Finite Element Formulation of Geometrically Nonlinear Beams

3.1 Introduction:

Large or small strain, have a significant effect on the load deformation behavior and result in geometric nonlinearity, also, includes deformation–dependent boundary conditions and loading.

As a direct consequence of geometric nonlinearity, the stiffness matrix of the finite element model is not constant. It is a function of the residual displacements. Therefore, an iterative procedure is required to obtain the equilibrium state. For a given load, there are two main formulations depending on the configuration to which the variables involved in each step are referred:

- The total Lagrangian formulation where all the variables are referred to initial configuration as shown in Figure (3.1). All integrals are evaluated with respect to the initial un deformed configuration.
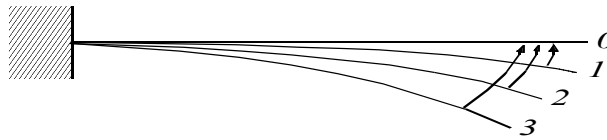

Figure 3.1: Total Lagrangian.

- The updated formulation where all the variables are referred to initial configuration at the beginning of the load step considered as shown in Figure (3.2). All integrals are evaluated with respect to the last completed iteration of the current increment.
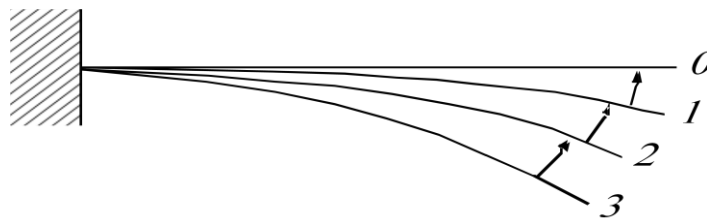

Figure 3.2: Updated Lagrangian

3.1.1 Basic Concepts of Nonlinear Analysis:

When a load causes significant changes in stiffness, the load-deflection curve becomes nonlinear as shown in Figure (3.3).

The challenge is to calculate the nonlinear displacement response using a linear set of equations.
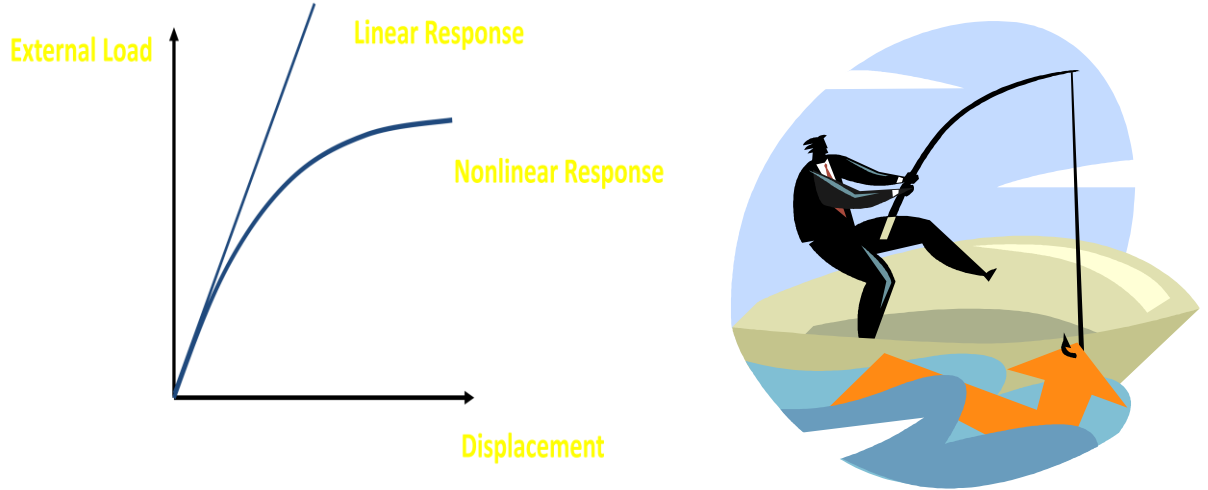

Figure 3.3: Linear and nonlinear displacement response.

