infty$. We consider the following family of functions

$$\mathcal{F} = \left\{f(x) = \lambda e^{-\lambda x}: A \leq \lambda \leq B\right\}.$$

We show that the family satisfies the assumptions of Theorem (4.2.3) for any bounded variable exponent $p \in \mathcal{P}(\Omega)$. Let $\widetilde{\mathcal{F}}$ and $\tilde{p}$ be the extensions of the family $\mathcal{F}$ and exponent p definedin Corollary (4.2.5) .

(i) Boundedness of $\mathcal{F}$ in $L^{p(.)}(\boldsymbol{\Omega})$:

$$\int_0^\infty (\lambda^{-\lambda})p^{(x)}\, dx = \int_{\{x>0:\lambda e-\lambda x\leq 1\}} (\lambda^{-\lambda})p^{(x)}\, dx + \int_{\{x>0:\lambda e-\lambda x\leq 1\}} (\lambda^{-\lambda})p^{(x)}\, dx \leq$$

$$\int_0^\infty (\lambda^{-\lambda})p^{-}\, dx + \int_0^\infty (\lambda^{-\lambda})p^{+}\, dx = \frac{\lambda^{p-^{-1}}}{p-} + \frac{\lambda^{p+^{-1}}}{p+} \leq \frac{B^{p-^{-1}}}{p-} + \frac{B^{p+^{-1}}}{p+} < \infty.$$

Uniform decay of translations: We fix h satisfying $|e^{-\lambda h}-1|<1$ for all $A< \lambda< B$. For such h we obtain

$$\int_{\mathbb{R}^n} |\widetilde{f}\ (x+h)-\widetilde{f}\ (x)|^{\tilde{p}\,(x)}dx = \int_{\{x<0:x+h\geq 0\}} |f(x+h)-f(x)|^{p(x)}\ dx +$$

$$\int_{\{x\geq 0:x+h<0\}} |f(x)\,|^{p(x)}\ dx + \int_{\{x<0:x+h\geq 0\}} |f(x+h)\ \,|^{\tilde{p}\,(x)}dx = I_{1+}\ I_{2+}I_3$$

Where $I_2 = 0$ for $h < 0$, $I_3 = 0$ for $h> 0$. Finally , direct calculations show that

$$I_1 = \int_{max\{0,h\}}^{\infty} |f(x+h)-f(x)|p^{(x)}dx \leq$$

$$|e^{-\lambda}-1|^{\,p-} \int_{max\{0,h\}}^{\infty} (\lambda^{-\lambda})^{\,p(x)}\,dx \leq = |e^{-\lambda}-1|^{\,p-}\int_0^{\infty}(\lambda^{-\lambda x})^{\,p(x)}\,dx \leq$$

$$|e^{-\lambda}-1|^{\,p-}\left(\int_{\{x>0:e^{-\lambda x}\leq 1\}}^{\infty} (\lambda^{-\lambda})^{\,p(x)}\,dx + \int_{\{x>0:e^{-\lambda x}\leq 1\}}^{\infty} (\lambda^{-\lambda x})^{\,p(x)}dx\right)$$

$$|e^{-\lambda}-1|^{\,p-}\left(\int_0^{\infty}(\lambda^{-\lambda x})^{\,p-}\,dx + \int_0^{\infty}(\lambda^{-\lambda x})^{\,p+}\,dx\right) \leq$$

$$\left|e^{-\lambda}-1\right|^{\,p-}\left(\frac{\lambda^{p-^{-}}}{p-}+\frac{\lambda^{p+^{-}}}{p+}\right) \xrightarrow{h\longrightarrow 0} 0$$

$$I_2 = \int_0^{-h} |f(x)\,|\mathrm{p}^{(\mathrm{x})}\ dx = \int_0^{-h} (\lambda^{-\lambda x})^{\,p(x)}\,dx \leq$$

$$\int_0^{-h} (\lambda^{-\lambda x})^{p+} dx + \int_0^{-h} (\lambda^{-\lambda x})^{p-} dx =$$

$$\frac{\lambda^{p+^{-1}}}{p+}(1 - e^{\lambda p+} + \frac{\lambda^{p+^{-1}}}{p-}\left(1 - e^{\lambda p-}\right).$$

$$I_3 \int_{-h}^{0} |f(x+h)|^{\mathrm{p}-} dx = \int_{-h}^{0} \left(\lambda e^{-\lambda(x+h}\right)^{p-} dx =$$

$$\frac{\lambda^{p-^{-1}}}{p-} e^{-\lambda hp-}\left(e^{\lambda hp-} - 1\right) \xrightarrow{h \to 0+} 0.$$

