Chapter 4

Integration and Loci of Integrability with Lebesgue Classes

In this chapter we define the Radon transform and show the corresponding inversion formula. We generalize the main result about the stability under integration of the class of constructible functions, by relaxing the conditions on integrability. Further, we give an interpolation result for constructible functions by constructible functions with maximal locus of integrability. For any $q > 0$ and constructible functions f and μ on $E \times \mathbb{R}^n$, we show a theorem describing the structure of the set

$$\{(x, p) \in E \times (0, \infty] : f(x,\cdot) \in L^p(|\mu|_x^q)\},$$

where $|\mu|_x^q$ is the positive measure on $\mathbb{R}^n$ whose Radon-Nikodym derivative with respect to the Lebesgue measure is $|\mu(x,\cdot)|^q : y \to |\mu(x, y)|^q$.

Sec(4.1): Positive Constructible Functions Against Euler Characteristic and Dimension

By a subanalytic set we will always mean a globally subanalytic subset $X \subset \mathbb{R}^n$, meaning that X is subanalytic in the classical sense inside $\mathbb{P}^n(\mathbb{R})$under the embeding$\mathbb{R}^n = \mathbb{A}^n(\mathbb{R}) \subseteq \mathbb{P}^n(\mathbb{R})$. By a subanalytic function we mean a function whose graph is a (globally) sub-analytic set.

By Sub we denote the category of subanalytic subsets $X \subset \mathbb{R}^n$ for all $n > 0$, with sub-analytic maps as morphisms. We work with the Euler characteristic $\chi : Sub \to \mathbb{Z}$ and the dimension $\mathrm{dim}: \mathrm{Sub} \to \mathbb{N}$ of subanalytic sets as defined for o-minimal structures in [173].
Note that if $X \in \mathrm{Sub}$, then, by the o-minimal triangulation theorem in [173], the o-minimal Euler characteristic $\chi(X)$ coincides with the Euler characteristic $\chi_{BM}(X)$ of X with respect to the Borel−Moore homology. If $X \in \mathrm{Sub}$ is locally compact, the o−minimal Euler characteristic $\chi(X)$ coincides with the Euler characteristic $\chi_c(X)$ of X with respect to sheaf cohomology of X with compact supports and constant coefficient sheaf.

By [173], the Euler characteristic χ: $\mathrm{Sub} \to \mathbb{Z}$ satisfies the following:

$$\chi(\emptyset) = 0,$$

$$\chi(X) = \chi(Y) \text{ if } X \text{ and } Y \text{ are isomorphic in Sub}$$

and

$$\chi(X \cup Y) = \chi(X) + \chi(Y)$$

whenever $X, Y \in \mathrm{Sub}$ are disjoint. The last equality for χ_{BM} and χ_c follows from the long exact (co)homology sequence. If we take X to be the unit circle in the plane $\mathbb{R}^2$ and Y a point in X, we see that this equality does not hold for the Euler characteristic associated with the topological singular (co)homology.

Thus we can think of χ: $\mathrm{Sub} \to \mathbb{Z}$ as a measure with values in the Grothendieck ring $K_0(\mathrm{Sub})$ of the category Sub and, for any $X \in \mathrm{Sub}$ and any function $f : X \to \mathbb{Z}$ with finite range and the property that $f^{-1}(a) \in \mathrm{Sub}$ for all $a \in \mathbb{Z}$ (constructible functions), one has an obvious definition for

$$\int_X f\chi$$

such that $\chi(X) = \int_X 1_X\chi$ (cf. [182]).

This measure and integration against Euler characteristic is what is considered by Viro [182], Shapira [180,181] and Bröcker [166]. However, for the measure $\chi : \mathrm{Sub} \to \mathbb{Z}$ it is not true that $\chi(X) = \chi(Y)$ if and only if X and Y are isomorphic in Sub. We construct the universal measure μ for the category Sub with values in the Grothendieck semi−ring $SK_0(\mathrm{Sub})$ of Sub such that $\mu(X) = \mu(Y)$ if and only if X and Y are isomorphic in Sub. Furthermore, we develop a direct image formalism for positive constructible functions, i.e., functions $f : X \longrightarrow SK_0(\mathrm{Sub})$ with finite range and the property that $f^{-1}(a) \in \mathrm{Sub}$ for all

$a \in SK_0(\mathrm{Sub})$. This formalism is generalized to arbitrary first−order logic models and is illustrated by several examples on the p-adics, on the Presburger structure and on o-minimal expansions of groups. Moreover, within this formalism, we define the Radon transform and show the corresponding inversion formula.

We start by pointing out that, instead of Sub, we can work with any o-minimal expansion of a field R using the category Def whose objects are definable sets and whose morphisms are definable maps.

By a semi-group we mean a commutative monoid with a unit element. Likewise, a semi-ring is a set equipped with two semi-group structures: addition and multiplication such that 0 is a unit element for the addition, 1 is the unit element for multiplication, and the two operations are connected by $x(y+z) = xy + xz$ and $0x = 0$. A morphism of semi-rings is a mapping compatible with the unit elements and the operations.

Let $A := \mathbb{Z} \times \mathbb{N}$ be the semi-ring where addition is given by $(a, b) + (a', b') = (a + a', \max(b, b'))$, the additive unit element is $(0, 0)$, multiplication is given by $(a, b)(a', b') = (aa', b + b')$, and the multiplicative unit is $(1, 0)$. Note that the ring generated by A by inverting additively any element of A is $\mathbb{Z}$ with the usual ring structure.

For $Z \in \mathrm{Sub}$, we define $C_+(Z)$ as the semi-ring of functions $Z \to A$ with finite image and whose fibers are subanalytic sets. We call $C_+(Z)$ the semi-ring of positive constructible functions on Z. In particular, $C_+(\{0\}) = A$.

If $Z \in \mathrm{Sub}$, then we denote by Sub_Z the category of subanalytic maps $X \to Z$ for $X \in \mathrm{Sub}$ with morphisms subanalytic maps that make the obvious diagrams commute. We define the Grothendieck semi-group $SK_0(\mathrm{Sub}_Z)$ as the quotient of the free abelian semi-group over symbols $[Y \to Z]$ with $Y \to Z$ in Sub_Z by relations

$$[\emptyset \to Z] = 0, \tag{1}$$

$$[Y \to Z] = [Y' \to Z] \tag{2}$$

If $Y \to Z$ is isomorphic to $Y' \to Z$ in Sub_Z and

$$[(Y \cup Y') \to Z] + [(Y \cap Y') \to Z] = [Y \to Z] + [Y' \to Z] \tag{3}$$

for Y and Y' subsets of some $X \to Z$. There is a natural semi-ring structure on $SK_0(\mathrm{Sub}_Z)$ where the multiplication is induced by taking fiber products over Z.

We write $SK_0(\mathrm{Sub}_Z)$ for $SK_0(\mathrm{Sub}_{\{0\}})$ and $[X]$ for $[X \to \{0\}]$. Note that any element of $SK_0(\mathrm{Sub}_Z)$ can be written as $[X \to Z]$ for some $X \in \mathrm{Sub}_Z$, because we can take disjoint unions in Sub corresponding to finite sums in $SK_0(\mathrm{Sub}_Z)$.

Proposition(4.1.1)[165]: For $Z \in \mathrm{Sub}$, there is a natural isomorphism of semi-rings

$$T : SK_0(\mathrm{Sub}_Z) \to C_+(Z)$$

induced by sending $[X \to Z]$ in Sub_Z to $Z \to A :\mapsto (\chi(X_z), \dim(X_z))$, where X_z is the fiber above z. By consequence, $SK_0(\mathrm{Sub}) = A$.

Proof: This follows immediately from the trivialisation property for definable maps in any o-minimal expansion of a field. See [173].

By means of this result, we may identify $SK_0(\mathrm{Sub}_Z)$ and $C_+(Z)$.

A general notion of positive measures on a Boolean algebra $\mathcal{S}$ of sets is a map $\mu : \mathcal{S} \to G$ with G a semi-group satisfying

$$\mu(X \cup Y) = \mu(X) + \mu(Y)$$

and

$$\mu(\emptyset) = 0$$

whenever $X, Y \in \mathcal{S}$ are disjoint. Often, one has a notion of isomorphisms between sets in $\mathcal{S}$ under which the measure should be invariant and which allows one to take disjoint unions of given sets in $\mathcal{S}$ (by taking disjoint isomorphic copies of the sets).

We let $\mu : \mathrm{Sub} \to A$ be the positive measure which sends X to $(\chi(X), \dim(X))$. This measure is a universal measure on Sub with the property that $\mu(X) = \mu(Y)$ whenever there exists a subanalytic bijection between X and Y and where universal means that any other positive measure with this property factorises through μ.

Note that μ measures, in some sense, the topological size since, by the cell decomposition theorem from [173], $\mu(A) = \mu(B)$ will hold for two subanalytic sets A, B if and only if, for any fixed $n \geq 0$, there exists a finite partition of A, resp. B, into subanalytic C^n-manifolds $\{A_i\}_{i=1}^m$, resp. $\{B_i\}_{i=1}^m$, and subanalytic maps $A_i \to B_i$ which are isomorphisms of C^n-manifolds.

Now we can define the integral of any positive function $f \in C_+(Z)$ as

$$\int_Z f\mu := \sum_i f_i\, \mu(Z_i)$$

where $\{Z_i\}$ is any finite partition of Z into subanalytic sets such that f is constant on each part Z_i with value f_i. To show that this is independent of the partition $\{Z_i\}$, we just note that there is a unique $[X \to Z]$ in $SK_0(Z)$ which corresponds to f under T and that $\sum_i f_i\, \mu(Z_i)$ corresponds to $[X] = (\chi(X), \dim(X))$ in $A = SK_0(\mathrm{Sub})$. This independence follows also from the cell decomposition theorem ([173]).

For $f: X \to Y$, there is an immediate notion of pushforward $f_! : C_+(X) \to C_+(Y)$, resp. $f_! : SK_0(\mathrm{Sub}_X) \to SK_0(\mathrm{Sub}_Y)$, which is given by

$$f_!\,(g)(y) = \int_{f^{-1}(y)} g_{|\,f^{-1}(y)}\, \mu$$

for $g \in C_+(X)$, resp. by

$$f_!\,([Z \to X]) = [Z \to Y],$$

for $Z \to X$ in Sub_X and where $Z \to Y$ is given by composition with $X \to Y$. Note that these pushforwards are compatible with T.

If $Y = \{0\}$, then $SK_0(\mathrm{Sub}_Y) = A$ and we write $\mu([Z \to X])$ for $f_!\,([Z \to X])$ which is the integral of $[Z \to X]$. Thus the functoriality condition $(h \circ f)_! = h_! \circ f_!$ can be interpreted as Fubini's Theorem, since

$$\int_X g\mu = \int_Y \left(\int_{f^{-1}(y)} g_{|\,f^{-1}(y)}\, \mu\right)\mu$$

for $g \in C_+(X)$ and $h : Y \to \{0\}$.

For $f: X \to Y$ a morphism in Sub, there is an immediate notion of pullback $f^*: C_+(Y) \to C_+(X)$, resp. $f^* : SK_0(\mathrm{Sub}_Y) \to SK_0(\mathrm{Sub}_X)$, which is given by

$$f^*\,(g) = g \circ f$$

for $g \in C_+(Y)$, resp. by

$$f^*\,([Z \to Y]) = [Z \otimes_Y X \to X],$$

for $Z \to Y$ in Sub_Y and where $Z \otimes_Y X \to X$ is the projection and $Z \otimes_Y X$ is the set-theoretical fiber product. Note that these pullbacks are also compatible with T and satisfy the functoriality property $(f \circ h)^* = h^* \circ f^*$.

Proposition(4.1.2)[165]: (Projection Formula). Let $f : X \to Y$ be a morphism in Sub and let g be in $C_+(X)$ and h in $C_+(Y)$. Then

$$f_!\left(gf^*\,(h)\right) = f_!\,(g)h.$$

Proof. This is immediate at the level of SK_0, since both the multiplication in SK_0 and the pullback are defined by the fiber product.

Let $S \subset X \times Y$, X, Y be subanalytic sets and write $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ for the projections and $q_X = \pi_{X|\,S}$ and $q_Y = \pi_{Y|\,S}$. For $g \in C_+(X)$, we define the Radon transform $\mathcal{R}_S(g) \in C_+(Y)$ by

$$\mathcal{R}_S(g) = q_{Y!} \circ q_X^*(g) = \pi_{Y!} \circ (\pi_X^*(g)1_S)$$

where 1_S is the characteristic function on S.

Example(4.1.3)[165]: Consider the case $X = \mathbb{R}^n, Y = \mathrm{Gr}(n)$ with $S = \{(p, \Pi): p \in \Pi\}$. Let $Z \subseteq \mathbb{R}^n$ be a subanalytic subset and $\sigma_Z : \mathrm{Gr}(n) \to A : \Pi \mapsto (\chi(\Pi \cap Z), \dim(\Pi \cap Z))$. Then $\sigma_Z = \mathcal{R}_S(1_Z)$.

Let $S' \subset Y \times X$ be another subanalytic set and put $q'_X = \pi_{X|S'}$ and $q'_Y = \pi_{Y|S'}$.

Proposition(4.1.4)[165]: (Inversion Formula). Let $r : S \otimes_Y S' \to X \times X$ be the projection and suppose that the following hypotheses hold:

(I) there exists $\lambda \in A$ such that $[r^{-1}(x, x')] = \lambda$ for all $x \neq x', x, x' \in X$;

(II) there exists $0 \neq \theta \in A$ such that $[r^{-1}(x, x)] = \theta + \lambda$ for all $x \in X$.

If g is in $C_+(X)$, then

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = \theta g + \lambda \int_X g\mu \tag{4}$$

and this is independent of the choice of θ.

Proof. Let h and h' be the projections from $S \otimes_Y S'$ to S and S', respectively. Then, by definition of fiber product, $q_Y \circ h = q'_Y \circ h'$, and so, by functoriality of pullback and push-forward, we have $h'_! \circ h^* = q'^*_Y \circ q_{Y!}$. Thus $\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = q'_{X!} \circ (q'_Y)^* \circ q_{Y!} \circ q_X^*(g) = q'_{X!} \circ h'_! \circ h^* \circ q_X^*(g)$.

The last formula is also equal to $p_{2!} \circ r_! \circ r^* \circ p_1^*(g)$, where $p_1, p_2: X \times X \to X$ are the projections onto the first and second coordinates respectively, since $q_X \circ h = p_1 \circ r$ and $q'_X \circ h' = p_2 \circ r$. The hypothesis shows that $r_!(1_{S\otimes_Y S'}) = \theta 1_{\Delta_X} + \lambda 1_{X\times X}$, more-over this expression is independent of the choice of θ. By the projection formula, $r_!\left(r^*\left(p_1^*(g)\right)\right) = r_!\left(1_{S\otimes_Y S'} r^*\left(p_1^*(g)\right)\right) = r_!\left(1_{S\otimes_Y S'}\right)p_1^*(g) = \left(\theta 1_{\Delta_X} + \lambda 1_{X\times X}\right)p_1^*(g)$ holds, hence we obtain $p_{2!}((\theta 1_{\Delta_X} + \lambda 1_{X\times X})\, p_1^*(g)) = \theta p_{2!}(1_{\Delta_X}\, p_1^*(g)) + \lambda p_{2!}(p_1^*(g)) = \theta g + \lambda \int_X g\mu$, as required.

We now show that the inversion formula is independent of the choice of θ. If $\theta + \lambda = \theta' + \lambda$ and $\theta \neq \theta'$, then necessarily $\lambda_2 > \theta_2$, $\lambda_2 > \theta'_2$ and $\theta_1 = \theta'_1$ with $\lambda = (\lambda_1, \lambda_2), \theta = (\theta_1, \theta_2)$ and $\theta' = (\theta'_1, \theta'_2)$. Hence, $\theta g + \lambda \int_X g\mu = \theta' g + \lambda \int_X g\mu$ for all $x \in X$.

Example(4.1.5)[165]: Consider the case $X = \mathbb{R}^n$, $Y = \mathrm{Gr}(n)$ with $S = \{(p, \Pi) : p \in \Pi\}$ and $S' = \{(\Pi, p): p \in \Pi\}$. Then $[r^{-1}[(x, x)] = [\mathbb{P}^{n-1}]$ and $[r^{-1}(x, x')] = [\mathbb{P}^{n-2}]$ for all $x, x' \in \mathbb{R}^n$ with $x \neq x'$. Since $[\mathbb{P}^n] = \left(\frac{1+(-1)^n}{2}, n\right)$, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = ((-1)^{n+1}, n-1)g + \left(\frac{1+(-1)^n}{2}, n-2\right)\int_X g\mu.$$

In particular, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(1_Z) = ((-1)^{n+1}, n-1)1_Z + \left(\frac{1+(-1)^n}{2}, n-2\right)[Z]$$

for every subanalytic subset Z of $\mathbb{R}^n$.

Let $\mathcal{M}$ be a model of a theory in a language $\mathcal{L}$ with at least two constant symbols c_1, c_2 satisfying $c_1, \neq c_2$. For Z a definable set, we define the category $\mathrm{Def}_Z(\mathcal{M})$, also written Def_Z for short, whose objects are definable sets X with a definable map $X \to Z$ and whose morphisms are definable maps that make the obvious diagram commute. We write $\mathrm{Def}(\mathcal{M})$ or Def for $\mathrm{Def}_{\{c_1\}}(\mathcal{M})$. In $\mathcal{M}$, one can pursue the usual operations of set theory like finite unions, intersections, Cartesian products, disjoint unions and fiber products.

We define the Grothendieck semi-group $SK_0(\mathrm{Def}_Z)$ as the quotient of the free abelian semi-group over symbols $[Y \to Z]$ with $Y \to Z$ in Def_Z by relations

$$[\emptyset \to Z] = 0, \tag{5}$$
$$[Y \to Z] = [Y' \to Z] \tag{6}$$
if $Y \to Z$ is isomorphic to $Y' \to Z$ in Def_Z and
$$[(Y \cup Y') \to Z] + [(Y \cap Y') \to Z] = [Y \to Z] + [Y' \to Z] \tag{7}$$
for Y and Y' subsets of some $X \to Z$. There is a natural semi-ring structure on $SK_0(\mathrm{Def}_Z)$ where the multiplication is induced by taking fiber products over Z. Note that any element of $SK_0(\mathrm{Def}_Z)$ can be written as $[X \to Z]$ for some $X \to Z \in \mathrm{Def}_Z$, because we can take disjoint unions in $\mathcal{M}$ corresponding to finite sums in $SK_0(\mathrm{Def}_Z)$.

The map $\mathrm{Def} \to SK_0(\mathrm{Def})$ sending X to its class $[X]$ is a universal positive measure with the property that two sets have the same measure if there exists a definable bijection between them. For $f : X \to Y$, there is an immediate notion of push forward $f_! : SK_0(\mathrm{Def}_X) \to SK_0(\mathrm{Def}_Y)$ given by
$$f_!([Z \to X]) = [Z \to Y],$$
for $Z \to X$ in Def_X and where $Z \to Y$ is given by composition with $X \to Y$.

If $Y = \{c_1\}$, then we write $\mu([Z \to X])$ for $f_!([Z \to X])$, which we call the integral of $[Z \to X]$; note that $\mu([Z \to X])$ is just $[Z]$ in $SK_0(\mathrm{Def})$. Thus the functoriality condition $(f \circ h)_! = f_! \circ h_!$ can be interpreted as Fubini's Theorem.

There is also an immediate notion of pullback $f^* : SK_0(\mathrm{Def}_Y) \to SK_0(\mathrm{Def}_X)$ given by
$$f^*([Z \to Y]) = [Z \otimes_Y X \to X],$$
for $Z \to Y$ in Def_Y and where $Z \otimes_Y X \to X$ is the projection and $Z \otimes_Y X$ the set-theoretical fiber product. The pullback is functorial, i.e., $(foh)^* = h^* o f^*$.

Proposition(4.1.6)[165]: (Projection Formula). Let $f : X \to Y$ be a morphism in Def and let g be in $SK_0(\mathrm{Def}_X)$ and h in $SK_0(\mathrm{Def}_Y)$. Then
$$f_!\left(gf^*(h)\right) = f_!(g)h.$$
Proof. Exactly the same proof as for the subanalytic sets above works.

One can also define the Radon transform in this context in exactly the same way as in the subanalytic case. Furthermore, the same argument as in the subanalytic case gives the corresponding inversion formula.

However, since, in general, there is no trivialisation theorem, the conditions (I) and (II) in Proposition(4.1.4) have to be replaced by global conditions. Using the embedding $SK_0(\mathrm{Def}) \to SK_0(\mathrm{Def}_U)$ sending $[W]$ to $[W \times U \to U]$ where $W \times U \to U$ is the projection, the statement becomes:

Let $r : S \otimes_Y S' \to X \times X$ be the projection and suppose that the following hypotheses hold:

(i) there exists Z_1 in Def such that in $SK_0(\mathrm{Def}_{X_1})$ we have
$$[B_1 \to X_1] = [Z_1],$$
(ii) there exists Z_2 in Def such that in $SK_0(\mathrm{Def}_{\Delta_X})$ we have
$$[B_2 \to \Delta_X] = [Z_1] + [Z_2]$$
where $X_1 = X \times X \setminus \Delta_X$, $B_1 = S \otimes_Y S' \setminus r^{-1}(\Delta_X)$, $B_2 = S \otimes_Y S' \cap r^{-1}(\Delta_X)$ and $B_1 \to X_1$ and $B_2 \to \Delta_X$ are the restrictions of the projection $r : S \otimes_Y S' \to X \times X'$.

If $Z \to X$ is in Def_X, then
$$\mathcal{R}_{S'} \mathrm{o}\, \mathcal{R}_S([Z \to X]) = [Z_2][Z \to X] + [Z_1][Z] \tag{8}$$
and this is independent of the choice of Z_2.

Example(4.1.7)[165]: (Semialgebraic and subanalytic sets in $\mathbb{Q}_p$)

For K any finite field extension of the field $\mathbb{Q}_p$ of p-adic numbers, one can calculate explicitly the semi-ring of semialgebraic sets $SK_0(K, \mathrm{Sem})$, resp. of globally subanalytic sets $SK_0(K, \mathrm{Sub})$, using work of [167] for semialgebraic sets, resp. using work of [169] for the subanalytic sets. In both cases it is a subset of $\mathbb{N} \times \mathbb{N}$, and the class of a semialgebraic set X, resp. a subanalytic set X, is $(\#X, 0)$ if X is finite and $(0, \dim X)$ if X is infinite. This is because there exists a semialgebraic bijection between two infinite semialgebraic sets if and only if they have the same dimension, and similarly for subanalytic sets. However, no trivialisation theorem is known, hence the relative semi-Grothendieck rings $SK_0(K, \mathrm{Sem}_Z)$, resp. $SK_0(K, \mathrm{Sub}_Z)$, for Z semialgebraic, resp. subanalytic, are expected to be much more complicated than maps $Z \to \mathbb{N} \times \mathbb{N}$ with finite image.

Example(4.1.8)[165]: (Presburger sets)

Consider the Presburger structure on $\mathbb{Z}$ by using the Presburger language

$$\mathcal{L}_{\mathrm{PR}} = \{+, -, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \mathbb{N}, n > 1\},$$

with $\equiv_n$ the equivalence relation modulo n. Again, one can can calculate explicitly the semi ring $SK_0(\mathbb{Z}, \mathcal{L}_{\mathrm{PR}})$, using work of [168].

It is a subset of $\mathbb{N} \times \mathbb{N}$, and the class of a Presburger set X is $(\#X, 0)$ if X is finite and $(0, \dim X)$ if X is infinite, where the dimension of [168] is used. Again, this is because there exists a Presburger bijection between two infinite Presburger sets if and only if they have the same dimension.

Again, no trivialisation theorem is known, hence the relative semi-Grothendieck rings are expected to be more complicated.

Example(4.1.9)[165]: (Semilinear sets)

Let $K = (K, 0, 1, +, \cdot, <)$ be an ordered field and consider the structure $\mathcal{M} = (K, 0, 1, +, (\lambda_c)_{c \in K}, <)$, where λ_c is the scalar multiplication by $c \in K$. The category Def in this case is the category of K-semilinear sets with K-semilinear maps.

By [177], the Grothendieck ring $K_0(\mathrm{Def})$ is isomorphic to $E = \mathbb{Z}[x]/(x(x+1))$ and there is a universal Euler characteristic $\epsilon : \mathrm{Def} \to E$ (see also [175]).

Let D be the set whose elements are of the form $\sum_{i=1}^{n} y^{k_i} z^{l_i} \in \mathbb{N}[y, z]$ with $k_i \leq l_i$ and, for $i \neq j$, $\neg(y^{k_i} z^{l_i} = y^{k_j} z^{l_j}) \wedge \neg(y^{k_i} z^{l_i} \prec y^{k_j} z^{l_j}) \wedge \neg(y^{k_j} z^{l_j} \prec y^{k_i} z^{l_i})$. Here, $y^{k_i} z^{l_i} \prec y^{k_j} z^{l_j}$ if and only if $k_i < k_j$ and $l_i < l_j$.

