Chapter (2)
Forms and Vector Calculus

Section (2.1) : Forms over oriented domains and IR^n

What really makes calculus work is the fundamental theorem of calculus: that differentiation, having to do with speeds, and integration, having to do with areas, are somehow inverse operations.
Obviously, we will want to generalize the fundamental theorem of calculus to higher dimensions. Unfortunately, we cannot do so using the techniques of Chapter (1) , where we integrated using $|d^n\mathrm{x}|$. The reason is that $|d^n x|$ always returns a positive number; it does not concern itself with the orientation of the subset over which it is integrating, unlike the dx of one dimensional calculus, which does:

$$\sum_{1 \le i1 < \ldots < ik \le n} a_{i1\ldots ik}\, dx_{i1} \Lambda \ldots \Lambda\, dx_{ik} \left(\vec{e}_j, \ldots, \vec{e}_{jk} \right) \qquad (2 - 1)$$

To get a fundamental theorem of calculus in higher dimensions, we need to introduce new tools. If we were willing to restrict ourselves to IR^2 and IR^3 we could use the techniques of vector calculus. We will use a different approach, forms, which work in any IR^n . Forms are integrands over oriented domains; they provide the theory of expressions containing *dx* or *dx dy* ... .
Because forms work in any dimension, they are the natural way to approach two towering subjects that are inherently four-dimensional: electromagnetism and the theory of relativity. They also provide a unified treatment of differentiation and of the fundamental theorem of calculus: one operator (the exterior derivative) works in all dimensions, and one short, elegant statement (the generalized Stokes's theorem) generalizes the fundamental theorem of calculus to all dimensions. In contrast, vector calculus requires special formulas, operators, and theorems for each dimension where it works.
On the other hand, the language of vector calculus is used in many science courses, particularly at the undergraduate level. So while in theory we could provide a unified treatment of higher dimensional calculus using only forms, this would probably not mesh well with other courses. If you are studying physics, for example, you definitely need to know vector calculus. In addition, the functions and vector fields of vector calculus are more intuitive

than forms. A vector field is an object that one can picture, as in Figure (2-1).

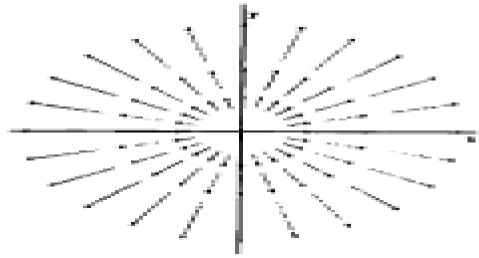
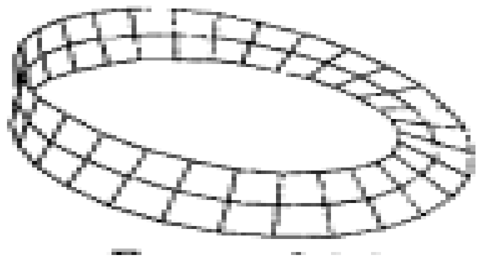


Figure (2-1).

Coming to terms with forms requires more effort . We can't draw you a picture of a form. A k-form is, as we shall see, something like the determinant: it takes k vectors, fiddles with them until it has a square matrix, and then takes its determinant. We said at the beginning of this chapter that the object of linear algebra "is at least in part to extend to higher dimensions the geometric language and intuition we have concerning the plane and space, familiar to us all from everyday experience." Here too we want to extend to higher dimensions the geometric language and intuition we have concerning the plane and space. We hope that translating forms into the language of vector calculus will help you do that Section (2.1) we introduce k-forms: integrands that take a little piece of oriented domain and return a number. In Section (2.1) we define oriented k-parallelograms The radial vector field and show how to integrate form fields-functions that assign a form at each point-over parametrized domains. Section (2.2) translates the language of forms on IR^3 into the language of vector calculus. Section (2.2) gives the definitions of orientation necessary to integrate form fields over oriented domains, while.

Now we will disucss forms as intergrands over oriented domains.

In Chapter (1) we showed how to integrate the integrand $|d^1\, x|$ (the element of arc length) over a curve, to determine its length, and how to integrate the integrand $|d^2\, x|$ over a surface, to determine its area. More generally, we saw how to integrate $|d^K\, x|$ over a k-dimensional manifold in IR^n , to determine its k-dimensional volume. Such integrands take a little piece (of curve, surface, or higher-dimensional manifold) and return a number. They require no mention of the orientation of the piece; non-orientable surfaces like the Moebius strip shown in Figure (2-2)



figure(2-2)

have a perfectly well-defined area, obtained by integrating $|d^2x|$ over them. The integrands above are thus fundamentally different from the integrand dx of one variable calculus, which requires oriented intervals. In one variable calculus, the standard integrand $f(x)\,dx$ takes a piece $[x_i, x_{j+1}]$ of the domain, and returns the number $f(x_i)(x_{i+1} - x_i)$: the area of a rectangle with height $f(x_i)$ and width x_{i+1} - x_i. Note that dx returns $x_{i+1} - x_i$, not $x_{i+1} - x_i$; that is why

$$\int_{-1}^{1} f(x)dx = -\int_{1}^{-1} f(x)dx \qquad (2\text{-}2)$$

In order to generalize the fundamental theorem of calculus to higher dimensions, We need integrands over oriented objects. Forms are such integrands.

Example (2-1-1):[**Flux form of a vector field:** $\phi\vec{F}$]

Suppose we are given a vector field $\vec{F}$ on some open subset U of IR^3. It may help to imagine this vector field as the velocity vector field of some fluid with a steady flow (not changing with time). Then the integran $\phi\vec{F}$ associates to a little piece of surface the flux of $\vec{F}$ through that piece; if you imagine the vector field as the flow of a fluid, then $\phi\vec{F}$ associates to a little piece of surface the amount of fluid that flows through it in unit time. But there's a catch: to define the flux of a vector field through a surface, you must orient the surface, for instance by coloring the sides yellow and blue, and counting how much flows from the blue side to the yellow side (counting the flow negative if the fluid flows in the opposite direction). It obviously does not make sense to calculate the flow of a vector field through a Moeblus strip.

Now we will study forms on IR^n .You should think of this section as a continuation . There we saw that there is a unique antisymmetric and multilinear function of n vectors in IR^n that gives 1 if evaluated on the standard basis vectors: the determinant.

Because of the connection between the determinants and volumes described the determinant is fundamental to multiple integrals, Here we will study the multilinear antisymmetric functions of k vectors in IR^n, where $k \leq 0$ may be any integer, though we will soon see that the only interesting case is when $k \leq n$. Again there is a close relation to volumes, and in fact these objects, called forms, are the right integrands for integrating over oriented domains.

Definition (2-1-2):

A k-form on IR^n is a function φ that takes k vectors in IR^n and returns a number, such that $\varphi(\vec{V}_1, \ldots, \vec{V}_K)$ is multilinear and antisymmetric. That is, a k-form φ is linear with respect to each of its arguments, and changes sign if two of the arguments are exchanged. It is rather hard to imagine forms, so we start with an example, which will turn out to be the fundamental example.

Example (2-1-3):

Let $i_1, \ldots, i_k$ be any k integers between 1 and n.

Then $dx_{i_1} \Lambda \ldots \Lambda\, dx_{ik}$ is that function of k vectors $\vec{v_1} \ldots.. \vec{v_k}$. in IR^n that puts these vectors side by side, making the *n* x *k* matrix

$$\begin{bmatrix} v_{1,1} & \cdots & v_{1,k} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,k} \end{bmatrix} \qquad (2 - 3)$$

and selects first the i_1 the row, then the i_2 row, etc, and finally the i_kth row, making the square *k* x *k* matrix

$$\begin{bmatrix} v_{i_1,1} & \cdots & v_{i_1,K} \\ \vdots & \ddots & \vdots \\ v_{i_K,1} & \cdots & v_{i_K,K} \end{bmatrix} \qquad (2\text{-}4)$$

and finally takes its determinant. For instance,

$$\underbrace{dx_1 \Lambda\, dx_2}_{2-Form}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}\right) = \det \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}}_{\substack{1st\ and\ 2nd\ rows \\ of\ original\ matrix}} = -8 \qquad (2\text{-}5)$$

$$\underbrace{dx_1 \Lambda\ dx_2 \Lambda\ dx_4}_{3-Form}\left(\begin{bmatrix}1\\2\\-1\\1\end{bmatrix},\begin{bmatrix}3\\-2\\1\\2\end{bmatrix},\begin{bmatrix}0\\1\\2\\1\end{bmatrix}\right)=\det\det\begin{bmatrix}1&3&0\\2&-2&1\\1&2&1\end{bmatrix}=-7 \qquad (2\text{-}6)$$

Remark(2-1-4):

The integrand $\left| d^k x \right|$ of Chapter (1) also takes k vectors in IR^n and gives a number:

$$|d^1\ x|\ (\vec{V}) = \left|\vec{V}\right| = \sqrt{\vec{V}^T\vec{V}} \quad ,$$

$$|d^2x\ |((\vec{V}_1),(\vec{V}_2)) = \sqrt{\det(\left[(\vec{V}_1,\vec{V}_2\right]^T\left[(\vec{V}_1,\vec{V}_2\right])} \qquad (2\text{-}7)$$

$$|d^kx\ |((\vec{V}_1),.......,(\vec{V}_K)) = \sqrt{\det\det(\left[\vec{V}_1,.......,\vec{V}_K\right]^T\left[\vec{V}_1,.......,\vec{V}_K\right]}$$

Unlike forms, these are not multilinear and not antisymmetric.

