Chapter One
Basic Fractional Calculus

1.1Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus and goes back to times when Leibniz and Newton invented differential calculus. One owes to Leibniz in a letter to L'Hˆospital, dated September 30, 1695, exact birthday of the fractional calculus and the idea of the fractional derivative. L'Hˆospital asked the question as to the meaning of $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$; i.e.,what if n is fractional? Leibniz replied that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx{:}\,x}$. In the letters to J. Wallis and J. Bernoulli (in 1697), Leibniz mentioned the possible approach to fractional-order differentiation in that sense that for non-integer values of n the definition could be the following:$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx}$. In 1730, Euler mentioned interpolating between integral orders of a derivative and suggested to use the following relationship: $\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$. where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt,\ \xi > 0$. Also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$; Euler obtained: $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \sqrt{\frac{4x}{\pi}} = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}$. , Laplace in [1812] defined a fractional derivative by means of an integral, and in 1819 there appeared the first discussion of a derivative of fractional order in a calculus text written by Lacroix. The first step to generalization of the notion of differentiation for arbitrary functions was done by Fourier (1822).The concept of derivative is traditionally associated to an integer; given a function, we can derive it one, two, three times and so on. It can be have an interest to investigate the possibility to derive a real number of times a function. The main idea is to examine the properties of the ordinary derivative and see where and how it is possible to generalize the concepts. As often happen there is not only a way to do that; we are going to use the most intuitive and, in a certain sense, less rigorous way. Let us consider the general properties of the derivative D_t^n for $n \in N$, where n is an integer. This operator is, in fact, defined to have the following properties, all of which we would like the fractional derivative to share. The first property of interest is that of association

$$D_t^n[Cf(t)] = CD_t^n[f(t)] \tag{1.1}$$

where C is a constant. The second property we would like to incorporate into the fractional calculus is the distributive law

$$D_t^n[f(t) \pm g(t)] = D_t^n[f(t)] \pm D_t^n[g(t)] \tag{1.2}$$

The final property is that the operator obeys Leibniz rule for taking the derivative of the product of two functions

$$D_t^n[f(t)g(t)] = \sum_{k=1}^{n} \binom{n}{k} D_t^{n-k}[f(t)]D_t^k[g(t)]$$

$$= \sum_t^n \binom{n}{k} D_t^{n-k}[g(t)] \tag{1.3}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. The above properties are certainly retained for the *nth* derivative of a monomial t^m with $m \in N$, so that :

$$D_t^n[t^m] = m(m-1)(m-2)\ldots(m-n+1)t^{m-n} \tag{1.4}$$

for $m > n$. Properties (1) and (2) establish that the operator D_t^n is linear and(1.4) enables us to compute the n^{th} derivative of an analytic function expressed in terms of a Taylor's series.

We now extend these considerations to fractional derivatives. Looking at Eq. (1.4) the most easily thing would be to replace the integer numbers with real numbers. The main difficulty is how to replace the factorial function that is defined for integer numbers.

Fortunately it is exists a special function, the gamma function, that has this property. The gamma function is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \quad (1.5)$$

The integral (1.5) is defined for $z > 0$ (or $Re[z] > 0$ if z is a complex number) and can be checked by elementary integration that for z integer this function coincide with the factorial; more precisely it holds:

$$\Gamma\,(n+1) = n!$$

Because

$$\Gamma\,(z+1) = \int_0^\infty t^z e^{-t} dt$$

Let

$$u = t^z \quad \rightarrow du = zt^{z-1}$$

$$dv = e^{-t} \rightarrow v = -e^{-t}$$

$$\Gamma\,(z+1) = -t^z e^{-t} \Big\{\begin{matrix}\infty \\ 0\end{matrix} - \int_0^\infty zt^{z-1}e^{-t}dt = z\int_0^\infty t^{z-1}\, e^{-t}dt$$

$$= z\,\Gamma\,(z)$$

Since we have

$\Gamma\,(z+1) = z\,\Gamma(z)$, for $z > 0$ & $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$

$\Gamma(z) = \frac{\Gamma(z+1)}{z}$,$\Gamma\,(1) =1$

$\Gamma\,(2) = \Gamma\,(1+1) = 1\Gamma\,(1) = 1.1 = 1!$

$\Gamma\,(3) = \Gamma\,(2+1) = 2\Gamma\,(2) = 2.1 = 2!$

$\Gamma\,(4) = \Gamma\,(3+1) = 3\Gamma\,(3) = 3.2.1= 3!$

$\Gamma\,(5) = \Gamma\,(4+1) = 4\Gamma\,(4) =4.3.2.1 =4!$

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$\Gamma\,(z+1) = z\,\Gamma(z) = z\,\Gamma(z-1) =z!$

We are ready now to define a real-indexed derivative, or more generally, a complex-indexed derivative D_t^α with $\alpha \epsilon R$ (or $\alpha \epsilon\ C$), of a monomial t^β as :

$$\frac{d^\alpha}{dt^\alpha}\left[t^\beta\right] = D_t^\alpha\left[t^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \quad , \beta + 1 \neq 0, -1, \dots, -n \qquad (1.6)$$

Some examples are given below:

(1) $Dx = 1$

From

$$D^\alpha x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta-\alpha}$$

$$\Rightarrow Dx = \frac{\Gamma(2)}{\Gamma(1)} x^0 = 1 \qquad .$$

(2) $D^{\frac{1}{2}}x = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = 2\sqrt{\frac{x}{\pi}}$

(3) $D^{\frac{1}{2}}x^2 = \frac{\Gamma(2+1)}{\Gamma\left(2-\frac{1}{2}+1\right)} x^{2-\frac{1}{2}} = \frac{2!x^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} = \frac{2x^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}+1\right)} = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)} = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$

$= \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}}$

In conclusion of this section we are going to check the intuitive idea that D^{-1} is the integration operator; examine the *C* (-*n*) constant and in particular the *C* (-*1*) constant. We first examine the operator D_t^{-1} applied to a monomial

$$D_t^{-1}\left[t^\beta\right] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 + 1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta + 1} \qquad (1.7)$$

from which we see that effectively this is the integral operator. Now again using the linearity property of the operator we know that we can take a sum of infinitesimals to obtain the standard definition of the integral and therefore in general we can write

$$D_t^{-1}[f(t)] = \int_0^t f(\tau)d\tau \tag{1.8}$$

The main objects of classical calculus are derivatives and integrals of functions – these two operations are inverse to each other in some sense. If we start with a function $f(t)$ and put its derivatives on the left-hand side and on the right-hand side we continue with integrals, we obtain a both side infinite sequence.

$$\ldots\frac{d^2f(t)}{dt^2}, \frac{df(t)}{dt}, f(t), \int_a^t f(\tau)d\tau, \int_a^t\int_a^{\tau_1} f(\tau)d\tau d\tau_1, \ldots$$

Fractional calculus tries to interpolate this sequence so this operation unifies the classical derivatives and integrals and generalizes them for arbitrary order. We will usually speak of differintegral, but sometimes the name α-derivative (α is an arbitrary real number) which can mean also an integral if α > 0, is also used, or we talk directly about fractional derivative and fractional integral.

