Chapter 6
A class of Non-Selfadjoint Quadratic Matrix Operator Pencils and Spectral Bounds

In this chapter the main results concern the structure and location of the spectrum and theorems about the minimality, completeness and basis properties of the eigenvectors and associated vectors corresponding to certain parts of the spectrum. Both Riesz basisness and Bari basisness results are obtained. The results are applied to a system of singular differential equations arising in the study of Hagen-Poiseuille flow with non-axisymmetric disturbances.

Sec(6.1): Quadratic Matrix Operator Pencils Arising In Elasticity Theory

We are going to study a class of damped non-selfadjoint quadratic operator pencils the coefficients of which are unbounded block operator matrices. The aim is to investigate the spectrum of such pencils, to study the properties of the eigenvectors and associated vectors corresponding to certain parts of the spectrum, and to apply the results to the problem of vibrations of a rotating beam with inner and outer damping in a possibly inhomogeneous outer medium. A fundamental tool here is factorization theorems by Markus, Matsaev and Russu ([45], [44], [32], [137]).

Let $\mathcal{H}$ be a separable (infinite dimensional) Hilbert space. We consider a quadratic operator pencil $\mathcal{L}$ acting in the product space $\mathcal{H} \times \mathcal{H}$ and given by the matrix representation

$$\mathcal{L}(\lambda) = \lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \lambda \begin{pmatrix} \alpha A + K_1 & 0 \\ 0 & \alpha A + K_2 \end{pmatrix} + \begin{pmatrix} A & \beta A \\ -\beta A & A \end{pmatrix}, \lambda \in \mathbb{C}. \qquad (1)$$

Here A is an unbounded self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$ having compact resolvent, $A \geq \delta I$ with some $\delta > 0$, α and β are positive constants, and K_1, K_2 are bounded operators, $0 \leq K_1, K_2 \leq \gamma I$ with some $\infty > \gamma \geq 0$. The domain of $\mathcal{L}(\lambda)$ is independent of λ and given by $\mathcal{D}(\mathcal{L}(\lambda)) = \mathcal{D}(A) \times \mathcal{D}(A)$.

An example for such a quadratic operator pencil arises in elasticity theory: The system of differential equations

$$EI\frac{\partial^4 u}{\partial z^4} + \omega\kappa EI\frac{\partial^4 v}{\partial z^4} + \kappa EI\frac{\partial^5 u}{\partial z^4 \partial t} + \varepsilon_1\frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} = 0, \qquad (2)$$

$$EI\frac{\partial^4 v}{\partial z^4} - \omega\kappa EI\frac{\partial^4 u}{\partial z^4} + \kappa EI\frac{\partial^5 v}{\partial z^4 \partial t} + \varepsilon_2\frac{\partial v}{\partial t} + m\frac{\partial^2 v}{\partial t^2} = 0, \qquad (3)$$

on the finite interval $[0, l]$ describes the vibrations of a rotating beam of length l and mass density m per unit length.

Here $EI > 0$ is the (constant) bending stiffness of the beam sections, $\omega > 0$ is the angular frequency of the rotation, $\kappa > 0$ is the coefficient of inner damping (Voigt material), and ε_1, ε_2 are nonnegative continuous functions on $[0, l]$ describing the outer viscous damping. In the general case when the outer medium is inhomogeneous, one has $\varepsilon_1 \not\equiv \varepsilon_2$ (see e.g. [49]).

The boundary conditions to be imposed e.g. in the case of hinged ends are

$$u(0,t) = u(l,t) = \frac{\partial^2 u}{\partial z^2}(0,t) = \frac{\partial^2 u}{\partial z^2}(l,t) = 0,$$
$$v(0,t) = v(l,t) = \frac{\partial^2 v}{\partial z^2}(0,t) = \frac{\partial^2 v}{\partial z^2}(l,t) = 0, \qquad (4)$$

For simplicity, we assume that $m \equiv 1$. Then separation of variables

$$\left(u(z,t), v(z,t)\right)^t = e^{\lambda t}\left(y_1(z), y_2(z)\right)^t, z \in [0, l], t \geq 0, \qquad (5)$$

leads to a spectral problem of the form

$$\mathcal{L}(\lambda)y = 0, \lambda \in \mathbb{C},$$

for $y = (y_1, y_2)^t$ in the Hilbert space $L_2(0,l) \times L_2(0,l)$ where the operators A and K in $L_2(0,l)$ are given by

$$Ay := EI\, y^{(4)}, \mathcal{D}(A) := \{y \in L_2(0,l): y(0) = y(l) = y''(0) = y''(l) = 0\}, \quad (6)$$

$$K_i y := \varepsilon_i y, \mathcal{D}(K_i) := L_2(0,l), i = 1,2, \quad (7)$$

and the constants α and β are given by $\alpha := \kappa$, $\beta := \omega\kappa$.

We first determine the structure of the spectrum of the quadratic operator pencil (1). We show that its essential spectrum consists of the points $-\frac{1+i\beta}{\alpha}, -\frac{1-i\beta}{\alpha}$, and that outside the essential spectrum $\mathcal{L}$ has 3 branches of eigenvalues accumulating at the points of the essential spectrum and at ∞. Secondly, we show a criterion for the stability of the pencil $\mathcal{L}$, that is, a criterion guaranteeing that the spectrum of $\mathcal{L}$ lies in the open left half plane.

We consider the case $K_1 = K_2$ where $\mathcal{L}$ in fact decomposes into two quadratic pencils in $\mathcal{H}$. In this case the eigenvalue branch accumulating at ∞ splits again into two branches. We derive theorems about the minimality, completeness and basis properties of the eigenvectors and associated vectors corresponding to the 4 branches of eigenvalues of $\mathcal{L}$.

We consider the case $K_1 \neq K_2$. We show a theorem about the minimality, completeness and basis properties of the eigenvectors and associated vectors corresponding to the branch of eigenvalues of $\mathcal{L}$ accumulating at ∞. Finally, we apply all results to the problem (2)–(4) of vibrations of a rotating beam.

We define the resolvent set $\rho(\mathcal{L})$ of the quadratic operator pencil $\mathcal{L}$ as

$$\rho(\mathcal{L}) := \{\lambda \in \mathbb{C} : \mathcal{L}(\lambda) : \mathcal{D}(A) \times \mathcal{D}(A) \to \mathcal{H} \times \mathcal{H} \text{ is bijective}, \ \mathcal{L}(\lambda)^{-1} \text{ is bounded}\}$$

and its spectrum $\sigma(\mathcal{L})$ as $\sigma(\mathcal{L}) := \mathbb{C}\backslash\rho(\mathcal{L})$. For $\lambda \in \mathbb{C}$, the operator $\mathcal{L}(\lambda)$ is called Fredholm if $\mathcal{L}(\lambda)$ is closed, its kernel is finite dimensional and its range is finite codimensional (see e.g. [48]). A point $\lambda_0 \in \mathbb{C}$ is said to be an eigenvalue of $\mathcal{L}$ if $\mathcal{L}(\lambda)$ is not injective. An eigenvalue $\lambda_0 \in \mathbb{C}$ of $\mathcal{L}$ is called normal (or of finite type) if λ_0 is isolated and $\mathcal{L}(\lambda_0)$ is Fredholm. The essential spectrum of $\mathcal{L}$ is defined as

$$\sigma_{ess}(\mathcal{L}) := \{\lambda \in \mathbb{C} : \mathcal{L}(\lambda) \text{ is not Fredholm}\}.$$

In order to determine the essential spectrum of the operator pencil $\mathcal{L}$, we consider the transformed pencil $\mathcal{L}_d$ given by $\mathcal{L}_d(\lambda) := S^{-1}\mathcal{L}(\lambda)S$ on $\mathcal{D}(A) \times \mathcal{D}(A)$ for $\lambda \in \mathbb{C}$ where the operator matrix S in $\mathcal{H} \times \mathcal{H}$ is of the form

$$S := \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

Then, for $\lambda \in \mathbb{C}$,

$$\mathcal{L}_d(\lambda) := \lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \lambda \begin{pmatrix} \alpha A + \frac{1}{2}(K_1 + K_2) & \frac{i}{2}(K_1 - K_2) \\ -\frac{i}{2}(K_1 - K_2) & \alpha A + \frac{1}{2}(K_1 + K_2) \end{pmatrix} + \begin{pmatrix} (1+i\beta)A & 0 \\ 0 & (1-i\beta)A \end{pmatrix},$$

and $\mathcal{L}_d(\lambda)$ is closed (Fredholm) if and only if $\mathcal{L}(\lambda)$ is closed (Fredholm).

Theorem(6.1.1)[51]: The essential spectrum of $\mathcal{L}$ consists of the two points

$$-\frac{1+i\beta}{\alpha}, -\frac{1-i\beta}{\alpha}.$$

The other points of the spectrum of $\mathcal{L}$ are normal eigenvalues which accumulate at most at the points $-\frac{1+i\beta}{\alpha}, -\frac{1-i\beta}{\alpha}$, and at ∞.

Proof. Let $\lambda \in \mathbb{C}$. If we write $\mathcal{L}_d(\lambda)$ in the form

$$\begin{pmatrix} (\lambda\alpha + (1+i\beta))A + \lambda^2 I + \lambda\frac{1}{2}(K_1+K_2) & \lambda\frac{i}{2}(K_1-K_2) \\ -\lambda\frac{i}{2}(K_1-K_2) & (\lambda\alpha + (1-i\beta))A + \lambda^2 I + \lambda\frac{1}{2}(K_1+K_2) \end{pmatrix},$$

it is not difficult to see that $\mathcal{L}_d(\lambda)$ (with domain $\mathcal{D}(A) \times \mathcal{D}(A)$) is closed if and only if $\lambda \neq -\frac{1 \pm i\beta}{\alpha}$. Now let $\lambda \neq -\frac{1 \pm i\beta}{\alpha}$. Since

$$\mathcal{L}_d(\lambda)\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} = \lambda^2\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} + \lambda\begin{pmatrix} \alpha I + \frac{1}{2}(K_1+K_2)A^{-1} & \frac{i}{2}(K_1-K_2)A^{-1} \\ -\frac{i}{2}(K_1-K_2)A^{-1} & \alpha I + \frac{1}{2}(K_1+K_2)A^{-1} \end{pmatrix}$$

$$+\begin{pmatrix} (1+i\beta)I & 0 \\ 0 & (1-i\beta)I \end{pmatrix} = \begin{pmatrix} (1+i\beta+\lambda\alpha)I & 0 \\ 0 & (1-i\beta+\lambda\alpha)I \end{pmatrix} + K(\lambda), \qquad (8)$$

where $K(\lambda)$ is a compact operator in $\mathcal{H} \times \mathcal{H}$, the operator on the left hand side of (8) is Fredholm (see e.g. [127]). Hence the same is true for $\mathcal{L}_d(\lambda)$. On the other hand, since $\mathcal{H}$ is infinite dimensional, it follows from (8) that $\mathcal{L}_d\left(-\frac{1 \pm i\beta}{\alpha}\right)$. is not Fredholm. Moreover, the operator in (8) is bijective for $\lambda = 0$. Now the theorem follows e.g. from a theorem about analytic Fredholm operator valued functions (see [127]).

Theorem(6.1.2)[51]: Assume that there exists a $\mu > 0$ with $K_1 \geq \mu I, K_2 \geq \mu I$, and such that

$$\mu\alpha \geq \delta \text{ and } \frac{\beta^2}{4\mu\alpha} < 1, \quad \text{or } \frac{\mu}{\alpha} < \delta \text{ and } \frac{\beta^2\delta}{(\alpha\delta+\mu)^2} < 1 \qquad (9)$$

Then the spectrum of $\mathcal{L}$ lies in the open left half plane.

Proof. If $\lambda_0 \in \sigma_{ess}(\mathcal{L})$, then obviously $Re(\lambda_0) < 0$ by the above theorem. Otherwise, if $\lambda_0 \in \sigma(\mathcal{L})\backslash\sigma_{ess}(\mathcal{L})$, then λ_0 is an eigenvalue of $\mathcal{L}$.

Let $Y = (y,z)^t \in \mathcal{D}(A) \times \mathcal{D}(A), \|y\|^2 + \|z\|^2 = 1$, be a corresponding eigenvector. Then $(\mathcal{L}(\lambda_0)Y, Y) = 0$ implies that

$$\mathrm{Re}(\lambda_0)^2 - \mathrm{Im}(\lambda_0)^2 + \mathrm{Re}(\lambda_0)\left(\alpha\big((Ay,y)+(Az,z)\big) + (K_1y,y) + (K_2z,z)\right)$$
$$+(Ay,y)+(Az,z) = 0, \qquad (10)$$

$$2\mathrm{Re}(\lambda_0)\,\mathrm{Im}(\lambda_0) + \mathrm{Im}(\lambda_0)\left(\alpha\big((Ay,y)+(Az,z)\big) + (K_1y,y) + (K_2z,z)\right)$$
$$+ 2\beta\mathrm{Im}(Az,y) = 0, \qquad (11)$$

Calculating $\mathrm{Im}(\lambda_0)$ from (11), using the estimate

$$|2\,\mathrm{Im}(Az,y)| \leq 2|(Az,y)| = 2\left|(A^{1/2}z, A^{1/2}y)\right|$$
$$\leq \left\|A^{1/2}z\right\|^2 + \left\|A^{1/2}y\right\|^2 = (Ay,y) + (Az,z)$$

and substituting it into (10), we arrive at

$$\mathrm{Re}(\lambda_0)^2 - \frac{\beta^2\big((Ay,y)+(Az,z)\big)^2}{\left(2\mathrm{Re}(\lambda_0) + \alpha\big((Ay,y)+(Az,z)\big) + (K_1y,y) + (K_2y,y)\right)^2}$$
$$+ \mathrm{Re}(\lambda_0)\left(\alpha\big((Ay,y)+(Az,z)\big) + (K_1y,y) + (K_2z,z)\right) + (Ay,y) + (Az,z) \leq 0. \quad (12)$$

The left hand side is monotonically increasing for $\mathrm{Re}(\lambda_0) \in [0,\infty)$. But the condition (9) and $A \geq \delta > 0$ imply that

$$-\frac{\beta^2\big((Ay,y)+(Az,z)\big)^2}{\left(\alpha\big((Ay,y)+(Az,z)\big) + (K_1y,y) + (K_2z,z)\right)^2} + (Ay,y) + (Az,z) > 0,$$

a contradiction to (12). Hence $Re(\lambda_0) < 0$.

In case $K_1 = K_2 = K$ the operator pencil $\mathcal{L}_d$ in $\mathcal{H} \times \mathcal{H}$ is the orthogonal sum of two quadratic operator pencils $\mathcal{L}_\pm$ in $\mathcal{H}$,

$$\mathcal{L}_d(\lambda) = \begin{pmatrix} \mathcal{L}_+(\lambda) & 0 \\ 0 & \mathcal{L}_-(\lambda) \end{pmatrix}, \lambda \in \mathbb{C},$$

where $\mathcal{L}_\pm(\lambda) := \lambda^2 I + \lambda(\alpha A + K) + (1 \pm i\beta)A, \lambda \in \mathbb{C}$.

In the following we are going to show minimality, completeness and basis results for the eigenvectors and associated vectors corresponding to various branches of eigenvalues of $\mathcal{L}$.

With regard to the eigenvalues which will show to accumulate at ∞, we introduce the auxiliary operator pencils $\mathcal{L}_\pm^1$ given by

$$\mathcal{L}_\pm^1(\lambda) := \frac{\lambda^2}{\alpha} A^{-1/2} \mathcal{L}_\pm\left(\frac{1}{\lambda}\right) A^{-1/2}$$
$$= \lambda^2 \frac{1 \pm i\beta}{\alpha} I + \lambda\left(I + \left(\frac{1}{\alpha}\right) A^{-1/2} K A^{-1/2}\right) + \left(\frac{1}{\alpha}\right) A^{-1}, \ \lambda \in \mathbb{C}.$$

Lemma(6.1.3)[51]: The spectrum and the essential spectrum of $\mathcal{L}_\pm^1$ are given by:

(i) $\sigma(\mathcal{L}_\pm^1) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(\mathcal{L}_\pm)\right\} \cup \{0\}$;

(ii) $\sigma_{ess}(\mathcal{L}_\pm^1) = \left\{0, \ \frac{-\alpha}{1 \pm i\beta}\right\}$.

Proof. The assertions are immediate from Theorem (6.1.1) and from the definition of $\mathcal{L}^1{}_\pm$.

The numerical range (or root domain) of a quadratic operator polynomial $\mathcal{T}$ in $\mathcal{H}$ is the set of all roots of all possible polynomials $(\mathcal{T}(\cdot)y, y)$, $y \in \mathcal{H}$, $y \neq 0$ (see [137]). It consists of at most two components. If the numerical range of $\mathcal{T}$ consists of two disjoint components (called the root zones of $\mathcal{T}$), then, clearly, for any $y \in \mathcal{H}$, $y \neq 0$, the polynomial $(\mathcal{T}(\cdot)y, y)$ has exactly one root in each component.

Lemma(6.1.4)[51]: The numerical range of $\mathcal{L}_+^1$ consists of two components $\Delta_+^1, \tilde{\Delta}_+^1$ which are bounded and separated by the strip

$$\left\{\lambda \in \mathbb{C} : 0 < \operatorname{Im}(\lambda) < \frac{\alpha\beta}{1+\beta^2}\right\}. \tag{13}$$

say Δ_+^1 below and $\tilde{\Delta}_+^1$ above this strip. There exists a simple contour Γ_+^1 surrounding $\overline{\Delta}_+^1 \cup \{0\}$ and separating $\overline{\Delta}_+^1 \cup \{0\}$ from $\tilde{\Delta}_+^1$, and such that

$$\inf_{\lambda \in \Gamma_+^1, \|y\|=1} |(\mathcal{L}_+^1(\lambda)y, y)| > 0.$$

Proof. The first assertion follows from the fact that $\mathcal{L}_+^1$ is a pencil of bounded operators. For the proof of the second statement, fix an element $y \in \mathcal{H}$, $\|y\| = 1$, consider the function

$$\varphi(\lambda, \eta) := \lambda^2 \frac{1+i\beta}{\alpha} + \lambda\left(1 + \eta \frac{1}{\alpha}\left(KA^{-1/2}y, A^{-1/2}y\right)\right) + \eta \frac{1}{\alpha}\left(A^{-1/2}y, A^{-1/2}y\right)$$

for $\lambda \in \mathbb{C}, \eta \in [0,1]$, and assume that there exists a point λ_0 in the strip (13) and an $\eta \in [0,1]$ such that $\varphi(\lambda_0, \eta) = 0$. The point λ_0 has a representation $\lambda_0 = t + id$ with $t \in \mathbb{R}$ and $0 < d < \frac{\alpha\beta}{1+\beta^2}$. Hence

$$(t^2 + 2itd - d^2)\frac{1+i\beta}{\alpha} + (t + id)\left(1 + \eta \frac{1}{\alpha}\left(KA^{-1/2}y, A^{-1/2}y\right)\right) + \eta \frac{1}{\alpha}\left\|A^{-1/2}y\right\|^2 = 0.$$

Taking the imaginary part and multiplying by $\frac{\alpha}{\beta}$ yields

$$t^2 + \frac{2\,td}{\beta} - d^2 + d\,\frac{\alpha}{\beta}\left(1 + \eta\frac{1}{\alpha}\left(KA^{-1/2}y, A^{-1/2}y\right)\right) = 0.$$

This is equivalent to

$$\left(t+\frac{d}{\beta}\right)^2=\frac{d}{\beta^2}\left(d(1+\beta^2)-\alpha\beta\left(1+\eta\,\frac{1}{\alpha}\left(KA^{-1/2}y,A^{-1/2}y\right)\right)\right),$$

which implies, since $K\geq 0$, that

$$d\geq\frac{\alpha\beta}{1+\beta^2},$$

a contradiction. Thus $\varphi(\lambda,\eta)\neq 0$, $\eta\in[0,1]$, and in particular$(\mathcal{L}_+^1(\lambda)y,y)=\|y\|^2\,\varphi(\lambda,1)\neq 0$ for all λ in the strip (13).

The zeros $\lambda_1(\eta),\lambda_2(\eta)\circ f\;\varphi(\cdot,\eta)=0$ depend continuously on the parameter η, and $\lambda_1(0)=0,\lambda_2(0)=-\frac{\alpha}{1+i\beta}$ are separated by the strip (13). Then, according to what was proved above, so are $\lambda_1(\eta),\lambda_2(\eta)$ for all $\eta\in[0,1]$. In particular, the zeros of $(\mathcal{L}_+^1(\cdot)y,y)=\|y\|^2\varphi(\cdot,1)$ are separated by the strip (13). This proves the second statement. The remaining assertions are immediate.

The subsequent corollary about the existence of a spectral root of $\mathcal{L}_+^1$ (or, equivalently, of a canonical factorization of $\lambda^{-1}(\mathcal{L}_+^1(\lambda))$ follows from the above lemma by some general results of Markus and Matsaev ([45], [44],[137]).

Corollary(6.1.5)[51]: The operator pencil $\mathcal{L}_+^1$ has a spectral root Z_+^1 such that

$$\sigma(Z_+^1)=\sigma(\mathcal{L}_+^1)\cap\Delta_+^1.$$

In the following let $\mathcal{S}_p\leq p\leq\infty$, denote the von Neumann-Schatten classes of compact operators (see [154], [137]). Further, for an operator $T\in\mathcal{S}_\infty$ we denote by $n(\tau,T)$ the sum of the algebraic multiplicities of the eigenvalues of Tin $\{\lambda\in\mathbb{C}:|\lambda|>\tau^{-1}\}$. For a quadratic operator pencil $\mathcal{T}$ the nonzero spectrum of which in some bounded domain G containing 0 consists of a sequence of eigenvalues of finite algebraic multiplicity converging to 0, we denote by $n(\tau,G,\mathcal{T})$ the sum of the algebraic multiplicities of the eigenvalues of $\mathcal{T}$ in $\{\lambda\in\mathbb{C}:|\lambda|>\tau^{-1}\}\cap G$ (see [137]). Using a theorem of Markus, Matsaev and Russu[32] , we obtain:

Theorem(6.1.6)[51]: (i) The set of eigenvectors and associated vectors Corresponding to the eigenvalues of $\mathcal{L}_+^1$ in Δ_+^1 is minimal in $\mathcal{H}$.

(ii) If $A^{-1}\in\mathcal{S}_p$ for some $p<\infty$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^1$ in Δ_+^1 is complete in $\mathcal{H}$. If, in addition, $n\left(\tau,\frac{1}{\alpha}A^{-1}\right)\sim c_1\tau^{c_2}$ *as* $\tau\to\infty$ with some $0<c_1$, $c_2<\infty$, then $n(\tau,G_+^1,\mathcal{L}_+^1)\sim c_1\tau^{c_2}$, where G_+^1 is the interior of the curve Γ_+^1.

(iii) If $n(\tau,A^{-1})=o(\tau^\gamma)$ for some $\gamma\in(0,\frac{1}{2}]$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^1$ in Δ_+^1 is a Riesz basis with parentheses in $\mathcal{H}$. If, in addition, $n\left(\tau,\frac{1}{\alpha}A^{-1}\right)=c_1\tau^{c_2}+O(\tau^\beta)$ for some $0<c_1,c_2<\infty,0\leq\beta<\alpha\leq\beta+\gamma$, then also $n\,(\tau,\,G_+^1,\,\mathcal{L}_+^1)=c_1\tau^{c_2}+O(\tau^\beta)$.

Proof. The theorem follows from a general result of Markus, Matsaev and Russu (see [32]) which is contained in [137]. To apply the statements therein, we choose $H=\frac{1}{\alpha}A^{-1},T=0$ for (ii) and $H=\frac{1}{\alpha}A^{-1},D_0=0,\;D_1=\left(\frac{1}{\alpha}\right)A^{-1/2}K\,A^{\frac{-1}{2+\gamma}}$ where $\gamma\left(0,\frac{1}{2}\right]$ for (iii).

Note that**:** Analogous assertions hold for the pencil $\mathcal{L}_-^1$.

With regard to the branches of eigenvalues possibly accumulating at $-\frac{1\pm i\beta}{\alpha}$, we first consider the pencils $\mathcal{L}_\pm$ themselves.

Lemma(6.1.7)[51]: the numerical rang of $\mathcal{L}_+$ consists of two components $\Delta_+, \tilde{\Delta}_+$ which are separated by the strip

$$\{\lambda \in \mathbb{C} : -\delta_2 < \operatorname{Im}(\lambda) < \delta_2\}$$

Where

$$\delta_2 := \beta((\alpha^2 + (2\alpha\|K\| + 4)\delta^{-1} + \|K\|^2\delta^{-2})^2 + 16\beta^2\delta^{-2})^{-\frac{1}{4}}.$$

The component located in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Im}(\lambda) \le -\delta_2\}$, say Δ_+, is bounded, and there exists a simple closed curve Γ_+ surrounding Δ_+ and separating it from $\tilde{\Delta}_+$ and such that

$$\inf_{\substack{\lambda\in\Gamma_+ \\ y\in\mathcal{D}(A), \|y\|=1}} |(\mathcal{L}_+(\lambda)y, y)| > 0 \quad .$$

Proof. Let $y \in \mathcal{D}(A)$, $\|y\| = 1$. Then the solutions $\lambda_\pm(y)$ *of* $((\mathcal{L}_+(\lambda)y, y)) = 0$ are given by

$$\lambda_\pm(y) = -\frac{\alpha(Ay, y) + (Ky, y)}{2} \pm \sqrt{\frac{\left(\alpha(Ay, y) + (Ky, y)\right)^2}{4} - (1 + i\beta)(Ay, y)}. \qquad (14)$$

Hence

$$\left|\operatorname{Im}\left(\lambda_\pm(y)\right)\right| = \frac{1}{2}\left|\operatorname{Im}\left(\left(\alpha(Ay, y) + (Ky, y)\right)^2 - 4(1 + i\beta)(Ay, y)\right)^{\frac{1}{2}}\right|$$

$$= \frac{1}{2\sqrt{2}}\left(4(Ay, y) - \left(\alpha(Ay, y) + (Ky, y)\right)^2\right.$$

$$\left. + \left(\left(\left(\alpha(Ay, y) + (Ky, y)\right)^2 - 4(Ay, y)\right)^2 + 16\beta^2(Ay, y)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$\ge \beta\left(\left(\frac{\left(\alpha(Ay, y) + (Ky, y)\right)^2 - 4(Ay, y)}{(Ay, y)^2}\right)^2 + \frac{16\beta^2}{(Ay, y)^2}\right)^{-\frac{1}{2}} \ge \delta_2.$$

This proves the first assertion. The second statement follows from the fact that the root $\lambda_+(y)$ lying in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Im}(\lambda) \le -\delta_2\}$ tends to $-\frac{1+i\beta}{\alpha}$ when (Ay, y) tends to infinity. The remaining assertions are then immediate.