Now we will express geometric meaning of k-forms that number. Evaluating the 2-form $dx_1 \wedge dx_2$ on the vectors $\vec{a}$ and $\vec{b}$, we have:

$$dx_1 \Lambda dx_2\left(\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix},\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}\right)=\det\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}=a_1b_2-a_2b_1 \qquad (2\text{-}8)$$

which can be understood geometrically. If we project $\vec{a}$ and $\vec{b}$ onto the (x_1, x_2)- plane, we get the vectors

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}\text{and}\begin{bmatrix}b_1\\b_2\end{bmatrix} \qquad (2\text{-}9)$$

the determinant in Equation (2- 7) gives the signed area of the parallelogram that they span, Thus $dx_1 \wedge dx_2$ deserves to be called the (x_1, x_2) component of signed area. Similarly, $dx_2 \wedge dx_3$ and $dx_1 \wedge dx_3$ deserve to be called the (x_2, x_3) and the (x_1,x_3) components of signed area.

We can now interpret Equations (2 - 4) and (2 - 5) geometrically. The 2 form

form $dx_1 \wedge dx_2$ tells us that the (x_1, x_2) component of signed area of the parallelogram spanned by the two vectors in Equation (2-4) is -8. The 3-form $dx_1 \wedge dx_2 \wedge dx_4$ tells us that the (dx_1, dx_2, dx_4) component of signed volume of the parallelepiped spanned by the three vectors in Equation (2-5) is-7. Similarly, the 1-form dx gives the x component of signed length of a vector, while dy gives its y component:

$$dx\left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right) = det\ 2 = 2 \text{ and } dy\left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right) = det(-3) = 3$$

More generally (and an advantage of k-forms is that they generalize so easily to higher dimensions), we see that

$$dx_i\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = \det[v_i] \qquad (2\text{-}10)$$

is the ith component of the signed length of $\vec{v}$, and that $dx_i \wedge \ldots \wedge dx_{ik}$,,, evaluated on $(\vec{V}_1, \ldots, \vec{V}_K)$ gives the $(x_{i1},, \ldots x_{ik})$ component of signed k-dimensional volume of the k-parallelogram spanned by $\vec{V}_1, \ldots, \vec{V}_K$.

Now we will illustrate elementary forms. There is a great deal of redundancy in the expressions $dx_i \wedge \ldots \wedge,, dx_{ik}$. Consider for instance $dx_1 \wedge dx_3 \wedge dx_1$ This 3-form takes three vectors in IR^n stacks them side by side to make an $n \times 3$ matrix , selects the first row, then the third, then the first again, to make a 3×3 and takes its determinant. So far, so good; but observe that the determinant in question is always 0, independent of what the vectors were; we have taken the determinant of a 3×3 matrix for which the third row is the same as the first; such a determinant is always . (Do you see why?') So

$$dx_1 \wedge dx_3 \wedge dx_1 = 0 \qquad (2\text{-}11)$$

it takes three vectors and returns 0.

But that is of course not the only way to write the form that takes three vectors and returns the number 0; both $dx_1 \wedge dx_1 \wedge dx_3$ and $dx_2 \wedge dx_3 \wedge dx_3$ do so as well, and there are others. More generally, if any two of the indices $i_1, \ldots, i_k$ are equal, we have $dx_1, \wedge \ldots \wedge dx_{ik}, = 0$: the k-form $dx_{i1} \wedge .. \wedge dx_{ik}$, where two indices are equal, is the k-form which takes k vectors and returns 0.

Next, consider $dx_1 \wedge dx_3$ and $dx_3 \wedge dx_1$. Evaluated on

$$\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad \text{we find}$$

$$dx_1 \Lambda dx_3(\vec{a},\vec{b}) = \det\begin{bmatrix} a_3 & b_3 \\ a_1 & b_1 \end{bmatrix} = a_3 b_1 - a_1 b_3$$

$$dx_3 \Lambda dx_1(\vec{a},\vec{b}) = \det\begin{bmatrix} a_3 & b_3 \\ a_1 & b_1 \end{bmatrix} = a_3 b_1 - a_1 b_3 \qquad (2\text{-}12)$$

.

Clearly $dx_l \Lambda dx_3 = -dx_3 \Lambda dx_l$; these two 2-forms, evaluated on the same two vectors, always return opposite numbers.

More generally, if the integers $i_l, \ldots, i_k$ and $j_1, \ldots, j_k$ are the same integers, just taken in a different order, so that $j_1, = i_{\sigma(1)}$, $j_2 = i_{\sigma(2)}$ …, $j_k = i_{\sigma(k)}$ for some permutation σ of $(1, \ldots, k)$, then

$$dx_{j1} \Lambda \ldots \Lambda dx_{jk} = \operatorname{sgn}(\sigma) dx_{i1}, \Lambda \ldots \Lambda dx_{ik} \qquad (2\text{-}13)$$

Indeed $dx_{j1} \Lambda \ldots \Lambda,, dx_{jk}$, computes the determinant of the same matrix as $dx_{i1} \Lambda \ldots \Lambda,, dx_{ik}$, only with the rows permuted by σ. For instance,

$$dx_1 \wedge dx_2 = - dx_2 \wedge dx_1 \text{ , and}$$
$$dx_1 \Lambda dx_2 \Lambda dx_3 = dx_2 \Lambda dx_3 \Lambda dx_1 = dx_3 \Lambda dx_1 \Lambda dx_2 \qquad (2\text{-}14)$$

To eliminate this redundancy, we make the following definition: an elementary k-form is of the form

$$dx_{i1} \Lambda \ldots \Lambda dx_{ik} \text{ with } 1 \le i_1 \le i_2 \ldots\ldots i_k \le n; \qquad (2\text{-}15)$$

putting the indices in increasing order selects one particular permutation for any set of distinct integers $j_1, \ldots, j_k$.

Definition (2-1-5) : (Elementary k-forms on IR^n)

A elementary k-form on IR^n is an expression of the form

$$dx_{i1}\Lambda ... \Lambda dx_{ik} \qquad (2\text{-}16)$$

where $1 \le i_1 < \ldots < i_k \le n$ (and $0 \le k \le n$). Evaluated on the vectors $\vec{V}_1, \ldots, \vec{V}_K$, it gives the determinant of the $k \times k$ matrix obtained by selecting the $i_1 \ldots i_k$ rows of the matrix whose columns are the vector $\vec{V}_1, \ldots, \vec{V}_K$.

The only elementary 0-form is the form, denoted 1, which evaluated on zero vectors returns 1.

Note that there are no elementary k forms on IR^n when k > n; indeed, there are no nonzero forms at all when $k > n$: there is no function φ that takes $k > n$ vectors in IR^n and returns a number, such that $\varphi(\vec{V}_1, \ldots, \vec{V}_K)$ is multilinear and antisymmetric. If $\vec{V}_1, \ldots, \vec{V}_K$ are vectors in R^n and k > n , then the vectors are not linearly independent, and at least one of them is a linear combination of the others, say

$$\vec{V}_K = \sum_{i=1}^{k-1} a_i \vec{V}_i \qquad (2\text{-}17)$$

Then if , φ is a k-form on IR^n , evaluation on the vectors $\vec{V}_1, \ldots, \vec{V}_K$ gives

$$\varphi(\vec{V}_1, \ldots, \vec{V}_K,) = \varphi(\vec{V}_1, \ldots, \sum_{i=1}^{k-1} a_i \vec{V}_i)$$

$$= \sum_{i=1}^{k-1} a_i \varphi(\vec{V}_1, \ldots \vec{V}_i \ldots, \vec{V}_i) \qquad (2\text{-}18)$$

Each term in this last sum will compute the determinant of a matrix, two columns of which coincide, and will give 0.

In terms of the geometric description, this should come as no surprise: you would expect any kind of three-dimensional volume in IR^2 to be zero, and more generally any k-dimensional volume in IR^n to be 0 when k > n. What elementary k-forms exist on IR^4

Now we will study all forms are liner combinations of elementary forms.k

We said above that $dx_{i1}\Lambda \ldots \Lambda dx_{ik}$ is the fundamental example of a k-form.

Now we will justify this statement, by showing that any k-form is a linear combination of elementary k-forms.

The following definitions say that speaking of such linear combinations makes sense: we can add k-forms and multiply them by scalars in the obvious way.

