Chapter (4)

Vector Calculus , Stoke's Theorem and Potential

Section(4.1):The Language of vector calculus and Stocke's Theorem

The operators grad, div, and curl are the workhorses of vector calculus. We will see that they are three different incarnations of the exterior derivative.

Definition (4-1-1): (Grad, curl and div)

Let $f : U \to IR$ be a C^1 function on an open set $U \subset IR^n$, and let F be a C^1 vector field on U. Then the grad of a function, the curl of a vector field, and the div of a vector field, are given by the formulas below

$$\text{Grad f} = \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix} = \vec{\nabla} f$$

$$Curl\ \vec{F} = curl \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \vec{\nabla} \times \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \underbrace{\begin{bmatrix} D_2F_3 - D_3F_2 \\ D_3F_1 - D_1F_3 \\ D_1F_2 - D_2F_1 \end{bmatrix}}_{\substack{\text{cross product of} \\ \vec{\nabla} \text{ and } \vec{F}}}$$

$$df = D_1 f dx + D_2 f dy + D_3 f dz = W_{\vec{\nabla} f} = w_{\begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}}$$

These operators all look kind of similar, some combination of partial derivatives. (Thus they are called differential operators.) We use the symbol *V* to make it easier to remember the above formulas, which we can summarize .

$$\begin{aligned} Grad f &= \vec{\nabla} f \\ Curl\ \vec{F} &= \vec{\nabla} \times \vec{F} \qquad (4\text{-}1) \\ div\ \vec{F} &= \vec{\nabla} . \vec{F} \end{aligned}$$

Example (4-1-2): (Curl and div)

Let $\vec{F}$ be the vector field

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -x \\ xz^2 \\ x+y \end{bmatrix}; ie F_1 = -x, F_2 = xz^2, F_3 = x+y$$

The partial derivative D_2F_3 is the derivative with respect to the second variable of the function F_3, i.e., $D_2(x+y)=1$. Continuing in this fashion we get

$$curl\left(\begin{bmatrix} -x \\ xz^2 \\ x+y \end{bmatrix}\right) = \begin{bmatrix} D_2(x+y) - D_3(xz^2) \\ D_3(-z) - D_1(x+y) \\ D_1(xz^2) - D_2(-z) \end{bmatrix} = \begin{bmatrix} 1-2zx \\ -2 \\ z^2 \end{bmatrix} \qquad (4\text{-}2)$$

The divergence of the vector field

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x+y \\ x^2z\ y \\ z\ y \end{bmatrix} is 1 + x^2 z + y$$

What is the grad of the function $f = x^2y + z$? What are the curl and div of the vector field

$$\vec{F} = \begin{bmatrix} -y \\ x \\ xz \end{bmatrix}$$

Check your answers below .The following theorem relates the exterior derivative to the work, flux and density form fields .

Theorem (4-1-3): (Exterior derivative of form fields on IR^3)

Let f be a function on IR^3 and let P be a vector field. Then we have the following three formulas .

(a) $df = W_{\vec{\nabla} f}$; i.e., df is the work form field of grad f ,

(b) $dW_{\vec{F}} = \Phi_{\vec{\nabla}\times\vec{F}}$; i.e., $dW_{\vec{F}}$ is the Bux form field of curl $\vec{F}$,

(c) $d\Phi_{\vec{F}} = \rho_{\vec{\nabla}.\vec{F}}$; i.e., $d\Phi_{\vec{F}}$ is the density form field of div $\vec{F}$.

Example (4-1-4): (Equivalence of df and $w_{grad\,f}$.)

In the language of forms , to compute the exterior derivative of a function in IR[3], we can use part (d) of Theorem (4-1-3) to compute d of the 0-form *f* :

$$df = D_1 f \ dx_1 + D_2 f dx_2 + D_3 f \ dx_3. \qquad (4\text{ - }3)$$

$$Gradf = \begin{bmatrix} 2xy \\ x^2 \\ 1 \end{bmatrix}; curl\ \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} -y \\ x \\ xz \end{bmatrix} = \begin{bmatrix} 0 \\ -x \\ 2 \end{bmatrix}; div\ \vec{F} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} . \begin{bmatrix} -y \\ x \\ xz \end{bmatrix}$$

Evaluated on the vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ this 1-form gives

$$df\ (\vec{v}) = D_1 f v_1 + D_2 f v_2 + D_3 f v_3 \qquad (4\text{ - }4)$$

In the language of vector calculus, we can compute $W_{gralf} = W_{\vec{\nabla} f} = w_{\begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}}$

which evaluated on $\vec{v}$ gives

$$W_{\vec{\nabla} f}\left(\vec{v}\right) = \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = D_1 f v_1 + D_2 f v_2 + D_3 f v_3 \qquad (4 - 5)$$

Example (4-1-5) :(Equivalence of $dW_{\vec{F}}$ and $\Phi_{\vec{\nabla} \times \vec{F}}$)

Let us compute the exterior derivative of the 1-form in it IR[3]

$$xydx + z\ dy + yz\ dz \qquad \text{ie} \quad w_{\vec{F}} \qquad \vec{F} = \begin{bmatrix} xy \\ z \\ yz \end{bmatrix}$$

In the language of forms,

$$d\,(xydx + z\ dy + yz\ dz) = d\,(xy)\Lambda dx + d\,(z)\Lambda dy + d\,(yz)\Lambda dz$$

$$= (D_1 xydx + D_2 xydy + D_3 xydz)\Lambda dx$$
$$+ (D_1 zdx + D_2 zdy + D_3 zdz)\Lambda dy + (D_1 yzdx + D_2 yzdy + D_3 yzdz)\Lambda dz$$

$$= -x\,(dx\,\Lambda dy) + (z - 1)(dy\,\Lambda dz) \qquad (4 – 6)$$

Since any 2-form in IR[3] can be written

$$\Phi_{\vec{G}} = G_1 dy \Lambda dz - G_2\, dx \Lambda dz + G_3 dx \Lambda dy$$

the last line of Equation (4 - 6) can be written $\Phi_{\vec{G}}$ for $\vec{G} = \begin{bmatrix} Z-1 \\ 0 \\ -X \end{bmatrix}$

This vector field is precisely the curl of $\vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} xy \\ x \\ yz \end{bmatrix} = \begin{bmatrix} D_2 zy - D_3 z \\ -D_1 zy + D_3 xy \\ D_1 z - D_2 xy \end{bmatrix} = \begin{bmatrix} Z-1 \\ 0 \\ -X \end{bmatrix}$$

Proof of Theorem (4-1-3):

The proof simply consists of using symbolic entries rather than the specific ones of Examples (4.1.4) and (4.1.5) For part (a), we find

$$df = D_1 f dx + D_2 f dy + D_3 f dz = W_{\vec{\nabla} f} = w_{\begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}} \qquad (4\text{ -}7)$$

For part (b), a similar computation gives

$$dW_{\vec{F}} = d(F_1 dx + F_2 dy + F_3 dz) = dF_1 \Lambda\, dx + dF_2 \Lambda\, dy + dF_3 \Lambda\, dz$$

$$\sum_{i=0}^{m-1} f\left(x_i + x_i\right) - f\left(x_i\right) = f\left(x_1\right) - f\left(x_0\right) + f\left(x_2\right) - f\left(x_1\right) \ldots . + f\left(x_m\right) - f\left(x_{m-1}\right)$$