In order to apply the results of Markus, Matsaev and Russu used before we need to consider pencils of bounded operators. Therefore we introduce

$$\mathcal{L}^2{}_\pm(\lambda) := \left(\frac{1}{\alpha}\right)A^{-1/2}\mathcal{L}_\pm\left(\lambda - \frac{1 \pm i\beta}{\alpha}\right)A^{-\frac{1}{2}}$$

$$= \frac{\lambda^2}{\alpha}A^{-1} + \lambda\left(I - \frac{2(1 \pm i\beta)}{\alpha^2}A^{-1} + \left(\frac{1}{\alpha}\right)A^{-1/2}KA^{-1/2}\right) + B_\pm$$

where

$$B_\pm = \frac{(1 \pm i\beta)^2}{\alpha^3}A^{-1} - \frac{1 \pm i\beta}{\alpha^2}A^{-1/2}K\,A^{-1/2}.$$

Corollary(6.1.8)[51]: The numerical range of $\mathcal{L}_+^2$ consists of two components $\Delta_+^2, \tilde{\Delta}_+^2$ which are bounded and separated by the strip

$$\left\{\lambda \in \mathbb{C} : \frac{\beta}{\alpha} - \delta_2 < \operatorname{Im}(\lambda) < \frac{\beta}{\alpha} + \delta_2\right\},$$

say Δ_+^2 below and $\tilde{\Delta}_+^2$ above this strip. There exists a simple closed curve Γ_+^2 surrounding $\Delta_+^2 \cup \{0\}$ and separating it from $\tilde{\Delta}_+^2$, and such that

$$\inf_{\lambda\in\Gamma_+^2,\|y\|=1} |(\mathcal{L}_+^2(\lambda)y,y)| > 0$$

Theorem(6.1.9)[51]: (i) The set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^2$ in Δ_+^2 is minimal in $\mathcal{H}$.

(ii) If $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_+^2$ in Δ_+^2 is complete in $\mathcal{H}$. If, in addition, $n(\tau, B_+) \sim c_1 \tau^{c_2}$ as $\tau \to \infty$ with some $0 < c_1, c_2 < \infty$, then $n(\tau, G_+^2, \mathcal{L}_+^2) \sim c_1 \tau^{c_2}$, where G_+^2 is the interior of the curve Γ_+^2.

Proof. As in the proof of Theorem (6.1.6), we use [137]. To apply the statements therein, we now choose $H = B_+, T = 0$ for (ii), and we note that if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, then also $B_+ \in \mathcal{S}_p$ (see e.g. [154]).

Note that**:** Analogous assertions hold for the pencil $\mathcal{L}_-^2$.

In order to formulate statements for the original pencil $\mathcal{L}$, we first note that

$$\sigma(\mathcal{L}) = \sigma(\mathcal{L}_d) = \sigma(\mathcal{L}_+) \cup \sigma(\mathcal{L}_-).$$

We denote by λ_k^1 and $\lambda_k^2, k = 1, 2, \ldots,$ the eigenvalues of the pencil $\mathcal{L}_+$ located in the upper and lower half plane, respectively (counted according to their algebraic multiplicities). It is not difficult to see that then the complex conjugates $\overline{\lambda_k^1}$ and $\overline{\lambda_k^2}$, $k = 1, 2, \ldots,$ are the eigenvalues of the pencil $\mathcal{L}_-$ located in the lower and upper half plane, respectively. We denote the corresponding eigenvectors and associated vectors of $\mathcal{L}_+$ by y_k^1 and y_k^2, and those of $\mathcal{L}_-$ by $\overline{y_k^1}$ and $\overline{y_k^2}$.

Then the eigenvalues of $\mathcal{L}_d$ (and hence those of $\mathcal{L}$) can be separated into the 4 branches $\{\lambda_k^1\}$,$\{\lambda_k^2\}$and $\{\overline{\lambda_k^1}\}$,$\{\overline{\lambda_k^2}$, which lie symmetrically to the real axis. The corresponding eigenvectors and associated vectors of $\mathcal{L}_d$ are of the form

$$(y_k^1, 0)^t, (y_k^2, 0)^t, \left(0, \overline{y_k^1}\right)^t, \left(0, \overline{y_k^2}\right)^t.$$

The respective eigenvectors of $\mathcal{L}$ are given by

$$(y_k^1, iy_k^1)^t, (y_k^2, iy_k^2)^t, \left(i\overline{y}_k^1, \overline{y}_k^1\right)^t, \left(i\overline{y_k^2}, \overline{y_k^2}\right)^t,$$

and there are analogous formulas for the associated vectors of $\mathcal{L}$.

Theorem(6.1.10)[51]:(i) The set of eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalues λ_k^1 and $\overline{\lambda_k^1}$, $k = 1,2, \ldots,$ is minimal in the space $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ where $\mathcal{H}_{A^{-1}} = (\mathcal{H}, (A^{-1}\cdot, A^{-1}\cdot))$, complete in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, and a Riesz basis with parentheses in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $n(\tau, A^{-1}) = O(\tau^{\gamma})$ for some $\gamma \in (0, \frac{1}{2}]$.

(ii) The set of eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalues λ_k^2 and $\overline{\lambda_k^2}, k = 1,2, \ldots,$ is minimal in the space $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ and complete in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$.

In particular, λ_k^1, $\overline{\lambda_k^1} \longrightarrow \infty$, $\lambda_k^2 \longrightarrow -\frac{1+i\beta}{\alpha}$, $\overline{\lambda_k^2} \longrightarrow -\frac{1-i\beta}{\alpha}$ for $k \longrightarrow \infty$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$.

Proof. The assertions in (i) and (ii) follow from Theorems (6.1.6) and (6. 1.9) The statement about the accumulation at the points of the essential spectrum follows from Theorem (6.1.1) together with the minimality and completeness from (i) and (ii).

Now we consider the general case of a quadratic block operator matrix pencil (1) with possibly different K_1, K_2. In this case $\mathcal{L}$ cannot be written as the orthogonal sum of two pencils in $\mathcal{H}$, and hence there exists no decomposition of the spectrum as it was used in above.

In order to guarantee a certain subdivision of the spectrum of $\mathcal{L}$ also here and to obtain minimality, completeness and basis results for the eigenvectors and associated vectors, we have to assume in addition that

$$\delta > 4/\alpha^2 \quad . \tag{15}$$

We choose $\rho > 0$ such that

$$\rho > \frac{\beta\alpha\delta}{\alpha^2\delta - 4} \quad . \tag{16}$$

Lemma(6.1.11)[51]: On the segment

$$\Gamma_1 := \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) = -\frac{2}{\alpha}, |\mathrm{Im}\,(\lambda)| \le \rho\}$$

we have the estimate

$$\inf_{\substack{\lambda \in \Gamma_1 \\ Y \in \mathcal{D}(A)\times\mathcal{D}(A), \|Y\|=1}} |(\mathcal{L}(\lambda)Y, Y)| > 0 \quad .$$

Proof. Let $\lambda = -\frac{2}{\alpha} + i\tau$ with $\tau \in \mathbb{R}, Y = (y, z)^t$ with $y,\ z \in \mathcal{D}(A)$ and $\|y\|^2 + \|z\|^2 = 1$. Then

$$|(\mathcal{L}(\lambda)Y, Y)| = \left| \begin{array}{c} \left(-\frac{2}{\alpha} + i\tau\right)^2 + \left(-\frac{2}{\alpha} + i\tau\right)\big(\alpha(Ay, y) + \alpha(Az, z) + (K_1 y, y) + (K_2 z, z)\big) \\ +(Ay, y) + (Az, z) + \beta(Az, y) - \beta(Ay, z) \end{array} \right|$$

$$\ge \left| \frac{4}{\alpha^2} - \tau^2 - (Ay, y) - (Az, z) - \frac{2}{\alpha}(K_1 y, y) - \frac{2}{\alpha}(K_2 z, z) \right| \ge \delta - \frac{4}{\alpha^2} > 0$$

by assumption (15).

Lemma(6.1.12)[51]: On the rays

$$\Gamma_2^{\pm} := \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) \ge -\frac{2}{\alpha}, \mathrm{Im}\,(\lambda) = \pm\rho\}$$

we have the estimate

$$\inf_{\substack{\lambda \in \Gamma_2^{\pm} \\ Y \in \mathcal{D}(A)\times\mathcal{D}(A), \|Y\|=1}} |(\mathcal{L}(\lambda)Y, Y)| > 0 \quad .$$

Proof. Let $\lambda = t + i\rho$ with $t \ge -\frac{2}{\alpha}, Y = (y, z)^t$ with $y, z \in \mathcal{D}(A)$ and $\|y\|^2 + \|z\|^2 = 1$. Then

$$\big|\mathcal{L}((\lambda)Y, Y)\big| \ge |\mathrm{Im}(\mathcal{L}(\lambda)Y, Y)|$$

$$= \big|2t\rho + \rho\big(\alpha(Ay, y) + \alpha(Az, z) + (K_1 y, y) + (K_2 z, z)\big) + 2\beta\,\mathrm{Im}(Az, y)\big|$$

$$= \big|2t + \big(\alpha(Ay, y) + \alpha(Az, z) + (K_1 y, y) + (K_2 z, z)\big)\big|$$

$$\cdot \left|\rho + \frac{2\beta\,\mathrm{Im}(Az, y)}{2t + \big(\alpha(Ay, y) + \alpha(Az, z) + (K_1 y, y) + (K_2 z, z)\big)}\right|$$

$$\ge \left(-\frac{4}{\alpha} + \alpha\delta\right)\left|\rho - \frac{2\beta\,|\mathrm{Im}(Az, y)|}{\big|2t + \big(\alpha(Ay, y) + \alpha(Az, z) + (K_1 y, y) + (K_2 z, z)\big)\big|}\right|$$

$$\ge \alpha\left(\delta - \frac{4}{\alpha^2}\right)\left|\rho - \frac{2\alpha\beta|(Az, y)|}{\alpha^2\delta - 4}\right| \ge \alpha\left(\delta - \frac{4}{\alpha^2}\right)\left|\rho - \frac{2\alpha\beta\|A^{1/2}z\|\|A^{1/2}y\|}{\alpha^2\delta - 4}\right|$$

$$\ge \alpha\left(\delta - \frac{4}{\alpha^2}\right)\left|\rho - \frac{\alpha\beta\left(\|A^{1/2}z\|^2 + \|A^{1/2}y\|^2\right)}{\alpha^2\delta - 4}\right| \ge \alpha\left(\delta - \frac{4}{\alpha^2}\right)\left|\rho - \frac{2\alpha\beta\delta}{\alpha^2\delta - 4}\right| > 0$$

by assumption (15) and the choice of ρ according to (16).

Lemma(6.1.13)[51]: On the segment

$$\Gamma_3 := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = \rho_1, |\operatorname{Im}(\lambda)| \leq \rho\} \text{ with } \rho_1 > \rho$$

we have

$$\inf_{\substack{\lambda\in\Gamma_3 \\ Y\in\mathcal{D}(A)\times\mathcal{D}(A), \|Y\|=1}} |(\mathcal{L}(\lambda)Y, Y)| > 0 .$$

Proof. Let $\lambda = \rho_1 + i\tau$ with $|\tau| \leq \rho, Y = (y, z)^t$ with $y, z \in \mathcal{D}(A)$ and $\|y\|^2 + \|z\|^2 = 1$. Then

$$|(\mathcal{L}(\lambda)Y, Y)| \geq |\operatorname{Re}\,(\mathcal{L}(\lambda)Y, Y)|$$
$$= \left| \begin{matrix} \rho_1^2 - \tau^2 + \rho_1\big(\alpha(Ay, y) + \alpha(Az, z) + (K_1y, y) + (K_2z, z)\big) \\ +(Ay, y) + (Az, z) \end{matrix} \right| > 2(\alpha\delta\rho_1 + \delta) > 0$$

since $\rho_1 > \rho$.

By Γ we now denote the rectangle the sides of which are Γ_1, Γ_3 and parts of the rays Γ_2^+, Γ_2^-. Further, we denote by G^- the interior and by G^+ the exterior of Γ (without the boundary of Γ).

Lemma(6.1.14)[51]: For any fixed $Y \in \mathcal{D}(A) \times \mathcal{D}(A)$, the polynomial $(\mathcal{L}(\lambda)Y, Y)$ has one root in G^- and one root in G^+.

Proof. Consider the auxiliary pencil

$$\mathcal{L}_1(\lambda) = \mathcal{L}\left(\lambda - \frac{2}{\alpha}\right), \lambda \in \mathbb{C},$$

and the corresponding polynomial $(\mathcal{L}_1(\lambda)Y, Y)$ for fixed $Y = (y, z)^t$ with $y \in \mathcal{D}(A), z \in \mathcal{D}(A), \|y\|^2 + \|z\|^2 = 1$. The roots of $(\mathcal{L}_1(\lambda)Y, Y) = 0$ are the roots of the equation

$$\lambda^2 + \lambda\left(-\frac{4}{\alpha} + \alpha(Ay, y) + \alpha(Az, z) + (K_1y, y) + (K_2z, z)\right) + \frac{4}{\alpha^2} - (Ay, y) - (Az, z)$$
$$-\frac{2}{\alpha}(K_1y, y) - \frac{2}{\alpha}(K_2z, z) + 2i\beta \operatorname{Im}(Az, y) = 0. \qquad (17)$$

Due to assumption (15) the quadratic equation

$$\lambda^2 + \lambda\left(\alpha\delta - \frac{4}{\alpha}\right) + \frac{4}{\alpha^2} - \delta = 0 \qquad (18)$$

possesses exactly one solution on the positive half axis and exactly one on the negative half axis (excluding 0). Now we consider A and K_i, $i = 1,2,$ as perturbations of δI and 0, respectively, i.e., we consider

$$A(\eta) := \eta(A - \delta I) + \delta I, K_i(\eta) = \eta K_i, \qquad i = 1, 2,$$

for $\eta \in [0,1]$ and the pencil $\mathcal{L}_1(\eta, \lambda)$ which arises if we substitute A and K_i, $i = 1, 2,$ in $\mathcal{L}_1(\lambda)$ by $A(\eta)$ and $K_i(\eta)$, $i = 1, 2$, i.e.,

$$\mathcal{L}_1(\eta, \lambda) := \lambda^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \lambda \begin{pmatrix} \alpha A(\eta) + K_1(\eta) - \frac{4}{\alpha} & 0 \\ 0 & \alpha A(\eta) + K_2(\eta) - \frac{4}{\alpha} \end{pmatrix}$$
$$+ \begin{pmatrix} -A(\eta) - \frac{2}{\alpha}K_1(\eta) + \frac{4}{\alpha^2} & \beta A \\ -\beta A & -A(\eta) - \frac{2}{\alpha} K_2(\eta) + \frac{4}{\alpha^2} \end{pmatrix},$$

$\eta \in [0, 1], \lambda \in \mathbb{C}$. Then $\mathcal{L}_1(1, \lambda) = \mathcal{L}_1(\lambda)$ and the equation $(\mathcal{L}_1(0, \lambda)Y, Y) = 0$ is just the equation (18). The quadratic form $(\mathcal{L}_1(\eta, \lambda)Y, Y)$ is analytic with respect to η and λ. The roots $\lambda_1(\eta)$ and $\lambda_2(\eta)$ of $(\mathcal{L}_1(\eta, \lambda)Y, Y) = 0$ are piecewise analytic, they may fail to be analytic in $[0, 1]$ only if for some $\eta \in [0,1]$, we have $\lambda_1(\eta) = \lambda_2(\eta)$. Since $\lambda_1(0)$ and $\lambda_2(0)$

are the roots of (18), one of them, say $\lambda_1(0)$, lies in the open left half plane and the other in the open right half plane. According to Lemmas (6.1.11), (6.1.12) and (6.1.13), we have

$$\inf_{\substack{\lambda\in\hat{\Gamma} \\ Y\in\mathcal{D}(A)\times\mathcal{D}(A),\|Y\|=1}} |(\mathcal{L}_1(\lambda)Y,Y)| > 0,$$

where $\hat{\Gamma} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) > 0\}$ is the rectangle $\hat{\Gamma} := \{\lambda + \frac{2}{\alpha} : \lambda \in \Gamma\}$. Now we choose $\rho_1 > \rho$ such that $\rho_1 > \lambda_2(0)$. Then $\lambda_2(\eta)$ remains inside $\hat{\Gamma}$ and $\lambda_1(\eta)$ outside $\hat{\Gamma}$ for all $\eta \in [0,1]$. The roots of $(\mathcal{L}(\lambda)Y,Y)$ are given by $\lambda_1(1) - \frac{2}{\alpha}$ and $\lambda_2(1) - \frac{2}{\alpha}$. Hence the first one lies outside Γ and the second one inside Γ.

In order to prove results about the eigenvectors and associated vectors corresponding to the branches of eigenvalues possibly accumulating at ∞, we introduce

$$\tilde{A}^{-1} := \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix},$$

and consider the pencil

$$\mathcal{L}_2(\lambda) := \frac{\lambda^2}{\alpha}\,\tilde{A}^{-1/2}\mathcal{L}\left(\frac{1}{\lambda}\right)\tilde{A}^{-1/2} = \lambda^2\,\frac{1}{\alpha}\begin{pmatrix} I & \beta \\ -\beta & I \end{pmatrix}$$
$$+\ \lambda\begin{pmatrix} I + \frac{1}{\alpha}A^{-1/2}K_1A^{-1/2} & 0 \\ 0 & I + \frac{1}{\alpha}A^{-1/2}K_2A^{-1/2} \end{pmatrix} + \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix}$$

for $\lambda \in \mathbb{C}$.

By $\tilde{\Gamma}$ we denote the closed simple curve obtained from the rectangle Γ aftere the transformation $\lambda \longrightarrow \frac{1}{\lambda}$. *Let* $\tilde{G}^+$ ($\tilde{G}^-$) denote the interior (exterior) of $\tilde{\Gamma}$.

Theorem(6.1.15)[51]: (i) The set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_2$ in $\tilde{G}^+$is minimal in $\mathcal{H} \times \mathcal{H}$.

(ii) If $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_2$ in $\tilde{G}^+$ is complete in $\mathcal{H} \times \mathcal{H}$. If, in addition, $n(\tau, \frac{1}{\alpha}A^{-1}) \sim c_1\tau^{c_2}$ *as* $\tau \longrightarrow \infty$ with some $0 < c_1, c_2 < \infty$, then $n(\tau, \tilde{G}^+, \mathcal{L}_2) \sim 2c_1\tau^{c_2}$.

(iii) If $n(\tau, A^{-1}) = O(\tau^{\gamma})$ for some $\gamma \in (0, \frac{1}{2}]$, then the set of eigenvectors and associated vectors corresponding to the eigenvalues of $\mathcal{L}_2$ in $\tilde{G}^+$ is a Riesz basis with parentheses in $\mathcal{H} \times \mathcal{H}$. If, in addition, $n(\tau, \frac{1}{\alpha}A^{-1}) = c_1\tau^{c_2} + O(\tau^{\beta})$ for some $0 < c_1, c_2 < \infty, 0 \le \beta < \alpha \le \beta + \gamma$, then also $n(\tau, \tilde{G}^+, \mathcal{L}_2) = 2c_1\tau^{c_2} + O(\tau^{\beta})$.

Proof. Again we invoke the results contained in [137] and apply them with

$$H = \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix},\ T = 0$$

for (ii) and

$$H = \frac{1}{\alpha}\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix},$$
$$D_0 = 0, \qquad D_1 = \frac{1}{\alpha}\begin{pmatrix} A^{-1/2}K_1A^{-1/2+\gamma} & 0 \\ 0 & A^{-1/2}K_2A^{-1/2+\gamma} \end{pmatrix}.$$

where $\gamma \in (0,12]$ for (iii).

Theorem(6.1.16)[51]: The set of eigenvectors and associated vectors of $\mathcal{L}$ corresponding to the eigenvalues in the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -\frac{2}{\alpha}\})$ is minimal in the space $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ where $\mathcal{H}_{A^{-1}} = (\mathcal{H}, (A^{-1}\cdot, A^{-1}\cdot))$. It is complete in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$, and a Riesz basis with parentheses in $\mathcal{H}_{A^{-1}} \times \mathcal{H}_{A^{-1}}$ if $n(\tau, A^{-1}) = O(\tau^{\gamma})$ for some $\gamma \in (0,\frac{1}{2}]$. In particular, the eigen-values in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -\frac{2}{\alpha}\})$ accumulate at ∞ if $A^{-1} \in \mathcal{S}_p$ for some $p < \infty$.

we are going to apply the previous results to the system (2), (3) of partial differential equations with boundary conditions (4). After the separation of variables (5) (and assuming $m \equiv 1$ for simplicity), it takes the form

$$EIy_1^{(4)} + \omega\kappa EIy_2^{(4)} + \lambda\left(\kappa EIy_1^{(4)} + \varepsilon_1 y_1\right) + \lambda^2 y_1 = 0, \quad (19)$$

$$EIy_2^{(4)} - \omega\kappa EIy_1^{(4)} + \lambda\left(\kappa EIy_2^{(4)} + \varepsilon_2 y_2\right) + \lambda^2 y_2 = 0, \quad (20)$$

with boundary conditions

$$y_1(0) = y_1(l) = y''_1(0) = y''_1(l) = 0,$$

$$y_2(0) = y_2(l) = y''_2(0) = y''_2(l) = 0. \quad (21)$$

Here the operators A and K_1, K_2 are determined by (6) and (7), and the constants α, β by $\alpha = \kappa$ (the coefficient of inner damping) and $\beta = \omega\kappa$ (where ω is the angular frequency of the rotation of the beam).

Obviously, the operator A has compact resolvent, and it is not difficult to see that the eigenvalues $\lambda_k, k = 1, 2, \ldots$, of A are all simple and given by

$$\lambda_k = EI\left(\frac{k\pi}{l}\right)^4, k = 1, 2, \ldots .$$

Hence the lower bound δ of A is its least eigenvalue,

$$\delta = EI\left(\frac{\pi}{l}\right)^4.$$

It is also easy to see that the number $n(\tau, A^{-1})$ of eigenvalues of A^{-1} greater than τ^{-1}, i.e., the number of eigenvalues of A less than τ satisfies

$$n(\tau, A^{-1}) \sim \frac{l}{\pi}\left(\frac{1}{EI}\right)^{1/4} \tau^{1/4}. \quad (22)$$

An immediate consequence of Theorem(6.1.1) is the following statement.

Theorem(6.1.17)[51]: The essential spectrum of the problem (19)–(21) consists of the two points

$$-\frac{1}{\kappa} - i\omega, -\frac{1}{\kappa} + i\omega.$$

The other points of the spectrum of the problem (19)–(21) are normal eigenvalues which accumulate at most at the points $-\frac{1}{\kappa} - i\omega$, $-\frac{1}{\kappa} + i\omega$, and at ∞.

From Theorem(6.1.2) we immediately get the following stability result.

Theorem(6.1.18)[51]: Set $\mu := \min_{i=1,2} \min_{x\in[0,l]}\{\varepsilon_i(x)\}$. Then the spectrum of the problem (19)–(21) lies in the open left half plane if

$$\mu \geq \kappa EI\left(\frac{\pi}{l}\right)^4 \text{ and } \mu > \frac{\omega^2\kappa}{4},$$

or if

$$\mu < \kappa EI\left(\frac{\pi}{l}\right)^4 \text{ and } \left(\kappa EI\left(\frac{\pi}{l}\right)^4 + \mu\right)^2 > \omega^2\kappa^2 EI\left(\frac{\pi}{l}\right)^4.$$

Concerning results about the minimality, completeness and basis properties of the eigenvectors of the problem (19)–(21) corresponding to certain branches of eigenvalues, we have to distinguish the case when the outer medium is homogeneous, i.e., $\varepsilon_1 \equiv \varepsilon_2 \equiv: \varepsilon$ and hence $K_1 = K_2$, and the case when the outer medium is inhomogeneous, i.e., $\varepsilon_1 \not\equiv \varepsilon_2$ and hence $K_1 \neq K_2$.