(*iii*) Uniform decay at infinity: For fixed $R > 0$ we have

$$\int_R^{\infty} \left(\lambda e^{-\lambda}\right)^{p(x)} dx = \int_{\{x \geq R: \lambda e^{-\lambda x} \leq 1\}} \left(\lambda e^{-\lambda}\right)^{p(x)} dx + \int_{\{x \geq R: \lambda e^{-\lambda x} \geq 1\}} \left(\lambda e^{-\lambda}\right)^{p(x)} dx \leq$$

$$\int_R^{\infty} \left(\lambda e^{-\lambda}\right)^{p-} dx + \int_R^{\infty} \left(\lambda e^{-\lambda}\right)^{p+} dx + dx = \frac{\lambda^{p-^{-1}}}{p-} e^{-\lambda p-R} + \frac{\lambda^{p+^{-1}}}{p+} e^{-\lambda p+R}$$

Example (4.2.7) [4]: Let $\Omega \subset \mathbb{R}^n$,$\mu(\Omega) < \infty$, $p, q \in \mathcal{P}(\Omega)$ *and* $p(x) \leq q(x)$ almost everywhere in Ω. Fix $s \in \mathcal{P}(\Omega)$ satisfying $\frac{1}{q(x)} + \frac{1}{s(x)} = \frac{1}{p(x)}$ almost everywhere in Ω. The Hölder inequality and Proposition (4.1.1) (*iii*) imply that

$$\|f\|_{L^{p(.)}(\Omega)} \leq 2\|f\|_{L^{q(.)}(\Omega)} \|1\|_{L^{s(.)}(\Omega)} \leq 2\big(1 + \mu(\Omega)\big)\|f\|_{L^{q(.)}(\Omega)}.$$

Let f be a bounded function with a compact support and $p_+ < \infty$. Then, there exists a compact set $k \subset \mathbb{R}^n$ such that sup $\big(f(.+h)\big) \subset k$ for all 0< h<1. We extend f by 0 on $K \backslash \Omega$ and we conclude

$$\|f(.+h) - f\|_{L^{p(.)}(\Omega \cup \mathrm{k})} \leq 2\big(1 + \mu(\Omega \cup \mathrm{k})\big)\|f(.+h)\|_{L^{p+}(\Omega \cup \mathrm{k})} \xrightarrow{h \to 0}.$$

Therefore, bounded functions with compact support satisfy $f(.+h) \xrightarrow{h \to 0} f$ $inL^{p(.)}(\Omega)$, provided $p_+ < \infty$.

We shall show a characterization of relatively compact sets $\boldsymbol{\ell}^{\{pn\}}$ in with $p_+ < \infty$, which is an extension of the Fréchet theorem.

Theorem (4.2.8)[4]: *Let* $\mathcal{A} \subset \ell^{\{p_n\}}$, $p_+ < \infty$. *Then , the set* $\mathcal{A} = \{a^i\}_{i \in I}$ *is precompact* in $\ell^{\{p_n\}}$ *If and only if the following conditions are satisfied*

(i) $\mathcal{A}$ *isbounded , i.e.* $\exists_{M>0}$ $\forall_{a^i \in \mathcal{A}}$ $\sum_{n=1}^{\infty} |a_n^i|^{p_n} \leq M$;

(ii) $\forall_{\varepsilon>0}$ $\exists_{K \in \mathbb{N}}$ $\forall_{a^i \in \mathcal{A}}$ $\sum_{n=K+1}^{\infty} |a_n^i|^{p_n} < \varepsilon$.

