# Chapter 5

# Quantitative and Spectral Stability Estimates

In this chapter we show that if $\Omega$ and $\Omega'$ are close enough for the complementary Hausdorff distance and their boundaries satisfy some geometrical and topological conditions then

$$|\lambda_1 - \lambda'_1| \leq C|\Omega\Delta\Omega'|^{\frac{\alpha}{N}}$$

where $\lambda_1$ $(resp.\ \lambda'_1)$ is the first Dirichlet eigenvalue of the Laplacian in $\Omega$ $(resp.\ \Omega')$ and $|\Omega\Delta\Omega'|$ is the Lebesgue measure of the symmetric difference. Here the constant $\alpha < 1$ could be taken arbitrary close to1(but strictly less) and $C$ is a constant depending on a lot of parameters including $\alpha$, dimension $N$ and some geometric properties of the domains. the argument also relies on suitable extension techniques and on an estimate on the decay of the eigenfunctions at the boundary which could be interpreted as a boundary regularity result.

## Sec(5.1): The First Dirichlet Eigenvalue in Reifenberg Flat Domains in $\mathbb{R}^N$

We show a stability result for the first Dirichlet eigenvalue of the Laplacian in some bounded open sets in $\mathbb{R}^N$. We estimate the difference

$$|\lambda_1 - \lambda'_1| \leq C|\Omega\Delta\Omega'|^{\frac{\alpha}{N}} \tag{1}$$

where $\lambda_1$(resp. $\lambda'_1$) is the first Dirichlet eigenvalue of the Laplacian in $\Omega$ (resp.$\Omega'$), and $|\Omega\Delta\Omega'|$ $:= |\Omega\backslash\Omega' \cup \Omega\backslash\Omega'|$ is Lebesgue measure of the symmetric difference between $\Omega$ and $\Omega'$. Stability results for the eigenvalues were studied a lot (see [155,152,95]) and have many applications, for instance in shape optimization problems (see [153,99]). On the other hand, as far as we know, estimates with a precise quantitative bound as (1) were only recently investigated [158, 156, 61], and always for regular domains, $C^{1,1}$ or at least Lipschitz domains. We would like to mention also [1] where a weaker inequality than (1) is proved for a very large class of domains including for instance bounded connected John sets with a "twisting external cone condition". The proof of Pang [1] uses a Brownian motion and is based on estimates on the Poisson kernel. In this section we present a simpler proof in the case of Reifenberg-flat domains.

We seek some geometrical conditions to impose on the domains in order to guarantee that (1) is true. What we obtain is that a "strong" −Reifenberg flat boundary is sufficient. In particular, domains with cracks are not permitted. Roughly, in terms of regularity, such domains have boundaries which are well approximated by hyperplanes at every scales (see Definition (5.1.1)). This is weak enough to permit Hölderian spirals or snowflake-like boundaries (in particular it is weaker than Lipschitz domains) but at the same time the geometry of a Reifenberg flat set is sufficiently under control in order to make some sharp estimates. See [76,81,16] for some earlier works on the analysis of operators in Reifenberg flat domains.

Notice that one could expect (1) to be true with $\alpha = 1$ for the case of Lipschitz domains. In this section, since we work with Reifenberg flat domains we only get (1) with $\alpha < 1$ which is optimal in the class of domains.

It is worth mentioning that (1) cannot be true without assuming any kind of regularity on the boundary of the domains. For example in $\mathbb{R}^2$, the domains $\Omega := B(0,1)\backslash\{x_1 = 0, x_2 \leq 0\}$ and $\Omega' := B(0,1)$ are such that $|\Omega\Delta\Omega'| = 0$ but clearly $\lambda_1 \neq \lambda'_1$. Inequality (1) can either not be true without adding some topological assumptions. Indeed, the Lipschitz domains $\Omega := B(0,1)\backslash$ $\{x_1 = 0\}$ and $\Omega' := B(0,1)$ are again such that $|\Omega\,\Delta\,\Omega'| = 0$ but $\lambda_1 \neq \lambda'_1$.

We denote $d_H$ the Hausdorff distance, namely for two compact sets $A$ and $B$

$$d_H(A,B) := \sup_{x\in A} \operatorname{dist}(x,B) + \sup_{y\in B} \operatorname{dist}(y,A).$$

We consider the case of Reifenberg flat Domains which are defined as follows.

**Definition(5.1.1)[80]:** An $(\varepsilon, r_0)$-Reifenberg-flat domain $\Omega \subset \mathbb{R}^N$ is an open and bounded set such that for each $x \in \partial\Omega$ and for any $r \le r_0$, $\Omega \cap B(x,r)$is connected and there exists a hyper plane $P(x,r)$ containing $x$ which satisfies

$$\frac{1}{r} d_H\left(\partial\Omega \cap B(x,r), P(x,r) \cap B(x,r)\right) \le \varepsilon. \qquad (2)$$

Let us mention the remarkable theorem of Reifenberg ([64], see also [60],[68]) known as the "topological disk theorem", which in the setting says the following.

**Theorem(5.1.2)[80]:** There exists a constant $\varepsilon_0$ depending only on the dimension, such that for any $(\varepsilon, r_0)$-Reifenberg-flat domain $\Omega \subset \mathbb{R}^N$ with $\varepsilon < \varepsilon_0$ and for any point $x \in \partial\Omega$, one has that $\partial\Omega \cap B(x, r_0/32)$ is a topological disk.

We devote to some preliminary results, especially a covering lemma and a geometrical fact saying that $d_H(\Omega'^c, \Omega^c)$ and $|\Omega\Delta\Omega'|^{\frac{1}{N}}$ are equivalent for two Reifenberg flat domains. Next we show some boundary estimates for both eigenfunctions and their gradients near the boundary of a Reifenberg flat domain. The more difficult part is to control the gradient. We first show a decay result on balls centered at the boundary and then use the covering lemma to estimate the gradient in a region close to the boundary. We show an extension result for functions in $H_0^1(\Omega)$. This extension lemma is a powerful tool which is used to compare two Dirichlet eigenvalues. We remark that the extension lemma implies a $\gamma$-convergence result from which we automatically obtain the stability for Dirichlet eigenvalues.

Finally we provide the Theorem (5.1.17) and it is proof using the Min-Max principle.

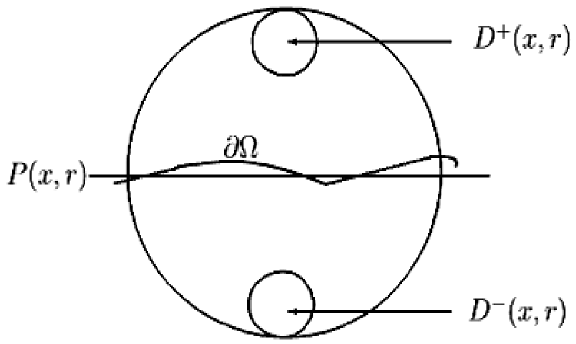


**Fig . 1.**

We start by giving some useful and classical facts about the Dirichlet eigenvalue problem for the Laplacian, that can be found for instance in [27].

**Proposition(5.1.3)[80]:** Let $\Omega$ be a domain in $\mathbb{R}^N$. Then $-\Delta$ has a countably infinite discrete set of eigenvalues, whose eigenfunctions span $H_0^1(\Omega)$. Moreover each eigenfunction $v$ belongs to $L^\infty(\Omega)$ and we have

$$\|v\|_\infty \le C\,(n, |\Omega|)\,\|v\|_2 .$$

For a Reifenberg-flat domain $\Omega$ and for any ball $B(x,r)$ centered at $\partial\Omega$ and radius $r \le r_0$, let us define the sets $D^\pm(x,r)$ in the following way. Let $P(x,r)$ be the hyperplane given by the definition of Reifenberg flatness of $\Omega$. Denote by $z^\pm(x,r)$ the two points that lie at distance $3r/4$ from $P(x,r)$ and whose orthogonal projection on $P(x,r)$ is equal to $x$. Then we set

$$D^{\pm}(x,r) := B(z^{\pm}(x,r), r/4) \tag{3}$$

as in Fig. 1.

We have the following useful fact regarding the sets $D^{\pm}$.

**Lemma(5.1.4)[80]:**Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain. Then for all $x \in \partial\Omega$ and $r < \frac{r_0}{2}$, the balls $D^+(x,r)$ and $D^-(x,r)$ lie in different connected components of $B(x,r)\backslash\partial\Omega$.

**Proof .** This can be seen as a consequence of the topological disk Theorem of Reifenberg [64]. Actually one could also prove it directly without using the whole result of Reifenberg but just the very beginning of Reifenberg's construction.

In the situation we find it convenient to simply apply the Theorem. More precisely, we use the statement of Theorem 1.1. in [68] (which holds for $N \neq 3$ for the case of hyperplanes) that gives for every $r < r_0$and $x \in \partial\Omega$ a hyperplane $P$ through $x$ and a continuous homeomorphism $f : B(x, \frac{3}{2}\, r) \to f(B(x, \frac{3}{2}r)) \subset B(x, 2r)$ such that

$$B(x,r) \subset f\left(B\left(x, \frac{3}{2}r\right)\right) \subset B(x, 2r), \tag{4}$$

$$\partial\Omega \cap B(x,r) \subset f\left(P \cap B\left(x, \frac{3}{2}r\right)\right) \subset \partial\Omega \cap B(x, 2r). \tag{5}$$

Now if we denote by $\nu$ any normal vector to $P$ and consider

$$P^+ := \{x \in \mathbb{R}^N\colon\ x\,.\nu > 0\}\,,\quad P^- := \{x \in \mathbb{R}^N\colon\ x\,.\nu < 0\}\,,$$

it is clear from(4) and (5) that $\partial\Omega$ separates the domains $f\left(P^{\pm} \cap B\left(x, \frac{3}{2}r\right)\right)$ and in particular the sets $D^{\pm}(x,r)$.

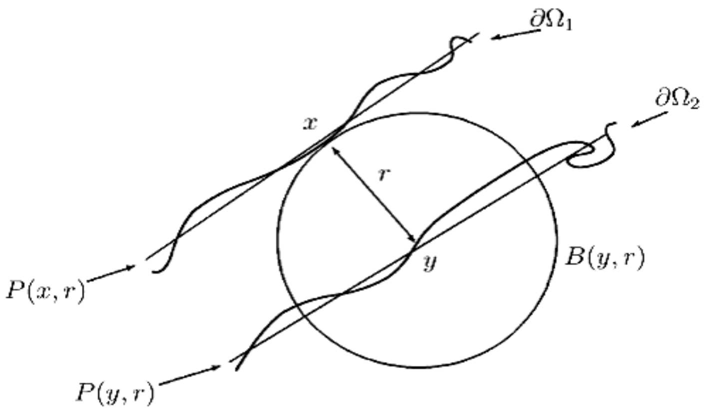


**Fig . 2.**

We have the following lemma.

**Lemma(5.1.5)[80]:** Let $\Omega_1$ and $\Omega_2$ be two $(\varepsilon, r_0)$-Reifenberg flat domains such that $d_H(\Omega_1^c, \Omega_2^c) \leq r_0/3$ . Then

$$d_H(\Omega_1^c, \Omega_2^c) \leq C|\Omega_1 \Delta\, \Omega_2|^{\frac{1}{N}}$$

where $C$ depends only on $N$.

**Proof .** Let $x \in \partial\Omega_1$ , be such that $r := \mathrm{dist}(x, \partial\Omega_2)$ is maximum, and let $y \in \partial\Omega_2$ be such that $\mathrm{dist}(x, \partial\Omega_2) = d(x,y) = r$. Let us set $D_1^{\pm} = D^{\pm}(x,r)$ and $D_2^{\pm} = D^{\pm}(y,r)$ as being the balls defined in (3). Under the assumptions we know that only one of $D_1^{\pm}$ lies in $\Omega_1$ and only one of $D_2^{\pm}$ lies in $\Omega_2$. Let us simply denote by $D_i$ those two balls.

Now by the definition of $y$, we know that $B(y,r) \cap \partial\Omega_1$ is empty. In particular, the two "approximating" hyperplanes $P(x,r)$ and $P(y,r)$ are almost parallel (with error less than $2.\varepsilon \leq 2.10^{-2}$ ) as in Fig. 2.

Then it is not difficult to show, considering also a similar situation in $B(x,3r)$ and $B(y,3r)$ with the corresponding selection of domains $D_i(3r) \in \{D^{\pm}(x,3r), D^{\pm}(y,3r)\}$, that whatever the positions of the $D_i$ and $D_i(3r)$ with respect to the lines $P(x,3r)$ and $P(y,3r)$ are, one can always find a ball of radius equivalent to $r$ that lies in the symmetric difference of $\Omega_1$ and $\Omega_2$. We conclude the proof by exchanging the role of $\Omega_1$ and $\Omega_2$ and using the same argument.

We provide the following elementary covering lemma.

**Lemma(5.1.6)[80]:** Let $\Omega \subset \mathbb{R}^N$ be an $(\varepsilon, r_0)$-Reifenberg flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega) = L < +\infty$. Then for every $r < r_0/2$ we can extract among $\{B(x,r)\}_{x\in\partial\Omega}$ a subfamily of at most $L/(C_N r^{N-1})$ balls that forms a covering of $\cup_{x\in\partial\Omega} B(x, \frac{8}{10} r)$ where $C_N$ is a dimensional constant. Moreover, for all $x$ we have that

$$\#\{i : x \in B_i\} \leq C \tag{6}$$

where $C$ is again a dimensional constant.

**Proof .** Since $r < r_0$, we have that

$$d_H(\partial\Omega \cap B(x,r), P(x,r) \cap B(x,r)) \leq 10^{-2} r . \tag{7}$$

We also know that $\partial\Omega$ separates $D^+(x,r)$ from $D^-(x,r)$ and since the set of minimal $\mathcal{H}^{N-1}$ area having this property and satisfying (7) is the corresponding part of a hyperplane, we deduce that there exists a dimensional constant $C_N$ such that for all $x \in \partial\Omega$ and all $r \leq r_0$

$$\mathcal{H}^{N-1}(\partial\Omega \cap B(x,r)) \geq C_N r^{N-1}.$$

Now let $B(x_i, r_i)$, be a subfamily of $\{B(x,r)\}_{x\in\partial\Omega}$ indexed by $i \in I$, maximal for the property that $\frac{1}{10} B_i \cap \frac{1}{10} B_j = \emptyset$. Using this fact (6) comes from a classical geometric argument in $\mathbb{R}^N$. Now we claim that $\#I$ is finite. Indeed, since $\frac{1}{10} B_i$ are disjoint balls we have

$$L \geq \mathcal{H}^{N-1}\left(\bigcup_{i\in I} \partial\Omega \cap \frac{1}{10} B_i\right) \geq \#I C_N r^{N-1} 10^{1-N}$$

thus

$$\#I \leq \frac{10^{N-1} L}{C_N r^{N-1}} . \tag{8}$$

Finally, it remains to prove that the family $\{B_i\}_{i\in I}$ forms a covering of $\cup_{x\in\partial\Omega} B(x, \frac{8}{10} r)$. Let $y \in \cup_{x\in\partial\Omega} B(x, \frac{8}{10} r)$ and let $x \in \partial\Omega$ be such that $y \in B(x, \frac{8}{10} r)$. Then by the maximality of the $\{B_i\}$, there exist an index $i$ and a point $z \in \frac{1}{10} B_i \cap B(x, r/10)$. Then if $x_i$ denotes the center of $B_i$ , we have

$$d(y,x_i) \leq d(y,x) + d(x,z) + d(z,x_i) \leq \frac{8}{10} r + \frac{2}{10} r = r$$

which proves that $y \in B(x_i, r)$.

We will need the following boundary version of the classical Sobolev inequality when $\Omega$ is a Reifenberg flat domain.

**Proposition(5.1.7)[80]:** Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$ and $u \in W_0^{1,p}(\Omega)$ for some $p \geq 1$. Then for all $x \in \partial\Omega$ and $r \leq r_0$ we have

$$\|u\|_{L^p(B(x,r)\cap\Omega)} \leq Cr\|\nabla u\|_{L^p(B(x,br)\cap\Omega)}$$

where $C := C(p, N)$ and $b := b(N)$.

**Proof .** The proof is a small modification of the classical proof of the Sobolev inequality that we will write here with full details for the convenience of the reader.

Without loss of generality, we may assume that $u \in C_0^1(\Omega)$, $x$ is the origin and that $P(x,r)$ is the hyperplane $\{x_1 = 0\}$. We shall show that

$$\|u\|_{L^p(\Omega\cap Q(x,r))} \leq Cr\|\nabla u\|_{L^p(\Omega\cap Q(x,r))}. \tag{9}$$

where $Q(x,r)$ is a cube centered at $x$, and with faces orthogonal to the axis of $\mathbb{R}^N$. Observe that (9) implies the desired inequality with constant $b$ coming from the comparison between cubes and euclidian balls in $\mathbb{R}^N$.

By changing the orientation of $x_1$ we can assume that $Q(x,r) \cap \Omega$ (which is connected by the assumptions) contains the upper part $Q(x,r) \cap \{x_1 > r/2\}$. It is clear that for any $u \in C_0^1(\Omega)$,

$$|u(x_1, x')| \leq \int_{-\infty}^{x_1} |D_1 u(t, x')|\, dt \leq \int_{-\infty}^{r} |D_1 u|(t, x')\, dt .$$

Integrating over $x_1$ we obtain

$$\int_{-\infty}^{r} |u(x_1, x')|\, dx_1 \leq 2r \int_{-\infty}^{r} |D_1 u(t, x')|\, dt .$$

Now integrating the last inequality between $-r$ and $r$ successively over each variable $x_2, \ldots, x_N$ we get

$$\int_{Q(x,r)\cap\Omega} |u(x)|\, dx \leq 2r \int_{Q(x,r)\cap\Omega} |D_1 u(x)|\, dx \leq Cr \int_{Q(x,r)\cap\Omega} \|Du\|(x)\, dx.$$

Then (9) follows if we apply this last inequality to $u^p$ and use the Hölder's inequality.

**Corollary(5.1.8)[80]:** Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$ and for $\delta \leq r_0/2$ set

$$A_\delta := \Omega \cap \{d(x, \partial\Omega) \leq \delta\}.$$

Then for any function $u \in W_0^{1,p}(\Omega)$ we have

$$\left(\int_{A_\delta} |u|^p\, dx\right)^{\frac{1}{p}} \leq C\delta \left(\int_{A_{2b\delta}} |\nabla u|^p\, dx\right)^{\frac{1}{p}}$$

where $b$ is the dimensional constant of Proposition (5.1.7).

**Proof .** Let $\{B_i\}_{i\in I}$ be the subfamily of balls $\{B(x, 2\delta)\}_{x\in\partial\Omega_1}$ given by Lemma (5.1. 6). Then

$$A_\delta := \Omega \cap \{x: d(x, \partial\Omega) \leq \delta\} \subset \bigcup_{x\in\partial\Omega} B\left(x, \frac{16}{10}\delta\right) \subset \bigcup_{i\in I} B_i .$$

Moreover the covering is bounded by a dimensional constant $C$. Then,

$$\int_{A_\delta} |u|^p\, dx \leq \sum_{i\in I} \int_{B_i\cap\Omega} |u|^p\, dx \quad .$$

and using Proposition (5.1.7), together with the fact that the $B_i$ are centered at $\partial\Omega$, we obtain

$$\int_{A_\delta} |u|^p \, dx \le C \sum_{i\in I} \delta^p \int_{bB_i\cap\Omega} |\nabla u|^p \, dx \le C\delta^p \int_{A_{2b\delta}} |\nabla u|^p \, dx$$

which proves the corollary.

**Proposition(5.1.9)[80]:**Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^{\boldsymbol{N}}$, and let $u$ be an eigenfunction for the Dirichlet Laplacian in $\Omega$, associated to the eigenvalue $\lambda$. Then for every $\beta > 0$ there is a constant $C_0$ depending on $N$, $|\Omega|$ and $\beta$ such that for every $x \in \partial\Omega$ and for all $r \le r_0$, we have that

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2 \, dx \le C_0 \lambda \|u\|^2_{L^2(\Omega)} \left(\frac{r}{r_0}\right)^{N-\beta}. \tag{10}$$

**Proof .** For a given $\beta > 0$, define

$$a := 2^{\frac{2}{\beta}}. \tag{11}$$

Without loss of generality we assume that $r_0 = 1$ and $\|u\|_2 = 1$. Now let $x \in \partial\Omega$. We will obtain the appropriate decay by showing that for $k \in \mathbb{N}$ and a specific selection of the constant $C_1$ we have

$$\int_{B(x,a^{-k})\cap\Omega} |\nabla u|^2 \, dx \le C_1 \lambda a^{-k(N-\beta)}. \tag{12}$$

We will prove (12) inductively. It is clear that(12) is true for $k = 0$ if $C_1 \ge 1$.

Suppose now that (12) is true for $k$ and denote by $v$ the "harmonic" replacement of $u$ in $S_k := B(x, a^{-k}) \cap \Omega$; that is a harmonic function $v \in H^1(S_k)$ which satisfies $u - v \in H^1_0(S_k)$. Such a function $v$ can be obtained by minimizing the Dirichlet integral, and since $u$ is a competitor we have that

$$\int_{S_k} |\nabla v|^2 \, dx \le \int_{S_k} |\nabla u|^2 \, dx \le C_1 \lambda \, a^{-k(N-\beta)}.$$

by the inductive hypothesis. On the other hand $w := |\nabla v|^2$ is subharmonic. Therefore by the classical mean value inequality applied to $w$ we have

$$\int_{S_{k+1}} |\nabla v|^2 \, dx \le a^{-N} \int_{S_k} |\nabla v|^2 \, dx,$$

that is

$$\int_{S_{k+1}} |\nabla v|^2 \, dx \le C_1 \lambda a^{-N} a^{-k(N-\beta)}. \tag{13}$$

Now we want to estimate $\int_{S_k} |\nabla(u - v)|^2 \, dx$. Notice that for all $k, u$ is the unique solution of the problem

$$\begin{cases} -\nabla w = \lambda w & \text{in } S_k, \\ w - u \in H^1_0 \, (S_k) \end{cases}$$

therefore $u$ is minimizing the energy

$$1/2 \int_{S_k} |\nabla w|^2 \, dx - \int_{S_k} \lambda u w \, dx .$$

among all functions $w$ such that $u - w \in H_0^1(S_k)$. Therefore we deduce that

$$1/2 \int_{S_k} |\nabla u|^2 - \lambda \int_{S_k} |u|^2 \leq 1/2 \int_{S_k} |\nabla v|^2 - \lambda \int_{S_k} uv .$$

Hence

$$\int_{S_k} |\nabla u|^2 - \int_{S_k} |\nabla v|^2 \leq 2\lambda \left( \int_{S_k} |u|^2 - \int_{S_k} uv \right) \leq C\lambda |S_k| \|u\|_\infty^2$$
$$\leq \lambda C(N, |\Omega|) a^{-kN} \leq \lambda C_2 \, a^{-kN}. \qquad (14)$$

where $|v|$ was estimated in terms of $\|u\|_\infty$by the maximum principle and $\|u\|_\infty \leq C(N, |\Omega|)$ by Proposition (5.1.3).