In the case of a homogeneous outer medium, according to above , the eigenvalues of the given problem (19)−(21) split into 4 branches $\{\lambda_k^1\} \cup \{\lambda_k^2\} \cup \left\{\overline{\lambda_k^1}\right\} \cup \{\overline{\lambda_k^2}\}$ where λ_k^1 and λ_k^2, $k = 1, 2, \ldots$, are the eigenvalues of the problem

$$\begin{cases} (1 + i\omega\kappa)EIy^{(4)} + \lambda\big(\kappa EIy^{(4)} + \varepsilon y\big) + \lambda^2 y = 0, \\ y(0) = y(l) = y''(0) = y''(l) = 0 \end{cases} \tag{23}$$

located in the upper and lower half plane, respectively (counted according to their algebraic multiplicities). The respective eigenfunctions and associated functions of the problem of (19)–(21) can be obtained from the eigenfunctions and associated functions y_k^1 and y_k^2 of the problem (23) and from the eigenfunctions and associated functions $\overline{y_k^1}$ and $\overline{y_k^2}$ of the problem

$$\begin{cases} (1 - i\omega\kappa)EIy^{(4)} + \lambda\big(\kappa EIy^{(4)} + \varepsilon y\big) + \lambda^2 y = 0, \\ y(0) = y(l) = y''(0) = y''(l) = 0. \end{cases} \tag{24}$$

For instance, the eigenfunctions of (19)–(21) are given by the formulas

$$(y_k^1, iy_k^1)^t, (y_k^2, iy_k^2)^t, \left(\overline{iy_k^1}, \overline{y_k^1}\right)^t, \left(\overline{iy_k^2}, \overline{y_k^2}\right)^t.$$

In the following we denote by $W_2^4(0, l)$ the Sobolev space of order 4 associated with $L_2(0, l)$.

Theorem(6.1.19)[51]: (i) The set of eigenfunctions and associated functions of problem (19)–(21) corresponding to the eigenvalues λ_k^1 and $\overline{\lambda_k^1}$, $k = 1, 2, \ldots$, forms a Riesz basis with parentheses in the space $W_2^4(0, l) \times W_2^4(0, l)$.

(ii) The eigenvalues λ_k^1 (and hence $\overline{\lambda_k^1}$) accumulate at ∞; if enumerated such that $|\lambda_k^1| \leq |\lambda_{k+1}^1|$, they satisfy the asymptotics

$$\lambda_k^1 = -\kappa EI \left(\frac{k\pi}{l}\right)^4 + i\omega + \xi_k^1 + i\eta_k^1, k \to \infty,$$

where ξ_k^1, η_k^1 are real and $\xi_k^1 = o(k^4), \eta_k^1 = o(1)$.

Proof. The assertion in (i) and the first assertion in (ii) follow from Theorem (6.1.10) which we can apply due to (22) with $\gamma = \frac{1}{4}$. From Theorem (6.1.10) we also obtain that

$$|\lambda_k^1| \sim \kappa EI \left(\frac{k\pi}{l}\right)^4, \quad k \to \infty. \tag{25}$$

Now let y_k^1, $\|y_k^1\| = 1$, be an eigenfunction of (23) (i.e., of $\mathcal{L}_+$) at λ_k^1. Then

$$(\lambda_k^1)^2 + \lambda_k^1(\alpha(Ay_k^1, y_k^1) + (Ky_k^1, y_k^1)) + (1 + i\beta)(Ay_k^1, y_k^1) = 0. \tag{26}$$

From this it follows that $|(Ay_k^1, y_k^1)| \to \infty, k \to \infty$, because otherwise

$$|\lambda_k^1(\lambda_k^1 + \alpha(Ay_k^1, y_k^1) + (Ky_k^1, y_k^1))|$$

would also be bounded, a contradiction to (25). Then the assertion follows from the formula for the solutions of the quadratic equation (26) (see (14)) and from (25).

Theorem(6.1.20)[51]: The set of eigenfunctions and associated functions of problem (19)-(21) corresponding to the eigenvalues λ_k^2 and $\overline{\lambda_k^2}$, $k = 1, 2, \ldots$, is minimal and complete in the space $W_2^4(0, l) \times W_2^4(0, l)$. In Particular, $\lambda_k^2 \to -\frac{1}{\kappa} - i\omega$, $\overline{\lambda_k^2} \to -\frac{1}{\kappa} + i\omega$ for $k \to \infty$.

In the case of an inhomogeneous outer medium we obtain:

Theorem (6.1.21)[51]: Assume that $EI\left(\frac{\pi}{l}\right)^4 > \frac{4}{\kappa^2}$. Then:

(i) The set of eigenfunctions and associated functions of problem (19)-(21) corresponding to the eigenvalues located in the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -\frac{2}{\kappa}\}$ forms a Riesz basis with parentheses in the space $W_2^4(0, l) \times W_2^4(0, l)$.

(ii) The eigenvalues λ_k of problem (19)-(21) in $\{\lambda \in \mathbb{C}: \mathrm{Re}(\lambda) < -\frac{2}{\kappa}\}$ accumulate at ∞; if enumerated such that $|\lambda_k| \leq |\lambda_{k+1}|$, they satisfy the asymptotics

$$\lambda_k = -\kappa EI \left(\frac{k\pi}{2l}\right)^4 + o(k^4), \quad k \to \infty.$$

Proof. The first assertion follows immediately from Theorem (6.1.16). The proof of the second statement is similar to the proof of Theorem (6.1.19) (ii).

Sec(6.2): Non-Selfadjoint Pencils with an Application to Hagen-Poiseuille Flow

In this section we show some eigenvalue enclosures and basisness results for eigen-and associated functions of non-selfadjoint linear operator pencil equations

$$\mathcal{A}u = \lambda \mathcal{B}u, \quad u \in \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B}). \tag{27}$$

The original motivation for this section is twofold. Firstly, it came from [111] in which proved that, for a certain operator pencil arising in the study of Hagen–Poiseuille flow, the essential spectrum is empty. Secondly, the refrernces [28, 26] showed that analytic spectral enclosures combined with interval arithmetic eigenvalue computations may be used to show linear instability results for non-selfadjoint spectral problems from hydrodynamics. How-ever linear stability results require not only eigenvalue enclosures, but also basisness results for the eigen-and associated functions. This is the problem we address in this section.

At the abstract level, the precise type of basisness result obtained depends only on the distribution of the eigenvalues. In all cases the eigenvalue spacing must tend to infinity and the imaginary parts of the eigenvalues must be bounded: in this case, at least a Riesz basisness result is obtained (i.e. equivalence to an orthonor-mal basis). If one has more information–for instance, if the eigenvalues are quadratically large with respect to their index – then a stronger Bari basisness result is obtained (i.e. quadratic closeness to an orthonormal basis). All these basisness results are obtained by reducing the pencil to an operator and invoking basisness results for non-selfadjoint operators having certain eigenvalue distributions.

The reduction of the pencil (27) to a linear operator in a suitable Hilbert space is therefore very important. However, as pointed out by Kato [17], at a formal level there are a number of different operators which might be considered as possible replacements for the pencil, with $\mathcal{B}^{-1}\mathcal{A}$, $\mathcal{A}\,\mathcal{B}^{-1}$,$\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}$, all being possible choices provided they exist. A priori the choice of operator seems somewhat arbitrary and the relationships between the spectra of these operators and the spectrum of the original pencil are, in general, unclear.

In this section we assume that the operator $\mathcal{B}$ on the right-hand side of (27) is uniformly positive, in which case all the above possibilities are available. Following ideas of [114] for the Orr–Sommerfeld problem, developed simultaneously and independently in [104], we introduce the Hilbert space $\mathcal{H}_{1/2} := \mathcal{D}(\mathcal{B}^{1/2})$ and consider therein the operator $\mathcal{B}^{-1}\mathcal{A}$. Thanks to an additional abstract hypothesis–which, in the Hagen–Poiseuille case, forces one to consider only the physical boundary conditions which are usually imposed – it turns out that $\mathcal{B}^{-1}\mathcal{A}$ is closable in the space $\mathcal{H}_{1/2}$ and, with $\mathcal{M} := \overline{\mathcal{B}^{-1}\mathcal{A}}$ denoting its closure, the nonclassical spectral problem (27) is equivalent to the classical spectral problem

$$\mathcal{M}u = \lambda u, \quad u \in \mathcal{D}(\mathcal{M}).$$

The spectral enclosures and basisness results are based on a decomposition of the operator $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$into a self-adjoint operator $\mathcal{M}_0$in $\mathcal{H}_{1/2}$ and a bounded perturbation $\mathcal{M}_1$. The Bari basisness is obtained by means of a result of A.S. Markus (see [154]) which gives a criterion for bases in terms of the convergence of a certain series involving only the eigen-values. An interesting aside is that the proof of the boundedness of $\mathcal{M}_1$ in $\mathcal{H}_{1/2}$ depends on a 1947 result of Krein (Lemma (6.2.6) below), originally stated for self-adjoint operators and later rediscovered and generalized by a number of authors (see the translation [115] and the historical remarks at the end).

Finally, in preparation for the eventual application to the linear stability of an incompressible flow in a circular pipe, we recall the coupled system of differential equations describing the so-called Hagen–Poiseuille flow with non-axisymmetric disturbances, even though these will not be seen again until mention it in this section . These equations have the form

$$\begin{pmatrix} \mathcal{T}(k^2r^2\,\mathcal{T}) + i\alpha RU\mathcal{T} + i\alpha R\frac{1}{r}\left(\frac{U'}{k^2r}\right)' & 2\alpha n\mathcal{T} \\ 2\alpha n\mathcal{T} - inR\frac{U'}{r} & \mathcal{S} + i\alpha RUk^2r^2 \end{pmatrix}\begin{pmatrix}\Phi\\ \Omega\end{pmatrix} = i\alpha Rc\begin{pmatrix}\mathcal{T} & 0\\ 0 & k^2r^2\end{pmatrix}\begin{pmatrix}\Phi\\ \Omega\end{pmatrix}$$

on the interval $(0,1]$ (see [105, 25]). Here $R \geq 0$ is the Reynolds number, $\alpha \in \mathbb{R}$ is the streamwise wave number, $n \in \mathbb{Z}$ is the azimuthal wave number, and c is a complex wave speed which result from an exponential dependence on the axial, angular, and space coordinate, respectively, of the form

$$\exp(i(\alpha x + n\Phi - \alpha ct)).$$

The function $U: [0,1] \to \mathbb{R}$ is the axial mean flow while the function $k: (0,1] \to \mathbb{R}$ is given by $k(r)^2 := \alpha^2 + \frac{n^2}{r^2}$, $r \in (0,1]$. The differential expressions $\mathcal{T}, \mathcal{S}$ have the form

$$\mathcal{T} := \frac{1}{r^2} - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{k(r)^2 r}\frac{d}{dr}\right), \qquad \mathcal{S} := k(r)^4r^2 - \frac{1}{r}\frac{d}{dr}\left(k(r)^2r^3\frac{d}{dr}\right).$$

With $\lambda = i\alpha RC$, and by choosing suitable operator realizations of the various differential expressions determined by boundary conditions, Hagen-Poiseuille flow problem can be cast in the form (27). We show that it satisfies all assumptions of the abstract theorems, we work out the corresponding basis results, and we establish explicit estimates for the real and imaginary parts of the eigenvalues in terms of the coefficients α, R, n, and the axial mean flow U; for parabolic U, the estimates yield new linear stability regions in the $(\alpha, R)-$plane.

In the following, we study linear operator pencils

$$\mathcal{L}_0(\lambda) = \mathcal{A}_0 - \lambda\mathcal{B};\ \ \mathcal{L}(\lambda) = \mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B};\ \lambda \in \mathbb{C}, \tag{28}$$

in a Hilbert space $\mathcal{H}$, for which the domains $\mathcal{D}(\mathcal{L}(\lambda))$ do not depend on λ, and are always equal to $\mathcal{D}_{\mathcal{L}} := \mathcal{D}(\mathcal{A}_0)$. The spectrum of $\mathcal{L}$, denoted $\sigma(\mathcal{L})$, is defined to be the set of $\lambda \in \mathbb{C}$ for which $0 \in \sigma(\mathcal{L}(\lambda))$, with corresponding definitions of point spectrum $\sigma_P(\mathcal{L})$, continuous spectrum $\sigma_c(\mathcal{L})$, residual spectrum $\sigma_r(\mathcal{L})$, essential spectrum $\sigma_{ess}(\mathcal{L})$, and resolvent set $\rho(\mathcal{L})$. Eigenvalue problems of the form

$$(\mathcal{A}_0 + \mathcal{A}_1)x = \lambda\mathcal{B}x,\ \ x \in \mathcal{D}(\mathcal{A}_0), \tag{29}$$

can then be written as $\mathcal{L}(\lambda)x = 0$, $x \in \mathcal{D}_{\mathcal{L}}$. Under the hypotheses, the spectra of $\mathcal{L}$ and $\mathcal{L}_0$ will turn out to be purely discrete. The aim is to show that the corresponding eigenfunctions and associated functions of $\mathcal{L}$ form a Bari basis, i.e., a basis quadratically close to an orthonormal basis, in a suitable Hilbert space.

We denote the scalar product and corresponding norm on $\mathcal{H}$ by $(\cdot\,,\,\cdot)$ and $\|\cdot\|$, respectively. The operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ are assumed to satisfy the following:

(i) $\mathcal{A}_0$ is self-adjoint;

(ii) $\mathcal{B}$ is self adjoint, uniformly positive, $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{B})$, and $\mathcal{B}$ is $\mathcal{A}_0$-compact;

(iii) $\mathcal{A}_1$ is closable with $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1)$ and $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1^*)$.

We start with several abstract lemmas.

Lemma(6.2.1)[31]: . Let E, F be reflexive Banach spaces, let T_0, T_1, S be linear operators from E to F, and let T_0 be closable. If S is T_0-compact, T_1 is S-bounded, and S is boundedly invertible, then T_1 is T_0-compact and S is $(T_0 + T_1)$-compact.

Proof. By the assumptions we have the inclusions $\mathcal{D}(T_0) \subset \mathcal{D}(S) \subset \mathcal{D}(T_1)$ and hence $\mathcal{D}(T_0 + T_1) = \mathcal{D}(T_0)$.

In order to prove that T_1 is T_0-compact, let $(x_j) \subset \mathcal{D}(T_0)$ be a sequence such that (x_j) and $(T_0 x_j)$ are bounded. Since S is T_0-compact, there exists a subsequence $(x_{jk}) \subset (x_j)$ such that (Sx_{jk}) converges. Since T_1 is S-bounded, the sequence $(T_1 x_{jk}) = (T_1 S^{-1} S x_{jk})$ converges as well.

In order to show that S is $(T_0 + T_1)$-compact, let $(x_j) \subset \mathcal{D}(T_0)$ be a sequence such that (x_j) and $((T_0 + T_1)x_j)$ are bounded, $\|x_j\| \le m$, $\|(T_0 + T_1)x_j\| \le M$ for $j \in \mathbb{N}$. Since T_1 is T_0-compact, E, F are reflexive and T_0 is closable, T_1 is T_0-bounded with relative bound 0 (see [62]). In particular, there exists $a \ge 0$ so that

$$\|T_1 x\| \le a\|x\| + \frac{1}{2}\|T_0 x\|, x \in \mathcal{D}(T_0) \subset \mathcal{D}(T_1).$$

Thus

$$\|T_0 x_j\| \le \|(T_0 + T_1)x_j\| + \|T_1 x_j\| \le M + am + \frac{1}{2}\|T_0 x_j\|, j \in \mathbb{N},$$

and hence $\|T_0 x_j\| \le 2(M + am)$. Now the T_0-compactness of S shows that there exists a subsequence $(x_{jk}) \subset (x_j)$ such that (Sx_{jk}) converges.

The following property of the adjoint of a sum of unbounded operators was first proved in [29] (see also [19] or [102]).

Lemma(6.2.2)[31]: Let E, F be Banach or Hilbert spaces and let T, S be densely defined linear operators from E to F. If T is Fredholm, S is $T-$ compact, and S^* is T^* $-$compact, then $(T + S)^* = T^* + S^*$.

Note that, if S is T-compact, then S^* need not be T^*-compact; in fact, it may happen that $\mathcal{D}(T^*) \cap \mathcal{D}(S^*) = \{0\}$ even if T is the inverse of a positive definite compact operator in a Hilbert space (see [29]).

In the next two propositions we use the preceding lemmas to deduce various properties of the operators $\mathcal{A}_0, \mathcal{A}_1$, and $\mathcal{B}$ from the assumptions (i)–(iii). First we consider several compactness properties of the operator pencil $\mathcal{L}$ and its essential spectrum which we define as

$$\sigma_{ess}(\mathcal{L}) := \{\lambda \in \mathbb{C} : 0 \in \sigma_{ess}(\mathcal{L}(\lambda))\} = \{\lambda \in \mathbb{C} : \mathcal{L}(\lambda) \text{ is not Fredholm}\}.$$

Here a linear operator T in a Banach space is called Fredholm if its kernel is finite-dimensional and its range is finite co-dimensional (and hence closed); the index of a Fredholm operator is defined as $\operatorname{ind} T := \operatorname{codim} \operatorname{ran} T - \dim \ker T$.

Proposition(6.2.3)[31]: Under the assumptions (i), (ii), and (iii) the following hold:

(I) $\mathcal{A}_0$ has compact resolvent;

(II) $\mathcal{A}_1$ and $\mathcal{A}_1^*$ are $\mathcal{A}_0$-compact;

(III) $\mathcal{B}$ is $(\mathcal{A}_0 + \mathcal{A}_1)$-compact;

(IV) $\sigma_{ess}(\mathcal{L}) = \sigma_{ess}(\mathcal{A}_0 + \mathcal{A}_1) = \sigma_{ess}(\mathcal{A}_0) = \emptyset$, and $\operatorname{ind} \mathcal{L}(\lambda) = \operatorname{ind} \mathcal{A}_0 = 0, \lambda \in \mathbb{C}$;

(V) $\sigma(\mathcal{L}) = \sigma_P(\mathcal{L})$.

Proof. (I) For $\mu \in \rho(\mathcal{A}_0)$, observe that $(\mathcal{A}_0 - \mu)^{-1} = \mathcal{B}^{-1}\mathcal{B}(\mathcal{A}_0 - \mu)^{-1}$ is compact because $\mathcal{B}$ is boundedly invertible and $\mathcal{B}(\mathcal{A}_0 - \mu)^{-1}$ is compact by assumption (ii).

(II) and (III) Since $\mathcal{B}$ is closed and $\mathcal{A}_1$ is closable, the inclusion $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1)$ already implies that $\mathcal{A}_1$ is $\mathcal{B}$ −bounded, and similarly for $\mathcal{A}_1^*$. Now both claims follow from Lemma (6.2.1).

(IV) The operator $\mathcal{A}_0$ is Fredholm with index 0 since $\mathcal{A}_0$ has compact resolvent by (I) and $\mathcal{A}_0$ is self-adjoint so that $(\operatorname{ran} \mathcal{A}_0)^{\perp} = \ker \mathcal{A}_0$. Now all claims follow from (II), (III), and (I) by means of the stability theorems for Fredholm operators (see [127]).

(V) By (IV) it follows that $\sigma_c(\mathcal{L}) \subset \sigma_{ess}(\mathcal{L}) = \emptyset$ and that $\dim\ker \mathcal{L}(\lambda) = \dim\left(\operatorname{ran} \mathcal{L}(\lambda)\right)^{\perp}$ for every $\lambda \in \mathbb{C}$, which implies that $\sigma_r(\mathcal{L}) = \emptyset$.

Proposition(6.2.4)[31]: Under the assumptions (i), (ii), and (iii) the following are true:

(I) $\mathcal{A}^* = (\mathcal{A}_0 + \mathcal{A}_1)^* = \mathcal{A}_0 + \mathcal{A}_1^*$,

(II) $(\mathcal{A} - \lambda\mathcal{B})^* = \mathcal{A}^* - \overline{\lambda}\mathcal{B}, \lambda \in \mathbb{C}$,

and hence

$$\mathcal{L}(\lambda)^* = \mathcal{A}_0 + \mathcal{A}_1^* - \overline{\lambda}\,\mathcal{B}, \quad \mathcal{D}(\mathcal{L}(\lambda)^*) = \mathcal{D}(\mathcal{A}_0), \ \lambda \in \mathbb{C}.$$

Proof. (I) We apply Lemma (6.2.2) with $S = \mathcal{A}_1$ and $T = \mathcal{A}$: as a consequence, (I) holds provided $\mathcal{A}_1$ and $\mathcal{A}_1^*$ are, respectively, $\mathcal{A}_0$-compact and $\mathcal{A}_0^*$ −compact. Since $\mathcal{A}_0$ is self-adjoint, the required relative compactness properties are both guaranteed by Proposition (6.2.3).

(II) This time we apply Lemma (6.2.2) with $S = \mathcal{B}$ and $T = \mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$. Since $\mathcal{B}$ is selfadjoint, it suffices to check that $\mathcal{B}$ is both $\mathcal{A}$-compact and $\mathcal{A}^*$-compact. Applying Lemma (6.2.1) and the result of part (I) we see that it is sufficient now to check that $\mathcal{B}$ is $\mathcal{A}_0$-compact, which is assumption (ii); and that $\mathcal{A}_1$ and $\mathcal{A}_1^*$ are $\mathcal{B}$-bounded, which is guaranteed by assumption (iii).

The last claim follows from (I), (II) and assumptions (ii), (iii).

Proposition(6.2.5)[31]: Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, let A_0, D_0 be self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let B_0 be a densely defined closable linear operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. Suppose that B_0^* is A_0-bounded with relative bound $\delta_{B_0^*}$, B_0 is D_0-bounded with relative bound δ_{B_0} and $\delta_{B_0^*}\,\delta_{B_0} < 1$. Then the block operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} A_0 & B_0 \\ B_0^* & D_0 \end{pmatrix}, \mathcal{D}(\mathcal{A}_0) := \mathcal{D}(A_0) \oplus \mathcal{D}(D_0),$$

is self-adjoint in $\mathcal{H}_1 \oplus \mathcal{H}_2$. If A_0 and D_0 are bounded below, then so is $\mathcal{A}_0$.

Proof. Since B_0 is closable, B_0^* is densely defined and hence so is $\mathcal{A}_0$. A direct calculation shows that $\mathcal{A}_0$ is symmetric. Therefore it is sufficient to find two points $\pm i\omega \in \rho(\mathcal{A}_0)$ with $\omega > 0$ or, in the case where A_0 and D_0 are bounded below, to show that there exists a $\zeta_0 \in \mathbb{R}$ such that $(-\infty, \zeta_0) \subset \rho(\mathcal{A}_0)$. The proof is closely modelled on [17].

Since $\delta_{B_0^*}\,\delta_{B_0} < 1$, we can choose $\beta > 0$ such that $\delta_{B_0^*}/\beta < 1$ and $\delta_{B_0}\beta < 1$. Observe that, for $\zeta \in \mathbb{C}$,

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}(\mathcal{A}_0 - \zeta)\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} + \begin{pmatrix} 0 & \beta B_0 \\ \frac{1}{\beta} B_0^* & 0 \end{pmatrix} - \zeta$$
$$=: T + S - \zeta. \tag{30}$$

Since, for $\zeta \in \rho(T)$,

$$T + S - \zeta = (I + S(T - \zeta)^{-1})(T - \zeta),$$

to show bounded invertibility of $T + S - \zeta$ it suffices to show that $\|S(T - \zeta)^{-1}\| < 1$.

In (30), due to the choice of β, the operator S is T-bounded with relative bound $\delta_S \le \max\{\delta_{B_0^*}/\beta, \delta_{B_0}\beta\} < 1$. Hence there exist $a_S \ge 0$ and $b_S \in (\delta_S, 1)$ such that

$$\|Sx\| \le a_S\|x\| + b_S\|Tx\|, \quad x \in \mathcal{D}(T).$$

In particular, for $\zeta \in \rho(T)$,

$$\|S(T-\zeta)^{-1}y\| \le a_S\|(T-\zeta)^{-1}y\| + b_S\|T(T-\zeta)^{-1}y\|, \; y \in \mathcal{H}_1 \oplus \mathcal{H}_2.$$

By the spectral theorem for the selfadjoint operator T,

$$\|S(T-\zeta)^{-1}\| \le a_S \sup_{\lambda\in\sigma(T)} \frac{1}{|\lambda-\zeta|} + b_S \sup_{\lambda\in\sigma(T)} \left|\frac{\lambda}{\lambda-\zeta}\right|. \tag{31}$$

If $\zeta = \pm i\omega$ with $\omega > 0$ is purely imaginary, then for all $\lambda \in \sigma(T) \subset \mathbb{R}$ we have

$$\frac{1}{|\lambda \mp i\omega|} \le \frac{1}{\omega}, \qquad \left|\frac{\lambda}{\lambda \mp i\omega}\right| \le 1,$$

and hence

$$\|S(T \mp i\omega)^{-1}\| \le \frac{a_S}{\omega} + b_S.$$

Since $b_S \in (0,1)$, we can choose ω sufficiently large to ensure that $a_S/\omega + b_S < 1$ and hence $\pm i\omega \in \rho(T+S) = \rho(\mathcal{A}_0)$.

If A_0 and D_0 are bounded from below, then so is T,

$$T \ge \min\{\min\sigma(A_0), \min\sigma(D_0)\} =: \gamma T.$$

Now (31) yields that, for $\zeta < \gamma T$,

$$\|S(T-\zeta)^{-1}\| \le \frac{a_S}{\gamma T-\zeta} + b_S \max\left\{1, \frac{|\gamma T|}{\gamma T-\zeta}\right\}. \tag{32}$$

Since $a_S/(\gamma T-\zeta) \to 0$ and $|\gamma T|/(\gamma T-\zeta) \to 0$ as $\zeta \to -\infty$, there exists $\zeta_0 < \gamma T$ such that $\|S(T-\zeta)^{-1}\| < 1$ for all $\zeta \le \zeta_0$ and hence $(-\infty, \zeta_0) \subset \rho(T+S) = \rho(\mathcal{A}_0)$.

We reduce the spectral problem $\mathcal{L}(\lambda)x = (\mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B})x = 0$ for the linear operator pencil $\mathcal{L}$ in the Hilbert space $\mathcal{H}$ to a classical spectral problem $(\mathcal{M}-\lambda)y = 0$ in another Hilbert space induced by the operator $\mathcal{B}$. Here we always assume that the operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ satisfy the pervious assumptions (i)–(iii).