1.2 The Riemann-Liouville Differintegral:

The Riemann-Liouville approach is based on the Cauchy formula (1.9) for the nth integral which uses only a simple integration so it provides a good basis for generalization.

$$I_a^n f(t) = \int_a^t \int_a^{\tau_{n-1}} \ldots \int_a^{\tau_1} f(\tau)d\tau d\tau_1 \ldots d\tau_{n-1}$$

$$= \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau \tag{1.9}$$

Proof: The formula (1.9) can be proven by the help of mathematical induction. The case

n = 1 is obviously fulfilled, so we show the case n = 2 which demonstrates the mechanism of the entire proof in a better way.
Let us substitute n = 2 into (1.9) and compute:

$$\frac{1}{1!}\int_a^t (t-\tau)f(\tau)d\tau = \begin{vmatrix} u = t-\tau & u' = -1 \\ v' = f(\tau) & v = \int_a^\tau f(r)dr \end{vmatrix} =$$

$$= \left[(t-\tau)\int_a^\tau f(r)dr\right]_{\tau=a}^{\tau=t} + \int_a^t \int_a^\tau f(r)dr$$

$$= I_a^2 f(t).$$

The first term is zero because in the upper limit the polynomial is zero while in the lower one we integrate over a set of measure zero.

Now we suppose the formula holds for general n. Then we integrate it once more and see what we obtain:

$$\int_a^t I_a^n f(r)dr = \int_a^t \frac{1}{(n-1)!}\int_a^r (r-\tau)^{n-1} f(\tau)d\tau dr$$

$$|change\ of\ order\ of\ integration|$$

$$= \frac{1}{(n-1)!}\int_a^t f(\tau)\int_\tau^t (r-\tau)^{n-1} dr d\tau$$

$$= \frac{1}{(n-1)!}\int_a^t f(\tau)\left[\frac{(r-\tau)^n}{n}\right]_\tau^t d\tau = \frac{1}{n!}\int_a^t (t-\tau)^n f(\tau)d\tau$$

$$= I_a^{n+1} f(t)$$

This completes the proof of the Cauchy formula (1.9).

Now it is obvious how to get an integral of arbitrary order. We simply generalize the Cauchy formula (1.9) the integer n is substituted by a positive real number α and the Gamma function is used instead of the factorial. Notice that the integrand is still integrable because

$\alpha - 1 > -1.$

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau) \qquad (1.10)$$

This formula represents the integral of arbitrary order $\alpha > 0$, but does not permit order $\alpha = 0$ which formally corresponds to the identity operator. This expectation is fulfilled under certain reasonable assumptions at least if we consider the limit for $\alpha \longrightarrow 0$
Hence, we extend the above definition by setting:

$$I_a^0 f(t) = f(t) \tag{1.11}$$

The definition of fractional integrals is very straightforward and there are no complications. A more difficult question is how to define a fractional derivative. There is no formula for the n^{th} derivative analogous to (1.9) so we have to generalize the derivatives through a fractional integral. First we perturb the integer order by a fractional integral according to (1.10) and then apply an appropriate number of classical derivatives.
We can always choose the order of perturbation less than 1
The result of these ideas is the following $(\alpha > 0)$:

$$\begin{aligned} D_a^\alpha f(t) &= \frac{d^n}{dt^n}[I_a^{n-\alpha} f(t)] \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \end{aligned} \tag{1.12}$$

where $n = [\alpha] + 1$. This formula includes even the integer order derivatives. If $\alpha = k$ and $k \in \mathbb{N}_0$then $n = k + 1$ and we obtain:

$$D_a^k f(t) = \frac{1}{\Gamma(1)} \frac{d^{k+1}}{dt^{k+1}} \int_a^t f(\tau) d\tau = \frac{d^k f(t)}{dt^k}$$

We can see that classical derivatives are something like singularities among differintegrals because the integration disappears and so there is no dependence on the lower bound a anymore. In this sense the classical derivatives are the only differintegrals which do not depend on history, i.e. are local.

If we put $D_a^{-\alpha} = I_a^{\alpha}$ and note that $f^{(0)}(t) = f(t)$ we can write both fractional integral and derivative by one expression and formulate the definition of the Riemann-Liouville differintegral.

Definition(1.1)(TheRiemann-Liouville differintegral):

Let a, T, α be real constant $(a < \mathrm{T})$,$n = \max(0, [\alpha] + 1)$ and $f(t)$ an integrable function on $\langle a, \mathrm{T})$. For $n > 0$ in additional assume that $f(t)$ is n-times differentiable on $\langle a, \mathrm{T})$ except on a set of measure zero. Then the Riemann-Liouville differintegral is defined for $t\epsilon\langle \alpha, \mathrm{T})$ by the formula:

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \qquad (1.13)$$

1.3 The Caputo Differintegral:

We will denote the Caputo differintegral by the capital letter with upper-left index c_D.The fractional integral is given by the same expression like before, so for $\alpha > 0$ we have

$$ {}^{c}D_a^{-\alpha} f(t) = D_a^{-\alpha} f(t) \qquad (1.14)$$

The difference occurs for fractional derivative. A non-integer-order derivative is again defined by the help of the fractional integral, but now we first differentiate $f(t)$ in the common sense and then go back by fractional integrating up to the required order. This idea leads to the following definition of the Caputo differintegral.

Definition 1.4 (The Caputo differintegral):

Let a, T, α be real constants $(\alpha < \mathrm{T})$, $n_c = \max(0, -[-\alpha])$ and $f(t)$ a function which is integrable on $\langle a, \mathrm{T})$ in case $n_c = 0$ and n_c-times differentiable on $\langle a, \mathrm{T})$ except on a set of measure zero in case $n_c > 0$. Then the Caputo differintegral is defined for $t\epsilon\langle a, \mathrm{T})$ by formula:

$$ {}^{c}D_{a}^{\alpha} f(t) = I_{a}^{n_c-\alpha}\left[\frac{d^{n_c} f(t)}{dt^{n_c}}\right] \tag{1.15} $$

Remark:

For $\alpha > 0$, $\alpha \notin \mathbb{N}_0$, formula (1.15) is often written in the form:

$$ {}^{c}D_{a}^{\alpha} f(t) = \frac{1}{\Gamma(n_c - \alpha)} \int_{a}^{t} (t-\tau)^{n_c-\alpha-1} f^{n_c}(\tau) d\tau \tag{1.16} $$

Chapter Two

Properties of Partial Fractional Calculus

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this thesis

2.1 Notations and Definitions:

Let $J := [0, a]X[0, b]$; $a, b > 0$ and $\rho > 0$. Denote $L^{\rho}(J.R^n)$ the space of Lebesgue-integrable functions $u: J \to R^n$ with the norm

$$\|u\|_{L^{\rho}} = \left(\int_0^a \int_0^b \|u(x,y)\| \, dydx\right)^{\frac{1}{\rho}}$$

