

Chapter(1)

Parallelograms, Fractals and Fractional Dimension

Section (1.1):Parallelograms, parametrizations and arc length

We specify a k-parallelogram in $\mathbb{R}^n$ by the point x where it is anchored, and the k vectors which span it.

Definition(1-1-1):

A k-parallelogram in $\mathbb{R}^n$ is the subset of $\mathbb{R}^n$.

$$p_x(\vec{V}_1, \dots, \vec{V}_k) = \{x + t_1 \vec{V}_1 + \dots + t_k \vec{V}_k\} \sin^{-1} \theta$$

where $x \in \mathbb{R}^n$ is a point and $\vec{V}_1, \dots, \vec{V}_k$ are k vectors. The corner x is part of the data, but the order in which the vectors are listed is not.

For example,

- (1) $p_x(\vec{V})$ is the line segment joining x to $x + \vec{v}$.
- (2) $p_x(\vec{V}_1, \vec{V}_2)$ is the (ordinary) parallelogram with its four vertices $x, x + \vec{v}_1, x + \vec{v}_2, x + \vec{v}_1 + \vec{v}_2$
- (3) $p_x(\vec{V}_1, \vec{V}_2, \vec{V}_3)$ is the (ordinary) parallelepiped with its eight vertices at.

$$\begin{aligned} & x, x + \vec{v}_1, x + \vec{v}_2, x + \vec{v}_3, x + \vec{v}_1 + \vec{v}_2 \\ & x + \vec{v}_1 + \vec{v}_3, x + \vec{v}_1 + \vec{v}_2, x + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \end{aligned}$$

Now we will discuss the volume of k-parallelograms.

Clearly the k-dimensional volume of a k-parallelogram $p_x(\vec{V}_1, \dots, \vec{V}_k)$ does not depend on the position of x in $\mathbb{R}^n$. But it isn't obvious how to compute this volume. Already the area of a parallelogram in $\mathbb{R}^3$ is the length of the cross product of the two vectors spanning it , and the formula is quite messy. How will we compute the area of a parallelogram in $\mathbb{R}^4$, where the cross product does not exist, never mind a 3-parallelogram in $\mathbb{R}^5$?

It comes as a nice surprise that there is a very pretty formula that covers all cases. The following proposition, which seems so innocent, is the key.

Proposition (1-1-2):

(Volume of a k-parallelogram in $\mathbb{R}^k$) Let $\vec{v}_1, \dots, \vec{v}_k$ be k vectors in $\mathbb{R}^k$, so that $T = [\vec{v}_1, \dots, \vec{v}_k]$ is a square $k \times k$ matrix. Then

$$\text{Vol}_k P(\vec{v}_1, \dots, \vec{v}_k) = \sqrt{\det(T^T T)} \quad (1-1)$$

Proof:

We have

$$\sqrt{\det(T^T T)} = \sqrt{\det \det(T) \det \det(T)} \sqrt{(\det T)^2} = |\det T| \quad (1-2)$$

$$\underbrace{\begin{bmatrix} \dots & \vec{v}_1^T & \dots \\ \dots & \vec{v}_2^T & \dots \\ \dots & \vdots & \dots \\ \dots & \vec{v}_K^T & \dots \end{bmatrix}}_{T^T} \underbrace{\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_K \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}}_T = \underbrace{\begin{bmatrix} |\vec{v}_1|^2 & \vec{v}_1 \cdot \vec{v}_2 & \dots & \vec{v}_1 \cdot \vec{v}_K \\ \vec{v}_1 \cdot \vec{v}_2 & |\vec{v}_2|^2 & \dots & \vec{v}_2 \cdot \vec{v}_K \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_1 \cdot \vec{v}_K & \vec{v}_2 \cdot \vec{v}_K & \dots & |\vec{v}_n|^2 \end{bmatrix}}_{T^T T} \quad (1-3)$$

The point of this is that the entries of $T^T T$ are all dot products of the vectors $\vec{v}_i$. In particular, they are computable from the lengths of the vectors $\vec{v}_1, \dots, \vec{v}_k$ and angles between these vectors; no further information about the vectors is needed.

Example (1-1-3):

(Computing the volume of parallelograms in $\mathbb{R}^2$ and $\mathbb{R}^3$) When $k = 2$, we have

$$\det(T^T T) = \det \begin{bmatrix} |\vec{v}_1|^2 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_1 \cdot \vec{v}_2 & |\vec{v}_2|^2 \end{bmatrix} = |\vec{v}_1|^2 \cdot |\vec{v}_2|^2 - (\vec{v}_1 \cdot \vec{v}_2)^2 \quad (1-4)$$

If you write $\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| |\vec{v}_2| \cos \theta$, this becomes

$$\det(T^T T) = |\vec{v}_1|^2 |\vec{v}_2|^2 (1 - \cos^2 \theta) = |\vec{v}_1|^2 |\vec{v}_2|^2 |\sin^2 \theta| \quad (1-5)$$

so that the area of the parallelogram spanned by $\vec{v}_1, \vec{v}_2$ is

$$vol_2 P(\vec{v}_1, \vec{v}_2) = \sqrt{\det(T^T T)} = |\vec{v}_1|^2 \cdot |\vec{v}_2|^2 \sin \theta \quad (1-6)$$

Of course, this should come as no surprise; But exactly the same computation in the case $n = 3$ leads to a much less familiar formula. Suppose $T = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$, and let us call θ_1 the angle between $\vec{v}_1$ and $\vec{v}_2$, θ_2 the angle between $\vec{v}_1$ and $\vec{v}_3$, and θ_3 the angle between $\vec{v}_2$ and $\vec{v}_3$. Then

$$T^T T = \begin{bmatrix} |\vec{v}_1|^2 & \vec{v}_1 \cdot \vec{v}_2 & \vec{v}_1 \cdot \vec{v}_3 \\ \vec{v}_1 \cdot \vec{v}_2 & |\vec{v}_2|^2 & \vec{v}_2 \cdot \vec{v}_3 \\ \vec{v}_1 \cdot \vec{v}_3 & \vec{v}_2 \cdot \vec{v}_3 & |\vec{v}_3|^2 \end{bmatrix} \quad (1-7)$$

and $\det T^T T$ is given by

$$\begin{aligned} & |\vec{v}_1|^2 \cdot |\vec{v}_2|^2 \cdot |\vec{v}_3|^2 + 2(\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{v}_3)(\vec{v}_1 \cdot \vec{v}_3) - |\vec{v}_1|^2 (\vec{v}_2 \cdot \vec{v}_3)^2 - |\vec{v}_2|^2 (\vec{v}_1 \cdot \vec{v}_3)^2 - |\vec{v}_3|^2 (\vec{v}_1 \cdot \vec{v}_2)^2 \end{aligned} \quad (1-8)$$

$$= |\vec{v}_1|^2 |\vec{v}_2|^2 |\vec{v}_3|^2 (1 + 2 \cos \theta_1 \cos \theta_2 \cos \theta_3 - \cos^2 \theta_1 - \cos^2 \theta_2 - \cos^2 \theta_3)$$

For instance, the volume of a parallelepiped spanned by three unit vectors, each making an angle of $\pi/4$ with the others, is

$$\sqrt{1 + 2 \cos 3 \frac{\pi}{4} - 3 \cos 2 \frac{\pi}{4}} = \sqrt{\frac{\sqrt{2}-1}{2}} \quad (1-9)$$

Thus we have a formula for the volume of a parallelepiped that depends only on the lengths and angles of the vectors that span it; we do not need to know what or where the vectors actually are. In particular, this formula is just as good for a k -parallelepiped in any $\mathbb{R}^n$, even (and especially) if $n > k$.

This leads to the following theorem.

