Chapter 3

Non-Selfadjoint Spectral Problems and Spectral Decomposition

In this chapter we establish completeness results for normal problems in certain finite co-dimensional subspaces of $W_2^k(0,1)$ which are characterized by means of Jordan chains in 0 of the adjoint of the compact operator $\mathbb{A} = \mathbb{H}\mathbb{K}^{-1}$. Under an additional assumption, the spectrum of $\tilde{A}$ consists of one part in the extended right and one part in the left half plane, and the corresponding spectral subspaces allow representations by means of angular operators. If the part of the spectrum of $\tilde{A}$ in the right half plane is discrete, a half range completeness statement follows. As an essential tool the quadratic numerical range of a block operator matrix is introduced.

Sec(3.1): Linear Pencils $N - \lambda P$ of Ordinary Differential Operators with λ-Linear Boundary Conditions

We consider boundary eigenvalue problems for pairs of ordinary differential operators of the form

$$N(y) - \lambda P(y) = 0, \tag{1}$$

$$U_j(y) - \lambda V_j(y) = 0, \qquad j = 1,2,\ldots,n, \tag{2}$$

on the finite interval [0,1] where the boundary conditions depend linearly on the eigenvalue parameter. Here N and P are ordinary linear differential expressions of order n and p, respectively, with $n > p \geq 0$, and U_j, V_j are boundary values in 0 and 1 of order $\leq n-1$ and $\leq p-1$, respectively.

The study of spectral problems which are not of the form $(T-\lambda)y = 0$ with some linear operator T is motivated by concrete examples. Both kinds of examples lead to nonself-adjoint boundary eigenvalue problems of the form (1), (2) where the order of P is greater than 0. In the first case the boundary conditions are independent of λ, in the second example they may depend on λ linearly.

The present section solves the problem of the completeness of the eigenfunctions and associated functions for non-selfadjoint problems $(1), (2)$. The completeness results of [147] for λ-independent boundary conditions are obtained as a particular case. However, if the boundary conditions depend on the eigenvalue parameter, a different operator approach has to be used. The operators cannot be settled in $L_2(0,1)$, they are defined on $L_2(0,1)$ but act to the larger space $L_2(0,1) \times \mathbb{C}^n$. A similar method of extending the space is often used in the selfadjoint case. However, following to some extent the lines of [138], the operators we associate with the given problem act between different spaces. Moreover, they involve the λ-dependent as well as the λ-independent boundary conditions.

These operators are introduced in next part below. The spectral problem (1), (2) is written equivalently as $\mathbb{K}y - \lambda\mathbb{H}y = 0$ where $\mathbb{K}$ and $\mathbb{H}$ act from $L_2(0,1)$ to space $L_2(0,1) \times \mathbb{C}^n$,

$$\mathbb{K}y = \begin{pmatrix} N(y) \\ (U_j(y)_1^n) \end{pmatrix}, \quad \mathbb{H}y = \begin{pmatrix} P(y) \\ (V_j(y)_1^n) \end{pmatrix}$$

Some well-known facts about the spectral properties of $(1), (2)$ are stated in terms of the linear pencil $\mathbb{L}(\lambda) = \mathbb{K} - \lambda\mathbb{H}$. If the resolvent set $\rho(\mathbb{L})$ is nonempty, then the spectrum $\sigma(\mathbb{L})$ is discrete and all eigenvalues have finite algebraic multiplicity.

The crucial step to show completeness of the corresponding eigenfunctions and associated functions in the Sobolev spaces $W_2^k\,(0\,,1)$ for $k = 0,1,\ldots,n$ is to define the subspaces $\mathcal{W}_{U,r}^k$.

In this section we establish these function spaces by means of the operator

$$\mathbb{A} = \mathbb{H}\mathbb{K}^{-1}$$

which acts in the space $L_2(0,1) \times \mathbb{C}^n$. The algebraic eigenspace $\mathrm{Ker}(\mathbb{A}^*)^r$ of the adjoint operator $\mathbb{A}^*$ corresponding to the eigenvalue 0 gives rise to a set of boundary conditions $\{\mathcal{U}_j\}$ which define the spaces $\mathcal{W}^k_{\mathcal{U},r}$. Here the λ-independent boundary conditions (2) arise from the geometric eigenspace $\mathrm{Ker}\,\mathbb{A}^*$. However, even in the case that all boundary conditions are independent of λ, the length of the Jordan chains of $\mathbb{A}^*$ in 0 may exceed 1, and some additional boundary conditions may occur.

This section provides an estimate for the growth of the adjoint of $\mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1}$ in $L_2(0,1) \times \mathbb{C}^n$ or, equivalently, of the resolvent of $I - \lambda\mathbb{A}^*$. This estimate is the second important step to show the completeness of the eigenfunctions and associated functions in the spaces $\mathcal{W}^k_{\mathcal{U},r}$. In this respect the theorem on the existence of an asymptotic fundamental system for the differential equation (1) is fundamental. Based on this theorem, we establish a detailed asymptotic expansion of the modified resolvent $\mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1}$. For Stone-regular problems (1), (2) we are then able to estimate the growth of $(\mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1})^*$ for large values of λ.

We show the completeness of the eigenfunctions and associated functions in the spaces $\mathcal{W}^k_{\mathcal{U},r}$ for normal problems (1), (2) where the number r is determined by the order of normality. To this end we write the spectral problem $(\mathbb{K} - \lambda\mathbb{H})y = 0$ in the equivalent form $(I - \lambda\mathbb{A})\mathbb{K}y = 0$ and we show that the eigenfunctions and associated functions of $I - \lambda\mathbb{A}$ are complete in the closure of the range of $\mathbb{A}^{r+1}$ using the so-called method of Keldysh [131].

We apply the results to some concrete examples. Starting with the simplest problem (1), (2) with $p > 0$ and ending with a concrete example from the theory of elasticity, we show how to calculate the boundary conditions $\{\mathcal{U}_j\}$, we determine the spaces $\mathcal{W}^k_{\mathcal{U},r}$ explicitly and formulate the respective completeness results.

We are going to consider boundary eigenvalue problems (1), (2) where N and P are linear differential operators on $[0,1]$ of order n and p, respectively, with $n > p \geq 0$ and sufficiently smooth coefficients (see Theorem (3.1.12),

$$N(y) = y^{(n)} + f_{n-1}\, y^{(n-1)} + \cdots + f_0\, y, \tag{3}$$

$$P(y) = y^{(p)} + g_{p-1}\, y^{(p-1)} + \cdots + g_0 y, \tag{4}$$

and U_j, V_j are two-point boundary conditions in 0 and 1 of order $\leq n-1$ and $\leq p-1$, respectively, with complex coefficients,

$$U_j(y) = \sum_{\mu=0}^{l_j} \left(\alpha^0_{j\mu} y^{(\mu)}(0) + \alpha^1_{j\mu} y^{(\mu)}(1)\right), \qquad j = 1,2,\ldots,n, \tag{5}$$

$$V_j(y) = \sum_{\mu=0}^{h_j} \left(\beta^0_{j\mu} y^{(\mu)}(0) + \beta^1_{j\mu} y^{(\mu)}(1)\right), \qquad j = 1,2,\ldots,n, \tag{6}$$

Where

$$0 \leq l_j \leq n-1, \left|\alpha^0_{jl_j}\right| + \left|\alpha^1_{jl_j}\right| > 0,\ \ 0 \leq h_j \leq p-1 \ \text{ if } \ V_j \not\equiv 0,$$

$$\left|\beta^0_{jh_j}\right| + \left|\beta^1_{jh_j}\right| > 0 \ \text{ and } \ h_j := -(n-p) \ \text{ if } \ V_j \equiv 0.$$

With regard to the substitution $\lambda = -\rho^{n-p}$ we define the order κ_j of a boundary condition $U_j(y) - \lambda V_j(y) = 0$ by

$$\kappa_j := \max\{l_j, n-p+h_j\}, \qquad j = 1, 2, \ldots, n,$$

and the total order κ of the boundary conditions (2) as

$$\kappa := \kappa_1 + \kappa_2 + \cdots + \kappa_n.$$

We assume that the boundary conditions (2) are normalized, that is, $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$, and for each equivalent system of boundary conditions of total order $\tilde{\kappa}$, we have $\kappa \leq \tilde{\kappa}$. For $k \in \mathbb{N}_0$, we denote by

$$W_2^k(0,1) := \{f \in L_2(0,1) : f^{(j)} \in L_2(0,1), \ j = 1, 2, \ldots, k\}$$

the Sobolev space of order k.

With the boundary eigenvalue problem (1), (2), we associate operators acting from $L_2(0,1)$) to $L_2(0,1) \times \mathbb{C}^n$ in the following way. We consider the linear pencil

$$\mathbb{K} - \lambda\mathbb{H} : L_2(0,1) \to L_2(0,1) \times \mathbb{C}^n, \quad \lambda \in \mathbb{C},$$

where the operators $\mathbb{K}, \mathbb{H}$ are given by $D(\mathbb{K}) := W_2^n(0,1), D(\mathbb{H}) := W_2^p(0,1)$ and

$$\mathbb{K}y = \begin{pmatrix} N(y) \\ (U_j(y)_1^n) \end{pmatrix}, \quad \mathbb{H}y = \begin{pmatrix} P(y) \\ (V_j(y)_1^n) \end{pmatrix}.$$

Moreover, we define

$$L^D(\lambda)y := N(y) - \lambda P(y),$$
$$L^R(\lambda)y := \left(U_j(y)\right)_1^n - \lambda\left(V_j(y)\right)_1^n,$$

so that

$$\mathbb{L}(\lambda) := \mathbb{K} - \lambda\mathbb{H} = \begin{pmatrix} L^D(\lambda) \\ L^R(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{C}. \tag{7}$$

Then the boundary eigenvalue problem (1), (2) can be written in the equivalent form

$$\mathbb{K}y - \lambda\mathbb{H}y = 0, \qquad y \in W_2^n(0,1). \tag{8}$$

The spectrum of the boundary eigenvalue problem (1), (2) coincides with the spectrum of (8), that is, with the spectrum $\sigma(\mathbb{L})$ of the corresponding linear pencil $\mathbb{L}$. Here $\sigma(\mathbb{L}) := \mathbb{C} \setminus \rho(\mathbb{L})$ where

$$\rho(\mathbb{L}) := \{\lambda \in \mathbb{C} : \mathbb{L}(\lambda) \text{ is boundedly invertible}\}$$

is the resolvent set of $\mathbb{L}$ (see also [137]), The problem (1), (2) is called nondegenerate if its spectrum is not the whole complex plane, that is, if the resolvent set $\rho(\mathbb{L})$ is not empty. In this case $\sigma(\mathbb{L})$ is a discrete subset of $\mathbb{C}$ (see [128], [138]). For the boundary eigenvalue operator function $\mathbb{L}$ is holomorphic in $\mathbb{C}$, and for each $\lambda \in \mathbb{C}$, the operator $\mathbb{L}(\lambda)$: $W_2^n(0,1) \to L_2(0,1) \times \mathbb{C}^n$ n is a Fredholm operator. Moreover, the spectrum of $\mathbb{L}$ consists only of eigenvalues of finite algebraic multiplicity, $\sigma(\mathbb{L}) = \sigma_p(\mathbb{L})$ where

$$\sigma_p(\mathbb{L}) := \{\lambda \in \mathbb{C} : \mathbb{L}(\lambda) \text{ is not injective}\},$$

and the inverse $\mathbb{L}^{-1}$, given by $\mathbb{L}^{-1}(\lambda) := \mathbb{L}(\lambda)^{-1}$, $\lambda \in \mathbb{C}$, is a meromorphic operator function whose poles are the eigenvalues of $\mathbb{L}$ or, equivalently, of (1), (2).