Definition (2-1-6): (Addition of k-forms)

Let φ and Ψ be two k-forms.Then

$$\varphi(\vec{V}_1,\ldots,\vec{V}_K,)+\Psi(\vec{V}_1,\ldots,\vec{V}_K,)=(\varphi+\Psi)(\vec{V}_1,\ldots,\vec{V}_K,)$$

DefinItion (2-1-7): (Multiplication of k-forms by scalars)

If φ is a k-form and a is a scalar, then

$$(a\varphi)(\vec{V}_1,\ldots,\vec{V}_K,)=a(\varphi\left(\vec{V}_1,\ldots,\vec{V}_K,\right))$$

Using these definitions of addition and multiplication by scalars, the space of k-forms in IR^n is a vector space. We will now show that the elementary k-forms form a basis of this space .

Definition (2-1-8):($A^k(IR^n)$)

The space of k-forms in IR^n is denoted $A^k\left(IR^n\right)$

Theorem (2-1-9):

The elementary k-forms form a basis for $A^k\left(IR^n\right)$.

In other words, every multilinear and antisymmetric function W of k vectors In IR^n can be uniquely written

$$\varphi=\sum_{1\le i_1<\ldots<i_k\le n} a_{i_1\ldots i_k}\,dx_{i_1}\Lambda\ldots\Lambda\, dx_{i_k} \qquad (2\text{-}19)$$

and in fact the coefficients are given by

$$a_{i_1\ldots i_k}=\varphi\left(\vec{e}_{i_1},\ldots,\vec{e}_{i_k}\right) \qquad (2\text{-}20)$$

Proof: Most of the work is already done, in the proof of Theorem (2-1-7), showing that the determinant exists and is uniquely characterized by its properties of multilinearity, antisymmetry, and normalization. (In fact, Theorem (2-1-9) is Theorem (2-1-7) when k = n.) We will illustrate it for the particular case of 2-forms on IR^3; this contains the idea of the proof while avoiding hopelessly complicated notation. Let φ be such a 2-form. Then, using multilinearity, we get the following computation. Forget about the coefficients, and notice that this equation says that φ is completely determined by what it does to the standard basis vectors

$$\varphi\left(\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\vec{v}}, \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\vec{w}}\right) = \varphi\left(\underbrace{v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3}_{\vec{v}} . \underbrace{w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3}_{\vec{w}}\right)$$

$$\varphi(v_1\vec{e}_1, \underbrace{w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3}_{\vec{w}}) + \varphi(v_2\vec{e}_2 . \underbrace{w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3}_{\vec{w}}) + \varphi(v_3\vec{e}_3 . \underbrace{w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3}_{\vec{w}}) \quad (2\text{-}21)$$

$$\varphi(v_1\vec{e}_1, w_1\vec{e}_1) + \varphi(v_1\vec{e}_1, w_2\vec{e}_2) + \varphi(v_1\vec{e}_1, w_3\vec{e}_3) + ...$$

$$= (v_1\, w_2 \text{-}\, v_2 w_1)\, \varphi(\, \vec{e}_1, \vec{e}_2\,) + (v_1\, w_3 \text{-}\, v_3 w_1)\, \varphi(\, \vec{e}_1, \vec{e}_3\,) + (v_2\, w_3 \text{-}\, v_3 w_2)\, \varphi(\, \vec{e}_2, \vec{e}_3\,)$$

An analogous but messier computation will show the same for any k and in φ is determined by its values on sequences $\vec{e}_i, ..., \vec{e}_{ik}$, with ascending indices. (The coefficients will be complicated expressions that give determinants, as in the case above, but you don't need to know that.) So any k-form that gives the same result when evaluated on every sequence $\vec{e}_i, ..., \vec{e}_{ik}$, with ascending indices coincides with φ. Thus it is enough to check that

$$\sum_{1 \le i1 < \ldots < ik \le n} a_{i1\ldots ik}\, dx_{i1} \Lambda \ldots \Lambda\, dx_{ik} \left(\vec{e}_j, ..., \vec{e}_{jk}\right) \quad (2\text{-}22)$$

This is fairly easy to see. If $i_1, ..., i_k \ne j_1, ...\, j_k$, then there is at least one *I* that dose not appear among the *j's*, so the corresponding dx_j, acting on the. Matrix $\vec{e}_j$, ... , $\vec{e}_{jk}$, selects a row of zeroes. Thus

$$\int_{\gamma(A)} \varphi = \sum_A W_{\vec{F}} \left(P^0_{\gamma(u)} \left(\underbrace{\vec{D}_1 \gamma(u)}_{\vec{\gamma}'(u)}\right)\right) |du| = \int_A \vec{F}\left(\gamma(u) . \vec{\gamma}'(u)\right) |du| \quad (2\text{-}23)$$

is the determinant of a matrix with a row of zeroes, so it vanishes. But

$$dx_{j1,\dots,jk}(\vec{e}_{j1},\dots,\vec{e}_{jk}) = 1 \qquad (2\text{-}24)$$

since it is the determinant of the identity matrix.

Theorem (2-1-10):(Dimension of $A^k(IR^n)$)

The space $A^k(IR^n)$ has diuleusion equal to the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (2\text{-}25)$$

Proof:

This is just a matter of counting the elements of the basis: i.e.. the number of elementary k-forms on IR^n . Not for nothing is the binomial coefficient called "*n* choose *k*".

Example (2-1-11): (Dimension of $A^k(IR^n)$)

The dimension of $A^\circ(IR^3)$ and of $A^3(IR^3)$ is 1, and the dimension of $A^1(IR^3)$ and of $A^2(R^3)$ is 3, because on R^3 we have

$$\binom{3}{0} = \frac{3!}{0!(3)!} = 1 \quad \text{elementary 0-form;}$$

$$\binom{3}{1} = \frac{3!}{1!(3)!} = 3 \quad \text{elementary 1-form;}$$

$$\binom{3}{2} = \frac{3!}{2!(3)!} = 3 \quad \text{elementary 0-form;}$$

$$\binom{3}{3} = \frac{3!}{3!(0)!} = 1 \quad \text{elementary 1-form;}$$

Now we will discuss forms on vector spaces.So far we have been studying *k* forms on IR^n . When defining orientation, we will make vital use of *k*-forms on a subspace $E \subset IR^n$. It is no harder to write the definition when *E* is an abstract vector space.

Definition (2-1-12) : (The space $\mathrm{A}^k(E)$)

Let E be a vector space. Then $\mathbf{A}^k(E)$ is the set of functions that take k vectors in E and return a number, and which are multilinear and anti-symmetric.The main result we will need is the following:

Proposition (2-1-13):

If E has dimension m, then $\mathbf{A}^k(E)$ has dimension $\binom{m}{k}$.

Proof:

We already know the result when $E = \mathrm{IR}^m$, and we will use a basis to translate from the concrete world of IR^m to the abstract world of E. Let $\underline{b}$,…..$\underline{b}$ Then the transformation $\Phi_{\{\bar{b}\}} : \mathrm{IR}^m \to E$ given by

$$\begin{bmatrix} a \\ . \\ a \end{bmatrix} \mapsto a_1\underline{b}_1 + \ldots. + a_m \underline{b}_1 \qquad (2\text{-}26)$$

is an invertible linear transformation, which performs the translation "concrete → abstract." We will use the inverse dictionary $\Phi^{-1}_{\{\underline{b}\}}$ We claim that the forms $\varphi_{i1,\ldots,ik}$, $1 \le i_1, < \ldots.. < i_k \le m$, defined by,

$$\varphi_{i1,\ldots ik}(\underline{v}_1 \ldots \underline{v}_k) = dx_{i1}\Lambda \ldots \Lambda dx_{ik}\left(\Phi^{-1}_{\{\bar{b}\}}\left(\bar{v}_1\right), \ldots, \Phi^{-1}_{\{\bar{b}\}}\left(\bar{v}_k\right)\right) \qquad (2\text{-}27)$$

form a basis of $\mathrm{A}^k(E)$. There is not much to prove: all the properties follow immediately from the corresponding properties in IR^m. One needs to check that the $\varphi_{i1,\ldots ik}$ are multilinear and antisymmetric, that they are linearly independent, and that they span $\mathrm{A}^k(E)$.

let us see for instance that the $\varphi_{i1,\ldots ik}$ are linearly independent suppose that

$$\sum_{1 \le i1 < \ldots < ik \le M} a_{i1\ldots ik}\varphi_{i1,\ldots ik} = 0 \qquad (2\text{-}28)$$

Then applied to the particular vectors

$$\underline{b}_{j1}, \ldots, \underline{b}_{jk} = \Phi_{\{\underline{b}\}}\left(\bar{e}_{j1}\right) \ldots\ldots \Phi_{\{\underline{b}\}}\left(\bar{e}_{jk}\right) \qquad (2\text{-}29)$$

We will still get 0 .But

$$\sum_{1\le i1<\ldots<ik\le M} a_{i1\ldots ik}\varphi_{i1,\ldots ik}(\Phi_{\{\bar{b}\}}\left(\vec{e}_{j1}\right)\ldots\ldots\Phi_{\{\bar{b}\}}\left(\vec{e}_{jk}\right)) = \sum_{1\le i1<\ldots<ik\le n} a_{i1\ldots ik}dx_{i1}\Lambda\ldots\Lambda\, dx_{ik}\left(\vec{e}_{j},\ldots,\vec{e}_{jk}\right)$$

(2-30)

So all the coefficients are 0, and the forms are linearly independent. The case of greatest interest to us is the case when $m = k$

Corollary (2-1-14):

If E is a k-dimensional vector space, then $A^{k}(E)$ is a vector space of dimension 1.