Now since $v$ is harmonic in $S_k$ and $u - v \in H_0^1(S_k)$, we deduce that $\nabla v$ and $\nabla(u - v)$ are orthogonal in $L^2(S_k)$ thus (14) implies

$$\int_{S_k} |\nabla u - \nabla v|^2 = \int_{S_k} |\nabla u|^2 - \int_{S_k} |\nabla v|^2 \leq \lambda C_2 a^{-kN}. \qquad (15)$$

Gathering (13) and (15) together we complete the induction as follows:

$$\int_{S_{k+1}} |\nabla u|^2 \, dx \leq 2 \int_{S_{k+1}} |\nabla v|^2 + 2 \int_{S_{k+1}} |\nabla(u - v)|^2 \, dx,$$
$$\leq 2C_1 \lambda a^{-N} \, a^{-k(N-\beta)} + 2\lambda C_2 a^{-kN} \leq C_1 \lambda a^{-(k+1)(N-\beta)}.$$

where the last inequality holds by the definition of $a$ and provided for instance that

$$C_1 \geq C_2 \, 2^{\frac{2(N+1)}{\beta}} . \qquad (16)$$

Now to finish the proof, since (12) is true, for every $r < 1$ one can find an integer $k$ such that $r \leq a^{-k} < ar$, thus

$$\int_{B(x,r)} |\nabla u|^2 \, dx \leq \int_{B(x,a^{-k})} |\nabla u|^2 \, dx \leq C_1 \lambda a^{-k(N-\beta)} \leq C_1 \, \lambda (ar)^{N-\beta}$$

so the proposition is true with $C_0 := a^{N-\beta} \; C_1$ .

A consequence of the above proposition is the following.

**Corollary(5.1.10)[80]:** Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^{\boldsymbol{N}}$ such that $0 < \mathcal{H}^{N-1}(\partial\Omega) = L < +\infty$. Then for any $\alpha < 1$ there is a constant $C_1 := C_1(|\Omega|, r_0, N, \alpha)$ such that for any eigenfunction $u$ for the Dirichlet Laplacian associated to the eigenvalue $\lambda$ in $\Omega$ and for any $\delta \leq r_0/2$ we have

$$\int_{\Omega \cap \{d(x,\partial\Omega_1) \leq \delta\}} |\nabla u|^2 \; dx \leq C_1 \lambda L \|u\|_{L^2(\Omega)}^2 \delta^\alpha.$$

**Proof .** We argue as in Corollary (5.1.8). Let $\{B_i\}_{i\in I}$ be the subfamily of balls $\{B(x, \; 2\delta)\}_{x\in\partial\Omega}$ given by Lemma (5.1.6). By (8) we know that

$$\#I \leq L / \, (2^{N-1} \; C_N \delta^{N-1})$$

and that

$$\Omega \cap \{x : \; d(x, \partial\Omega) \leq \delta\} \subset \bigcup_{x\in\partial\Omega} B(x, 16/10 \, \delta) \subset \bigcup_{i\in I} B_i \; .$$

Moreover the covering is bounded by a dimensional constant $C$. Then,

$$\int_{\Omega\cap\{d(x,\partial\Omega)\le\delta\}} |\nabla u|^2 \, dx \le \sum_{i\in I} \int_{B_i\cap\Omega} |\nabla u|^2 \, dx \; .$$

and using Proposition (5.1.9) (applied with $r = \delta$ and $\beta = 1-\alpha$) together with the fact that the $B_i$ are centered at $\partial\Omega$ we obtain

$$\int_{\Omega\cap\{d(x,\partial\Omega)\le\delta\}} |\nabla u|^2 \, dx \le C\lambda\|u\|^2_{L^2(\Omega)} \sum_{i\in I} \delta^{N-1+\alpha}$$

$$\le C\lambda\|u\|^2_{L^2(\Omega)} \# I \delta^{N-1+\alpha} \le C\, L\lambda\|u\|^2_{L^2(\Omega)}\, \delta^{\alpha}.$$

where $C = C(r_0, N, \alpha, |\Omega|)$ and the proof is complete.

We need the following extension lemma for Sobolev functions in Reifenberg flat domains. The proof relies on a Whitney extension which is now well established. A first result of this kind (but however slightly different) is probably due to P. Jones in [88] which has been used by several authors, particularly by scientists working on quasi-conformal maps (see for instance [157]). However, in the case the extension will be much more simpler than the original one of P. Jones since we allow ourselves to modify the function inside the domain in a small neighborhood of the boundary. We would like to mention that [81] contains a lemma similar to the following one but for Neumann extensions and for domains with cracks. One can also find again the same sort of extension lemma used together with a stopping time argument to show some thin convergence results in [74].

**Lemma(5.1.11)[80]:**Let $\Omega_1$ and $\Omega_2$ be two $(\varepsilon, r_0)$-Reifenberg flat domains such that

$$d_H(\Omega_1^c, \Omega_2^c) \le \delta \le (100b)^{-1} r_0,$$

where $b$ is the dimensional constant of Proposition (5.1.7) and set

$$A_\delta := \{x : d(x, \partial\Omega_1) \le \delta\} .$$

Then for any $v \in W_0^{1,p}(\Omega_1)$ there exists a function $\tilde{v} \in W_0^{1,p}(\Omega_2)$ such that $v = \tilde{v}$ in $\Omega_1 \setminus A_{2\delta}$ and

$$\|\tilde{v}\|_{L^p(\Omega_2)} \le \|v\|_{L^p(\Omega_1)}, \qquad (17)$$

$$\|\nabla\tilde{v}\|^p_{L^p(\Omega_2)} \le \|\nabla v\|^p_{L^p(\Omega_1)} + C\|\nabla v\|^p_{L^p(\Omega_1\cap A_{4b\delta})}. \qquad (18)$$

**Proof .** Let $\{B_i\}_{i\in I}$ be the subfamily of balls $\{B(x, 2\delta)\}_{x\in\partial\Omega_1}$ given by Lemma (5.1.6). We will denote by $x_i$ the center of $B_i$ and $r_i$ its radius. Since $\Omega_1 \Delta \Omega_2 \subset \bigcup_{i\in I} B_i$, to define a function $\tilde{v} \in W^{1,2}(\Omega_2)$, it is sufficient to define an extension of $v$ in $\Omega_2 \cap \bigcup_{i\in I} B_i$.

For all $i$, define a function $\varphi_i \in C_c^1(5B_i)$, such that $\varphi_i = 1$ in $2B_i$, $|\nabla\varphi| \le \delta^{-1}$ and let $\varphi_0$ be a function that is equal to 1 in $\Omega_1 \setminus \bigcup_{i\in I} 4B_i$, $\varphi_0 = 0$ in $\bigcup_{i\in I} 2B_i$ and $\varphi_0 + \sum_{j\in J}\varphi_j \ge 1$ in $\Omega_1 \setminus \bigcup_{j\in J} 5B_i$ .Moreover, we can assume that for all $x \in 4B_i \setminus 2B_i$, $|\nabla\varphi_0(x)| \le \delta^{-1}$. Indeed, such a function $\varphi_0$ can be obtained by setting

$$\varphi_0(x) := \prod_{i\in I} l\left(\frac{d(x, x_i)}{\delta}\right).$$

where $l$ is a Lipschitz function equal to 0 in $[0,2]$, equal to 1 in $[4, +\infty)$ and $l'(x) \le 1$. Finally, define

$$\theta_i := \frac{\varphi_i}{(\varphi_0 + \sum_{i\in I}\varphi_i)} \quad \text{for } i \in I \cup \{0\} .$$

This allows us to obtain a partition of the unity in $\Omega_1 \cup \bigcup_{i\in I} 5B_i$.

Next we simply define $\tilde{v}$ by

$$\tilde{v}(x) := \theta_0(x)v(x). \tag{19}$$

in such a way that $\tilde{v}(x)$ vanishes on $\bigcup_{i\in I} 2B_i \supset \partial\Omega_2$. We claim that $\tilde{v} \in W_0^{1,p}(\Omega_2)$ and that (17), (18) are satisfied. The first estimate (17) comes directly from the fact that $\theta_0(x) \le \chi_{\Omega_1}$. So we only have to prove (18), which will also imply that $\tilde{v} \in W^{1,p}(\Omega_2)$.

We have that

$$\nabla\tilde{v}(x) = v(x)\nabla\theta_0(x) = \theta_0(x)\nabla v(x).$$

thus

$$\begin{aligned}\|\nabla\tilde{v}(x)\|_{L^p(\Omega_2)} &\le \left\|\nabla v(x)\chi_{supp(\theta_0)}\right\|_{L^p(\Omega_2)} + \|v(x)\nabla\theta_0(x)\|_{L^p(\Omega_2)} \\ &\le \|\nabla v(x)\|_{L^p(\Omega_1)} + \|v(x)\nabla\theta_0(x)\|_{L^p(\Omega_2)}.\end{aligned}$$

therefore it is enough to prove that

$$\|v(x)\nabla\theta_0(x)\|_{L^p(\Omega_2)} \le C\|\nabla v(x)\|_{L^p(A)} \; . \tag{20}$$

with

$$A := \Omega_1 \cap \bigcup_{i\in I} 4bB_i \subset A_{4b\delta}.$$

On the other hand, from the construction of $\theta_0$ we have

$$|\nabla\theta_0\,(x)| \le \sum_{i\in I} \chi_{4B_i}(x)\delta^{-1}. \tag{21}$$

Therefore, since the sum in (21) is locally finite we conclude that

$$\begin{aligned}\|v(x)\nabla\theta_0(x)\|^p_{L^p(\Omega_2)} &\le \int_{\Omega_2} \left| v(x)\sum_{j\in J}\chi_{4B_i}(x)\delta^{-1}\right|^p \\ &\le C\sum_{i\in I}\delta^{-p}\int_{4B_i}|v(x)|^p.\end{aligned} \tag{22}$$

Now since $B_i$ is centered on $\partial\Omega_1$ , from Proposition (5.1.7) we get

$$\int_{4B_i}|v|^p \;\; dx \le C\delta^{\,p}\int_{4bB_i}|\nabla u|^p \;\; dx$$

(the definition of $v$ outside $\Omega_1$ is considered being 0). Then,

$$\|v(x)\nabla\theta_0(x)\|^p_{L^p(\Omega_2)} \le C\sum_{i\in I}\int_{4bB_i}|\nabla v(x)|^p \le C\int_{A_{4b\delta}\cap\Omega_1}|\nabla v(x)|^p \quad .$$

which concludes the proof.

As in [81], the extension Lemma will imply the Mosco-convergence of $H_0^1\,(\Omega_n)$ to $H_0^1\,(\Omega)$ while $\Omega_n$ tends to $\Omega$ for the complementary Hausdorff distance. It is well known that this notion is equivalent to the $\gamma$-convergence of $\Omega_n$ to $\Omega$ which will in particular imply a stability result for eigenvalues.

For $u \in H_0^1(\Omega)$ we will identify $u$ as a function in $H^1(\mathbb{R}^N)$ by extending them being zero outside $\Omega$.

**Definition(5.1.12)[80]:** (Mosco-convergence). Let $\Omega_n$ and $\Omega$ be open subsets of $\mathbb{R}^N$. We say that $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco if the following two properties hold:
**(M1)** for every $u \in H_0^1(\Omega)$, there exists a sequence $u_n \in H_0^1(\Omega_n)$ such that $u_n$ converges to $u$ strongly in $H^1(\mathbb{R}^N)$;
**(M2)** if $h_k$ is a sequence of indices converging to $+\infty$, $u_k$ is a sequence such that $u_k \in H_0^1(\Omega_{h_k})$ for every $k$, and $u_k$ converges weakly in $H^1(\mathbb{R}^N)$ to a function $\emptyset$, then $\emptyset \in H_0^1(\Omega)$.

The Mosco convergence is a great tool to study stability for elliptic problems. Indeed, for any bounded open set $\Omega \subset \mathbb{R}^N$ and any $f \in H^{-1}(\Omega)$ let us denote by $u_\Omega^f \in H_0^1(\Omega)$ the unique solution of the equation $-\Delta u = f$ in $\Omega$.

**Definition(5.1.13)[80]:** Let $D \subset \mathbb{R}^N$ be bounded. We say that the sequence of open sets $\Omega_n \subset D$ $\gamma$-converges to $\Omega \subset D$ if for any $f \in H^{-1}(D)$ we have that $u_{\Omega_n}^f$ strongly converges to $u_\Omega^f$ in $H_0^1(D)$.

The following classical result shows the link between Mosco convergence and $\gamma$-convergence (see [99]).

**Proposition(5.1.14)[80]:** $\Omega_n$ $\gamma$-converges to $\Omega$ if andonly if $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.

**Theorem(5.1.15)[80]:**Let $r_0, \varepsilon > 0$ and let $\{\Omega_n\}_{n\in\mathbb{N}}$ and $\Omega$ be some $(\varepsilon, r_0)$-Reifenberg flat domains. Assume that $\Omega_n$ converges to $\Omega$ for the complementary Hausdorff distance. Then $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.

**Proof .** The proof is the same as for Theorem 11 in [81], using this time the extension lemma for $H_0^1$ (Lemma (5.1.11)).

A useful consequence of $\gamma$-convergence is the stability of eigenvalues, which is again very standard (see [153,99]).

**Proposition(5.1.16)[80]:** Let $r_0$, $\varepsilon > 0$ and let $\{\Omega_n\}_{n\in\mathbb{N}}$ and $\Omega$ be $(\varepsilon, r_0)$ $-$Reifenberg flat. Assume that $\Omega_n$ converges to $\Omega$ for the complementary Hausdorff distance. Then the $k$-th eigenvalue in $\Omega_n$ converges to the $k$-th eigenvalue in $\Omega$.

We are now ready to show Theorem (5.1.17). Notice that the theorem contains in particular a second proof of Proposition (5.1.16) for the case of the first eigenvalue.

Next we present the main result concerning the Dirichlet eigenvalues.

**Theorem(5.1.17)[80]:** Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$ such that
$$0 < \mathcal{H}^{N-1}(\partial\Omega) = L < +\infty.$$
Let $B$ be a ball such that $10B$ is contained in $\Omega$ and let $\gamma_1$ be the first eigenvalue of $B$. Then for every $\alpha < 1$ and for every $M > L$ there is a constant $C$ depending on $\alpha, N, |\Omega|, \gamma_1, r_0$ and $M$ such that the following holds. Let $\Omega'$ be an $(\varepsilon, r_0)$-Reifenberg flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega') \leq M$ and let $\lambda_1$(resp.$\lambda_1'$) be the first eigenvalue for the Dirichlet Laplacian in $\Omega$ (resp. $\Omega'$). If
$$d_H(\Omega'^c, \Omega^c) \leq C^{-1}$$
then
$$|\lambda_1 - \lambda_1'| \leq C|\Omega \Delta \Omega'|^{\frac{\alpha}{N}}.$$

The proof relies on a different approach than the technics in [158] and [1]. The principal idea is to obtain some estimates on the behavior of eigenfunctions near the boundary and combine them with the Min-Max principle using a good extension lemma to compare two functions defined on different domains.

**Proof.** Let $\alpha, M$ and $\gamma_1$ be given as in the statement of the theorem. Let $u_1$ $(resp. u_1')$ be an eigenfunction of unit $L^2$-norm associated to the first eigenvalue $\lambda_1$ $(resp. \lambda_1')$ in $\Omega (resp. \Omega')$. We denote by $\delta := d_H\,(\Omega^c, \Omega'^c)$. Let $\mu_1$ be the first eigenvalue of the Laplacian in a ball contained in both $\Omega$ and $\Omega'$, in such a way that the inequality $\max(\lambda_1, \lambda_1') \le \mu_1 \le \gamma_1$ holds by the monotonicity property for Dirichlet eigenvalues. We finally denote by $C_2 := \min(C_0, C_1)$ where $C_0$ and $C_1$ are the constants of Corollary (5.1.8) and Corollary(5.1.10), depending on $N, \alpha$ and $\max(|\Omega|, |\Omega'|) \le 10|\Omega|$ (provided that the constant $C$ in the statement of the theorem is big enough).

We know that

$$\lambda_1 := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \int_\Omega |\nabla u_1|^2 .$$

Let $\tilde{u}_1' \in H_0^1(\Omega)$ be the extension of $u_1'$ given by Lemma (5.1.11). In particular we have

$$\int_\Omega |\nabla \tilde{u}_1'|^2 \, dx \le \int_{\Omega'} |\nabla u_1'|^2 \, dx + C \int_{\Omega' \cap A_{4b\delta}} |\nabla u_1'|^2 \, dx$$

$$\le \lambda_1' + C\lambda_1' L \delta^\alpha \le \lambda_1' + C\gamma_1 L \delta^\alpha \le \lambda_1' + C\delta^\alpha . \qquad (23)$$

where (23) comes from Corollary (5.1.10) and here $A_\tau := \{d(x, \partial\Omega) \le \tau\}$. Further , since $u_1' = \tilde{u}_1'$ in the complement of $\Omega' \cap A_{4\delta}$ we also have that

$$\int_\Omega |\tilde{u}_1'|^2 \ge \int_{\Omega'} |u_1'|^2 - C \int_{\Omega' \cap A_{4\delta}} |u_1'|^2 . \qquad (24)$$

which implies, using this time Corollary (5.1.8) and then Corollary (5.1.10) again,

$$\int_\Omega |\tilde{u}_1'|^2 \ge 1 - C\delta^2 \int_{\Omega' \cap A_{4b\delta}} |\nabla u_1'|^2 = 1 - C\delta^2 C_1 \lambda_1' \, L\delta^\alpha \ge 1 - C\delta^{2+\alpha}. \qquad (25)$$

Now using (24) and (25) we can compute

$$\lambda_1 = \int_\Omega |\nabla u_1|^2 \le \frac{\int_\Omega |\nabla \tilde{u}_1'|^2 \, dx}{\int_\Omega |\tilde{u}_1'|^2 \, dx} \le \frac{\lambda_1' + C\delta^\alpha}{1 - C\delta^{2+\alpha}}$$

$$\le \lambda_1' + \frac{C\delta^\alpha - C\lambda_1' \delta^{2+\alpha}}{1 - C\delta^{2+\alpha}} \le \lambda_1' + C\delta^\alpha . \qquad (26)$$

provided that $\delta$ is small enough depending on $C, \alpha$ and $\gamma_1$. The constant $C$ in (26) depends on $\gamma_1$ , $C_2$ , $N, \alpha, |\Omega|$, and $M$. Then by the same argument and exchanging the role of $\lambda_1$ and $\lambda_1'$ we get the desired inequality, namely

$$|\lambda_1 - \lambda_1'| \le C\delta^\alpha .$$

We conclude the proof by observing that the estimate involving $|\Omega_1 \Delta \Omega_2'|^{\frac{1}{N}}$ is a direct consequence of Lemma (5.1.5).

**Corollary(5.1.18)[206]:** Let $\Omega_{m-1}$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$ such that

$$0 < \mathcal{H}^{N-1}(\partial\Omega_{m-1}) = L < +\infty .$$

Let $B$ be a ball such that $10B$ is contained in $\Omega_{m-1}$ and let $\gamma_m$ be the first eigenvalue of $B$. Then for every $\varepsilon > 0$ and for every $M > L$ there is a constant $C_{m-2}$ depending on $\varepsilon, N, |\Omega_{m-1}|$, $\gamma_m$ , $r_0$ and $M$ such that the following holds. Let $\Omega_{m-1}'$ be an $(\varepsilon, r_0)$-Reifenberg flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega_{m-1}') \le M$ and let $\lambda_m$(resp.$\lambda_m'$) be the first eigenvalue for the Dirichlet

Laplacian in $\Omega_{m-1}$ (resp. $\Omega'_{m-1}$). If

$$d_H\left(\Omega'^{c}_{m-1}, \Omega^{c}_{m-1}\right) \leq C_{m-2}^{-1}$$

then

$$\left|\lambda_m - \lambda'_m\right| \leq C_{m-2}|\Omega_{m-1} \Delta \Omega'_{m-1}|^{\frac{1-\varepsilon}{N}}.$$

**Proof.**

Let $\varepsilon$, $M$ and $\gamma_m$ be given as in the statement of the theorem. Let $u_m$ $(resp. u'_m)$ be an eigen function of unit $L^2$-norm associated to the first eigenvalue $\lambda_m (resp. \lambda'_m)$ in $\Omega_{m-1}(resp. \Omega'_{m-1})$. We denote by $\delta := d_H\ (\Omega^c_{m-1}, \Omega'^{c}_{m-1})$. Let $\mu_m$ be the first eigenvalue of the Laplacian in a ball contained in both $\Omega_{m-1}$ and $\Omega'_{m-1}$, in such a way that the inequality $\max(\lambda_m, \lambda'_m) \leq \mu_m \leq \gamma_m$ holds by the monotonicity property for Dirichlet eigenvalues. We finally denote by $C_{m+1} := \min(C_{m-1}, C_m)$ where $C_{m-1}$ and $C_m$ are the constants of Corollary (5.1.23) and Corollary (5.1.25), depending on $N, \varepsilon$ and $\max(|\Omega_{m-1}|, |\Omega'_{m-1}|) \leq 10\ |\Omega_{m-1}|$ (provided that the constant $C_{m-2}$ in the statement of the theorem is big enough).

We know that

$$\lambda_m := \inf_{u_{m-1} \in H^1_0(\Omega_{m-1})} \frac{\int_{\Omega_{m-1}} |\nabla u_{m-1}|^2}{\int_{\Omega_{m-1}} |u_{m-1}|^2} = \int_{\Omega_{m-1}} |\nabla u_m|^2.$$

Let $\tilde{u}'_m \in H^1_0(\Omega_{m-1})$ be the extension of $u'_m$ given by Corollary (5.1.26).
In particular we have

$$\int_{\Omega_{m-1}} |\nabla \tilde{u}'_m|^2\ dx_{m-1} \leq \int_{\Omega'_{m-1}} |\nabla u'_m|^2\ dx_{m-1} + C_{m-2} \int_{\Omega'_{m-1} \cap A_{4b\delta}} |\nabla u'_m|^2\ dx_{m-1}$$

$$\leq \lambda'_m + C_{m-2}\,\lambda'_m\, L\, \delta^{1-\varepsilon} \leq \lambda'_m + C_{m-2}\,\gamma_m\, L\, \delta^{1-\varepsilon} \leq \lambda'_m + C_{m-2}\, \delta^{1-\varepsilon}. \quad (23)^*$$

where (23)* comes from Corollary (5.1.25) and here $A_\tau := \{d(x_{m-1}, \partial\Omega_{m-1}) \leq \tau\}$. Further, since $u'_m = \tilde{u}'_m$ in the complement of $\Omega'_{m-1} \cap A_{4\delta}$ we also have that

$$\int_{\Omega_{m-1}} |\tilde{u}'_m|^2 \geq \int_{\Omega'_{m-1}} |u'_m|^2 - C_{m-2} \int_{\Omega'_{m-1} \cap A_{4\delta}} |u'_m|^2. \quad (24)^*$$

which implies, using this time Corollary (13) and then Corollary (15) again,

$$\int_{\Omega_{m-1}} |\tilde{u}'_m|^2 \geq 1 - C_{m-2}\, \delta^2 \int_{\Omega'_{m-1} \cap A_{4b\delta}} |\nabla u'_m|^2 = 1 - C_{m-2}\, \delta^2 C_m \lambda'_m\, L \delta^{1-\varepsilon}$$

$$\geq 1 - C_{m-2}\, \delta^{3-\varepsilon}. \quad (25)^*$$

Now using (24)* and (25)* we can compute

$$\lambda_m = \int_{\Omega_{m-1}} |\nabla u_m|^2 \leq \frac{\int_{\Omega_{m-1}} |\nabla \tilde{u}'_m|^2\ dx_{m-1}}{\int_{\Omega_{m-1}} |\tilde{u}'_m|^2\, dx_{m-1}} \leq \frac{\lambda'_m + C_{m-2}\, \delta^{1-\varepsilon}}{1 - C_{m-2}\, \delta^{3-\varepsilon}}$$

$$\leq \lambda'_m + \frac{C_{m-2}\, \delta^{1-\varepsilon} - C_{m-2}\, \lambda'_m \delta^{3-\varepsilon}}{1 - C_{m-2}\, \delta^{3-\varepsilon}} \leq \lambda'_m + C_{m-2}\, \delta^{1-\varepsilon}. \quad (26)^*$$

provided that $\delta$ is small enough depending on $C_{m-2}, \varepsilon$ and $\gamma_m$ . The constant $C_{m-2}$ in (26)* depends on $\gamma_m$ , $C_{m+1}$ , $N, \varepsilon, |\Omega_{m-1}|$, and $M$. Then by the same argument and exchanging the role of $\lambda_m$ and $\lambda'_m$ we get the desired inequality, namely

$$|\lambda_m - \lambda'_m | \leq C_{m-2}\, \delta^{1-\varepsilon}.$$

We conclude the proof by observing that the estimate involving $|\Omega_m \,\Delta\, \Omega_{m+1}|^{\frac{1}{N}}$ is a direct consequence of Lemma (10).

