Sudan University of Science and Technology

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Boundary Quotients of C^*-algebras of LCM Semigroups with Irreversible Algebraic Dynamical Systems

القواسم الحدوديه لجبر-C^* لشبه زمر LCM مع الأنظمه الحركيه الجبريه اللانعكاسيه

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Dedication:

To my:

FatherMother

To my:

Sisters and Brothers

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Abstract

The known examples of a nonuniquely ergodic minimal diffeomorphism of an odd dimensional sphere are given. For every such minimal dynamical system (S^n,β) there is a Cantor minimal system (X,α) such that the corresponding product system $(X \times S^n, \alpha \times \beta)$ is minimal and the resulting crossed product C^*-algebra$C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$is tracially approximately an interval algebra. This entails classification for such C^*-algebras.We study the saturation properties of several classes of C^*-algebras. Saturation has been shown by Farah and Hart to unify the proofs of several properties of coronas of σ-unital C^*- algebras; we extend their results by showing that some coronas of non-σ-unital C^*-algebras are countably degree-1 saturated. We study C^*-algebras associated to right LCM semigroups, that is, semigroups which are left cancellative and for which any two principal right ideals are either disjoint or intersect in another principal right ideal. We develop a general notion of independence for commuting group endomorphisms. Based on this concept, we initiate the study of irreversible algebraic dynamical systems, which can be thought of as irreversible analogues of the dynamical systems considered by Schmidt.

الخلاصة

تم اعطاء الامثلة المعروفة للدفومورفيزم الاصغرى الارقودلو غير الوحيد لكرة البعد الفردى. لأجل أى مثل هذا النظام الحركى الاصغرى (S^n, β) يوجد نظام اصغرى كانتور(X, α) حيث أن نظام الضرب التقابلى $(X \times S^n, \alpha \times \beta)$هو اضغرى و نتاج الضرب الاتجاهى لجبر -C^* $C(X \times S^n) \rtimes_{\alpha \times \beta} \mathbb{Z}$هو تقريبى اثري جبر فترة هذا يستتبع تصنيف لمثل جبر-C^*. درسنا خواص التشبع لعائلات متعددة لجبر-C^*. اوضح التشبع بواسطة فاراه وهارت لتوحيد الاثباتات للخواص المتعددة لكورونا لجبر-C^* الوحدوى-σ و مددنا نتائجها بواسطة توضيح ان بعض كورونا لجبر-C^* غير الوحدوى هو تشبع درجة – 1 قابلة للعد. درسنا جبر-C^* المشارك الى شبه زمرLCM الايمن ، اى، شبه الزمر التى هى يسرى الابغاء و لأجل انه لاى مثالين يمينين رئسيين هما اما منفصلين او متقاطعين فى مثالى ايمن رئيسى آخر. طورنا فكرة عامة. لأجل استقلال اندوفورمترمات الزمر التبديلية، بناءاً على هذا المفهوم بدأنا دراسة للانظمة الحركية الجبرية اللانعكاسية والتى يمكن ان نفكر فيها كنظائر لانعكاسية للانظمة الحركية المغيرة بواسطة شميدت.

The contents

CHAPTER 1

C^*-Algebeas of Minimal Dynamical Systems of a Cantor Set and Odd Dimensional Sphere.

The minimal Cantor system (X, α) is such that each tracial state on $\mathcal{C}(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$ induces the same state on the K_0-group and such that the embedding of $\mathcal{C}(S^n) \rtimes_{\beta} \mathbb{Z}$into $\mathcal{C}(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$preserves the tracial state space. This implies $\mathcal{C}(S^n) \rtimes_{\beta} \mathbb{Z}$ is TAI after tensoring with the universal UHF algebra, which in turnshows that the C^*-algebras of these examples of minimal diffeomorphisms of odd dimensionalspheres are classified by their tracial state spaces.

Section (1.1): Breaking the Orbit at Afibere and Existence of Minimal Product Homeomorphisms

The classification programme for C^*-algebrasaims to classify simple separable unital nuclear C^*-algebras by their Elliott invariants, an invariant which can be assigned to any unital C^*-algebras consisting of K-theory, the tracial state space, and a map which pairs the tracial state space with theK_0-group. The programme was initiated by George Elliott with his classification of the approximately finite-dimensional (AF)C^*-algebras. Helater conjectured that all simple unital nuclear C^*-algebras should be classified by these invariants, that is, that an isomorphism of Elliott invariants should be liftable to a^*-isomorphism of C^*-algebras.

It is now known that the conjecture does not hold in full generality: various counterexamples have been constructed of nonisomorphic simple separable unital nuclear C^*-algebras which nevertheless are indistinguishable at the level of their Elliott invariants. A short explanation of the failure of the conjecture can be attributed to the construction of the Jiang–Su algebra$Z$ by Xinhui Jiangand Hongbing Su .The Jiang-Su algebra is a simple separable unital nuclear C^*-algebras with no nontrivial projections that is infinite-dimensional yet has the same Elliott invariant as $\mathbb{C}$ The conjecture thenpredictsthat for any C^*-algebra A which falls within the scope of the programme we should have that A is Z-stable, that is,$A \cong A \otimes Z$ .Counterexamples thus far produced all show failure of Z-stability.

The study of C^*-algebras has long been influenced by the subject of dynamical systems. The case where (X, α) is a minimal dynamical system is of particular interest to the classification programme for C^*-algebras, as the crossed product $\mathrm{C(X)} \rtimes_{\alpha} \mathbb{Z}$ is an elegant example of a simple nuclear C^*-algebra. In the case of a minimal homeomorphism of a Cantor set, one has the striking classification result of Ian Putnam, Thierry Giordano and Christian Skau which shows anisomorphism of$K$-theory of the associated C^*-algebras implies the existence of a $*$-isomorphism of the C^*-algebras themselves. Moreover, in this setting, isomorphisms of K-theoryhence also $*$-isomorphisms of C^*-algebras imply topological strong orbit equivalence of the underlying dynamical systems.

For spaces of higher dimension, classification becomes more complicated. While many of the techniques used to tackle the examples coming from Cantor systems can be generalized to an arbitrary infinite compact metrizable space X, the fact that X is no longer completely disconnected means that there are not necessarily many projections in the C^*-algebra. Nevertheless, substantial progress has been made in this more general setting. A particularly wide-reaching result follows from the work of Andrew Toms and Wilhelm Winter (which uses a special case of a more general theorem found), where they give classification for the class of C^*-algebrasof minimal dynamical systems (X, α) of infinite finitedimensional metric spaces under the additional assumption that projections in the C^*-algebras separate their tracial states.

The correspondence between tracial states on $C(X) \rtimes_\alpha \mathbb{Z}$and α-invariant Borel probability measures on X means that the class covered by Toms and Winter's classification includes those C^*-algebras associated to uniquely ergodic minimal dynamical systems. However, notable examples of nonuniquely ergodic minimal dynamical systems have been constructed, and moreover, the resulting C^*-algebras need not have projections separating tracial states. Perhaps the most famous such examples come from minimal diffeomorphisms of n-spheres, with $n \geq 3$ odd. Albert Fathi and Michael Herman give an argument that proves the existence of uniquely ergodic diffeomorphisms of S^n. Generalizing this result (based on the so-called "fast approximationconjugation"technique of Dmitri Anosov and Anatole Katok), Alistair Windsor shows that one can construct a minimal diffeomorphism on S^{n} with any prescribed number (finite, countable or continuum) of ergodic measures . In all these cases, Alain Connes shows that the resulting C^*-algebras have no nontrivial projections. Thus as soon as one has more than one ergodic measure the associated C^*-algebras lie beyond current classification results. Previous classification techniques for the C^*-algebras of minimal dynamical systems have used a large C^*-subalgebra, originally introduced by Ian Putnam, which, in a sense, breaks the orbit of the homeomorphism at a single point in the underlying space. Let (X, α) be a minimal dynamical system and u the canonical unitary implementing α in the C^*-algebras $C(X) \rtimes_\alpha \mathbb{Z}$. For any point $x \in X$ one defines a C^*-subalgebra$C^*(C(X), u\, C_0(X \setminus \{x\})) \subset C(X) \rtimes_\alpha \mathbb{Z}$(that is, the C^*-subalgebra generated by $C(X)$ and$u\, C_0(X \setminus \{x\}))$This subalgebra retains a lot of information about $C(X) \rtimes_\alpha \mathbb{Z}$while having a significantly more tractable structure.

In the case of the minimal Cantor systems, these subalgebras turn out to be AF algebras. More generallN. Christopher Phillips and Qing Lin show that for arbitrary minimal dynamical systems of infinite compact metrizable spaces these subalgebras can be written as inductive limits of so-called recursive subhomogeneous (RSH) subalgebras. In the case where projections separate tracial states, then after tensoringC^*-$C(X), uC_0(X \setminus \{x\}))$with the universal UHF algebra$Q$, they are moreover tracially approximately finite-dimensional (TAF), as defined by Huaxin Lin . This is then shown to pass to $C(X) \rtimes_\alpha \mathbb{Z}$and entails classificatio.Unfortunately, this technique is so far insufficient in the case of the odd dimensional spheres. For these examples, the invariant suggests that they should be tracially approximately interval algebras (TAI) after tensoring with Q. Using the results of [1] and Winter, it would be enough to show that the largeC^*-subalgebra is TAI after tensoring with Q, as this would then pass to the whole C^*-algebra. However, at present, showing that an approximately RSH algebra is TAI after tensoring with Q requires certain restrictions which are unlikely to be satisfied in this case.Let (S^n, β) be a minimal dynamical system as constructed by Windsor.In this section, instead of tensoring our C^*algebra with a UHF algebra, we consider the related dynamical system given by the product action with a minimal Cantor set.For a minimal system (S^n, β), we show that we can find a minimal Cantor system (X, α)such that $(X \times S^n, \alpha \times \beta)$ is also minimal. Instead of considering the large subalgebra given by breaking the orbit of $\alpha \times \beta$ at a point, we break the action at a fibre $\{x\} \times S^n$. This subalgebra is TAI and we show this implies the C^*-algebra $C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$ is also TAI. Since the subalgebra breaks the orbit at a fibre, most of the difficult work, such a Berg's technique, takes place within the setting of the Cantor minimal system (X, α). In this way, after taking the product with the Cantor system, the crossed product is easier to handle. This entails classification of these product systems. The structure of the arguments given are mostly based on the results of Huaxin Lin and Hiroki Matui's work on crossed products of minimal dynamical systems on $(X \times T)$ as well as the work of H. Lin and Phillips on orbit-breaking subalgebras

Furthermore, we are able to ensure that the embedding $\mathcal{C}(S^n) \rtimes_\beta \mathbb{Z} \hookrightarrow \mathcal{C}(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$preserves the tracial state space and that each tracial state $T\big(\mathcal{C}(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}\big)$induces the same state on $K_0\big(\mathcal{C}(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}\big)$. It then follows fromthat $\big(\mathcal{C}(S^n) \rtimes_\beta \mathbb{Z}\big)\otimes\mathcal{Q}$is in fact TAI. This implies classification, by tracial state spaces, for the C^*-algebras of minimal diffeomorphisms of odd dimensional spheres constructed. That this should be true was conjectured by Phillips. Finally, we provide some discussion about how these results might be extended from minimal dynamical systems (S^n, β) to more general systems (Y, β).

A- Let (X, α) be a Cantor dynamical system and $\mathcal{C}(\mathrm{X}) \rtimes_\alpha \mathbb{Z}$ the associated C^*-algebra with canonical unitary u. A Kakutani–Rokhlin partition is a family of nonempty clopen subsets of X, index by a finite subset V , denoted

$$P = \{X(v,k) \mid v \in V, 1 \leq k \leq h(v)\},$$

satisfying the following:

(i) $\mathcal{P}$ partitions X,

(ii) for every$v \in V$and$k = 1,2,\ldots,h(v) - 1$
we have$\alpha(X(v,k)) = X(v,k+1)$. The clopen subset
$R(\mathcal{P}) := \cup_{v\in V} X(v,h(v))$ is called the roof set of P.

If $\mathcal{Y} \in \mathrm{X}$ then, for the construction), we may find a sequence $(\mathcal{P}_k)_{k\in\mathbb{N}}$ of Kakutani– Rokhlin partitions, $\mathcal{P}_k = \{X(k,v,j) \mid v \in V_k, j = 1,\ldots,h_k(v)\}$where the union generates the topology of X and such that the roof sets $R(\mathcal{P}_k)$ are nested decreasing clopen sets shrinking to the singleton $\{\mathcal{Y}\}$. Such a partition is used to show that C^*- $\big(\mathcal{C}(X), u\mathcal{C}_0(X\,\{\mathcal{Y}\})\big) \cong \lim_{\rightarrow} A_k$

Where the A_k are AFalgebras given by

$$A_k = \bigoplus_{v\in V_k} M_{h_k(v)}(\mathcal{C}(X_{k,v,h_k(v)})).$$

B-In what follows, n will always be an odd number with $n \geq 3$. Let $\alpha : X \to X$ and $\beta : S^n \to S^n$ be homeomorphisms and consider the product homeomorphism $\alpha \times \beta : X \times S^n \to X \times S^n$ given by

$$\alpha \times \beta(x, \mathcal{Y}) = (\alpha(x), \beta(\mathcal{Y})).$$

We will denote

$$A = \mathcal{C}(X \times S^n \rtimes_{\alpha\times\beta} \mathbb{Z}.$$

Let u denote the canonical unitary implementing α × β, that is, theunitary u satisfying $ufu^* = f \circ (\alpha \times \beta)^{-1}$ for every

$f \in \mathcal{C}(X \times S^n)$. Note that we have canonical embeddings

$$\mathcal{C}(X) \rtimes_\alpha \mathbb{Z} \hookrightarrow A$$

And

$$\mathcal{C}(S^n) \rtimes_\beta \mathbb{Z} \hookrightarrow A$$

C- For a clopen subset$Y \subset X$, we denote

$$A_Y = \mathcal{C}^*(C(X \times S^n), u\mathcal{C}_0((X \setminus Y) \times S^n)).$$

Proposition(1.1.1)[1]: Let $\mathcal{Y} \in X$. Then $A_{\{\mathcal{Y}\}}$is an inductive limit of AH algebras with no dimension growth.

Proof:

The after replacing $\mathbb{T}$ with S^n.For every $k \in \mathbb{N}$, let

$$P_k = \{X(k, v, j) \mid v \in V_j, j = 1, \dots, h_k(v)\},$$

be Kakutani-Rokhlin partitions corresponding to the minimal Cantor system (X, α) such that the roof sets

$$R(P_k) = \cup_{v\in V} X\big(k, v, h_k(v)\big)$$

shrink to the singleton $\{\mathcal{Y}\}$ and the union of P_k generate the topology on X. Let

$$A_k = \mathcal{C}^*(C(X \times S^n), u\mathcal{C}_0((X \setminus R(P_k)) \times S^n)).$$

Since the roof sets shrink to $\{\mathcal{Y}\}$, we have

$$A_{\{\mathcal{Y}\}} = lim_{\rightarrow} A_k.$$

For$i, j \in \{1, \dots, h_k(v)\}$define $e_{i,j}^{(k,v)} = u^{i-j} 1_{X_{(k,v,j)\times S^n}}$. It is straightforward to check that by the choice of the partition, each $e_{i,j}^{(k,v)} \in A_k$. One checks that the $e_{i,j}^{(k,v)}$ satisfy the relations of matrix units, theC^*-subalgebra $\mathcal{C}^*(e_{i,j}^{(k,v)}, \{(f, 1_{S^n}) \,|f|_{X(k,v,j)} \in \mathbb{C}\}) \cong M_{h_k(v)}$

is finite-dimensional. Furthermore,

$$\bigoplus_{v\in V_k} \bigoplus_{j=1}^{h_{k(v)}} e_{i,i}^{(k,v)} A_k e_{i,i}^{(k,v)} \cong \bigoplus_{v\in V_k} \bigoplus_{j=1}^{h_{k(v)}} C(X(k,v,j) \times S^n),$$

hence A_k is a direct sum of homogeneous C^*-algebras with topological dimensional most n whence $A_{\{y\}} = \lim_{\to} A_k$ is an AH algebra with no dimension growth.

Proposition(1.1.2)[1]: Suppose that $\alpha : X \to X$ and $\beta : S^n \to S^n$are minimal homeomorphisms. Then the K-theory of A is given by

$$K_i(A) \cong \mathbb{Z} \oplus K_0(C(X) \rtimes_\alpha \mathbb{Z})$$

for $i = 0$ and $i = 1$, where

$$K_0(C(X) \rtimes_\alpha \mathbb{Z}) \cong C(X,\mathbb{Z})/\{f - f \circ \alpha^{-1} | f \in C(X,\mathbb{Z})\}.$$

Proof:

By the Künneth Theorem for tensor products, we have

$$K_0(C(X \times S^n)) \cong C(X,\mathbb{Z}), K_1(C(X \times S^n)) \cong C(X,\mathbb{Z}).$$

Since n is odd and $\beta : S^n \to S^n$has no fixed points, its degree is

$\deg(\beta) = (-1)^{n+1} = 1$ thus β is homotopic to the identity map by the Hopf Theorem. It follows that the homeomorphism $\alpha \times \beta$ induces the map

$\alpha_i$on the $K_i(C(X \times S^n))$ given by $\alpha_i(f) = f \circ \alpha \; for \; f \in C(X,\mathbb{Z})$. The Pimsner–Voiculescu six term exact sequence is

$$\begin{array}{ccccc} C(X,\mathbb{Z}) & \xrightarrow{id-\alpha_0} & C(X,\mathbb{Z}) & \xrightarrow{l_0} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \underset{l_1}{\leftarrow} & C(X,\mathbb{Z}) & \underset{id-\alpha_1}{\longleftarrow} & C(X,\mathbb{Z}) \end{array}$$

Let $f \in ker(id - \alpha_i)$, where $i = 0$ or 1. Then $f(x) = f \circ \alpha(x)$ *for all* $x \in X$. Since α is minimal, this implies that f is constant, thus $\ker(id - \alpha_i) \cong \mathbb{Z}$. We also have $C(X,\mathbb{Z})/\{f - f \circ \alpha^{-1} \mid f \in C(X,\mathbb{Z})\} \cong \mathrm{coker}(id - \alpha_i)$. Since $\mathbb{Z} = \ker(id - \alpha_{i+1})$ is free abelian ,$K_i(A) = \mathrm{coker}(id - \alpha_i) \oplus \ker(id - \alpha_{i+1})$, that is,

$$K_i(A) \cong \mathbb{Z} \oplus K_0(C(X) \rtimes_\alpha \mathbb{Z})$$

for $i = 0$ and $i = 1$.

We now restrict to the case that $\alpha \times \beta$ is a minimal homeomorphism. Starting with a minimal homeomorphism $\beta : S^n \to S^n$ it is not necessarily the case that the product $\alpha \times \beta$ will be minimal, even if α is itself a minimal homeomorphism. The next result establishes that such an α indeed exists and moreover can be chosen to be uniquely ergodic. The fact that an odometer system would suffice was suggested by Taylor Hines, who, at the same time as the writing of following proposition, also provided the author with a very similar proof. This result is probably known, but we couldn't find a reference so include the proof here.

A. Recall that an odometer is a minimal Cantor system defined as follows: Let $(m_i)_{i\in\mathbb{N}}$ be a sequence such that $2 \geq m_i$ divides m_{i+1}Then there are homomorphisms $\mathbb{Z}/m_{i+1} \to \mathbb{Z}/m_i$ given by congruence modulo mi and the resulting inverse limit $\lim_{\leftarrow} \mathbb{Z} / m_j$ is a Cantor set. The odometer action is given by $\alpha(x) = x + 1$; it is minimal and equicontinuous. A minimal dynamical system (Y,γ) is called totally minimal if $\gamma^k : Y \to Y$ is minimal for every $k \in \mathbb{N}$. If $\beta : S^n \to S^n$ is minimal, then it is totally minimal. This follows from the fact that for any minimal dynamical system (Y, γ) and any k, there is a partition of Y into mutually disjoint clopen minimal γ^k −invariant subsets . Since S^n is connected, this implies there is only one nonempty β^k-invariant subset, namely S^n.

Proposition(1.1.3)[1]: Let $\beta : S^n \to S^n$ be a minimal homeomorphism. Then there is a uniquely ergodic minimal homeomorphism $\alpha : X \to X$ such that the homeomorphism $\alpha \times \beta : X \times S^n \to X \times S^n$ is minimal.

Proof:

We show that β is disjoint from any odometer, that is, for any odometer system (X, α) the product system $(X \times S^n, \alpha \times \beta)$ is minimal. Let $(lim_{\leftarrow} \mathbb{Z} / m_j , \alpha)$, $m_j \in \mathbb{N}$, m_j divides m_{j+1}, $j \in \mathbb{N}$, be an odometer system. Set a metric d on $X \times S^n$ and let d_X and d_{S^n} be the restrictions to X and S^n, respectively. Suppose $(x_0, \mathcal{Y}_0) \in X \times S^n$. We will show that the orbit of $(x_0, \mathcal{Y}_0)$ is dense in $X \times S^n$. Let $\epsilon > 0$ and $(x, \mathcal{Y}) \in X \times Y$. Since α is equicontinuous, there is a $\delta > 0$ such that $d_X(x'_0 , x_0) < \delta$ implies $d_X(\alpha^m x'_0 , \alpha^m x_0) < \epsilon/2$ for every $m \in \mathbb{N}$. Take j sufficiently large to find elements $x'_0, x' \in \mathbb{Z}/m_j$ such that $d_X(x_0, x'_0) < \delta \; and \; d_X(x, x') < \epsilon/2$. Since $\mathbb{Z}/m_j$ is finite, there is a $k \in \mathbb{Z}$ such that $\alpha^k(x'_0) = x'$. Since β^{m_j} is minimal, there is $l \in \mathbb{N}$ such that

$$d_{S^n}(\beta^{lm_j}(\beta^k(\mathcal{Y}_0)), \mathcal{Y}) < \epsilon.$$

Then we have

$$\begin{aligned} d\left((\alpha \times \beta)^{lm_j+k}(x_0, \mathcal{Y}_0), (x, \mathcal{Y})\right) &< d\left(\left(\alpha^{lm_j+k}(x_0), \beta^{lm_j+k}(\mathcal{Y}_0)\right), (x, \mathcal{Y})\right) \\ &= max\{d_X(\alpha^{lm_j+k}(x'_0) , x) \; + \; \varrho, \varrho\} \\ &= \; max\{dX(d_X(\alpha^{lm_j}(x') , x') \; + \; \epsilon, \epsilon\} = \epsilon. \end{aligned}$$

Proposition (1.1.4) [1]: Let $\mathcal{Y} \in X$ and let $\alpha \times \beta : X \times S^n \to X \times S^n$ be minimal. Then $A_{\{\mathcal{Y}\}}$ is simple.

Proof:

For a straightforward direct proof, one may proceed after replacing each instance of X with $X \times S^n$ and $U \subset X$ with $U \times S^n$. Alternatively, the result follow by regarding A as groupoid C^*-algebra generated by the orbit equivalence relation, $A_{\{\mathcal{Y}\}}$ the groupoid C^*-algebra generated by a dense subequivalence relation and applying .

Proposition (1.1.5)[1]: Let $\alpha \times \beta : X \times S^n \to X \times S^n$ be a minimal homeomorphism where (X, α) is an odometer system. Then every tracial state $\tau \in T(A)$ comes from the product of the unique tracial state $\tau_0 \in T(C(X) \rtimes_\alpha \mathbb{Z})$ and a tracial state $\tau_1 \in T(C(S^n) \rtimes_\beta \mathbb{Z})$.

Proof:

It is obvious that $\tau_0 \otimes \tau_1(f \circ (\alpha \times \beta)^{-1}) = \tau_0 \otimes \tau_1(f)$ for every $f \in C(X \times S^n)$, so that $\tau_0 \otimes \tau_1 \in T(A)$, but we need to show that every tracial state must be of this form. Since tracial states correspond to $(\alpha \times \beta)$-

invariant probability measures, it suffices to show that every such measure on $X \times S^n$ comes from a product measure. Let μ be an $(\alpha \times \beta)-$invariant Borel probability measure on$X \times S^n$. Define μ_0 and μ_1 by $\mu_0(B) = \mu(B \times S^n)$ for every Borel set $B \subset X$ and $\mu_1(B) = \mu(X \times B)$for every Borel set $B \subset S^n$. Then μ_0 and μ_1 are Borel probability measures on X and S^n respectively. It is easy to see that$\mu_1$is α-invariant and that μ_1 is β-invariant. Consider the measure-preserving dynamical systems $(X, \mu_0), (S^n, \mu_1)$and $(X \times S^n, \mu)$. Since β is totally minimal and (X, α) is an odometer, it follows from that(X, μ_0) and(S^n, μ_1) are disjoint as measurable dynamical systems, that is, $\mu = \mu_0 \times \mu_1$.

Proposition(1.1.6)[1]: Let $\alpha \times \beta : X \times S^n \to X \times S^n$ be a minimal homeomorphism where (X, α) is an odometer system. Then $K_0(C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}) = \mathbb{Z} \oplus C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}\}$ has the strictordering from the second coordinate and every tracial state $\tau \in T(C(C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z})$ induces the same state on$K_0(C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z})$.

Proof.

Since $\dim(X) = 0$it follows that $H^1(X, \mathbb{Z}) = 0$. Since $H^1(S^n, \mathbb{Z}) = 0$ as well, from the Ku¨nneth Theorem we get $H^1(S^n, \mathbb{Z}) = 0$. In this case, to determine the range of a state τ^*induced by any tracial state $\tau \in T(A)$ it is enough to determine the range of τ^*on $K_0(C(X \times S^n)$. By the previous proposition, every $\tau \in T(A)$ is of the form$\tau_X \otimes \tau_{S^n}$ where τ_X is the unique $\alpha-$invariant tracial state on $C(X)$ andτ_{S^n} is a $\beta-$invariant tracial state on $C(S^n)$. By,the range of any $(\tau_{S^n})_*$is $\mathbb{Z}$ and any two τ_{S^n},τ'_{S^n} are homotopic via the map $t \mapsto [(1 - t)\tau_{S^n} + t\tau'_{S^n}]_*$. Thus the range of $\tau_X \otimes \tau_{S^n}$is $\tau_X(K_0(C(X \times \mathbb{Z}))) =\tau_X(K_0(C(X))$and since the order of $K_0(A)$ is determined by the states (for example, since A is Z -stable) it follows that the order on $K_0(A)$is determined by the second coordinate. Finally, for any two τ_{S^n},τ'_{S^n}, we clearly have $\tau'_{S^n} \otimes \tau_{S^n}$ homotopic to $\tau_X \otimes \tau'_{S^n}$by the above. It follows that $(\tau_X \otimes \tau_{S^n})_* = (\tau_X \otimes \tau'_{S^n})_*$.

Proposition (1.1.7) [1]: Let $\alpha \times \beta : X \times S^n \to X \times S^n$ be a minimal product homeomorphism. Let $\mathcal{Y} \in X$ and let u denote the canonical unitary in $A = C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$. Then every tracial state$\tau \in T(A_{\{\mathcal{Y}\}}$satisfies

$$\tau(f \circ (\alpha \times \beta)^{-1}) = \tau(f)$$

For every$f \in C(X \times S^n)$.In particular, the map $\tau \to \tau \circ \iota$ is a bijection between $T(A)$ and $T(A_{\{\mathcal{Y}\}})$.

Proof.

The proof is the same as the proof upon replacing T with S^n.

In this section, we will show that the C^*-algebras of the homeomorphisms of S^n constructed by Windsor can be embedded into a simple unital tracially approximately interval algebra in a trace-preserving way. Let us briefly recall the definition for tracially approximately S, where S is a given class of unital algebras.

Section (1.2): Tracia l States and Main Results

Definition(1.2.1) [1]: Let S denote a class of separable unital C^*-algebras. Let A be a simple unital C^*-algebra. Then A is tracially approximately S (or TAS) if the following holds. For every finite subset $F \subset A$, every $\epsilon > 0$, and every nonzero positive element $c \in A$, there exist a projection $p \in A$ and a unital C^*-subalgebra $B \subset pAp$ with $1_B = p$ and $B \in S$ such that:

(i) $\|pa - ap\| < \epsilon$ *for all* $a \in F$,
(ii) $dist(pap, B) < \epsilon$ *for all* $a \in F$,
(iii) $1_A - p$ is Murray–von Neumann equivalent to a projection in $\overline{cAc}$.
The C^*-algebra B is in the class I of interval algebras if it is of the form

$$B = \bigotimes_{n=1}^{N} C(X_n) \otimes M_{r_n}$$

for some $N \in \mathbb{N} \setminus \{0\}$, where $X_n = [0,1]$ or X_n is a single point, and $r_n \in N \setminus \{0\}, 0 \leq n \leq N$.

We now restrict to the case of homeomorphisms $\beta : S^n \to S^n$ that can be written as the limit of a sequence $(T_i)_{i \in N}$ of homeomorphisms such that $T_i : S^n \to S^n$ has period M_i, each M_i divides M_{i+1}, and

$$\sup_{\substack{t \in S^n \\ j=1,\dots,M_i}} \left|\beta^j(t) - T_i^j(t)\right| \to 0 \; as \; i \to \infty.$$

In particular, this holds for the nonuniquely ergodic homeomorphisms constructed by Windsor.

Lemma (1.2.2)[1]: There is an odometer system (X, α) such that the following holds: For any $\mathcal{Y} \in X$, any $\epsilon > 0$, any $N_0 \in R_+$, and any pair of finite sets $\mathcal{F}_X \subset C(X), \mathcal{F}_{S^n} \subset C(S^n)$ there are $M > N_0 \in \mathbb{N}$ and $\mathcal{Y} \in U \subset X$ a clopen subset and a partial isometry $W \in A_{\{\mathcal{Y}\}}$ such that

(i) $\alpha^{-M}(U), \ldots, \alpha^{-1}(U), U, \alpha(U), \ldots, \alpha^{M}(U)$ are pairwise disjoint

(ii) $w^*w = 1_{U \times S^n}$ and $ww* = 1_{\alpha^{M(U)} \times S^n}$,

(iii) $|wa - aw| < \epsilon$ for every $a \in \{f \otimes 1_{S^n} | f \in \mathcal{F}_X\} \cup \{1_X \otimes f \mid f \in \mathcal{F}_{S^n}\}$.

Proof.

By the assumption on β, we have a sequence $(M_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ with M_j dividing M_{j+1} and corresponding periodic homeomorphisms $T^j : S^n \to S^n$ with period $M_j, j \in \mathbb{N}$. Let (X, α) be the odometer system corresponding to the sequence $(M_j)_{j\in\mathbb{N}}$ (which exists since M_j divides M_{j+1}. In this case, we may write a particular element $x \in X = lim_{s \leftarrow} \mathbb{Z} / M_j$ as $x = (x_j)_{j\in\mathbb{N}} \subset \prod_{j\in\mathbb{N}} \mathbb{Z}/M_j$ such that $x_{j+1} = x_j \bmod M_j$. Let $\mathcal{Y} = (\mathcal{Y}_j)_{j\in\mathbb{N}} \subset X$.

Following the construction at the beginning, we can find an approximation of $A_i \subset A_{\{\mathcal{Y}\}}$ by choosing increasingly fine Kakutani-Rokhlin partitions in X, as. In this case, the roof sets $Z_i \subset X$ are given by the cylinder sets $Z_i = \{x \in X \mid x_j = \mathcal{Y}_j, 0 \le j \le i\}$ and the single first return time to Z_i *is* M_i. The matrix units from are then given by $e_{k,l} = u^{k-l} l_{\alpha^l (Z_i) \times S^n}, k, l = 1, \ldots, M_i$.

By assumption on β there is a strictly decreasing sequence of positive numbers $(\epsilon_i)_{i\in\mathbb{N}}$ with $\epsilon_i \to 0$, such that for every $t \in S^n$ we have $d_{S^n}(\beta^{M_i}(t), t) < \epsilon_i$ for every $t \in S^n$, or correspondingly, we have that $d_{S^n}(\beta^{M_i}(\beta^{-M_i}(t)), \beta^{M_i}(t)) < \epsilon_i$ for every $\beta^{M_i}(t) \in S^n$. Let $\delta > 0$ be sufficiently small so that $d_X(x, x') < \delta$ implies $|f(x) - f(x')| \epsilon/4$ for every $x, x' \in X$ and every $f \in \mathcal{F}_X$. By the assumptions on β as a limit of periodic homeomorphisms together with the choice of the odometer (X, α), by going sufficiently far out in the sequences of periods $\{M_i\}_{i\in\mathbb{N}}$, we can find $M > N_0$ such that

$$d_X(\alpha^M(\mathcal{Y}), \mathcal{Y}) < \frac{\delta}{4},$$

and so that $d_{S^n}(t, \beta^{-M})$ is sufficiently small to have

$$\|f \circ \beta^{-M} - f\| < \frac{\epsilon}{2},$$

for every $f \in \mathcal{F}_{S^n}$.

By continuity of α^M there is a clopen set $U \subset X$ containing y of the form $\{x \in X \mid x_j = \mathcal{Y}_j, 0 \leq j \leq k\}$ for some $k \in N$ such that $d_X(\alpha^M(x), x) < \delta/2$

for every $x \in U$. Shrinking U if necessary, we may furthermore assume that $U, \alpha(U), \ldots, \alpha^M(U)$ are pairwise disjoint and $d_X(x, x') < \delta/4$ for every $x, x' \in U$ and $x, x' \in \alpha^M(U)$.

Again by our assumptions on β, as remarked above, for sufficiently large i we may arrange $d_{S^n}(t, \beta^{M_i}(t))$ is sufficently small so that

$$\|f \circ \beta^{-M} \circ \beta^{M_i} - f \circ \beta^{-M}\| \quad =$$

$$sup_{t \in S^n} |f \circ \beta^{-M} \circ \beta^{M_i}(t) - f \circ \beta^{-M}(t)| < \epsilon/2$$

for every $f \in \mathcal{F}_{S^n}$. By increasing i if necessary, we may also arrange that the roof set of the i[th] Kakutani–Rokhlin partition $Z_i \subset U$.Analogously, define $w_1 \in A_{\{y\}}$ by

$$w_1 = e_{1,M_i} + \sum_{k=2}^{M_i} e_{k,k-1}$$

$$= 1_{\alpha(U) \times S^n}(1_{\alpha(Z_i) \times S^n} u^{1-M_i} + u 1_{(X \setminus Z_i) \times S^n}).$$

Note that $w_1^* w_1 = 1_{U \times S^n}$ and $w_1 w_1^* = 1_{\alpha(U) \times S^n}$.

Also, for $f \in \mathcal{F}_{S^n}$, we have

$$w_1(1_X \otimes f) w_1^* = 1_{\alpha(Z_i)} \otimes f \circ \beta^{-1} \circ \beta^{M_i} + 1_{\alpha(U \setminus Z_i)} \otimes f \circ \beta^{-1}.$$

For $2 \leq m \leq M$, define $w_m \in A_{\{y\}} by\, w_m = u 1_{\alpha^{m-1}(U) \times S^n}$.

Put $w = w_M w_{M-1} \cdots w_1$.

Then it is easy to check that $w^* w = 1_{U \times S^n}\, and ww^* = 1_{\alpha^M(U) \times S^n}$.

For $f \in C(S^n)$ we have

$$w(1_X \otimes f) w^* = 1_{\alpha^M(Z_i)} \otimes f \circ \beta^{-M} \circ \beta^{M_i} + 1_{\alpha^M(U \times Z_i)} \otimes f \circ \beta^{-M},$$

so for $f \in \mathcal{F}_{S^n}$,

$$\begin{aligned}\|w(1_X \otimes f) - (1_X \otimes f)w\| &= \left\|w(1_U \otimes f)w^* - 1_{\alpha^M(U)} \otimes f\right\| \\ &= \|f \circ \beta^{-M} - f\| + \epsilon/2 < \epsilon.\end{aligned}$$

Now let $f \in \mathcal{F}_X$. Then $|f(x) - f(x')| < \epsilon/4$ for every $x, x' \in Z_i$ by choice of U. Thus, since Z_i is clopen in X, we can approximate f by $\tilde{f} \in C(X)$ which is constant on Z_i, equal to f on $X \setminus Z_i$, and satisfies $\|f - \tilde{f}\| < \epsilon/4$. Furthermore, since $d(\alpha^M(x), x) < \delta/2$ for every $x \in U$ and both U and $\alpha^M(U)$ have width less than $\delta/4$, we may assume that $\left\|\tilde{f}|_U - \tilde{f}|_{\alpha^M(U)}\right\| < \epsilon/2$. Since $\tilde{f}$ is constant on Z_i we have $w_1 \tilde{f} w_1^* = \tilde{f} \circ \alpha^{-1}$ and therefore,

$$\begin{aligned}&\|wf \otimes 1_{S^n} - f \otimes 1_{S^n} w\| \\ &= \left\|w\tilde{f} \otimes 1_{S^n} - \tilde{f} \otimes 1_{S^n} w\right\| + \epsilon/2 \\ &= \left\|w\, \tilde{f}|_U \otimes 1_{S^n} w^* - \tilde{f}|_{\alpha^M(U)} \otimes 1_{S^n}\right\| + \epsilon/2 \\ &< \left\|\tilde{f} \circ \alpha^{-M} 1_{\alpha^M(U)} - \tilde{f}|_{\alpha^M(U)}\right\| + \epsilon/2 \\ &< \left\|\tilde{f}|_U - \tilde{f}|_{\alpha^M(U)}\right\| + \epsilon/2 < \epsilon.\end{aligned}$$

Lemma(1.2.3) [1]: Let $\beta : S^n \to S^n$ be as above. There is an odometer system (X, α) such that the following holds:

Let $\mathcal{Y} \in X$. Then, for any finite subset $\mathcal{F} \subset A = C(X \times S^n) \rtimes_{\alpha \times \beta} \mathbb{Z}$ and any $\epsilon > 0$, there is a projection p in $A_{\{y\}}$ such that

(i) $\|pa - ap\| < \epsilon$ *for all* $a \in \mathcal{F}$

(ii) $dist(pap, pA_{\{y\}}p) < \epsilon$ for all $a \in \mathcal{F}$,

(iii) $\tau(1_A - p) < \epsilon$ for all $\tau \in T(A)$.

Proof:-

Let $\epsilon > 0$. We may assume $\mathcal{F}$ is of the form $\mathcal{F} = \mathcal{G} \cup \{u\}$ where $\mathcal{G} := \mathcal{G}_X \cup \mathcal{G}_{S^n}$ and $\mathcal{G}_X$, $\mathcal{G}_{S^n}$ are finite subsets of $C(X)^1$ and $C(S^n)^1$ respectively. Let $M_0 \in \mathbb{N}$ such that $\pi/(2M_0) < \epsilon/4$. By Lemma (1.2.4) applied to $\epsilon/4$, $M_0 + 1$, and the finite subsets $\bigcup_{m=0}^{M_0} u^m \mathcal{G}_X u^{-m}$ and $\bigcup_{m=0}^{M_0} u^m \mathcal{G}_{S^n} u^{-m}$, we find $M \in \mathbb{N}$ with $M > M_0 + 1$ and a clopen set $U \subset X$ containing y such that

$$\alpha^{-M_0}(U), \alpha^{-M_0+1}(U), \ldots, U, \alpha(U), \ldots, \alpha^M(U)$$

are all disjoint, and a partial isometry $w \in A_{\{\mathcal{Y}\}}$ satisfying $w^*w = 1_{U\times S^n} := q_0$ and $ww^* = 1_{\alpha^M(U)\times S^n} := q_M$, and $\|wf - fw\| < \frac{\epsilon}{4}$ for all $f \in \bigcup_{k=0}^{M_0} u^k \mathcal{G} u^{-k}$.

Let $R > (M + M_0 + 1)/\min(1, \epsilon)$.

Shrinking U if necessary, we may may assume that

$$\alpha^{-M_0}(U), \alpha^{-M_0+1}(U), \ldots, U, \alpha(U), \ldots, \alpha^R(U)$$

are pairwise disjoint. To apply Berg's technique, we only need the sets $\alpha^M(U)\ for\ -M_0 \leq m \leq M$, however we require R to be larger in order to satisfy property (iii) of the lemma.

Define projections $q_m = 1_{\alpha^m(U)\times S^n} = u^m q_0 u^{-m}$. This gives us pairwise orthogonal projections in A$\{\mathcal{Y}\}$ for which conjugation by u is the shift. We now perform Berg's technique to splice along the pairs of indices $(-M_0, M - M_0), (-M_0 + 1, M - (M_0 - 1)), \ldots, (0, M)$ to obtain a loop of length M where conjugation by u is approximately the cyclic shift. For $t \in R$, define

$$v(t) = cos(\pi t/2)(q_0 + q_M) + sin(\pi t/2)(w - w^*),$$

which has matrix representation given by

$$v(t) = \begin{pmatrix} cos\left(\frac{\pi t}{2}\right) & -sin\left(\frac{\pi t}{2}\right) \\ sin\left(\frac{\pi t}{2}\right) & cos\left(\frac{\pi t}{2}\right) \end{pmatrix}.$$

For each $t \in R$, is a unitary in the corner $(q_0 + q_M)A_{\{\mathcal{Y}\}}(q0 + qM)$. For $0 \leq k \leq M_0$ define

$$w_k = u^{-k} v(k/M_0) u^k.$$

Each $w_k \in A_{\{\mathcal{Y}\}}$ for $0 \leq k \leq M_0$. The proof of this is the same. Estimating the matrix entries we get

$$\begin{aligned} &\|v((k+1)/M_0) - v(k/M_0)\| \\ &\qquad \leq 2|cos(\pi(k+1)/2M_0) - cos(\pi k/2M_0)| + 2|sin(\pi(k+1)/2M_0) - sin(\pi k/2M_0)| \\ &\qquad = 2\pi/2M_0|sin(\xi_1)| + 2\pi/2M_0|cos(\xi_2)|, for\ some\ \xi_1, \xi_2 \in (k, k+1) \leq 2\pi/M_0 < \epsilon/2. \end{aligned}$$

From this it follows that

$$\|uw_{k+1}u^* - w_k\| \leq \| v((k+1)/M_0) - v(k/M_0)\| < \epsilon/2.$$

For $0 \leq m \leq M - M_0$ define $e_m = q_m$ and for $M - M_0 \leq m \leq M$ define projections

$$e_m = w_{M-m}q_{-(M-m)}w^*_{M-m}.$$

It is straightforward to check that the two definitions agree for e_{M-M_0} and that $e_0 = e_M$. For $1 \leq m \leq M - M_0$ conjugating e_{m-1} by u gives

$$ue_{m-1}u^* = uq_{m-1}u^* = q_m = e_m$$

and for conjugation of e_M we have

$$ue_Mu^* = ue_0u^* = uq_0u^* = q_1 = e_1.$$

When $M - M_0 \leq m \leq M$ we have

$$\begin{aligned}
&\|ue_{m-1}u^* - e_m\| \\
&\quad = \left\|uw_{M-(m-1)}q_{-(M-(m-1))}w^*_{M-(m-1)}u^* - w_{M-m}q_{-(M-m)}w^*_{M-m}\right\| \\
&\quad = \left\|uw_{M-(m-1)}u^*q_{-(M-m)}uw^*_{M-(m-1)}u^* - w_{M-m}q_{-(M-m)}w^*_{M-m}\right\| \\
&\quad \leq \left\|uw^*_{M-(m-1)}u^* - w^*_{M-m}\right\| + \left\|uw_{M-(m-1)}u^* - w_{M-m}\right\| \\
&\quad = 2\left\|uw^*_{M-(m-1)}\, u^* - w^*_{M-m}\right\| < \epsilon.
\end{aligned}$$

Hence conjugation of the e_m by u is approximately a cyclic shift. By the definition of e_m, $1 \leq m \leq M$, it is clear that each $e_m \in A_{\{y\}}$. Set $e = \sum_{m-1}^{M} e_m$ and $p = 1 - e$.