Theorem (1- 1- 4): (Volume of a k -parallelepiped in $\mathbb{R}^n$)

Let $\vec{v}_1, \dots, \vec{v}_k$ be k vectors in $\mathbb{R}^n$, and T be the $n \times k$ matrix with these vectors as its columns:

$T = [\vec{v}_1, \dots, \vec{v}_k]$. Then the k -dimensional volume of $P_x(\vec{v}_1, \dots, \vec{v}_k)$ is

$$vol_k P(\vec{v}_1, \vec{v}_2) = \sqrt{\det(T^T T)} \quad (1-10)$$

Proof:

If we compute $T^T T$, we find

$$T^T T = \begin{bmatrix} \left| \vec{v}_1 \right|^2 & \vec{v}_1 \cdot \vec{v}_2 & \dots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_1 \cdot \vec{v}_2 & \left| \vec{v}_2 \right|^2 & \dots & \vec{v}_2 \cdot \vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_1 \cdot \vec{v}_n & \vec{v}_2 \cdot \vec{v}_n & \dots & \left| \vec{v}_n \right|^2 \end{bmatrix} \quad (1-11)$$

which is precisely our formula for the k-dimensional volume of a k-parallelotope in terms of lengths and angles.

Example (1- 1- 5):

What is the volume of the 3-parallelotope in $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Set $T = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ then

$$T^T T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \det T^T T = 4$$

so the volume is 2.

Now in the following in this section we are going to relax our definition of a parametrization. In we said that a parametrization of a manifold M is a C^1 mapping φ from an open subset $U \subset \mathbb{R}^n$ to M , which is one to one and onto, and whose derivative is also one to one.

The problem with this definition is that most manifolds do not admit a parametrization. Even the circle does not; neither does the sphere, nor the torus. On the other hand, our entire theory of integration over manifolds is going to depend on parametrizations, and we cannot simply give up on most examples.

Let us examine what goes wrong for the circle and the sphere. The most obvious parametrization of the circle is $\gamma : t \rightarrow \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. The problem is choosing a domain: If we choose $(0, 2\pi)$, then γ is not onto. If we choose $[0, 2\pi]$, the domain is not open, and γ is not one to one. If we choose $(0, 2\pi)$, the domain is not open.

For the sphere, spherical coordinates

$$\gamma : \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \rightarrow \begin{bmatrix} \cos \varphi \cos \theta \\ \cos \varphi \sin \theta \\ \sin \varphi \end{bmatrix} \quad (1-12)$$

present the same sort of problem. If we use as domain $(-\pi/2, \pi/2) \times (0, 2\pi)$, then γ is not onto; if we use $[-\pi/2, \pi/2] \times [0, 2\pi]$, then the map is not one to one, and the derivative is not one to one at points where $\varphi = \mp\pi/2, \dots$

The key point for both these examples is that the trouble occurs on sets of volume 0, and therefore it should not matter when we integrate. Our new definition of a parametrization will be exactly the old one, except that we allow things to go wrong on sets of k-dimensional volume 0 when parametrizing k-dimensional manifolds.

Now we will study sets of k-dimensional volume 0 in $\mathbb{R}^n$

Let X be a subset of $\mathbb{R}^n$. We need to know when X is negligible as far as k-dimensional integrals are concerned. Intuitively it should be fairly clear what this means: points are negligible for 1-dimensional integrals or higher, points and curves are negligible for 2-dimensional integrals, etc.

It is possible to define the k-dimensional volume of an arbitrary subset $X \subset \mathbb{R}^n$. That definition is quite elaborate; it is considerably simpler to say when such a subset has k-dimensional volume 0.

Definition (1- 1- 6):

A bounded subset of $\mathbb{R}^n$ has k-dimensional volume 0 if

$$\lim_{N \rightarrow \infty} \sum_{C \in DN} \left(\frac{1}{2^N}\right)^K = 0 \quad (1-13)$$

An arbitrary subset X has k -dimensional volume 0 if for all R , the bounded set $X \cap B_R(0)$ has k -dimensional volume 0.

Definition (1- 1-7):(parametrization of manifold)

Let $M \subset \mathbb{R}^n$ be a k -dimensional manifold and U be a subset of $\mathbb{R}^k$ with boundary of k -dimensional volume 0;

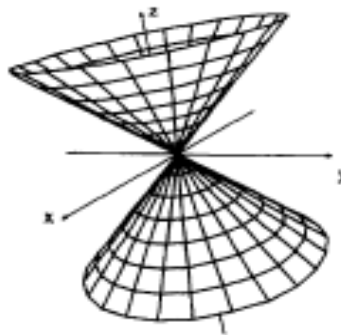
let $X \subset U$ have k -dimensional volume 0, and let $U - X$ be open. Then a continuous mapping $\gamma: U \rightarrow \mathbb{R}^n$ parametrizes M if

- (1) $\gamma(U) \supset M$;
- (2) $\gamma(U - X) \subset M$;
- (3) $\gamma(U - X) \rightarrow M$ is one to one, of class C^1 , with locally Lipschitz derivative;
- (4) the derivative $[D \gamma(u)]$ is one to one for all u in $U - X$;
- (5) $\gamma(X)$ has k -dimensional volume 0.

Often condition (1) will be an equality; for example, if M is a sphere and U a closed rectangle mapped to M by spherical coordinates, then $\gamma(U) = M$. In that case, X is the boundary of U , and $\gamma(X)$ consists of the poles and half a great circle (the international date line, for example), giving $\gamma(U - X) \subset M$ for condition (2).

Example(1-1-8):(Parametrization of a cone)

The subset of $\mathbb{R}^3$ of equation $x^2 + y^2 = z^2$, shown in Figure(1-1) , is not a manifold in the neighborhood of the vertex, which is at the origin. However, the subset



Figure(1-1)

Parametrization of a cone

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x^2 + y^2 - z^2 = 0, 0 < z < 1 \right\} \quad (1- 14)$$

is a manifold. Consider the map $\gamma: [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ given by

$$\gamma: \begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ r \end{pmatrix} \quad (1-15)$$

If we let $U = [0, 1] \times [0, 2\pi]$, and $X = \partial U$, then γ is a parametrization of M . Indeed, $\gamma: ([0, 1] \times [0, 2\pi]) \supset M$ (it contains the vertex and the circle of radius 1 in the plane $z = 1$, in addition to M), and γ does map $([0, 1] \times [0, 2\pi])$ into M (this time, it omits the line segment $x = z, y = 0$). The map is one to one on $([0, 1] \times [0, 2\pi])$, and so is its derivative.

Now we will discuss a small catalog of parametrizations.

As we will see below, essentially all manifolds admit parametrizations with the new definition. But it is one thing to construct such a parametrization using the implicit function theorem, and another to write down a parametrization explicitly.

Below we give a few examples, which frequently show up in applications and exam problems.

If U is an open subset of $\mathbb{R}^n$ with boundary all of k -dimensional volume 0, and $f: U \rightarrow \mathbb{R}^{n-k}$ is a C^1 mapping, then the graph of f is a manifold in $\mathbb{R}^n$, and the map

$$X \rightarrow \begin{pmatrix} x \\ f(x) \end{pmatrix} \quad (1-16)$$

is a parametrization.

There are many cases where the idea of parametrizing as a graph still works, even though the conditions above are not satisfied: those where you can "solve" the defining equation for $n - k$ of the variables in terms of the other k .

