Chapter (3)

Orientation , Integration and Exterior Derivative

Section (3.1) : Orientation and Integration of Form Fields

Now we will discuss compatible orientations of parametrized manifolds
We have discussed how to integrate k-form fields over k-dimensional parametrized domains. We have seen that where integrands like $|d^K u|$ are concerned, the integral does not depend on the parametrization. Is this still true for form fields? The answer is "not quite": for two parametrizations to give the same result, they have to induce the same orientation on the image. Let us see this by trying to prove the (false) statement that the integral does not depend on the parametrization, and discovering where we go wrong. Let $M \subset IR^n$ be a k-dimensional manifold U, V be subsets of IR^K , and $\gamma_1 : V \to M$, $\gamma_2 : V \to M$ be two parametrizations, each inducing its own orientation.
Let W be a k-form on a neighborhood of M.

Define as in Theorem (1-1-12) the "change of parameters" map

$$\Phi = \gamma_2^{(-1)} o \gamma_1 : U^{ok} \to V^{ok} .$$

Then Definition (2-2-1) (integrating a k-form field over a parametrized domain)
and the change of variables formula, give

$$\int_{\gamma_1(V)} \varphi = \int_V \varphi\left(P_{\gamma_2(V)}\left(\vec{D}_1 \gamma_1(V), \ldots, \vec{D}_k\, \gamma_2(V)\right)\right) |d^K V| \qquad (3\text{-}1)$$

$$(d\,(f dx_i \Lambda \ldots \Lambda dx_{ik}) = d\,(df\ \Lambda dx_i \Lambda \ldots \Lambda dx_{ik}) =$$

We want to express everything in terms of γ_1 . There is no trouble with the point $(\gamma_2 o\, \Phi(u))(u) = \gamma_1(u)$ where the parallelogram is anchored, but the vectors which span it are more troublesome, and will require the following lemma.

Lemma (3-1-1):

If $\vec{w}_1, \ldots, \vec{w}_k$ are any k vectors in IR^K, then

$$\varphi\left(P_{\gamma_2(V)}\left(\vec{D}_1\gamma_1(V),\ldots,\vec{D}_k\,\gamma_2(V)\right)\right)\det\left[\vec{w}_1\ldots\vec{w}_k\right]$$

$$=\varphi\left(P_{\gamma_2(V)}\left(D\gamma_1(V)\vec{w}_1,\ldots,D\gamma_2(V)\vec{w}_k\right)\right) \qquad (3\text{-}2)$$

Proof:

Since the vectors $[D\gamma_1(V)\vec{w}_1,\ldots,D\gamma_2(V)\vec{w}_k]$ $\vec{w}_k$ in the second line of Equation (3-2) depend on $\vec{w}_1\ldots\vec{w}_k$, we can consider the entire right-hand side of that line as a function of v and $\vec{w}_1\ldots\vec{w}_k$, multilinear and antisymmetric with respect to the $\vec{w}$. The latter are k vectors in IR^k, so the right-hand side can be written as a multiple of the determinant: a(v) $\det\left[\vec{w}_1\ldots\vec{w}_k\right]$ for some function a(v).

To find a(v), we set $\vec{w}_1\ldots\vec{w}_k=\vec{e}_1\ldots\vec{e}_K$ .Since $\left[D\gamma_2(V)\right]\vec{e}_1=\vec{D}_1\gamma_2(V)$, Substituting $\vec{e}_1 \ldots \vec{e}_K$ for $\vec{w}_1 \ldots \vec{w}_k$ in second line of Equation (3-2) gives

$$\varphi\left(P_{\gamma_2(V)}\left(D\gamma_2(V)\vec{e}_1,\ldots,D\gamma_2(V)\vec{e}_k\right)\right)=\varphi\left(P_{\gamma_2(V)}\left(\vec{D}_1\gamma_2(V),\ldots,\vec{D}_k\,\gamma_2(V)\right)\right)$$

$$=a(v)\det\det\left[\vec{w}_1\ldots\vec{w}_k\right]=a(v) \qquad (3\text{-}3)$$

So

$$\varphi\left(P_{\gamma_2(V)}\left(D\gamma_2(V)\vec{w}_1,\ldots,D\gamma_2(V)\vec{w}_k\right)\right)=a(v)\det\left[\vec{w}_1\ldots\vec{w}_k\right]=$$
$$\varphi\left(P_{\gamma_2(V)}\left(\vec{D}_1\gamma_2(V),\ldots,\vec{D}_k\,\gamma_2(V)\right)\right)\det\left[\vec{w}_1\ldots\vec{w}_k\right] \qquad (3\text{-}4)$$

Now we write down the function being integrated on the second line of Equation (3-1) , except that we take $\det[D\Phi(u)]$ out of absolute value signs, so that we will be able to apply Lemma (3-1-1) to go from the second to the third line :

$$\varphi\left(P_{\gamma_2 o\Phi(u)}\left(\vec{D}_1\gamma_1(\Phi(u)),\ldots,\vec{D}_k\,\gamma_2(\Phi(u))\right)\right)\det\left[D\Phi(u)\right]$$

$$\varphi\left(P_{\gamma_2 o\Phi(u)}\left(\vec{D}_1\gamma_1(\Phi(u)),\ldots,\vec{D}_k\,\gamma_2(\Phi(u))\right)\right)\det\det\left[\underbrace{\vec{D}_1(\Phi(u)),\ldots,\vec{D}_k(\Phi(u))}_{\vec{w}_1\ldots\vec{w}_k}\right]$$

$$=\varphi\left(P_{\gamma_2 o\Phi(u)}\left(\left[D\gamma_2\underbrace{\Phi(u)}_{v}\right]\vec{D}_1\Phi(u),\ldots,\left[D\gamma_2\Phi(u)\right]\underbrace{\vec{D}_K\,\Phi(u)}_{\vec{w}_k}\right)\right)$$

$$=\varphi\left(P_{\gamma_1(u)}\left(\vec{D}_1\gamma_1(u),\ldots,\vec{D}_k\,\gamma_1(u)\right)\right) \qquad (3\text{-}5)$$

To pass from the second to the third line of Equation (3-5) we use Lemma (3-1-1), setting $\vec{w}_j = D_j\Phi(u)$ and $v=\Phi(u)$. (We have marked some of these correspondences with underbraces.) We use the chain rule to go from the third to the fourth line.

Now we come to the key point. The second line of Equation (3-1) has $\det[D\Phi(u)]$, while the first line of Equation (3-5) has $\det[D\Phi(u)]$. Therefore the integral

$$\int_U\varphi\left(P_{\gamma_1(u)}\left(\vec{D}_1\gamma_1(u),\ldots,\vec{D}_k\,\gamma_1(u)\right)\right)\left|d^K u\right| \qquad (3\text{-}6)$$

obtained using γ_1 and the integral

$$\int_V\varphi\left(P_{\gamma_2(V)}\left(\vec{D}_1\gamma_1(V),\ldots,\vec{D}_k\,\gamma_2(V)\right)\right)\left|d^K V\right| \qquad (3\text{-}7)$$

obtained using γ_2 will be the same only if $\left|\det\left[D\Phi(u)\right]\right| = \det[D\Phi(u)]$. That is, they will be identical if $\det[D\Phi]>0$ for all $u \in U$, and otherwise probably not. $If\ det[D\Phi] < 0$ for all $u \in U$ then

$$\int_{\gamma_1(U)} \varphi = \int_{\gamma_2(V)} \varphi \qquad (3\text{-}8)$$

If $\det[D\Phi(u)]$is positive in some regions of U and negative in others, then the integrals are probably unrelated.

If $\det\left[D\Phi(u)\right] > 0$, we say that the two parametrizations of M induce compatible orientations of M.

Definition (3-1-2):(Compatible orientation)

Let γ_1 and γ_2 be two parametrizations, with the "change of parameters" map $\Phi = \gamma_2^{(-1)} o \gamma_1$. The two parametrizations γ_1 and γ_2 are compatible if $\det[D\Phi] > 0$.This leads to the following theorem.

Theorem (3-1-3):(Integral independent of compatible parametrizations)

Let $M \subset \mathrm{IR}^n$ be a k-dimensional oriented manifold, U, V open subsets of IR^k, and $\gamma_1 : \mathrm{U} \to I\mathrm{R}^n$ and $\gamma_2 : \mathrm{U} \to I\mathrm{R}^n$ be two parametrizations of *M* that induce compatible orientations of *M*. Then for any k-form φ defined on a neighborhood of *M*,

$$\int_{\gamma_1(U)} \varphi = \int_{\gamma_2(V)} \varphi \qquad (3\text{-}9)$$

Now we will study orientation of manifolds.

When using a parametrization to integrate a k-form field over an oriented domain, clearly we must take into account the orientation induced by the parametrization. We would like to be able to relate this to some characteristic of the domain of integration itself. What kind of structure can we bestow on an oriented curve, surface, or higher-dimensional manifold that would enable us to decide how to check whether a parametrization is appropriate?