**Corollary(5.1.19)[206]:** Let $\Omega_{m-1}$ be an $(\varepsilon, r_0)$-Reifenberg flat domain. Then for all $x_{m-1} \in \partial\Omega_{m-1}$ and $r_0 < 2\varepsilon$, the balls $D^+(x_{m-1}, r_0 - \varepsilon)$ and $D^-(x_{m-1}, r_0 - \varepsilon)$lie in different connected components of $B(x_{m-1}, r_0 - \varepsilon) \setminus \partial\Omega_{m-1}$.

**Proof .** This can be seen as a consequence of the topological disk Theorem of Reifenberg [64]. Actually one could also prove it directly without using the whole result of Reifenberg but just the very beginning of Reifenberg's construction. In our situation we find it convenient to simply apply the Theorem. More precisely, we use the statement of Theorem1.1. in [68] (which holds for $N \neq 3$ for the case of hyperplanes) that gives for every $\varepsilon > 0$ and $x_{m-1} \in \partial\Omega_{m-1}$ a hyperplane $P$ through $x_{m-1}$ and a continuous homeomorphism $f_m : B(x_{m-1}, \frac{3}{2}(r_0 - \varepsilon)) \to f_m(B(x_{m-1}, \frac{3}{2}(r_0 - \varepsilon))) \subset B(x_{m-1}, 2(r_0 - \varepsilon))$ such that

$$B(x_{m-1}, r_0 - \varepsilon) \subset f_m\left(B\left(x_{m-1}, \frac{3}{2}(r_0 - \varepsilon)\right)\right) \subset B\big(x_{m-1}, 2(r_0 - \varepsilon)\big), \qquad (4)^*$$

$$\partial\Omega_{m-1} \cap B(x_{m-1}, r_0 - \varepsilon) \subset f_m\left(P \cap B\left(x_{m-1}, \frac{3}{2}(r_0 - \varepsilon)\right)\right) \subset \partial\Omega_{m-1} \cap B(x_{m-1}, 2(r_0 - \varepsilon)). \qquad (5)^*$$

Now if we denote by $\nu_{m-1}$ any normal vector to $P$ and consider

$$P^+ := \{x_{m-1} \in \mathbb{R}^N \colon\ x_{m-1} \,.\, \nu_{m-1} > 0\}\,,\quad P^- := \{x_{m-1} \in \mathbb{R}^N \colon\ x_{m-1} \,.\, \nu_{m-1} < 0\}\,,$$

it is clear from (4)* and(5)* that $\partial\Omega_{m-1}$ separates the domains $f_m\left(P^\pm \cap B\left(x_{m-1},\ \frac{3}{2}(r_0 - \varepsilon)\right)\right)$ and in particular the sets $D^\pm(x_{m-1}, r_0 - \varepsilon)$ .

**Corollary(5.1.20)[206]:** Let $\Omega_m$ and $\Omega_{m+1}$ be two $(\varepsilon, r_0)$-Reifenberg flat domains such that $d_H(\Omega_m^c,\ \Omega_{m+1}^c) \leq r_0/3$ . Then

$$d_H(\Omega_m^c, \Omega_{m+1}^c) \leq C_{m-2} |\Omega_m \Delta\, \Omega_{m+1}|^{\frac{1}{N}}$$

where $C_{m-2}$ depends only on $N$.

**Proof .** Let $x_{m-1} \in \partial\Omega_m$ , be such that $r_0 - \varepsilon := \mathrm{dist}(x_{m-1}, \partial\Omega_{m+1})$ is maximum, and let $y_{m-1} \in \partial\Omega_{m+1}$ be such that $\mathrm{dist}(x_{m-1}, \partial\Omega_{m+1}) = d(x_{m-1}, y_{m-1}) = r_0 - \varepsilon$. Let us set $D_1^\pm = D^\pm(x_{m-1}, r_0 - \varepsilon)$ and $D_2^\pm = D^\pm(y_{m-1}, r_0 - \varepsilon)$ as being the balls defined in (3)*. Under the assumptions we know that only one of $D_1^\pm$ lies in $\Omega_m$ and only one of $D_2^\pm$ lies in $\Omega_{m+1}$. Let us simply denote by $D_i$ those two balls.

Now by the definition of $y_{m-1}$, we know that $B(y_{m-1}, r_0 - \varepsilon) \cap \partial\Omega_m$ is empty. In particular, the two "approximating" hyperplanes $P(x_{m-1}, r_0 - \varepsilon)$ and $P(y_{m-1}, r_0 - \varepsilon)$ are almost parallel (with error less than $2.\varepsilon \le 2.10^{-2}$ ) as in Fig. 2.

Then it is not difficult to show, considering also a similar situation in $B(x_{m-1}, 3(r_0 - \varepsilon))$ and $B(y_{m-1}, 3(r_0 - \varepsilon))$ with the corresponding selection of domains $D_i(3(r_0 - \varepsilon)) \in \{D^\pm(x_{m-1}, 3(r_0 - \varepsilon)), D^\pm(y_{m-1}, 3(r_0 - \varepsilon))\}$, that whatever the positions of the $D_i$ and $D_i(3(r_0 - \varepsilon))$ with respect to the lines $P(x_{m-1}, 3(r_0 - \varepsilon))$ and $P(y_{m-1}, 3(r_0 - \varepsilon))$ are, one can always find a ball of radius equivalent to $r_0 - \varepsilon$ that lies in the symmetric difference of $\Omega_m$ and $\Omega_{m+1}$. We conclude the proof by exchanging the role of $\Omega_m$ and $\Omega_{m+1}$ and using the same argument.

**Corollary(5.1.21)[206]:** Let $\Omega_{m-1} \subset \mathbb{R}^N$ be an $(\varepsilon, r_0)$ −Reifenberg flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega_{m-1}) = L < +\infty$. Then for every $r_0 < 2\varepsilon$ we can extract among $\{B(x_{m-1}, r_0 - \varepsilon)\}_{x_{m-1} \in \partial\Omega_{m-1}}$ a subfamily of at most $L/(C_{m+N-1}(r_0 - \varepsilon)^{N-1})$ balls that forms a covering of $\bigcup_{x_{m-1} \in \partial\Omega_{m-1}} B\left(x_{m-1}, \frac{8}{10}(r_0 - \varepsilon)\right)$ where $C_{m+N-1}$ is a dimensional constant. Moreover, for all $x_{m-1}$ we have that

$$\#\{i : x_{m-1} \in B_i\} \le C_{m-2} \tag{6)*$$

where $C_{m-2}$ is again a dimensional constant.

**Proof .** Since $\varepsilon > 0$, we have that

$$d_H\left(\partial\Omega_{m-1} \cap B(x_{m-1}, r_0 - \varepsilon), P(x_{m-1}, r_0 - \varepsilon) \cap B(x_{m-1}, r_0 - \varepsilon)\right) \le 10^{-2}(r_0 - \varepsilon). \quad (7)^*$$

We also know that $\partial\Omega_{m-1}$ separates $D^+(x_{m-1}, r_0 - \varepsilon)$ from $D^-(x_{m-1}, r_0 - \varepsilon)$ and since the set of minimal $\mathcal{H}^{N-1}$ area having this property and satisfying (7)* is the corresponding part of a hyperplane, we deduce that there exists a dimensional constant $C_{m+N-1}$ such that for all $x_{m-1} \in \partial\Omega_{m-1}$ and all $\varepsilon \ge 0$

$$\mathcal{H}^{N-1}(\partial\Omega_{m-1} \cap B(x_{m-1}, r_0 - \varepsilon)) \ge C_{m+N-1}(r_0 - \varepsilon)^{N-1}.$$

Now let $B(x_{m+i-1}, (r_0 - \varepsilon)_i)$, be a subfamily of $\{B(x_{m-1}, r_0 - \varepsilon)\}_{x_{m-1} \in \partial\Omega_{m-1}}$ indexed by $i \in I$, maximal for the property that $\frac{1}{10}B_i \cap \frac{1}{10}B_j = \emptyset$. Using this fact (6)* comes from a classical geometric argument in $\mathbb{R}^N$. Now we claim that $\#I$ is finite. Indeed, since $\frac{1}{10}B_i$ are disjoint balls we have

$$L \ge \mathcal{H}^{N-1}\left(\bigcup_{i \in I} \partial\Omega_{m-1} \cap \frac{1}{10}B_i\right) \ge \#I\, C_{m+N-1}(r_0 - \varepsilon)^{N-1}\, 10^{1-N}$$

thus

$$\#I \le \frac{10^{N-1} L}{C_{m+N-1}(r_0 - \varepsilon)^{N-1}}. \tag{8)*$$

Finally, it remains to prove that the family $\{B_i\}_{i \in I}$ forms a covering of $\bigcup_{x_{m-1} \in \partial\Omega_{m-1}} B\left(x_{m-1}, \frac{8}{10}(r_0 - \varepsilon)\right)$. Let $y_{m-1} \in \bigcup_{x_{m-1} \in \partial\Omega_{m-1}} B\left(x_{m-1}, \frac{8}{10}(r_0 - \varepsilon)\right)$ and let $x_{m-1} \in \partial\Omega_{m-1}$ be such that $y_{m-1} \in B\left(x_{m-1}, \frac{8}{10}\ (r_0-\varepsilon)\right)$. Then by the maximality of the $\{B_i\}$, there exist an index $i$ and a point $z_{m-1} \in \frac{1}{10} B_i \cap B(x_{m-1}, \frac{r_0-\varepsilon}{10})$. Then if $x_{m+i-1}$ denotes the center of $B_i$ , we have

$$d(y_{m-1}, x_{m+i-1}) \le d(y_{m-1}, x_{m-1}) + d(x_{m-1}, z_{m-1}) + d(z_{m-1}, x_{m+i-1})$$
$$\le \frac{8}{10}\ (r_0-\varepsilon) + \frac{2}{10}\ (r_0-\varepsilon) = r_0-\varepsilon$$

which proves that $y_{m-1} \in B(x_{m+i-1}, r_0-\varepsilon)$.

**Corollary(5.1.22)[206]:** Let $\Omega_{m-1}$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^{\boldsymbol{N}}$ and $u_{m-1} \in W_0^{1,1+\varepsilon}(\Omega_{m-1})$ for some . Then for all $x_{m-1} \in \partial\Omega_{m-1}$ and $\varepsilon \ge 0$ we have

$$\|u_{m-1}\|_{L^{1+\varepsilon}\,(B(x_{m-1}, r_0-\varepsilon)\cap\Omega_{m-1})} \le C_{m-2}(r_0-\varepsilon)\|\nabla u_{m-1}\|_{L^{1+\varepsilon}\,(B(x_{m-1}, b(r_0-\varepsilon))\cap\Omega_{m-1})}$$

where $C_{m-2} := C_{m-2}(\varepsilon, N)$ and $b := b(N)$.

**Proof .** The proof is a small modification of the classical proof of the Sobolev inequality that we will write here with full details for the convenience of the reader.

Without loss of generality, we may assume that $u_{m-1} \in C^1_{m-1}(\Omega_{m-1})$, $x_{m-1}$ is the origin and that $P(x_{m-1}, r_0-\varepsilon)$ is the hyper plane $\{x_m = 0\}$. We shall prove that

$$\|u_{m-1}\|_{L^{1+\varepsilon}\,(\Omega_{m-1}\cap Q(x_{m-1}, r_0-\varepsilon))} \le C_{m-2}(r_0-\varepsilon)\|\nabla u_{m-1}\|_{L^{1+\varepsilon}\,(\Omega_{m-1}\cap Q(x_{m-1}, r_0-\varepsilon))}. \quad (9)^*$$

where $Q(x_{m-1}, r_0-\varepsilon)$ is a cube centered at $x_{m-1}$, and with faces orthogonal to the axis of $\mathbb{R}^{\boldsymbol{N}}$. Observe that (9)* implies the desired inequality with constant $b$ coming from the comparison between cubes and euclidian balls in $\mathbb{R}^{N}$.

By changing the orientation of $x_m$ we can assume that $Q(x_{m-1}, r_0-\varepsilon) \cap \Omega_{m-1}$ (which is connected by our assumptions) contains the upper part $Q(x_{m-1}, r_0-\varepsilon) \cap \{x_m > (r_0-\varepsilon)/2\}$.

It is clear that for any $u_{m-1} \in C^1_{m-1}\,(\Omega_{m-1})$,

$$|u_{m-1}(x_m, x'_{m-1})| \le \int_{-\infty}^{x_m} |D_1 u_{m-1}(t, x'_{m-1})|\, dt \le \int_{-\infty}^{r_0-\varepsilon} |D_1 u_{m-1}|(t, x'_{m-1})\, dt\,.$$

Integrating over $x_m$ we obtain

$$\int_{-\infty}^{r_0-\varepsilon} |u_{m-1}(x_m, x'_{m-1})|\, dx_m \le 2(r_0-\varepsilon) \int_{-\infty}^{r_0-\varepsilon} |D_1 u_{m-1}(t, x'_{m-1})|\, dt\,.$$

Now integrating the last inequality between $-(r_0-\varepsilon)$ and $(r_0-\varepsilon)$ successively over each variable $x_{m+1}, \ldots, x_{m+N-1}$ we get

$$\int_{Q(x_{m-1}, r_0-\varepsilon)\cap\Omega_{m-1}} |u_{m-1}(x_{m-1})|\, dx_{m-1}$$
$$\le 2(r_0-\varepsilon) \int_{Q(x_{m-1}, r_0-\varepsilon)\cap\Omega_{m-1}} |D_1 u_{m-1}(x_{m-1})|\, dx_{m-1}$$

$$\leq C_{m-2}(r_0-\varepsilon)\int_{Q(x_{m-1},r_0-\varepsilon)\cap\Omega_{m-1}}\|Du_{m-1}\|(x_{m-1})\ dx_{m-1}.$$

Then (9)* follows if we apply this last inequality to $u_{m-1}^{1+\varepsilon}$ and use the Hölder's inequality.

**Corollary(5.1.23)[206]:** Let $\Omega_{m-1}$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$ and for $\delta \leq r_0/2$ set

$$A_\delta := \Omega_{m-1}\cap\{d(x_{m-1},\partial\Omega_{m-1})\leq\delta\}.$$

Then for any function $u_{m-1}\in W_0^{1,1+\varepsilon}(\Omega_{m-1})$ we have

$$\left(\int_{A_\delta}|u_{m-1}|^{1+\varepsilon}\ dx_{m-1}\right)^{\frac{1}{1+\varepsilon}} \leq C_{m-2}\,\delta\left(\int_{A_{2b\delta}}|\nabla u_{m-1}|^{1+\varepsilon}\ dx_{m-1}\right)^{\frac{1}{1+\varepsilon}}$$

where $b$ is the dimensional constant of Corollary (5.1.22).

**Proof .**Let $\{(B_m)_i\}_{i\in I}$ be the subfamily of balls $\{B(x_{m-1},2\delta)\}_{x_{m-1}\in\partial\Omega_m}$ given by Corollary (5.1.21). Then

$$A_\delta := \Omega_{m-1}\cap\{x_{m-1}: d(x_{m-1},\partial\Omega_{m-1})\leq\delta\}\subset\bigcup_{x_{m-1}\in\partial\Omega_{m-1}}B\left(x_{m-1},\frac{16}{10}\delta\right)\subset\bigcup_{i\in I}(B_m)_i\,.$$

Moreover the covering is bounded by a dimensional constant $C_{m-2}$. Then,

$$\int_{A_\delta}|u_{m-1}|^{1+\varepsilon}\ dx_{m-1}\leq\sum_{i\in I}\int_{(B_m)_i\cap\Omega_{m-1}}|u_{m-1}|^{1+\varepsilon}\ dx_{m-1}\quad .$$

and using Corollary (5.1.22), together with the fact that the $(B_m)_i$ are centered at $\partial\Omega_{m-1}$, we obtain

$$\int_{A_\delta}|u_{m-1}|^{1+\varepsilon}\ dx_{m-1}\leq C_{m-2}\sum_{i\in I}\delta^{1+\varepsilon}\int_{b(B_m)_i\cap\Omega_{m-1}}|\nabla u_{m-1}|^{1+\varepsilon}\,dx_{m-1}$$

$$\leq C_{m-2}\,\delta^{1+\varepsilon}\int_{A_{2b\delta}}|\nabla u_{m-1}|^{1+\varepsilon}\ dx_{m-1}$$

which proves the corollary.

**Corollary(5.1.24)[206]:** Let $\Omega_{m-1}$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$, and let $u_{m-1}$ be an eigenfunction for the Dirichlet Laplacian in $\Omega_{m-1}$, associated to the eigenvalue $\lambda_{m-1}$. Then for every $\beta > 0$ there is a constant $C_{m-1}$ depending on $N$, $|\Omega_{m-1}|$ and $\beta$ such that for every $x_{m-1}\in\partial\Omega_{m-1}$ and for all $\varepsilon\geq 0$, we have that

$$\int_{B(x_{m-1},r_0-\varepsilon)\cap\Omega_{m-1}}|\nabla u_{m-1}|^2\ dx_{m-1}\leq C_{m-1}\,\lambda_{m-1}\|u_{m-1}\|^2_{L^2(\Omega_{m-1})}\left(\frac{r_0-\varepsilon}{r_0}\right)^{N-\beta}. \qquad (10)^*$$

**Proof .** For a given $\beta > 0$, define

$$a := 2^{\frac{2}{\beta}}. \qquad (11)^*$$

Without loss of generality we assume that $r_0 = 1$ and $\|u_{m-1}\|_2 = 1$. Now let $x_{m-1} \in \partial\Omega_{m-1}$. We will obtain the appropriate decay by showing that for $k \in \mathbb{N}$ and a specific selection of the constant $C_m$ we have

$$\int_{B(x_{m-1},a^{-k})\cap\Omega_{m-1}} |\nabla u_{m-1}|^2 \; dx_{m-1} \le C_m \, \lambda_{m-1} a^{-k(N-\beta)}. \qquad (12)^*$$

We will prove (12)* inductively. It is clear that (12)* is true for $k = 0$ if $C_m \ge 1$.

Suppose now that (12)* is true for $k$ and denote by $v_{m-1}$ the "harmonic" replacement of $u_{m-1}$ in $S_k := B(x_{m-1}, a^{-k}) \cap \Omega_{m-1}$; that is a harmonic function $v_{m-1} \in H^1(S_k)$ which satisfies $u_{m-1} - v_{m-1} \in H_0^1(S_k)$. Such a function $v_{m-1}$ can be obtained by minimizing the Dirichlet integral, and since $u_{m-1}$ is a competitor we have that

$$\int_{S_k} |\nabla v_{m-1}|^2 \; dx_{m-1} \le \int_{S_k} |\nabla u_{m-1}|^2 \; dx_{m-1} \le C_m \lambda_{m-1} \, a^{-k(N-\beta)}.$$

by the inductive hypothesis. On the other hand $w_{m-1} := |\nabla v_{m-1}|^2$ is subharmonic. Therefore by the classical mean value inequality applied to $w_{m-1}$ we have

$$\int_{S_{k+1}} |\nabla v_{m-1}|^2 \; dx_{m-1} \le a^{-N} \int_{S_k} |\nabla v_{m-1}|^2 \; dx_{m-1},$$

that is

$$\int_{S_{k+1}} |\nabla v_{m-1}|^2 \; dx_{m-1} \le C_m \, \lambda_{m-1} \, a^{-(N+k(N-\beta))}. \qquad (13)^*$$

Now we want to estimate $\int_{S_k} |\nabla(u_{m-1} - v_{m-1})|^2 \; dx_{m-1}$. Notice that for all $k, u_{m-1}$ is the unique solution of the problem

$$\begin{cases} -\nabla w_{m-1} = \lambda_{m-1} w_{m-1} & \text{in } S_k, \\ w_{m-1} - u_{m-1} \in H_0^1\,(S_k) \end{cases}$$

therefore $u_{m-1}$ is minimizing the energy

$$1/2 \int_{S_k} |\nabla w_{m-1}|^2 \; dx_{m-1} - \int_{S_k} \lambda_{m-1} u_{m-1} w_{m-1} \; dx_{m-1} \, .$$

among all functions $w_{m-1}$ such that $u_{m-1} - w_{m-1} \in H_0^1(S_k)$. Therefore we deduce that

$$1/2 \int_{S_k} |\nabla u_{m-1}|^2 - \lambda_{m-1} \int_{S_k} |u_{m-1}|^2 \le 1/2 \int_{S_k} |\nabla v_{m-1}|^2 - \lambda_{m-1} \int_{S_k} u_{m-1} v_{m-1} \, .$$

Hence

$$\int_{S_k} |\nabla u_{m-1}|^2 - \int_{S_k} |\nabla v_{m-1}|^2 \le 2\lambda_{m-1} \left( \int_{S_k} |u_{m-1}|^2 - \int_{S_k} u_{m-1} v_{m-1} \right)$$

$$\le C_{m-2} \lambda_{m-1} |S_k| \|u_{m-1}\|_\infty^2 \le \lambda_{m-1} C_{m-2}(N, |\Omega_{m-1}|) a^{-kN}$$

$$\leq \lambda_{m-1} C_{m+1}\, a^{-kN}. \qquad (14)^*$$

where $|v_{m-1}|$ was estimated in terms of $\|u_{m-1}\|_\infty$ by the maximum principle and $\|u_{m-1}\|_\infty \leq C_{m-2}(N, |\Omega_{m-1}|)$ by Proposition (5.1.3).

Now since $v_{m-1}$ is harmonic in $S_k$ and $u_{m-1} - v_{m-1} \in H_0^1(S_k)$, we deduce that $\nabla v_{m-1}$ and $\nabla(u_{m-1} - v_{m-1})$ are orthogonal in $L^2(S_k)$ thus (14)* implies

$$\int_{S_k} |\nabla u_{m-1} - \nabla v_{m-1}|^2 = \int_{S_k} |\nabla u_{m-1}|^2 - \int_{S_k} |\nabla v_{m-1}|^2 \leq \lambda_{m-1} C_{m+1}\, a^{-kN}. \qquad (15)^*$$

Gathering (13)* and (15)* together we complete the induction as follows:

$$\int_{S_{k+1}} |\nabla u_{m-1}|^2\, dx_{m-1} \leq 2 \int_{S_{k+1}} |\nabla v_{m-1}|^2 + 2 \int_{S_{k+1}} |\nabla (u_{m-1} - v_{m-1})|^2\, dx_{m-1},$$

$$\leq 2 C_m \lambda_{m-1}\, a^{-(N+k(N-\beta))} + 2\lambda_{m-1} C_{m+1} a^{-kN} \leq C_m \lambda_{m-1} a^{-(k+1)(N-\beta)}.$$

where the last inequality holds by the definition of $a$ and provided for instance that

$$C_m \geq C_{m+1}\, 2^{\frac{2(N+1)}{\beta}} . \qquad (16)^*$$

Now to finish the proof, since (12)* is true, for every $\varepsilon > 0$ one can find an integer $k$ such that $1 - \varepsilon \leq a^{-k} < a(1-\varepsilon)$ thus

$$\int_{B(x_{m-1},1-\varepsilon)} |\nabla u_{m-1}|^2\, dx_{m-1} \leq \int_{B(x_{m-1},a^{-k})} |\nabla u_{m-1}|^2\, dx_{m-1} \leq C_m \lambda_{m-1} a^{-k(N-\beta)}$$

$$\leq C_m\, \lambda_{m-1} \big(a(1-\varepsilon)\big)^{N-\beta}$$

so the proposition is true with $C_{m-1} := a^{N-\beta}\, C_m$ .