Linear finite element analysis is based on:

Linearized geometrical equations (strain-displacement relations):

$$\{\varepsilon\} = [B]\{d\} \tag{3.1}$$

Linearized constitutive equations (stress-strain relations):

$$\{\sigma\} = [C]\{\varepsilon\} = [C][B]\{d\} \tag{3.2}$$

Equations of equilibrium: $R^i = R^e$

$$[K]\{D\} = \{R^e\} \tag{3.3}$$

3.2 Nonlinear solution techniques:

Most of the approaches to perform a nonlinear finite element analysis divide the total load in steps. Iterations are, then, used within load steps to acquittaly track the equilibrium path.

3.2.1 Incremental method:

The pure incremental approach is to apply the load gradually by dividing it into a series of increments and adjusting the stiffness matrix at the end of each increment.

The problem with this approach is that errors accumulate with each load increment, causing the final results to be out of equilibrium.
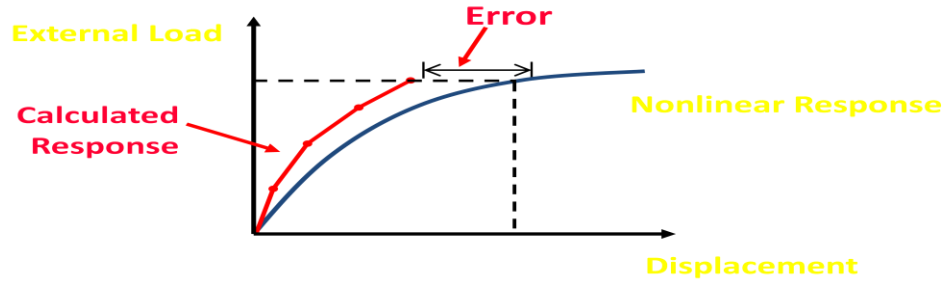

Figure 3.4: Pure Incremental Method.

The load is divided into a set of small increments ΔF_i as shown in Figure (3.5). Increments of displacements are calculated from the set of linear simultaneous equations:

$$K_{T(i-1)} * \Delta d_i = \Delta F_i \tag{3.4}$$

Where $K_{T(i-1)}$ is tangent stiffness matrix computed from displacements $d_{(i-1)}$ obtained in previous incremental step.

Nodal displacements after force increment of ΔF_i are:

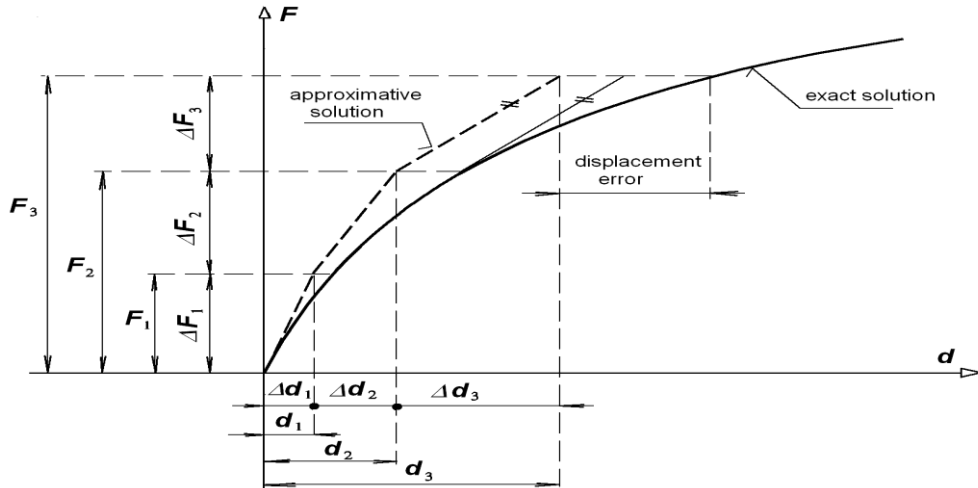$$d_i = d_{i-1} + \Delta d_i \tag{3.5}$$

Figure 3.5: Incremental Method.