There are two ways to approach the somewhat challenging topic of orientation. One is the ad hoc approach: to limit the discussion to points, curves, surfaces, and three-dimensional objects. This has the advantage of being more concrete, and the disadvantage that the various definitions appear to have nothing to do with each other. The other is the unified approach: to discuss orientation of k-dimensional manifolds, showing how orientation of points, curves, surfaces, etc., are embodiments of a general definition. This has the disadvantage of being abstract. We will present the ad hoc approach first, followed by the unified theory.

Now we will study the ad hoc world, orienting the objects.

We will treat orientations of the objects first, followed by orientation-preserving parametrizations.

Definition (3-1-4):(Orientation of a point)

An orientation of a point is a choice of ±: an oriented point is "plus the point" or "minus the point." It is easy to understand orientations of curves (in any IR^n): give a direction to go along the curve. The following definition is a more formal way of saying the same thing; it is illustrated in Figure (3-1) . By "unit tangent vector field" we mean a field of vectors tangent to the curve and of length 1.

Figure (3-1)**Unit Tangent vector field**

Definition (3-1-5):(Orientation of a curve in IR^n)

An orientation of acurve $C \subset IR^3$ is the choice of a unit tangent vector field f that depends continuously on x.

We orient a surface $C \subset IR^3$ by choosing a normal vector at every point, as shown in Figure (3-2) and defined more formally below.

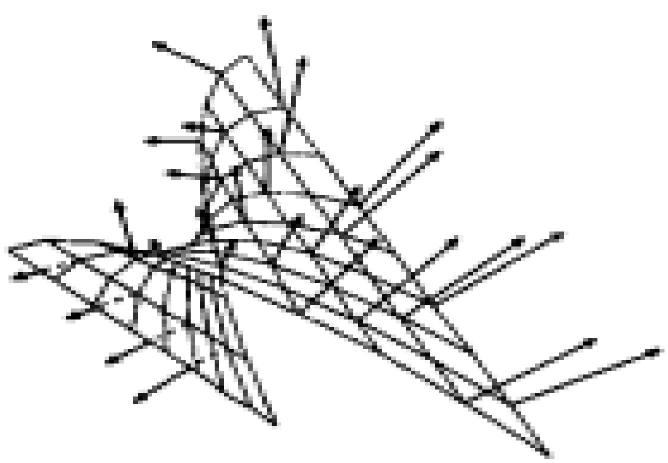


Figure (3-2)

Definition (3-1-6):(Orientation of a surface in IR^3)

To orient a surface in IR^3, choose a unit vector field $\vec{N}$ orthogonal to the surface. At each point x there are two vectors $\vec{N}$(x); choose one at each point, so that the vector field $\vec{N}$ depends continuously on the point.

This is possible for an orientable surface like a sphere or a torus: choose either the outer-pointing normal or the inward-pointing normal. But it is impossible on a Moebius strip. This definition does not extend at all easily to a surface in IR^4: at every point there is a whole normal plane, and choosing a normal vector field does not provide an orientation.

Definition(3-1-7):(Orientation of open subsets of IR^3)

One orientation of an open subset X of IR^3 is given by det; the opposite orientation is given by - det. The standard orientation is by det.

We will use orientations to say whether three vectors $\vec{v}_1,\vec{v}_2,\vec{v}_3$ form a direct basis of IR^3; with the standard orientation, $\vec{v}_1,\vec{v}_2,\vec{v}_3$ being direct means that $\det\left[\vec{v}_1,\vec{v}_2,\vec{v}_3\right] > 0$. If we have drawn $\vec{e}_2,\vec{e}_2,\vec{e}_3$ in the standard way, so that they fit the right hand, then $\vec{v}_1,\vec{v}_2,\vec{v}_3$ will be direct precisely if those vectors also satisfy the right-hand rule.

Now we will discuss the unified approach, orienting the objects

All three notions of orientation are reasonably intuitive, but they do not appear to have anything in common. Signs of points, directions on curves, normals to surfaces, right hands: how can we make all four be examples of a single construction?

We will see that orienting manifolds means orienting their tangent spaces, so before orienting manifolds we need to see how to orient vector spaces. We saw in (Corollary (2-1-13)) that for any k-dimensional vector space E, the space $A^k(E)$ of k-forms in h' has dimension one. Now we will use this space to show that the different definitions of orientation we gave at the beginning of this section are all special cases of a general definition.

Definition(3-1-8):(Orienting the space $A^k(E)$)

The one-dimensional space $A^k(E)$ is oriented by choosing a nonzero element w of $A^k(E)$. An element aw, with $a > 0$, gives the same orientation as w, while bw, with $b < 0$, gives the opposite orientation.

Definition(3-1-9):(Orienting a finite-dimensional vector)

An orientation of a k-dimensional vector space E is specified by a nonzero element of $A^k(E)$. Two nonzero elements specify the same orientation if one is a multiple of the other by a positive number.

Definition(3-1-9) :makes it clear that every finite-dimensional vector space (in particular every subspace of IR^n) has two orientations.

Now we will express Equivalence of the ad hoc and the unified approaches for subspaces of IR^3.

Let $E \subset IR^n$ be a line, oriented in the ad hoc sense by a nonzero vector $\vec{v} \in E$, and oriented in the unified sense by a nonzero element w $A^1(E)$. Then these two orientations coincide precisely $if\ w(\vec{v}) > 0$.

For instance, if $E \subset IR^2$ is the line of equation $x + y = 0$, then the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ defines an ad hoc orientation, whereas dx provides a unified orientation. They do coincide: $dx \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 > 0$. The element of $A^n(E)$ corresponding to dy also defines an orientation of E, in fact the opposite orientation. Why does $dx + dy$ not define an orientation of this line?

Now suppose that $E \subset IR^3$ is a plane, oriented "ad hoc" by a normal $\vec{n}$ and oriented "unified" by $w \subset A^2(E)$. Then the orientations coincide if for any two vectors $\vec{v}_1, \vec{v}_2 \subset E$, the number $w(\vec{v}_1, \vec{v}_2)$ is a positive multiple of $\det\left[\vec{n}, \vec{v}_1, \vec{v}_2\right]$ For instance, suppose E is the plane of equation $x + y + z = 0$, oriented "ad hoc" by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and oriented "unified" by $dx \wedge dy$. Any two vectors in E can be written

$$\begin{bmatrix} a \\ b \\ -a-b \end{bmatrix}, \begin{bmatrix} c \\ d \\ -c-d \end{bmatrix} \qquad (3\text{-}10)$$

So we have

Unified approach :
$$dx \wedge dy \left(\begin{bmatrix} a \\ b \\ -a-b \end{bmatrix}, \begin{bmatrix} c \\ d \\ -c-d \end{bmatrix} \right) = ad - bc \qquad (3\text{-}11)$$

Ad hoc approch :
$$\det \begin{bmatrix} 1 & a & c \\ 1 & b & d \\ 1 & -a-b & -c-d \end{bmatrix} - 3(ad - bc) \qquad (3\text{-}12)$$

These orientations coincide, since 3 > 0. What if we had chosen $dy \wedge dz$ or

$dx \wedge dz$ as our nonzero element of $A^2(E)$?

We see that in most cases the choice of orientation is arbitrary: the choice of one nonzero element of $A^k(E)$ will give one orientation, while the choice of another may well give the opposite orientation. But IR^n itself and $\{0\}$ (the zero subspace of IR^n), are exceptions; these two trivial subspaces of P." do have a standard orientation. For $\{\vec{0}\}$, we have $A^{\circ}(\{\vec{0}\}) = IR$, so one orientation is specified by +1, the other by -1; the positive orientation is standard. The trivial subspace IR^n is oriented by ω = det; and det > 0 is standard.

Now we will illustrate Orienting manifolds. Most often we will be integrating a form over a curve, surface, or higher dimensional manifold, not simply over a line, plane, or IR^3. A k-manifold is oriented by orienting T_xM, the tangent space to the manifold at x, for each $x \in M$: we orient the manifold M by choosing a nonzero element of $A^k(T_xM)$.

Definition(3-1-10):(Orientation of a k-dimensional manifold)

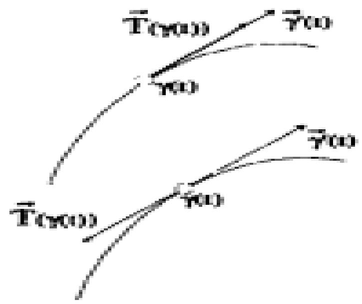
Art orientation of a k-dimensional manifold $M \subset IR'$ is an orientation of the tangent space T_xM at every point $x \in M$, so that the orientation varies continuously with x. To orient the tangent space, we choose a nonzero element of $A^k(T_XM)$.

Once again, we use a linearization (the tangent space) in order to deal with nonlinear objects (curves, surfaces, and higher-dimensional manifolds).