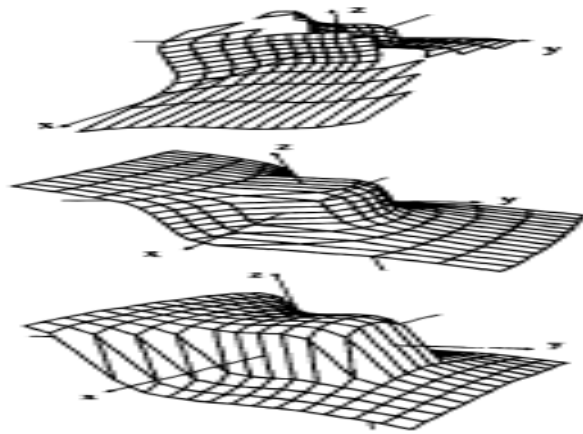
Example (1-1-9): (Parametrizing as a graph)

Consider the surface in $\mathbb{R}^3$ of equation $x^2 + y^3 + z^5 = 1$. In this case you can "solve" for x as a function of y and z :

$$x = \pm \sqrt{1 - y^3 - z^5} \quad (1-17)$$

You could also solve for y or for z , as a function of the other variables, and the three approaches give different views of the surface, as shown in Figure

(1-2). Of course, before you can call any of these parametrizations, you have to specify exactly what the domain is. When the equation is solved for x , the domain is the subset of the (y, z) -plane where $1 - y^3 - z^5 > 0$. When solving for y , remember that every number has a unique cube root, so the function $y = (1 - x^2 - z^5)^{1/3}$ is defined at every point, but it is not differentiable when $x^2 + z^5 = 1$, so this curve must be included in the set X of trouble points that can be ignored (using the notation of Definition (1-1-7))



Figure(1-2)
Parametrization of a graph

Now we will illustrate Surfaces of revolution. The graph of a function $f(x)$ is the curve C of equation $y = f(x)$.

Let us suppose that f takes only positive values, and rotate C around the x -axis, to get the surface of revolution of equation

$$y^2 + z^2 = (f(x))^2. \quad (1-18)$$

This surface can be parametrized by:

$$\gamma: \begin{pmatrix} x \\ \varphi \end{pmatrix} \rightarrow \begin{bmatrix} x \\ f(x)\cos \theta \\ f(x)\sin \theta \end{bmatrix} \quad (1-19)$$

Again, to be precise one must specify the domain of γ . Suppose that f

$(a, b) \rightarrow \mathbb{R}$ is defined and continuously differentiable on (a, b) . Then the domain of γ is $(a, b) \times (0, 2\pi)$ and γ is one to one, with derivative also one to one on $(a, b) \times (0, 2\pi)$

If C is a parametrized curve, (not necessarily a graph), say parametrized by $t \rightarrow \begin{pmatrix} u(t) \\ u(t) \end{pmatrix}$, the surface obtained by rotating C can still be parametrized by:

$$\begin{pmatrix} t \\ \phi \end{pmatrix} \rightarrow \begin{bmatrix} u(t) \\ u(t) \cos \theta \\ u(t) \sin \theta \end{bmatrix} \quad (1-20)$$

Spherical coordinates on the sphere of radius R are a special case of this construction: If C is the semi-circle of radius R in the (x, z) -plane, parametrized by:

$$\begin{pmatrix} x = R \cos \phi \\ z = R \sin \phi \end{pmatrix} \quad -\pi/2 \leq \phi \leq \pi/2 \quad (1-21)$$

then the surface obtained by rotating this circle is precisely the sphere of radius R centered at the origin in $\mathbb{R}^3$, parametrized by:

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} \rightarrow \begin{bmatrix} R \cos \phi \cos \theta \\ R \cos \phi \sin \theta \\ R \sin \phi \end{bmatrix} \quad (1-22)$$

the parametrization of the sphere by latitude and longitude

Example (1-1-10):

Consider the surface obtained by rotating the curve of equation $(1 - x)^3 = z^2$ in the (x, z) -plane around the z -axis. This surface has the equation

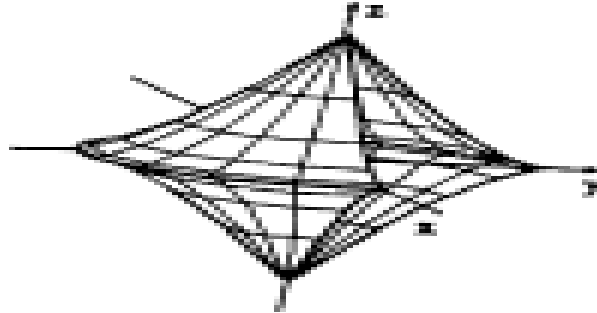
$$(1 - \sqrt{X^2 + Y^2})^3 = z^2.$$

The curve can be parametrized by:

$$\begin{pmatrix} X \\ Y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 1-t^2 \\ t^3 \\ t^3 \end{pmatrix} \quad (1-23)$$

so the surface can be parametrized by

$$\begin{pmatrix} t \\ \theta \end{pmatrix} \rightarrow \begin{bmatrix} (1-t^2)\cos\theta \\ (1-t^2)\sin\theta \\ t^3 \end{bmatrix} \quad (1-24)$$



Figure(1-3)

Represent the image of parametrization

Figure(1-3) represents the image of the parametrization $|t| < 1, 0 \leq \theta \leq 3\pi/2$. It can be guessed from the picture (and proved from the formula) that the subset of $[-1, 1] \times [0, 3\pi/2]$ where $t = \pm 1$ are trouble points (they correspond to the top and bottom "cone points"), and so is the subset $\{0\} \times [0, 3\pi/2]$, which corresponds to a "curve of cusps."

Now we will discuss the existence of parametrizations.

Since our entire theory of integrals will be based on parametrizations, it would be nice to know that manifolds, or at least some fairly large class of manifolds, actually can be parametrized.

Remark (1-1-11):

There is here some ambiguity as to what "actually" means. In the above examples, we came up with a formula for the parametrizing map, and that is what you would always like, especially if you want to evaluate an integral. Unfortunately, when a manifold is given by equations (the usual situation), it is usually impossible to find formulas for parametrizations. the parametrizing mappings only exist in the sense that the implicit function theorem guarantees their existence. If you really want to know the value of the mapping at a point, you will need to solve a system of nonlinear equations, presumably using Newton's method; you will not be able to find a formula .

Theorem (1-1-12): (What manifolds can be parametrized)

Let $M \subset \mathbb{R}^n$ be a manifold, such that there are finitely many open subsets $U_i \subset M$ covering M , corresponding subsets $V \subset \mathbb{R}^k$ all with boundaries of k -dimensional

volume 0, and continuous mappings $\gamma_i : \vec{V} \rightarrow \vec{M}$ which are one to one on V_i , with derivatives which are also one to one. Then M can be parametrized.

It is rather hard to think of any manifold that does not satisfy the hypotheses of the theorem, hence be parametrized. Any compact manifold satisfies the hypotheses, as does any open subset of a manifold with compact closure. We will assume that our manifolds can all be parametrized. The proof of this theorem is technical and not very interesting.

Now we will study change of parametrization.

Our theory of integration over manifolds will be set up in terms of parametrizations, but of course we want the quantities computed (arc length, surface area, fluxes of vector fields, etc.), to depend only on the manifold and the integrand, not the chosen parametrization. We show that the length of a curve, the area of a surface and, more generally, the volume of a manifold, are independent of the parametrization used in computing the length, area or volume. In all three cases, the tool we use is the change of variables formula for improper integrals, we set up a change of variables mapping and apply the change of variables formula to it. We need to justify this procedure, by showing that our change of variables mapping is something to which the change of variables formula can be applied.