Since $\mathcal{B}$ is uniformly positive by assumption (ii), we may introduce two new Hilbert spaces, $\mathcal{H}_{1/2}$ and $\mathcal{H}_{-1/2}$, as follows. The space $\mathcal{H}_{1/2}$ with scalar product $[\cdot,\cdot]_{1/2}$ and corresponding norm $\|\cdot\|_{1/2}$ is defined by

$$\mathcal{H}_{1/2} := \mathcal{D}(\mathcal{B}^{1/2}), \; [\cdot,\cdot]_{1/2} := (\mathcal{B}^{1/2}\,\cdot, \mathcal{B}^{1/2}\,\cdot); \tag{33}$$

the space $\mathcal{H}_{-1/2}$ is defined as the completion of $\mathcal{H}$ with respect to the norm induced by the inner product $[\cdot,\cdot]_{-1/2}$ given by

$$[\cdot,\cdot]_{-1/2} = (\mathcal{B}^{-1/2}\,\cdot, \mathcal{B}^{-1/2}\,\cdot). \tag{34}$$

In the following, in view of the formal identity

$$\mathcal{L}(\lambda) = \mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B} = \mathcal{B}(\mathcal{B}^{-1}\mathcal{A}_0 + \mathcal{B}^{-1}\mathcal{A}_1 - \lambda), \; \lambda \in \mathbb{C}, \tag{35}$$

we consider the operators $\mathcal{B}^{-1}\mathcal{A}_0, \mathcal{B}^{-1}\mathcal{A}_1$ in the Hilbert space $\mathcal{H}_{1/2}$. To this end, from now on we additionally assume that

(iv) $\mathcal{D}(\mathcal{A}_0)$ is a core for $\mathcal{B}^{1/2}$ and $\mathcal{D}(\mathcal{A}_1) \subset \mathcal{D}(\mathcal{B}^{1/2})$.

Note that the property that $\mathcal{D}(\mathcal{A}_0)$ is a core of $\mathcal{B}^{1/2}$ does not follow from the previous hypotheses (i),(ii), and(iii). To see this one may choose $\mathcal{B}$ to be the shifted one-dimensional Neumann Laplacian $\mathcal{B} = -\frac{d^2}{dx^2} + 1$ in $L^2(0,1)$ with domain equipped with boundary conditions $u'(0) = 0 = u'(1)$, while $\mathcal{A}_0 = \frac{d^4}{dx^4} + 1$ in $L^2(0,1)$ equipped with boundary conditions $u(0) = u'(0) = 0 = u(1) = u'(1)$. Both $\mathcal{A}_0$ and $\mathcal{B}$ are self-adjoint and uniformly positive and $\mathcal{B}$ is $\mathcal{A}_0$-compact. The closure of $\mathcal{D}(\mathcal{A}_0)$ in the graph norm of $\mathcal{B}^{1/2}$ is $\mathcal{H}_0^1(0,1)$, but the domain of $\mathcal{B}^{1/2}$ is $\mathcal{H}^1(0,1)$.

Lemma(6.2.6)[31]: Let E be a Banach space with norm $\|\cdot\|$ and let $[\cdot,\cdot]$ be a positive definite continuous inner product on E (i.e. $|[x,y]| \le \gamma\|x\| \ \|y\|$, $x,y \in E$, for some $\gamma > 0$). Let S be a bounded operator in E and suppose that there exists a bounded operator S^+ in E such that $[Sx,y] = [x,S^+y], x,y \in E$. Then:

(I) S is bounded with respect to the norm generated by the inner product $[\cdot,\cdot]$ and $[Sx,Sx] \le \max\{\|S\|,\|S^+\|\}[x,x], x \in E$.

(II) If S is compact in E, then S is compact with respect to the norm generated by the inner product $[\cdot,\cdot]$. In this case, if $\tilde{S}$ denotes the continuous extension of S to the completion $\tilde{E}$ of E with respect to the norm generated by $[\cdot,\cdot]$, then $\sigma(S) = \sigma(\tilde{S})$ and the algebraic eigenspaces of S in E and of $\tilde{S}$ in $\tilde{E}$ corresponding to the same non-zero eigenvalue coincide.

In the sequel we use Lemma (6.2.6) with $E = \mathcal{H}$ equipped with the norm $\|\cdot\|$ of $\mathcal{H}$, with the inner product $[\cdot,\cdot] = [\cdot,\cdot]_{-\frac{1}{2}}$ and with $\tilde{E} = \mathcal{H}_{-\frac{1}{2}}$.

Proposition(6.2.7)[31]: Under the assumptions (i), (ii), (iii), and (iv) the following hold:

(I) $\mathcal{B}^{-1/2}\mathcal{A}_0\mathcal{B}^{-1/2}$ is densely defined;

(II) $\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}$ is densely defined and bounded with

$$\|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\| \le \max\{\|\mathcal{A}_1\mathcal{B}^{-1}\|,\|\mathcal{A}_1^*\mathcal{B}^{-1}\|\}.$$

Proof. (I) The domain of $\mathcal{B}^{-1/2}\mathcal{A}_0\mathcal{B}^{-1/2}$ is the set $\mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_0)$. To see that this set is dense in $\mathcal{H}$, let $x \in \mathcal{H}$ and $\varepsilon > 0$ be arbitrary and set $y = \mathcal{B}^{-1/2}x \in \mathcal{D}(\mathcal{B}^{1/2})$. Since $\mathcal{D}(\mathcal{A}_0)$ is a core of the closed operator $\mathcal{B}^{1/2}$ by assumption (iv), there exists a $z \in \mathcal{D}(\mathcal{A}_0)$ such that $\|y - z\| < \varepsilon$ and $\|x - \mathcal{B}^{1/2}z\| = \|\mathcal{B}^{1/2}y - \mathcal{B}^{1/2}z\| < \varepsilon$.

(II) The domain of $\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}$ is the set $\mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1)$. By assumption (iii) we have $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1)$ and hence $\mathcal{B}^{\frac{1}{2}}\mathcal{D}(\mathcal{A}_1) \supset \mathcal{B}^{\frac{1}{2}}\mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{B}^{\frac{1}{2}})$, which is dense.

In order to prove the remaining claims, we apply Lemma (6.2.6) with $E = \mathcal{H}$ and $\tilde{E} = \mathcal{H}_{-1/2}$. Since $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1)$ and $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1^*)$ by assumption (iii), the operators $S := \mathcal{A}_1\mathcal{B}^{-1}$ and $S^+ := \mathcal{A}_1^*\mathcal{B}^{-1}$ are both bounded on $\mathcal{H}$. Moreover,

$$\begin{aligned}[\mathcal{A}_1\mathcal{B}^{-1}x,y]_{-1/2} &= (\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1}x,\mathcal{B}^{-1/2}y) = (\mathcal{A}_1\mathcal{B}^{-1}x,\mathcal{B}^{-1}y)\\ &= (\mathcal{B}^{-1}x,\mathcal{A}_1^*\mathcal{B}^{-1}y) = (\mathcal{B}^{-1/2}x,\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1}y)\\ &= [x,\mathcal{A}_1^*\mathcal{B}^{-1}y]_{-1/2}, \quad x,y \in \mathcal{H}.\end{aligned}$$

Now Lemma (6.2.6) (I) shows that $\mathcal{A}_1\mathcal{B}^{-1}$ is bounded with respect to the norm induced by $[\cdot,\cdot]_{-1/2}$ and, for $y \in \mathcal{D}(\mathcal{B}^{1/2})$,

$$\begin{aligned}\|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}y\|^2 &= [\mathcal{A}_1\mathcal{B}^{-1}\mathcal{B}^{1/2}y,\mathcal{A}_1\mathcal{B}^{-1}\mathcal{B}^{1/2}y]_{-1/2}\\ &\le C\,[\mathcal{B}^{1/2}y,\mathcal{B}^{1/2}y]_{-1/2} = C\|y\|^2\end{aligned}$$

with $C = \max\{\|\mathcal{A}_1\mathcal{B}^{-1}\|, \|\mathcal{A}_1^*\mathcal{B}^{-1}\|\}$. We have already seen that $\mathcal{D}(\mathcal{B}^{1/2})$ is dense in $\mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2})$, the estimate holds for all $y \in D(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2})$.

Proposition(6.2.8)[31]: Under the assumptions (i), (ii), (iii), and (iv) with $\beta := \min\sigma(\mathcal{B})$ (> 0), the following hold:

(I) $\mathcal{B}^{-1}\mathcal{A}_0$ is densely defined and essentially self-adjoint in $\mathcal{H}_{1/2}$; for its self-adjoint closure $\mathcal{M}_0 := \overline{\mathcal{B}^{-1}\mathcal{A}_0}$ we have

$$\mathcal{A}_0 \ge \alpha_0 \Rightarrow \mathcal{M}_0 \ge \min\{0,\alpha_0/\beta\}.$$

(II) $\mathcal{B}^{-1}\mathcal{A}_1$ is densely defined and bounded in $\mathcal{H}_{1/2}$; for its bounded closure $\mathcal{M}_1 := \overline{\mathcal{B}^{-1}\mathcal{A}_1}$ we have

$$\|\mathcal{M}_1\| \leq \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\| \ .$$

Proof. (I) Since $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{B}^{1/2}) = \mathcal{H}_{1/2}$, the domain of $\mathcal{B}^{-1}\mathcal{A}_0$ as an operator in $\mathcal{H}_{1/2}$ is $\mathcal{D}(\mathcal{A}_0)$. The set $\mathcal{D}(\mathcal{A}_0)$ is dense in $\mathcal{H}_{1/2}$ if and only if the set $\mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_0)$ is dense in $\mathcal{H}$. The latter was shown in Proposition (6.2.7) (I) and its proof.

The symmetry of $\mathcal{B}^{-1}\mathcal{A}_0$ in $\mathcal{H}_{1/2}$ is immediate from the self-adjointness of $\mathcal{A}_0$ and $\mathcal{B}$ in $\mathcal{H}$:

$$[\mathcal{B}^{-1}\mathcal{A}_0 x, x]_{\frac{1}{2}} = \left(\mathcal{B}^{\frac{1}{2}}\mathcal{B}^{-1}\mathcal{A}_0 x, \mathcal{B}^{\frac{1}{2}}x\right) = (\mathcal{A}_0 x, x) \in \mathbb{R}, \ \ x \in \mathcal{D}(\mathcal{A}_0). \tag{36}$$

Hence for the first claim it remains to be shown that $\left(\operatorname{ran}(\mathcal{B}^{-1}\mathcal{A}_0 \mp i)\right)^{\perp_{\frac{1}{2}}} = \{0\}$, where $\perp_{1/2}$ denotes the orthogonal complement in $\mathcal{H}_{1/2}$. By the definition of $[\cdot,\cdot]_{1/2}$ in (33) and by Proposition (6.2.4) (with $\mathcal{A}_1 = 0$), we have

$$\begin{aligned}\left(\operatorname{ran}(\mathcal{B}^{-1}\mathcal{A}_0 \mp i)\right)^{\perp_{1/2}} &= \left(\operatorname{ran}(\mathcal{A}_0 \mp i\mathcal{B})\right)^{\perp} = \ker((\mathcal{A}_0 \mp i\mathcal{B})^*)\\ &= \ker(\mathcal{A}_0 \pm i\mathcal{B}).\end{aligned}$$

If $y \in \ker(\mathcal{A}_0 \pm i\mathcal{B})$, then

$$0 = \left((\mathcal{A}_0 \pm i\mathcal{B})y, y\right) = (\mathcal{A}_0 y, y) \pm i(\mathcal{B}y, y).$$

Since $\mathcal{A}_0$ and $\mathcal{B}$ are selfadjoint, we conclude that $(\mathcal{B}y, y) = 0$; as $\mathcal{B}$ is uniformly positive, this implies $y = 0$.

The last claim in (I) is immediate from (36) and from the estimate

$$(\mathcal{A}_0 x, x) \geq \alpha_0(x, x) \geq \min\{0, \alpha_0/\beta\}[x, x]_{1/2} \ , x \in \mathcal{D}(\mathcal{A}_0).$$

(II) Since $\mathcal{D}(\mathcal{A}_1) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}})$ by assumption (iv), the domain of $\mathcal{B}^{-1}\mathcal{A}_1$ as an operator in $\mathcal{H}_{1/2}$ is $\mathcal{D}(\mathcal{A}_1)$. The set $\mathcal{D}(\mathcal{A}_1)$ is dense in $\mathcal{H}_{1/2}$ if and only if the set $\mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1)$ is dense in $\mathcal{H}$. The latter was shown in Proposition (6.2.7) (II) and its proof.
By Proposition (6.2.7) (II) the operator $\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}$ is bounded. Hence, by the definition of the norm of $\mathcal{H}_{1/2}$ in (33),

$$\begin{aligned}\|\mathcal{B}^{-1}\mathcal{A}_1 x\|_{1/2} &= \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}(\mathcal{B}^{1/2}x)\|\\ &\leq \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\| \ \|\mathcal{B}^{1/2}x\|\\ &= \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\| \ \|x\|_{1/2}\end{aligned}$$

for $x \in \mathcal{D}(\mathcal{A}_1) \subset \mathcal{D}(\mathcal{B}^{1/2})$ This implies that $\mathcal{B}^{-1}\mathcal{A}_1$ is bounded in $\mathcal{H}_{1/2}$ and hence so is its closure $\mathcal{M}_1$.

Proposition (6.2.9)[31]: In the Hilbert space $\mathcal{H}_{1/2}$ define the linear operator $\mathcal{M}$ by

$$\mathcal{M} := \mathcal{M}_0 + \mathcal{M}_1 = \overline{\mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1)}, \tag{37}$$

where the closure on the right-hand side is taken in $\mathcal{H}_{1/2}$. Then

$$\mathcal{M} - \lambda = \mathcal{B}^{-1/2}\,\overline{\mathcal{B}^{-1/2}\,\mathcal{L}(\lambda)\,\mathcal{B}^{-1/2}}\mathcal{B}^{1/2} \ , \lambda \in \mathbb{C}, \tag{38}$$

where the closure on the right-hand side is taken in $\mathcal{H}$ and the operator $\mathcal{B}^{1/2}$ is regarded as an isometric isomorphism from $\mathcal{H}_{1/2}$ to $\mathcal{H}$ with inverse $\mathcal{B}^{-1/2}$.

Proof. Since $\mathcal{B}^{-\frac{1}{2}}\mathcal{B}^{\frac{1}{2}} = I|_{\mathcal{D}(\mathcal{B}^{1/2})}$ and $\mathcal{D}(\mathcal{L}(\lambda)) \subset \mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}})$, we can rewrite the operator identity (35) as

$$\mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1) - \lambda = \mathcal{B}^{-1/2}\,\mathcal{B}^{-1/2}\mathcal{L}(\lambda)\,\mathcal{B}^{-1/2}\,\mathcal{B}^{1/2} \ , \lambda \in \mathbb{C}. \tag{39}$$

The middle factor $\mathcal{B}^{-1/2}\,\mathcal{L}(\lambda)\,\mathcal{B}^{-1/2}$ on the right-hand side of (39) can be written as

$$\mathcal{B}^{-1/2}\,\mathcal{L}(\lambda)\mathcal{B}^{-1/2} = \mathcal{B}^{-1/2}\,\mathcal{A}_0\,\mathcal{B}^{-1/2} + \mathcal{B}^{-1/2}\,\mathcal{A}_1\,\mathcal{B}^{-1/2} - \lambda.$$

Here the operator $\mathcal{B}^{-1/2}\mathcal{A}_0\mathcal{B}^{-1/2}$ is densely defined by Proposition (6.2. 7) (I) and clearly symmetric in $\mathcal{H}$, thus closable in $\mathcal{H}$, while $\mathcal{B}^{-1/2}\mathcal{A}_1\,\mathcal{B}^{-1/2}$ is densely defined and bounded in $\mathcal{H}$ by Proposition (6.2.7) (II), thus also closable in $\mathcal{H}$. Hence $\mathcal{B}^{-1/2}\mathcal{L}(\lambda)\mathcal{B}^{-1/2}$ is a densely defined closable operator in $\mathcal{H}$.

The factors $\mathcal{B}^{-1/2}$, $\mathcal{B}^{1/2}$ bordering the operator $\mathcal{B}^{-1/2}\,\mathcal{L}(\lambda)\,\mathcal{B}^{-1/2}$ on the right-hand side of (39) are, respectively, an isometric isomorphism from $\mathcal{H}_{1/2}$ to $\mathcal{H}$, and an isometric isomorphism from $\mathcal{H}$ to $\mathcal{H}_{1/2}$. Taking closures in $\mathcal{H}_{1/2}$in (39), and bearing in mind the definition (37), we obtain (38).

Note that**:** In the proof of Proposition (6.2.9), we have shown that $\mathcal{B}^{1/2}\mathcal{L}(\lambda)^{-1}\mathcal{B}^{1/2}$ is bounded in $\mathcal{H}$ with dense domain $\mathcal{D}(\mathcal{B}^{1/2})$, while $\mathcal{B}\,\mathcal{L}(\lambda)^{-1}$ is bounded in $\mathcal{H}_{-1/2}$ with dense domain $\mathcal{H}$. Since $\mathcal{B}^{1/2}$ is an isometric isomorphism from $\mathcal{H}$ to $\mathcal{H}_{-1/2}$, the relation,

$$\mathcal{B}^{1/2}\mathcal{L}(\lambda)^{-1}\mathcal{B}^{1/2} = \mathcal{B}^{-1/2}\,\mathcal{B}\,\mathcal{L}(\lambda)^{-1}\mathcal{B}^{1/2}$$

shows that the closure of $\mathcal{B}^{1/2}\,\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{1/2}$ in $\mathcal{H}$ satisfies

$$\overline{\mathcal{B}^{1/2}\,\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{1/2}} = \mathcal{B}^{-1/2}\,(\mathcal{B}\,\mathcal{L}(\lambda)^{-1})^{\sim}\,\mathcal{B}^{1/2} \tag{40}$$

where $(\mathcal{B}\,\mathcal{L}(\lambda)^{-1})^{\sim}$ is the continuous extension of $\mathcal{B}\,\mathcal{L}(\lambda)^{-1}$ to the space $\mathcal{H}_{-1/2}$.

In Proposition (6.2.3) (V) we have shown that, under the assumptions (i), (ii), and (iii), the spectrum of $\mathcal{L}$ consists only of eigenvalues, but not that $\sigma(\mathcal{L})$ is discrete. If we additionally suppose assumption (iv), we are able to prove that the operator $\mathcal{M}$ in $\mathcal{H}_{1/2}$ has compact resolvent and hence discrete spectrum and that the spectra of $\mathcal{M}$ and of the linear operator pencil $\mathcal{L}$ coincide. Moreover, we employ Lemma (6.2.6) (II) to show that the algebraic eigenspaces of $\mathcal{L}$ and of $\mathcal{M}$ at each eigenvalue coincide.

Recall that a sequence $(x_j)_0^l \subset \mathcal{D}(\mathcal{A}_0)$ with $l \in \mathbb{N}_0 \cup \{\infty\}$ is a Jordan chain of $\mathcal{L}$ at an eigenvalue $\lambda \in \sigma_P(\mathcal{L})$ corresponding to an eigenvector x_0 if

$$(\mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B})x_j = \mathcal{B}x_{j-1},\; j = 0, 1, \ldots, l, \tag{41}$$

where we have set $x_{-1} := 0$ in the eigenvalue relation (see [137)] and note that $\mathcal{L}'(\lambda) = -\mathcal{B}$ and $\mathcal{L}^{(j)}(\lambda) = 0$ for $j > 2$ since $\mathcal{L}$ is linear).

Theorem(6.2.10)[31]: Under the assumptions (i), (ii), (iii), and (iv), the operator $\mathcal{M}$ has compact resolvent in $\mathcal{H}_{1/2}$ and

$$\sigma_P(\mathcal{L}) = \sigma(\mathcal{L}) = \sigma(\mathcal{M}) = \sigma_P(\mathcal{M});$$

moreover, the algebraic eigenspaces of $\mathcal{M}$ at an eigenvalue λ_0 and of the linear operator pencil $\mathcal{L}$ at λ_0 coincide.

Proof. First we prove that $\rho(\mathcal{L}) \subset \rho(\mathcal{M})$. To this end, let $\lambda \in \rho(\mathcal{L})$. By (38), we have $\lambda \in \rho(\mathcal{M})$ if $\overline{\mathcal{B}^{-1/2}\mathcal{L}(\lambda)\mathcal{B}^{-1/2}}$ is boundedly invertible in $\mathcal{H}$. To prove the latter, we first show that $\mathcal{B}^{1/2}\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{1/2}$, which has dense domain $\mathcal{D}(\mathcal{B}^{1/2})$, is bounded in $\mathcal{H}$. Since $\mathcal{D}(\mathcal{L}(\lambda)) = \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{D}(\mathcal{B})$, the operator $\mathcal{B}\,\mathcal{L}(\lambda)^{-1}$ is bounded in $\mathcal{H}$. We apply Lemma (6.2.6) (I) with the inner product $[x,y]_{-\frac{1}{2}} := (\mathcal{B}^{-1/2}x, \mathcal{B}^{-1/2}y)$, $x, y \in \mathcal{H}$, defined in (34). Since $\mathcal{D}(\mathcal{L}(\lambda)^*) = \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{D}(\mathcal{B})$ by Proposition(6.3.5), the operator $\mathcal{B}\,\mathcal{L}(\lambda)^{-*}$is bounded on $\mathcal{H}$ and we have

$$[\mathcal{B}\,\mathcal{L}(\lambda)^{-1}x, y]_{-1/2} = (\mathcal{B}^{-1/2}\,\mathcal{B}\,\mathcal{L}(\lambda)^{-1}x, \mathcal{B}^{-1/2}\,y) = (\mathcal{L}(\lambda)^{-1}x, y)$$
$$= (x, \mathcal{L}(\lambda)^{-*}y) = (\mathcal{B}^{-1/2}\,x, \mathcal{B}^{-1/2}\,\mathcal{B}\,\mathcal{L}(\lambda)^{-*}y) = [x, \mathcal{B}\,\mathcal{L}(\lambda)^{-*}y]_{-1/2}, x, y \in \mathcal{H}.$$

Now Lemma (6.2.6) shows that $\mathcal{B}\,\mathcal{L}(\lambda)^{-1}$ is bounded with respect to the norm induced by $[\cdot,\cdot]_{-1/2}$. Thus there exists $C \geq 0$ such that

$$\left\|\mathcal{B}^{-1/2}\,\mathcal{B}\,\mathcal{L}(\lambda)^{-1}y\right\|^2 \leq C\left\|\mathcal{B}^{-1/2}\,y\right\|^2, y \in \mathcal{H};$$

setting $\mathcal{B}^{-1/2}\,y = x$, we obtain

$$\left\|\mathcal{B}^{1/2}\,\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{1/2}\,x\right\|^2 \leq C\|x\|^2, x \in \mathcal{D}(\mathcal{B}^{1/2}). \quad (42)$$

Since $\mathcal{D}(\mathcal{B}^{\frac{1}{2}})$ is dense in $\mathcal{H}$, it follows that the operator $\mathcal{B}^{\frac{1}{2}}\mathcal{L}(\lambda)^{-1}\mathcal{B}^{\frac{1}{2}}$ is closable in $\mathcal{H}$ with everywhere defined bounded closure $\overline{\mathcal{B}^{\frac{1}{2}}\,\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{\frac{1}{2}}}$.

From the relation

$$\mathcal{B}^{1/2}\mathcal{L}(\lambda)^{-1}\mathcal{B}^{1/2}\mathcal{B}^{-1/2}\mathcal{L}(\lambda)\,\mathcal{B}^{-1/2} = \mathcal{B}^{1/2}\mathcal{L}(\lambda)^{-1}\mathcal{L}(\lambda)\,\mathcal{B}^{-1/2} = \mathcal{B}^{1/2}\,I|_{\mathcal{D}(\mathcal{A}_0)}\mathcal{B}^{-1/2}$$
$$= I|_{\mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_0)},$$

it follows that the operator $G(\lambda) = \mathcal{B}^{-\frac{1}{2}}\,\mathcal{L}(\lambda)\,\mathcal{B}^{-\frac{1}{2}}$ in $\mathcal{H}$, which is closable by Proposition (6.2.9), is injective. By (42), the inverse $G(\lambda)^{-1} = \mathcal{B}^{\frac{1}{2}}\mathcal{L}(\lambda)^{-1}\mathcal{B}^{\frac{1}{2}}$ is bounded with dense domain, thus closable. Hence (see, e.g.,[101]) the closure $\overline{G(\lambda)} = \overline{\mathcal{B}^{-\frac{1}{2}}\mathcal{L}(\lambda)\,\mathcal{B}^{-\frac{1}{2}}}$ is also injective with inverse $\overline{G(\lambda)}^{-1} = \overline{G(\lambda)^{-1}} = \overline{\mathcal{B}^{\frac{1}{2}}\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{\frac{1}{2}}}$, which is bounded and everywhere defined.