Proof: Assume first that conditions (i) – (ii) are satisfied. Let us fix any $\varepsilon > 0$. We choose $K \in \mathbb{N}$ from condition (ii) for $\frac{\varepsilon}{2^{p_+ +1}}$ .We define an operator which is a projection of an element $a^i \in \mathcal{A}$ on its first K coordinates, i.e.

We note that boundedness of $\mathcal{A}$ implies boundedness of $P[\mathcal{A}]$. Moreover, $P[\mathcal{A}]$is totally bounded since $P[\mathcal{A}] \subset \mathbb{R}^K$. . It follows from condition(ii) that

$$\sum_{n=1}^{\infty} |a_n^i|^{pn} = \sum_{n=1}^{\infty} |a_n^i|^{pn} + \sum_{n=k+1}^{\infty} |a_n^i|^{pn} < \sum_{n=1}^{\infty} | p(Q_i|^{pn} + \frac{\varepsilon}{2p^{++1}}.$$

$$for\ Q^i, b^i \in A\ satisfying \sum_{n=1}^{\infty} p\big|(a^i)\,n - p(a^i)\,\big|^{pn} < \frac{\varepsilon}{2} wehave$$

$$\sum_{n=1}^{\infty} |a_n^i - b_n^i|^{pn} = \sum_{n=1}^{\infty} \big|p(a^i)_n - p(b^i)_n\big|^{pn} + = \sum_{n=k+1}^{\infty} |a_n^i - b_n^i|^{pn} \leq$$

$$\sum_{n=1}^{\infty} \left|p\left(a^{i}\ \right)_{n}-p\left(b^{i}\ \right)_{n}\right|^{pn}+2^{p}+{}^{-1}\sum_{n=k+1}^{\infty}\left(\left|a_{n}^{i}\right|^{pn}+\ \left|\ b_{n}^{i}\ \right|^{pn}\right)\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

By virtue of Lemma (4.2.4), $\mathcal{A}.$ is totally bounded and therefore precompact.

It remains to proof the converse. Assume that the family $\mathcal{F}$ is relatively compact. We shall show that conditions (*i*) and (*ii*) are satisfied.

(i) It is obvious.

(ii) We fix $\varepsilon > 0$ and we choose $\{a^1, a^2, \dots . a^m\} \subset \mathcal{A} - an\frac{\varepsilon}{2^{p+}} -$ net in $\mathcal{A}.$ For any $j = 1, \dots, m$ *we choose* $k_i \in \mathbb{N}$ satisisfy

$$\sum_{n=k_{i+1}}^{\infty} \left|a_n^i\right| p^n \leq \frac{\varepsilon}{2^{p+}}.Let k = max\{k_i : j = 1, \dots, m\}.$$

For any fixed $a^0 \in \mathcal{A}$ there exists a^i from the net such that

$$\sum_{n=1}^{\infty} |a_n^0 - a_n^0| p^n \leq \frac{\varepsilon}{2^{p+}}. \quad then\, we\, have$$

and the proof is complete.

In this section we apply Theorem (4.1.7) to prove compact embedding of Hajłasz–Sobolev space $M^{1,p(\cdot)}(X,\mu)$ into $L^{p(\cdot)}(X,\mu)$.

Theorem (4.2.9) [4]: *Let* (X, ϱ, μ) *be a compact metric space with doubling measure and let* $p \in \mathcal{P}_{log}(X, \mu)$ *satisfy* $1 < p_- \leq p_+ < \infty$ *.Then*

$$M^{\mathbf{1},\boldsymbol{p}(.)}(X, \mu) \hookrightarrow\hookrightarrow L^{p(.)}(X, \mu).$$

Let us mention that the above theorem is a very special case of the much more general result. Nevertheless, the following proof is straight forward.