(4 - 8)

$$= (D_1 F_2 - D_2 F_1) dx \Lambda dy + (D_1 F_3 - D_3 F_1) dx \Lambda dz + (D_2 F_3 - D_3 F_2) dy \Lambda dz$$

$$= \Phi_{\begin{bmatrix} D_2 F_3 - D_3 F_2 \\ D_3 F_1 - D_1 F_3 \\ D_1 F_2 - D_2 F_1 \end{bmatrix}} = \Phi_{\vec{\nabla} \times \vec{F}}$$

For part (c), the computation gives

$$d\Phi_{\vec{F}} = d\,(F_1 dy \Lambda dz + F_2 dz \Lambda dx + F_3 dx \Lambda dy\,)$$

$$= (D_1 F_1 dx + D_2 F_2 dy + D_3 F_1 dz\,) \Lambda dy \Lambda dz + (D_1 F_2 dx + D_2 F_2 dy + D_3 F_2 dz\,) \Lambda dz \Lambda dx$$

$$+ (D_1 F_3 dx + D_2 F_3 dy + D_3 F_3 dz\,) \Lambda dx \Lambda dy \qquad (4\text{ - }9)$$

$$= (D_1 F_1 + D_2 F_2 + D_3 F_3) dx \Lambda dy \Lambda dz = p_{\vec{\nabla} . \vec{F}} \,.$$

Theorem (4-1-3) says that the three incarnations of the exterior derivative in IR^3 are precisely grad, curl, and div. Grad goes from 0-form fields to 1-form fields , curl goes from 1-form fields to 2-form fields, and div goes from 2-form fields to 3-form fields. This is summarized by the diagram in Table (2) , which you should learn.

Vector Calculus in R^3		**From Fields in R^3**
Functions	=	**0-form fields**
↓ gradient		**↓ d**
Vector fields	$\xrightarrow{work\ w}$	**1-form fields**
↓ curl		**↓ d**
Vector fields	$\xrightarrow{flux\ \Phi}$	**2-form fields**
↓ div		**↓ d**
Functions	$\xrightarrow{density\ p}$	**3-form fields**

Table (2). In IR^3, 0-form fields and 3-form fields can be identified with functions, and 1-form fields and 2-form fields can be identified with vector fields. The operators grad, curl, and div are three incarnations of the exterior derivative d, which takes a k-form field and gives a (k + 1)-form field .

Now we will discuss geometric interpretation of the exterior derivative in IR^3.We already knew how to compute the exterior derivative of any k-form, and we had an interpretation of the exterior derivative of a k-form was integrating W over the oriented boundary of a (k + 1)-parallelogram. Why did we bring in grad, curl and div?

One reason is that being familiar with grad, curl, and div is essential in many physics and engineering courses. Another is that they give a different perspective on the exterior derivative in IR^3, with which many people are more comfortable.

Now we will express geometric interpretation of the gradient.

The gradient of a function, abbreviated grad, looks a lot like the Jacobian matrix. Clearly grad $f(x) = [Df(x)]^T$; the gradient is gotten simply by putting the entries of the line matrix $[Df(x)]$ in a column instead of a row. In particular,

$$grad f(x).\vec{v} = [Df(x)]\vec{v} \qquad (4-10)$$

the dot product of v with the gradient is the directional derivative in the direction $\vec{v}$. If θ is the angle between grad *f (x)* and v, we can write

$$gradf\ (x).\vec{v} = |grad\ f\ (x)||\vec{v}|\, cos\theta\ , \qquad (4 - 11)$$

which becomes $|grad\ f(x)|\ \cos\theta$ if v is constrained to have length 1. This is maximal when $\theta = 0$, giving grad $f\ (x) = |grad\ f\ (x)|$. So we see that .The gradient of a function f at x points in the direction in which f increases the fastest, and has a length equal to its rate of increase in that direction.

Remark (4-1-6):

Some people find it easier to think of the gradient, which is a vector, and thus an element of R^n, than to think of the derivative, which is a line matrix, and thus a linear function IR^n. They also find it easier to think that the gradient is orthogonal to the curve (or surface, or higher-dimensional manifold) of equation $f\ (x) - c = 0$ than to think that ker[*D f (x)]* is the tangent space to the curve (or surface or manifold).

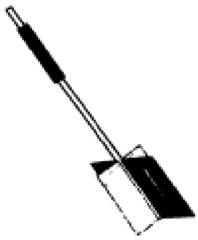
Since the derivative is the transpose of the gradient, and vice versa, it may not seem to make any difference which perspective one chooses. But the derivative has an advantage that the gradient lacks: as Equation (4-10) makes clear, the derivative needs no extra geometric structure on IR^n, whereas the gradient requires the dot product. Sometimes (in fact usually) there is no natural dot product available. Thus the derivative of a function is the natural thing to consider.

But there is a place where gradients of functions really matter: in physics, gradients of potential energy functions are force fields, and we really want to think of force fields as vectors. For example, the gravitational force field is the vector $\begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix}$, which we saw in Equation (2-57) this is the gradient of the height function (or rather, minus the gradient of the height function).

As it turns out, force fields are conservative exactly when they are gradients of functions, called potentials . However, the potential is not observable, and discovering whether it exists from examining the force field is it big chapter in mathematical physics.

Now we will study geometric interpretation of the curl.

The peculiar mixture of partials that go into the curl seems impenetrable. We aim to justify the following description.The curl probe. Consider an axis, free to rotate in a bearing that you hold,
and having paddles attached, as in Figure (4 -2).


Figure(4-2)

We will assume that the bearing is packed with a viscous fluid, so that its angular speed (not acceleration) is proportional to the torque exerted by the paddles. If a fluid is in constant motion with velocity vector field F, then the curl of the velocity vector field at *x*, $(\vec{\nabla}\ x\ \vec{F})(x)$, is measured as follows:

The curl of a vector field at a point x points in the direction such that if You insert the paddle of the curl probe with its axis in that direction, it will spin the fastest. The speed at which it spins is proportional to the magnitude of the curl.

Why should this be the case? Using Theorem (4-1-3) (b) and Definition (3-2-19) of the exterior derivative, we see that

$$\underset{dw_{\vec F}}{\Phi_{\vec\nabla\times\vec F}}(p_x^0(\vec v_1,\vec v_2)) = \lim_{n\to 0}\frac{1}{h^2}\int_{\vartheta p_x^0(h\vec v_1,h\vec v_2)} W_{\vec F} \qquad (4 - 12)$$

measures the work of $\vec{F}$ around the parallelogram spanned by $\vec{V}_1$, and $\vec{V}_2$ (ie , over its oriented boundary). IF $\vec{V}_1$ and $\vec{V}_2$ are unit vectors orthogonal to the axis of the probe and to each other. this work is approximately proportional to the torque to which the probe will be subjected.

Theorems (3-2-25) and (4-1-3) have the following important consequence in IR^3: If f is a C^2 function on an open subset $U \subset IR^3$, then curl grad $f = 0$.

Therefore, in order for a vector field to be the gradient of a function, its curl must be zero. This may seem obvious in terms of a falling apple; gravity does not exert any torque and cause the apple to spin. In more complicated settings, it is less obvious; if you observed the motions of stars in a galaxy, you might be tempted to think, there was some curl, but there isn't . (We will see in Section (4.2) that having curl zero does not quite guarantee that a vector field is the gradient of a function.)

Now we will illustrate geometric interpretation of the divergence.

The divergence is easier to interpret than the curl. If you put together the formula of Theorem (4-1-3) (c) and Definition (3-2-19) of the exterior derivative, we see that the divergence of $\vec{F}$ at a point x is proportional to the flux of $\vec{F}$ through the boundary of a small box around *x*, i.e., the net flow out of the box. In particular, if the fluid is incompressible, the divergence of its velocity vector field is 0: exactly as much must flow in as out. Thus, the divergence measures the extent to which flow along the vector field changes the density.