The set D can be equipped with a semi-ring structure in the following way: the zero element 0_D is $\sum_{i=1}^{0} y^{k_i} z^{l_i}$, the identity element 1_D is $y^0 z^0$, the addition is given by

$$\sum_{i=1}^{n} y^{k_i} z^{l_i} +_D \sum_{i=1}^{m} y^{k_i'} z^{l_i'}$$

$$= \sum \max_{\prec} \left\{ y^k z^l : y^k z^l \text{ a monomial in } \sum_{i=1}^{n} y^{k_i} z^{l_i} + \sum_{i=1}^{m} y^{k_i'} z^{l_i'} \right\}$$

and multiplication is given by

$$\sum_{i=1}^{n} y^{k_i} z^{l_i} \cdot_D \sum_{i=1}^{m} y^{k_i'} z^{l_i'}$$

$$= \sum \max_{\prec} \left\{ y^k z^l : y^k z^l \text{ a monomial in } \sum_{i=1}^{n} y^{k_i} z^{l_i} \cdot \sum_{i=1}^{m} y^{k_i'} z^{l_i'} \right\}$$

where the symbol $\sum \max_{\prec} S$ mean that we sum up the $\prec$-maximal elements of the finite set S.

By [177], there is a universal abstract dimension δ: Def $\to D$ and two sets in Def are isomorphic in Def if and only if they have the same universal Euler characteristic and the same universal abstract dimension. Thus, if A is the semi-ring $E \times D$, then the Grothendieck semi-ring SK_0(Def) is isomorphic to A and the map μ: Def $\to A$ given by $\mu(X) = (\epsilon(X), \delta(X))$ is the positive universal measure on Def.

Note that the results that we used above from [177] were proved in the field of real numbers, but the same arguments hold in any arbitrary ordered field K.

Example(4.1.10)[165]: (Semibounded sets)

Let $K = (K, 0, 1, +, \cdot, <)$ be a real closed field and consider the structure $\mathcal{M} = (K, 0, 1, +, (\lambda_c)_{c\in K}, B, <)$, where λ_c is the scalar multiplication by $c \in K$ and B is the graph of multiplication on a bounded interval. The category Def in this case is the category of K-semibounded sets with K-semibounded maps. By [176], all bounded semialgebraic subsets are in Def and, by [179], $\mathcal{M}$ is, up to definability, the only o-minimal structure properly between $(K, 0, 1, +, (\lambda_c)_{c\in K}, <)$ and $(K, 0, 1, +, \cdot, <)$.

By [177], the Grothendieck ring K_0(Def) is isomorphic to $E = \mathbb{Z}[x]/(x(x+1))$ and there is a universal Euler characteristic $\epsilon : Def \to E$ (see also [175]). Furthermore, if D is the semi-ring of Example (4.1.9), then there is a universal abstract dimension δ: Def $\to D$ and two sets in Def are isomorphic in Def if and only if they have the same universal Euler characteristic and the same universal abstract dimension. Thus, if A is the semi-ring $E \times D$, then the Grothendieck semi-ring SK_0(Def) is isomorphic to A and the map μ : Def $\to A$ given by $\mu(X) = (\epsilon(X), \delta(X))$ is the positive universal measure on Def. The results that we used above from [177] were proved in the field of real numbers and are based on Peterzil's [178] structure theorem for semibounded sets in the real numbers. However, the same arguments hold in any arbitrary real closed field K using the structure theorem from [174].

Sec(4.2): Zero Loci, and Stability Under Integration for Constructible Functions on Euclidean Space with Lebesgue Measure

We define and study loci of integrability of certain (families of) functions. A recent insight into parameterized integrals is that, for functions f belonging to certain classes of functions on certainproduct measure spaces $E \times T$, a set of the form

$$\{x \in E \mid T \to \mathbb{C} : t \mapsto f(x,t) \text{ is measurable and integrable over } T\}, \qquad (9)$$

is in fact equal to the zero locus of a function on E belonging to the same class of functions; see [187]. If we call the set in (9) the locus of integrability of f in E, then we can rephrase the recent insight as a link between loci of integrability and zero loci for certain kinds of functions.

In this section, we give such a link for the class of constructible functions on Euclidean spaces with the Lebesgue measure; see Theorem (4.2.3). We follow the terminology of [188]: a constructible function is by definition a sum of products of globally subanalytic functions and of logarithms of globally subanalytic functions. The advantage of the class of constructible functions is that it is closed under integration. Indeed, in Cluckers and Miller [188] the authors prove that if f is constructible on $\mathbb{R}^n \times \mathbb{R}^m$ such that $y \mapsto f(x,y)$ is integrable over $\mathbb{R}^m$for each $x \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^m} f(x,y)\, dy$$

is constructible on $\mathbb{R}^n$, which generalizes results of [190]. We extend this stability result by relaxing the conditions on integrability; see Theorem (4.2.5). Further, we give an interpolation result, Theorem (4.2.4), of constructible functions by constructible functions with maximal locus of integrability.

Recall that a function $f: X \subset \mathbb{R}^n \to \mathbb{R}$ is called globally subanalytic if its graph is a globally subanalytic set, and a set $A \subset \mathbb{R}^n$ is called globally subanalytic if its image under the natural embedding of $\mathbb{R}^n$ into n-dimensional real projective space, namely $\mathbb{R}^n \longrightarrow \mathbb{P}^n(\mathbb{R})$: $(x_1, \ldots, x_n) \longmapsto (1: x_1 : \cdots : x_n)$, is a subanalytic subset of $\mathbb{P}^n(\mathbb{R})$ in the classical sense; see Definition (4.2.6) below for a self-contained definition.
From now on in this section, we write "subanalytic" instead of "globally subanalytic" (see again Definition (4.2.6)).
Definition(4.2.1)[183]: For each subanalytic set X, let $C(X)$ be the $\mathbb{R}$-algebra of real-valued functions on X generated by all subanalytic functions on X and all the functions $x \mapsto \log f(x)$, where $f: X \longrightarrow (0, +\infty)$ is subanalytic. Functions in $C(X)$ are called constructible functions on X and $C(X)$ is called the algebra of constructible functions on X.
In the whole section, we use the Lesbegue measure on $\mathbb{R}^n$. We introduce the locus of integrability of a function, as follows.
Definition(4.2.2)[183]: For E a set, and for $f: E \times \mathbb{R}^n \longrightarrow \mathbb{C}$ a function, define the locus of integrability *of* f in E as the set

$$\mathrm{Int}(f, E) := \{x \in E | f(x,\cdot) \text{ is measurable and integrable over } \mathbb{R}^n\},$$

where $f(x,\cdot)$ is the function sending $y \in \mathbb{R}^n$ to $f(x, y)$, and where the Lebesgue measure is used on $\mathbb{R}^n$.

We have the following three theorems, for which we will give relatively short and simple proofs.
Theorem(4.2.3)[183]: Let f be in $C(E \times R)$ for some subanalytic set E. Then there exists h in $C(E)$ such that

$$\mathrm{Int}(f, E) = \{x \in E \mid h(x) = 0\}. \tag{10}$$

Conversely, for every h in $C(E)$ there exists f in $C(E \times \mathbb{R})$ such that (10) holds.
Theorem (4.2.3) thus gives a correspondence between loci of integrability and zero loci of constructible functions, at least when integration is in dimension 1, that is, over $\mathbb{R}$. One should not misunderstand Theorem (4.2.3): zero loci of constructible functions are much more general than, say, Zariski closed sets, and for example, a zero locus of $h \in C(E)$ can easily be dense in E. Indeed, the characteristic function of any subanalytic subset of E lies in $C(E)$. Note that when f in Theorem (4.2.3) is moreover subanalytic, then one can take h to be a subanalytic function as well by the main result of [190]. Theorem (4.2.3) implies Theorem 1.4 of [188]. In Cluckers and Miller [189] they treat a higher dimensional variant of Theorem (4.2.3), also treating L^p-integrability for various p.

Constructible functions allow an interpolation by constructible functions with maximal locus of integrability, as follows.
Theorem(4.2.4)[183]: Let f be in $C(E \times \mathbb{R})$ for some subanalytic set E. Then there exists $g \in C(E \times \mathbb{R})$ with

$$\mathrm{Int}(g, E) = E$$

and such that, for all $x \in \mathrm{Int}(f, E)$ and all $y \in \mathbb{R}$, one has

$$g(x, y) = f(x, y).$$

Proofs of Theorems (4.2.3) and (4.2.4). Let f be in $C(E \times \mathbb{R})$, with $E \subset \mathbb{R}^n$ for some n. Apply Corollary (4.2.10) to the collection of functions consisting only of f. Consider a 1-cell A over $\mathbb{R}^n$ in the obtained partition, with center θ, and write f as in (12). By regrouping the terms and using the notation of (12), we may suppose, for each i, that either $|\tilde{y}|^{\alpha_i}$ is integrable over A_x, or that (α_i, ℓ_i) is different from the $(\alpha_j, \ell_j)$for all $j \neq i$. Let I be those indices i such that $|\tilde{y}|^{\alpha_i}$ is not integrable over A_x. Now define Q_A as the set $\{x \in B \mid d_i(x) = 0 \text{ for } i \in I\}$ and define, for $(x, y) \in A$, the constructible function

$$g(x,y) = \sum_{i \notin I} d_i(x) S_i(x,y) |\tilde{y}|^{\alpha_i} (\log|\tilde{y}|)^{\ell_i}.$$

Note that

$$\{x \in B : f(x, \cdot) \text{ is integrable over } A_x\} = Q_A,$$

because of condition (a) in Corollary (4.2.10), and because we have taken the exponent pairs (α_i, ℓ_i) mutually different for nonintegrable terms. Do the above construction for each occurring 1-cell A over $\mathbb{R}^n$. On any 0-cell A' over $\mathbb{R}^n$ in the partition, define $g(x,y)$ as $f(x,y)$. Then g is as desired by Theorem (4.2.4). Now note that a finite union of zero loci of constructible functions h_i equals the zero locus of a single constructible function by taking the product of the h_i. Similarly, a finite intersection of zero loci of constructible functions h_i equals the zero locus of a single constructible function by taking the sum of the squares of the h_i . Now one is done for $\mathrm{Int}(f, E)$. Indeed, $\mathrm{Int}(f, E)$ equals the finite intersection

$$\cap_A Q'_A,$$

where A runs over all 1-cells over $\mathbb{R}^n$ in the partition, and where, for any such 1-cell A, Q'_A equals the set $Q_A \cup (E \backslash B)$. Note that $E \backslash B$ is a sub-analytic set and each of the Q'_A equals thus the zero locus of a constructible function on E. For the converse statement of Theorem (4.2.3), given h, it suffices to put $f(x,y) = h(x)y$ for all $(x,y) \in E \times \mathbb{R}$.

We can integrate in any dimension m to find the following generalization of the principal result, Theorem 1.3, of [188].

Theorem(4.2.5)[183]: Let f be in $\mathrm{C}(E \times \mathbb{R}^m)$ for some subanalytic set E and some $m > 0$. Then there exists $g \in C(E)$ such that, for each $x \in \mathrm{Int}(f, E)$, one has

$$g(x) = \int_{y \in \mathbb{R}^m} f(x,y)\, dy.$$

Proof. Consider f in $C(E \times \mathbb{R}^m)$ for some $m > 0$. If $m = 1$, then apply Theorem (4.2.4) to f to find g_0 in $C(E \times \mathbb{R})$ with $\mathrm{Int}(g_0, E) = E$ and such that $g_0(x,y) = f(x,y)$ for all $x \in \mathrm{Int}(f, E)$ and all $y \in \mathbb{R}$. Now Apply Theorem (1.3) of [188] to g_0, which states that, if one defines, for $x \in E$,

$$g(x) := \int_{\mathbb{R}} g_0(x,y)\, dy ,$$

then g lies in $C(E)$. Then this g is as desired. The result for general m now follows from Fubini's Theorem.

Alternatively to deriving Theorem (4.2.5) for $m = 1$ from Theorem 1.3 of [188], one can also derive the case $m = 1$ from Corollary (4.2.10) by the integration procedure by Lion and Rolin of [196], which is also used and explained in Cluckers and Miller [188]. This self-contained approach for obtaining Theorem (4.2.5) is simpler than the approaches of [188, 190,196], which moreover only yielded special forms of Theorem (4.2.5).

The above theorem is proved in Cluckers and Miller [188] under the extra condition that $\mathrm{Int}(f, E)$ equals E (which in turn generalized main results from [190, 196]). Note that integrals of constructible functions are related to what one could call families of periods; see [192–194]. In several special cases, explicit formulas for parameterized integrals of constructible functions are given in [184, 199]. Parameterized integrals of constructible functions are often used for the study of singularities, as in [185, 197, 198]. For context on subanalytic functions we refer to [186, 191].

The present section is inspired by Cluckers et al. [187] which contains several p-adic and motivic analogues of this section, where [188] was more closely inspired on p-adic and motivic results of [169, 172]. The results and proofs of this section can be used to replace some of the technical difficulties encountered in Cluckers and Miller [188].

We recall a basic form of the subanalytic preparation theorem from [195] (see also [200]), we fix some notation, and we give a new preparation result for constructible functions.

Definition(4.2.6)[183]: Call a function $f: X \subset \mathbb{R}^{\ell} \longrightarrow \mathbb{R}^{k}$ analytic if it extends to an analytic function on an open neighborhood of X. A restricted analytic function is a function $f: \mathbb{R}^{n} \longrightarrow \mathbb{R}$ such that the restriction of f to $[-1,1]^{n}$ is analytic and $f(x) = 0$ on $\mathbb{R}^{n} \backslash [-1,1]^{n}$. Call a set or a f unction subanalytic i f and only if it is definable in the expansion of the real field by all restricted analytic functions. Thus in this section,“subanalytic” is an abbreviation of “globally subanalytic”, and in this meaning, the natural logarithm $\log : (0, +\infty) \longrightarrow \mathbb{R}$ is not subanalytic.

We fix an ordered list of variables $x_1, \dots, x_{n+1}$, where $n \geq 0$, and we write x for $(x_1, \dots, x_n)$ and write y for x_{n+1}, since the variable x_{n+1} will play a special role.

Definition(4.2.7)[183]: (Cells with a base)**.** Consider subanalytic sets $A \subset \mathbb{R}^{n+1}$ and $B \subset \mathbb{R}^{n}$ and an analytic subanalytic function $\theta : B \longrightarrow \mathbb{R}$. Then A is called a 0-cell over $\mathbb{R}^{n}$ with base B if A equals the graph of an analytic subanalytic function

$$c : B \longrightarrow \mathbb{R} : x \longmapsto y = c(x).$$

Call A a 1-cell over $\mathbb{R}^{n}$ with base B and with center θ if there are analytic sub-analytic functions $a : B \longrightarrow \mathbb{R}$ and $b : B \longrightarrow \mathbb{R}$, with $a < b$ on B, such that A is of the following form:

$$A = \{(x, y) \in B \times \mathbb{R} : \ a(x) \quad _1 y \quad _2 b(x)\},$$

with $\ _i$ either $<$ or no condition for each $i = 1, 2$, and such that the graph $\Gamma(\theta)$ of θ satisfies either

$$\Gamma(\theta) \subset \bar{A} \backslash A, \ \text{or}, \ \Gamma(\theta) \cap \bar{A} = \emptyset,$$

where $\bar{A}$ is the topological closure of A inside $\mathbb{R}^{n+1}$. In any case, A is called a cell over $\mathbb{R}^{n}$.

Definition(4.2.8)[183]: (Strong functions) **.** Let A be a 1-cell over $\mathbb{R}^{n}$ with base B and with center θ. A basic function with center θ is a function $\varphi : A \rightarrow \mathbb{R}^{N+2}$, for some $N \geq 0$, with bounded image and which is of the form

$$\varphi(x, y) = (a_1(x), \dots, a_N(x), b_1(x)| y - \theta(x)|^{\frac{1}{p}}, b_2(x)| y - \theta(x)|^{-\frac{1}{p}}), \qquad (11)$$

where $a_1, \dots, a_N, b_1, b_2$ are analytic subanalytic functions from B to $\mathbb{R}$ and p is a positive integer. A strong function on A with center θ is a function $A \rightarrow \mathbb{R}$ of the form $F \circ \varphi$, where φ is a basic function with center θ and where the function F is given by a single power series that converges on an open neighborhood of the image of φ . Note that strong functions are automatically subanalytic functions.

Theorem(4.2.9)[183]: (Preparation of subanalytic functions [195, 200]) **.** Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $X \subset \mathbb{R}^{N+2}$. Then there exists a finite partition of X into cells over $\mathbb{R}^{n}$ such that the following holds for any 1-cell A over $\mathbb{R}^{n}$ in this partition**:**

There exists a center θ for A such that each $f \in \mathcal{F}$ can be written in the form

$$f(x, y) = g(x)|y - \theta(x)|^{r} S(x, y)$$

on A, where g is an analytic subanalytic function on the base of A, r is a rational number, and S is a strong function on A with center θ, and such that, moreover, $S > \epsilon$ on A for some $\varepsilon > 0$.

The last part in the following corollary is new and simplifies the proofs concerning integration and integrability when compared with [188].

Corollary(4.2.10)[183]: (Preparation of constructible functions) **.** Let $\mathcal{F}$ be a finite set of constructible functions on a subanalytic set $X \subset \mathbb{R}^{N+2}$. Then there exists a finite partition of X into cells over $\mathbb{R}^n$ such that for each 1-cell A over $\mathbb{R}^n$ with base B in this decomposition, there exists a center θ such that, the following holds for each $f \in \mathcal{F}$ and all $(x, y) \in A$, and with $\tilde{y} := y - \theta(x)$:

$$f(x, y) = \sum_{i=1}^{M} d_i(x) S_i(x, y) |\tilde{y}|^{\alpha_i} (log\ |\tilde{y}|)^{\ell_i} , \qquad (12)$$

for some $M \geq 0$, functions $d_i \in C(B)$, rational numbers α_i, integers $\ell_i \geq 0$, and strong functions S_i on A with center θ. Moreover, one can ensure for each i that at least one of the following two conditions holds:

(a) $S_i(x, y) = 1$ on A;

(b) $y \mapsto |\tilde{y}|^{\alpha_i}$ is integrable over A_x for all $x \in B$.

Proof. Let $\mathcal{F}'$ be a finite collection of subanalytic functions such that each $f \in \mathcal{F}$ is a finite sum of products of functions in $\mathcal{F}'$ and of logarithms of functions in $\mathcal{F}'$. Apply Theorem (4.2.9) to $\mathcal{F}'$. Note that $\log(S)$ is a strong function with center θ if S is a strong function with center θ satisfying $S > \epsilon$ for some $\epsilon > 0$. Hence, we are done with the first part of the statement by writing logarithms of products as sums of logarithms, and since the product of strong functions with center θ is a strong function with center θ.

Suppose now that, for some occurring term $S_i(x, y) d_i(x) |\tilde{y}|^{\alpha_i}$on some cell A with center θ and base B, one has that $y \mapsto |\tilde{y}|^{\alpha_i}$ is not integrable over A_x for some (and hence for all) $x \in B$. Then, by the supposed presence of this nonintegrable term and by partitioning the cells slightly further, we may suppose that exactly one of the following two conditions holds:

(i) The graph of the center θ lies in $\bar{A}$ and $A_x$is bounded in $\mathbb{R}$ for each value of $x \in B$.

(ii) The graph of the center θ is disjoint from $\bar{A}$ and A_x is not contained in a compact subset of $\mathbb{R}$ for any value of $x \in B$.

Since the argument is completely similar in both cases, let us suppose (i) holds. Then, writing the strong function S_i as $F_i \circ \varphi$ with φ a basic function with center θ, as in (11), and F_i a converging power series, and by recalling that the image of φ is bounded, one sees that $b_2(x) = 0$ for all $x \in B$, with notation from (11). Moreover, $\tilde{y}$ is bounded on A_x for each x, and thus, $|\tilde{Y}|^{q+\alpha_i}$ is integrable over A_x for all $x \in B$ as soon as $q \in \mathbb{Q}$ is sufficiently large. For any $s > 0$ we can develop finitely many terms of F_i in $|\tilde{y}|^{1/p}$ plus the remaining series in $|\tilde{Y}|^{1/p}$, as follows:

$$S_i(x, y) = \left(\sum_{j=0}^{s-1} c_j(x)\ |\tilde{y}|^{j/p} \right) + \left(\sum_{j \geq s} c_j(x)\ |\tilde{y}|^{j/p} \right). \qquad (13)$$

By pulling out the factor $|\tilde{y}|^{s/p}$ from the last term, by writing out $S_i(x, y) d_i(x)\ |\tilde{y}|^{\alpha_i}$ using distributivity and (13), and by taking s large enough, the first s such terms will be as in part (a) of the corollary, and the last term will be integrable as in (b). This completes the proof.

Sec(4.3): Preparation of Real Constructible Functions

The Lebesgue spaces, $L^p(\mu)$ for $p \in (0, \infty]$, are ubiquitous in many areas of mathematical analysis and its applications. Much of the research about the Lebesgue spaces has been conducted in a very general measure-theoretic framework, with the focus being on discovering a host of relationships between the various L^p spaces. A number of the classical theorems are inequalities that explain how various function operations behave with respect to the Lebesgue spaces. For example, for addition there is Minkowski's inequality; for multiplication there is Hölder's inequality; for convolutions there is Young's convolution inequality; for Fourier transforms of periodic functions there is the Hausdorff-Young inequality. Other classical theorems explain the structure of linear maps between the various L^p spaces, such as the duality of Lebesgue spaces with conjugate exponents and the Riesz Thorin interpolation theorem.

This section explores theorems about the Lebesgue spaces of a rather different sort. We use geometric techniques to study the structure of the Lebesgue classes of parameterized families of functions, along with a related preparation theorem. The starting point of the investigation is the observation that, although much of the utility of the Lebesgue spaces-and more generally, of the theory of integration as a whole -stems from the generality of the measure-theoretic framework in which it has been developed, it is many times applied to study integrals of very special functions that arise naturally in real analytic geometry. And, if we focus the attention on studying the L^p properties of these very special functions, we should be able to obtain rather strong theorems that cannot be proven, or even reasonably formulated, in a very general measure-theoretic framework. This is because by focusing on special functions, we can supplement the very general tools from mathematical analysis with much more specialized tools from real analytic geometry and o-minimal structures. Similar approaches have been followed in the context of p-adic and motivic integration; see e.g. [172].

The o-minimal framework is still a bit too general for the purposes, and we choose to focus on the constructible functions, by which we mean the real-valued functions that have globally subanalytic domains and that can be expressed as sums of products of globally subanalytic functions and logarithms of positively-valued globally sub-analytic functions. The study of constructible functions largely originated in the work of Lion and Rolin, [196], where these functions naturally arose in their study of integration of globally subanalytic functions. (In the context of p-adic integration, analogues of constructible functions arose from the work by J. Denef [203].) The integration theory of globally subanalytic and constructible functions was then further developed by Comte, Lion and Rolin in [190] and also in [188], [183]. Much of the utility of the constructible functions stems from the fact that they are stable under integration-from which it follows that they are the smallest class of functions that is stable under integration and contains the subanalytic functions-and that they have very simple asymptotic behavior (see Theorem(1.3) and Proposition(1.5) in [188]). In fact, these results have typically lagged behind the motivic and p-adic developments. In this section, the real situation takes the lead over the p-adic and motivic results.

We obtain two main theorems about the constructible functions; see Theorems (4.3.1) and (4.3.2). The first theorem considers a constant $q > 0$ and constructible functions f and μ on $E \times \mathbb{R}^n$, and it describes the structure of the set

$$LC(f, |\mu|^q, E) := \{(x, p) \in E \times (0, \infty]: f(x,\cdot) \in L^p(|\mu|_x^q)\}, \qquad (14)$$

where $|\mu|_x^q$ is the positive measure on $\mathbb{R}^n$ whose Radon-Nikodym derivative with respect to the Lebesgue measure is $|\mu(x,\cdot)|^q: y \longmapsto |\mu(x, y)|^q$. The theorem and its corollaries show that the set of all fibers of $LC(f, |\mu|^q, E)$ over E is a finite set of open subintervals of $(0, \infty]$, and that the set of all fibers of $LC(f, |\mu|^q, E)$ over $(0, \infty]$ is a finite set of subsets of E, each of which is the zero locus of a constructible function on E. This theorem therefore relates analysis with geometry, in the sense that Lebesgue classes are an object of study in analysis, while zero loci of functions are widely studied in analytic geometry. A similar link between geometry and analysis (but with $\mu = 1$ and with focus on L^1-integrability) is obtained in p-adic and motivic contexts in [187].