Proof: We shall show that any sequence $\{f_n\} \subset M^{\mathbf{1},\boldsymbol{p}(.)}(X, \mu)$ for which

$$\exists_{M>0}\ \forall_{n\in\mathbb{N}}\ \|f_n\|_{M^{\mathbf{1},\boldsymbol{p}(.)}(X,\mu)} < M$$

Has a subsequence converging strongly in $L^{p(.)}(X,\mu)$. Let $\{g_n\}$ be a sequence of weak gradients corresponding to $\{f_n\}$ satisfying

$$\forall_{n\in\mathbb{N}}\ \|f_n\|_{M^{1,p(.)}(X,\mu)} \leq \|f_n\|_{L^{p(.)}(X,\mu)} + \|f_n\|_{L^{p(.)}(X,\mu)} < M$$

We apply theorem(4.1.7). Boundedness of $\{f_n\}$ in $L^{p(.)}(X,\mu)$ is straightforward from the definition of the norm in $M^{1,p(.)}(X,\mu)$ Compactness of X implies condition (*iii*) of theorem (4.1.7). It remains to prove

$$\lim_{r\to 0}\int_x \left|f_n(x) - (f_n)_{B(x,r)}\right|^{p(x)} du(x) = 0 \text{ uniformly with respect } n \in \mathbb{N}. \quad (4)$$

Since $\sup_{n\in\mathbb{N}}\|f_n\|_{M^{1,p(.)}(X,\mu)} \leq M$ then inparticular $\sup_{n\in\mathbb{N}}\|g_n\|_{L^{p(.)}(X,\mu)} \leq M$

And from Theorem (4.1.7) we obtain

$$\sup_{n\in\mathbb{N}}\|M(g_n)\|_{L^{p(.)}(X,\mu)} \leq \frac{Cp_-}{p_- - 1}\sup_{n\in\mathbb{N}}\|g_n\|_{L^{p(.)}(X,\mu)} \leq \frac{Cp_-}{p_- - 1}M.$$

Instead of $\{f_n\}$ and $\{g_n\}$ we consider functions $u_n = \frac{(p_- - 1)f_n}{C_p - M}$ and its weak gradients $w_n = \frac{(p_- - 1)g_n}{C_p - M}$. Then $\|w_n\|_{L^{p(.)}(X,\mu)} \leq 1$ and $\|M(w_n)\|_{L^{p(.)}(X,\mu)} \leq 1$It suffices to prove (4) for $\{u_n\}$ For 0< r <1 we have

$$\int_X \left|u_n(x) - (u_n)_{B(x,r)}\right|^{p(x)} d\mu(x) \leq \int_X \left|\frac{1}{\mu(B(x,r))}\right| \int_{B(x,r)} |u_n(x) - u_n(y)|\, d\mu(y)\Bigg|^{p(x)} d\mu \leq$$

$$\int_X \left|\frac{r}{\mu(B(x,r))}\int_{B(x,r)} |\omega_n(x + \omega_n(y)| d\mu(y)\right|^{p(x)} d\mu(x) \leq$$

$$\int_X \left| r\left(|\omega_n(x)| + \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |\omega_n(y)\,| d\mu(y)\right)\right|^{p(x)} d\mu(x) \leq$$

$$2^{p} + {}^{-1}r^{p-}\left(\int_X |\omega_n(x)|^{p(x)} d\mu(x) + \int_X \left|\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |\omega_n(y)\,| d\mu(y)\right|^{p(x)} d\mu(y)\right) \leq$$

$$2^{p} + {}^{-1} r^{p-} \left(\int_X |\omega_n(x)|^{p(x)} d\mu(x) + \int_X |\mathcal{M}(\omega_n)(x)|^{p(x)} d\mu(x) \right)$$

Since $\|w_n\|_{L^{p(.)}} \leq 1$ and $\|M(w_n)\|_{L^{p(.)}} \leq 1$, then from Proposition (4.1.1) (i) and theorem (4.1.6) we get

$$\int_X \left| u_n(x) - (u_n)_{B(x,r)} \right|^{p(x)} d\mu(x) \leq$$

$$2^{p+-1} r^{p-} (\|w_n\|_{L^{p(.)}(X,\mu)} + \|M(w_n)\|_{L^{p(.)}(X,\mu)} \leq 2^{p+} r^{p-} \quad .$$

Therefore, theorem (4.1.7) completes the proof.

List of symbles :

J.E . littlewood	John Edens or littlewood
(r.v.m)	real – Value measurable

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