Suppose we have a k -dimensional manifold M and two parametrizations

$$\gamma_1 : \overline{U}_1 \rightarrow M \quad \text{and} \quad \gamma_2 : \overline{U}_2 \rightarrow M \quad (1-25)$$

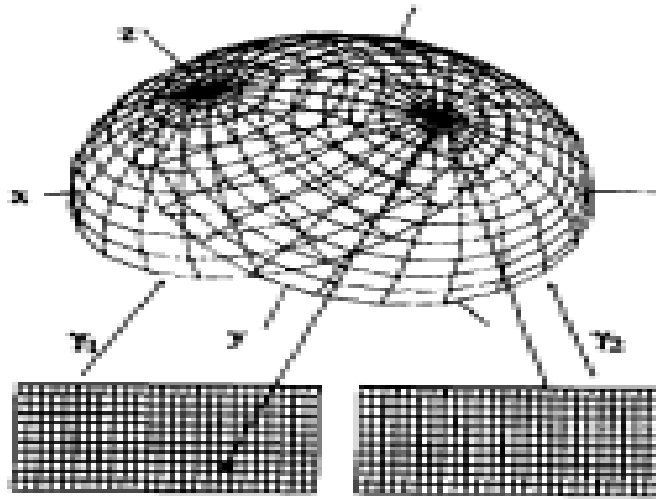
where U_1 and U_2 are subsets of $\mathbb{R}^k$. Our candidate for the change of variables

mapping is $\Phi = \gamma_2^{-1} \circ \gamma_1$, i.e.,

$$\overline{U} \xrightarrow{\gamma_1} M \xrightarrow{\gamma_2^{-1}} \overline{U} \quad (1-26)$$

Example (1-1-13):(Problems when changing parametrizations)

Let γ_1 and γ_2 be two parametrizations of S^2 by spherical coordinates, but with different poles. Call P_1, P'_1 (the poles for γ_1 and P_2, P'_2 the poles for γ_2 . Then $\gamma_2^{-1} \circ \gamma_1$ is not defined at $\gamma_2^{-1}(\{P_2, P'_2\})$. Indeed, some one point in the domain of γ_1 maps to P_2 . But as shown in Figure(1-4) , γ_2 maps a whole segment to P_2 , so that $\gamma_2^{-1} \circ \gamma_1$ maps a point to a line segment, which is nonsense. The only way to deal with this is to remove $\gamma_2^{-1}(\{P_2, P'_2\})$ from the domain of γ_1 , and hope that the boundary still has k-volume 0. In this case this is no problem: we just removed two points from the domain, and two points certainly have area 0.



Figure(1-4)

Changing parametrization

Let us set up our change of variables with a bit more precision. Let U_1 and U_2 be subsets of $\mathbb{R}^k$. Following the notation of Definition (1-1-7), denote by X_1 the negligible "trouble spots" of γ_1 , and by X_2 the trouble spots of γ_2 . In Example (1-1-11) , X_1 and X_2 consist of the points that are mapped to the poles

(i.e., the lines marked in bold in Figure(1-4) , 'If P_2 happens ables to be on the date line with respect to γ_1 , two points map to P_2 : in Figure, a point on the right-hand boundary of the rectangle, and the corresponding point on the left-hand boundary

$$y_1 = (\gamma_1^{-1} \circ \gamma_1)(x_1) \text{ and } y_2 = (\gamma_1^{-1} \circ \gamma_2)(X_2) \quad (1-27)$$

In Figure(1-5) , the dark dot in the rectangle at left is γ_2 , which is mapped by γ_1 to a pole of γ_2 and then by γ_2^{-1} to the dark line at right; γ_1 is the (unmarked) dot in the right rectangle that maps to the pole of γ_1 .

Set

$$U_1^{ok} = U_1 - (X_1 U Y_2) \quad \text{and} \quad U_2^{ok} = U_2 - (X_2 U Y_1) \quad (1-28)$$

i.e., we use the superscript "ok" to denote the domain or range of a change of mapping with any trouble spots of volume 0 removed

Theorem (1-1-14):

Both U_1^{ok} and U_2^{ok} are open subsets of $\mathbb{R}^k$ with boundaries of k-dimensional volume 0, and

$$\phi: U_1^{ok} \rightarrow U_2^{ok} = \gamma_2^{-1} \circ \gamma_1$$

is a C^1 diffeomorphism with locally Lipschitz inverse.

Theorem (1-1-14) says that Φ is something to which the change of variables formula applies.

Now we will study integrand $|d^1x|$, called the element of arc length, is an integrand to be integrated over curves. As such, it should take a 1-parallelotope $P_x(\vec{V})$ in $\mathbb{R}^n$ (i.e., a line segment) and return a number, and that is what it does:

$$|d^1x| (P_x(\vec{V})) = |\vec{V}| \quad (1-29)$$

More generally, if f is a function on $\mathbb{R}^n$, then the integrand $f |d^1x|$ is defined by the formula

$$f |d^1x| (P_x(\vec{V})) = f(x) |\vec{V}| \quad (1-30)$$

If $C \subset \mathbb{R}^n$ is a smooth curve, the integral

$$\int_C |d^1x| \quad (1-31)$$

is the number obtained by the following process: approximate C by little line segments as in Figure(1-5), apply $|d^1x|$ to each to get its length, and add. Then let the approximation become infinitely fine; the limit is by definition the length of C . We carried out this computation when I is the interval $[a, b]$,

and $C \subset \mathbb{R}^3$ is a smooth curve parametrized by $\gamma: I \rightarrow \mathbb{R}^3$, and showed that the limit is given by

$$\int_1 \left| d^1 x \right| P_{\gamma(t)}(\gamma'(t)) |dt| = \int_I |\gamma'(t)| |dt| \quad (1-32)$$

In particular, the integral

$$\int_I |\gamma'(t)| |dt| \quad (1-33)$$

depends only on C and not on the parametrization.

Example (1-1-15):

The graph of a C^1 function $f(x)$, for $a \leq x \leq b$, is parametrized by

$$X \rightarrow \begin{pmatrix} x \\ f(x) \end{pmatrix} \quad (1-34)$$

and hence its arc length is given by the integral

$$\int_{[a,b]} \left| \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} \right| dx = \int_a^b \sqrt{1 + (f'(x))^2} dx \quad (1-35)$$

Because of the square root, these integrals tend to be unpleasant or impossible to calculate in elementary terms. The following example, already pretty hard, is still one of the simplest. The length of the arc of parabola $y = ax^2$ for $0 \leq x \leq A$ is given by

$$\int_0^A \sqrt{1 + 4a^2 x^2} dx \quad (1-36)$$

A table of integrals will tell you that

$$\text{Area of } \gamma(C \cap U) \text{vol}_2(c) \sqrt{\det [D\gamma(u)]^T [D\gamma(u)]} \quad (1-37)$$

Setting $2ax = u$, this leads to

$$\begin{aligned}\int_0^A \sqrt{1+4a^2x^2} dx &= \frac{1}{4a} \left[2ax\sqrt{1+4a^2x^2} + \log \left| 2ax + \sqrt{1+4a^2x^2} \right| \right]_0^A \\ &= \frac{1}{2a} \left(2aA\sqrt{1+4a^2A^2} + \log \log \left| 2aA + \sqrt{1+4a^2A^2} \right| \right)\end{aligned}\quad (1-38)$$

Moral: if you want to compute arc length, brush up on your techniques of integration and dust off the table of integrals.

Curves in $\mathbb{R}^n$ have lengths even when $n > 3$, as the following example illustrates.

Example (1-1-16) :

Let p, q be two integers, and consider the curve in $\mathbb{R}^4$ parametrized by

$$\gamma(t) = \begin{pmatrix} \cos pt \\ \sin pt \\ \cos qt \\ \sin qt \end{pmatrix}, 0 \leq t \leq 2\pi \quad (1-39)$$

Its length is given by

$$\begin{aligned}\int_0^{2\pi} \sqrt{(-p \sin pt)^2 + (p \cos pt)^2 + (-q \sin qt)^2 + (q \cos qt)^2} dt \\ = 2\pi \sqrt{p^2 + q^2}\end{aligned}\quad (1-40)$$

can also measure data other than pure arc length, using the integral

$$\int_c \left(f(x) |dx| \right) \stackrel{\text{def}}{=} \int_1 \left(f(\gamma(t)) |\gamma'(t)| |dt| \right) \quad (1-41)$$

for instance if $f(x)$ gives the density of a wire of variable density, the integral above would give the mass of the wire. In other contexts (particularly surface area), it will be much harder to define the analogs of "arc length" independently of a parametrization. So here we give a direct

proof that the arc length given by Equation (1- 33) does not depend on the chosen parametrization; later we will adapt this proof to harder problems.