The second theorem is a closely related preparation result that expresses f and μ as finite sums of terms of a very simple form that naturally reflect the structure of $\mathrm{LC}(f, |\mu|^q, E)$. This theorem can be most easily appreciated through the historical context in which it was developed, starting with the following simple preparation result for constructible functions, which is a rather direct consequence of Lion and Rolin's preparation theorem for globally subanalytic functions:

Let $f: E \times \mathbb{R}^n \longrightarrow \mathbb{R}$, be constructible with $E \subset \mathbb{R}^m$ and write $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ for the standard coordinates on $E \times \mathbb{R}^n$. Then f can be piecewise written on subanalytic sets as finite sums $\sum_{k \in K} T_k(x, y)$ where up to performing translations in y be by globally subanalytic functions of a triangular form, each term is of the form $T_k(x, y) = g_k(x) \left(\prod_{j=1}^{n} |y_i|^{r_{k,j}} (\log|y_j|)^{s_{k,j}} \right) u_k(x, y)$ for some constructible function g_k, rational numbers $r_{k,j}$, natural numbers $s_{k,j}$ and globally subanalytic unit u_k which is of the special form as given by the globally subanalytic perparation theorem. (15)

Lion and Rolin [195] used (15) when proving that any parameterized integral of a constructible function is piecewise given by constructible functions, but on pieces that need not be globally subanalytic sets. Comte, Lion and Rolin [190] also used (15) when proving that any parameterized integral of a globally subanalytic function is a constructible function. Then subsumed both of these results in [188] by showing that $F(x) = \int_{\mathbb{R}^n} f(x, y) dy$ is a constructible function on E if $f : E \times \mathbb{R}^n \to \mathbb{R}$ is a constructible function such that $f(x, \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in E$. The key to doing this was to improve (15) by showing that in the special case of $n = 1$, if $f(x, \cdot) \in L^1(\mathbb{R})$ for every $x \in E$, then the sums can be constructed in such a way so that each term $T_k(x, y)$ is also integrable in y for every $x \in E$. This alleviated various analytic considerations employed in [195] and [190] to get around the awkward fact that (15) allows the possibility of expressing integrable functions as sums of nonintegrable functions. In [183] improved upon (15) in the special case of $n = 1$ by dropping the assumption that $f(x, y)$ be integrable in y for every $x \in E$, and then showing that the set $\mathrm{Int}(f, E) := \{x \in E : f(x, \cdot) \in L^1(\mathbb{R})\}$ is the zero locus of a constructible function on E, and that the sums in (15) can be constructed so that each term $T_k(x, y)$ is integrable in y for every $x \in E$, provided that we only require the equation $f(x, y) = \sum_k T_k (x, y)$ to hold for those values of (x, y) with $x \in \mathrm{Int}(f, E)$.

The preparation theorem of this section strengthens this line of results even further by considering an arbitrary positive integer n, not just $n = 1$, and by considering all L^p classes simultaneously, not just L^1. In order to convey the main idea of the theorem without getting bogged down in technicalities, let us use the Lebesgue measure on $\mathbb{R}^n$ (thus $\mu = 1$, where μ is the function from (14)), and let us also only consider the L^p classes for finite values of p. Under these simplifying assumptions, the preparation theorems states that the sums $\sum_{k \in K} T_k (x, y)$ in (15) can be constructed in such a way so that there is a partition $\{K_i\}_i$ of the finite index set K such that for each $x \in E$ and $p \in (0,\infty)$ with $f(x,\cdot) \in L^p(\mathbb{R}^n)$, and for each i, either $T_k(x,\cdot)$ is in L^p for all $k \in K_i$, or else $\sum_{k\in K_i} T_k(x,y) = 0$ for all y. So, for instance, if for some fixed value of p the function $f(x,\cdot)$ happened to be in $L^p(\mathbb{R}^n)$ for every $x \in E$, then the sums in (15) can be constructed so that each term $T_k(x,\cdot)$ is in L^p for every $x \in E$, for we may simply omit the remaining terms in the sum because they collectively sum to zero.

Part of the interest in developing a good integration theory for constructible functions comes from a desire to study various integral transforms in the constructible setting. And, to summarize, we now have three main tools at the disposal to conduct such studies: the constructible functions are stable under integration, they have simple asymptotic behavior, and they have a multivariate preparation theorem with good analytic properties. We apply these three tools to the field of harmonic analysis in [202] by proving a theorem that bounds the decay rates of parameterized families of oscillatory integrals. This is an adaptation of a classical theorem found in Stein [204] but with different assumptions. The classical theorem bounds a single oscillatory integral with an amplitude function that is smooth and compactly supported and a phase function that is smooth and of finite type. In contrast, we give a uniform bound on a parameterized family of oscillatory integrals with an amplitude function that is constructible and integrable and a phase function that is globally subanalytic and satisfies a certain "hyperplane condition" (which closely relates to the notion of "finite type" in the setting). Thus by restricting the attention to the special classes of constructible and globally subanalytic functions, we obtain a much more global, parameterized version of the classical theorem with significantly weaker analytic assumptions. This application of the preparation theorem was, in fact, the initial stimulus for the work in this section.

We begin by fixing some notation to be used throughout the section.

Denote the set of natural numbers by $\mathbb{N} = \{0,1,2,3,\dots\}$.

Denote the subset and proper subset relations by $\subset$ and $\subsetneq$, respectively. Write $x = (x_1,\dots, x_m)$ and $y = (y_1,\dots,y_n)$ for the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If $f = (f_1,\dots,f_n)\colon D \to \mathbb{R}^n$ is a differentiable map with $D \subset \mathbb{R}^{m+n}$, write

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{\partial f_i}{\partial y_j}(x,y)\right)_{(i,j)\in\{1,\dots,n\}^2}$$

for its Jacobian matrix in y. Define the coordinate projection $\Pi_m\colon \mathbb{R}^{m+n} \to \mathbb{R}^n$ by

$$\Pi_m(x,y) = x.$$

For any $D \subset \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$, define the fiber of D over x by

$$D_x = \{y \in \mathbb{R}^n : (x,y) \in D\}.$$

For any $d \in \{0,\dots,n\}$ and $\in \{<,\le,>,\ge\}$, define $y_{\ d} = (y_i)_{i\ d}$. For example, $y_{\le d} = (y_1,\dots,y_d)$, and in accordance with the above notation for coordinate projections, the maps $\Pi_d : \mathbb{R}^n \to \mathbb{R}^d$ and $\Pi_{m+d} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+d}$ are given by $\Pi_d(y) = y_{\le d}$ and $\Pi_{m+d}(x,y) = (x,y_{\le d})$. More generally, if $\lambda : \{1,\dots,d\} \to \{1,\dots,n\}$ is an increasing map, define $\Pi_{m,\lambda} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+d}$ by

$$\Pi_{m,\lambda}(x,y) = (x,y_\lambda),$$

where $y_\lambda = (y_{\lambda(1)},\dots,y_{\lambda(d)})$.

For any set $D \subset \mathbb{R}^n$, call a function $f : D \to \mathbb{R}^m$ analytic if it extends to an analytic function on a neighborhood of D in $\mathbb{R}^n$. A restricted analytic function is a function $f: \mathbb{R}^n \to \mathbb{R}$ such that the restriction of f to $[-1,1]^n$ is analytic and $f(x) = 0$ on $\mathbb{R}^n \setminus [-1,1]^n$. We shall henceforth call a set or function subanalytic if, and only if, it is definable (in the sense of first-order logic) in the expansion of the real field by all restricted analytic functions. Thus in this section, the word "subanalytic" is an abbreviation for the phrase "globally sub analytic", and in this meaning, the natural logarithm $\log : (0,\infty) \to \mathbb{R}$ is not subanalytic. For any subanalytic set D, let $C(D)$ denote the $\mathbb{R}$-algebra of functions on D generated by the functions of the form $x \longmapsto f(x)$ and $x \longmapsto \log g(x)$, where $f: D \to \mathbb{R}$ and $g : D \to (0,\infty)$ are subanalytic. A function that is a member of $C(D)$ for some sub-analytic set D is called a constructible function.

Consider a Lebesgue measurable set $D \subset \mathbb{R}^{m+n}$ and Lebesgue measurable functions $f: D \to \mathbb{R}$ and $\nu : D \to [0,\infty)$, and put $E = \Pi_m(D)$. Define the diagram of Lebesgue classes of f over E with respect to ν to be the set

$$\mathrm{LC}(f,\nu,E) = \{(x,p) \in E \times (0,\infty] : f(x,\cdot) \in L^p(\nu_x)\},$$

where ν_x is the positive measure on D_x defined by setting

$$\nu_x(Y) = \int_Y \nu(x,y)dy \tag{16}$$

for each Lebesgue measurable set $Y \subset D_x$, where the integration in (16) is with respect to the Lebesgue measure on $\mathbb{R}^n$. Thus for each $x \in E$, when $0 < p < \infty$, the function $f(x,\cdot)$ is in $L^p(\nu_x)$ if and only if

$$\int_{D_x} |f(x,y)|^p \, \nu(x,y)dy < \infty ,$$

and the function $f(x,\cdot)$ is in $L^\infty(\nu_x)$ if and only if there exist a constant $M > 0$ and a Lebesgue measurable set $Y \subset D_x$ such that $\nu_x(Y) = 0$ and $|f(x,y)| \le M$ for all $y \in D_x \setminus Y$.

The fibers of $\mathrm{LC}(f,\nu,E)$ over E and over $(0,\infty]$ are both of interest, so we give them special names. For each $x \in E$, define the set of Lebesgue classes of f at x with respect to ν to be the set

$$\mathrm{LC}(f,\nu,x) = \{p \in (0,\infty] : f(x,\cdot) \in L^p(\nu_x)\}.$$

For each $p \in (0,\infty]$, define the L^p-locus of f in E with respect to ν to be the set

$$\mathrm{Int}^p(f,\nu,E) = \{x \in E : f(x,\cdot) \in L^p(\nu_x)\}.$$

When $\nu = 1$ (which is the case of most interest because it means we are simply using the n-dimensional Lebesgue measure on D_x), it is convenient to simply write $\mathrm{LC}(f, E)$, $\mathrm{LC}(f,x)$ and $\mathrm{Int}^p(f,E)$and to drop the phrase "with respect to ν" in the names of theses sets. Also when $\nu = 1$, we shall write $L^p(D_x)$ rather than $L^p(\nu_x)$.

We order the set $[0,\infty]$ in the natural way, and we topologize $(0,\infty]$ by letting

$$\{(a,b) : 0 \le a < b < \infty\} \cup \{\{\infty\}\}$$

be a base for its topology. A convex subset of $(0,\infty]$ is called a subinterval of $(0,\infty]$. The endpoints of a subinterval of $(0,\infty]$ are its supremum and infimum in $[0,\infty]$. Note that the empty set is a subinterval of $(0,\infty]$, and that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

It is elementary to see that $\mathrm{LC}(f,\nu,x)$ is a subinterval of $(0,\infty]$ for each $x \in E$. Much more can be said when f and ν are assumed to be constructible functions or their powers.

Theorem(4.3.1)[201]: (The Structure of Diagrams of Lebesgue Classes). Let $q > 0$ and $f,\mu \in C(D)$ for some subanalytic set $D \subset \mathbb{R}^{m+n}$, and put $E = \Pi_m(D)$ and $\mathfrak{T} = \{LC(f,|\mu|^q, x) : x \in E\}$. Then $\mathfrak{T}$ is a finite set of open subintervals of $(0,\infty]$ with endpoints in $\mathrm{span}_{\mathbb{Q}}\{1, q\} \cap [0,\infty) \cup \{\infty\}$, and for each $I \in \mathfrak{T}$ there exists $g_I \in C(E)$ such that

$$\{x \in E : I \subset \mathrm{LC}(f,|\mu|^q,x)\} = \{x \in E : g_I(x) = 0\}. \tag{17}$$

Moreover, if f and μ are subanalytic, then each of the functions g_I can be taken to be sub analytic.

Proof: in the Subanalytic Case. Suppose that $q > 0$ and that f and μ are real-valued sub analytic functions on $D \subset \mathbb{R}^{m+n}$. Put $E = \Pi_m(D)$ and $\mathfrak{T} = \{LC(f, |\mu|^q, x): x \in E\}$. Apply Proposition (4.3.23) to $\mathcal{F} = \{f, \mu\}$. This constructs an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$, there exist a subanalytic analytic isomorphism $F: B \to A$ over $\mathbb{R}^m$ and a rectilinear rational monomial map ψ on B over $\mathbb{R}^m$ such that $f \circ F, \mu \circ F$ and det $\frac{\partial F}{\partial y}$ are ψ-prepared.

Focus on one $A \in \mathcal{A}$, along with its associated maps $F: B \to A$ and ψ on B, where ψ is l-rectilinear over $\mathbb{R}^m$. Define $\nu : B \to \mathbb{R}$ by

$$\nu(x, y) = |\mu \circ F(x, y)|^q \left| det \frac{\partial F}{\partial y}(x, y) \right|.$$

On B write

$$f \circ F(x, y) = a(x) y^\alpha u(x, y),$$
$$\nu(x, y) = b(x) y^\beta v(x, y),$$

for some analytic subanalytic functions a and b, tuples $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in (\text{span}_{\mathbb{Q}}\{1, q\})^n$, and ψ-units u and v. We may assume that a and b have constant sign. If $a = 0$ or $b = 0$, let $I_A = (0, \infty]$. Otherwise, let I_A be the set consisting of all $p \in (0, \infty)$ such that $\alpha_i p + \beta_i > -1$ for all $i \in \{l + 1, \dots, n\}$, and also consisting of ∞ if $\alpha_i \geq 0$ for all $i \in \{l + 1, \dots, n\}$. Note that I_A is a subinterval of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$. Also note that by Corollary (4.3.25),

$$\text{LC}(f|_A, |\mu|^q|_A, \Pi_m(A)) = \text{LC}(f \circ F, \nu, \Pi_m(A)) = \Pi_m(A) \times I_A.$$

Now, for each $x \in E$, the set $\text{LC}(f, \mu, x)$ is a subinterval of $(0, \infty]$ with endpoints in $\left(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)\right) \cup \{\infty\}$ because it equals the intersection of the sets I_A for all $A \in \mathcal{A}$ with $x \in \Pi_m(A)$. This, and the fact that $\mathcal{A}$ is finite, also implies that $\mathfrak{T}$ is finite. To finish, let $I \in \mathfrak{T}$, and note that $\{x \in E : I \subset \text{LC}(f, \mu, x)\}$ equals

$$\{x \in E : I \subset I_A \text{ for all } A \in \mathcal{A} \text{ with } x \in \Pi_m(A)\},$$

which is a subanalytic set, and hence is the zero locus of a subanalytic function.

Theorem (4.3.1) has been formulated in such a way so as to make it adaptable to a variety of situations. This section contains an extensive list of corollaries that further explain how the theorem elucidates the structure of $\text{LC}(f, |\mu|^q, E)$, and how it can be easily adapted to give analogous theorems about local L^p spaces, complex measures, and measures defined from differential forms on subanalytic sets, all within the context of constructible functions.

The proof of Theorem (4.3.1) is intimately linked to the proof of a preparation theorem for constructible functions that is stated in full strength in present section given below, where it is showed. Here we state only a simple version of the preparation theorem that is sufficient for the application to oscillatory integrals in [202]. But first, we need one more definition: a cell over $\mathbb{R}^m$ is a subanalytic set $A \subset \mathbb{R}^{m+n}$ such that for each $i \in \{1, \dots, n\}$, the set $\Pi_{m+i}(A)$ is either the graph of an analytic subanalytic function on $\Pi_{m+i-1}(A)$, or

$$\Pi_{m+i}(A) = \{(x, y_{\leq i}): (x, y_{<i}) \in \Pi_{m+i-1}(A), a_i(x, y_{<i}) \ _1\, y_i \ _2\, b_i(x, y_{<i})\} \qquad (18)$$

for some analytic subanalytic functions a_i, $b_i : \Pi_{m+i-1}(A) \to \mathbb{R}$ for which $a_i(x, y_{<i}) < b_i(x, y_{<i})$ on $\Pi_{m+i-1}(A)$, where $_1$ and $_2$ denote either $<$ or no condition.

Theorem(4.3.2)[201]: (Preparation of Constructible Functions-Simple Version). Let $p \in (0, \infty)$ and $f \in C(D)$ for some subanalytic set $D \subset \mathbb{R}^{m+n}$, and assume that $\text{Int}^p(f, \Pi_m(D)) = \Pi_m(D)$. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ whose fibers over $\Pi_m(A)$ are open in $\mathbb{R}^n$, we may write f as a finite sum

$$f(x,y) = \sum_k T_k(x,y)$$

on A, with $\mathrm{Int}^p(T_k, \Pi_m(A)) = \Pi_m(A)$ for each k, as follows: there exists a bounded function $\varphi : A \to (0,\infty)^M$ of the form

$$\varphi(x,y) = \left(c_i(x)\prod_{j=1}^{n}\left|y_j - \theta_j(x, y_{<j})\right|^{\gamma_{i,j}}\right)_{i\in\{1,\dots,M\}},$$

and for each k,

$$T_k(x,y) = g_k(x)\left(\prod_{i=1}^{n}|y_i - \theta_i(x,y_{<i})|^{r_{k,i}}(\log|y_i - \theta_i(x,y_{<i})|)^{s_{k,i}}\right)U_k \circ \varphi(x,y), \quad (19)$$

where the $g_k : \Pi_m(A) \to \mathbb{R}$ are constructible, the $c_i : \Pi_m(A) \to (0,\infty)$ and $\theta_i \colon \Pi_{m+i-1}(A) \to \mathbb{R}$ are analytic subanalytic functions, the graph of each θ_i is disjoint from $\Pi_{m+i}(A)$, the $\gamma_{i,j}$ and $r_{k,i}$ are rational numbers, the $s_{k,i}$ are natural numbers, and the U_k are positively-valued analytic functions on the closure of the range of φ.

In addition, the fact that $\mathrm{Int}^p(T_k, \Pi_m(A)) = \Pi_m(A)$ only depends on the values of the $r_{k,i}$, and not the values of $s_{k,i}$, in the following sense: we have $\mathrm{Int}\,(T'_k, \Pi_m(A)) = \Pi_m(A)$ for any function T'_k on A of the form

$$T'_k(x,y) = \prod_{i=1}^{n}|y_i - \theta_i(x,y_{<i})|^{r_{k,i}}(\log|y_i - \theta_i(x,y_{<i})|)^{s'_{k,i}},$$

where the $r_{k,i}$, are as in (19) and the $s'_{k,i}$are arbitrary natural numbers.

The key aspect of Theorem (4.3.2) that is of interest, and what makes its proof nontrivial, is that the piecewise sum representation of f can be constructed so that each of its terms $T_k(x,\cdot)$ are in the same L^p class as $f(x,\cdot)$; namely, $\mathrm{Int}^p(T_k, \Pi_m(A)) = \Pi_m(A)$ for each A and T_k, provided that $\mathrm{Int}^p(f, \Pi_m(D)) = \Pi_m(D)$. There is an analog of Theorem (4.3.2) for $p = \infty$, but then one must replace (19) with the more complicated form

$$T_k(x,y) = g_k(x)\left(\prod_{i=1}^{n}|y_i - \theta_i(x,y_{<i})|^{r_{k,i}}\left(\log\prod_{j=1}^{n}\left|y_j - \theta_j(x,y_{<j})\right|^{\beta_{i,j}}\right)^{s_{k,i}}\right)U_k \circ \varphi(x,y), \quad (20)$$

where the $\beta_{i,j}$ are rational numbers and everything else is as before, and where the fact that $\mathrm{Int}^\infty(f, \Pi_m(A)) = \Pi_m(A)$ now depends on all the values of the $r_{k,i}$, $s_{k,i}$ and $\beta_{i,j}$, not just the values of the $r_{k,i}$ alone.

We show a theorem on the fiberwise vanishing of constructible functions and a theorem on parameterized rectilinearization of subanalytic functions, given below.

Theorem(4.3.3)[201]: (Fiberwise Vanishing of Constructible Functions). If $f \in C(D)$ for a subanalytic set $D \subset \mathbb{R}^{m+n}$ and $E = \Pi_m(D)$, then there exists $g \in C(E)$ such that

$$\{x \in E : f(x,y) = 0 \text{ for all } y \in D_x\} = \{x \in E : g(x) = 0\}.$$

Proof: Let $f \in C(D)$ for a subanalytic set $D \subset \mathbb{R}^{m+n}$, and put $E = \Pi_m(D)$. Write $V = \{x \in E : f(x,y) = 0 \text{ for all } y \in D_x\}$. We proceed by induction on n.

First suppose that $n = 1$. By Corollary (4.3.10) we may fix a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that the restriction of f to A is analytic for each $A \in \mathcal{A}$. We claim that for each $A \in \mathcal{A}$ there exists $g_A \in C(\Pi_m(A))$ such that

$$\{x \in \Pi_m(A) \colon f(x,y) = 0 \text{ for all } y \in A_x\} = \{x \in \Pi_m(A) \colon g_A(x) = 0\}.$$

The theorem (with $n = 1$) follows from the claim, for then

$$V = \left\{x \in E : \sum_{A \in \mathcal{A}} \left(g'_A(x)\right)^2 = 0\right\},$$

where $g'_A : E \to \mathbb{R}$ is the extension of g_A by 0 on $E \setminus \Pi_m(A)$. To show the claim, fix $A \in \mathcal{A}$. We may assume that A is open over $\mathbb{R}^m$, else the claim is trivial. Since $f(x,\cdot)$ is analytic on A_x for each $x \in \Pi_m(A)$, and since $f|_A$ is definable in the expansion of the real field by all restricted analytic functions and the exponential function, which is o-minimal (see Van den Dries, Macintyre and Marker [205], or Lion and Rolin [195]), it follows that we may fix a positive integer N such that for each $x \in \Pi_m(A), f(x,y) = 0$ for all $y \in A_x$ if and only if there exist distinct $y_1, \ldots, y_N \in A_x$ such that $f(x,y_1) = \cdots = f(x,y_N) = 0$. So fix subanalytic functions $\xi_1, \ldots, \xi_N : \Pi_m(A) \to \mathbb{R}$ whose graphs are disjoint subsets of A. Then the claim holds for the function

$$g_A(x) = \sum_{i=1}^{N} \left(f\left(x, \xi_i(x)\right)\right)^2.$$

This establishes the theorem when $n = 1$.

Now suppose that $n > 1$, and inductively assume the theorem holds with k in place of n for each $k < n$. The set V is defined by the formula

$$(x \in E) \wedge \forall y \in \mathbb{R}^n((x,y) \in D \to f(x,y) = 0).$$

Applying the induction hypothesis twice shows that that this formula is equivalent to

$$(x \in E) \wedge \forall y_1 \in \mathbb{R}((x,y_1) \in \Pi_{m+1}(D) \to h(x,y_1) = 0)$$

for some $h \in C(\Pi_{m+1}(D))$, which in turn is equivalent to

$$(x \in E) \wedge (g(x) = 0)$$

for some $g \in C(E)$. Thus $V = \{x \in E : g(x) = 0\}$.

The parameterized rectilinearization theorem requires some additional terminology to state. For any sets $D \subset \mathbb{R}^{m+n}$ and $B \subset \mathbb{R}^{m+d}$, we call a map $f = (f_1, \ldots, f_{m+n}) : B \to A$ an analytic isomorphism over $\mathbb{R}^m$ if f is a bijection, f and f^{-1} are both analytic, and $f_1(x,z) = x_1, \ldots, f_m(x,z) = x_m$, where $z = (z_1, \ldots, z_d)$.

For $l \in \{0, \ldots, d\}$, we say that a set $B \subset \mathbb{R}^{m+d}$ is l-rectilinear over $\mathbb{R}^m$ if B is a cell over $\mathbb{R}^m$ such that for each $x \in \Pi_m(B)$, the fiber B_x is an open subset of $(0,1)^d$ of the form

$$B_x = \Pi_l(B_x) \times (0,1)^{d-l},$$

where the closure of $\Pi_l(B_x)$ is a compact subset of $(0,1]^l$. When $B \subset \mathbb{R}^{m+d}$ is l-rectilinear over $\mathbb{R}^m$, we call a function u on B an l-rectilinear unit if it may written in the form $u = U \circ \psi$, where $\psi : B \to (0,\infty)^{N+d-l}$ is a bounded function of the form

$$\psi(x,z) = \left(c_1(x) \prod_{j=1}^{l} z_j^{\gamma_{1,j}}, \ldots, c_N(x) \prod_{j=1}^{l} z_j^{\gamma_{1,j}}, z_{l+1}, \ldots, z_d\right) \tag{21}$$

for some positively-valued analytic subanalytic functions ci and rational numbers $\gamma_{i,j}$, and where U is a positively-valued analytic function on the closure of the range of ψ.

Theorem(4.3.4)[201]:(Parameterized Rectilinearization of Subanalytic Functions). Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists a finite partition $\mathcal{A}$ of D into subanalytic sets such that for each $A \in \mathcal{A}$ there exist $d \in \{0, \ldots, n\}$, $l \in \{0, \ldots, d\}$ and a subanalytic map $F : B \to A$ such that F is an analytic isomorphism over $\mathbb{R}^m$, the set $B \subset \mathbb{R}^{m+d}$ is l–rectilinear over $\mathbb{R}^m$, and each function g in the set $\mathcal{G}$ defined by

$$\mathcal{G} = \begin{cases} \{f \circ F\}_{f \in \mathcal{F}}, & \text{if } d < n, \\ \{f \circ F\}_{f \in \mathcal{F}} \cup \left\{\det \frac{\partial F}{\partial y}\right\}, & \text{if } d = n, \end{cases}$$

may be written in the form

$$g(x, z) = h(x)\left(\prod_{j=1}^{d} z_j^{r_j}\right)u(x, z) \quad (22)$$

on B for some analytic subanalytic function h, rational numbers r_j, and l-rectilinear unit u.