Where $\|.\|$ denotes a suitable complete norm on R^n Also $L^1(J, \mathbb{R}^n)$ is endowed with norm $\|.\|_{L^1}$ defined by

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\| dydx$$

Let $L^{\infty}(J, \mathbb{R}^n)$ be the Banach space of measurable functions $u: J \longrightarrow \mathbb{R}^n$ which are bounded, equipped with the norm

$$\|u\|_{L^{\infty}} = inf\{c > 0: \|u(x,y)\| \leq c, a.e\ (x,y) \in J\}$$

As usual, by AC$(J, \mathbb{R}^n)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$; and $C(J, \mathbb{R}^n)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|u\|_{\infty} = sup_{(x,y)\in J}\|u(x,y)\|$$

Also $C(J, \mathbb{R})$ is endowed with norm $\|.\|_{\infty}$ defined by

$$\|u\|_{\infty} = sup_{(x,y)\in J}|u(x,y)|$$

Define a multiplication "." by
$(u.v)(x,y) = u(x,y).v(x,y)$, for $(x,y) \in J$.
Then $C(J, \mathbb{R})$ is a Banach algebra with above norm and multiplication.

if $u \in C([-\alpha, a]X[-\beta, b], \mathbb{R}^n)$; $a, b, \alpha, \beta > 0$ then for any $(x, y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t) = u(x+s, y+t),$$

for $(s,t) \in C([-\alpha, 0]X[-\beta, 0]$. Here $u_{(x,y)}(.,.)$ represents the history of the state from time $(x-\alpha, y-\beta)$ up to the present time(x, y).

Definition (2.1)[22]:

The Riemann–Liouville fractional integral of order $\alpha \in (0, \infty)$ of a function $h \in L^1([0, b], \mathbb{R}^n)$; $b > 0$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

Definition (2.2) [22]:

The Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ of a function $h \in L^1([0, b], \mathcal{R}^n)$; defined by

$$D_0^\alpha h(t) = \frac{d}{dt} I^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds;$$

for almost all $t \in [0, b]$.

Definition (2.3) [22]:

The Caputo fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1([0, b], \mathbb{R}^n$ is defined by

$${}^cD_0^\alpha h(t) = I_0^{1-\alpha} \frac{d}{dt} h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} h(s);$$

for almost all $t \in [0, b]$.

Definition (2.4) [17, 22]:

Let $\alpha \in (0, \infty)$ and $u \in L^1(J, \mathbb{R}^n)$. The partial Riemann- Liouville integral of order α of $u(x, y)$ with respect to x is defined by the expression

$$I_{0,x}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y) ds;$$

for almost all $(x, y) \in J$.

Analogously, we define the integral

$$I_{0,y}^{\alpha}u(x,y) = \frac{1}{\Gamma(\alpha)}\int_{0}^{y}(y-t)^{\alpha-1}u(x,t)dt;$$

for almost all $(x,y) \in J$.

Definition (2.5) [17, 22]:

Let $\alpha \in (0,1]$ and $u \in L^1(J,\mathbb{R}^n)$ The Riemann–Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$$\left(D_{0,x}^{\alpha}u\right)(x,y) = \frac{\partial}{\partial x}I_{0,x}^{1-\alpha}u(x,y);$$

for almost all $(x,y) \in J$.

Analogously, we define the derivative

$$\left(D_{0,y}^{\alpha}u\right)(x,y) = \frac{\partial}{\partial y}I_{0,y}^{1-\alpha}u(x,y);$$

for almost all $(x,y) \in J$.

Definition (2.6) [17, 22]:

Let $\alpha \in (0,1]$ and $u \in L^1(J,\mathbb{R}^n)$ .The Caputo fractional derivative of order α of $u(x,y)$ with respect to x is defined by the expression

$${}^{c}D_{0,x}^{\alpha}u(x,y) = I_{0,x}^{1-\alpha}\frac{\partial}{\partial x}u(x,y);$$

for almost all $(x,y) \in J$.

Analogously, we define the derivative

$${}^{c}D_{0,y}^{\alpha}u(x,y) = I_{0,y}^{1-\alpha}\frac{\partial}{\partial y}u(x,y);$$

for almost all $(x,y) \in J$.

Definition (2.7)[23]:

Let $r = (r_1,r_2) \in (0,\infty)X(0,\infty)$ and $u \in L^1(J,\mathbb{R}^n)$.The left-sided mixed Riemann–Liouville integral of order r of u is defined by

$$(I_0^r u)(x,y) = \frac{1}{\Gamma(r_1)}\frac{1}{\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}u(x,y)dtds$$

In particular,

$$(I_0^0 u)(x,y) = u(x,y), \quad (I_0^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)dtds$$

for almost all $(x,y) \in J$, where $\sigma = (1,1)$.

for instance, $I_0^r u$ exists for all $r_1, r_2 > 0$, when $u \in L^1(J, \mathbb{R}^n)$. Note also that when $u \in C(J, \mathbb{R}^n)$, then $(I_0^r u) \in C(J, \mathbb{R}^n)$; moreover,

$$(I_0^r u)(x,0) = (I_0^r)(0,y) = 0; \; x \in [0,a], y \in [0,b].$$

Example (2.8):

Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,\infty)X(0,\infty)$, then

$$I_0^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2};$$

for all most all $(x,y) \in J$.

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0.1)X[0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second-order partial derivative.

Definition (2.9)[21]:

Let $r \in (0,1]X(0,1]$ and $u \in L^1(J, \mathbb{R}^n)$. The Caputo fractional-order derivative of order r of u is defined by the expression $({}^c D_0^r u)(x,y) = \left(I_0^{1-r} D_{xy}^2 u\right)(x,y)$ and the mixed fractional Riemann–Liouville derivative of order r of u is defined by the expression

$(D_0^r u)(x,y) = \left(D_{xy}^2 I_0^{1-r} u\right)(x,y)$

The case $\sigma = (1,1)$ is included and we have

$$({}^c D_0^\sigma u)(x,y) = (D_0^\sigma u)(x,y) = \left(D_{xy}^2 u\right)(x,y);$$

.

for almost all $(x,y) \in J$.

Example (2.10):
Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,1]X(0,1]$ then

$$D_0^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2};$$

for almost all $(x, y) \in J$.