Now we will illustrate the wedge product.We have used the wedge $\wedge$ to write down forms; now we will see what it means:

it denotes the wedge product, also known as the exterior product.

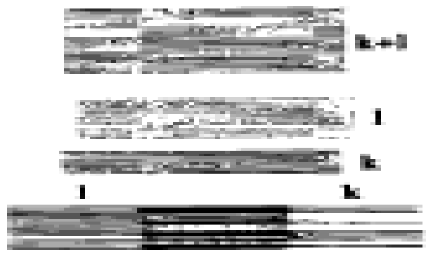
Definition (2-1-15):(Wedge product)

Let φ be a k-form and ψ be a l-form, both on IR^{n}. Then their wedge product $\varphi\wedge\psi$ is a (k + 1)-form that acts on k + l vectors. It is defined by the following sum, where the summation is over all permutations σ of the numbers 1, 2, 3, ... , k + 1 such that $\sigma(1)<\sigma(2)<\ldots<\sigma(k)\, and <\sigma(k+1)\ldots<\sigma(k+1)$:_

$$\overbrace{\varphi\Lambda\psi(\vec{v}_{1},\vec{v}_{2},\ldots\ldots\ldots\ldots,\vec{v}_{k+1})}^{\text{wedg product evaluted on } k+1 \text{vectors}} = \sum_{\substack{shuffles\\ \sigma\in perm(k,1)}} \operatorname{sgn}(\sigma)\underbrace{(\vec{v}_{\sigma(1)},\ldots\vec{v}_{\sigma(k)}}_{k\ \ vector}\psi\underbrace{(\vec{v}_{\sigma(k+1)},\ldots\vec{v}_{\sigma(k+1)}}_{i\ \ vector}$$

We start on the left with a (k + l)-form evaluated on k + l vectors. Oil the right we have a somewhat complicated expression involving a k-form φ acting on k vectors, and a l-form ip acting on l *vectors*. To understand the right-hand side, first consider all possible permutations of the k+l vectors $\vec{v}_{1},\vec{v}_{2},\ldots\ldots,\vec{v}_{k+1}$ dividing each permutation with a bar line $|$ so that there are k vectors to the left and l vectors to the right, since φ acts on k vectors and ψ acts on l vectors. (For example, if k = 2 and l = 1, one permutation would be written $\vec{v}_{1},\vec{v}_{2}|\vec{v}_{3}$, another would be written $\vec{v}_{2},\vec{v}_{3}|\vec{v}_{1}$, and a third $\vec{v}_{3},\vec{v}_{2}|\vec{v}_{1}$.)

Next, chose only those permutations where the indices for the k-form (to the left of the dividing bar) and the indices for the l-form (to the right of the bar) are each, separately and independently, in ascending order, as illustrated by Figure (2-3)



Figure(2-3)

(For k = 2 and l = 1, the only allowable choice is $\vec{v}_1, \vec{v}_2 | \vec{v}_3$ We assign each chosen permutation its sign.

Example (2-1-16):(The wedge product of two 1-forms)

If φ and ψ are both 1-forms, we have two permutations, $\vec{v}_1 | \vec{v}_2$ and $\vec{v}_2 | \vec{v}_1$, both allowable under our "ascending order" rule. The sign for the first is positive, since

$\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \to \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$ gives the permutation matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, with determinant + 1.

The sign for the second is negative, since $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \to \begin{bmatrix} \vec{v}_2 \\ \vec{v}_1 \end{bmatrix}$ gives the permutation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with determinat -1.So in this case the equation of Definition(2-1-14)becomes

$$(\varphi \Lambda \psi)(\vec{v}_1, \vec{v}_2) = \varphi\left(\vec{v}_1\right)\psi\left(\vec{v}_2\right) - \varphi\left(\vec{v}_2\right)\psi(\vec{v}_1) \qquad (2\text{-}31)$$

We see that the 2-form $dx_1 \wedge dx_2$

$$dx_1 \Lambda dx_2(\vec{a}, \vec{b}) = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1 \qquad (2\text{-}32)$$

is indeed equal to the wedge product of the 1-forms dx_1 and dx_2, which, evaluated on the same two vectors, gives

$$dx_1 \Lambda dx_2(\vec{a},\vec{b}) = dx_1(\vec{a})dx_2(\vec{b}) - dx_1(\vec{b})dx_2(\vec{a}) = a_1 b_2 - a_2 b_1 \qquad (2\text{-}33)$$

So our use of the wedge in naming the elementary forms is coherent with its use to denote this special kind of multiplication .

Example (2-1-17) : (The wedge product of a 2-form and a 1-form)

If φ is a 2-form and is a 1-form, then we have the six permutations

$$\vec{v}_1\vec{v}_2|\vec{v}_3 \;, \vec{v}_1\vec{v}_3|\vec{v}_2 \;, \vec{v}_2\vec{v}_3|\vec{v}_1, \vec{v}_3\vec{v}_1|\vec{v}_2, \vec{v}_2\vec{v}_1|\vec{v}_3 \text{ and } \vec{v}_3\vec{v}_2|\vec{v}_1 \qquad (2\text{-}34)$$

The first three are in ascending order, so we have three permutations to sum

$$+(\vec{v}_1,\vec{v}_2|\vec{v}_3) - (\vec{v}_1,\vec{v}_3|\vec{v}_2) + (\vec{v}_2,\vec{v}_3|\vec{v}_1) \qquad (2\text{-}35)$$

giving the wedge product

$$\varphi \,\Lambda\, \psi\; (\vec{v}_1,\vec{v}_2,\vec{v}_3) = \varphi(\vec{v}_1,\vec{v}_2)\psi(\vec{v}_3) - \varphi(\vec{v}_1,\vec{v}_3)\psi(\vec{v}_2) + \varphi(\vec{v}_2,\vec{v}_3)\psi(\vec{v}_1) \qquad (2\text{-}36)$$

Again, let's compare this result with what we using Definition(2-1-5) setting φ = $dx_1 \wedge dx_2$ and $\psi = dx_3$; and $\psi = dx_3$; to avoid double indices we will rename the vectors $\vec{v}_1,\vec{v}_2,\vec{v}_3$ calling them $\vec{u}$, $\vec{v}$, and $\vec{w}$. Using Definition (2-1-5) we get

$$\underbrace{dx_1 \Lambda\, dx_2}_{\varphi} \Lambda \underset{\psi}{dx_3}\; (\vec{u},\vec{v},\vec{w}\;) = \det\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \qquad (2\text{-}37)$$

$$= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2\; v_3 w_1 - u_3 v_1 w_2 - u_3 v_2 w_1$$

If instead we use Equation (2-35) for the wedge product, we get

$$\left(dx_1 \Lambda\; dx_2\right)\Lambda dx_3(\vec{u},\vec{v},\vec{w}) = \left(dx_1 \Lambda\; dx_2\right)\left(\vec{u},\vec{v}\right)dx_3\left(\vec{w}\right)$$

$$-\left(dx_1 \Lambda\; dx_2\right)\left(\vec{u},\vec{w}\right)dx_3\left(\vec{v}\right) + \left(dx_1 \Lambda\; dx_2\right)\left(\vec{v},\vec{w}\right)dx_3\left(\vec{u}\right)$$

$$= \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} w_3 - \det\begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix} v_3 + \det\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} u_3 \qquad (2\text{-}38)$$

$= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 \; v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1$.