3.2.2- Iterative methods:

a) Standard Newton-Raphson (NR) method:

Consider that d_i is estimation of nodal displacement. As it is only an estimation, the condition of equilibrium would not be satisfied

$$R(d_i) \neq F \tag{3.6}$$

This means that conditions of equilibrium of internal and external nodal forces are not satisfied and in nodes are unbalanced forces

$$r_i = R(d_i) - F \tag{3.7}$$

Correction of nodal displacements can be then obtained from the set of linear algebraic equations

$$K_{T(i)\,\Delta}\, d_i = r_i \tag{3.8}$$

and new, corrected estimation of nodal displacements is:

$$d_{i+1} = d_i + \Delta d_i \tag{3.9}$$

The procedure is repeated until the sufficiently accurate solution is obtained

The first estimation is obtained from linear analysis

$$K\, d_i = F \tag{3.10}$$

Applies the load gradually, in increments.

Also performs equilibrium iterations at each load increment to drive the incremental solution to equilibrium. Solves the equation

$$[K_T]\{\Delta u\} = \{F\} - \{F^{nr}\} \tag{3.11}$$

Where

$[K_T]$ = tangent stiffness matrix

$\{\Delta u\}$ = displacement increment

$\{F\}$ = external load vector

$\{F^{nr}\}$ = internal force vector

Iterations continue until $\{F\}$ - $\{F^{nr}\}$ (Difference between external and internal loads) is within a tolerance. Some nonlinear analyses have trouble converging. Advanced analysis techniques are available in such cases (covered in the Structural Nonlinearities training course).

b) **Modified Newton-Raphson (MNR) method:**

Modified Newton-Raphson (MNR) method - the same stiffness matrix is used in all iterations
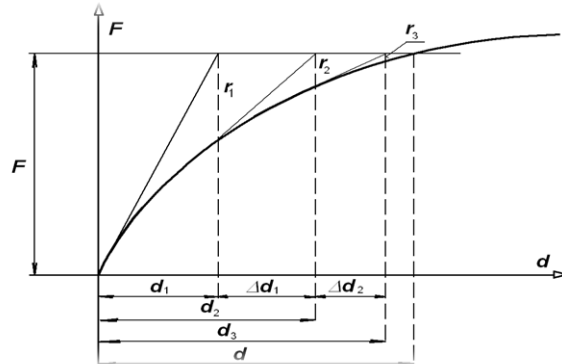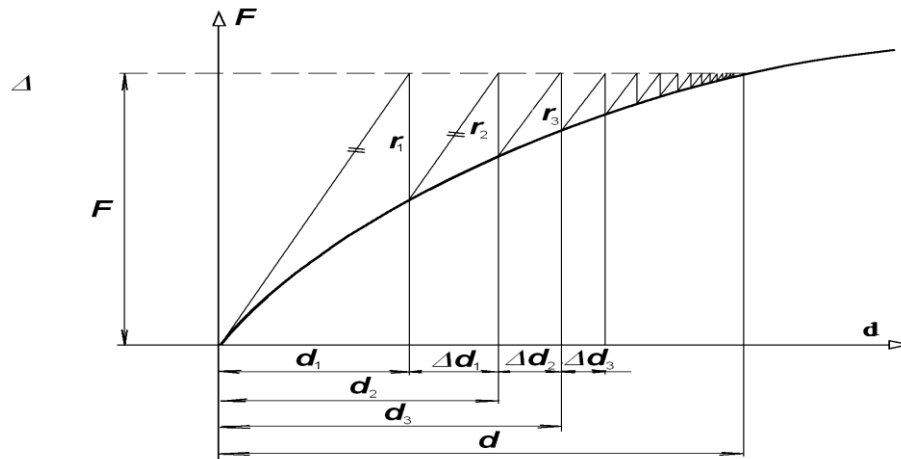

Figure 3.6: Standard Newton-Raphson (MNR) method.