Again, Theorems (3-2-25) and (4-1-3)have the following consequence: If $\vec{F}$ is a C^2 vector field on an open subset $U \subset IR^3$, then div curl $\vec{F} = 0$.

Remark (4-1-7) :

Theorem (3-2-25) says nothing about

$$div \text{ grad } f, \text{ grad div } \vec{F}, \text{ or curl curl} \vec{F},$$

which are also of interest (and which are not 0); they are three incarnations of the Laplacian.

Now we will express the generalized Stokes's theorem . We worked pretty hard to define the exterior derivative, and now we are going to reap some rewards for our labor: we are going to see that there is a higherdimensional analog of the fundamental theorem of calculus, Stokes's theorem. It covers in one statement the four integral theorems of vector calculus . Recall the fundamental theorem of calculus:

Theorem (4-1-8):(Fundamental theorem of calculus)

If f is a C^1 function on a neighborhood of [a, b], then

$$\int_a^b f^{1}(t)dt = f(b) - f(a) \tag{4-13}$$

Restate this as

$$\int_{[a,b]} df = \int_{\vartheta[a,b]} f , \tag{4-14}$$

i.e., the integral of df over an oriented interval is equal to the integral off over the oriented boundary of the interval. In this form, the statement generalizes to higher dimensions

Theorem (4-1-9):(Generalized Stokes's theorem)

Let X be a compact piece-with-boundary of a (k + 1)-dimensional oriented manifold $M \subset IR^n$.

Give the boundary ∂X of X the boundary orientation, and let φbe a k-form defined on a neighborhood of X. Then

$$\int_{\partial X} \varphi = \int_X d\,\varphi \tag{4-15}$$

This beautiful, short statement is the main result of the theory of forms.

Example (4-1-10):(Integrating over the boundary of a square)
You apply Stokes's theorem every time you use anti-derivatives to compute an integral: to compute the integral of the 1-form *f (x)* dx over the oriented line segment [a, b], you begin by finding a function g(x) such that $dg(x) = f(x)\,dx$, and then say

$$\int_a^b f(x)\,dx = \int_{[a,b]} dg = \int_{\partial[a,b]} g = g(b) - g(a) \qquad (4\text{-}16)$$

This isn't quite the way it is usually used in higher dimensions, where "looking for anti-derivatives" has a different flavor.

For instance, to compute the integral fo $\int_c x\,dy - y\,dx$, where C is the boundary of the square S described by the inequalities $|x|, |y| \le 1$, with the boundary orientation, one possibility is to parametrize the four sides of the square (being careful to get the orientations right), then to integrate $x\ dy\ - y\ dx$ over all four sides and add. Another possibility is to apply Stokes's theorem:

$$\int_c x\,dy - y\,dx = \int_s (dx \Lambda dy - dy \Lambda dx) = \int_s 2dx \Lambda dy = 8 \qquad (4\text{-}17)$$

What is the integral over C of *x dy* + *y dx*? Check below.

Example (4-1-11): (Integrating over the boundary of a cube)
Let us integrate the 2-form

$$\varphi = (x - y_2 + z_3)(dy \Lambda dz + dx \Lambda dz + dx \Lambda dy) \qquad (4\text{-}18)$$

over the boundary of the cube C_a. given by $0 \le x,\ y,\ z \le a$. It is quite possible to do this directly, parametrizing all six faces of the cube, but Stokes's theorem simplifies things substantially.
Computing the exterior derivative of φ gives

$$d\varphi = dx \Lambda dy \Lambda dz - 2y dy \Lambda dx \Lambda ds + 3z\ 2dz \Lambda dx \Lambda dy = (1 + 2y + 3z\ 2) dx \Lambda dy \Lambda dz \qquad (4\text{-}19)$$

$$\int_{\partial c} \varphi = \int_{c_a} (1 + 2y + 3z^2) dx \Lambda dy \Lambda dz,$$

$$\int_0^a\int_0^a\int_0^a \left(1+2y+3z^2\right) dx\ dy\ dz, \qquad (4\text{-}20)$$

$$a^2\left(\left[x\right]_0^a+\left[y^2\right]_0^a+\left[z^3\right]_0^a\right)=a^2(a+a^2+a^3)$$

Example (4-1-12) : (Stokes's theorem: a harder example)
Now let's try something similar to Example (4.1.11), but harder, integrating

$$\varphi=\left(x_1-x_2^2+x_3^3-\mp x_n^n\right)\left(\sum_{t=1}^{n} dx_1\Lambda\ldots\Lambda dx_i\Lambda\ldots dx_n\right) \qquad (4\text{-}21)$$

over the boundary of the cube C_a. given by $0\le x_j\le a,\ j = 1,\ldots,n.$

This time, the idea of computing the integral directly is pretty awesome: parametrizing all *2n* faces of the cube, etc. Doing it using Stokes's theorem is also pretty awesome, but much more manageable.
We know how to compute $d\varphi$, and it comes out to

$$d\varphi = (1+2x_2+3x_3^2+\ldots+ nn_n^{n-1})\ dx_1 \wedge \ldots \wedge dx_n \qquad (4\text{-}22)$$

The integral of $j\ x_j^{j-1}\ dx_1 \wedge . . \wedge dx_n$ over C_a is

$$\int_0^a\ldots\int_0^a x_j^{j-1}\left|d^m x\right| = a^{j+n-1} \qquad (4\text{-}23)$$

so the whole integral is $a^n\left(1\ +\ a\ +\ \ +\ a^{n-1}\right)$.

The examples above bring out one unpleasant feature of Stokes's theorem: it only relates the integral of a k -1 form to the integral of a k-form if the former is integrated over a boundary. It is often possible to skirt this difficulty, as in the example below.

Example (4-1-13):(Integrating over faces of a cube)
Let S be the union of the faces of the cube C given by $-1\le x,\ y,\ z\le 1$ except the top face, oriented by the outward pointing normal. What is

$$\int_s \Phi_{\vec{F}}\ \vec{F}=\begin{bmatrix}X\\Y\\Z\end{bmatrix}?$$

The integral of $\Phi_{\vec{F}}$ over the whole boundary ∂C is by Stokes's theorem the integral over C of $\Phi_{\vec{F}}$ = div $\vec{F}$ $dx \Lambda dy \Lambda dz = 3\, dx \Lambda dy \Lambda dz$, so

$$\int_{\partial C} \Phi_{\vec{F}} = \int_{C} div\, \vec{F}\, dx \Lambda dy \Lambda dz = 3\int_{C} dx \Lambda dy \Lambda dz = 24 \qquad (4\text{-}24)$$

Now we must subtract from that the integral over the top. Using the obvious

Parametrization $\begin{pmatrix} s \\ t \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ gives

$$\int_{-1}^{1}\int_{-1}^{1} \det \begin{bmatrix} s & 1 & 0 \\ t & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

So the whole integral is 24 - 4 = 20.

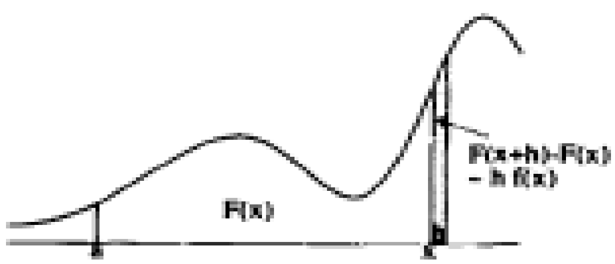
Now we will study Proof of the generalized Stokes's theorem.
Before starting the proof of the generalized Stokes's theorem, we want to sketch two proofs of the fundamental theorem of calculus, Theorem (4-1-7). You probably saw the first in first-year calculus, but it is the other that will generalize to prove Stokes's theorem.