In the following we study properties of the wedge product.
The wedge product behaves much like ordinary multiplication, except that one needs to be careful about the sign, because of skew commutativity:

Proposition (2-1-18):(Properties of the wedge product)

The wedge product has the following properties:

(1) distributivity : $\varphi \Lambda (\psi_1 + \psi_2) = \varphi \Lambda \psi_1 + \varphi \Lambda \psi_2$ (2-39)

(2) associativity. : $(\varphi_1 \Lambda \varphi_2)\varphi_3 = \varphi_1(\varphi_2 \Lambda \varphi_3)$ (2-40)

(3) skew commutativity : If φ is a k-Form and ψ is an L-Form , Then

$$\varphi \Lambda \psi = (-1)^{k1} \psi \Lambda \varphi \tag{2-41}$$

Note that in Equation (2-40) the φ and ψ change positions. For example, if $\varphi = dx_l \wedge dx_2$ and $\psi = dx_3$, skew commutativity says that

$$(dx_1 \Lambda \; dx_2)\Lambda dx_3 = (-1)^2 dx_3 \Lambda (dx_1 \Lambda \; dx_2), \text{ i.e}$$

$$\det\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = \det\begin{bmatrix} u_3 & v_3 & w_3 \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{bmatrix}, \tag{2-42}$$

which you can confirm either by observing that the two matrices differ by two exchanges of rows (changing the sign twice) or by carrying out the computation.
Now we will study integrating form field over parametrized domains.
The objective of this chapter is to define integration and differentiation over oriented domains. We now make our first stab at defining integration of forms; we will translate these results into the language of vector calculus in Section (2.1) and will return to orientation and integration of form fields in Section(2.1) We say that k linearly independent vectors $\vec{v}_1,....,\vec{v}_k$ in IR^k form a direct basis of IR^k if $\det[\vec{v}_1,....,\vec{v}_k] > 0$, otherwise an indirect basis. Of

course, this depends on the order in which the vectors $\vec{v}_1,....,\vec{v}_k$ are taken. We want to think of things like the k-parallelogram $P_x(\vec{v}_1,....,\vec{v}_k)$ in IR^k (which is simply a subset of IR^k) plus the information that the spanning vectors form a direct or an indirect basis.

The situation when there are k vectors in IR^n and $k \neq n$ is a little different. Consider a parallelogram in IR^3 spanned by two vectors, for instance

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad (2\text{-}43)$$

This parallelogram has two orientations, but neither is more "direct" than the other. Below we define orientation for such objects.

An oriented k-parallelogram in IR^n denoted $\pm p_0^x(\vec{v}_1,....,\vec{v}_k)$ a k-parallelogram as defined in Definition (1.1.1), except that this time all the symbols written are part of the data: the anchor point, the vectors $\vec{v_i}$, and the sign. As usual, the sign is usually omitted when it is positive.

Definition(2-1-19):

An oriented k-parallelogram $\pm p_0^x(\vec{v}_1,....,\vec{v}_k)$ is a k-parallelogram in which the sign and the order of the vectors are part of the data. The oriented k-parallelograms

$$p_0^x(\vec{v}_1,....,\vec{v}_k) \text{ and } -p_0^x(\vec{v}_1,....,\vec{v}_k)$$

have opposite orientations, as do two oriented k-parallelogram $p_0^x(\vec{v}_1,....,\vec{v}_k)$ if two of the vectors are exchanged.

Two oriented k-parallelograms are opposite if the data for the two is the same, except that either (1) the sign is changed, or (2) two of the vectors are exchanged (or, more generally, there is an odd number of transpositions of vectors). They are equal if the data is the same except that (1) the order of the vectors differs by an even number of transpositions, or (2) the order differs by an odd number of transpositions, and the sign is changed. For example

$$p_0^x(\vec{v}_1,\vec{v}_2) \quad \text{and} \quad p_0^x(\vec{v}_2,\vec{v}_1) \quad \text{are opposite ;}$$

$$p_0^x(\vec{v}_1,\vec{v}_2) \quad \text{and} \quad -p_0^x(\vec{v}_2,\vec{v}_1) \quad \text{are equal ;}$$

$p_0^x(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ and $p_0^x(\vec{v}_2,\vec{v}_1,\vec{v}_3)$ are opposite ;

$p_0^x(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ and $-p_0^x(\vec{v}_2,\vec{v}_3,\vec{v}_1)$ are opposite ;

$p_0^x(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ and $p_0^x(\vec{v}_2,\vec{v}_3,\vec{v}_1)$ are equal .

Are $p_0^x(\vec{v}_1,\vec{v}_2,\vec{v}_3)$, $p_0^x(\vec{v}_3,\vec{v}_2,\vec{v}_1)$, $-p_0^x(\vec{v}_2,\vec{v}_3,\vec{v}_1)$ equal or opposite

Now we will express form fields.
Most often, rather than integrate a. k-form, we will integrate a k-form field. A k-form field V on an open subset U of IR^n assigns a k-form $\varphi(x)$ to every point x in U. While the number returned by a k-form depends only on k vectors.
the number returned by a k-form field depends also on the point at which is evaluated: a k-form is a function of k vectors, but a k-form field is a function of an oriented k-parallelogram $P_x(\vec{v}_1,....,\vec{v}_k)$, which is anchored at x.

Definition (2-1-20):

A k-form field on an open subset $U \subset IR^n$ is a function that takes k vectors $\vec{v}_1,....,\vec{v}_k$ anchored at a point $x \in IR^n$, and which returns a number. It is multilinear and antisymmetric as a function of the $\vec{V}$s.
We already know how to write k-form fields: it is any expression of the form

$$\varphi = \sum_{1\le i1<\ldots<ik\le M} a_{i1\ldots ik}(x)dx_i \Lambda \ldots \Lambda\, dx_{ik} \qquad (2\text{–}44)$$

where the $a_{i1,..,ik}$ are real-valued functions of $x \in U$.

Example (2-1-21):(A 2-form field on IR^3)

The form field $\cos(xz)\, dx \Lambda dy$ is a 2-form field on IR^3. Below it is evaluated twice, each time on the same vectors, but at different points

$$\cos(xz)dx\ \Lambda\ dy\left(P_{\begin{pmatrix}1\\2\\\pi\end{pmatrix}}^{0}\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)\right) = (\cos(1.\pi))\det\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} = -2.$$

$$\cos(xz)dx \wedge dy\left(P^{0}_{\begin{pmatrix}\frac{1}{2}\\2\\\pi\end{pmatrix}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)\right)=\left(\cos\left(\frac{1}{2}.\pi\right)\right)\det\begin{bmatrix}1&2\\0&2\end{bmatrix}=0$$

Section (2.2): Integrating form fields and vector calculus

Now we will discuss integrrating form filds over parametrized domains.

Before we can integrate form fields over oriented domains, we must define the orientation of domains;. Here, as an introduction, we will show how to integrate form fields over domains that come naturally equipped with orientation-preserving parametrizations: parametrized domains.

A parametrized *k*-dimensional domain in IR^n is the image $\gamma(A)$ of a C^1 mapping γ that goes from a payable subset A of IR^K to IR^n . Such a domain $\gamma(A)$ may well not be a smooth manifold ; a mapping γ always parametrizes some thing or other in IR^n, but $\gamma(A)$ may have horrible singularities (although it is more likely to be mainly a k-dimensional manifold with some bad points). If we had to assign orientation to $\gamma(A)$ this would be a problem; we will see in Section (2.2) how to assign orientation to a manifold, but we don't know how to assign orientation to something that is "mainly a k-dimensional manifold with some bad points."

Fortunately, for our purposes here it doesn't matter how nasty the image is. We don't need to know what $\gamma(A)$ looks like, and we don't have to determineits orientation. We are not thinking of $\gamma(A)$ in its own right, but as "the result of γ acting on A." A parametrization by a mapping ry automatically carries an orientation: γ maps an oriented k-parallelogram $p_0^x(\vec{v}_1,....,\vec{v}_k)$ to a curvilinear parallelogram that can be approximated by $P^0_{\gamma(x)}\left(\vec{D}_1\gamma(x),\ldots,\vec{D}_k\gamma(x)\right)$; the order of the vectors in this k-parallelogram depends on the order of the variables in IR^k. To the extent that $\gamma(A)$ has an orientation, it is oriented by this order of vectors.

The image $\gamma(A)$ comes with a natural decomposition into little pieces: take some N, and decompose $\gamma(A)$ into the little pieces $\gamma(C\cap A)$, where $C\in D_N(IR^n)$ Such a piece $\gamma(C\cap A)$, is naturally well approximated by a k-parallelogram: if u ∈ R^K is the lower left-hand corner of C, the parallelogram

$$P^0_{\gamma(u)}\left(\frac{1}{2N}\vec{D}_1\gamma(u),\ldots,\frac{1}{2N}\vec{D}_k\,\gamma(u)\right) \quad (2\text{-}45)$$

is the image of *C* by the linear approximation

$$w \to \gamma(u)\left[D\gamma(u)\right](w-u)\,to\ \gamma\ at\ u \quad (2\text{-}46)$$

So if φ is a k-form field on ' (or at least on a neighborhood of $\gamma(\mathrm{A})$ an approximation to

$$\int_{\gamma(A)}\varphi \quad (2\text{-}47)$$

should be

$$\sum_{\substack{c\in D_{N}(R^n)\\ A\cap c\neq\varnothing}}\varphi(P^0_{\gamma(u)}\left(\frac{1}{2N}\vec{D}_1\gamma(u),\ldots,\frac{1}{2N}\vec{D}_k\,\gamma(u)\right))$$

$$vol_K(c)\sum_{\substack{c\in D_{N}(R^n)\\ A\cap c\neq\varnothing}}\varphi(P^0_{\gamma(u)}\left(\vec{D}_1\gamma(u),\ldots,\vec{D}_k\,\gamma(u)\right)) \quad (2\text{-}48)$$