Then p is a projection in $A_{\{y\}}$.

We now show that p satisfies $(\mathrm{i}) - (\mathrm{iii})$ with respect to the set $\mathcal{F}$. We have

$$\|p - upu^*\| = \left\|\sum_{m=M-M_0+1}^{M} ue_{m-1}\, u^* - e_m\right\|$$

The terms in the sum are pairwise orthogonal and have norm less than ϵ, hence$\|p-upu^*\| < \epsilon$, proving (i) for u. Also, we have $p \le 1-q_0 = 1-1_U \in A_{\{y\}}$ so that $pup \in A_{\{y\}}$, showing (ii) for$u$.

For $f \in \mathcal{G}$we have $\|wu^kfu^{-k}-u^kfu^{-k}w\| \le \epsilon/4$ and therefore

$$\begin{aligned}\|w_kf - fw_k\| &= \left\|u^kv\left(\frac{k}{M_0}\right)u^{-u}f - fu^kv\left(\frac{k}{M_0}\right)u^{-k}k\right\| \\ &= \|sin(\pi k/(2M_0))(w-w^*)u^{-k}fu^k \\ &- u^{-k}fu^k sin(\pi k/(2M_0))(w-w^*)\| \\ &= 2\|wu^{-k}fu^k - u^{-k}fu^kw\| < \frac{\epsilon}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\|pf-fp\| = &\left\|\sum_{m=M-M_0+1}^{M}(w_{M-m}q_{-(M-m)}w^*_{M-m}f\right. \\ &\left. - fw_{M-m}q_{-(M-m)}w^*_{M-m})\right\| \\ &< \max_{mM}\|w_{M-m}q_{-(M-m)}w^*_{M-m}f - w_{M-m}q_{-(M-m)}fw^*_{M-m)}\| \\ &+ \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Since $f \in C(X \times S^n)$ we have that $pfp \in p\,A_{\{y\}}p$. This shows (1) and(2) for $\mathcal{G}$. For (3), we have

$$\tau(1-p) = \sum_{m=1}^{M}\tau(e_m) = \sum_{m=1}^{M}\tau(q_m) = \tau(q_0) < M/R < \epsilon.$$

for every $\tau \in T(A)$.

Lemma(1.2.4)[1]: Let A be a simple unitalC^*-algebra with strict comparison. Suppose that for every finite subset $F \subset A$, every$\epsilon > 0$, and every nonzero positive $c \in A$, there exists a projection $p \in A$ and a simple unitalC^*-subalgebra $B \subset pAp$ which is TAI, satisfies$1_B = p$and

(i) $\|pa - ap\| < \epsilon$ for all $a \in \mathcal{F}$,
(ii) $\mathrm{dist}(pap, B) < \epsilon$ for all $a \in \mathcal{F}$,
(iii) $1_A - p$is Murray–on Neumann equivalent to a projection in $\overline{cAc}$.
Then A is TAI.

Proof:

A has property (SP) and by assumption, strict comparison. After noting this, the proof is essentially the same, replacing the C^*-subalgebra of tracial rank zero (respecivelyTAS) with the TAI C^*-subalgebra$B$, and replacing the finite-dimensional (respectively in the class S) C^*-subalgebra with a C^*-subalgebra from the class I.

Theorem (1.2.5)[1]: There is an odometer system (X, α) such that $C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$ is TAI.

Proof:

Set $A: = C(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}$. *Let* $\epsilon > 0$ and nonzero$c \in A_+^1$be given. Again we may assume $\mathcal{F}$is of the form

$$\mathcal{F} = G \cup \{u\}$$

where $\mathcal{G} := \mathcal{G}_X \cup \mathcal{G}_{S^n}$ and$\mathcal{G}_X$,$\mathcal{G}_{S^n}$ are finite subsets of $C(X)^1$and $C(S^n)^1$srespectively. Since A is Z -stable , A has strict comparison. Thus we need only verify the conditions. Let $\mathcal{Y} \in$ X and use Lemma (1.2.4) with respect to $\mathcal{F}$andϵ_0 =min$\{\epsilon, \min_{\tau\in T(A)} \tau(c)\}$ to find a projection $p \in A_{\{\mathcal{Y}\}}$. Let $B = pA_{\{\mathcal{Y}\}}p$. Since $pA_{\{\mathcal{Y}\}}p$ is a simple AH algebra with no dimension growth it is. We have satisfied (i) and (ii) by the choice of p. Since $\tau(1 - p) < \epsilon_0$ for every $\tau \in T(A)$, condition (iii) follows from strict comparison.

We are now able to use the above to apply entails classification of the minimal dynamical systems of odd dimensional spheres constructed by Windsor.

Corollary(1.2.6)[1]: Suppose$\beta_1, \beta_2 : S^n \to S^n$are minimal homeomorphisms. Then$A: = C(S^n) \rtimes_{\beta_1} \mathbb{Z} \cong C(S^n) \rtimes_{\beta_2} \mathbb{Z} = : B$ if and only if$T(A) \cong T(B)$.

In particular, this holds if β_1 and β_2 come from the constructions.

Proof:

By the above, there are minimal Cantor systems (X, α_i) such that $C(S^n) \rtimes_{\beta_i} \mathbb{Z}$ embeds in a trace-preserving way into $C(X \times S^n) \rtimes_{\alpha_i\times\beta_i} \mathbb{Z}, i \in \{1,2\}$. Moreover, $(\tau_i)^* = (\tau_i')^*$ for every tracial state $\tau_i \in T((X \times S^n) \rtimes_{\alpha_i\times\beta_i} \mathbb{Z}), i \in \{1,2\}$. $C(X \times S^n) \rtimes_{\alpha_i\times\beta_i} \mathbb{Z}$ is TAI, $C(S^n) \rtimes_{\beta_i} \mathbb{Z} \otimes Q$ is TAI, where Q denotes the universal UHF algebra. Since A and B satisfy the UCT, classification up to Z -stability by Elliott invariants follows from.

Since $\dim(S_n) < \infty$, both A and B are Z -stable . It follows from A and B have isomorphic K-theory, thus the Elliott invariant collapses to the tracial state space.

A close look at the method of proof suggests that one can generalize to other minimal systems of the form $(X \times Y, \alpha \times \beta)$ with X a Cantor set. The particular requirements of $\beta : S^n \to S^n$ are that it is totally minimal and that β is a limit of periodic homeomorphisms in the sense of.

Total minimality holds at least for any connected space Y. In fact, one does not need total minimality, but rather only requires that (Y, β) is disjoint from some odometer system (X, α); in this case $\alpha \times \beta$ will be minimal and $T(\mathcal{C}(X \times S^n) \rtimes_{\alpha\times\beta} \mathbb{Z}) \cong T(\mathcal{C}(S^n) \rtimes_{\beta} \mathbb{Z})$. This will hold in other situations, for example, if (Y, β) is weakly mixing.

The restriction that $\beta$is a limit of periodic homeomorphisms in the sense of will be the case if β, for example, is constructed by the "fast approximation-conjugation" technique as initiated, provided the periods of the periodic homeomorphisms increase quickly enough.

Otherwise, observe that the structure for β is only required in two places. First, it is used to find the unitary we required for Berg's technique. One should be able to get around this by showing approximate unitary equivalence of the maps $\phi_1, \phi_2 : C(S^n) \to q_0 A_{\{y\}} \otimes \mathcal{Q} q_0$given by $\phi_1(f) = v^* u^M q_0 f q_0 u^{-M} v$ and $\phi_2(f) = q_0 f q_0$, where v is the unitary which comes from Berg's technique applied only to the Cantor minimal system, that is, $v q_0 v^* = q_M$. This is the technique appled. Since $q_0 A_{\{y\}} \otimes \mathcal{Q} q_0$ is TAI, there are various results of *H*. Lin which may be applied . Secondly, we have the technical requirement that $\|f \circ \beta^M - f\|$can be made small for every f in the given finite subset, which at present is more difficult to remove. Nevertheless, it appears the result will hold in greater generality.

CHAPTER 2

Elementary Equivalence of C^*-Algebrasand Saturation

We relate saturation of the abelian C^*-algebra $C(X)$, where X is 0-dimensional, to topological properties of X, particularly the saturation of $CL(X)$. We also characterize elementary equivalence of the algebras $C(X)$ in terms of $CL(X)$ when X is 0-dimensional, and show that elementary equivalence of the generalized Calkin algebras of densities $\aleph_\alpha$ and $\aleph_\beta$implies elementary equivalence of the ordinals α and β.

Section(2.1):Countable Degree-1 Saturation and Coronas of Non-σ-unital Algebras:

In this sction we examine the extent to which several classes of operator algebras are saturated in the sense of model theory. In fact, few operator algebras are saturated in the full model-theoretic sense, but in this setting there are useful weakenings of saturation that are enjoyed by a variety of algebras. The main results of this sectoin show that certain classes of C^*-algebras do have some degree of saturation, and as a consequence, have a variety of properties previously considered in the operator algebra literature. For all the definitions involving continuous model theory for metric structures (or in particular of C^*-algebras), we refer. Different degrees of saturation and relevant concepts will be defined . Among the weakest possible kinds of saturation an operator algebra may have, which nevertheless has interesting consequences, is beingcountably degree-1 saturated. This property was introduced by Farah and Hart, where it was shown to imply a number of important consequences. It was also shown that countable degree-1 saturation is enjoyed by a number of familiar algebras, such as coronas of σ-unitalC^*-algebras and all non-trivial ultraproducts and ultrapowers of C^*-algebras. Further examples were found by Voiculescu. Countable degree-1 saturation can thus serve to unify proofs about these algebras. We extend the results of Farah and Hart by showing that a class of algebras which is broader than the class of σ-unital ones have countably degree-1 saturated coronas.For the definitions of σ-unital C^*-algebras and essential ideals.

Theorem (2.1.1)[1]: Let Mbe a unital C^*-algebra, and let $A \subseteq M$ be an essential ideal.Suppose that there is an increasing sequence of positive elements in A whose supremum is 1_M, and suppose that any increasing uniformly bounded sequence converges in M. Then M/A is countably degree-1 saturated.Theorem(2.1.1) is proved as Theorem (2.1.28) below.

One interesting class of examples of a non-σ-unital algebra to which our result applies is the following. Let N be a II_1 factor, H a separable Hilbert space and$\mathcal{K}$ be the unique two-sided closed ideal of the von Neumann tensor product $N \overline{\otimes} \mathcal{B}(H)$. Then $(N \overline{\otimes} \mathcal{B}(H))/\mathcal{K}$is countably degree-1 saturated. These results are the contents. We consider generalized Calkin algebras of uncountable weight, as well as $\mathcal{B}(H)$ where H has uncountable density. Considering their complete theories as metric structures, we obtain the following (Theorem (2.2.3) below).

Theorem(2.1.2)[2]: Let $\alpha \neq \beta$ be ordinals, $H_\alpha$the Hilbert space of density $\aleph_\alpha$. Let $\mathcal{B}_\alpha = \mathcal{B}(H_\alpha)$ and $\mathcal{B}_\alpha = \mathcal{B}_\alpha/\mathcal{K}$the Calkin algebra of density $\aleph_\alpha$. Then the projections of the algebras $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ as posets with respect to the Murray-von Neumann order are elementary equivalent if and only if $\alpha = \beta$ modω^ω, where ω^ω is computed by ordinal exponentiation, as they are the infinite projections of $\mathcal{B}_\alpha$ and $\mathcal{B}_\alpha$. Consequently, if $\alpha \not\equiv \beta$ then $\mathcal{B}_\alpha \not\equiv \mathcal{B}_\beta$ and $\mathcal{C}_\alpha \not\equiv \mathcal{C}_\beta$. Elementary equivalence of C^*-algebras$A$ and B can be understood, via the Keisler-Shelah theorem for metric structures, as saying that A and B have isomorphic ultrapowers (see Theorem (2.1.12) below for a more precise description). For our second group of results we consider (unital) abelianC^*-algebras, which are all of the form $C(X)$ for some compact Hausdorff space X. We focus in particular on the real rank zero case, which corresponds to$X$ being 0-dimensional. In Section (2.2) we first establish a correspondence between the Boolean algebra (The Boolean algebras we have seen so far have all been concrete, consisting of bit vectors or equivalently of subsets of some set. Such a Boolean algebra consists of a set and operations on that set which can be shown to satisfy the laws of Boolean algebra) [6] of the clopen set of X and the theory of $C(X)$(see Theorem (2.2.11)).

Theorem(2.1.3)[2]: Let $X$and Y be compact 0-dimensional Hausdorffspaces.Then$C(X)$and $C(Y)$are elementarily equivalent if and only if the Boolean algebras $CL(X)$and $CL(Y)$ are elementarily equivalent.

We obtain several corollaries of the above theorem. For example, we show that many familiar spaces have function spaces which are elementarily equivalent, and hence have isomorphic ultrapowers. Finally, we study saturation properties in the abelian setting. We find that if$C(X)$is countably degree-1 saturated then X is a subStonean space without the countable chain condition and which is not Rickart. In the 0-dimensional setting we describe the relation between the saturation of $C(X)$and the saturation of $CL(X)$.While some implications hold in general, a complete characterization occurs in the case where Xhas no isolated points. The following is a special case of Theorems (2.2.16) and (2.2.17) in Section (2.2).

Theorem (2.1.4)[2]: Let X be a compact 0-dimensional Hausdorff space without isolated points. Then the following are equivalent:

i-$C(X)$is countably degree-1 saturated,

ii-$C(\mathrm{X})$is countably saturated,

iii-$C\mathrm{L}(\mathrm{X})$is countably saturated.

We wish to give further illustrations of the importance of the saturation properties we will be considering, particularly the full model-theoretic notion of saturation (see Definition (2.1.9) below). For countable degree-1 saturation we refer to Theorem (2.1.15) for a list of consequences. The following fact follows directly from the fact that axiomatizable properties are preserved to ultrapowers, which are countably saturated.

Fact (2.1.5)[2]: Let P be a property that may or may not be satisfied by a C^*-algebra. Suppose that countable saturation implies the negation of P. Then P is not axiomatizable.

Other interesting consequences follow when the Continuum Hypothesis is also assumed. In this case, all ultrapowers of a separable algebra by a non-principal ultrafilter on $\mathbb{N}$are isomorphic. In fact, all that is needed is that the ultrapowers are countably saturated and elementarily equivalent:

Fact (2.1.6)[2]: Assume the Continuum Hypothesis. Let A and Bbe two elementary equivalent countably saturated C^*algebra of density $\aleph_1$. Then$A \cong B$.

Applying Parovicenko's, the above fact immediately yields that under the Continuum Hypothesis if X and Y are locally compact Polish 0-dimensional spaces then $C(\beta X \backslash X) \cong C(\beta Y \backslash Y)$. Saturation also has consequences for the structure of automorphism groups:

Fact (2.1.7)[2]: Assume the Continuum Hypothesis.Let$A$ be a countably saturated C^*-algebra of density $\aleph_1$.Then A has $2^{\aleph_1}$-many automorphisms.In particular, A has outer automorphisms.

It is known that for Fact (2.1.7) the assumption of countable saturation can be weakened in some particular cases, and the property of having many automorphisms under the Continuum Hypothesis is shared by many algebras that are not even quantifier free saturated (for example the Calkin algebra). In particular it is plausible that the assumption of countable saturation in Fact (2.1.7) can be replaced with a lower degree of saturation. In light of this, and since the consistency of the existence of nontrivial homeomorphisms of spaces of the form $\beta\mathbb{R}^n \backslash \mathbb{R}^n$ is still open (for$n \geq 2$), it makes sense to ask about the saturation of $C(\beta\mathbb{R}^n \backslash \mathbb{R}^n)$. In the opposite direction, the Proper Forcing Axiom has been used to show the consistency of all automorphisms of certain algebras being inner.For more on this topic.

In this Section we describe the notions we will be considering throughout the remainder of the section. We also take this opportunity to fix notation that will be used in subsequent sections. The main topics of this sectionare several weakenings of the model-theoretic notion of saturation. We begin by reviewing the definition and basic properties. Since finite-dimensional C^*-algebras have full model-theoretic saturation, and hence have all of the weakenings in which we are interested, we assume throughout the section that all C^*-algebras under discussion are infinite dimensional unless otherwise specified.

Notation(2.1.8)[2]: For a compact set $K \subseteq R$ and $\epsilon > 0$, we denote the thickening of $K$by$(K)_\epsilon = \{x \in \mathbb{R}: d(x, K) < \epsilon\}$. We will be considering C^*-algebras as structures for the continuous logic formalism (or, for the more specific case of operator algebras) .Nevertheless, for many of our results it is not necessary to be familiar with that logic. Informally, a formula is an expression obtained from a finite set of norms of $*$-polynomials with complex coefficients by applying continuous functions and taking suprema and infima over some of the variables. A formula is quantifier-free if it does not involve suprema or infima. A formula is a sentence if every variable appears in the scope of a supremum or infimum. Given a C^*-algebra A we will denote as $A_{\leq 1}$, A_1 and A^+ the closed unit ball of A, its boundary, and the cone of positive elements respectively.

Definition (2.1.9)[2]: Let A be a C^*-algebra, and let Φ be a collection of formulas in the language of C^*-algebras. We say that A is countablyΦ-saturated if for every sequence $(\phi_n)_{n\in\mathbb{N}}$of formulas from Φ with parameters from $A_{\leq 1}$, and sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets, the following are equivalent:

(i) There is a sequence $(b_k)_{k\in\mathbb{N}}$of elements of $A_{\leq 1}$such that $\phi_n^A(\bar{b}) \in K_n$ for all $n \in \mathbb{N}$,

(ii) For every $\epsilon > 0$ and every finite $\Delta \subset \mathbb{N}$ there is $(b_k)_{k\in\mathbb{N}} \subseteq A_{\leq 1}$, depending on ϵ and Δ, such that $\phi_n^A(\bar{b}) \in (K_n)_\epsilon$ for all $n \in \Delta$. The three most important special cases for us will be the following:

a- If Φ contains all 1-degree $*$-polynomials, we say that A is countably 1-degree saturated.

b- If Φcontains all quantifier free formulas, we say that A is quantifierfree saturated.

c- If Φ is the set of all formulas we say that the algebra A is countably saturated.

Clearly condition (i) in the definition always implies condition (ii), but the converse does not always hold. We recall the (standard) terminology for the various parts of the above definition. A set of conditions satisfying (ii) in the definition is called a type; we say that the conditions are approximately finitely satisfiable or consistent. When condition (i) holds, we say that the type is realized (or satisfied) by $(b_k)_{k\in\mathbb{N}}$.

An equivalent definition of quantifier-free saturation is obtained by allowing only $*$-polynomials of degree . By (model-theoretic) compactness the concepts defined by Definition (2.1.9) are unchanged if each compact set $K_n$is assumed to be a singleton.

In the setting of logic for C^*-algebras, the analogue of a finite discrete structure is a C^*-algebra with compact unit ball, that is, a finitedimensional algebra. The following fact is then the C^*-algebra analogue of a well-known result from discreet logic.

Fact(2:1:10)[2]: Every ultraproduct of C^*-algebras over a countably incomplete ultrafilter is countably saturated. In particular, every finite-dimensional C^*-algebra is countably saturated. The second part of the fact follows from the first because any ultrapower of a finite-dimensional C^*-algebra is isomorphic to the original algebra .

A condition very similar to the countable saturation of ultraproducts was considered by Kirchberg and Rørdam under the name "-test". Before returning to the analysis of the different degrees of saturation, we give definitions for two well-known concepts that we are going to use strongly, but that may not be familiar to a C^*-algebraist.

Definition (2.1.11)[2]:The theory of a C^*-algebra A is the set of all sentences in the language of C^*-algebras which have value 0 when evaluated in A. We say that C^*-algebras A and B are elementary equivalent, written $A \equiv B$, if their theories are equal. Elementary equivalence can be defined without reference to continuous logic by way of the following result, which is known as the KeislerShelah theorem for metric structures. The version we are using is stated, and was originally proved in an equivalent setting.

Theorem (2.1.12)[2]: Let $A$and B be C^*-algebras. Then $A \equiv B$if and only if there is an ultrafilter$\mathcal{U}$ (over a possibly uncountable set) such that the ultrapowers $A^{\mathcal{U}}$ and $\mathrm{B}^{\mathcal{U}}$are isomorphic.

Definition (2.1.13)[2]: Let A be a C^*-algebra. We say that the theory of Ahas quantifier elimination if for any formula $\phi(\bar{x})$ and any $\epsilon > 0$ there is quantifier-free formula $\psi(\bar{x})$ such that for every C^*-algebra B satisfying $A \equiv B$, and any $\bar{b} \subseteq B$ (of the appropriate length) we have that in B,

$$\left|\phi(\bar{b}) - \psi(\bar{b})\right| \leq \epsilon.$$

Countable degree-1 saturation is the weakest form of saturation that we will consider in this Section. Even this modest degree of saturation for a C^*-algebra has interesting consequences. In particular it implies several properties (see the detailed definition before) that were shown to hold in coronas of σ-unital algebras in

Definition (2.1.14)[2]: Let A be a C^*-algebra. Then A is said to be

(a)SAW^*if any two σ-unital subalgebras C, B are orthogonal (i.e., $bc = 0$ for all $b \in B$and$c \in C$) if and only if are separated (i.e., there is $x \in A$ such that $xbx = b$ for all$b \in B$and$xc = 0$for all$x \in C$).

(b)$\mathrm{AA-CRISP}$ if for any sequences of positive elements $(a_n), (b_n)$ such that for all n we have $a_n \leq a_{n+1} \leq \ldots \leq b_{n+1} \leq b_n$ and any separable$D \subseteq A$ such that for all $d \in D$ we have $lim_n \|[d, a_n]\| = 0$, there is $c \in A^+$such that $a_n \leq c \leq b_n$ for any n and for all $d \in D$ we have $[c, d] = 0$.

(c)σ-sub-Stonean if whenever $C \subseteq A$ is separable and $a, b \in A^+$ are such that $\mathrm{aCb} = \{0\}$ then there are contractions $f, g \in C_0 \cap A$ such that $fg = 0, fa = a$ and $gb = b$, $C' \cap A$ denoting the relative commutant of C inside A.

Theorem (2.1.15)[2]: Let A be a countably degree-1 saturated C^*-algebra. Then:

(i)A is SAW*,

(ii) A is AA − CRISP,

(iii) A satisfies the conclusion of Kasparov's technical theorem,

(iv) A is σ-sub-Stonean,

(v) every derivation of a separable subalgebra of A is of the form $\delta_b(x) = bx - xb$for some $b \in A$

(vi) A is not the tensor product of two infinite dimensional C^*-algebras (this is a consequence of being SAW^*).

It is useful to know that when a degree-1 type can be approximately finitely satisfied by elements of a certain kind then the type can be realized by elements of the same kind.

Lemma (2.1.16)[2]: Let A be a countably degree-1 saturated C^*-algebra. If a type can be finitely approximately satisfied by selfadjoint elements then it can be realized by self-adjoint elements, and similarly with "self-adjoint" replaced by "positive".

We will also make use of the converse of the preceding lemma, which says that to check countable degree-1 saturation it is sufficient to check that types which are approximately finitely satisfiable by positive elements are realized by positive elements.

Lemma (2.1.17)[2]: Suppose that A is a C^*-algebra that is not countably degree-1 saturated. Then there is a countable degree-1 type which is approximately finitely satisfiable by positive elements of Abut is not realized by any positive element of$A$.

Proof:

Let $(P_n(\bar{x}))_{n=\mathbb{N}}$ be degree-1 polynomials, and $(K_n)_{n\in\mathbb{N}}$compact sets, such that the type$\{\|P_n(\bar{x})\| \in K_n : n \in \mathbb{N}\}$is approximately finitely satisfiable but not satisfiable in A. For each variable x_k, we introduce new variables v_k,w_k,x_k, and Z_k. For each n, let $Q_n(\bar{v}, \bar{w}, \bar{\mathcal{Y}}, \bar{z})$ be the polynomial obtained by replacing each x_k in $P_n$by $\mathcal{V}_k + iw_k - \mathcal{Y}_k - i\mathcal{Z}_k$. Since every$x \in A$can be written as$x = v + iw - \mathcal{Y} - i\mathcal{Z}$ where $v, w, y, z \in A^+$, it follows that $\{\|Q_n(v, w, y, z)\| \in K_n : n \in \mathbb{N}\}$ is approximately finitely satisfiable (respectively, satisfiable) by positive elements in A if and only if $\{\|P_n(\bar{x})\| \in K_n : n \in \mathbb{N}\}$ is approximately finitely satisfiable (respectively, satisfiable).

The first example of an algebra which fails to be countably degree-1 saturated is$\mathcal{B}(H)$, where H is an infinite dimensional separable Hilbert space . In fact, no infinite dimensional separable C^*-algebra can be countably degree-1 saturated; this was observed. We include here a proof of the slightly stronger result, enlarging the class of algebras that are not countably degree-1 saturated.

Definition (2.1.18)[2]: A C^*-algebra A has few orthogonal positive elements if every family of pairwise orthogonal positive elements of A of norm 1 is countable.

Lemma (2.1.19)[2]: If an infinite dimensional C^*-algebra A has few orthogonal positive elements, then A is not countably degree-1 saturated.

Proof:-

Suppose to the contrary that A has few orthogonal positive elements and is countably degree-1 saturated. Using Zorn's lemma (The terms used in the statement of the lemmaare defined as follows. Suppose (P, ≤) is a partially ordered set. A subset T is totally ordered if for any s, t in T we have s ≤ t or t ≤ s. Such a set T has an upper bound u in P if t ≤ u for all t in T. Note that u is an element of P but need not be an element of T. An element m of P is called a maximal element (or non-dominated) if there is no element x in P for which m < x. Note that P is not explicitly required to be non-empty. However, the empty set is a chain (trivially), hence is required to have an upper bound, thus exhibiting at least one element of P. An equivalent formulation of the lemma is therefore: Suppose a non-empty partially ordered set P has the property that every non-empty chain has an upper bound in P. Then the set P contains at least one maximal element)[6], find a set $Z \subseteq A_1^+$which is maximal (under inclusion) with respect to the property that if $x, \mathcal{Y} \in Z$ and $x \neq \mathcal{Y}$ then $x\mathcal{Y} = 0$. By hypothesis ,the set Z is countable list it as $Z = \{a_n\}_n \in N$. For each $n \in \mathbb{N}$ define $P_n(x) = a_n x$, and let$K_n = \{0\}$.Let $P_{-1}(x) = x$, and $K_{-1} = \{1\}$. The type $\{\|P_n(x)\| \in K_n : n \geq -1\}$ is finitely satisfiable. Indeed, by definition of Z for any $m \in \mathbb{N}$ and any $0 \leq n \leq m$ we have $\|P_n(a_{m+1})\| = \|a_n a_{m+1}\| = 0$, and$\|a_{m+1}\| = 1$. By countable degree-1 saturation there is a positive element $b \in A_1^+$ such that $\|P_n(b)\| = 0$ for all $n \in N$. This contradictsthemaximality of Z Subalgebras of $B(H)$clearly have few positive orthogonal elements, whenever H is separable. As a result, we obtain the following.

Corollary (2.1.20)[2]: No infinite dimensional subalgebra of $B(H)$, with H separable, can be countably degree-1 saturated.

Corollary (2.1.21)[2]:shows that many familiar C^*-algebras fail to be countably degree-1 saturated. In particular, it implies that no infinite dimensional separable C^*-algebra is countable degree-1 saturated. Corollary (2.1.21) also shows that the class of countably degree-1 saturated algebras is not closed under taking inductive limits (consider, for example, the $CAR \otimes_{i=1}^{\infty} M_2(\mathbb{C})$, or any AF algebra) or subalgebras. On the other hand, several examples of countably degree-1 saturated algebras are known. It isthat every corona of a σ-unital algebra is countably degree-1 saturated. Recently Voiculescu found examples of algebras which are not C^*-algebras, but which have the unexpected property that their coronas are countably degree-1 saturated C^*-algebras. The results of the following section expand the list of examples of countably degree-1 saturated C^*-algebras. Our goal in this section is to give examples of coronas of non-σ-unital algebras which are countably degree-1 saturated. For convenience, we recall some definitions which we will need:

Definition (2.1.21)[2]: A C^*-algebra A is σ-unital if it has a countable approximate identity, that is, a sequence $(e_n)_{n\in\mathbb{N}}$ such that for all $x \in A$,

$$\lim_{n\to\infty} \|e_n x - x\| = \lim_{n\to\infty} \|x e_n - x\| = 0.$$

A closed ideal $I \subseteq A$is essential if it has trivial annihilator, that is, if$\{x \in A : Ix = \{0\}\} = \{0\}$.

Motivating example.

Notation (2.1.22)[2]: Let $\mathcal{R}$ be the hyperfinite II_1 factor. Let $M = \mathcal{R} \overline{\otimes} B(H)$ be the unique hyperfinite II_∞ factor associated to $\mathcal{R}$, and let τ be its unique trace. We denote by $\mathcal{K}_{\mathrm{M}}$ the unique norm closed two-sided ideal generated by the positive elements of finite trace in M. Note that M is the multiplier algebra of $\mathcal{K}_M$, so the quotient $M/\mathcal{K}_M$is the corona of M.

Any ideal in a von Neumann algebra is generated, as a linear space, by its projections, hence $\mathcal{K}_{\mathrm{M}}$ is the closure of the linear span in M of the set of projections of finite trace. In particular, $\mathcal{R} \otimes \mathcal{K}(H) \subset \mathcal{K}_M$. To see that the inclusion is proper, fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$for H, and choose$(p_n)_{n\in\mathbb{N}}$fromRsuch that $\tau(p_{n)} = 2^{-2}$for all $n \in \mathbb{N}$.

For each n, let$q_n \in B(H)$ be the projection onto e_n, and let

$q = \sum_n p_n \otimes q_n$. Then $q \in M$ is a projection of finite trace, but $q \notin \mathcal{R} \otimes \mathcal{K}(H)$.

We recall few well known properties of this object.

Proposition(2.1.23)[2]:

a. $\mathcal{R} \neq Mp(\mathcal{R})$ for every prime number p.

Consequently $M_n(\mathcal{R}) \cong M_m(\mathcal{R})$ for every $m, n \in \mathbb{N}$.

b . $K_0(\mathcal{K}_M) = \mathbb{R} = K_1(M/\mathcal{K}_M)$.

c. $\mathcal{K}_M$ is not σ-unital.

d. $\mathcal{K}_M \otimes \mathcal{K}(H)$ is not isomorphic to $\mathcal{R} \otimes \mathcal{K}(H)$.

Proof:-

a. This is because $M_p(\mathcal{R})$ is hyperfinite and $\mathcal{R}$ is the unique hyperfinite II1-factor.

b. Note that $\mathcal{K}_0(M) = 0 = \mathcal{K}_1(M)$and apply the exactness of the six term K-sequence.

c. Suppose to the contrary that $(x_n)_{n\in\mathbb{N}}$ is a countable approximate identity in$\mathcal{K}_\mathrm{M}$ formed by positive elements such that $0 \leq x_n \leq 1$ for all n. Using spectral theory, we can find projections $p_n \in \mathcal{K}_M$ such that $\|p_n x_n - x_n\| \leq \frac{1}{n}$ for each n. Then $(p_n)_{n\in\mathbb{N}}$ is again a countable approximate identity for $\mathcal{K}_\mathrm{M}$. For each $n \in \mathbb{N}$ define

$q_n = sup_{k\leq n} p_k \in \mathcal{K}_M$, and by passing to a subsequence we can suppose that (qn)n∈N is strictly increasing. For each n ∈ ℕ find a projection $r_n \leq q_{n+1} - q_n$ such that $\tau(r_n) \leq \frac{1}{2^n}$. Then $r = \sum_{n\in\mathbb{N}} r_n \in \mathcal{K}_M$, and we have that for all $n \in N$,

$$\|q_n r - r\| = 1.$$

This contradicts that $(q_n)_{n\in\mathbb{N}}$ is an approximate identity.

d. This follows from (3), since $\mathcal{R} \otimes \mathcal{K}(H)$ has a countable approximate identity and $\mathcal{K}_{\mathrm{M}} \otimes \mathcal{K}(\mathrm{H})$ does not. To see this, suppose that $(x_n)_{n\in\mathbb{N}}$ is a countable approximate identity for $\mathcal{K}_M \otimes \mathcal{K}(H)$, and let p be a rank one projection in $\mathcal{K}(\mathrm{H})$. Then $((1 \otimes p)x_n(1 \otimes p))_{n\in\mathbb{N}}$ is be a countable approximate identity for $\mathcal{K}_M \otimes p$, but $\mathcal{K}_M \otimes p \cong \mathcal{K}_M$, so this contradicts (3).

There are many differences between the Calkin algebra and $M/\mathcal{K}_M$. Some of them are already clear from the K-theory considerations above, or from the fact that $\mathcal{K}(H)$ is separable. Another difference, a little bit more subtle, is given by the following:

Proposition(2.1.24)[2]: Let H be a separable Hilbert space, and let Q be the canonical quotient map onto the Calkin algebra. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for H, and let $S \in (H)$ be the unilateral shift in $B(H)$ defined by $S(e_n) = e_{n+1}$ for all n. Then neither S nor $Q(\mathrm{S})$ has a square root, but $1 \otimes S \in \mathcal{R} \otimes (H)$ does have a square root.

Proof:-

Suppose that $Q(T) \in C(H)$ is such that $Q(\mathrm{T})^2 = Q(\mathrm{S})$. Since Q (S) is invertible in the Calkin algebra so is $Q(\mathrm{T})$. The Fredholm index of S is -1, so if $\mathrm{n} \in \mathbb{Z}$ is the Fredholm index of T then $2n = -1$, which is impossible. Therefore $Q(\mathrm{S})$ has no square root, and hence neither does. For the second assertion recall that $Q \cong \mathrm{M}_2(\mathrm{R})$, and so $R \overline{\otimes} B(H) \cong M_2(R \overline{\otimes} B(H)) = R \overline{\otimes} (M_2 \overline{\otimes} B(H))$.

Weview $\mathrm{B(H)}$ as embedded in $M_2 \otimes B(H) = B(H')$ for another Hilbert space H'; find $(f_n)_{n\in\mathbb{N}}$ such that $\{e_n, f_n : n \in \mathbb{N}\}$ is an orthonormal basis for H'. Let $\mathrm{S}' \in \mathrm{B}(\mathrm{H}')$ be defined such that $S'(e_n) = f_n$ and $S'(f_n) = e_{n+1}$ for all.Then $T = 1 \otimes S' \in R \overline{\otimes} B(H')$, and $T^2 = 1 \otimes S$.

A consequence of the previous proof, and of the fact that $\mathrm{R} \cong \mathrm{M_p}(\mathrm{R})$ for any integer p, is the following

Corollary (2.1.25) [2]: $1 \otimes \mathrm{S} \in \mathrm{M}$ has a q^{th}-root for every rational q. With the motivating example in mind, we turn to establishing countable degree-1 saturation of a class of algebras containing M/K_M.

A weakening of the σ-unital assumption. We recall the following result, which may be found

Lemma (2.1.26)[2]: Let A be a C^*-algebra, $S \in A_1$ and$T \in A^+_{\leq 1}$. Then

$$\|[S, T]\| = \epsilon \leq \frac{1}{4} \Rightarrow \left\|[S, T^{1/2}]\right\| \leq \frac{5}{4}\sqrt{\epsilon}$$

The following lemma is the key technical ingredient of Theorem (2:1:29) below. It is a strengthening of the construction used as if A is σ-unital and $M = M(A)$ is the multiplier algebra of A, then M and A satisfy the hypothesis of our lemma.

Lemma(2.1.27)[2]: Let M be a unital C^*-algebra, let $A \subseteq M$ be an essential ideal, and let $\pi: M \to M/A$ be the quotient map. Suppose that there is an increasing sequence $((g_n)_{n\in\mathbb{N}} \subset A$of positive elements whose supremum is 1_M,and suppose that any increasing uniformly bounded sequence converges in M.

Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of the unit ball of M and $(\epsilon_n)_{n\in\mathbb{N}}$be a decreasing sequence converging to 0, with $\epsilon_0 < 1/4$.

Then there is an increasing sequence$(e_n)_{n\in\mathbb{N}} \subset A^+_{\leq 1}$ such that, for all $n \in \mathbb{N}$ and $a \in F_n$, the following conditions hold, where $f_n = (e_{n+1} - e_n)^{1/2}$:

(i) $|\|(1 - e_{n-2})a(1 - e_{n-2})\| - \|\pi(a)\|| < \epsilon_n \; for \; all \; n \geq 2,$
(ii) $\|[f_n, a]\| < \epsilon_n \; for \; all \; n,$
(iii) $\|f_n(1 - e_{n-2}) - f_n\| < \epsilon_n for \; all \; n \geq 2,$
(iv) $\|f_n f_m\| < \epsilon_n \; for \; all \; m \geq n + 2,$
(v) $\|k[fn, fn + 1]k\| < n + 1 \; for \; all \; n,$
(vi) $\|f_n a f_n\| \geq \|\pi(a)\| - \epsilon_n \; for \; all \; n,$
(vii) $\sum_{n\in\mathbb{N}} f_n^2 = 1;$

and further, whenever $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence from M, the following conditions also hold:

(viii) the series$\sum_{n\in\mathbb{N}} f_n x_n f_n$ converges to an element of M,
(ix) $\|\sum_{n\in\mathbb{N}} f_n x_n f_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\|;$
(x) Whenever$\limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|x_n f_n^2\|$we*have* $lim_{n\to\infty} \, sup\|x_n f_n^2\| \leq \|\pi(\sum_{n\in\mathbb{N}} x_n f_n^2)\|$

Proof:

For each $n \in \mathbb{N}$et$\delta_n = 10^{-100}\epsilon_n^2$, and let$(g_n)_{n\in\mathbb{N}}$ be an increasing sequence in A whose weak limit is 1. We will build a sequence$(e_n)_{n\in\mathbb{N}}$ satisfying the following conditions:

(a)$\big|\|(1-e_{n-2})a(1-e_{n-2})\| - \|\pi(a)\|\big| < \epsilon_n$ for all $n \geq 2$ and $a \in F_n$,

(b) $0 \leq e_0 \leq \ldots \leq e_n \leq e_{n-1} \leq \ldots \leq 1$, and for all n we have $e_n \in A$,

(c)$\|e_n e_k - e_k\| < \delta_{n+1}$ for all $n > k$,

(d)$\|[e_n, a]\| < \delta_n$ for all $n \in \mathbb{N}$ *and* $a \in F_{n+1}$,

(e)$\|(e_{n+1} - en)a\| \geq \|\pi(a)\| - \delta_n$ *for all* $n \in \mathbb{N}$ *and* $a \in F_n$

(f)$\left\|(e_{m+1} - e_m)^{1/2} e_n (e_{m+1} - e_m)^{1/2} - (e_{m+1} - e_m)\right\| < \delta_{n+1}$

for all $n > m + 1$,

(g)$e_{n+1} \geq g_{n+1}$ for all $n \in \mathbb{N}$.

We claim that such a sequence will satisfy (i)–(vii). Conditions (i) and (a) are identical. Condition (d) implies condition (ii). Condition (c) and the C^*-identity imply condition (iii), which in turn implies conditions (iv) and (v). We have also that conditions (e) and (g) imply respectively conditions (vi) and (vii), so the claim is proved. After the construction we will show that (viii)–(x) also hold.

Take $\Lambda = \{\lambda \in A^+ : \lambda \leq 1\}$ to be the approximate identity of positive contractions (indexed by itself) and let Λ' be a subnet of Λ that is quasicentral for M.

Since A is an essential ideal of M, there is a faithful representation β on an Hilbert space H such that

$$1_H = SOT - \lim_{\lambda \in \Lambda'} \{\beta(\lambda)\},$$

Consequently, for every finite $F \subset M$, $\epsilon > 0$ and $\lambda \in \Lambda'$ there is $\mu > \lambda$ such that for all $a \in F$,

$$\nu \geq \mu \Rightarrow \|(\nu - \lambda)a\| \geq \|\pi(a)\| - \epsilon.$$

We will proceed by induction. Let $e_{-1} = 0$ and $\lambda_0 \in \Lambda'$ be such that for all $\mu > \lambda'$ and $a \in F_1$ we have $\|[\mu, a]\| < \delta_0$. By cofinality of Λ' in Λ we can find a $e_0 \in \Lambda'$ such that $e_0 > \lambda_0, g_0$. Find now $\lambda_1 > e_0$ such that for all $\mu > \lambda_1$ and $a \in F_2$ we have

$$\|[\mu, a]\| < \delta_1, \|(\mu - e_0)a\| \geq \|\pi(a)\| - \delta_1.$$

Since we have that

$$\|\pi(a)k\| = \lim_{\lambda\in\Lambda'}\|(1-\lambda)a(1-\lambda)\|$$

we can also ensure that for all $a \in F_3$ and all $\mu > \lambda_1$, condition (1) is satisfied.

Picking $e_1 \in \Lambda'$ such that $e_1 > \lambda_1, g_1$ we have that the base step is completed.

Suppose now that $e_0, \ldots, e_n, f_0, \ldots, f_{n-1}$ are constructed.

We can choose λ_{n+1}so that for all $\mu > \lambda_{n+1}$, with $\mu \in \Lambda'$, we have $\|[\mu, a]\| < \frac{\delta_{n+1}}{4}$ and$\|(\mu - e_n)a\| \geq \|\pi(a)\| - \delta_n$ for $a \in F_{n+2}$. Moreover, by the fact that Λ' is an approximate identity for A we can have that$\|f_m\mu f_m - f_m^2\| < \delta_{n+2}$ for every $m < n$ and that

$\|\mu e_k - e_k\| < \delta_{n+2}$for all$k \leq n$. By equation (i) we can also ensure that for all $a \in F_{n+2}$and all $\mu > \lambda_{n+1}$, condition (i) is satisfied.