Now we are ready to prove that $\mathcal{M}$ has compact resolvent. Let $\lambda \in \rho(\mathcal{L}) \subset \rho(\mathcal{M})$. Since $\mathcal{B}$ is $(\mathcal{A}_0 + \mathcal{A}_1)$-compact by Proposition (6.2.3) (III) and, clearly, $\mathcal{B}$-bounded, Lemma (6.2.1) implies that $\mathcal{B}$ is $\mathcal{L}(\lambda)$-compact. Now we apply Lemma (6.2.6) (II) to the compact operator $\mathcal{B}\,\mathcal{L}(\lambda)^{-1}$ in $\mathcal{H}$ (see above). As a consequence, we obtain that the continuous extension $(\mathcal{B}\,\mathcal{L}(\lambda)^{-1})^{\sim}$ of $\mathcal{B}\,\mathcal{L}(\lambda)^{-1}$ to the space $\mathcal{H}_{-1/2}$ is compact in $\mathcal{H}_{-1/2}$ and hence, by (38) and (40),

$$(\mathcal{M}-\lambda)^{-1} = \mathcal{B}^{-1/2}\overline{\mathcal{B}^{1/2}\,\mathcal{L}(\lambda)^{-1}\,\mathcal{B}^{1/2}}\,\mathcal{B}^{1/2} = \mathcal{B}^{-1}(\mathcal{B}\mathcal{L}(\lambda)^{-1})^{\sim}\,\mathcal{B} \quad (43)$$

is compact in $\mathcal{H}_{1/2}$.

Next we show the equality of the spectra and point spectra of $\mathcal{L}$ and $\mathcal{M}$. Due to (37), (41) and the inclusion $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{M})$, if $\lambda_0 \in \sigma_P(\mathcal{L})$ with algebraic eigenspace $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{A}_0)$, then $\lambda_0 \in \sigma_P(\mathcal{M})$ with the same algebraic eigenspace $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{M})$. Together with the inclusion $\rho(\mathcal{L}) \subset \rho(\mathcal{M})$ proved above and the fact that the spectrum of $\mathcal{L}$ consists only of eigenvalues by Proposition (6.2.3) (V), we find

$$\sigma_P(\mathcal{L}) \subset \sigma_P(\mathcal{M}) \subset \sigma(\mathcal{M}) \subset \sigma(\mathcal{L}) = \sigma_P(\mathcal{L}),$$

and hence equality prevails everywhere.

It remains to be proved that if $\lambda_0 \in \sigma_P(\mathcal{M}) = \sigma_P(\mathcal{L})$ and $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{M}) \subset \mathcal{H}_{1/2}$ is the algebraic eigenspace of $\mathcal{M}$ at λ_0, then $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{L}(\lambda_0))$; then, since $\mathcal{M}|_{\mathcal{D}(\mathcal{A}_0)} - \lambda_0 = \mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1) - \lambda_0$, it is immediate that $Z_{\lambda_0}$is the algebraic eigenspace of $\mathcal{L}$ at λ_0. Let $\mu \in \rho(\mathcal{M})$ be arbitrary, but fixed. It is not very difficult to show that $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{M}) \subset \mathcal{H}_{1/2}$ is the algebraic eigenspace of $\mathcal{M}$ at λ_0 if and only if Z_{λ_0} is the algebraic eigenspace of the operator $(\mathcal{M}-\mu)^{-1}$ at $(\lambda_0-\mu)^{-1}$. By (43) we see that $\mathcal{B}Z_{\lambda_0} \subset \mathcal{H}_{-1/2}$ is the algebraic eigenspace of the operator $(\mathcal{B}\mathcal{L}(\mu)^{-1})^{\sim}$ in $\mathcal{H}_{-1/2}$ at $(\lambda_0-\mu)^{-1}$; in fact, the relation

$$((\mathcal{M}-\mu)^{-1} - (\lambda_0-\mu)^{-1})\,x_j = x_{j-1}, \quad j = 0,1,\ldots,l,$$

for a Jordan chain $(x_j)_0^l \subset \mathcal{H}_{1/2} = \mathcal{D}(\mathcal{B}^{1/2})$ of $(\mathcal{M}-\mu)^{-1}$ at $(\lambda_0-\mu)^{-1}$ can be written as

$$\mathcal{B}^{-1}((\mathcal{B}\,\mathcal{L}(\lambda)^{-1})^{\sim} - (\lambda_0-\mu)^{-1})\,\mathcal{B}x_j = \mathcal{B}^{-1}\,\mathcal{B}x_{j-1}, \quad j = 0,1,\ldots,l.$$

Now Lemma (6.2.6)(II) yields that $\mathcal{B}Z_{\lambda_0}$is the algebraic eigenspace of the operator $\mathcal{B}\mathcal{L}(\mu)^{-1}$ in $\mathcal{H}$ at $(\lambda_0-\mu)^{-1}$ and hence, in particular, $\mathcal{B}Z_{\lambda_0} \subset \mathcal{H}$. The latter implies that $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{B})$ and that Z_{λ_0} is the algebraic eigenspace of the operator $\mathcal{L}(\mu)^{-1}\mathcal{B}$ in $\mathcal{H}$ at $(\lambda_0-\mu)^{-1}$. The corresponding relation for Jordan chains

$$(\mathcal{L}(\mu)^{-1}\mathcal{B} - (\lambda_0-\mu)^{-1})\,x_j = x_{j-1}, \quad j = 0,1,\ldots,l,$$

shows that $x_0 = (\lambda_0 - \mu)\,\mathcal{L}(\mu)^{-1}\,\mathcal{B}x_0 \in \mathcal{D}(\mathcal{L}(\mu)) = \mathcal{D}(\mathcal{A}_0)$ and hence, by induction, $x_j = (\lambda_0 - \mu)(\mathcal{L}(\mu)^{-1}\,\mathcal{B}x_j + x_{j-1}) \in \mathcal{D}(\mathcal{L}(\mu)) = \mathcal{D}(\mathcal{A}_0), j = 1, 2, \ldots, l$. This proves that $Z_{\lambda_0} \subset \mathcal{D}(\mathcal{A}_0)$.

Theorem(6.2.11)[31]: Under the assumptions (i), (ii), (iii), and (iv), the spectrum of the linear operator pencil $\mathcal{L}(\lambda) = \mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B},\ \lambda \in \mathbb{C}$, is discrete, consists only of eigenvalues of finite algebraic multiplicity and lies in the horizontal strip

$$\sigma_P(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : |\mathfrak{I}(\lambda)| \leq \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\|\} .$$

Proof. All claims follow from Theorem (6.2.10) and from the fact that $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$ is a perturbation of the selfadjoint operator $\mathcal{M}_0$ in $\mathcal{H}_{1/2}$ by the operator $\mathcal{M}_1$ which is bounded in $\mathcal{H}_{1/2}$ with norm $\|\mathcal{M}_1\| \leq \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\|$ by Proposition (6.2.8) (II).

The above spectral inclusion may be improved if e.g. $\mathcal{A}_0$ is semi-bounded and the structure of the perturbation $\mathcal{A}_1$ is taken into account using a numerical range argument.

Theorem(6.2.12)[31]: Let the assumptions (i), (ii), (iii), and (iv) be satisfied and let $b_1, b_2 \in \mathbb{R}$ be such that for the bounded operator $\mathcal{B}^{-1/2}\mathcal{A}_1\,\mathcal{B}^{-1/2}$

(I) $b_1 \leq \mathfrak{I}(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\,y, y) \leq b_2, y \in \mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1), \|y\| = 1.$

Then the spectrum of the linear operator pencil $\mathcal{L}(\lambda) = \mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B}, \lambda \in \mathbb{C}$, is contained in the horizontal strip

$$\sigma_P(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : b_1 \leq \mathfrak{I}(\lambda) \leq b_2\};$$

if $a_1 \in \mathbb{R}$ is such that

(II) $\mathfrak{R}(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\,y, y) \geq a_1, y \in \mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1), \|y\| = 1,$

and we suppose, in addition, that $\mathcal{A}_0$ is semi-bounded with $a_0 \in \mathbb{R}$ such that

(III) $\mathcal{B}^{-1/2}\mathcal{A}_0\,\mathcal{B}^{-1/2} \geq a_0$,

then the spectrum of $\mathcal{L}$ is contained in the horizontal semi-strip

$$\sigma_P(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \mathfrak{R}(\lambda) \geq a_0 + a_1, b_1 \leq \mathfrak{I}(\lambda) \leq b_2\}.$$

Proof. Let $\lambda \in \sigma(\mathcal{L}) = \sigma_P(\mathcal{L})$. Then there is an $x \in D(\mathcal{L}(\lambda)) = \mathcal{D}(\mathcal{A}_0), x \neq 0$, with

$$(\mathcal{A}_0 x, x) + (\mathcal{A}_1 x, x) = \lambda(\mathcal{B}x, x).$$

As $\mathcal{B}$ is uniformly positive, we have $y := \mathcal{B}^{1/2}x \neq 0, y \in \mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_0) \subset \mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1)$, and

$$\lambda = \frac{(\mathcal{B}^{-1/2}\mathcal{A}_0\mathcal{B}^{-1/2}y, y)}{\|y\|^2} + \frac{(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}y, y)}{\|y\|^2} .$$

Since $\mathcal{A}_0$ is self-adjoint, all claims follow if we take real and imaginary parts.

Note that: The spectral inclusions in Theorem (6.2.12) also follow if we note that $\sigma_P(\mathcal{L}) = \sigma_P(\mathcal{M}) \subset W(\mathcal{M})$ and that the numerical range of $\mathcal{M} = \overline{\mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1)}$ as an operator in $\mathcal{H}_{1/2}$ satisfies $W(\mathcal{M}) \subset \overline{W(\mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1))}$ and

$$W(\mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1)) = \left\{\frac{(\mathcal{B}^{-1}(\mathcal{A}_0 + \mathcal{A}_1)x, x)_{\frac{1}{2}}}{\|x\|^2_{\frac{1}{2}}} : x \in \mathcal{D}(\mathcal{A}_0), x \neq 0\right\}$$
$$= \left\{\frac{((\mathcal{A}_0 + \mathcal{A}_1)x, x)}{\|\mathcal{B}^{1/2}x\|^2} : x \in \mathcal{D}(\mathcal{A}_0), x \neq 0\right\}.$$

Hence, according to Proposition (6.2.8), one can always choose

$$a_0 = \min\left\{0, \frac{\alpha_0}{\beta}\right\}, \quad a_1 = b_1 = b_2 = \|\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}\| .$$

We study the basis properties of the eigenvectors and associated vectors of the linear operator pencil $\mathcal{L}(\lambda) = \mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B}, \lambda \in \mathbb{C}$, by means of the linear operator $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$ introduced in the previous part.

Throughout this part we assume that the operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ satisfy the assumptions (i), (ii), (iii) and assumption (iv). We begin by showing that a system of eigen−and associated vectors of $\mathcal{L}$ can be chosen to form a so-called Riesz basis in $\mathcal{H}_{1/2}$, that is, a basis equivalent to an orthonormal basis (see [154]). Here we use the following perturbation result for semi-bounded self-adjoint operators.

Theorem(6.2.13)[31]: Let T_0 be a semi-bounded self-adjoint operator in a Hilbert space G with compact resolvent whose eigenvalues $\lambda_1^0 \leq \lambda_2^0 \leq \cdots$ are eventually simple and satisfy $\lambda_l^0 - \lambda_{l-1}^0 \to \infty, l \to \infty$, and let T_1 be a bounded operator in G. Then the operator $T := T_0 + T_1$ has compact resolvent, its eigenvalues are eventually simple, and its eigen-and associated functions can be chosen to form a Riesz basis in G.

Theorem(6.2.14)[31]: Suppose that the eigenvalues $\lambda_1^0 \leq \lambda_2^0 \leq \cdots$ of the linear operator pencil $\mathcal{L}_0(\lambda) = \mathcal{A}_0 - \lambda\mathcal{B}$ are eventually simple and satisfy $\lambda_l^0 - \lambda_{l-1}^0 \to \infty, l \to \infty$. Then the eigen- and associated functions of the operator pencil $\mathcal{L}(\lambda) = \mathcal{A}_0 + \mathcal{A}_1 - \lambda\mathcal{B}$ can be chosen to form a Riesz basis in $\mathcal{H}_{1/2}$.

Proof. Due to Theorem (6.2.10), the eigenvalues of $\mathcal{L}$ and $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$ and the corresponding algebraic eigenspaces coincide. By Proposition (6.2.8), $\mathcal{M}_0 = \overline{\mathcal{B}^{-1}\mathcal{A}_0}$ is self-adjoint in $\mathcal{H}_{1/2}$, while $\mathcal{M}_1 = \overline{\mathcal{B}^{-1}\mathcal{A}_1}$ is bounded and everywhere defined in $\mathcal{H}_{1/2}$. Now Theorem (6.2.13) applied to the operators $\mathcal{M}_0$ and $\mathcal{M}_1$ in $\mathcal{H}_{1/2}$ yields the claim.

The next aim is to show that the eigen- and associated vectors of the linear pencil $\mathcal{L}$ even form a Bari basis, that is, a basis quadratically close to an orthonor-mal basis (see [154]). To this end, we employ a result due to A.S. Markus (see [154], [114]) which requires more detailed knowledge about the asymptotic behaviour of the eigenvalues.

Recall that an operator T in a Hilbert space G is called dissipative if

$$\Im(Tx, x) \geq 0, \qquad x \in \mathcal{D}(T),$$

and quasi-dissipative if there exists a $\gamma \in \mathbb{R}$ such that $T + i\gamma$ is dissipative.

Theorem(6.2.15)[31]: Suppose that T is a bounded dissipative operator in a Hilbert space G with compact imaginary part which has a sequence of eigenvalues $(\lambda_\nu)_1^\infty, \lambda_\nu \neq \lambda_\mu$ for $\nu \neq \mu$. Let N_{λ_ν} and L_{λ_ν} denote, respectively, the geometric and algebraic eigenspaces of T at λ_ν and define $n_\nu := \dim L_{\lambda_\nu}$ .Suppose that there exists $\nu_0 \in \mathbb{N}$ such that $N_{\lambda_\nu} = L_{\lambda_\nu}$ for $\nu \geq \nu_0$, and suppose that the system of all eigen-and associated vectors of T is complete in G. If

$$\sum_{\substack{\nu,\mu=1 \\ \nu\neq\mu}}^{\infty} \min\{n_\nu, n_\mu\} \frac{\Im(\lambda_\nu)\Im(\lambda_\mu)}{\left|\lambda_\nu - \overline{\lambda_\mu}\right|^2} < \infty,$$

then a sequence consisting of orthonormal bases of all the algebraic eigenspaces L_{λ_ν} forms a Bari basis of G.

In the following we show that there exists a point $\zeta \in \mathbb{C}$ such that the operator $(\mathcal{M} - \zeta)^{-1}$ satisfies the assumptions of Theorem (6.2.15). To this end, we show the following auxiliary result.

Theorem(6.2.16)[31]: Suppose that the eigenvalues $\lambda_1^0 \leq \lambda_2^0 \leq \cdots$ of the linear operator pencil $\mathcal{L}_0(\lambda) = \mathcal{A}_0 - \lambda\mathcal{B}$ are eventually simple and satisfy $\lambda_l^0 - \lambda_{l-1}^0 \to \infty, l \to \infty$. Let $(\lambda_l)_1^\infty \subset \mathbb{C}$ denote the eigenvalues of $\mathcal{L}$ and hence of $\mathcal{M}$. Choose $\zeta \in \mathbb{C}$ such that $(\mathcal{M} - \zeta)^{-1}$ exists and is dissipative and set $\tilde{\lambda}_l := (\lambda_l - \zeta)^{-1}, l \in \mathbb{N}$. If

$$\sum_{\substack{\nu,\mu=1 \\ \nu\neq\mu}}^{\infty} \frac{\Im(\tilde{\lambda}_\nu)\Im(\tilde{\lambda}_\mu)}{\left|\tilde{\lambda}_\nu - \overline{\tilde{\lambda}_\mu}\right|^2} < \infty, \tag{44}$$

then the eigen-and associated functions of the linear operator pencil $\mathcal{L}$ can be chosen to form a Bari basis in $\mathcal{H}_{1/2}$.

Proof. By Proposition (6.2.8), the operator $\mathcal{M}_1$ is bounded in $\mathcal{H}_{1/2}$. Hence we can choose $\zeta \in \mathbb{C}$ such that $(\mathcal{M} - \zeta)^{-1}$ exists and

$$Y := -\Im(\zeta) + \frac{1}{2i}(\mathcal{M}_1 - \mathcal{M}_1^*) \leq 0.$$

By Theorem (6.2.10), $(\mathcal{M} - \zeta)^{-1}$ is compact and has, thus, also compact imaginary part. Moreover, a short calculation shows that

$$\Im((\mathcal{M} - \zeta)^{-1}) = -(\mathcal{M} - \zeta)^{-*} \quad Y(\mathcal{M} - \zeta)^{-1} \geq 0.$$

Theorem (6.2.14) also yields that the eigenvalues of $\mathcal{M}$ are eventually simple and the eigen-and associated vectors of $\mathcal{M}$ are complete in $\mathcal{H}_{1/2}$. Hence Theorem (6.2.15) applies to the operator $(\mathcal{M} - \zeta)^{-1}$. It is not difficult to see that the points $\tilde{\lambda}_l$ are the eigenvalues of $(\mathcal{M} - \zeta)^{-1}$ and the algebraic eigenspaces of $\mathcal{M}$ at λ_l and of $(\mathcal{M} - \zeta)^{-1}$ at $\tilde{\lambda}_l$ coincide. Now Theorem (6.2.15), together with the fact that the eigenvalues and algebraic eigen-spaces of $\mathcal{L}$ and $\mathcal{M}$ coincide by Theorem (6.2.10), completes the proof.

In view of applications to differential equations, the next observation is useful.

Proposition(6.2.17)[31]: Let $(\lambda_l)_1^\infty \subset \mathbb{C}$ be a sequence and suppose that there exist constants $c, d \in \mathbb{R}$ such that $\lambda_l = cl^2 + dl + O(1)$ for $l \to \infty$. Let $\zeta \in \mathbb{C}$ and set $\tilde{\lambda}_l := (\lambda_l - \zeta)^{-1}, l \in \mathbb{N}$. Then the sequence $(\tilde{\lambda}_l)_1^\infty$ satisfies the condition (44).

The Hagen-Poiseuille problem arises in the linear stability analysis of incompressible flow in a circular pipe subject to non-axisymmetric disturbances (see [105]); it leads to a spectral problem for a system of singular differential equations of the form

$$\begin{pmatrix} \mathcal{T}(k^2r^2\,\mathcal{T}) + i\alpha RU\mathcal{T} + i\alpha R\frac{1}{r}\left(\frac{U'}{k^2r}\right)' & 2\alpha n\mathcal{T} \\ 2\alpha n\mathcal{T} - inR\frac{U'}{r} & \mathcal{S} + i\alpha RUk^2r^2 \end{pmatrix}\begin{pmatrix}\Phi \\ \Omega\end{pmatrix}$$
$$= i\alpha Rc\begin{pmatrix}\mathcal{T} & 0 \\ 0 & k^2r^2\end{pmatrix}\begin{pmatrix}\Phi \\ \Omega\end{pmatrix} \qquad (45)$$

on the interval $(0,1]$ where $R \geq 0$ is the Reynolds number, $\alpha \in \mathbb{R}$ is the streamwise wave number, $n \in \mathbb{Z}$ is the azimuthal wave number, c is a complex wave speed, and $\lambda = i\alpha Rc$ is considered as the spectral parameter. The differential expressions $\mathcal{T}$ and $\mathcal{S}$ are given by

$$\mathcal{T} := \frac{1}{r^2} - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{k(r)^2r}\frac{d}{dr}\right),$$
$$\mathcal{S} := k(r)^4r^2 - \frac{1}{r}\frac{d}{dr}\left(k(r)^2r^3\frac{d}{dr}\right), \qquad (46)$$

where

$$k(r)^2 = \alpha^2 + n^2/r^2, \qquad r \in (0,1], \qquad (47)$$

and the axial mean flow $U : [0,1] \to \mathbb{R}$ is assumed to be twice differentiable with

$$\lim_{r\to 0} U'(r) = 0, \qquad U'' \text{ bounded on } [0,1]. \qquad (48)$$

In the applications literature the system (45) is supplemented by what we shall term 'physical' boundary conditions, such as those in [105]:

$$\Phi(1) = \Phi'(1) = \Omega(1) = 0,$$
$$\lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Phi'(r) = 0 \text{ if } n = 0, \qquad (49)$$
$$\Phi(1) = \Phi'(1) = \Omega(1) = 0, \lim_{r\to 0}\Phi(r) = 0,$$
$$\lim_{r\to 0}\Phi'(r) \text{ finite}, \lim_{r\to 0}\Omega(r) = 0 \text{ if } n = \pm 1, \qquad (50)$$

$$\Phi(1) = \Phi'(1) = \Omega(1) = 0,$$
$$\lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Phi'(r) = \lim_{r\to 0}\Omega(r) = 0 \text{ if } |n| \geq 2. \qquad (51)$$

Reference [111] discussed the extent to which these conditions appear in the construction of operator realizations of the differential expressions. It turns out that some of the conditions at $r = 0$ need not be imposed, as they are satisfied automatically in the natural Hilbert space setting. In order to understand these and other issues, we now establish some notation and review results from [111]. The characterization of the operators involved, and their properties, depends on whether or not $n = 0$ because the coefficient $k(r)$ behaves rather differently as $r \longrightarrow 0$ for $n \neq 0$ and for $n = 0$.

In the Hilbert space $L^2((0,1];\ r)$ let the operator $\tilde{T}_0$ be defined by $\mathcal{D}(\tilde{T}_0) = C_0^\infty(0,1)$ with $\tilde{T}_0 u = \mathcal{T}u$ for $u \in \mathcal{D}(\tilde{T}_0)$. Integration by parts shows that $\tilde{T}_0$ is symmetric in $L^2((0,1];r)$ and hence closable. It is a well-known result in the standard theory of Sturm-Liouville problems (see e.g. [24]) that its closure T_0, called the minimal operator associated with $\mathcal{T}$, has domain

$$\mathcal{D}(T_0) = \{u \in \mathcal{D}(T_0^*): u(1) = u'(1) = 0;\ \forall v \in \mathcal{D}(T_0^*), \{u,v\} = 0\}, \qquad (52)$$

where

$$\mathcal{D}(T_0^*) = \left\{u \in L^2((0,1];\ r): u, \frac{u'}{k^2 r} \in AC_{loc}([0,1]), \mathcal{T}u \in L^2((0,1];\ r)\right\}, \qquad (53)$$

$$\{u,v\} := \lim_{r\to 0}\begin{vmatrix} u(r) & v(r) \\ \frac{1}{k^2 r}u'(r) & \frac{1}{k^2 r}v'(r)\end{vmatrix}. \qquad (54)$$

If $n \neq 0$ then $r = 0$ is a limit point singularity and so the condition $\{u,v\} = 0$ is satisfied for all u and v in $\mathcal{D}(T_0^*)$; it can therefore be dropped.

Lemma(6.2.18)[31]: For every $n \in \mathbb{Z}$ the minimal operator T_0 associated with $\mathcal{T}$ in $L^2((0,1];\ r)$ is uniformly positive and admits a self-adjoint Friedrichs extension T_1 with domain

$$\mathcal{D}(T_1) = \{u \in \mathcal{D}(T_0^*): u(1) = 0\} \quad \text{if} \quad n \neq 0, \qquad (55)$$
$$\mathcal{D}(T_1) = \{u \in \mathcal{D}(T_0^*): u(1) = 0, \{u,v\} = 0\} \quad \text{if} \quad n = 0, \qquad (56)$$

(Here v is the solution of the differential equation $\mathcal{T}v = 0$ having the asymptotic behavior $v(r) = r^2 + ar^4 + \cdots$ when $r \longrightarrow 0$.) A lower bound for T_1 is given by

$$T_1 \geq \tau_{\alpha,n} := \min\sigma(T_1) \geq \tilde{\tau}_{\alpha,n} := \begin{cases} 1 + \frac{j_{0,1}^2}{\alpha^2 + n^2} & \text{if } n \neq 0, \\ 1 + \frac{\pi^2}{\alpha^2} & \text{if } n = 0, \end{cases} \qquad (57)$$

Moreover, T_1 has compact resolvent.

Proof. Since T_0 and T_1 have the same lower bound, we may consider $y \in \mathcal{D}(T_0)$. By [111] we know that $\lim_{r\to 0} y(r) = y(1) = 0$ for every $n \in \mathbb{Z}$.

For $n \neq 0$ we use the estimate $k(r)^2 r^2 = \alpha^2 r^2 + n^2 \leq \alpha^2 + n^2, r \in [0,1]$, to obtain

$$\begin{aligned}(T_0 y, y)_{L^2((0,1];\ r)} &= \int_0^1 \left(\frac{|y(r)|^2}{r} + \frac{r|y'(r)|^2}{k(r)^2 r^2}\right) dr \\ &\geq \|y\|^2_{L^2((0,1];\ r)} + \frac{1}{\alpha^2 + n^2}\int_0^1 r|y'(r)|^2\ dr \\ &\geq \left(1 + \frac{\nu_0}{\alpha^2 + n^2}\right) \|y\|^2_{L^2((0,1];\ r)},\end{aligned} \qquad (58)$$

where ν_0 is the smallest eigenvalue of the spectral problem

$$-(ry')' = \lambda ry, \ \ y(0)\text{finite}, \ \ y(1) = 0, \ \ y \in L^2((0,1];\, r). \tag{59}$$

Eq. (59) is a Bessel differential equation for which $\nu_0 = j_{0,1}^2$ is the square of the first zero of the Bessel function J_0 (see [18]).

For $n = 0$ we use the estimate $k(r)^2 r = \alpha^2 r \le \alpha^2, r \in [0,1]$, to obtain

$$\begin{aligned}(T_0 y, y)_{L^2((0,1];\, r)} &= \int_0^1 \left(\frac{|y(r)|^2}{r} + \frac{|y'(r)|^2}{\alpha^2 r}\right) dr \\ &\ge \|y\|^2_{L^2((0,1];\, r)} + \frac{1}{\alpha^2}\int_0^1 |y'(r)|^2 \, dr \\ &\ge \left(1 + \frac{\pi^2}{\alpha^2}\right) \|y\|^2_{L^2((0,1];\, r)},\end{aligned}$$

where π^2 is the smallest eigenvalue of the Sturm–Liouville problem

$$-y'' = \lambda y, \ \ y(0) = y(1) = 0, \ \ y \in L^2((0,1];\, 1).$$

The compactness of the resolvent of T_1 was proved in [111].