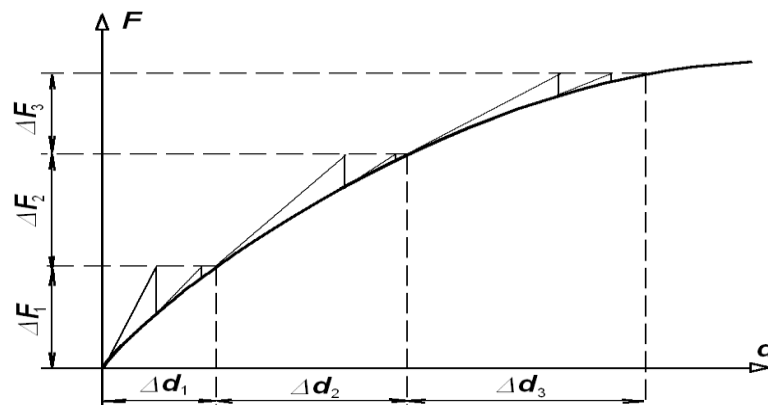*Figure 3.7: Modified Newton-Raphson (MNR) method.*

3.2.3 Combination of Newton-Raphson and incremental methods:

Figure 3.8: Combination of Newton-Raphson and incremental methods.

3.3 Geometrically Non-Linear Formulation for Thin Beams:

3.3.1 Non-Linear Strain displacement relations:

* For Beams, the Green strains are: [7]

$$e = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$x = -\frac{\partial^2 v}{\partial x^2}\left(1 + \frac{\partial u}{\partial x}\right) + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} \tag{3.12}$$

$$x = -\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

Where:

e is the direct strain.

x is the curvature.

Equation (3.12) can be written as:

$$\{\varepsilon\} = \begin{Bmatrix} e \\ x \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ -\frac{\partial^2 v}{\partial x^2} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \\ -\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} \end{Bmatrix} \tag{3.13}$$

$$= \{\varepsilon_0\} + \{\varepsilon_l\}$$

Where:

$$\{\varepsilon_0\} = [B_0]\{a\} \tag{3.14}$$

Are the infinitesimal strains, and $\{a\}$ are the nodal variables. The nonlinear strain component may be written as:

$$\{\varepsilon_l\} = \frac{1}{2}\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & 0 & 0 \\ -\frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 u}{\partial x^2} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ -\frac{\partial^2 v}{\partial x^2} \\ \frac{\partial^2 u}{\partial x^2} \end{Bmatrix} \tag{3.15}$$

$$= \frac{1}{2}[B_L(a)]\{a\}$$

$$= \frac{1}{2}[A_\theta]\{\theta\}$$

Where:

$$[A_\theta] = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & 0 & 0 \\ -\dfrac{\partial^2 v}{\partial x^2} & \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \end{bmatrix}$$

and

$$\{\theta\} = \left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, -\frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 u}{\partial x^2} \right\}^T = [G]\{a\} \tag{3.16}$$

$[B_L(a)]$ is the linear strain matrix.

From (3.12) the variation in strain $\delta\{\varepsilon\} = [B]\delta\{a\})$ gives the strain matrix

as:

$$[B] = [[B_0] + B_L(\{a\})]] \tag{3.17}$$

3.3.2 The Stress Resultant:

The beam stress resultants are defined as:

$$\{S\} = \begin{Bmatrix} F \\ M \end{Bmatrix} = [D]\{\varepsilon\} \tag{3.18}$$

Where F is axial force and M is the bending moment and the modulus

matrix:

$$[D] = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix} \tag{3.19}$$

In which E is young's modulus of beam and A is x-section area of beam

and I the second moment of area.