Once this λ_{n+1} is picked we may choose

$$e_{n+1} \in \Lambda', e_{n+1} > \lambda_{n+1}, g_{n+1},$$

to end the induction. It is immediate from the construction that the sequence $(e_n)_{n\in\mathbb{N}}$ chosen in this way satisfies conditions (a) - (g). To complete the proof of the lemma we need to show that conditions (viii), (xi) and (x) are satisfied by the sequence$\{f_n\}$.

To prove (viii), we may assume without loss of generality that eachx_n is a contraction. Recall that every contraction in M is a linear combination (with complex coefficients of norm 1) of four positive elements of norm less than 1, and addition and multiplication by scalar are weak operator continuous functions. It is therefore sufficient to consider a sequence (x_n) of positive contractions. By positivity of x_n, we have that $(\sum_{i\leq n} f_i x_i f_i)_{n\in\mathbb{N}}$is an increasing uniformly bounded

$$\sum_{i\leq n} f_i x_i f_i \leq \sum_{i\leq n} f_i^2 \ and \ f_n x_n f_n \geq 0.$$

Hence $(\sum_{i\leq n} f_i x_i f_i)_{n\in\mathbb{N}}$ converges in weak operator topology to an element of M of bounded norm, namely the supremum of the sequence, which is$\sum_{n\in\mathbb{N}} f_n x_n f_n$.

For (xi), consider the algebra $\prod_{k\in\mathbb{N}} M$with the sup norm and the map$\phi_n\colon \prod_{k\in\mathbb{N}} M \to M$ such that $\phi_n((x_i)) = f_n x_n f_n$. Each ϕ_n is completely positive, and since $f_n^2 \leq \sum_{i\in\mathbb{N}} f_i^2 = 1$, also contractive. For the same reason the maps $\psi_n\colon \prod_{k\in\mathbb{N}} M \to M$ defined as $\psi_n((x_i)) = \sum_{j\leq n} f_j x_j f_j$ are completely positive and contractive. TakeΨ to be the supremum of the maps ψ_n. Then $\Psi((x_n)) = \sum_{i\in\mathbb{N}} f_i x_i f_i$. This map is a completely positive map of norm 1, because $\|\Psi\| = \|\Psi(1)\|$, and from this condition (9) follows.

For (x), we can suppose $\limsup_{i\to\infty}\|x_i\| = \limsup_{i\to\infty}\left\|x_i f_i^2\right\| = 1$.

Then for all $\epsilon > 0$ there is a sufficiently large $m \in \mathbb{N}$ and a unit vector $\xi_m \in H$ such that $\|x_m f_m^2(\xi_m)\| \geq 1-\epsilon$.

Since $\|x_i\| \leq 1$ for all i, we have that $\|f_m(\xi_m)\| \geq 1-\epsilon$, that is, $|(f_m^2\xi_m \mid \xi_m)| \geq 1-\epsilon$. In particular we have that $\|\xi_m - f_m^2(\xi_m)\| \leq \epsilon$.

Since$\sum f_i^2 = 1$ we have that ξ_m and ξ_n constructed in this way are almost orthogonal for all n, m. In particular, choosing small enough at every step, we are able to construct a sequence of unit vectors $\{\xi_m\}$ such that $|(\xi_m \mid \xi_n)| \leq 1/2^m \; for \; m > n$. But this means that for any finite projection $P \in M$only finitely many ξ_m are in the range of P up to for every $\epsilon > 0$. In particular, if I is the set of all convex combinations of finite projections, we have that that

$$\lim_{\lambda\in I}\left\|\sum_{i\in\mathbb{N}} x_i f_i^2 - \lambda\left(\sum_{i\in\mathbb{N}} x_i f_i^2\right)\right\| \geq 1.$$

Since I is an approximate identity for A we have that

$$\left\|\pi\left(\sum_{i\in\mathbb{N}} x_i f_i^2\right)\right\| = \lim_{\lambda\in I}\left\|\sum_{i\in\mathbb{N}} x_i f_i^2 - \lambda\left(\sum_{i\in\mathbb{N}} x_i f_i^2\right)\right\|,$$

as desired.

We can then proceed with the proof of the main theorem.

Theorem(2.1.28): Let M be a unital C^*-algebra, and let $A \subseteq M$ be an essential ideal. Suppose that there is an increasing sequence $(g_n)_{n\in\mathbb{N}} \subset A$ of positive elements whose supremum is 1_M, and suppose that any increasing uniformly bounded sequence converges in M. Then M/A is countably degree-1 saturated.

Proof:-

Let $\pi : M \to M/A$ be the quotient map. Let $(P_n(\bar{x}))_{n\in\mathbb{N}}$ be a collection of $*$-polynomial of degree 1 with coefficients in M/A, and for each $n \in N$ let $r_n \in R^+$. Without loss of generality, reordering the polynomials and eventually adding edundancy if necessary, we can suppose that the only variables occurring in P_n are$x_0, \ldots, x_n$.

Suppose that the set of conditions $\{\|P_n(x_0, \ldots, x_n)\| = r_n : n \in N\}$ is approximately finitely satisfiable. As wenoted immediately after Definition (2.1.9), it is sufficient to assume that the partial solutions are all in $(M/A) \leq 1$, and we must find a total solution also in $(M/A)_{\leq 1}$. So we have partial solutions

$$\{\pi(x_{k,i})\}_{k\leq i} \subseteq (MA)_{\leq 1}$$

such that for all $i \in N$ and $n \leq i$ we have

$$\left\|P_n(\pi(x_{0,i}), \ldots, \pi(x_{n,i}))\right\| \in (r_n)_{1/i}.$$

For each $n \in \mathbb{N}, let\ Q_n(x_0, \ldots, x_n)$ be a polynomial whose coefficients are liftings of the coefficients ofP_n to M, and let F_n be a finite set that contains

(i) all the coefficients of Q_k, for $k \leq n$
(ii) $x_{k,i}, x_{k,i}^*$for$k \leq i \leq n.$
(iii) $Q_k(x_{0,i}, \ldots, x_{k,i})\ for\ k \leq i \leq n.$

Let $\epsilon_n = 4^{-n}$. Find sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ satisfying the conclusion of Lemma (2:1:28) for these choices of $(F_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$.

Let$\bar{x}_{n,i} = (x_{0,i}, \ldots, x_{n,i})$, $\mathcal{Y}_k = \sum_{i\geq k} f_i x_{k,i} f_i$,$\bar{\mathcal{Y}}_n = (\mathcal{Y}_0, \ldots, \mathcal{Y}_n)$ and

$\bar{\mathcal{Z}}_n = \pi(\bar{\mathcal{Y}}_n)$. Fix $n \in \mathbb{N};$ we will prove that $\|P_n(\mathcal{Z}_n)\| = r_n$. First, since $x_{k,i} \in M_{\leq 1}$, as a consequence of condition (ix) of Lemma (2.1.28), we have that $\mathcal{Y}_i \in M_{\leq 1}$ for all (i). Moreover, since Q_n is a polynomial whose coefficients are lifting of those of P_n we have

$\|P_n(\bar{\mathcal{Z}}_n)\| = \|\pi(Q_n(\bar{\mathcal{Y}}_n))\|$.

We claim that

$$Q_n(\bar{\mathcal{Y}}_n) - \sum_{j\in\mathbb{N}} f_j Q_n(\bar{x}_{n,j}) f_j \in A.$$

It is enough to show that

$$\sum_{j\in\mathbb{N}} f_j a x_{k,j} b f_j - \sum_{j\in\mathbb{N}} a f_j x_{k,j} f_j b \ \in A,$$

where a, b are coefficients of a monomial in Q_n, since Q_n is the sum of finitely many of these elements (and the proof for monomials of the form $ax^*_{k,j}, b$ is essentially the same as the one for $ax_{k,j}b$). By construction we have $a, b \in F_n$, and hence by condition (ii) of Lemma (2.1.27), for j sufficiently large,

$$\forall x \in M_{\leq 1} \left(\left\|af_j x f_j b - f_j a x b f_j\right\| \leq 2^{-j}\left(\left\|\|a\| + \|b\|\right\|\right)\right).$$

Therefore $\sum_{j\in\mathbb{N}}(f_j a x_{k,j} b f_j - a f_j x_{k,j} f_j b)$ is a series of elements in A that is converging in norm, which implies that the claim is satisfied. In particular,

$$\|P_n(\bar{\mathcal{Z}}_n)\| = \left\|\pi\Big(\sum_{j\in\mathbb{N}} f_j Q_n\big(\bar{x}_{n,j}\big) f_j\Big)\right\|$$

For each $j \geq 2$, let $a_j = (1 - e_{j-2}) Q_n(\bar{x}_{n,j})(1 - e_{j-2})$. By condition (i) of Lemma (2:1:27), the fact that $Q_n(\bar{x}_{n,j}) \in F_n$, and the originalchoice of the $x_{n,j}$'s, we have that limsup$\|a_j\| = r_n$. Similarly to the above, but this time using condition (iii) of Lemma (2:1:27), we have

$$\left\|\pi\Big(\sum_{j\in\mathbb{N}} f_j Q_n\big(\bar{x}_{n,j}\big) f_j\Big)\right\| = \left\|\pi\Big(\sum_{j\in\mathbb{N}} f_j a_j f_j\Big)\right\| \leq \left\|\sum_{j\in\mathbb{N}} f_j a_j f_j\right\|$$

Using condition (xi) of Lemma (2:1:27) and the fact that $Q_n(\bar{x}_{n,j}) \in F_j$ we have that

$$\left\|\sum_{j\in\mathbb{N}} f_j a_j f_j\right\| \leq \lim_{j\to\infty} \sup \|a_j\| = r_n$$

Combining the calculations so far, we have shown

$$\|P_n(\bar{\mathcal{Z}}_n)\| = \left\|\pi(\sum_{j\in\mathbb{N}} f_j Q_n(\bar{x}_{n,j}) f_j))\right\| = \left\|\pi\left(\sum_{j\in\mathbb{N}} f_j a_j f_j\right)\right\| \leq r_n.$$

Since $Q_n(\bar{x}_{n,j}) \in F_j$ for all j, condition (vi) of Lemma (2.1.27) implies

$$r_n \leq \lim_{j\to\infty} \sup \|f_j Q_n(\bar{x}_{n,j}) f_j\|.$$

It now remains to prove that

$$\lim_{j\to\infty} \sup \|f_j a_j f_j\| \leq \left\|\pi(\sum_{j\in\mathbb{N}} f_j a_j f_j)\right\|$$

so that we will have

$$r_n \leq \lim_{j\to\infty} \sup \|f_j Q_n(\bar{x}_{n,j}) f_j\|$$

$$= \lim_{j\to\infty} \sup \|f_j a_j f_j\|$$

$$\leq \left\|\pi\left(\sum_{j\in\mathbb{N}} f_j a_j f_j\right)\right\|$$

$$= \|P_n(\overline{\mathcal{Z}_n})\|.$$

We have $Q_n(\bar{x}_{n,j}) \in F_j$, so by condition (ii) of Lemma(2.1.27), we have $\lim_{j\to\infty} \sup \|f_j a_j f_j\| = lim_{j\to\infty}\, sup\, \|a_j f_j^2\|$,

and hence

$\sum_{j\in\mathbb{N}} f_j a_j f_j - \sum_{j\in\mathbb{N}} a_j f_j^2 \in A.$

The final required claim will then follow by condition (x) of Lemma (2:1:27), once we verify

$$\lim_{j\to\infty} \sup \|a_j f_j^2\| = \underset{j\to\infty}{lim}\, sup\, \|a_j\|.$$

We clearly have that for all j,

$$\|a_j f_j^2\| \le \|a_j\|$$

On the other hand,

$$\limsup_{j\to\infty} \|a_j f_j^2\| = \limsup_{j\to\infty} \|f_j a_j f_j\| = \limsup_{j\to\infty} \|f_j Q_n(\bar{x}_{n,j}) f_j\|$$

by condition (iii)

$$\ge r_n = \limsup_{j\to\infty} \|a_j\|.$$

The theorem above applies, in particular, to coronas of σ-unital algebras. The following result is due to Farah and Hart, but unfortunately their proof has a technical error. Specifically, uses the same strategy, but avoids their equation (x), which is incorrect.

Corollary (2.1.29)[2]: If A is a σ-unital C^*-algebra, then its corona $C(A)$ is countably degree-1 saturated.

We also obtain countable degree-1 saturation for the motivating example from the beginning of this section.

Corollary (2:1:30)[2]: Let N be a II_1 factor, H a separable Hilbert space and $M = N \overline{\otimes} \mathcal{B}(H)$ be the associated II_∞ factor. Let $\mathcal{K}_{\mathrm{M}}$ be the unique two-sided closed ideal of M, that is the closure of the elements of finite trace. Then $M/\mathcal{K}_M$ is countably degree-1 saturated. In particular, this is the case when $N = \mathcal{R}$, the hyperfinite II_1 factor.

More generally, recall that a von Neumann algebra M is finite if there is not a projection that is Murray-von Neumann equivalent to 1_{M}, and σ-finite if there is a sequence of finite projections weakly converging to 1_M.

Corollary (2.1.31)[2]: Let M be a σ-finite but not finite tracial von Neumann algebra, and let A be the ideal generated by the finite trace projections. Then M/A is countably degree-1 saturated.

Section(2.2): Generalized Calkin Algebras and Abelian C^*-algebra:-

Notation (2.2.1)[2]: Let α be an ordinal and $H_\alpha = \ell^2(\aleph_\alpha)$ be the unique (up to isomorphism) Hilbert space of density character $\aleph_\alpha$. Let $\mathcal{B}_\alpha = \mathcal{B}(\mathrm{H}_\alpha)$. Let $\mathcal{K}_\alpha$ be the ideal of compact operators in $\mathcal{B}_\alpha$. The quotient $\mathcal{C}_\alpha = \mathcal{B}_\alpha/\mathcal{K}_\alpha$ is called the generalized Calkin algebra of weight $\aleph_\alpha$.

Note that, when H is separable, the ideal of compact operators in $\mathcal{B}(\mathrm{H})$ is separable, and in particular σ-unital, so it follows that the Calkin algebra is countably degree-1 saturated. We are going to give explicit results on the theories of the generalized Calkin algebras. It is known that the Calkin algebra is not countably quantifier-free saturated; we show that the generalized Calkin algebras also fail to have this degree of saturation. This follows immediately from the fact that the Calkin algebra is isomorphic to a corner of the generalized Calkin algebra and that if A is a C^*-algebra that is Φ-saturated, where Φ include all $*$-polynomials of degree 1, then every corner of A is Φ-saturated. On the other hand, the proof shown below is direct and much easier than the proof in the separable case. It is worth noting, however, that the method we will use does not apply to the Calkin algebra $\mathcal{C}_0$ itself.

Lemma (2.2.2) [2]: Let $\alpha \geq 1$ be an ordinal. Then $\mathcal{C}_\alpha$ is not countably quantifier-free saturated.

Proof:-

Fix $\{A_n\}_{n\in\mathbb{N}}$ a countable partition of $\aleph_\alpha$ in disjoint pieces of size $\aleph_\alpha$ and a base $(e_\beta)_{\beta<\aleph_\alpha}$ for $\mathrm{H}_{\aleph_\alpha}$. For each $n \in \mathbb{N}$ let P_n be the projection onto $\overline{\mathrm{span}s(e_\beta : \beta \in A_n)}$.

Claim A: If $\mathcal{Q}$ is a projection in $\mathcal{B}_\alpha$ such that $\mathcal{Q}P_n \in \mathcal{K}_\alpha$ for all n then $\mathcal{Q}$ has range of countable density.

Proof:-

We have that for any $n \in N$ and $\epsilon > 0$ there is a finite $\mathrm{C}_{\epsilon,n} \subseteq \aleph_\alpha$ such that

$$\beta \notin C_{\epsilon,n} \Rightarrow \|\mathcal{Q}P_n e_\beta\| < \epsilon.$$

Let $D = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} C_{1/m,n}$. If $\beta \notin \mathrm{D}$ then for all $n \in \mathbb{N}$ we have $\|\mathcal{Q}P_n e_\beta\| = 0$ and since there is n such that $e_\beta \in P_n$, we have that $\|\mathcal{Q}e_\beta\| = 0$. Since D is countable, $\mathcal{Q}$ is identically zero on a subspace of countable codimension.

Let $\mathcal{Q}_{-4} = xx^* - 1$, $\mathcal{Q}_{-3} = x^*x - \mathcal{Y}$, $\mathcal{Q}_{-2} = \mathcal{Y} - \mathcal{Y}^*$, $\mathcal{Q}_{-1} = \mathcal{Y} - \mathcal{Y}^2$, and $\mathcal{Q}_n = \mathcal{Y}P_n$. The type $\{\|\mathrm{Q}_\mathrm{i}\| = 0\}_{-4\leq \mathrm{i}}$ admits a partial solution, but not a total solution.

We are going to have a further look at the theories of $\mathcal{C}_\alpha$. In particular we want to see if it is possible to distinguish between the theories of $\mathcal{C}_\alpha$ and of $\mathcal{C}_\beta$, whenever $\alpha \neq \beta$. Of course, since there are at most $2^{\aleph_0}$ many possible theories, we have that there are ordinals $\alpha \neq \beta$ such that $\mathcal{C}_\alpha \equiv \mathcal{C}_\beta$. As we show in the next theorem, this phenomenon cannot occur whenever α and β are sufficiently small, and similarly for $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$.

Theorem (2.2.3)[2]: Let $\alpha \neq \beta$ be ordinals, and H_α the Hilbert space of density $\aleph_\alpha$. Then the projections of the algebras $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ as posets with respect to the Murray-von Neumann order are elementary equivalent if and only if $\alpha = \beta \bmod \omega^\omega$, where ω^ω is computed by ordinal exponentiation, as they are the infinite projections of $\mathcal{B}_\alpha$ of $\mathcal{B}_\beta$. Consequently, if $\alpha \not\equiv \beta$ then $\mathcal{B}_\alpha \not\equiv \mathcal{B}_\beta$ and $\mathcal{C}_\alpha \not\equiv \mathcal{C}_\beta$.

Proof:-

The key fact is that $\alpha \equiv \beta$ (as first-order structures with only the ordering) if and only if $\alpha = \beta \bmod \omega^\omega$. Hence the proof will be complete as soon as we notice that the ordinal α is interpretable in both $\mathcal{C}_\alpha$ (as the set of projections under Murray-von Neumann equivalence) and inside $\mathcal{B}_\alpha$ (as the set of infinite projections under Murray-von Neumann equivalence).

To see this note that there is a formula ϕ such that $\phi(p, q) = 0$ if $p \sim_{M_vN} q$ and p, q are projections and $\phi(p, q) = 1$ otherwise, and that being an infinite projection is axiomatizable (since p is an infinite projection if and only if $\psi(p) = 0$ if and only if $\psi(p) < 1/4$, where $\psi(x) = \|x - x^*\| + \|x - x^2\| + inf_{\mathcal{Y}}(\|\mathcal{Y}\mathcal{Y}^* - x\|) + \|\mathcal{Y}^*\mathcal{Y}x - \mathcal{Y}^*\mathcal{Y}\| + (1 - \|\mathcal{Y}^*\mathcal{Y} - x\|)$ where $\mathcal{Y}$ ranges over the set of partial isometries. Since we have that to any projection we can associate the density of its range (both in $\mathcal{C}_\alpha$ and $\mathcal{B}_\alpha$), and that we have that $p \sim_{M_vN} q$ if and only if the density of p is less or equal than the range of q. Since every possible value for the density is of the form $\aleph_\beta$, for $\beta < \alpha$, the theorem is proved.

In this Section we consider abelian C^*-algebras, and particularly the theories of real rank zero abelian C^*-algebras. In the first part of this section we give a full classification of the complete theories of abelian real rank zero C^*-algebras in terms of the (discrete first-order) theories of Boolean algebras (recall that a theory is complete if whenever $M \models T$ and $N \models T$ then $M \equiv N$).As an immediate consequence of this classification we find that there are exactly $\aleph_0$ distinct complete theories of abelian real rank zero C^*-

algebras. We also give a concrete description of two of these complete theories. In the second part of the section we return to studying saturation. We show how saturation of abelianC^*-algebras is related to the classical notion of saturation for Boolean algebras. We begin by recalling some well-known definitions and properties.

Notation(2.2.4)[2]:A topological space X is said sub-Stonean if any pair of disjoint open σ-compact sets has disjoint closures; if, in addition, those closures are open and compact,X is said Rickart. A space X is said to be totally disconnected if the only connected components of X are singletons and 0-dimensional if X admits a basis of clopen sets. A topological space X such that every collection of disjoint nonempty open subsets of X is countable is said to carry the countable chain condition.

Note that for a compact space being totally disconnected is the same as being 0-dimensional, and this corresponds to the fact that $C(X)$ has real rank zero. Moreover any compact Rickart space is 0-dimensional and sub-Stonean, while the converse is false (take for example $\beta\mathbb{N}\setminus\mathbb{N}$). The space X carries the countable chain condition if and only if $C(X)$ has few orthogonal positive elements (see Definition (2.1.18)).

Notation(2.2.5)[2]: A Boolean algebra is atomless if $\forall a \neq 0$ there is b such that $0 < b < a$. For $Y, Z \subset B$ we say that $Y < Z$if$\forall(\mathcal{Y}, z) \in Y \times Z$ we have $\mathcal{Y} < z$.

Note that, for a 0-dimensional space, $CL(X)$ is atomless if and only if X does not have isolated points. In particular $|\{a \in CL(X)\colon \text{a is an atom}\}| = |\{x \in X\colon x\text{is isolated}\}|$.

Definition (2.2.6)[2]: Let κ be an uncountable cardinal. A Boolean algebra B is said to be κ-saturated if every finitely satisfiable type of cardinalit$\mathcal{Y} < \kappa$ in the first-order language of Boolean algebras is satisfiable.

For atomless Boolean algebras this model-theoretic saturation can be equivalently rephrased in terms of increasing and decreasing chains:

Theorem(2.2.7)[2]: Let B be an atomless Boolean algebra, and κ an uncountable cardinal. Then B is κ-saturated if and only if for every directed $Y < Z$ such that $|Y| + |Z| < \kappa$ there is $c \in B$ such that $Y < c < Z$.

Notation(2.2.8)[2]: Let $\mathcal{U}$ be an ultrafilter (over a possibly uncountable index set). If A is a C^*-algebra, we denote theC^*-algebraic ultrapower of

A by $\mathcal{U}$ by $A^{\mathcal{U}}$. Similarly, if B is a Boolean algebra we denote the classical model-theoretic ultrapower of B by $B^{\mathcal{U}}$. If X is a topological space we denote the ultracopower by $\sum_{\mathcal{U}} X$.

Lemma (2.2.9)[2]: Let X be a compact Hausdorff space, and let $\mathcal{U}$ be an ultrafilter. Then $C(X)^{\mathcal{U}} \cong C(\sum_{\mathcal{U}} X)$, and $CL(X)^{\mathcal{U}} \cong CL(\sum_{\mathcal{U}} X)$.

Theorem (2.2.10) [2]: Let A and B be abelian, unital, real rank zero C^*-algebras. Write $A = C(X)$ and$B = C(Y)$, where X and Y are 0-dimensional compact Hausdorff spaces. Then$A \equiv B$ as metric structures if and only if $CL(X) \equiv CL(Y)$ as Boolean algebras.

Proof:-

Suppose that $A \equiv B$. By the Keisler-Shelah theorem (Theorem (2.1.12)) there is an ultrafilter U such that$A^{\mathcal{U}} \cong B^{\mathcal{U}}$. By Lemma (2.2.10)$A^{\mathcal{U}} \cong C(\sum_{\mathcal{U}} X)$. Thus we have $C(X_{\mathcal{U}}) \cong C(Y_{\mathcal{U}})$, and hence by GelfandNaimark$X_{\mathcal{U}}$ is homeomorphic to $Y_{\mathcal{U}}$. Then$(\sum_{\mathcal{U}} X) \cong CL(\sum_{\mathcal{U}} Y)$.Applying Lemma (2.2.10)again, we have$CL(\sum_{\mathcal{U}} X) = CL(X)^{\mathcal{U}}$, so weobtain $CL(X)^{\mathcal{U}} \cong CL(Y)^{\mathcal{U}}$, and in particular, $CL(X) \equiv CL(Y)$. The converse direction is similar, starting from the Keisler-Shelah theorem for first-order logic.

It is interesting to note that the above result fails when $C(X)$ is considered only as a ring in first-order discrete logic .

Corollary (2.2.11)[2]: There are exactly$\aleph_0$ distinct complete theories of abelian, unital, real rank zero C^*-algebras.

Proof:-

There are exactly $\aleph_0$ distinct complete theories of Boolean algebras;for a description of these theories.

Corollary (2.2.12)[2]: If X and Y are infinite, compact, 0-dimensional spaces both with the same finite number of isolated points or both having a dense set of isolated points, then $C(X) \equiv C(Y)$.

In particular, let α be any infinite ordinal. Then $C(\alpha + 1) \equiv C(\beta\omega)$. Moreover, if α is a countable limit, $C(2^{\omega}) \equiv C(\beta\omega \backslash \omega) \equiv C(\beta\alpha \backslash \alpha)$.

Proof:-

Given X, Y as in the hypothesis , we have that $CL(X) \equiv CL(Y)$.

This section is dedicated to the analysis of the relations between topology and countable saturation of abelianC^*-algebras. In particular, we want to study which kind of topological properties the compact Hausdorff space X has to carry in order to have some degree of saturation of the metric structure $C(X)$ and, conversely, to establish properties that are incompatible with the weakest degree of saturation of the corresponding algebra. From now on X will denote an infinite compact Hausdorff space (note that if X is finite then $C(X)_{\leq 1}$ is compact, and so $C(X)$ is fully saturated).

Lemma(2.2.13)[2]: Let X be an infinite compact Hausdorff space, and suppose that X satisfies one of the following conditions:

(i) X has the countable chain condition;
(ii) X is separable;
(iii) $X$is metrizable;
(iv) $X$is homeomorphic to aproduct of two infinite compact Hausdorff spaces;
(v) X is not sub-Stonean;
(vi) X is Rickart. Then $C(X)$ is not countably degree-1 saturated.

Proof:-

First, note that (iii) ⇒ (ii) ⇒ (i). The fact that (i) implies that C(X) is not countably degree-1 saturated is an instance. Failure of countable degree-1 saturation for spaces satisfying (iv) follows from Theorem (2.1.15), while for those satisfying (v). It remains to consider (vi).

Let X be Rickart. The Rickart condition can be rephrased as saying that any bounded increasing monotone sequence of self-adjoint functions in $C(X)$ has a least upper bound in C(X). Consider a sequence $(a_n)_{n\in\mathbb{N}} \subseteq C(X)_1^+$ of positive pairwise orthogonal elements, and let $b_n = \sum_{i\leq n} a_i$. Then $(b_n)_{n\in\mathbb{N}}$ is a bounded increasing sequence of positive operators, so it has a least upper bound b. Since $\|b_n\| = 1$ for all n, we also have $\|b\| = 1$. The type consisting of $P_{-3}(x) = x$, with $K_{-3} = \{1\}$, $P_{-2}(x) = b - x$ with$K_{-2} = [1,2]$, $P_{-1}(x) = b - x - 1$ with $K_{-1} = \{1\}$ and $P_n(x) = x - b_n - 1$with $K_n = [0,1]$ is consistent with partial solution b_{n+1} (for $\{P_{-3}, \dots, P_n\}$). This type cannot have a positive solution y, since in that case we would have that $\mathcal{Y} - b_n \geq 0$ for all $n \in N$, yet $b - \mathcal{Y} > 0$, a contradiction to X being Rickart.

Note that the preceding proof shows that the existence of a particular increasing bounded sequence that is not norm-convergent but does have a least upper bound (a condition much weaker than being Rickart) is sufficient to prove that $C(X)$ does not have countable degree-1 saturation. Moreover, the latter argument does not use that the ambient algebra is abelian. We will compare the saturation of $C(X)$ (in the sense of Definition (2.1.9)) with the saturation of $CL(X)$, in the sense of the above theorem. The results that we are going to obtain are the following:

Theorem(2.2.14)[2]: Let X be a compact 0-dimensional Hausdorff space. Then $C(X)$ is countably saturated $\Rightarrow CL(X)$ is countably saturated and $CL(X)$ is countably saturated $\Rightarrow$ C(X) is countably q.f. saturated.

Theorem (2.2.15)[2]: Let X be a compact 0-dimensional Hausdorff space, and assume further that X has a finite number of isolated points. If $C(X)$ is countably degree-1 saturated, then $CL(X)$ is countably saturated. Moreover, if X has no isolated points, then countable degree-1 saturation and countable saturation coincide for C(X).

Proof:-

Countable saturation of $C(X)$ (for all formulas in the language of C^*-algebras) implies saturation of the Boolean algebra, since being a projection is a weakly-stable relation, so every formula in $CL(X)$ can be rephrased in a formula in C(X); to do so, write sup for $\forall$, inf for $\exists$, $\|x-\mathcal{Y}\|$ for $x \neq \mathcal{Y}$, and so forth, restricting quantification to projections. This establishes the first implication. The second implication will require more effort. To start, we will to need the following Proposition, relating elements of $C(X)$ to certain collections of clopen sets:

Proposition(2.2.16)[2]: Let X be a compact 0-dimensional space and $f \in C(X)_{\leq 1}$. Then there exists a countable collection of clopen sets $\tilde{Y}_f = \{Y_{n,f} : n \in \mathbb{N}\}$ which completely determines f, in the sense that for each $x \in X$, the value $f(x)$ is completely determined by $\{n: x \in Y_{n,f}\}$.

Proof:-

Let $\mathbb{C}_{m,1} = \{\frac{j_1+\sqrt{-1}j_2}{m} : j_1, j_2 \in \mathbb{Z} \wedge \|j_1 + \sqrt{-1}_{j_2}\| \leq m\}$.

For every $\mathcal{Y} \in \mathbb{C}_{m,1}$ consider $Y_{\mathcal{Y},f} = f^{-1}(B_{1/m}(\mathcal{Y}))$. We have that each $X_{\mathcal{Y},f,n}$ is a σ-compact open subset of X, so is a countable union of clopen sets $Y_{n,f,1},\ldots,Y_{n,f,n}\ldots \in CL(X)$. Note that $\bigcup_{\mathcal{Y}\in\mathbb{C}_{m,1}} \bigcup_{n\in\mathbb{N}} X_{\mathcal{Y},f} = X$. Let $\tilde{X}_{m,f} = \{X_{\mathcal{Y},f,n}\}_{(\mathcal{Y},n)\in\mathbb{C}_{m,1}\times\mathbb{N}} \subseteq CL(X)$.

We claim that $\tilde{X}_f = \bigcup_m \tilde{X}_{m,f}$ describes f completely. *Fix* $x \in X$.For every $m \in \mathbb{N}$ we can find a (not necessarily unique) pair $(\mathcal{Y}, n) \in \mathbb{C}_{m,1}$ such that $x \in X_{\mathcal{Y},f,n}$. Note that, for any $m, n_1, n_2 \in \mathbb{N}$ and $\mathcal{Y} \neq \mathcal{Z}$, we have that $X_{\mathcal{Y},f,n_1} \cap X_{\mathcal{Y},f,n_2} \neq \emptyset$ implies$|\mathcal{Y} - \mathcal{Z}| \leq \sqrt{2}/m$. In particular, for every $m \in \mathbb{N}$ and $x \in X$we have

$$2 \leq |\{\mathcal{Y} \in \mathbb{C}_{m,1}: \exists n(x \in X_{\mathcal{Y},f,n})\}| \leq 4.$$

Let $A_{x,m} = \{\mathcal{Y} \in \mathbb{C}_{m,1}: \exists n(x \in X_{\mathcal{Y},f,n})\}$and choose $a_{x,m} \in A_{x,m}$ to have minimal absolute value. Then $f(x) = lim_m a_{x,m}$ so the collection $\tilde{X}_f$completely describes f in the desired sense.

The above proposition will be the key technical ingredient in proving the second implication in Theorem(2.2.14) . We will proceed by first obtaining the desired result under the Continuum Hypothesis, and then showing how to eliminate the set-theoretic assumption.

Lemma (2.2.17)[2]: Assume the Continuum Hypothesis. Let B be a countably saturated Boolean algebra of cardinality $2^{\aleph_0} = \aleph_1$. Then C(S(B)) is countably saturated.

Proof:-

Let $B' \preccurlyeq B$ be countable, and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. By the uniqueness of countably saturated models of size $\aleph_1$, and the continuum hypothesis, we have $B'^{\mathcal{U}} \cong B$. By Lemma (2.2.9) we therefore have$C(S(B)) \cong C(S(B'))^{\mathcal{U}}$, and hence $C(S(B))$is countably saturated.

Theorem (2.2.18)[2]: Assume the Continuum Hypothesis. Let X be a compact Hausdorff 0-dimensional space. If CL(X) is countably saturated as a Boolean algebra, then C(X) is quantifier free saturated.

Proof:-

Let $\|P_n\| = r_n$ be a collection of conditions, where each P_n is a 2-degree $*$-polynomial in $x_0, \ldots, x_n$, such that there is a collection $F = \{f_{n,i}\}_{n \leq i} \subseteq C(X)_{\leq 1}$, with the property that for all i we have $\|P_n(f_{0,i}, \ldots, f_{n,i})\| \in (r_n)_{1/i}$ for all $n \leq i$. For any n, we have that P_n has finitely many coefficients. Consider G the set of all coefficients of every P_n and L the set of all possible 2-degree $*$-polynomials in $F \cup G$. Note that for any $n \leq i$ we have that

$P_n(f_{0,i},\ldots,f_{n,i}) \in L$ and that L is countable. For any element $f \in L$ consider a countable collection $\tilde{X}_f$of clopen sets describing f, as in Proposition(2.2.16).

Since $CL(X)$ is countably saturated and $2^{\aleph_0} = \aleph_1$we can find a countably saturated Boolean algebra $B \subseteq CL(X)$ such that $\emptyset, X \in B$, for all $f \in L$ we have $\tilde{X}_f \subseteq B$, and$|B| = \aleph_1$.

Let $\iota: B \to CL(X)$ be the inclusion map. Then ι is an injective Boolean algebra homomorphism and hence admits a dual continuous surjection $g_\iota: X \to S(B)$.

Claim B: For every $f \in L$ we have that$\bigcup \tilde{X}_f = S(B)$.

Proof:-

Recall that

$\bigcup \tilde{X}_f = X$.

By compactness of X, there is a finite $C_f \subseteq \tilde{X}_f$ such that $\bigcup C_f = X$.In particular every ultrafilter on B (i.e., a point of $S(B)$), corresponds via g_ι to an ultrafilter on CL(X) (i.e., a point of X), and it has to contain an element of C_f. So $\bigcup \tilde{X}_f = S(B)$.

From g_ι as above, we can define the injective map $\phi: C(S(B)) \to C(X)$ defined as $\phi(f)(x) = f(g_\iota^{-1}(x))$. Note that ϕ is norm preserving: Since ϕ is a unital $*$-homomorphism of C^*-algebra we have that $\|\phi(f)\| \leq \|f\|$. For the converse, suppose that $x \in S(B)$ is such that $|f(x)| = r$, and by surjectivity take $\mathcal{Y} \in X$ such that $g_\iota(\mathcal{Y}) = x$. Then

$$|\phi(f)(\mathcal{Y})| = |f(g_\iota(g_\iota^{-1}(x)))| = |f(x)|.$$

For every$f \in L$consider the function f_0 defined by $\tilde{X}_f$and construct the corresponding $*$-polynomials P'_n.

Claim C:

$a. f = \phi(f')$ for all $f \in L$.

b. $\|P'_n(f'_{0,i},\ldots,f'_{n,i})\| \in (r_n)_{1/i}$ for all i and $n \leq i$.

Proof:-

Note that, since $f_{n,i} \in L$ and every coefficient of P_n is in L, wehave that $P_n(f_{0,i}, \ldots, f_{n,i}) \in L$. It follows that condition (a), combinedwith the fact that ϕ is norm preserving, implies condition (b).

Recall that $g = g_\iota$ is defined by Stone duality, and is a continuous surjective map $g: X \to Y$. In particular g is a quotient map. Moreover by definition, since $X_{q,f,n} \in CL(Y) = B \subseteq CL(X)$, we have that if $x \in Y$ is such that $x \in X_{q,f,n}$for some$(q, f, n) \in \mathbb{Q} \times L \times \mathbb{N}$, then for all $\mathcal{Z}$ such that $g(\mathcal{Z}) = x$ we have $\mathcal{Z} \in X_{q,f,n}$. Take f and $x \in X$ such that$f(x) \neq \phi(f')(x)$. Consider m such that $|f(x) - \phi(f')(x)| > 2/m$. Pick $\mathcal{Y} \in C_{m,1}$ such that there is k for which $x \in X_{\mathcal{Y},f,k}$and find $\mathcal{Z} \in Y$such that $g(\mathcal{Z}) = x$.Then$\mathcal{Z} \in X_{\mathcal{Y},f,k}$,that implies $f'(\mathcal{Z}) \in B_{1/m}(\mathcal{Y})$and so$\phi(f')(x) = f'(\mathcal{Z}) \in B_{1/m}(\mathcal{Y})$ contradicting

$$|f(x) - \phi(f')(x)| \geq 2/m$$

Consider now $\{\|P_n'(x_0, \ldots, x_n)\| = r_n\}$. This type is consistent typein $\mathrm{C(S(B))}$ by condition (b), and $\mathrm{C(S(B))}$ is countably saturated, so there is a total solution g. Then $h_j = \phi(g_j)$ will be such that$\|P_n(\bar{h})\| = r_n$, since φ is norm preserving, proving quantifier-free saturation for $C(X)$.

To remove the Continuum Hypothesis from Theorem (2.2.20) we will show that the result is preserved by σ-closed forcing. We first prove a more general absoluteness result about truth values of formulas. For more examples of absoluteness of model-theoretic notions . Our result will be phrased in terms of truth values of formulas of infinitary logic for metric structures. Such a logic, in addition to the formula construction rules of the finitary logic we have been considering, also allows the construction of $\sup_n \phi_n$ and $\inf_n \phi_n$ as formulas when the ϕ_n are formulas with a total of finitely many free variables. Two such infinitary logics have been considered in the literature. The first, introduced by Ben Yaacov and Iovino, allows the infinitary operations only when the functions defined by the formulas ϕ_n all have a common modulus of uniform continuity; this ensures that the resulting infinitary formula is again uniformly continuous. The second , does not impose any continuity restriction on the formulas ϕ_n when forming countable infima or suprema; as a consequence, the infinitary formulas of this logic may define discontinuous functions. The following result is valid in both of these logics; the only complication is that we must allow metric structures to be based on incomplete metric space, since a complete metric space may become incomplete after forcing.

Lemma (2.2.19)[2]: Let$M$ be a metric structure, $\phi(\bar{x})$ be a formula of infinitary logic for metric structures, and $\bar{a}$be a tuple from M of the appropriate length. Let$P$ be any notion of forcing. Then the value $\phi^M(\bar{a})$ is the same whether computed in V or in the forcing extension $V\,[G]$.

Proof:-

The proof is by induction on the complexity of formulas; the key point is that we consider the structure M in $V\,[G]$ as the same set as it is in. The base case of the induction is the atomic formulas, which are of the form $P(\bar{x})$ for some distinguished predicate P. In this case since the structure M is the same in V and in $V\,[G]$, the value of $P^M(\bar{a})$ is independent of whether it is computed in V or $V\,[G]$.

The next case is to handle the case where ϕ is $f(\psi_1, \dots, \psi_n)$,where each ψ_i is a formula and $f : [0,1]^n \to [0,1]$ is continuous. Since the formula ϕ is in, so is the function f. By induction hypothesis each $\psi_i^M(\bar{a})$ can be computed either in V or $V\,[G]$, and so the same is true of $\phi^M(\bar{a}) = f(\psi_1^M(\bar{a}), \dots, \psi_n^M(\bar{a}))$. A similar argument applies to the case when ϕ is $\sup_n \psi_n$ or infn ψ_n. Finally, we consider the case where $\phi(\bar{x}) = \inf_{\mathcal{Y}} \psi(\bar{x}, \mathcal{Y})$ (the case with sup instead of inf is similar). Here we have that for every $b \in M$, $\psi^M(\bar{a}, b)$ is independent of whether computed in Vor $V\,[G]$ by induction. In both$V$ and $V\,[G]$ the infimum ranges over the same set M, and hence $\phi^M(\bar{a})$ is also the same whether computed in V or $V\,[G]$.

We now use this absoluteness result to prove absoluteness of countable saturation under σ-closed forcing.

Proposition (2.2.20)[2]: Let $\mathbb{p}$ be a σ-closed notion of forcing. Let M be a metric structure, and let Φ be a set of (finitary) formulas. Then M is countably Φ-saturated in V if and only if M is countably Φ-saturated in the forcing extension $V[G]$.

Proof:-

First, observe that since $\mathbb{P}$ is σ-closed, forcing with $\mathbb{P}$ does not introduce any new countable set. In particular, the set of types whichmust be realized for M to be countablyΦ-saturated are the same in V and in $V\,[G]$.

Let $t(x)$ be a set of instances of formulas from Φ with parameters from a countable set $A \subseteq M$. Add new constants to the language for each $a \in A$, so that we may view t as a type without parameters. Define $\phi(\bar{x}) = inf\{\psi(\bar{x}) : \psi \in t\}$.

Note that $\phi^M(\bar{a}) = 0$ if and only if $\bar{a}$ satisfies t in M. This Φ is a formula in the infinitary logic ,for any $\bar{a}$from M we have that $\phi^M(\bar{a}) = 0$ in Vif and only if $\phi^M(\bar{a}) = 0$ in $V\ [G]$. As the same finite tuples $\bar{a}$ from M exist in V and in $V\ [G]$, this completes the proof.

Lemma(2.2.21)[2]: The Continuum Hypothesis can be removed from the hypothesis of Theorem (2.2.18).

Proof:-

Let X be a 0-dimensional compact space such that $CL(X)$ is countably saturated, and suppose that the Continuum Hypothesis fails. Let $\mathbb{P}$be a σ-closed forcing which collapses $2^{\aleph_0}$ to $\aleph_1$. Let $A = C(X)$. Observe that since $\mathbb{P}$ is σ-closed we have that X remains a compact 0-dimensional space in $V[G]$, and we still have $A = C(X)$ in $V[G]$. By Proposition (2.2.20) we have that $CL(X)$ remains countably saturated in $V[G]$. Since $V[G]$ satisfies the Continuum Hypothesis we can apply Theorem (2.2.18) to conclude that A is countably quantifier-free saturated in V [G], and hence also in V by Proposition (2.2.20).