Now we will illustrate first proof of the fundamental theorem of calculus

Set $F(x) = \int_{0}^{x} f(t)\,dt$. We will show that

$$F'(x) = f(x), \qquad (4\text{-}25)$$

as Figure (4-3) suggests. Indeed,



Figure(4-3)

$$F'(x) = \lim_{h \to 0} \frac{1}{h}\left(\int_{0}^{x+h} f(t)\,dt - \int_{0}^{x} f(t)\,dt \right)$$

$$\lim_{h \to 0} \frac{1}{h} \underbrace{\int_{0}^{x+h} f\ (t)dt}_{\mathrm{hf(x)}} \qquad (4\text{-}26)$$

(The last integral is approximately *hf(x)*; the error disappears in the limit.) Now consider the function

$$\mathrm{f\,(x)} - \underbrace{\int_{0}^{x} f\ (t)dt}_{\text{with deny.f}'(\mathrm{x})} \qquad (4\text{-}27)$$

The argument above shows that its derivative is zero, so it is constant; evaluating the function at *x* = *a*, we see that the constant is *f (a)*. Thus

$$f\ (b) - \int_{a}^{b} f\ (t)dt = f\ (a) \qquad (4\text{-}28)$$

Now we will discuss Second proof of the fundamental theorem of calculus. Here the appropriate drawing is the Riemann sum drawing of Figure (4-4)

By the very definition of the integral

Figure(4-4**)**

$$\int_{a}^{b} f\ '(x)dx \approx \sum_{i} f\ (x_i)(x_i + x_i) \qquad (4\text{-}29)$$

where $x_0 < x_i < . . < x_m$,,, decompose [a, b] into m little pieces, with a = x_0 and b = x_m

By Taylor's theorem

$$f\ (x_i + 1) f\ (x_i)\ + f\ '(x_i)(x_{i+1} - x_i). \qquad (4\text{-}30)$$

These two statements together give

$$\int_{a}^{b} f\ '(x)dx \approx \sum_{i} f\ (x_i)(x_i + x_i) \approx \sum_{i} f\ (x_i + x_i) - f\ (x_i) \qquad (4\text{-}31)$$

In the far right-hand term all the interior xi's cancel

$$\sum_{i=0}^{m-1} f\left(x_i + x_i\right) - f\left(x_i\right) = f\left(x_1\right) - f\left(x_0\right) + f\left(x_2\right) - f\left(x_1\right) \ldots + f\left(x_m\right) - f\left(x_{m-1}\right) \tag{4-32}$$

Leaving $f\left(x_m\right) - f\left(x_o\right).\ i.e.,\ f\left(b\right) - f\left(a\right).$

Let us analyze a little more closely the errors we are making at each step; we are adding more and more terms together as the partition becomes finer, so the errors had better be getting smaller faster, or they will not disappear in the limit. Suppose we have decomposed the interval into m pieces. Then when we replace the integral in Equation (4.32) by the first sum, we are making m errors, each bounded as follows. The first equality uses the fact that $A\left(b - a\right) = \int_a^b A$.

$$\left| \int_{x_i}^{x_{i+1}} f'\left(x\right)dx - \overbrace{f'(x)}^{A}\,\overbrace{x_{i-1} - x_i}^{b-a} \right| = \left| \int_{x_i}^{x_{i+1}} \left(f'\left(x\right) - \overbrace{f'(x_i)}^{A} \right) dx \right|$$

$$\leq \int_{x_i}^{x_{i+1}} \sup\left|f''\right|(x - x_i)dx \tag{4-33}$$

$$= \sup\left|f''\right| \int_{x_i}^{x_{i+1}} \left(x - x_i\right)dx$$

$$\sup\left|f''\right| \frac{\left(x_{i+1} - x_i\right)}{2} = \sup\left|f''\right| \frac{(b-a)^2}{2m^2}$$

We also need to remember the error term from Taylor's theorem, Equation(4-31) , which turns out to be about the same. So all in all. we made m errors, each of which is $\leq C_1/m^2$, where Cr is a constant that does not depend on m. Multiplying that maximal error for each piece by the number m of pieces leaves an m in the denominator, and a constant in the numerator, so the error tends to 0 as the decompositions becomes finer and finer.

Now we will express An informal proof of Stokes's theorem Suppose you decompose X into little pieces that are approximated by oriented (k + 1)-parallelograms p_i^0:

$$P_i^0 = P_x^0\left(\vec{v}_{1,i}, \vec{v}_{2,i}, \ldots \vec{v}_{k+1,i}\right) \tag{4-34}$$

Then

$$\int_X d\varphi \approx \sum_i d\varphi\left(P_i^0\right) \approx \sum_i \int_{\vartheta P_1^0} \varphi \approx \int_{\partial X} \varphi \tag{4-35}$$

The first approximate sign is just the definition of the integral; the tt becomes an equality in the limit as the decomposition becomes infinitely fine. The second approximate sign comes from our definition of the exterior derivative When we add over all the P,°, all the internal boundaries cancel, leaving $\int_{\partial X} \varphi$.

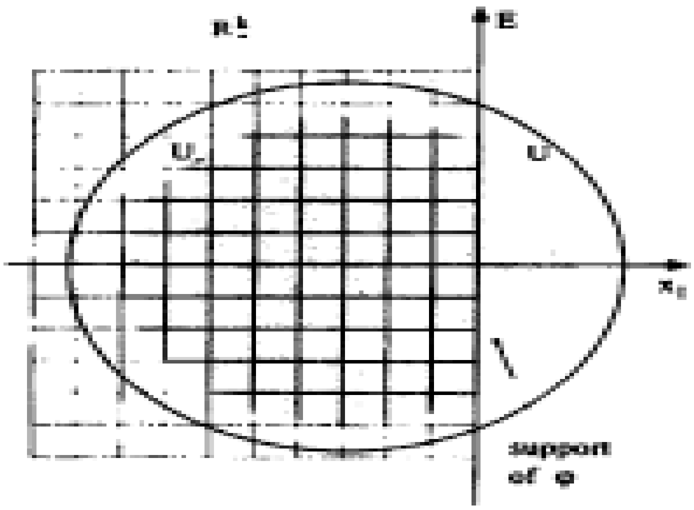
As in the case of Riemann sums, we need to understand the errors that are signaled by our signs. If our parallelograms P, have side c, then there are approximately $\in^{-(k+I)}$ such parallelograms. The errors in the first and second replacements are of order $\in^{k+2}$. For the first, it is our definition of the integral, and the error becomes small as the decomposition becomes infinitely fine. For the second, from the definition of the exterior derivative

$$d\varphi\left(P_i^0\right) = \int_{\partial P_1^0} \varphi + terms\ \ of\ \ order \qquad (4\text{-}36)$$

so indeed the errors disappear in the limit.

Now we will illustrate A situation where the easy proof works

We will now describe a situation where the proof in Section (4.1) really does work. In this simple case, we have a (k - 1)-form in IR^k, and the boundary of the piece we will integrate over is simply the subspace $E \subset IR^k$ of equation $x_1 = 0$. There are no manifolds, nothing curvy. Figur (4-5)illustrates Proposition (4-1-14) .



Figure(4-5)

Proposition (4-1-14) :

Let U be a bounded open subset of R^k, and let U - be the subset of U where the first coordinate is non-positive (i.e., $x_1 \leq 0$). Give U the standard orientation of R^K (by det), and give the boundary orientation to $\partial U\_ = U \cap E$. Let φ be a (k - 1)-form on fl of class C^2, which vanishes identically outside U. Then

$$\int_{\partial U-} \varphi = \int_{U-} d\varphi \qquad (4\text{-}37)$$

Proof:

We will repeat the informal proof above, being a bit more careful about the bounds. Choose $\in > 0$, and denote by IR^k the subset of IR^k, where $x_1 \geq 0$.