For $n \ne 0$, we work in the space $L^2((0,1]; r)$ and we define S_0 to be the minimal operator associated with $\mathcal{S}$ in the same way as we defined the minimal operator T_0 associated with $\mathcal{T}$.

Lemma(6.2.19)[31]: For $n \ne 0$ the minimal operator S_0 associated with $\mathcal{S}$ in $L^2((0,1];\, r)$ is uniformly positive and admits a self-adjoint Friedrichs extension S_1 with domain

$$\mathcal{D}(S_1) = \left\{\begin{matrix} y \in L^2((0,1];\, r) : y, k^2 r^3 y' \in AC_{\text{loc}}((0,1]), \\ \mathcal{S}y \in L^2((0,1];\, r), y(1) = 0 \end{matrix}\right\}.$$

satisfying $\mathcal{D}(S_1) \subset \mathcal{D}(T_1) \subset \mathcal{D}(T_0^*)$, and lower bound given by

$$\begin{aligned} S_1 \ge \sigma_{\alpha,n} &= \min \sigma(S_1) \ge \tilde{\sigma}_{\alpha,n} \\ &:= \alpha^2 + n^2 j_{0,1}^2 + \begin{cases} (\alpha^2 + n^2)^2 & \text{if } \alpha < n \\ 4\alpha^2 n^2 & \text{if } \alpha \ge n \end{cases}. \end{aligned} \tag{60}$$

(Here $j_{0,1}$ is the first zero of the Bessel function J_0.) Moreover, S_1 has compact resolvent.

Proof. By [111], we know that $\lim_{r\to 0} y(r) = 0$ for every $y \in \mathcal{D}(S_1)$. From the differential expression $\mathcal{S}$ defining S_1 (see (46)),

$$(S_1 y, y)_{L^2((0,1];\, r)} = \int_0^1 r h(r)^2 |y(r)|^2 \, dr + \int_0^1 h(r)\, |r y'(r)|^2 \, dr, \tag{61}$$

in which $h(r) := k(r)^2 r = \alpha^2 r + \frac{n^2}{r}$, $r \in (0,1]$. We immediately bound the first integral from below by $(\min_{r\in(0,1]} h(r)^2)\|y\|^2_{L^2((0,1];\, r)}$ and we easily check that

$$\min_{r\in(0,1]} h(r) = \begin{cases} h(1) = \alpha^2 + n^2 & \text{if } \alpha < n, \\ h\left(\frac{n}{\alpha}\right) = 2\alpha n & \text{if } \alpha \ge n, \end{cases}$$

For the second integral we note that $r^2 h(r) = \alpha^2 r^3 + n^2 r, r \in (0,1]$, and that

$$\begin{aligned}\int_0^1 h(r)\, |r y'(r)|^2 \, dr &= \alpha^2 \int_0^1 r^3\, |y'(r)|^2 \, dr + n^2 \int_0^1 r\, |y'(r)|^2 \, dr \\ &\ge (\alpha^2 \mu_0 + n^2 \nu_0)\|y\|^2_{L^2((0,1];\, r)},\end{aligned}$$

in which $\nu_0 = j_{0,1}^2$ is the smallest eigenvalue of the spectral problem (59) and μ_0 is the infimum of the spectrum of the Sturm–Liouville problem

$$-(r^3 y')' = \lambda r y, \ \ y(1) = 0, \ \ y \in L^2((0,1];\, r), \tag{62}$$

Eq.(62) can be solved explicitly in powers of r to show that the spectrum is purely essential and covers the semi-axis $[1,\infty)$, whence $\mu_0 = 1$.

The compactness of the resolvent of S_1 was proved in [111].

For $n = 0$, the system of differential equations (45) decouples, but in the second equation the coefficient k^2r^2 of the spectral parameter $\lambda = i\alpha R$ is no longer uniformly positive. Therefore, in this case we work in the space $L^2\big((0,1];\ r^3\big)$. We shall need the following results.

Lemma(6.2.20)[31]: For $n = 0$ the Sturm–Liouville differential expression $\frac{1}{\alpha^2r^2}\ \mathcal{S}$ is formally symmetric in the space $L^2\big((0,1];r^3\big)$. It is regular at $r = 0$ and limit point (see, e.g., [24]) at $r = 0$. Moreover, the non-trivial solutions of the differential equation $\frac{1}{\alpha^2r^2}\ \mathcal{S}u = \lambda u, \lambda \in \mathbb{R}$, always have at most finitely many zeros in $(0,1]$.

Proof. A standard Frobenius ansatz $u(r;\ \lambda) = r^{\nu}\sum_{j=0}^{\infty} a_j r^j$ yields the indicial equation $\nu(\nu + 2) = 0$ together with recurrence relations for the coefficients a_j of the form

$$(\nu + j)(\nu + j + 2)a_j = (\alpha^2 - \lambda)a_{j-2}.$$

For $\nu = 0$ we obtain a solution $u_0(r;\lambda)$ in even powers of j which is an entire function of r. For $\nu = -2$ only the trivial solution is obtained; however Liouville's formula for a second solution (see [103]),

$$u_1(r;\ \lambda) = u_0(r;\ \lambda)\int^{r}\frac{1}{s^3u_0(s;\ \lambda)}\ ds$$

yields a second solution with the asymptotic behaviour $u(r;\lambda) \sim r^{-2}$ for small r. Since $u_0(\cdot\ ;\ \lambda) \in L^2\big((0,1];\ r^3\big)$ and $u_1(\cdot\ ;\ \lambda) \notin L^2\big((0,1];\ r^3\big)$, the singular point at $r = 0$ is of limit point type. Moreover since $u_0(\cdot;\ \lambda)$ has only finitely many zeros in $(0,1)$, for each λ, the same is true for any non-trivial solution $\hat{u}(\cdot,\alpha)$, for any real λ, by the Sturm Interlacing Theorem.

In the Hilbert space $L^2\big((0,1];\ r^3\big)$ we consider the operator $\tilde{R}_0$ given by $\mathcal{D}(\tilde{R}_0) = C_0^\infty(0,1)$ and $\tilde{R}_0 u = \frac{1}{\alpha^2r^2}\ \mathcal{S}u$ for $u \in \mathcal{D}(\tilde{R}_0)$. Integration by parts shows that $\tilde{R}_0$ is symmetric in $L^2\big((0,1];\ r^3\big)$ and hence closable; its closure, denoted by R_0, is the so-called minimal operator associated with $\frac{1}{\alpha^2r^2}\ \mathcal{S}$ in $L^2\big((0,1];\ r^3\big)$.

Lemma(6.2.21)[31]: For $n = 0$ the minimal operator R_0 associated with $\frac{1}{\alpha^2r^2}\ \mathcal{S}$ in the space $L^2\big((0,1];\ r^3\big)$. is uniformly positive and admits a self-adjoint Friedrichs extension R_1 with domain and lower bound given by

$$\mathcal{D}(R_1) = \left\{u \in L^2\big((0,1];\ r^3\big): u, r^3u' \in AC_{\text{loc}}\big((0,1]\big), \frac{1}{r^2}\mathcal{S}u \in L^2\big((0,1];\ r^3\big), u(1) = 0\right\},$$

$$R_1 \geq \mathrm{r}_\alpha := \min\sigma(R_1) \geq \tilde{r}_\alpha := \alpha^2 + j_{1,1}^2. \tag{63}$$

Moreover, R_1 has compact resolvent.

Proof. Integration by parts shows that, for $u \in \mathcal{D}(R_0)$,

$$(R_0u,u)_{L^2\left((0,1];\,r^3\right)} = \int_0^1 \left(\alpha^2r^3\,|u(r)|^2 + r^3|u'(r)|^2\right)\ dr. \tag{64}$$

It is not difficult to see that the smallest eigenvalue of the spectral problem

$$-(r^3u')' = \lambda r^3u,\quad u(0)\ \text{finite},\quad u(1) = 0,$$

is the square of the first zero $j_{1,1}$ of the Bessel function J_1 (see [18]). This means that in (64) we can estimate $\int_0^1 r^3|u'(r)|^2\ dr \geq j_{1,1}^2\int_0^1 r^3|u(r)|^2\,dr$ and hence that $R_0 \geq \alpha^2 + j_{1,1}^2$ is a uniformly positive symmetric operator. The same lower bound holds for R_1 , which is the Friedrichs extension of R_0.

Standard Sturm-Liouville theory (see e.g. [109, 110]) guarantees that at the regular endpoint $r = 1$, functions $u \in \mathcal{D}(R_1)$ satisfy the boundary condition $u(1) = 0$; no boundary condition is obtained at $r = 0$ since this is a limit point endpoint by Lemma (6.2.20) (comp. [111]).

It was proved in Lemma (6.2.20) that, by [23], the solutions of $\frac{1}{\alpha^2 r^2}\,\mathcal{S}u = \lambda u,\ \lambda \in \mathbb{R}$, are non-oscillatory. By [23], this implies that $\sigma_{ess}(R_1) = \emptyset$. Together with the selfadjointness of R_1 , this shows that R_1 has compact resolvent.

For $n \neq 0$ there are many possibilities, including $T_1\, k^2\, r^2\, T_1$ and $T_0^*\, k^2\, r^2\, T_0 = (krT_0)^*\,(krT_0)$. The operator $T_0^*\, k^2 r^2 T_0$ is the one whose domain carries the boundary conditions $u(1) = 0 = u'(1)$ specified in the physical problem, whereas $T_1 k^2 r^2 T_1$ involves the boundary conditions $u(1) = 0 = (\mathcal{T}u)(1)$. It turns out that $T_0^*\, k^2 r^2 T_0$ is the Friedrichs extension of the minimal operator associated with the expression $\mathcal{T}k^2 r^2 \mathcal{T}$, and is uniformly positive with $T_0^*\, k^2 r^2 T_0 \geq n^2$.

For $n = 0$, multiplication by kr is no longer boundedly invertible so krT_0 may not be closed; thus $(krT_0)^*(krT_0)$ may not be self-adjoint. The self-adjoint Friedrichs realization of the expression $\mathcal{T}k^2 r^2 \mathcal{T}$ is described in [111], where it is denoted L_1. Following the results at the end of the proof of [111],

$$\mathcal{D}(L_1) = \left\{ \begin{matrix} y \in L^2\big((0,1];\, r\big) : \mathcal{T}k^2 r^2 \mathcal{T}y \in L^2\big((0,1]; r\big), \\ y(1) = 0 = y'(1),\ \ y(r) = O(r^2),\ \ y'(r) = O(r)(r \longrightarrow 0) \end{matrix} \right\}. \tag{65}$$

Because L_1 is a Friedrichs extension, we can calculate a lower bound for L_1 by using the fact that it has the same lower bound on $C_0^\infty(0,1)$-functions as on elements of its domain. Integration by parts then yields no boundary terms, and

$$\begin{aligned}(L_1 y, y)_{L^2((0,1];\, r)} &= (\alpha r\, \mathcal{T}y, \alpha r\, \mathcal{T}y)_{L^2((0,1];\, r)} = \int_0^1 r \left| \frac{\alpha}{r}\, y(r) - \alpha \frac{d}{dr}\left(\frac{1}{\alpha^2 r}\, \frac{d}{dr}\right) y(r) \right|^2 dr \\ &= \int_0^1 \frac{\alpha^2}{r}\, |y(r)|^2\, dr + 2\int_0^1 \frac{|y'(r)|^2}{r}\, dr + \int_0^1 r\alpha^2 \left| \frac{d}{dr}\left(\frac{1}{\alpha^2 r}\, \frac{d}{dr}\right) y(r) \right|^2 dr \\ &\geq (\alpha^2 + 2\pi^2)\|y\|^2_{L^2((0,1];\, r)},\end{aligned} \tag{66}$$

where, in the last line, we have omitted the third integral and used a Poincaré estimate of the second. A tighter lower bound for L_1 is given in Lemma (6.2.37) below.

The system of differential equations (45) with the boundary conditions (49), (50), and (51) can now be written as a spectral problem for an operator pencil

$$(\mathcal{A}_0 + \mathcal{A}_1)u = \lambda \mathcal{B}u,$$

in the Hilbert space $L^2((0,1]; r) \oplus L^2((0,1];\, r)$ where the coefficients $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ are block operator matrices given by

$$\mathcal{A}_0 := \begin{pmatrix} T_0^*\, k^2 r^2 T_0 & 2\alpha n T_0^* \\ 2\alpha n T_0 & S_1 \end{pmatrix},\quad \mathcal{D}(\mathcal{A}_0) := \mathcal{D}\big(T_0^*\, k^2 r^2 T_0\big) \oplus \mathcal{D}(S_1); \tag{67}$$

$$\mathcal{A}_1 := i\alpha R \begin{pmatrix} UT_1 + \frac{1}{r}\left(\frac{U'}{k^2 r}\right)' & 0 \\ -\frac{n}{\alpha}\frac{U'}{r} & Uk^2 r^2 \end{pmatrix},\quad \mathcal{D}(\mathcal{A}_1) := \mathcal{D}(T_1) \oplus L^2\big((0,1]; r\big); \tag{68}$$

$$\mathcal{B} := \begin{pmatrix} T_1 & 0 \\ 0 & k^2 r^2 \end{pmatrix},\quad \mathcal{D}(\mathcal{B}) := \mathcal{D}(T_1) \oplus L^2((0,1];\, r). \tag{69}$$

Remark(6.2.22)[31]: To justify this choice of domains we need to make two observations.

(I) By hypothesis (48), $U'(r) = \int_0^r U''(s)\, ds, r \in [0,1]$, and U'' is essentially bounded. Thus also $\frac{U'}{r}$ is essentially bounded with $L^\infty((0,1])$-norm bound

$$\left\|\frac{U'}{r}\right\|_\infty \le \|U''\|_\infty. \tag{70}$$

(II) As established in [111], the function $\frac{1}{r}\left(\frac{U'}{k^2 r}\right)'$ is essentially bounded when $n \neq 0$; in fact,

$$\left\|\frac{1}{r}\left(\frac{U'}{k^2 r}\right)'\right\|_\infty = \left\|\frac{1}{\alpha^2 r^2 + n^2} U'' + \frac{-\alpha^2 r^2 + n^2}{(\alpha^2 r^2 + n^2)^2} \frac{U'}{r}\right\|_\infty \le \frac{2}{n^2}\|U''\|_\infty, \tag{71}$$

where we have used some simple analysis to show that the coefficients of $U''$and U'/r are both bounded by $1/n^2$.

Remark(6.2.23)[31]: The operator $\mathcal{A}_0$ can also be written as

$$\mathcal{A}_0 = \begin{pmatrix} T_0^* k^2 r^2 T_0 & 2\alpha n T_1 \\ 2\alpha n T_1 & S_1 \end{pmatrix}. \tag{72}$$

In fact, in (67), the $(2,1)$ term acts on functions in $\mathcal{D}(T_0^* k^2 r^2 T_0) \subseteq \mathcal{D}(T_0)$, while for $n \neq 0$ the $(1,2)$ term acts on functions in $\mathcal{D}(S_1) \subseteq \mathcal{D}(T_1)$. Thus the operator T_0 in the off-diagonal corners may be replaced by its extension T_1.

Lemma(6.2.24)[31]:(Checking assumption (i) for $n \neq 0$). The operator $\mathcal{A}_0$ is self-adjoint.

Proof. We shall use Proposition (6.2.5). Since $A_0 := T_0^* k^2 r^2 T_0 = (k r T_0)^* (k r T_0)$, it is sufficient to show that T_0 is $(krT_0)^*(krT_0)$ –bounded with relative bound zero, and that T_0^* is S_1 –bounded. The fact that $\mathcal{D}(S_1) \subseteq \mathcal{D}(T_1)$ was established in [111]; a fortiori $\mathcal{D}(S_1) \subseteq \mathcal{D}(T_0^*)$ and so T_0^* is S_1-bounded. We shall prove slightly more than required by showing that T_0 is $(krT_0)^*(krT_0)$-compact.

The operator A_0 is uniformly positive, $A_0 \ge n^2$, and the quadratic form of A_0 is given (see [17]) by

$$\mathfrak{t}_{A_0}(y) = \|krT_0 y\|^2_{L^2((0,1];r)} = \int_0^1 r\,|k(r) r(\mathcal{T}y)(r)|^2\, dr, \quad y \in \mathcal{D}(\mathfrak{t}_{A_0}) = \mathcal{D}\left(A_0^{\frac{1}{2}}\right), \tag{73}$$

and its domain $\mathcal{D}(\mathfrak{t}_{A_0})$ consists of those y which can be approximated in the norm defined by (73) by elements of $\mathcal{D}(A_0)$. Since $\mathcal{D}(A_0) \subseteq \mathcal{D}(\mathrm{T}_0)$ and since kr is boundedly invertible, this immediately establishes that $\mathcal{D}(\mathfrak{t}_{A_0}) \subset \mathcal{D}(T_0)$, and hence T_0 is $A_0^{1/2}$ –bounded. But A_0 has compact inverse, so $A_0^{-1/2}$ is compact. Hence T_0 is A_0 –compact.

Lemma(6.2.25)[31]: (Checking assumption (ii) for $n \neq 0$). The operator $\mathcal{B}$ is selfadjoint, uniformly positive with $\mathcal{B} \ge \min \sigma(\mathcal{B}) = \min\{\tau_{\alpha,n}, n^2\} \ge 1$, and $\mathcal{A}_0$-compact.

Proof. The operator T_1 is selfadjoint and $T_1 \ge \tau_{\alpha,n} \ge 1$ by Lemma (6.2.18). Because $k(r)^2 r^2 = \alpha^2 r^2 + n^2 \ge n^2$, $r \in (0,1]$, the corresponding multiplication operator in $L^2((0,1]; r)$ is also selfadjoint and uniformly positive with $\min \sigma(k^2 r^2) = n^2 \ge 1$ for $n \neq 0$. This proves the first two claims about $\mathcal{B}$.

Using the alternative form of $\mathcal{A}_0$ in (72) and [111] it is sufficient to check the following: $\mathcal{D}(T_0^* k^2 r^2 T_0) \subseteq \mathcal{D}(T_1)$; $\mathcal{D}(S_1) \subseteq \mathcal{D}(k^2 r^2)$; T_1 is $T_0^* k^2 r^2 T_0$-compact; $k^2 r^2$ is S_1-compact.

Clearly, $\mathcal{D}(T_0^* k^2 r^2 T_0) \subseteq \mathcal{D}(T_0) \subseteq \mathcal{D}(T_1)$. The fact that T_1 is $T_0^* k^2 r^2 T_0$-compact is immediate from the fact that T_1 is a finite-dimensional extension of T_0 and, as established in the proof of Lemma (6.2.24), T_0 is $T_0^* k^2 r^2 T_0$-compact.

Finally, it was proved in [111] that S_1 has compact resolvent. Since $k(r)^2r^2 = \alpha^2r^2 + n^2, r \in (0,1)$, the multiplication operator k^2r^2 is bounded and everywhere defined in $L^2((0,1];\, r)$. Thus $\mathcal{D}(S_1) \subseteq \mathcal{D}(k^2r^2)$ and k^2r^2 is S_1-compact.

Lemma(6.2.26)[31]: (Checking assumption (iii) for $n \neq 0$). The operator $\mathcal{A}_1$ is closable and satisfies $\mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}_1), \mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1^*)$.

Proof. To prove that $\mathcal{A}_1$ is closable it is sufficient to check that $\mathcal{A}_1^*$ is densely defined. Thus it suffices to prove the claims for the domains.

By the definitions in (68) and (69), we have $\mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}_1)$. According to Remark (6.2.22), the multiplication operators by U, $\frac{1}{r}\left(\frac{U'}{k^2r^2}\right)'$, $\frac{U'}{r}$, and Uk^2r^2 which occur in $\mathcal{A}_1$ by (68) are bounded, everywhere defined and self-adjoint in $L^2((0,1];\, r)$. Moreover, T_1 is self-adjoint so that we have the operator identity $(UT_1)^* = T_1U$. Hence $\mathcal{A}_1^*$ is given by

$$\mathcal{A}_1^* = -i\alpha R \begin{pmatrix} T_1U + \frac{1}{r}\left(\frac{U'}{k^2r}\right)' & -n\frac{U'}{\alpha r} \\ 0 & Uk^2r^2 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_1^*) = \mathcal{D}(T_1U) \oplus L^2\left((0,1];\, r\right). \tag{74}$$

In order to show that $\mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{A}_1^*)$, we thus have to show that $\mathcal{D}(T_1) \subset \mathcal{D}(T_1U)$. To this end, let $y \in \mathcal{D}(T_1)$. Since $U \in C^2([0,1])$, we clearly have Uy, $\frac{1}{k^2r}(Uy)' = \frac{1}{k^2r}(Uy' + U'y) \in AC_{loc}([0,1])$ and

$$\mathcal{T}Uy = UT_1y - 2\frac{U'}{r}\frac{y'}{k^2r} - \frac{1}{r}\left(\frac{U'}{k^2r}\right)' y. \tag{75}$$

According to the assumptions on U and the properties deduced from them in Remark (6.2.22), the first and the last terms belong to $L^2((0,1];\, r)$ and the coefficient $2\frac{U'}{r}$ of the second term is essentially bounded. Since $y(1) = 0$ for $y \in \mathcal{D}(T_1)$, we trivially have $(Uy)(1) = 0$ also, and so to show that $Uy \in \mathcal{D}(T_1)$ it suffices to show that $\frac{y'}{k^2r} \in L^2((0,1]; r)$. In fact, if we use $k(r)^2 \geq \alpha^2 + n^2$, $r \in (0,1]$, and (58), we find

$$\left\|\frac{y'}{k^2r}\right\|^2_{L^2((0,1];r)} = \int_0^1 r\left|\frac{y'(r)}{k(r)^2r}\right|^2 dr = \int_0^1 \frac{1}{k(r)^2}\frac{|y'(r)|^2}{k(r)^2r}\, dr$$

$$\leq \frac{1}{\alpha^2+n^2}\int_0^1 \frac{|y'(r)|^2}{k(r)^2r}\, dr \leq \frac{1}{\alpha^2+n^2}(T_1y, y)_{L^2((0,1];r)} < \infty. \tag{76}$$

Remark(6.2.27)[31]: The operators $\mathcal{A}_1\mathcal{B}^{-1}$ and $\mathcal{A}_1^*\mathcal{B}^{-1}$ are bounded with the following bounds, in which $\tau_{\alpha,n} = \min \sigma(T_1) \geq 1$ is the positive lower bound on T_1:

$$\|\mathcal{A}_1\mathcal{B}^{-1}\| \leq |\alpha|R\left(\|U\|_\infty + \frac{1}{\tau_{\alpha,n}}\max\left\{\frac{|n|}{|\alpha|}, \frac{2}{n^2}\right\}\|U''\|_\infty\right), \tag{77}$$

$$\|\mathcal{A}_1^*\mathcal{B}^{-1}\| \leq |\alpha|R\left(\|U\|_\infty + \left(\frac{2}{\sqrt{\tau_{\alpha,n}}}\frac{1}{\sqrt{\alpha^2+n^2}} + \frac{1}{|n||\alpha|}\right)\|U''\|_\infty\right), \tag{78}$$

These bounds follow if we use the explicit forms of $\mathcal{A}_1\mathcal{B}^{-1}$ and $\mathcal{A}_1^*\mathcal{B}^{-1}$ according to (68) and (74), (75), respectively, together with the bounds for U, $\frac{U'}{r}$, $\frac{1}{r}\left(\frac{U'}{k^2r}\right)'$ given in Remark (6.2.22), the estimate for $\frac{y'}{k^2r}$ derived in (76), and the inequality

$$(T_1y, y)_{L^2((0,1];\, r)} = (f, T_1^{-1}f)_{L^2((0,1];\, r)} \leq \frac{1}{\tau_{\alpha,n}}\|f\|^2_{L^2((0,1];r)} \tag{79}$$

for $y = T_1^{-1}f \in \mathcal{D}(T_1)$.

Lemma(6.2.28)[31]: (Checking assumption (iv) for $n \neq 0$). $\mathcal{D}(\mathcal{A}_0)$ is a core for $\mathcal{B}^{1/2}$ and we have the domain inclusion $\mathcal{D}(\mathcal{A}_1) \subseteq \mathcal{D}(\mathcal{B}^{1/2})$.

Proof. The operator $\mathcal{B}$ has domain $\mathcal{D}(T_1) \oplus L^2((0,1];r)$ and is block diagonal, and so $\mathcal{D}(\mathcal{B}^{1/2}) = \mathcal{D}(T_1^{1/2}) \oplus L^2((0,1];r)$.

Since T_1 is the Friedrichs extension of the (symmetric) minimal operator T_0, which in turn is the closure of $\tilde{T}_0$ having domain $\mathcal{D}(\tilde{T}_0) = C_0^\infty[0,1]$, the set $C_0^\infty[0,1]$ is a core for $T_1^{1/2}$. Thus $C_0^\infty[0,1] \oplus L^2((0,1];r)$ is a core for $\mathcal{B}^{1/2}$. Because $\mathcal{D}(\mathcal{A}_0) \supseteq C_0^\infty[0,1] \oplus L^2((0,1];r)$, the first part of assumption (iv) is checked.

Since the domains of $\mathcal{A}_1$ and $\mathcal{B}$ coincide by definition, it is immediate that $\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{B}) \subset \mathcal{D}(\mathcal{B}^{1/2})$.