Proposition (1-1-17): (Arc length independent of parametrization)

Suppose $\gamma_1: I_1 \rightarrow \mathbb{R}^3$ and $\gamma_2: I_2 \rightarrow \mathbb{R}^3$ are two parametrizations of the same curve $C \in \mathbb{R}^3$. Then

$$\int_{I_1} \left| \vec{\gamma}'_1(t_1) \right| |dt_1| = \int_{I_2} \left| \vec{\gamma}'_2(t_2) \right| |dt_2| \quad (1- 42)$$

Example (1-1-18): (Parametrizing a half-circle)

We can parametrize the upper half of the unit circle by

$$x \rightarrow \left(\begin{array}{c} x \\ \sqrt{1-x^2} \end{array} \right), -1 \leq x \leq 1, \text{ or by } t \rightarrow \left(\begin{array}{c} \cos t \\ \sin t \end{array} \right), 0 \leq t \leq \pi \quad (1- 43)$$

In both cases we get length π . With the first parametrization we get

$$\int_{[-1,1]} \left| \left(\begin{array}{c} 1 \\ -x \\ \sqrt{1-x^2} \end{array} \right) \right| |dx| = \int_{[-1,1]} \sqrt{1 + \frac{x^2}{1-x^2}} |dx| \quad (1-44)$$

$$\int_{[-1,1]} \frac{1}{\sqrt{1-x^2}} |dx| = [\arcsin x]_{(-1)}^1 = \pi / 2 - (-\pi / 2) = \pi$$

The second gives

$$\int_{[0,\pi]} \left| \left(\begin{array}{c} -\sin t \\ \cos t \end{array} \right) \right| |dt| = \int_{[0,\pi]} \sqrt{\sin^2 t + \cos^2 t} |dt| = \int_{[0,\pi]} |dt| = \pi \quad (1- 45)$$

Proof of Proposition (1-1- 17):

Denote by $\Phi = \gamma_2^{-1} \circ \gamma_1: I_1^{ok} \rightarrow I_2^{ok}$ the "change of parameters map" such that $\Phi(t_1) = t_2$. This map Φ goes from an open subset of I_1 to an open subset of I_2 by way of the curve; γ_1 takes a point of I_1 to the curve, and then γ_2^{-1} takes it "backwards" from the curve to I_2 . Substituting γ_2 for f , Φ for g and T_1 for a in Equation of the chain rule gives

$$|D(\gamma_2 \circ \Phi)(t_1)| = |D(\gamma_2(\Phi(t_1)))| \circ |D\Phi(t_1)| \quad (1- 46)$$

$$\gamma_2 \circ \Phi = \gamma_2 \overbrace{\gamma_2^{-1} \circ \gamma_1}^{\phi} = \gamma_1 \quad (1-47)$$

we can substitute γ_1 for $(\gamma_1 \circ \Phi)$ in Equation (1-46) to get

$$\underbrace{|D\gamma_1(t_1)|}_{\gamma_1^{-1}(t_1)} = \underbrace{|D\gamma_2(\Phi(t_1))|}_{\gamma_2^{-1}(t_2)} \underbrace{|D\Phi(t_1)|}_{\gamma_2^{-1} \circ \gamma_1} \quad (1-48)$$

$$|D\gamma_1(t_1)| = \gamma_1^{-1}(t_1) \quad \text{and} \quad |D\gamma_2(\Phi(t_1))| = \gamma_2^{-1}(\Phi(t_1)) = \gamma_2^{-1}(t_2) \quad (1-49)$$

are really column vectors (they go from $\mathbb{R}$ to $\mathbb{R}^3$) and that $[D\Phi(t_1)]$, which goes from $\mathbb{R}$ to $\mathbb{R}$, is a 1×1 matrix, i.e., really a number. So when we take absolute values,

$$|[D\gamma_1(t_1)]| = |[D\gamma_2(\Phi(t_1))]| |[D\Phi(t_1)]| \quad (1-50)$$

$$\begin{aligned} \int_{I_2} \left| \vec{\gamma}'_2(t_2) \right| |dt_2| &= \int_{I_1} |[D(\gamma_2 \circ \Phi)(t_1)]| |dt_1| = \int_{I_1} \overbrace{|[D\Phi(t_1)]|}^{\det|[D\Phi(t_1)]|} |dt_1| \\ &= \int_{I_1} |[D\gamma_1(t_1)]| |dt_1| = \int_{I_1} \left| \vec{\gamma}'_1(t_1) \right| |dt_1| \end{aligned} \quad (1-51)$$

Section (1.2) : Surface , Manifolds and Fractional Dimension

The integrand $|d^2x|$ takes a parallelogram $P_x(\vec{v}_1, \vec{v}_2)$ and returns its area. In $\mathbb{R}^3$, this means .

$$|d^2x|(P_x(\vec{V}_1, \vec{V}_2)) = \left| \vec{V}_1 \times \vec{V}_2 \right| \quad (1-52)$$

the general formula, which works in all dimensions, and which is a special case of Theorem (1-1-4), is

$$|d^2x|(P_x(\vec{V}_1, \vec{V}_2)) = \sqrt{\det(\det(\vec{V}_1, \vec{V}_2)^T \vec{V}_1, \vec{V}_2)} \quad (1-53)$$

To integrate $|d^2x|$ over a surface, we wish to break up the surface into little parallelograms, add their areas, and take the limit as the decomposition becomes fine. Similarly, to integrate $f |d^2x|$ over a surface, we go through the same process, except that instead of adding areas we add $f(x) \times$ area of parallelogram.

But while it is quite easy to define arc length as the limit of the length of inscribed polygons, and not much harder to prove that Equation (1-33) computes it, it is much harder to define surface area. In particular, the obvious idea of taking the limit of the area of inscribed triangles as the triangles become smaller and smaller only works if we are careful to prevent the triangles from becoming skinny as they get small, and then it isn't obvious that such inscribed polyhedral exist at all. The difficulties are not insurmountable, but they are daunting.

Definition(1-2-1):(Surface area)

Let $S \subset \mathbb{R}^3$ be a smooth surface parametrized by $\gamma : U \rightarrow S$, where U is an open subset of $\mathbb{R}^2$. Then the area of S is

$$\int_U |d^2x|(P_{\gamma(u)}(D_1 \gamma(u), D_2 \gamma(u))) |d^2u| = \int_U \sqrt{\det(|D\gamma(u)|^T |D\gamma(u)|)} |d^2u| \quad (1-54)$$

Let us see why this ought to be right. The area

$$\lim_{N \rightarrow \infty} \sum_{C \in D(\mathbb{R}^2)} \text{Area of } \gamma(C \cap U) \quad (1-55)$$

That is, we make a dyadic decomposition of $\mathbb{R}^2$ and see how γ maps to S the dyadic squares C that are in U or straddle it. We then sum the areas of $\gamma(C \cap U)$, which, for $C \subset U$, is the same as $\gamma(C)$; for C that straddle U , we add to the sum the area of the part of C that is in U . The side length of a square C is $1/2^N$, so at least when $C \subset U$, the set $\gamma(C \cap U)$ is, as shown in Figure (1-5), approximately the parallelogram.

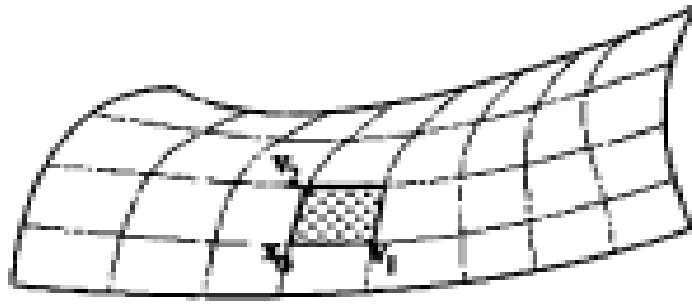


Figure (1-5)
Surface area

$$p_{\gamma(u)} \cdot \left(\frac{1}{2N} \vec{D}_1 \gamma(u), \frac{1}{2N} \vec{D}_2 \gamma(u) \right) \quad (1-56)$$

where u is the lower left hand corner of C .