Recall from the proof of Theorem (3-2-21) that there exists" a constant K and $\delta > 0$ such that when $|h| < \delta$,

$$\left| d\varphi\left(P_x^0\left(h\vec{e}_1,\ldots,h\vec{e}_k\right)\right) - \int_{\partial P_x^0\left(h\vec{e}_1,\ldots,h\vec{e}_k\right)} \varphi \right| \leq kh^{k+1} \qquad (4\text{-}38)$$

That is why we required φ to be of class C^2, so that the second derivatives of the coefficients of φ are bounded. Take the dyadic decomposition $D_N\left(R^k\right)$, where $h = 2'N$. By taking N sufficiently large, we can guarantee that the difference between the integral of d(p over U_ and the Riemann sum is less than $\epsilon/2$:

$$\left| \int_{U-} d\varphi - \sum_{C \in D_N(R^k)} d\varphi(C) \right| < \frac{\epsilon}{2} \qquad (4\text{-}39)$$

Now we replace the k-parallelograms of Equation (4-38) by dyadic cubes, and evaluate the total difference between the exterior derivative of V over the cubes C, and ,p over the boundaries of the C. The number of cubes of $D_N\left(R^k\right)$ that intersect the support of φ is at most $L2^{kN}$ for some constant L, and since $h = 2^{-N}$, the bound for each error is now $K2^{-N(k+1)}$ so

$$\left| \sum_{C \in D_N\left(R^k\right)} d\varphi(C) - \sum_{C \in D_N\left(R^k\right) BC} \int \varphi \right| \underbrace{L\,2^{KN}}_{No.of\ cubes} \underbrace{K\,2^{-N(K+1)}}_{\substack{bound\ for \\ each\ error}} \leq= LK\,2^{-N} \qquad (4\text{-}40)$$

This can also be made < ∈/2 by taking N sufficiently large-to be precise, by taking

$$N \geq \frac{\log 2LK - \log \in}{\log 2} \qquad (4\text{-}41)$$

Putting these inequalities together, we get

$$\overbrace{\left| \int_{U-} d\varphi - \sum_{C \in D_N(R^k)} d\varphi(C) \right|}^{\leq t/2} + \overbrace{\left| \sum_{C \in D_N(R^k)} d\varphi(C) - \sum_{C \in D_N(R^k)} \int_{BC} \varphi \right|}^{\leq t/2} \in \qquad (4\text{-}42)$$

so in particular, when N is sufficiently large we have

$$\left| \int_{U1} d\varphi - \sum_{C \in D_N(R^k)} \int_{\partial c} \varphi \right| \leq \in \qquad (4\text{-}43)$$

Finally, all the internal boundaries in the sum

$$\sum_{C \in D_N(R^k)} \int_{\partial c} \varphi \qquad (4\text{-}44)$$

cancel, since each appears twice with opposite orientations. The only boundaries that count are those in R^{K-1} . So (using C' to denote cubes of the dyadic composition of R^{K-1})

$$\sum_{C \in D_N(R^k)} \int_{\partial c} \varphi = \sum_{C \in D_N(R^k)} \int_{C'} \varphi = \int_E \varphi \int_{\partial U-} \varphi \qquad (4\text{-}45)$$

(We get the last equality because W vanishes identically outside U, and therefore outside U_1.) So

$$\left| \int_{U-} d\varphi - \int_{\partial U-} \varphi \right| < \in \qquad (4\text{-}46)$$

Since a is arbitrary, the proposition follows

Section(4 .2): The Integral Theorem and potential

The four forms of the generalized Stokes's theorem that make sense IR^2 and IR^3 don't say anything that is not contained in that theorem, but each is of great importance in many applications; these theorems should all become personal friends, or at least acquaintances. They are used everywhere in electromagnetism, fluid mechanics, and many other fields.

Theorem (4-2-1) : (Fundamental theorem for line integrals)

Let C be an oriented curve in R^2 or R^3 (or for that matter any IR^n), with oriented boundary (P_b^0 - P_a^0), and let f be a function defined on a neighborhood of C. Then

$$\int_c df = f(b) - f(a) \qquad (4\text{-}47)$$

Now we will express Green's theorem and Stokes's theorem
Green's theorem is the special case of Stokes's theorem for surface integrals when the surface is flat.

Theorem (4-2-2): (Green's theorem)
Let S be a bounded region of IR^2, bounded by a curve C (or several curves C), carrying the boundary orientation as described in Definition (3-2-14). Let P bea vector field defined on a neighborhood of S. Then

$$\int_s dW_{\vec{F}} = \int_C dW_{\vec{F}} \; OR \int_s dW_{\vec{F}} = \sum_i \int_C dW_{\vec{F}} \qquad (4\text{-}48)$$

This is traditionally written

$$\int_s \left(D_1 g - D_2 f\right) dxdy = \int_C f \; dx + g \; dy \qquad (4\text{-}49)$$

To see that the two versions are the same, write $W_{\vec{F}} = f\begin{pmatrix} x \\ y \end{pmatrix} dx + g\begin{pmatrix} x \\ y \end{pmatrix} dy$ and use Theorem (3.2.21) to compute its exterior derivative:

$$dW_{\vec{F}} = d(f \; dx + g \; dy) = df \wedge dx + dg \wedge dy = (D_1 f \; dx + D_2 f \; dy) \wedge dx + (D_1 g \; dx + D_2 g \; dy) \wedge dy$$

$$= D_2 f \; dy \wedge dx + D_1 g \; dx \wedge dy = (D_1 g - D_2 f) dx \wedge dy \qquad (4\text{-}50)$$

Example (4-2-3): (Green's theorem)
What is the integral

$$\int_{\partial D} 2xy \, dy + x^2 dx \qquad (4\text{-}51)$$

where U is the part of the disk of radius *R* centered at the origin where $y \geq 0$, with the standard orientation?
This corresponds to Green's theorem, with $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2$ and $g\begin{pmatrix} x \\ y \end{pmatrix} = 2xy$, so that $D_1 g = 2y$ and $D_2 f = 0$. Green's theorem say

$$\int_{\partial D} 2xy \, dy + x^2 dx = \int_s \left(D_1 g - D_2 f\right) dxdy = \int_U 2yxdy \qquad (4\text{-}52)$$

$$= \int_0^{\pi} \int_0^{R} \left(2r \sin\theta\right) r \; dr \; d\theta = \frac{2R^3}{3} \int_0^{\pi} \sin\theta d\theta = \frac{4R^3}{3}$$

What happens if we integrate over the boundary of the entire disk?