**Corollary(5.1.25)[206]:** Let $\Omega_{m-1}$ be an $(\varepsilon, r_0)$-Reifenberg flat domain in $\mathbb{R}^N$ such that $0 < \mathcal{H}^{N-1}(\partial\Omega_{m-1}) = L < +\infty$. Then for any $\varepsilon > 0$ there is a constant $C_m := C_m(|\Omega_{m-1}|, r_0, N, \varepsilon)$ such that for any eigen-function $u_{m-1}$ for the Dirichlet Laplacian associated to the eigen value $\lambda_{m-1}$ in $\Omega_{m-1}$ and for any $\delta \leq r_0/2$ we have

$$\int_{\Omega_{m-1}\cap\{d(x_{m-1},\partial\Omega_m)\leq\delta\}} |\nabla u_{m-1}|^2\ dx_{m-1} \leq C_m\, \lambda_{m-1}\, L \|u_{m-1}\|^2_{L^2(\Omega_{m-1})} \delta^{1-\varepsilon}.$$

**Proof .** We argue as in Corollary (5.1.23). Let $\{(B_m)_i\}_{i\in I}$ be the subfamily of balls $\{B(x_{m-1}, 2\delta)\}_{x_{m-1}\in\partial\Omega_{m-1}}$ given by Corollary (5.1.21). By (8)* we know that

$$\# I \leq L / \left(2^{N-1}\ C_{m+N-1} \delta^{N-1}\right)$$

and that

$$\Omega_{m-1} \cap \{x_{m-1} : d(x_{m-1}, \partial\Omega_{m-1}) \leq \delta\} \subset \bigcup_{x_{m-1}\in\partial\Omega_{m-1}} B(x_{m-1}, 16/10\,\delta) \subset \bigcup_{i\in I} (B_m)_i .$$

Moreover the covering is bounded by a dimensional constant $C_{m-2}$. Then,

$$\int_{\Omega_{m-1}\cap\{d(x_{m-1},\partial\Omega_{m-1})\le\delta\}} |\nabla u_{m-1}|^2 \, dx_{m-1} \le \sum_{i\in I} \int_{(B_m)_i\cap\Omega_{m-1}} |\nabla u_{m-1}|^2 \, dx_{m-1} .$$

and using Corollary (5.1.24) (applied with $r_0-\varepsilon=\delta$ and $\beta=\varepsilon$) together with the fact that the $(B_m)_i$ are centered at $\partial\Omega_{m-1}$ we obtain

$$\int_{\Omega_{m-1}\cap\{d(x_{m-1},\partial\Omega_{m-1})\le\delta\}} |\nabla u_{m-1}|^2 \, dx_{m-1} \le C_{m-2}\lambda_{m-1}\|u_{m-1}\|^2_{L^2(\Omega_{m-1})} \sum_{i\in I}\delta^{N-\varepsilon}$$
$$\le C_{m-2}\lambda_{m-1}\|u_{m-1}\|^2_{L^2(\Omega_{m-1})}\#I\delta^{N-\varepsilon}$$
$$\le C_{m-2}\, L\lambda_{m-1}\|u_{m-1}\|^2_{L^2(\Omega_{m-1})}\,\delta^{1-\varepsilon}.$$

where $C_{m-2}=C_{m-2}(r_0,N,\varepsilon,|\Omega_{m-1}|)$ and the proof is complete.

**Corollary(5.1.26)[206]:** Let $\Omega_m$ and $\Omega_{m+1}$ be two $(\varepsilon,r_0)$-Reifenberg flat domains such that

$$d_H\,(\Omega_m^c,\Omega_{m+1}^c)\le \delta \le (100b)^{-1}\, r_0,$$

where $b$ is the dimensional constant of Corollary (5.1.22) and set

$$A_\delta := \{x_{m-1}: d(x_{m-1},\partial\Omega_m)\le\delta\}.$$

Then for any $v_{m-1}\in W_0^{1,1+\varepsilon}(\Omega_m)$ there exists a function $\tilde v_{m-1}\in W_0^{1,1+\varepsilon}(\Omega_{m+1})$ such that $v_{m-1}=\tilde v_{m-1}$ in $\Omega_m\setminus A_{2\delta}$ and

$$\|\tilde v_{m-1}\|_{L^{1+\varepsilon}(\Omega_{m+1})} \le \|v_{m-1}\|_{L^{1+\varepsilon}(\Omega_m)}, \qquad (17)^*$$

$$\|\nabla\tilde v_{m-1}\|^{1+\varepsilon}_{L^{1+\varepsilon}(\Omega_{m+1})} \le \|\nabla v_{m-1}\|^{1+\varepsilon}_{L^{1+\varepsilon}(\Omega_m)} + C_{m-2}\|\nabla v_{m-1}\|^{1+\varepsilon}_{L^{1+\varepsilon}(\Omega_m\cap A_{4b\delta})}. \qquad (18)^*$$

**Proof .**Let $\{(B_m)_i\}_{i\in I}$ be the subfamily of balls $\{B(x_{m-1},2\delta)\}_{x_{m-1}\in\partial\Omega_m}$ given by Corollary (5.1.21). We will denote by $x_{m+i-1}$ the center of $(B_m)_i$ and $(r_0-\varepsilon)_i$ its radius. Since $\Omega_m\Delta\,\Omega_{m+1}\subset\bigcup_{i\in I}(B_m)_i$ ,to define a function $\tilde v_{m-1}\in W^{1,2}(\Omega_{m+1})$, it is sufficient to define an extension of $v_{m-1}$ in $\Omega_{m+1}\cap\bigcup_{i\in I}(B_m)_i$ .

For all $i$, define a function $(\varphi_{m-2})_i\in C^1_{m+c-1}(5(B_m)_i)$, such that$(\varphi_{m-2})_i=1$ in $2(B_m)_i$ , $|\nabla\varphi_{m-2}|\le\delta^{-1}$ and let $\varphi_{m-1}$ be a function that is equal to 1 in $\Omega_m\setminus\bigcup_{i\in I}4(B_m)_i$, $\varphi_{m-1}=0$ in $\bigcup_{i\in I}2(B_m)_i$ and $\varphi_{m-1}+\sum_{j\in J}(\varphi_{m-2})_j\ge 1$ in $\Omega_m\setminus\bigcup_{j\in J}5(B_m)_i$ .Moreover, we can assume that for all $x_{m-1}\in 4(B_m)_i\setminus 2(B_m)_i$, $|\nabla\varphi_{m-1}(x_{m-1})|\le\delta^{-1}$.Indeed, such a function $\varphi_{m-1}$ can be obtained by setting

$$\varphi_{m-1}(x_{m-1}) := \prod_{i\in I} l\left(\frac{d(x_{m-1},x_{m+i-1})}{\delta}\right).$$

where $l$ is a Lipschitz function equal to $0$ in $[0,2]$, equal to 1 in $[4,+\infty)$ and $l'(x_{m-1})\le 1$. Finally, define

$$(\theta_{m-2})_i := \frac{(\varphi_{m-2})_i}{(\varphi_{m-1}+\sum_{i\in I}(\varphi_{m-2})_i)} \quad \text{for } i\in I\cup\{0\}.$$

This allows us to obtain a partition of the unity in $\Omega_m\cup\bigcup_{i\in I}5(B_m)_i$.

Next we simply define $\tilde v_{m-1}$ by

$$\tilde v_{m-1}(x_{m-1}) := \theta_{m-1}(x_{m-1})v(x_{m-1}). \qquad (19)^*$$

in such a way that $\tilde{v}_{m-1}(x_{m-1})$ vanishes on $\bigcup_{i\in I} 2(B_m)_i \supset \partial\Omega_{m+1}$. We claim that $\tilde{v}_{m-1} \in W_0^{1,1+\varepsilon}(\Omega_{m+1})$ and that (17)*, (18)* are satisfied. The first estimate (17)* comes directly from the fact that $\theta_{m-1}(x_{m-1}) \le \chi_{\Omega_m}$. So we only have to prove (18)*, which will also imply that $\tilde{v}_{m-1} \in W^{1,1+\varepsilon}(\Omega_{m+1})$.

We have that

$$\nabla\tilde{v}_{m-1}(x_{m-1}) = v_{m-1}(x_{m-1})\nabla\theta_{m-1}(x_{m-1}) = \theta_{m-1}(x_{m-1})\nabla v_{m-1}(x_{m-1}).$$

thus

$$\begin{aligned}&\|\nabla\tilde{v}_{m-1}(x_{m-1})\|_{L^{1+\varepsilon}(\Omega_{m+1})}\\ &\quad\le \left\|\nabla v_{m-1}(x_{m-1})\chi_{supp(\theta_{m-1})}\right\|_{L^{1+\varepsilon}(\Omega_{m+1})} + \|v_{m-1}(x_{m-1})\nabla\theta_{m-1}(x_{m-1})\|_{L^{1+\varepsilon}(\Omega_{m+1})}\\ &\quad\le \|\nabla v_{m-1}(x_{m-1})\|_{L^{1+\varepsilon}(\Omega_m)} + \|v_{m-1}(x_{m-1})\nabla\theta_{m-1}(x_{m-1})\|_{L^{1+\varepsilon}(\Omega_{m+1})}.\end{aligned}$$

therefore it is enough to prove that

$$\|v_{m-1}(x_{m-1})\nabla\theta_{m-1}(x_{m-1})\|_{L^{1+\varepsilon}(\Omega_{m+1})} \le C_{m-2}\|\nabla v_{m-1}(x_{m-1})\|_{L^{1+\varepsilon}(A)} \quad . \qquad (20)^*$$

with

$$A := \Omega_m \cap \bigcup_{i\in I} 4b(B_m)_i \subset A_{4b\delta}.$$

On the other hand, from the construction of $\theta_{m-1}$ we have

$$|\nabla\theta_{m-1}(x_{m-1})| \le \sum_{i\in I}\chi_{4(B_m)_i}(x_{m-1})\delta^{-1}. \qquad (21)^*$$

Therefore, since the sum in (21)* is locally finite we conclude that

$$\begin{aligned}\|v_{m-1}(x_{m-1})\nabla\theta_{m-1}(x_{m-1})\|^{1+\varepsilon}_{L^{1+\varepsilon}(\Omega_{m+1})} &\le \int_{\Omega_{m+1}}\left|v_{m-1}(x_{m-1})\sum_{j\in J}\chi_{4(B_m)_i}(x_{m-1})\delta^{-1}\right|^{1+\varepsilon}\\ &\le C_{m-2}\sum_{i\in I}\delta^{-(1+\varepsilon)}\int_{4(B_m)_i}|v_{m-1}(x_{m-1})|^{1+\varepsilon}. \qquad (22)^*\end{aligned}$$

Now since $(B_m)_i$ is centered on $\partial\Omega_m$ , from Corollary (5.1.22) we get

$$\int_{4(B_m)_i}|v_{m-1}|^{1+\varepsilon}\ dx_{m-1} \le C_{m-2}\,\delta^{\,1+\varepsilon}\int_{4b(B_m)_i}|\nabla u_{m-1}|^{1+\varepsilon}\ dx_{m-1}$$

(the definition of $v_{m-1}$ outside $\Omega_m$ is considered being 0). Then,

$$\begin{aligned}\|v_{m-1}(x_{m-1})\nabla\theta_{m-1}(x_{m-1})\|^{1+\varepsilon}_{L^{1+\varepsilon}(\Omega_{m+1})} &\le C_{m-2}\sum_{i\in I}\int_{4b(B_m)_i}|\nabla v_{m-1}(x_{m-1})|^{1+\varepsilon}\\ &\le C_{m-2}\int_{A_{4b\delta}\cap\,\Omega_m}|\nabla v_{m-1}(x_{m-1})|^{1+\varepsilon} \quad .\end{aligned}$$

which concludes the proof.

**Corollary(5.1.27)[206]:** by considering the assumptions appear in Corollary (5.1.18) show the sharp estimate

$$|\lambda_m - \lambda'_m| \le C_{N-\beta}(1+\varepsilon)\delta^{1-\varepsilon}.$$

Where $N \le \beta$.

**Proof:** From Corollary (5.1.24) we have

$$C_{m-1} = \left(2^{\frac{2}{\beta}}\right)^{N-\beta} C_m,$$

Where $a = 2^{\frac{2}{\beta}}$ and $C_{m-2} = \frac{(C_{m-1})^2}{C_m}$, $C_m = (1+\varepsilon)$, $\varepsilon \ge 0$.

From the proof of Corollary (5.1.18) we have

$$|\lambda_m - \lambda'_m| \le C_{N-\beta}(1+\varepsilon)\delta^{1-\varepsilon}.$$

Where $C_{N-\beta} = \left(2^{\frac{2}{\beta}}\right)^{2(N-\beta)}$

Hence the result follows when $N \le \beta$.

## Sec(5.2): The Dirichlet and Neumann Laplacien in Rough Domains.

We deal with the eigenvalues of the Laplace operator for the Dirichlet and the Neumann problems in rough domains. Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set. If $\partial\Omega$ satisfies suitable mild regularity assumptions, then we can apply the classical results concerning the spectrum of compact operators and infer that both the Dirichlet and the Neumann problems admit a sequence of nonnegative eigenvalues, which we denote by

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \le \ldots \to +\infty$$

and

$$0 = \mu_1(\Omega) \le \mu_2(\Omega) \le \cdots \le \mu_k(\Omega) \le \ldots \to +\infty,$$

respectively. Each eigenvalue is counted according to its multiplicity.

The problem of investigating the way that the eigenvalues $\lambda_k$ and $\mu_k$ depend on the domain $\Omega$ has been widely studied. We refer to Bucur and Buttazzo [59] and by Henrot [52]. See also the expository work by Hale [152]. In the present section we establish new stability estimates concerning the dependence of the eigenvalues on domain perturbations. The most relevant features of the section are the following.

We apply to both the Dirichlet and the Neumann problem, while many of the previous results were concerned with the Dirichlet problem only. The key argument in the present section is based on an abstract lemma (Lemma (5.2.12)) which is sufficiently general to apply to both Dirichlet and Neumann problems. Lemma (5.2. 12) has an elementary proof which uses ideas due to Birkhoff, de Boor, Swartz and Wendroff [58]. Although in this section we choose to mainly focus on domains satisfying a specific regularity condition, first introduced by E. R. Reifenberg [64], Lemma (5.2.12) can be applied to other classes of domains. As an example, we consider the case of Lipschitz domains, see Theorem (5.2.4).

We impose very weak regularity conditions on the domains $\Omega_a$ and $\Omega_b$. The exact definition of the regularity assumptions we impose is given later, here we just mention that Reifenberg flatness is a property weaker than Lipschitz continuity and that Reifenberg flat domains are relevant for the study of minimal surfaces [64] and of other problems, see Toro [72].

In the present section we establish quantitative estimates, while much of the analysis discussed in [59, 52] aimed at proving existence and convergence results. More precisely, we will obtain estimates of the following type:

$$|\lambda_k(\Omega_a) - \lambda_k(\Omega_b)| \le C d_H(\Omega_a^c, \Omega_b^c)^\alpha,$$
$$|\mu_k(\Omega_a) - \mu_k(\Omega_b)| \le C \max(d_H(\Omega_a^c, \Omega_b^c), d_H(\Omega_a, \Omega_b))^\alpha. \tag{27}$$

In the previous expressions, $\Omega^C := \mathbb{R}^N \backslash \Omega$ stands for the complement of the set $\Omega$ and $d_H$ for the Hausdorff distance, namely

$$d_H(X, Y) := \max\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\}. \tag{28}$$

In the following we will discuss both the admissible values for the exponent α and the reasons why the quantity $d_H(\Omega_a, \Omega_b)$ only appears in the stability estimate for the Neumann eigen values.

Note that quantitative stability results were established in a series of references by Burenkov, Lamberti, Lanza De Cristoforis and collaborators (see [155] for an overview). However, the regularity assumptions we impose on the domains are different than those in [155] and, moreover, the approach relies on different techniques. Indeed, the analysis in [155] is based on the notion of transition operators, while as mentioned before the argument combines real analysis techniques with an abstract lemma whose proof is based on elementary tools.

We also refer to a very recent work by Colbois, Girouard and Iversen [57] for other quantitative stability results concerning the Dirichlet problem.

As a final remark, we point out that Lemma 19 in [63] ensures that, given two sufficiently close Reifenberg flat domains $\Omega_a, \Omega_b \subseteq \mathbb{R}^N$, the Hausdorff distance is controlled by the Lebesgue measure of the symmetric difference, more precisely

$$d_H(\Omega_a^c, \Omega_b^c) \le C|\Omega_a \vartriangle \Omega_b|^{\frac{1}{N}}, \quad d_H(\Omega_a, \Omega_b) \le C|\Omega_a \vartriangle \Omega_b|^{\frac{1}{N}},$$

where the constant $C$ only depends on the dimension $N$ and on a regularity parameter of the domains. Hence, an immediate consequence of (27) is that the corresponding estimates in terms of $|\Omega_a \vartriangle \Omega_b|^{\frac{1}{N}}$. Before introducing the results, we specify the exact regularity assumptions we impose on the sets $\Omega_a$ and $\Omega_b$.

**Definition(5.2.1)[65]:** Let $\varepsilon, r_0$ be two real numbers satisfying $0 < \varepsilon < 1/2$ and $r_0 > 0$. An $(\varepsilon, r_0)$-Reifenberg-flat domain $\Omega \subseteq \mathbb{R}^N$ is a nonempty open set satisfying the following two conditions:

i) for every $x \in \partial\Omega$ and for every $r \le r_0$, there is a hyperplane $P(x, r)$ containing $x$ which satisfies

$$\frac{1}{r} d_H\big(\partial\Omega \cap B(x, r), P(x, r) \cap B(x, r)\big) \le \varepsilon. \tag{29}$$

ii) For every $x \in \partial\Omega$, one of the connected component of

$$B(x, r_0) \cap \{x : \mathrm{dist}(x, P(x, r_0)) \ge 2\varepsilon r_0\}$$

is contained in $\Omega$ and the other one is contained in $\mathbb{R}^N \setminus \Omega$.

A direct consequence of the definition is that if $\varepsilon_1 < \varepsilon_2$, then any $(\varepsilon_1, r_0)$-Reifenberg flat domain is also an $(\varepsilon_2, r_0)-$Reifenberg flat domain. Note also that, heuristically speaking, condition i) ensures that the boundary is well approximated by hyperplanes at every small scale, while condition ii) is a separating requirement equivalent to those in [20, 70, 71, 16]. The notion of Reifenberg flatness is strictly weaker than Lipschtiz continuity and appears in many areas like free boundary regularity problems and geometric measure theory (see [66, 67, 68, 20, 70, 71, 74, 75, 80, 81, 16, 64, 72, 73]).

We now state the main result concerning the Dirichlet problem. We denote by $\mathcal{H}^{N-1}$ the Hausdorff $(N-1)$-dimensional measure.

**Theorem(5.2.2)[65]:**(Dirichlet Problem). Let $B_0, D \subseteq \mathbb{R}^N$ be two given balls satisfying $B_0 \subseteq D$ and denote by $(\gamma_n)_{n\in\mathbb{N}}$ the spectrum of the Dirichlet Laplacian in $B_0$ and by $R$ the radius of $D$. For any $\alpha \in ]0,1[$ there is $\varepsilon = \varepsilon(\alpha)$ such that the following holds. For any $n \in \mathbb{N}, r_0 > 0$ and $L_0 > 0$ there are constants $\delta_0 = \delta_0(\gamma_n, n, \alpha, r_0, N, L_0, R)$ and $C = C(\alpha, r_0, N)$ such that whenever $\Omega_a$ and $\Omega_b$ are two $(\varepsilon, r_0)$-Reifenberg flat domains in $\mathbb{R}^N$ such that

$(a)$ $B_0 \subseteq \Omega_a \cap \Omega_b$ and $\Omega_a \cup \Omega_b \subseteq D$;

$(b)$ $L := \max(\mathcal{H}^{N-1}(\partial\Omega_a), \mathcal{H}^{N-1}(\partial\Omega_b)) \leq L_0$;

$(c)$ $d_H(\Omega_a^c, \Omega_b^c) \leq \delta_0$,

then

$$|\lambda_n^a - \lambda_n^b| \leq Cn\gamma_n\left(1 + \gamma_n^{\frac{N}{2}}\right)Ld_H(\Omega_a^c, \Omega_b^c)^{\alpha}, \tag{30}$$

where $\{\lambda_n^a\}$ and $\{\lambda_n^b\}$ denote the sequences of eigenvalues of the Dirichlet Laplacian in $\Omega_a$ and $\Omega_b$, respectively.

**Proof**. First, we recall that $\max\{\lambda_n^a, \lambda_n^b\} \leq \gamma_n$. Next, we fix $n \in \mathbb{N}$ and we denote by $u_1^b, \ldots, u_n^b$ the first $n$ eigenfunctions of the Dirichlet Laplacian in $H_0^1(\Omega_b)$. Given $u = \sum_{k=1}^n c_k u_b^k$, by applying Proposition (5.2.15) we get

$$\left\|\nabla u - \nabla P_{\Omega_a}^{\mathcal{D}} u\right\|_{L^2(D)}^2 = \left\|\sum_{k=1}^n c_k\left(\nabla u_b^k - \nabla P_{\Omega_a}^{\mathcal{D}} u_b^k\right)\right\|_{L^2(D)}^2 \leq n\sum_{k=1}^n c_k^2 \left\|\nabla u_b^k - \nabla P_{\Omega_a}^{\mathcal{D}} u_b^k\right\|_{L^2(D)}^2$$

$$\leq n\sum_{k=1}^n c_k^2 C\gamma_n\left(1 + \gamma_n^{\frac{N}{2}}\right)L\delta^{\alpha}\left\|u_b^k\right\|_{L^2(D)}^2 = nC\gamma_n(1 + \gamma_n^{\frac{N}{2}})L\delta^{\alpha}\|u\|_{L^2(D)}^2. \tag{31}$$

To get the last equality we have used the fact that the eigenfunctions associated with different eigenvalues are orthogonal with respect to the standard scalar product in $L^2$.

We use (31) and the Sobolev-Poincar´e inequality in the ball $D$, which has radius $R$, and we conclude that the assumptions of Lemma (5.2.12) are verified provided that:

(i) $A = nC\gamma_n(1 + \gamma_n^{\frac{N}{2}})L\delta^{\alpha}$, $B = C(N,R)A$ and

(ii) $B < 1$.

Hence, we fix a threshold $\delta_0(N, n, \gamma_n, r_0, \alpha, R, L)$ satisfying $nC\gamma_n(1 + \gamma_n^{\frac{N}{2}})L\delta_0^{\alpha} < \frac{1}{4}$, and we get that for any $\delta \leq \delta_0$ one has $1/(1-\sqrt{B}) \leq 2$. By applying Lemma (5.2.12) we then get

$$\lambda_n^a - \lambda_n^b \leq \frac{A}{1-\sqrt{B}} \leq 2A \leq 2nC\gamma_n(1 + \gamma_n^{\frac{N}{2}})L\delta^{\alpha}.$$

The theorem follows by exchanging the roles of $\Omega_a$ and $\Omega_b$ .

Before stating the precise result, we have to introduce the following definition.