With the continuum hypothesis removed from Theorem (2.2.18), we have completed the proof of Theorem (2.2.12). It would be desirable to improve this result to say that if $CL(X)$ is countably saturated then $C(X)$ is countably saturated. We note that if the map ϕ in Theorem (2.2.18) could be taken to be an elementary map then the same proof would give the improved conclusion.

Proposition (2.2.22)[2]: If X is a 0-dimensional compact space with finitely many isolated points such that $C(X)$ is countably degree-1 saturated, then the Boolean algebra CL(X) is countably saturated.

Proof:-

Assume first that X has no isolated points. In this case we get that $CL(X)$ is atomless, so it is enough to see that $CL(X)$ satisfies the equivalent condition of Theorem (2.2.8). Let $Y < Z$ be directed such that $|Y| + |Z| < \aleph_1$. Assume for the moment that both Y and Z are infinite. Passing to a cofinal increasing sequence in Z and a cofinal decreasing sequence in, we can suppose that $Z = \{U_n\}_{n\in\mathbb{N}} and\, Y = \{V_n\}_{n\in\mathbb{N}}$, where

$$U_1 \subsetneq \cdots \subsetneq U_n \subsetneq U_{n+1} \subsetneq \cdots \subsetneq V_{n+1} \subsetneq V_n \subsetneq \ldots \subsetneq V_1.$$

If $\bigcup_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} V_n$ then $\bigcup_{n\in\mathbb{N}} U_n$ is a clopen set, so by the remark following the proof of Lemma (2.2.15), we have a contradiction to the countable degree-1 saturation of C(X). For each $n \in N$, let $p_n = \chi U_n$ and $q_n = \chi V_n$, where χA denotes the characteristic function of the set A. Then $p_1 < \ldots < p_n < p_{n+1} < \ldots < q_{n+1} < q_n < \ldots < q_1$ and by countable degree-1 saturation there is a positive r such that $p_n < r < q_n$ for every n. In particular $A = \{x \in X: r(x) = 0\}$ and $C = \{x \in X: r(x) = 1\}$ are two disjoint closed sets such that $\overline{\bigcup_{n\in\mathbb{N}} U_n} \subseteq C$ and $X \setminus \overline{\bigcap_{n\in\mathbb{N}} V_n} \subseteq A$. We want to find a clopen set D such that $A \subseteq D \subseteq X \setminus C$. For each $x \in A$ pick W_x a clopen neighborhood contained in $X \setminus C$. Then $A \subseteq \bigcup_{x\in A} W_x$. By compactness we can cover A with finitely many of these sets, say $A \subseteq \bigcup_{i\le n} W_{x_i} \subseteq X \setminus C, so\ D = \bigcup_{i\le n} W_{x_i}$ is the desired clopen set.

Essentially the same argument works when either Y or Z is finite. We need only change some of the inequalities from < with ≤, noting that a finite directed set has always a maximum and a minimum. If X has a finite number of isolated points, write $X = Y \cup Z$, where Y has no isolated points and Z is finite. Then $C(X) = C(Y) \oplus C(Z)$ and $CL(X) = CL(Y) \oplus CL(Z)$. The above proof shows that $CL(Y)$ is countably saturated, and $CL(Z)$ is saturated because it is finite, so $CL(X)$ is again saturated.

To finish the proof of Theorem (2.2.17) it is enough to show that when X has no isolated points the theory of X admits elimination of quantifiers. By Corollary (2.2.12) we have that $C(X) \equiv C(\beta\mathbb{N}\setminus\mathbb{N})$ for such X, so it suffices to show that the theory of $C(\beta\mathbb{N}\setminus\mathbb{N})$ eliminates quantifiers.

Definition (2.2.23)[2]: Let $a_1, \ldots, a_n \in C(X)$ (more generally, one can consider commuting operators on some Hilbert space H). We say that $a = (a_1, \ldots, a_n)$ is non-singular if the polynomial $\sum_{i=1}^{n} a_i x_i = I$ has a solution $x_1, \ldots, x_n$ *in* $C(X)$. We define the joint spectrum of $a_1, \ldots, a_n$ to be

$$j\sigma(\bar{a}) = \{\bar{\lambda} \in \mathbb{C}^n: (\lambda_1 - a_1, \ldots, \lambda_n - a_n) \text{ is singular}\}$$

Proposition (2.2.24)[2]: Fix $a_1, \ldots, a_n \in C(X)$. Then $\lambda \in j\sigma(\bar{a})$ if and only if $\sum_{i\le n} |\lambda_i - a_i|$ is not invertible.

Proof:-

We have that $\bar{\lambda} \in j\sigma(\bar{a})$ if and only if there is $x \in X$ such that $a_i(x) = \lambda_i$ for all $i \le n$. In particular, $\bar{\lambda} \in j\sigma(\bar{a})$ if and only if $0 \in$

$\sigma(\sum|\lambda_i - a_i|)$ if and only if there is x such that $\sum_{i\leq n}|\lambda_i - a_i|(x) = 0$. Since each $|\lambda_i - a_i|$ is positive we have that this is possible if and only if there is x such that for all $i \leq n$, $|\lambda_i - a_i|(x) = 0$.

Proposition(2.2.25)[2]: The joint spectrum of an abelian C^*-algebra A is quantifier free-definable.

Proof:-

First of all recall that, when $\bar{a} = a$, then $j\sigma(\bar{a}) = \sigma(a)$, hence the two definitions coincide for elements. We want to define a quantifierfree definable function $F: A \times \mathbb{C} \to [0,1]$ such that $F(a,\lambda) = 0$ if and only if $\lambda \in \sigma(a)$. Since we showed that $\bar{\lambda} \in \sigma(\bar{a})$ if and only if $0 \in \sigma \sum_{i\leq n}|\lambda_i - a_i|$, so, in light of this, we can define a function

$$F_n: A^n \times \mathbb{C}^n \to [0,1]$$

as $F_n(\bar{a},\bar{\lambda}) = F(\sum|\lambda_i - a_i|, 0)$, hence we have that $F_n(\bar{a},\bar{\lambda}) = 0$ if and only if $\bar{\lambda} \in j\sigma(\bar{a})$, that implies that the joint spectrum of $a \in A^n$ is quantifier-free definable.

To define $\sigma(\bar{a})$, recall that, for $f \in A$, the absolute value of f is quantifier-free definable as $|f| = \sqrt{ff^*}$, and for a self-adjoint $f \in A$, its positive part is quantifier-free definable as the function $f_+ = max(0,f)$. Then $F(a,\lambda) = |1 - \|(1 - |a - \lambda \cdot 1|)_+\||$ is the function we were seeking.

Theorem (2.2.26)[2]: The theory of $C(\beta\mathbb{N}\backslash\mathbb{N})$ has quantifier elimination. Consequently the theory of real rank zero abelian C^*-algebras without minimal projections has quantifier elimination.

Proof:-

It is enough to prove that for any $n \in N$ and $\bar{a},\bar{b} \in C(\beta\mathbb{N}\backslash\mathbb{N})^n$ that have the same quantifier-free type over $\emptyset$ there is an automorphism of $C(\beta\mathbb{N}\backslash\mathbb{N})$ sending a_i to b_i, for all $i \leq n$. Since a and b have the same quantifier-free type, we have that $K = j\sigma(\bar{a}) = j\sigma(\bar{b})$. Consider D be a countable dense subset of K and pick $f_1,\dots,f_2,g_1,\dots,g_n \in C(\beta\mathbb{N}) = \ell^\infty(\mathbb{N})$ such that $\forall(d_1,\dots,d_n) \in D$ we have that $F_d = \{m \in \mathbb{N}: \forall i \leq n(f_i(m) = d_i)\}$ and $G_d = \{m \in \mathbb{N}: \forall i \leq n(g_i(m) = d_i)\}$ are infinite, $\pi(f_i) = a_i, \pi(g_i) = b_i$ and for $m \in \mathbb{N}$ we have that $(f_1(m),\dots,f_n(m)),(g_1(m),\dots,g_n(m)) \in D$.

In particular we have that $\mathbb{N} = \bigcup_{d \in D} F_d = \bigcup_{d \in D} G_d$ and that for all $d \neq d'$ we have $F_d \cap F_{d'} = \emptyset = G_d \cap G_{d'}$, then there is a permutation σ on $\mathbb{N}$ (that induces an automorphism of $C(\beta\mathbb{N} \setminus \mathbb{N})$) such that $f_i \circ \sigma = g_i$ for all $i \leq n$.

The proof of Theorem (2.2.15) is now complete by combining Theorem (2.2.14), Proposition (2.2.22).

In light of our result related to the Breuer ideal of a II_1 factor, we can ask whether or not this quotient structure carries more saturation than countable degree-1 saturation, and if the degree of saturation may depend on the structure of the II_1 factor itself. The proof of the failure of quantifier-free saturation for the Calkin algebra, involved the notion of the abelian group Ext, a notion which has not been developed for quotients with an ideal that is not σ-unital, so it cannot be easily modified to obtain a similar result for our case (and in general, for the case of a tracial Von Neumann algebra modulo the ideal of finite projections).

CHAPTER 3

C^*-Algebras of Right LCM Semigroups ofBoundary Quotients

If P is such a semigroup, its C^*-algebra admits a natural boundary quotient $Q(P)$. We show that $Q(P)$ is isomorphic to the tight C^*-algebra of a certain inverse semigroup associated to P, and thus is isomorphic to the C^*-algebra of an ´etale groupoid. We use this to give conditions on P which guarantee that $Q(P)$is simple and purely infinite, and give applications to self-similar groups and Zappa-Sz´ep products of semigroups.

Section(3.1): Background and the Boundary Q(P)as the Tight C^*-Algebra of an Inverse Semigroup:-

A semigroup$P$ is left cancellative if $pq = ps$ implies that $q = s$, and C^*-algebras associated to such semigroups are an active topic of research in operator algebras. Li's construction] of a C^*-algebra $C^*(P)$ from a left cancellative semigroup P generalizes Nica's quasi-lattice ordered semigroups and encompass a great deal of interesting C^*-algebras, including the Cuntz algebras (the **Cuntz algebra**$\mathcal{O}_n$(after Joachim Cuntz) is the universal C^*--algebra generated by n isometries satisfying certain relations. It is the first concrete example of a separable infinite simple C^*-algebra)[7] and the C^*-algebra of the $ax + b$ semigroup.Many semigroups of interest can be embedded into groups, and represents a comprehensive study of the C^*-algebras of such semigroups. Another interesting class of semigroups (which has some overlap with the previous)are the right LCM semigroups, which are semigroups in which two principal right ideals are either disjoint or intersect in another principal right ideal,considers the C^*-algebras of such semigroups, and obtains many results about how the properties of P influence the properties of$C^*(P)$in this case.

We are concerned with boundary quotients of such algebras, we define a boundary quotient $Q(P)$ of $C^*(P)$ when Pis a right LCMsemigroup, and this is the principal object of study in this section. Quotients of this type are worthy of singling out because they frequently give examples of simple C^*-algebra where the original would not (unless of course they are equal), and examples of simple C^*-algebras are of interest to C^*-algebra classification.

It turns out that this boundary quotient can be studied by using work of Exel on inverse semigroupC^*-algebras. An inverse semigroup is a semigroup S such that for each $s \in S$ there is a unique$s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$, and one can define a universal C^*-algebra for representations of S. Further work of Norling determined that for a left cancellativesemigroup$P$, $C^*(P\ )$ is isomorphic to the universal C^*-algebra of a certain inverse semigroup (denoted S in the sequel) obtained from P. Exel discovered a natural quotient for Paterson's inverse semigroup algebra called the tight C^*-algebra of an inverse semigroup. This C^*-algebra is universal for so called tight representations of of the inverse semigroup; representations of this kind enforce a kind of nondegeneracy condition. This construction has been studied by many authors. Our first main result, states that the tight C^*-algebra of S is isomorphic to the boundary quotient $Q(\mathrm{P}\ )$. This generalizes a combination of from the case of self-similar groups, and in fact it was the desire togeneralize this result to other types of semigroups which was the motivation for this work.

Both Paterson's and Exel'sC^*-algebras can be presented as the C^*-algebras of certain ´etalegroupoids, and so can be analyzed by using the many results concerning ´etalegroupoids in the literature.There is a small difficulty in doing so however, because the groupoids which arise in this way can be non-Hausdorff, and a majority of the results in the literature about the structure of ´etalegroupoidC^*-algebras assumes the Hausdorff property.One condition which guarantees Hausdorff is right cancellativity (in addition to the left cancellativity already assumed), but one can weaken this a bit to obtain a condition on P which is equivalent to the groupoid being Hausdorff, see Proposition (3.2.1). Here, we employ the results to find conditions on P which guarantee that $Q(P\ )$ is simple and purely infinite,because that section is concerned with ´etale groupoids arising from inversesemigroup actions. We note that in the recent work .Steinberg independently comes to many of the same conclusions, and many of the results we use from also appear, but throughout this article we will reference their appearance.

If X is a right ideal of P,then for all $p \in P$ the sets

$$pX = \{px | x \in X\}, \quad p^{-1}X = \{\mathcal{Y} \in P \mid p\mathcal{Y} \in X\}$$

are also right ideals. We let $\mathcal{J}(P)$ denote the smallest set of right ideals which contains $P$and ϕ, is closed under intersections, and such that $\mathrm{X} \in \mathcal{J}(\mathrm{P})$and $p \in P$ implies that bothpX and $p^{-1}X$ are in $\mathcal{J}(P)$. Then $\mathcal{J}(\mathrm{P})$ is a semilattice under intersection, and is called the semilattice of constructible ideals. For a left cancellative semigroup P, Lie constructs a C^*-algebra$C^*(\mathrm{P}\ )$.

Definition (3.1.1)[3]: Let P be a left cancellative semigroup, and let $\mathcal{J}(P)$ denote the set of constructible ideals of P.Then $C^*(P)$ is defined to be the universal C^*-algebra generated by a set of isometries$\{v_p | p \in P\}$and a set of projections $\{e_X | X \mathcal{J}(P)\}$ subject to the following:

$(L1)$ $v_p v_q = v_{pq}$for all $p, q \in P$,

$(L2)$ $v_p e_X v_p^* = e_p X$ for all $p \in P$ and $X \in \mathcal{J}$,

$(L3) e_P = 1$ and$e_\phi = 0$, and

$(L4)$ $e_X e_Y = e_{Y \cap Y}$for all $X, Y \in \mathcal{J}$.

It is clear that $\mathcal{J}(P)$ contains every principal right ideal. In this section we consider the following class of semigroups for which $\mathcal{J}(P)$ is equal to the set of principal right ideals.

Definition(3.1.2)[3]: A semigroupP is called a right LCM semigroup if it is left cancellative and the intersection of any two principal right ideals is either empty or another principal right ideal.

Semigroups of this type have gone by other names in the literature. Lawson considers the dual definition (ie, right cancellativesemigroups such that two principal right ideals are either disjoint or intersect in another principal right ideal) and calls these CRM monoids (named for Clifford, Reilly and McAlister). In other works such as , such semigroups are said to satisfy Clifford's condition.

Let P be a right LCM semigroup with identity, and let $U(P)$ denote the invertible elements of P (invertible elements of P are also sometimes called the units of P). Then if we have $p, q \in P$ such that $pP \cap qP = rP$, we see that every element of P which is right multiple of both p and q is also a right multiple of r, and we say that r is a right least common multiple (or right LCM) of p and q. If $rP = sP$, then a short calculation shows that there must exist $u \in U(P)$ such that $ru = s$. Hence, if r is a right LCM of p and q then so is ru for all $u \in U(P)$.Also, if $pP \cap qP = rP$, then there exist$p', q' \in P$ such that $pp' = qq' = r$. This right least common multiple property is the source of the terminology "right LCM".

Let P be a right LCMsemigroup and suppose that we have $p, q \in P$ such that $pP \cap qP = rP$ with $pp' = qq' = r$. Then it is straightforward to show that $p^{-1}qP = p'P$, and so the set of principal right ideals is in fact equal to the set of constructible ideals.

We shall be concerned with groupoids constructed from semigroups. Recall that a groupoid consists of a set $\mathcal{G}$ together with a subset $\mathcal{G}(2) \subset \mathcal{G} \times \mathcal{G}$,called the set ofcomposable pairs, a product map $\mathcal{G}(2) \to \mathcal{G}$ with $(\gamma, \eta) \mapsto \gamma\eta$, and an inverse map from $\mathcal{G}$ to $\mathcal{G}$ with $\gamma \mapsto \gamma^{-1}$such that

(a)$(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$,

(b)If $(\gamma, \eta), (\eta, \nu) \in \mathcal{G}^{(2)}$, then $(\gamma\eta, \nu), (\gamma, \eta\nu) \in \mathcal{G}^{(2)} \mathit{and}\ (\gamma\eta)\nu = \gamma(\eta\nu)$,

(c)$(\gamma, \gamma^{-1}), (\gamma^{-1}, \gamma) \in \mathcal{G}^{(2)}$, *and* $\gamma^{-1} = \eta, \xi\gamma\gamma^{-1}$*for all* η, ξ *with* $(\gamma, \eta), (\eta, \xi) \in \mathcal{G}^{(2)}$.

The set of units of $\mathcal{G}$ is the subset $\mathcal{G}^{(0)}$of elements of the form $\gamma\gamma^{-1}$. The maps $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$and $d : \mathcal{G} \to \mathcal{G}^{(0)}$ given by

$$r(\gamma) = \gamma\gamma^{-1}, \quad d(\gamma) = \gamma^{-1}\gamma$$

are called the range and source maps respectively. It is straightforward to check that $(\gamma, \eta) \in \mathcal{G}^{(2)}$ is equivalent to $r(\eta) = d(\gamma)$.

One thinks of a groupoid$\mathcal{G}$ as a set of "arrows" between elements of $\mathcal{G}^{(0)}$. Given $x \in \mathcal{G}^{(0)}$,let

$$\mathcal{G}^x := r^{-1}(x), \quad \mathcal{G}_x := d^{-1}(x), \quad \mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x),$$

which are thought of, respectively, as the arrows ending at x, the arrows beginning at x, and all the arrows both ending and beginning at x. For all $x \in \mathcal{G}^{(0)}$,$\mathcal{G}_x^x$ is a group withidentity x when given the operations inherited from $\mathcal{G}$, and is called the isotropy group of x. The set Iso$(\mathcal{G}) = \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x$ is called the isotropy group bundle of $\mathcal{G}$. The orbit of $x \in \mathcal{G}^{(0)}$is the set $\mathcal{G}(u) := r(\mathcal{G}_x) = s(\mathcal{G}^x)$.

A topological groupoid is a groupoid which is a topological space where the inverse and product maps are continuous, where we are considering $\mathcal{G}^{(2)}$ with the product topology inherited from $\mathcal{G} \times \mathcal{G}$. Two topological groupoids are said to be isomorphic if there is a homeomorphism between them which preserves the inverse and product operations. A groupoid with topology $\mathcal{G}$ is called ´etale if it is locally compact, second countable, and the maps r and d are local homeomorphisms. These properties imply that $\mathcal{G}^{(0)}$is open in G and that for all x $\in$ $\mathcal{G}^{(0)}$ the spaces $\mathcal{G}^{x}$ and $\mathcal{G}_{x}$ are discrete.

For subsets $S, T \subset \mathcal{G}$, let$ST = \{\gamma\eta \mid \gamma \in S, \eta \in T, d(\gamma) = r(\eta)\}$. A subset $S \subset \mathcal{G}$ of a topological groupoid is called a bisection if the restrictions of r and d to S are both injective. In an ´etale groupoid $\mathcal{G}$, the collection of open bisections forms a basis for the topology of $\mathcal{G}$. If S and T are bisections in an ´etale groupoid, then so is ST.

A subset $U \subset \mathcal{G}^{(0)}$ is called invariant if for all $\gamma \in \mathcal{G}, r(\gamma) \in U$ implies that $s(\gamma) \in U$. A topological groupoid is called minimal if the only nonempty open invariant subset of $\mathcal{G}^{(0)}$ is $\mathcal{G}^{(0)}$. We say that $\mathcal{G}$ is topologically principal if the set of $x \in \mathcal{G}^{(0)}$ for which $\mathcal{G}_{x}^{x} = \{x\}$ is dense. We will say that $\mathcal{G}$ is essentially principal if the interior of Iso $(\mathcal{G})$ is equal to $\mathcal{G}^{(0)}$, and we will say that $\mathcal{G}$ is effective if the interior of Iso $(\mathcal{G}) \setminus \mathcal{G}^{(0)}$ is empty. When $\mathcal{G}$is alocally compact, second countable, Hausdorff, ´etalegroupoid, then

$\mathcal{G}$ topologically principal $\Leftrightarrow$ $\mathcal{G}$essentially principal $\Leftrightarrow$ $\mathcal{G}$ effective,

In a construction, to an ´etalegroupoid$\mathcal{G}$ one can associate C^{*}-algebras$C^{*}(\mathcal{G})$ and$\mathcal{G}_{r}^{*}(\mathcal{G})$, called the C^{*}-algebra of G and the reduced C^{*}-algebra of $\mathcal{G}$ respectively. To build these C^{*}-algebras one starts with$C_{c}(\mathcal{G})$, the continuous compactly supported functions on $\mathcal{G}$, which becomes a complex $*$-algebra when given the convolution product and involution given by

$$f \star g(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2)\, f^{*}(\gamma) = \overline{f\,(\gamma^{-1})}.$$

We note that the fact that $\mathcal{G}$ is étale implies that the sum above is finite. In this work we are not concerned with the specifics, but both $C^{*}(\mathcal{G})$and $C_{r}^{*}(\mathcal{G})$ are completions of $C_{c}(\mathcal{G})$in suitable norms, and $C_{r}^{*}(\mathcal{G})$ is always a quotient of $\mathrm{C}^{*}(\mathcal{G})$. In this section we construct étale groupoids from certain semigroups, and most of our results concern the case when $C^{*}(\mathcal{G}) = C_{r}^{*}(\mathcal{G})$. This happens for instance when $\mathcal{G}$ is amenable.

We will use the following result to deduce properties of the C^{*}-algebras we construct.

Theorem (3:1:3)[3]: Let $\mathcal{G}$ be a second countable locally compact Haus-dorffétale groupoid. Then $C^*(\mathcal{G})$is simple if and only if the following conditions aresatisfied:

i. $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$,
ii. $\mathcal{G}$ is topologically principal, and
iii. $\mathcal{G}$ is minimal.

An étale groupoid is called locally contracting if for every nonempty open subset $U \subset \mathcal{G}^{(0)}$, there exists an open subset $V \subset U$ and an open bisection $S \subset \mathcal{G}$ such that $\bar{V} \subset S^{-1} S$and $SV S^{-1} \subsetneq V$, if $C^*(\mathcal{G})$ is simple and $\mathcal{G}$ is locally contracting, then $C^*(\mathcal{G})$ is purely infinite. We assume knowledge of C^*-algebras, but for the unfamiliar an excellent reference for the undefined terms above.

Recall that a semigroup$S$is called regular if for all $s \in S$ there exists an element $t \in S$ such that $tst = t$ and $sts = s$. Such an element t is often called an inverse of s, though even if S has an identity we need not have $ts = 1$. However, we always have $(ts)^2 = tsts = ts$, that is to say that ts is idempotent. We let $E(S) = \{e \in S \,|e^2 = e\}$ denote the set of all idempotent elements of S. A regular semigroup is called an inverse semigroup if each element shas a unique inverse,denoted s^*.It is a fact that a regular semigroup is an inverse semigroup if and only if elements of $E(S)$ commute, and we note that in this case $E(S)$ is closed under multiplication.

Example(3:1:4)[3]: We give an important and fundamental example of an inverse Isemigroup. Let X be a set. Consider

$$I(X) = \{f: U \to V \mid U, V \subset X, f \text{ is bijective}\}.$$

Then$I(X)$ is an inverse semigroup when given the operation of function composition on the largest domain possible. The inverse of an element $f: U \to V$ is the inverse function $f^* = f^{-1} : V \to U$. One sees that the identity function is the identity for this inverse semigroup, and more generally every idempotent is the identity on some subset.If we have $f, g \in I(X)$ such that the range of f does not intersect the domain of g, then the composition$g \circ f$ on the largest domain possible is equal to the empty function, which acts as a zero element in $I(X)$. It is an important fact in semigroup theory that every inverse semigroup can be embedded into $I(X)$ for some set X–this is known as the Wagner-Preston theorem.

This example demonstrates that many inverse semigroups naturally contain a zero element. Because of this, the two algebraic objects we consider in this section, namely “right LCMsemigroups” and “inverse semigroups” should be thought of as quite different types of objects, as left cancellativity in a right LCMsemigroup eliminates the possibility of a zero element in nontrivial cases.

Now, given a right LCM semigroupP we will construct an inverse semigroup S. We define an equivalence relation $\sim$ on $P \times P$ by saying that$(p,q) \sim (a,b)$ if and only if there exists $u \in U(P)$such that $au = p$and$bu = q$. In other words, the equivalence class of (p,q) consists of all elements of the form $(pu,qu)$with $u \in U(P)$. Denote by $[p,q]$ the equivalence class of (p,q).The following proposition is essentially, though Lawson presents the dual construction for right cancellative left LCMsemigroups.

Proposition (3:1:5)[3]: Let P be a right LCMsemigroup with identity 1_P , and let

$$S := \{[p,q] \mid p,q \in P\} \cup \{0\}. \qquad (1)$$

Then Sbecomes an inverse semigroup with identity $1_S = [1_P, 1_P]$ when given the operation

$$[a,b][c,d] = \begin{cases} ([ab',dc'] \text{ if } cP \cap bP = rP \text{ and } cc' = bb' = r \\ 0 \qquad \text{if} \quad cP \cap bP = \emptyset \end{cases}$$

and $s0 = 0s = 0$ for all $s \in S$. In this case, we have that$[a,b]^* = [b,a]$ and

$$E(S) = \{[a,a] \mid a \in P\} \cup \{0\}.$$

Proof:-

Before we start, we note that although this proof is straightforward it is long and tedious. However, it may be valuable if one wishes to get a feel for right LCMsemigroups.

We first show that the multiplication above is well-defined. Suppose that $[a,b],[c,d] \in S$. Then if $u,v \in U(P)$, we know that $buP = bP$ and $cvP = cP$, and so $[a,b][c,d] = 0$ if andonly if $[au,bu][cv,dv] = 0$. So, suppose that $[a,b][c,d] = [ab',dc']$, where $bP \cap cP = rP$ with$bb' = cc' = r$. Then $buP \cap cvP = bP \cap cP = rP$, and so there exist $b'',c'' \in P$ such that $bub'' = cvc'' = r$. Because $bb' = cc' = r$ and P is left cancellative, we have that $ub'' = b'$ and $vc'' = c'$. Hence

$$[au, bu][cv, dv] = [aub'', dvc''] = [ab', dc'] = [a, b][c, d]$$

and so the multiplication is well-defined.

We now show that the given multiplication is associative. Take $[a, b], [c, d], [e, f] \in S$ and first suppose that $[a, b]([c, d][e, f]) \neq 0$. Then there must be $r_1, d_1, e_1 \in P$ such that $dP \cap eP = r_1P$, $dd_1 = ee_1 = r_1$, and $[c, d][e, f] = [cd_1, fe_1]$. Since we assumed that $[a, b][cd_1, fe_1] = 0$, we now must have that there exist $r_2, b_1, c_1 \in P$ such that $bP \cap cd_1P = r_2P$, $bb_1 = cd_1c_1 = r_2, and$

$$[a, b]([c, d][e, f]) = [ab_1, fe_1c_1] \neq 0.$$

Since $bP \cap cd_1P = \emptyset$, we must have that $bP \cap cP = \emptyset$, so there exists $r_3, b_2, c_2 \in P$ such that $bP \cap cP = r_3P$, $bb_2 = cc_2 = r_3$, and $[a, b][c, d] = [ab_2, dc_2]$. In addition, we have that $r_2P \subset r_3P$, and so there exists q ∈ P such that $r_2 = r_3q$. Now we have that

$$cd_1c_1 = r_2 = r_3q = cc_2q \quad \Rightarrow \quad d_1c_1 = c_2q \qquad (2)$$

and so we have

$$ee_1c_1 = dd_1c_1 = dc_2q \quad \Rightarrow \quad dc_2P \cap eP = \emptyset.$$

Hence $([a, b][c, d])[e, f] = [ab_2, dc_2][e, f] = 0$. Furthermore, there exist $r_4, d_2, e_2 \in P$ such that $dc_2P \cap eP = r_4P, dc_2d_2 = ee_2 = r_4$, and

$$([a, b][c, d])[e, f] = [ab_2, dc_2][e, f] = [ab_2d_2, fe_2].$$

In this case, we have that $r_4P \subset r_1P$, and so there exists $p \in P$ such that $r_4 = r_1p$. Also, similar to (2), we have that $c_2d_2 = d_1p$. If instead we started by insisting that $([a, b][c, d])[e, f] \neq 0$, then a similar argument gives that $[a, b]([c, d][e, f]) \neq 0$. Thus to show associativity we can assume both products are nonzero and that we have elements $b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, r_1, r_2, r_3, r_4, q, p \in P$ such that

$dP \cap eP = r_1P \qquad dd_1 = ee_1 = r_1r_2 = r_3q$

$bP \cap cd_1P = r_2P \qquad bb_1 = cd_1c_1 = r_2 \qquad r4 = r_1p$

$bP \cap cP = r_3P \qquad bb_2 = cc_2 = r3 \qquad d_1c_1 = c_2q$

$dc_2P \cap eP = r_4 P \qquad dc_2d_2 = ee_2 = r_4c_2d_2 = d_1p$

Now, we have that $r_1c_1 = ee_1c_1 = dd_1c_1 = dc_2q$, and so $r_1c_1\ dc_2P \cap eP = r_4P$, meaning that there exists $k_1 \in P$ such that $r_1c_1 = r_4k_1 = r_1pk_1$, and because P is left cancellative we have that $pk_1 = c_1$.

Similarly, $r_3d_2 = bb_2d_2 = cc_2d_2 = cc_1d_1, and\ so\ r_3d_2 \in bP \cap cd_1P = r2P$. This means that there exists $k_2 \in P$ such that $r_3d_2 = r_2k_2 = r_3qk_2$, and since P is left cancellative we have that $d_2 = qk_2$.

We claim that $k_1k_2 = 1 = k_2k_1 = 1_P$, and hence $k_1, k_2 \in U(P)$. We calculate

$$d_1pk_1k_2 = d_1c_1k_2 = c_2qk_2 = c_2d_2 = d_1p \quad \Rightarrow \quad k_1k_2 = 1_P ,$$

$$c_2qk_2k_1 = c_2d_2k_1 = d_1pk_1 = d_1c_1 = c_2q \quad \Rightarrow \quad k_2k_1 = 1_P .$$

Now, $bb_1 = r_2 = r_3q = bb_2q$, and so $b_1 = b_2q$. Similarly, $e_2 = e_1p$. Thus,

$$[a, b]([c, d][e, f\ ]) = [ab_1\ , fe_1c_1] = [ab_2q, fe_1c_1],$$

$$([a, b][c, d])[e, f\ ] = [ab_2d_2, fe_2] = [ab_2d_2, fe_1p],$$

and

$$ab_2d_2k_1 = ab_2qk_2k_1 = ab_2q,\ \ fe_1pk_1 = fe_1c_1.$$

Hence $[a, b]([c, d][e, f\ ]) = ([a, b][c, d])[\ e, f\ ]$ as required.

Suppose now that $[p, q][p, q] = [p, q]$. *Then* $pP \cap qP = rP$ and there exist p', q' such that $pp' = qq' = r$, and $[p, q] = [p, q][p, q] = [pp', qq' = [r, r]$. Hence there exists $u \in U(P)$ such that $p = ru = q$. Hence the only idempotent elements of S are elements of the form $[p, p]$, together with the 0 element. Now suppose that we have $p, q \in P$ such that $[p, p][q, q] = 0$. Then $pP \cap qP = rP$ for some $r \in P$ and there exist p', q' such that $pp' = qq' = r$, and *[p, p][q, q] = [pp', qq'] = [r, r].* It is clear that this is equal to $[q, q][p, p]$, and that $[p, p][q, q] = 0$ if and only if $[q, q][p, p] = 0$. Hence the idempotents of S commute. It is obvious that $[p, q][q, p][p, q] = [p, q]$ and $[q, p][p, q][q, p] = [q, p]$. Hence each element of S has an inverse (0 is the inverse of 0), and so $\mathcal{S}$ is regular. As above, the idempotents of $\mathcal{S}$ commute, hence $\mathcal{S}$ is an inverse semigroup.There is another formulation of the semigroup $\mathcal{S}$ above, considered for example. Consider $I(P\ )$ as in Example (3.1.4). Since P is assumed to be left cancellative, the map

$\lambda_p : P \to pP$ defined by $\lambda_p(q) = pq$ is a bijection, and hence an element of (P). Let $I_l(P)$ denote the inverse semigroup generated by the elements $\{\lambda_p\}_{p\in P}$ inside $I(P)$. This is sometimes called the left inverse hull of P. Then the map from S to $I_l(P)$ given by $[p,q] \mapsto \lambda_p\lambda_q^{-1}$ is an isomorphism.

The main result of this section is an isomorphism between two C^*-algebras, denoted in the sequel $Q(P)$ and $C^*_{tight}(S)$. We begin by defining $Q(P)$. A finite set $F \subset P$ is called a foundation set if for all $p \in P$ there exists $f \in F$ such that f $P \cap pP \neq \emptyset$. The following

Definition (3.1.6)[3]: Let P be a right LCMsemigroup. The boundary quotient of $C^*(P)$, denoted $Q(P)$ is the universal C^*-algebra generated by sets $\{v_p | p \in P\}$ and $\{e_X | X \in \mathcal{J}\}$ subject to the relations (L1)–(L4) in Definition (3.1.1) and

$\prod_{p\in F}(1 - e_pP) = 0$ for all foundation sets $F \subset P$.

Now that we have defined $Q(P)$, we define the second algebra which concerns us. Let A be a C^*-algebra and let S be an inverse semigroup with zero. Then a representation of S is a map $\pi : S \to A$ such that for all $s, t \in S$ we have $\pi(st) = \pi(s)\pi(t), \pi(s^*) = \pi(s)^*$, and $\pi(0) = 0$. The universal C^*-algebra of S, denoted $C^*_u(S)$, is the universal C^*-algebra generated by one element for each element of S such that the standard map $\pi_u : S \to C^*_u(S)$ is a representation. Note that this implies that $\pi_u(s)$ is a partial isometry for each $s \in S$.

Let S be an inverse semigroup,let $\pi : S \to A$ be a representation, and let D_π denotsethe C^*-subalgebra of A generated by $\pi(E(S))$. Since $E(S)$ is commutative, D_π must be a commutative C^*-algebra. The set

$$\mathfrak{B}_\pi = \{e \in D_\pi \mid e^2 = e\}$$

is a Boolean algebra when given the operations

$$e \wedge f = ef \quad e \vee f = e + f - ef \quad \neg e = 1 - e$$

We will recall a subclass of representations defined . Let S be an inverse semigroup and let $F \subset Z \subset E(S)$. We say that F is a cover of Z if for every nonzero $\mathcal{Z} \in Z$ there is $f \in Z$ such that $f\mathcal{Z} = 0$. If $x \in E(S)$ and F is a cover for $\{\mathcal{Y} \in E(S) \mid \mathcal{Y}x = x\}$, then we say that F is a cover of x. For finite sets $X, Y \subset E(S)$,let

$$E(S)^{X,Y} = \{e \in E(S) | ex = e \text{ for all } x \in X \text{ and } e\mathcal{Y} = 0 \text{ for all } \mathcal{Y} \in Y\}$$

A representation $\pi : S \to A$ is called tight if for every pair of finite sets $X, Y \subset E(S)$ and every finite cover Z of $E(S)^{X,Y}$, we have

$$\bigvee_{z \in Z} \pi(z) = \prod_{x \in X} \pi(x) \prod_{\mathcal{Y} \in Y} (1 - \pi(y)) .$$

The tight C^*-algebra of S, denoted $C^*_{tight}(S)$,is the universal C^*-algebra generated by one element for each element of S subject to the relations saying that the standard map $\pi_t : S \to C^*_{tight}(S)$is a tight representation.

Lemma (3.1.7)[3]: Let P be a right LCM semigroup, and let $\{v_p \mid p \in P\}$and$\{e_X \mid X \in \mathcal{J}(P)\}$be as in Definition (3.1.1). Then for all $p, q \in P$ we have

$$v_p^* v_q = \begin{cases} v_{p'} v_{q'}^* & \text{if } pP \cap qP = rP \text{ and } r = pp' = qq' \\ 0 & \text{if } pP \cap qP = \phi. \end{cases}$$

Proof:-

Suppose that $pP \cap qP = rP$ and $r = pp' = qq'$. Then

$$v_p^* v_q = (v_p^* e_{pP})(e_{qP} v_q) = v_p^* e_{rP} v_q$$

$$= v_p^* v_r v_r^* v_q = v_p^* (v_p v_{p'})(v_q v_{q'})^* v_q$$

$$= (v_p^* v_p) v_{p'} v_{q'}^* (v_{q'}^* v_{p'}) = v_{p'} v_{q'}^* .$$

The second equality above shows that $v_p^* v_q = 0$ if $pP \cap qP = \emptyset$.

Lemma (3.1.8)[3]: Let P be a right LCMsemigroup with identity, and let S be as in (1). Then the map $\pi : S \to \mathcal{Q}(P)$ defined by

$$\pi([p, q]) = v_p v_q^*$$

$$\pi(0) = 0$$

is a tight representation of S.

Proof:-

It is easy to see from Lemma (3.1.7) and (1) that the map π above is a representation of S. Now, suppose that $F$is a foundation set. Then, by de Morgan's laws in a Booleanalgebra we have

$$0 = \prod_{f\in F} (1 - e_{fP}) = \bigwedge_{f\in F}(\neg\pi([f,f])) = \neg\left(\bigvee_{f\in F}\pi([f,f])\right)$$
$$= 1 - \bigvee_{f\in F}\pi([f,f])$$

and so $\bigvee_{f\in F}\pi([f,f]) = 1$.To show that π is tight we need only check that for every $[p,p] \in E(S)$ and every finite cover$Z$ of $[p,p]$, we have theequality

$$\bigvee_{z\in Z}\pi(z) = e_{pP}.$$

So, take $p \in P$ and suppose that Z is a finite cover of $[p,p]$. For$Z$ to be a finite cover of $[p,p]$, we must have that for all $z \in Z, zP \subset pP$ and whenever we have $q \in P$ such that $qP \subset pP$, there exists $z \in Z$ such that $qP \cap zP = \emptyset$. The first condition implies that for all $z \in Z$, there exists $az \in P$ such that $z = pa_z$. We claim that $\{a_z \mid z \in Z\}$ is a foundation set. Indeed, for every $q \in P$, we have that $pqP \subset pP$, and so there exists $z \in Z$ such that $zP \cap pqP = pa_zP \cap pqP = \emptyset$, and so $a_zP \cap qP = \emptyset$. Hence $1 = \bigvee_{z\in Z} e_{a_z}P$. Hence we have

$$e_{pP} = v_p 1_{v_p^*} = v_p\left(\bigvee_{z\in Z} e_{a_z}P\right)v_p^* = \bigvee_{z\in Z} v_p e_{a_z}P\, v_p^* = \bigvee_{z\in Z} e_{pa_z}P$$
$$= \bigvee_{z\in Z}\pi(z)$$

Lemma (3.1.9)[3]: Let P be a right LCMsemigroup with identity, let S be as in (1), and let πbe any tight representation of S. Then for every foundation set $F \subset P$,

$$\prod_{f\in F} (1 - \pi([f,f]))$$

Proof:-

Let $F \subset P$ be a foundation set. Again, by de Morgan’s laws in a Boolean algebra,we have

$$\prod_{f\in F}(1-\pi([f,f]))=\bigwedge_{f\in F}(\neg\pi([f,f]))=\neg\left(\bigvee_{f\in F}\pi([f,f])\right)$$
$$=1-\bigvee_{f\in F}\pi([f,f])$$

Hence we will be done if we can show that $\bigvee_{f\in F}\pi_t([f,f])=1$. *Let* $X=\{1_S\}$and$Y=\emptyset$.

Then

$$E(S)^{X,Y}=\{e\in E(S)\mid e1_S=e\}=E(S)$$

and since $F$is a foundation set, $Z=\{[f,f]\mid f\in F\}$ is a finite cover for $E(S)$. Thus wehave

$$\bigvee_{z\in Z}\pi(z)=\prod_{x\in X}\pi(x)\prod_{y\in Y}\pi(1-\pi(y))\Rightarrow\bigvee_{f\in F}\pi([f,f])=\pi(1)$$
$$=1.$$

The above two lemmas combine to give the main result of this section.

Theorem (3.1.10)[3]: Let P be a right LCMsemigroup with identity, and let S be as in (1).

Then there is an isomorphism $\Phi:\mathcal{Q}(P)\to C^*_{tight}(S)$such that $\Phi(v_pv_q^*)=\pi_t([p,q])$for all $p,q\in P$.

Proof:-

By Lemma (3.1.8) and the fact that $C^*_{\mathrm{tight}}(S)$, there exists a $*$-homomorphism $\Phi_\pi: C^*_{tight}(S)\to\mathcal{Q}(P)$such that $\Phi_\pi\circ\pi_t([p,q])=v_pv_q^*$. Conversely, by Lemma (3.1.9) and the universal property of $\mathcal{Q}(P)$, there exists a $*$- homomorphism $\Phi:\mathcal{Q}(P)\to C^*_{tight}(S)$such that $\Phi(v_pv_q^*)=\pi_t([p,q])$. Hence $\Phi\circ\Phi_\pi$ is the identity on $\mathcal{Q}(P)$, $\Phi_\pi\circ\Phi$ is the identity on $C^*_{\mathrm{tight}}(S)$, and so Φ is an isomorphism.

One of the consequences of Theorem (3.1.10) is that $\mathcal{Q}(P)$ is isomorphic to the C^*-algebra of an ètale groupoid, and we may therefore study $\mathcal{Q}(P)$ by studying the groupoid.

Section (3.2):Proerties of $\mathcal{Q}(\mathrm{P})$ and Examples:-

We now review the construction of the tight groupoid of an inverse semigroup. Let S be an inverse semigroup. There is a natural partial order on S given by $s \leq t$ if and only if $s = ts^*s$. If $e, f \in E(S), e \leq f$ if and only if $ef = e$. This partial order is best understood in the context of the inverse semigroup$I(X)$ – here we have $\varphi \leq \psi$ if and only if ψ extends φ as a function.

In this order, each pair $e, f \in E(S)$ has a unique greatest lower bound, namely their product ef . Hence, with the order above $E(S)$ is a semilattice. If S has an identity 1_S , then it is the unique maximal element of E(S), and if S has a zero element it is the unique minimal element of E(S).