But this last sum is a Riemann sum for the integral

$$\sum_A\varphi(P^0_{\gamma(u)}\left(\vec{D}_1\gamma(u),\ldots,\vec{D}_k\,\gamma(u)\right)\left|d^Ku\right| \quad (2\text{-}49)$$

To be rigorous, we define $\int_{\gamma(A)}\varphi$ to be the above integral

Definition (2-2-1): (Integrating a k-form field over a parametrized domain)

Let $\mathrm{A}\subset \mathrm{IR}^k$ be a payable set and $\gamma : A\to \mathrm{IR}^n$ be a C^1 mapping. Then the integral of the k-form field φ over $\gamma(\mathrm{A})$ is

$$\int_{\gamma(A)}\varphi=\sum_A\underbrace{\varphi(P^0_{\gamma(u)}\left(\vec{D}_1\gamma(u),\ldots,\vec{D}_k\,\gamma(u)\right)}_{This\ is\ function\ of\ u}\left|d^Ku\right| \quad (2\text{-}50)$$

Example (2-2-2): (Integrating a 1-form field over a parametrized curve)

Consider a case where k = 1, n = 2. We will use $\gamma(u) = \begin{pmatrix} R\cos u \\ R\sin u \end{pmatrix}$ and will $R \sin u$ take A to be the interval [0, a], for some a > 0. If we integrate the 1-form field $x\ dy\ -y\ dx$ over $\gamma(A)$ using the above definition, we find

$$\int_{\gamma(A)} (x\ dy - y\ dx) = \int_{[0,a]} (x\ dy - y\ dx)\left(P^{0}_{\begin{pmatrix} R\cos\cos u \\ R\sin\sin u \end{pmatrix}} \begin{bmatrix} -R\sin\sin u \\ R\cos\cos u \end{bmatrix} \right) |du|$$

$$\int_{[0,a]} \left(R\cos u R\cos u - (R\sin u)(-R\sin u) \right) |du| = \int_{[0,a]} R^{3} |du|$$

$$\int_{0}^{a} R^{2} |du| = R^{2} \quad (2\text{-}51)$$

What would we have gotten if a < 0? Until the bottom line, everything is the same. But then we have to decide how to interpret [0, a]. Should we write

$$\int_{0}^{a} R^{2} du \quad \text{or} \quad \int_{a}^{0} R^{2} du \quad (2\text{-}52)$$

We have to choose the second, because we are now integrating over an oriented interval, and we must choose the positive orientation. So the answer is still R^2 a, which is now negative.

Example (2-2- 3):(Another parametrized curve)

In Example (2.2.2) , you probably saw that y was parametrizing an arc of circle. To carry out the sort of computation we are discussing, the image need not be a smooth curve. For that matter, we don't need to have any idea what $\gamma(A)$ looks like.

Take for instance $\gamma(t) = \begin{pmatrix} 1 + t^2 \\ \arctan t \end{pmatrix}$, set A = [0,a] for some a > 0 and $\varphi = x\ dy$. then

$$\int_{\gamma(A)} \varphi = \int_{[0,a]} x\ dy \left(P_{\begin{pmatrix} 1+t^2 \\ \arctan\arctan t \end{pmatrix}} \begin{bmatrix} 2t \\ \frac{1}{1+t^2} \end{bmatrix} \right) \quad (2\text{-}53)$$

$$=\int_0^a \frac{1+t^2}{1+t^2}|dt| = a$$

Example (2-2-4): (Integrating a 2-form field over a parametrized surface in IR^3)

Let us compute

$$\int_{\gamma(C)} dx \wedge dy + y dx \wedge dz \qquad (2\text{-}54)$$

Over the parametrized domain $\gamma(C)$ where

$$\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}, C = \left\{ \begin{pmatrix} s \\ t \end{pmatrix} \middle| \ 0 \le s, t \le 1 \right\} \qquad (2\text{-}55)$$

Applying Definition (2-2-1) , we find

$$\int_{\gamma(C)} dx \wedge dy + y dx \wedge dz$$

$$= \int_0^1\int_0^1 \left(dx \wedge dy + y dx \wedge dz\right)\left(P_{\begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}}\left(\begin{bmatrix} 1 \\ 2s \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2t \end{bmatrix} \right)\right)$$

$$\int_0^1\int_0^1 \left(-2s + s^2(2t)\right)|ds \ dt|$$

$$= \int_0^1 \left[s^2 + \frac{s^3}{3} 2t \right]_{s=0}^{s=1} |dt| = \int_0^1 \left(-1 + \frac{2t}{3} \right)|dt| \qquad (2\text{-}56)$$

$$= \left[-t + \frac{t^2}{3} \right]_0^1 = -\frac{2}{3}.$$

In the following we discuss forms and vector calculus .

The real difficulty with forms is imagining what they are. What "is" $dx_l \wedge dx_2 + dx_3 \wedge dx_4$? We have seen that it is the function that takes two vectors in IR^4, projects them first onto the (x_l, x_2)-plane and takes the signed area of the resulting parallelogram, then projects them onto the (x_3, x_4)-plane,

takes the signed area of that parallelogram, and finally adds the two signed volumes.
But that description is extremely convoluted, and although it isn't too hard to use it in computations, it hardly expresses understanding.
However, in IR^3, it really is possible to visualize all forms and form fields, because they can be described in terms of functions and vector fields. There are four kinds of forms on IR^3: 0-forms, 1-forms, 2-forms, and 3-forms. Each has its own personality.
0-form fields. In IR^3 and in any IR^n, a 0-form is simply a number, and a 0 form field is simply a function. If f is a function on an open subset $U \subset R^n$ and $f : U \to IR$ is a function, then the rule $f\left(P_x^0\right) = f(x)$ makes f into a 0-form field. The requirement of antisymmetry then says that

$$-f(P_x^0) = -f(x).$$

1-form fields. Let $\vec{F}$ be a vector field on an open subset $U \subset IR^n$. We can then associate to $\vec{F}$ a 1-form field $W_{\vec{F}}$, which we call the work form field:

Definition(2-2-5):(Work form field)

The work form field $W_{\vec{F}}$ of a vector Field $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$ is the 1-form field defined by

$$W_{\vec{F}}\left(P_x^0\left(\vec{v}\right)\right) = \vec{F}(x).\vec{v} \qquad (2\text{-}57)$$

This can also be written in coordinates: the work form field $W_{\vec{F}}$ of a vector Field $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$ *is the* $1 - form\ field\ F_1 dx_1 + \cdots + F_n dx_n$ indeed

$$\left(F_1 dx_1 + \ldots + F_n dx_n\right)\left(P_x^0\left(\vec{v}\right)\right) = \left(F_1(x)dx_1 + \ldots + F_n(x)dx_n\right)\begin{bmatrix} v_1 \\ \vdots \\ v_2 \end{bmatrix}$$

$$= F_1(x)dx_1 + \ldots + F_n(x)dx_n = \vec{F}(x).\vec{v}$$

In this form, it is clear from Theorem (2-1-8) that every 1-form on U is the work of some vector field. What have we gained by saying that that a 1-form

-field is the work form field of a vector field? Mainly that it is quite easy to visualize $W_{\vec{F}}$ and to understand what it measures: if $\vec{F}$ is a force field, its work form field associates to a little line segment the work that the force field does along the line segment. To really understand this you need a little bit of physics, but even without it you can see what it means. Suppose for instance that $\vec{F}$ is the force field of gravity. In the absence of friction, it requires no work to push a wagon of mass m horizontally from a to b; the vector $\overrightarrow{b-a}$ and the constant vector field representing gravity are orthogonal to each other, with dot product zero:

$$\begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix} \cdot \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ 0 \end{bmatrix} = 0 \qquad (2\text{-}58)$$

But if the wagon rolls down an inclined plane, the force field of gravity does "work" on the wagon equal to the dot product of gravity and the displacement vector of the wagon :

$$\begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix} \cdot \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix} = -gm\left(b_3 - a_3\right) \qquad (2\text{-}59)$$

which is positive, since b_3 - a_3 is negative. If you want to push the wagon back up the inclined plane, you will need to furnish the work, and the force field of gravity will do negative work.

For what vector field $\vec{F}$ can the 1-form field $x_2 dx_1 + x_2 x_4 dx_2 + x_1^2 dx_4$ be written as $W_{\vec{F}}$?"