**Definition(5.2.3)[65]:** Let $\Omega \subseteq \mathbb{R}^N$ be an open set, then $\Omega$ satisfies a uniform $(\rho, \theta)$-cone condition if for any $x \in \partial\Omega$ there is a unit vector $\nu \in \mathbb{R}^{N-1}$, possibly depending on $x$, such that

$$B(x, 3\rho) \cap \Omega - C_{\rho,\theta}(\nu) \subseteq \Omega \text{ and } B(x, 3\rho)\backslash\mathbb{R}^N + C_{\rho,\theta}(\nu) \subseteq \Omega\backslash\mathbb{R}^N,$$

where $B(x, 3\rho)$ denotes the ball centered at $x$ with radius $3\rho$ and $C_{\rho,\theta}(\nu)$ is the cone with height $\rho > 0$ and opening $\theta \in ]0, \pi]$,

$$C_{\rho,\theta}(\nu) := \{h \in \mathbb{R}^N : h \cdot \nu > |h| \cos\theta\} \cap B(\vec{0}, \rho).$$

We now state the stability result for Lipschitz domains.

**Theorem(5.2.4)[65]:**(Dirichlet Problem in Lipschitz domains). Let $B_0, D \subseteq \mathbb{R}^N$ be two given balls satisfying $B_0 \subseteq D$ and denote by $(\gamma_n)_{n\in\mathbb{N}}$ the spectrum of the Dirichlet Laplacian in $B_0$ and by $R$ the radius of $D$.

For an $\rho > 0, \theta \in ]0, \pi]$ there are constants $C = C(\rho, \theta, n, N, \gamma_n, R)$ and $\delta_0 = \delta_0(\rho, \theta, n, N, \gamma_n, R)$ such that the following holds. Let $\Omega_a$ and $\Omega_b$ be two open sets satisfying a $(\rho, \theta)$-cone condition and the following properties:

(d) $B_0 \subseteq \Omega_a \cap \Omega_b$ and $\Omega_a \cup \Omega_b \subseteq D$;

(e) $\delta := d_H(\Omega_a^c, \Omega_b^c) \leq \delta_0$.

Then

$$|\lambda_n^a - \lambda_n^b| \leq C\delta.$$

**Proof**. Let $\Omega_a$ and $\Omega_b$ be as in the statement of Theorem. We fix $n \in \mathbb{N}$ and $k \leq n$ and we denote by $u_a^k \in H_0^1(\Omega_a)$ and $u_b^k \in H_0^1(\Omega_b)$ the eigenfunctions associated with $\lambda_k^a$ and $\lambda_b^k$, respectively. In particular, $u_b^k$ solves $-\Delta u = \lambda_b^k u_b^k$ in $\Omega_b$. Let $\overline{u}_b \in H_0^1(\Omega_a)$ be the distributional solution of

$$\begin{cases} -\Delta u = \lambda_b^k u_b & \text{in } \Omega_a \\ u = 0 & \text{on } \partial\Omega_a \end{cases}.$$

We can now apply Theorem 1 in the reference by Savar´e and Schimperna [61]. Then formula (3.4) in [61] yields

$$\left\|\nabla u_b^k - \nabla P_{\Omega_a}^{\mathcal{D}} u_b^k\right\|_{L^2(D)}^2 \leq \left\|\nabla u_b^k - \nabla \overline{u}_b\right\|_{L^2(D)}$$
$$\leq C(\rho, N, R)\left\|\lambda_b^k u_b^k\right\|_{L^2(D)} \left\|\lambda_b^k u_b^k\right\|_{H^{-1}(D)} \frac{d_H(\Omega_a^c, \Omega_b^c)}{\rho \sin\theta},$$

where the projection $P_{\Omega_a}^{\mathcal{D}}$ is the same as in (68). Then, by using the definition of eigenfunction, we get

$$\left\|\lambda_b^k u_b^k\right\|_{H^{-1}(D)} \leq C(N, R)\sqrt{\lambda_b^k}\left\|u_b^k\right\|_{L^2(D)}.$$

We recall that $\lambda_b^k \leq \gamma_n$ and that $S_b^n$ is the eigenspace of $H_0^1(\Omega_b)$ generated by the first $n$ eigenfunctions and by arguing as (31), we get that, for any $u \in S_b^n$,

$$\left\|\nabla u - \nabla P_{\Omega_a}^{\mathcal{D}} u\right\|_{L^2(D)}^2 \leq C(\theta, \rho, \gamma_n, N, R, n) d_H(\Omega_a^c, \Omega_b^c)\|u\|_{L^2(D)}^2.$$

Hence, by proceeding as in the proof of Theorem (5.2.2) we can conclude.

We now state the stability result concerning the Neumann problem.

**Theorem(5.2.5)[65]:**(Neumann Problem). For any $\alpha \in ]0,1[$ there is $\varepsilon = \varepsilon(\alpha)$ such that the following holds. Let $\Omega_a$ and $\Omega_b$ be two bounded, connected, $(\varepsilon, r_0)$-Reifenberg flat domains in $\mathbb{R}^N$ such that

(a) $L := \max(\mathcal{H}^{N-1}(\partial\Omega_a), \mathcal{H}^{N-1}(\partial\Omega_b)) \leq L_0$;

(b) both $\Omega_a$ and $\Omega_b$ are contained in the ball $D$, which has radius $R$.

Let $\mu_n^a$ and $\mu_n^b$ be the corresponding sequences of Neumann Laplacian eigenvalues and denote by $\mu_n^* := \max\{\mu_n^a, \mu_n^b\}$. For any $n \in \mathbb{N}$, there are constants $\delta_0 = \delta_0(\mu_n^*, \alpha, r_0, n, N, R, L_0)$ and $C = C(N, r_0, \alpha, R)$ such that if

$$\max\{d_H(\Omega_a^c, \Omega_b^c), d_H(\Omega_a, \Omega_b)\} \leq \delta_0,$$

then

$$|\mu_n^a - \mu_n^b| \leq Cn\left(1 + \sqrt{\mu_n^*}\right)^{2\gamma(N)+2} L(\max(d_H(\Omega_a^c, \Omega_b^c), d_H(\Omega_a, \Omega_b))^{\alpha}), \tag{32}$$

where $\gamma(N) = \max\left\{\frac{N}{2}, \frac{2}{N-1}\right\}$.

**Proof**. By comparing (100) with Proposition (5.2.17) and by arguing as in (31) we get that the hypotheses of Lemma (5.2.12) are satisfied provided that $A = B = nC(1+\mu)^{2\gamma(N)+2}\, L\delta^{\alpha}$, and hence by repeating the same argument as in the Dirichlet case, we conclude.

We start by quoting a covering lemma that we need in the following.

**Lemma(5.2.6)[65]:** Let $\Omega \subseteq \mathbb{R}^N$ be an $(\varepsilon, r_0)$-Reifenberg flat domain such that $0 < \mathcal{H}^{N-1}(\partial\Omega) < +\infty$. Given $r < r_0/2$, consider the family of balls $\{B(x,r)\}_{x\in\partial\Omega}$. We can extract a subfamily $\{B(x_i,r)\}_{i\in I}$ satisfying the following properties:

(i) $\{B(x_i,r)\}_{i\in I}$ is a covering of $\bigcup_{x\in\partial\Omega} B(x, \frac{4}{5} r)$ ;

(ii) we have the following bound: $\#I \le C(N)\mathcal{H}^{N-1}(\partial\Omega)/r^{N-1}$;

(iii) the covering is bounded, namely

$$B\left(x_i, \frac{r}{10}\right) \cap B\left(x_j, \frac{r}{10}\right) = \emptyset \quad \text{if} \quad i \neq j. \tag{33}$$

Note also that, by applying a similar argument, we get the estimate

$$\text{for any} \quad x \in \bigcup_{i\in I} B(x_i, r), \qquad \#\{i:\ x \in B(x_i, 2r)\} \le C(N). \tag{34}$$

In the following we need cut-off functions $\theta_0, \ldots, \theta_{\#I}$ satisfying suitable conditions. The construction of these functions is standard, but for completeness we provide it.

**Lemma(5.2.7)[65]:** Under the same hypothesis as in Lemma (5.2.6), there are Lipschitz continuous cut-off function $\theta_i : \mathbb{R}^N \to R, i = 0, \ldots, \#I$, that satisfy the following conditions:

$$0 \le \theta_i(x) \le 1, \forall\, x \in \mathbb{R}^N \quad, |\nabla\theta_i(x)| \le \frac{C(N)}{r} \quad a.e.\, x \in \mathbb{R}^N, i = 0, \ldots, \#I$$

$$\theta_0(x) = 0 \text{ if } x \in \bigcup_{i\in I} B(x_i, r) \quad , \theta_0(x) = 1 \text{ if } x \in \mathbb{R}^N \setminus \bigcup_{i\in I} B\,(x_i, 2r) \tag{35}$$

$$\theta_i(x) = 0 \text{ if } x \in \mathbb{R}^N \setminus B(x_i, 2r) \quad , i = 1, \ldots, \#I\, , \sum_{i=0}^{\#I} \theta_i(x) = 1, \forall\, x \in \mathbb{R}^N.$$

**Proof**. Let $\ell, h : [0, +\infty) \to [0,1]$ be the Lipschitz continuous functions defined as follows:

$$\ell(t) := \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ 2(t-1) & \text{if } 1 \le t \le \frac{3}{2} \\ 1 & \text{if } t \ge \frac{3}{2} \end{cases} \qquad h(t) := \begin{cases} 1 & \text{if } 0 \le t \le \frac{3}{2} \\ -2(t-2) & \text{if } \frac{3}{2} \le t \le 2 \\ 0 & \text{if } t \ge 2 \end{cases}$$

First, we point out that $|\ell'|, |h'| \le 2$. Next, we set

$$\psi_0(x) := \prod_{i=1}^{\#I} \ell\left(\frac{d(x, x_i)}{r}\right) \qquad \psi_i(x) := h(d(x, x_i)/r)\, , i = 1, \ldots, \#I.$$

The goal is now showing that

$$1 \le \sum_{i=0}^{\#I} \psi_i(x) \le\ C(N), \forall\, x\ \in\ \mathbb{R}^N. \tag{36}$$

To establish the bound from below, we make the following observations: first, all the functions $\psi_i, i = 0, \dots, \#I$ take by construction only nonnegative values. Second,

$$\psi_0(x) \equiv 1 \ \forall \ x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{\#I} B\left(xi, \frac{3r}{2}\right),$$

and hence the lower bound holds on that set. If $x \in \bigcup_{i=1}^{\#I} B\left(x_i, \frac{3r}{2}\right)$, then at least one of the $\psi_i, \ i = 1, \dots, \#I$, takes the value 1. This establishes bound from below in (36).

To establish the bound from above, we point out that, for any $i = 1, \dots, \#I$, $\psi_i(x) = 0$ if $x \in \mathbb{R}^N \setminus B(x_i, 2r)$. By recalling (34) and that $\psi_i(x) \leq 1$ for every $x \in \mathbb{R}^N$ and $i = 0, \dots, \#I$, we deduce that

$$\sum_{i=0}^{\#I} \psi_i(x) \ \leq \ C(N) + 1 \leq C(N), \forall \ x \in \mathbb{R}^N$$

and this concludes the proof of (36).

Due to the lower bound in (36), we can introduce the following definitions:

$$\theta_i(x) := \frac{\psi_i(x)}{\sum_{i=0}^{\#I} \psi_i(x)} \ , i = 0, \dots, \#I, x \in \mathbb{R}^N.$$

We now show these functions satisfy (35): the only nontrivial point is establishing the bound on the gradient. To this end, we first point out that

$$|\nabla \psi_i(x)| \leq \frac{2}{r} \ , \forall \ x \in \mathbb{R}^N, i = 0, \dots, \#I.$$

Next, we recall that $\psi_i(x) = 0$ for every $x \in \mathbb{R}^N \setminus B(x_i, 2r)$ and every $i = 1, \dots, \#I$. By combining these observations with inequalities (34) and (36) we get

$$|\nabla \theta_i(x)| = \left| \frac{\nabla \psi_i(x)}{\sum_{i=0}^{\#I} \psi_i(x)} - \frac{\psi_i(x) \sum_{i=0}^{\#I} \nabla \psi_i(x)}{\left(\sum_{i=0}^{\#I} \psi_i(x)\right)^2} \right| \leq \frac{2}{r} + \frac{C(N)}{r} \ \leq \frac{C(N)}{r} \ ,$$
$$\forall \ x \ \in \mathbb{R}^N, \ \ i = 0, \dots, \#I.$$

This concludes the proof of the lemma.

We now state a result ensuring that the classical Rellich-Kondrachov Theorem applies to Reifenberg flat domains. The proof is provided in [63]. For simplicity, here we only give the statement in the case when the summability index is $p = 2$, but the result hold in the general case, see [63].

**Proposition(5.2.8)[65]:** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, connected, $(\varepsilon, r_0)$-Reifenberg flat domain and assume that $\varepsilon \leq 1/600$. Then the following properties hold:

(i) if $N > 2, H^1(\Omega)$ is continuously embedded in $L^{2^*}(\Omega), 2^* := \frac{2N}{(N-2)}$, and it is compactly embedded in $L^q(\Omega)$ for any $q \in [1, 2^*[$. Also, the norm of the embedding operator only depends on $N, r_0, q$ and on the diameter $\mathrm{Diam}(\Omega)$.

(ii) if $N = 2, H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ for every $q \in [1, +\infty[$. Also, the norm of the embedding operator only depends on $r_0, q$ and $\mathrm{Diam}(\Omega)$.

A consequence of Proposition(5.2.8) is that Neumann eigenfunctions defined in Reifenberg flat domains are bounded (see again [63] for the proof).

**Proposition(5.2.9)[65]:** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, connected, $(\varepsilon, r_0)$-Reifenberg flat domain and let $u$ be a Neumann eigenfunction associated with the eigenvalue μ. If $\varepsilon \leq 1/600$, then $u$ is bounded and

$$\|u\|_{L^\infty(\Omega)} \leq C\left(1+\sqrt{\mu}\right)^{\gamma(N)} \|u\|_{L^2(\Omega)} \;, \tag{37}$$

where $\gamma(N) = \max\left\{\frac{N}{2}, \frac{2}{N-1}\right\}$ and $C = C(N, r_0, \mathrm{Diam}(\Omega))$.

The following result ensures that Dirichlet eigenfunctions satisfy an inequality similar to (37). Note that in this case no regularity requirement is imposed on the domain.

**Proposition(5.2.10)[65]:**Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let $v$ be an eigenfunction for the Dirichlet Laplacian in $\Omega$ and let $\lambda$ be the associated eigenvalue. Then $v$ is bounded and

$$\|v\|_{L^\infty(\Omega)} \leq \left(\frac{\lambda e}{2\pi N}\right)^{\frac{N}{4}} \|v\|_{L^2(\Omega)} \,. \tag{38}$$

To conclude this part we quote a result from [74] concerning harmonic functions satisfying mixed Neumann-Dirichlet conditions.

More precisely, let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg-flat domain and let $u \in H^1(\Omega)$. Given $x \in \partial\Omega$ and $r < r_0$, consider the problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \cap B(x,r) \\ v = u & \text{on } \partial B(x,r) \cap \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \cap B(x,r) \end{cases} . \tag{39}$$

The existence of a weak solution of (39) can be obtained by considering the variational formulation

$$\min\left\{ \int_{B(x,r)\cap\Omega} |\nabla w|^2 \, dx : w \in H^1(B(x,r)\cap\Omega), \quad w = u \text{ on } \partial B(x,r)\cap\Omega \right\} \tag{40}$$

and by then applying the argument in [55]. We state the result that we need in the following.

**Theorem(5.2.11)[65]:** For every $\beta > 0$ and $a \in ]0,1/2[$ there is a positive $\varepsilon = \varepsilon(\beta, a)$ such that the following holds. Let $\Omega$ be an $(\varepsilon, r_0)$-Reifenberg flat domain, $u \in H^1(\Omega)$, $x \in \partial\Omega$ and $r \leq r_0$. Then the solution of the minimum problem (40) satisfies

$$\int_{B(x,ar)\cap\Omega} |\nabla v|^2 \, dx \leq a^{N-\beta} \int_{B(x,r)\cap\Omega} |\nabla v|^2 \, dx. \tag{41}$$

We study the abstract eigenvalue problem and we establish Lemma (5.2.12), which roughly speaking says that controlling the behavior of suitable projections is sufficient to control the difference between eigenvalues. The abstract framework is sufficiently general to apply to both the Dirichlet and the Neumann problem: these applications are discussed in this section.

The abstract framework is composed of the following objects:

(i) $H$ is a real, separable Hilbert space with respect to the scalar product $\mathcal{H}(\cdot,\cdot)$, which induces the norm $\|\cdot\|_{\mathcal{H}}$ .

(ii) $h : H \times H \to \mathbb{R}$ is a function satisfying:

$$h(\alpha u + \beta v, z) = \alpha h(u,z) + \beta h(v,z), h(u,v) = h(v,u), h(u,u) \geq 0, \tag{42}$$

for all $u, v, z \in H$ and $\alpha$, $\beta \in \mathbb{R}$. Note that we are not assuming that $h$ is a scalar product, namely $h(u,u) = 0$ does not necessarily imply $u = 0$. We also assume that there is a constant $C_H > 0$ satisfying

$$h(u,u) \leq C_H \mathcal{H}(u,u), \qquad \forall\, u \in H. \tag{43}$$

(iii) $V$ is a closed subspace of the Hilbert space $H$ such that the restriction of the bilinear form $h(\cdot,\cdot)$ to $V \times V$ is actually a scalar product on $V$, namely

$$\text{for every } v \in V,\ h(v,v) = 0 \text{ implies } v = 0. \tag{44}$$

We denote by $\overline{V}$ the closure of $V$ with respect to the norm $\|\cdot\|_h$ induced by $h$ on $V$ and we also assume that the inclusion

$$i : (V, \|\cdot\|_{\mathcal{H}}) \to (\overline{V}, \|\cdot\|_h). \qquad u \to u \tag{45}$$

is compact.

Now, we introduce an abstract eigenvalue problem associated with the function $h$ defined above: we are interested in eigencouples $(u,\lambda) \in V\backslash\{0\} \times \mathbb{R}$ satisfying

$$\mathcal{H}(u,v) = \lambda\, h(u,v), \tag{46}$$

for every $v \in V$.

We first check that (46) admits a solution. In view of (42) and (43) we obtain that, for any fixed $f \in \overline{V}$, the map

$$v \to h(f,v)$$

is a linear, continuous operator defined on the Hilbert space $(V, \|\cdot\|_{\mathcal{H}})$. As a consequence, Riesz's Theorem ensures that there is a unique element $u_f \in V$ such that

$$\mathcal{H}(u_f, v) = h(f,v), \qquad \forall v \in V. \tag{47}$$

Consider the map

$$T : (\overline{V}, \|\cdot\|_h) \to (V, \|\cdot\|_{\mathcal{H}}) \qquad f \to u_f$$

which is linear and since by setting $v = u_f$ in (47) and by using (43) one gets $\|u_f\|_{\mathcal{H}} \leq \sqrt{C_H}\, \|f\|_h$. Let $i$ be the same map as in (45), then the composition $T \circ i$ is compact.

Also, in view of (47) we infer that it is self-adjoint. By relying on classical results on compact, selfadjoint operators we infer that there is a sequence of eigencouples $(\nu_n, u_n)$ such that

$$u_n \neq 0,\ T u_n = \nu_n u_n,\ \lim_{n\to+\infty} \nu_n = 0,\ \mathcal{H}(u_n, u_m) = 0 \ \text{ if } \ m \neq n.$$

As usual, each eigenvalue is counted according to its multiplicity. Note that $\nu_n > 0$ for every $n$ by (44) and (47), and that $(u_n, \lambda_n = 1/\nu_n)$ is a sequence of eigencouples for the eigenvalue problem (46).

As a final remark, we recall that the value of $\lambda_n$ is provided by the so-called Reyleigh min-max principle, namely

$$\lambda_n = \min_{S\in\mathcal{S}_n} \max_{u\in S\{0\}} \frac{\mathcal{H}(u,u)}{h(u,u)} = \max_{u\in S_n\{0\}} \frac{\mathcal{H}(u,u)}{h(u,u)} = \frac{\mathcal{H}(u_n,u_n)}{h(u_n,u_n)},\ \forall n \in \mathbb{N}. \tag{48}$$

In the previous expression, $\mathcal{S}_n$ denotes the set of subspaces of $V$ having dimension equal to $n$ and $S_n$ denotes the subspace generated by the first $n$ eigenfunctions.

We now consider two closed subspaces $V_a, V_b \subseteq H$ satisfying assumptions (44) and (45) above and we denote by $(u_n^a, \lambda_n^a)$ and $(u_n^b, \lambda_n^b)$ the sequences solving the corresponding eigenvalue problem (46) in $V = V_a$ and $V = V_b$ respectively. We fix $n \in \mathbb{N}$ and as before we denote by $S_n^b$ the subspace

$$S_n^b = \mathrm{span}\langle u_1^b, \ldots, u_n^b\rangle. \tag{49}$$

We denote by $P: H \to V_a$ the projection of $H$ onto $V_a$, namely

$$\forall\, u \in H, \quad \mathcal{H}(u - Pu, v) = 0\,, \qquad \forall\, v \in V_a. \tag{50}$$

By relying on an argument due to Birkhoff, De Boor, Swartz and Wendroff [58] we get the main result of the present part.

**Lemma(5.2.12)[65]:**Fix $n \in \mathbb{N}$ and assume there are constants $A$ and $B$, possibly depending on $n$, such that $A > 0, 0 < B < 1$ and, for every $u \in S_n^b$,

$$\|Pu - u\|_{\mathcal{H}}^2 \leq A\|u\|_h^2. \tag{51}$$

and

$$\|Pu - u\|_h^2 \leq B\|u\|_h^2. \tag{52}$$

Then

$$\lambda_n^a \leq \lambda_n^b + \; A/(1 - \sqrt{B})^2\,. \tag{53}$$

**Proof**. We proceed in several steps.

⋄ Step 1. From (52) we get that for every $u \in S_n^b$ one has

$$\|Pu\|_h = \|Pu - u + u\|_h \geq \|u\|_h - \|Pu - u\|_h \geq \|u\|_h\big(1 - \sqrt{B}\big)\,. \tag{54}$$

In particular the restriction of the projection $P$ to $S_n^b$ is injective because $S_n^b \subseteq V_b$ and hence $\|u\|_h = 0$ implies $u = 0$. Hence, from the min-max principle (48) we get

$$\lambda_n^a \leq \max_{u \in S_n^b \setminus \{0\}} \; \mathcal{H}(Pu, Pu)/h(Pu, Pu). \tag{55}$$

⋄ Step 2. To provide an estimate on the right hand side of (55),we introduce the auxiliary function $p_n: H \to S_n^b$ defined as follows: for every $z \in H, p_n z \in S_n^b$ is the unique solution of the minimum problem

$$h(z - p_n z, z - p_n z) = \inf_{v \in S_n^b} \{h(z - v, z - v)\}. \tag{56}$$

Loosely speaking, $p_n$ is the orthogonal projection of $H$ onto $S_n^b$ with respect to the bilinear form $h$. The argument to show that the minimum problem (56) admits a solution which is also unique is standard, but for completeness we provide it.