What does it mean to say that the "orientation varies continuously with x"? This is best understood by considering a case where you cannot choose such an orientation, a Moebius strip. If you imagine yourself walking along the surface of a Moebius strip, planting a forest of normal vectors, one at each point, all pointing "up" (in the direction of your head), then when you get back to where you started there will be vectors arbitrarily close to each other, pointing in opposite directions.

Now we will discuss The ad hoc world: when does a parametrization preserve orientation?

We can now define what it means for a parametrization to preserve orientation. For a curve, this means that the parameter increases in the specified direction: a parametrization $\gamma:[a,b] \to C$ preserves orientation if C is oriented from $\gamma(a)$ to $\gamma(b)$. The following definition spells this out; it is illustrated by Figure (3-3) .



Figure(3-3)

Definition(3-1-11):(Orientation-preserving parametrization of a curve) Let $C \subset IR$ be a curve oriented by the choice of unit tangent vector field T. Then the parametrization $\gamma : [a,b] \to C$ is orientation preserving if at every $t \in (a, b)$, we have

$$\vec{\gamma}'(t).\vec{T}\left(\gamma(t)\right) > 0 \qquad (3\text{-}13)$$

Equation (3-13) says that the velocity vector of the parametrization points in the same direction as the vector orienting the curve. Remember that

$$\vec{v}_1.\vec{v}_2 = (\cos\theta)\left|\vec{v}_1\right|\left|\vec{v}_2\right| \qquad (3\text{-}14)$$

where θ is the angle between the two vectors. So the angle between $\vec{\gamma}^1$(t) and $\vec{T}(\gamma(t))$ is less than 90°. Since the angle must be either 0 or 180°, it is 0. It is harder to understand what it means for a parametrization of an oriented surface to preserve orientation. In Definition (2-1-12) , $\vec{D}_1\gamma\ (u)$ and $\vec{D}_2\gamma\ (u)$ are two vectors tangent to the surface at $\gamma\ (u)$.

Definition(3-1-12):(Orientation-preserving parametrization of a surface)

Let $S \subset IR^3$ be a surface oriented by a choice of normal vector field $\vec{N}$. Let $U \subset IR^2$ be open and γ : $U \to S$ be a parametrization. Then γ is orientation preserving If at every u ∈ U,

$$\det\left[\vec{N}\left(\gamma(u)\right), \vec{D}_1\gamma(u), \vec{D}_2\gamma(u)\right] > 0 \qquad (3\text{-}15)$$

Definition(3-1-13):

An open subset U of IR^3 carries a standard orientation, defined by the determinant. If V is another open subset of IR^3, and $\gamma : V \to U$ is a parametrization (i.e., a change of variables), then y is orientation preserving if $\det[D\gamma(v)] > 0$ for all $v \in V$.

Now we will illusterate the unified approach: when does a parametrization preserve orientation**?**

First let us define what it means for a linear transformation to be orientation preserving.

Definition(3-1-14):(Orientation-preserving linear transformation)

If $V \subset IR^n$ is a k-dimensional subspace oriented by $w \in A^k(V)$ and $T : IR^k \to V$ is a linear transformation, T is orientation-preserving if

$$\omega\left(T\left(\vec{e}_1\right),\ldots,T\left(\vec{e}_k\right)\right) > 0 \qquad (3\text{-}16)$$

It is orientation reversing if

$$\omega\left(T\left(\vec{e}_1\right),\ldots,T\left(\vec{e}_k\right)\right) < 0 \qquad (3\text{-}17)$$

Note that for a linear transformation to preserve orientation, the domain and the range must have the same dimension, and they must be oriented. As usual, faced with a nonlinear problem, we linearize it: a (nonlinear) parametrization of a manifold is orientation preserving if the derivative of the parametrization is orientation preserving .

Definition(3-1-15):(Orientation-preserving parametrization of a manifold)

Let M be an oriented k-dimensional manifold, $U \subset IR^n$ be an open set, and $\gamma : U \to M$ be a parametrization. Then γ is orientation preserving if $[D\gamma(u)] : IR^k \to T_{\gamma(u)}M$ is orientation preserving for every $u \in U$, i.e., if

$$\omega\left(\left[D\gamma(u)\right]\left(\vec{e}_1\right),\ldots,\left[D\gamma(u)\right]\left(\vec{e}_K\right)\right) = \omega\left(\vec{D}_1\gamma(u),\ldots,\vec{D}_K\gamma(u)\right) > 0$$

Example(3-1-16): (Orientation-preserving parametrization)

Consider the surface S in C^3 parametrized by

$$z \to \begin{pmatrix} z \\ z^2 \\ z^3 \end{pmatrix}, |z| < 1 \qquad (3\text{-}18)$$

We will denote points in C^3 by $\begin{pmatrix} z \\ z^2 \\ z^3 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \end{pmatrix}$

Orient S, using $\text{w} = dx_1 \Lambda dy_1$

If we parametrize the surface by

$$\gamma : \begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} \\ \\ \\ \\ x_1 = r\cos\theta \\ y_1 = r\sin\theta \\ x_2 = r^2\cos 2\theta \\ y_2 = r^2\sin 2\theta \\ x_3 = r^3\cos 3\theta \\ y_3 = r^3\sin 3\theta \end{pmatrix} \qquad (3\text{-}19)$$

does that parametrization preserve orientation? It does, since

$$dx_1 \Lambda dy_1 \left(D_1\gamma(u), D_2\gamma(u)\right) = dx_1 \Lambda dy_1 \begin{bmatrix} & \\ \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \qquad (3\text{-}20)$$

$$= \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0.$$

Now we will study Compatibility of orientation-preserving parametrizations Theorem (3-1-3) said the result of integrating a k-form over an oriented manifold does not depend on the choice of parametrization, as long as the parametrizations induce compatible orientations. Now we show that the integral is independent of parametrization if the parametrization is orientation preserving. Most of the work was done in proving Theorem

(3-1-3). The only thing we need to show is that two orientation-preserving parametrizations define compatible orientations.

Theorem(3-1-17):(Orientation-preserving parametrizations define compatible orientations)

If M is an oriented k-manifold, U_1 and U_2 are open subsets of iR^n, and $\gamma_1 : U_1 \to M$, $\gamma_2 : U_2 \to M$ are orientation preserving parametrizations, then they define compatible orientations.

Proof:

Consider two points $u_1 \in U_1, u_2 \in U_2$ such that $\gamma_1(u_1) = \gamma_2(u_2) = x \in M$ The derivatives then give us maps

$$\mathrm{IR}^k \xrightarrow{D_1\gamma(u_1)} T_x M \xrightarrow{D_2\gamma(u_2)} \mathrm{IR}^k \qquad (3\text{-}21)$$

where both derivatives are one to one linear transformations. Moreover, we have $\omega(x) \neq 0$ in the one-dimensional vector space $A^k(T_X M)$. What we must show is that if

$$\omega(x)\left(\vec{D}_1\gamma_1(u_1),\ldots,\vec{D}_k\,\gamma_1(u_1)\right) > 0 \ \ and \ \ \omega(x)\left(\vec{D}_1\gamma_2(u_2),\ldots,\vec{D}_k\,\gamma_2(u_2)\right) > 0$$

$$then\ \det\left(\left[D_2\gamma(u_2)\right]\right)^{-1}\left[D_1\gamma(u_1)\right] > 0$$

Note that

$$\omega(x)\left(\left[D\gamma_1(u_1)\right]\left(\vec{v}_1\right),\ldots,\left[D\gamma_1(u_1)\right]\left(\vec{v}_k\right)\right) = \alpha \det\left[\vec{v}_1,\ldots,\vec{v}_k\right]$$

$$\omega(x)\left(\left[D\gamma_2(u_2)\right]\left(\vec{w}_1\right),\ldots,\left[D\gamma_2(u_2)\right]\left(\vec{w}_k\right)\right) = \beta \det\left[\vec{w}_1,\ldots,\vec{w}_k\right] \qquad (3\text{-}22)$$

for some positive numbers a and P. Indeed, both left-hand sides are nonzero elements of the one-dimensional vector space $A^k\left(\mathrm{IR}^k\right)$, hence nonzero multiples of the determinant, and they return positive values if evaluated on the standard basis vectors. Now write

$$\alpha = \omega(x)\left(\vec{D}_1\gamma_1(u_1)\left(\vec{v}_1\right),\ldots,\vec{D}_K\,\gamma_1(u_1)\left(\vec{v}_k\right)\right)$$

$$= \omega(x)\left(\left[D\gamma_1(u_1)\right]\vec{e}_1,..,\left[D\gamma_1(u_1)\right]\vec{e}_k\right)$$

$$= \omega(x)\left(\left[D\gamma_2(u_2)\right]\left(\left[D\gamma_2(u_2)\right]\right)^{-1}\left[D\gamma_1(u_1)\right]\vec{e}_1,\ldots,\left[D\gamma_2(u_2)\right]\left(\left[D\gamma_2(u_2)\right]\right)^{-1}\left[D\gamma_1(u_1)\right]\vec{e}_k\right)$$

$$= \beta \det\left(\left(\left[D\gamma_2(u_2)\right]\right)^{-1}\left[D\gamma_1(u_1)\right]\vec{e}_1,\ldots,\left(\left[D\gamma_2(u_2)\right]\right)^{-1}\left[D\gamma_1(u_1)\right]\vec{e}_k\right)$$

$$= \beta \det\left[\left(\left[D\gamma_2(u_2)\right]\right)^{-1}\left[D\gamma_1(u_1)\right]\right]\det\left[\vec{e}_1,\ldots,\vec{e}_k\right] \qquad (3\text{-}23)$$

$$= \beta \det\left(\left(\left[D\gamma_2(u_2)\right]\right)^{-1}\left[D\gamma_1(u_1)\right]\right)$$

Corollary(3-1-18):(Integral independent of orientation-preserving parametrizations)

Let M be an oriented k-manifold, U and V be open subsets of IR^k, and $\gamma_1 : V \to M$, $\gamma_2 : V \to M$ be orientation-preserving parametrizations of M. Then for any k-form φ defined on a neighborhood of M, we have

$$\int_{\gamma_{1(U)}} \varphi = \int_{\gamma_{2(V)}} \varphi \qquad (3\text{-}24)$$

Now we will discuss Integrating form fields over oriented manifolds.