Definition (2.11):
for a function $u: J \longrightarrow \mathbb{R}^n$ we set

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order
$r = (r_1, r_2) \in (0,1]X(0,1]$ of a function $u(x,y)$, we name
The function $\overline{D_0^r} u(x,y) = D_0^r q(x,y)$.The function

$$\overline{D_{0,x}^{r_1}} u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

Is called the partial r_1- order regularized derivative of the function $u(x,y): J \longrightarrow \mathbb{R}^n$ with respect to the variable x.
Analogously, we define the derivative

$$\overline{D_{o,y}^{r_2} u(x,y)} = D_{0,y}^{r_1}[u(x,y) - u(x,o)].$$

Let $a_1 \in [0,a]$, $z^+ = (a_1, 0) \in J$, $J_z = [a_1, a]X[0,b]$. For $w \in L^1(J_z, \mathbb{R}^n)$, the expression

$$\left(I_{z^+}^r w\right)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} w(s,t)\,dt\,ds,$$

is called the left-sided mixed Riemann–Liouville integral of order r of w. The Caputo fractional-order derivative of order r of w is defined by$({}^c D_{z^+}^r w)(x,y) = \left(I_{z^+}^{1-r} D_{xy}^2 w\right)(x,y)$
and the mixed fractional Riemann–Liouville derivative of order r of w is defined by $\left(D_{z^+}^r w\right)(x,y) = \left(D_{xy}^2 I_{z^+}^{1-r} w\right)(x,y)$.

Let $f, g \in L^1(J, \mathbb{R}^n)$.

Lemma (2.12)[1,2]:

A function $u \in AC(J, \mathbb{R}^n)$ such that its mixed derivative D_{xy}^2 exists and is integrable on J is a solution of problems

$$\begin{cases} ({}^cD_0^r u)(x,y) = f(x,y); (x,y) \in J, \\ u(x,0) = \varphi(x); x \in [0,a], \ \ u(0,y) = \psi(y); y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu(x,y) + (I_0^r f)(x,y); (x,y) \in J,$$

where $\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$.

Lemma (2.13) [5]:

A function $u \in AC(J, \mathbb{R}^n)$ such that the mixed derivative $D_{xy}^2(u-g)$ exists and is integrable on J is a solution of problems

$$\begin{cases} {}^cD_0^r[u(x,y)-g(x,y)]=f(x,y); \ (x,y)\in J \\ u(x,0) = \varphi(x); x \in [0,a], u(0,y) = \psi(y); y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

If and only if $u(x,y)$ satisfies

$$\mu(x,y) = g(x,y) - g(x,0) - g(0,y) + g(0,0) + I_0^r(f)(x,y); \quad (x,y) \in J.$$

Let $h \in C([x_k, x_{k+1}]X[0,b], R^n), z_k = (x_k, 0), \ 0 = x_0 < x_1 < . . . < x_m < x_{m+1} = a$

$$\mu_k(x,y) = u(x,0) + u(x_k^+, y) - u(x_k^+, 0); \ \ k = 0, \ . . . , m.$$

Lemma (2.14) [3, 4]:

A function $u \in AC([x_k, x_{k+1}]X[0,b], \mathbb{R}^n); \ \ k = 0, \ . . . , m$ whose r-derivative exists on $[x_k, x_{k+1}]X[0,b], \ k = 0, \ldots, m$ is a solution of the differential equation :

$$({}^cD_{z_k}^r u)(x,y) = h(x,y); \ (x,y) \in [x_k, x_{k+1}]X[0,b],$$

if and only if $u(x,y)$ satisfies:

$$u(x,y) = \mu_k(x,y) + \left(I_{z_k}^r h\right)(x,y); \ \ (x,y) \in [x_k, x_{k+1}]X[0,b].$$

2.2 Fixed Point Theorems:

By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively. Let us start by stating a well-known result, the nonlinear alternative.

Theorem (2.15)[8]:

(Nonlinear alternative of Leray Schauder Type)

Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T: \overline{U} \longrightarrow \mathrm{C}$ continuous and compact operator.

Then either

(a) T has fixed points or

(b) There exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda \mathrm{T}(\mathrm{u})$

The multivalued version of nonlinear alternative:

Lemma (2.16)[8]:

Let X be a Banach space and C a nonempty convex subset of. Let U a nonempty open subset of C with $0 \in U$ and $T: \overline{U} \longrightarrow C$ an upper semicontinuous and compact multivalued operator.

Then either

(a) T has fixed points or

(b) There exist $u \in \partial U$ and $\lambda \in (0.1)$ with $u \in \lambda T(u)$

Theorem (2.17) (Schaefer):

Let X be a Banach space and $N: X \longrightarrow X$ completely continuous operator. If the set

$$E(N) = \{u \in X : u = \lambda N(u) \; for\ some\ \lambda \in (0,1)\}$$

Is bounded, then N has fixed points.

2.3 Gronwall Lemmas:

In the sequel we will make use of the following generalizations of Gronwall's lemmas for two independent variables and singular kernel

Lemma (2.18):

Let $v: J \to [o, \infty 0$ be a real function and $\omega(.,.)$ be a Nonnegative, locally integrable function on J. If there are constants $c > 0$ and $o < r_1, r_2 < 1$ such that

$$v(x,y) \leq \omega(x,y) + c \int_0^x \int_0^y \frac{v(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dtds$$

Then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$v(x,y) \leq \omega(x,y) + \delta c \int_0^x \int_0^y \frac{\omega(x,y)}{(x-s)^{r_1}(y-t)^{r_2}} dtds$$

for every $(x,y) \in J$.

Lemma (2.19):

Let $v, \omega: J \to [o, \infty)$ be nonnegative, locally integrable functions on J. if there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that

$$v(x,y) \leq \omega(x,y) + c \int_0^x \int_0^y \frac{v(x,y)}{(x-s)^{r_1}(y-t)^{r_2}} dtds$$

Then, for every $(x,y) \in J$,

$$v(x,y) = \omega(x,y) + \int_0^x \int_0^y \sum_{j=1}^{\infty} \frac{\left(c\Gamma(r_1)\Gamma(r_2)\right)^j \omega(s,t)dtds}{\Gamma(jr_1)\Gamma(jr_2)(x-s)^{1-jr_1}(y-t)^{1-jr_2}}.$$

If $\omega(x,y) = \omega$ constant on J, then the inequality(3.8) is reduced to
$v(x,y) \leq \omega E_{(r_1,r_2)}(c\Gamma(r_1)\Gamma(r_2)x^{r_1}y^{r_2})$,
Where $E_{(r_1,r_2)}$ is the Mittag-Leffler function [17], defined by

$$E_{(r_1,r_2)}(z) := \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kr_1+1)\Gamma(kr_2+1)}; \; r_j z \in \mathbb{C}, \Re(r_j) > 0; j = 1,2.$$