In the following we will only consider nondegenerate problems. We denote the set of eigenvalues by $\{\lambda_\nu\}$ and, correspondingly, the system of eigenfunctions and associated functions by $\{y_\nu^s\}$. Here the eigenvalues are enumerated according to their geometric multiplicities, y_ν^0 are the eigenvectors corresponding to λ_ν and $y_\nu^s, s = 1, 2, \ldots, p_\nu$, are the associated vectors,

$$\mathbb{K}y_\nu^s - \lambda_\nu\mathbb{H}y_\nu^s = \mathbb{H}y_\nu^{s-1}, \qquad s = 0, 1, \ldots, p_\nu, \tag{9}$$

where $y_\nu^{-1} := 0$ for $s = 0$.

It is immediate from (9) that the eigenfunctions and associated functions satisfy all boundary conditions (2) which do not depend on λ . However, to show completeness of the system $\{y_\nu^s\}$ in subspaces of the Sobolev spaces $W_2^k(0,1)$, $k = 0,1,\dots,n$, it is not enough to impose these λ-independent boundary conditions.
Even in the case that the boundary conditions (2) do not depend on λ at all, it may happen that a certain additional finite dimensional defect occurs (see [147]). For the more general situation of λ-linear boundary conditions, this phenomenon will be describeed by introducing the spaces $\mathcal{W}_{\mathcal{U},r}^k$.

Since the problem is nondegenerate, we may assume without loss of generality that $\mathbb{K}$ is invertible. Then the operator $\mathbb{A} : L_2(0,1) \times \mathbb{C}^n \to L_2(0,1) \times \mathbb{C}^n$ given by

$$\mathbb{A} := \mathbb{H}\mathbb{K}^{-1}$$

is well-defined on $L_2(0,1) \times \mathbb{C}^n$.

Lemma (3.1.1)[117]: The operator $\mathbb{A}$ is compact in $L_2(0,1) \times \mathbb{C}^n$

Proof. The operators

$$\mathbb{K}^{-1} : L_2(0,1) \times \mathbb{C}^n \to W_2^n(0,1), \qquad \mathbb{H} : W_2^p(0,1) \to L_2(0,1) \times \mathbb{C}^n$$

are bounded. Since $n > p$ by assumption, the identity operator

$$I : W_2^n(0,1) \to W_2^p(0,1)$$

is compact by the Lemma of Sobolev. Hence $\mathbb{A}$ is compact.
According to the definition of $\mathbb{A}$, the spectral problem (8) is equivalent to

$$(I - \lambda\mathbb{A})z = 0, \qquad z \in L_2(0,1) \times \mathbb{C}^n, \tag{10}$$

in the following sense: The spectrum of (10) coincides with the spectrum of $\mathbb{L}$, λ_ν, is an eigenvalue of $\mathbb{L}$ if and only if λ_ν is an eigenvalue of the linear pencil $I - \lambda\mathbb{A}$, and y_ν^s is an eigenfunction or an associated function of $\mathbb{L}$ at λ_ν, if and only if $z_\nu^s := \mathbb{K}y_\nu^s$ is an eigenfunction or an associated function of $I - \lambda\mathbb{A}$ corresponding to λ_ν, that is,

$$(I - \lambda_\nu\mathbb{A})z_\nu^s = \mathbb{A}z_\nu^{s-1}, \qquad s = 0,1,\dots,p_\nu, \tag{11}$$

where $z_\nu^{-1} := \mathbb{K}y_\nu^{-1} = 0$. The resolvent

$$(I - \lambda\mathbb{A})^{-1} = \mathbb{K}(\mathbb{K} - \lambda\mathbb{H})^{-1}$$

is a meromorphic operator function whose poles are the eigenvalues of $\mathbb{L}$.

Lemma(3.1.2)[117]: The principal part of the Laurent representation of $(I - \lambda\mathbb{A})^{-1}$ in a neighbourhood of an eigenvalue μ is of the form

$$\sum_{j=1}^{m}\sum_{s=0}^{p_j}\frac{(\cdot, v_j^0)z_j^s + (\cdot, v_j^1)z_j^{s-1} + \cdots + (\cdot, v_j^s)z_j^0}{(\lambda-\mu)^{p_j+1-s}} \tag{12}$$

Where $\left\{z_j^0, z_j^1, \dots, z_j^{p_j}\right\}_{j=1}^m$ is a canonical system of eigenfunctions and associated functions of $\mathbb{A}$ in μ, and $\left\{v_j^0, v_j^1, \dots, v_j^{p_j}\right\}_{j=1}^m$ is a canonical system of eigenfunctions and associated functions of $\mathbb{A}^*$ in μ uniquely determined by the system $\left\{z_j^0, z_j^1, \dots, z_j^{p_j}\right\}_{j=1}^m$ and such that

$$\left(z_j^s, v_k^{p_k-t}\right) = -\delta_{jk}, \delta_{st}, \quad s = 0,1,\dots,p_j, \quad t = 0,1,\dots,p_k, \quad j,k = 1,2,\dots,m.$$

Analogously, the principal part of $\left(I - \lambda\mathbb{A}^*\right)^{-1}$ in a neighbourhood of an eigenvalue μ of $\mathbb{A}^*$ is of the form

$$\sum_{j=1}^{m}\sum_{s=0}^{p_j}\frac{(\cdot,z_j^0)v_j^s+(\cdot,z_j^1)v_j^{s-1}+\cdots+(\cdot,z_j^s)v_j^0}{(\lambda-\mu)^{p_j+1-s}} \qquad (13)$$

Proposition(3.1.3)[117]: Let $r\in\mathbb{N}$. Then we have $\dim\mathrm{Ker}(\mathbb{A}^*)^r=\dim\mathrm{Ker}(\mathbb{A}^r)^*<\infty$ and $\mathrm{Ker}\ \mathbb{H}^*=\mathrm{Ker}\ \mathbb{A}^*\subset\mathrm{Ker}\ (\mathbb{A}^*)^r$.

Proof. Since $\mathbb{H}$ and $\mathbb{K}^{-1}$ are Fredholm operators and $\mathbb{H}$ is densely defined, $\mathbb{A}^r$ is Fredholm for $r\in\mathbb{N}$ (see [127]). Since $\mathbb{A}^*$ is compact, we have $(\mathbb{A}^*)^r=(\mathbb{A}^r)^*$ and hence

$$\dim\mathrm{Ker}(\mathbb{A}^*)^r=\dim\mathrm{Ker}(\mathbb{A}^r)^*=\ \mathrm{codim}\ \mathrm{Im}(\mathbb{A}^r)<\infty.$$

As $\mathrm{Im}\mathbb{K}^{-1}=W_2^n(0,1)$ and $W_2^{n-p}(0,1)$ is dense in $L_2(0,1)$, it follows

$$\mathrm{Ker}\ \mathbb{A}^*=\ (\mathrm{Im}\mathbb{A})^{\perp}=(\mathrm{Im}\mathbb{H}\ )^{\perp}=\mathrm{Ker}\ \mathbb{H}^*.$$

This completes the proof of the proposition.

We are now ready to introduce the subspaces $\mathcal{W}_{\mathcal{U},r}^k$ of the Sobolev spaces $W_2^k(0,1)$ and to establish some of their crucial properties.

Definition(3.1.4)[117]: Let $r\in\mathbb{N}$. Suppose $\{\varphi_j^k\}_{k=0,j=1}^{r-1\ l_k}\subset L_2(0,1)\times\mathbb{C}^n$ to be a basis of $\mathrm{Ker}\ (\mathbb{A}^*)^r$ such that

i) $\{\varphi_j^0\}_{j=1}^{l_0}$ is a basis of $\mathrm{Ker}\ \mathbb{A}^*$,

ii) $\{\varphi_j^k\}_{j=1}^{l_k}$ is a basis of $\mathrm{Ker}\ (\mathbb{A}^*)^{k+1}\ominus\mathrm{Ker}\ (\mathbb{A}^*)^k,\ k=1,2,\dots,r-1$

Then we introduce linear forms U_j^k by

$$U_j^k(y):=(\mathbb{K}y,\varphi_j^k),y\in W_2^n(0,1),\quad k=0,1,\dots,r-1,\quad j=1,2,\dots,l_k,$$

where $\langle\cdot,\ \cdot\rangle$ denotes the canonical scalar product in $L_2(0,1)\times\mathbb{C}^n$. The linear forms U_j^k are of the form

$$U_j^k(y)=\widehat{U}_j^k(y)+\int_0^1 y(x)\overline{u_j^k(x)}\,dx,k=0,1,\dots,r-1,\ \ j=1,2,\dots,l_k,$$

with some two point boundary conditions $\widehat{U}_j^k(y)$ and certain functions $u_j^k\in L_2(0,1)$. The order of a linear form U_j^k is defined by the order of $\widehat{U}_j^k$ if $\widehat{U}_j^k\not\equiv 0$, and by -1 if $\widehat{U}_j^k\equiv 0$,.

On normalizing the system $\{U_j^k\}_{k=0,j=1}^{r-1\ l_k}$, we obtain an equivalent system of linear forms $\{\mathcal{U}_j\}_1^m$, and we define the spaces $\mathcal{W}_{\mathcal{U},r}^k$ by

$$\mathcal{W}_{\mathcal{U},r}^k:=\{y\in W_2^k(0,1):\mathcal{U}_j(y)=0\text{ if ord }\mathcal{U}_j\le k-1\}$$

for $k=0,1,\dots,n$.

Lemma(3.1.5)[117]: We have $\{y_\nu^s\}\subset\mathcal{W}_{\mathcal{U},r}^k$ for $k=0,1,\dots,n,\ r\in\mathbb{N}$.