Proof. Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. We proceed by induction on n. The base case of $n = 0$ is trivial, so assume that $n > 0$ and that the theorem holds with k in place of n for all $k < n$. Let $\mathcal{A}$ be the open partition of D over $\mathbb{R}^m$ given by applying Proposition (4.3.23) to $\mathcal{F}$, and let $D' = \bigcup \mathcal{A}$. Thus the theorem holds for $\mathcal{F}|_{D'}$. It follows from the induction hypothesis that the theorem also holds for $\mathcal{F}|_{D \setminus D'}$, since $D \setminus D'$ may be partitioned into cells over $\mathbb{R}^m$, and each of these cells projects via an analytic isomorphism into $\mathbb{R}^{m+d}$ for some $d < n$.

Note that if one desires, one can take the $\gamma_{i,j}$ in (21) and the r_j in (22) to all be integers. To do this, simply pull back each map F in Theorem (4.3.4) by a map $(x, z) \mapsto (x,\ z_1^{k_1},\ \ldots,\ z_d^{k_d})$ for a suitable choice of positive integers $k_1, \ldots, k_d$.

We formulate a version of the subanalytic preparation theorem of Lion and Rolin [195]. We begin with some multi-index notation.

For any tuples $y = (y_1, \ldots, y_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{R}^n$, define

$|y| = (|y_1|, \ldots, |y_n|)$,

$\log y = (\log y_1, \ldots, \log y_n)$, provided that $y_1, \ldots, y_n > 0$,

$y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ provided that this is defined,

$|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$\operatorname{supp}(\alpha) = \{i \in \{1, \ldots, n\} : \alpha_i \neq 0\}$, which is called the support of α.

There is a conflict of notation between this use of $|y|$ and $|\alpha|$, but the context will always distinguish the meaning: if α is a tuple of exponents of a tuple of real numbers, then $|\alpha|$ means $\alpha_1 + \cdots + \alpha_n$; if y is a tuple of real numbers not used as exponents, then $|y|$ means $(|y_1|, \ldots, |y_n|)$. These notations may be combined, such as with $|y|^\alpha = |y_1|^{\alpha_1} \ldots |y_n|^{\alpha_n}$ and $(\log|y|)^\alpha = (\log|y_1|)^{\alpha_1} \cdots (\log|y_n|)^{\alpha_n}$.

Definitions(4.3.5)[201]: Consider a subanalytic set $A \subset \mathbb{R}^{m+n}$. We say that A is open over $\mathbb{R}^m$ if A_x is open in $\mathbb{R}^n$ for all $x \in \Pi_m(A)$.

We call a function $\theta = (\theta_1, \ldots, \theta_n) : A \to \mathbb{R}^n$ a center for A over $\mathbb{R}^m$ if A is open over $\mathbb{R}^m$, and if for each $i \in \{1, \ldots, n\}$ the component θ_i is an analytic subanalytic function $\theta_i : \Pi_{m+i-1}(A) \to \mathbb{R}$ with the following two properties.

I. The range of θ_i is contained in either $(-\infty, 0), \{0\}$ or $(0, \infty)$. And, when θ_i is non-zero, the closure of the set $\{y_i / \theta_i(x, y_{<i}) : (x, y) \in A\}$ is a compact subset of $(0, \infty)$.

II. Let $\tilde{y}_i = y_i - \theta_i(x, y_{<i})$. The set $\{\tilde{y}_i : (x, y) \in A\}$ is a subset of either $(-\infty, -1)$, $(-1, 0), (0, 1)$ or $(1, \infty)$.

We call $(x, \tilde{y}) := (x, \tilde{y}_1, \ldots, \tilde{y}_n)$ the coordinates on A with center θ.

A rational monomial map on A over $\mathbb{R}^m$ with center θ is a bounded function $\varphi : A \to \mathbb{R}^M$ of the form

$$\varphi(x, y) = (c_1(x)|\tilde{y}|^{\gamma_1}, \ldots, c_M(x)|\tilde{y}|^{\gamma_M}), \quad (23)$$

where $c_1, \ldots, c_M$ are positively-valued analytic subanalytic functions on $\Pi_m(A)$ and $\gamma_1, \ldots, \gamma_M$ are tuples in $\mathbb{Q}^n$. Note that $\varphi(A) \subset (0, \infty)^M$. If $A \subset \mathbb{R}^m \times (0, 1)^n$ and $\theta = 0$, we say that φ is basic.

An analytic function is called a unit if its range is contained in either $(-\infty, 0)$ or $(0, \infty)$. A function $f: A \to \mathbb{R}$ is called a φ-function if $f = F \circ \varphi$ for some analytic function F whose domain is the closure of the range of φ; if F is also a unit, then we call f a φ-unit.

A function $f : A \to \mathbb{R}$ is φ-prepared if

$$f(x, y) = g(x)|\tilde{y}|^{\alpha}\, u(x, y)$$

on A for some analytic subanalytic function g, tuple $\alpha \in \mathbb{Q}^n$ and φ-unit u.

Definition(4.3.6)[201]: To any rational monomial map $\varphi: A \to \mathbb{R}^M$ over $\mathbb{R}^m$ with center θ, we associate a basic rational monomial map over $\mathbb{R}^m$. denoted by φ_θ, as follows. For each $i \in \{1, \ldots, n\}$, the set $\{\tilde{y}_i : (x,\ y) \in A\}$ is contained in either $(-\infty, -1), (-1,\ 0), (0, 1)$ or $(1, \infty)$, so there exist unique $\varepsilon_i, \zeta_i \in \{-1, 1\}$ such that $0 < \varepsilon_i \tilde{y}_i^{\zeta_i} < 1$ for all $(x, y) \in A$.

Define an analytic isomorphism $T_\theta : A \to A_\theta$ by

$$T_\theta(x, y) = \left(x, \varepsilon_1 \tilde{y}_1^{\zeta_1}, \ldots, \varepsilon_n \tilde{y}_n^{\zeta_n}\right).$$

Define $\varphi_\theta := \varphi \circ T_\theta^{-1} : A_\theta \to \mathbb{R}^M$.

Write $\varphi_\theta(x, y) = (c_1(x)y^{\gamma_1}, \ldots, c_M(x)y^{\gamma_M})$ for some $\gamma_1, \ldots, \gamma_M \in \mathbb{Q}^n$. For each $i \in \{0, \ldots, n\}$, define $\varphi_{\theta,i}$ to be the function on $\Pi_{m+i}(A)$ consisting of the components $c_j(x)y_i^{\gamma_j}$ of φ_θ such that $\operatorname{supp}(\gamma_j) \subset \{1, \ldots, i\}$, and when $i > 0$, such that $i \in \operatorname{supp}(\gamma_j)$. Thus

$$\varphi_\theta(x, y) = (\varphi_{\theta,0}(x), \varphi_{\theta,1}(x, y_1), \ldots, \varphi_{\theta,n}(x, y_1, \ldots, y_n)).$$

For each $i \in \{0, \ldots, n\}$ and $\ \in \{<, \leq, >, \geq\}$, define $\varphi_{\theta,\ i} = \left(\varphi_{\theta,j}\right)_{j\ \ i}$ on its appropriate domain. For example, $\varphi_{\theta,\leq i}$ is the function on $\Pi_{m+i}(A)$ given by

$$\varphi_{\theta,\leq i}(x, y_{\leq i}) = (\varphi_{\theta,0}(x), \varphi_{\theta,1}(x, y_1), \ldots, \varphi_{\theta,i}(x, y_1, \ldots, y_{\leq i})).$$

Definition(4.3.7)[201]: If $C \subset \mathbb{R}^{m+n}$ is a cell over $\mathbb{R}^m$, then there exists a unique increasing map $\lambda : \{1, \ldots, d\} \to \{1, \ldots, n\}$ whose image consists of the set of all $i \in \{1, \ldots, n\}$ for which $\Pi_{m+i}(C)$ is of the form (18). We call C a λ-cell.

Note that $\Pi_{m,\lambda}$ defines an analytic isomorphism from a λ-cell C onto $\Pi_{m,\lambda}(C)$, and $\Pi_{m,\lambda}(C)$ is a cell over $\mathbb{R}^m$ that is open over $\mathbb{R}^m$.

Definition(4.3.8)[201]: We say that φ is prepared over $\mathbb{R}^m$ if A is a cell over $\mathbb{R}^m$ such that for each $i \in \{1, \ldots, n\}$, if we write

$$\Pi_{m+i}(A_\theta) = \{(x, y_{\leq i}): (x, y_{<i}) \in \Pi_{m+i-1}(A_\theta), a_i(x, y_{<i}) < y_{i <} b_i(x, y_{<i})\},$$

then the functions a_i, b_i and $b_i - a_i$ are $\varphi_{\theta,<i}$-prepared, and a_i is either identically zero or is strictly positively-valued.

Proposition(4.3.9)[201]: (Subanalytic Preparation). Suppose that $\mathcal{F}$ is a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$, if A is a λ-cell over $\mathbb{R}^m$ and we write $g : \Pi_{m,\lambda}(A) \to A$ for the inverse of the projection $\Pi_{m,\lambda}|_A : A \to \Pi_{m,\lambda}(A)$, then there exists a prepared rational monomial map $\varphi : \Pi_{m,\lambda}(A) \to \mathbb{R}^M$ over $\mathbb{R}^m$ such that $f \circ g$ is φ-prepared for each $f \in \mathcal{F}$.

Corollary(4.3.10)[201]: Suppose that $\mathcal{F}$ is a finite set of constructible functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ and $f \in \mathcal{F}$, the restriction of f to A is analytic. Moreover, if each function in $\mathcal{F}$ is subanalytic, then $\mathcal{A}$ can be chosen so that $f(A)$ is contained in either $(-\infty, 0), \{0\}$ or $(0, \infty)$ for each $A \in \mathcal{A}$ and $f \in \mathcal{F}$.

Proof: When $\mathcal{F}$ consists entirely of subanalytic functions, this follows directly from Proposition (4.3.9). In the general constructible case, fix a finite set $\mathcal{F}'$ of subanalytic functions such that each function in $\mathcal{F}$ is a sum of products of functions of the form $(x, y) \longmapsto f(x, y)$ and $(x, y) \longmapsto \log g(x, y)$ with $f, g \in \mathcal{F}'$. Now apply the result of the subanalytic case to $\mathcal{F}'$.

Definition(4.3.11)[201]: If $\mathcal{S}$ is a set of subsets of a set X, we say that a partition $\mathcal{A}$ of X is compatible with S if for each $A \in \mathcal{A}$ and each $S \in \mathcal{S}$, either $A \subset S$ or $A \subset X \backslash S$.

Note that in Proposition(4.3.9) and Corollary(4.3.10), the partition $\mathcal{A}$ can be made to be compatible with any prior given finite set of subanalytic subsets of D.

Corollary(4.3.12)[201]: For each $I \in \mathfrak{T}$,

$$\{x \in E: \mathrm{LC}(f, |\mu|^q, x) = I\} = \left\{x \in E : (g_I(x) = 0) \wedge \left(\bigwedge_{J \in \mathcal{T}_I} g_J(x) \neq 0 \right) \right\}, \quad (24)$$

where $\mathfrak{T}_{\mathrm{I}} = \{J \in \mathfrak{T} : I \subsetneq J\}$.

Proof: This follows from (17) and from the fact that for each $x \in E$, $\mathrm{LC}(f, \mu, x) = I$ if and only if $I \subset \mathrm{LC}(f, \mu, x)$ and $J \not\subset \mathrm{LC}(f, \mu, x)$ for all $J \in \mathfrak{T}_{\mathrm{I}}$.

The final sentence of Theorem (4.3.1) shows that when f is subanalytic, so is the set (24). Note that: The set $\mathrm{LC}(f, |\mu|^q, E)$ can be expressed as the disjoint union

$$\bigcup_{I \in \mathfrak{T}} (\{x \in E : \mathrm{LC}(f, |\mu|^q, x) = I\} \times I) \quad (25)$$

and as the (not necessarily disjoint) union

$$\bigcup_{I \in \mathfrak{T}} (\{x \in E : I \subset \mathrm{LC}(f, |\mu|^q, x) = I\} \times I) \quad (26)$$

The fact that $\mathrm{LC}(f, |\mu|^q, E)$ equals (25), and that (25) is contained in (26), are both clear. To see that (26) is contained in (25), note that if (x, p) is such that $I \subset \mathrm{LC}\ (f, |\mu|^q, x)$ and $p \in I$, then $J = \mathrm{LC}(f, |\mu|^q, x)$ and $p \in J$ for some $J \in \mathfrak{T}$ with $I \subset J$. Observe that (24) and (17) show how to use the functions $\{g_I\}_{I \in \mathfrak{T}}$ to define the sets occurring in (25) and (26).

Corollary(4.3.13)[201]: For each $P \subset (0, \infty]$ there exists $G_P \in C(E)$ such that

$$\{x \in E : P \subset \mathrm{LC}(f, |\mu|^q, x)\} = \{x \in E : G_P(x) = 0\}. \quad (27)$$

Proof. Define G_P to be the product of the g_I for all $I \in \mathfrak{T}$ with $P \subset I$. Then (27) follows from (17) and from the fact that for each $x \in E$, we have $P \subset \mathrm{LC}(f, |\mu|^q, x)$ if and only if $\mathrm{LC}(f, |\mu|^q, x) = I$ for some $I \in \mathfrak{T}$ with $P \subset I$.

For each $p \in (0, \infty]$, taking $P = \{p\}$ in (27) shows that $\mathrm{Int}^p(f, |\mu|^q, E)$ is the zero locus of a constructible function. A very elementary proof of this fact is given in [183] for the special case when $\mu = 1, p = 1$ and $n = 1$.

Corollary(4.3.14)[201]: The set $\{\mathrm{Int}^p(f, |\mu|^q, E) : p \in (0, \infty]\}$ is finite.

Proof: Since $\mathfrak{T}$ is finite by Theorem (4.3.1), we may fix a finite partition $\mathcal{J}$ of $(0, \infty]$ compatible with $\mathfrak{T}$. If $J \in \mathcal{J}$ and $p \in J$, then for each $I \in \mathfrak{T}$, $p \in I$ if and only if $J \subset I$; so $\mathrm{Int}^p(f, |\mu|^q, E) = \{x \in E : J \subset LC(f, |\mu|^q, x)\}$. Therefore

$$\{\mathrm{Int}^p(f, |\mu|^q, E): p \in (0, \infty]\} = \{\{x \in E : J \subset LC(f, |\mu|^q, x) : J \in \mathcal{J}\},$$

which is finite because $\mathcal{J}$ is finite.

Corollary (4.3.15)[201]: There exists $g \in C(E)$ such that

$$\{x \in E : f(x, \cdot) \text{ is bounded on } D_x\} = \{x \in E : g(x) = 0\}.$$

Proof: Zero loci of constructible functions are closed under intersections and unions (by taking sums of squares and by taking products, respectively), so we may assume by Corollary (4.3.10) that D is a cell over $\mathbb{R}^m$ and that f is analytic. By projecting into a lower dimensional space, we may further assume that D is open over $\mathbb{R}^m$. Thus $f(x, \cdot)$ is bounded on D_x if and only if it is in $L^\infty(D_x)$, so we are done by applying Corollary (4.3.13) with $P = \{\infty\}$.

Corollary(4.3.16)[201]: There exist $g, h \in C(E)$ such that

$$\{x \in E : f(x,y) = 0 \text{ for all } y \in D_x\} = \{x \in E : g(x) = 0\}$$

and

$$\{x \in E: f(x,y) = 0 \; for \; |\mu|_x \text{ −almost all } y \in D_x\} = \{x \in E: h(x) = 0\}.$$

Proof. Define $F : D \times \mathbb{R} \to \mathbb{R}$ by $F(x,y,z) = zf(x,y)$. Note that for each $x \in E$, $f(x,y) = 0$ for all $y \in D_x$ if and only if $(y,z) \mapsto F(x,y,z)$ is bounded on $D_x \times \mathbb{R}$, and that $f(x,y) = 0$ for $|\mu|_x$-almost all $y \in D_x$ if and only if $(y,z) \mapsto F(x,y,z)$ is in $L^\infty(\nu_x)$, where $\nu : D \times \mathbb{R} \to [0,\infty)$ is defined by $\nu(x,y,z) = |\mu(x,y)|$. So we are done by applying Corollaries (4.3.15) and (4.3.13) (with $P = \{\infty\}$) to F.

The following result generalizes [188].

Corollary(4.3.17)[201]: Let $q > 0, P \subset (0,\infty]$, and $F, \nu \in C(X \times Y \times \mathbb{R}^k)$ for some subanalytic sets X and Y. Suppose that for each $x \in X$, the set $\{y \in Y : P \subset \mathrm{LC}(F, |\nu|^q, (x,y))\}$ is dense in Y. Then there exists a subanalytic set $C \subset X \times Y$ such that $C \times P \subset \mathrm{LC}(F, |\nu|^q, X \times Y)$ and C_x is dense in Y for each $x \in X$.

Proof: Assume that $X \subset \mathbb{R}^m$. We may assume that $Y = \mathbb{R}^n$ because the case of a general subanalytic set Y follows from this special case by arguing as in [188]. By Corollary (4.3.13) we may fix $g \in C(X \times \mathbb{R}^n)$ such that

$$\{(x,y) \in X \times \mathbb{R}^n : P \subset \mathrm{LC}(F, |\nu|^q, (x,y))\} = \{(x,y) \in X \times \mathbb{R}^n : g(x,y) = 0\}. \quad (28)$$

By Corollary (4.3.10) we may fix a partition $\mathcal{A}$ of $X \times \mathbb{R}^n$ into subanalytic cells over $\mathbb{R}^m$ such that g restricts to an analytic function on each $A \in \mathcal{A}$. Let C be the union of the members of $\mathcal{A}$ that are open over $\mathbb{R}^m$. Then C is subanalytic, $\Pi_m(C) = X$, and C_x is open and dense in $\mathbb{R}^n$ for each $x \in X$. If there exists $(a,b) \in C$ such that $g(a,b) \neq 0$, then $\{y \in C_a : g(a,y) = 0\}$ would be a proper analytic subset of the open set C_a, so $\{y \in \mathbb{R}^n : g(a,y) = 0\}$ would not be dense in $\mathbb{R}^n$, contradicting (28) and the assumption on F and $|\nu|^q$. Therefore $g(x,y) = 0$ for all $(x,y) \in C$, which by (28) proves the corollary.

We now show how Theorem(4.3.1) adapts easily to the study of local integrability, complex measures, and measures defined from constructible differential forms on subanalytic sets.

We only discuss the analogs of Theorem (4.3.1) itself, but it follows that analogs of the previous list of corollaries of this theorem hold as well, via the same proofs.

Suppose that $Y \subset \mathbb{R}^n$ and $f: Y \to \mathbb{R}$ are Lebesgue measurable, that ν is a positive measure on Y that is absolutely continuous with respect to the n-dimensional Lebesgue measure, and that $p \in (0,\infty]$. We say that f is locally in $L^p(\nu)$, written as $f \in L^p_{\mathrm{loc}}(\nu)$, if for each $y \in Y$ there exists a neighborhood U of y in Y such that $f|_U$ is in $Lp(\nu|_U)$. Similarly, we say that f is locally bounded on Y if for each $y \in Y$ there exists a neighborhood U of y in Y such that $f(U)$ is bounded.

For measurable functions $f : D \to \mathbb{R}$ and $\nu : D \to [0,\infty)$, where $D \subset \mathbb{R}^{m+n}$ and $E = \Pi_m(E)$, define the sets $\mathrm{LC}_{\mathrm{loc}}(f,\nu,E), \mathrm{LC}_{\mathrm{loc}}(f,\nu,x)$ and $\mathrm{Int}^p_{\mathrm{loc}}(f,\nu,E)$ analogously to how $\mathrm{LC}(f,\nu,E)$, $\mathrm{LC}(f,\nu,x)$ and $\mathrm{Int}^p(f,\nu,E)$ were defined in above, but replacing the condition $f(x,\cdot) \in L^p(\nu_x)$ with $f(x,\cdot) \in L^p_{\mathrm{loc}}(\nu x)$.

Proposition(4.3.18)[201]: The local analog of Theorem (4.3.1) holds, which describes the structure of $\mathrm{LC}_{\mathrm{loc}}(f, |\mu|^q, E)$ rather than $\mathrm{LC}(f, |\mu|^q, E)$.

Proof. By extending f and μ by 0 on $(E \times \mathbb{R}^n)\backslash D$, we may assume that $D = E \times \mathbb{R}^n$.

Define functions F and ν on $E \times \mathbb{R}^n \times [-1,1]^n$ by $F(x,y,z) = f(x,y+z)$ and $\nu(x,y,z) = |\mu(x,y+z)|^q$.

The compactness of $[-1,1]^n$ implies that for each $x \in E$ and $p \in (0,\infty], f(x,\cdot) \in L^p_{\mathrm{loc}}(|\mu|^q_x)$ if and only if $F(x,y,\cdot) \in L^p(\nu_{(x,y)})$ for all $y \in \mathbb{R}^n$ Therefore

$$\mathrm{LC}_{\mathrm{loc}}(f,|\mu|^q,x) = \bigcap_{y\in\mathbb{R}^n} \mathrm{LC}(F,\nu,(x,y)) .$$

Theorem (4.3.1) shows that $\{\mathrm{LC}(F,\nu,(x,y)) : (x,y) \in E \times \mathbb{R}^n\}$ is a finite set of subintervals of $(0,\infty]$ with endpoints in $(\mathrm{span}_{\mathbb{Q}}\{1,q\} \cap [0,\infty)) \cup \{\infty\}$, so the set

$$\mathrm{T}_{\mathrm{loc}} := \{\mathrm{LC}_{\mathrm{loc}}(f,|\mu|^q,x) : x \in E\}$$

is of this form as well. Let $I \in \mathrm{T}_{\mathrm{loc}}$. By Corollary (4.3.13) we may fix $g \in C(E \times \mathbb{R}^n)$ such that

$$\{(x,y) \in E \times \mathbb{R}^n : I \subset \mathrm{LC}(F,\nu,(x,y))\} = \{(x,y) \in E \times \mathbb{R}^n : g(x,y) = 0\}.$$

Thus

$$\{x \in E : I \subset \mathrm{LC}_{\mathrm{loc}}(f,|\mu|^q,x)\} = \{x \in E : g(x,y) = 0 \text{ for all } y \in \mathbb{R}^n\},$$

and this set is the zero locus of a constructible function by Theorem (4.3.3) (or Corollary (4.3.16)).

Suppose that f and ν are complex-valued Lebesgue measurable functions on a measurable set $D \subset \mathbb{R}^{m+n}$ such that $\nu(x,\cdot)$ is Lebesgue integrable on D_x for all $x \in E$, where $E = \Pi_m(D)$. For each $x \in E$, define a complex measure ν_x on D_x by setting

$$\nu_x(Y) = \int_Y \nu(x,y)\,dy$$

for each Lebesgue measurable set $Y \subset D_x$. The notion of an L^p-class with respect to a complex-measure is defined using the absolute variation of the measure, so we define $\mathrm{LC}(f,\nu,E) := \mathrm{LC}(|f|,|\nu|,E)$, $\mathrm{LC}(f,\nu,x) := \mathrm{LC}(|f|,|\nu|,x)$ for each $x \in E$, and $\mathrm{Int}^p(f,\nu,E) := \mathrm{Int}^p(|f|,|\nu|,E)$ for each $p \in (0,\infty]$.

Proposition(4.3.19)[201]: The complex analog of Theorem (4.3.1) holds with $q = 1$, which describes the structure of $\mathrm{LC}(f,\mu,E)$ for complex-valued functions f and μ on a subanalytic set $D \subset \mathbb{R}^{m+n}$ whose real and imaginary parts are constructible, where $\mu(x,\cdot)$ is Lebesgue integrable on D_x for all x in $E = \Pi_m(D)$.

Proof. Apply Theorem (4.3.1) to the constructible functions $|f|^2$ and $|\mu|^2$ with $q = \frac{1}{2}$. Then note that for any $p \in (0,\infty]$, $|f| \in L^p(|\mu|_x)$ if and only if $|f|^2 \in L^{p/2}(|\mu|_x)$.