2-forms. If $\vec{F}$ is a vector field on an open subset U ⊂ IR^3, then we can associate to it a 2-form field on U called its flux form field $\Phi_{\vec{F}}$, which we first saw in Example (2-1-1)

Definition(2-2-6):

The flux form field $\Phi_{\vec{F}}$ is the 2-form field defined by

$$\Phi_{\vec{F}}\left(P_x^0\left(\vec{v},\vec{w}\right)\right) = \det\left[\vec{F}\left(x\right),\vec{v},\vec{w}\right]. \qquad (2\text{-}60)$$

In coordinates, this becomes $\Phi_{\vec{F}} = F_1 dy \wedge dx - F_2 dx \wedge dz + F_3 dx \wedge dy$:

$$\left(F_1 dy \wedge dx - F_2 dx \wedge dz + F_3 dx \wedge dy\right) P_x^0 \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) \qquad (2\text{-}61)$$

$$F_1(x)(v_2 w_3 - v_3 w_2) - F_2(x)(v_1 w_3 - v_3 w_1) + F_3(x)(v_1 w_2 - v_2 w_1) = \det\left[\vec{F}(x), \vec{v}, \vec{w}\right]$$

In this form, it is clear, again from Theorem (2-1-8) , that all 2-form fields on IR^3 are flux form fields of a vector field : the flux form field is a linear combination of all the elementary 2-forms on IR^3, so it is just a question of using the coefficients of the elementary forms to make a vector field.
Once more, what we have gained is an ability to visualize, as suggested by Figure (2-4) : the flux form field of a vector field associates to a parallelogram the flow of the vector field through the parallelogram

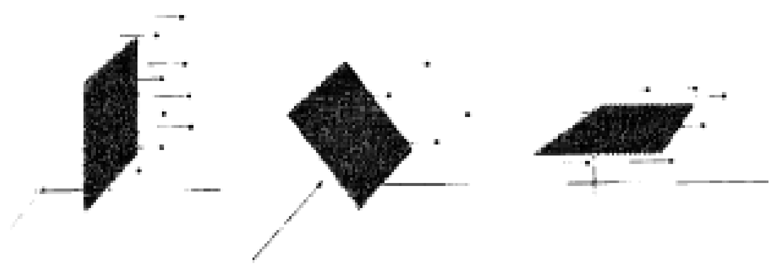


Figure (2- 4)

Figure (2-4) . The flow of $\vec{F}$ through a surface depends on the angle between $\vec{F}$ and the surface. Left: $\vec{F}$ is orthogonal to the surface, providing maximum flow. This corresponds to $\vec{F}(x)$ being perpendicular to the parallelogram spanned by, $\vec{v}, \vec{w}$. (The volume of the parallelepiped is $\det[\vec{F}, \vec{v}, \vec{w}] = \vec{F}(\vec{v} \times \vec{w})$ which is greatest when the angle θ between $\vec{F}$ and $\vec{v} \times \vec{w}$ is 0, since $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta$.) Middle : $\vec{F}$ is not orthogonal to the surface, allowing less flow. Right: $\vec{F}$ is parallel to the surface; the flow is 0. In this case P_x^0 ($\vec{F}$ (x) , $\vec{v}$, $\vec{w}$) is flat. This corresponds to $\vec{F}(\vec{v} \times \vec{w}) = 0$ i.e., $\vec{F}$ is perpendicular to $\vec{v} \times \vec{w}$ and therefore parallel to the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

If ? is the velocity vector field of a fluid, the integral of its flux form field over a surface measures the amount of fluid flowing through the surface. Indeed, the fluid which flows through the parallelogram $P_x^0(\vec{v},\vec{w})$ in unit time will fill the parallelepiped P_x^0 ($\vec{F}$(x) , $\vec{v}$, $\vec{w}$): the particle which at time 0 was at the corner x is now at $x + \vec{F}(x)$. The sign is positive if $\vec{F}$ is on the same side of the parallelogram as $\vec{v} \times \vec{w}$, otherwise negative (and 0 if $\vec{F}$ is parallel to the parallelogram; indeed, nothing flows through it then).

3-forms. Any 3-forms on an open subset of IR^3 is the 3-forms $dx \wedge dy \wedge dz$ (alias the determinant) multiplied by a function: we will denote by pf the 3-forms $f\, dx \wedge dy \wedge dz$, and call it the density of f.

Definition(2-2-7):(Density form of a function)

Let $U \subset$ IR^3 be open. The density form ρ_f of a function $f : U \to$ R is the 3-forms defined by:

$$\underbrace{\rho_f}_{\substack{density\ form\\ of\ f}} \quad P_x^0(\vec{F}(\mathrm{x}),\vec{v},\vec{w})P_x^0(\vec{v}_1,\vec{v}_2,\vec{v}_3) = f(x)\underbrace{\det\left(\vec{v}_1,\vec{v}_2,\vec{v}_3\right)}_{signed\ volume\ of\ P} \tag{2-62}$$

Summary: work, flux, and density forms on IR^3

Let f be a function on IR^3 and $\vec{F} = \begin{bmatrix} F_3 \\ F_2 \\ F_1 \end{bmatrix}$ be a vector field. Then

$$W_{\vec{F}} = F_1 dx + F_2 dy + F_3 dz \tag{2-63}$$

$$\Phi_{\vec{F}} F_1 dx \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy \tag{2-64}$$

$$\rho_f = f\ dx \wedge dy \wedge dz \tag{2-65}$$

Now we will discuss integrating work, flux and density form fields over parametrized domains .

Now let us translate Definition (2-2-1) (integrating a k-form field over a parametrized domain) into the language of vector calculus.

Example(2-2-8):

(Integrating a work form field over a parametrized curve) When integrating the work form field over a parametrized curve $\gamma(A) = C$, the equation of Definition (2-2-1) :

$$\int_{\gamma(A)} \varphi = \sum_{A} \varphi(P^{0}_{\gamma(u)} \left(\vec{D}_1 \gamma(u), \ldots, \vec{D}_k \gamma(u) \right) \left| d^K u \right| \qquad (2\text{-}66)$$

Becomes

$$\int_{\gamma(A)} \varphi = \sum_{A} W_{\vec{F}} (P^{0}_{\gamma(u)} \left(\underbrace{\vec{D}_1 \gamma(u)}_{\vec{\gamma}'(u)} \right) |du| = \int_{A} \vec{F} \left(\gamma(u) . \vec{\gamma}'(u) \right) |du| \qquad (2\text{-}67)$$

This integral measures the work of the force field $\vec{F}$ along the curve.

Example (2-2-9):

(Integrating a work form field over a helix) What is the work of the vector field

$$\underbrace{\rho_f}_{\substack{density\ form \\ of\ f}} \quad P_x^0(\vec{F}(\mathrm{x}), \vec{v}, \vec{w}) P_x^0(\vec{v}_1, \vec{v}_2, \vec{v}_3) = f(x) \underbrace{\det\left(\vec{v}_1, \vec{v}_2, \vec{v}_3 \right)}_{signed\ volume\ of\ P}$$

over the helix parametrized by (2- 67)

$$\gamma(t) = \begin{bmatrix} \cos\cos t \\ \sin\sin t \\ t \end{bmatrix}, 0 < \mathrm{t} < 4\pi \,? \qquad (2\text{-}68)$$

By Equation (2-67) this is

$$\int_0^{4\pi} \begin{bmatrix} \sin\sin t \\ -\cos\cos t \\ 0 \end{bmatrix} . \begin{bmatrix} -\sin -\sin t \\ \cos\cos t \\ 1 \end{bmatrix} dt = \int_0^{4\pi} \left(-\sin^2 t - \cos^2 t \right) dt = -4\pi \qquad (2\text{-}69)$$

Example (2-2-10):(Integrating a flux form field over a parametrized surface)

Let U be a subset of IR^2 , $\gamma : \mathrm{U} \to \mathrm{IR}^3$ be a parametrized domain, and $\vec{F}$ a vector field defined on a neighborhood of S. Then

$$\int_{\gamma(U)} \Phi_{\vec{F}} \left(P^0_{\gamma(u)} \left(\vec{D}_1 \gamma(u), \vec{D}_2 \gamma(u) \right) \left| d^2u \right| \right) \qquad (2\text{-}70)$$

$$= \int_U \det[\vec{F}\, \gamma(u), \vec{D}_1 \gamma(u), \vec{D}_2 \gamma(u)] \left| d^2u \right|$$

If $\vec{F}$ is the velocity vector field of a fluid, this integral measures the amount of fluid flowing through the surface S .