Proof. Obviously, $\{y_\nu^s\}\subset W_2^n(0,1)$. By (9) and with $z_\nu^s=\mathbb{K}y_\nu^s$, we obtain

$$U_j^k(y_\nu^s)=\langle\mathbb{K}y_\nu^s,\varphi_j^k\rangle=\lambda_\nu\langle\mathbb{H}y_\nu^s,\varphi_j^k\rangle+\langle\mathbb{H}y_\nu^{s-1},\varphi_j^k\rangle$$
$$=\lambda_\nu\langle\mathbb{A}z_\nu^s,\varphi_j^k\rangle+\langle\mathbb{A}z_\nu^{s-1},\varphi_j^k\rangle=\lambda_\nu\langle z_\nu^s,\mathbb{A}^*\varphi_j^k\rangle+\langle z_\nu^{s-1},\mathbb{A}^*\varphi_j^k\rangle=0$$

for $k=0,1,\dots,r-1,\ j=1,2,\dots,l_k$, since $\mathbb{A}^*\varphi_j^k\in\mathrm{Ker}\ (\mathbb{A}^*)^{r-1}=\mathrm{Ker}(\mathbb{A}^{r-1})^*=(\mathrm{Im}\mathbb{A}^{r-1})^{\perp}$ and $z_\nu^s,z_\nu^{s-1}\in\mathrm{Im}\mathbb{A}^{r-1}$ because, by induction,

$$z_\nu^s=\mathbb{A}^{r-1}\left(\sum_{k=\max\{0,s-r+1\}}^{s}\binom{r-1}{s-k}\lambda_\nu^{k-(s-r+1)}z_\nu^k\right)$$

for $s=0,1,\dots,p_\nu$. This completes the proof of the Lemma.

Definition(3.1.6)[117]: Let $\eta \in \{0, 1, \ldots, n\}$ be the number of λ-independent boundary conditions (2), and let $\Lambda := \{m_1, m_2, \ldots, m_\eta\} := \{j \in \{1,2, \ldots, n\} : V_j \equiv 0\}$. Then we define

$$W_U^k := \{y \in W_2^k(0,1) : U_{m_j}(y) = 0, \quad j = 1,2, \ldots, \eta, \quad \text{if ord } U_m \le k-1\}$$

for $k = 0,1, \ldots, n$.

Remark(3.1.7)[117]: The inclusion $W_U^k \subset \mathcal{W}_{\mathcal{U},1}^k \subset \mathcal{W}_{\mathcal{U},r}^k$ holds for $k = 0,1, \ldots, n, r \in \mathbb{N}$.

Proof. From $V_{m_j} \equiv 0$ *for* $j = 1, 2, \ldots, \eta$, we infer

$$\left\{\left(0\ e_{m_j}\right)^t\right\}_{j=1}^{\eta} \in (\text{Im}\mathbb{H})^\perp = \text{Ker } \mathbb{H}^* = \text{Ker } \mathbb{A}^*.$$

by Proposition (3.1.3) where $\left(0\ e_j\right)^t \in L_2(0,1) \times \mathbb{C}^n$, e_j denoting the j-th unit vector in $\mathbb{C}^n$.

Hence we can choose $\varphi_j^0 = \left(0\ e_{m_j}\right)^t$, $j = 1,2, \ldots, \eta$, and we obtain

$$U_j^0 = \langle \mathbb{K}y, \left(0\ e_{m_j}\right)^t \rangle = U_{m_j}(y), \quad j = 1,2, \ldots, \eta,$$

and hence the assertion follows.

For λ-independent boundary conditions, the spaces $W_{\mathcal{U}}^k$ have been introduced in [147] by

$$W_{\mathcal{U}}^k := \{y \in W_2^k(0,1) : \mathcal{U}_j(y) = 0, \quad \text{if ord } \mathcal{U}_j \le k-1\}$$

for $k = 0,1, \ldots, n$ where $\{\mathcal{U}_j\}_1^m$ is a system of linear forms equivalent to $\{U_j\}_1^n \cup \{\widetilde{U}_j\}_1^l$, normalized in case of need, with

$$\widetilde{U}_j(y) := \left(N(y), \varphi_j\right)_{L_2(0,1)}, \quad y \in W_2^n(0,1), \quad j = 1,2, \ldots, l.$$

Here $\{\varphi_j\}_1^l$ is a basis of $\text{Ker } P^*$ where P^* is the adjoint of the operator $P : L_2(0,1) \to L_2(0,1)$, $D(P) = W_U^p, Py = P(y)$.

Proposition(3.1.8)[117]: Suppose the boundary conditions (2) to be λ-independent. Then $\mathcal{W}_{\mathcal{U},2}^k = W_{\mathcal{U}}^k$ for $k = 0, 1, \ldots, n$.

Proof. Let $g = (g_1 g_2)^t \in L_2(0,1) \times \mathbb{C}^n$. Then

$$\begin{aligned}
(\mathbb{A}^*)^2 g = 0 \Leftrightarrow \mathbb{A}^* g \in \text{Ker}\mathbb{H}^* \Leftrightarrow \langle \mathbb{A}^* g, \begin{pmatrix} v \\ 0 \end{pmatrix} \rangle = 0, \qquad & v \in L_2(0,1) \\
\Leftrightarrow \quad \langle g, \mathbb{H}\mathbb{K}^{-1}, \begin{pmatrix} v \\ 0 \end{pmatrix} \rangle = 0, \quad & v \in L_2(0,1) \\
\Leftrightarrow \quad \langle g, \begin{pmatrix} PN^{-1}v \\ 0 \end{pmatrix} \rangle = 0, \quad & v \in L_2(0,1) \\
\Leftrightarrow \quad (g_1, P\omega)_{L_2(0,1)} = 0, \quad & \omega \in W_2^p(0,1) \\
\Leftrightarrow \quad g_1 \in \text{Ker } P^* &
\end{aligned}$$

since $\text{Im } N^{-1} = W_2^n(0,1)$ where $N : L_2(0,1) \to L_2(0,1)$, $D(N) = W_U^n$, and $Ny = N(y)$ and $W_2^{n-p}(0,1)$ is dense in $L_2(0,1)$.

Consequently, a basis of $\text{Ker}(\mathbb{A}^*)^2$ is given by $\left\{\left(0\ e_j\right)^t\right\}_1^n \cup \left\{\left(\varphi_j\ 0\right)^t\right\}_1^l =: \{\varphi_j^0\}_1^n \cup \{\varphi_j^1\}_1^l$

where $\{\varphi_j\}_1^l \subset L_2(0,1)$ is a basis of $\text{Ker } P^*$. Thus we obtain

$$U_j^0(y) = \langle \mathbb{K}y, \left(0\ e_j\right)^t \rangle = U_j(y), \quad j = 1, 2 \ldots, n,$$

$$U_j^1(y) = \langle \mathbb{K}y, \left(\varphi_j\ 0\right)^t \rangle = \widetilde{U}_j(y), \quad j = 1, 2 \ldots, l.$$

This completes the proof of the proposition.

We have showed now that the spaces $\mathcal{W}_{\mathcal{U},r}^k$ involve all given λ-independent boundary conditions and some additional boundary conditions which are, however, also fulfilled by the eigen functions and associated functions.

It is our aim to show that the system $\{y_\nu^s\}$ of eigenfunctions and associated functions is even complete in the spaces $\mathcal{W}_{\mathfrak{U},r}^k$ by proving completeness of the system $\{z_\nu^s\} = \{\mathbb{K}y_\nu^s\}$ of eigen-functions and associated functions of the linear pencil $I - \lambda\mathbb{A}$. For this we first need to estimate the growth of the resolvent of $I - \lambda\mathbb{A}$ and hence of the modified resolvent $\mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1}$ since

$$\begin{aligned}(I - \lambda\mathbb{A})^{-1} &= (\mathbb{K} - \lambda\mathbb{H} + \lambda\mathbb{H})(\mathbb{K} - \lambda\mathbb{H})^{-1} \\ &= I + \lambda\mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1} \end{aligned} \tag{14}$$

The resolvent $(\mathbb{K} - \lambda\mathbb{H})^{-1} :\ L_2(0,1) \times \mathbb{C}^n \to L_2(0,1)$ is given by (comp.[138])

$$(\mathbb{K} - \lambda\mathbb{H})^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = R_1(\lambda)f_1 + R_2(\lambda)f_2, \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L_2(0,1) \times \mathbb{C}^n \tag{15}$$

where

$$\begin{aligned} R_1(\lambda)f_1 &:= \int_0^1 G(\cdot,\xi,\lambda)\, f_1(\xi)\, d\xi, \qquad && f_1 \in L_2(0,1) \\ R_2(\lambda)f_2 &:= \big(y_1(\cdot,\lambda) \ldots y_n(\cdot,\lambda)\big)M(\lambda)^{-1}f_2, && f_2 \in \mathbb{C}^n \end{aligned} \tag{16}$$

Here $\{y_1(\cdot,\lambda) \ldots y_n(\cdot,\lambda)\}$ is a fundamental system of $N(y) - \lambda P(y) = 0$,

$$M(\lambda) := \big(L^R(\lambda)y_1(\cdot,\lambda) \ldots L^R(\lambda)y_n(\cdot,\lambda)\big)$$

is the corresponding characteristic matrix function and

$$G(\cdot,\xi,\lambda) := \frac{(-1)^n}{\Delta(\lambda)} \det\begin{pmatrix} y_1(\cdot,\lambda) \ldots y_n(\cdot,\lambda) & g(\cdot,\xi,\lambda) \\ M(\lambda) & L^R(\lambda)g(\cdot,\xi,\lambda) \end{pmatrix}, \quad \xi \in [0.1], \tag{17}$$

is the Green's function of (1), (2) where

$$\Delta(\lambda) := \det M(\lambda)$$

denotes the characteristic determinant. The function g is given by

$$g(x,\xi,\lambda) := \frac{1}{2}\operatorname{sign}(x-\xi)\sum_{\nu-1}^{n} y_\nu(x,\lambda)z_\nu(\xi,\lambda), \qquad x,\xi \in [0.1], \tag{18}$$

where $\{z_1(\cdot,\lambda) \ldots z_n(\cdot,\lambda)\}$ is the fundamental system of the formally adjoint differential equation of $N(y) - \lambda P(y) = 0$ defined by

$$z_\nu(\cdot,\lambda) := \frac{W_\nu(\cdot,\lambda)}{W(\cdot,\lambda)}, \quad \nu = 1, 2, \ldots, n,$$

with $W(\lambda)$ denoting the Wronskian determinant of$\{y_1(\cdot,\lambda), \ldots, y_n(\cdot,\lambda)\}$ and $W_\nu(\lambda)$ being the algebraic complement of $y_\nu^{(n-1)}(\cdot,\lambda)$ in $W(\lambda)$.