Chapter Three

Existence of Solutions for Differential Equations with Fractional Order

3.1 Introduction

In this chapter, we shall present existence results for some classes of IVP for partial differential equations with fractional order.
$({}^cD_0^r u)(x,y) = f\big(x,y,u_{(x,y)}\big),$
$(x,y) \in J := [0,a] \times [0,b], \qquad u(x,y) = \emptyset(x,y)$, (3.1)
If $(x,y) \in \tilde{J} := [-\alpha,a] \times [-\beta,b] \setminus (0,a] \times (0,b],$ (3.2)
$(x,0) = \varphi(x), x \in [0,a], u(0,y) = \psi(y), y \in [0,b],$ (3.3)
where $\alpha,\beta,a,b > 0$, $c_{D_0^r}$ is the standard Caputo fractional derivative of order $r = (r_1,r_2) \in (0,a] \times (0,b], f: J \times C \to \mathbb{R}^n$
is a given continuous function $\phi: \tilde{J} \to \mathbb{R}^n$, $\psi: [0,b] \to \mathbb{R}^n$, are given absolutely continuous functions with
$\varphi(x) = \phi(x,0), \psi(y) = \phi(0,y)$ for each $x \in [0,a], y \in [0,b]$
And $\mathsf{C} := \mathsf{C}([-\alpha,0] \times [-\beta,0], \mathbb{R}^n)$ is the space of continuous Functions on $\big[-\alpha,0\big] \times [-\beta,0]$.
Next we consider the following nonlocal initial value problem
$({}^cD_0^r u)(x,y) = f\big(x,y,u_{(x,y)}\big)$, if $(x,y) \in J$ (3.4)
$u(x,y) = \varphi(x,y)$, if $(x,y) \in \tilde{J}$ (3.5)

$u(x,0) + \mathbb{Q}(u) = \varphi(x), x\epsilon[0,a], u(0,y) + K(y) = \psi(y), y \in [0,b],$

(3.6)

Where $f,\emptyset,\varphi,\psi$ are as in problem (3.1)–(3.2) and $\mathbb{Q}, K: \mathsf{C}(J,\mathbb{R}^n) \to \mathbb{R}^n$ are continuous functions.

3.2 Existence of Solutions:

Let us start by defining what we mean by a solution of the problem (3.1)–(3.3).

Definition (3.1):

A function $u \in C_{(a,b)} := C([-\alpha, a] \times [-\beta, b], \mathbb{R}^n)$ whose mixed derivative D_{xy}^2 exists and is integrable is said to be a solution of (3.1)–(3.3) if u satisfies (3.1) and (3.3) on J and the condition (3.2) on Further, we present conditions for the existence and uniqueness of a solution of
problem (3.1)–(3.3) on $\tilde{J}$.
Further, we present conditions for the existence and uniqueness of a solution of problem (3.1)–(3.3).

Theorem (3.2):

Assume that the following hypotheses hold:
(3.2.1) $f: J \times C \to R^n$ is a continuous function.
(3.2.2) For any $u, v \in C$ and $(x, y) \in J$, there exists $k > 0$ such that

$$\|f(x, y, u) - f(x, y, v)\| \le k\|u - v\|_C.$$

if

$$\frac{k a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1 \qquad (3.7)$$

then there exists a unique solution for IVP (3.1)–(3.3) on $[-\alpha, a] \times [-\beta, b]$.

Proof:

Transform the problem (3.1)–(3.3) into a fixed-point problem. Consider the operator $N: C_{(a,b)} \to C_{(a,b)}$ defined by,

$$N(u)(x, y) = \begin{cases} \emptyset(x, y), & (x, y) \in \tilde{J} \\ \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} & \\ \quad \times f(s, t, u_{(s,t)}) dt ds, & (x, y) \in J \end{cases} \qquad (3.8)$$

Let $v, w \in C_{(a,b)}$. Then, for $(x,y) \in [-\alpha, a] \times [-\beta, b]$,

$\|N(v)(x,y) - N(w)(x,y)\|$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y \left\| f(s,t,v_{(st)}) - f(s,t,w_{(s,t)}\right\|$$

$$\times |(x-s)^{r_1-1}||(y-t)^{r_2-1}| dtds$$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$

$$\times \|v(s,t) - w(s,t)\|_C dtds$$

$$\leq \frac{k}{\Gamma(r_1)\Gamma(r_2)} \|v - w\|_J \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds$$

$$\leq \frac{kx^{r_1}y^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|v - w\|_J.$$

Consequently,

$$\|N(v) - N(w)\|_J \leq \frac{kx^{r_1}y^{r_2}}{\Gamma(r_1+1)(r_2+1)} \|v - w\|_J.$$

By (3.7), N is a contraction, and hence N has a unique fixed point by Banach's contraction principle.

Theorem (3.3):

Assume that (3.2.1) and the following hypothesis hold:

(3.3.1) There exist $p, q \in C(J, R_+)$ such that

$$\|f(x,y,u\| \leq q(x,y) + p(x,y)\|u\|_C,$$

for $(x,y) \in J$ and each $u \in C$ Then the IVP (1.3)-(3.3) have at least one solution on $[-\alpha, a] \times [-\beta, b]$.

Proof:

Transform the problem (3.1)–(3.3) into a fixed point problem. Consider the operator N defined in (3.8). We shall show that the operator N is continuous and completely continuous.

Step1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$.let $\eta > 0$ be such that $\|u_n\| \leq \eta$. Then

$$\|N(u_n)(x,y) - N(u)(x,y)\| \leq$$

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y |x-s|^{r_1-1}|y-t|^{r_2-1}$$

$$\times \left\|f\left(x,y,u_{n(s,t)}\right) - f\left(x,y,u_{(s.t)}\right)\right\|dtds$$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y |x-s|^{r_1-1}|y-t|^{r_2-1}$$

$$\times\, sup_{(s,t)\in J}\left\|f\left(x,y,u_{n(s,t)}\right) - f\left(x,y,u_{(s,t)}\right)\right\|dtds$$

$$\leq \frac{a^{r_1}b^{r_2}\left\|f\left(.,.,u_{n(.,.)}\right) - f\left(.,.,u_{(.,.)}\right)\right\|_\infty}{r_1 r_2\Gamma(r_2)\Gamma(r_2)}.$$

Since f is a continuous function, we have

$$\|N(u_n) - N(u)\|_\infty \leq \frac{a^{r_1}b^{r_2}\left\|f\left(.,.,u_{n(.,.)}\right) - f\left(.,.,u_{(.,.)}\right)\right\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$\longrightarrow 0 \; as \; n \longrightarrow \infty$$

Step 2: N maps bounded sets into bounded sets in $C_{(a,b)}$. Indeed, it is enough to show that, for any $\eta^* > 0$, there exists a positive constant $\tilde{\ell}$ such that, for each $u \in B_{\eta^*} = \left\{u \in C_{(a,b)}: \|u\|_\infty \leq \eta^*\right\}$, we have $\|N(u)\|_\infty \leq \tilde{\ell}$ By (H_3) we have for each $(x,y) \in J$,

$$\|N(u)(x,y)\| \leq \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$

$$\times \left\|f\left(s,t,u_{(s,t)s}\right)\right\|dtds$$

$$\leq \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\left\|\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}p(s,t)dtds\right\|$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}q(s,t)\left\|u_{(s,t)}\right\|_\infty dtds$$

$$\leq \|\mu(x,y)\| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds$$

$$+ \frac{\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds.$$

Thus $\|N(u)\|_\infty \leq \|\mu\|_\infty + \frac{\|p\|_\infty\|q\|_\infty\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}a^{r_1}b^{r_2} := \tilde{\ell}$.