Theorem (4-2-4):(Stokes's theorem)

Let S be an oriented surface in IR^3, bounded by a curve C that is given the boundary orientation. Let φ be a 1-form field defined on a neighborhood of S. Then

$$\int_S d\varphi = \int_C \varphi \qquad (4\text{-}53)$$

Again, let's translate this into classical notation. First, and without loss of generality, we can write $\varphi = W_{\vec{F}}$, so that Theorem (4-2- 4) becomes

$$\int_S dW_{\vec{F}} \int_S \Phi_{curl\,\vec{F}} = \sum_i \int_{C1} W_{\vec{F}} \qquad (4\text{-}54)$$

This still isn't the classical notation. Let $\vec{N}$ be the normal unit vector field on S defining the orientation, and $\vec{T}$ be the unit vector field on the C_i defining the orientation there. Then

.

$$\iint_S \left(curl\ \vec{F}(x) \right) . \vec{N}(x) \left| d^2 x \right| = \sum_i \int_{C_i} \vec{F}(x) . \vec{T}(x) \left| d^2 x \right| \qquad (4\text{-}55)$$

The left-hand side of Equation (4-55) is discussed in the margin. Here let's compare the right-hand sides of Equations (4-54) and (4-55) . Let us set $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ An the right-hand side of Equation (4-54), the integrand is = $W_{\vec{F}} = F_1 dx + F_2 dy + F_3 dz$; given a vector 9, it returns the number $F_1 v_1 + F_2 v_2 + F_3 v_3$. In Equation (4-55) , $\vec{T}(x)\ |d^1 x|$ is a complicated way of expressing the identity: given a vector $\vec{v}$, it returns $\vec{T}(x)$ times the length of $\vec{v}$. Since $\vec{T}(x)$ is a unit vector, the result is a vector with length $\vec{v}$, tangent to the curve. When integrating, we are only going to evaluate the integrand on vectors tangent to the curve and pointing in the direction of $\vec{T}$, so this process just takes such a vector and returns precisely the same vector. So $F(x)\ \vec{T}(x)\ |d^1 x|$ takes a vector $\vec{v}$ and returns the number

$$\underbrace{(\vec{F}(x) . \vec{T}(x) \left| d^1 x \right|)\left(\vec{v} \right)}_{\vec{v}} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = F_1 v_1 + F_2 v_2 + F_3 v_3 = W_{\vec{F}} \left(\vec{v} \right)$$

(4-56)

Example (4-2-5):(Stokes's theorem)

Let C be the intersection of the cylinder of equation $x^2 + y^2 = 1$ with the surface of equation $z = \sin^{2} z$. Orient C so that the polar angle decreases along C. What is the work over C of the vector field

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y^3 \\ x \\ z \end{bmatrix} ? \qquad (4\text{-}57)$$

It's not so obvious how to visualize C, much less integrate over it. Stokes's theorem says there is an easier approach: compute the integral over the subsurface S consisting of the cylinder $x^2 + y^2 = 1$ bounded at the top by C and at the bottom by the unit circle C_1 in the (x, y) −plane, oriented counterclockwise.

By Stokes's theorem, the integral over C plus the integral over C_1 equals the integral over S, so rather than integrate over the irregular curve C_1, we will integrate over S and then subtract the integral over C_1. First we integrate over S:

$$\int_C W_{\vec{F}} + \int_{C_1} W_{\vec{F}} = \int_S \Phi_{curl\,\vec{F}} = \int_S \Phi_{\begin{bmatrix} 0 \\ 0 \\ 1-3y^2 \end{bmatrix}} = 0 \qquad (4\text{-}58)$$

This last equality comes from the fact that the vector field is vertical, and has no flow through the vertical cylinder. Finally parametrize C_1 in the obvious way:

$$t \rightarrow \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad (4\text{-}59)$$

which is compatible with the counterclockwise orientation of C_1, and compute

$$\int_{C_1} W_{\vec{F}} = \int_0^{\pi} \begin{bmatrix} (\sin t)^3 \\ \cos t \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt \qquad (4\text{-}60)$$

$$= \int_0^{\pi} (-\sin t)^4 + \cos^2 t \; dt = \frac{3}{4}\pi + \pi = \frac{7}{4}\pi$$

So the work over C is

$$\int_C W_{\vec{F}} = -\frac{7}{4}\pi \qquad (4\text{-}61)$$

Now we will illustrate the divergence theorem .The divergence theorem is also known as Gauss's theorem.

Theorem (4-2-6):(The divergence theorem)
Let M be a bounded domain in IR^3 with the standard orientation of space, and let its boundary ∂M be a union of surfaces S_i, each oriented by the outward normal. Let φ be a 2-form field defined on a neighborhood of M. Then

$$\int_M d\varphi = \sum_i \int_{S_i} \varphi \qquad (4\text{-}62)$$

Again, let's make this look a bit more classical. Write $\varphi = \Phi_{\vec{F}}$, so that $d\varphi = d\Phi_{\vec{F}} = \rho_{div\vec{F}}$, and let $\vec{N}$ be the unit outward-pointing vector field on the S_i; then Equation (4 - 63) can be rewritten

$$\iiint_M div\ \vec{F}\, dx\ \ dy\ \ dz = \sum_i \iint_{s_i} \vec{F}\,.\vec{N}\ \left|d^2x\right| \qquad (4\text{-}63)$$

When we discussed Stoker's theorem, we saw that $\vec{F}\ .\ \vec{N}$, evaluated on a parallelogram tangent to the surface, is the same thing as the flux of $\vec{F}$ evaluated on the same parallelogram. So indeed Equation (4 - 64) is the same as

$$\int_M d\Phi_{\vec{F}} = \int_M \rho_{div\,\vec{F}} = \sum_i \int_{S_i} \Phi_{\vec{F}} \qquad (4\text{-}64)$$

Remark(4-2-7) :
We think Equations (4 - 55) and (4 - 63) are a good reason to avoid the classical notation. For one thing, they bring in N, which will usually involve dividing by the square root of the length; this is messy, and also unnecessary, since the $|d^2x|$ term will cancel with the denominator. More seriously, the classical notation hides the resemblance of this special Stokes's theorem and the divergence theorem to the general one, Theorem (4-1-9). On the other hand, the classical notation has a geometric immediacy that really speaks to people who are used to it.

Example (4-2-8) : (Divergence theorem)
Let Q be the unit cube. What is

the flux of the vector field $\begin{bmatrix} x^2y \\ -2yz \\ x^3y^2 \end{bmatrix}$ through the boundary of Q if Q carries the standard orientation of IR^3 and the boundary has the boundary orientation?

The divergence theorem asserts that

$$\int_{\partial Q} \Phi_{\begin{bmatrix} x^2y \\ -2yz \\ x^3y^2 \end{bmatrix}} = \int_{Q} \rho_{\,div \begin{bmatrix} x^2y \\ -2yz \\ x^3y^2 \end{bmatrix}} = \int_{Q} (2xy - 2z)\left|d^3x\right| \qquad (4\text{-}65)$$

This can readily be computed by Fubini's theorem:

$$\int_0^1\int_0^1\int_0^1 (2xy - 2z)\,dx\ dy\ dz = \frac{1}{2} - 1 = -\frac{1}{2} \qquad (4\text{-}66)$$

Example (4-2-9): (The principle of Archimedes)

Archimedes is said to have been asked by Creon, the tyrant of Syracuse, to determine whether hi crown was really made of gold. Archimedes discovered that by weighing the crown when suspended in water, he could determine whether or not it was counterfeit. According to legend, he made the discovery in the bath, and proceeded to run naked through the streets, crying "Eureka" ("I have found it").

The principle he claimed is the following: A body immersed in a fluid receives a buoyant force equal to the weight of the displaced fluid.

We do not understand how he came to this conclusion, and the derivation

we will give of the result uses mathematics that was certainly not available to Archimedes.

The force the fluid exerts on the immersed body is due to pressure. Suppose that the body is M, with boundary ∂M made up of little oriented parallelograms P_1^{o}. The fluid exerts a force approximately

$$P_1\ (x_i)Area\left(P_1\right)\vec{n}, \qquad (4\text{-}67)$$

where $\vec{n}$ is an inner pointing unit vector perpendicular to P_1^{o} and x_i is a point of P_1^{o}this becomes a better and better approximation as P_1^{o} becomes small so that the pressure on it becomes approximately constant. The total force exerted by the fluid is the sum of the forces exerted on all the little pieces of the boundary.