The basis for the asymptotic expansion of the resolvent $(\mathbb{K} - \lambda\mathbb{H})^{-1}$ is a theorem on the existence of an asymptotic fundamental system for the differential equation $N(y) = \lambda P(y)$. This theorem has its roots in early works of G.D. Birkhoff [119,120]. Later contributions to a more detailed structure of these asymptotic expansions are due to W. Eberhard and G. Freiling [126] as well as R. Mennicken and M. Moller[138].

Using the standard substitution $\lambda = -\rho^{n-p}$, we define sectors $S_\kappa = e^{|\kappa\pi/(2(n-p))}S_0$ for $\kappa = 0, 1, 2, \ldots, 4(n-p) - 1$ in the complex plane where

$$S_0 := \left\{ z \in \mathbb{C} :\ 0 \le \arg z \le \frac{\pi}{2(n-p)} \right\},$$

We denote the $(n-p)$-th roots of -1 by $\omega_1, \omega_2, \ldots, \omega_{n-p}$ and enumerate them in such a way that

$$\Re(\rho\omega_1) \le \cdots \le \Re(\rho\omega_d) \le 0 \le \Re(\rho\omega_{d+1}) \le \ldots \le \Re(\rho\omega_{n-p}), \rho \in S_0, \tag{19}$$

where $d \in \{1,2,\ldots,n-p\}$, and accordingly for all sectors S_κ.

To arrange the subsequent asymptotic expansions clearly, we introduce the following abbreviation.

Let $l \in \mathbb{N}_0$. Suppose g to be a function on some sector S of the complex plane with values in a Banach space E. If g has an asymptotic expansion

$$g(\rho) = f(\rho) + O\left(\frac{1}{\rho^{l+1}}\right), f(\rho) = f_0 + \frac{f_1}{\rho} + \ldots + \frac{f_l}{\rho^l},$$

with $f_0, f_1, \ldots, f_l \in E$ for $\rho \in S$ with respect to the norm in E, then we write

$$g(\rho) = [f(\rho)]_l,$$

omitting the index l if $l = 0$.

This notation will be applied with $E = \mathbb{C}$ and $E = C[0,1]$. Whenever precise information is unnecessary, we will write $[\cdot(\rho)]_l$ and $[\cdot(x,\rho)]_l$, respectively, where x, for example, indicates the independent variable if $E = C[0,1]$.

Theorem(3.1.9)[117]: Let $l \in \mathbb{N}_0$, $\kappa \in \{0,1,\ldots,4(n-p)-1\}$, and let $z_0 \in \mathbb{C}$. Suppose that $f_\nu, g_\nu \in W_2^{\nu+l+s}(0,1)$ where $s := \max\{n-1, 2(p-1)\}$. Then there exists a fundamental system $\{y_1(\cdot,\lambda) \ldots, y_n(\cdot,\lambda)\}$ of $N(y) = \lambda P(y)$ in the sector $S = z_0 + S_\kappa$ such that

$$\begin{cases} y_\nu^{(j)}(x,\rho) = h_\nu^{(j)}(x) + \dfrac{h_{\nu 1}^{(j)}(x)}{\lambda} + \cdots + \dfrac{h_{\nu L}^{(j)}(x)}{\lambda^L} + O\left(\dfrac{1}{\lambda^{L+1}}\right), & \nu = 1,2,\ldots,p \\ y_\nu^{(j)}(x,\rho) = \rho^j e^{\rho\omega_{\nu-p}x}[\cdot(x,\rho)]_l & \nu = p+1,\ldots,n \end{cases} \tag{20}$$

for $j = 0,1,\ldots,n-1, x \in [0,1]$, where $\lambda = -\rho^{n-p}$, $L := \left[\frac{l}{n-p}\right]$, $\{h_1, \ldots, h_p\}$ is a fundamental system of $P(y) = 0$, and

$$\begin{cases} z_\nu^{(j)}(\xi,\rho) = \dfrac{1}{\lambda}\left(h_\nu^{*(j)}(\xi) + \dfrac{h_{\nu 1}^{*(j)}(\xi)}{\lambda} + \cdots + \dfrac{h_{\nu L}^{*(j)}(\xi)}{\lambda^L} + O\left(\dfrac{1}{\lambda^{L+1}}\right)\right), & \nu = 1,2,\ldots,p \\ z_\nu^{(j)}(\xi,\rho) = \dfrac{1}{\rho^{n-1-j}}\rho^j e^{\rho\omega_{\nu-p}\xi}[\cdot(\xi,\rho)]_l & \nu = p+1,\ldots,n \end{cases} \tag{21}$$

for $j = 0,1,\ldots,n-1, \xi \in [0,1]$, where $\{h_1^*, \ldots, h_p^*\}$ is a fundamental system of the formally adjoint differential equation $P^*(z) = 0$.

In the following we will always assume that the coefficients f_ν, g_ν of the differential equation (1) fulfill the smoothness conditions of Theorem (3.1.9) with some $l \in \mathbb{N}_0$. Then, with the aid of this theorem, the asymptotic expansion of the characteristic determinant Δ of the problem (1), (2) can be established.

Proposition(3.1.10)[117]: For $\rho \in S_0$, we have

$$\Delta(\rho) = \rho^\gamma e^{\rho\Omega}([\Theta_0(\rho)]_l + [\Theta_1(\rho)]_l e^{\rho\omega_k}) \quad \text{if } n-p = 2k-1,$$

and

$$\Delta(\rho) = \rho^\gamma e^{\rho\Omega}([\Theta_0(\rho)]_l + [\Theta_1(\rho)]_l e^{\rho\omega_k} + [\Theta_2(\rho)]_l e^{2\rho\omega_k}) \quad \text{if } n-p = 2k,$$

with

$$\gamma := \sum_{j=p+1}^{n} \kappa_j, \quad \Omega := \omega_{k+1} + \cdots + \omega_{n-p},$$

and polynomials Θ_i in ρ^{-1},

$$\Theta_i(\rho) = \Theta_{i0} + \frac{\Theta_{i1}}{\rho} + \cdots + \frac{\Theta_{il}}{\rho^l}, \qquad i = 0,1,2,$$

Analogous representations hold in all other sectors S_κ.

Now the notions of Stone-regularity and normality can be formulated as follows.

Definition(3.1.11)[117]: A problem (1), (2) is called Stone-regular if there exists an integer $m \in \mathbb{N}_0$ such that

$$\left(\sum_{s=0}^{m} |\Theta_{0s}|\right)\left(\sum_{s=0}^{m} |\Theta_{1s}|\right) \neq 0 \quad \text{if } n-p = 2k-1, \qquad (22)$$
$$\sum_{s=0}^{m} |\Theta_{0s}| \neq 0 \qquad \text{if } n-p = 2k$$

If m is minimal, then the problem is said to be Stone-regular of order m.

The problem is called normal of order m if either $n-p \leq 2$ and (22) holds for some $m \in \mathbb{N}_0$ or if $n-p > 2$ and

$$\sum_{s=0}^{m} (|\Theta_{0s}| + |\Theta_{1s}|) \neq 0 \qquad (23)$$

where m is minimal.

For abbreviation we introduce functions ψ_ν on $[0,1]$ by

$$\psi_\nu(x) := \begin{cases} \omega_{\nu-p}x, & \nu = p+1, \ldots, p+d, \\ \omega_{\nu-p}(x-1), & \nu = p+d+1, \ldots, n \end{cases} \qquad x \in [0,1], \qquad (24)$$

where d is given by (19).

Now we are ready to establish the asymptotic expansion of $\mathbb{K}(\mathbb{K}-\lambda\mathbb{H})^{-1}$.

Theorem(3.1.12)[117]: The modified resolvent $\mathbb{K}(\mathbb{K}-\lambda\mathbb{H})^{-1} : L_2(0,1) \times \mathbb{C}^n \to L_2(0,1) \times \mathbb{C}^n$ is of the form

$$\mathbb{K}(\mathbb{K}-\lambda\mathbb{H})^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \int_0^1 G_1^D(\cdot,\xi,\lambda)\, f_1(\xi)d\xi + G_2^D(\cdot,\lambda)^t f_2 \\ \left(\int_0^1 G_{1j}^R(\xi,\lambda)\, f_1(\xi)d\xi + G_{2j}^R(\lambda)^t f_2\right)_{j=1}^{n} \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L_2(0,1) \times \mathbb{C}^n,$$

where G_1^D, G_{1j}^R are scalar functions and G_2^D, G_{2j}^R are vector valued functions of size $n \times 1$ having the following asymptotic representations:

i) For sufficiently large $\rho \in S_0$ and $x, \xi \in [0,1]$, we have

$$G_1^D(x,\xi,\rho) = \frac{1}{\rho^{n-p}} \sum_{i=0}^{4} H_i^D(x,\xi,\rho)$$

where H_0^D *is* a holomorphic and bounded function with respect to ρ,

$$H_1^D(x,\xi,\rho) = \rho \sum_{\nu,\mu=p+1}^{n} [\cdot\,(x)][\cdot\,(\zeta)]\, e^{\rho\psi_\nu(x)}\; e^{\rho\psi_\mu(1-\xi)} R_{\nu,\mu}(\rho),$$

$$H_2^D(x,\xi,\rho) = \rho^{p-\kappa_{p+1}} \sum_{\nu=p+1}^{n} \sum_{\mu=1}^{p} [\cdot\,(x)][\cdot\,(\xi)]\, e^{\rho\psi_\nu(x)}\; R_{\nu,\mu}(\rho),$$

$$H_3^D(x,\xi,\rho) = \rho^{\kappa_p-n+1}\sum_{\nu=1}^{p}\sum_{\mu=p+1}^{n}[\cdot(x)][\cdot(\xi)]\ e^{\rho\psi_\mu(1-\xi)}R_{\nu,\mu}(\rho),$$

$$H_4^D(x,\xi,\rho) = \frac{1}{\rho^{n-p}}\sum_{\nu,\mu=1}^{p}[\cdot(x)][\cdot(\xi)]\,R_{\nu,\mu}(\rho),$$