Step 3: N maps bounded sets into equicontinuous sets in $C_{(a,b)}$. let $(x_1,y_1),(x_2,y_2) \in (0,a] \times (0,b], x_1 < x_2, y_1 < y_2$. B_{η^*} be a bounded set of $C_{(a,b)}$ as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\|N(u)(x_2,y_2) - N(u)(x_1,y_1)\| = \|\mu(x_1,y_1) - \mu(x_2,y_2)\|$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\left\|\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\right.$$

$$-(x_1-s)^{r_1-1}(y_1-t)^{r_1-1}]\times f\big(s,t,u_{(s,t)}\big)dtds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2$$

$$-t)^{r_2-1}f\big(x,y,u_{(s,t)}\big)dtds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2$$

$$-t)^{r_2-1}f\big(s,t,u_{(s,t)}\big)dtds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2$$

$$\left.-1)^{r_2-1}f\big(s,t,u_{(s,t)}\big)dtds\right\|$$

$$\leq\|\mu(x_1,y_1)-\mu(x_2,y_2)\|+\frac{\|p\|_\infty\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}-(x_2-s)^{r_{s1}-1}(y_{s2}-t)^{r_2-1}]dtds$$

$$+\frac{\|p\|_\infty\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds$$

$$+\frac{\|p\|_\infty\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds$$

$$+\frac{\|p\|_\infty\|q\|_\infty\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds$$

$$\leq \|\mu(x_1, y_1) - \mu(x_2, y_2)\| + \frac{\|p\|_\infty \|q\|_\infty \eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \left[2y_2^{r_2}(x_2 - x_1)^{r_1} + 2x_2^{r_1}(y_2 - y_1)^{r_2} + x_1^{r_1} y_1^{r_2} - x_2^{r_1} y_2^{r_2} - 2(x_2 - x_1)^{r_1}(y_2 - y_1)^{r_2}\right].$$

As $x_1 \longrightarrow x_2, y_1 \longrightarrow y_2$ the righthand side of the above inequality tends to zero. The e quicontinuity for the cases $x_1 < x_2 < 0, y_1 < y_2 < 0$ and $x_1 \leq 0 \leq x_2, y_1 \leq 0 \leq y_2$ is obvious. As a consequence of Steps 1–3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Step 4: A priori bounds. We now show there exists an open set $U \subseteq C_{(a,b)}$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C_{(a,b)}$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x, y) \in J$,

$$u(x, u) = \lambda\mu(x, y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} (x - s)^{r_1 - 1}(y - t)^{r_1 - 1} dtds.$$

This implies by (3.3.1) that, for each $(x, y) \in J$, we have

$$\|u(x, y)\| = \|\mu(x, y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_2} (x - s)^{r_1 - 1}(y - t)^{r_1 - 1} \times \left[p(s, t) + q(s, t)\left\|u_{(s,t)S}\right\|_C\right] dtds$$

$$\leq \|\mu(x, y)\| + \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} a^{r_1} b^{r_2} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} (x - s)^{r_1 - 1}(y - t)^{r_1 - 1} q(s, t)\left\|u_{(s,t)}\right\|_C dtds$$

We consider the function τ defined by

$$\tau(x, y) = sup\{u(s, t): -\alpha \leq s \leq x, -\beta \leq t \leq y, 0 \leq x \leq a, 0 \leq y \leq b\}.$$

Let $(x^*, y^*) \in [-\alpha, x] \times [-\beta, y]$ be such that $\tau(x, y) = \|u(x^*, y^*)\|$. if $(x^*, y^*) \in J$ then by the previous inequality, we have for $(x, y) \in J$,

$$\tau(x, y) = \|\mu(x, y)\| + \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} a^{r_1} b^{r_2}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}q(s,t)\tau(s,t)dtds$$

$$\leq \|\mu(x,y)\| + \frac{\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}a^{r_1}b^{r_2}$$

$$+\frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\tau(s,t)dtds.$$

If $(x^*,y^*)\in\widetilde{J}$, then $\tau(x,y)=\|\phi\|_C$ and the previous inequality holds. By Lemma (2.18) there exists a constant $\delta=(r_1,r_2)$ such that

$$\tau(x,y)\leq\left[\|\mu\|_\infty+\frac{a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]$$

$$\times\left[1+\frac{\delta\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\right]$$

$$\leq\left[\|\mu\|_\infty+\frac{a^{r_1}b^{r_2}\|p\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]$$

$$\times\left[1+\frac{\delta a^{r_1}b^{r_2}\|q\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]\coloneqq M.$$

Since for every $(x,y)\in J$, $\left\|u_{(x,y)}\right\|_C\leq\tau(x,y)$, we have

$$\|u\|_\infty\leq max(\|\phi\|_C,M)\coloneqq R.$$

Set

$$U=\{u\in C_{(a,b)}:\|u\|_\infty<R+1\}.$$

By our choice of U there is no $u\in\partial U$ such that $u=\lambda N(u)$, for $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type, we deduce that N has a fixed point u in U which is a solution to problem (3.1)–(3.3).

Now we present two similar existence results for the nonlocal problem (3.4)–(3.6).

Definition (3.4):

A function $u\in C_{(a,b)}$ is said to be a solution of (3.4)–(3.6) if u satisfies (3.4) and (3.6) on J and the condition (3.5) on $\tilde{J}$

Theorem (3.5):

Assume that (3.2.1), (3.2.2), and the following conditions, (3.5.1) There exists $\tilde{k} > 0$ such that
$\|\mathbb{Q}(u) - \mathbb{Q}(v)\| \leq \tilde{k}\|u - v\|_\infty$, for any $u, v \in C(J, R^n)$,
(3.5.2) There exists $k^* > 0$ such that
$\|K(u) - K(v)\| \leq k^*\|u - v\|_\infty$, for any $u, v \in C(J, R^n)$
Hold. If

$$\tilde{k} + k^* + \frac{k a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_1 + 1)} < 1, \tag{3.9}$$

Then there exists a unique solution for IVP (3.4)–(3.6) on

$$[-\alpha, a] \times [-\beta, b].$$

Theorem (3.6):

Assume that (3.2.1), (3.3.1), and the following conditions
(3.6.1) There exists $d^{\sim} > 0$ such that
$\|\mathbb{Q}(u)\| \leq \tilde{d}(1 + \|u\|_\infty)$, for any $u \in C(J, R^n)$,
(3.6.2) There exists $d^* > 0$ such that
$\|K(u)\| \leq d^*(1 + \|u\|_\infty)$, for any $u \in C(J, R^n)$ hold, then there exists at least one solution for IVP (3.4)–(3.6) on $[-\alpha, a] \times [-\beta, b]$

An Example (3.7):