We consider a subanalytic set $D \subset \mathbb{R}^{m+n}$ such that for each x in $E := \Pi_m(D)$, the fiber D_x is a smooth k-dimensional submanifold of $\mathbb{R}^n$. For each $x \in E$, consider a smooth k-form ω_x on D_x , such that moreover there exist constructible functions $\omega_{i_1,\dots,i_k}(x,y)$ on D with $1 \le i_1 < \cdots < i_k \le n$ such that

$$\omega_x(y) = \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1,\dots,i_k}(x,y)\,dy_{i_1} \wedge \cdots \wedge dy_{i_k}.$$

For each $x \in E$, write $|\omega_x|$ for the measure on D_x associated to the smooth k-form ω_x. For $f \in C(D)$, consider

$$\mathrm{LC}(f,\omega_x,x) = \{p \in (0,\infty] : f(x,\cdot) \in L^p(|\omega_x|)\},$$

and

$$\mathrm{LC}(f,\omega,E) = \{(x,p) \in E \times (0,\infty] : f(x,\cdot) \in L^p(|\omega_x|)\},$$

where ω stands for the family $(\omega_x)_{x\in E}$.

Proposition(4.3.20)[201]: With the above notation for $D,\omega,$ and E, and with $f \in C(D)$, the analog of Theorem (4.3.1) holds for $\mathrm{LC}(f,\omega,E)$. To adapt the last sentence of Theorem (4.3.1) to $\mathrm{LC}(f,\omega,E)$, the extra assumption that μ be subanalytic should be replaced by the condition that the $\omega_{i_1,\dots,i_k}$ be subanalytic.

Proof. Because D is subanalytic, basic o-minimality implies that there exists a finite family $\mathcal{U}$ of subanalytic subsets of D which covers D and is such that the following hold for each

$U \in \mathcal{U}$:

I. for every $x \in \Pi_m(U)$, the fiber U_x is open in D_x;

II. there exists an increasing function $\lambda^U: \{1, \dots, k\} \to \{1, \dots, n\}$ such that for each $x \in \Pi_m(U)$, the projection Π_{λ^U} is injective on U_x and has constant rank k.

For each $U \in \mathcal{U}$, let $G^U(x, z) = (x, g^U(x, z))$ be the inverse of $\Pi_{m,\lambda^U} : U \to \Pi_{m,\lambda^U}(U)$, where $z = (z_1, \dots, z_k)$. Then for each $U \in \mathcal{U}$, the functions $f \circ G^U$ and

$$\omega^U(x, z) := \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1, \dots, i_k}(x, g^U(x, z)) \frac{\partial\left(g^U_{i_1}, \dots, g^U_{i_k}\right)}{\partial(z_1, \dots, z_k)}(x, z)$$

are both constructible functions on U, and in the case that f and all the $\omega_{i_1, \dots, i_k}$ are sub-analytic, the ω^U and $f \circ G^U$ also are. Hence, Theorem (4.3.1) applies to $\mathrm{LC}(f \circ G^U, |\omega^U|, \Pi_m(U))$. The proposition now follows relatively easily from this and from the fact that

$$\mathrm{LC}(f|_U, \omega|_U, \Pi_m(U)) = \mathrm{LC}(f \circ G^U, |\omega^U|, \Pi_m(U))$$

for each $U \in \mathcal{U}$.

Definition(4.3.21)[201]: Consider $l \in \{0, \dots, n\}$ and a rational monomial map ψ on B over $\mathbb{R}^m$, where $B \subset \mathbb{R}^{m+n}$. We say that ψ is l-rectilinear over $\mathbb{R}^m$ if B is l-rectilinear over $\mathbb{R}^m$ (as defined prior to Theorem 4.3.4) and if ψ is of the form

$$\psi(x, y) = c_1(x) y_{\le l}^{\gamma_1}, \dots, c_N(x) y_{\le l}^{\gamma_N}, y_{l+1}, \dots, y_n$$

for some positively-valued analytic subanalytic functions $c_1, \dots, c_N$ on $\Pi_m(B)$ and tuples $\gamma_1, \dots, \gamma_N$ in $\mathbb{Q}^l$. We say that set B, or a rational monomial map ψ on B over $\mathbb{R}^m$, is rectilinear over $\mathbb{R}^m$ to mean that it is l-rectilinear over $\mathbb{R}^m$ for some l.

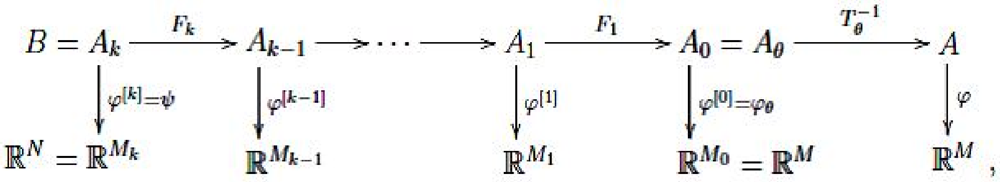
Definition(4.3.22)[201]: For a subanalytic set $D \subset \mathbb{R}^{m+n}$, an open partition of D over $\mathbb{R}^m$ is a finite family $\mathcal{A}$ of disjoint subanalytic subsets of D that are open over $\mathbb{R}^m$ and are such that $\dim(D \setminus \bigcup \mathcal{A})\, x < n$ for all $x \in \Pi_m(D)$.

Proposition(4.3.23)[201]: Let $\mathcal{F}$ be a finite set of subanalytic functions on a sub-analytic set $D \subset \mathbb{R}^{m+n}$. Then there exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exists a subanalytic analytic isomorphism $F : B \to A$ over $\mathbb{R}^m$ with $B \subset \mathbb{R}^{m+n}$, and there exist rational monomial maps φ on A and ψ on B over $\mathbb{R}^m$ with the following properties.

I. Pullback Property: Each function in $\{f \circ F\}_{f \in \mathcal{F}} \cup \left\{\det \frac{\partial F}{\partial y}\right\}$ *is* ψ-prepared, and ψ is rectilinear over $\mathbb{R}^m$.

II. Pushforward Property: The components of F^{-1} are φ-prepared, and $\psi \circ F^{-1}$ is a φ-function.

Proof: Let $\mathcal{F}$ be a finite set of subanalytic functions on $D \subset \mathbb{R}^{m+n}$. Apply Proposition (4.3.9) to $\mathcal{F}$, and focus on one rational monomial map $\varphi: A \to \mathbb{R}^M$ over $\mathbb{R}^m$ that this gives for which A is open over $\mathbb{R}^m$. Thus φ is prepared, and each function in $\mathcal{F}$ restricts to a φ-prepared function on A. Let θ be the center of φ. We will first construct finitely many sequences of maps diagrammed as follows,



(29)

where for each $i \in \{1, \dots, k\}$ the maps F_i and $\varphi^{[i]}$ are a pullback construction for $\varphi^{[i-1]}$ of one of the six types listed above, the map ψ is rectilinear over $\mathbb{R}^m$, and the ranges of the maps $F: B \to A$ given by $F = T_\theta^{-1} \circ F_1 \circ \cdots \circ F_k$ for all such sequences (29) constructed form an open partition of A over $\mathbb{R}^m$. Doing this shows the pullback property. We will construct

(29) to also have the following property.

$$\text{For each } j \in \{1, \ldots, n\}, \text{at most one map } F_i \text{ in (29) is a flip in } y_j\,. \tag{30}$$

Assuming we can construct (29) as such, to prove the pushforward property it suffices to define $A' = F(B)$, to inductively define $B_k = B$ and $B_{i-1} = F_i(B_i)$ for each $i \in \{1, \ldots, k\}$, and to show that we can construct maps diagrammed as follows,

$$\begin{array}{ccccccccc} B = B_k & \xrightarrow{F_k} & B_{k-1} & \longrightarrow \cdots \longrightarrow & B_1 & \xrightarrow{F_1|_{B_1}} & B_0 & \xrightarrow{T_\theta^{-1}|_{B_0}} & A' \\ \downarrow{\scriptstyle \psi^{[k]}=\psi} & & \downarrow{\scriptstyle \psi^{[k-1]}} & & \downarrow{\scriptstyle \psi^{[1]}} & & \downarrow{\scriptstyle \psi^{[0]}} & & \downarrow{\scriptstyle \varphi'=\psi^{[0]}\circ T_\theta|_{A'}} \\ \mathbb{R}^N = \mathbb{R}^{N_k} & & \mathbb{R}^{N_{k-1}} & & \mathbb{R}^{N_1} & & \mathbb{R}^{N_0} & & \mathbb{R}^{M'} = \mathbb{R}^{N_0}\,, \end{array} \tag{31}$$

where for each $i \in \{1, \ldots, k\}, \psi^{[i-1]}$ is a pushforward construction for $F_i|_{B_i} : B_i \to B_{i-1}$ and $\psi^{[i]}$. (Thus the map $\varphi : A \to \mathbb{R}^M$ in the statement of the theorem is being denoted by $\varphi' : A' \to \mathbb{R}^{M'}$ here in the proof.) These pushforward constructions will be possible because if a map F_i in (29) is a flip in y_j, we can ensure that $\psi^{[i]}$ is of the form (33). Indeed, from among the six types of pullback and pushforward constructions we use, only blowups in one of the variables $y_j, \ldots, y_n$ can possibly destroy the form (33). So and (32) imply that, in fact, all the maps $\varphi^{[i]}, \ldots, \varphi^{[k]}$ and $\psi^{[k]}, \ldots, \psi^{[i]}$ are of the form (33).

So it remains to construct the sequences (29). This is done by an induction, and to simplify notation we will write $\varphi : A \to \mathbb{R}^M$ instead of the more cumbersome $\varphi^{[i]} : A_i \to \mathbb{R}^{M_i}$. (So we are now assuming that φ is basic.) Let $d \in \{1, \ldots, n\}$, and inductively assume that $\varphi_{<d}$ is l-rectilinear over $\mathbb{R}^m$ for some $l \in \{0, \ldots, d-1\}$ and that φ is prepared over $\mathbb{R}^{m+d-1}$. Thus A is a cell over $\mathbb{R}^m$, so we use the definition (4.3.21). To complete the construction, it suffices to show that after taking an open partition of A over $\mathbb{R}^m$ and pulling back φ, we may reduce to the case that $\varphi_{\leq d}$ is rectilinear and φ is prepared over $\mathbb{R}^{m+d}$.

By pulling back by a blowup in y_d and then by power substitutions in $y_{l+1}, \ldots, y_d$, and using Lemma (4.3.26), we may assume that $b_d = 1$ and that all the powers of $y_{l+1}, \ldots, y_d$ occurring in the components of φ are natural numbers, and when $a_d > 0$, that all the powers of $y_{l+1}, \ldots, y_{d-1}$ in the monomials occurring outside the units in the $\varphi_{<d}$-prepared forms of a_d and $1 - a_d$ are also natural numbers. There are two cases that can be handled very easily.

Case 1: $a_d = 0$.

In this case, $\Pi_{m+d}(A)$ is l-rectilinear, so we are done after using Lemma (4.3.27.I) to adjust φ.

Case 2: The closure of $\{y_d : (x, y) \in A\}$ is contained in $(0, 1]$.

In this case, use Lemma (4.3.27.II) to adjust φ to assume that φ is of the form (33), and then apply a flip in y_d to reduce to Case 1.

(Note that if we reduce to either of these two cases, we need not require that $b_d = 1$ or that the requisite powers of $y_{l+1}, \ldots, y_d$ are natural numbers, because the blowup and power substitutions mentioned just prior to these cases can be applied if needed.) So assume that $a_d > 0$, and write

$$a_d(x, y_{<d}) = \hat{a}(x) y_{<d}^{\alpha}\, u(x, y_{<d})$$

for some analytic subanalytic function $\hat{a}$, tuple of rational numbers $\alpha = (\alpha_1, \ldots, \alpha_{d-1})$, and $\varphi_{<d}$-unit u. We proceed by induction on $|\mathrm{supp}(\alpha_{>l})|$, the cardinality of the set $\mathrm{supp}(\alpha_{>l})$.

Suppose that $\mathrm{supp}(\alpha_{>l})$ is empty, and write $y_{\leq l}^{\alpha}$ instead of $y_{<d}^{\alpha}$. Fix a constant C that is greater than the supremum of the range of u. Construct a partition of $\Pi_{m+l}(A)$ into cells over $\mathbb{R}^m$ compatible with the condition $\hat{a}(x) y_{\leq l}^{\alpha} C = 1$. By considering the restriction of φ to $A \cap (B \times \mathbb{R}^{n-l})$ for each cell B from this partition that is open over $\mathbb{R}^m$, we may assume

that either $\hat{a}(x)y^{\alpha}_{\le l}C > 1$ on A or $\hat{a}(x)y^{\alpha}_{\le l}C < 1$ on A. If $\hat{a}(x)y^{\alpha}_{\le l}C > 1$ on A, then a_d is bounded below by a positive constant, and we are in Case 2. So assume that $\hat{a}(x)y^{\alpha}_{\le l}C < 1$ on A.

Consider the two sets

$$\{(x,y) \in A: a_d(x,y_{<d}) < y_d < \hat{a}(x)y^{\alpha}_{\le l}C\} \text{ and } \{(x,y) \in A : \hat{a}(x)y^{\alpha}_{\le l}C < y_d < 1\}.$$

By restricting φ to the first set and then pulling back by a blowup in y_d, we reduce to Case 2. By restricting φ to the second set and then swapping the coordinates y_{l+1} and y_d, we reduce to the case that $\varphi_{\le d}$ is $(l+1)$-rectilinear and φ is prepared over $\mathbb{R}^{m+d}$, and we are done. This completes the proof when $\operatorname{supp}(\alpha_{>l})$ is empty.

Now suppose that $\operatorname{supp}(\alpha_{>l})$ is nonempty. By pulling back by a swap, we may assume that $l+1 \in \operatorname{supp}(\alpha_{>l})$. By pulling back by the power substitution $y_d \mapsto y_d^{\alpha_{l+1}}$,we may also assume that $\alpha_{l+1} = 1$. Let y' and α' be the tuples indexed by $\{1,\dots,d-1\}\backslash\{l+1\}$ that are respectively obtained from $y_{<d}$ and α by omitting their $(l+1)$-th components, and write $y_{<d} = (y', y_{l+1})$; thus $\alpha_{>l} = (1, \alpha_{>l+1})$ and $\alpha'_{>l} = \alpha_{>l+1}$. Fix a constant $C > 1$ that is greater than the supremum of the range of $\hat{a}(x)(y')^{\alpha'}u(x,y',y_{l+1})$; this may may done because $\hat{a}(x)(y')^{\alpha'}y_{l+1}$ is bounded (since it equals $a_d\big((x,y_{<d})/u(x,\ y_{<d})\big)$ and y_{l+1} may freely approach 1 independently of the other variables. Thus

$$a_d(x,y',y_{l+1}) = \hat{a}(x)(y')^{\alpha'}y_{l+1}u(x,y',y_{l+1}) < Cy_{l+1}$$

on A. Consider the three sets,

$$\{(x,y) \in A: C^{-1} < y_{l+1} < 1\},$$
$$\{(x,y) \in A : 0 < y_{l+1} < C^{-1} \text{ and } a(x,y',y_{l+1}) < y_d < Cy_{l+1}\}$$

and

$$\{(x,y) \in A: 0 < y_{l+1} < C^{-1} \text{ and } Cy_{l+1}\, y_d < 1\}.$$

By restricting φ to the first set, we reduce to the case that $\varphi_{\le d}$ is $(l+1)$-rectilinear, and we are done by the induction hypothesis since $|\operatorname{supp}(\alpha_{>l+1})| < |\operatorname{supp}(\alpha_{>l})|$. If we restrict φ to either the second or third set, we may pull back by a blowup in $y_{l+1}$to assume that $C = 1$. On the second set, we may then pull back by a blowup in y_d, and we are done by the induction hypothesis since $|\operatorname{supp}(\alpha'_{>l})| < |\operatorname{supp}(\alpha_{>l})|$. The third set can also be written as $\{(x,y) \in A : 0 < y_d < 1, 0 < y_{l+1} < y_d\}$, so we may reduce to Case 1 by swapping the coordinates y_{l+1} and y_d.

The purpose of the pushforward property is that it ensures that for each sub-analytic function $h: B \to \mathbb{R}$ that is ψ-prepared, $h \circ F^{-1}$ is φ-prepared. This proposition is essentially Theorem (4.3.4), the only differences being that the theorem does not mention the pushforward property and that the theorem deals with an actual partition of D rather than just an open partition of D over $\mathbb{R}^m$. In the proposition we use open partitions over $\mathbb{R}^n$, rather than actual partitions, because it allows the proof of the proposition to be stated somewhat more simply since we may ignore subsets of D whose fibers over $\mathbb{R}^m$ have dimension less than n, and doing so is of no loss to the study of L^p-spaces on D_x .

The following lemma of one-variable calculus, and its corollary, are apparent.

Lemma(4.3.24)[201]: Let $\alpha \in \mathbb{R}$ and $\beta \ge 0$. Then the function $t \mapsto t^{\alpha}(\log t)^{\beta}$ is

I. integrable on $(0,1)$ if and only if $\alpha > -1$;
II. bounded on $(0,1)$ if and only if $\alpha > 0$ or $\alpha = \beta = 0$.

Corollary(4.3.25)[201]: Suppose that $A \subset \mathbb{R}^n$ is l-rectilinear over $\mathbb{R}^0$, and let $\alpha(\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$and $\beta = (\beta_1,\dots,\beta_n) \in [0,\infty)^n$. Then the function $y \mapsto y^{\alpha}|\log y|^{\beta}$ is

I. integrable on A if and only if for all $i \in \{l+1,\dots,n\}, \alpha_i > -1$;
II. bounded on A if and only if for all $i \in \{l+1,\dots,n\}, \alpha_i > 0$ or $\alpha_i = \beta_i = 0$.

Note that if $A \subset \mathbb{R}^{m+n}$ is l-rectilinear over $\mathbb{R}^m$, then by applying Corollary (4. 3.25) to each of the fibers A_x, we see that $y \longmapsto y^\alpha |\log y|^\beta$ is integrable on A_x either for all $x \in \Pi_m(A)$ or for no $x \in \Pi_m(A)$, according to whether the condition given in clause I of the corollary holds; and likewise for boundedness and clause II.

Lemma(4.3.26)[201]: Let $A \subset \mathbb{R}^n$ be l-rectilinear over $\mathbb{R}^0$, and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$.

I. If $\{y^\alpha : y \in A\}$ is bounded, then $\alpha_{l+1}, \dots, \alpha_n \geq 0$.

II. Let $\beta \in \mathbb{Q}$ and $B = \{(y, z) \in A \times \mathbb{R} : a(y) < z < 1\}$, where $0 \leq a(y) < 1$ for all $y \in A$. If $\{y^\alpha z^\beta : (y, z) \in B\}$ is bounded, then $\alpha_{l+1}, \dots, \alpha_n \geq 0$.

Proof: Statement I is clear. Statement II follows from statement I because $\{y^\alpha : y \in A\}$ is in the closure of the set $\{y^\alpha z^\beta : (y, z) \in B\}$, so $\{y^\alpha : y \in A\}$ is bounded if $\{y^\alpha z^\beta : (y, z) \in B\}$ is bounded.

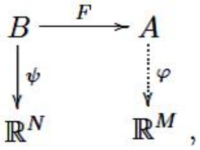
The following lemma is apparent.

Lemma(4.3.27)[201]: Let $\varphi : A \to \mathbb{R}$ be a basic rational monomial map over $\mathbb{R}^m$, where $A \subset \mathbb{R}^{m+n}$ and $\varphi(x, y) = c(x) y^\alpha$.

I. If A is l-rectilinear over $\mathbb{R}^m$ and $\alpha \in \mathbb{Q}^l \times \mathbb{N}^{n-l}$, then $c(x) y_{\leq l}^{\alpha \leq l}$ is bounded on $\Pi_{m+l}(A)$, and φ is a $(c(x) y_{\leq l}^{\alpha \leq l}, y_l, \dots, y_n)$-function.

II. Let $j \in \{1, \dots, n\}$, and put $y' = (y_{<j}, y_{>j})$ and $\alpha' = (\alpha_{<j}, \alpha_{>j})$. If the closure of $\{y_j : (x, y) \in A\}$ is contained in $(0,1]$, then $c(x)(y')^{\alpha'}$ is bounded on A, and φ is a $\left(c(x)(y')^{\alpha'}, y_j\right)$-function.

Definition(4.3.28)[201]: Suppose we are given a basic rational monomial map $\varphi : A \to \mathbb{R}^M$ over $\mathbb{R}^m$, where $A \subset \mathbb{R}^{m+n}$ is a cell over $\mathbb{R}^m$. A pullback construction for φ consists of a subanalytic map $F : B \to A$ and a basic rational monomial map $\psi : B \to \mathbb{R}^N$ over $\mathbb{R}^m$, diagrammed as follows,



where $B \subset \mathbb{R}^{m+n}$ is a cell over $\mathbb{R}^m$, $F : B \to F(B)$ is an analytic isomorphism over $\mathbb{R}^m$, $\det \frac{\partial F}{\partial y}$ and the components of F are ψ-prepared, and $\varphi \circ F$ is a ψ-function.

Observe that these properties ensure that if h is any φ-prepared function, then $h \circ F$ is ψ-prepared.

We will use the six types of pullback constructions listed below, where

$$\Pi_{m+j}(A) = \left\{(x, y_{\leq j}) : (x, y_{<j}) \in \Pi_{m+j-1}(A), a_j(x, y_{<j}) < y_j < b_j(x, y_{<j})\right\} \quad (32)$$

for each $j \in \{1, \dots, n\}$. When defining F below, we only specify its action on coordinates on which it acts nontrivially.

I. Adjustment: This means that F is the identity map (but ψ may be different from φ).

II. Restriction: This means that F is an inclusion map and $\psi = \varphi|_B$.

III. Power Substitution in y_j: This means that F sends $y_j \longmapsto y_j^p$ for some positive integer p, and $\psi = \varphi \circ F$.

IV. Blowup in y_j:This means that we are assuming that $\varphi_{\leq j}$is prepared over $\mathbb{R}^{m+j-1}$ that F sends $y_j \longmapsto y_j b_j(x, y_{<j})$, and that ψ is the pullback of $\overline{\varphi}$ by the transformation sending $y_j \longmapsto y_j \hat{b}(x) y_{<j}^\beta$, where $b_j(x, y_{<j}) = \hat{b}(x) y_{<j}^\beta u(x, y_{<j})$ is the $\varphi_{<j}$-prepared form of b_j and $\overline{\varphi}$ is the natural extension of φ to $\Pi_m(A) \times (0, \infty)^n$

V. Flip in y_j: This means we are assuming that φ is prepared over $\mathbb{R}^{m+j-1}$, that the closure of $\{y_j : (x, y) \in A\}$ is contained in $(0,1]$, that $b_j = 1$, and that φ is of the form

$$\varphi(x,y) = \left(\varphi_{<j}(x,y_{<j}), y_j, \varphi_{>j}(x,y_{<j},y_{>j})\right); \tag{33}$$

F is the transformation sending $y_j \mapsto 1 - y_j$, and ψ is defined by the formula on the right side of (33), but on B rather than on A.

VI. Swap in y_i and y_j :This means that F is the transformation sending $(y_i, y_j) \mapsto (y_j, y_i)$ and $\psi = \varphi \circ F$, provided that the resulting set B is still a cell over $\mathbb{R}^m$.

Note that when (F,ψ) is a flip in y_j, we always assume that φ is prepared over $\mathbb{R}^{m+j-1}$ and that the closure of $\{y_j : (x,y) \in A\}$ is contained in $(0,1]$. We may therefore additionally assume that for each $i \in \{j+1,\ldots,n\}$, the monomials in $y_{<i}$ occurring outside the units in the prepared forms of a_i, b_i and $b_i - a_i$ do not contain any nonzero powers of y_j, because any nonzero powers of y_j may be included in the units.

Definition(4.3.29)[201]: Suppose that we are given a basic rational monomial map $\psi : B \to \mathbb{R}^N$ over $\mathbb{R}^m$ and a subanalytic analytic isomorphism $F: B \to A$ over $\mathbb{R}^m$, where $A, B \subset \mathbb{R}^{m+n}$. A pushforward construction for ψ and F is a basic rational monomial map $\varphi: A \to \mathbb{R}^M$ over $\mathbb{R}^m$, diagrammed as follows,

$$\begin{array}{ccc} B & \xrightarrow{F} & A \\ \downarrow{\scriptstyle\psi} & & \vdots{\scriptstyle\varphi} \\ \mathbb{R}^N & & \mathbb{R}^M \end{array},$$

where the components of $F^{-1}$are φ-prepared and $\psi \circ F^{-1}$ is a φ-function.

Observe that these properties ensure that if h is any ψ-prepared function, then $h \circ F^{-1}$ is φ-prepared.

If $F : B \to A$ is a map from any one of the six types of pullback constructions described above, $\psi' : B' \to \mathbb{R}^{N'}$ is a basic rational monomial map over $\mathbb{R}^m$ with $B' \subset B$, and $A' = F(B')$, then the maps $F|_{B'} : B' \to A'$ and ψ' have an obvious pushforward construction $\varphi' : A' \to \mathbb{R}^{M'}$, provided that when F is a flip in y_j, the map ψ' is of the form $\psi'(x,y) = \left(\psi'_{<j}(x,y_{<j}), y_j, \psi'_{<j}(x,y_{<j},y_{>j})\right)$.

We now come to show the Proposition (4.3.33), which is a preparation result for constructible functions in transformed coordinates on rectilinear sets.