3.3.3 The tangent stiffness matrix:

The nonlinear equilibrium equation are:

$$\{\varphi\} = \int [B]^T \{S\}^{i+1} dl - \{R\} = \{0\}$$

Where $\{R\}$ =vector of equivalent applied nodal loads.

From which the   tangent stiffness matrix ($[K_T] = \dfrac{\partial\{\varphi\}}{\partial\{a\}}$), is given by:

$$[K_T] = [K_0] + [K_L(\{a\})] + [K_\sigma] \tag{3.20}$$

Where

$$[K_0] + [K_L(\{a\})] = \int [B]^T [D][B] dl$$

$$= \int [B_0]^T [D][B_0]dl + \int [B_0]^T [D][B_L]dl + \int [B_L]^T [D][B_0]dl$$

$$+ \int [B_L]^T [D][B_L]dl \tag{3.21}$$

And the initial stress stiffness matrix:

$$[K_\sigma] = \int [G]^T [P_i][G]dl \tag{3.22}$$

Where the initial stress matrix:

$$[P_i] = \begin{bmatrix} F[I] & M[I] \\ M[I] & 0 \end{bmatrix} \tag{3.23}$$

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.24}$$

In which F =axial force, M= bending moment.

3.3.4 Incremental strains and stress resultants:

* Nodal displacements for iteration $(i + 1)$ are:

$$\{a\}^{i+1} = \{a\}^i + \{\Delta a\}^i \tag{3.25a}$$

Where: $\{\Delta a\}^i = -[K_T]^{-1} * \{\varphi\}^i \tag{3.25b}$

Are the displacement increments

* From (3.2) and (3.14) the incremental strains are:

$$\{\Delta \varepsilon\}^i = \left[[B_0] + [B_L(\{a\}^i)] + \frac{1}{2} [B_L(\{\Delta a\}^i)] \right] \{\Delta a\}^i \tag{3.26}$$

* The increment of the stress resultants (2nd Piola-Kirchhff stress) are given by:

$$\{\Delta S\}^i = [D]\{\Delta \varepsilon\}^i \tag{3.27}$$

*The total stress resultants are, there defined as:

$$\left\{ {F \atop M} \right\}^{i+1} = \{S\}^{i+1} = \{S\}^i + \{\Delta S\}^i \tag{3.28}$$

* The nodal residual forces for the next iteration are:

$$-\{\varphi\}^{i+1} = \{R\} - \int [B]^T \{S\}^{i+1} dl \tag{3.29}$$

And

$$[B] = [B_0] + [B_L(\{a\})^{i+1}] \tag{3.30}$$

3.3.5 Formulation the beam Finite Element (Figure 3.9):

X-section Area = A, 2nd moment of area = I, Young's Modules = E,
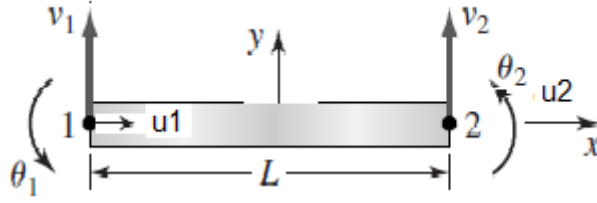

Figure 3.9: Beam Finite Element.

The element displacement function are:

$$U = \left(\frac{1}{2} - \frac{X}{L}\right) u_1 + \left(\frac{1}{2} + \frac{X}{L}\right) u_2 \tag{3.31}$$

$$V = \left(\frac{1}{2} - \frac{3x}{2l} + \frac{2x^3}{l^3}\right) v_1 + \left(\frac{l}{8} - \frac{x}{4} - \frac{x^2}{2l} + \frac{x^3}{l^2}\right) \theta_1 + \left(\frac{1}{2} + \frac{3x}{2l} - \frac{2x^3}{l^3}\right) v_2 +$$

$$\left(-\frac{l}{8} - \frac{x}{4} + \frac{x^2}{2l} + \frac{x^3}{l^2}\right) \theta_2 \tag{3.32}$$