Example (2-2-11):

The flux of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{bmatrix} x \\ y^2 \\ z \end{bmatrix}$ through the parametrized domain

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}, 0 \le u, v \le 1$$

$$\int_0^1 \int_0^1 \det \begin{bmatrix} u^2 & 2u & 0 \\ u^2 v^2 & v & u \\ v^2 & 0 & 2v \end{bmatrix} du\ dv = \int_0^1 \int_0^1 \left(2u^2 v^2 - 4u^3 v^3 + 2u^2 v^2 \right) du dv$$

$$\int_0^1 \left[\frac{4}{3} u^3 v^2 - u^4 v^3 \right]_{u=0}^1 dv = \int_0^1 \left(\frac{4}{3} v^2 - v^3 \right) = -\frac{1}{4} \qquad (2\text{-}71)$$

Example (2-2-12):(Integrating a density form field over a parametrized piece of IR^3)

Let U, V $\subset$ IR^3 be open sets, and γ: U$\to$V be a C^1 mapping. If f: V $\to$ IR is a function then

$$\int_{\gamma(U)} \rho f = \int_U \rho f \left(P^0_{\gamma(u)} \left(\vec{D}_1 \gamma(u), \vec{D}_2 \gamma(u), \vec{D}_3 \gamma(u) \right) \left| d^3u \right| \right)$$

$$\int_U f \left(\gamma(u) \right) \det \det \left[D\gamma(u) \right] \left| d^3u \right| \qquad (2\text{-}72)$$

There is a particularly important special case of such a mapping y : U - V:

the case where $V = U$ and $\gamma(\mathrm{x}) = \mathrm{x}$ is the identity. In that case, the formula for integrating a density form field becomes

$$\int_{\gamma(U)} \rho f = \int_U f(u)\left|d^3u\right|, \qquad (2\text{-}73)$$

i.e., the integral of ρf is simply what we had called the integral of f in section (2.1) If f is the density of some object, then this integral measures its mass

Example (2-2-13): (Integrating a density form)

Let f be the function

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 \qquad (2\text{-}74)$$

and for $r < R$, let $T_{r,R}$ be the torus obtained by rotating the circle of radius r centered at $\binom{R}{0}$ in the (x, z)-plane around the z-axis, shown in Figure (2 - 5).

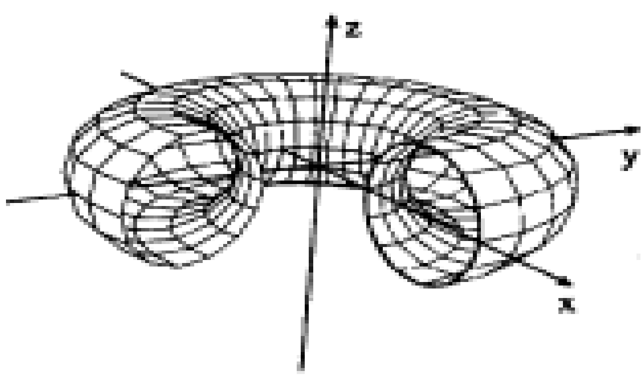


Figure (2 - 5)

Compute the integral of ρf over the region bounded by $T_{r,R}$ (i.e., the inside of the torus). Here, using the identity parametrization would lead to quite a clumsy integral. The following parametrization, with $0 \le \mathrm{u} \le \mathrm{r},\ 0 \le \mathrm{v},\ \mathrm{w} \le 2\pi$, is better adapted:

$$\gamma\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} (R+ucosv)cosw \\ (R+ucosv)sinw \\ usinv \end{pmatrix}. \qquad (2\text{-}75)$$

The integral becomes

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{r} -(R+ucosv)^2 u\left(R+ucosv\right)dudvdw$$

$$2\pi\int_0^{2\pi}(R^3+3R^2u^2cosv+3R^3cosv+u^4\cos^3 v)dudv \qquad (2\text{-}76)$$

$$=2\pi\int_0^{2\pi}\left(\frac{R^3r^2}{2}+R^2r^3cosv+\frac{3Rr^4\cos^2 v}{4}+\frac{r^5\cos^3 v}{5}\right)dv$$

$$=-\pi\left(2R^3r^2+\frac{3Rr^4}{2}\right)$$

You might wonder whether this has anything to do with the integral we would have obtained if we had used the identity parametrization. A priori, it doesn't , but actually if you look carefully, you will see that there is a computation of det[D_γ], and therefore that the change of variables formula might well say that the integrals are equal, and this is true. But the absolute value that appears in the change of variables formula isn't present here (or needed, since the determinant is positive). Really figuring out whether the absolute value is needed will be a lengthy story, involving a precise definition of orientation.

Now we will study work, flux and density in IR^n.

In all dimensions,

(1) 0-form fields are functions.

(2) Every 1-form field is the work form field of a vector field.

(3) Every (n - 1)-form field is the flux form field of a vector field

(4) Every n-form is the density form field of a function.

We've already seen this for 0-form fields and 1-form fields. In IR^3, the flux form field is of course a 2 = (n - 1)-form field; its definition can be generalized:

Definition (2-2-14):(Flux form field on IR^n)

If $U \subset R^n$ is an open subset and $\vec{F}$ is a vector field on U, then the flux form field $\Phi_{\vec{F}}$ is the (n -1)-form field defined by the formula

$$\Phi_{\vec{F}}\, p_0^x(\vec{v}_1. ...,\vec{v}_{n-1}) = \det\left[\vec{F}(x),\vec{v}_1 \ldots,\vec{v}_{n-1}\right] \qquad (2\text{-}77)$$

In coordinates, this becomes

$$\Phi_{\vec{F}} = \sum_{i=1}^{n} (-1)^{i-1} F_i d\ x_i \Lambda \ldots \Lambda\ dx_i \Lambda \ldots \Lambda\ dx_n$$

$$= F_1 d\ x_2 \Lambda \ldots \Lambda\ dx_n - F_2 d\ x_1 \Lambda\ dx_3 \Lambda \ldots \Lambda\ dx_n + \ldots + (-1)^{n-1} F_n d\ x_1 \Lambda\ dx_2 \Lambda \ldots \Lambda\ dx_{n-1}$$

where the term under the hat is omitted

For instance, the flux of the radial vector field $\vec{F} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\Phi_{\vec{F}} = (x_1 d\ x_2 \Lambda \ldots \Lambda\ dx_n) - (x_2 d\ x_1 \Lambda \ldots \Lambda\ dx_n) + \ldots \pm (x_n d\ x_2 \Lambda \ldots \Lambda\ dx_{n-1})$$

where the last term is positive if n is odd, and negative if it is even.

In any dimension n, n-form fields are multiples of the determinant, so all n-form fields are densities of functions:

Definition (2-2-15):(Density form field on IR^n)

Let $U \subset IR^n$ be open. The density form field ρf of a function $f : U \to IR$ is given by $\rho f = f\ dx_1 \Lambda\ ...\ \Lambda\ dx_n$.

The correspondences between form fields, functions and vectors, summarized in Table (2-1) , explain why vector calculus works in IR^3-and why it doesn't work in higher dimensions than 3. For k-forms on IR^n , when k is anything other than $0, 1, n - 1$, or n, there is no interpretation of form fields in terms of functions or vector fields.

A particularly important example is the electromagnetic field, which is a 6-component object, and thus cannot be represented either as a function (a 1-component object) or a vector field (in IR^4, a 4-component object).

The standard way of dealing with the problem is to choose coordinates x, y, z, t, in particular choosing a specific space-like subspace and a specific time like subspace, quite likely those of your laboratory. Experiment indicates the following force law: there are two vector fields, $\vec{E}$ (the electric field) and $\vec{B}$ (the magnetic field), with the property that a charge q at (x, t) and with velocity v (in the laboratory coordinates) is subject to the force

$$q\left(\vec{E}(x) + \frac{\vec{V}}{C} \times \vec{B}(x)\right) \qquad (2\text{-}78)$$

.

But $\vec{E}$ and $\vec{B}$ are not really vector fields. A true vector field keeps its individuality when you change coordinates. In particular, if a vector field is $\vec{0}$ in one coordinate system, it will be $\vec{0}$ in every coordinate system. This is not true of the electric and magnetic fields. If in one coordinate system the charge is at rest and the electric field is $\vec{0}$, then the particle will not be accelerated in those coordinates. In another system moving at constant velocity with respect to the first (on a train rolling through the laboratory, for instance) it will still not be accelerated. But it now feels a force from the magnetic field, which must be compensated for by an electric field, which cannot now be zero.
Is there something natural that the electric field and the magnetic field together represent? The answer is yes: there is a 2-form field on IR^4, namely

$$E_x dx \wedge cdt + E_y dy \wedge cdt + E_3 dz \wedge cdt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$
$$= W_{\vec{E}} \wedge cdt + \Phi_{\vec{B}} \qquad (2\text{-}79)$$

This 2-form field, which the distinguished physicists Charles Misner, Kip Thorne, and J. Archibald Wheeler call the Faraday (in their book Gravitation, the bible of general relativity), is really a natural object, the same in every inertial frame. Thus form fields are really the natural language in which to write Maxwell's equations

Form fields	**Vector Calculus**	
	IR^3	**IR^2**
0-form field	**Function**	**Function**
1-form field	**Vector field(via work form field)**	**Vector field**
(n-2)-form field	**Same as 1-form**	**No Equivalent**
(n-1)-form field	**Vector field (via flux form field)**	**Vector field**
n-form field	**Function (via density form field)**	**Function**

Table (1)**Correspondence between forms and vector calculus**.
In all dimensions, 0-form fields, 1-form fields, (n-1)-form fields, and n-form fields can be identified to a vector field or a function. Other form fields haveno equivalence in vector calculus.