For any set $E \subset \mathbb{R}^m$, let $\emptyset_E$ denote the ring of all analytic germs on E, and let $\emptyset_E[y]$ denote the ring of all polynomials in $y = (y_1,\ldots,y_n)$ with coefficients in $\emptyset_E$. Each member of $\emptyset_E[y]$ is an equivalence class of functions defined on neighborhoods of $E \times \mathbb{R}^n$ in $\mathbb{R}^{m+n}$, and hence defines a function on $E \times \mathbb{R}^n$. For each $\mathcal{F} \subset \emptyset_E[y]$, define the variety of $\mathcal{F}$ by

$$\mathbb{V}(\mathcal{F}) = \{(x,y) \in E \times \mathbb{R}^n : f(x,y) = 0 \text{ for all } f \in \mathcal{F}\}.$$

For each $x \in \mathbb{R}^m$, the ring $\emptyset_{\{x\}}$ is Noetherian, so $\emptyset_{\{x\}}[y]$ is as well. This implies that when E is compact, the varieties of $\emptyset_E[y]$ form the collection of closed subsets of a Noetherian topological space on $E \times \mathbb{R}^n$; in other words, for any $\mathcal{F} \subset \emptyset_E[y]$ there exists a finite $\mathcal{F}' \subset \mathcal{F}$ such that $\mathbb{V}(\mathcal{F}') = \mathbb{V}(\mathcal{F})$.

We partially order $\mathbb{N}^k$ by defining $\alpha \le \beta$ if and only if $\alpha_j \le \beta_j$ for all $j \in \{1,\ldots,k\}$, where $\alpha = (\alpha_1,\ldots,\alpha_k)$ and $\beta = (\beta_1,\ldots,\beta_k)$. For any $\alpha \in \mathbb{N}^k$ write $[\alpha] = \{\beta \in \mathbb{N}^k : \beta \ge \alpha\}$, and for any $A \subset \mathbb{N}^k$ write $[A] = \bigcup_{\alpha \in A}[\alpha]$ for the upward closure of A. If $A \subset \mathbb{N}^k$ is nonempty, define min A to be the set of minimal members of A, and define min $\emptyset = \emptyset$. Dickson's lemma states that min A is finite for every $A \subset \mathbb{N}^k$.The following is a parameterized version of Dickson's lemma.

Lemma(4.3.30)[201]: Let $E \subset \mathbb{R}^m$ be compact and $\{f_\alpha\}_{\alpha\in\mathbb{N}^k} \subset \emptyset_E[y]$. Then the set

$$\bigcup_{(x,y)\in E\times \mathbb{R}^n} \min\{\alpha \in \mathbb{N}^k : f_\alpha(x,y) \neq 0\} \quad (34)$$

is finite.

Proof. The proof is by induction on k, with the base case of $k = 0$ being trivial. Forthe inductive step, use topological Noetherianity to fix $\beta \in \mathbb{N}^k$ such that $\mathbb{V}(\{f_\alpha\}_{\alpha\le\beta}) = \mathbb{V}(\{f_\alpha\}_{\alpha\in\mathbb{N}^k})$. Then (34) is finite because it is contained in

$$\bigcup_{i=1}^{k}\bigcup_{j=0}^{\beta_i}\left(\bigcup_{(x,y)\in E\times \mathbb{R}^n} \min\{\alpha \in \mathbb{N}^n : f_\alpha(x,y) \neq 0, \alpha_i = j\}\right), \quad (35)$$

and each of the sets in parenthesis in (35) is finite by the induction hypothesis.

Lemma(4.3.31)[201]: Let $M \subset \mathbb{N}^k$ be finite. Then there exists a finite partition of $[M]\backslash M$ that is compatible with $\{[\alpha]\}_{\alpha\in M}$ and is such that each member of the partition has a unique minimal member.

Proof. Define $\epsilon = (\epsilon_1,\ldots,\epsilon_k)$ by $\epsilon_i = \max\{\alpha_i : \alpha \in M\}$ for each $i \in \{1,\ldots,k\}$. Let the partition of $[M]\backslash M$ consist of all the singletons $\{\alpha\}$ with $\alpha \in \left(\prod_{i=1}^{k}[0,\epsilon_i]\right) \cap [M]\backslash M$ and all sets of the form

$$\left\{\alpha \in \mathbb{N}^k : \left(\bigwedge_{i\in N} \alpha_i > \epsilon_i\right) \wedge \left(\bigwedge_{j\in\{1,\ldots,k\}\backslash N} \alpha_j = \beta_j\right)\right\},$$

for each nonempty $N \subset \{1\ldots k\}$ and $\beta = (\beta_1\ldots\beta_k)$ in $\left(\prod_{i=1}^{k}[0,\epsilon_i]\right) \cap [M]$

Lemma(4.3.32)[201]: Let $E \subset \mathbb{R}^m$ be compact, and suppose that f is represented by a convergent power series

$$f(x,y,z) = \sum_{\alpha\in\mathbb{N}^k} f_\alpha(x,y) z^\alpha$$

on $E \times \mathbb{R}^n \times [0,1]^k$, where $f_\alpha \in \emptyset_E[y]$ for each $\alpha \in \mathbb{N}^k$. Then we may write

$$f(x,y,z) = \sum_{\alpha\in M^{\mathrm{cr}}} z^\alpha f_\alpha(x,y) + \sum_{\beta\in M^{\mathrm{nc}}} z^\beta f_\beta(x,y,z) \quad (36)$$

on $E \times \mathbb{R}^n \times [0,1]^k$, where the sets $M^{\mathrm{cr}}, M^{\mathrm{nc}} \subset \mathbb{N}^k$ are finite and disjoint, each f_β with $\beta \in M^{\mathrm{cr}}$ is represented by a subseries of $\sum_{\alpha\ge\beta} f_\alpha(x,y) z^{\alpha-\beta}$, and for each $(x,y) \in E \times \mathbb{R}^n$ and each $\beta \in M^{\mathrm{nc}}$, if $f_\beta(x,y,z) \neq 0$ for some $z \in [0,1]^k$, then $f_\alpha(x,y) \neq 0$ for some $\alpha \in M^{\mathrm{cr}}$ with $\alpha \le \beta$.

Proof. Let M^{cr} be the set defined in (34), let S be the partition of $[M^{\mathrm{cr}}]\backslash M^{\mathrm{cr}}$ given by Lemma(4.3.31) , and let M^{nc} be the set of minimal members of the sets in S. For each $\beta \in M^{\mathrm{nc}}$, write S_β for the unique member of S whose minimal member is β, and define $f_\beta(x,y,z) = \sum_{\alpha\in S_\beta} f_\alpha(x,y) z^{\alpha-\beta}$. Then (36) holds. Consider $\beta \in M^{\mathrm{nc}}$ and $(x,y) \in E \times \mathbb{R}^n$ such that $f_\beta(x,y,z) \neq 0$ for some $z \in [0,1]^k$. Then $f_\gamma(x,y) \neq 0$ for some $\gamma \in S_\beta$. Fix $\alpha \in M^{\mathrm{cr}}$ such that $f_\alpha(x,y) \neq 0$ and $\alpha \le \gamma$. Thus $S_\beta \cap [\alpha]$ is nonempty, so $S_\beta \subset [\alpha]$ by the compatibility property of S, and hence $\alpha \le \beta$.

Proposition(4.3.33)[201]: Let $\mathcal{F}$ be a finite set of constructible functions on a sub-analytic set $D \subset \mathbb{R}^{m+n}$. There exists an open partition $\mathcal{A}$of D over $\mathbb{R}^m$such that for each $A \in \mathcal{A}$ there exist a subanalytic analytic isomorphism $F = (F_1,\ldots,F_{m+n}): B \to A$ over $\mathbb{R}^m$, rational monomial maps φ on A and ψ on B over $\mathbb{R}^m$, and $l \in \{0,\ldots,n\}$ with the following properties.

I. Pullback Property: The map ψ is l-rectilinear over $\mathbb{R}^m$, $\det\frac{\partial F}{\partial y}$ is ψ-prepared, and for every $f \in \mathcal{F}$ we may write $f \circ F$ in the form

$$f \circ F(x,y) = \sum_{s\in S} (\log y_{>l})^s \left(\sum_{r\in R_s^{\mathrm{cr}}} y_{>l}^r f_{r,s}(x,y) + \sum_{r\in R_s^{\mathrm{nr}}} y_{>l}^r f_{r,s}(x,y_{\le l}) \right) \tag{37}$$

on B, where the sets $S \subset \mathbb{N}^{n-l}$ and $R_s^{\mathrm{cr}}, R_s^{\mathrm{nc}} \subset \mathbb{Z}^{n-l}$ are finite with $R_s^{\mathrm{cr}} \cap R_s^{\mathrm{nc}} = \emptyset$ for each s, and each function $f_{r,s}$ may be written as a finite sum

$$\begin{cases} f_{r,s}(x,y_{\le l}) = \sum_j g_j(x) y_{\le l}^{\alpha_j} (\log y_{\le l})^{\beta_j} h_j(x,y_{\le l}), \text{ if } r \in R_s^{\mathrm{cr}} \\ f_{r,s}(x,y) = \sum_j g_j(x) y_{\le l}^{\alpha_j} (\log y_{\le l})^{\beta_j} h_j(x,y), \text{ if } r \in R_s^{\mathrm{nc}} \end{cases} \tag{38}$$

where $g_j \in C(\Pi_m(A)), \alpha_j \in \mathbb{Z}^l, \beta_j \in \mathbb{N}^l, h_j$ is either a $\psi_{\le l}$-function or a ψ-function according to whether r is in R_s^{cr} or R_s^{nc}, and the following holds:

$$\begin{cases} \text{For each some } s \in S, r' \in R_s^{nr} \text{ and } (x,y_{\le l}) \in \Pi_{m+l}(B), \text{ if } f_{r',s}(x,y_{\le l},y_{>l}) \neq 0 \\ \text{for some } y_{>l} \in (0,1)^{n-l} \text{ then } f_{r,s}(x,y_{\le l}) \neq 0 \text{ for some } r \in R_s^{cr} \text{ with } r \le r' \end{cases} \tag{39}$$

II. Pushforward Property: The components of F^{-1} are φ-prepared, and $\psi \circ F^{-1}$ is a φ-function.

The superscripts "cr" and "nc" in the notation R_s^{cr} and R_s^{nc} stand for critical and noncritical. We will use (39) to see that the L^p-classes of $f(x,\cdot)$ are determined by which of the terms $f_{r,s}(x,\cdot)$ with $r \in R_s^{cr}$ are identically zero, so in this sense these are the "critical" terms.

In the degenerate case of $l = n$, (37) and (38) simply mean that

$$f \circ F(x,y) = \sum_j g_j(x) y^{\alpha_j} (\log y)^{\beta_j} h_j(x,y)$$

for some constructible functions g_j, tuples $\alpha_j \in \mathbb{Z}^n$ and $\beta_j \in \mathbb{N}^n$, and ψ-functions h_j. To see this, note that if $f \circ F$ is nonzero and $l = n$, then $S = \mathbb{N}^0 = \{0\}$ and $R_0^{cr}, R_0^{nc} \subset \mathbb{Z}^0 = \{0\}$ with $R_0^{cr} \cap R_0^{nc} = \emptyset$, so $R_0^{cr} = \{0\}$ and $R_0^{nc} = \emptyset$ by (39).

Proof. For each $f \in \mathcal{F}$ write $f(x,y) = \sum_i f_i(x,y) \prod_j \log f_{i,j}(x,y)$ for finitely many subanalytic functions $f_i: D \to \mathbb{R}$ and $f_{i,j}: D \to (0,\infty)$. Apply Proposition (4.3.23) to $\bigcup_{f\in\mathcal{F}}\{f_i, f_{i,j}\}_{i,j}$, and focus on one set A in the open partition of D over $\mathbb{R}^m$ that this gives, along with its associated maps $F : B \to A$, φ on A, and ψ on B, where ψ is l-rectilinear over $\mathbb{R}^m$. Thus $\det\frac{\partial F}{\partial y}$ is ψ-prepared, and we may write

$$f \circ F(x,y) = \sum_i a_i(x) y^{\alpha_i} u_i(x,y) \prod_j \log a_{i,j}(x) y^{\alpha_{i,j}} u_{i,j}(x,y)$$

on B for some analytic subanalytic functions a_i and $a_{i,j}$, tuples α_i and $\alpha_{i,j}$ in $\mathbb{Q}^n$, and ψ-units u_i and $u_{i,j}$. By expanding the logarithms and distributing, we may rewrite this in the form

$$f \circ F(x,y) = \sum_i g_i(x) y^{\alpha_i} (\log y)^{\beta_i} h_i(x,y) \tag{40}$$

for some constructible functions g_i, tuples $\alpha_i \in \mathbb{Q}^n$ and $\beta_i \in \mathbb{N}^n$, and ψ-functions h_i. Bypulling back by power substitutions in y, we may assume that $\alpha_i \in \mathbb{Z}^n$ for each α_i in (40).

Write $h_i(x,y) = H_i(\psi_{\le l}(x,y_{\le l}), y_{>l})$ for some analytic function $H_i(X,y_{>l})$ on the closure of the image of ψ.

We are done if $l = n$, so assume that $l < n$ and work by induction on $n - l$. Since the closure of the range of $\psi_{\leq l}$ is compact, we may fix $\epsilon > 0$ such that each function H_i is given by a single convergent power series in $y_{>l}$ with analytic coefficients in $(X, y_{\leq l})$, say

$$H_i(X, y_{>l}) = \sum_{\gamma \in \mathbb{N}^{n-l}} H_{i,\gamma}(X) y_{>l}^{\gamma} \ , \tag{41}$$

for all X in the closure of the range of $\psi_{\leq l}$ and all $y_{>l}$in $[0, \epsilon]^{n-l}$. For each $j \in \{l + 1, \ldots n\}$, by restricting ψ to $\{(x, y) \in B : y_j > \epsilon\}$ and swapping the coordinates y_{l+1} and y_j, we may reduce to the case that ψ is $(l + 1)$-rectilinear, in which case we are done by the induction on $n - l$. So it suffices to restrict ψ to $B \cap (\mathbb{R}^{m+l} \times (0, \epsilon)^{n-l})$. After pulling back by the maps sending $y_j \longmapsto \epsilon y_j$ for each $j \in \{l + 1, \ldots, n\}$, and again expanding the logarithms $\log y_j \epsilon = \log y_j + \log \epsilon$ and distributing, we may assume that $\epsilon = 1$. We are now done pulling back ψ. The pushforward property of the proposition we are showing follows from the fact that φ satisfies the pushforward property of Proposition (4.3.23), because we have only applied some very simple pullback constructions to the map ψ originally given by Proposition (4.3.23). It remains to show that we can express $f \circ F$ as a sum in the desired form.

By grouping terms in (40) according to like powers of log $y_{>l}$, factoring out suitable monomials in y, and absorbing any remaining monomials in $y_{>l}$ with nonnegative powers inside of ψ-functions, we may rewrite (40) in the form

$$f \circ F(x, y) = \sum_{s \in S} (\log y_{>l})^s y^{\delta_s} \sum_{j \in J_s} g_j(x) y_{\leq l}^{\alpha_j} (\log y_{\leq l})^{\beta_j} \, h_j(x, y) \tag{42}$$

for some finite $S \subset \mathbb{N}^{n-l}$ and finite index sets J_s, constructible functions g_j, tuples $\delta_s \in \mathbb{Z}^n$ and $\alpha_j, \beta_j \in \mathbb{N}^l$, and ψ-functions h_j, which we still write as $h_j = H_j \circ \psi$ with H_j written as a power series (41). For each $s \in S$ write

$$G_s \circ \Psi s(x, y) = \sum_{j \in J_s} g_j(x) y_{\leq l}^{\alpha_j} (\log y_{\leq l})^{\beta_j} \, h_j(x, y) \, ,$$

where

$$\Psi s(x, y) = \left(\psi_{\leq l}(x, y_{\leq l}), \log y_{\leq l} \, , \left(g_j(x) \right)_{j \in J_s} , y \right) ,$$

$$G_s(X, Y, Z_s, y) = \sum_{j \in J_s} Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} \, H_j(x, y_{>l}) \, ,$$

with $Z_s = (Z_j)_{j \in J_s}$ and $Y = (Y_1, \ldots, Y_l)$. By computing

$$\sum_{j \in J_s} \left(Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} \sum_{\gamma \in \mathbb{N}^{n-l}} H_{j,\gamma}(X) y_{>l}^{\gamma} \right) = \sum_{\gamma \in \mathbb{N}^{n-l}} \left(\sum_{j \in J_s} Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} H_{j,\gamma}(X) \right) y_{>l}^{\gamma} \tag{43}$$

we may write

$$G_s(X, Y, Z_s, y_{>l}) = \sum_{\gamma \in \mathbb{N}^{n-l}} G_{s,\gamma}(X, Y, Z_s, y_{\leq l}) y_{>l}^{\gamma}$$

with each

$$G_{s,\gamma}(X, Y, Z_s, y_{\leq l}) = \sum_{j \in J_s} Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} H_{j,\gamma}(X) \, .$$

Note that each $G_{s,\gamma}$ is a polynomial in $(Y, Z_s, y_{\leq l})$ with analytic coefficients in X, and X ranges over a compact set. So we may apply Lemma (4.3.32) to get

$$G_s(X,Y,Z_s,y) = \sum_{\gamma\in R_s^{cr}} y_{>l}^{\gamma} G_{s,\gamma}(X,Y,Z_s,y_{\le l}) + \sum_{\gamma\in R_s^{nc}} y_{>l}^{\gamma} G_{s,\gamma}^{\mathrm{nc}}(X,Y,Z_s,y),$$

where R_s^{cr} and R_s^{nc} are disjoint subsets of $\mathbb{N}^{n-l}$, each $G_{s,\gamma}^{\mathrm{nc}}$ is an analytic function represented by a subseries of $\sum_{\delta\ge\gamma} y_{>l}^{\delta-\gamma} G_{s,\delta}(X,Y,Z_s,y_{\le l})$, and for each choice of $(X,Y,Z_s,y_{\le l})$ and $\gamma' \in R_s^{nc}$, if $G_{s,\gamma'}^{nc}(X,Y,Z_s,y_{\le l},y_{>l}) \ne 0$ for some $y_{>l}\in[0,1]^{n-l}$, then there exists $\gamma \in R_s^{cr}$ such that $G_{s,\gamma}(X,Y,Z_s,y_{\le l}) \ne 0$ and $\gamma \le \gamma'$. Write

$$f\circ F(x,y) = \sum_{s\in S} (\log y_{>l})^s\, y^{\delta_s} \left(\sum_{\gamma\in R_s^{cr}} y_{>l}^{\gamma} G_{s,\gamma}\circ \Psi_{s,\le l}(x,y_{\le l}) + \sum_{\gamma\in R_s^{nc}} y_{>l}^{\gamma} G_{s,\gamma}^{\mathrm{nc}} \circ \Psi_s(x,y) \right), \quad (44)$$

where $\Psi_{s,\le l}$ is the map obtained from Ψ_s by omitting its components $y_{>l}$. By distributing each y^{δ_s} and expressing each function $G_{s,\gamma}^{\mathrm{nc}}$ as a sum of terms indexed by $j\in J_s$, via a computation analogous to what was done in (43) for G_s (but going from right to left rather than from left to right), we see that (44) expresses $f\circ F$ in the desired form.

Definition(4.3.34)[201]: Consider a finite set $\mathcal{F}$of constructible functions on a subanalytic set $D\subset\mathbb{R}^{m+n}$, and let $\mathcal{A}$ be an open partition of D over $\mathbb{R}^m$ obtained by applying Proposition (4.3.33) to $\mathcal{F}$. Focus on one $A\in\mathcal{A}$, along with its associated maps $F=(F_1,\dots,F_{m+n}) : B\to A, \varphi$ on A, and ψ on B, where ψ is l-rectilinear over $\mathbb{R}^m$, as in the statement of the proposition. Write $(x,\tilde{y})$ for the coordinates on A with center θ, where θ is the center of φ. Write

$$\det \frac{\partial F}{\partial y}(x,y) = H(x) y^{\gamma} U(x,y)$$

on B for some analytic subanalytic function H, tuple $\gamma=(\gamma_1,\dots,\gamma_n)$ in $\mathbb{Q}^n$, and φ-unit U. For each $f\in\mathcal{F}$ write equation (34) as

$$f\circ F(x,y) = \sum_{(r,s)\in\Delta(f,A)} \hat{f}_{r,s}(x,y)$$

on B, where

$$\Delta^{\mathrm{cr}}(f,A) = \{(r,s) : s\in S \text{ and } r\in R_{\mathrm{s}}^{\mathrm{cr}}\},$$
$$\Delta^{\mathrm{nc}}(f,A) = \{(r,s) : s\in S \text{ and } r\in R_{\mathrm{s}}^{\mathrm{nc}}\},$$
$$\Delta(f,A) = \Delta^{\mathrm{cr}}(f,A) \cup \Delta^{\mathrm{nc}}(f,A),$$
$$\hat{f}_{r,s}(x,y) = \begin{cases} y_{>l}^{r} (\log y_{>l})^s f_{r,s}(x,y_{\le l}), & \text{if } (r,s)\in\Delta^{\mathrm{cr}}(f,A),\\ y_{>l}^{r} (\log y_{>l})^s f_{r,s}(x,y), & \text{if } (r,s)\in\Delta^{\mathrm{nc}}(f,A),\end{cases}$$

for the sets $S, R_{\mathrm{s}}^{\mathrm{cr}}$ and $R_{\mathrm{s}}^{\mathrm{nc}}$ and the functions $f_{r,s}$ defined from f and A in Proposition (4. 3.33). For each $f\in\mathcal{F}$ and $x\in\Pi_m(A)$, define

$$\Delta^{\mathrm{cr}}(f,A,x) = \{(r,s)\in\Delta^{\mathrm{cr}}(f,A) : f_{r,s}(x,y_{\le l}) \ne 0 \text{ for some } y_{\le l}\in\Pi_l(Bx)\},$$
$$\Delta^{\mathrm{nc}}(f,A,x) = \{(r,s)\in\Delta^{\mathrm{nc}}(f,A) : f_{r,s}(x,y) \ne 0 \text{ for some } y\in B_x\},$$
$$\Delta(f,A,x) = \Delta^{\mathrm{cr}}(f,A,x) \cup \Delta^{\mathrm{nc}}(f,A,x),$$
$$\Omega(f,A,x) = \{y_{\le l}\in\Pi_l(B_x) : f_{r,s}(x,y_{\le l}) \ne 0 \text{ for all } (r,s)\in\Delta^{\mathrm{cr}}(f,A,x)\}.$$

For each $x\in\Pi_m(A)$ and $i\in\{l+1,\dots,n\}$, define

$$\bar{r}_i(f,A,x) = \inf\{r_i : (r,s)\in\Delta^{\mathrm{cr}}(f,A,x)\},$$
$$\bar{s}_i(f,A,x) = \sup\{s_i : (r,s)\in\Delta^{\mathrm{cr}}(f,x) \text{ and } r_i = \bar{r}_i(f,A,x)\},$$

under the convention that $\bar{r}_i(f,A,x)=\infty$ and $\bar{s}_i(f,A,x)=0$ when $\Delta^{\mathrm{cr}}(f,A,x)$ is empty.

Remarks(4.3.35)[201]: Consider the situation described in Definition (4.3.34), and let $f \in \mathcal{F}$.

I. For each $x \in \Pi_m(A)$, the set $\Omega(f, A, x)$ is dense and open in $\Pi_l(B_x)$.

Proof. This follows from the fact that for each $x \in \Pi_m(A)$ and $(r, s) \in \Delta^{\mathrm{cr}}(f, A, x)$, $f_{r,s}(x,\cdot)$ is a nonzero analytic function on $\Pi_l(B_x)$, and $\Pi_l(B_x)$ is connected and open in $\mathbb{R}^l$.

II. For each $x \in \Pi_m(A)$,, the set $\Delta^{\mathrm{cr}}(f, A, x)$ is empty if and only if $f(x, y) = 0$ for all $y \in A_x$.

Proof. If $\Delta^{\mathrm{cr}}(f, A, x)$ is empty, then (39) implies that $f(x,\cdot)$ is identically zero on A_x . If $\Delta^{\mathrm{cr}}(f, A, x)$ is nonempty, then the following lemma implies that $f(x,\cdot)$ is not identically zero on A_x.

Lemma(4.3.36)[201]: Consider the situation described in Definition (4.3.34). Fix $f \in \mathcal{F}, i \in \{l + 1, \dots, n\}, x \in \Pi_m(A)$ with $\Delta^{\mathrm{cr}}(f, A, x) \neq \emptyset$, and $y_{\le l} \in \Omega(f, A, x)$. For any tuple $y_{>l} = (y_{l+1}, \dots, y_n)$, write $y' = (y_j)_{j\in\{l+1,\dots,n\}\setminus\{i\}}$ and $y_{>l} = (y', y_i)$. Then the limit

$$\lim_{y_i\to 0} \frac{f \circ F(x, y)}{y_i^{\bar{r}_i(f,A,x)} (\log y_i)^{\bar{s}_i(f,A,x)}} \tag{45}$$

exists for all $y' \in (0, 1)^{n-l-1}$, and the set

$$\{ y' \in (0, 1)^{n-l-1} : (45) \text{ is nonzero}\} \tag{46}$$

is dense and open in $(0, 1)^{n-l-1}$.