A filter in $E(S)$ is a proper subset $\xi \subset E(S)$ which is downwards directed in the sensen that$e, f \in \xi$ implies that $ef \in \xi$, and upwards closed in the sense that if $e \in \xi, f \in E(S)$and$e \leq f$ implies that $f \in \xi$. If a subset $\xi \subset E(S)$ is proper and downwards directed it iscalled a filter base, and the set

$$\xi = \{e \in E(S) \mid f \leq e \text{ for some } f \in \xi\},$$

called the upwards closure of ξ, is a filter. A filter is called an ultrafilter if it is not properly contained in another filter. Ultrafilters always exist by Zorn's Lemma.

We let $\hat{E}_0(S)$ denote the set of filters in $E(S)$. This set has a natural topology given by seeing it as a subspace of $\{0,1\}^{E(S)}$with the product topology. There is a convenient basis for this topology: for finite sets $X, Y \subset E(S)$, let

$$U(X, Y) = \{\xi \in \hat{E}_0(S) | x \in \xi \text{ } for\ all\ x \in X, \mathcal{Y} \notin \xi \text{ } for\ all\ \mathcal{Y} \in Y\}.$$

These sets are open and closed and generate the subspace topology on $\hat{E}_0(S)$ as X and Yrange over all finite subsets of E(S).Let $\hat{E}_\infty(S)$ denote the subspace of ultrafilters. We shall denote by $\hat{E}_{tight}(S)$the closure of $\hat{E}_\infty(S) in \hat{E}_0(S)$and call this the space of tight filters.

If X is a topological space and S is an inverse semigroup, recall that an action of S on X is a pair $(\{D_e\}_{e \in E(S)}, \{\theta_s\}_{s \in S})$such that each $D_e \subset X$is open, the union of the D_e coincides with X, each map $\theta_s : D_{s^*s} \to D_{ss^*}$ is continuous and bijective, and for all $s, t \in S$ we have $\theta_s \circ \theta_t = \theta_{st}$, where composition is on the largest domain possible. These properties imply that $\theta_{s^*} = \theta_s^{-1}$ and so each θ_s is actually a homeomorphism.

There is a canonical way to construct an ètale groupoid from an inverse semigroup action. Let $\theta = (\{D_e\}_{e\in E(S)}, \{\theta_s\}_{s\in S})$ be an action of an inverse semigroup S on a space X, and let

$$S \times_\theta X := \{(s,x) \in S \times X \mid x \in D_{s^*s}\}.$$

For $(s,x),(t,\mathcal{Y}) \in S \times_\theta X$ we write $(s,x) \sim (t,\mathcal{Y})$ if $x = \mathcal{Y}$ and there exists $e \in E(S)$ such that $x \in D_e$ and $se = te$. It is straightforward to check that $\sim$ is an equivalence relation. We write $[s,x]$ for the equivalence class of (s,x) and we let $\mathcal{G}(S,X,\theta)$ denote the set of all such equivalence classes. This set becomes a groupoid when given the operations

$$[s,x]^{-1} = [s^*,\theta_s(x)], \quad r([s,x]) = \theta_s(x), \quad d([s,x]) = x, \quad [t,\theta_s(x)][s,x] = [ts,x].$$

For $s \in S$ and U an open subset of D_{s^*s}, let

$$\Theta(s,U) = \{[s,x] \mid x \in U\}.$$

As s and U vary, these sets form a basis for an ètale topology on $\mathcal{G}(S,X,\theta)$; with this topology $\mathcal{G}(S,X,\theta)$ is called the groupoid of germs of the action θ. In this topology, $\Theta(s,U)$is an open bisection, and if U is in addition closed (resp. compact), $\Theta(s,U)$is a clopen (resp.compact open) bisection. It is easy to see that the orbit of a point $x \in X$under the groupoid of germs is the set $\{\theta_s(x) \mid s \in S\}$.

An inverse semigroup acts naturally on $\hat{E}_{tight}(S)$.Let $D_e = \{\xi \in \hat{E}_{tight}(S) \mid e \in \xi\}$, and define $\theta_s : D_{s^*s} \to D_{ss^*}$ by

$$\theta_s(\xi) = \overline{s\xi s^*} = \overline{\{ses^* \mid e \in \xi\}}.$$

The groupoid of germs of this action is denoted

$$\mathcal{G}_{tight}(S) := \mathcal{G}(S, \hat{E}_{tight}, \theta)$$

and is called the tight groupoid of S. The C^*-algebra of $\mathcal{G}$tight(S) is naturally isomorphic $C^*_{tight}(S)$; in particular the map $\pi : S \to C^*(\mathcal{G}_{tight\mathcal{G}})$ given by $\pi(s) = \chi_{\Theta(s,D_{s^*s}}$ is a tight representation.

If P is a right LCMsemigroup and S is as in (1), then it is easy to see that $\mathcal{J}(\mathrm{P})$ and $\mathrm{E}(\mathrm{S})$ are isomorphic as semilattices, with the isomorphism being the map $pP \longmapsto [p,p]$. The spaces of filters and ultrafilters in $\mathcal{J}(\mathrm{P})$ were considered in previous study of C^*-algebras associated to P, though filters were termed directed and hereditary subsets of $\mathcal{J}(\mathrm{P})$while the ultrafilters were called maximal directed hereditary subsets. In what follows, we will consider the elements of $\hat{E}_{tight}(S)$ as tight filters in $\mathcal{J}(\mathrm{P})$, and will shorten $D_{[p,p]}$toD_p

(noting that $D_p = D_{pu}$ for all $u \in U(P)$. We will then have,

$$\theta_{[p,q]} : D_q \to D_p$$

$$\theta_{[p,q]}(\xi) = \{p(q^{-1}rP) \mid rP \in \xi\},$$

for any$p, q \in P$ and $\xi \in \hat{E}_{tight}(S)$ with $qP \in \xi$.

We now characterize when $Q(\mathrm{P})$ is simple using the fact that it is isomorphic to the C^*-algebra of the ètale groupoid $\mathcal{G}_{\text{tight}}(\mathrm{S})$. To use the characterization of Theorem (3.1.3),(we need conditions which guarantee that$\mathcal{G}_{\text{tight}}(\mathrm{S})$ is Hausdorff, minimal, and topologically principal. We begin with Hausdorff. As noted ,$\mathcal{G}_{\text{tight}}(\mathrm{S})$is Hausdorff if S is E^*-unitary, that is, for all $s \in S$ and$e \in E(S) \setminus \{0\}, e \leq \mathrm{s}$ implies that $s \in E(S)$. Norling notes that if P is a right LCMsemigroup with identity and S is as in (1), then P is cancellative if and only if S is E^*-unitary. Thus if P is cancellative , $\mathcal{G}_{\text{tight}}(\mathrm{S})$is Hausdorff. However,by we can do a little bit better. We are more precise below, but what we prove is essentially that $\mathcal{G}_{\text{tight}}(\mathrm{S})$is Hausdorff if and only if the counterexamples to right cancellativity in P have a "finite cover" in some sense.

For $p, q \in P$, $let P_{p,q} = \{b \in P \mid pb = qb\}. If\ P_{p,q}$is nonempty then it is a right ideal of P , and in this case we say that p and q meet. We introduce the following condition that P may satisfy.

(H) For every $p, q \in P$ which meet, there is a finite set $F \subset P$ such that $pf = qf$ for all $\mathrm{f} \in \mathrm{F}$ and whenever we have$b \in P$ such that$pb = qb$, there is an $f \in F$ such that$fP \cap bP \neq \emptyset$.

One sees that (H) is weaker than right cancellativity, since if P is right cancellative p only meets $q$when$p = q$, and in this case $P_{p,q} = P$ and the finite set $F = \{1_P\}$ verifies (H).

Proposition(3.2.1)[3]: Let P be a right LCMsemigroup with identity, and let S be as in (1). Then $\mathcal{G}_{\text{tight}}(S)$is Hausdorff if and only if P satisfies condition (H).

Proof:-

We shall show that condition (H) is equivalent to the set

$$J_{[p,q]} = \{rP \mid [r,r] \leq [p,q]\}$$

either being empty or having a finite cover for all $p,q \in P$. If we do this, then the conclusion will follow from. Notice that if we have $p,q,r \in P$ such that$[r,r] \leq [p,q]$, then$[p,q][r,r] = [r,r]$. If $rP \cap qP = kP$ and $rr' = qq' = k$, then we obtain that $[pq', rr'] = [r,r]$implying that $r' \in U(P)$. This means that $rP = kP$, and so we may assume (perhaps by rechoosing q′) that $[pq', r] = [r,r]$, and so $pq' = r = qq'$. Thus, for each element $rP \in J_{[p,q]}$, there is an element $p_r := q'$such that $pp_r = r = qp_r$.

First, assume that for all $p,q \in P$ the set$J_{[p,q]}$ is empty or has a finite cover. Suppose that $p,q \in P$ meet, that is, there exists $b \in P$such that $pb = qb$. Then

$$[p,q][qb,qb] = [pb,qb] = [qb,qb]$$

and *so* $qbP = pbP \in J_{[p,q]}$, meaning that $J_{[p,q]}$ is not empty. Hence there is a finite set $F \subset P$ such that $f\,P \in J_{[p,q]}$ for all $f \in F$ and for all $rP \in J_{[p,q]}$there exists$f \in F$such that $rP \cap fP = \emptyset$. By the above, there exists $p_f \in P$ such that $pp_f = f = qp_f$.We now see that the finite set $\{p_f\}_{f\in F}$ verifies (H), because if we have d such that $pd = qd$, there is $f \in F$ such that$fP \cap pdP \neq \emptyset$, which implies that $p_fP \cap dP = \emptyset$.

Conversely, suppose Psatisfies condition(H). Ifp,qdo not meet, then the above discussion shows that $J_{[p,q]}$is empty. If p, q do meet, let F be the finite set guaranteed by (H), and consider the finite set

$$pF = qF = \{pf \mid f \in F\}.$$

If $rP \in J_{[p,q]}$, then again by the above there exists r'such that $pr' = qr' = r$. So, there exists $f \in F$ such that$fP \cap r'P \neq \emptyset$, which implies that$pfP \cap pr'P = pfP \cap rP \neq \emptyset$ and we are done.

Now that we have addressed when $\mathcal{G}_{\text{tight}}(\mathsf{S})$is Hausdorff, we turn to minimality.

Lemma (3.2.2)[3]: Let P be a right LCMsemigroup with identity, and let S be as in (1).Then$\mathcal{G}_{\text{tight}}(\mathsf{S})$is minimal.

Proof:-

We will show that for every ultrafilterξ and open set $U(X, Y) \subset \hat{E}_{tight}(S)$,there is a $[p, q] \in S$ such that $\theta_{[p,q]}(\xi) \in U(X, Y)$. The set U(X, Y) is open, so it contains an ultrafilter η which must have the property that $xP \in \eta$ for all $x \in X$ and, for all $\mathcal{Y} \in Y$, there exists $p_{\mathcal{Y}} \in P$ such that $p_{\mathcal{Y}}P \in \eta$ and $p_{\mathcal{Y}}P \cap \mathcal{Y}P = \emptyset$. Because η is closed under intersection, there must be $r \in P$ such that

$$\left(\bigcap_{x \in X} xP\right) \cap \left(\bigcap_{\mathcal{Y} \in Y} p_{\mathcal{Y}}P\right) = rP.$$

Now, $\zeta := \theta_{[r, 1_P]}(\xi)$ is an ultrafilter which contains rP. Since ζ is upwards directed,xP, $p_{\mathcal{Y}}$P ∈ ζ for all$x \in$ X,$\mathcal{Y} \in$ Y , and so $\zeta \in U(X, Y)$. Thus, the orbit of every ultrafilter is dense. Each open set contains an ultrafilter, so the only nonempty open invariant subset of $\hat{E}_{tight}(S)$ *is* $\hat{E}_{tight}(S)$, and so $\mathcal{G}(S, \hat{E}_{tight}, \theta)$ is minimal.

Lastly, we discuss conditions which guarantee that $\mathcal{G}_{\text{tihgt}}(\mathsf{S})$is topologically principal. We use the following concepts. For an action $(\{D_e\}_{e \in E(S)}, \{\alpha_s\}_{s \in S})$ of an inverse semigroup S on a locally compact Hausdorff space X,and s ∈ S, let

$$F_s = \{x \in X \mid \alpha_s(x) = x\}$$

and call this the set of fixed points for s. Also let

$$\mathrm{TF}_s = \{x \in \mathrm{X} \mid \text{there exists } e \in \mathrm{E(S)} \text{such that } 0 \neq e \leq s \text{ and} \mathrm{x} \in \mathrm{D_e} = \bigcup_{\mathrm{e \leq s}} \mathrm{D_e} \quad (3)$$

and call this the set of trivially fixed points for s, the groupoid of germs $\mathcal{G}(S, X, \alpha)$ is Hausdorff if and only if TF_s is closed in $\mathrm{D}_{s^*s}$for all s ∈ S.

Definition (3.2.3)[3]: An action $(\{D_e\}_{e\in E(S)}, \{\alpha_s\}_{s\in S})$ of an inverse semigroup S on a locally compact Hausdorff space X is said to be topologically free if the interior of F_s is contained in TF_s for all $s \in S$.

We note that if x is trivially fixed by some s with $e \le s$ and $x \in D_e$, we have $\alpha_e(x) = \alpha_s(\alpha_e(x)) = \alpha_{se}(x) = \alpha_e(x) = x$, so x is fixed, that is to say that

$$TF_s \subset F_s \text{for all} s \in S.$$

Also, by (3), TF_s is open and so is contained in the interior of F_s . Hence stating that α is topologically free is equivalent to saying that $TF_s = F^{\circ}{}_s$

Theorem(3.2.4)[3]: An action $(\{D_e\}_{e\in E(S)}, \{\alpha_s\}_{s\in S})$ of an inverse semigroup S on a locally compact Hausdorff space X is topologically free if and only if $\mathcal{G}(S, X, \alpha)$ is essentially principal.

We now show that we can characterize when the canonical action of S on $\widehat{E}_{\text{tight}}(S)$ is topologically free by considering the behaviour of a subsemigroup of P which generalizesone originally considered .

Proposition(3.2.5)[3]: Let P be a right LCMsemigroup with identity. Then the set

$$P_0 := \{p \in P \mid pP \cap qP = \emptyset \; for \; all \; q \in P\} \qquad (4)$$

is a subsemigroup of P which contains the identity. Furthermore,

(i) $pq \in P_0$ implies that $p, q \in P_0$, and
(ii) $p, q \in P_0$ and $pP \cap qP = rP$ implies that $r \in P_0$.

Proof:-

The details of this proof are almost identical. For instance, take $p, q \in P_0$, and $r \in P$. A short calculation shows that $pqP \cap rP = p(qP \cap p^{-1}(pP \cap rP))$, and since $p, q \in P_0$ this must be nonempty, whence $pq \in P_0$.

Definition(3.2.6)[3]: Let P be a right LCMsemigroup with identity. Then the subsemigroup $P_0 \subset P$ from (4) is called the core of P .

The subsemigroup P_0 was defined when P is quasi-lattice ordered, though it still makes sense in our context. We note that for all $p \in P_0$, the singleton $\{p\}$is a foundation set, and so v_p is a unitary in $\mathcal{G}(P)$. We also note that $U(P) \subset P_0$, though this inclusion may be proper(1)

Now let

$$S_0 := \{[p,q] \in S \mid p,q \in P_0\}.$$

We will also call this the core ofS. For $a,b,c,d \in P_0$, we see that $[a,b][c,d] = [ab',dc']$where $bP \cap cP = rP$ and $bb' = cc' = r$ implies that $r \in P_0$ and hence$b',c' \in P_0$. Thus, S_0 is an inverse subsemigroup of S. We note that S_0 does not contain the zero element.

Proposition(3.2.7)[3]: Let P be a right LCMsemigroup with identity which satisfies condition (H), and let S be as in (1). Then $\mathcal{G}_{\mathrm{tihgt}}(S)$ is essentially principal if and only if for all $s \in S_0$, each interior fixed point of s is trivially fixed.

Proof:-

The "only if " direction is obvious. S_0, assume that for each $s \in S_0$, each interior fixed point of s is trivially fixed, and suppose that $\mathcal{G}_{\mathrm{tihgt}}(S)$ is not essentially principal. Then there must exist $[p,q] \in S$ such that $TF_{[p,q]}($ $F^{\circ}_{[p,q]}$. Since P satisfies condition (H), $TF_{[p,q]}$ is closed in D_q and so $F^{\circ}_{[p,q]} \setminus TF_{[p,q]}$ is open, so we can find an open set$U \subset F^{\circ}_{[p,q]} \setminus T\,F_{[p,q]}$. Since U is open, it must contain an ultrafilter $\xi \in U$. If we are able to find a $b \in P$ such that $bP \in \xi$ and $[1_P,b][p,q][b,1_P] \in S_0$, then D_b would contain ξ, and *so* $\theta_{[1_p,b]}(\xi)$ is fixed by $[1_P,b][p,q][b,1_P]$. By assumption, it must be in $TF[1_P,b][p,q][b,1_P]$, so we can find a nonzero idempotent $[r,r]$ such that $[r,r] \le [1_P,b][p,q][b,1_P]$ with $rP \in \theta_{[1_P,b]}(\xi)$. A short calculation shows that this implies that $[b,1_P][r,r][1_P,b] = [br,br]$is a nonzero idempotent less than $[p,q]$. Furthermore, $rP \in \theta[1_P,b](\xi)$ implies that $brP \in \xi$, and so $\xi \in D_{br}$ which would imply that ξ is trivially fixed by $[p,q]$, a contradiction. So finding such $a\ b \in P$ would prove theproposition.

So, supose that $[1_P,b][p,q][b,1_P] \notin S_0$ for all $b \in P$such that $bP \in \xi$, and fix an element $bP \in \xi$. Because ξ is fixed by $[p,q]$, we have that $p(q^{-1}(bP)) \in \xi$. Hence, thereexist$b_1,q_1,r_1 \in P$ such that $bP \cap qP = r_1P$and $bb_1 = qq_1 = r_1$, and so we have $p(q^{-1}(bP) = pq_1P \in \xi$. Since $bP, pq_1P \in \xi$, there exist $p_1, b_2, r_2 \in P$ such that $pq_1P \cap bP = r_2P$and $pq_1p_1 = bb_2 = r_2$. Upon redefining $a := b_2$and $c := b_1p_1$, a short calculation shows that $[1_P,b][p,q][b,1_P] = [a,c]$.

Since we are assuming that this is not an element of S_0 , we must have that one of a or c is not an element of P_0. Suppose for the moment that $a \notin P_0$, which means there exists $\mathcal{Z} \in P$ such that $\mathcal{Z}P \cap aP = \emptyset$. Letting $\xi_{bP} = \theta_{[bz,1]}(\xi)$, we see that $bP \in \xi_{bP}$ (because $bzP \in \xi_{bP}$ andξ_{bP} is upwards closed). However, ξ_{bP} is not fixed by$[p,q]$ (whether $\theta_{[p,q]}(\xi_{bP})$ is defined or not) because any filter containing bP and fixed by $[\mathrm{p},\mathrm{q}]$ must contain $r_2P = baP$by the same reasoning as above, and since ξ_{bP} is closed under intersections this would mean that it contains $baP \cap b\mathcal{Z}P$, which is empty by assumption. In a similar fashion, we can construct an ultrafilter ξ_{bP} containing bP not fixed by $[p,q]$ if we instead assume that $c \notin P_0$. Hence we have constructed a net $\{\xi_b\}_{bP\in\xi}$ of ultrafilters each of which contains bP but none of which is fixed by $[\mathrm{p},\mathrm{q}]$. This net converges to ξ, and so U contains a point in this net, which is impossible since$\mathrm{U}$ is fixed by $[p,q]$. Hence, we are forced to conclude that we can find $b \in P$such that $bP \in \xi$ and $[1_P , b][p,q][b,1_P] \in S_0$, and so we are done.

We would like an algebraic condition on P which guarantees $\mathcal{G}_{\mathrm{tihgt}}$ (S) is essentially principal. To do this we recall some terminology .

Definition(3.2.8)[3]: Let S be an inverse semigroup, let $s \in S$. If $e \in E(S)$ is an idempotent with $e \leq s^*s$ it is said that

(i) e is fixed by s if $e \leq s$ $(ie, se = e)$, and

(ii) e is weakly fixed by s if $sfs^*f \neq 0$ for every nonzero idempotent $\mathrm{f} \leq \mathrm{e}$. We translate this terminology to our situation in the following lemma.

Lemma (3.2.9):-

Let P be a rightLCMsemigroup with identity, and take $p,q,r \in P$ such that $r = qk$for some$k \in P$. Then

(a) $[r,r] = [qk,qk]$ is fixed by $[p,q]$ if and only if $pk = r = qk$, and

(b) $[r,r] = [qk,qk]$ is weakly fixed by $[p,q]$ if and only if for all $a \in P$;

$qkaP \cap pkaP = \emptyset;$

Also, if $[r,r]$ is fixed by $[p,q]$,it is weakly fixed by $[p,q]$.

Proof:-

Note that in the statement of the lemma, we only consider r's which are right multiples of q because this is exactly what it means to have $[r,r] \leq [p,q]^*[p,q]$.

A. If $[r,r]$ is fixed by $[p,q]$,we have $[p,q][r,r] = [r,r]$ and so$[r,r] = [pk,r]$.*Thus* $pk = r = qk$. On the other hand, if $pk = qk = r$, then $[p,q][r,r] = [p,q][qk,qk] = [pk,qk] = [r,r]$,and so$[r,r]$ is fixed by $[p,q]$.

B. Suppose that $[r,r]$ is weakly fixed by $[p,q]$. An idempotent is below $[r,r]$ if and only if it is of the form $[ra,ra]$. Thus $[r,r]$ being weakly fixed by $[p,q]$ implies that for every $a \in P$,

$$0 \neq [p,q][ra,ra][q,p][ra,ra] = [p,q][qka,qka][q,p][ra,ra] \\ = [pka,pka][ra,ra].$$

Hence $raP \cap pkaP \neq \emptyset$. Conversely, if $raP \cap pkaP \neq \emptyset$for all$a \in P$, then with the same calculation above we see that for all $a \in P$the product $[p,q][ra,ra][q,p][ra,ra] \neq 0$,and so $[r,r]$ is weakly fixed by $[p,q]$.

As in the general situation, it is clear that each fixed idempotent is weakly fixed. The following statement is implicit , though we spell it out here for emphasis.

Lemma(3.2.10)[3]: Let S be an inverse semigroup, and suppose that either

(i) every tight filter in $E(S)$ is an ultrafilter, or
(ii) for every $s \in S$, the set $J_s = \{e \in E(S) \mid e \leq s\}$ has a finite cover.

Then for each $s \in S, F^{\circ}{}_s \subset TFs$ if and only if for all e weakly fixed by s, there is a finite cover for J_e consisting of fixed idempotents.

The following is a rephrasing of the above result for our situation.

Lemma (3.2.11)[3]: Let P be a right LCMsemigroup with identity which satisfies condition (H), or such that the only tight filters in $J(P)$ are ultrafilters. Then $F^{\circ}{}_{[p,q]} \subset TF_{[p,q]}$ if and only if $[p,q]$ satisfies the following condition:

(EP) for all $[qk,qk]$ weakly fixed by $[p,q]$, there exists a foundation set $F \subset P$ such that $qkf = pkf$ for all $f \in F$.

We note that any idempotent $[p,p]$ satisfies (EP) using the foundation set $\{1_P\}$.

One notices that we still fall slightly short of being able to apply Theorem(3.1.3), because we have only given conditions under which $\mathcal{G}_{\mathrm{tihgt}}(\mathrm{S})$ is essentially principal, not topologically principal, these two notions are equivalent when $\mathcal{G}_{tihgt}(S)$is Hausdorff and second countable. We are considering only countable semigroups$P$, and one easily sees that this guarantees that $\mathcal{G}_{\mathrm{tihgt}}(\mathrm{S})$is second countable.

We now come to the main result of this section.

Theorem(3.2.12)[3]: Let P be a right LCM semigroup with identity which satisfies condition (H), let P_0 be the core of P , and let S be as in (1). Then $\mathcal{G}(P)$ is simple if and only if

(i) $\mathcal{G}(P) \cong C^*(\mathcal{G}_{tihgt}(S)\ )$, and

(ii)for all $p, q \in P_0$, the element $[p, q]$ satisfies condition (EP).

Proof:-

By Proposition (3.2.1), $\mathcal{G}_{\mathrm{tihgt}}(\mathrm{S})$is Hausdorff, so we can apply Theorem(3.2.3). By Lemma (3.2.2), $\mathcal{G}_{\mathrm{tihgt}}(\mathrm{S})$is always minimal.Proposition (3.2.7), and Lemma (3.2.11), $\mathcal{G}_{\mathrm{tihgt}}(\mathrm{S})$ is topologically principal if and only if we have (ii) above. The result follows.

Definition(3.2.13)[3]: An inverse semigroup$S$ is called locally contracting if for every nonzero $e \in E(S)$there exists $s \in S$ and a finite set $F = \{f_0, f_1, \dots, f_n\} \subset E(S) \setminus \{0\}$ with $n \geq 0$ such that for all $0 \leq i \leq 1$we have

i. $f_i \leq es^*s$,
ii. there exists $f \in F$such that $sf_is^*f = 0$, and
iii. $f_0sf_i = 0$.

As one might guess from the name, if S is locally contracting then $\mathcal{G}_{\mathrm{tihgt}}(\mathrm{S})$is locally contracting .

Lemma (3.2.14)[3]: Let P be a right LCM semigroup with identity and let S be as in (1). Then S is locally contracting if and only if $P = P_0$.

Proof:-

The "only if " direction is trivial, because if $P = P_0$ we could not satisfy part iii of Definition (3.2.13), as the product of the two idempotents $f_0$and sf_is^* could not be zero.

For the "if " direction, we suppose that $P \neq P_0$, and hence we can find $p, q \in P$such that$pP \cap qP = \emptyset$, we will be done if for every $r \in P$ we can find $a \in P$ and $f_0, f_1 \in P$ such that $f_0P \subset f_1P \subset rP$, $af_1 P \subset f_1P$and $[f_0, f_0][a, 1_P][f_1 , f_1] = 0$.

To this end, let

$$a = f_1 = rp \qquad f_0 = rprq.$$

Then clearly$f_0P \subset f_1P \subset rP$, and $af_1P = rprpP \subset f_1P$. We also have that

$$\begin{aligned}[f_0 , f_0][a, 1_P][f_1, f_1] &= [rprq, rprq][rp, 1_P][rp, rp] \\ &= [rprq, rprq][rprp, rp]\end{aligned}$$

and since $rprqP \cap rprpP = \emptyset$ by assumption, this product is zero.

Theorem (3.2.15)[3]: Let P be a right LCM semigroup with identity which satisfies condition (H), and suppose that $\mathcal{Q}(P)$ is simple. Then $\mathcal{Q}(P)$ is purely infinite if and only if $\mathcal{G}_{\text{tihgt}}(\mathsf{S})$is not the trivial (one-point) groupoid.

Proof:-

The "only if " direction is clear, because if $\mathcal{G}_{\text{tihgt}}(\mathsf{S})$is one point, its C^*-algebra is isomorphic to $\mathbb{C}$, which is not purely infinite.

On the other hand, if $\mathcal{G}_{\text{tihgt}}(\mathsf{S})$is not the one-point groupoid, we have two cases. If$\widehat{\mathsf{E}}_{\text{tight}}(\mathsf{S})$is one point then there are no points with trivial isotropy, and so $\mathcal{G}_{\text{tihgt}}(\mathsf{S})$is nottopologically principal, contradicting Theorem(3.1.3). If $\widehat{\mathsf{E}}_{\text{tight}}(\mathsf{S})$has more than one point,then there are at least two distinct ultrafilters in $\mathcal{J}(\mathsf{P})$.Hence we can find and an ultrafilterξ such that $pP \notin \xi$, and since ξ is an ultrafilter there must be $qP \in \xi$ such that $pP \cap qP = \emptyset$. Thus neither p nor q is in P_0, and so $\mathsf{P} = \mathsf{P}_0$ implying that S is locallycontracting by Lemma (3.2.14). Hence, $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_{\text{tight}}(\mathsf{S})) = \mathcal{Q}(\mathsf{P})$ is purely infinite.

Hence, in the presence of simplicity,pure infiniteness of $\mathcal{Q}(\mathsf{P})$ follows automatically in all but the most trivial cases.

Let X be a finite set, and letX^{n}denote the set of words of length n in X, with X^0 consisting of a single empty word, $\emptyset$. Let

$$X^* = \bigcup_{n\leq 0} X^n \quad .$$

Then X^* becomes a semigroup with the operation of concatenation: if $\alpha = \alpha_1\alpha_2 \dots \alpha_k$ and $\beta = \beta_1\beta_2 \dots \beta_l$ then their product is $\alpha\beta = \alpha_1\alpha_2 \dots \alpha_k\beta_1\beta_2 \dots \beta l$, while the empty word is the identity. If $\alpha \in X^n$, we write $|\alpha| = n$ and say that the length of α is n. The core of this semigroup is $X^* = U\,(X^*) = \{\emptyset\}$. If we have $\alpha, \beta \in X^*$, then $\alpha X^* = \beta X^*$if and only if $\alpha = \beta$. Furthermore, X^* is left cancellative, and either $\alpha X^* \cap \beta X^* = \emptyset$ or one is included in the other, so X^* is right LCM.

From the relations (L1)-(L4) it follows easily that $C^*(X^*)$ is the universal unital C^*- algebra generated by isometries $v_1, \dots, v_{|X|}$ such that

$$v_i^* v_j = \begin{cases} 1 & if\, i = j \\ 0 & otherwise \end{cases}$$

that is, $C^*(X^*)$ is isomorphic to the Toeplitz algebra $\mathcal{TO}_{|X|}$. Furthermore, the set $X = X^1 \subset X^*$ is a foundation set, and so in $\mathcal{Q}(X^*)$ we have

$$0 = \prod_{x\in X}(1 - e_x X^*) = 1 - \bigvee_{x\in X} v_x v_x^* = 1 - \sum_{x\in X} v_x v_x^*$$

$$\Rightarrow \sum_{x\in X} v_x v_x^* = 1$$

and since the Cuntz algebra $\mathcal{O}_{|X|}$is the universal C^*-algebra generated by such elements, there is a surjective $*$-homomorphism from $\mathcal{O}_{|X|}$ to $\mathcal{Q}(X^*)$ which must be an isomorphism because $\mathcal{O}_{|X|}$is simple.

Principal left ideals of X^* are either disjoint or comparable by inclusion, and hence ultrafilters are maximal well-ordered subsets of $\mathcal{J}(X^*)$. The space of ultrafilters can be identified with the compact space Σ_{X} of right-infinite words in X via the homeomorphism

$$\alpha \in \Sigma_X \mapsto \{X^*, \alpha_1 X^*, \alpha_1\alpha_2 X^*, \alpha_1\alpha_2\alpha_3 X^*, \dots\} \in \hat{E}_{tight}(S).$$

Here every tight filter is an ultrafilter. Because $X^*$is right cancellative (in fact, it can be embedded in the free group on $|X|$elements) it satisfies condition (H).The inverse semigroup S from (1) is known in the literature as the polycyclic monoid on$|X|$ generators. For all $\alpha \in X^*$, the idempotent $[\alpha, \alpha]$ is weakly fixed by $[\emptyset, \emptyset]$, and $[\emptyset, \emptyset]$ trivially satisfies condition .

In Lie builds upon his earlier work and makes a comprehensive study of the C^*-algebras of semigroups which may be embedded into groups. There the semilattice of constructible ideals $\mathcal{J}(\mathrm{P})$is considered, though itis not always equal to the set of all principal right ideals. There, the set of filters is denoted Σ, the set of ultrafilters is denoted $\Sigma_{\max}$ and its closure is denoted $\partial\Sigma$. an inverse semigroup analogous to our S from (1) is defined; this inverse semigroup acts on Σ. The groupoid of germs of this action is the universal groupoid for S, and the C^*-algebra of its reduction to $\partial\Sigma$ (that is to say, $\mathcal{G}_{\text{tight}}$ (S)), is identified as a suitable boundary quotient. Our only contribution to the literature for this situation would seem to be the isomorphism between this boundary quotient and the one defined .We do note that our result Proposition (3.2.7), and indeed it seems both were inspired .

The following is a construction considered .Let $U$and A be semigroups with identities 1_U and 1_A respectively and suppose there exist maps $A \times U \to U$ given by $(a, u) \mapsto a \cdot u$, and $A \times U \to A$ given by $(a, u) \mapsto a|u$ which satisfy

$$(\mathrm{ZS1})\, 1_A \cdot u = u \qquad (ZS5)\ a|1_U = a$$

$$(ZS2)\, (ab) \cdot u = a \cdot (b \cdot u) \qquad (ZS6)\ a|uv = a|u\,|v$$

$$(ZS3)\ a \cdot 1_U = 1_U \qquad (ZS7)\ 1_A\,|_u = 1_A$$

$$(ZS4)\ a \cdot (uv) = (a \cdot u)(a|_u \cdot v) \quad (ZS8)\ ab|_u = a|_{b \cdot u}\, b|_u$$

for all $u, v \in U$ and $a, b \in A$. Then $U \times A$ becomes a semigroup with identity $(1_U, 1_U)$ when given the operation

$$(u, a)(v, b) = (u(a \cdot v), a|_v b).$$

This is called the Zappa-Szèp product of U and A, and is denoted $\mathrm{U} \bowtie \mathrm{A}$. If in addition to the above, we have that

(i) $U$and A are both left cancellative,

(ii) U is right LCM,

(iii) $\mathcal{J}(A)$ is totally ordered by inclusion, and

(iv) the map $u \mapsto a \cdot u$is a bijection on U for each $a \in A$,

then $U \bowtie A$ is a right LCMsemigroup as well.

By Theorem (3.1.10), the boundary quotient $\mathcal{Q}(\mathsf{U} \bowtie \mathsf{A})$ defined is isomorphic to the C^*-algebra of an ètale groupoid $\mathcal{G}_{tight}(S)$(where S is as in (1)) whose unit space is homeomorphic to the space of tight filters in$\mathcal{J}(\mathsf{U} \bowtie \mathsf{A})$.To use Theorems (3.2.12) and (3.2.15) requires that we know the nature of the core of our semigroup, and in this case the core has an easily describable form. Firstly, each element $(1_{\mathrm{U}}, \mathsf{a})$ is in the core of $U \bowtie A$. Furthermore, for $u, v \in U$ and $a, b \in A$, we have

$$(u, a)U \bowtie A \cap (v, b)U \bowtie A = \emptyset \quad \Leftrightarrow \quad uU \cap vU = \emptyset.$$

Therefore, $\{(u, a)\}$ is a one-point foundation set in $U \bowtie A$ if and only if $\{u\}$ is a foundation set in U . Hence the core of $U \bowtie A$ is

$$(U \bowtie A)_0 = \{(u, a) \in U \bowtie A \mid u \in U_0\}.$$

By Proposition (3.2.5) this is a subsemigroup of $U \bowtie A$, and in particular, we have that for all $u \in U_0$, $a \cdot u \in U_0$ for all $a \in A$. Thus we are justified writing $(U \bowtie A)_0 = U_0 \bowtie A$. Without having more information about U and A we cannot say much more, though in the sequel we consider a specific example for which we can.

We close with an example which is a specific case of the situation. The conclusions we come to in this section are known, and combine the results. Indeed, generalizing the results implicit in combining (which was a preliminary version was a major inspiration for this work. We present what follows to illustrate our results in the context of this interesting example.

Let X be a finite set, let G be a group, and let X^*. Suppose that we have a length-preserving action of G on $\mathsf{X}^*$with$(g, \alpha) \mapsto g \cdot \alpha$, such that for all $g \in G, x \in X$ there exists a unique element of G, denoted $g|x$, such that for all $\alpha \in X^*$

$$g(x\alpha) = (g \cdot x)(g|_x \cdot \alpha).$$

In this case, the pair (G, X) is called a self-similar group. Nekrashevy7ch associates a C^*-algebra to (G, X),denoted $\mathcal{O}_{\mathrm{G,X}}$, which is the universal C^*-

algebra generated by a set of isometries $\{s_x\}_{x\in X}$ and a unitary representation $\{u_g\}_{g\in G}$ satisfying

(i) $s_x^* s_y = 0 \; if \, x = y$,

(ii) $\sum_{x\in X} s_x s_x^* = 1$,

(iii) $u_g s_x = s_{g\cdot x} u_g|_x$.

If one defines, for $\alpha \in X^*$ and $g \in G$,

$$g|_\alpha := g|_{\alpha_1}|_{\alpha_2} \cdots |_{\alpha_{|\alpha|}}$$

then the free semigroup X^* and the group G (viewed as a semigroup) together with the maps $(g, \alpha) \longmapsto g \cdot \alpha$ and $(g, \alpha) \mapsto g|_\alpha$ satisfy the condition, and so we may form the Zappa-Sz´ep product $X^* \bowtie G$. Furthermore, are easily seen to hold, so $X^* \bowtie G$ is a right LCM semigroup. The semilattice of principal right ideals of $X^* \bowtie G$ is isomorphic to that of X^* via the map $(\alpha, g)X^* \bowtie G \longmapsto \alpha X^*$, and so one may identify $\mathcal{J}(X^* \bowtie G)$ with $\mathcal{J}(X^*)$, with inclusion order given by

$$\alpha X^* \subset \beta X^* \Leftrightarrow \beta \text{ is a prefix of } \alpha.$$

as before, principal right ideals are either disjoint or are comparable by inclusion. Hence as before, the unit space of the tight groupoid is homeomorphic to Σ_X , which is homeomorphic to the Cantor set. $\mathcal{Q}(X^* \bowtie G) \cong \mathcal{O}_{G,X}$.

In general, $X^* \bowtie G$ is not cancellative, although it is embeddable into a group if and only if it is cancellative . We recall the following concepts. Let $\alpha \in X^*$, and $g \in G$. Then α is said to be strongly fixed by g if $g \cdot \alpha = \alpha$ and $g|_\alpha = 1_G$, and we let

$SF_g = \{\alpha \in X^* |_\alpha \text{strongly fixed by} g\}$.

Of course, if $\alpha \in SF_g$, then $\alpha\gamma \in SF_g$ for every word $\gamma \in X^*$. We will say that a strongly fixed word α is minimal by g if $\alpha \in SF_g$ and no prefix of α is strongly fixed by g, and will denote this set by

$MSF_g = \{\alpha \in X^* |_\alpha \text{minimal strongly fixed}\} \subset SF_g$.

The self-similar group (G,X) is said to be pseudo-free if SF_g is empty for all $g \neq 1_G$. A short calculation shows that $X^* \bowtie G$ is cancellative if and only if *(G, X)* is pseudo-free .As mentioned earlier, our condition (H) is slightly weaker than right cancellativity, so one hopes that we can give conditions on (G,X) which are equivalent to (H).*If* $(\alpha,g),(\beta,h) \in X^* \bowtie G$ meet, then there exists $(\gamma,k) \in X^* \bowtie G$such that

$$\begin{aligned}(\alpha,g)(\gamma,k) &= (\beta,h)(\gamma,k)\\ (\alpha(g\cdot\gamma),g|_\gamma k) &= (\beta(h\cdot\gamma),h|_\gamma k),\end{aligned}$$

and since the action of G on X^* fixes lengths, we must have that $\alpha = \beta, g\cdot\gamma = h\cdot\gamma$and $g|_\gamma = h|_\gamma$. After noticing that the definition of a self-similar group implies that $v = k^{-1}|_{k\cdot v}$for all $k \in G, v \in X^*$, this is easily seen to imply thatγ is strongly fixed by $g^{-1}h$. Hence, $(X^* \bowtie G)(\alpha,g),(\alpha,h)$ is only nonempty when $g^{-1}h$ has a strongly fixed word,and

$$(X^* \bowtie G)_{(\alpha,g),(\alpha,h)} = \{(\gamma,k)\,|k \in G, \gamma \in SF_{g^{-1}h}\} = (X^* \bowtie G)_{(\emptyset,g),(\emptyset,h)}.$$

Thus, $X^* \bowtie G$ will satisfy condition (H) if we can show that for all $g \in G\backslash\{1_G\}$, there exists a finite set $F \subset SF_g$ such that for all $\alpha \in SF_g$ there exists $\beta \in F$ such that $\beta X^* \cap \alpha X^* \neq \emptyset$.The following result appears gives conditions for when this occurs.

Lemma (3.2.16)[3]: Let (G,X)be a self-similar group.Then $X^* \bowtie G$satisfies condition (H) if and only if, for all $g \in G \setminus \{1_G\}$,the set MSF_g is finite.

Proof:-

One easily sees that if MSF_g is finite, it will satisfy the above condition, as each strongly fixed word must have a prefix which is minimal. Conversely, if such a finite F exists, and MSF_g is infinite, find a $\gamma \in MSF_g$ such that $|\gamma| > \max_{\alpha\in F}|\alpha|$. Then there must exist $\alpha \in F$ such that $\alpha X^* \cap \gamma X^* \neq \emptyset$, and since $|\gamma| > |\alpha|$, α must be a prefix of γ. But α is strongly fixed, and γ is supposed to be minimal, so we have a contradiction. Hence MSF_g is finite.

We now address condition (EP). In this example, the core of $X^* \bowtie G$ coincides with the group of units of $X^* \bowtie G$, which is

$$(X^* \bowtie G)_0 = U(X^* \bowtie G) = \{(\emptyset,g)\,|_g \in G\}$$

and can be identified with the group G. The inverse semigroup (1) has been previouslyconstructed, and generalizing their results there was an inspiration for this work.Let

$$S_{G,X} = \{(\alpha, g, \beta) \mid \alpha, \beta \in X^*, g \in G\}.$$

This set becomes an inverse semigroup when given the operation

$$(\alpha, g, \beta)(\gamma, h, \nu) = \begin{cases} (\alpha(g \cdot \gamma'), g|_{\gamma'} h, \nu), & if\ \gamma = \beta\gamma', \\ (\alpha, g(h^{-1}|_{\beta'})^{-1}, \nu(h^{-1} \cdot \beta')), & if\ \beta = \gamma\beta', \\ 0 & otherwise \end{cases}$$

With

$$(\alpha, g, \beta)^* = (\beta, g^{-1}, \alpha).$$

Then the map from our S to$S_{G,X}$given by

$$[(\alpha, g), (\beta, h)] \longmapsto (\alpha, gh^{-1}, \beta)$$

is an isomorphism of inverse semigroups, so from now on we will use this identification to discuss condition (EP). We note at this point that it follows,$C^*_{tight}(S_{G,X}) \cong \mathcal{O}_{G,X}$, and so in this case our Theorem 3.7 is already known.