We introduce a minimizing sequence $\{v_k\} \subseteq S_n^b$ , then we have that the sequence $h(z - v_k, z - v_k)$ is bounded. From (42) we infer that the bilinear form $h$ satisfies the Cauchy-Schwarz inequality and hence that

$$\sqrt{h(z - v_k, z - v_k)} \geq \sqrt{h(v_k, v_k)} - \sqrt{h(z, z)},$$

which implies that the sequence $h(v_k, v_k)$ is also bounded. Since $S_n^b \subseteq V_b$, then by (44) we have that the bilinear form $h$ is actually a scalar product on $S_n^b$. Since $S_n^b$ has finite dimension, from the bounded sequence $\{v_k\}$ we can extract a converging subsequence $\{v_{k_j}\}$, namely $h(v_{k_j} - v_0, v_{k_j} - v_0) \to 0$ as $j \to +\infty$, for some $v_0 \in S_n^b$. Hence,

$$h(z - v_{k_j}, z - v_{k_j}) \to h(z - v, z - v_0)$$

as $j \to +\infty$. This implies that $v_0$ is a solution of (56) and we set $v_0 = p_n z$. To establish uniqueness, we first observe that, given $z \in H$, any $p_n z$ solution of the minimization problem (56) satisfies the Euler-Lagrange equation

$$h(z - p_n z, v) = 0, \qquad \forall\, v \in S_n^b\,. \tag{57}$$

Assume by contradiction that (56) admits two distinct solutions $p_n^1 z$ and $p_n^2 z$, then from (57) we deduce that $h(p_n^1 z - p_n^2 z, v) = 0$ for every $v \in S_n^b$. By taking $v = p_n^1 z - p_n^2 z$ and in view of (44) we obtain $p_n^1 z = p_n^2 z$.

⋄ Step 3. Next we show that

$$\lambda_n^a \leq \lambda_n^b + \max_{u\in S_n^b\setminus\{0\}} \frac{\mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu)}{h(Pu, Pu)}, \tag{58}$$

where $p_n \circ P$ denotes the composition of $p_n$ and $P$. To obtain(58) we observe that for every $u \in S_n^b$ we have

$$\begin{aligned}\mathcal{H}(Pu, Pu) &= \mathcal{H}(Pu - p_n \circ Pu + p_n \circ Pu, Pu - p_n \circ Pu + p_n \circ Pu)\\ &= \mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu)\\ &\quad + \mathcal{H}(p_n \circ Pu, p_n \circ Pu),\end{aligned} \tag{59}$$

where we have used that

$$\mathcal{H}(Pu - p_n \circ Pu, p_n \circ Pu) = 0. \tag{60}$$

Note also that

$$h(Pu, Pu) \geq h(p_n \circ Pu, p_n \circ Pu), \tag{61}$$

for every $u \in S_n^b$. We assume for the moment that (60) and (61) are true as their proof will be provided in Step 5 and Step 6.

Combining (59) and (61) we obtain that for every $u \in S_n^b\setminus\{0\}$ such that $p_n \circ Pu \neq 0$ we have

$$\begin{aligned}\frac{\mathcal{H}(Pu, Pu)}{h(Pu, Pu)} &= \frac{\mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu)}{h(Pu, Pu)} + \frac{\mathcal{H}(p_n \circ Pu, p_n \circ Pu)}{h(Pu, Pu)}\\ &\leq \frac{\mathcal{H}(Pu - p_n \circ Pu, P\,u - p_n \circ Pu)}{h(Pu, Pu)} + \frac{\mathcal{H}(p_n \circ Pu, p_n \circ Pu)}{h(p_n \circ Pu, p_n \circ Pu)}\\ &\leq \frac{\mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu)}{h(Pu, Pu)} + \lambda_n^b.\end{aligned} \tag{62}$$

In the previous chain of inequalities we have used that $p_n \circ P$ attains values in $S_n^b$ and that

$$\lambda_n^b = \max_{w\in S_n^b\setminus\{0\}} \frac{\mathcal{H}(w, w)}{h(w, w)}.$$

If $p_n \circ Pu = 0$, we have

$$\frac{\mathcal{H}(Pu, Pu)}{h(Pu, Pu)} \leq \frac{\mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu)}{h(Pu, Pu)} + \lambda_n^b, \tag{63}$$

since $\lambda_n^b > 0$. By combining (62) and (63) with (55) we eventually get (58).

⋄ Step 4. We now conclude the argument by establishing (53). First, observe that, for every $u \in S_n^b$, one has

$$\begin{aligned}&\mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu)\\ &\quad\leq \mathcal{H}(Pu - p_n \circ Pu, Pu - p_n \circ Pu) + \mathcal{H}(p_n \circ Pu - u, p_n \circ P\,u - u),\\ &\quad= \mathcal{H}\big(Pu - p_n \circ Pu + p_n \circ Pu - u, Pu - p_n \circ Pu + p_n \circ Pu - u\big),\\ &\quad= \mathcal{H}(Pu - u, Pu - u).\end{aligned}$$

In the previous formula we have used the equality

$$H(Pu - p_n \circ Pu, p_n \circ Pu - u) = 0, \tag{64}$$

which is proved in Step 5. Combining (63), (64) along with (51) and (54) we finally obtain

$$\begin{aligned}\lambda_n^a &\leq \lambda_n^b + \max_{u\in S_n^b\setminus\{0\}} \frac{\mathcal{H}(Pu - u, Pu - u)}{h(Pu, Pu)}\\ &\leq \lambda_n^b + \frac{Ah(u, u)}{\big(1 - \sqrt{B}\big)h(u, u)},\end{aligned} \tag{65}$$

which concludes the proof of Lemma (5.2.12) provided that (60), (61) and (64) are true.

⋄ Step 5. We establish the proof of (60) and (64). Observe that, if $z \in V_b$, then the following implication holds true:

$$h(z, v) = 0, \qquad \forall\, v \in S_n^b \Longrightarrow \mathcal{H}(z, v) = 0, \qquad \forall\, v \in S_n^b\,. \tag{66}$$

Indeed, $v$ is a linear combination of the eigenfunctions $u_1^b, \ldots, u_n^b$ and hence

$$\mathcal{H}(z, v) = \sum_{i=1}^{n} v_i \mathcal{H}(z, u_i^b) = \sum_{i=1}^{n} \lambda_i^b\, v_i h(z, u_i^b) = 0.$$

To establish (60), we observe that $p_n \circ Pu \in S_n^b$. Also, by the property (57) of the projection $p_n$ we have $h(Pu - p_n \circ Pu, v) = 0$ for every $v \in S_n^b$. Hence by applying (66) we obtain (60). To prove (64) we can repeat the same argument. Indeed, if $u \in S_n^b$, then $[u - p_n \circ Pu] \in S_n^b$ and (64) follows from (66).

⋄ Step 6. Finally we establish (61). Note that (57) implies

$$h(p_n \circ Pu, Pu - p_n \circ Pu) = 0$$

for every $u \in S_n^b$. Hence

$$\begin{aligned} h(p_n \circ Pu, p_n \circ Pu) &\le h(p_n \circ Pu, p_n \circ Pu) + h(Pu - p_n \circ Pu, Pu - p_n \circ Pu) \\ &= h(Pu, Pu) \end{aligned}$$

for every $u \in S_n^b$.

Given an open and bounded set $\Omega$, $(u, \lambda) \in (H_0^1(\Omega)\backslash\{0\}) \times \mathbb{R}$ is an eigencouple for the Dirichlet Laplacian in $\Omega$ if

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \lambda \int_\Omega u(x)\, v(x)\, dx, \qquad \forall\, v \in H_0^1(\Omega). \tag{67}$$

We apply the abstract framework to study how the eigenvalues $\lambda$ satisfying (67) depend on $\Omega$. Let $\Omega_a$ and $\Omega_b$ two Reifenberg flat domains and denote by $D$ a ball containing both $\Omega_a$ and $\Omega_b$. We set

$$H := H_0^1(D), \mathcal{H}(u, v) := \int_D \nabla u(x) \cdot \nabla v(x)\, dx, h(u, v) := \int_D u(x) v(x)\, dx.$$

Note that (43) is satisfied because of Poincar´e- Sobolev inequality. We also set

$$V_a := H_0^1(\Omega_a), \qquad V_b := H_0^1(\Omega_b)$$

and observe that they can be viewed as two subspaces of $H_0^1(D)$ by extending the functions of $H_0^1(\Omega_a)$ and $H_0^1(\Omega_b)$ by $0$ outside $\Omega_a$ and $\Omega_b$ respectively. The compactness of the inclusion (45) is guaranteed by Rellich-Kondrachov Theorem. Also, the projection $P = P_{\Omega_a}^{\mathcal{D}}$ defined by (50) satisfies in this case

$$\left\|\nabla\left(u - P_{\Omega_a}^{\mathcal{D}} u\right)\right\|_{L^2(D)} = \min_{v \in H_0^1(\Omega_a)} \left\{\|\nabla(u - v)\|_{L^2(D)}\right\}. \tag{68}$$

By using the above notation, the Dirichlet problem (67) reduces to (46) and hence it is solved by a sequence of eigencouples $(\lambda_n, u_n)$ with $\lambda_n > 0$ for every $n$ and $\lim_{n\to+\infty} \lambda_n = +\infty$.

We employ the same notation as in above and we denote by $\{\lambda_n^a\}$ and $\{\lambda_n^b\}$ the eigenvalues of the Dirichlet problem in $\Omega_a$ and $\Omega_b$, respectively. By applying Lemma (5.2. 12), we infer that to control the difference $|\lambda_n^a - \lambda_n^b|$ it is sufficient to provide an estimate on suitable projections. This is the content of the following part.

Boundary estimate on the gradient of Dirichlet eigenfunctions. We now establish a decay result for the gradient of Dirichlet eigenfunctions. The statement is already in [ 80], but the proof contains a gap. This is why we hereafter give a different and complete proof, which is based on techniques from Alt, Caffarelli and Friedman [106]. We begin with a monotonicity Lemma.

**Lemma(5.2.13)[65]:** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let $x_0 \in \partial\Omega$. Given a radius $r > 0$, we denote by $\Omega_r^+ := B(x_0, r) \cap \Omega$, by $S_r^+ := \partial B(x_0, r) \cap \Omega$ and by $\sigma(r)$ the first Dirichlet eigenvalue of the Laplace operator on the spherical domain $S_r^+$. If there are constants $r_0 > 0$ and $\sigma^* \in ]0, N-1[$ such that

$$\inf_{0<r<r_0} (r^2\sigma(r)) \geq \sigma^*,$$

then the following holds. If $(u, \lambda)$ is an eigencouple for the Dirichlet Laplacian in $\Omega$, then the function

$$r \to \left( \frac{1}{r^\beta} \int_{\Omega_r^+} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \, dx \right) + C_0 r^{2-\beta} . \tag{69}$$

is non decreasing on $]0, r_0[$. In the previous expression, the exponent $\beta \in ]0, 2[$ is defined by the formula

$$\beta := \sqrt{(N-2)^2 + 4\sigma^*} - (N-2)$$

and $C_0$ is a suitable constant satisfying

$$C_0 = (\beta C(N)/(2-\beta))\lambda \, \|u\|_\infty^2 . \tag{70}$$

We also have the bound

$$\int_{\Omega_{r_0/2}^+} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} dx \leq C(N, r_0, \beta)\lambda \left(1 + \lambda^{\frac{N}{2}} \right) \|u\|_2^2 , \tag{71}$$

where

$$C(N, r_0, \beta) = C(N) 2^\beta r_0^{-\beta} \left[ r_0^{2-N} \frac{1}{\beta} + r_0^{3\beta} + \frac{\beta}{2-\beta} \; 2^{-\beta} r_0^2 \right] .$$

**Proof.** We assume without loss of generality that $x_0 = 0$ and to simplify notation we denote by $B_r$ the ball $B(0, r)$. Also, in the following we identify $u \in H_0^1(\Omega)$ with the function $u \in H^1(\mathbb{R}^N)$ obtained by setting $u(x) = 0$ if $x \in \Omega^c$.

The proof is based on the by now standard monotonicity Lemma of Alt, Caffarelli and Friedman [106] and is divided in the following steps.

◇ Step 1. We prove the following inequality: for a.e. $r > 0$,

$$2 \int_{\Omega_r^+} |\nabla u|^2 |x|^{2-N} \, dx$$
$$\leq r^{2-N} \int_{S_r^+} 2u \frac{\partial u}{\partial \nu} \, dS + (N-2) r^{1-N} \int_{S_r^+} u^2 \, dS + 2\lambda \int_{\Omega_r^+} u^2 |x|^{2-N} \, dx \tag{72}$$

We recall that the eigenfunction $u \in C^\infty(\Omega)$ (see the book by Gilbarg and Trudinger [27 ]). Although (72) can be formally obtained through an integration by parts, the rigorous proof is slightly technical. Given $\varepsilon > 0$, we set

$$|x|_\varepsilon := \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2 + \varepsilon},$$

so that $|x|_\varepsilon$ is a $C^\infty$ function. A direct computation shows that

$$\Delta(|x|_\varepsilon^{2-N}) = (2-N)N \, \frac{\varepsilon}{|x|_\varepsilon^{N+2}} \leq 0,$$

in other words $|x|_\varepsilon^{2-N}$ is superharmonic.

Let $u_n \in C_c^\infty(\Omega)$ be a sequence of functions converging in $H^1(\mathbb{R}^N)$ to $u$. By using the equality

$$\Delta(u_n^2) = 2|\nabla u_n|^2 + 2u_n\Delta u_n \tag{73}$$

we deduce that

$$2\int_{\Omega_r^+} |\nabla u_n|^2 |x|_\varepsilon^{2-N} = \int_{\Omega_r^+} \Delta(u_n^2)|x|_\varepsilon^{2-N} - 2\int_{\Omega_r^+} (u_n\Delta u_n)|x|_\varepsilon^{2-N}. \tag{74}$$

Since $\Delta(|x|_\varepsilon^{2-N}) \leq 0$, the Gauss-Green Formula yields

$$\int_{\Omega_r^+} \Delta(u_n^2)|x|_\varepsilon^{2-N}\,dx = \int_{\Omega_r^+} u_n^2\Delta(|x|_\varepsilon^{2-N})\,dx + I_{n,\varepsilon}(r) \leq I_{n,\varepsilon}(r), \tag{75}$$

where

$$I_{n,\varepsilon}(r) = (r^2+\varepsilon)^{\frac{2-N}{2}} \int_{\partial\Omega_r^+} 2u_n \frac{\partial u_n}{\partial \nu}\,dS + (N-2)\frac{r}{(r^2+\varepsilon)^{\frac{N}{2}}} \int_{\partial\Omega_r^+} u_n^2\,dS.$$

In other words, (74) reads

$$2\int_{\Omega_r^+} |\nabla u_n|^2 |x|_\varepsilon^{2-N}\,dx \leq I_{n,\varepsilon}(r) - 2\int_{\Omega_r^+} (u_n\Delta u_n)|x|_\varepsilon^{2-N}\,dx. \tag{76}$$

We now want to pass to the limit, first as $n \to +\infty$, and then as $\varepsilon \to 0^+$. To tackle some technical problems, we first integrate over $r \in [r, r+\delta]$ and divide by $\delta$, thus obtaining

$$\frac{2}{\delta}\int_r^{r+\delta} \left( \int_{\Omega_\rho^+} |\nabla u_n|^2 |x|_\varepsilon^{2-N} dx \right) d\rho \leq A_n - R_n, \tag{77}$$

where

$$A_n = \frac{1}{\delta}\int_r^{r+\delta} I_{n,\varepsilon}(\rho)\,d\rho$$

and

$$R_n = 2\frac{1}{\delta}\int_r^{r+\delta} \left( \int_{\Omega_\rho^+} (u_n\Delta u_n)|x|_\varepsilon^{2-N}\,dx \right) d\rho.$$

First, we investigate the limit of $A_n$ as $n \longrightarrow +\infty$: by applying the coarea formula, we rewrite $A_n$ as

$$A_n = \frac{1}{\delta}\begin{pmatrix} 2\displaystyle\int_{\Omega_{r+\delta}^+ \setminus \Omega_r^+} (|x|^2+\varepsilon)^{\frac{2-N}{2}}\, u_n \nabla u_n \cdot \frac{x}{|x|}\,dx + \\ (N-2)\displaystyle\int_{\Omega_{r+\delta}^+ \setminus \Omega_r^+} \frac{|x|}{(|x|^2+\varepsilon)^{\frac{N}{2}}}\, u_n^2 dx \end{pmatrix}.$$

Since $u_n$ converges to $u$ in $H^1(\mathbb{R}^N)$ when $n \to +\infty$, then by using again the coarea formula we get that

$$A_n \longrightarrow \frac{1}{\delta}\int_r^{r+\delta} I_\varepsilon(\rho)d\rho \qquad n \to +\infty,$$

where

$$I_\varepsilon(\rho) = (\rho^2+\varepsilon)^{\frac{2-N}{2}} \int_{\partial\Omega_\rho^+} 2u\,\frac{\partial u}{\partial \nu}\,dS + (N-2)\frac{r}{(\rho^2+\varepsilon)^{\frac{N}{2}}}\int_{\partial\Omega_\rho^+} u_n^2\,dS.$$

Next, we investigate the limit of $R_n$ as $n \to +\infty$.By using Fubini's Theorem, we can rewrite $R_n$ as

$$R_n = \int_\Omega (u_n\Delta u_n) f(x)\,dx,$$

where

$$f(x) = |x|_\varepsilon^{2-N}\ \frac{2}{\delta}\int_r^{r+\delta} \mathbf{1}_{\Omega_\rho^+}(x)d\rho\,.$$

Since

$$\frac{1}{\delta}\int_r^{r+\delta} \mathbf{1}_{\Omega_\rho^+}(x)d\rho = \begin{cases} 1 & \text{if } x \in \Omega_r^+ \\ \dfrac{r+\delta-|x|}{\delta} & \text{if } x \in \Omega_{r+\delta}^+ \setminus \Omega_r^+ \\ 0 & \text{if } x \notin \Omega_{r+\delta}^+ \end{cases},$$

then $f$ is Lipschitz continuous and hence by recalling $u_n \in C_c^\infty(\Omega)$ we get

$$\begin{aligned}\left|\int_\Omega (u_n\Delta u_n - u\Delta u)\,f dx\right| &\le \left|\int_\Omega (\Delta u_n - \Delta u)u_n f dx\right| + \left|\int_\Omega (u_n-u)\Delta u f dx\right| \\ &= \left|\int_\Omega (\nabla u_n - \nabla u)(u_n\nabla f + \nabla u_n f)dx\right| \\ &+ \left|\int_\Omega \nabla u((\nabla u_n-\nabla u)f + (u_n-u)\nabla f)\,dx\right| \\ &\le \|\nabla u_n - \nabla u\|_{L^2(\Omega)}\left(\|u_n\|_{L^2(\Omega)}\|\nabla f\|_{L^\infty(\Omega)} + \|\nabla u_n\|_{L^2(\Omega)}\|f\|_{L^\infty(\Omega)}\right) \\ &+ \|\nabla u\|_{L^2(\Omega)}\left(\|\nabla u_n - \nabla u\|_{L^2(\Omega)}\|f\|_{L^\infty(\Omega)} + \|u_n-u\|_{L^2(\Omega)}\|\nabla f\|_{L^\infty(\Omega)}\right)\end{aligned}$$

and hence the expression at the first line converges to 0 as $n \to +\infty$.

By combining the previous observations and by recalling that $u$ is an eigenfunction we infer that by passing to the limit $n \to \infty$ in (77) we get

$$\frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+} |\nabla u|^2|x|_\varepsilon^{2-N}dx\right)dr \le \frac{1}{\delta}\int_r^{r+\delta} I_\varepsilon(\rho)d\rho - \frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+}\lambda u^2|x|_\varepsilon^{2-N}dx\right)d\rho.$$

Finally, by passing to the limit $\delta \to 0^+$ and then $\varepsilon \to 0^+$ we obtain (72).

⋄ Step 2. We provide an estimate on the right hand side of (72). First, we observe that

$$\int_{\Omega_r^+} u^2|x|^{2-N}dx \le \|u\|_\infty^2 \int_{\Omega_r^+}|x|^{2-N}dx \le C(N)\|u\|_\infty^2 r^2. \qquad (78)$$

and we recall that $C(N)$ denotes a constant only depending on $N$, whose exact value can change from line to line.

Next, we point out that the definition of $\sigma^*$ implies that

$$\int_{S_r^+} u^2 \, dS \le \frac{1}{\sigma^*} r^2 \int_{S_r^+} |\nabla_\tau u|^2 dS \quad r \in ]0, r_0[, \tag{79}$$

where $\nabla_\tau$ denotes the tangential gradient on the sphere. Also, let $\alpha > 0$ be a parameter that will be fixed later, then by combining Cauchy-Schwarz inequality, (79) and the inequality $ab \le \frac{\alpha}{2} a^2 + \frac{1}{2\alpha} b^2$, we get

$$\begin{aligned}\left| \int_{S_r^+} u \frac{\partial u}{\partial \nu} \, dS \right| &\le \left( \int_{S_r^+} u^2 \, dS \right)^{\frac{1}{2}} \left( \int_{S_r^+} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \right)^{\frac{1}{2}} \\ &\le \frac{r}{\sqrt{\sigma^*}} \left( \int_{S_r^+} |\nabla_\tau u|^2 \, dS \right)^{\frac{1}{2}} \left( \int_{S_r^+} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \right)^{\frac{1}{2}} \\ &\le \frac{r}{\sqrt{\sigma^*}} \left( \frac{\alpha}{2} \int_{S_r^+} |\nabla_\tau u|^2 \, dS + \frac{1}{2\alpha} \int_{S_r^+} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \right).\end{aligned} \tag{80}$$

Hence,

$$\begin{aligned} & r^{2-N} \int_{S_r^+} 2u \frac{\partial u}{\partial \nu} \, dS + (N-2) r^{1-N} \int_{S_r^+} u^2 \, dS \\ & \quad \le r^{2-N} \frac{2r}{\sqrt{\sigma^*}} \left[ \frac{\alpha}{2} \int_{S_r^+} |\nabla_\tau u|^2 \, dS + \frac{1}{2\alpha} \int_{S_r^+} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \right] \\ & \quad + (N-2) r^{1-N} \frac{1}{\sigma^*} r^2 \int_{S_r^+} |\nabla_\tau u|^2 \, dS \\ & \quad \le r^{3-N} \left[ \left( \frac{\alpha}{\sqrt{\sigma^*}} + \frac{N-2}{\sigma^*} \right) \int_{S_r^+} |\nabla_\tau u|^2 \, dS + \frac{1}{\alpha \sqrt{\sigma^*}} \int_{S_r^+} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \right]. \end{aligned} \tag{81}$$

Next, we choose $\alpha > 0$ in such a way that

$$\frac{\alpha}{\sqrt{\sigma^*}} + \frac{N-2}{\sigma^*} = \frac{1}{\alpha \sqrt{\sigma^*}},$$

namely

$$\alpha = \frac{1}{2\sqrt{\sigma^*}} \left[ \sqrt{(N-2)^2 + 4\sigma^*} - (N-2) \right].$$

Hence, by combining (72), (79) and (81) we finally get

$$\int_{\Omega_r^+} |\nabla u|^2 |x|^{2-N} dx \le r^{3-N} \gamma(N, \sigma^*) \int_{S_r^+} |\nabla u|^2 \, dS + C(N) \lambda \|u\|_\infty^2 r^2, \tag{82}$$

where

$$\gamma(N,\sigma^*) = \left[\sqrt{(N-2)^2 + 4\sigma^*} - (N-2)\right]^{-1}.$$

⋄ Step 3. We establish the monotonicity property. We set

$$f(r) = \int_{\Omega_r^+} |\nabla u|^2 |x|^{2-N} dx$$

and we observe that

$$f'(r) = r^{2-N} \int_{S_r^+} |\nabla u|^2 \quad a.e.\, r \in ]0, r_0[\,,$$

hence (82) implies that

$$f(r) \leq \gamma r f'(r) + K r^2, \tag{83}$$

with $\gamma = \gamma(N,\sigma^*)$ and $K = C(N)\lambda\|u\|_\infty^2$. This implies that

$$\left(\frac{f(r)}{r^\beta} + \frac{K\beta}{(2-\beta)} r^{2-\beta}\right)' \geq 0,$$

with $\beta = 1/\gamma = \sqrt{(N-2)^2 + 4\sigma^*} - (N-2) \in ]0,2[$. This establishes the monotonicity result.