That parallelogram has area

$$\frac{1}{2^2 N} \sqrt{\det [D\gamma(u)]^T [D\gamma(u)]} \quad (1-57)$$

So it seems reasonable to expect that the error we make by replacing

$$\text{Area of } \gamma(C \cap U) \text{ by } \text{vol}_2(c) \sqrt{\det [D\gamma(u)]^T [D\gamma(u)]} \quad (1-58)$$

Will disappear in the limit as $N \rightarrow \infty$. And the area given by Equation(1-58) is precisely a Riemann sum for the integral giving surface area:

$$\int_0^1 \left(\frac{\sqrt{5+9y^4}}{2} + \frac{1+9y^4}{4} \log \log \frac{2+\sqrt{5+9y^4}}{\sqrt{1+9y^4}} \right) dy$$

$$\underbrace{\int_U \sqrt{\det [D\gamma(u)]^T [D\gamma(u)]} |d^2u|}_{\text{area of surface by Eq(1-54)}} \quad (1-59)$$

Unfortunately, this argument isn't entirely convincing. The parallelograms above can be imagined as some sort of tiling of the surface, gluing small flat tiles at the corners of a grid drawn on the surface, a bit like using ceramic tiles to cover a curved counter. It is true that we get a better and better fit

by choosing smaller and smaller tiles, but is it good enough? Our definition involves a parametrization γ ; only when we have shown that surface area is independent of parametrization can we be sure that Definition (1-2-1) is correct. We will verify this after computing a couple of examples of surface integrals.

Example (1-2-2): (Area of a torus)

Choose $R > r > 0$. We obtain the torus shown in Figure (1-6) by taking the circle of radius r in the (x, z) -plane that is centered at $x = R, z = 0$, and rotating it around the z -axis.

This surface is parametrized by

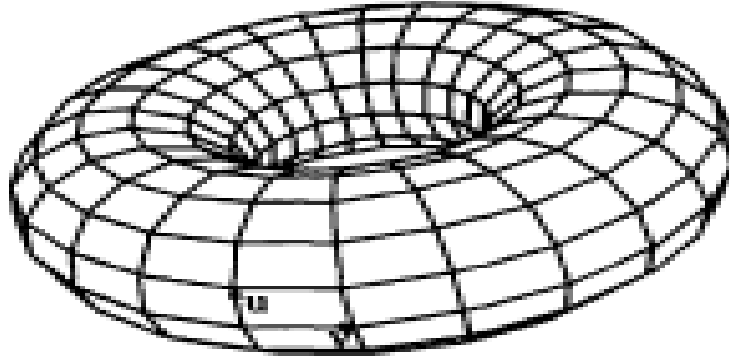


Figure (1-6)
Area of a torus

$$\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (R + r \cos u) \cos v \\ (R + r \cos u) \sin v \\ r \sin u \end{pmatrix} \quad (1-60)$$

Then the surface area of the torus is given by the integral

$$\int_{[0,2\pi][0,2\pi]} \left| \begin{array}{c} \overbrace{\begin{bmatrix} -r \sin u \cos v \\ -r \sin u \sin v \\ r \cos u \end{bmatrix}}^{\bar{D}_1 \gamma} \times \overbrace{\begin{bmatrix} (R + r \cos u) \sin v \\ (R + r \cos u) \cos v \\ 0 \end{bmatrix}}^{\bar{D}_2 \gamma} \end{array} \right|$$

$$r \int_0^{2\pi} \int_0^{2\pi} (R + r \cos u) du dv = 4\pi^2 r R \quad (1-61)$$

Example (1-2-3): (Surface area: a harder problem)

What is the area of the graph of the function $x^2 + y^3$ above the unit square $Q \subset \mathbb{R}^2$?

Applying Equation (1-16), we parametrize the surface by

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ x^2 + y^3 \end{pmatrix}, \text{ and apply Equation (1-54)}$$

$$3^n \left(\frac{1}{2n} \right)^p$$

$$\int_Q \left| \begin{pmatrix} \vec{D}_1 \gamma \\ \vec{D}_2 \gamma \end{pmatrix} \right| dx dy = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 9y^4} dx dy \quad (1-62)$$

The integral with respect to x is one we can calculate (just barely in our case, checking our result with a table of integrals). First we get

$$\int \sqrt{u^2 + a^2} du = \frac{u\sqrt{u^2 + a^2}}{2} + \frac{a^2 \log(u + \sqrt{u^2 + a^2})}{2} \quad (1-63)$$

This leads to the integral

$$\int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 9y^4} dx dy$$

$$= \int_0^1 \left[\frac{x\sqrt{1 + 4x^2 + 9y^4}}{2} + \frac{1 + 9y^4 \log \log(2x + \sqrt{1 + 4x^2 + 9y^4})}{4} \right]_0^1 dy$$

$$\int_0^1 \left(\frac{\sqrt{5+9y^4}}{2} + \frac{1+9y^4}{4} \log \log \frac{2+\sqrt{5+9y^4}}{\sqrt{1+9y^4}} \right) dy \quad (1-64)$$

It is hopeless to try to integrate this mess in elementary terms: the first term requires elliptic functions, and we don't know of any class of special functions in which the second term could be expressed. But numerically, this is no big problem; Simpson's method with 20 steps gives the approximation 1.93224957.....

Now we will illustrate surface area is independent of the choice of parametrization., it is quite difficult to give a rigorous definition of surface area that does not rely on a parametrization. In Definition (1-2-1) we defined surface area using a parametrization; now we need to show that two different parametrizations of the same surface give the same area. Like Proposition (1-1-15) (the analogous statement for curves), this is an application of the change of variables formula .

Proposition (1-2-4):(Surface area independent of parametrization)

Let S be a smooth surface in $\mathbb{R}^3$ and $\gamma_1: U \rightarrow \mathbb{R}^3$, $\gamma_2: V \rightarrow \mathbb{R}^3$ be two parametrizations. Then

$$\int_U \sqrt{\det [D\gamma_1(u)]^T [D\gamma_1(u)]} |d^2u| = \int_V \sqrt{\det [D\gamma_2(v)]^T [D\gamma_2(v)]} |d^2v| \quad (1-65)$$

Proof :

We begin as we did with the proof of Proposition (1-2-4) . Define

$\phi = \gamma_2^{-1} \circ \gamma_1 : U \rightarrow V$ to be the "change of parameters" map such that

$$v = \Phi(u).$$

Notice that the chain rule applied to the equation $\gamma_1 = \gamma_2 \circ \phi$ Gives

$$[D(\gamma_2 \circ \Phi)(u)] = [D\gamma_1(u)] = [D\gamma_2(\Phi(u))][D(\Phi(u))] \quad (1-66)$$

if we apply the change of variables formula, we find

$$\int_U \sqrt{\det [D\gamma_1(u)]^T [D\gamma_1(u)]} |d^2u| = \int_V \sqrt{\det [D\gamma_2(v)]^T [D\gamma_2(v)]} |d^2v|$$

$$\begin{aligned}
&= \int_U \sqrt{\det([D(\gamma_2 \circ \Phi)(u)]^T [D(\gamma_2 \circ \Phi)(u)]) |\det[D(\Phi(u))]|} d^2u \\
&= \int_U \sqrt{\det([D\gamma_2(\Phi(u))]^T [D\gamma_2(\Phi(u))]) \sqrt{\det([D(\Phi(u))]^T [D(\Phi(u))])}} d^2u \\
&= \int_U \sqrt{\det \det([D\gamma_2(\Phi(u))]^T [D\gamma_2(\Phi(u))][D(\Phi(u))]^T [D(\Phi(u))])} d^2u \\
&= \int_U \sqrt{\det([D(\gamma_2 \circ \Phi)(u)]^T [D(\gamma_2 \circ \Phi)(u)])} d^2u \\
&= \int_v \sqrt{\det[D\gamma_1(u)]^T [D\gamma_1(v)]} d^2u \tag{1-67}
\end{aligned}$$

Now we will study areas of surfaces in $\mathbb{R}^n$, $n > 3$. A surface (i.e., a two-dimensional manifold) embedded in $\mathbb{R}^n$ should have an area for any n , not just for $n = 3$.