Proof. Define

$\Delta_i(f, A, x) = \{(r, s) \in \Delta(f, A, x) : r_i = \bar{r}_i(f, x) \text{ and } s_i = \bar{s}_i(f, x)\}$,

$\Delta_i^{cr} (f, A, x) = \Delta_i(f, A, x) \cap \Delta^{cr}(f, A)$,

$\Delta_i^{nc} (f, A, x) = \Delta_i(f, A, x) \cap \Delta^{nc}(f, A)$.

It follows from (39) that for each $(r, s) \in \Delta(f, A, x)$, either $r_i > \bar{r}_i(f, A, x)$, or $r_i = \bar{r}_i(f, A, x)$ and $s_i \le \bar{s}_i(f, A, x)$. Therefore the limit (45) exists and equals $g(y')$, where $g: (0, 1)^{n-l-1} \to \mathbb{R}$ is the analytic function defined by

$$g(y') = \sum_{(r,s)\in\Delta_i^{cr}(f,A,x)} (y')^{r'} (\log y')^{s'} f_{r,s}(x, y_{\le l}) + \sum_{(r,s)\in\Delta_i^{nc}(f,A,x)} (y')^{r'} (\log y')^{s'} f_{r,s}(x, y_{\le l}, y', 0) .$$

So to show that (46) is dense and open in $(0, 1)^{n-l-1}$, it suffices to show that g is not identically zero. To do that we will show that $g \circ \eta$ is not identically zero, where $\eta : \Lambda \times (0, 1) \to (0,1)^{n-l-1}$ is defined by

$$\eta(\lambda, t) = \left(t^{\lambda_j}\right)_{j\in\{l+1,\dots,n\}\setminus\{i\}}$$

for some suitably chosen open set $\Lambda \subset (0, \infty)^{n-l-1}$.

Note that

$$g \circ \eta(\lambda, t) = \sum_{(r,s)\in\Delta_i^{\mathrm{cr}} (f,A,x)} t^{\lambda\cdot r'} \lambda^{s'} (\log t)^{|s'|} f_{r,s}(x, y_{\le l}) + \sum_{(r,s)\in\Delta_i^{\mathrm{nc}} (f,A,x)} t^{\lambda\cdot r'} \lambda^{s'} (\log t)^{|s'|} f_{r,s}(x, y_{\le l}, \eta(\lambda, t), 0).$$

We may choose Λ so that there exist $\bar{r}' \in \{r' : (r, s) \in \Delta_i^{\mathrm{cr}}(f, A, x)\}$ and $c > 0$ such that for all $(r, s) \in \Delta_i^{\mathrm{cr}}(f, A, x)$ with $r' \neq \bar{r}'$,

$$\lambda \cdot \bar{r}' + c < \lambda \cdot r' \text{ for all } \lambda \in \Lambda. \tag{47}$$

By (39), for each $(r, s) \in \Delta_i^{\mathrm{nc}}(f, A, x)$ there exists ρ such that $(\rho, s) \in \Delta_i^{\mathrm{cr}} (f, A, x)$ and $\rho \le r$ (and necessarily $\rho \neq r$), so $\lambda \cdot \rho' < \lambda \cdot r'$ for all $\lambda \in \Lambda$. Therefore by shrinking Λ and

c, we can ensure that (47) also holds for all $(r,s) \in \Delta_i^{\mathrm{nc}}(f,A,x)$. So by defining

$$\bar{s}' = \max\{|s'|: (r,s) \in \Delta_i^{\mathrm{cr}}(f,A,x) \text{ and } r' = \bar{r}'\},$$

$$\Delta_{i,\Lambda}^{\mathrm{cr}}(f,A,x) = \{(r,s) \in \Delta_i^{\mathrm{cr}}(f,A,x): r' = \bar{r}' \text{ and } |s'| = \bar{s}'\},$$

we see that as t tends to 0, $g \circ \eta(\lambda,t)$ is asymptotic with

$$t^{\lambda\cdot\bar{r}'}(\log t)^{\bar{s}'}\left(\sum_{(r,s)\in\Delta_{i,\Lambda}^{\mathrm{cr}}(f,A,x)} \lambda^{s'} f_{r,s}(x, y_{\le l})\right),$$

which is not identically zero because the sum in parentheses is a nonzero polynomial in λ.

To show the next lemma, we need the following inequality:

$$(x_1 + \cdots + x_k)^p \le x_1^p + \cdots + x_k^p \text{ if } x_1,\ldots,x_k \ge 0 \text{ and } 0 < p \le 1. \qquad (48)$$

The inequality (48) can be verified when $k = 2$ by considering $f(t) = (x_1 + t)^p$ and $g(t) = x_1^p + t^p$, where $x_1 \ge 0$ and $0 < p \le 1$, and then showing that $f(0) = g(0)$ and $f'(t) \le g'(t)$ for all $t > 0$. The general case then follows by induction on k.

Lemma(4.3.37)[201]: Let ν be a positive measure on a set Y, let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be finite families of real-valued ν-measurable functions on Y, and let $p,q > 0$. Put $M = \max\{p,q\}$. Then

$$\int_Y \left|\sum_{i\in I} f_i\right|^p \left|\sum_{j\in J} g_j\right|^q d\nu \le \begin{cases} \displaystyle\sum_{(i,j)\in I\times J} \int_Y |f_i|^p |g_j|^q d\nu & \text{if } M < 1, \\ \displaystyle\left(\sum_{(i,j)\in I\times J}\left(\int_Y |f_i|^p |g_j|^q d\nu\right)^{1/M}\right)^M, & \text{if } M \ge 1 \end{cases}$$

Proof. By symmetry we may assume that $p \ge q$. Then

$$\begin{aligned}
\int_Y \left|\sum_{i\in I} f_i\right|^p \left|\sum_{j\in J} g_j\right|^q d\nu &\le \int_Y \left(\sum_{i\in I}|f_i|\right)^p \left(\sum_{j\in J}|g_j|\right)^q d\nu \\
&= \int_Y \left(\left(\sum_{i\in I}|f_i|\right)\left(\sum_{j\in J}|g_j|\right)^{q/p}\right)^p d\nu \\
&\le \int_Y \left(\left(\sum_{i\in I}|f_i|\right)\left(\sum_{j\in J}|g_j|^{q/p}\right)\right)^p d\nu \text{ by (48)} \\
&= \int_Y \left(\sum_{(i,j)\in I\times J} |f_i||g_j|^{q/p}\right)^p d\nu, \\
&\le \begin{cases} \displaystyle\sum_{(i,j)\in I\times J} \int_Y |f_i|^p |g_j|^q d\nu, & \text{if } p < 1 \\ \displaystyle\left(\sum_{(i,j)\in I\times J}\left(\int_Y |f_i|^p |g_j|^q d\nu\right)^{1/p},\right)^p & \text{if } p \ge 1 \end{cases}
\end{aligned}$$

with the last inequality following from (48) when $p < 1$ and from the triangle inequality for $L^p(\nu)$ when $p \ge 1$.

Lemma(4.3.38)[201]: Consider the situation described in Definition(4.3.34), and suppose that $f, \mu \in \mathcal{F}, q > 0$ and $x \in \Pi_m(A)$. Then

$$\mathrm{LC}(f|_A, |\mu|^q|_A, x) \cap (0,\infty) = \bigcap_{i=l+1}^{n} \{p \in (0,\infty): \bar{r}_i(f,A,x)p + \bar{r}_i(\mu,A,x)q + \gamma_i > -1\}.$$

And, $\infty \in \mathrm{LC}(f|_A, |\mu|^q|_A, x)$ if and only if either $\Delta^{\mathrm{cr}}(\mu,A,x)$ is empty or else for each $i \in \{l+1,\dots,n\}, \bar{r}_i(f,A,x) > 0$ or $\bar{r}_i(f,A,x) = \bar{s}_i(f,A,x) = 0$.

Proof. Let $x \in \Pi_m(A)$. The conclusion is clear from Remark (4.3.35.II) when either $\Delta^{\mathrm{cr}}(f,A,x)$ or $\Delta^{\mathrm{cr}}(\mu,A,x)$ is empty, for then $\mathrm{LC}(f,|\mu|^q,x) = (0,\infty]$ and either $\bar{r}_i(f,A,x) = \infty$ for all $i \in \{l+1,\dots,n\}$ (when $\Delta^{\mathrm{cr}}(f,A,x)$ is empty), or $\bar{r}_i(\mu,A,x) = \infty$ for all $i \in \{l+1,\dots,n\}$ (when $\Delta^{\mathrm{cr}}(\mu,A,x)$ is empty). So we assume that $\Delta^{\mathrm{cr}}(f,A,x)$ and $\Delta^{\mathrm{cr}}(\mu,A,x)$ are both nonempty. Let $p \in (0,\infty)$.

Suppose that

$$\bar{r}_i(f,A,x)p + \bar{r}_i(\mu,A,x)q + \gamma_i > -1 \tag{49}$$

for all $i \in \{l+1,\dots,n\}$. Then

$$r_i p + r_i' q + \gamma_i > -1$$

for all $i \in \{l+1,\dots,n\}, (r,s) \in \Delta(f,A,x)$ and $(r',s') \in \Delta(\mu,A,x)$. By applying Lemma (4.3.37) to the sums $f \circ F = \sum_{(r,s)} \hat{f}_{r,s}$ and $\mu \circ F = \sum_{(r,s)} \hat{\mu}_{r,s}$ using the measure defined from the Jacobian of F in y, and then by applying Corollary (4.3.25), we see that $p \in \mathrm{LC}(f|_A, |\mu|^q|_A, x)$.

Conversely, suppose that $p \in \mathrm{LC}(f|_A, |\mu|^q|_A, x)$, and let $i \in \{l+1,\dots,n\}$. Fubini's theorem and Remark (4.3.35.I) imply that there exist $y_{\le l} \in \Omega(f,A,x) \cap \Omega(\mu,A,x)$ and y' in then set (46) such that

$$yi \longmapsto |f \circ F(x,y)|^p\, |\mu \circ F(x,y)|^q \det \frac{\partial F}{\partial y}(x,y)$$

is integrable on $(0,1)$. So (49) holds by Lemmas (4.3.24) and (4.3.36).

The L^∞ case is similar. Indeed, suppose that $\bar{r}_i(f,A,x) > 0$ or $\bar{r}_i(f,A,x) = \bar{s}_i(f,A,x) = 0$ for all $i \in \{l+1,\dots,n\}$. Then $r_i > 0$ or $r_i = s_i = 0$ for all $i \in \{l+1,\dots,n\}$ and $(r,s) \in \Delta(f,A,x)$. So applying Corollary (4.3.25) to each term of the sum $f \circ F = \sum_{(r,s)} \hat{f}_{r,s}$ shows that $f \circ F(x,\cdot)$ is bounded on B_x, and hence $\infty \in \mathrm{LC}(f|_A, |\mu|^q|_A, x)$.

Conversely, suppose that $\infty \in \mathrm{LC}(f|_A, |\mu|^q|_A, x)$. Then $f \circ F(x,\cdot)$ is bounded on B_x. So for each $i \in \{l+1,\dots,n\}$ we may choose $y_{\le l} \in \Omega(f,A,x)$ and y' in the set (46), and thereby conclude that $\bar{r}_i(f,A,x) > 0$ or $\bar{r}_i(f,A,x) = \bar{s}_i(f,A,x) = 0$ by Lemmas (4.3.24) and (4.3.36).

Proof of Theorem (4.3.1) in the Constructible Case. Let $f, \mu \in C(D)$ for a subanalytic set $D \subset \mathbb{R}^{m+n}$, fix $q > 0$, and write $E = \Pi_m(D)$. Apply Proposition (4.3.33) to $\mathcal{F} = \{f, \mu\}$, and use definition (4.3.34). We claim that for each $A \in \mathcal{A}$, the set

$$\mathfrak{I}_A := \{\mathrm{LC}(f|_A, |\mu|^q|_A, x) : x \in \Pi_m(A)\}$$

is a finite set of open subintervals of $(0,\infty]$ with endpoints in $(\mathrm{span}_{\mathbb{Q}}\{1,q\} \cap [0,\infty)) \cup \{\infty\}$, and that for each $I \in \mathfrak{I}_A$ there exists $g_{A,I} \in C(\Pi_m(A))$ such that

$$\{x \in \Pi_m(A) : I \subset \mathrm{LC}(f|_A, |\mu|^q|_A, x)\} = \{x \in \Pi_m(A): g_{A,I}(x) = 0\}.$$

The claim implies the theorem because for each $x \in E$,

$$LC(f,|\mu|^q,x) = \bigcap_{\substack{A \in \mathcal{A} \text{ s.t.} \\ x \in \Pi_m(A)}} \mathrm{LC}(f|_A, |\mu|^q|_A, x),$$

so the claim shows that $\mathfrak{I}$ is a finite set of open subintervals of $(0,\infty]$ with endpoints in

$(span_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, and that for each $I \in \mathfrak{I}$,

$$\{x \in E : I \subset \mathrm{LC}(f, |\mu|^q, x)\} = \{x \in E : I \subset \mathrm{LC}(f|_A, |\mu|^q|_A, x) \text{ for all } A \in \mathcal{A} \text{ with } x \in \Pi_m(A)\}$$

$$= \left\{ x \in E : \sum_{A \in \mathcal{A}} \sum_{\substack{J \in \mathfrak{I}_A \text{ s.t.} \\ I \subset J}} \left(g'_{A,J}(x)\right)^2 = 0 \right\},$$

where each $g'_{A,J}: E \to \mathbb{R}$ is defined by extending $g_{A,J}$ by 0 on $E \setminus \Pi_m(A)$.

To show the claim, focus on one $A \in \mathcal{A}$. Lemma (4.3.38) shows that each member of $\mathfrak{I}_A$ is an open subinterval of $(0, \infty]$ with endpoints in $(\mathrm{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, and that $\mathfrak{I}_A$ is finite because

$$L \subset \mathrm{LC}(f|_A, |\mu|^q|_A, x) = \mathrm{LC}(f|_A, |\mu|^q|_A, x')$$

for all $x, x' \in \Pi_m(A)$ such that $\Delta^{\mathrm{cr}}(f, A, x) = \Delta^{\mathrm{cr}}(f, A, x')$ and $\Delta^{\mathrm{cr}}(\mu, A, x) = \Delta^{\mathrm{cr}}\Delta(\mu, A, x')$. Fix $I \in \mathfrak{I}_A$ .We may define $g_{A,I} = 0$ if I is empty, so assume that I is nonempty. Let $a = \inf I$ and $b = \sup I$. Lemma (4.3.38) implies that for any $x \in \Pi_m(A)$, when the infimum of $\mathrm{LC}(f|_A, |\mu|^q|_A, x)$ is finite, this infimum is determined by the inequalities (49) for all $i \in \{l + 1, \ldots, n\}$ for which $\bar{r}_i\ (f,\ A,\ x)$ is positive; and similarly, when the supremum of $\mathrm{LC}(f|_A, |\mu|^q|_A, x)$ is finite, this supremum is determined by the inequalities (49) for all $i \in \{l + 1, \ldots, n\}$ for which $\bar{r}_i(f, A, x)$ is negative. Therefore $I \subset \mathrm{LC}(f|_A, |\mu|^q|_A, x)$ if and only if each of the following two conditions hold.

I. If $I \cap (0, \infty)$ is nonempty, then

$$f_{r,s}(x, y_{\leq l}) = 0 \text{ and } \mu_{r',s'}(x, y_{\leq l}) = 0 \text{ for all } y_{\leq l} \in \Pi_l(B_x),$$

for every $(r, s) \in \Delta^{\mathrm{cr}}(f, A)$ and $(r', s') \in \Delta^{\mathrm{cr}}(\mu, A)$ such that for all $i \in \{l + 1, \ldots, n\}$,

$$\begin{cases} r_i a + r'_i q + \gamma_i < -1, & \text{if } r_i > 0, \\ r'_i q + \gamma_i \leq -1, & \text{if } r_i = 0, \\ r_i b + r'_i q + \gamma_i < -1, & \text{if } r_i < 0, \end{cases}$$

with the understanding that we are allowing computations in the extended real number system since a or b could be ∞.

II. If $\infty \in I$, then at least one of the following two conditions hold.

(a) We have

$$\mu_{r',s'}(x, y_{\leq l}) = 0 \text{ for all } y_{\leq l} \in \Pi_l(B_x),$$

for every $(r', s') \in \Delta^{\mathrm{cr}}(\mu, A)$.

(b) We have

$$f_{r,s}(x, y_{\leq l}) = 0 \text{ for all } y_{\leq l} \in \Pi_l(B_x),$$

for every $(r, s) \in \Delta^{\mathrm{cr}}(f, A)$ such that for all $i \in \{l + 1, \ldots, n\}$, either $r_i < 0$, or else $r_i = 0$ and $s_i > 0$.

Therefore $g_{A,I}$ can be constructed using Theorem (4.3.3).

We now turn the attention to stating and showing the preparation theorem.

Definition(4.3.39)[201]: When considering the situation described in Definition (4.3.34), we shall now also write $G = (G_1, \ldots, G_{m+n}): A \to B$ for the inverse of F, and for each $j \in \{l + 1, \ldots, n\}$ write

$$G_{m+j}(x, y) = H_j(x)|\tilde{y}|^{\beta_j} V_j(x, y)$$

on A, where H_j is an analytic subanalytic function, $\beta_j \in \mathbb{Q}^n$, and V_j is a φ-unit.

Lemma(4.3.40)[201]: Consider the situation described in Definition (4.3.34) and (4.3.39) Let $f \in \mathcal{F}$and $(r, s) \in \Delta(f, A)$, where $r = (r_{l+1}, \ldots, r_n)$ and $s = (s_{l+1}, \ldots, s_n)$.

We may express $\hat{f}_{r,s} \circ G$ in the form

$$\hat{f}_{r,s} \circ G(x,y) = \sum_{k\in K_{r,s}(f,A)} T_k(x,y) \quad (50)$$

on A, where $K_{r,s}(f,A)$is a finite index set and for each $k \in K_{r,s}(f,A)$,

$$T_k(x,y) = g_k(x)G_{>m}(x,y)^{R_k} \left(\prod_{j=1}^{n}\left(\log|\tilde{y}|^{\beta_j}\right)^{S_{k,j}}\right)u_k(x,y) \quad (51)$$

for some $g_k \in C(\Pi_m(A))$, tuples $R_k = (R_{k,1},\ldots,R_{k,n}) \in \mathbb{Q}^n$and $S_k = (S_{k,1},\ldots,S_{k,n}) \in \mathbb{N}^n$ satisfying $R_{k,j} = r_j$ and $S_{k,j} \le s_j$ for all $j \in \{l+1,\ldots,n\}$, and φ-units u_k.

Proof. By (38) we may write $\hat{f}_{r,s}\,(x,y)$ as a finite sum of terms of the form

$$g(x)y^R(\log y)^S h(x,y) \quad (52)$$

on B, where $g \in C(\Pi_m(A))$, the tuples $R = (R_1,\ldots,R_n) \in \mathbb{Q}^n$ and $S = (S_1,\ldots,S_n) \in \mathbb{N}^n$ satisfy $R_j = r_j$ and $S_j = s_j$ for all $j \in \{l+1,\ldots,n\}$, and h is a ψ-function. Pulling back (52) by G gives

$$g(x)G_{>m}(x,y)^R \,(\log G_{>m}(x,y))^S\; h\circ G(x,y)$$

on A. In the above equation, by writing

$$\log G_{m+j}(x,y) = \log H_j(x) + \log|\tilde{y}|^{\beta_j} + \log V_j(x,y)$$

for each $j \in \{1,\ldots,n\}$, and then distributing, we obtain the desired form given in (50) and (51), except that each u_k is only a φ-function, not necessarily a φ-unit. But then by writing $u_k = (u_k - c) + c$ for some sufficiently large constant c so that $u_k - c$ and c are both units, and then separating each term in (50) into two terms, we may further assume that each $u_k$in (50) is a φ-unit.

Lemma(4.3.41)[201]: Consider a single term T_k given in (51). We may express $T_k \circ F$ as a finitesum

$$T_k \circ F(x,y) = \sum_{\zeta} g_\zeta(x)y^{R_k}(\log y)^{S_\zeta} h_\zeta(x,y) \quad (53)$$

on B for some $g_\zeta \in C(\Pi_m(A))$, tuples $S_\zeta = (S_{\zeta,1},\ldots,S_{\zeta,n}) \in \mathbb{N}^n$ satisfying $S_{\zeta,j} \le S_{k,j}$ for each $j \in \{1,\ldots,n\}$, and bounded functions h_ζ.

Proof. Since

$$|\tilde{y}|^{\beta_j} = \frac{G_{m+j}(x,y)}{H_j(x)V_j(x,y)}$$

for each $j \in \{1,\ldots,n\}$, it follows from (51) that

$$T_k\circ F(x,y) = g_k(x)y^{R_k}\left(\prod_{j=1}^{n}\left(\log\frac{y_j}{H_j(x)V_j\circ F(x,y)}\right)^{S_{k,j}}\right)u_k\circ F(x,y)$$

on B. In the above equation, write

$$\log\frac{y_j}{H_j(x)V_j\circ F(x,y)} = \log y_j - \log H_j(x) - \log V_j\circ F(x,y)$$

for each $j \in \{1,\ldots,n\}$, and then distribute.

Theorem(4.3.42)[201]: (Preparation of Constructible Functions-Full Version). Let Φ be a finite subset of $C(D)\times C(D)\times(0,\infty)$ for some subanalytic set $D \subset \mathbb{R}^{m+n}$. For each $(f,\mu,q) \in \Phi$ let

$$\mathfrak{T}(f,\mu,q) = \{\mathrm{LC}(f,|\mu|^q,x) : x \in \Pi_m(D)\},$$

and let $\mathcal{F} = \{f,\mu : (f,\mu,q) \in \Phi\}$. Then there exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ into subanalytic cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exist a rational monomial map φ on A over $\mathbb{R}^m$ and rational numbers $\beta_{i,j}$, where $i,j \in \{1,\ldots,n\}$, for which we may express each $f \in \mathcal{F}$ in the form

$$f(x,y) = \sum_{k\in K(f,A)} T_k(x,y) \tag{54}$$

on A, where $K(f,A)$ is a finite index set and for each $k \in K(f,A)$,

$$T_k(x,y) = g_k(x)\left(\left(\prod_{i=1}^{n}|\tilde{y}_i|^{r_{k,i}}\right)\left(\log\prod_{j\backslash=1}^{n}|\tilde{y}_j|^{\beta_{i,j}}\right)^{s_{k,i}}\right)u_k(x,y) \tag{55}$$

for some $g_k \in C(\Pi_m(A))$, rational numbers $r_{k,i}$, natural numbers $s_{k,i}$, and φ-units u_k, where we are writing $(x,\tilde{y})$ for the coordinates on A with center θ, with θ being the center for φ. Moreover, for each $f \in \mathcal{F}$ and $A \in \mathcal{A}$ there exists a partition $\mathcal{P}(f,A)$ of $K(f,A)$ described as follows. For each $A \in \mathcal{A}$, $(f,\mu,q) \in \Phi$, $K \in \mathcal{P}(f,A)$, $\Lambda \in \mathcal{P}(\mu,A)$, and $I \in \mathfrak{T}(f,\mu,q)$, at least one of the following two statements holds:

I. for all $(\kappa,\lambda) \in K \times \Lambda$, we have $\Pi_m(A) \times I \subset \mathrm{LC}(T_\kappa,|T_\lambda|^q,\Pi_m(A))$;

II. for all $x \in \Pi_m(A)$ such that $I \subset \mathrm{LC}(f,|\mu|^q,x)$, either $\sum_{\kappa\in K} T_\kappa(x,y) = 0$ for all $y \in A_x$ or $\sum_{\lambda\in\Lambda} T_\lambda(x,y) = 0$ for all $y \in A_x$; and if statement II does not hold, then

$$\Pi_m(A) \times (I\backslash\{\infty\}) \subset \mathrm{LC}\left(T'_\kappa,|T'_\lambda|^q,\Pi_m(A)\right) \tag{56}$$

for all $(\kappa,\lambda) \in K \times \Lambda$ and all functions T'_κ and T'_λ of the form

$$T'_\kappa(x,y) = \prod_{i=1}^{n}|\tilde{y}_i|^{r_{\kappa,i}}\left(\log\prod_{j=1}^{n}|\tilde{y}_i|^{\beta'_{\kappa,i,j}}\right)^{s'_{\kappa,i}}$$

$$T'_\lambda(x,y) = \prod_{i=1}^{n}|\tilde{y}_i|^{r_{\lambda,i}}\left(\log\prod_{j=1}^{n}|\tilde{y}_i|^{\beta'_{\lambda,i,j}}\right)^{s'_{\lambda,i}}$$

where the $\beta'_{\kappa,i,j}, \beta'_{\lambda,i,j} \in \mathbb{Q}$ and $s'_{\kappa,i}, s'_{\lambda,i} \in \mathbb{N}$ are arbitrary and the $r_{\kappa,i}, r_{\lambda,i}$ are as in (55).