⋄ Step 4. We establish (71).

First, we observe that by combining the coarea formula with Chebychev inequality we get that for any $a > 0$

$$\mathcal{H}^1\left(\{t \in [r_0/2, r_0] : \int_{S_t^+} |\nabla u|^2 \, dS \geq a\}\right) \leq \frac{1}{a} \int_{\Omega_{r_0}^+ \setminus \Omega_{r_0/2}^+} |\nabla u|^2 \, dx\,.$$

By applying this inequality with $a = \frac{4}{r_0} \int_{\Omega_{r_0}^+ \setminus \Omega_{r_0/2}^+} |\nabla u|^2 \, dx$ we get that there is at least a radius $r_1 \in [r_0, r_0/2]$ such that

$$\int_{S_{r_1}^+} |\nabla u|^2 \, dS \leq \frac{4}{r_0} \int_{\Omega_{r_0}^+} |\nabla u|^2 \, dx \leq \frac{4}{r_0} \|\nabla u\|_{L^2(\Omega)}^2\,.$$

By combining (82) with the fact that $r_0/2 \leq r_1 \leq r_0$ we infer

$$\int_{\Omega_{r_1}^+} |\nabla u|^2 |x|^{2-N} dx \leq C(N) r_0^{2-N} \gamma \|\nabla u\|_{L^2(\Omega)}^2 + C(N)\lambda\|u\|_\infty^2 r_0^2$$

and by monotonicity we have that

$$\left(\frac{r_0}{2}\right)^{-\beta} \int_{\Omega_{r_0/2}^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \, dx \leq r_1^{-\beta} \int_{\Omega_{r_1}^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \, dx + C_0 r_1^{2-\beta}$$

$$\leq 2^\beta C(N) r_0^{2-N-\beta} \gamma \|\nabla u\|_{L^2(\Omega)}^2 + 2^\beta C(N)\lambda\|u\|_\infty^2 r_0^{2\beta} + C_0 r_0^{2-\beta}.$$

Finally, by using that $C_0 = \frac{\beta C(N)}{(2-\beta)} \lambda \|u\|_\infty^2$, that $\|u\|_\infty \leq C(N)\lambda^{\frac{N}{4}} \|u\|_{L^2(\Omega)}$ (see Proposition (5.2.10)), that $\|\nabla u\|_{L^2(\Omega)}^2 \leq \lambda \|u\|_{L^2(\Omega)}^2$, and that $\max\{\lambda, \lambda^{1+N/2}\} \leq \lambda(1 + \lambda^{\frac{N}{2}})$ we get (71).

We now provide an estimate on the energy of a Dirichlet eigenfunction near the boundary.

**Proposition(5.2.14)[65]:** For every $\eta \in ]0,1[$ there is a positive constant $\varepsilon := \varepsilon(\eta)$ such that the following holds. Given $r_0 \in ]0,1[$, let $\Omega \subseteq \mathbb{R}^N$ be an $(\varepsilon, r_0)$ −Reifenberg flat domain, $x_0 \in \partial\Omega$ and let $u$ be a Dirichlet eigenfunction in $\Omega$ associated with the eigenvalue $\lambda$. Then

$$\int_{B(x_0,r)\cap\Omega} |\nabla u|^2 \, dx \leq C\lambda(1 + \lambda^{\frac{N}{2}})r^{N-\eta}\|u\|_2^2 \; \forall\, r \in ]0, r_0/2[\,, \tag{84}$$

for a suitable positive constant $C = C(N, r_0, \eta)$.

**Proof**. We fix $\eta \in ]0,1[$ and we recall that the first eigenvalue of the spherical Dirichlet Laplacian on a half sphere is equal to $N-1$. For $t \in ]-1,1[$, let $S_t$ be the spherical cap $S_t := \partial B(0,1) \cap \{x_N > t\}$, so that $t = 0$ corresponds to a half sphere. Let $\lambda_1(S_t)$ be the first Dirichlet eigenvalue in $S_t$. In particular, $t \to \lambda_1(S_t)$ is monotone in $t$. Therefore, since $\eta < 1$ and $\lambda_1(S_t) \to 0$ as $t \to -1$, there is $t^*(\eta) < 0$ such that

$$\lambda_1(t^*) \leq N - 1 - \frac{\eta}{4}(2N - \eta).$$

By relying on Lemma 5 in [63](see also [20]), we infer that, if $\varepsilon < t^*(\eta)/2$, then $\partial B(x_0, r) \cap \Omega$ is contained in a spherical cap homothetic to $S_{t^*}$ for every $r \leq r_0$. Since the eigenvalues scale of by factor $r^2$ when the domain expands of a factor $1/r$, by the monotonicity property of the eigenvalues with respect to domains inclusion, we have

$$\inf_{r<r_0} r^2\lambda_1(\partial B(x_0, r) \cap \Omega) \geq \lambda_1(S_{t^*}) \geq N - 1 - \frac{\eta}{4}(2N - \eta). \tag{85}$$

As a consequence, we can apply Lemma (5.2.13) which ensures that, if $u$ is a Dirichlet eigen function in $\Omega$ and $x_0 \in \partial\Omega$, then (69) is a non decreasing function of $r$, provided that

$$\beta = \sqrt{(N-2)^2 + 4(N-1) - \eta(2N-\eta)} - (N-2) = 2 - \eta, \tag{86}$$

and $C_0$ is the same as in (70). In particular, by monotonicity we know that for every $r \leq r_0 < 1$,

$$\frac{1}{r^{N-2+\beta}} \int_{\Omega_r^+} |\nabla u|^2 \, dx \leq \left( \frac{1}{r^\beta} \int_{\Omega_r^+} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \, dx \right) + C_0 r^{2-\beta}$$

$$\leq \left( \frac{1}{r_0^\beta} \int_{\Omega_{r_0}^+} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \, dx \right) + C_0 r_0^{2-\beta}, \tag{87}$$

and we conclude that for every $r \leq r_0/2$,

$$\int_{\Omega_r^+} |\nabla u|^2 \, dx \leq K r^{N-2+\beta},$$

with

$$K = \left( \frac{1}{(r_0/2)^\beta} \int_{\Omega_{r_0/2}^+} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \, dx \right) + C_0 (r_0/2)^{2-\beta}.$$

Let us now provide an estimate on $K$. By using equations (86) and (70) and Proposition (5.2. 10) we get

$$C_0 \leq \frac{2C(N)}{\eta} \lambda \|u\|_\infty^2 \leq C(N)\eta^{-1}\lambda^{1+\frac{N}{2}}\|u\|_{L^2(\Omega)}^2. \tag{88}$$

To estimate the first term in $K$ we use (71) and the fact that $\beta = 2 - \eta$ to obtain

$$\frac{1}{(r_0/2)^{\beta}} \int_{\Omega^+_{r_0/2}} \frac{|\nabla u|^2}{|x - x_0|^{N-2}}\, dx \le C(N, r_0, \eta)\lambda \left(1 + \lambda^{\frac{N}{2}}\right) \|u\|_2^2. \qquad (89)$$

Finally, by combining (88) and (89) we obtain (84) and this concludes the proof.

Note that: By relying on an argument similar, but easier, to the one used in the proofs of Lemma (5.2.13) and Proposition (5.2.14), we get the following estimate. Let $\eta, r_0, \Omega$ and $x_0$ be as in the statement of Proposition (5.2.14), and assume that $u$ is an harmonic function in $\Omega$ satisfying homogeneous Dirichlet conditions on the boundary $\partial\Omega \cap B(x_0, r)$. For every $\eta \in ]0,1[$ we have

$$\int_{B(x_0,r)\cap\Omega} |\nabla u|^2\, dx \le C r^{N-\eta} \|u\|_2^2, \quad \forall\, r \in \left]0, \frac{r_0}{2}\right[, \qquad (90)$$

for a suitable positive constant $C = C(N, r_0, \eta)$.

Note that the decay estimate (90) is sharp, in the sense that we cannot take $\eta = 0$, as the following example shows. Let $N = 2$ and let $\Omega$ be an angular sector with opening angle $\omega$,

$$\Omega := \{(r, \theta) : -\omega/2 < \theta < \omega/2\}.$$

By recalling that the Laplacian in polar coordinates is

$$\Delta = \frac{\partial^2}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta},$$

we get that the function $u(r, \theta) := r^d \cos(\theta \frac{\pi}{\omega})$ satisfies the homogeneous Dirichlet condition on $\partial\Omega$ and is harmonic provided that $d = \pi/\omega$. Also, by computing the gradient in the circular Frenet basis $(\tau, \nu)$ and by using formulas

$$\frac{\partial u}{\partial \tau} = \frac{1}{r}\frac{\partial u}{\partial \theta} = -r^{\frac{\pi}{\omega}-1}\ \frac{\pi}{\omega} \sin\left(\theta\ \frac{\pi}{\omega}\right) \text{and } \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial r} = \frac{\pi}{\omega}\, r^{\frac{\pi}{\omega}-1} \cos(\theta\ \frac{\pi}{\omega}),$$

we get

$$\int_{B(0,r)\cap\Omega} |\nabla u|^2 = \int_{B(0,r)\cap\Omega} \left|\frac{\partial u}{\partial \tau}\right|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2 = \left(\frac{\pi}{\omega}\right)^2 \int_0^r \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} s^{2\left(\frac{\pi}{\omega}-1\right)} s\, ds d\theta = \frac{\pi}{2}\, r^{2\frac{\pi}{\omega}},$$

which leads to the following remarks:

(i) if $\Omega$ is the half-space (i.e., if $\omega = \pi$), then the Dirichlet integral decays like $r^2 = r^N$.

(ii) If the opening angle $\omega < \pi$, then we have a good decay of the order $r^{\alpha}$ for some $\alpha > 2$.

(iii) The most interesting behavior occurs when the opening angle $\omega > \pi$, then the Dirichlet integral decays like $r^{\alpha}$ with $\alpha < 2$. Note, moreover, that if we want that $\alpha$ gets closer and closer to $2$, we have to choose $\omega$ closer and closer to $\pi$ and this amounts to require that $\Omega$ is an $(\varepsilon, r_0)-$Reifenberg flat domain for a smaller and smaller value of $\varepsilon$.

Difference between the projection of eigenfunctions. By relying on the analyis in [80] we get the following result. Note that we use the convention of identifying any given $u \in H_0^1(\Omega)$ with the function defined on the whole $\mathbb{R}^N$ by setting $u = 0$ outside $\Omega$.

**Proposition(5.2.15)[65]:** For any $\alpha \in ]0,1[$ there is threshold $\varepsilon(\alpha) \in ]0,1/2[$ such that the following holds. Let $\Omega_a$ and $\Omega_b$ be two $(\varepsilon, r_0)$-Reifenberg flat domains in $\mathbb{R}^N$, both contained in the disk $D$, which has radius $R$. If

$$d_H(\Omega_a^c, \Omega_b^c) \leq \delta \leq \frac{r_0}{8\sqrt{N}}, \tag{91}$$

then there is a constant $C = C(N, r_0, \alpha)$ such that, if $u \in H_0^1(\Omega_b)$ is a Dirichlet eigenfunction associated with the eigenvalue $\lambda$, then

$$\left\|\nabla u - \nabla P_{\Omega_a}^{\mathcal{D}}(u)\right\|_{L^2(D)}^2 \leq C\lambda\left(1 + \lambda^{\frac{N}{2}}\right)\delta^\alpha L \|u\|_{L^2(\Omega_b)}^2, \tag{92}$$

where $L := \mathcal{H}^{N-1}(\partial\Omega_b)$ and the projection $P_{\Omega_a}^{\mathcal{D}}$ is defined by (68) .

**Proof**. We proceed in two steps.

⋄ Step 1. We first fix $u \in H_0^1(\Omega_b)$ and construct $\tilde{u} \in H_0^1(\Omega_a)$ which is "close" to $u$, in the sense specified in the following.

To begin with, we point out that (91) implies that $\{B(x, 2\delta)\}_{x\in\partial\Omega_b}$ is a covering of $\Omega_b \backslash \Omega_a$. Indeed, by contradiction assume there is $y \in \Omega_b \backslash \Omega_a$ such that

$$2\delta < d(y, \partial\Omega_b) = d(y, \Omega_b^c) \leq \sup_{y\in\Omega_a^c} d(y, \Omega_b^c).$$

This would contradict (91) and hence the implication holds true.

By applying Lemma (5.2.6) with $r = 5\delta/2$ we can find a finite set $I$, such that $\#I \leq C(N)L/\delta^{N-1}$ and $\{B(x_i, 5\delta/2)\}_{i\in I}$ is a covering of $\Omega_b \backslash \Omega_a$.

Next, we use the function $\theta_0$ given by Lemma (5.2.7) (with $r = 5\delta/2$) and we set $\tilde{u}(x) := \theta_0(x)u(x)$. We observe that, since

$$\Omega_a^c \subseteq \Omega_b^c \cup (\Omega_b \backslash \Omega_a) \subseteq \Omega_b^c \cup \bigcup_{i\in I} B(x_i, 5\delta/2),$$

then $\tilde{u} \in H_0^1(\Omega_a)$. Also, $\nabla\tilde{u} = u\nabla\theta_0 + \theta_0\nabla u$ and hence

$$\|\nabla\tilde{u} - \nabla u\|_{L^2(D)} \leq \|(1 - \theta_0)\nabla u\|_{L^2(D)} + \|u\nabla\theta_0\|_{L^2(D)} \tag{93}$$

Next, by recalling that $\theta_0 \equiv 1$ outside the union of the balls $\{B(x_i, 5\delta)\}_{i\in I}$, we get

$$\|(1 - \theta_0)\nabla u\|_{L^2(D)}^2 = \int_{\cup_{i\in I} B(x_i, 5\delta)} (1 - \theta_0)^2 |\nabla u|^2 \, dx \leq \sum_{i\in I} \int_{B(x_i, 5\delta)} |\nabla u|^2 \, dx. \tag{94}$$

Also, by recalling that $|\nabla\theta_0|(x) \leq C(N)/\delta$, we obtain

$$\begin{aligned}\|u\nabla\theta_0\|_{L^2(D)}^2 &= \int_{\cup_i B(x_i, 5\delta)} |\nabla\theta_0|^2 u^2 \, dx \leq \frac{C(N)}{\delta^2} \sum_i \int_{B(x_i, 5\delta)} u^2 \, dx \\ &\leq \frac{C(N)}{\delta^2} \sum_i (5\delta)^2 \int_{B(x_i, 5\sqrt{N}\delta)} |\nabla u|^2 \, dx.\end{aligned} \tag{95}$$

To get the last inequality we have used [80, Proposition 12] and the fact that one can take $b(N) = \sqrt{N}$ in there.

⋄ Step 2. We now restrict to the case when $u$ is an eigenfunction for the Dirichlet Laplacian, and $\lambda$ is the associated eigenvalue. By using Proposition (5.2.14) and Lemma (5.2.6) we get

$$\sum_i \int_{B(x_i, 4\sqrt{N}\delta)} |\nabla u|^2 \, dx \leq \sum_i C\lambda\left(1 + \lambda^{\frac{N}{2}}\right)\delta^{N-\eta} \|u\|_2^2 \leq C\lambda\left(1 + \lambda^{\frac{N}{2}}\right)L\delta^{1-\eta}\|u\|_2^2, \tag{96}$$

with $C = C(N, r_0, \eta)$ and provided that $\delta \leq r_0/8\sqrt{N}$.

By choosing $\eta := (1-\alpha)$, inserting (96) into (94) and (95) and recalling (93) we finally get

$$\left\|\nabla u - \nabla P_{\Omega_a}^{\mathcal{D}}(u)\right\|_{L^2(D)}^2 \leq \|\nabla u - \nabla \tilde{u}\|_{L^2(D)}$$
$$\leq C(N, r_0, \alpha)\lambda(1 + \lambda^{\frac{N}{2}})L\delta^{\alpha}\|u\|_2^2.$$

Given an open and bounded set $\Omega$, $(u, \mu) \in H^1(\Omega) \setminus \{0\} \times \mathbb{R}$ is an eigencouple for the Neumann Laplacian in $\Omega$ if

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x)\, dx = \mu \int_{\Omega} u(x)\, v(x) dx \;\; \forall v \in H^1(\Omega). \tag{97}$$

We apply the abstract framework to study how the eigenvalues $\mu$ satisfying (97) depend on the domain $\Omega$. We set

$$H = L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R}^N)$$

and we equip it with the scalar product

$$\mathcal{H}((u_1, v_1), (u_2, v_2)) = \int_{\mathbb{R}^N} u_1(x) u_2(x) dx + \int_{\mathbb{R}^N} v_1(x) \cdot v_2(x)\; dx. \tag{98}$$

Also, we set

$$h((u_1, v_1), (u_2, v_2)) = \int_{\mathbb{R}^N} u_1(x) u_2(x) dx. \tag{99}$$

Note that $h$ is a symmetric, positive bilinear form (i.e., it satisfies properties of abstract framework), although it is not a scalar product on $H$. Inequality (43) is trivially satisfied.

As before, $\Omega_a$ and $\Omega_b$ are two Reifenberg-flat domains contained in $\mathbb{R}^N$ and we denote the Sobolev spaces by $H^1(\Omega_a)$ and $H^1(\Omega_b)$. The spaces $V_a$ and $V_b$ are defined by considering the map

$$j_{\Omega} : H^1(\Omega) \to L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N, \mathbb{R}^N)$$
$$u \to (u\mathbf{1}_{\Omega}, \nabla u \mathbf{1}_{\Omega}),$$

where $\mathbf{1}_{\Omega}$ denotes the characteristic function of $\Omega$. Note that the ranges $V_a = j_{\Omega_a}(H^1(\Omega_a))$ and $V_b = j_{\Omega_b}(H^1(\Omega_b))$ are closed and that (44) is satisfied. Note also that the inclusion (45) is compact because, in virtue of Proposition (5.2.8), we can apply Rellich's Theorem.

The Neumann problem (97) reduces to (46) provided that $\lambda = \mu + 1$ and hence it is solved by a sequence of eigencouples $(\mu_n, u_n)$ with $\lim_{n \to +\infty} \mu_n = +\infty$. By relying on Lemma (5.2.12), we deduce that to control the difference $|\mu_n^a - \mu_n^b|$ it is sufficient to provide an estimate on the projection operator defined by (50), which can be identified by a map $P_{\Omega_a}^{\mathcal{N}} : H \to H^1(\Omega_a)$ satisfying the following property: $\forall\, u \in H^1(\Omega_b)$,

$$\left\| j_{\Omega_b}(u) - P_{\Omega_a}^{\mathcal{N}}\, j_{\Omega_b}(u)\right\|_{\mathcal{H}}^2$$
$$= \left\|\mathbf{1}_{\Omega_b} u - \mathbf{1}_{\Omega_a} P_{\Omega_a}^{\mathcal{N}} u\right\|_{L^2(\mathbb{R}^N)}^2 + \left\|\mathbf{1}_{\Omega_b} \nabla u - \mathbf{1}_{\Omega_a} \nabla\left[P_{\Omega_a}^{N} u\right]\right\|_{L^2(\mathbb{R}^N)}^2$$
$$= \min_{v \in H^1(\Omega_a)} \left\{\left\|\mathbf{1}_{\Omega_b} u - \mathbf{1}_{\Omega_a} v\right\|_{L^2(\mathbb{R}^N)}^2 + \left\|\mathbf{1}_{\Omega_b} \nabla u - \mathbf{1}_{\Omega_a} \nabla v\right\|_{L^2(\mathbb{R}^N)}^2\right\}. \tag{100}$$

To provide an estimate on (100) we first establish a preliminary result concerning the decay of the gradient of a Neumann eigenfunction.

**Proposition(5.2.16)[65]:** For every $\eta > 0$ there is a positive constant $\varepsilon := \varepsilon(\eta)$ such that, for every connected, $(\varepsilon, r_0)$-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^N$, the following holds. Let $u$ be a Neumann eigenfunction in $\Omega$ associated with the eigenvalue $\mu$, let $x \in \partial\Omega$ and let $r \leq \min\{r_0, 1\}$. Then there is a constant $C = C(N, r_0, \eta, \mathrm{Diam}(\Omega))$ such that

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2 \; dx \leq C\mu(1 + \sqrt{\mu})^{2\gamma(N)} \, \|u\|^2_{L^2(\Omega)} \left(\frac{r}{\min\{r_0, 1\}}\right)^{N-\eta} \tag{101}$$

where $\gamma(N) = \max\left\{\frac{N}{2}, \frac{2}{N-1}\right\}$ o as in the statement of Proposition (5.2.9).

**Proof**. For a given $\eta > 0$ we choose $\beta$ in such a way that $0 < \beta < \eta$ and that

$$a := 8^{\frac{1}{\beta-\eta}} < \frac{1}{2}. \tag{102}$$

Also, we choose $\varepsilon$ smaller or equal to the constant given by Theorem (5.2.11) with this choice of $a$ and $\beta$. Note that $\varepsilon$ only depends on $\eta$.

We now consider a Neumann eigencouple $(u, \mu)$, while the point $x \in \partial\Omega$ is fixed. We may assume without losing generality that $r_0 \leq 1$, up to redefine $r_0$ by $\min\{1, r_0\}$.

We first use the induction principle in order to show that, for a suitable constant $C_4 = C_4(N, r_0, \mathrm{Diam}(\Omega))$ that will be chosen later, and for any $k \in \mathbb{N}$,

$$\int_{B(x,a^k r_0)\cap\Omega} |\nabla u|^2 \; dx \; \leq \; C_4\mu(1 + \sqrt{\mu})^{2\gamma(N)} \, a^{k(N-\eta)} \; \|u\|^2_{L^2(\Omega)}. \tag{103}$$

If $k = 0$ the inequality (103) is satisfied provided that $C_4 \geq 1$ because $\int_\Omega |\nabla u|^2 \, dx = \mu \, \|u\|_2^2$. Next, we consider the inductive step and we assume that (103) holds for a given $k \geq 0$. We term $v$ the solution of Problem (80) in $\Omega_k := B(x, a^k r_0) \cap \Omega$. Then Theorem (5.2.11) gives

$$\int_{\Omega_{k+1}} |\nabla v|^2 \; dx \; \leq \; a^{N-\beta} \int_{\Omega_k} |\nabla v|^2 \; dx. \tag{104}$$

Note that since $a < 1$, then $\Omega_{k+1} \subseteq \Omega_k$. We now compare $\nabla u$ and $\nabla v$ in $B(x, a^{k+1} r_0)$ by using the inequality $\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$:

$$\begin{aligned} \int_{\Omega_{k+1}} |\nabla u|^2 \; dx &\leq 2 \int_{\Omega_{k+1}} |\nabla u|^2 \, dx + 2 \int_{\Omega_{k+1}} |\nabla (u - v)|^2 \, dx \\ &\leq 2a^{N-\beta} \int_{\Omega_k} |\nabla v|^2 \; dx + 2 \int_{\Omega_k} |\nabla (u - v)|^2 \; dx \\ &\leq 2a^{N-\beta} \int_{\Omega_k} |\nabla u|^2 \; dx + 2 \int_{\Omega_k} |\nabla (u - v)|^2 \; dx. \end{aligned} \tag{105}$$

Since $v$ is harmonic and $u$ is a competitor, then $\nabla(u - v)$ is orthogonal to $\nabla v$ in $L^2(\Omega_k)$ and hence

$$\int_{\Omega_k} |\nabla (u - v)|^2 \, dx = \int_{\Omega_k} |\nabla u|^2 \, dx - \int_{\Omega_k} |\nabla v|^2 \, dx. \tag{106}$$

Moreover, $u$ minimizes the functional $w \to \int_{\Omega_k} |\nabla w|^2 \, dx - 2\mu \int_{\Omega_k} wu \, dx$ with its own Dirichlet conditions on $\partial B(x, a^k r_0) \cap \Omega$, and hence by taking $v$ as a competitor we obtain

$$\int_{\Omega_k} |\nabla u|^2\, dx \;-\; \int_{\Omega_k} |\nabla v|^2\, dx \;\le\; 2\mu \int_{\Omega_k} [u^2 - vu] \;\; dx \;\le\; 4\mu\omega_N a^{Nk} \, \|u\|_\infty^2. \qquad (107)$$

To get the above inequality we have used that $r_0 \le 1$ and the estimate

$$\|v\|_{L^\infty(\Omega)} \;\le\; \|u\|_{L^\infty(\Omega)}\,, \qquad (108)$$

which can be established arguing by contradiction. Indeed, set $M := \|u\|_{L^\infty(\Omega)}$. If (108) is violated, the truncated function $w := \min\{M, \max\{v, -M\}\}$ would be a competitor of $v$ satisfying $\|\nabla w\|^2_{L^2(\Omega)} < \|\nabla v\|^2_{L^2(\Omega)}$, which contradicts the definition of $v$.