A first difficulty is that it is hard to imagine such surfaces, and perhaps impossible to visualize them. But it isn't particularly hard to describe them mathematically.

For instance, the subset of $\mathbb{R}^4$ given by the two equations

$x_1^2 + x_2^2 = r_1^2$, $x_3^2 + x_4^2 = r_2^2$, is a surface; it corresponds to two equations in four unknowns. This surface is discussed in Example (1-2-5). More generally, we saw in Section(1-1) that the set $X \subset \mathbb{R}^n$ defined by the $n - k$ equations in n variables

$$f_1 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0, \dots, f_m \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \tag{1-68}$$

defines a k -dimensional manifold if $[Df(x)] : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto for each $x \in X$.

Example (1-2-5) : (Area of a surface in $\mathbb{R}^4$)

The surface described above, the subset of $\mathbb{R}^4$ given by the two equations
 $x_1^2 + x_2^2 = r_1^2$, and $x_3^2 + x_4^2 = r_2^2$ is
 parametrized by

$$\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r \cos \cos u \\ r \sin \sin u \\ r \cos \cos v \\ r \sin \sin v \end{pmatrix}, 0 \leq u, v \leq 2\pi \quad (1-69)$$

and since

$$\begin{aligned} & \left[D_\gamma \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) \right]^T \left[D_\gamma \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) \right] \\ &= \begin{bmatrix} -r_1 \sin r_1 \sin u & r_1 \cos \cos u & 0 & 0 \\ 0 & 0 & -r_2 \sin r_2 \sin v & r_2 \cos \cos v \end{bmatrix} \begin{bmatrix} -r_1 \sin r_1 \sin u & 0 \\ r_1 \cos \cos u & 0 \\ 0 & -r_2 \sin r_2 \sin v \\ 0 & r_2 \cos \cos v \end{bmatrix} \\ &= \begin{bmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{bmatrix} \end{aligned} \quad (1-70)$$

Equation (1-54) tells us that its area is given by

$$\begin{aligned} & \int_{[0,2\pi],[0,2\pi]} \sqrt{\det \left(\left[D_\gamma \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) \right]^T \left[D_\gamma \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) \right] \right)} |du \, dv| \\ &= \int_{[0,2\pi],[0,2\pi]} \sqrt{r_1^2 \cdot r_2^2} |du \, dv| \end{aligned} \quad (1-71)$$

Another class of surfaces in $\mathbb{R}^4$ which is important in many applications, and which leads to remarkably simpler computations than one might expect, uses complex variables. Consider for instance the graph of the function $f(z) = z^2$, where z is complex. This graph has the equation $z_2 = z_1^2$ in $\mathbb{C}^2$, or

$$x^2 = x_1^2 - y_1^2, y_2 = 2x_1y_1 \text{ in } \mathbb{R}^4. \quad (1-72)$$

Equation (1-54) tells us how to compute the areas of such surfaces, if we manage to parametrize them. If $S \subset \mathbb{R}^n$ is a surface, $U \subset \mathbb{R}^2$ is an open subset, and $\gamma : U \rightarrow \mathbb{R}^n$ is a parametrization of S , then the area of S is given by

$$\int_S |d^2x| = \int_{[0,2\pi] \times [0,2\pi]} \sqrt{\det([D\gamma(u)]^T [D\gamma(u)])} |d^2u| \quad (1-73)$$

Example (1-2-6): (Area of a surface in $\mathbb{C}^2$)

Let us tackle the surface in $\mathbb{C}^2$ of Equation (1-73). More precisely, let us compute the area of the part of the surface of equation $z_2 = z_1^2$, where $|z_1| \leq 1$. Polar coordinates for z_1 give a nice way to parametrize the surface:

$$\gamma \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ r^2 \cos 2\theta \\ r^2 \sin 2\theta \end{pmatrix}, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \quad (1-74)$$

Again we need to compute the area of the parallelogram spanned by the two partial derivatives. Since

$$\left[D\gamma \begin{pmatrix} r \\ \theta \end{pmatrix} \right]^T \left[D\gamma \begin{pmatrix} r \\ \theta \end{pmatrix} \right]$$

$$\begin{aligned}
&= \begin{bmatrix} \cos \theta & \sin \theta & 2r \cos 2\theta & 2r \sin 2\theta \\ -r \sin \theta & r \cos \theta & -r^2 \sin 2\theta & r^2 \cos 2\theta \end{bmatrix} \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \\ 2r \cos 2\theta & -r^2 \sin 2\theta \\ 2r \sin 2\theta & r^2 \cos 2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 + 4r^2 & 0 \\ 0 & r^2(1 + 4r^2) \end{bmatrix} \tag{1-75}
\end{aligned}$$

$$\int_{[0,1] \times [0,2\pi]} \sqrt{\det \left(\begin{bmatrix} D\gamma \left(\begin{smallmatrix} r \\ \theta \end{smallmatrix} \end{bmatrix} \right)^T \begin{bmatrix} D\gamma \left(\begin{smallmatrix} r \\ \theta \end{smallmatrix} \end{bmatrix} \right) \end{bmatrix} \right)} |dr \, d\theta| \tag{1-76}$$

$$\int_{[0,1] \times [0,2\pi]} \underbrace{\sqrt{r^2(1 + 4r^2)}}_{\text{square root of a perfect square}} |du \, dv| = 2\pi \left[\frac{r^2}{2} + r^4 \right]_0^3 = 3\pi$$

Now we will express volume of manifolds . Everything we have done so far in this chapter works for a manifold M of any dimension k , embedded in any $\mathbb{R}^n$. The k -dimensional volume of such a manifold is written

$$\int_M |d^k x|$$

where $|d^k x|$ is the integrand that takes a k -parallelogram and returns its k dimensional volume. Heuristically, this integral is defined by cutting up the manifold into little k -parallelograms, adding their k -dimensional volumes and taking the limits of the sums as the decomposition becomes infinitely fine.

The way to do this precisely is to parametrize M . That is, we find a subset.

$U \subset \mathbb{R}^k$ and a mapping $\gamma: U \rightarrow M$ which is differentiable, one to one and onto, with a derivative which is also one to one.

Then the k -dimensional volume is defined to be

$$\int_U \sqrt{\det \left([D\gamma(u)]^T [D\gamma(u)] \right)} |d^k u| \tag{1-77}$$

The independence of this integral from the chosen parametrization also goes through without any change at all.