For sufficiently large $\rho \in S_0$ and $x \in [0,1]$, we have

$$G_2^D(x,\rho) = \frac{1}{\rho^{n-p}}\sum_{i=1}^{2}F_i^D(x,\rho)$$

where

$$F_1^D(x,\rho) = \left(\sum_{\nu=1}^{p}[\cdot(x)][\cdot]\,R_{\nu k}(\rho)\right)_{k=1}^{n},$$

$$F_2^D(x,\rho) = \rho^{n-\kappa_{p+1}}\left(\sum_{\nu=p+1}^{n}e^{\rho\psi_\nu(x)}[\cdot(x)][\cdot]\,R_{\nu k}(\rho)\right)_{k=1}^{n}.$$

ii) For sufficiently large $\rho \in S_0$ and $\xi \in [0,1]$, we have, with $\Lambda = \{j \in \{1,2,\dots,n\}: V_j \equiv 0\}$

$$G_{1j}^R(\xi,\rho) = 0, \qquad j \in \Lambda$$

$$G_{1j}^R(\xi,\rho) = \frac{1}{\rho^{n-p}}\sum_{i=0}^{4}H_{ij}^R(\xi,\rho), \quad j \in \{1,2,\dots,n\} \setminus \Lambda$$

where, for $j \in \{1,2,\dots,n\} \setminus \Lambda$, H_{0j}^R are holomorphic and bounded functions with respect to ρ,

$$H_{1j}^R(\xi,\rho) = \rho^{l_j-n+1}\sum_{\nu,\mu=p+1}^{n}[\cdot][\cdot(\xi)]\ e^{\rho\psi_\mu(1-\xi)}R_{\nu\mu j}(\rho),$$

$$H_{2j}^R(\xi,\rho) = \rho^{l_j-\kappa_{p+1}-(n-p)}\sum_{\nu=p+1}^{n}\sum_{\mu=1}^{p}[\cdot][\cdot(\xi)]\,R_{\nu\mu j}(\rho),$$

$$H_{3j}^R(\xi,\rho) = \rho^{\kappa_p-n+1}\sum_{\nu=1}^{p}\sum_{\mu=p+1}^{n}[\cdot][\cdot(\xi)]e^{\rho\psi_\mu(1-\xi)}\,R_{\nu\mu j}(\rho),$$

$$H_{4j}^R(\xi,\rho) = \frac{1}{\rho^{n-p}}\sum_{\nu,\mu=1}^{p}[\cdot][\cdot(\xi)]\,R_{\nu\mu j}(\rho).$$

For sufficiently large $\rho \in S_0$, we have

$$G_{2j}^R(\rho) = 0, \qquad j \in \Lambda$$

$$G_{2j}^R(\rho) = \frac{1}{\rho^{n-p}}\sum_{i=1}^{2}F_{ij}^R(\rho), \quad j \in \{1,2,\dots,n\} \setminus \Lambda$$

where, for $j \in \{1,2,\dots,n\} \setminus \Lambda$,

$$F_{1j}^R(\rho) = \rho^{n-p}\left(\sum_{\nu=1}^{p}[\cdot]\, R_{\nu kj}(\rho)\right)_{k=1}^{n},$$

$$F_{2j}^R(\rho) = \rho^{h_j-\kappa_{p+1}+n-p}\left(\sum_{\nu=p+1}^{n}[\cdot]\, R_{\nu kj}(\rho)\right)_{k=1}^{n}.$$

In all cases the functions $R_{\nu\mu}, R_{\nu k}, R_{\nu\mu j}$ and $R_{\nu kj}$ are asymptotically of the form

$$\begin{cases} \dfrac{[\cdot] + [\cdot]e^{\rho\omega_k}}{[\Theta_0(\rho)]_l + [\Theta_1(\rho)]_l e^{\rho\omega_k}} & \text{if } n-p = 2k-1, \\ \dfrac{[\cdot] + [\cdot]e^{\rho\omega_k} + [\cdot]e^{2\rho\omega_k}}{[\Theta_0(\rho)]_l + [\Theta_1(\rho)]_l e^{\rho\omega_k} + [\Theta_2(\rho)]_l e^{2\rho\omega_k}} & \text{if } n-p = 2k, \end{cases}$$

Analogous representations hold in all other sectors S_κ.

Proof. Let $(f_1 f_2)^t \in L_2(0,1) \times \mathbb{C}^n$ and $\rho \in S_0$. Then we have

$$\mathbb{H}(\mathbb{K} + \rho^{n-p}\mathbb{H})^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \int_0^1 G(\cdot,\xi,\rho)\, f_1(\xi)d\xi + \big(y_1(\cdot,\rho),\dots,y_n(\cdot,\rho)\big)M(\rho)^{-1}f_2.$$

For simplicity we keep the notations for the Green's function G, for the fundamental system $\{y_\nu(\cdot,\rho)\}_{\nu=1}^n$ and for the characteristic matrix M also after the substitution $\lambda = -\rho^{n-p}$.
In view of the different asymptotic structure of the fundamental system $\{y_\nu(\cdot,\rho)\}_{\nu=1}^n$ for $\nu = 1,2,\dots,p$ and $\nu = p+1,\dots,n$, we write the function g defined in (18) as

$$g(x,\xi,\rho) = g_1(x,\xi,\rho) + g_2(x,\xi,\rho)$$

where

$$g_1(x,\xi,\rho) = \begin{cases} 0, & x > \xi, \\ -\sum_{\nu=1}^{p} y_\nu(x,\rho)z_\nu(\xi,\rho) & x < \xi, \end{cases}$$

$$g_2(x,\xi,\rho) = \begin{cases} \sum_{\nu=p+1}^{p+d} y_\nu(x,\rho)z_\nu(\xi,\rho) & x > \xi, \\ -\sum_{\nu=p+d+1}^{n} y_\nu(x,\rho)z_\nu(\xi,\rho), & x < \xi, \end{cases}$$

If we expand the determinant in the numerator of the Green's function G with respect to its first row, this implies the decomposition

$$G(x,\xi,\rho) = g_1(x,\xi,\rho) + g_2(x,\xi,\rho) - \sum_{i=1}^{2}\sum_{\nu=1}^{n} y_\nu(x,\rho)\frac{\Delta_\nu^i(\xi,\rho)}{\Delta(\rho)}. \tag{25}$$

Here the functions $\Delta_\nu^i(\cdot,\rho)$ are determined by

$$\Delta_\nu^i(\xi,\rho) = \det\big(L^R(\rho)y_1(\cdot,\rho)\big)\dots \overset{\nu-\text{th column}}{L^R(\rho)g_i(\cdot,\xi,\rho)} \dots L^R(\rho)y_n(\cdot,\rho)\big),$$

that is, by substituting the ν-th column $L^R(\rho)y_\nu(\cdot,\rho)$ of the characteristic determinant $\Delta(\rho)$ by $L^R(\rho)g_i(\cdot,\xi,\rho)$ (see also [130], [151]).

i) Suppose $\rho \in S_0$ to be sufficiently large. According to the decomposition (25), we have

$$G_1^D(x,\xi,\rho) = \frac{1}{\rho^{n-p}} \sum_{i=0}^{4} H_i^D(x,\xi,\rho), \qquad G_2^D(x,\rho) = \frac{1}{\rho^{n-p}} \sum_{i=1}^{2} F_i^D(x,\rho),$$

For $x, \xi \in [0,1]$ with

$$H_0^D(x,\xi,\rho) := \rho^{n-p} P_x\big(g_1(x,\xi,\rho) + g_2(x,\xi,\rho)\big),$$

$$H_1^D(x,\xi,\rho) := -\rho^{n-p} \sum_{\nu=p+1}^{n} Py_\nu(x,\rho), \frac{\Delta_\nu^2(\xi,\rho)}{\Delta(\rho)},$$

$$H_2^D(x,\xi,\rho) := -\rho^{n-p} \sum_{\nu=p+1}^{n} Py_\nu(x,\rho), \frac{\Delta_\nu^1(\xi,\rho)}{\Delta(\rho)},$$

$$H_3^D(x,\xi,\rho) := -\rho^{n-p} \sum_{\nu=1}^{p} Py_\nu(x,\rho), \frac{\Delta_\nu^2(\xi,\rho)}{\Delta(\rho)},$$

$$H_4^D(x,\xi,\rho) := -\rho^{n-p} \sum_{\nu=1}^{p} Py_\nu(x,\rho), \frac{\Delta_\nu^1(\xi,\rho)}{\Delta(\rho)},$$

and

$$F_1^D(x,\rho)^t := \rho^{n-p}(Py_1(x,\rho) \ \cdots \ Py_p(x,\rho) \ 0 \ \cdots \ 0) M(\rho)^{-1},$$

$$F_2^D(x,\rho)^t := \rho^{n-p}(0 \ \cdots \ 0 \ Py_{p+1}(x,\rho) \cdots \ Py_n(x,\rho)) M(\rho)^{-1},$$

First let $\nu \in \{p+1,\dots,n\}$. By Theorem (3.1.9) we have

$$Py_\nu(x,\rho) = \rho^p e^{\rho\omega_{\nu-p}x}[\cdot(x)],$$

$$\frac{\Delta_\nu^1(\xi,\rho)}{\Delta(\rho)} = \frac{\rho^{-\kappa_{p+1}}}{\rho^{n-p}} e^{\rho\psi_\nu(0)} \sum_{\mu=1}^{p} [\cdot(\xi)]\, R_{\nu\mu}(\rho),$$

$$\frac{\Delta_\nu^2(\xi,\rho)}{\Delta(\rho)} = \frac{1}{\rho^{n-1}} e^{\rho\psi_\nu(0)} \sum_{\mu=p+1}^{n} [\cdot(\xi)] e^{\rho\psi_\nu(1-\xi)}\, R_{\nu\mu}(\rho),$$

The last two representations are obtained by expanding the determinants $\Delta_\nu^1(\xi,\rho)$ with respect to the $n-p-1$ columns $p+1,\dots,\nu-1,\nu+1,\dots,n$, and $\Delta_\nu^2(\xi,\rho)$ with respect to the last $n-p$ columns and using the asymptotic expansion of the characteristic determinant $\Delta(\rho)$ given in Proposition (3.1.10).

Now let $\nu \in \{1,\dots,p\}$. Then we have, again by Theorem (3.1.9),

$$Py_\nu(x,\rho) = \frac{1}{\rho^{n-p}}[\cdot(x)],$$

$$\frac{\Delta_\nu^1(\xi,\rho)}{\Delta(\rho)} = \frac{1}{\rho^{n-p}} \sum_{\mu=1}^{p} [\cdot(\xi)]\, R_{\nu\mu}(\rho),$$

$$\frac{\Delta_\nu^2(\xi,\rho)}{\Delta(\rho)} = \frac{\rho^{\kappa_p}}{\rho^{n-1}} \sum_{\mu=p+1}^{n} [\cdot(\xi)] e^{\rho\psi_\mu(1-\xi)}\, R_{\nu\mu}(\rho),$$

on expanding $\Delta_\nu^1(\xi,\rho)$ with respect to the last $n-p$ columns, $\Delta_\nu^2(\xi,\rho)$ with respect to the $n-p+1$ columns $\nu, p+1,\dots,n$ and using Proposition (3.1.10).