An element of $S_{G,X}$is an idempotent if and only if it is of the form $(\alpha, 1_G, \alpha)$.Identifying the core with G, we see that an idempotent $(\alpha, 1_G, \alpha)$is weakly fixed by $g \in G$ if and only if, for all $\gamma \in X^*$, $(g \cdot \alpha)\gamma X^* \cap \alpha\gamma X^* \neq \emptyset$. By length considerations, this is equivalent tosaying that $g \cdot \alpha \neq \alpha$ and for all $\gamma \in X^*$, $g|_\alpha \cdot \gamma = \gamma$. If the action of G on $X^*$is faithful (which is to say that for all $g \in G \setminus \{1_G\}$ there exists $\alpha \in X^*$ such that $g \cdot \alpha \neq \alpha$), then this is equivalent to saying that α is strongly fixed by g. So, in the presence of faithfulness, $(\alpha, 1_G, \alpha)$weakly fixed by $(\emptyset, g, \emptyset) \Leftrightarrow \alpha$ strongly fixed by $g \Leftrightarrow (\alpha, 1_G, \alpha)$ fixed by $(\emptyset, g, \emptyset)$.

Hence we have the following.

Lemma(3.2.17)[3]: Let (G, X)be a faithful self-similar group, and let $g \in G$. Then $(\emptyset, g, \emptyset)$satisfies condition (EP).

We now come to the following result on self-similar groups. We note that it is not original to this work .

Theorem(3.2.18)[3]: Let (G,X) be a faithful self-similar group, suppose G is amenable, and suppose that for all $g \in G \setminus \{1_G\}, MSF_g$ is finite. Then $\mathcal{O}_{G,X} \cong \mathcal{Q}(X^* \bowtie G)$ is nuclear simple, and purely infinite.

Proof:-

Let S be as in (1) for the semigroup $X^* \bowtie G$. Because MSF_g is finite for all $g \in G \setminus \{1_G\}, X^* \bowtie G$satisfies condition (H) by Lemma (3.2.16). Because $\mathcal{G}$is amenable, we may applingto get that $\mathcal{Q}(X^* \bowtie G)$is nuclear.Since nuclearity passes to quotients, this implies that $C_r^*\left(\mathcal{G}_{\text{tight}}(S)\right)$is nuclear.Thus,$\mathcal{G}_{\text{tight}}(S)$is amenable and so $C_r^*(\mathcal{G}_{tight}(S)) \cong C^*(\mathcal{G}_{tight}(S)) \cong \mathcal{Q}(X^* \bowtie G)$.By Lemma (3.2.17), every element of S_0 satisfies (EP).To concludethat $\mathcal{Q}(X^* \bowtie G)$is simple. Since we are assuming that $|X| > 1$, $X^* \bowtie G \neq (X^* \bowtie G)_0$implying that $\mathcal{G}_{\text{tight}}(S)$is not the trivialgroupoid, and so by Theorem (3.2.15) we have that $\mathcal{Q}(X^* \bowtie G)$ is purely infinite.

Example (3.2.19)[3]: The Odometer and Modified Odometer

We will give two examples of faithful self-similar groups. For the first, let $X = \{0,1\}$, *let* $\mathbb{Z} = \langle \mathcal{Z} \rangle$ be the group of integers with identity e written multiplicatively. The 2-odometer is the self-similar group $(\mathbb{Z}, X)$ determined by

$$\mathcal{Z} \cdot 0 = 1 \qquad \mathcal{Z}|_0 = e$$

$$\mathcal{Z} \cdot 1 = 0 \qquad \mathcal{Z}|_1 = \mathcal{Z}.$$

If one views a word $\alpha \in X^*$ as a binary number (written backwards), then $z \cdot \alpha$ is the same as 1 added to the binary number for α, truncated to the length of α if needed. If such truncation is not needed, $\mathcal{Z}|_\alpha = e$, but if truncation is needed, $\mathcal{Z}|_\alpha = \mathcal{Z}$. This self-similar group is faithful and pseudo-free. Hence $(\mathbb{Z}, X)$satisfies the hypotheses of Theorem (3.2.18), and so $\mathcal{Q}(X^* \bowtie \mathbb{Z})$ is nuclear, simple, and purely infinite. In fact, this C^*-algebra was shown to be isomorphic to the C^*-algebra $\mathcal{Q}_2$, and there we that it is nuclear, simple, and purelyinfinite. We showed that this C^*-algebra isisomorphic to a partial crossed product of the continuous functions on the Cantor set by the Baumslag-SolitargroupB.

Since pseudo-freeness is stronger than what is needed to imply condition (H),we give a modified version of this example which is not pseudo-free but whose Zappa-Szèp product does satisfy condition (H). To this end, let $X_B = \{0, 1, B\}$, and let $\mathbb{Z}$ be written multiplicatively as before. Define

$$\mathcal{Z} \cdot 0 = 1 \qquad \mathcal{Z}|_0 = e$$

$$\mathcal{Z} \cdot 1 = 0 \qquad \mathcal{Z}|_1 = \mathcal{Z}.$$

$$\mathcal{Z} \cdot B = B \qquad \mathcal{Z}|_B = e.$$

One notices that the first two lines above are the same as in the previous example, but we have added a new symbol B which is fixed by every group element; in fact, the word B is strongly fixed by each group element. If one is wondering why we are calling this symbol "B", one could think of it as a Brick wall past which no group element can travel or, if one pictures the odometer as acting like a car odometer, one could think of it as a Broken digit. In any case, our new self-similar group $(X_B, \mathbb{Z})$ is not pseudo-free.

If $\alpha \in MSF_{\mathcal{Z}^m}$ with $m > 0$, then one quickly sees that $\alpha = \beta B$ for some $\beta \in \{0, 1\}^*$, such that $\mathcal{Z}^m \cdot \beta = \beta$. Due to the description of the action as adding in binary, one sees that for a given $\beta \in \{0, 1\}^*$, this equation is satisfied if and only if $k2^{|\beta|} = m$ for some $k > 0$. Hence for a fixed m, only words β of length less than $log_2(m)$ could possibly satisfy $\mathcal{Z}^m \cdot \beta = \beta$.There are only finitely many such words, so for all $m > 0$ the set $MSF_{\mathcal{Z}^m}$is finite. The case $m < 0$ is similar. Hence$X_B^* \bowtie \mathbb{Z}$ satisfies condtion (H).Direct application of Theorem (3.2.18) gives that$\mathcal{Q}(X_B^* \bowtie \mathbb{Z})$is nuclear, simple, and purely infinite.

CHAPTER 4

Irreversible Algebraic Dynamical Systems forC^*-Algebeas

we discuss the structure of the core subalgebra, which turns out to be closely related to generalized Bunce-Deddens algebras in the sense of Orfanos. We also construct discrete product systems of Hilbert bimodules for irreversible algebraic dynamical systems which allow us to view the associated C^*-algebras as Cuntz-Nica-Pimsner algebras. Besides, we prove a decomposition theorem for semigroup crossed products of unitalC^*-algebras by semidirect products of discrete, left cancellative monoids.

Section (4.1): Irreversible Algebraic Dynamical Systems and the Dual in the Commutative Case:-

Let G be a countable discrete group and $(\xi_g)_{g\in G}$denote the standard orthonormal basis of the Hilbert space $\ell^2(G)$. Suppose φ is an injective group endomorphism of G. Then $S_\varphi \xi_g = \xi_{\varphi(g)}$ defines an isometry on $\ell^2(G)$. For$g \in G$, let U_g denote the canonical unitary on $\ell^2(G)$ given by left translation. Then $S_\varphi U_g = U_\varphi(g)\, S_\varphi$ holds for all $g \in G$. This leads to the C^*algebra $\mathcal{O}_r[\varphi]$ generated by the isometry S_φ and the unitaries $(U_g)_{g\in G.}$ A natural object to study within this context is a universal model for $\mathcal{O}_r[\varphi]$, which is a C^*algebra $\mathcal{O}[\varphi] = C^*\left(\left\{s_\varphi\, , \left(u_g\right)_{g\in G} \middle| \mathcal{R}\right\}\right)$generated by an isometry s_φ and unitaries ug satisfying a suitable set of relations $\mathcal{R}$.

Suppose$\varphi$is a group automorphism of G which generates an effective $\mathbb{Z}$-action on G. Then$C^*\left(s_\varphi\, , \left(u_g\right)_{g\in G}\right)$is the crossed product $C_r^*(G) \rtimes_\alpha \mathbb{Z}$, where $\alpha(u_g\,) = u_{\phi(g)}$. It is well-known that this crossed product is canonically isomorphic to the reduced group C^*-algebra of the semidirect product $G \rtimes_\varphi \mathbb{Z}$. Hence, the universal model for $\mathcal{O}_r[\varphi]$ is given by the full group C^*-algebra of $G \rtimes_\varphi \mathbb{Z}$, provided that G is amenable. The structure of these C^*-algebhas already been studied extensively.In stark contrast, the situation for an injective, but non-surjective group endomorphism φ has started to receive more attention in the recent past. The most elementaryexamples of such endomorphisms are $\times\, 2\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ and the one-sided shift on $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}\, for\, n \ge 2$.Restricting to the case where G is amenable and $G/\varphi(G)$ is finite, Ilan Hir- shberg introduced a universal C^*-algebraic model $\mathcal{O}[\varphi]$ for $\mathcal{O}_r[\varphi]$. He showed that the core $\mathcal{F} \subset \mathcal{O}[\varphi]$, which is the fixed pointalgebra under the canonical gauge action, is simple if $(\varphi^n\,(G))_{n\in N}$ sepa- rates the points in G, that is, $\bigcup_{n\in\mathbb{N}} \varphi^n(G) =$

$\{1_G\}$. Using simplicity of $\mathcal{F}$, he concluded that $\mathcal{F}$ is the crossed product of a natural commutative subalgebra $\mathcal{D}$, called the diagonal, by G. Assuming that the family ofsubgroups $(\varphi^n(G))_{n\in N}$ separates the points in G and consists of normal subgroups of G, Hirshberg established that $\mathcal{O}[\varphi]$is simple and therefore iso- morphic to $\mathcal{O}_r[\varphi]$. Additionally, he computed the K-theory of $\mathcal{O}[\varphi]$ basedon the K-theory of $\mathcal{F}$ and the Pimsner-Voiculescu six-term exact sequencefor $\times n: \mathbb{Z} \longrightarrow \mathbb{Z}, n \geq 2$, the shifton $\bigoplus_{\mathbb{N}} H$, where H is a finite group, and

$\varphi : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}, a \longmapsto bab, b \longmapsto aba$, where a, b denote the standard generators of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

A decade later, Felipe Vieira extended Hirshberg's results to the case where G is amenable and $(\varphi^n(G))_{n\in N}$separates the points in G. His approach used techniques for semigroup crossed products as well as partial group crossed products. One remarkable outcome of his work is the connection to semigroupC^*-algebras for left cancellative semigroups as introduced by Xin LieIf G is amenable, $(\varphi^n(G))_{n\in N}$separates the points in G, and $G/\varphi(G)$ is infinite, then $\mathcal{O}[\phi]$ is canonically isomorphic to the full semigroup C^*-algebra of $G \rtimes_\varphi \mathbb{N}$. Furthermore, Vieira showed that this is the same as the reduced semigroupC^*-algebra of $G \rtimes_\varphi \mathbb{N}$.

At about the same time, Joachim Cuntz and Anatoly Vershik examined the case where G is abelian, $G/\varphi(G)$is finite, and $(\varphi^n(G))_{n\in\mathbb{N}}$separates the points in G.They proved that $\mathcal{O}[\varphi]$is a UCT Kirchberg algebra and provided a general method to compute the K-theory of $\mathcal{O}[\varphi]$.In addition, they found that the spectrum of the diagonal $\mathcal{D}$ is a compact abelian group G_φ, which can be interpreted as a completion of G with respect to φ. Another interesting outcome is the fact that $\mathcal{F} \cong C(G_\varphi) \rtimes G$is also isomorphic to $C(\hat{G}) \rtimes \hat{G}_\varphi$.

Summarizing the current status, it is fair to say that a lot is known about the C^*-algebras $\mathcal{O}[\varphi], \mathcal{F}$ and $\mathcal{D}$ associated to a single injective, non-surjective group endomorphism $\varphi$of a countably infinite, discrete group G. Indeed, in many cases we are able, at least in principle, to compute the K-theory for $\mathcal{O}[\varphi]$, which is known to be a complete invariant due to the celebrated classification theorem by EberhardKirchberg and Christopher N. Phillips. Thus, by computing the K-theory of $\mathcal{O}[\varphi]$, we can recover the information on the dynamical system (G,φ) that is encoded in $\mathcal{O}[\varphi]$ It is therefore natural to ask whether analogous results hold for similar dynamical systems involving more than one transformation.

To motivate this question, let us mention an important example which showcases some interesting phenomena for such dynamical systems. Hillel Furstenberg proved the following result, which applies for instance to $\times 2, \times 3 \colon \mathbb{T} \longrightarrow \mathbb{T}$, the Pontryagin dual of $\times 2, \times 3 \colon \mathbb{Z} \longrightarrow \mathbb{Z}$, Every closed subset of $\mathbb{T}$, which is invariant under the action of a non-lacunarysubsemigroup of $\mathbb{Z}^{\times}$, is either finite or equals $\mathbb{T}$.

Coming back to $\times p, \times q : \mathbb{T} \longrightarrow \mathbb{T}$for relatively prime integers $p, q \geq 2$, it is natural to ask: What are the essential features of this dynamical systems? By Pontryagin duality, it corresponds to $\times p$, $\times q : \mathbb{Z} \longrightarrow \mathbb{Z}$. The condition that p and q are relatively prime is mirrored both by $p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}\; and\; p\mathbb{Z} \cap q\mathbb{Z} = pq\mathbb{Z}$. These simple facts led Joachim Cuntz and Anatoly Vershikto define the notion of independence for pairs of commuting injective group endomorphisms φ and ψ of a discrete abelian group G with the restriction that $G/\varphi(G)$ and $G/\psi(G)$ be finite, φ and ψ are said to be independent if $\varphi(G) \cap \psi(G) = \varphi\psi(G)$.It is shown that independence is equivalent to $\varphi(G) + \psi(G) = G$ as well as to the statement that the inclusion$\varphi(G) \hookrightarrow G$ induces an isomorphism $\varphi(G)/(\varphi(G) \cap \psi(G)) \cong G/\psi(G)$.

In this Secation, we will extend the notion of independence to the general case of two commuting injective group endomorphisms$\varphi$and ψ of a discrete group G. In particular, we show that the last equivalence still holds if we onlyask for a bijection$\varphi(G)/(\varphi(G) \cap \psi(G)) \longrightarrow G/\psi(G)$, see Proposition (4.1.1). But $\varphi(G) \cap \psi(G) = \varphi\psi(G)$ turns out to be weaker than $\varphi(G)\psi(G) = G$, where $\varphi(G)\psi(G) = \{\varphi(g)\psi(g') \mid g, g' \in G\}$, see Example (4.1.9). We will therefore differentiate between independence and what we call strong independence, see Definition(4.1.3). An equivalent characterisation of independence can be given in terms of the isometries$S_{\varphi}, S_{\psi} \in \ell^2(G)$: The commuting en-domorphisms φ and ψ are independent if and only if $S_{\varphi}^* S_{\psi} = S_{\psi} S_{\varphi}^*$holds.

With this notion of independence for commuting injective group endomor- phisms of discrete groups at our disposal, we can think of $\times 2, \times 3 : \mathbb{Z} \longrightarrow \mathbb{Z}$in an abstract way as a dynamical system (G, P, θ) given by

(A) a countably infinite, discrete group G with unit 1_G ,

(B) a countably generated, free abelianmonoid$P$with unit 1_P ,

(C) a P–actionθ on G by injective group endomorphisms for which $\theta_p$and θ_q are independent if and only if p and q are relatively prime.

We will refer to triples (G, P, θ) satisfying the three requirements stated above as irreversible algebraic dynamical systems. The term irreversible is chosen because $\theta_p \in \mathrm{Aut}(G)$ implies $p = 1_P$, and algebraic emphasizes the contrast to topological dynamical systems, since the imposed conditions arepurely algebraic. More specifically, such dynamical systems can be regarded as irreversible analogues of algebraic dynamical systems as introduced by Klaus Schmidt, and the references therein.

We specialise to the case where the group G is abelian. Using Pontryagin duality and some fact about annihilators, we arrive at a notion of independence for commuting surjective group endomorphisms of an arbitrary group, see Definition (4.1.18).This allows us to describe commutative irreversible algebraic dynamical systems (G, P, θ) entirely in terms of their dual models $(\hat{G}, P, \hat{\theta})$, see Proposition (4.1.19). It is precisely this description which represents the close connection irreversible algebraic dynamical systems and irreversible$*$-commuting dynamical systems, for more information. Is devoted to the construction and study of a universal C^*- algebra $\mathcal{O}[G, P, \theta]$ associated to each irreversible algebraic dynamical system (G, P, θ). This C^*-algebra is a direct generalisation of the C^*-algebra $\mathcal{O}[\varphi]$ that appeared and we show that the structural properties of $\mathcal{O}[G, P, \theta]$ are in good accordance with the ones that have been found for $\mathcal{O}[\varphi]$. More precisely, we prove that the spectrum G_θ of the (commutative) diagonal subalgebra $\mathcal{D}$ of $\mathcal{O}[G, P, \theta]$ can be interpretedas a completion of G with respect to θ if (G, P, θ) is minimal in the sense the$\bigcap_{p\in P} \theta_p(G) = \{1_G\}$ see Lemm (4.2.7).is a direct extension .

The C^*-algebra $\mathcal{O}[G, P, \theta]$is then identified with the semigroup crossed product $\mathcal{D} \rtimes (G \rtimes_\theta P)$, where $(g, p). d = u_g s_p d(u_g s_p)^*$, see Proposition(4.2.14). Using the decomposition theorem for crossed productsbysemidirect products of monoids provided, the isomorphism between $\mathcal{O}[G, P, \theta]$and $\mathcal{D} \rtimes (G \rtimes_\theta P)$ yields an isomorphism of $\mathcal{F}$ and $C(G_\theta) \rtimes_\tau G$, where $g. d = u_g d u_g^*$, see Corollary (4.2.19).

As a next step, we show that minimality of (G, P, θ) and amenability of the G-action $\hat{\tau}$ on G_θ are sufficient for simplicity and pure infiniteness of $\mathcal{O}[G, P, \theta]$, see Theorem (4.2.22). The general idea of the proof of this result goes back, but the technical details are more involved compared to the singly generated case. But with this result at hands, we get that minimality of (G, P, θ) and amenability of $\hat{\tau}$ imply that $\mathcal{O}[G, P, \theta]$is a UCT Kirchberg algebra, hence classifiable by K-theory due, see Corollary (4.2.24). Unfortunately, the computation of the K-theory of $\mathcal{O}[G, P, \theta]$,beyond the case of a single group endomorphismfor which this has been accomplished, is a hard problem, at least with the techniques currently available.

In Section (2.2), we restrict our focus to the case where $G / \theta_p(G)$ is finite for all $p \in P$. We find that, in case G is amenable and (G, P, θ) is minimal, the core $\mathcal{F}$ is a generalised Bunce-Deddens algebra ,see Proposition (4.2.30). In this case, $\mathcal{F}$ is classified by its Elliott invariant due to a combination of results ,see Corollary (4.2.31). In addition, we find an intriguing chain of isomorphisms$\mathcal{F} \cong C(G_\theta) \rtimes_\tau G \cong C(\hat{G}) \rtimes_{\bar{\tau}} \hat{G}_\theta$ in the case where (G, P, θ) is minimal and G is commutative, see Corollary(4.2.32) The corresponding result for the case of a single group endomorphism was established. Section(1.1) provides an alternative approach to the C^*-algebra $O[G, P, \theta]$ as the Cuntz-Nica-Pimsner algebra of a discrete product systems of Hilbertbimodules naturally associated to (G, P, θ), see Theorem(4.2.45). Discrete prod- uct systems form a generalisation of the original construction introduced by MihaiPimsner for a single Hilbert bimodule. We refer for more information on the subject. One interesting aspect is that the product system $\mathcal{X}$ associated to(G, P, θ) comes with a canonical system of orthonormal bases on its fibres $\mathcal{X}_p$, obtained by choosing a transversal for $G / \theta_p(G)$, see Proposition (4.2.38)..

A particular advantage of realizing $\mathcal{O}[G, P, \theta]$ as the Cuntz-Nica-Pimsner algebra of the product system $\mathcal{X}$ is that it has a natural Toeplitz extension, called the Nica-Toeplitz algebra. We show that the Nica-Toeplitz algebra associated to an irreversible algebraic dynamical system (G, P, θ) is canonically isomorphic to the (full) semigroupC^*-algebra $C^*(G \rtimes_\theta P)$ in the sense of Xin Lie. In fact, we will prove this in a more general context where P may be an arbitrary right LCM semigroup in the sense. Moreover, this C^*-algebra coincides with$\mathcal{O}[G, P, \theta]$ for irreversible algebraic dynamical systems of infinite type (G, P, θ), that is, $G / \theta_p(G)$ is infinite for all $p \neq 1_P$. This sheds new light on the results mentioned in the beginning.

The purpose of this section is to familiarize with the primary object of interest called irreversible algebraic dynamical system in its most generalform

Vaguely speaking, such a dynamical system is given by a countably infinite, discrete group G and at most countably many commuting injective, non- surjective group endomorphisms $(\theta_i)_{i\in I}$ of G that are independent in the sense that the intersection of their images is as small as possible. Additionally, we will introduce a minimality condition stating that the intersection of the images of the group endomorphisms from the semigroup generated by $(\theta_i)_{i\in I}$is trivial. In other words, the group endomorphisms $(\theta_i)_{i\in I}$ (more precisely, finite products of these) separate the points in G. At a later stage, namely in Theorem(4.2.26), this condition is shown to be intimately connected to simplicity of the C^*-algebra $\mathcal{O}[G,P,\theta]$ associated to such a dynamical system in Definition (4.2.1).

The following observation is an extension of the concept of independence introduced. In contrast to the situation , we will require neither the group G to be abelian nor the cokernels of the injective group endomorphisms of G to be finite.

Proposition (4:1:1)[4]: Suppose G is a group. Consider the following statements for two commuting injective group endomorphisms θ_1 and θ_2 of G:

(i) $\theta_1(G)\theta_2(G) = G$.

(ii) The map$\theta_1(G)/(\theta_1(G)\cap\theta_2(G)) \longrightarrow G/\theta_2(G)$ $induced\ by\ the\ inclusion$ $\theta_1(G) \hookrightarrow G$ $is\ a\ bijection.$

(ii')The map $\dfrac{\theta_2(G)}{\theta_1(G)\cap\theta_2(G)} \longrightarrow \dfrac{G}{\theta_1(G)}$ $induced\ by\ the\ inclusion$ $\theta_2(G) \hookrightarrow G$ $is\ a\ bijection.$

(iii) $\theta_1(G)\cap\theta_2(G) = \theta_1\theta_2(G)$.

Then (i), (ii), and (ii') are equivalent and imply (iii). If either of the sub- groups $\theta_1(G)$ or $\theta_2(G)$ is of finite index in G, then (i)–(iii) are equivalent.

Proof:-

Note that we always have $\theta_1(G)\theta_2(G) \subset G$ *and* $\theta_1(G) \cap \theta_2(G) \supset \theta_1\theta_2(G)$. Moreover, in condition (ii), the inclusion $\theta_1(G) \hookrightarrow G$ induces an injective map $\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \longrightarrow G/\theta_2(G)$.

The corresponding statement holds for (ii').

If (i) holds true, then $G \ni g = \theta_1(g_1)\theta_2(g_2)$ for suitable $g_i \in G$. Hence, the left-coset of $\theta_1(g_1)$ maps to the left-coset of g and (ii) follows.

Conversely, suppose (ii) is valid and pick $g \in G$. Then there is $g_1 \in G$ such that $\theta_1(g_1)\,(\theta_1(G) \cap \theta_2(G)) \longmapsto g\theta_2(G)$ via the map from (ii). But since this map comes from the inclusion $\theta_1(G) \hookrightarrow G$, we have $g\theta_2(G) = \theta_1(g_1)\theta_2(G)$. Thus, there is $g_2 \in G$ such that $g = \theta_1(g_1)\theta_2(g_2)$ showing (i).The equivalence of (i) and (ii') is obtained from the previous argument by swapping θ_1 and θ_2. Given (ii), that is,

$$f_1 : \theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \longrightarrow G/\theta_2(G)$$

is a bijection (induced by $\theta_2(G) \hookrightarrow G$), composing f_1^{-1} with the bijection

$$f_2 : \theta_1(G)/(\theta_1\theta_2(G)) \longrightarrow G/\theta_2(G)$$

obtained from injectivity of θ_1 yields a bijection

$$f_1^{-1} : \theta_1(G)/(\theta_1\theta_2(G)) \longrightarrow \theta_1(G)/(\theta_1(G) \cap \theta_2(G)).$$

Let us assume $\theta_1\theta_2(G) \subsetneqq \theta_1(G) \cap \theta_2(G)$. This means, that there is $g \in \theta_1(G)$ such that $g\theta_1\theta_2(G) \neq \theta_1\theta_2(G)$ but $g\theta_1(G) \cap \theta_2(G) = \theta_1(G) \cap \theta_2(G)$. Noting that $f_1^{-1}f_2$ maps a left-coset $g'\theta_1\theta_2(G)$ *to* $g'\theta_1(G) \cap \theta_2(G)$, this contradicts injectivity of $f_1^{-1}f_2$. Hence, we must have $\theta_1(G) \cap \theta_2(G) = \theta_1\theta_2(G)$. Similarly, (iii) follows from (ii').

Finally, suppose (iii) holds. By injectivity of θ_1, we have

$$\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) = \theta_1(G)/\theta_1\theta_2(G) \cong G/\theta_2(G).$$

So if $[G : \theta_2(G)]$ is finite, then the injective map from (ii) is necessarily a bijection. If $[G : \theta_1(G)]$ is finite, we get (ii') in the same manner.

Definition (4.1.2)[4]: Let G be a group and θ_1, θ_2 commuting, injective group endomorphisms of G. Then θ_1 and θ_2 are said to be independent, if they satisfy condition (iii) from Proposition (4.1.1). $\theta_1$andθ_2

are said to be strongly independent, if they satisfy the condition (i) from Proposition (4.1.1).

Note that (strong) independence holds if θ_1 or θ_2 is an automorphism.

Lemma (4:1:3)[4]: Let G be a group and suppose $\theta_1, \theta_2, \theta_3$ are commuting, injective group endomorphisms of G. $\theta_1$is (strongly) independent of$\theta_2\theta_2$ if and only if $\theta_1$is (strongly) independent of both θ_2 and θ_3.

Proof:-

If θ_1 and $\theta_2\theta_3$are strongly independent, then

$$\theta_1(G)\theta_2(G) \supset \theta_1(G)\theta_2\,(\theta_3(G)) = G$$

shows that $\theta_1$and θ_2 are strongly independent. As $\theta_2$and θ_3 commute, $\theta_1$is also strongly independent of θ_3. Conversely, if θ_1 is strongly independent of both $\theta_2$andθ_3, then

$$G = \theta_1(G)\theta_2(G) = \theta_1(G)\theta_2\,(\theta_1(G)\theta_3\,(G))$$
$$= \theta_1(G\theta_2\,(G))\theta_2\,(\theta_3(G)) \subset \theta_1\,(G)\theta_2\theta_3(G),$$

so θ_1 and $\theta_2\theta_3$ are strongly independent since the reverse inclusion is triv- ial. If θ_1 and $\theta_2\theta_3$ are independent, then commutativity of $\theta_1, \theta_2$and θ_3 in combination with injectivity of θ_3^{-1} $yield$

$$\theta_1(G) \cap \theta_2(G) = \theta_3^{-1}\big(\theta_1\theta_3(G) \cap \theta_2\theta_3(G)\big) \subset \theta_3^{-1}\big(\theta_1(G) \cap \theta_2\theta_3(G)\big)$$
$$= \theta_3^{-1}\big(\theta_1\theta_2\theta_3(G)\big) = \theta_1\theta_2(G).$$

Since the reverse inclusion is always true, we conclude that $\theta_1$and $\theta_2$are independent. Exchanging the role of $\theta_2$and θ_3 shows independence of $\theta_1$and θ_3. Finally, if θ_1 is independent of both θ_2 and θ_3, we get

$$\theta_1(G) \cap \theta_2\theta_3\,(G) = \theta_1(G) \cap \theta_2(G) \cap \theta_2\theta_3(G) = \theta_1\theta_2(G) \cap \theta_2\theta_3\,(G)$$
$$= \theta_2(\theta_1(G) \cap \theta_3(G)) = \theta_1\theta_2\theta_3\,(G)$$

by injectivity of θ_2. Thus θ_1 and $\theta_2\theta_3$ are independent.

If $(P, \leq)$ is a lattice-ordered monoid with unit 1_P , we shall denote the least common multiple and the greatest common divisor of two elements $p, q \in P$ by $p \vee q$ and $p \wedge q$, respectively. p and q are said to be relatively prime (in P) if $p \wedge q = 1_P$or, equivalently, $p \vee q = pq$. Simple examples of such monoids are countably generated free abelian monoids since such monoids areeitherisomorphic to$\mathbb{N}^k$ for some $k \in \mathbb{N}$or $\bigoplus_{\mathbb{N}} \mathbb{N}$.

Definition(4.1.4)[4]: An irreversible algebraic dynamical system (G, P, θ) is

(A) a countably infinite, discrete group G with unit 1_G ,

(B) a countably generated, free abelianmonoid$P$ with unit 1_P , and

(C) a P -action θ on G by injective group endomorphisms for which θ_p and θ_q are independent if and only if p and q are relatively prime.

An irreversible algebraic dynamical system (G, P, θ) is said to be

· minimal, if $\bigcap_{p\in P} \theta_p(G) = \{1_G\}$,

· commutative, if G is commutative,

· of finite type, if $[G : \theta_p(G)]$ is finite for all $p \in P$, and

· of infinite type, if $[G : \theta_p(G)]$ is infinite for all $p = 1_P$.

Examples(4.1.5)[4]: There are various examples for commutative irreversible algebraic dynamical systems and most of them are of finite type. Let us recall that it suffices to check independence of the endomorphisms on the generators of P according to Lemma (4.1.3).

(a) Choose a family

$$(p_i)_{i\in I} \subset \mathbb{Z}^{\times}\backslash\mathbb{Z}^{*} = \mathbb{Z}\backslash\{0, \pm 1\}\ and\ let\ P = |(p_i)_{i\in I}$$

act on $G = \mathbb{Z}$ by $\theta_{p_i}(g) = p_i g$. Since $\mathbb{Z}$ is an integral domain, each θ_{p_i} is an injective group endomorphism of G with $\left[G{:}\,\theta_{p_i}(G)\right] = p_i. For\ i = j, \theta_{p_i}\ and\ \theta_{p_j}$ are independent if and only if $p_i$and $p_j$are relatively prime in $\mathbb{Z}$. Thus,we get a commutative irreversiblealgebraic dynamical system of finite type if and only if $(p_i)_{i\in I}$ consists of relatively prime integers. Since the number of factors in its prime factorization is finite for every integer, such irreversible algebraic dynamical systems are automatically minimal.

(b) Let $I \subset \mathbb{N}$, choose relatively prime integers $\{q\} \cup (p_i)_{i\in I} \subset \mathbb{Z}^{\times}\backslash\mathbb{Z}^{*} and\ let\ G = \mathbb{Z}[1/q]$. $As\ \mathbb{Z}[1/q] = \lim_{\rightarrow} \mathbb{Z}$ with connecting maps given by multiplication with q, and q is relatively prime to each pi, the arguments from (a) carry over almost verbatim. Thus we get minimal commutative irreversible algebraic dynamical systems of finite type (G, P, θ) which generalise .

(c) Let K be a countable field and let $G = \mathbb{K}[T\ ]$ denote the polynomial ring in a single variable T over $\mathbb{K}$. Choose non-constant polynomials $p_i \in \mathbb{K}[T\ ], i \in I$. Multiplying by p_i defines an endomorphism θ_{p_i} of G with $[G\ : \theta_{p_i}(G)] = |K|^{\deg(p_i)}$, where $\deg(p_i)$ denotes the degree of $p_i \in \mathbb{K}[T\ ]$. Thus, if we let$P \coloneqq (p_i)_{i\in I}\rangle$,then the index of $\theta_p(G)$in G is finite for all $p \in P$ if and only if $\mathbb{K}$ is finite. It is clear that θ_{p_i} and $\theta_{p_j}$are independent if and only if $(p_i) \cap (p_j) = (p_i p_j\ )$ holds for the principal ideals (whenever $i = j$). Since every $g \in \mathbb{K}[T\ ]$has finite degree, (G, P, θ) is automatically minimal. Thus, provided $(p_i)_{i\in I}$ has been chosen accordingly, we obtain a minimal commutative irreversible algebraic dynamical system which is of finite type if and only if $\mathbb{K}$ is finite, compare .

Example (4.1.6)[4]: For $G = \mathbb{Z}^d\ with\ d \geq 1$, themonoid of injective group endo-morphisms of G is isomorphic to the monoid of invertible integral matrices $M_d(\mathbb{Z}) \cap Gl_d\ (\mathbb{Q})$. For each such endomorphism, the index of its image in G is given by the absolute value of the determinant of the corresponding matrix. In particular, their images always have finite index in G and an endomorphism of G is not surjective precisely if the absolute value of the determinant of the matrix exceeds 1. So let $(T_i)_{i\in I} \subset M_d(\mathbb{Z}) \cap Gl_d\ (\mathbb{Q})$ be a family of commuting matrices satisfying $|\det T_i| > 1$ for all $i \in I$ and set

$$P = |(T_i)_{i\in I}\rangle, a\text{s well as}\theta_i(g) = T_i g.\ For\ i \neq j,$$

it is easier to check strong independence of θ_i and θ_j instead of independence. Indeed, since we are dealing with a finite type case, the two conditions are equivalent and strong independence takes the form $T_i(\mathbb{Z}^d\ ) + T_j(\mathbb{Z}^d) = \mathbb{Z}^d$, see Proposition(4:1:1). This condition can readily be checked. Moreover, minimality is related to generalised eigenvalues and we note that, in the case where P is singly generated, the generating integer matrix has to be a dilation matrix.This situation has been studied extensively .

Example (4:1:8)(a) can be generalised to the case of rings of integers:

Example (4.1.7)[4]: Let $\mathcal{R}$ be the ring of integers in a number field and denote by $\mathcal{R}^\times = \mathcal{R} \setminus \{0_\mathcal{R}\}$ the multiplicative subsemigroup as well as by

$\mathcal{R}^* \subset \mathcal{R}^\times$ the group of units in $\mathcal{R}$. Take $G = \mathcal{R}$ and choose a (countable) family$(p_i)_{i\in I} \subset \mathcal{R}^\times \setminus \mathcal{R}^*$. $If\ we\ set\ P\ =\ |(p_i)_{i\in I}\rangle$, then this monoid acts on G by multiplication, i.e.

$$\theta_p(g)\ =\ pg\ for\ g \in G, p \in P\ .\ For\ \ i\ \neq\ j, \theta_{p_i}\ \ and\ \ \theta_{p_j}$$

areindependent if and only if the principal ideals (p_i) and (p_j) in $\mathcal{R}$ have no common prime ideal. If this is the case, (G, P, θ) constitutes a commutative irreversible algebraic dynamical system of finite type. Since the number of factors in the (unique) prime ideal factorization of (g) in $\mathcal{R}$ is finite for every $g \in G$, minimality is once again automatically satisfied.

As a matter of fact, the construction from Example (4.1.7) is applicable to Dedekind domains $\mathcal{R}$. Next, we would like to mention the following ex-ample even though, having singly generated P , it has nothing to do with independence. The reason is that Joachim Cuntz and Anatoly Vershik ob- served, that the C^*-algebra $\mathcal{O}[G, P, \theta]$ associated to this irreversible algebraic dynamical system is isomorphicto $\mathcal{O}_n$.

Example(4:1:8)[4]: For $n\ \geq\ 2$, consider the unilateral shift θ_1 acting on $G = \bigoplus_{\mathbb{N}}\ \mathbb{Z}/n\mathbb{Z}$ by $(g_0\ , g_1\ , \ldots)\ \mapsto (0, g_0\ , g_1\ , \ldots)$. Since θ_1 is an injective group endomorphism with $[G : \theta_1(G)]\ =\ n$, (G, P, θ) with $P\ =\ |\theta_1\rangle$ is a minimal commutative irreversible algebraic dynamical system of finite type.

Example(4.1.9)[4]: Generalising Example (4.1.8), suppose P is as required in condition (B) of Definition (4.1.4) and let G_0 be a countable group. Let us assume that G_0 has at least two distinct elements. Then P admits a shift action θ on $G \coloneqq \bigoplus_P G_0$ given by

$$G_0\left(\theta_p\left(\left(g_q\right)_{q\in P}\right)\right)_r\ ((P\ ))r\ =\ \chi_p P\ (r) g_{p^{-1}r}\ for\ all\ p, r\ \in P.$$

It is apparent that $\theta_p\theta_q\ =\ \theta_q\theta_p$ holds for all $p, q \in P$ and that θ_p is an injective group endomorphism for all $p\ \in\ P$. The index $[G\ :\ \theta p(G)]$ is finite for $p \in P \setminus \{1_P\}$ if and only if G_0 is finite and P is singly generated. Indeed, if $p\ \neq\ 1_P$, then each element of $\bigoplus_{q\in P\setminus pP} G_0$ yields a distinct left-coset in$G\ /\ \theta_p(G)$. Clearly, this group is finite if andonly if G_0 is finite and P is singly generated. Given relatively prime p and q in

$P \setminus \{1_P\}, \theta_p(G)\theta_q(G) \neq G$ since $g_{1_P} = 1_{G_0}$ for $all\ (g_r)_{r\in P} \in \theta_p(G)\theta_q(G) as\ 1_P \notin pP \cup qP$. Thus, unless P is singly generated, θ does not satisfy the strong independence condition . However,the independence condition is satisfied because$g = (g_r)_{r\in P} \in (g_r)_{r\in P} \in \theta_p(G) \cap \theta_q(G)$

implie sthat $g_r \neq 1_{G_0}$onlyif $r \in pP \cap qP = pqP$ and thus $g \in \theta_{pq}(G)$.

We have seen in Example (4.1.9)that one cannot expect strong independence for irreversible algebraic dynamical systems of infinite type in general. On the other hand, there are some examples where the subgroups in question have infinite index and the endomorphisms are strongly independent

Example (4.1.10)[4]: Given a family $(G^{(i)}, P, \theta^{(i)})_{i\in\mathbb{N}}$ of irreversible algebraic dynamical systems, we can consider $G := \bigoplus_{i\in\mathbb{N}} G^{(i)}$. If P acts on G component-wise, i.e. $\theta_p(g_i)_{i\in\mathbb{N}} := (\theta_p^{(i)}(g))_{i\in\mathbb{N}}$,then (G, P, θ) is an irreversible algebraic dynamical system and $[G : \theta_p(G)]$ isinfinite unless $p = 1_P$ minimal if and only if each $(G^{(i)}, P, \theta^{(i)})$ is minimal.If each $(G^{(i)}, P, \theta^{(i)})$ satisfies the strong independence condition, then θ inherits this property as well.

As a final example, we provide more general These examples are neither commutative irreversible algebraic dynamical systems nor of finite type.

Example (4:1:11)[4]: For $2 \leq n \leq \infty$, let $\mathbb{F}_n$ be the free group in n generators $(a_k)_{1\leq k\leq n}$. Fix $1 \leq d \leq n$ and choose for each $1 \leq i \leq d$ an n-tuple $(m_{i,k})_{1\leq k\leq n} \subset \mathbb{N}^\times$ such that

a) there exists k such that $m_{i,k} > 1$ for each $1 \leq i \leq d$, and

b) $m_{i,k}$ and $m_{j,k}$ are relatively prime for all $i \neq j, 1 \leq k \leq n$

Then $\theta_i(a_k) = a_k^{m_{i,k}}$defines a group endomorphism of $\mathbb{F}_n$for each $1 \leq i \leq d$. Noting that the length of an element of $\mathbb{F}_n$ in terms of the generators $(a_k)_{1\leq k\leq n}$and their inverses is non-decreasing under θ_i, we deduce that θ_i is injective. It is clear that $\theta_i\theta_j = \theta_j\theta_i$ holds for all i and j. For every $1 \leq i \leq d$, the index $[\mathbb{F}_n : \theta_i(\mathbb{F}_n)]$ is infinite. Indeed, take $1 \leq k \leq n$such that $m_{i,k} > 1$ according to (a) andpick $1 \leq \ell \leq n$ with $\ell \neq k$. Then the family $((a_k a_\ell)^j)_{j\geq 1}$ yields pairwise distinct leftcosets in $\mathbb{F}_n / \theta_i(\mathbb{F}_n)$ sincereduced words of the form $a_k a_\ell b \ldots$ with $b \neq a_\ell^{-1}$ are not contained in $\theta_i(\mathbb{F}_n)$.

A similar argument shows that $\theta_i$and θ_j are not strongly independent for $i \neq j$: By 1), there are$1 \leq k, \ell \leq n$such that $m_{i,k} > 1$ and$m_{j,\ell} > 1$. This forces $a_k a_\ell \notin \theta_i(\mathbb{F}_n)\theta_j(\mathbb{F}_n)$.

Nonetheless, θ_i and θ_j are independentdue to (b).Thus, $G = \mathbb{F}_n and\ P = |(\theta_i)1 \leq i \leq d_i$ acting on G in the obviousway constitutes an irreversible algebraic dynamical system which is neithercommutative nor of finite type. Minimality of such irreversible algebraic dynamical systems can easily be characterized by:

c) For each $1 \leq k \leq n$, there exists $1 \leq i \leq d$ satisfying $m_{i,k} > 1$.

In addition to the presented spectrum of examples, we would like to mention that there are also examples of minimal, commutative irreversible algebraic dynamical systems of finite type arising from cellular automata.

We close this section with two preparatory lemmas which are relevant for the C^*-algebraic considerations in Section (4.1). The first lemma reflects a crucial feature of the independence assumption.

Lemma (4.1.12)[4]: If (G, P, θ) is an irreversible algebraic dynamical system,

$$g\theta_p(G) \cap h\theta_q(G) = \begin{cases} (g\theta_p(h')\theta_{p\vee q}(G) & if\ g^{-1}h \in \theta_p(G)\theta_q(G), \\ \emptyset & else \end{cases}$$

holds for all $g, h \in G, p, q \in P$, where h' is uniquely determined by $g\theta_p(h') \in h\theta_q(G)$ up to multiplication from the right by elements from $\theta_{p^{-1}(p\vee q)}(G)$.

Proof:-

If there exist $g_1, g_2 \in G$ such that $g\theta_p(g_1) = h\theta_q(g_2)$,then$g^{-1}h = \theta_p(g_1)\theta_q(g_2^{-1}) \in \theta_p(G)\theta_q(G)$ follows because$G$is group.