Now we know everything we need to know in order to integrate form fields over oriented manifolds. We saw in Section (1-2) how to integrate form fields over parametrized domains. Corollary (2-1-18) says that we can use the same formula to integrate over oriented manifolds, as long as we use an orientation-preserving parametrizations. This gives the following:

Definition(3-1-19):(Integral of a form field over an oriented manifold) Let M be a

k-dimensional oriented manifold, φ be a k-form field on a neighborhood of M, and $\gamma : U \to M$ be any orientation-preserving parametrization of M.

$$\int_M \varphi = \int_{\gamma(U)} \varphi = \int_U P^0_{\gamma(u)}\left(\vec{D}_1\gamma\ (u\ ),\ldots,\vec{D}_k\ \gamma\ (u\ )\right)\left|d^k u\right|$$

Example(3-1-20):(Integrating a flux form over an oriented surface)

What is the flux of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y \\ -x \\ z \end{bmatrix}$ through the piece of the plane P defined by $x + y + z = 1$ where $x, y, z > 0$, and which is oriented by the normal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

This surface is the graph of $z = 1 - x - y$, so that

$$\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ y \\ 1 - z - y \end{pmatrix} \qquad (3\text{-}25)$$

is a parametrization, if x and y are in the triangle T ⊂ IR^2 given by $x, y \geq 0$, $x + y \leq 1$. Moreover, this parametrization preserves orientation (see Definition(3-1-12), since $\det\left[\vec{N}\left(\gamma(u)\vec{D}_1\gamma(u)\vec{D}_2\gamma(u)\right)\right]$ is

$$\det\left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right] = 1 > 0 \qquad (3\text{-}26)$$

By Definition (2-2-11) , the flux is

$$\int_{\rho} \Phi_{\begin{bmatrix} y \\ -x \\ z \end{bmatrix}} = \int_{T} \det\left[\overbrace{\begin{bmatrix} y \\ -x \\ 1 - x - y \end{bmatrix}}^{\vec{F}\gamma(\vec{u})}, \overset{\vec{D}_1\gamma}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \overset{\vec{D}_2\gamma}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}\right] |dx\ dy|$$

$$= \int_{T} (1 - 2x)|dx\ dy| = \int_0^1 \left(\int_0^y (1 - 2x)\,dx\right) dy \qquad (3\text{-}27)$$

$$=\int_0^1 \left[x - x^2\right]_0^y dy = \int_0^1 \left(y - y^2\right) dy = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{6}$$

Example (3-1-21) :

Consider again the surface S in C^3 of Example (3-1-16) what is

$$\int_S dx_1 \Lambda dy_1 + dx_2 \Lambda dy_2 + dx_3 \Lambda dy_3 \qquad (3\text{-}28)$$

As in Example (3-1-16) , parametrize the surface by

$$\gamma : \begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x_1 = r\cos\theta \\ y_1 = r\sin\theta \\ x_2 = r^2\cos 2\theta \\ y_2 = r^2\sin 2\theta \\ x_3 = r^3\cos 3\theta \\ y_3 = r^3\sin 3\theta \end{pmatrix} \qquad (3\text{-}29)$$

which we know from that example preserves orientation. Then

$$\left(dx_1 \Lambda dy_1 + dx_2 \Lambda dy_2 + dx_3 \Lambda dy_3\right)\left(P^0_{\gamma\begin{pmatrix} r \\ \theta \end{pmatrix}}\left(\vec{D}_1\gamma\begin{pmatrix} r \\ \theta \end{pmatrix}, \dots, \vec{D}_2\gamma\begin{pmatrix} r \\ \theta \end{pmatrix}\right)\right)$$

$$\det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} + \det\begin{bmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{bmatrix}$$

$$+\det\begin{bmatrix} 3r^2\cos 3\theta & -3r^3\sin 3\theta \\ 2r\sin 3\theta & 3r^3\cos 3\theta \end{bmatrix}$$

$$r + 4r^3 + 9r^5 \qquad (3\text{-}30)$$

Finally . we find for our intergral

$$2\pi\int_0^1\left(r+4r^3+9r^5\right) \qquad (3\text{-}31)$$

For completeness, we show the case where is a 0-form field:

Example(3-1-22):(Integrating a 0-form over an oriented point)

Let x be an oriented point, and f a function (i.e., a 0-form field) defined in some neighborhood of x. Then

$$\int_{+x} f = +f(x) \quad \text{and} \quad \int_{-x} f = -f(x) \qquad (3\text{-}32)$$

Example(3-1-23): (Integrating over an oriented point).

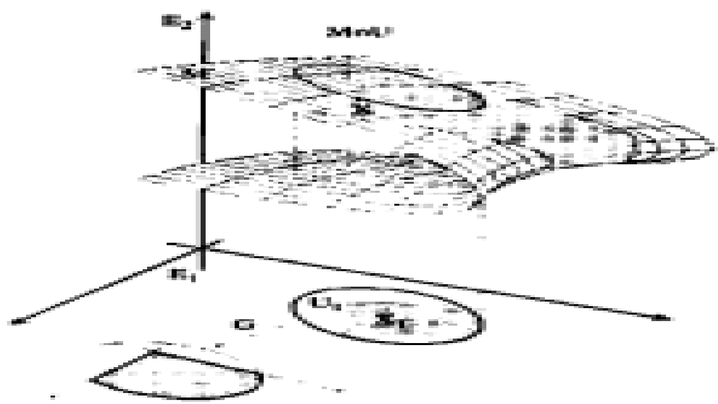
$$\int_{+\{+2\}} x^2 = 4 \quad \text{and} \quad \int_{-\{+2\}} x^2 = -4 \qquad (3\text{-}33)$$

Section (3.2) : Boundary orientation and exterior derivative

In the following we will discuss boundary orientation Stokes's theorem, the generalization of the fundamental theorem of calculus, is all about comparing integrals over manifolds and integrals over their boundaries. Here we will define exactly what a "manifold with boundary" is; we will see moreover that if a "manifold with boundary" is oriented, its boundary carries a natural orientation, called, naturally enough, the boundary orientation You may think of a "piece-with-boundar" of a k-dimensional manifold as a piece one can carve out of the manifold, such that the boundary of the piece is part of it (the piece is thus closed). However, the boundary can't have any arbitrary shape. In many treatments the boundaries are restricted to being smooth. In such a treatment, if the manifold is three-dimensional. the boundary of a piece of the manifold must be a smooth surface; if it is two-dimensional. The boundary must be a smooth curve.

We will be less restrictive, and will allow our boundaries to have corners. There are two reasons for this. First, in many cases, we wish to apply Stokes theorem to things like the region in the sphere where in spherical coordinates. $0 \le \theta \le \frac{\pi}{2}$, and such a region has corners (at the poles). Second. we would like k-parallelograms to be manifolds with boundary, and they most definitely have corners. Fortunately, allowing our boundaries to have corners doesn't make any of the proofs more difficult.

However, we won't allow the boundaries to be just anything: the boundary can't be fractal, like the Koch snowflake we saw in Section (1.2); neither can it contain cusps. (fractals would really cause problems: cusps would he acceptable, but would make our definitions too involved.) You should think that a region of the boundary either is smooth or contains a corner. Being smooth means being a manifold: locally the graph of a function of some variables in terms of others. What do we mean by corner? Roughly (we will be painfully rigorous below) if you should think of the kind of curvilinear "angles" you can get if you drew the (x, y)-plane on a piece of rubber and stretched it, or if your squashed a cube made of foam rubber.
Definition (3-2-1) is illustrated by Figure (3-4)



Figure(3-4)

(illustrates our-defnition of a piece-with boundary of a manifold)

Definition (3-2-1):(Piece-with-boundary of a manifold)

Let $M \to IR^n$ be a k-dimensional manifold. A subset $X \subset M$ will be called a piece with boundary if for every, $x \in X$, there exist

(1) Open subsets $U_1 \subset E_1$ and $U \subset IR^n$ with $x \subset U$ and $: U_1 \to E_2$ a C^l mapping such that m n u is the graph of f.