By Cramer's rule we obtain

$$F_1^D(x,\rho)^t = \rho^{n-p}\left(\sum_{\nu=1}^{n} P y_\nu(x,\rho)\frac{1}{\Delta(\rho)}, \det M_{\nu k}(\rho)\right)_{k=1}^{n},$$

$$F_2^D(x,\rho)^t = \rho^{n-p}\left(\sum_{\nu=p+1}^{n} P y_\nu(x,\rho)\frac{1}{\Delta(\rho)}, \det M_{\nu k}(\rho)\right)_{k=1}^{n},$$

where

$$M_{\nu k}(\rho) = \left(L^R(\rho)y_i(\cdot,\rho) \ldots \overset{\nu\text{-th column}}{e_k} \ldots L^R(\rho)y_n(\cdot,\rho)\right),$$

arises from the characteristic matrix M by replacing the ν-th column by the k-th unit vector e_k of $\mathbb{C}^n$. Inserting the asymptotic representations for $P\,y_\nu(x,\rho)$ given above and expanding $\det M_{\nu k}(\rho)$ with respect to the last $n-p$ columns for $\nu = 1,2,\ldots,p$ and with respect to the $n-p-1$columns $p+1,\ldots,\nu-1,\nu+1,\ldots,n$ for $\nu = p+1,\ldots,n$, we see

$$\frac{\det\ M_{\nu k}(\rho)}{\Delta(\rho)} = [\cdot]R_{\nu k}(\rho), \qquad \nu = 1,2,\ldots,p$$

$$\frac{\det\ M_{\nu k}(\rho)}{\Delta(\rho)} = \rho^{-\kappa_{p+1}}e^{\rho\psi_\nu(0)}[\cdot]R_{\nu k}(\rho), \qquad \nu = 1,2,\ldots,n,$$

for $k = 1,2,\ldots,n$. This completes the proof of part i).

ii) Let again $\rho \in S_0$ be sufficiently large. Then we have

$$G_{1j}^R(\xi,\rho) = V_j\big(G(\cdot,\xi,\rho)\big), \qquad G_{2j}^R(\rho) = \frac{1}{\rho^{n-p}}\sum_{i=1}^{2} F_{ij}^R(\rho),$$

for $j = 1,2,\ldots,n$ where

$$F_{1j}^R(\rho)^t := \rho^{n-p}\Big(V_j\big(y_1(\cdot,\rho)\big)\cdots V_j\big(y_p(\cdot,\rho)\big) \quad 0 \quad \cdots \quad 0\Big)M(\rho)^{-1}$$

$$F_{2j}^R(\rho)^t := \rho^{n-p}\Big(0 \quad \cdots \quad 0 \quad V_j(y_{p+1}(\cdot,\rho))\cdots V_j(y_n(\cdot,\rho))\Big)M(\rho)^{-1}$$

Since $G(\cdot,\xi,\rho)$fulfills the boundary conditions (2) and $V_j \equiv 0$ for $j \in \Lambda$, we know

$$G_{1j}^R(\xi,\rho) = 0, \qquad j \in \Lambda$$

$$G_{1j}^R(\xi,\rho) = \frac{1}{\rho^{n-p}}U_j\big(G(\cdot,\xi,\rho)\big) \qquad j \in \{1,2,\ldots,n\}\setminus\Lambda$$

Let $j \in \{1,2,\ldots,n\}\setminus\Lambda$. According to (25) we have

$$G_{1j}^R(\xi,\rho) = \frac{1}{\rho^{n-p}}\sum_{i=0}^{4} H_{ij}^R(\xi,\rho)$$

for $\xi \in [0,1]$ with

$$H_{0j}^R(\xi,\rho) := U_j\big(g_1(\cdot,\xi,\rho) + g_2(\cdot,\xi,\rho)\big),$$

$$H_{1j}^R(\xi,\rho) := -\sum_{\nu=p+1}^{n} U_j(y_\nu(\cdot,\rho))\frac{\Delta_\nu^2(\xi,\rho)}{\Delta(\rho)},$$

$$H_{2j}^R(\xi,\rho) := -\sum_{\nu=p+1}^{n} U_j(y_\nu(\cdot,\rho))\frac{\Delta_\nu^1(\xi,\rho)}{\Delta(\rho)},$$

$$H_{3j}^R(\xi,\rho) := -\sum_{\nu=1}^{p} U_j(y_\nu(\cdot,\rho))\frac{\Delta_\nu^2(\xi,\rho)}{\Delta(\rho)},$$
$$H_{4j}^R(\xi,\rho) := -\sum_{\nu=1}^{p} U_j(y_\nu(\cdot,\rho))\frac{\Delta_\nu^1(\xi,\rho)}{\Delta(\rho)}.$$

From Theorem (3.1.9), it follows

$$U_j(y_\nu(\cdot,\rho)) = [\cdot], \qquad \nu = 1,2,\dots,p$$
$$U_j(y_\nu(\cdot,\rho)) = \rho^{l_j} e^{-\rho\psi_\nu(0)}[\cdot], \qquad \nu = p+1,\dots,n,$$

Together with the asymptotic expansions of $\Delta_\nu^i(\xi,\rho)/\Delta(\rho)$ derived in the proof of part i), this proves the asserted asymptotic representations of $H_{ij}^R(\xi,\rho)$ for $i = 1,2,3,4$.

The asymptotic expansions of $F_{ij}^R(\rho)^t$ for $i = 1,2$ are obtained from $V_j \equiv 0$ for $j \in \Lambda$ and from

$$V_j(y_\nu(\cdot,\rho)) = [\cdot], \qquad \nu = 1,2,\dots,p$$
$$V_j(y_\nu(\cdot,\rho)) = \rho^{h_j} e^{-\rho\psi_\nu(0)}[\cdot], \qquad \nu = p+1,\dots,n,$$

for $j \in \{1,2,\dots,n\}\backslash\Lambda$ following the same lines as in the end of the proof of part i).

This completes the proof of Theorem (3.1.12).

For the proof of the completeness theorem we need an estimate for the adjoint of the modified resolvent which can be deduced from the preceding theorem if the problem (1), (2) is regular in some sense.

Theorem(3.1.13)[117]: Suppose the problem (1), (2) to be Stone-regular of order l.

Let $\{\rho_m\}_{m=1}^{\infty}$ be the set of zeros of the characteristic determinant $\Delta(\rho)$, and let $\delta > 0$.

Then we have

$$\rho^{n-p}(\mathbb{H}(\mathbb{K} + \overline{\rho^{n-p}}\mathbb{H})^{-1})^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = o(\rho^{n-p+1+l} + \rho^{n-\kappa_{p+1}+l}), \quad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L_2(0,1) \times \mathbb{C}^n,$$

with respect to the norm in $L_2(0,1) \times \mathbb{C}^n$ for $\rho \in \mathbb{C}$, $|\rho - \rho_m| > \delta$, $k = 1,2,\dots$ If the problem is normal of order l, then there exists at least one ray Γ in each sector S_κ such that the above estimate holds for $\rho \in \Gamma$.

Proof: The adjoint of the modified resolvent is given by

$$\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (\mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1})^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\rangle = \langle \mathbb{H}(\mathbb{K} - \lambda\mathbb{H})^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\rangle$$
$$= \int_0^1 \int_0^1 G_1^D(x,\xi,\lambda)\, f_1(\xi)\, d\xi\, \overline{g_1(x)}\, dx + \int_0^1 (G_2^D(x,\lambda)^t f_2)\overline{g_1(x)}\, dx$$
$$+ \left(\left(\int_0^1 G_{1j}^R(\xi,\lambda)\, f_1(\xi)\, d\xi\right)_{j=1}^{n}\right)^t \overline{g_2} + \left(\left(G_{2j}^R(\lambda)^t\right)_{j=1}^{n} f_2\right)^t \overline{g_2}$$
$$= \int_0^1 f_1(\xi)\, \overline{\int_0^1 \overline{G_1^D(x,\xi,\lambda)}\, g_1(x)\, dx}\, d\xi + f_2^t \overline{\left(\int_0^1 e_\iota^t \overline{G_2^D(x,\lambda)} g_1(x)\, dx\right)_{\iota=1}^{n}}$$
$$+ \int_0^1 f_1(\xi)\, \overline{\left(\overline{G_{1j}^R(\xi,\lambda)}\right)_{j=1}^{n\ t} g_2}\, d\xi + f_2^t \overline{\left(\overline{G_{2j}^R(\lambda)^t}\right)_{j=1}^{n\ t} g_2}$$

for $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L_2(0,1) \times \mathbb{C}^n$ and hence

$$\left(\mathbb{H}\left(\mathbb{K}-\bar{\lambda}\mathbb{H}\right)^{-1}\right)^{*}\binom{g_1}{g_2}=\begin{pmatrix}\int_0^1\overline{G_1^D(x,\xi,\bar{\lambda})}\,g_1(x)dx+\left(\overline{G_{1j}^R(\xi,\bar{\lambda})}\right)_{j=1}^{n\ t}g_2\\ \left(\int_0^1 e_i^t\overline{G_2^D(x,\bar{\lambda})}\,g_1(x)dx\right)_{i=1}^{n}+\left(\overline{G_{2j}^R(\bar{\lambda})^t}\right)_{j=1}^{n\ t}g_2\end{pmatrix}.$$

First we consider the sector S_0. Since the problem is Stone-regular of order l, we have

$$R_{\nu\mu}(\rho)=O(\rho^l),\ \ R_{\nu k}(\rho)=O(\rho^l),\ \ R_{\nu\mu j}(\rho)=O(\rho^l),\ \ R_{\nu kj}(\rho)=O(\rho^l),$$

for $\rho\in S_0, |\rho-\rho_m|>\delta,\ m=1,2,\ldots$,(see [133] or [145]).

Moreover, we have $\Re(\psi_\nu(x)\le 0, \nu=p+1,\ldots,n,$ according to the definition of the function ψ_ν in (24) and hence

$$\int_0^1 e^{\rho\psi_\nu(x)}f(x)dx = o(1),\qquad f\in L_2(0,1),$$

for $\rho\in S_0$. Observing $0\le l_j\le n-1, h_j=-(n-p)$ if $V_j\equiv 0, 0\le h_j\le p-1$ if $V_j\not\equiv 0$ and, consequently, $0\le\kappa_j=\max\{l_j,n-p+h_i\}\le n-1$, we obtain the asserted estimate in S_0 by Theorem (3.1.12) In the same way the estimates in the sectors S_κ follow.

If the problem is normal of order l, then the asserted estimate holds on the ray Γ given by $\Gamma:=\{\rho\in S_\kappa:\ \Re(\rho\omega_k)=0\}$. This completes the proof of the theorem.