Thus the force is naturally a surface integral, and in fact is really an integral of a 2-form field, since the orientation of ∂M matters. But we can't think of it as a single 2-form field: the force has three components, and we have to think of each of them as a 2-form field. In fact, the force is

$$\begin{bmatrix} \int_{\partial M} P\Phi_{\vec{e}_1} \\ \int_{\partial M} P\Phi_{\vec{e}_2} \\ \int_{\partial M} P\Phi_{\vec{e}_3} \end{bmatrix} \tag{4-68}$$

$$\begin{bmatrix} P\Phi_{\vec{e}_1} \\ P\Phi_{\vec{e}_2} \\ P\Phi_{\vec{e}_3} \end{bmatrix}\left(P_x\left(\vec{v}_1,\vec{v}_2\right)\right) = p\left(x\right) = \begin{bmatrix} \det\left[\vec{e}_1,\vec{v}_1,\vec{v}_2\right] \\ \det\left[\vec{e}_2,\vec{v}_1,\vec{v}_2\right] \\ \det\left[\vec{e}_3,\vec{v}_1,\vec{v}_2\right] \end{bmatrix} \tag{4-69}$$

$$p(x)(\vec{v}_1,\vec{v}_2) = P_x\left(x_i\right)Area\left(P_x\left(\vec{v}_1,\vec{v}_2\right)\right)\vec{n}$$

In an incompressible fluid on the surface of the earth, the pressure is of the form $p(x) = -\mu gz$, where s is the density, and g is the gravitational constant. Thus the divergence theorem tells us that if ∂M is oriented in the standard way, i.e., by the outward normal, then

$$Total\ \ Force = \begin{bmatrix} \int_{\partial M} \mu gz\Phi_{\vec{e}_1} \\ \int_{\partial M} \mu gz\Phi_{\vec{e}_2} \\ \int_{\partial M} \mu gz\Phi_{\vec{e}_3} \end{bmatrix} = \begin{bmatrix} \int_M \rho\vec{\nabla}.\left(\mu gz\Phi_{\vec{e}_1}\right) \\ \int_M \rho\vec{\nabla}.\left(\mu gz\Phi_{\vec{e}_2}\right) \\ \int_M \rho\vec{\nabla}.\left(\mu gz\Phi_{\vec{e}_3}\right) \end{bmatrix} \tag{4-70}$$

The divergences are

$$\vec{\nabla}.(\mu gz\Phi_{\vec{e}_1}) = \vec{\nabla}.(\mu gz\Phi_{\vec{e}_2}) = 0, \vec{\nabla}.(\mu gz\Phi_{\vec{e}_3}) = \mu g \tag{4-71}$$

Thus the total force is

$$\begin{bmatrix} 0 \\ 0 \\ \int_M \rho \mu g \end{bmatrix} \qquad (4\text{-}72)$$

and the third component is the weight of the displaced fluid; the force is oriented upwards. This proves the Archimedes principle

Now we will study Potentials . A very important question that constantly comes up in physics is: when is a vector field conservative? The gravitational vector field is conservative: if you climb from sea level to an altitude of 500 meters by bicycle and then return to your starting point, the total work against gravity is zero, whatever your actual path. Friction is not conservative, which is why you actually get tired during such a trip. A very important question that constantly comes up in geometry is: when does a space have a "hole" in it?

We will see in this section that these two questions are closely related

In the following we will discuss Conservative vector fields and their potentials Asking whether a vector field is conservative is equivalent to asking whether it is the gradient of a function.

Theorem (4-2-10) :

A vector field is the gradient of a function if and only if it is conservative: i.e., if and only if the work of the vector field along any path depends only on the endpoints, and not on the oriented path joining them.

Proof:

Suppose $\vec{F}$ is the gradient of a function f: $\vec{F} = \vec{\nabla}.f$ of. Then by Theorem (4-1-9), for any parametrized path

$$\gamma : [a,b] \rightarrow R^n \qquad (4\text{-}73)$$

we have (Theorem (4-2-1))

$$\int_{\gamma:[a,b]} W_{\vec{\nabla}.f} = f\left(\gamma(b)\right) - f\left(\gamma(a)\right) \qquad (4\text{-}74)$$

Clearly, the work of a vector field that is the gradient of a function depends only on the endpoints: the path taken between those points doesn't matter.

It is a bit harder to show that path independence implies that the vector field is the gradient of a function. First we need to find a candidate for the function f, and there is an obvious choice: choose any point xo in the domain of F, and define

$$f(x) = \int_{\gamma(x)} W_{\vec{F}} \qquad (4\text{-}75)$$

where $\gamma(x)$ is an arbitrary path from x_o to x: our independence of path condition guarantees that the choice does not matter.
Now we have to see that $\vec{F} = \vec{\nabla}.f$, or alternatively that $W_{\vec{F}} = df$. We know That

$$df\left(P_x\left(\vec{v}\right)\right) = \lim_{h\to 0}\frac{1}{h}\left(f\left(x + h\vec{v}\right) - f(x)\right) \qquad (4\text{-}76)$$

and (remembering the definition of f in Equation (4-75)) $f(x+h) - f(x)$ is the work of F first from x back to xo, then from xo to $x + h\vec{v}$. By independence of path, we may replace this by the work from x to $x + h\vec{v}$ along the straight line. Parametrize the segment in the obvious way (by $\gamma : t \to x + t\vec{v}$, with ($0 \le t \le h$) to get

$$df\left(P_x\left(\vec{v}\right)\right) = \lim_{h\to 0}\frac{1}{h}\left(\int_0^h \underbrace{\vec{F}\left(x + h\vec{v}\right).\vec{v}}_{F'(\gamma(t))\gamma'(t)}\, dt\right) = \vec{F}(x).\vec{v}, \qquad (4\text{-}77)$$

i.e $, df = W_{\vec{F}}$

Definition (4-2-11) :
A function f such that grad f = $\vec{F}$ is called a potential of $\vec{F}$. A vector field has more than one potential, but pretty clearly, two such potentials f and g differ by a constant, since

$$grad\,(f - g) = grad f - grad g = \vec{F} - \vec{F} = 0 \qquad (4\text{-}78)$$

the only functions with gradient 0 are the constants.
So when does a vector field have a potential, and how do we find it? The first question turns out to be less straightforward than might appear. There is a necessary condition: in order for a vector field f to be the gradient of a function, it must satisfy

$$curl\,\vec{F} = 0 \qquad (4\text{-}79)$$

This follows immediately from Theorem (3-2-25): *ddf* = 0. Since $df = W_{\vec{\nabla}.f}$ then
IF $\vec{F} = \vec{\nabla}.f$,

$$dW_{\vec{F}} = \Phi_{curl\,\vec{F}} = ddf = 0 \qquad (4 - 80)$$

the flux of the curl of $\vec{F}$ can be 0 only if the curl is 0.
Some textbooks declare this condition to be sufficient also, but this is not true, as the following example shows.