Now suppose that$g_3, g_4 \in G$ satisfy $g\theta_p(g_3) = h\theta_q(g_4)$ as well.

Since this implies $\theta_p(g_1^{-1}g_3) = \theta_q(g_2^{-1}g_4)$, we deduce $\theta_p(g_1^{-1}g_3) \in \theta_{p\vee q}(G)$.Using injectivity of θ_p , this isequivalent to$g_1^{-1}g_3 \in \theta_{p^{-1}(p\vee q)}(G)$.

$$h' = g_1$$

is unique up to right

Therefore,multiplication by elements from $\theta_{p^{-1}(p\vee q)}(G)$.

we will need the following auxiliary result, which relies on irreversibility of the dynamical system:

Lemma (4.1.13)[4]: Suppose (G,P,θ) is an irreversible algebraic dynamical system and we have $n \in \mathbb{N}, g_i \in G, p_i \in P \setminus \{1_P\}\, for\ 0 \leq i \leq n$. Then,there exist**s**

$$g \in g_0\theta_{p_0}(G), p \in p_0P$$

Satisfying

$$g\theta_p(G) \subset G \bigcup_{1\leq i\leq n}\left(g_i \bigcap_{m\in\mathbb{N}} \theta_{p_i^m}(G)\right)$$

Proof:-

We proceed by induction starting with $n = 1$. As $p_1 \neq e$,wecan find $m \in \mathbb{N}$ such that $p_0 \notin p_1^m P$. Thus we have $p_0 \vee p_1^m \gneqq p_0$.

By Lemma (4.1.12),

$$(g_0\theta p_0(G)) \cap (g_1\theta_{p_1^m}(G)) \\ = \begin{cases}(g_0\theta_{p_0}(\tilde{g}_1)\theta_{p_0\vee p_1^m}(G) & if\ g_0^{-1}g_1 \in \theta_p(G)\theta_{p_1^m}(G), \\ \emptyset & else\end{cases}$$

Where$\tilde{g}_1$ is uniquely determined up to$\theta_{p_0^{-1}(p_0\vee p_1^m)}(G)$.

While $g := g_0$ works inthe second case we need

$$g \in (g_o\theta_{p_0}(G)) \setminus g_0\theta_{p_0}(\tilde{g}_1)\theta_{p_0\vee p_1^m}(G)$$

in the first case. Note that such a g exists as $p_0 \vee p_1^m \gneqq p_0$ by the choice of m and we set $p := p_0 \vee p_1^m$.

The induction step from n to $n + 1$ is just a verbatim repetition of the first step: Assume that the statement holds for fixed n. This means that there exist $h \in g_o\theta_{p_0}(G)\ and\ q \in p_0P$ such that

$$h\theta_q(G) \quad \subset G \setminus \bigcup_{1\le i\le n}\left(g_i \bigcap_{m\in\mathbb{N}} \theta_{p_i^m}(G)\right)$$

As $p_{n+1} \neq e$, we can find $m \in \mathbb{N}$ suchthat $q \notin p_{n+1}^m P$. In other words, we have$q \vee p_{n+1}^m \gneqq q$Recall that

$$(h\theta_q(G)) \cap (g_{n+1}\theta_{p_{n+1}^m}(G)) = \begin{cases} (h\theta_q(\tilde{g}_{n+1})\theta_{p_0 \vee p_{n+1}^m}(G) & if\ h^{-1}g_{n+1} \notin \theta_q(G)\theta_{p_{n+1}^m}(G), \\ \emptyset & else \end{cases}$$

where $\tilde{g}_{n+1}$ is uniquely determined up to$\theta_{q^{-1}(q\vee p_{n+1}^m)}(G)$.In the second case,take $g := h$. For the first case, we choose

$$g \in (h\theta_q(G)) \setminus h\theta_q(\tilde{g}_{n+1})\theta_{q\vee p_{n+1}^m}(G)$$

Note that such a g exists as $q \vee p_{n+1}^m \gneqq q$ by the choice of m.

Finally, let

$$p := q \vee p_{n+1}^m$$

Then, it is clear from the construction that we indeed have

$$g\theta_p(G) \subset G \setminus \bigcup_{1\le i\le n}\left(g_i \bigcap_{m\in\mathbb{N}} \theta_{p_i^m}(G)\right)$$

We will now restrict our focus to commutative irreversible algebraic dynami- cal systems (G,P,θ): Injective group endomorphisms θ_p of a discrete abelian group G correspond to surjective group endomorphisms $\hat{\theta}_p$of its Pontryagin dual $\hat{G}$, which is a compact abelian group. Moreover, the cardinality of ker $\hat{\theta}_p$ is equal to the index $[G : \theta_p(G)]$. Via duality, we arrive at a definition of (strong) independence for commuting surjective group endomorphismsη_1 and η_2 of an arbitrary group K, see Definition(4.1.18).With this notion of independence, we then recast the conditions for an irreversible algebraic dynamical system (G,P,θ) with commutative G in terms of its dual model $(\hat{G},P,\hat{\theta})$, see Proposition (4.1.19).This provides a new perspective on irreversible algebraic dynamical systems: If G is commutative and (G,P,θ) is of finite type, it can be regarded as an irreversible topological dynamical system. More precisely, it arises from surjective local homeomorphisms $\hat{\theta}_p$ of the compact Hausdorff space $\hat{G}$, for details.

We start with a short review of basic facts about characters on groups, for details and further information. Recall that a character χ on a locally compact abelian group G is a continuous group homomorphism $\chi : G \longrightarrow \mathbb{T}$. The set of characters on G forms a locally compact abelian group $\hat{G}$ when equipped with the topology of uniform convergence on compac subsets of G.

Pontryagin duality states that $\hat{\hat{G}} \cong G$. Forthis result, we interpret $g \in G$ as a character on $\hat{G}$ via $g(\chi) := \chi(g)$. If Gdiscrete, then $\hat{G}$ is compact and vice versa.

Definition (4.1.14)[4]: Let G be a locally compact abelian group. For a subset$H \subset G$, the annihilator of H is given by $H^{\perp} := \{ \chi \in G\hat{} \ |\chi|_H = 1 \}$.

Lemma (4.1.15)[4]: Let G be a locally compact abelian group and $\eta : G \longrightarrow G$ a group endomorphism. Then $\hat{\eta}(\chi)(g) := \chi \circ \eta(g)$ defines a group endomor- phism $\hat{\eta} : \hat{G} \longrightarrow \hat{G}$ which is continuous if and only if η is and we have:

i) $\hat{\eta} = \eta$.
ii) $\eta(G)^{\perp} = ker\, \hat{\eta}$.
iii) $\hat{\eta}(\hat{G}) \subset \hat{G}$ is dense if and only if η is injective.
iv) $\widehat{ker\, \hat{\eta}} \cong coker\, \eta$ if $\eta(G)$ is closed.

In particular, if G is discrete, then ii) states that $\hat{\eta} : \hat{G} \longrightarrow \hat{G}$ is surjective if and only if $\eta : G \longrightarrow G$ is injective. Moreover, $\eta(G)$ isalways closed. If, in addition, cokerη is finite, thenker $\hat{\eta} \cong \widehat{ker\, \hat{\eta}} \cong coker$ follows from (iv).

Lemma(4.1.16)[4]: If G is a locally compact abelian group and $H_1, H_2 \subset G$ aresubgroups, then:

i) $(H_1 \cdot H_2)^{\perp} = H_1^{\perp} \cap H_2^{\perp}$.
ii) $(H_1 \cap H_2)^{\perp} = H_1^{\perp} \cdot H_2^{\perp}$holds if$H_1$and $H_2$are closed.

Proposition(4.1.17)[4]: Let G be a discrete abelian group andθ_1, θ_2 be commuting, injective endomorphisms of G. Then the following statements hold:

i) θ_1 and θ_2 are strongly independent if and only if ker $\hat{\theta}_1$ and ker $\hat{\theta}_2$ intersect trivially.

ii) θ_1 and θ_2 are independent if and only if ker$\hat{\theta}_1\cdot$ ker$\hat{\theta}_2$= $ker\ \widehat{\theta_1\theta_2}$.

Proof:-

For strong independence, we compute

$$\theta_1(G)\theta_2\,(G))^{\perp} = \theta_1(G)^{\perp} \cap \theta_2(G)^{\perp} = ker\hat{\theta}_1 \cap ker\hat{\theta}_2$$

Therefore, $\theta_1(G)\theta_2\,(G) = G$ is equivalent to $ker\hat{\theta}_1 \cap ker\hat{\theta}_2 = \{1_{\hat{G}}\}$. Similarly,we get

$$\theta_1(G) \cap \theta_2\,(G))^{\perp} = \theta_1(G)^{\perp} \cdot \theta_2(G)^{\perp} = ker\hat{\theta}_1 \cdot ker\hat{\theta}_2$$

On the other hand, Lemma (4.1.19) ii) gives $ker\ \widehat{\theta_1\theta_2} = \theta_1\theta_2(G)^{\perp}$.

This motivates the following definition in analogy to Definition (4.1.1):

Definition(4.1.18)[4]: Two commuting,surjective group endomorphismsη_1 and η_2 of a group K are said to be strongly independent, if ker η_1 and ker η_2 intersect trivially. η_1 and $\eta_2$are called independent, if ker $\eta_1\cdot$ ker $\eta_2 = \ker \eta_1\eta_2$.

It is clear that we have an equivalence between the statements:

(i) η_1 and $\eta_2$are strongly independent.
(ii) η_1 is an injective group endomorphism of ker η_2.

(ii')η_2 is an injective group endomorphism of ker η_1.

If both kerη_1 ker $\eta_2$are finite, then strong independence and independence coincide. Therefore, this definition is consistent with, where the case of endomorphisms (of a compact abelian group K) with finite kernels is treated. Note that there is no conflict with (strong) independence for injective group endomorphisms, see Definition (4.1.2), as all these conditions are trivially satisfied by group automorphisms.

With the observations from Lemma (4.1.15) and Lemma (4.1.16)at hands, we can now translate the setup from Definition (4.14)for commutative irreversible algebraic dynamical systems:

Proposition (4.1.19)[4]: For a discrete abelian group G, a triple (G, P, θ) is acommutative irreversible algebraic dynamical system if and only if

(A) $\hat{G}$ is a compact abelian group,

(B) P is a countably generated,free, abelian monoid (with unit 1P), an$P$ on $\hat{G}$ by surjective group

(C) $\hat{\theta}$ is an action oftheendomorphisms with property that $\hat{\theta}_p$ and $\hat{\theta}_q$ are independent if and only if p and q are relatively prime in P.

(G, P, θ) is minimal if and only if $\bigcup_{p\in P} ker\, \hat{\theta}_p \subset \hat{G}$ is dense. It is of finite (infinite) type if and only if ker $\hat{\theta}_p$ is (infinite) finite for all $p \in P$ $(p \neq 1_P)$.

Proof:-

Conditions (A) and (B) of this characterization follow directly from Lemma(4.1.19). Moreover, for an$p \in P$, the equation $(\ker\hat{\theta}_p)^{\perp} = \mathrm{im}\theta_p$ *yields* an isomorphism between coker θ_p and the Pontryagin dual of ker $\hat{\theta}_p$. Combining Lemma (4.1.15) (iii) and Proposition (4.1.14) yields (C). Note that we have $\theta_q(G) \subset \theta_p(G)$ and, correspondingly, $ker\, \hat{\theta}_p \subset ker\, \hat{\theta}_q$ whenever $q \in pP$. Since P is directed, Lemma (4.1.16) (i) and Lemma(4.1.19)(ii) yield the equivalence between minimality of (G, P, θ) and$\bigcup_{p\in P} ker\, \hat{\theta}_p$ being dense in$\hat{G}$.

For the last claim, we recall that a locally compact abelian group is finite if and only ifits dual group is finite. Thus ker$\hat{\theta}_p$ is finite if and only if cokerθ_p is finite.

We will now revisit some of the examples from Section (4.1) to present their dual models:

Example (4.1.20) [4]: The following list corresponds to the one in

(a) For $G = \mathbb{Z}$, a family of relatively prime numbers $(p_i)_{i\in I} \subset \mathbb{Z}^{\times} \backslash \mathbb{Z}^{*}$ generates a monoid $P = |(p_i)_{i\in I}\rangle \subset \mathbb{Z}^{\times}$which acts by $\theta_{p_i}(g) = p_i g$. In this case, $\hat{G} = \mathbb{T}$ and $\hat{\theta}_p(t) = t^p$ for all $t \in \mathbb{T}$ and $p \in P$.

(b) For $I \subset \mathbb{N}, 0 \in I$, let $q, (p_i)_{i\in I} \subset \mathbb{Z}^{\times} \backslash \mathbb{Z}^{*}$be relatively prime numbers and set $P = |(p_i)_{i\in I}\rangle$ as well as $G = \mathbb{Z}[1/q] = \lim_{\rightarrow} \mathbb{Z}$ with connecting maps given by multiplication with q. Then this constitutes a minimal commutative irreversible algebraic dynamical system of finite type see Example(4.1.5) (b). Then $\hat{G}$ is the solenoid on

$$\mathbb{Z}_q = \lim_{\longleftarrow} \mathbb{Z}/q^k \mathbb{Z}$$

which $\hat{\theta}_p$ is given by multiplication with p.

(c) For a finite field $\mathbb{K}$, let $p_i \in \mathbb{K}[T], i \in I$ (for an index set I) be polynomials in $G = \mathbb{K}[T]$ with the property that $(p_i) \cap (p_j) = (p_i p_j)$ holds for all $i = j$. Then the action θ of $P := |(p_i)_{i\in I}\rangle$ given by multiplication with the polynomial itself yields a commutative irreversible algebraic dynamical system of finite type, see Example (4.1.5) (c). Then $\hat{G}$is the ring of formal power series $\mathbb{K}[[T]]$ over $\mathbb{K}$, compare , and $\hat{\theta}_p$ is given by multiplication with p in $\mathbb{K}[[T]]$.

Example(4.1.21)[4]: Recall that, in Example (4.1.6), we considered $G = \mathbb{Z}^d$ for some $d \geq 1$, a family of pairwise commuting matrices $(T_i)_{i\in I} \subset M^d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$satisfying $|\,det\, T_i| > 1$ for all $i \in I$ and set$P = |(T_i)_{i\in I}\rangle$with

$$\theta_{T_i}(g) = T_i\, g$$

In this case, we have$\hat{G} = \mathbb{T}^d$ and the endomorphism $\hat{\theta}_p$ isgiven by the matrix corresponding to θ_p interpreted as an endomorphism of

$$\mathbb{R}^d/\mathbb{Z}^d \cong \mathbb{T}^d$$

Example (4.1.22)[4]: The dual model for the unilateral shift on $G = \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$ from Exampl (4.1.8) is given by the shift

$$(x_k)_{k\in\mathbb{N}} \longmapsto (x_{k+1})_{k\in\mathbb{N}}\ on\ \hat{G} = \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^{\mathbb{N}}.$$

The discussion for Example (4.1.9) with the restriction that G_0 be abelian is analogous, where $\mathbb{N}$ is replaced by P and $\mathbb{Z}/n\mathbb{Z}$ by G_0.

Example(4.1.23): Inthe situation of Example(4.1.10), where we will now require that $(G_n, P, \theta^{(i)})_{i\in\mathbb{N}}$be a family of commutative irreversible algebraicdynamical systems, $G = \bigoplus_{i\in\mathbb{N}} G_i$turns into $\hat{G} = \prod_{i\in\mathbb{N}} \hat{G}_i$for each$p \in P$the group endomorphism $\hat{\theta}_p$ is given by applying $\theta_p^{(i)}$ to the i-th component of $\hat{G}$. ker $\hat{\theta}_p$ is infinite for all $p \in P \setminus \{1P\}$. If each $\theta^{(i)}$ satisfies the strong independence condition from Definition (4.1.3), $\hat{\theta}$ satisfies the strong indepen- dence condition from Definition (4.1.18)due to Proposition (4.1.17).

Section (4.2):Structure of the Associated C^*-algebras and the Product Systems Perspective:

In this Section, we associate a universal C^*-algebra $\mathcal{O}[G,P,\theta]$ to every irreversible algebraic dynamical system (G,P,θ).The general approach is inspired by the methods for the case of a single group endomorphism with finite cokernel of a discrete abelian group. Partly, these ideas can even be traced back.Note however, that we are going to use a different spanning family than the one used.

We will examine structural properties of $\mathcal{O}[G,P,\theta]$ as well as of two nested subalgebras: the core $\mathcal{F}$ and the diagonal$\mathcal{D}$. In Lemma (4.2.9), a description of the spectrum G_θ of the diagonal $\mathcal{D}$ is provided, which allows us to regard G as a completion of G with respect to θ in the case where (G,P,θ) is minimal, compare .

Based on the description of G_θ , the action $\hat{\tau}$ of G on G_θ coming from $\tau_g\,(e_{h,p}) = e_{gh,p}$ is shown to be always minimal. Moreover, we prove that topological freeness of $\hat{\tau}$corresponds to minimality of (G,P,θ), see Proposition (4.2.10). As an immediate consequence we deduce that $\mathcal{D} \rtimes_\tau G$ is simple if and only if (G,P,θ) is minimal and $\hat{\tau}$ is amenable, see Corollary (4.2.11). This crossed product is actually isomorphic to $\mathcal{F}$, see Corollary (4.2.14).

We remark that our strategy of proof differs from the one because we start by establishing an ,we deduce that $\mathcal{O}[G,P,\theta]$ is isomorphic to the semigroup crossed product $\mathcal{F} \rtimes P$.

So we get

$$\mathcal{O}[G,P,\theta] \cong \mathcal{D} \rtimes (G \rtimes_\theta P) \cong \mathcal{F} \rtimes P$$

One advantage of this strategy is that we are able to establish these iso- morphisms in greater generality, i.e.withoutminimalityof (G,P,θ) andamenability of $\hat{\tau}$ which would give simplicity of both $\mathcal{F}$ and $\mathcal{O}[G,P,\theta]$, we conclude that, whenever (G,P,θ) is minimal and the G-action$\hat{\tau}$ on G_θ is amenable, the C^*-algebra $\mathcal{O}[G,P,\theta]$ is a unital UCT Kirchberg algebra, see Theorem (4.2.22)and Corollary (4.2.24). Thus $\mathcal{O}[G,P,\theta]$ is classified by its K-theory in this case due to the important classification results of Christopher Phillips and Eberhard Kirchberg.Throughout this section, (G, P, θ) will represent an irreversible algebraic dy- namical system unless specified otherwise.

Let $(\xi_g)_{g\in G}$ denote the canonical orthonormal basis of $\ell^2(G)$.

For $g \in G$ and $p \in P$, define operators U_g and S_p on $\ell^2(G)$ by $U_g\ (\xi_{g'})\colon= \xi_{gg'}$ and $S_p\ (\xi_{g'}\ ) := \ \xi_{\theta_p(g')}$ for $g' \in G$.

Then $(U_g)_{g\in G}$is a unitary representation of the group G and$S_p^*(\xi_{g'}) = \chi_{\theta_p(G)}(g')\xi_{\theta_p^{-1}(g')}$ for $g' \in G$,so $\left(S_p\right)_{p\in P}$is a representation of the semigroup P by isometries. Furthermore, these operators satisfy.

$$(CNP\ \ 1) S_pU_g\left(\xi_{g'}\right) \ = \ \xi_{\theta_p(gg')} \ = \ U_{\theta_p(g)}S_p\left(\xi_{g'}\right),$$

and

$$(CNP\ \ 3) \quad [g] \in \frac{G}{\theta p}(G) \sum_{[g]\in G/\theta_p(G)} E_{g,p}\left(\xi_{g'}\right) = \xi_{g'}\ if\ [G : \theta_p(G)] < \infty,$$

where $E_{g,p} = U_gS_pS_p^*U_g^*$. In fact, (CNP 3) holds also in the case of an infinite index $[G : \theta_p(G)], as\ (\sum_{[g]\in F} E_{g,p})_{F\subset G/\theta_p(G)}$converges to the identity on$\ell^2(G)\ as\ F \nearrow\ G/\theta_p(G)$ with respect to the strong operator topology. But this convergence does not hold in norm because each $E_{g,p}$ is a non-zero projection. In view of our motivation to construct a universal C^*-algebra based on this model, it is therefore reasonable to restrict this relation to the case where $[G : \theta_p\ (G)]$ is finite.

As the numbering indicates, we are interested in an additional relation which will increase the accessibility of the universal model: If G was trivial, this would simply be the condition that S_p and S_q doubly commute for all relatively prime p and q in P , i.e. $S_p^*S_q = S_qS_p^*$. This condition has been employed successfully for quasi-lattice ordered groups, for more information. But as G is an infinite group, this will not be sufficient.

Moreover, we want to ensure that, within the universal model to be built, an expression corresponding to $S_p^*U_qS_p$ belongs to $C^*(G)$. This property has been used extensively in the context of semigroup crossed products involving transfer operators.

An entirely different way to put it is that we aim for a better understanding of the structure of the commutative subalgebra$C^*(\{ E_{g,p} \mid g \in G,\ p \in P \})$ inside $L(\ell^2(G))$. In a much more general framework, this has been con- sidered by Xin Li, and resulted in a new definition of semigroup C^*-algebras for discrete left cancellative semigroups with identity. One particular strength of his notion is the close connection between amenability of semigroups and nuclearity of their C^*-algebras .

All of these three instances suggest that a closer examination of the terms

$S_p^* U_g S_q$ is in order. For $g = \theta_p(g_1)\theta_q(g_2)$ with $g_1, g_2 \in G$,

we get $S_p^* U_g S_q = U_{g_1} S_{(p\wedge q)^{-1}q} S^*_{(p\wedge q)^{-1}p} U_2$ .On the other hand, $g \notin \theta_p(G)\theta_p(G)$ is equivalent

to $g\theta_q(G) \cap \theta_p(G) = \emptyset$, which forces$S_p^* U_g S_q = 0$. Thus we get

$$(CNP\ 2) S_p^* U_g S_q = \begin{cases} U_{g_1} S_{(p\wedge q)^{-1}q} S^*_{(p\wedge q)^{-1}p} U_2 & if\ g = \theta_p(g_1)\theta_q(g_2) \\ 0 & else \end{cases}$$

for all $g \in G, p, q \in P$. These observations motivate the following definition:

Definition (4.2.1)[4]:$\mathcal{O}[G, P, \theta]$ is the universal C^*algebra generated by a unitary representation $(u_g)_{g\in G}$ of the group G and a representation $(s_p)_{p\in P}$ of the semigroup P by isometries subject to the relations:

$$(CNP\ 1) \qquad s_p u_g = u_{\theta_p(g)} s_p$$

$$(CNP\ 2) s_p^* u_g s_q = \begin{cases} u_{g_1} s_{(p\wedge q)^{-1}q} s^*_{(p\wedge q)^{-1}p} u_2 & if\ g = \theta_p(g_1)\theta_q(g_2) \\ 0, & else \end{cases}.$$

$$(CNP\ 3) \qquad 1 = \sum_{[g]\in G/\theta_p(G)} e_{g,p} \qquad if\ [G : \theta_p(G)] < \infty,$$

where $e_{g,p} = u_g s_q s_p^* u_p^*$.

Proposition(4.2.2)[4]:$\mathcal{O}[G,P,\theta]$ has acanonical non-trivial representation on$\ell^2(G)$ given by $u_g \mapsto U_g$, $s_p \mapsto S_p$. In particular, $\mathcal{O}[G,P,\theta]$ is non-zero.

for all $g \in G, p \in P$.

One immediate benefit is the following lemma, whose straightforward proof is omitted.

Lemma (4.2.3)[4]: The linear span of $(u_g s_p s_q^* uh)_{g,h\in G,p,q\in P}$ is dense in $\mathcal{O}[G,P,\theta]$.

Lemma(4.2.4)[4]: The projections$(e_{g,p})_{g\in G,p\in P}$ommute.More precisely, for

$g, h \in G$ and $p, q \in P$, we have

$$e_{g,p}e_{h,q} = \begin{cases} e_{g\theta_p(h'),p\vee q} & if\ g^{-1}h \in \theta_p(G)\theta_q(G) \\ 0 & else, \end{cases}$$

where $h' \in G$ is determined uniquely up to multiplication from the right by elements of $\theta_{p^{-1}(p\vee q)}(G)$ by the condition that $g\theta_p(h') \in h\theta_q(G)$.

Proof:-

For $g, h \in G$ and $p, q \in P$, the product $e_{g,p}e_{h,q}$ is non-zero only if $g^{-1}h \in \theta_p(G)\theta_q(G)$ by (CNP 2). So let us assume that $g^{-1}h \in \theta_p(G)\theta_q(G))$ holds. Then there are $g', h' \in G$ such that $g^{-1}h \in \theta_p(h')\theta_q(g')$. As G is a group, this is equivalent to $h\theta_q(g')^{-1} = g\theta_p(h')$. Thus we get

$$e_{g,p}e_{h,q} = u_{g\theta_p(h')} s_p s_{(p\wedge q)^{-1}q} s^*_{(p\wedge q)^{-1}p} s_q^* u^*_{h\theta_q(g')^{-1}} = e_{g\theta_p(h'),p\vee q} .$$

Clearly, this also proves that the two projections commute. The uniqueness assertion follows from .

Definition(4.2.5)[4]: The C^*-subalgebra D of $\mathcal{O}[G,P,\theta]$ generated by the com- muting projections $(e_{g,p})_{g\in G,p\in P}$is called the diagonal.

In addition, let

$$\mathcal{D}_p := C^*(\{e_{g,q} \mid [g] \in G/\theta_p(G), p \in qP\})$$

Denotes the C^*-subalgebra of $\mathcal{D}$ corresponding to $p \in P$.

We note the following obvious fact:

lemma(4.2.6)[4]: For all $p, q \in P$,$p \in qP$ implies $\mathcal{D}_q \subset \mathcal{D}_p$. $\mathcal{D}$ is the closure of

$\bigcup_{p \in P} \mathcal{D}_p$. If $[G : \theta_p(G)]$ is finite, then

$$\mathcal{D}_p = \text{span}\{e_{g,p} \mid [g] \in G/\theta_p(G)\} \cong \mathbb{C}^{[G:\theta_p(G)]}.$$

Let us make the following non-trivial observation:

Lemma (4.2.7)[4]: Suppose $g \in G, p \in P$ and a finite subset F of $G \times P$ arechosen in such a way that $\prod_{(h,q)\in F}(1 - e_{h,q}) = 1$is non-zero.

Then thereexist $g' \in G$ and $p' \in P$satisfying

$$e_{g',p'} \leq e_{g,p} \prod_{(h,q)\in F} (1 - e_{h,q}).$$

Proof:-

If F is empty, then $\prod_{(h,q)\in F}(1 - e_{h,q}) = 1$ by convention, so there is nothing to show. Now let F be non-empty. For $(h, q) \in F$, let us decompose$q$ uniquely as $q = q^{(fin)}q^{(inf)}$, where $[G : \theta_{q^{(fin)}}(G)]$ is finite and we require that, for each $r \in P$ with $q \in rP$, finiteness of $[G : \theta_r(G)]$ implies $q^{(fin)} \in rP$. In other words,$G : \theta_r(G)]$ is infinite for every $r \neq 1_P$ with $q^{(inf)} \in rP$, for $q^{(fin)}$ and Lemma (4.2.3), we compute

$$1 - e_{h,q} = \left(1 - e_{h,q^{(fin)}} e_{h,q^{(inf)}}\right) \sum_{[k]\in \frac{G}{\theta_{q^{(fin)}}(G)}} e_{k,p^{(fin)}}$$

$$= e_{h,q^{(fin)}}\left(1 - e_{h,q^{(inf)}}\right) + \sum_{\substack{[k]\in G/\theta_{q^{(fin)}}(G) \\ [k]\neq[h]}} e_{k,p^{(fin)}}$$

Therefore, we can rewrite the initial product as

$$e_{g,p} \prod_{(h,q)\in F} (1 - e_{h_i,q_i}) = \sum_{(\tilde{g},\tilde{p})\in\tilde{F}} e_{\tilde{g},\tilde{p}} \prod_{(h,q)\in F_{(\tilde{g},\tilde{p})}} (1 - e_{h,q}),$$

where

· $\tilde{F}$ is a finite subset of $G \times P$,

·$e_{\tilde{g},\tilde{p}} \leq e_{g,p}$ for all $(\tilde{g},\tilde{p}) \in \tilde{F}$

· the projections $(e_{\tilde{g},\tilde{p}})_{(\tilde{g},\tilde{p})\in\tilde{F}}$ are mutually orthogonal,

· for each $(\tilde{g},\tilde{p}) \in \tilde{F}$, $F_{(\tilde{g},\tilde{p})}$ is a finite subset of $G \times P$, and

. each $(h,q) \in F_{(\tilde{g},\tilde{p})}$satisfies $q = q^{(inf)}$ and $\tilde{p} \notin qP.$

Since the product $e_{g,p}\prod_{(h,q)\in F}(1 - e_{h_i,q_i})$on the left hand side is non-zero,

here is $(g_0, p_0) \in \tilde{F}$such that

$$e_{g_0,p_0}\prod_{(h,q)\in F_{(g_0,p_0)}}(1 - e_{h,q}),$$

is non-zero.

Without loss of generality, we may assume that $e_{g_0,p_0}e_{h,g}$ is non-zero for all$(h,q) \in F_{(g_0,p_0)}$.Consider

$$F_P := \{p_0 \vee q \mid (h,q) \in F_{(g_0,p_0)}$$

for some $h \in G\}$. Pick $p_1 \in F_P$ which is minimal in the sense that for any other$r \in F_P, p_1 \in rP$ implies $r = p_1$.Let$(h_1, q_1), \ldots, (h_n, q_n) \in F_{(g_0,p_0)}$ denote the elements satisfying$p_0 \vee q_i = p_1$, we have$e_{g_0,p_0}e_{h_i,q_i} = e_{g_0,\theta_{p_0}(g_i'),p_1}$for a suitable

$$g_i' \in G\ (for\ i = 1, \ldots, n)$$

Note that$p_0^{-1}p_1 \neq 1_P$ and $q_1 = q^{(inf)} \in p_0^{-1}p_1Pso[G : p_{p_0^{-1}p_1}(G)]$, is infinite. Hence there exists $g_1 \in g_0\theta_{p_0}(G)$ with$e_{g_1,p_1} \leq e_{g_0,p_0}$and $e_{g_1,p_1}e_{h_i,q_i} = 0\ for\ i = 1, \ldots, n.$

Setting

$$F(g_1, p_1) := \{(h,q) \in F_{(g_0,p_0)} \mid e_{h,q}e_{g_1,p_1} \neq 0\} \subsetneqq F_{(g_0,p_0)},$$

we observe that

$$e_{g_1,p_1} \prod_{(h,q)\in F_{(g_1,p_1)}} (1 - e_{h,q}) \neq 0$$

follows from the initial statement for (g_0, p_0) and $F_{(g_0,p_0)}$since we have chosen p_1 in a minimal way. Indeed, if the product was trivial, then therewould be $(h, q) \in F_{(g_1,p_1)}$ with $e_{h,q} \geq e_{g_1,p_1}$. By Lemma (4.1.32), this would force$p_1 \in qP$ and therefore $p_1 \in (p_1 \vee q)P \subset (p_0 \vee q)P$, which cannot be truesince p_1 was chosen in a minimal way.

Thus, we can iterate the process used to obtain (g_1, p_1) and $F_{(g_1,p_1)}$ for(g_0, p_0) and $F_{(g_0,p_0)}$. After finitely many steps, we arrive at an element $(g_n, p_n) =: (g', p')$ with the property that$e_{h',q'} \leq e_{g_0,p_0}$ is orthogonal to $e_{h,q}$for all $(h, q) \in F_{(g_0,p_0)}$. This establishes the claim.

The possibility of passing to smaller subprojections that avoid finitely many defect projections provided through Lemma (4.2.6) will be crucial for the proofof pure infiniteness and simplicity of $\mathcal{O}[G, P, \theta]$, see Theorem (4.2.21)and inparticular Lemma (4.2.20). A first application of this observation lies in the determination of the spectrum of $\mathcal{D}$:

Lemma(4.2.8)[4]: The spectrum of $\mathcal{D}$, denoted by G_θ, is a totally disconnected, compact Hausdorff space. A basis for the topology on G_θ is given by the cylinder sets

$$Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)} = \{\chi \in G_\theta \mid \chi(e_{g,p}) = 1, \chi(e_{h_i,q_i}) = 0 \; for\ all\ i\},$$

where $n \in \mathbb{N}, g, h_1, \ldots, h_n \in G$ and $p, q_1, \ldots, q_n \in P$. Moreover

$$\iota(g) \in Z_{(g',p),(h_1,q_1),\ldots,(h_n,q_n)} \Leftrightarrow g \in g'\theta_p(G) \; and\ g \notin h_i\theta_{q_i}(G)$$

for all $i$defines a map $\iota : G \longrightarrow G_\theta$ with dense image. ι is injective if and only if(G, P, θ) is minimal.

Proof:-

G_θ is a totally disconnected, compact Hausdorff space since $\mathcal{D}$ is a unital C^*-algebra generated by commuting projections.The statement concerning the basis for the topology on G_θ follows from Lemma (4.2.5). To see that ι has dense image, let $\chi \in G_\theta$. As the cylinder sets form a basisfor the topology of G_θ , every open neighbourhood of χ contains a cylinder set$Z_{(g,p),(h_1,q_1),\dots,(h_n,q_n)}$with $\chi \in Z_{(g,p),(h_1,q_1),\dots,(h_n,q_n)}$.

This means that

$$e_{g,p} \coprod_{i=1}^{n} (1 - e_{h_i,q_i})$$

is non-zero. Hence we can obtain

$$(g', p') \in G \times P$$

Satisfying

$$e_{g',p'} \leq e_{g,p} \coprod_{i=1}^{n} (1 - e_{h_i,q_i})$$

In other words

$$\iota(g') \in Z_{(g,p),(h_1,q_1),\dots,(h_n,q_n)}$$

so $\iota(G)$ is a dense subset of G_θ.

Now given$g, h \in G$, we observe that $\iota(g) = \iota(h)$ is equivalent $to\ g^{-1}h \in \bigcap_{p\in P} \theta_p(G)$because the cylinder sets form a basis of the topology on the Hausdorff spaceG_θ . Therefore ι is injective precisely if (G, P, θ) is minimal.

Definition(4.2.9)[4]: Let X be a topological space and G a group. A G-action on X is said to be topologically free, if the set $X^g = \{x \in X \mid g.x = x\}$ has empty interior for $g \in G \setminus \{1_G\}$.

Definition (4.2.10) [4]: Let X be a topological space and G a group. A G-action on X is said to be minimal, if the orbit $\mathcal{O}(x) = \{g.x \mid g \in G\}$ is dense in X for every $x \in X$ .Equivalently, an action is minimal if the only invariant open (closed) subsets of X are $\emptyset$and X .

Proposition (4.2.11)[4]: If (G, P, θ) is an irreversible algebraic dynamical system, then the action G-action $\hat{\tau}$ on G_θ is minimal. It is topologically free if and only if (G, P, θ) is minimal.

Proof:

On $\iota(G)$, which is dense in G_θ by Lemma (4.2.7), $\hat{\tau}$ is simply given bytranslation from the left.

Hence$\hat{\tau}$is minimal. For the second part, wenote that $\tau_g = \mathrm{id}_{\mathcal{D}}$ holds for every$g \in \bigcap_{p\in P}\theta_p(G)$Thus, if (G,P,θ) is not minimal, there is $g \neq 1_G$ such that $G_\theta^g = G_\theta$, so $\hat{\tau}$ is not topologicallyfree. If (G,P,θ) is minimal, then $\hat{\tau}$ acts freely on $\iota(G)$ because ι is injectiveand G is left-cancellative. Since $\iota(G)$ is dense in G_θ, we conclude that $\hat{\tau}$ is topologically free.

Corollary(4.2.12)[4]: The crossed product $\mathcal{D} \rtimes_\tau G$ is simple if and only if (G,P,θ) is minimal and $\hat{\tau}$ is amenable.

Proof:-

Due to a central result amenability of the action is equivalent to regularity of the crossed product. Hence Proposition (4.2.9)establish the claim.

Definition (4.2.13)[4]: The core $\mathcal{F}$ is the C^*-subalgebra of $\mathcal{O}[G,P,\theta]$ generatedby $\mathcal{D}$ and $(u_g)_{g\in G}$.

Lemma (4.2.14)[4]: The linear span of $\left(u_g s_p s_p^* u_h^*\right)_{g,h\in G,p\in P}$ is dense in $\mathcal{F}$.

Proposition(4.2.15)[4]:Let $\left(v_{(g,p)}\right)_{(g,p)\in G\rtimes_\theta P}$denote the family of isometries in $\mathcal{D} \rtimes (G \rtimes_\theta P\,)$ implementing the action of the semigroup $G \rtimes_\theta P$ on D given by (g,p).

$$e_{h,q} = e_{g\theta_p(h),pq}, that\ \ is, v_{(g,p)}e_{h,q}v_{(g,p)}^* = e_{g\theta_p(h),pq}$$

Then the map

$$\mathcal{O}[G,P,\theta] \xrightarrow{\varphi} \mathcal{D} \rtimes (G \rtimes \theta\ P\,)$$

$$u_g s_p \longmapsto v_{(g,p)}$$

is an isomorphis

Proof:-

Recall from Definition (4.2.1)that $O[G,P,\theta]$ is the universal C^*-algebragenerated by a unitary representation $(u_g)_{g\in G}$ of the group G and a semi- group of isometries $(s_p\,)_{p\in P}$subject to the relations . Hence, in order to show thatφ defines a surjective $*$-homomorphism, itsuffices to show that for every $g \in G$, the isometry $v_{(g,1_P)}$ is a unitary, andthat the families $(v_{(g,1_P)})_{g\in G}, (v_{(1_G,p)})_{p\in P}$ satisfy :

$$v_{(g,1_P)}v_{(g^{-1},1_P)} = v_{(g,1_P)(g^{-1},1_P)} = v_{(1_G,1_P)} = 1$$

$$v_{(1_G,p)}v_{(g,1_P)} = v_{(1_G,p)(g,1_P)} = v_{(\theta_p(g),p)} = v_{(\theta_p(g),1_p)}v_{(1_G,p)}$$

$$v^*_{(1_G,p)}v_{(g,1_P)}v_{(1_G,p)} = \chi_{\theta_p(G)\theta_q(G)}(g)\, v_{(g_1,((p\wedge q)^{-1})q)}v^*_{(g_2^{-1},((p\wedge q)^{-1})p)}$$

where $g = \theta_p(g_1)\theta_q(g_2)$.

$$\Leftrightarrow v_{(1_G,p)}v^*_{(1_G,p)}v_{(g,q)}v^*_{(g,q)} = \chi_{\theta_p(G)\theta_q(G)}(g)\, v_{(\theta_p(g_1),p\vee q)}v^*_{(g\theta_q(g_2^{-1}),p\vee q)}$$

$$\Leftrightarrow e_{1_G,p}e_{g,q} = \chi_{\theta_p(G)\theta_q(G)}(g)e_{(g\theta_q(g_2^{-1}),p\vee q)}$$

as $g = \theta_p(g_1)\theta_q(g_2)$ $gives\ \theta_p(g_1) = g\theta_q(g_2^{-1})$.

This last equation holds byLemma (4.2.3), is satisfied as well. is a relation that is encoded inside $\mathcal{D}$, so it is satisfied as the range projection of the isometry$v_{(g,p)}$coincides with $e_{(g,p)}$. Injectivity of φ follows from the fact that the isometries ugsp satisfy the covariance relation for the action of $G \rtimes_\theta P$ on $\mathcal{D}$ since $u_gs_pe_{h,q}(u_gs_p)^* = e_{g\theta_p(h),pq} = (g,p).e_{h,q}$. Indeed, in this case there isa surjective$*$-homomorphism from $\mathcal{D} \rtimes (G \rtimes_\theta P)$ to $\mathcal{O}[G,P,\theta]$ sending $v_{(g,p)}$to u_gs_p and the two $*$-homomorphisms are mutually inverse, so φ is anisomorphism.

This description of $\mathcal{O}[G,P,\theta]$ allows us to deduce several relevant properties of $\mathcal{O}[G,P,\theta]$ and its core subalgebra $\mathcal{F}$.

Corollary(4.2.16)[4]: The isomorphismφ from Proposition (4.2.14)restricts to an isomorphism between $\mathcal{F}$ and $\mathcal{D} \rtimes G$. In particular, we have a canonical isomorphism $\mathcal{O}[G,P,\theta] \cong \mathcal{F} \rtimes P$.

Proof:-

The first claim follows immediately from Proposition (4.2.14).The second assertion is implied by Lemma (4.2.12).

Proposition(4.2.17) [4]: If the G-action $\hat{\tau}$on G_θ is amenable, then both F and$\mathcal{O}[G,P,\theta]$ are nuclear and satisfy the universal coefficient theorem (UCT).

Proof:

As $\mathcal{F} \cong \mathcal{D} \rtimes G$, $\hat{\tau}$is amenable, $\mathcal{F}$ is nuclear by results of Claire Anatharaman-Delaroche. Similarly, amenability of $\hat{\tau}$passes to the corresponding transformation groupoid $\mathcal{G}$. Thus, we can rely on results of Jean-Louis Tu, to deduce that $\mathcal{F} \cong \mathcal{D} \rtimes_{\tau} G \cong C^*(\mathcal{G})$ satisfies the UCT. The class of separablenuclear C^*-algebras that satisfy the UCT is closed under crossed products by $\mathbb{N}$ and inductive limits. Recall that either $P \cong \mathbb{N}^k$ for some $k \in \mathbb{N}\, or\, P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$ according to condition (B) of Definition (4.1.6). Hence the claims concerning $\mathcal{O}[G, P, \theta]$ follow from $\mathcal{O}[G, P, \theta] \cong \mathcal{F} \rtimes P$.

Corollary (4.2.18)[4]: The map $E_2(u_g s_p s_q^* u_h^*) := \delta_{gh} e_{g,p}$ defines a conditionalexpectation $E_2 : \mathcal{F} \longrightarrow \mathcal{D}$ which is faithful if and only if $\hat{\tau}$is amenable.

Proof:-

Due to Corollary (4.2.15), $\mathcal{F}$ is canonically isomorphic to $\mathcal{D} \rtimes_{\tau} G$. Since $G$is discrete, the reduced crossed product $\mathcal{D} \rtimes_{\tau,r} G$ has a faithful conditionalexpectation given by evaluation at 1_G. The map E_2 is nothing but thecomposition of

$$F \cong \mathcal{D} \rtimes_{\tau} G \twoheadrightarrow \mathcal{D} \rtimes_{\tau,r} G \xrightarrow{ev_{1_G}} \mathcal{D}.$$

The canonical surjection $\mathcal{D} \rtimes_{\tau} G \twoheadrightarrow \mathcal{D} \rtimes_{r,\tau} G$ is an isomorphism if and only if$\hat{\tau}$ is amenable.