We are going to show that under certain regularity assumptions, the system of eigenfunctions and associated functions of problem (1), (2) is complete in the spaces $\mathcal{W}_{\mathcal{U},r}^k$ introduced in this section, where the number r is determined by the order of normality.

To this end we follow the lines of the so-called method of Keldysh [131] which is based on the representation of the principal parts of the Laurent expansion of the resolvent given in this section and on the estimates of the growth of the resolvent provided also in this section.

Proposition(3.1.14)[117]: Suppose the problem (1),(2) to be normal of order l, and let $r(l)\in\mathbb{N}$ be defined by

$$r(l):=\left[\frac{n-p+l+\max\{0,p-1-\kappa_{p+1}\}}{n-p}\right]$$

Then we have

$$\operatorname{Ker}(\mathbb{A}^*)^s=\operatorname{Ker}(\mathbb{A}^*)^{r(l)+1},\qquad s\ge r(l)+1.$$

Proof: It is sufficient to prove $\operatorname{Ker}(\mathbb{A}^*)^{r(l)+2}\subset\operatorname{Ker}(\mathbb{A}^*)^{r(l)+1}$. Assume there exists an element $f\in\operatorname{Ker}(\mathbb{A}^*)^{r(l)+2}$ such that $(\mathbb{A}^*)^{r(l)+1}f\ne 0$. If we let

$$\phi_0:=f,\qquad \phi_i:=\mathbb{A}^*\phi_{i-1},\qquad i=1,2,.,r(l)+1,$$

then we have $\phi_{r(l)+1}=(\mathbb{A}^*)^{r(l)+1}f\ne 0$ and

$$f=(I-\lambda\mathbb{A}^*)\left(\phi_0+\phi_1\lambda+\ldots+\phi_{r(l)}\lambda^{r(l)}+\phi_{r(l)+1}\lambda^{r(l)+1}\right).$$

Substituting $\lambda=-\rho^{n-p}$ we obtain for $\rho\in\mathbb{C},\ |\rho-\rho_m|>\delta$ for some $\delta>0$,

$$\begin{aligned}(I+\rho^{n-p}\mathbb{A}^*)^{-1}f&=\phi_0+\phi_1\lambda+\ldots+\phi_{r(l)}\lambda^{r(l)}+\phi_{r(l)+1}\lambda^{r(l)+1}\\&=o\left(\rho^{n-p+l+\max\{0,p-1-\kappa_{p+1}\}+1}\right)+\phi_{r(l)+1}\lambda^{r(l)+1}\end{aligned}\qquad(26)$$

by definition of $r(l)$. Here $\{\rho_m\}_{m=1}^\infty$ are the zeros of $\Delta(\rho)$. On the other hand, since the problem is normal of order l, Theorem (3.1.13) shows

$$(I+\rho^{n-p}\mathbb{A}^*)^{-1}f=o\left(\rho^{n-p+l+\max\{0,p-1-\kappa_{p+1}\}+1}\right)\qquad(27)$$

As $(r(l)+1)(n-p)>n-p+l+\max\{0,p-1-\kappa_{p+1}\}$, it ensues from (26) and (27) that $\phi_{r(l)+1}=0$ in contradiction with $f\notin\operatorname{Ker}(\mathbb{A}^*)^{r(l)+1}$.

This completes the proof of the proposition.

Corollary(3.1.15)[117]: Let r_0 be the maximal length of the Jordan chains of $\mathbb{A}^*$ corresponding to the eigenvalue 0. Then $r_0 \le r(l) + 1$.

The preceding proposition and its corollary show that the length of the Jordan chains of $\mathbb{A}^*$ corresponding to the eigenvalue 0 is $\le r(l) + 1$ where $r(l)$ is determined by the order l of normality.

In this way, the spaces $\mathcal{W}^k_{\mathcal{U},r(l)+1}$ are associated with the spectral problem (1), (2).

Theorem(3.1.16)[117]: Suppose the problem (1), (2) to be normal of order l, and let $r(l) \in \mathbb{N}$ be defined by

$$r(l) := \left[\frac{n-p+l+\max\{0, p-1-\kappa_{p+1}\}}{n-p}\right].$$

Then the system $\{y^s_\nu\}$ of eigenfunctions and associated functions is complete in the spaces $\mathcal{W}^k_{\mathcal{U},r(l)+1}$ for $k = 0, 1 \ldots, n$.

Proof: By Lemma (3.1.5) we have $\{y^s_\nu\} \subset \mathcal{W}^k_{\mathcal{U},r(l)+1}$ for $k = 0,1,\ldots,n$.

As $\mathcal{W}^n_{\mathcal{U},r(l)+1}$ is dense in $\mathcal{W}^k_{\mathcal{U},r(l)+1}$ for $k < n$ (see [145]), it is sufficient to prove completeness in $\mathcal{W}^n_{\mathcal{U},r(l)+1}$.

By definition of $\mathbb{A}$, an element y^s_ν is an eigenfunction or an associated function of $\mathbb{L}$ or, equivalently, of (1), (2) if and only if $z^s_\nu = \mathbb{K}y^s_\nu$ is an eigenfunction or an associated function, respectively, of $I - \lambda\mathbb{A}$. The operator $\mathbb{K} : W^n_2(0,1) \to L_2(0,1) \times \mathbb{C}^n$ is bounded and invertible and hence $\mathbb{K}^{-1}$ is also bounded. According to the definition of the space $\mathcal{W}^k_{\mathcal{U},r(l)+1}$ and we have $\mathcal{W}^k_{\mathcal{U},r(l)+1} = \mathbb{K}^{-1}\left(\left(\operatorname{Ker}(\mathbb{A}^*)^{r(t)+1}\right)^{\perp}\right)$ Consequently,

$$\mathbb{K} : \mathcal{W}^n_{\mathcal{U},r(l)+1} \to \overline{\operatorname{Im} \mathbb{A}^{r(l)+1}}$$

is a bijection. Thus the system $\{y^s_\nu\}$ is complete in $\mathcal{W}^n_{\mathcal{U},r(l)+1}$ if and only if the system $\{z^s_\nu\}$ is complete in $\overline{\operatorname{Im} \mathbb{A}^{r(l)+1}} \subset L_2(0,1) \times \mathbb{C}^n$.

In order to prove the completeness of $\{z^s_\nu\}$, let $f \in \overline{\operatorname{Im} \mathbb{A}^{r(l)+1}}$ such that $f \perp \{z^s_\nu\}$ with respect to the scalar product in $L_2(0,1) \times \mathbb{C}^n$. Then the function

$$\varphi(\lambda) := (I - \lambda\mathbb{A}^*)^{-1} f, \qquad \lambda \in \mathbb{C},$$

is holomorphic in the whole complex plane by Lemma (3.1.2). Using (14) we see

$$(I - \lambda\mathbb{A}^*)^{-1} = \left(\left(I - \bar{\lambda}\mathbb{A}\right)^{-1}\right)^* = \left(I - \bar{\lambda}\mathbb{H}\left(\mathbb{K} - \bar{\lambda}\mathbb{H}\right)^{-1}\right)^*$$

$$= I + \lambda\left(\mathbb{H}\left(\mathbb{K} - \bar{\lambda}\mathbb{H}\right)^{-1}\right)^*.$$

Thus we obtain for $\chi(\rho) := \varphi(\rho^{n-p})$,

$$\chi(\rho) = (I + \rho^{n-p}\mathbb{A}^*)^{-1} f = f - \rho^{n-p}(\mathbb{H}\,(\mathbb{K} + \overline{\rho^{n-p}}\,\mathbb{H})^{-1})^* f.$$

Since the given problem is normal of order l, Theorem (3.1.13) implies

$$\chi(\rho) = o\left(\rho^{n+l-\min\{p-1,\kappa_{p+1}\}}\right) \tag{28}$$

on a ray belonging to the sector S_0 if $n - p > 2$ and for $\rho \in \mathbb{C}$, $|\rho - \rho_m| > \delta$, if $n - p \le 2$, where $\{\rho_m\}^\infty_{m=1}$ are the zeros of $\Delta(\rho)$. Since χ is an entire function of order 1 in ρ, the estimate (28) holds for all $\rho \in \mathbb{C}$ according to the Theorem of PhragmenLindelof (see [136]). Thus φ is a polynomial of degree

$$\left[\frac{n+l-\min\{p-1,\kappa_{p+1}\}-1}{n-p}\right] = r(l)$$

in λ, that is,

$$(I-\lambda\mathbb{A}^*)^{-1}f = \phi_0 + \phi_1\lambda + \ldots + \phi_{r(l)}\lambda^{r(l)}.$$

Comparing coefficients, we see

$$f = \phi_0, \ \ \phi_i = \mathbb{A}^*\phi_{i-1}, \ \ i = 1,2,\ldots,r(l), \ \ \mathbb{A}^*\phi_{r(l)} = 0.$$

Consequently, $f \in \mathrm{Ker}\,(\mathbb{A}^*)^{r(l)+1} = \left(\mathrm{Im}\mathbb{A}^{r(l)+1}\right)^{\perp}$ and hence $f = 0$. This completes the proof of the theorem.

For λ-independent boundary conditions, a completeness theorem was proved for normal problems of order l under the additional restrictions $l \le n-p-1$ and $2p-n-l_{p+1} \le 0$. Under these assumptions we have $r(l) = 1$, $\mathcal{W}^k_{U,2} = W^k_U$ according to Proposition (3.1. 8).

Corollary(3.1.17)[117]: Suppose the boundary conditions (2) to be independent of λ, and assume the problem (1), (2) to be normal of order l with $l \le n-p-1$ and $2p-n-l_{p+1} \le 0$. Then the system $\{y^s_\nu\}$ of eigenfunctions and associated functions is complete in $\mathcal{W}^k_{U,2} = W^k_U$ for $k = 0,1,\ldots,n$.

Nevertheless, even in the case of λ-independent boundary conditions, the completeness theorem of the present section is an extension of the completeness result in [147] since we do not require $l \le n-p-1$ and $2p-n-l_{p+1} \le 0$ here.

We consider four examples of boundary eigenvalue problems (1), (2). The first example is the simplest problem of this form with $p > 0$, its boundary conditions are independent of λ. The same differential equation but with $\lambda-$linear boundary conditions is Studied as a second example. These examples have been considered before by Everitt [125], and recently in [134] in a self−adjoint setting. The third example concerns the non−selfadjoint case with $\lambda-$linear boundary conditions. It is the adjoint problem of a boundary eigenvalue problem with $\lambda-$independent boundary conditions which was studied in [147], [148].

Finally, we consider an example arising in the theory of elasticity: the eigenvalue problem for the buckling loads of a clamped-free elastic bar subjected to a certain load at its free end, the so-called Petterson-Kӧnig′s rod.