Example (4-2-12) : (Necessary but not sufficient)

Consider the vector field

$$\vec{F} = \frac{1}{x^2 + y^2}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \qquad (4\text{-}81)$$

on IR^3 with the z-axis removed. Then

$$curl\,\vec{F} = \begin{bmatrix} 0 \\ 0 \\ D_1 \dfrac{X}{x^2 + y^2} - D_1 \dfrac{-Y}{x^2 + y^2} \end{bmatrix} \qquad (4\text{-}82)$$

and the third entry gives

$$\frac{\left(x^2 + y^2\right) - 2x^2}{\left(x^2 + y^2\right)^2} + \frac{\left(x^2 + y^2\right) - 2y^2}{\left(x^2 + y^2\right)^2} = 0 \qquad (4\text{-}83)$$

But f cannot be written $\vec{\nabla} f$ for any function f : (IR^3 - z-axis) → IR. Indeed,
using the standard parametrization

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \qquad (4\text{-}84)$$

the work of f around the unit circle oriented counterclockwise gives

$$\int_{S^1} W_{\vec{F}} = \int_0^{2\pi} \overbrace{\underbrace{\frac{1}{\cos^2 t + \sin^2 t}\begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}}_{\vec{F}(\gamma(t))} . \underbrace{\begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}}_{\gamma'(t)}}^{1} dt = 2\pi \qquad (4\text{-}85)$$

This cannot occur for work of a conservative vector field: we started at one point and returned to the same point, so if the vector field were conservative, the work would be zero.
We will now play devil's advocate. We claim

$$\vec{F} = \vec{\nabla}\left(\arctan\frac{y}{x}\right) \qquad (4\text{-}86)$$

Why doesn't this contradict the statement above, that $\vec{F}$ cannot be written $\vec{\nabla} f$? The answer is that

$$\arctan\frac{y}{x} \qquad (4\text{-}87)$$

is not a function, or at least, it cannot be defined as a continuous function on IR^3 minus the z-axis. Indeed, it really is the polar angle 0, and the polar angle cannot be defined on IR minus the z-axis; if you take a walk counterclockwise on a closed path around the origin, taking your polar angle with you, when you get back where you started your angle will have increased by 2π.

Example (4-2-13) :

shows exactly what is going wrong. There isn't any problem with $\vec{F}$, the problem is with the domain. We can expect trouble any time we have a domain with holes in it (the hole in this case being the z-axis, since $\vec{F}$ is not defined there). The function f such that $\vec{\nabla} f = \vec{F}$ is determined only up to an additive constant, and if you go around the hole, there is no reason to think that you will not add on a constant in the process. So to get a converse to Equation (4-80), we need to restrict our domains to domains without holes. This is a bit complicated to define, so instead we will restrict them to convex domains.

Definition (4-2-14) : (Convex domain)
A domain $U \subset IR^n$ is convex if for any two points x and y of U, the straight line segment $[x, y]$ joining x to y lies entirely in U.
Theorem.(4-2-15) :
If $U \subset IR^3$ is convex, and if is a vector field on U, then $\vec{F}$ is the gradient of a function f defined on U if and only if curl$\vec{F}$ = 0
Proof:
The proof is very similar to the proof of Theorem (4-2-10). First we need to find a candidate for a function f, and there is again an "obvious" choice. Choose a point x_0 U, and set

$$f(x) = \int_{\gamma(x)} W_{\vec{F}} \ , \qquad (4 - 88)$$

where this time γ(x) is specifically the straight line joining xn to x. Note that this is because U is convex; if U were a pond with an island, the straight line might go through the island (where the vector field is undefined).
Now we need to show that $\vec{\nabla} f = \vec{F}$. Again,

$$\vec{\nabla} f\left(x\right).\vec{v} = \lim_{h \to 0} \frac{1}{h}\left(f\left(x + h\vec{v}\right) - f\left(x\right)\right), \qquad (4\text{-}89)$$

and f(x+h$\vec{v}$) –f(x) is the work of $\vec{F}$ along the path that goes straight from x to x_0 and then straight on to x + h$\vec{v}$. We wish to replace this by the path that goes straight from x to x+h$\vec{v}$. We don't have path independence to allow this, but we can do it by Stokes's theorem. Indeed, the three oriented segments
[x, x_0], [x_0, x + $h\vec{v}$], and [x + h$\vec{v}$, x] together bound a triangle T, so the work of $\vec{F}$ around the triangle is equal to zero

$$\int_{\partial T} W_{\vec{F}} = \int_{T} dW_{\vec{F}} = \int_{T} \Phi_{curl\,\vec{F}} = 0 \qquad (4\text{-}90)$$

We can now rewrite equation(4-90)

$$\vec{\nabla} f(x).\vec{v} = \lim_{h\to 0}\frac{1}{h}\left(\underbrace{\int_{[x,x_0]} W_{\vec{F}}}_{-f(x)} + \underbrace{\int_{[x_0,Xh\vec{v}]} W_{\vec{F}}}_{f(x+h\vec{v})}\right) = \lim_{h\to 0}\frac{1}{h}\int_{\left[x,x+h\vec{v}\right]} W_{\vec{F}} \qquad (4\text{-}91)$$

The proof finishes as above (Equation(4-78)).

Example (4-2-16): (Finding the potential of a vector field)

Let us carry out the computation in the proof above in one specific case. Consider the vector field

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y^2 / 2y^z \\ x(y+z) \\ xy \end{bmatrix} \qquad (4\text{-}92)$$

Whose curl is indeed 0:

$$\vec{\nabla}\times\vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} y^2 / 2y^z \\ x(y+z) \\ xy \end{bmatrix} = \begin{bmatrix} x-x \\ -y=y \\ y+z-(y+z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4\text{-}93)$$

Since $\vec{F}$ is defined on all of IR^3, which is convex, Theorem (4-2-14) asserts that $\vec{F} = \vec{\nabla}f$, where

$$f(a) = \int_{\gamma a} W_{\vec{F},} \quad \text{for } \gamma_a(t) = ta, 0 \le t \le 1, \qquad (4\text{-}94)$$

i.e., γ_a, is a parametrization of the segment joining 0 to a. If we set a = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}, this\ ieads\ to$

$$f\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \int_0^1 \begin{bmatrix} (tb)^2 / 2 + t^2 bc \\ (tb+tc)ta \\ t^2 ab \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} dt \qquad (4\text{-}95)$$

$$= \left| \int_{x_i}^{x_{i+1}} f'(x)dx - \overbrace{f'(x)}^{A}\overbrace{x_{i-1} - x_i}^{b-a} \right| = \left| \int_{x_i}^{x_{i+1}} \left(f'(x) - \overbrace{f'(x_i)}^{A} \right) dx \right|$$

This means that

$$f\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{xy^2}{2}+xyz\ , \qquad (4\text{-}96)$$

and it is easy to chech that $\vec{\nabla}f = \vec{F}$

References

[1] John Hamal ,Vector calculus, liner Algebra, and Differentiable forms, cornell university,1991

[2] Henri cartan , Differential forms, Hermann Paris; H.Miffin Co.Boston ,1970

[3] Jean Dieudonne' Infintesimal calculus,Houghton miffin company Boston,1971

[4] Stanley Grossman, Multivariable calculus, linear Algebra and Differential Equation, Harcourt Brace College Publishers for worth 1995

[5] Lyrm Loomis and Shlomo sternber, Advanced calculus, Addison-wesly publishing company, Reading MA, 1968

[6] Jerrola Marsden and Anthony Tromba,Vector calculus, fourth edition,W.H. freemen and company ,New York1 .1996

[7] Michael Spivak, calculus , second edition, publish or perish, Inc, Wilmington,DE,1980

[8] Michael Spivak, calculus on manifolds, W.A. Benjamin ,Inc, New York, 1965

[9] John Milnor , Morse Theory Princeton university press,1963

[10] Jean –Luc Dorier , ed, L'enseignement de L'Algebre Lineaire en Question la pensea sauvage Edition, 1997

[11] Frank Morgan ,Riemannian Geometry, ABeginner's , second edition, AK, Ltd, wellesly,MA,1998

[12] John Stillwell , Mathematices and Its History, Springer – Verlag, New York, 1989

[13] Manfredop – do carmo, Differential Geometry of curves and surface , prentice Hall, Inc ,1976