By plugging (106) and (107) in (105) and using the inductive hypothesis (103), we get the estimate

$$\int_{\Omega_{k+1}} |\nabla u|^2 \;\le\; 2a^{N-\beta} C_4 \mu\big(1 \;+\; \sqrt{\mu}\big)^{2\gamma(N)}\; a^{k(N-\eta)} \|u\|_2^2 \;+\; 8\mu\omega_N a^{Nk} \|u\|^2_{L^\infty(\Omega)}\;,$$

where $\omega_N$ is the measure of the unit ball in $\mathbb{R}^N$. By using (37), the above expression reduces to

$$\int_{\Omega_{k+1}} |\nabla u|^2 \le \mu(1 + \sqrt{\mu})^{2\gamma(N)} a^{(k+1)(N-\eta)} \|u\|_2^2 \big[2C_4 a^{\eta-\beta} + C_5 a^{\eta(k+1)-N}\big]\,, \qquad (109)$$

for some constant $C_5 = C_5(N, r_0, \mathrm{Diam}(\Omega))$. We claim that by choosing in (103)

$$C_4 \;=\; 8C_5 a^{\eta-N} \qquad (110)$$

then the right hand side of (109) is less than $C_4\mu(1 + \sqrt{\mu})^{2\gamma(N)} a^{(k+1)(N-\eta)}$, which proves (103). Indeed, the choice of $a$ implies $a^{\eta-\beta} = 1/8$ and hence (110) implies

$$2C_4 a^{\eta-\beta} + C_5 a^{\eta(k+1)-N} \;=\; C_4 \frac{2}{8} + (8C_5 a^{\eta-N}) \frac{a^{k\eta}}{8} \le\; C_4 \frac{2}{8} + (8C_5 a^{\eta-N}) \frac{1}{8} \;=\; \frac{3}{8}\, C_4, \quad (111)$$

which concludes the proof of (103).

To conclude the proof of the Proposition we observe that, given $r \le\; r_0$, we can select an integer $k \ge 0$ such that $r < a^k r_0 \le r a^{-1}$, which yields

$$\begin{aligned}\int_{B(x,r)\cap\Omega} |\nabla u|^2 \;\; dx \;&\le\; \int_{B(x,a^k r_0)\cap\Omega} |\nabla u|^2 \;\; dx \\ &\le\; C_4 \mu\big(1 + \sqrt{\mu}\big)^{2\gamma(N)}\; a^{k(N-\eta)} \|u\|^2_{L^2(\Omega)} \\ &\le\; C_4 \mu(1 + \sqrt{\mu})^{2\gamma(N)} \|u\|^2_{L^2(\Omega)} \; \left(\frac{r}{a r_0}\right)^{N-\eta}\end{aligned}$$

and this implies (101) provided that $C := \; C_4 / a^{N-\eta}$.

By combining a covering argument from [81] with the previous proposition we establish the projection estimate provided by the following result.

**Proposition(5.2.17)[65]:** For any $\alpha \in\, ]0{,}1[$ there is a constant $\varepsilon = \varepsilon(\alpha) \le 1/600$ such that the following holds. Let $\Omega_a$ and $\Omega_b$ be two connected $(\delta, r_0)$-Reifenberg flat domains of $\mathbb{R}^N$ satisfying

$$\max\{d_H(\Omega_a, \Omega_b);\; d_H(\Omega_a^c, \Omega_b^c)\} \le\; \delta, \qquad (112)$$

where $0 < \delta \le \min\{r_0/5, 1\}$. Then there is a constant $C = C(N, r_0, \mathrm{Diam}(\Omega_b), \alpha)$ such that, if $u \in H^1(\Omega_b)$ is a Neumann eigenvector associated with the eigenvalue $\mu$, then there is $\tilde{u} \in H^1(\Omega_a)$ satisfying

$$\big\|\mathbf{1}_{\Omega_a}\tilde{u} - \mathbf{1}_{\Omega_b} u\big\|^2_{L^2(\mathbb{R}^N)} \;+\; \big\|\mathbf{1}_{\Omega_a}\nabla\tilde{u} - \mathbf{1}_{\Omega_b}\nabla u\big\|^2_{L^2(\mathbb{R}^N)} \;\le\; C(1 \;+\; \sqrt{\mu})^{2\gamma(N)+1}\, L\delta^\alpha \, \|u\|^2_{L^2(\Omega_b)}\,,$$

where $L := \mathcal{H}^{N-1}(\partial\Omega_b)$ and $\gamma(N) = \max\left\{\frac{N}{2}, \frac{2}{N-1}\right\}$ as in the statement of Proposition (5.2.9).

**Proof**. The goal of the first part of the proof is to construct a function $\tilde{u}$ which only differs from $u$ in a narrow strip close to the boundary: this is done by relying on a covering argument similar to those in [81]. The proof is then concluded by relying on Theorem (5.2.9) and Proposition (5.2.16). The details are organized in the following steps.

⋄ Step 1. We construct a partition of unity. First, we observe that (112) implies that $\{B(x, 2\delta)\}_{x\in\partial\Omega_b}$ is a covering of $\Omega_a \triangle \Omega_b$ and by relying on Lemma (5.2. 6) we can find a finite set $I$, such that (i) $\#I \leq C(N)L/\delta^{N-1}$; (ii) $\{B(x_i, 5\delta/2)\}_{i\in I}$ is a covering of $\Omega_a \triangle \Omega_b$. To simplify the exposition, in the following we use the notations

$$B_i := B\left(x_i, \frac{5\delta}{2}\right) \quad 2B_i := B(x_i, 5\delta) \quad 6B_i := B(x_i, 15\delta) \quad W = \bigcup_{i=1}^{\#I} 2B_i \, .$$

Note that property (33) in the statement of Lemma (5.2.6) implies that, for every $x \in \mathbb{R}^N$, $\#\{i : x \in 2B_i\} \leq C(N)$ and hence, in particular, that, for every integrable function $v$,

$$\sum_{i=1}^{\#I} \int_{2B_i} |v(x)| \, dx \leq C(N) \int_{\cup_i 2B_i} |v(x)| \, dx. \tag{113}$$

Next, we apply Lemma (5.2.7) with $r := 5\delta/2$ and we obtain Lipschtz continuous functions $\theta_0, \theta_1, \dots, \theta_{\#I} : \mathbb{R}^N \to [0,1]$ satisfying

$$\begin{cases} |\nabla\theta_i(x)| \leq \dfrac{C(N)}{\delta} \; a.e. \, x \in \mathbb{R}^N, \qquad i = 0, \dots, \#I \\ \theta_0(x) = 0 \text{ if } x \in \bigcup_{i\in I} B_i, \;\; \theta_0(x) = 1 \text{ if } x \in \mathbb{R}^N \setminus \bigcup_{i\in I} 2B_i \\ \theta_i(x) = 0 \text{ if } x \in \mathbb{R}^N \setminus 2B_i, \; i = 1, \dots, \#I, \; \sum_{i=0}^{\#I} \theta_i(x) = 1 \text{ for every } x \in \mathbb{R}^N. \end{cases} \tag{114}$$

⋄ Step 2. We define the function $\tilde{u}$. For $i \in I$ we term $Y_i$ the point in $\Omega_b \cap 2B_i$ such that $d(Y_i, x_i) = 3\delta$ and the vector $Y_i - x_i$ is orthogonal to $P(x_i, 5\delta)$. Note that such a point exists provided $5\delta \leq r$ and $\varepsilon \leq 3/10$ due to Lemma (5) in [63] (see also [20]). Then we define the domain $D_i := B(Y_i, \delta) \subseteq \Omega_b \cap 2B_i$ and we set

$$m_i := \frac{1}{|D_i|} \int_{D_i} u(x) \, dx \qquad i = 1, \dots, \#I.$$

Note that $\left(\mathbf{1}_{\Omega_b} u(x)\right) \theta_0(x)$ is well defined for $x \in \mathbb{R}^N$ and belongs to $H^1(\mathbb{R}^N)$ because $\theta_0(x) = 0$ in a neighborhood of $\partial\Omega_b$. This allows us to define

$$\tilde{u}(x) := \left(\mathbf{1}_{\Omega_b} u(x)\right) \theta_0(x) + \sum_{i=1}^{\#I} m_i \theta_i(x) \quad \forall x \in \mathbb{R}^N,$$

so that $\tilde{u} \in H^1(\Omega_a)$.

⋄ Step 3. We provide an estimate on $\left\|\tilde{u}\mathbf{1}_{\Omega_a} - u\mathbf{1}_{\Omega_b}\right\|^2_{L^2(\mathbb{R}^N)}$. First, we point out that $\Omega_a \triangle \Omega_B \subseteq W$ and that $\tilde{u}\mathbf{1}_{\Omega_b} = u\mathbf{1}_{\Omega_a}$ in $\mathbb{R}^N \setminus W$, so by recalling the definition of $\tilde{u}$ we have

$$\left\|\tilde u\mathbf{1}_{\Omega_a}-u\mathbf{1}_{\Omega_b}\right\|^2_{L^2(\mathbb{R}^N)} = \int_{\Omega_a\cap\Omega_b\cap W}|\tilde u-u|^2(x)\,dx+\int_{\Omega_b\setminus\Omega_a}u^2(x)\,dx+\int_{\Omega_a\setminus\Omega_b}\tilde u^2(x)\,dx$$

$$\leq 2\int_{W\cap\Omega_b}u^2(x)\,dx+2\int_W\left(\sum_{i=1}^{\#I}m_i\theta_i(x)\right)^2dx$$

By using the convexity of the square function, Jensen inequality and estimate (113) we have

$$\int_W\left(\sum_{i=1}^{\#I}m_i\theta_i(x)\right)^2dx \leq \int_W\sum_{i=1}^{\#I}m_i{}^2\theta_i(x)\,dx \leq \int_W\sum_{i=1}^{\#I}\frac{1}{|D_i|}\left(\int_{D_i}u^2(y)\,dy\right)\theta_i(x)\,dx$$

$$\leq \sum_{i=1}^{\#I}\frac{1}{\omega_N\delta^N}\left(\int_{D_i}u^2(y)\,dy\right)\int_{2B_i}\theta_i(x)dx \leq \sum_{i=1}^{\#I}\frac{1}{\omega_N\delta^N}\left(\int_{2B_i\cap\Omega_b}u^2(y)\,dy\right)\omega_N 5^N\delta^N$$

$$\leq C(N)\int_{\cup_i 2B_i\cap\Omega_b}u^2(y)\,dy.$$

In the previous expression, $\omega_N$ denotes as usual the Lebesgue measure of the unit ball in $\mathbb{R}^N$. By combining the previous two estimates we conclude that

$$\left\|\tilde u\mathbf{1}_{\Omega_a}-u\mathbf{1}_{\Omega_b}\right\|^2_{L^2(\mathbb{R}^N)} \leq C(N)\int_W u^2(y)\,dy \tag{115}$$

⋄ Step 4. We introduce some notations we need in Step 5. Given $i_0\in I$ we denote by $J_{i_0}$ be the finite set of indices $j\in I$ such that $2B_j\cap 2B_{i_0}\neq\emptyset$. Note that $\#J_{i_0}\leq C(N)$ by property (33) in the statement of Lemma (5.2.6) and, also, that any ball $2B_j$ is contained in $6B_{i_0}$ if $j\in J_{i_0}$.

Let $P_0$ be the hyperplane in $6B_{i_0}$ provided by the definition of Reifenberg flatness and let $\nu_0$ denote its unit normal vector, oriented in such a way that $x_{i_0}+15\delta\nu_0\in\Omega_b$. Also, let $Y_j$ and $D_j$ be as in Step 2, and let $P_j$ denote the hyperplane $P(x_j,5\delta)$. For any $j\in J_{i_0}$ we have

$$d_H\left(P_j\cap 2B_j,P_0\cap 2B_j\right) \leq d_H\left(P_j\cap 2B_j,\partial\Omega_b\cap 2B_j\right)+d_H\left(\partial\Omega_b\cap 2B_j,P_0\cap 2B_j\right)$$
$$\leq 5\delta\varepsilon+15\delta\varepsilon.$$

Hence,

$$d(Y_j,P_0)\geq d(Y_j,P_j)-d_H(P_j\cap 2B_j,P_0\cap 2B_j)\geq 3\delta-20\varepsilon\delta\geq 2\delta,$$

provided that $\varepsilon\leq 1/20$. This shows that

$$\bigcup_{j\in J_{i_0}}D_j\subseteq\widehat D_{i_0}:=B(x_{i_0},15\delta)\cap\{x\,;\,(x-x_{i_0})\cdot\nu_0\geq\delta\}\subseteq\Omega_b,$$

where the last inclusion holds by Lemma 5 in [63] (see also [20]) since $30\varepsilon\delta\leq\delta$.

The key point in this construction is that $\widehat D_{i_0}$ is a Lipschitz domain and satisfies the Poincar´e-Sobolev inequality with constant $C(N)\delta$. Hence, by setting

$$\widehat m_{i_0}:=\frac{1}{|\widehat D_{i_0}|}\int_{\widehat D_{i_0}}u(y)\,dy$$

and by using Jensen inequality we get that, for every $j\in J_{i_0}$, we have

$$|m_j - \widehat{m}_{i_0}|^2 = \left( \frac{1}{|D_j|} \int_{D_j} \left(u(x) - \widehat{m}_{i_0}\right) dx \right)^2 \leq \frac{1}{|D_j|} \int_{D_j} \left|u(x) - \widehat{m}_{i_0}\right|^2 dx$$

$$\leq \frac{1}{|D_j|} \int_{\widehat{D}_{i_0}} \left|u(x) - \widehat{m}_{i_0}\right|^2 dx \leq \frac{C(N)\delta^2}{|D_j|} \int_{\widehat{D}_{i_0}} |\nabla u(x)|^2 dx$$

$$\leq \frac{C(N)}{\delta^{N-2}} \int_{6B_{i_0} \cap \Omega_b} |\nabla u(x)|^2 \, dx. \tag{116}$$

On the other hand, by definition of $\widehat{D}_{i_0}$, we have that, for any $y \in 6B_{i_0} \cap \Omega_b \setminus \widehat{D}_{i_0}$,

$$d(y, \partial\Omega_b) \leq d(y, P_0) + d_H(P_0 \cap 6B_{i_0}, \partial\Omega_b \cap 6B_{i_0}) \leq \delta + 6\varepsilon\delta \leq 2\delta,$$

which implies that $y \in \bigcup_{x \in \partial\Omega_b} B(x, 2\delta)$ . In particular,

$$2B_{i_0} \cap \Omega_b \setminus \widehat{D}_{i_0} \subseteq \bigcup_{x \in \partial\Omega_b} B(x, 2\delta) \subseteq \bigcup_{i \in I} B_i$$

and this implies that $\operatorname{supp}(\theta_0) \cap 2B_{i_0} \subseteq \widehat{D}_{i_0}$ .

⋄ Step 5. We provide an estimate on $\left\|\mathbf{1}_{\Omega_a} \nabla\tilde{u} - \mathbf{1}_{\Omega_b} \nabla u\right\|_{L^2(\mathbb{R}^N)}$. First, we recall that $\Omega_a \triangle \Omega_b \subseteq W$ and we observe that

$$\left\|\mathbf{1}_{\Omega_a} \nabla\tilde{u} - \mathbf{1}_{\Omega_b} \nabla u\right\|^2_{L^2(\mathbb{R}^N)}$$

$$= \int_{\Omega_a \cap \Omega_b \cap W} |\nabla\tilde{u} - \nabla u|^2(x)\, dx + \int_{\Omega_a \setminus \Omega_b} |\nabla\tilde{u}|^2(x)\, dx + \int_{\Omega_b \setminus \Omega_a} |\nabla u|^2(x)\, dx$$

$$\leq 2 \int_{W \cap \Omega_b} |\nabla u|^2(x)\, dx + 2 \int_{\Omega_a \cap \Omega_b \cap W} |\nabla\tilde{u}|^2(x)\, dx + \int_{\Omega_a \setminus \Omega_b} |\nabla\tilde{u}|^2(x)\, dx. \tag{117}$$

The first term in the last line of the above expression satisfies

$$2 \int_{W \cap \Omega_b} |\nabla u|^2(x)\, dx \leq \sum_{i \in I} \int_{B_i \cap \Omega_b} |\nabla u|^2(x)\, dx$$

$$\leq \sum_{i \in I} \int_{6B_i \cap \Omega_b} |\nabla u|^2(x)\, dx. \tag{118}$$

To establish an estimate on the second term in (117), we start by observing that, if $x \in \Omega_b$, then

$$\tilde{u}(x) := u(x)\theta_0(x) + \sum_{j \in I} m_i \theta_j(x) \quad .$$

Next, we fix $i_0$ in such a way that $x \in B_{i_0}$ and we observe that, since $\nabla\theta_0 + \sum_{j \in I} \nabla\theta_j = 0$, we have

$$\nabla \tilde{u}(x) = \theta_0(x)\nabla u(x) + \nabla\theta_0(x)u(x) + \sum_{j \in J_{i_0}} m_j \nabla\theta_j(x)$$
$$= \theta_0(x)\nabla u(x) + \underbrace{\left(u(x) - \widehat{m}_{i_0}\right)\nabla\theta_0(x)}_{f_1} + \underbrace{\sum_{j \in J_{i_0}} \left(m_j - \widehat{m}_{i_0}\right)\nabla\theta_j(x)}_{f_2} .$$

Next, we point out that, for any $i = 1, \ldots, \#I$ we have

$$\int_{2B_i \cap \Omega_b} |\theta_0(x)\nabla u(x)|^2 \, dx \leq \int_{2B_i \cap \Omega_b} |\nabla u(x)|^2 \, dx. \tag{119}$$

Also, we recall that $\operatorname{supp}(\theta_0) \cap 2B_{i_0} \subseteq \widehat{D}_{i_0}$ and that $|\nabla\theta_{i_0}| \leq C(N)/\delta$. We then recall that the Poicar´e-Sobolev constant of $\widehat{D}_{i_0}$ is bounded by $C(N)\delta$ and by combining these observations we get

$$\int_{2B_{i_0} \cap \Omega_b} |f_1(x)|^2 \leq \frac{C(N)}{\delta^2} \int_{\widehat{D}_{i_0}} \left|u(x) - \widehat{m}_{i_0}\right|^2 \leq C(N) \int_{\widehat{D}_{i_0}} |\nabla u|^2 \leq C(N) \int_{6B_{i_0} \cap \Omega_b} |\nabla u|^2 . \tag{120}$$

Finally, by recalling (116) we have

$$\int_{2B_{i_0} \cap \Omega_b} |f_2|^2 \, dx \leq C(N) \int_{2B_{i_0} \cap \Omega_b} \sum_{j \in J_{i_0}} \frac{1}{\delta^2} \left(m_j - \widehat{m}_{i_0}\right)^2 \leq C(N) \int_{6B_{i_0} \cap \Omega_b} |\nabla u|^2 \, dx \tag{121}$$

and by combining (119), (120) and (121) we infer

$$\int_{\Omega_a \cap \Omega_b \cap W} |\nabla \tilde{u}|^2(x) \, dx \leq C(N) \sum_{i \in I} \int_{6B_i \cap \Omega_b} |\nabla u|^2 \, dx. \tag{122}$$

To provide a bound on the third term in (117), we observe that, if $x \in \Omega_a \backslash \Omega_b \subseteq \bigcup_{i \in I} B_i$, then

$$\tilde{u}(x) = \sum_{i \in I} m_i \theta_i(x) \qquad \theta_0(x) = 0.$$

Hence, if we choose $i_0$ in such a way that $x \in B_{i_0}$, we get

$$\nabla \tilde{u}(x) = \sum_{j \in J_{i_0}} m_j \nabla\theta_j(x) = \sum_{j \in J_{i_0}} \left(m_j - \widehat{m}_{i_0}\right)\nabla\theta_j(x) .$$

By arguing as in (121) we get

$$\int_{B_{i_0} \backslash \Omega_b} |\nabla \tilde{u}|^2 \, dx \leq C(N) \int_{6B_{i_0} \cap \Omega_b} |\nabla u|^2 \, dx,$$

which implies

$$\int_{\Omega_a \backslash \Omega_b} |\nabla \tilde{u}|^2 (x) dx \leq \sum_{i \in I} \int_{B_i \backslash \Omega_b} |\nabla \tilde{u}|^2 \, dx \leq C(N) \sum_{i \in I} \int_{6B_i \backslash \Omega_b} |\nabla u|^2 \, dx. \tag{123}$$

Finally, by combining (117), (119), (122) and (123) we conclude that

$$\left\| \mathbf{1}_{\Omega_a} \nabla \tilde{u} - \mathbf{1}_{\Omega_b} \nabla u \right\|^2_{L^2(\mathbb{R}^N)} \leq C(N) \sum_{i \in I} \int_{6B_i \backslash \Omega_b} |\nabla u|^2 \, dx. \tag{124}$$

⋄ Step 6. We conclude the proof of the Proposition by relying on Propositions (5.2.9) and (5.2.16). First, by combining Proposition (5.2.9) and (115) we get

$$\left\|\tilde{u}\mathbf{1}_{\Omega_a} - u\mathbf{1}_{\Omega_b}\right\|^2_{L^2(\mathbb{R}^N)} \leq C(N)\sum_{i\in I}\int_{2B_i\ \setminus\Omega_b} u^2(x)\,dx \ \leq\ C(N)\sum_{i\in I}\delta^N\ \|u\|^2_{L^\infty(\Omega_b)}$$
$$\leq\ C\big(N, r_0, \mathrm{Diam}(\Omega_b)\big)\delta^N\big(1+\sqrt{\mu}\big)^{2\gamma(N)}\ \|u\|^2_{L^2(\Omega_b)}\ \#I$$
$$\leq\ C\big(N, r_0, \mathrm{Diam}(\Omega_b)\big)L\big(1+\sqrt{\mu}\big)^{2\gamma(N)}\ \delta\|u\|^2_{L^2(\Omega_b)}\ . \tag{125}$$

Next, we combine Proposition (5.2.16) with (124) and we obtain

$$\left\|\mathbf{1}_{\Omega_a}\nabla\tilde{u} - \mathbf{1}_{\Omega_b}\nabla u\right\|^2_{L^2(\mathbb{R}^N)}\ \leq\ C\mu\big(1+\sqrt{\mu}\big)^{2\gamma}\delta^{N-\eta}\|u\|^2_{L^2(\Omega_b)}\ \#I$$
$$\leq\ CL\mu\big(1+\sqrt{\mu}\big)^{2\gamma}\delta^{\alpha}\|u\|^2_{L^2(\Omega_b)}\ , \tag{126}$$

provided that $\alpha = 1-\eta$. In the previous expression, $C = C\big(N, r_0, \alpha, \mathrm{Diam}(\Omega_b)\big)$.
By combining (125) and (126) we conclude the proof.