(2) A diffeomorphism $G = \begin{pmatrix} G_1 \\ \vdots \\ G_K \end{pmatrix} : U_1 \to IR^K$

such that $X \cap U$ is $f\ (X_1)$, where $X_1 \subset U_1$ is the subset where $G_1 \geq 0 ..., G_k \geq 0$.

Example(3-2-2):

A k-parallelogram $P_x^0\left(\vec{v}_1,\ldots,\vec{v}_k\right)$ in IR^n is a piece-with-boundary of an oriented k-dimensional submanifold of IR^n when the vectors $\vec{v}_1,\ldots,\vec{v}_k$ are linearly independent. Indeed, if $M \subset \mathrm{IR}^n$ is the set parametrized by

$$\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \to x + t_1\vec{v}_1 + t_k\vec{v}_k \qquad (3\text{-}34)$$

then M is a k-dimensional manifold in IR^n. It is the translation by x of the subspace spanned by $\vec{v}_1,\ldots,\vec{v}_k$ (it is not itself a subspace because it doesn't contain the origin). For every a ∈ M, the tangent space T_aM is the space spanned by $\vec{v}_1,\ldots,\vec{v}_k$. The manifold M is oriented by the choice of a nonzero element $w \in A^k(T,M)$, and w gives the standard orientation if

$$\omega\left(\vec{v}_1,\ldots,\vec{v}_k\right) > 0 \qquad (3\text{-}35)$$

The k-parallelogram $P_x^0\left(\vec{v}_1,\ldots,\vec{v}_k\right)$ is a piece-with-boundary of *M*, and thus it carries the orientation of M.

Definition (3-2-3):(Boundary of a piece-with-boundary of a manifold)

If X is a piece-with-boundary of a manifold M, its boundary ∂X is the set of points where at least one of the $G_i = 0$; the smooth boundary is the set where exactly one of the G, vanishes.

Remark(3-2-4):

We can think of a piece-with-boundary of a k-dimensional manifold as composed of strata of various dimensions: the interior of the piece and the various strata of the boundary, just as a cube is stratified into its interior and its two-dimensional faces, one-dimensional edges, and 0-dimensional vertices. When integrating a k-form over a piece-with-boundary of a k-dimensional manifold, we can disregard the boundary; similarly, when integrating a $(k-1)$- form over the boundary, we can ignore strata of dimension less than $k-1$. More precisely, the m-dimensional stratum of the boundary is the set where exactly $k - m$ of the G_i of Definitions (3-2-1) and (3-2-3) vanish, so the inside of the piece is the k-dimensional stratum,

the smooth boundary is the (k-1)-dimensional stratum, etc. The m dimensional stratum is an m-dimensional manifold in IR^n, hence has m' dimensional volume 0 for any m' > m it can be ignored when integrating m' -forms.
Now we will study Boundary orientation, the ad hoc world
The faces of a cube are oriented by the outward-pointing normal, but the other strata of the boundary carry no distinguished orientation at all: there is no particularly natural way to draw an arrow on the edges. More generally, we will only be able to orient the smooth boundary of a piece-with-boundary. The oriented boundary of a piece-with-boundary of an oriented curve is simply its endpoint minus its beginning point.

Definition(3-2-5):

Let C be a curve oriented by the unit tangent vector $\vec{T}$, and let $P \subset C$ be a piece-with-boundary of C. Then the oriented boundary of P consists of the two endpoints of P, taken with sign +1 if the tangent vector points out of P at that point, and with sign -1 if it points in.
If the piece-with-boundary consists of several such P;, its oriented boundary is the sum of all the endpoints, each taken with the appropriate sign.

Definition(3-2-6):(Oriented boundary of a piece-with-boundary of IR^2)

If $U \subset IR^2$ is a two-dimensional piece-with-boundary, then its boundary is a union of smooth curves C_i. We orient all the C_i so that if you walk along them in that direction, U will be to your left, as shown in Figure (3-5).

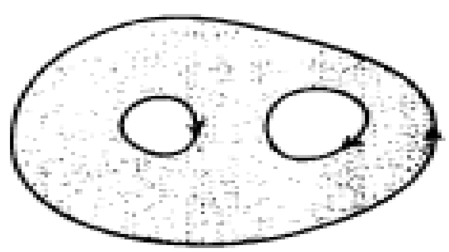


Figure (3-5)**The boundary of the shaded region of IR^2**

When IR^2 is given its standard orientation $by + \det$, Definition (3-2-6) says that when you walk on the curves, your head is pointing in the direction of the z-axis. With this definition, the boundary of the unit disk $\{x\,2 + y\,2 \leq 1\}$ is the unit circle oriented counterclockwise.

For a surface in IR^3 oriented by a unit normal, the normal vector field tells you on which side of the surface to walk. Let $S \subset IR^3$ be a surface oriented by a normal vector field $\vec{N}$, and let U be a piece-with-boundary of S, bounded by some union of curves C_i. An obvious example is the upper hemisphere bounded by the equator. If you walk along the boundary so that your head points in the direction of $\vec{N}$, and U is to your left, you are walking in the direction of the boundary orientation. Translating this into mathematically meaningful language gives the following, illustrated by Figure (3-6).

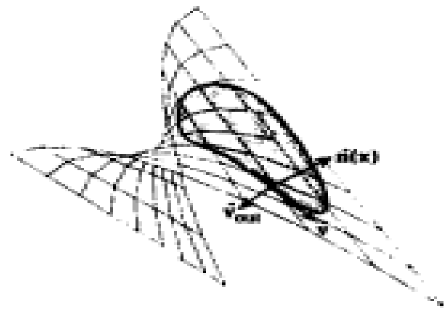


Figure (3-6) **The shaded area is the piece-with-bounndary**

Definition(3-2-7):

Let $S \subset IR^3$ be a surface oriented by a normal vector field $\vec{N}$, and let S_1 be a piece-with-boundary of S, bounded by some union of closedcurves C_i. At a point $x \in C_i$, let $\vec{v}_{out}$ be a vector tangent to S and pointing out of S_1. Then the boundary orientation is defined by the unit vector $\vec{v}$ tangent to C_i , chosen so that

$$\det\left[\vec{N}\left(x\right),\vec{v}_{out},\vec{v}\right] > 0 \qquad (3\text{-}36)$$

Since the system composed of your head, your right arm, and your left arm also satisfies the right-hand rule, this means that to walk in the direction of aS1, you should walk with your head in the direction of $\vec{N}$, and the surface to your left. Finally let's consider the three-dimensional case.

Definition(3-2-8):(Oriented boundary of a piece-with-boundary of IR^3)

Let $U \subset IR^3$ be piece-with-boundary of IR^3 , whose smooth boundary is a union of surfaces S_i. We will suppose that U is given the standard orientation of IR^3 . Then the orientation of the boundary of U (i.e., the orientation of the surfaces) is specified by the outward-pointing normal .

Now we will see that our ad hoc definitions of oriented boundaries of curves, surfaces, and open subsets of IR^3 are all special cases of a general definition.We need first to define outward-pointing vectors.

Let $M \subset IR^n$ be a manifold , $X \subset M$ a piece-with-boundary, and $x \in \partial X$ a point of the smooth boundary of X. At x, the tangent space $T_x(\partial X)$ is a subspace of T_X whose dimension is one less than the dimension of and which subdivides the tangent space into the outward-pointing vectors and the inward-pointing vectors

Definition(3-2-9):(Outward-pointing and inward-pointing vectors)

Let $\vec{v} \in T_x(\partial X_1)$ and write

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \quad \text{with} \quad \vec{v}_1 \in E_1, \vec{v}_2 \in E_2 \tag{3-37}$$

Outward pointing if $[D_g(x_1)]\vec{v}_1 > 0$ and

inward pointing if $[D_g(x_1)]\vec{v}_1 < 0$

Definition(3-2-10):(Oriented boundary of piece-with-boundary of an oriented manifold)

Let M be a k-dimensional manifold oriented by ω, and P be a piece-with-boundary of M. Let x be in ∂P, and $\vec{v}_{out} \in T_x M$ be an outward-pointing vector tangent to M. Then, at x, the boundary ∂P of P is oriented by ω_∂, where

$$\overbrace{\omega_\partial\left(\vec{v}_1, \ldots, \vec{v}_{k-1}\right)}^{oriented\ boundary} = \overbrace{\omega_\partial\left(\vec{v}_{out}, \vec{v}_1, \ldots, \vec{v}_{k-1}\right)}^{oriented\ manifold} \tag{3-38}$$

Example(3-2-11):(Oriented boundary of a piece-with-boundary of an oriented curve)

If C is a curve oriented by ω, and P is apiece-with-boundary of C, then at an endpoint x of P (i.e., a point in ∂P), with an outward-pointing vector $\vec{v}_{out}$, anchored at x, the boundary point x is oriented by the nonzero number $\omega_\partial = \omega(\vec{v}_{out})$. Thus it has the sign +1 if we is positive, and the sign - 1 if ω_∂ is negative. (In this case, w takes only one vector.)