Corollary (4.2.19)[4]: The map $E(u_g s_p s_q^* u_h^*) := \delta_{pq} \delta_{gh} e_{g,p}$ defines a conditional expectation $E : \mathcal{O}[G, P, \theta] \longrightarrow \mathcal{D}$ which is faithful if and only if $\hat{\tau}$is amenable.

Proof:

Clearly,$E = E_2 \circ E_1$, so the result follows from andCorollary(4.2.22).

Note that if G happens to be amenable, the faithful conditional expectation E can be obtained directly by showing that the left Ore semigroup $G \rtimes_{\theta} P$has an amenable enveloping group. Before we can turn to simplicity of$\mathcal{O}[G, P, \theta]$, we need the following general observations:

Definition (4.2.20)[4]: Given a family of commuting projections $(E_i)_{i \in I}$ in a unital C^*-algebra B and finite subsets $A \subset F$ of I , let

$$Q_{F,A}^{E} := \prod_{i \in A} E_i \prod_{j \in F \setminus A} (1 - E_j) .$$

Products indexed by $\emptyset$are treated as 1 by convention.

Lemma (4.2.21)[4]: Suppose $((E_i)_{i \in I}$ is a family of commuting projections in unitalC^*-algebra $B, A \subset F$ are finite subsets of I . Then each $Q_{F,A}^{E}$

Isprojection,$\sum_{A \subset F} Q_{F,A}^{E} = 1$and, for all $\lambda_i \in \mathbb{C}, i \in F$, we have

$$\sum_{i \in F} \lambda_i E_i = \sum_{A \subset F} \left(\sum_{i \in A} \lambda_i \right) Q_{F,A}^{E}$$

and

$$\left\| \sum_{i \in F} \lambda_i E_i \right\| = \max_{\substack{A \subset F \\ Q_{F,A}^{E} \neq 0}} \left| \sum_{i \in A} \lambda_i \right| = \sum_{A \subset F} Q_{F,A}^{E}$$

The twoequations from the claim follow immediately from this.

Proof

Since the projections $E_i$commute, $Q_{F,A}^{E}$ is a projection.

The second assertion is obtained via $1 = \prod_{i \in F} (E_i + 1 - E_i)$.

Lemma (4.2.22): For $d = \sum_{i=1}^{n} \lambda_i e_{g_i,p_i} \in \mathcal{D}_+$ with $\lambda_i \in \mathbb{C}$ and $(g_i, p_i) \in G \times P$there exists$(g, p) \in G \times P$satisfying $d e_{g,p} = \|d\|_{e_{g,p}}$.

Proof:

d is contained in $C^* \left(\left\{ Q_{F,A}^{e} | A \subset F = \{(g_i, p_i) | 1 \leq i \leq n\} \right\} \right)$, which his commutative by Lemma (4.2.7). Then Lemma (4.2.21) says that there exists$A \subset F$ such that $Q_{F,A}^{e}$ is non-zeroand $d Q_{F,A}^{e} = \|d\| Q_{F,A}^{e}$.In particular$\prod_{(q,p) \in A} e_{g,p}$is non-zero, so Lemma (4.2.22) implies that there exist $g_A \in G$ and$p_A \in P$such that$\prod_{(g,p) \in A} e_{g,p} = e_{g_A,p_A}$.Thus, we can apply Lemma (4.2.8) to $e_{g_A,p_A} \prod_{(h,q) \in F \setminus A} (1 - e_{h,q}) = Q_{F,A}^{e} \neq 0$and the proof is complete.

Note that the hard part of the proof for Lemma (4.2.29)is hidden in Lemma (4.2.9).

Theorem (4.2.23)[4]: If (G, P, θ) is minimal and the action $\hat{\tau}$is amenable, then$\mathcal{O}[G, P, \theta]$is purely infinite and simple.

Proof:

The linear span of $\left(u_g s_p s_q^* u_h^*\right)_{g,h\in G,p,q\in P}$is dense in $\mathcal{O}[G, P, \theta]$accord$\mathcal{O}[G, P, \theta]$. Every element z from this linear span is of the form

$$z = \sum_{i=1}^{m_1} c_i e_{g_i,p_i} + \sum_{i=m_1+1}^{m_3} c_i u_{g_i} s_{p_i} s_{p_i}^* u_{h_i}^* + \sum_{i=m_2+1}^{m_3} c_i u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^*,$$

Where $c_i \in \mathbb{C}$,

a) $g_i \neq h_i$ for $m_1 + 1 \leq i \leq m_2$, and
b) $p_i \neq q_i$ for $m_2 + 1 \leq i \leq m_3$.

By Corollary (4.2.22), we have$E(z) = \sum_{i=1}^{m_1} c_i e_{g_i,p_i} \in \mathcal{D}$ we assume z to benon-zero and positive, which we will do from now on, then $E(z) > 0$as$E$is a faithful conditional expectation. Applying Lemma (4.2.3)to $E(z)$yields$(g, p) \in G \times P$ such that

c) $E(z)e_{g,p} = \|E(z)\|e_{g,p}$.

In order to prove simplicity and pure infiniteness of $O[G, P, \theta]$, it suffices to establish the following claim: There exist $(\tilde{g}, \tilde{p}) \in G \times P$satisfying

(i) $e_{\tilde{g},\tilde{g}} \leq e_{g,p}$
(ii) $e_{\tilde{g},\tilde{g}} u_{g_i} s_{p_i} s_{p_i}^* u_{h_i}^* e_{\tilde{g},\tilde{g}}$=0 for $m_1 + 1 \leq i \leq m_2$ and
(iii) $e_{\tilde{g},\tilde{p}} u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^* e_{\tilde{g},\tilde{p}} = 0$ for $m_2 + 1 \leq i \leq m_3$.

Indeed, if this can be done, then we get

$$e_{\tilde{g},\tilde{p}} z e_{e_{\tilde{g},\tilde{p}}} \overline{\overline{(b),(c)}} e_{\tilde{g},\tilde{p}} E(z) e_{\tilde{g},\tilde{p}} \overline{\overline{(c),(a)}} \|E(z)\| e_{\tilde{g},\tilde{p}}.$$

Now for $x \in \mathcal{O}[G, P, \theta]$positive and non-zero, let $\epsilon > 0$and choose a positive, non-zero element z, which is a finite linear combination of elements$u_{q'}s_{p'}s_{q'}^{*}u_{h'}^{*}$, to approximate $x$up to ϵ. Then $\left|\|E(z)\|\right|$is a non-zero positive element of $\mathcal{D}$. Thus, choosing $e_{\tilde{g},\tilde{p}}$as above, we see that $e_{\tilde{g},\tilde{p}}ze_{\tilde{g},\tilde{p}} = \|E(z)\|e_{\tilde{g},\tilde{p}}$is invertible in $e_{\tilde{g},\tilde{p}}\mathcal{O}[G,P,\theta]e_{\tilde{g},\tilde{p}}$. If $\|x-z\|$ is sufficiently small, this impliesthat $e_{\tilde{g},\tilde{p}}xe_{\tilde{g},\tilde{p}}$is positive and invertible in $e_{\tilde{g},\tilde{p}}\mathcal{O}[G,P,\theta]e_{\tilde{g},\tilde{p}}$as wellbecause$\|E(z)\|\overrightarrow{\epsilon \to 0}\|E(x)\| > 0$. Hence, if we denote its inverse by$y$, then

$$\left(y^{\frac{1}{2}}u_{\tilde{g}}s_{\tilde{p}}\right)^{*} e_{\tilde{g},\tilde{p}}xe_{\tilde{g},\tilde{p}} \left(y^{\frac{1}{2}}u_{\tilde{g}}s_{\tilde{p}}\right) = 1.$$

We claim that there is a pair $(\tilde{g},\tilde{p}) \in G \times P$satisfying (a)–(c).Let $(g',p') \in g\theta_p(G) \times pP$and $m_1 + 1 \le i \le m_2$. Noting that $u_{g_i}s_{p_i}s_{p_i}^{*}u_{h_i}^{*} = u_{g_ih_i^{-1}}e_{h_i,e_i}$ Lemma(4.2.5) implies

$$\begin{aligned} e_{g',p'}u_{g,h_i^{-1}}e_{h_i,p_i}e_{g',p'} &= e_{g',p'}u_{g,h_i^{-1}}e_{g',p'}e_{h_i,p_i} \\ &= \chi_{\theta_{p'}(G)}((g')^{-1}g_ih_i^{-1}g')u_{g_i,h_i^{-1}}e_{g',p'}e_{h_i,p_i}. \end{aligned}$$

According to a), we have $(g')^{-1}g_ih_i^{-1}g' \neq I_G$. Thus,minimalist of $(G,P,\theta\ )$ provide $p_i' \in pP$ with the property that $(g')^{-1}g_ih_i^{-1}g' \notin \theta_{p_i'}(G)$. So if we take$p^{(b)} = \bigvee_{i=m_1+1}^{m_2} p_i'$then (a) and (b) of the claim hold for all$(g',p') \in y\,\theta_p(G) \times p^{(b)}P$. Let us assume that$p' \ge p^{(b)} \vee \bigvee_{i=m_2+1}^{m_3} p_i \vee q_i$ Then condition (c) holds for (g',p')if and only if

$$0 = s_{p'}^{*}u_{(g')^{-1}g_i}s_{p_i}u_{h_i^{-1}g'}s_{p'}$$
$$= \chi_{\theta_{p_i}(G)}((g')^{-1}g_i)\chi_{\theta_{q_i}(G)}(h_i^{-1}g')s_{p_i^{-1}p'}^{*}u_{\theta_{P_i}^{-1}}((g')^{-1}g_i)\theta_{q_i}^{-1}(h_i^{-1}g')^{s}q_i^{-1}p'$$

is valid for all $m_2 + 1 \le i \le m_3$. This is precisely the case if at least one ofthe conditions

(I) $(g')^{-1}g_i \in \theta_{p_i}(G)$,
(II) $(g')^{-1}h_i \in \theta_{q_i}(G)$, or
(III) $\theta_{p_i}^{-1}((g')^{-1}g_i)\theta_{q_h}^{-1}(h_i^{-1}g') \in \theta_{(p_i \vee q_i)^{-1}p'}(G)$.

fails for each i. Suppose, we have an index i for which the first two conditions are satisfied. Using injectivity of $\theta_{p_i \vee q_i}$, the third condition is equivalent$\theta_{\tau_q}(g')^{-1}g_i)\theta_{r_p}(h_i^{-1}g') \in \theta_{p'}(G)$ where $r_p := (p_i \wedge q_i)^{-1}p_i$ and $r_q := (p_i \wedge q_i)^{-1}q_i$. Condition b) implies $r_p \wedge r_q = 1P \neq r_pr_q$. Moreover, we have

$$\theta_{r_q}((g')^{-1}g_i)\theta_{r_p}(h_i^{-1}g') = 1_G \Leftrightarrow \theta_{r_q}(g')\theta_{r_p}(g')^{-1} = \theta_{r_q}(g_i)\theta_{r_p}(h_i^{-1}).$$

Let us examine the range of the map $f_i: G \to G$that is defined by$g \to \theta_{r_q}(g)\theta_{r_p}(g)^{-1}$.Note that $f_i$need notbe a group homomorphismunless $G$is abelian, in which case the following part can be shortened. If$k_1, k_2 \in G$ have the same image under f_i, then $\theta_{r_p}(k_2^{-1}k_1) = \theta_{r_q}(k_2^{-1}k_1)$. ByC1) from Definition(4.1.4), this gives $k_2^{-1}k_1 \in \theta_{r_p}(G) \cap \theta_{r_q}(G) = \theta_{r_p, r_q}(G)$. But if $k_2^{-1}k_1 = \theta_{r_p r_q}(k_3)$ holds for some $k_3 \in G$, then $\theta_{r_p}(k_2^{-1}k_1) = \theta_{r_p}(k_2^{-1}k_1) = \theta_{r_q}(k_2^{-1}k_1)$implies that $\theta_{r_p}(k_3) = \theta_{r_p r_q}(k_3)$holds as well because $P$is commutative and$\theta_{q_{i,1}q_{i,2}}$is injective. By induction, we get$k_2^{-1}k_1 \in \cap_{n\in\mathbb{N}} \theta_{(r_p r_q)^n}(G)$.

Hence $f_i^{-1}\left(\theta_{r_p}(h_i)\theta_{r_q}(g_i^{-1})\right)$is either empty, in which case there is nothing to do, or it is of the form$\tilde{g}_i \cap_{n\in\mathbb{N}} \theta_{(r_p r_q)^n}(G)$for a suitable$\tilde{g}_i \in G$. But for the collection of those i for which the preimage in question isnon-empty, we can obtain $\tilde{g} \in g\theta_{p'}(G)$such that$f_i(\tilde{g}) \neq \theta_{r_p}(h_i)\theta_{r_q}(g_i^{-1})$ for all relevant i.

By condition (C2) from Definition (4.1.4), we can choose $\tilde{p} \geq p'$large enoughso that these elements are still different modulo $\theta_{(p_i \vee q_i)^{-1}\tilde{p}}(G)$for all i. In this case, we get

$$\theta_{p_i}^{-1}(\tilde{g}^{-1}g_i)\theta_{q_i}(h_i^{-1}\tilde{g}) \notin \theta_{(p_i \vee q_i)^{-1}\tilde{p}}(G) \;\; for\ all\ \ m_2 + 1 \leq i \leq m_3,$$

so $(\tilde{g}, \tilde{p})$satisfies (c). In other words, we have proven that the pair $(\tilde{g}, \tilde{p})$ satisfies (a)–(c). Thus, $O[G, P, \theta]$is purely infinite and simple.

From this result, we easily get the following corollaries:

Corollary (4.2.24)[4]: If $(G, P, \theta)$is minimal and $\hat{\tau}$is amenable, then the representation$\lambda: \mathcal{O}[G, P, \theta] \to \mathcal{L}(\ell^2(G))$from Proposition (4.2.2)is faithful.

Proof:

This follows readily from Proposition (4.2.2)and simplicity of $O[G, P, \theta]$.

Combining Lemma (4.2.3), Theorem (4.2.23)and Proposition (4.2.17), we get:

Corollary (4.2.25)[4]: If (G,P,θ) is minimal and $\hat{\tau}$ is amenable, then $\mathcal{O}[G,P,\theta]$ is a unital UCT Kirchberg algebra.

Thus, minimal irreversible algebraic dynamical systems (G,P,θ) for which the action $\hat{\tau}$ is amenable yield C^*-algebras $\mathcal{O}[G,P,\theta]$ that are classified by their K-theory. Let us come back to some of the examples from Section (1.1) and briefly describe the structure of the C^*-algebras obtained in the various cases:

Examples (4.2.26)[4]:

(a) Let $G = \mathbb{Z}, (p_i)_{i\in I} \subset \mathbb{Z}\backslash\{0,\pm 1\}$ be a family of relatively prime integers, and set $P = |(p_i)_{i\in I} \subset \mathbb{Z}^\times|$, which acts on G by $\theta_i(g) = p_i g$. We know from the considerations in Example (4.1.5)(a) that (G,P,θ) is minimal, so $\mathcal{O}[G,P,\theta]$ is a unital UCT Kirchberg algebra. If we denote $p := \prod_{i\in I}|p_i| \in \mathbb{N}\cup\{\infty\}$, then G_θ can be identified with the p-adic completion $\mathbb{Z}_p = \lim_{\leftarrow}\left(\frac{\mathbb{Z}}{q\mathbb{Z}},\theta_q\right)_{q\in P}$ of $\mathbb{Z}$. Moreover, $\mathcal{F}$ is the Bunce-Deddens algebra of type p^∞, for the classification of Bunce-Deddens algebras by supernatural numbers.

(b) Let $I \subset \mathbb{N}$, choose $\{q\}\cup(p_i)_{i\in I} \subset \mathbb{Z}\backslash\mathbb{Z}^*$ relatively prime, $P = |(p_i)_{i\in I}|$, set $G = \mathbb{Z}\left[\frac{1}{q}\right]$, and let $\theta_p(g) = pg$ for $g \in G, p \in P$. As in (a), $\mathcal{O}[G,P,\theta]$ is a UCT Kirchberg algebra by the considerations in Example (4.1.5) (b) and Corollary (4.1.3): If $p := \prod_{i\in I}|p_i| \in \mathbb{N}\cup\{\infty\}$, then G_θ can be thought of as a p-adic completion of $\mathbb{Z}\left[\frac{1}{q}\right]$ and we obtain $\mathcal{F} \cong C(G_\theta)\alpha_\tau\,\mathbb{Z}\left[\frac{1}{q}\right]$.

Example (4.2.27)[4]: We have seen in Example (4.1.11) that for $n \geq 2$, the dynamical system given by the unilateral shift on $G = \bigoplus_{\mathbb{N}}\mathbb{Z}/n\mathbb{Z}$ is a minimal commutative irreversible algebraic dynamical system of finite type. It has been observed that $\mathcal{O}[G,P,\theta]$ is isomorphic to $\mathcal{O}_n$ in a canonical way: If $e_1 = (1,0,0,\dots) \in, s \in \mathcal{O}[G,P,\theta]$ denotes the generating isometry for P and $s_1,\dots,s_n$ are the generating isometries of $\mathcal{O}_n$, then this isomorphism is given by $u_{ke_1}s \to s_k$ for $k = 1,\dots,n$. In particular, $\mathcal{F}$ is the UHF algebra of type n^∞ and G_θ is homeomorphic to the space of infinite words using the alphabet $\{1,\dots,n\}$.

Example (4.2.28): Given a family $\left(G^{(i)},P,\theta^{(i)}\right)$, where each $\left(G^{(i)},P,\theta^{(i)}\right)$ is an irreversible algebraic dynamical system, we can consider $G := \bigoplus_{i\in\mathbb{N}} G^{(i)}$ on which P acts component-wise. Assume that each

$\left(G^{(i)}, P, \theta^{(i)}\right)$and hence$(G, P, \theta)$is minimal, compare Example (4.1.10). We have $G_\theta \cong \prod_{i\in I} G^{(i)}_{\theta(i)}$.Thus the G_i-action $\hat{\tau}_i$on $G^{(i)}_{\theta(i)}$ is amenable. As G is commutative (amenable) if and only if each $G^{(i)}$ is, there are various cases where amenability of $\hat{\tau}$is for granted.In such situations, $\mathcal{O}[G, P, \theta]$is a unital UCT Kirchberg algebra.

Example (4.2.29): For the examples arising from free group $\mathbb{F}_n$with $2 \leq n \leq \infty$, see Example (4.1.7):, we are able to provide criteria (i)–(iii) to ensurethat we obtain minimal irreversible algebraic dynamical systems. Hence G_θ can be interpreted as a certain completion of $\mathbb{F}_n$ with respect to θ. Now $\mathbb{F}_n$is far from being amenable, but the action $\hat{\tau}$could still be amenable:The free groups are known to be exact. By a famous result of NarutakaOzawa, exactness of a discrete group is equivalent to amenability of the left translation action on its Stone-Cech (In the mathematical discipline of general topology,**Stone–Čech compactification** is a technique for constructing a universal map from a topological space X to a compactHausdorff spaceβX. The Stone–Čech compactification βX of a topological space X is the largest compact Hausdorff space "generated" by X, in the sense that any map from X to a compact Hausdorff space factors through βX (in a unique way). If X is a Tychonoff space then the map from X to its image in βX is a homeomorphism, so X can be thought of as a (dense) subspace of βX. For general topological spaces X, the map from X to βX need not be injective[8])compactification,. Recently, MehrdadKalantar and Matthew Kennedy have shown that exactness of a discrete group is also determined completely by amenability of the naturalaction on its Furstenberg boundary, for details. The latter space is usually substantially smaller than the Stone-Cˇ echcompactification and their methods may give some insights into the question of amenability inthe context of the examples presented here. The finite type case revisited.

This section provides a more detailed presentation of the case where $(G, P, \theta)$is of finite type. In particular, we exhibit additional structural properties of the spectrum $G_\theta$of the diagonal Din $\mathcal{O}[G, P, \theta]$. For instance, the assumption that $\theta_p(G) \subset G$is normal for every $p \in P$causes $H_\theta$to inheritthe group structure from G. This turns $G_\theta$into a profinite group. If, in addition, $(G, P, \theta)$is minimal and G is amenable, then $\mathcal{F}$falls into the classof generalized Bunce-Deddens algebras, for details. They belong to a large class of C^*-algebras that can be classified by K-theory.

We are particularly interested in the case where G is abelian.

For such dynamical systems, the situation is significantly easier as $\theta_p(G) \subset G$isnormal for all $p \in P$and the action$\hat{\tau}$is always amenable. In fact, thestructure of $D$and $\mathcal{F}$is quite similar to the one discovered in the singlygenerated case, $G_\theta$is a compact abelian group and we have a chain of isomorphisms$\mathcal{F} \cong C(G_\theta)\alpha_\tau G \cong C(\hat{G})\alpha_{\bar{\tau}}\bar{G}_\theta$.

Throughout this section, we will assume that $(G, P, \theta)$is an irreversible algebraic dynamical system of finite type.

Proposition (4.2.30)[4]: Suppose $(G, P, \theta)$is minimal and G is amenable. Then $\mathcal{F}$ is a generalized Bunce-Deddens algebra.

Proof

This follows directly from the construction of the generalized Bunce-Deddens algebras presented.Choose an arbitrary,increasing, cofinal sequence $(p_n)_{n\in\mathbb{N}} \subset P$, where cofinal means that, for every $q \in P$, there exists an $n \in \mathbb{N}$such that $p_n \in qP$. Then $\left(\theta_{p_n}(G)\right)_{n\in\mathbb{N}}$is a family of nested, normalsubgroups of finite index in G. This family is separating for G by minimality of (G, P, θ).

In particular, these assumptions force $\mathcal{F}$to be unital, nuclear, separable, simple, quasidiagonal, and to have real rank zero, stable rank one, strict comparison for projections as well as a unique tracical . As the combination of real rank zero and strict comparison for projections yields strict comparison (for ositive elements), the prerequisites are met, so $\mathcal{F}$ also has finite decomposition rank. This establishes the remaining step to achieve classification of the core $\mathcal{F}$by means of its Elliott invariant $\left(K_0(\mathcal{F}), K_0(\mathcal{F})_+, [1_\mathcal{F}], K_1(\mathcal{F})\right)$thanks to results of Huaxin Lin and Wilhelm Winter.

Corollary (4.2.31)[4]: Let (G_i, P_i, θ_i)be minimal and $G_i$be amenable for $i = 1,2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the respective cores, then $\mathcal{F}_1 \cong \mathcal{F}_2$holds if and only if

$$\left(K_0(\mathcal{F}_1), K_0(\mathcal{F}_1)_+, \left[1_{\mathcal{F}_1}\right], K_1(\mathcal{F}_1)\right) \cong \left(K_0(\mathcal{F}_2), K_0(\mathcal{F}_2)_+, \left[1_{\mathcal{F}_2}\right], K_1(\mathcal{F}_2)\right).$$

We close this section by presenting an intriguing isomorphism of group crossed products on the level of F .

Corollary (4.2.32)[4]: Let (G,P,θ) be commutative and minimal. Then there is a $\hat{G}_\theta$-action $\bar{\tau}$ on $C(\hat{G})$ for which $\mathcal{F} \cong C(G_\theta)\alpha_\tau G \cong C(\hat{G})\alpha_{\bar{\tau}}\hat{G}_\theta$.

Proof.

The first isomorphism has been achieved in Corollary (4.2.19). For the second part, let $\bar{\tau}_{\chi_\theta}(\chi)(g) := \chi_\theta(\iota(g))\chi(g)$ for $\chi_\theta \in \hat{G}$ and $g \in G$. Since $\iota: G \to G_\theta$ is a group homomorphism, $\bar{\tau}_{\chi_\theta}(\chi)$ defines a character of G. Clearly, $\hat{\tau}$ is compatible with the group structure on $\hat{G}_\theta$, the group homomorphism ι identifies G with a dense subgroup of G_θ. In this case the characters on G_θ are in one-to-one correspondence with the characters on G. Note that this correspondence is precisely givenby regarding characters on G_θ as characters on G using ι. Therefore, $\bar{\tau}$ defines an action of $\hat{G}_\theta$ by homeomorphisms of the compact space $\hat{G}$. Oncewe know that $\bar{\tau}$ defines an action, we readily see that there is a canonical surjective∗-homomorphism $C(G_\theta)\alpha_\tau G \twoheadrightarrow C(\hat{G})\alpha_{\bar{\tau}} G$. As $C(G_\theta)\alpha_\tau G$ is simple, this map is an isomorphism.

This Section is designed to provide a product system of Hilbert bimodules for each irreversible algebraic dynamical system (G,P,θ). The features of (G,P,θ) result in a particularly well-behaved product system χ. Therefore, it is possible to obtain a concrete presentation of $\mathcal{O}_\chi$ from the data of the dynamical system. In the case of irreversible algebraic dynamical systems of finite type, this algebra is shown to be isomorphic to $\mathcal{O}[G,P,\theta]$.

The corresponding result in the general case, that is, allowing for thepresence of group endomorphisms θ_p of G with infinite index, requires a more involved argument. The reason is that the prerequisites fare not met, so one has to deal with Nica covariance of representations. Since this is more closely related to the Nica-Toeplitz algebra NT_χ, we will only treat the finite type case and refer to for the strategy in the general case. In fact, the proof reveals a close connection between Nica covariance, relying essentially on independence of group endomorphisms for relatively prime elements of P. More precisely, it shows that, for χ associated to (G,P,θ), Nica covariance boils down to its original form. A representation φ of the product system χ is Nicacovariant if and only if $\varphi_p(1_{C^*(G)})$ and $\varphi_q(1_{C^*(G)})$ are doubly commuting isometries whenever p and q are relatively prime in P.

We start with a brief recapitulation of the necessary definitions for product systems and Cuntz-Nica-Pimsner covariance.

Definition (4.2.33)[4]: A product system of Hilbert bimodules over a monoid P with coefficients in a C^*-algebra A is a monoid X together with a monoidal homomorphism $\rho: \chi \to P$ such that:

(a) $\chi_p \coloneqq \rho^{-1}(p)$ is a Hilbert bimodule over A for each $p \in P$,

(b) $\chi_{1_p} \cong_{id} A_{id}$ as Hilbert bimodules and

(c) for all $p, q \in P$, we have $\chi_p \otimes \chi_q \cong \chi_{pq}$ if $p \neq 1_p$, and $\chi_{1_p} \otimes_A \chi_q \cong \overline{\phi_q(A)\chi_q}$.

Definition (4.2.34)[4]: Let H be a Hilbert bimodule over a C^*-algebra A and $(\xi_i)_{i\in I} \subset \mathcal{H}$. Consider the following properties:

(i) $\langle \xi_i, \xi_j \rangle = \delta_{ij} 1_A$ for all $i, j \in I$.

(ii) $\eta = \sum_{i\in I} \xi_i \langle \xi_i, \eta \rangle$ for all $\eta \in \mathcal{H}$.

If $(\xi_i)_{i\in I}$ satisfies (i) and (ii), it is called an orthonormal basis for $\mathcal{H}$.

Lemma (4.2.35)[4]: Let $\mathcal{H}$ be a Hilbert bimodule. If $(\xi_i)_{i\in I} \subset \mathcal{H}$ is an orthonormal basis, then $\left(\Theta_{\xi_i,\xi_j}\right)_{i,j\in I}$ is a system of matrix units and $\sum_{i\in I} \Theta_{\xi_i,\xi_i} = 1_{\mathcal{L}(\mathcal{H})}$. If $\mathcal{H}$ admits a finite orthonormal basis, then $\mathcal{K}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$.

Definition (4.2.37)[4]: Let χ be a product system over P and suppose B is a C^*-algebra. A map $\varphi: \chi \to B$, whose fibre maps $\chi_P \to B$ are denoted by φ_p, is called a Toeplitz representation of χ, if:

(a) φ_{1_P} is a $*$-homomorphism.

(b) φ_p is linear for all $p \in P$.

(c) $\varphi_p(\xi)^* \varphi_p(\eta) = \varphi_{1_P}(\langle \xi, \eta \rangle)$ for all $p \in P$ and $\xi \in \chi_{p,\eta} \in \chi_q$.

(d) $\varphi_p(\xi)\varphi_q(\eta) = \varphi_{pq}(\xi\eta)$ for all $p, q \in P$ and $\xi \in \chi_P, \eta \in \chi_q$.

A Toeplitz representation will be called a representation whenever there is no ambiguity. Given a representation φ of χ in B, it induces $*$-homomorphisms $\psi_{\varphi,p}: \mathcal{K}(\chi_p) \to B$ for $p \in P$ characterised by $\Theta_{\xi,\eta} \to \varphi_p(\xi)\varphi_p(\eta)^*$. If χ is compactly aligned, the representation φ is said to be

Nica covariant, if $\psi_{\varphi,p}(k_p)\psi_{\varphi,q}(k) = \psi_{\varphi,p\wedge q}\left(\imath_p^{p\wedge q}(k_p)\imath_q^{p\vee q}(k_q)\right)$holds for all $p, q \in P$and$k_p \in \mathcal{K}(\chi_p), k_q \in \mathcal{K}(\chi_q)$. Concerning the choice of an appropriate notion of Cuntz-Pimsner covariance for product systems, there have been multiple attempts:

Definition (4.2.38) [4]: Let B be a C^*-algebra and suppose $\chi$is a compactly aligned product system of Hilbert bimodules over $P$with coefficients in A.

(CP_F)A representation $\varphi: \chi \to B$called Cuntz-Pimsner covariantin, if it satisfies

$$\psi_{\varphi,p}(\phi_p(a)) = \varphi_{1_p}(a) \text{for all } p \in P \text{ and } a \in \phi_p^{-1}\left(\mathcal{K}(\chi_p)\right) \subset A.$$

(CP) A representation $\varphi: \chi \to B$is called Cuntz-Pimsner covariant , if the following holds: Suppose $F \subset P$is finite and we fix $k_p \in \mathcal{K}(\chi_p)$for each $p \in F$. If, for every $r \in P$, there is $s \geq r$such that

$\sum_{p\in F} \iota_p^t(k_p) = 0$holds for all $t \geq s$,

then $\sum_{p\in P} \psi_{\varphi,p}(k_p) = 0$ holds true.

A representation $\varphi: \chi \to B$is said to be Cuntz-Nica-Pimsnercovariant, if it is Nica covariant and (CP)-covariant.

Fortunately, it was observed that the different notions are closely related and that (CP_F)implies Nica covariance in the cases of interest to us.

Proposition (4.2.39)[4]: Suppose $(G, P, \theta)$is an irreversible algebraic dynamical system. Let $(u_g)_{g\in G}$denote the standard unitaries generating $C^*(G)$and αbe the action of Pon $C^*(G)$ induced by θ, i.e. $\alpha_p(u_g) = u_{\theta_p(g)}$for $p \in P$and $g \in G$. Then $\chi_p := C^*(G)_{\alpha p}$, with left action $\phi_p$given by multiplication in$C^*(G)$and inner product $\langle u_g, u_h\rangle_p = \chi_{\theta_p(G)}(g^{-1}h)u_{\theta_p^{-1}(g^{-1}h)}$ is an essential Hilbert bimodule. The union of all $\chi_p$forms a product system χover $P$with coefficients in $C^*(G)$. $\chi$is a product system with orthonormal bases.It is of finite type if $(G, P, \theta)$is of finite type.

Proof:

It is straightforward to show that χdefines a product system of essential Hilbert bimodules and we omit the details. For $p \in P$, we claimthat every complete set of representatives $(g_i)_{i\in I}$for $G/\theta_p(G)$gives rise to an orthonormal basis of χ_p. Indeed, if we fix such a transversal $(g_i)_{i\in I}$and pick $g \in G$, then$\langle u_{g_i}, u_g\rangle_p = \chi_{\theta_p(G)}(g_i^{-1}g)u_{\theta_p^{-1}(g_i^{-1}g)}$ equals 0 for all but one$j \in I$, namely the one representing the left-coset $[g]$in $G/\theta_p(G)$.Thus, the family $(u_{g_i})_{i\in I}$consists of orthonormal elements with respect to $\langle\cdot,\cdot\rangle_p$, and $u_{g_i}\alpha_p(\langle u_{g_i}, u_g\rangle) = \delta_{ij}u_g$, so $(u_{g_i})_{i\in J}$satisfies (4.2.37) (b).

Lemma (4.2.40)[4]: Suppose $(G, P, \theta)$is an irreversible algebraic dynamical system and χ denotes the associated product system from Proposition (4.2.42). Then the rank-one projection $\Theta_{u_g,u_g} \in \mathcal{K}(\chi_p)$depends only on the equivalence class of g in $G/\theta_p(G)$. Moreover, if $\varphi$is a Nica covariant representation of χ, then

$$\psi_{\varphi,p}\left(\Theta_{u_{g_1},u_{g_1}}\right)\psi_{\varphi,q}\left(\Theta_{u_{g_2},u_{g_2}}\right)$$

$$= \begin{cases} \psi_{\varphi,p\vee q}\left(\Theta_{u_{g_3},u_{g_3}}\right) & if\ g_1^{-1}g_2 = \theta_p(g_3)\theta_q(g_4)\text{for some}g_3, g_4 \in G \\ 0, & else \end{cases}$$

holds for all $g_1, g_2 \in G$and $p, q \in P$.

Proof:

If $g_1 = g\theta_p(g_2)$for some $g_2 \in G$, then

$$\Theta_{u_{g_1},u_{g_1}}(u_h) = \chi_{\theta_p(G)}\left(\theta_p(g_2^{-1})g^{-1}h\right)u_h = \chi_{\theta_p(G)}(g^{-1}h)u_h = \Theta_{u_g,u_g}(u_h)$$

for all $h \in G$and hence $\Theta_{u_{g_1},u_{g_1}} = \Theta_{u_g,u_g}$. For the second claim, Nicacovariance of $\imath O_\chi$implies

$$\psi_{\varphi,p}\left(\Theta_{u_{g_1},u_{g_1}}\right)\psi_{\varphi,p}\left(\imath_p^{p\vee q}\left(\Theta_{u_{g_1},u_{g_1}}\right)\imath_p^{p\vee q}\left(\Theta_{u_{g_2},u_{g_2}}\right)\right).$$

If we denote $p' \coloneqq (p \wedge q)^{-1}p$and $q' \coloneqq (p \wedge q)^{-1}$, then

$$\imath_p^{p\vee q}\left(\Theta_{u_{g_1},u_{g_1}}\right) = \sum_{[g_4]\in G/\theta_{q'}(G)} \Theta_{u_{g_1}\theta_p(g_4),u_{g_1}\theta_p(g_3)} \in \mathcal{L}(\chi_{p\vee q})$$

And

$$\iota_p^{p\vee q}\left(\Theta_{u_{g_2},u_{g_2}}\right) = \sum_{[g_4]\in G/\theta_{p'}(G)} \Theta_{u_{g_2}\theta_p(g_4),u_{g_2}\theta_p(g_4)} \in \mathcal{L}(\chi_{p\vee q})$$

hold. We observe that

$$\Theta_{u_{g_1}\theta_p(g_3),u_{g_1}\theta_p(g_3)}\Theta_{u_{g_2}\theta_q(g_4),u_{g_2}\theta_q(g_4)}$$

is non-zero if and only if $[g_1\theta_p(g_3)] = [g_2\theta_q(g_4)] \in G/\theta_{p\vee q}(G)$. In particular, this is always zero if $g_1^{-1}g_2^{-1} \notin \theta_p(G)\theta_q(G)$.Let us assume that there are$g_3, \dots, g_8 \in G$such that

$$\theta_p(g_3^{-1})g_1^{-1}g_2\theta_q(g_4) = \theta_{p\vee q}(g_7)$$

And

$$\theta_p(g_5^{-1})g_1^{-1}g_2\theta_q(g_6) = \theta_{p\vee q}(g_8)$$

Rearranging the first equation to insert it into the second, we get

$$\theta_p(g_5^{-1}g_3)\theta_{p\vee q}(g_7)\theta_q(g_4^{-1}g_6) = \theta_{p\vee q}(g_8).$$

By injectivity of $\theta_{p\wedge q}$ this is equivalent to

$$\theta_{p'}(g_5^{-1}g_3)\theta_{(p\wedge q)^{-1}(p\vee q)}(g_7)\theta_{q'}(g_4^{-1}g_6) = \theta_{(p\wedge q)^{-1}(p\vee q)}(g_8).$$

From this equation we can easily deduce $g_5^{-1}g_3 \in \theta_{q'}(G)$and $g_4^{-1}g_6 \in \theta_{p'}(G)$ from independence of $\theta_{p'}$and $\theta_{q'}$, see Definition (4.1.5) (C). Thus, if there are $g_3, g_4 \in G$such that $\theta_p(g_3^{-1})g_1^{-1}g_2\theta_q(g_4) \in \theta_{p\vee q}$, then they are unique upto $\theta_{q'}(G)$and $\theta_{p'}(G)$, respectively. This completes the proof.

Theorem (4.2.41): Let (G, P, θ)be an irreversible algebraic dynamical system of finite type and $\chi$the product system. Then$u_g s_p \to \iota O_{\chi,p}(u_g)$, defines an isomorphisms$\varphi: O[G, P, \theta] \to O_\chi$.

Proof:

The idea is to exploit the respective universal property on both sides. We begin by showing that$\left(\iota O_{\chi,1p}(u_g)\right)_{u\in G}$is a unitary representation of $G$and$\left(\iota O_{\chi,p}(1_{C^*(G)})\right)_{p\in P}$ is a representation of the monoid$P$ by isometries satisfying compare. $\imath O_{\chi,1p}$ is a $*$-homomorphism, so we get a unitary representation of G. In addition,

$$\imath O_{\chi,p}\left(1_{C^*(G)}\right) = \imath O_{\chi,1p}\left(\langle 1_{C^*(G)}, 1_{C^*(G)}\rangle_p\right)$$
$$= \imath O_{\chi,1p}\left(1_{C^*(G)}\right) = 1_{O_\chi}$$

and

$$\imath O_{\chi,p}\left(1_{C^*(G)}\right)\imath O_{\chi,q}\left(1_{C^*(G)}\right) = \imath O_{\chi,pq}\left(1_{C^*(G)}\alpha_p\left(1_{C^*(G)}\right)\right) = \imath O_{\chi,pq}\left(1_{C^*(G)}\right)$$

show that we have a representation of P by isometries. (CNP 1) follows from

$$\imath O_{\chi,p}\left(1_{C^*(G)}\right)\imath O_{\chi,1p}\left(u_g\right) = \imath O_{\chi,p}\left(u_{\theta_p(g)}\right) = \imath O_{\chi,1p}\left(u_{\theta_p(g)}\right)\imath O_{\chi,p}\left(1_{C^*(G)}\right).$$

Let $p,q \in P$and $g \in G$. Then follows easily from applying Lemma

$$\imath O_{\chi,p}\left(1_{C^*(G)}\right)^*\imath O_{\chi,1p}\left(u_g\right)\imath O_{\chi,q}\left(1_{C^*(G)}\right)$$
$$= \imath O_{\chi,p}\left(1_{C^*(G)}\right)^*\psi_{\imath O_{\chi},p}\left(\Theta_{1,1}\right)\psi_{\imath O_{\chi},q}\left(\Theta_{u_g,u_g}\right)\imath O_{\chi,q}\left(u_g\right).$$

Finally, we observe that

$$\imath O_{\chi,1p}\left(u_g\right)\imath O_{\chi,p}\left(1_{C^*(G)}\right)\imath O_{\chi,p}\left(1_{C^*(G)}\right)^*\imath O_{\chi,1p}\left(u_g\right)^* = \psi_{\imath O_{\chi},p}\left(\Theta_{u_g,u_g}\right),$$

and the computation

$$\sum_{[g]\in G/\theta_p(G)}\psi_{\imath O_{\chi},p}\left(\Theta_{u_g,u_g}\right) = \psi_{\imath O_{\chi},p}\left(1_{\mathcal{L}(\chi)}\right) = \psi_{\imath O_{\chi},p}\left(\phi_p\left(1_{C^*(G)}\right)\right)$$
$$= O_{\chi,1p}\left(1_{C^*(G)}\right) \quad = 1_{O_\chi},$$

yield Thus we conclude that $\varphi: O[G,P,\theta] \to O_\chi$defines a surjective$*$-homomorphism. For the reverse direction, we show that

$$\varphi_{CNP}: \chi \to O[G,P,\theta]$$
$$\xi_{p,q} \to u_g s_p$$

defines a representation of χ, where $\xi_{p,g}$denotes the representative for $u_g$in χ_p. To do so, we have to verify (a)–(d) from Definition (4.2.37)and the

(CNP)-covariance condition. (a) and (b) are obvious. Using (CNP2) to compute

$$\varphi_{CNP,p}(\xi_{p,g_1})^*\varphi_{CNP,p}(\xi_{p,g_2}) = s_p^* u_{g_1}^{-1} g_2^{s_p} \quad = \chi_{\theta_{p(G)}}(g_1^{-1}g_2)u_{\theta_p^{-1}}(g_1^{-1}g_2)$$
$$= \varphi_{CNP,1p}(\langle \xi_{p,g_1}, \xi_{p,g_2}\rangle),$$

we get (c). (d) follows as

$$\varphi_{CNP,p}(\xi_{p,g_1})^*\varphi_{CNP,q}(\xi_{q,g_2}) = u_{g_1}s_p u_{g_2 s_q} = u_{g_1}\theta_p(g_2)^{s_{pq}}$$
$$= \varphi_{CNP,pq}\left(\xi_{p,g_1}\alpha_p(\xi_{q,g_2})\right).$$

Thus, we are left with the covariance condition. But since $\chi$is aproduct system of finite type , we only have to show that$\varphi CNP$is (CP_F)-covariant due . Noting that$\varphi_p^{-1}\left(\mathcal{K}(\chi_p)\right) = C^*(G)$ for all $p \in P$, we obtain

$$\psi_{\varphi CNP,p}\left(\phi_p(u_g)\right) = \psi_{\varphi CNP,p}\left(\sum_{[h]\in G/\theta_p(G)} \Theta_{u_{gh},u_h}\right) = u_g \sum_{[h]\in G/\theta_p(G)} e_{h,p}$$
$$= u_g = \varphi_{CNP,1P}(\xi_{1P,g}).$$

Thus, φ_{CNP} is a covariant representation of χ. By the universal property of O_χ, there exists a *-homomorphism $\bar{\varphi}_{CNP}: O_\chi \to O[G,P,\theta]$ such that $\bar{\varphi}_{CNP} \circ \imath O_\chi = \bar{\varphi}_{CNP}$. It is apparent that $\bar{\varphi}_{CNP}$ and φ are inverse to each other, so φ is an isomorphism.

List of Symbols

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