As an application of our results we consider the following partial hyperbolic functional differential equations of the form:

$$({}^cD_0^r u)(x, y) = \frac{1}{(3e^{x+y+2})(1 + |u(x - 1, y - 2)|)}$$

$$\text{If}(x, y) \in [0,1] \times [0,1] \tag{3.10}$$

$$u(x, 0) = x, u(0, y) = y^2, x, y \in [0,1] \tag{3.11}$$

$$u(x, y) = x + y^2, (x, y) \in [-1,1] \times [-2,1] \setminus (0,1] \times (0,1] \tag{3.12}$$

Set

$$f(x, y, u_{(x,y)}) = \frac{1}{(3 + e^{x+y+2})(1 + u(x - 1, y - 2))}$$

for $(x, y) \in [0,1] \times [0,1]$.
for each $u, \overline{u} \in R$ and $(x, y) \in [0,1] \times [0,1]$ we have

$$\left|f\left(x,y,u_{(x,y)}\right)-f\left(x,y,\overline{u}_{(x,y)}\right)\right|\leq\frac{1}{3e^2}\|u-\overline{u}\|_C.$$

Hence condition (3.2.2) is satisfied with $k=\frac{1}{3e^2}$.We shall show that condition (3.7) holds with $a=b=1$. Indeed

$$\frac{ka^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}=\frac{1}{3e^2\Gamma(r_1+1)\Gamma(r_2+1)}<1$$

Which is satisfied for each $(r_1,r_2)\in(0,1]\times(0,1]$. Consequently, Theorem 3.2 implies that problem (3.10)–(3.12) has a unique solution defined on $[-1,1]\times[-2,1]$.

Chapter Four

The Modified Variational Iteration Method

4.1 Introduction:

The Lagrange multipliers technique [16] was widely used to solve a number of nonlinear problems which arise in mathematical physics and another related areas, and it was developed into a powerful analytical method, i.e. the varianional iteration method [6, 7], [9-15] for solving differential equations. The method has been applied to initial boundary problems [18-20], fractal initial value problems [24], q-difference equations, etc.

Generally, in applications of variational iteration method to initial value problems of differential equations, one usually follows the fallowing three steps:

(a) Establishing the correction functional

(b) Identifying the Lagrange multipliers

(c) Determining the initial iteration

The step (b) is very crucial. Applications of the method to fractional differential equations (FDEs) mainly and directly used the Lagrange multipliers in ordinary differential equations (ODEs).the present article conceives a method how the Lagrange multiplier has to be defined from Laplace transform. The technique can be readily and universally extended to solve both differential equations and FDEs with initial value conditions.

4.2 Basics of the variation iteration method:

In order to illustrate the basic idea of the technique, consider the following general nonlinear system:

$$\frac{d^m u(t)}{dt^m} + R[u(t)] + N[u(t)] = g(t) \tag{4.1}$$

Where R is a linear operator and N is a nonlinear operator and *g(t)* is a given continuous function and $\frac{d^m u}{dt^m}$ is the term of the highest-order derivative.

The basic concept of the method is to construct a correction functional for the system (4.1), which reads

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda(t,\tau)\{Ru_n(\tau) + Nu_n(\tau) - g(\tau)\}d\tau, \qquad (4.2)$$

Where $\lambda(t,\tau)$ is a general Lagrange multipliers [12,13,16] that can be identified optimally via variational theory, u_n is the n^{th} approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta u_n = 0$, where δ is the variational derivative.

To illustrate how restricted variation works in the variational iteration method.

4.3 New identification of the Lagrange multipliers:

Let us revisit the original idea of the Lagrange multipliers in the case of an algebraic equation. Firstly, an iteration formula for finding the solution of the algebraic equation $f(x) = 0$ can be constructed as:

$$x_{n+1} = x_n + \lambda f(x_n) \qquad (4.3)$$

The optimality condition for the extreme $\frac{\delta x_{n+1}}{\delta x_n} = 0$ leads to

$$\lambda = -\frac{1}{f'(x_n)} \qquad (4.4)$$

Where δ is the classical variational operator. From (4.3) and (4.4), for a given initial value x_0 we can find the approximate solution x_{n+1} by the iterative scheme for (4.4)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f'(x_n) \neq 0, n = 0,1,2,\ldots \qquad (4.5)$$

This algorithm is well known as the Newton-Raphson method and has quadratic convergence.

Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the Laplace transform to Equation (4.1), then the linear part is transformed into an algebraic equation as follows:

$$s^m u(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)}u(0) + L\big[R[u]\big] + L\big[N[u]\big] - L[g(t)] = 0 \qquad (4.6)$$

where

$$u(s) = L[u(t)] = \int_0^\infty e^{-st}u(t)dt.$$

The iteration formula of (4.6) can be used to suggest the main iterative scheme involving the Lagrange multiplier as:

$$u_{n+1}(s) = u_n(s) + \lambda(s)\big[s^m u_n(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)}u(0) + L\big(R[u_n] + N[u_n] - g(t)\big)\big] \qquad (4.7)$$

Considering $L(R[u_n] + N[u_n])$ as restricted terms, one can derive a Lagrange multiplier as:

$$\begin{aligned} \delta u_{n+1}(s) &= \delta u_n(s) + s^m \delta\lambda(s) u_n(s) \\ 0 &= 1 + s^m \lambda(s) \\ \lambda &= -\frac{1}{s^m} \end{aligned} \qquad (4.8)$$

With Equation (4.8) and the inverse-Laplace transform L^{-1},
The iteration formula (4.7) can be explicitly given as:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^m}\left[s^m u_n(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)}u(0) + L\big(R[u_n] + N[u_n] - g(t)\big)\right]\right]$$

$$= L^{-1}\left(\frac{1}{s^m}u^{(m-1)}(0) + \cdots + \frac{u(0)}{s} - \frac{1}{s^m}L\big(R[u_n] + N[u_n] - g(t)\big)\right) \qquad (4.9)$$

Where the initial iteration $u_0(t)$ can be determined by

$$u_0(t) = L^{-1}\left(\frac{1}{s^m}u^{(m-1)}(0) + \cdots + \frac{u(0)}{s}\right)$$
$$= u(0) + u'(0)t + \cdots + \frac{u^{(m-1)}t^{m-1}}{(m-1)!} \qquad (4.10)$$

Equation (4.10) also explained why the initial iteration in the classical VIM is determined by the Taylor series.
So Consequently, the solution

$$u(t) = \lim_{n\to\infty} u_n(t).$$

4.4 Illustrative examples:

We now consider the applications of the modified VIM to both ODEs and FDEs.