This is consistent with the ad hoc definition (Definition(3-2-5)). If $\omega\left(\vec{v}\right) = \vec{t}.\vec{v}$ then the condition $\omega_\partial > 0$ means exactly that $\vec{t}(x)$ points out of P.

Example (3-2-12):(Oriented boundary of a piece-with-boundary of IR^2)

Let the smooth curve C be the smooth boundary of a piece-with-boundary S of IR^2 . If IR^2 is oriented in the standard way (i.e., by det), then at a point $x \in C$, the boundary C is oriented by

$$\omega_{\partial}\left(\vec{v}\right)=\det(\vec{v}_{out},\vec{v}) \tag{3-39}$$

Suppose we have drawn the standard basis vectors in the plane in the standard way, with $\vec{e}_2$ counterclockwise from et. Then

$$\det\left(\vec{v}_{out},\vec{v}\right)>0 \tag{3-40}$$

if, when you look in the direction of $\vec{v}$, the vector $\vec{v}_{out}$is on your right. In this case S is on your left, as was already shown in Figure (3.5)

Example (3-2-13):(Oriented boundary of a piece-with-boundary of an oriented surface in IR^3)

Let $S_1 \subset S$ be a piece-with-boundary of an oriented surface S. Suppose that at $x \in \partial S_i$, S is oriented by $\omega \in A^2\left(T_x, t(S)\right)$ and that $\vec{v}_{out} \in T_x$ S is tangent to S at x but points out of S_1. Then the curve ∂S_1 is oriented by

$$\omega_{\partial}\left(\vec{v}\right)=\omega_{\partial}\left(\vec{v}_{out},\vec{v}\right) \tag{3-41}$$

This is consistent with the ad hoc definition, illustrated by Figure (3-6). In the ad the ad hoc definition, where S is oriented by a normal vector field $\vec{N}$, the corresponding ω is

$$\omega_{\partial}\left(\vec{v}_1,\vec{v}_2\right)=\det\det\left[\vec{N}\left(x\right),\vec{v}_{out},\vec{v}_1,\vec{v}_2\right] \tag{3-42}$$

$$\omega_{\partial}\left(\vec{v}\right)=\det\det\left[\vec{N}\left(x\right),\vec{v}_{out},\vec{v}\right] \tag{3-43}$$

Thus if the vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$ are drawn in the standard way, satisfying the right-hand rule, then V defines the orientation of ∂S_1 if $\vec{N}\left(x\right),\vec{v}_{out},\vec{v}$ satisfy the right-hand rule also.

Example (3-2-14):(Oriented boundary of a piece-with-boundary of IR^3)

Suppose U is a piece-with-boundary of IR^3 with boundary $\partial U = S$, and U is oriented in the standard way, by det. Then S is oriented by

$$\omega_{\partial}\left(\vec{v}_1,\vec{v}_2\right)=\det\left[\vec{v}_{out},\vec{v}_1,\vec{v}_2\right] \qquad (3\text{-}44)$$

If we wish to think of orientating S in the ad hoc language, i.e., by a field of normals $\vec{N}$, this means exactly that for any $x \in S$ and any two vectors $\vec{v}_1,\vec{v}_2 \in T_X S$, the two numbers

$$\det\left[\vec{N}(x),,\vec{v}_1,\vec{v}_2\right] \quad \text{and} \quad \det\left[\vec{v}_{out},\vec{v}_1,\vec{v}_2\right] \qquad (3\text{-}45)$$

should have the sable sign, i.e., $\vec{N}(x)$ should point out of U.

Now we will discuss the oriented boundary of an oriented k-parallelogram We saw above that an oriented k-parallelogram $P_x^0\left(\vec{v}_1,\ldots,\vec{v}_k\right)$ is a piece-with boundary of an oriented manifold if the vectors $\vec{v}_1,\ldots,\vec{v}_k$ are linearly independent (i.e., the parallelogram is not squished flat). As such its boundary carries an orientation.

Proposition(3-2-15):(Oriented boundary of an oriented k-parallelogram)

The oriented boundary of an oriented k-parallelogram $P_x^0\left(\vec{v}_1,\ldots,\vec{v}_k\right)$ is given by

$$\partial P_x^0\left(\vec{v}_1,\ldots,\vec{v}_k\right)=\sum_{i=1}^{k}(-1)^{i-1}\left(P^0_{x+\vec{v}_i}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)-P_x^0\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)\right) \qquad (3\text{-}46)$$

where a bat over a term indicates that it is being omitted.

This business of hats indicating an omitted term may seem complicated. Recall that the boundary of an object always has one dimension less than the object itself: the boundary of a disk is a curve, the boundary of a box consists of the six rectangles making up its sides, and so on. The boundary of a k-dimensional parallelogram is made up of (k - 1)-parallelograms, so omitting a vector gives the right number of vectors. For the faces of the form $P_x\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)$, each of the k vectors has a turn at being omitted. (In Figure (3-7) , these faces are the three faces that include the point x.) For the faces of the type $P^0_{x+\vec{v}_i}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)$, the omitted vector is the vector added to the point x.

Before the proof, let us give some examples, which should make the formula easier to read.

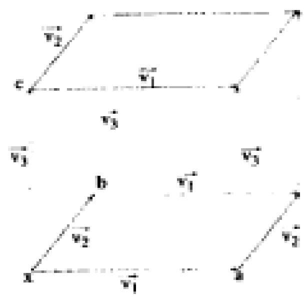


Figure (3-7)

Example (3-2-16): (The boundary of an oriented 1-parallelogram)

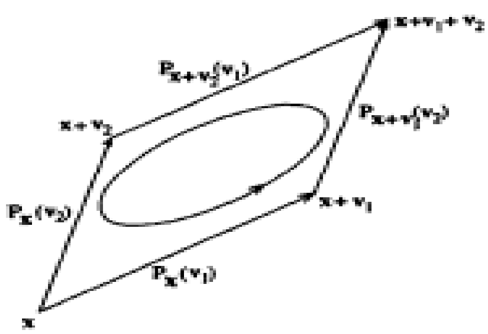
The boundary of $P_x^0\left(\vec{v}\right)$ is

$$\partial P_x^0\left(\vec{v}\right) = P^0_{x+\vec{v}} - P_x^0 \qquad (3\text{-}47)$$

So the boundary of an oriented line segment is its end minus its beginning, as you probably expect.

Example (3-2-17):(The boundary of an oriented 2-parallelogram)

A look at Figure (3-8) will probably lead you to guess that the boundary of an oriented parallelogram is Figure (3.8)



Figure(3-8)

$$\underbrace{\partial P_x^0\left(\vec{v}_1,\vec{v}_2\right)}_{boundary} = \underbrace{P_x^0\left(\vec{v}_1\right)}_{let\ side} + \underbrace{P^0_{x+\vec{v}_1}\left(\vec{v}_2\right)}_{2nd\ side} - \underbrace{P^0_{x+\vec{v}_2}\left(\vec{v}_1\right)}_{3nd\ side} - \underbrace{P_x^0\left(\vec{v}_2\right)}_{4tb\ side} \qquad (3\text{-}48)$$

which agrees with Proposition (3-2-15)

Example (3-2-18): (Boundary of a cube)

For the faces of a cube shown in Figure (3-7) we have

$$\left(i=1 so\left(-1\right)^{-1}=1\right);+\underbrace{P^{0}_{x+\vec{v}_1}\left(\vec{v}_2,\vec{v}_3\right)}_{right\ side}-\underbrace{P^{0}_{x}\left(\vec{v}_2,\vec{v}_3\right)}_{left\ side}$$

$$\left(i=2 so\left(-1\right)^{-1}=-1\right);+\underbrace{P^{0}_{x+\vec{v}_2}\left(\vec{v}_1,\vec{v}_3\right)}_{back}-\underbrace{P^{0}_{x}\left(\vec{v}_1,\vec{v}_3\right)}_{front} \qquad (3\text{- }49)$$

$$\left(i=3 so\left(-1\right)^{-1}=1\right);+\underbrace{P^{0}_{x+\vec{v}_3}\left(\vec{v}_1,\vec{v}_2\right)}_{top}-\underbrace{P^{0}_{x}\left(\vec{v}_1,\vec{v}_2\right)}_{bottom}$$

How many "faces" make up the boundary of a 4-parallelogram? What is each face? How would you describe the boundary following the format used for the cube in Figure (3-7)? Check your answer below.?