Proof. Apply Proposition (4.3.33) to $\mathcal{F}$. Fix $A \in \mathcal{A}$ and use the notation found in Definition (4.3.34) and (4.3.39) and in Lemmas (4.3.40) and (4.3. 41). Lemma (4. 3.40) shows that each $f \in \mathcal{F}$ may be written in the form given in (54) and (55), where each T_k is defined as in (51) and

$$K(f,A) = \bigcup_{(r,s)\in\Delta(f,A)} K_{r,s}(f,A).$$

For each $f \in \mathcal{F}$, define

$$\mathcal{P}(f,A) = \{K_{r,s}(f,A)\}_{(r,s)\in\Delta(f,A)}.$$

Now also fix $(f,\mu,q) \in \Phi, K \in \mathcal{P}(f,A), \Lambda \in \mathcal{P}(\mu,A)$ and $I \in \mathfrak{T}(f,\mu,q)$.Write $K = K_{r,s}(f, A)$ and $\Lambda = K_{r',s'}(\mu,A)$ for some $(r,s) \in \Delta(f,A)$ and $(r',s') \in \Delta(\mu,A)$. We are done if statement II in the last sentence of the theorem holds, so assume otherwise. Therefore we may fix $x_0 \in \Pi_m(A)$ such that $I \subset \mathrm{LC}(f,|\mu|^q,x_0), (r,s) \in \Delta(f,A,x_0)$ and $(r',s') \in \Delta(\mu, A,x_0)$. Lemma (4.3.35) gives the following.

$$\begin{cases}\text{For all } p \in I \cap (0,\infty) \text{ and all } i \in \{l+1,\dots,n\}\\ \quad \bar{r}_i(f,A,x_0)p + \bar{r}_i(\mu,A,x_0)q + \gamma_i > -1.\end{cases} \tag{57}$$

$$\begin{cases}\text{If } \infty \in I, \text{ then For all } i \in \{l+1,\dots,n\}\\ \quad \bar{r}_i(f,A,x_0) > 0 \text{ or } \bar{r}_i(f,A,x_0) = \bar{s}_i(f,A,x_0) = 0.\end{cases} \tag{58}$$

Let $\kappa \in K$ and $\lambda \in \Lambda$. Write T_κ and T_λ as in (51) with $k = \kappa$ and $k = \lambda$, respectively, and write

$$T_\kappa \circ F(x,y) = \sum_\zeta g_\zeta(x) y^{R_\kappa} (\log y)^{S_\zeta} h_\zeta(x,y) \tag{59}$$

$$T_\lambda \circ F(x,y) = \sum_\eta g_\eta(x) y^{R_\lambda} (\log y)^{S_\eta} h_\eta(x,y) \tag{60}$$

as in (53). Note that for each $i \in \{l+1, \ldots, n\}$,

$$R_{\kappa,i} = r_i \geq \bar{r}_i(f, A, x_0) \text{ and } R_{\lambda,i} = r_i' \geq \bar{r}_i(\mu, A, x_0). \tag{61}$$

So (57) holds with $R_{\kappa,i}$ and $R_{\lambda,i}$ in place of $\bar{r}_i(f, A, x_0)$ and $\bar{r}_i(\mu, A, x_0)$, respectively. Therefore by Corollary (4.3.25), Lemma (4.3.37), (59) and (60), it follows that

$$\Pi_m(A) \times (I \setminus \{\infty\}) \subset \mathrm{LC}(T_k, |T_\lambda|^q, \Pi_m(A)).$$

Note that the proof of this fact depends only the values of r and r', being independent the values of $\beta_1, \ldots, \beta_n, s$ and s', so (56) follows.

Now suppose that $\infty \in I$. Note that for each ζ and $i \in \{l+1, \ldots, n\}$, we have $S_{\zeta,i} \leq S_{\kappa,i} \leq s_i$. Combining this with (61) shows that for each $i \in \{l+1, \ldots, n\}$, either $R_{\kappa,i} > 0$ or else $R_{\kappa,i} = S_{\zeta,i} = 0$ for all ζ. Therefore Corollary (4.3.25) and (59) show that $T_\kappa \circ F(x, \cdot)$ is bounded on B_x
for each $x \in \Pi_m(A)$. So $\Pi_m(A) \times \{\infty\} \subset \mathrm{LC}(T_k, |T_\lambda|^q, \Pi_m(A))$.

This completes the proof of the theorem, except for the fact that A need not be a cell over $\mathbb{R}^m$. To remedy this, simply construct an open partition of A over $\mathbb{R}^m$ consisting of cells over $\mathbb{R}^m$ (for instance, using Proposition (4.3.9)), and then restrict to each of these cells.

Theorem (4.3.42) was formulated in such a way so as to be as strong and general as possible, but at the cost of having a technical formulation that may obscure the fact that it implies the simpler Theorem (4.3.2). The corollary of Theorem(4.3.42) given below directly implies Theorem (4.3 .2) and its analog for $p = \infty$ described in (20), and it generalizes the interpolation theorem [183].

The proof of the corollary makes use of the following observation: for the set $\mathcal{F}$ from Theorem (4.3.42), if $f \in \mathcal{F}$ is subanalytic, then the restriction of f to A is φ-prepared(as opposed to being in the more general form allowed by (54)and(55)). This observation follows from the way the proof of Theorem (4.3.42) uses Proposition (4. 3.33), and from the way the proof of Proposition (4.3.33) uses Proposition (4.3.23).

Corollary(4.3.43)[201]: Suppose that $P \subset (0, \infty]$, that $D \subset \mathbb{R}^{m+n}$ is subanalytic, and that Φ is a finite set of triples (f, μ, q) for which $f: D \to \mathbb{R}$ is constructible, $\mu: D \to \mathbb{R}$ is subanalytic, and $q > 0$. Define $E = \Pi_m(D)$ and $\mathcal{F} = \{f : (f, \mu, q) \in \Phi\}$. Then to each $f \in \mathcal{F}$ we may associate a function $f^* \in C(D)$ in such a way so that the following statements hold.

I. There exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exists a rational monomial map φ on A over $\mathbb{R}^m$ such that for every $(f, \mu, q) \in \Phi$, the function μ *is* φ-prepared and we may express f^* as a finite sum

$$f^*(x,y) = \sum_k T_k(x,y) \tag{62}$$

on A, where each function T_k is of the form (55).

II. The following hold for all $(f, \mu, q) \in \Phi$.

(a) We have $f = f^*$ on $\{(x,y) \in D : P \subset \mathrm{LC}(f, |\mu|^q, x)\}$.

(b) For all $A \in \mathcal{A}$ and all terms T_k in the sum (62), we have $\Pi_m(A) \times P \subset \mathrm{LC}(T_k, |\mu|^q, \Pi_m(A))$. (Hence $E \times P \subset \mathrm{LC}(f^*, |\mu|^q, E)$.).

III. If $\infty \notin P$, then we may take each function T_k *to* be of the simpler form

$$T_k(x,y) = g_k(x)\left(\prod_{i=1}^{n}|\tilde{y}_i|^{r_{k,i}}(\log|\tilde{y}_i|)^{s_{k,i}}\right)^{s_{k,i}} u_k(x,y), \tag{63}$$

and the fact that $\Pi_m(A) \times P \subset \mathsf{LC}(T_k, |\mu|^q, \Pi_m(A))$ only depends on the values of the $r_{k,i}$, and not the values of the $s_{k,i}$, in the following sense: we have $\Pi_m(A) \times P \subset \mathsf{LC}(T'_k, |\mu|^q, \Pi_m(A))$ for any function T'_k on A of the form

$$T'_k(x,y) = \prod_{i=1}^{n}|\tilde{y}_i|^{r_{k,i}}(\log|\tilde{y}_i|)^{s'_{k,i}},$$

where the $r_{k,i}$ are as in (63) and the $s'_{k,i}$ are arbitrary natural numbers.

Proof. Let $\mathcal{A}$ be the open partition of D obtained by applying Theorem (4 .3.42) to Φ; we use the notation of the theorem. Because μ is subanalytic for every $(f,\mu,q) \in \Phi$, it follows that we may partition the members of $\mathcal{A}$ further in the x-variables to assume that for each $A \in \mathcal{A}$ and each $(f,\mu,q) \in \Phi$, either $\mu(x,y) = 0$ for all $(x,y) \in A$, or else for each $x \in \Pi_m(A)$ there exists $y \in A_x$ such that $\mu(x,y) \neq 0$. Therefore for all $(f,\mu,q) \in \Phi, I \in \mathfrak{I}(f,\mu,q), A \in \mathcal{A}$ and $K \in \mathcal{P}(f,A)$, at least one of the following two statements holds.

I. For every $k \in K$ we have $\Pi_m(A) \times I \subset \mathsf{LC}(T_k, |\mu|^q, \Pi_m(A))$.

II. We have $\sum_{k\in K} T_k(x,y) = 0$ on $\{(x,y) \in A : I \subset LC(f, ||\mu|^q, x)\}$.

For each $(f,\mu,q) \in \Phi$ and $A \in \mathcal{A}$, define $K^*(f,A)$ to be the union of all $K \in \mathcal{P}(f,A)$ for which there exists $I \in \mathfrak{I}(f,\mu,q)$ such that $P \subset I$ and the above statement I holds. For each $(f,\mu,q) \in \Phi$, define f^* by

$$f^*(x,y) = \begin{cases} \sum_{k\in K^*(f,A)} T_k(x,y) & \text{if } (x,y) \in A \text{ with } A \in \mathcal{A}, \\ f(x,y) & \text{if } (x,y) \in D \backslash \cup \mathcal{A}. \end{cases}$$

Observe that statements I and II of the corollary hold.

To prove statement III, suppose that $\infty \notin P$. By writing

$$\log\prod_{j=1}^{n}|\tilde{y}_j|^{\beta_{i,j}} = \sum_{j=1}^{n}\beta_{i,j}\log|\tilde{y}_j|$$

in (55) and then distributing, we may write each term T_k as a finite sum of terms of the form (63) with the same values of the $r_{k,i}$ but possibly different values of the the $s_{k,i}$. But only the values of the $r_{k,i}$ are relevant by (56) since $\infty \notin P$.

The analog of Theorem (4.3.2) for $p = \infty$ mentioned can be stated as follows: if $D \subset \mathbb{R}^{m+n}$ is subanalytic and $f \in C(D)$ is such that $\mathsf{Int}^\infty(f, \Pi_m(D)) = \Pi_m(D)$, then there exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ into cells over $\mathbb{R}^m$ such that for every $A \in \mathcal{A}$ we may express f as a finite sum $f(x,y) = \sum_k T_k\ (x,y)$ on A for terms $T_k$with $\mathsf{LC}^\infty(T_k, \Pi_m(A)) = \Pi_m(A)$ that are of the form

$$T_k(x,y) = g_k(x)\left(\prod_{i=1}^{n}|\tilde{y}_i|^{r_i}\left(\log\prod_{j=1}^{n}|\tilde{y}_j|^{\beta_{i,j}}\right)^{s_{k,i}}\right)u_k(x,y), \tag{64}$$

as denoted in the above. This statement was proven in Corollary (4.3. 43). A more literal analog of Theorem (4.3.2) for $p = \infty$ would require the terms T_k to be of the simpler form

$$T_k(x,y) = g_k(x)\left(\prod_{i=1}^{n}|\tilde{y}_i|^{r_i}(\log|\tilde{y}_i|)^{s_{k,i}}\right)u_k(x,y); \tag{65}$$

however, this more literal analog is false, and the purpose of this part is to prove this by giving a counterexample. It follows that in Statement III of Corollary (4.3.43), one may not drop the assumption that $\infty \notin P$; and in Theorem (4.3.42), one may not replace (56) with the statement $\Pi_m(A) \times I \subset \mathrm{LC}(T'_\kappa, |T'_\lambda|^q, \Pi_m(A))$.

We write $(x, y) = (x, y_1, y_2)$ for coordinates on $\mathbb{R}^3$, and define $f: D \to \mathbb{R}$ by

$$f(x, y) = \log\left(\frac{y_1}{y_2}\right), \tag{66}$$

where

$$D = \{(x, y) \in \mathbb{R}^3 : 0 < x < 1, 0 < y_1 < 1, xy_1 < y_2 < y_1\}. \tag{67}$$

Note that the function $f(x,\cdot)$ is bounded on D_x for every $x \in (0,1)$, and that the function f is already a single term of the form given in (64) on D. The obvious way to express f as a sum of terms of the form (65) is to write

$$f(x, y) = \log y_1 - \log y_2$$

on D; however, the terms log y_1 and log y_2 now become unbounded on each fiber D_x. It should therefore seem feasible that f is a counterexample for the more literal analog of Theorem (4.3.2) for $p = \infty$. To show that this is in fact the case, we show an assertion below.

Lemma(4.3.44)[201]: Let $A = \{(x, z) \in \mathbb{R}^2 : 0 < x < 1,\ x < z < 1\}$, and define an analytic isomorphism $\eta : (0,1)^2 \to A$ by

$$\eta(x, t) = (x, x^t).$$

Suppose that $g: A \to \mathbb{R}$ is a function of the form

$$g(x, z) = \sum_{i \in I} (\log x)^i x^{\alpha_i} z^{\beta_i} g_i(x, z) \tag{68}$$

where $I \subset \mathbb{N}$ is finite and nonempty, the α_i and β_i are integers, and each g_i is a function on A that is not identically zero and is of the form

$$g_i(x, z) = G_i\left(x, z, \frac{x}{z}\right)$$

for an analytic function G_i on $[0,1]^3$ represented by a single convergent power series, say

$$G_i(X) = \sum_{\gamma \in \mathbb{N}^3} G_{i,\gamma} X^\gamma, \quad \text{for} \quad X \in [0,1]^3.$$

Then there exist $\epsilon \in (0,1]$, a nonzero real number a, a natural number r, and integers p and q such that for all $t \in (0, \epsilon)$,

$$\lim_{x \to 0} \frac{g \circ \eta(x, t)}{x^{p+qt} (\log x)^r} = a. \tag{69}$$

Proof. By factoring out the lowest powers of x and z in (68), we may assume that the α_i and β_i are all natural numbers. But then each monomial $x^{\alpha_i} z^{\beta_i}$ can be incorporated into the function g_i, so we may in fact assume that the numbers α_i and β_i are all zero. For each

$$\begin{aligned} g_i \circ \eta(x, t) &= G_i(x, x^t, x^{1-t}) \\ &= \sum_{\gamma \in \mathbb{N}^3} G_{i,\gamma}\, x^{\gamma_1 + t\gamma_2 + (1-t)\gamma_3} \sum_{k=0}^{\infty} \sum_{l=-k}^{\infty} G_i^{[k,l]}\, x^{k+lt}, \end{aligned}$$

where

$$G_i^{[k,l]} = \sum_{\substack{\gamma \in \mathbb{N}^3 \text{s.t} \\ \gamma_1 + \gamma_3 = k, \gamma_2 - \gamma_3 = l}} G_{i,\gamma}.$$

So

$$g \circ \eta(x,t) = \sum_{i \in I} (\log x)^i g_i \circ \eta(x,t)$$
$$= \sum_{i \in I} \sum_{k=0}^{\infty} \sum_{l=-k}^{\infty} G_i^{[k,l]} x^{k+lt} (\log x)^i. \quad (70)$$

Note that for each $i \in I$, the function g_i is not identically zero and η is a bijection, so $g_i \circ \eta$ is not identically zero, which implies that $G_i^{[k,l]} \neq 0$ for some k and l.

Let (p,q) be the lexicographically minimum member of the set

$$\bigcup_{i \in I} \{(k,l) \in \mathbb{N} \times \mathbb{Z} : k + l \geq 0 \text{ and } G_i^{[k,l]} \neq 0\}, \quad (71)$$

and define $r = \max\{i \in I : G_i^{[p,q]} \neq 0\}$, $a = G_r^{[p,q]}$, and $\epsilon = \frac{1}{p+q+1}$. We claim that for all $(k,l) \neq (p,q)$ in the set (71) and all $t \in (0,\epsilon)$,

$$k + lt > p + qt. \quad (72)$$

The claim and (70) together imply (69). To prove the claim, consider $(k,l) \neq (p,q)$ in (71). If $k = p$, then $l > q$, in which case (72) holds for all $t > 0$. So suppose that $k \geq p+1$. Simplifying the inequality $(p+1)(1-t) > p + qt$ shows that it is equivalent to the inequality $t < \epsilon$. So for all $t \in (0,\epsilon)$,

$$k + lt = k(1-t) + (k+l)t \geq (p+1)(1-t) + 0t > p + qt,$$

which proves the claim.

Assertion(4.3.45)[201]: For the function $f: D \to \mathbb{R}$ defined in (66) and (67), there does not exist an open cover $\mathcal{A}$ of D over $\mathbb{R}$ such that for each $A \in \mathcal{A}, f$ may be written as a finite sum of terms T_k of the form (65) with each $T_k(x,\cdot)$ bounded on A_x for all $x \in \Pi_m(A)$.

In the following proof, we shall say that two functions $g, h : A \to \mathbb{R}\backslash\{0\}$ are equivalent on A if the range of g/h is contained in a compact subset of $(0,\infty)$.

Proof: Suppose for a contradiction that there exists an open cover $\mathcal{A}'$ of D over $\mathbb{R}$ such that for each $A' \in \mathcal{A}'$, f may be written as a finite sum $f(x,y) = \sum_k T_k(x,y)$ on A' for terms $T_k$of the form (65) with each $T_k(x,\cdot)$ bounded on A'_x for all $x \in \Pi_m(A')$; note that we associate to A' a certain rational monomial map φ' on A' over $\mathbb{R}$ that is used to defined the terms T_k. By Proposition (4.3.9) there exists an open cover $\mathcal{A}$ of D over $\mathbb{R}^0$ such that for each $A \in \mathcal{A}$ there exist a unique $A' \in \mathcal{A}'$ containing A and a prepared rational monomial map φ on A over $\mathbb{R}^0$ such that for each function g_k occurring in (65), say of the form

$$g_k(x) = \sum_i g_{k,i}(x) \prod_j \log g_{k,i,j}(x) \quad (73)$$

for subanalytic functions $g_{k,i}$ and $g_{k,i,j}$, the functions $g_{k,i}$ and $g_{k,i,j}$ are all $\varphi_{\leq 1}$-prepared on $\Pi_1(A)$. The functions xy_1 and y_1 are not equivalent for x near 0, so we may fix $A \in \mathcal{A}$ of the form

$$A = \{(x,y): 0 < x < b_0, 0 < y_1 < b_1(x), a_2(x,y_1) < y_2 < b_2(x,y_1)\}$$

with $a_2$and b_2 not equivalent on $\Pi_2(A)$. Let φ be the rational monomial map on A over $\mathbb{R}^0$ associated with A. Note that x is not equivalent on $\Pi_1(A)$ to a constant, that y_1 is not equivalent on $\Pi_2(A)$ to a function of x, and that y_2 is not equivalent on A to a function of (x,y_1), so φ must have center 0. For the same reason, if A' is the unique member of $\mathcal{A}'$ containing A, and if φ' is the rational monomial map over $\mathbb{R}$ associated with A', then φ' must also have center 0. We are only interested in the restriction of f to A, so we may therefore simply assume that $A' = A$ and $\varphi = \varphi'$. So we may write

$$\log\left(\frac{y_1}{y_2}\right) = \sum_k g_k(x) y_1^{r_{k,1}} y_2^{r_{k,2}}, (\log y_1)^{s_{k,1}} (\log y_2)^{s_{k,2}} u_k(x, y_1, y_2) \tag{74}$$

on A for the constructible functions g_k given in (73), rational numbers $r_{k,1}$ and $r_{k,2}$, natural numbers $s_{k,1}$ and $s_{k,2}$, and φ-units u_k; and we may write

$$a_2(x, y_1) = x^\alpha y_1 u(x, y_1) \text{ and } b_2(x, y_1) = x^\beta y_1 v(x, y_1)$$

on $\Pi_2(A)$ for some rational numbers α and β satisfying $0 \le \beta < \alpha \le 1$ and some $\varphi_{\le 2}$-units u and v.

Fix positive constants c and d satisfying $c > u(x, y_1)$ and $d < v(x, y_1)$ on $\Pi_2(A)$. Since $\alpha > \beta$, by shrinking b_0 we may assume that

$$A = \{(x, y) : 0 < x < b_0, 0 < y_1 < b_1(x), cx^\alpha y_1 < y_2 < dx^\beta y_1\}.$$

Pulling back the equation (74) by the map $(x, y_1, y_2) \mapsto (x, y_1, y_1 y_2)$ gives

$$\log\left(\frac{1}{y_2}\right) = \sum_k g_k(x) y_1^{r_{k,1}+r_{k,2}} y_2^{r_{k,2}} (\log y_1)^{s_{k,1}+s_{k,2}} \left(1 + \frac{\log y_2}{\log y_1}\right)^{s_{k,2}} u_k(x, y_1, y_1 y_2) \tag{75}$$

on the set

$$\{(x, y_1, y_2) : 0 < x < b_0, 0 < y_1 < b_1(x), cx^\alpha < y_2 < dx^\beta\}.$$

By assumption, each term of (75) is bounded for each fixed value of x, so letting y_1 tend to 0 for each fixed value of (x, y_2) shows that for each k, either $r_{k,1} + r_{k,2} > 0$ or $r_{k,1} + r_{k,2} = s_{k,1} + s_{k,2} = 0$ (and $s_{k,1} + s_{k,2} = 0$ means that $s_{k,1} = s_{k,2} = 0$). So letting y_1 tend to 0 in (75) gives

$$\log\left(\frac{1}{y_2}\right) = \sum_k g_k(x) y_2^{r_{k,2}} v_k(x, y_2) \tag{76}$$

on

$$\{(x, y_2) : 0 < x < b_0, cx^\alpha < y_2 < dx^\beta\},$$

where each v_k is a ψ-unit with ψ defined by $\psi(x, y_2) = \lim_{y_1 \to 0} \varphi(x, y_1, y_1 y_2)$.

By pulling back (76) by the map $(x, y_2) \mapsto \left(x, cx^\beta y_2^{\alpha-\beta}\right)$ and expanding logarithms using (73), we may write

$$\log y_2 = \sum_i (\log x)^i \, x^{\alpha_i} \, y_2^{\beta_i} f_i(x, y_2) \tag{77}$$

on

$$\{(x, y_2) : 0 < x < b_0, \ x < y_2 < C\} \tag{78}$$

for some $C > 0$, rational numbers α_i and β_i, and ψ-functions f_i (for an appropriately modified ψ), where i ranges over some finite set of natural numbers. By pulling back by $(x, y_2) \mapsto (x^r, y_2^r)$ for a suitable positive integer r, we may further assume that all the α_i and β_i are integers, and that the components of $\psi(x, y_2)$ are also all monomial in (x, y_2) with integer powers. Thus each component of ψ is either of the form x^p for some positive integer p, is of the form y_2^q for some positive integer q, or is of the form $\frac{x^p}{y_2^q} = x^{p-q}(x/y_2)^q$ for some positive integers p and q with $p \ge q$. So we may assume that $\psi(x, y_2) = (x, y_2, x/y_2)$, and therefore write $f_i(x, y_2) = F_i(x, y_2, x/y_2)$ for some analytic function F_i defined on the closure of $\{(x, y_2, x/y_2) : (x, y_2) \in A\}$. Fix $\delta > 0$ sufficiently small so that

$$\{(x, y_2) : 0 < x < \delta^2, 0 < y_2 < \delta, x/y_2 < \delta\} \tag{79}$$

is contained in (78) and that F_i is represented by a single convergent power series on $[-\delta^2, \delta^2] \times [-\delta, \delta] \times [-\delta, \delta]$. Thus restricting to (79) and then pulling back by $(x, y_2) \mapsto (\delta^2 x, \delta y_2)$ gives an equation of the form

$$\log y_2 = \sum_i (\log x)^i \, x^{\alpha_i} \, y_2^{\beta_i} F_i\left(x, y_2, \frac{x}{y_2}\right) \tag{80}$$

on

$$\{(x, y_2): 0 < x < 1, x < y_2 < 1\},$$

with each F_i represented by a single convergent power series on $[-1,1]^3$ centered at the origin.

Applying Lemma (4.3.44) to the right side of (80) shows that there exist $\epsilon \in (0,1]$, a nonzero real number a, a natural number r, and integers p and q such that for all $t \in (0, \epsilon)$,

$$\lim_{x \to 0} \frac{t \log x}{x^{p+qt} (\log x)^r} = a.$$

Considering this limit for any fixed value of $t \in (0, \epsilon)$ shows that $r = 1$ and that $p + qt = 0$, so in fact $p = q = 0$ since $t \in (0, \epsilon)$ is arbitrary.

But then $t = a$ for all $t \in (0, \epsilon)$, which is a contradiction that completes the proof.