Ordinary Differential Equation:

Example 1:

Consider the following differential equation:

$$\frac{du}{dt} + u = 0, \qquad u(0) = u_0,$$

Which has the exact solution:

$$u(t) = u_0 e^{-t}$$

We can obtain the successive approximate solutions as:

$$u_{n+1}(t) = L^{-1}\left(\frac{u^{(m-1)}(0)}{s^m} + \cdots + \frac{u(0)}{s} - \frac{1}{s^m}L[u_n(t)]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_n]\right)$$

$$u_0(t) = u(0) = u_0$$

$$u_1(t) = L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_0]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{u_0}{s}L[1]\right) = L^{-1}\left(\frac{u_0}{s} - \frac{u_0}{s^2}\right)$$

$$= u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s^2}\right) = u_0(1-t)$$

$$u_2(t) = L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_1]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L\big(u_0(1-t)\big)\right) = u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s}L[1-t]\right)$$

$$= u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s}\left(\frac{1}{s} - \frac{1}{s^2}\right)\right) = u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)$$

$$= u_0\left(1 - t + \frac{t^2}{2}\right)$$

$$\vdots$$

$$= u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \cdots + (-1)^{n+1}\frac{1}{s^{n+1}}\right)$$

$$u_n(t) = u_0\left[\sum_{k=1}^{n}\frac{(-t)^n}{n!}\right]$$

For $n \to \infty$, $u_n(t)$ tends to the exact solution $u_0 e^{-t}$.
We note that the integration by parts is not used and the calculation of the Lagrange multiplier here is much simpler. Furthermore, the VIM can be easily extended to FDEs and this is the main purpose of this thesis

Fractional Differential Equations:

Let us consider the FDE

$$\frac{du}{dt} + {}_0^cD_t^\alpha u = g(t,u), \qquad 0 < t, \qquad 0 < \alpha < 1,$$

And the variational iteration formula is given as

$$u_{n+1}(t) = u_n + \int_0^t \lambda(t,\tau)\left(\frac{du_n}{dt} + {}_0^cD_t^\alpha u_n - g(\tau,u_n)\right)d\tau,$$

Where ${}_0^cD_t^\alpha u$ is the Caputo derivative and $g(\tau,u_n)$ is a nonlinear term:

$${}_0^cD_t^\alpha u + R[u] + N[u] = g(t)$$

$$u^{(k)}(0) = a_k, \qquad 0 < t, 0 < \alpha, m = [\alpha] + 1, k = 0, \dots, m-1. \tag{4.11}$$

Now, we consider the application of the modified VIM.
The following Laplace transform of the term ${}_0^cD_t^\alpha u$ holds

$$L[{}_0^cD_t^\alpha u] = s^\alpha u(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-1-k}, m-1 < \alpha \leq m. \tag{4.12}$$

Taking the above Laplace transform to both sides of (4.11), the iteration formula of equation (4.11) can be constructed as:

$$u_{n+1}(s) = u_n(s) + \lambda(s) \times$$

$$\left[s^{\alpha}u_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} + L\big(R[u_n] + N[u_n] - g(t)\big)\right]$$

As a result, after the identification of a Lagrange multiplier

$$\lambda(s) = -\frac{1}{s^{\alpha}},$$

then**:**

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^{\alpha}}\left[s^{\alpha}u_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} + L\big(R[u_n] + N[u_n] - g(t)\big)\right]\right]$$

$$= L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} - \frac{1}{s^{\alpha}}L\big(R[u_n] + N[u_n] - g(t)\big)\right),$$

$$m-1 < \alpha \le m \quad (4.13)$$

and

$$u_0(t) = L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1}\right)$$

$$= u(0) + u'(0) + \cdots + \frac{u^{(m-1)}(0)t^{m-1}}{(m-1)!} \quad (4.14)$$

Let us apply the above VIM to solve FDEs of Caputo type.

Example 2:

Consider the following linear initial value problem

$$D^{\alpha}u + u = 0,\ u(0) = 1, u'(0) = 0, 0 < \alpha < 2. \quad (4.15)$$

After taking the Laplace transform to both sides of Equation (4.15) we get the following iteration formula:

$$u_{n+1}(s) = u_n(s) + \lambda(s^\alpha u_n(s) - u(0)s^{\alpha-1} - u'(0)s^{\alpha-2} + L[u_n(t)]) \quad (4.16)$$

Setting $L[u_n(t)]$ as a restricted variation, $\lambda(s)$ can be identified as

$$\lambda(s) = -\frac{1}{s^\alpha} \quad (4.17)$$

The approximate solution of Equation (4.15) can be given as:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^\alpha}(s^\alpha u_n(s) - u(0)s^{\alpha-1} - u'(0)s^{\alpha-2} + L[u_n(t)])\right]$$

$$= L^{-1}\left(\frac{1}{s^\alpha}(u(0)s^{\alpha-1} - L[u_n(t)])\right)$$

$$u_{n+1}(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_n(t)]\right)$$

$$u_0(t) = 1$$

$$u_1(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_0(t)]\right) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[1]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}}\right) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_1(t)]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}}\right)$$

$$u_2(t) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\begin{aligned} u_3(t) &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L[u_2(t)]\right) \\ &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}L\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right) \\ &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^\alpha}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}}\right)\right) \\ &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}} - \frac{1}{s^{3\alpha+1}}\right) \end{aligned}$$

$$u_3(t) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Then $u_n(t)$ rapidly tends to the exact solution for $n \to \infty$:

$$u(t) = \mathrm{E}_\alpha(-t)^\alpha$$

Since E_α is the Mittag-Leffler function defind as :

$$\mathrm{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$$

Solution of Equation (4.15) by Using Laplace Transform:

$$L[D^\alpha f(t)] = \frac{s^m F(s) - s^{m-1}F(0) - \cdots - F^{(m-1)}(0)}{s^{m-\alpha}}$$

$$m-1 < \alpha \leq m.$$

Then equation (4.15) will be:

$$\frac{s^2 u(s) - su(0)}{s^{2-\alpha}} + u(s) = 0$$

$$\frac{su(s) - 1}{s^{1-\alpha}} + u(s) = 0$$

$$s^{\alpha}u(s) - s^{\alpha-1} + u(s) = 0$$

$$u(s)[s^{\alpha} + 1] = s^{\alpha-1}$$

$$u(s) = \frac{s^{\alpha-1}}{s^{\alpha} + 1} = \frac{1}{s}\left(\frac{1}{1 + \frac{1}{s^{\alpha}}}\right)$$

$$= \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{-1}{s^{\alpha}}\right)^{n} = \sum_{n=0}^{\infty}\frac{(-1)^{n}}{s^{n\alpha+1}}$$

By taking the inverse Laplace transform for both two sides give:

$$u(t) = \sum_{n=0}^{\infty}\frac{(-1)^{n}t^{n\alpha}}{\Gamma(n\alpha + 1)} = \sum_{n=0}^{\infty}\frac{(-t)^{n\alpha}}{\Gamma(n\alpha + 1)} = \mathrm{E}_{\alpha}(-t)^{\alpha}$$

Conclusion:

Variational iteration method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this thesis, we have presented the modified variation iteration method which included Lagrange multiplier that easily identify by Laplace transform to give an analytical solutions of fractional differential equations, All examples showed that the results of the modified variational iteration method are in excellent agreement with those obtained by the Laplace transform method, but the results showed that the modified variational iteration method is more effective than the results of Laplace transform method because the inverse of Laplace transform some-times may be difficult to compute.

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