Proof of Proposition (3-2-15):

As in Example (3-2-1) , denote by M the manifold of which $P^{0}_{x}\left(\vec{v}_1,\ldots,\vec{v}_k\right)$ is a piece-with-boundary. The boundary $\partial P^{0}_{x}\left(\vec{v}_1,\ldots,\vec{v}_k\right)$ is composed of its 2k faces (four for a parallelogram, six for a cube ...), each of the form

$$P^{0}_{x+\vec{v}}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right) or P^{0}_{x}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right) \qquad (3\text{-}50)$$

where a hat over a term indicates that it is being omitted. The problem is to show that the orientation of this boundary is consistent with Definition (3-2-10) of the oriented boundary of a piece-with-boundary

Let $\omega \in A^{k}(M)$ defin e the orientation of M, so that $\omega\left(\vec{v}_1,\ldots,\vec{v}_k\right)>0$. At a point of $P^{0}_{x+\vec{v}}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)$, the vector $\vec{v}_i$, is outward pointing, whereas at a point of $P^{0}_{x}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)$, the vector -v; is outward pointing, Thus the standard orientation of $P^{0}_{x+\vec{v}}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)$ is consistent with the boundary orientation of $P^{0}_{x}\left(\vec{v}_1,\ldots,\hat{\vec{v}}_i,\ldots,\vec{v}_k\right)$ precisely if

$$\omega\left(\vec{v}_i, \vec{v}_1, \ldots, \hat{\vec{v}}_i, \ldots, \vec{v}_k\right) > 0$$

i.e.. precisely if the permutation σ_i on k symbols which consists of taking the ith element and putting it in first position is a positive permutation. But the signature of σ_i is $(-1)^{i-1}$ because you can obtain σ_i by switching the ith symbol first with the $(-1)^{i-1}$th, then the *(i - 2)*th, etc., and finally the first, doing i - I transpositions. This explains why $P^0_{x+\vec{v}}\left(\vec{v}_1, \ldots, \hat{\vec{v}}_i, \ldots, \vec{v}_k\right)$ occurs with sign $(-1)^{i-1}$.

A similar argument holds for $P^0_x\left(\vec{v}_1, \ldots, \hat{\vec{v}}_i, \ldots, \vec{v}_k\right)$ This oriented parallelogram has orientation compatible with the boundary orientation precisely if $\omega\left(-\vec{v}_i, \vec{v}_1, \ldots, \hat{\vec{v}}_i, \ldots, \vec{v}_k\right) > 0$, which occurs if the permutation ai is odd. This explains why $P^0_x\left(\vec{v}_1, \ldots, \hat{\vec{v}}_i, \ldots, \vec{v}_k\right)$ occurs in the sum with sign $(-1)^{i-1}$.

Now we will discuss the exterior derivative . In which we differentiate forms.

Now we come to the construction that gives the theory of forms its power, making possible a fundamental theorem of calculus in higher dimensions. We have already discussed integrals for forms. A derivative for forms also exists. This derivative, often called the exterior derivative, generalizes the derivative of ordinary functions. We will first discuss the exterior derivative in general; later we will see that the three differential operators of vector calculus (div, curl, and grad) are embodiments of the exterior derivative.

Now we will illustrate reinterpreting the derivative .

What is the ordinary derivative? Of course, you know that

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left(f(x+h) - f(x)\right) \qquad (3\text{-}51)$$

but we will reinterpret this formula as

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \int_{\partial P_x^0(h)} f \qquad (3\text{-}52)$$

What does this mean? We are just using different words and different notation to describe the same operation. Instead of saying that we are evaluating f at the two points $x + h$ and x, we say that we are integrating the 0-form f over the boundary of the oriented segment $[x, x+h] = P_x^0(h)$. This boundary consists of the two oriented points $+P_{x+h}^0$ and $-P_x^0$. The first point is the endpoint of $P_x^0(h)$, and the second its beginning point; the beginning point is taken with a minus sign, to indicate the orientation of the segment. Integrating the 0-form f over these two oriented points means evaluating f on those points (Definition (3-1-22)). So Equations (3-51) and (3-52) say exactly the same thing.

It may seem absurd to take Equation (3-51) , which everyone understands perfectly well, and turn it into Equation (3-52), which is apparently just a more complicated way of saying exactly the same thing. But the language generalizes nicely to forms.

Now we will exepress Defining the exterior derivative. The exterior derivative d is an operator that takes a k-form and gives a $(k + 1)$-form, do. Since a $(k + 1)$-form takes an oriented $(k + 1)$ dimensional parallelogram and gives a number, to define the exterior derivative of a k-form :p, we must say what number it gives when evaluated on an oriented $(k + I)$-parallelogram.

Definition(3-2-19): (Exterior derivative)

The exterior derivative d of a k-form φ, denoted $d\varphi$, takes a k + 1-parallelogram and returns a number, as follows:

$$\underset{\substack{(k+1)-\\ form}}{d\varphi}\Bigg(\overbrace{P_x^0\left(\vec{v}_1,\ldots,\vec{v}_{k+1}\right)}^{(k+1)-parallelogram}\Bigg) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int \overbrace{\underbrace{\partial P_x^0\left(h\vec{v}_1,\ldots,h\vec{v}_{k+1}\right)}_{\substack{boundary\ of\ k+1-parallelgoram\\ smallar\ and\ smallar\ as\ h\to 0}}}^{integrating\ \varphi\ over\ boundary} \qquad (3\text{-}53)$$

This isn't a formula that you just look at and say-"got it." We will work quite hard to see what the exterior derivative gives in particular cases, and to see how to compute it. That the limit exists at all isn't obvious. Nor is it. obvious that the exterior derivative is a (k + 1)-form: we can see that $d\varphi$ is a function of k + 1 vectors, but it's not obvious that it is multilinear and alternating. Two of Maxwell's equations say that a certain 2-form on IR^4 has exterior derivative zero; a course in electromagnetism might well spend six months trying to really understand what this means. But observe that the definition makes sense; $P_x^0\left(\vec{v}_1,\ldots,\vec{v}_{k+1}\right)$ is (k + 1)-dimensional, its boundary is

k-dimensional, so it is something over which we can integrate the k-form φ. Notice also that when k = 0, this boils down to Equation (3-50), as restated in Equation (3-51).

Remark (3-2-20):

Here we see why we had to define the boundary of a piece-with boundary as we did in Definition (3-2-10). The faces of the (k + 1)-parallelogram $P_x^0\left(\vec{v}_1,\ldots,\vec{v}_{k+1}\right)$ are k-dimensional. Multiplying the edges of these faces by h should multiply the integral over each face by h^k. So it may seem that the limit above should not exist, because the individual terms behave like h^k/h^{k+l} = 1/h. But the limit does exist, because the faces come in pairs with opposite orientation, according to Equation (3-45), and the terms in h^k from each pair cancel, leaving something of order h^{k+1}.

This cancellation is absolutely essential for a derivative to exist-, that is why we have put so much emphasis on orientation

Now we will study computing the exterior derivative

Theorem (3-2-21): (Computing the exterior derivative of a k-form)

(a) If the coefficients a of the k-form

$$\varphi = \sum_{1\le i_1<\ldots<i_k\le n} a_{i_1\ldots i_k} dx_{i_1} \Lambda \ldots \Lambda dx_{i_k} \tag{3-54}$$

are C^2 functions on $U \subset IR^n$, then the limit in Equation (3-53) exists, and defines a (k + 1)-form.

(b) The exterior derivative is linear over R: if φ and Ψ are k-forms on $U \subset IR^n$, and a and b are numbers (not functions), then

$$d\left(a\varphi + b\Psi\right) = ad\,\varphi + bd\,\Psi \tag{3-55}$$

(c) The exterior derivative of a constant form is 0.

(d) The exterior derivative of the 0-form (i.e., function) f is given by the Formula

$$df = \left[Df\right] = \sum_{i=1}^{n}\left(D_i f\right)dx_{i_k} \tag{3-56}$$

(e) If f is a function, then

$$d\left(f\ dx_{i_1}\Lambda\ldots\Lambda dx_{i_k}\right)=df\ \Lambda dx_{i_1}\ldots.\Lambda dx_{i_k} \qquad (3\text{-}57)$$

Theorem (3-2-21) is proved in section (3.2).
These rules allow you to compute the exterior derivative of any k-form, as shown below for any k-form and as illustrated in the margin :

$$d\varphi = d\overbrace{\sum_{1\le i_1<\ldots<i_k\le n} a_{i_1\ldots i_k}dx_{i_1}\Lambda\ldots\Lambda dx_{i_k}}^{writing\ \varphi\ in\ full}$$

$$\underset{(b)}{=}\underbrace{\sum_{1\le i_1<\ldots<i_k\le n} d(a_{i_1\ldots i_k})dx_{i_1}\Lambda\ldots\Lambda dx_{i_k})}_{exterior\ derivative\ of\ sum\ equals\ sum\ of\ exterior\ derivattive} \qquad (3\text{-}58)$$

$$\underset{(e)}{=}\underbrace{\sum_{1\le i_1<\ldots<i_k\le n} d\ (\underbrace{a_{i_1\ldots i_k}}_{f})dx_{i_1}\Lambda\ldots\Lambda dx_{i_k})}_{problem\ reduced\ computing\ ext\ deriv\ of\ function}$$