As a consequence of Lemmas (6.2.24), (6.2.25), (6.2.26), and (6.2.28), all the hypothesis of the abstract study are satisfied by the Hagen–Poiseuille system for $n \neq 0$. So there is only one obstacle left to applying Theorem (6.2.16) to obtain a Bari basisness result for the eigen-and associated functions of the Hagen-Poiseuille operator: we must verify that the spectral asymptotics given in Proposition (6.1.17) are satisfied.

To this end we note that in the purely self-adjoint case $U \equiv 0$, which is equivalent to $\mathcal{A}_1 \equiv 0$, the eigenvalues were calculated explicitly as long ago as 1969 by Burridge and Drazin in [25, 19]. In particular, the asymptotics were also calculated

$$\lambda_l^{(0)} = \frac{1}{4}\left(l + n + \frac{1}{2}\right)^2 \pi^2 + O(1), l \to \infty. \tag{80}$$

Lemma(6.2.29)[31]: (Checking eigenvalue asymptotics in Proposition (6.2.17)). Both for $n = 0$ and for $n \neq 0$, the eigenvalues of the Hagen–Poiseuille problem (45), (50), (51) satisfy

$$\lambda_l = \frac{1}{4}\left(l + n + \frac{1}{2}\right)^2 \pi^2 + O(1), l \to \infty. \tag{81}$$

Proof. By Theorem (6.2.10) the eigenvalues $(\lambda_l)_1^\infty$ of the Hagen−Poiseuille problem coincide with the eigenvalues of the operator $\mathcal{M}$; in the case $\mathcal{A}_1 = 0$, the eigenvalues $\left(\lambda_l^{(0)}\right)_1^\infty$ coincide with those of the self-adjoint operator $\mathcal{M}_0$. Since $\mathcal{M}$ is a bounded perturbation of $\mathcal{M}_0$, (81) is immediate from (80).

Theorem(6.2.30)[31]: For $n \neq 0$, the eigen- and associated functions of the Hagen–Poiseuille problem (45), (50), (51) may be chosen to form a Bari basis of the Hilbert space $\mathcal{H}_{1/2} = \mathcal{D}(T_1^{1/2}) \oplus L^2((0,1];r)$ equipped with the scalar product

$$\left(\begin{pmatrix}\Phi\\ \Omega\end{pmatrix}, \begin{pmatrix}\Phi\\ \Omega\end{pmatrix}\right)'_{1/2} = \left(\frac{\Phi}{r}, \frac{\Phi}{r}\right)_{L^2((0,1];r)} + \left(\frac{\Phi'}{kr}, \frac{\Phi'}{kr}\right)_{L^2((0,1];r)} + (\Omega, \Omega)_{L^2((0,1];r)}. \tag{82}$$

Proof. Since all assumptions have been verified, Theorem (6.2.16) yields the Bari basis property in $\mathcal{H}_{1/2} = \mathcal{D}(\mathcal{B}^{1/2}) = \mathcal{D}(T_1^{1/2}) \oplus L^2((0,1];r)$ equipped with the scalar product $(\cdot,\cdot)_{1/2} = (\mathcal{B}^{1/2}\,\cdot, \mathcal{B}^{1/2}\,\cdot)$ given by

$$\left(\begin{pmatrix}\Phi\\ \Omega\end{pmatrix}, \begin{pmatrix}\Phi\\ \Omega\end{pmatrix}\right)_{1/2} = \left(T_1^{1/2}\Phi, T_1^{1/2}\Phi\right)_{L^2((0,1];r)} + (kr\Omega, kr\Omega)_{L^2((0,1];r)}.$$

By (58) the first term on the right-hand side coincides with the first two terms on the right-hand side of (82). Since $k^2r^2\cdot$ is a bounded and boundedly invertible operator in $L^2((0,1];r)$, the scalar product $(kr\,\cdot, kr\,\cdot)_{L^2((0,1];r)}$ is equivalent to the usual scalar product in $L^2((0,1];r)$. Hence the scalar product $(\cdot,\cdot)'_{1/2}$ in (82) is equivalent to the original scalar product $(\cdot,\cdot)_{1/2}$ on $\mathcal{H}_{1/2}$.

In case $n = 0$ we have $k(r)^2 = \alpha^2, r \in [0,1]$, and the system of differential equations (45) decouples into the two scalar differential equations

$$\left(\mathcal{T}(\alpha^2 r^2 \mathcal{T}) + i\,\alpha R\left(U\,\mathcal{T} + \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)'\right)\right)\Phi = \lambda\,\mathcal{T}\Phi,$$

$$(\mathcal{S} + i\,\alpha R U)\,\Omega = \alpha^2\, r^2\,\lambda\,\Omega,$$

with $\lambda = i\alpha R$. While the abstract results still apply to the linear operator pencil induced by the first equation in $L_2((0,1];\ r)$, they no longer apply to the linear operator pencil induced by the second equation since the coefficient $\alpha^2 r^2$ of λ does not induce a uniformly positive multiplication operator in $L_2((0,1];\ r)$. Thus we obtain basis properties of the eigen- and associated functions for the second component in a different Hilbert space, formally by dividing by $\alpha^2 r^2$.

A convenient formulation in terms of operators in the Hilbert spaces $L_2((0,1];r)$ and $L_2((0,1];\ r^3)$, respectively, is now the following

$$\left(L_1 + i\alpha R\left(UT_1 + \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)'\right)\right)\Phi = \lambda T_1\Phi,\ \ \Phi \in \mathcal{D}\big((\alpha r T_0)^*(\alpha r T_0)\big), \tag{83}$$

$$(R_1 + i\alpha R U)\Omega = \lambda\Omega, \qquad \Omega \in \mathcal{D}(R_1), \tag{84}$$

where L_1 , T_1 , and R_1 are the operators introduced in the present section .

We treat the two components separately. For the first spectral problem we apply the abstract results for linear operator pencils developed in the preceding parts; for the second problem, which is a classical spectral problem, we apply the abstract results in the particular case $\mathcal{B} = I$.

We begin with the problem (83) and introduce the linear operator pencil $\mathcal{L}_1$ in the Hilbert space $L_2((0,1];\ r)$ by

$$\mathcal{L}_1(\lambda) := A_0 + A_1 - \lambda B,\ \ \mathcal{D}\big(\mathcal{L}_1(\lambda)\big) := \mathcal{D}(A_0),\ \lambda \in \mathbb{C}, \tag{85}$$

where

$$A_0 := L_1,\ \ A_1 := i\alpha R\left(UT_1 + \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)'\right),\ \ B := T_1. \tag{86}$$

The following proposition shows that the linear operator pencil $\mathcal{L}_1$ satisfies the abstract assumptions (i)-(iv) and of Proposition (6.2.17) (i.e. the results of Lemmas (6.2.26), (6.2. 24), (6.2.25), (6.2. 28), and Proposition (6.2.4) all continue to hold with $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ replaced by A_0, A_1 , and B, respectively).

Lemma(6.2.31)[31]: (Checking assumptions (i)–(iv) for $n = 0$). The operator A_0 is self-adjoint with $A_0 \geq \alpha^2 + 2\pi^2$; $B = T_1$ is uniformly positive with $T_1 \geq \tau_{\alpha,0} \geq 1$, has compact inverse, and is A_0-compact; A_1 is closable with $\mathcal{D}(B) = \mathcal{D}(A_1)$ and $\mathcal{D}(B) \subseteq \mathcal{D}(A_1^*)$; $\mathcal{D}(A_0)$ is a core for $B^{1/2}$ and $\mathcal{D}(A_1) \subseteq \mathcal{D}(B^{1/2})$.

Proof. The first claim was established in above and (66). All claims for $B = T_1$ except for the last were showed in Lemma (6.2.18).

To check that B is A_0-compact, first observe that $\tilde{L}_1 := (T_1\alpha r)(\overline{\alpha r T_1})$ is self-adjoint by Von Neumann's theorem, because $\left(\overline{\alpha r T_1}\right)^* = T_1^*\alpha r = T_1\alpha r$. We also note that both L_1 and $\tilde{L}_1$ are selfadjoint realizations of the same ordinary differential expression $\mathcal{T}k^2r^2\mathcal{T}$. This means that the difference of their resolvents is a finite rank operator, and so it suffices to check that $T_1\left(\tilde{L}_1 - \lambda\right)^{-1}$ is compact. Following the method in the proof of [111], we observe that $0 \in \rho(\tilde{L}_1)$ and that $T_1\tilde{L}_1^{-1} = \left(\tilde{L}_1 T_1^{-1}\right)^{-1}$. However, since $(\overline{\alpha r T_1})|_{\mathcal{D}(T_1)} = \alpha r T_1$, we have

$$\tilde{L}_1 T_1^{-1} = (T_1\alpha r)(\overline{\alpha r T_1})T_1^{-1} = T_1\alpha^2 r^2.$$

Thus $T_1\tilde{L}_1^{-1} = \frac{1}{\alpha^2 r^2} T_1^{-1}$. This is compact by [111].

Unlike in the case $n \neq 0$, the multiplication operator $i\alpha R \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)' \cdot$ is no longer bounded in $L^2((0,1];\ r)$. But, by [111], we have

$$\mathcal{D}(T_1) \subset \mathcal{D}\left(i\alpha R \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)' \cdot\right), \tag{87}$$

which shows that $\mathcal{D}(B) = \mathcal{D}(T_1) = \mathcal{D}(A_1)$ and also $\mathcal{D}(A_1) \subset \mathcal{D}(B^{1/2})$.

As in the proof of Lemma (6.2.26), to prove that $\mathcal{A}_1$ is closable it is sufficient to check that $\mathcal{D}(B) \subset \mathcal{A}_1^*$ which implies that $\mathcal{A}_1^*$ is densely defined. To this end, let $y \in \mathcal{D}(T_1) = \mathcal{D}(B)$. As in the case $n \neq 0$ (see the proof of Lemma (6.2.26), we have $Uy, \frac{1}{k^2 r}(Uy)' = \frac{1}{k^2 r}(U\,y' + U'\,y) \in AC_{loc}([0,1])$ and formula (75) now reads, because $k\,(r)^2 = \alpha^2,\ r \in [0,1]$,

$$\mathcal{T}Uy = U\mathrm{T}_1 y - 2\frac{U'}{r}\frac{y'}{\alpha^2 r} - \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)' y. \tag{88}$$

By the assumptions on U and the properties deduced from them in Remark (6.2. 22), the first term belongs to $L^2((0,1]\,;\,r)$, the coefficient $2\,\frac{U\prime}{r}$ of the second term is essentially bounded, and the last term belongs to $L^2((0,1],\ r)$ because of (87). The proof that $\frac{y\prime}{\alpha^2 r} \in L^2((0,1], r)$ is analogous to the proof in the case $n \neq 0$ (comp. (76)):

$$\left\|\frac{y'}{\alpha^2 r}\right\|^2_{L^2((0,1];r)} = \frac{1}{\alpha^2}\int_0^1 \frac{|y'(r)|^2}{\alpha^2 r}\,dr \leq \frac{1}{\alpha^2}(T_1 y, y)_{L^2((0,1];r)} < \infty.$$

Thus it remains to be shown that Uy satisfies the boundary condition $\{Uy, v\} = 0$ for v with $v(r) = O(r^2), r \longrightarrow 0$ (see remarks following (6.2.27)). By (54) we have

$$\{Uy, v\} = \lim_{r\longrightarrow 0}(U(r))\{y, v\} - \lim_{r\longrightarrow 0}\left(\frac{1}{\alpha^2 r}U'(r)y(r)v(r)\right),$$

and $\{y,\ v\} = 0$ since $y \in D(T_1)$. We know that $U'(0) = 0$ by hypothesis on U. The condition $y \in D(T_1)$ implies that $y(0) = 0$ through the Cauchy-Schwarz inequality

$$|y(r)| \leq \left(\int_0^1 \frac{|y'(s)|^2}{s}\,ds\right)^{1/2}\left(\int_0^r s\ ds\right)^{1/2} \leq \frac{r}{\sqrt{2\alpha^2}}(T_1 y, y)_{L^2((0,1];r)}.$$

Hence $\lim_{r\longrightarrow 0}\left(\frac{1}{\alpha^2 r}U'(r)y(r)v(r)\right) = 0$ and so $\{Uy, v\} = 0$ as required.

The fact that $\mathcal{D}(A_0)$ is a core for $B^{1/2}$follows in the same way as for the case $n \neq 0$ in the proof of Lemma (6.2.28): precisely because $B = T_1$ is the Friedrichs extension of T_0, the operator $B^{1/2}$ has $C_0^\infty\,[0,1]$ as a core, and hence has $\mathcal{D}(A_0)$ as a core.

As a consequence of Lemma (6.2.31) and of the eigenvalue asymptotics in Lemma (6.2.29) , the pencil $\mathcal{L}_1(\lambda) = A_0 + A_1 - \lambda B,\ \lambda \in \mathbb{C}$, in (83) satisfies all the hypotheses of the abstract study; in particular, Proposition (6.2.3) shows that $\mathcal{L}_1$ has empty essential spectrum (comp. [111]). Theorem (6.2.16) now yields the following basis result in the Hilbert space $\mathcal{H}_{1,1/2} = \mathcal{D}(B^{\ 1/2})$ if we note that $B = T_1$ and observe formula (58).

Theorem(6.2.32)[31]: For $n = 0$, the eigen- and associated functions of the linear operator pencil $\mathcal{L}_1(\lambda) = A_0 + A_1 - \lambda\,B,\ \lambda \in \mathbb{C}$, may be chosen to form a Bari basis of the space $\mathcal{H}_{1,1/2} = \mathcal{D}(T_1^{1/2})$ equipped with the scalar product

$$(\Phi,\Phi)_{1,1/2} = \left(T_1^{1/2}\,\Phi, T_1^{1/2}\,\Phi\right)_{L^2((0,1];r)} = \left(\frac{\Phi}{r},\frac{\Phi}{r}\right)_{L^2((0,1];r)} + \left(\frac{\Phi'}{\alpha r},\frac{\Phi'}{\alpha r}\right)_{L^2((0,1];r)}.$$

Finally we consider the second spectral problem (84).

Theorem(6.2.33)[31]: For $n = 0$, the eigen- and associated functions of the linear operator $L_2 := R_1 + i\alpha RU$ in $L^2((0,1];r^3)$ may be chosen to form a Bari basis of $L^2((0,1];\, r^3)$.

Proof. By Lemma (6.2.21), R_1 is a selfadjoint semi-bounded operator in $L^2((0,1];r^3)$ with compact resolvent. The eigenvalue asymptotics (80) for the case $\mathcal{A}_1 = 0$ and $n = 0$ imply a fortiori that the eigenvalues of R_1, being a subset of $\left(\lambda_l^{(0)}\right)_0^\infty$, have the asymptotic behaviour $O(l^2)$. Since $U \in C^2([0,1])$ by assumption, the multiplication operator $i\alpha RU\cdot$ is bounded in $L_2((0,1];\, r^3)$.

Theorem (6.2.13) readily shows that the eigen-and associated functions of the operator L_2 may be chosen to form a Riesz basis of $L^2((0,1];\, r^3)$. In order to see that they even form a Bari basis, we note that the linear operator pencil $\mathcal{L}_2(\lambda) := L_2 - \lambda,\ \lambda \in \mathbb{C}$, trivially satisfies the abstract assumptions (i)–(iv) with $\mathcal{B} = I$ in $L_2((0,1];\, r^3)$ because R_1 has compact inverse and $\mathcal{A}_1 = i\alpha RU\cdot$ is bounded and everywhere defined. Now Theorem (6.2. 16) yields the claim.

While Burridge and Drazin give information about the asymptotic location of eigenvalues in [25], Theorems (6.2.11) and (6.2.12) yield global bounds for the whole set of eigenvalues, which depend on estimates for the operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ respectively. We estimate these eigenvalue bounds for the Hagen–Poiseuille problem in terms of the parameters α, R, n, and of the axial mean flow U.

Here we consider the case $\alpha \geq 0$; the formulation of the results for $\alpha < 0$ requires only minor modifications.

For the refined spectral inclusion in Theorem (6.2.12) we first estimate the constants a_0, a_1, b_1, and b_2 for the operators $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{B}$ introduced in (67), (68), and (69).

Lemma(6.2.34)[31]: (Bounds involving $\mathcal{A}_0$). The operator $\mathcal{A}_0$ is semi-bounded with

$$\mathcal{B}^{-1/2}\,\mathcal{A}_0\,\mathcal{B}^{-1/2} \geq a_{\alpha,n} \geq \tilde{a}_{\alpha,n} > 0 \tag{89}$$

Where

$$a_{\alpha,n} := \frac{1}{2}\left(n^2\tau_{\alpha,n} + \frac{\sigma_{\alpha,n}}{\alpha^2+n^2} - \sqrt{\left(n^2\tau_{\alpha,n} - \frac{\sigma_{\alpha,n}}{\alpha^2+n^2}\right)^2 + \frac{16\alpha^2n^2}{\alpha^2+n^2}}\right) \tag{90}$$

and $\tilde{a}_{\alpha,n}$ arises from $a_{\alpha,n}$ by replacing $\tau_{\alpha,n}, \sigma_{\alpha,n}$ by their lower bounds $\tilde{\tau}_{\alpha,n}, \tilde{\sigma}_{\alpha,n}$ defined in (57) and (60), respectively.

Proof. Suppose that $u = (f,g)^T \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(T_0^* k^2 r^2 T_0) \oplus \mathcal{D}(S_1) \subset \mathcal{D}(T_1) \oplus \mathcal{D}(S_1)$. A direct calculation yields

$$\begin{aligned}
&(\mathcal{A}_0 u, u)_{L^2((0,1];r)}\\
&= \int_0^1 r\,|k(r)r(T_0 f)(r)|^2\,dr + \int_0^1 r(S_1 g)(r)\overline{g(r)}dr + 4\alpha n\,\Re\left(\int_0^1 r(T_0 f)(r)\overline{g(r)}\,dr\right)\\
&\geq n^2\|T_1 f\|^2_{L^2((0,1];r)} + \sigma_{\alpha,n}\|g\|^2_{L^2((0,1];r)} - 4\alpha|n|\left|(T_1 f, g)_{L^2((0,1];r)}\right|\\
&\geq (n^2 - 4\alpha|n|\epsilon)\|T_1 f\|^2_{L^2((0,1];r)} + \left(\sigma_{\alpha,n} - \frac{\alpha|n|}{\epsilon}\right)\|g\|^2_{L^2((0,1];r)},
\end{aligned} \tag{91}$$

where we have used Young's inequality $|ab| \leq \epsilon|a|^2 + |b|^2/(4\epsilon)$ with some $\epsilon > 0$ in the last step. Now let $u = \mathcal{B}^{-1/2}\,v$, i.e. $v = \mathcal{B}^{1/2}\,u = (x,\ y)^T \in \mathcal{B}^{1/2}\,\mathcal{D}(\mathcal{A}_0)$ and $(f,g)^T = \left(T_1^{-1/2}\,x, \frac{1}{kr}y\right)^T$. Using the lower bounds

$$T_1 \geq \tau_{\alpha,n}; \quad \frac{1}{k(r)^2 r^2} \geq \frac{1}{\alpha^2 + n^2}, \quad r \in (0,1],$$

in (91), we obtain

$$\left(\mathcal{A}_0 \mathcal{B}^{-1/2} v, \mathcal{B}^{-1/2} v\right)_{L^2((0,1];r)}$$
$$\geq (n^2 - 4\alpha|n|\epsilon)\tau_{\alpha,n}\|x\|^2_{L^2((0,1];r)} + \frac{\sigma_{\alpha,n} - \alpha|n|/\epsilon}{\alpha^2 + n^2}\|y\|^2_{L^2((0,1];r)}. \quad (92)$$

By the intermediate value theorem it is easy to see that the equation

$$(n^2 - 4\alpha|n|\epsilon)\tau_{\alpha,n} = \frac{\sigma_{\alpha,n} - \alpha|n|/\epsilon}{\alpha^2 + n^2}$$

always has a solution $\epsilon \in (0, \infty)$; this solution can easily be found explicitly,

$$\epsilon = \epsilon_{\alpha,n} = \frac{1}{8\alpha|n|}\left(n^2 - \frac{\sigma_{\alpha,n}}{\tau_{\alpha,n}(\alpha^2 + n^2)} + \sqrt{\left(n^2 - \frac{\sigma_{\alpha,n}}{\tau_{\alpha,n}(\alpha^2 + n^2)}\right)^2 + \frac{16\alpha^2 n^2}{\tau_{\alpha,n}(\alpha^2 + n^2)}}\right).$$

Then $\left(n^2 - 4\alpha|n|\epsilon_{\alpha,n}\right)\tau_{\alpha,n} = a_{\alpha,n}$ with $a_{\alpha,n}$ as in (90) and hence the best lower bound in (92) is $\left(a_{\alpha,n}\|x\|^2_{L^2((0,1];r)} + \|y\|^2_{L^2((0,1];r)}\right)$, as required.

Finally, it is easy to see that $a_{\alpha,n}$ is monotonically increasing as a function of both $\tau_{\alpha,n}$ and $\sigma_{\alpha,n}$, which gives $a_{\alpha,n} \geq \tilde{a}_{\alpha,n}$; the inequality $\tilde{a}_{\alpha,n} > 0$ follows, e.g. by contradiction, if we note that $\tilde{\tau}_{\alpha,n} \geq 1$ and $\tilde{\sigma}_{\alpha,n} \geq 4\alpha^2$.

Lemma(6.2.35)[31]: (Bounds involving $\mathcal{A}_1$). The bounded (and densely defined) operator $\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2}$ satisfies the estimates

(I) $\Re(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2} v, v) \geq -\alpha R b_{\alpha,n}\|U''\|_\infty$,

(II) $\alpha R\left(U_{\min} - b_{\alpha,n}\|U''\|_\infty\right) \leq \Im\left(\mathcal{B}^{-1/2}\mathcal{A}_1\mathcal{B}^{-1/2} v, v\right) \leq \alpha R(U_{\max} + b_{\alpha,n}\|U''\|_\infty)$,

for $v \in \mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1), v = 1$, with

$$U_{\min} := \min_{r\in[0,1]} U(r), \quad U_{\max} := \max_{r\in[0,1]} U(r),$$

$$b_{\alpha,n} := \frac{1}{\sqrt{\tau_{\alpha,n}}}\,\frac{1}{2}\left(\frac{1}{\sqrt{\alpha^2 + n^2}} + \sqrt{\frac{1}{\alpha^2 + n^2} + \frac{1}{\alpha^2}}\right) \leq \tilde{b}_{\alpha,n}$$

where $\tau_{\alpha,n} = \min\sigma(T_1) \geq \tilde{\tau}_{\alpha,n}$ is defined as in Lemma (6.2.18) and $\tilde{b}_{\alpha,n}$ arises from $b_{\alpha,n}$ if we replace $\tau_{\alpha,n}$ by its lower bound $\tilde{\tau}_{\alpha,n}$.