From which:

$$\frac{\partial U}{\partial X} = -\frac{1}{l} u_1 + \frac{1}{l} u_2 \tag{3.33}$$

$$\frac{\partial^2 U}{\partial X^2} = 0 \tag{3.34}$$

$$\frac{\partial V}{\partial X} = \left(-\frac{3}{2l} + \frac{6x^2}{l^3}\right) v_1 + \left(-\frac{1}{4} - \frac{x}{l} + \frac{3x^2}{l^2}\right) \theta_1 + \left(\frac{3}{2l} - \frac{6x^2}{l^3}\right) v_2 +$$

$$\left(-\frac{1}{4} + \frac{x}{l} + \frac{3x^2}{l^2}\right) \theta_2 \tag{3.35}$$

$$\frac{\partial^2 V}{\partial X^2} = \left(\frac{12x}{l^3}\right) v_1 + \left(-\frac{1}{l} + \frac{6x}{l^2}\right) \theta_1 + \left(-\frac{12x}{l^3}\right) v_2 + \left(\frac{1}{l} + \frac{6x}{l^2}\right) \theta_2 \tag{3.36}$$

From which the strain component matrix are obtained as follows:

$$[B_0] =$$

$$\left[\begin{array}{cccccc} -\frac{1}{l} & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & \left(\frac{12x}{l^3}\right) & \left(-\frac{1}{l} + \frac{6x}{l^2}\right) & 0 & \left(-\frac{12x}{l^3}\right) & \left(\frac{1}{l} + \frac{6x}{l^2}\right) \end{array} \right] \tag{3.37}$$

$$\left[B_{0\left(x=\frac{-l}{2}\right)} \right] =$$

$$\left[\begin{array}{cccccc} -\frac{1}{l} & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & \left(\frac{-6}{l^2}\right) & \left(\frac{-4}{l^2}\right) & 0 & \left(\frac{6}{l^2}\right) & \left(\frac{-2}{l}\right) \end{array} \right] \tag{3.38}$$

$$\left[B_{0\left(x=\frac{l}{2}\right)}\right] =$$

$$\begin{bmatrix} -1/l & 0 & 0 & 1/l & 0 & 0 \\ 0 & \left(\frac{6}{l^2}\right) & \left(\frac{2}{l}\right) & 0 & \left(\frac{-6}{l^2}\right) & \left(\frac{4}{l}\right) \end{bmatrix} \quad (3.39)$$

$$[G] =$$

$$\begin{bmatrix} -\frac{1}{l} & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ 0 & \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & 0 & \frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.40)$$

$$[G] =$$

$$\begin{bmatrix} -\frac{1}{l} & 0 & 0 & \frac{1}{l} & 0 & 0 \\ 0 & \left(-\frac{3}{2l}+\frac{6x^2}{l^3}\right) & \left(-\frac{1}{4}-\frac{x}{l}+\frac{3x^2}{l^2}\right) & 0 & \left(\frac{3}{2l}-\frac{6x^2}{l^3}\right) & \left(-\frac{1}{4}+\frac{x}{l}+\frac{3x^2}{l^2}\right) \\ 0 & \left(\frac{12x}{l^3}\right) & \left(\frac{-1}{l}+\frac{6x}{l^2}\right) & 0 & \left(\frac{-12x}{l^3}\right) & \left(\frac{1}{l}+\frac{6x}{l^2}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(3.41)$$

$$A_\theta = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & 0 & 0 \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 u}{\partial x^2} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{bmatrix} \quad (3.42)$$

$$[B_l] = [A_\theta][G] \quad (3.43)$$

The element modulus matrix is:

$$[D] = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix} \quad (3.44)$$

The element tangent stiffness matrix is:

$$\boldsymbol{K_T = K_0 + K_\sigma} \quad (3.45)$$

And the explicit form of its components are obtained as follows:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \qquad (3.46)$$