Going from the first to the second line reduces the computation to computing exterior derivatives of elementary forms; going from the second to the third line reduces the computation to computing exterior derivatives of functions. In applying (e) we think of the coefficients $a_{i_1\ldots i_k}$ as the function f. We compute the exterior derivative of the function $f = a_{i_1\ldots i_k}$ from part (d):

$$da_{i_1\ldots i_k} = \sum_{j=1}^{n} D_j a_{i_1\ldots i_k} dx_{i_k} \qquad (3\text{-}59)$$

For example, if f and g are functions in the three variables x, y and z, then

$$df = D_1 f dx + D_2 f dy + D_3 f dz \qquad (3\text{-}60)$$

So

$$df\ \Lambda dx\ \Lambda dy = (D_1 f dx + D_2 f dy + D_3 f dz)dx\ \Lambda dy$$
$$= D_1 f\ \underbrace{dx\ \Lambda dx\ \Lambda dy}_{0} + D_2 f\ \underbrace{dy\ \Lambda dx\ \Lambda dy}_{0} + D_3 f dz\ \Lambda dx\ \Lambda dy$$
$$D_3 f dz\ \Lambda dx\ \Lambda dy = D_3 f dz\ \Lambda dx\ \Lambda dy \qquad (3\text{-}61)$$

Example (3-2-22): (Computing the exterior derivative of an elementary 2-form on IR[4])

Computing the exterior derivative of $(x_1x_3)\left(dx_2\Lambda dx_4\right)$ gives $d\left(x_1x_3\right)\Lambda dx_2\Lambda dx_4$

$$=\underbrace{(\overbrace{D_1(x_2x_3)}^{0}dx_1+D_2(x_2x_3)dx_2+D_3(x_2x_3)dx_3+\overbrace{D_4(x_2x_3)}^{0}dx_4)}_{d\left(x_1x_3\right)}\Lambda dx_2\Lambda dx_4$$

$$\left(x_3dx_2+x_2dx_3\right)\Lambda dx_2\Lambda dx_4=\left(x_3dx_2\Lambda dx_2\Lambda dx_4\right)+\left(x_2dx_2\Lambda dx_2\Lambda dx_4\right)$$

$$=\underbrace{x_2\left(dx_3\Lambda dx_2\Lambda dx_4\right)}_{dx's\ out\ of\ order}=-\underbrace{x_2\left(dx_3\Lambda dx_2\Lambda dx_4\right)}_{\substack{sign\ changes\ as\ order\\ is\ corrected}} \qquad (3\text{-}62)$$

What is the exterior derivative of the 2-form on IR[3] $x_l\ x_3^2\ dx_l\ \Lambda\ d_2$? Check your answer below.

Example(3-2-23): (Computing the exterior derivative of a 2-form). Compute the exterior derivative of the 2-form on IR[4],

$$\Psi=x_1x_3dx_2\Lambda dx_4-x_2^2dx_2\Lambda dx_4 \qquad (3\text{-}63)$$

which is the sum of two elementary 2-forms. We have

$$d\Psi=d\left(x_1x_3dx_2\Lambda dx_4\right)-d\left(x_2^2dx_2\Lambda dx_4\right)$$

$$\begin{aligned}&=(D_1(x_1x_2)dx_1)+D_2(x_1x_2)dx_2+D_3(x_1x_2)dx_3+D_4(x_1x_2)dx_4)\Lambda dx_2\Lambda dx_4\\&-(D_1(x_2^2)dx_1)+D_2(x_2^2)dx_2+D_3(x_2^2)dx_3+D_4(x_2^2)dx_4)\Lambda dx_2\Lambda dx_4\end{aligned}$$

$$\left(x_2dx_1+x_1dx_2\right)\Lambda dx_2\Lambda dx_4-\left(2x_2dx_2\Lambda dx_3\Lambda dx_4\right)$$

$$=x_2dx_1\Lambda dx_2\Lambda dx_4+\underbrace{x_1dx_2\Lambda dx_2\Lambda dx_4}_{=0}-2x_2dx_2\Lambda dx_3\Lambda dx_4 \qquad (3\text{-}64)$$

$$=x_2dx_1\Lambda dx_2\Lambda dx_4-2x_2dx_2\Lambda dx_3\Lambda dx_4$$

Example (3-2-23): (Element of angle)

The vector fields

$$\vec{F}_2 = \frac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \end{bmatrix} and\ \vec{F}_3 = \frac{1}{(x^2+y^2+z^2)^{3/2}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (3\text{-}65)$$

satisfy the property that dw $_{\vec{F}_2}$ = 0 and d$\Phi_{\vec{F}_2}$ = 0. The forms $W_{\vec{F}_3}$ and $\Phi_{\vec{F}_3}$ can be called respectively the "element of polar angle" and the "element of solid angle"; the latter is depicted in Figure (3-10) .

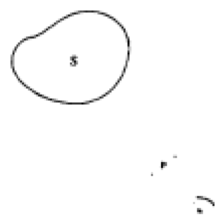


Figure (3-9)

We will now find the analogs in any dimension. Using again a hat to denote a term that is omitted in the product, our candidate is the (n-1)-form on IR^n:

$$\omega_n = \frac{1}{(x_1^2+\ldots+x_n^2)^{n/2}}\sum_{i=1}^{n}(-1))^{i-1}x_i dx_1\Lambda\ldots\Lambda \hat{dx_i}\ \Lambda\ldots\Lambda dx_n \qquad (3\text{ - }66)$$

which can also be thought of as the flux of the vector field

$$\vec{F}_n = \frac{1}{(x_1^2+\ldots+x_n^2)^{n/2}}\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$ which can be written $\frac{\vec{x}}{|\vec{x}|^n}$.

It is clear from the second description that the integral of the flux of this vector field over the unit sphere S^{n-1} is positive; at every point, this vector field points outwards, In fact, the flux is equal to the (n - 1)-dimensional volume of S^{n-1}

The computation in Equation (3-66) below shows that d_{ω_n} = 0:

$$\omega_n = d\left(\frac{1}{(x_1^2+\ldots+x_n^2)^{n/2}}\sum_{i=1}^{n}(-1)^{i-1}x_i dx_1\Lambda\ldots\Lambda \hat{dx_1}\ \Lambda\ldots\Lambda dx\right)$$

$$= \sum_{i=1}^{n}(-1)^{i-1}D_i\ \frac{x_i}{(x_1^2+\ldots+x_n^2)^{n/2}}dx_i\Lambda\ dx_1\Lambda\ldots\Lambda \hat{dx_i}\ \Lambda\ldots\Lambda dx_n$$

$$\sum_{i=1}^{1n} D_i \left(\frac{x_i}{\left(x_1^2+\ldots+x_n^2\right)^{n/2}} \right) dx_i \Lambda \ldots . \Lambda dx_n$$

$$\sum_{i=1}^{n} \left(\frac{\left(x_1^2+\ldots+x_n^2\right)^{n/2} - n x_i^2 (x_1^2+\ldots+x_n^2)^{n/2-1}}{(x_1^2+\ldots+x_n^2)^n} \right) dx_1 \Lambda \ldots \Lambda dx_n$$

$$\sum_{i=1}^{n} \left(\frac{x_1^2+\ldots+x_n^2 - n x_1^2}{(x_1^2+\ldots+x_n^2)^{n/2+1}} \right) dx_1 \Lambda \ldots \Lambda dx_n = 0, \qquad (3\text{-}67)$$

We get the last equality because the sum of the numerators cancel. For instance, when n = 2 we have $x_1^2 + x_2^2 - 2x_1^2 + x_1^2 + x_2^2 - 2x_2^2 = 0$

Now we will discuss taking the exterior derivative twice The exterior derivative of a *k*-form is a (*k* + 1)-form; the exterior derivative of that (*k* + 1)-forth is a (*k* + 2)-form. One remarkable property of the exterior derivative is that if you take it twice, you always get 0. (To be precise, we must specify that Sp be twice continuously differentiable.)

Theorem (3-2- 25):

For any *k*-form on $U \subset R^2$ of class C^2, we have d(dφ) = 0.

Proof:

This can just be computed out. Let us see it first for 0-forms

$$ddf = \left(\sum_{i=1}^{n} D_i f dx_i \right) = \sum_{i=1}^{n} d\,(D_i f dx_i)$$

$$\sum_{i=1}^{n} dD_i f \, \Lambda dx_i = \sum_{i=1}^{n} \sum_{j=1}^{n} D_j D_i f dx_j \Lambda dx_i = 0. \qquad (3\text{-}68)$$

If *k* > 0, it is enough to make the following computation:

$$(d\,(f dx_i \Lambda \ldots \Lambda dx_{ik}) = d\,(df \, \Lambda dx_i \Lambda \ldots \Lambda dx_{ik}) =$$

$$\underbrace{(ddf)}_{=0} \Lambda dx_{i1} \Lambda \ldots \Lambda dx_{ik} = 0 \qquad (3\text{-}69)$$