Proposition (1-2-7) : (Volume of manifold independent of parametrization)

If U_1, U_2 are subsets of $\mathbb{R}^k$ and $\gamma_1: U \rightarrow M, \gamma_2: V \rightarrow M$ are two parametrizations of M , then

$$\int_U \sqrt{\det([D\gamma(u)]^T [D\gamma(u)])} |d^k u| = \int_V \sqrt{\det([D\gamma(v)]^T [D\gamma(v)])} |d^k v|$$

Example (1-2-8) : (Volume of a three-dimensional manifold in $\mathbb{R}^4$)

Let $U \subset \mathbb{R}^3$ be an open set, and $f: U \rightarrow \mathbb{R}$ be a C^1 function. Then the graph of f is a three-dimensional manifold in $\mathbb{R}^4$, and it comes with the natural parametrization

$$\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ f \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} \quad (1-78)$$

We then have

$$\det \left(\left[D\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]^T \left[D\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] \right)$$

$$\det \left(\begin{bmatrix} 1 & 0 & 0 & D_1 f \\ 0 & 1 & 0 & D_2 f \\ 0 & 0 & 1 & D_3 f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ D_1 f & D_2 f & D_3 f \end{bmatrix} \right) \quad (1-79)$$

$$d \text{ et } \begin{bmatrix} 1+(D_1 f)^2 & (D_1 f)(D_2 f) & (D_1 f)(D_3 f) \\ (D_1 f)(D_2 f) & 1+(D_2 f)^2 & (D_2 f)(D_3 f) \\ (D_1 f)(D_3 f) & (D_2 f)(D_3 f) & 1+(D_3 f)^2 \end{bmatrix}$$

$$= 1 + (D_1 f)^2 + (D_2 f)^2 + (D_3 f)^2$$

So the three-dimensional volume of the graph of f is

$$\int_U \sqrt{1 + (D_1 f)^2 + (D_2 f)^2 + (D_3 f)^2} |d^3 x| \quad (1-80)$$

It is a challenge to find any function for which this can be integrated in elementary terms. Let us try to find the area of the graph of

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} (x^2 + y^2 + z^2) \quad (1-81)$$

above the ball $B_R(0)$ of radius R centered at the origin. Using spherical coordinates, this leads to.

$$\begin{aligned} \int_{B_0(R)} \sqrt{1 + x^2 + y^2 + z^2} |d^3 x| &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^R \sqrt{1 + r^2} r^2 \cos \varphi \cos \theta dr d\varphi d\theta \\ &= 4\pi \int_0^R \sqrt{1 + r^2} r^2 dr \end{aligned} \quad (1-82)$$

$$= \pi \left(R (1 + R^2)^{3/2} - \frac{1}{2} \log \log \left(R + \sqrt{1 + R^2} \right) - \frac{1}{2} R \sqrt{1 + R^2} \right)$$

Now we will discuss fractals and fractional dimension . In 1919, Felix Hausdorff showed that dimensions are not limited to length, area, volume, . . we can also speak of fractional dimension. This discovery acquired much greater significance with the work of Benoit Mandelbrot showing that many objects in nature (the lining of the lungs, the patterns of frost on windows, the patterns formed by a film of gasoline on water, for example) are fractals, with fractional dimension

Example (1-2-10) : (Koch snowflake)

We construct the Koch snowflake curve K as follows. Start with a line segment, say $0 < x < 1$, $y = 0$ in $\mathbb{R}^2$. Replace its middle third by the top of an equilateral triangle, as shown in Figure (1-8). This gives four segments, each one-third the length of the original segment.

Now replace the middle third of each by the top of an equilateral triangle, and 1, so on .

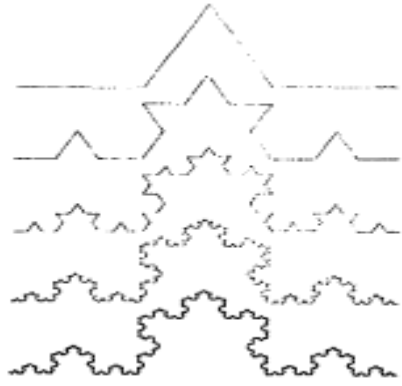


Figure (1-7)

Koch snowflake

What is the length of this "curve"? At resolution $N = 0$, we get length 1. At resolution $N = 1$, when the curve consists of four segments, we get length $4 \cdot 1/3$. At the next resolution, the length is $16 \cdot 1/9$. As our decomposition "Length" is the wrong word to apply to the Koch snowflake, which is neither a curve nor a surface. It is a fractal, with fractional dimension: the Koch snowflake has dimension $\log 4 / \log 3 \approx 1.26$

Let us see why this might be the case. Call A the part of the curve constructed on $(0, 1/3)$, and B the whole curve, as in Figure (1-8). Then B consists of four copies of A . (This is true at any level, but it is easiest to see at the firsting the Koch snowflake. Its length level, the top graph in Figure (1-7). Therefore, in any dimension d , it should be infinite, but length is the wrong word to use. It is true that $vol_d(B) = 4vol_d(A)$.



Figure (1-8)

However, if you expand A by a factor of 3, you get B . (This is true in the limit, after the construction has been carried out infinitely many times.)

According to the principle that area goes as the square of the length, volume goes as the cube of the length, etc., we would expect d-dimensional volume to go as the dth power of the length, which leads to

$$vol_d(B) = 3^d vol_d(A). \quad (1-83)$$

If you put this equation together with $vol_d(B) = vol_d(A)$, you will see that the only dimension in which the volume of the Koch curve can be different from 0 or ∞ is the one for which $4 = 3^d$, i.e., $d = \log 4 / \log 3$.

If we break up the Koch curve into the pieces built on the sides constructed at the nth level (of which there are 4^n , each of length $1/3^n$), and raise their side-lengths to the dth power, we find

$$4^n \left(\frac{1}{3} \right)^{n \log 4 / \log 3} = 4^n e^{n \frac{\log 4}{\log 3} (\log \frac{1}{3})} = 4^n e^{-n \log 4} = \frac{4^n}{4^n} = 1 \quad (1-84)$$

(In Equation (1-85) we use the fact that $a^x = e^{x \log a}$.) Although the terms have not been defined precisely, you might expect the computation above to mean

.

$$\int_K \left| dx^{\frac{\log 4}{\log 3}} \right| = 1 \quad (1-85)$$

Example (1-2-11) : (Sierpinski gasket)

While the Koch snowflake looks like a thick curve, the Sierpinski gasket looks more like a thin surface. This is the subset of the plane obtained by taking a filled triangle of side length 1, removing the central inscribed subtriangle, then removing the central subtriangles from the three triangles that are left, then removing the central subtriangles from the nine triangles that are left, and so on; the process is sketched in Figure

(1-10). We claim that this is a set of dimension $\log 3 / \log 2$: at the nth stage of the construction, sum, over all the little pieces, the side-length to the power p:

$$3^n \left(\frac{1}{2^n} \right)^p \quad (1-86)$$

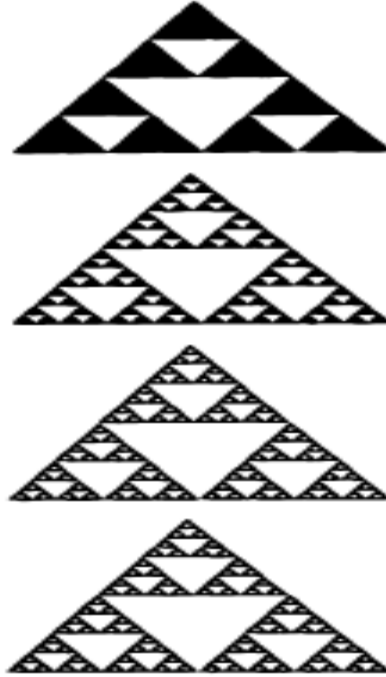


Figure (1-9)

Sierpinski gasket

(If measuring length, $p = 1$; if measuring area, $p = 2$.) If the set really had a length, then the sum would converge when $p = 1$, as $n \rightarrow \infty$; in fact, the sum is infinite. If it really had an area, then the power $p = 2$ would lead to a finite limit; in fact, the sum is 0. But when $p = \log 3 / \log 2$, the sum converges to $L^{\log 3 / \log 2} \rightarrow L^{1.58}$. This is the only dimension in which the Sierpinski gasket has finite, nonzero measure; in dimensions greater than $\log 3 / \log 2$, the measure is 0, and in dimensions less than $\log 3 / \log 2$ it is infinite.