Neglecting the effect of initial displacements the initial stress stiffness matrix is obtained as follows:

$$K_\sigma = \int_{-\frac{l}{2}}^{\frac{l}{2}} [G]^T [P_i][G] dx \qquad (3.47)$$

$$= \begin{bmatrix} K_{\sigma 11} & \cdots & K_{\sigma 16} \\ \vdots & \ddots & \vdots \\ K_{\sigma 61} & \cdots & K_{\sigma 66} \end{bmatrix}$$

$$K_{\sigma 11} = \frac{F_1}{2l} + \frac{F_2}{2l}$$

$$K_{\sigma 12} = K_{\sigma 21} = \frac{3}{2l^2}(M_1 - M_2)$$

$$K_{\sigma 13} = K_{\sigma 31} = \frac{1}{4l}(5M_1 - M_2)$$

$$K_{\sigma 14} = K_{\sigma 41} = \frac{-1}{2l}(F_1 + F_2)$$

$$K_{\sigma 15} = K_{\sigma 51} = \frac{3}{2l^2}(M_2 - M_1)$$

$$K_{\sigma 16} = K_{\sigma 61} = \frac{1}{4l}(M_1 - 5M_2)$$

$$K_{\sigma 22} = \frac{3}{5l}(F_1 + F_2)$$

$$K_{\sigma 23} = K_{\sigma 32} = \frac{1}{160}(23F_2 - 7F_1)$$

$$K_{\sigma 24} = K_{\sigma 42} = \frac{3}{2l^2}(M_2 - M_1)$$

$$K_{\sigma 25} = K_{\sigma 52} = -\frac{3}{5l}(F_1 + F_2)$$

$$K_{\sigma 26} = K_{\sigma 62} = \frac{1}{160}(23F_1 - 7F_2)$$

$$K_{\sigma 33} = \frac{l}{480}(17F_2 + 47F_1)$$

$$K_{\sigma 34} = K_{\sigma 43} = \frac{1}{4l}(M_2 - 5M_1)$$

$$K_{\sigma 35} = K_{\sigma 53} = \frac{1}{160}(-23F_2 + 7F_1)$$

$$K_{\sigma 63} = K_{\sigma 36} = -\frac{l}{60}(F_1 + F_2)$$

$$K_{\sigma 44} = \frac{1}{2l}(F_1 + F_2)$$

$$K_{\sigma 45} = K_{\sigma 54} = \frac{3}{2l^2}(M_1 - M_2)$$

$$K_{\sigma 46} = K_{\sigma 64} = \frac{1}{4l}(5M_2 - M_1)$$

$$K_{\sigma 55} = \frac{3}{5l}(F_1 + F_2)$$

$$K_{\sigma 56} = K_{65} = \frac{1}{160}(7F_2 - 23F_1)$$

$$K_{\sigma 66} = \frac{l}{480}(47F_2 + 17F_1) \tag{3.48}$$

The components of the internal forces for calculation of residuals are:

$$\{R^{\varphi}\} = \int_{-\frac{l}{2}}^{\frac{l}{2}} [B]^T \{S\}dl = \{R_1{}^{\varphi} \dots R_6{}^{\varphi}\}^T \tag{3.49}$$

And

$$R_1{}^{\varphi} = -\frac{1}{2}(F_1 + F_2)$$

$$R_2{}^{\varphi} = -\frac{3}{2l}(M_1 - M_2)$$

$$R_3{}^{\varphi} = -\frac{1}{4}(5M_1 - M_2)$$

$$R_4{}^{\varphi} = \frac{1}{2}(F_1 + F_2)$$

$$R_5{}^{\varphi} = \frac{3}{2l}(M_1 - M_2)$$

$$R_6{}^{\varphi} = -\frac{1}{4}(M_1 + 5M_2) \tag{3.50}$$

The initial stress stiffness matrix, taking into account the effect of initial displacements is shown in Appendix A.

Appendix B contains the linear displacement stiffness matrix.