Proof. Let $v = (x, y)^T \in \mathcal{B}^{1/2}\mathcal{D}(\mathcal{A}_1)$, $\|v\| - L^2((0,1]; r) = 1$, and set $u := \mathcal{B}^{-1/2}v := (f, g)^T \in \mathcal{D}(A_1) = \mathcal{D}(T_1) \oplus L^2((0,1]; r)$. Then $f = T_1^{-1/2}x, g = \frac{1}{kr}y$ and we have

$$\|f\|_{L^2((0,1];r)} = \left\|T_1^{-1/2}x\right\|_{L^2((0,1];r)} \leq \frac{1}{\sqrt{\tau_{\alpha,n}}}\|x\|_{L^2((0,1];r)}, \quad (93)$$

$$\|g\|_{L^2((0,1];r)} = \left\|\frac{1}{kr}y\right\|_{L^2((0,1];r)} \leq \frac{1}{|n|}\|y\|_{L^2((0,1];r)}. \quad (94)$$

By the definition of $\mathcal{A}_1$ in (68),

$$(\mathcal{A}_1 u, u)_{L2((0,1];r)} = i\alpha R\left(\left(\left(UT_1 + \frac{1}{r}\left(\frac{U'}{k^2 r}\right)'\right)f, f\right)_{L^2((0,1];r)} - \left(\frac{\alpha}{n}\frac{U'}{r}f, g\right)_{L^2((0,1];r)}\right.$$
$$\left. + (Uk^2r^2 g, g)_{L^2((0,1];r)}\right). \quad (95)$$

Integrating by parts in the first term, we find

$$\left(\left(UT_1 + \frac{1}{r}\left(\frac{U'}{k^2 r}\right)'\right) f, f\right)_{L^2((0,1];r)}$$

$$= \int_0^1 rU(r)\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{r}\frac{1}{k(r)^2 r}\left|f'^{(r)}\right|^2\right) dr - \int_0^1 \frac{U'(r)}{k(r)^2 r} f(r)\overline{f'(r)}\, dr. \quad (96)$$

The first term in (96) and the last term within the brackets in (95) are both real. Using the Cauchy-Schwarz inequality and the estimates (70), (93), (94), the second term within the brackets in (95) can be estimated by

$$\left|\left(\frac{\alpha}{n}\frac{U'}{r} f, g\right)_{L^2((0,1];r)}\right| \leq \frac{|n|}{\alpha} \|U''\|_\infty \|f\|_{L^2((0,1];r)} \|g\|_{L^2((0,1];r)}$$
$$\leq \frac{1}{\alpha} \|U''\|_\infty \frac{1}{\sqrt{\tau_{\alpha,n}}} \|x\|_{L^2((0,1];r)} \|y\|_{L^2((0,1];r)}$$
$$\leq \frac{1}{2\alpha} \|U''\|_\infty \frac{1}{\sqrt{\tau_{\alpha,n}}} \left(\delta\|x\|^2_{L^2((0,1];r)} + \frac{1}{\delta}\|y\|^2_{L^2((0,1];r)}\right) \quad (97)$$

for some $\delta > 0$. Using the Cauchy–Schwarz inequality and the estimates (70), $k(r)^2 \geq \alpha^2 + n^2, r \in (0,1]$, and (93), we estimate the second term in (96) by

$$\left|\int_0^1 r\,\frac{U'(r)}{k(r)^2 r^2} f(r)\overline{f'(r)}\, dr\right| \leq \|f\|_{L^2((0,1];r)} \left(\int_0^1 r\left|\frac{U'(r)}{k(r)^2 r^2} f'(r)\right|^2 dr\right)^{1/2}$$
$$= \|f\|_{L^2((0,1];r)} \left(\int_0^1 \frac{1}{k(r)^2}\left|\frac{U'(r)}{r}\right|^2 \frac{|f'(r)|^2}{k(r)^2 r}\, dr\right)^{1/2}$$
$$\leq \frac{1}{\sqrt{\alpha^2 + n^2}} \|U''\|_\infty \|f\|_{L^2((0,1];r)} \left(\int_0^1 \frac{|f'(r)|^2}{k(r)^2 r}\, dr\right)^{1/2}$$
$$\leq \frac{1}{\sqrt{\alpha^2 + n^2}} \|U''\|_\infty \|f\|_{L^2((0,1];r)} (T_1 f, f)^{1/2}_{L^2((0,1];r)}$$
$$\leq \frac{1}{\sqrt{\alpha^2 + n^2}} \|U''\|_\infty \frac{1}{\sqrt{\tau_{\alpha,n}}} \|x\|^2_{L^2((0,1];r)}. \quad (98)$$

Hence, if we take the real part in (95), use (97), (98), and recall that we have assumed that $\alpha \geq 0$, we arrive at

$$\Re\left(\mathcal{A}_1 \mathcal{B}^{-1/2} v, \mathcal{B}^{-1/2} v\right)_{L^2((0,1];r)}$$
$$= -\alpha R \Im\left(-\left(\frac{\alpha}{n}\frac{U'}{r} f, g\right)_{L^2((0,1];r)} - \int_0^1 \frac{U'(r)}{k(r)^2 r} f(r)\overline{f'(r)}\, dr\right)$$
$$\geq -\alpha R\left(\left|\left(\frac{\alpha}{n}\frac{U'}{r} f, g\right)_{L^2((0,1];r)}\right| + \left|\int_0^1 \frac{U'(r)}{k(r)^2 r} f(r)\overline{f'(r)}\, dr\right|\right)$$
$$\geq -\alpha R \|U''\|_\infty \frac{1}{\sqrt{\tau_{\alpha,n}}} \left(\frac{\delta}{2\alpha} + \frac{1}{\sqrt{\alpha^2 + n^2}}\right) \|x\|^2_{L^2((0,1];r)} + \frac{1}{2\alpha\delta} \|y\|^2_{L^2((0,1];r)}.$$

This lower bound is optimized by the choice

$$\delta = -\frac{\alpha}{\sqrt{\alpha^2 + n^2}} + \sqrt{\frac{\alpha^2}{\alpha^2 + n^2} + 1}$$

and, using $\|v\|^2_{L^2((0,1];r)} = \|x\|^2_{L^2((0,1];r)} + \|y\|^2_{L^2((0,1];r)} = 1$, we obtain

$$\Re\left(\mathcal{A}_1 \mathcal{B}^{-1/2} v, \mathcal{B}^{-1/2} v\right)_{L^2((0,1];r)} \geq -\alpha R \|U''\|_\infty \frac{1}{\sqrt{\tau_{\alpha,n}}} \frac{1}{2}\left(\frac{1}{\sqrt{\alpha^2 + n^2}} + \sqrt{\frac{1}{\alpha^2 + n^2} + \frac{1}{\alpha^2}}\right).$$

Taking the imaginary part in (95) and using (97), (98) again yields

$$\Im\left(\mathcal{A}_1 \mathcal{B}^{-1/2} v, \mathcal{B}^{-1/2} v\right)_{L^2((0,1];r)} = \alpha R \left(\begin{array}{c} \int_0^1 rU(r)\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{r}\frac{1}{k(r)^2 r}|f'(r)|^2\right) dr \\ +(Uk^2r^2 g, g)_{L^2((0,1];r)} \\ +\Re\left(-\left(\frac{\alpha}{n}\frac{U'}{r} f, g\right)_{L^2((0,1];r)} - \int_0^1 \frac{U'(r)}{k(r)^2 r} f(r)\overline{f'(r)}\, dr\right)\end{array}\right). \qquad (99)$$

In order to show the estimate from above, we use the inequality $U(r) \leq U_{\max}, r \in [0,1]$, to estimate the first two terms in (99) by

$$\int_0^1 rU(r)\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{k(r)^2 r^2}|f'(r)|^2\right) dr$$
$$\leq U_{\max} \int_0^1 r\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{k(r)^2 r^2}|f'(r)|^2\right) dr = U_{\max}(T_1 f, f)_{L^2((0,1];r)}$$
$$= U_{\max}\|x\|^2_{L^2((0,1];r)}$$

and

$$(Uk^2r^2 g, g)_{L^2((0,1];r)} \leq U_{\max}(k^2r^2 g, g)_{L^2((0,1];r)} = U_{\max}\|y\|^2_{L^2((0,1];r)}.$$

For the last two terms we proceed in the same way as for the term $\Re\left(\mathcal{A}_1\mathcal{B}^{-1/2}v, \mathcal{B}^{-1/2}v\right)_{L^2((0,1];r)}$ and we conclude that

$$\Im\left(\mathcal{A}_1\mathcal{B}^{-1/2}v, \mathcal{B}^{-1/2}v\right)_{L^2((0,1];r)} \leq \alpha R\left(U_{max} + \frac{\|U''\|_\infty}{2\sqrt{\tau_{\alpha,n}}}\left(\frac{1}{\sqrt{\alpha^2 + n^2}} + \sqrt{\frac{1}{\alpha^2 + n^2} + \frac{1}{\alpha^2}}\right)\right)$$

The proof of the estimate from below is analogous.

Using the above estimates we now obtain the following spectral inclusion.

Theorem(6.2.36)[31]: Suppose $n \neq 0$. Then the set of eigenvalues $\lambda = i\alpha Rc \in \sigma_P(\mathcal{L})$ of the Hagen– Poiseuille problem (45), (49), (50),(51)satisfies the inclusion

$$\sigma_P(\mathcal{L}) \subset \left\{\lambda \in \mathbb{C} : \Re(\lambda) \geq a_{\alpha,n} - \alpha R b_{\alpha,n}\|U''\|_\infty, \alpha R\left(U_{\min} - b_{\alpha,n}\|U''\|_\infty\right) \leq \Im(\lambda) \leq \alpha R\left(U_{\max} + b_{\alpha,n}\|U''\|_\infty\right)\right\}$$

with $a_{\alpha,n}, b_{\alpha,n}, U_{\min}, U_{\max}$ defined as in Lemmas (6.2.34) and (6.2.35).
The inclusion continues to hold with $a_{\alpha,n}, b_{\alpha,n}$ replaced by their lower and upper, respectively, bounds $\tilde{a}_{\alpha,n}, \tilde{b}_{\alpha,n}$.

Proof. The claim is immediate from Theorem (6.2.12) and Lemmas (6.2.34) and (6.2.35).

In case $n = 0$ the problem splits into the two uncoupled problems, (83) and (84). We first estimate the bounds needed for the application of Theorem (6.2.12) to the first spectral problem (83).

Lemma(6.2.37)[31]:(Bounds involving L_1). The operator $T_1^{-\frac{1}{2}} L_1 T_1^{-\frac{1}{2}}$, with T_1 , L_1 defined in above, is semi-bounded with

$$l_\alpha := \min \sigma(L_1) \geq \tilde{l}_\alpha := \alpha^2 + j_{1,1}^2 \frac{\alpha^2 + j_{0,1}^2}{\alpha^2 + j_{1,1}^2} \tag{100}$$

where $j_{0,1}$, $j_{1,1}$ are the smallest zeros of the Bessel functions J_0 and J_1, respectively.

Proof. Let $y \in \mathcal{D}(L_1) \subset \mathcal{D}(T_1)$ (see [111]). In above (see (66)) it was shown that

$$\begin{aligned}(L_1 y, y)_{L^2((0,1];r)} &= \int_0^1 \frac{\alpha^2}{r} |y(r)|^2 \, dr + 2\int_0^1 \frac{|y'(r)|^2}{r} dr \\ &+ \int_0^1 r\alpha^2 \left|\frac{d}{dr}\left(\frac{1}{\alpha^2 r}\frac{d}{dr}\right) y(r)\right|^2 dr = \alpha^2 (T_1 y, y)_{L^2((0,1];r)} + \int_0^1 \frac{|y'(r)|^2}{r} dr \\ &+ \int_0^1 r\alpha^2 \left|\frac{d}{dr}\left(\frac{1}{\alpha^2 r}\frac{d}{dr}\right) y(r)\right|^2 dr \end{aligned} \tag{101}$$

where we have used that

$$(T_1 y, y)_{L^2((0,1];r)} = \int_0^1 \frac{|y(r)|^2}{r} dr + \frac{1}{\alpha^2} \int_0^1 \frac{|y'(r)|^2}{r} dr. \tag{102}$$

In order to estimate the third integral in the last line of (101), we use that the smallest eigenvalue of the spectral problem (59) is $j_{0,1}^2$ (see [18]). Letting $u(r) = y'(r)/r$ in (59) and using this estimate in (101), we obtain

$$(L_1 y, y)_{L^2((0,1];r)} \geq \alpha^2 (T_1 y, y)_{L^2((0,1];r)} + \left(\alpha^2 + j_{0,1}^2\right) \frac{1}{\alpha^2} \int_0^1 \frac{|y'(r)|^2}{r} dr.$$

We split the last integral into two parts weighted by $(1 - \mu)$ and μ, for some $\mu \in [0,1]$. The first part is estimated by observing that the lowest eigenvalue of the problem

$$-\left(\frac{1}{r} u'\right)' = \lambda \frac{1}{r} u, \quad u(0) \text{ finite}, \quad u(1) = 0, \tag{103}$$

is $j_{1,1}^2$ (see [18]), and so with the additional help of (102) we obtain

$$\begin{aligned}\frac{1}{\alpha^2} \int_0^1 \frac{|y'(r)|^2}{r} dr &\geq (1 - \mu) \frac{j_{1,1}^2}{\alpha^2} \int_0^1 \frac{|y(r)|^2}{r} dr + \mu \frac{1}{\alpha^2} \int_0^1 \frac{|y'(r)|^2}{r} dr \\ &\geq \min\left\{(1 - \mu)\frac{j_{1,1}^2}{\alpha^2}, \mu\right\} (T_1 y, y)_{L^2((0,1];r)}\end{aligned}$$

for arbitrary $\mu \in [0,1]$. If we choose $\mu = j_{1,1}^2/(\alpha^2 + j_{1,1}^2)$, then the above minimum becomes maximal and we arrive at the estimate

$$(L_1 y, y)_{L^2((0,1];r)} \geq \left(\alpha^2 + \left(\alpha^2 + j_{0,1}^2\right) \frac{j_{1,1}^2}{\alpha^2 + j_{1,1}^2}\right) (T_1 y, y)_{L^2((0,1];r)} \tag{104}$$

from which the result is immediate.

Lemma(6.2.38)[31]: (Bounds involving A_1). The bounded (and densely defined) operator $B^{-1/2}A_1B^{-1/2}$ satisfies the estimates

(I) $\Re(B^{-1/2}A_1B^{-1/2}v,v) \geq -\alpha R b_{\alpha,0}\|U''\|_\infty$,

(II) $\alpha R\big(U_{\min} - b_{\alpha,0}\|U''\|_\infty\big) \leq \Im\big(B^{-1/2}\,A_1\,B^{-1/2}v,v\big)$
$$\leq \alpha R\big(U_{\max} + b_{\alpha,0}\|U''\|_\infty\big),$$

for $v \in B^{\frac{1}{2}}\mathcal{D}(A_1), \|v\| = 1$, where
$$U_{\min} := \min_{r\in[0,1]} U(r), \quad U_{\max} := \max_{r\in[0,1]} U(r),$$
$$b_{\alpha,0} := \frac{1}{\sqrt{\tau_{\alpha,0}}}\,\frac{1}{\alpha} \leq \frac{1}{\sqrt{\tilde{\tau}_{\alpha,0}}}\,\frac{1}{\alpha} \leq \tilde{b}_{\alpha,0},$$
with $\tau_{\alpha,0} \geq \tilde{\tau}_{\alpha,0}$ defined as in Lemma (6.2.18).

Proof. We proceed in a way similar to the proof of Lemma (6.2.35). Let $x \in B^{\frac{1}{2}}\mathcal{D}(A_1)$, $\|x\| = 1$, and set $f := B^{\frac{1}{2}}x \in \mathcal{D}(A_1) = \mathcal{D}(T_1)$. Then we have the estimates
$$\|f\|_{L^2((0,1];r)} = \left\|T_1^{\frac{-1}{2}}x\right\|_{L^2((0,1];r)} \leq \frac{1}{\sqrt{\tau_{\alpha,0}}}\|x\|_{L^2((0,1];r)}. \tag{105}$$
By the definition of A_1 in (86) and by integration by parts, we have
$$\big(A_1B^{-1/2}x, B^{-1/2}x\big)_{L^2((0,1];r)}$$
$$= (A_1f,f)_{L^2((0,1];r)} = i\alpha R\left(\left(UT_1 + \frac{1}{r}\left(\frac{U'}{\alpha^2 r}\right)'\right)f,f\right)_{L^2((0,1];r)}$$
$$= i\alpha R\left(\begin{array}{c}\displaystyle\int_0^1 rU(r)\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{r}\frac{1}{\alpha^2 r}|f'(r)|^2\right)dr \\ \displaystyle -\int_0^1 \frac{U'(r)}{\alpha^2 r}f(r)\overline{f'(r)}\,dr\end{array}\right). \tag{106}$$
The second integral within the brackets can be estimated by
$$\left|\int_0^1 r\,\frac{U'(r)}{\alpha^2 r^2}f(r)\overline{f'(r)}\,dr\right| \leq \|f\|_{L^2((0,1];r)}\left(\int_0^1 r\left|\frac{U'(r)}{\alpha^2r^2}f'(r)\right|^2 dr\right)^{\frac{1}{2}}$$
$$= \frac{1}{\alpha}\|f\|_{L^2((0,1];r)}\left(\int_0^1 r\left|\frac{U'(r)}{r}\right|^2\frac{|f'(r)|^2}{\alpha^2r^2}\,dr\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{\alpha}\|U''\|_\infty\|f\|_{L^2((0,1];r)}\left(\int_0^1\frac{|f'(r)|^2}{\alpha^2 r}\,dr\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{\alpha}\|U''\|_\infty\|f\|_{L^2((0,1];r)}(T_1f,f)^{\frac{1}{2}}_{L^2((0,1];r)} \leq \frac{1}{\alpha}\|U''\|_\infty\frac{1}{\sqrt{\tau_{\alpha,0}}}\|x\|^2_{L^2((0,1];r)}.$$
The first term within the brackets in (106) is real and can be estimated from above using the inequality $U(r) \leq U_{\max}, r \in [0,1]$:

$$\int_0^1 rU(r)\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{\alpha^2 r^2}|f'(r)|^2\right) dr$$
$$\leq U_{\max}\int_0^1 r\left(\frac{1}{r^2}|f(r)|^2 + \frac{1}{\alpha^2 r^2}|f'(r)|^2\right) dr = U_{\max}(T_1 f, f)_{L^2((0,1];r)}$$
$$= U_{\max}\|x\|^2_{L^2((0,1];r)};$$

the estimate from below is analogous. Taking real and imaginary parts in (106) yields the claimed bounds.

Theorem(6.2.39)[31]: Suppose $n = 0$. Then the set of eigenvalues $\lambda = i\alpha Rc \in \sigma_P(\mathcal{L})$ of the Hagen-Poiseuille problem (45), (49), (50), (51) satisfies the inclusion

$$\sigma_P(\mathcal{L}) \subset \{\lambda \in \mathbb{C}: \Re(\lambda) \geq \min\{l_\alpha - \alpha R b_{\alpha,0}\|U''\|_\infty, r_\alpha\}, \alpha R(U_{\min} - b_{\alpha,0}\|U''\|_\infty) \leq \Im(\lambda) \leq \alpha R(U_{\max} + b_{\alpha,0}\|U''\|_\infty)\}$$

where l_α, $b_{\alpha,0}$, $U_{\min}$ and $U_{\max}$ are defined as in Lemmas (6.2.37) and (6. 2.38) and r_α is defined in Lemma(6.2.21). The inclusion continues to hold with $b_{\alpha,0}$ replaced by its upper bound $\tilde{b}_{\alpha,0}$ and with

$$\min\{l_\alpha - \alpha R b_{\alpha,0}\|U''\|_\infty, r_\alpha\},$$

replaced by its lower bound

$$\min\{\tilde{l}_\alpha - \alpha R\tilde{b}_{\alpha,0}\|U''\|_\infty, \tilde{r}_\alpha\} = \tilde{l}_\alpha - \alpha R\tilde{b}_{\alpha,0}\|U''\|_\infty.$$

Proof. We have $\sigma_P(\mathcal{L}) = \sigma_P(\mathcal{L}_1) \cup \sigma_P(L_2)$ where the linear operator pencil $\mathcal{L}_1$ is given by $\mathcal{L}_1(\lambda) = L_1 + A_1 - \lambda T_1, \lambda \in \mathbb{C}$, (see (85),(86)) and L_2 is the linear operator $L_2 = R_1 + i\alpha RU$ with R_1 defined in Lemma (6.2.21).

For $\lambda \in \sigma_P(\mathcal{L}_1)$ we use Lemmas (6.2.37) and (6.2.38) which yield that

$$\Re(\lambda) \geq l_\alpha - \alpha R b_{\alpha,0}\|U''\|_\infty$$

and that $\Im(\lambda)$ satisfies the claimed estimate. For $\lambda \in \sigma_P(L_2)$, we use Lemma (6.2.21) to conclude that $\Re(\lambda) \geq r_\alpha$, while the bounds $\alpha R U_{\min} \leq \Im(\lambda) \leq \alpha R U_{\max}$ are immediate from (84). Hence both $\sigma_P(\mathcal{L}_1)$ and $\sigma_P(L_2)$ satisfy the claimed inclusion. For the last claim we note that $j_{0,1} < j_{1,1}$ for the smallest zeros of the Bessel functions J_0 and J_1, respectively, and hence $\tilde{l}_\alpha < \tilde{r}_\alpha$ by (134) and (63).

It is important to emphasize that the spectral bounds obtained in the preceding part, based on numerical range estimates, are rather coarse. However, it should also be stressed that numerical approximations of eigenvalues for non-selfadjoint spectral problems may not be reliable. A promising way to address this problem is to combine analytical estimates with validated numerics. This was done successfully in [26] for plane Poiseuille flow where a linearly unstable eigenvalue was proved to exist at the Reynolds number 5772.221818, thus providing the first guaranteed upper bound for the critical Reynolds number.

Besides numerical ranges, which we have discussed here, other approaches are possible, e.g. methods based on attempting to bound the resolvent of an appropriate operator. With the latter approach, for the particular case of a parabolic flow profile $U(r) = 1 - r^2$, Åsén and Kreiss [30] showed that all eigenvalues must satisfy $\Re(\lambda) > 0$ if either $16|\alpha R| \leq \max\{1, n^2\}$ *or* $|\alpha|^2 \leq R$.

Figs. 1 and 2 below compare the neutral (linear) stability regions for parabolic flow profiles to those of Åsén and Kreiss for the case $n = 0$ and for $n = 1$ since the latter is believed to be the least stable case. Here we have used Theorems (6.2.39) and (6.2.36), respectively, with the lower bounds established in this section. It turns out that the stability regions are quite different from theirs.

In the case $n = 0$, the stability region contains the part of the plane not yet covered by Åsén and Kreiss, thus rendering all pairs (α, R) linearly stable. In the case $n = 1$, the stability region improves that of Åsén and Kreiss for all $\alpha \gtrapprox 5.25$; the region in the right upper corner in Fig. 2 remains uncovered by either set of results.

However, Fig. 2 suggests that the results yield a critical Reynolds number $R_{n,crit}$ for $n \neq 0$ such that all pairs (α, R) with $R \leq R_{n,crit}$ are linearly stable, while the estimates of Åsén and

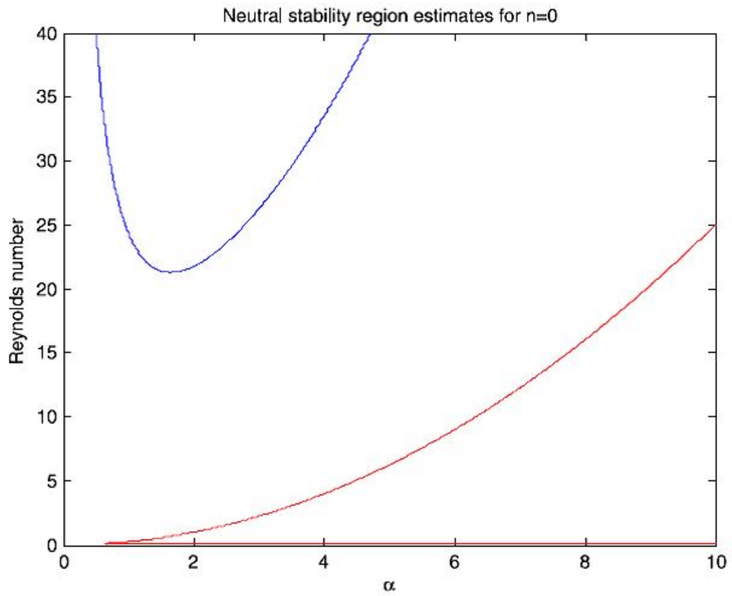

Fig. 1. $n = 0$: Area below the blue curve is stable by Theorem (6.2.39). Area to the left of the red curve is stable by [30]. Thus all pairs (α, R) are linearly stable.

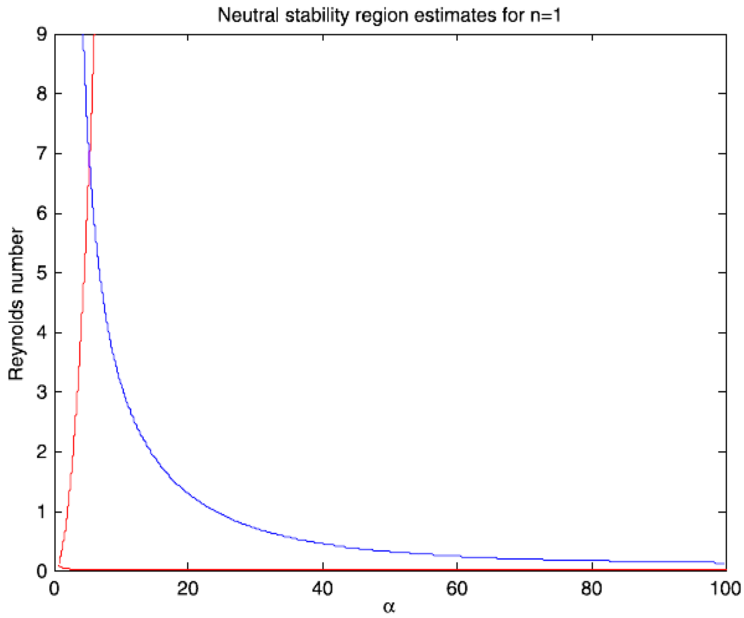

Fig. 2. $n = 1$: Area below the blue curve is stable by Theorem (6.2.36). Area to the left of the red curve is stable by [30]. The region in the top right-hand corner is not guaranteed stable by either set of estimates.

Kreiss require that $R \to 0$ when $\alpha \to \infty$. In fact, the estimates can be used to show that for $n \neq 0$,

$$R_{n,crit} \geq \lim_{\alpha\to\infty} \frac{\tilde{a}_{\alpha,n}}{\|U''\|_\infty \alpha \tilde{b}_{\alpha,n}} = \frac{1}{\|U''\|_\infty} \frac{5n^2 + 1 - \sqrt{(3n^2+1)^2 + 16}}{1+\sqrt{2}} > 0.$$

This limit is increasing in n which supports that $n = 1$ is the least (linearly) stable case; in particular, for $n = 1$ and parabolic U we obtain, in agreement with Fig. 2,

$$R_{n,crit} \geq \frac{1}{2} \frac{6-4\sqrt{2}}{1+\sqrt{2}} \sim 0.0711, \qquad n \neq 0.$$

Note that all the estimates apply not only to pipe Poiseuille flow, but to any other flow profile U that is twice differentiable with $U'(r) \to 0$ for $r \to 0$ and bounded U'', e.g. velocity profiles with inflection points.

List of symbols

symbol	Page
$W^{1,p}$: Sobolev space	1
L^2 : Hilbert space	1
dist : distant	1
Sup : Supremum	1
L^∞ : essential Lebesque space	6
min : minimum	9
S. E. I. D : strong energy image density	10
L^1 : Lebesque integral on the real line	11
H^1 : Sobolev space	12
L^p : Lebesque space	17
Loc : locally	17
det : determinant	18
max : maximum	21
Rad : radius	29
Diam : diameter	38
L^q : Lebesque dual space	41
a. e : almost everywhere	69
Ker : kernel	69
Im : imaginary	69
dim : dimension	69
codim : codimension	69
$\ominus$: direct difference	69
sign : signature	71
arg : argument	71
sub : the category of subanalytic subset	101
$\otimes$: tensor product	103
sem : semialgebraic	106
$\equiv_n$: equivalence relation modulo	106
int : interior	108
supp : support	125
inf : infimum	176
ess : essential	193

Re : real	194
$\oplus$: direct sum	207