Example(3.1.18)[117]: We consider the boundary eigenvalue problem

$$-y'' = \lambda i y',$$

$$y(0) = 0, \qquad y(1) = 0.$$

The eigenvalues and eigenfunctions of this problem are

$$\lambda_\nu = 2\pi\nu, \qquad \nu = \pm1, \pm2, \ldots,$$

$$y_\nu(x) = \frac{1}{2\pi\nu}(e^{2\pi i\nu x}-1), \qquad x \in [0,1], \qquad \nu = \pm1,\pm2,\ldots$$

Here the operators $\mathbb{K},\mathbb{H} : L_2(0,1) \to L_2(0,1) \times \mathbb{C}^2$ are given by $D(\mathbb{K}) = W^2_2(0,1)$, $D(\mathbb{H}) = W^1_2(0,1)$ and

$$\mathbb{K}y = \begin{pmatrix} y'' \\ y(0) \\ y(1) \end{pmatrix}, \quad \mathbb{H}y = \begin{pmatrix} y' \\ 0 \\ 0 \end{pmatrix},$$

so that the given eigenvalue problem can be written equivalently as $(\mathbb{K}+i\lambda\mathbb{H})y = 0$. It is not difficult to see that this problem is normal of order 0 with $r(0) = 1$. For we have

$$\kappa_1 = l_1 = 0, \qquad \kappa_2 = l_2 = 0, \qquad \kappa = \kappa_1 + \kappa_2 = 0,$$

and, with $i\lambda = \rho$ and $\omega_1 = -1$,

$$\Delta(\rho) = \begin{vmatrix} 1 & 1 \\ 1 & e^{-\rho} \end{vmatrix} = -1 + e^{-\rho}.$$

In order to determine the spaces $\mathcal{W}^k_{\mathcal{U},2}$, we have to calculate the adjoint of the operator $\mathbb{A} = \mathbb{H}\mathbb{K}^{-1}$.
Note that $\mathbb{K}$ is invertible, its inverse $\mathbb{K}^{-1} : L_2(0,1) \times \mathbb{C}^2 \to L_2(0,1)$ being given by

$$(\mathbb{K}^{-1}f)(x) = \int_0^x \int_0^t f_1(\tau)d\tau dt + \left(-\int_0^1 \int_0^t f_1(\tau)d\tau dt + f_3 - f_2\right)x + f_2,$$
$$x \in [0,1],\ f \in L_2(0,1) \times \mathbb{C}^2,$$

where $f = (f_1 f_2 f_3)^t$. Consequently, $\mathbb{A} : L_2(0,1) \times \mathbb{C}^2 \to L_2(0,1) \times \mathbb{C}^2$ is determined by

$$(\mathbb{A}f)(x) = \begin{pmatrix} \int_0^x f_1(\tau)dr - \int_0^1 \int_0^t f_1(\tau)d\tau dt + f_3 - f_2 \\ 0 \\ 0 \end{pmatrix},\quad x \in [0,1],\quad f \in L_2(0,1) \times \mathbb{C}^2$$

with $f = (f_1 f_2 f_3)^t$. An easy calculation shows that $\mathbb{A}^* : L_2(0,1) \times \mathbb{C}^2 \to L_2(0,1) \times \mathbb{C}^2$ is given by

$$(\mathbb{A}^*g)(x) = \begin{pmatrix} \int_0^x g_1(\tau)d\tau - \left(\int_0^1 g_1(\tau)d\tau\right)x \\ \int_0^1 g_1(\tau)d\tau \\ \int_0^1 g_1(\tau)d\tau \end{pmatrix},$$
$$x \in [0,1],\qquad g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in L_2(0,1) \times \mathbb{C}^2$$

Hence a basis of $\mathrm{Ker}(\mathbb{A}^*)^2$ is $\{\varphi_1^0, \varphi_2^0, \varphi_1^1\}$ where

$$\varphi_1^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\qquad \varphi_2^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\qquad \varphi_1^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

According to Definition (3.1.4), we obtain

$$\mathcal{U}_1(y) = U_1^0(y) = \langle \mathbb{K}y, \varphi_1^0 \rangle_{L_2(0,1)\times\mathbb{C}^2} = y(0),$$
$$\mathcal{U}_2(y) = U_2^0(y) = \langle \mathbb{K}y, \varphi_2^0 \rangle_{L_2(0,1)\times\mathbb{C}^2} = y(1),$$
$$\mathcal{U}_3(y) = U_1^1(y) = \langle \mathbb{K}y, \varphi_1^1 \rangle_{L_2(0,1)\times\mathbb{C}^2} = \int_0^1 y''(x)dx = y'(1) - y'(0),$$

and

$$\mathcal{W}^0_{\mathcal{U},2} = L_2(0,1),$$
$$\mathcal{W}^1_{\mathcal{U},2} = \{y \in W_2^1(0,1) : y(0) = 0, y(1) = 0\}.$$
$$\mathcal{W}^2_{\mathcal{U},2} = \{y \in W_2^2(0,1) : y(0) = 0, y(1) = 0, y'(1) - y'(0) = 0\}.$$

By Theorem (3.1.16), the system $\{y_\nu\}_{\nu\in\mathbb{Z},\nu\neq 0}$ is complete in $L_2(0,1)$, $\mathcal{W}^1_{\mathcal{U},2}$ and $\mathcal{W}^2_{\mathcal{U},2}$.

Since the boundary conditions of the preceding example are independent of λ, the theory developed in [147], [148] can also be applied here to show that $\{y_\nu\}_{\nu\in\mathbb{Z},\nu\neq 0}$ is complete and even a Riesz basis in the spaces $W^k_{\mathcal{U}} = \mathcal{W}^k_{\mathcal{U},2}$ for $k = 0, 1, 2$.
Moreover, this example shows that the additional defect of completeness due to the boundary condition $y'(1) - y'(0) = 0$ is not a consequence of a lack of selfadjointness. For in [134], a selfadjoint operator A is associated with Example (3.1.18), given by

$$D(A) = \{y \in H : y \in W_2^2(0,1), \quad y'(1) - y'(0) = 0\},$$
$$(Ay)(x) = \mathrm{i}y'(x) - \mathrm{i}y'(0), \quad y \in D(A).$$

Here the space H is defined as

$$H = \{f \in W_2^1(0,1) : f(0) = 0, f(1) = 0\},$$
$$(f,g)_H = \int_0^1 f' \, \overline{g'} dx, \; f,g \in H.$$

In [134] it is proved that $\{y_\nu\}_{\nu\in\mathbb{Z},\nu\neq 0}$ is an orthonormal basis in $(H, (\cdot,\cdot)_H)$ and hence a Riesz basis in $\mathcal{W}^1_{\mathcal{U},2}$ since $H = \mathcal{W}^1_{\mathcal{U},2}$ and the norm on H is equivalent to the usual Sobolev norm on $W_2^1(0,1)$. It should be noted that here the additional boundary condition $y'(1) - y'(0) = 0$ arises as a restriction on the domain of A, that is, $D(A) = \mathcal{W}^2_{\mathcal{U},2}$, which guarantees $AH \subset H$.

Example(3.1.19)[117]: The next example we study is

$$-y'' = \lambda i y',$$
$$y(0) = 0, \quad y'(1) + \lambda \frac{1}{2} y(1) = 0.$$

Here the eigenvalues and eigenfunctions are

$$\lambda_\nu = (2\nu - 1)\pi, \qquad \nu \in \mathbb{Z},$$
$$y_\nu(x) = \frac{1}{(2\nu-1)\pi}\left(e^{2(\nu-1)\pi \mathrm{i}x} - 1\right), \quad x \in [0,1], \quad \nu \in \mathbb{Z},$$

The operators $\mathbb{K}, \mathbb{H} : L_2(0,1) \to L_2(0,1) \times \mathbb{C}^2$ with $D(\mathbb{K}) = W_2^2(0,1), D(\mathbb{H}) = W_2^1(0,1)$ are now given by

$$\mathbb{K}y = \begin{pmatrix} y'' \\ y(0) \\ y'(1) \end{pmatrix}, \quad \mathbb{H}y = \begin{pmatrix} y' \\ 0 \\ \frac{1}{2}y(1) \end{pmatrix},$$

The problem is normal of order 0 with $r(0) = 1$ since

$$\kappa_1 = l_1 = 0, \quad \kappa_2 = \max\{l_2, n - p + h_2\} = 1, \quad \kappa = 1,$$

and, with $\mathrm{i}\lambda = \rho$,

$$\Delta(\rho) = \begin{vmatrix} 1 & 1 \\ \frac{1}{2}\rho & e^{-\rho}\left(\frac{1}{2}\rho - 1\right) \end{vmatrix} = \frac{1}{2}\rho\left(-1 + \left(1 - \frac{2}{\rho}\right)e^{-\rho}\right).$$

The inverse of $\mathbb{K}$ is given by

$$(\mathbb{K}^{-1}f)(x) = \int_0^x \int_0^t f_1(\tau)d\tau dt + \left(-\int_0^1 f_1(\tau)d\tau + f_3\right)x + f_2,$$
$$x \in [0,1], \qquad f \in L_2(0,1) \times \mathbb{C}^2,$$

where $f = (f_1 f_2 f_3)^t$. Hence we obtain

$$(\mathbb{A}f)(x) = \begin{pmatrix} \int_0^x f_1(\tau)d\tau - \int_0^1 f_1(\tau)d\tau + f_3 \\ 0 \\ \frac{1}{2}\left(\int_0^1 \int_0^t f_1(\tau)d\tau dt - \int_0^1 f_1(\tau)d\tau + f_3 + f_2\right) \end{pmatrix},$$
$$x \in [0,1], \quad f \in L_2(0,1) \times \mathbb{C}^2$$

with $f = (f_1 f_2 f_3)^t$. The adjoint $\mathbb{A}^*$ is given by

$$(\mathbb{A}^*g)(x) = \begin{pmatrix} -\int_0^x g_1(\tau)d\tau - \frac{1}{2}g_3 x \\ \frac{1}{2}g_3 \\ \int_0^1 g_1(\tau)d\tau + \frac{1}{2}g_3 \end{pmatrix}, \quad x \in [0,1], \quad g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in L_2(0,1) \times \mathbb{C}^2$$

Thus a basis $\{\varphi_1^0, \varphi_1^1\}$ of $\operatorname{Ker}(\mathbb{A}^*)^2$ is determined by

$$\varphi_1^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varphi_1^1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

Then the boundary conditions $\mathcal{U}_j$ introduced in Definition (3.1.4) read

$$\mathcal{U}_1(y) = U_1^0(y) = \langle \mathbb{K}y, \varphi_1^0 \rangle_{L_2(0,1)\times\mathbb{C}^2} = y(0),$$

$$\mathcal{U}_2(y) = U_1^1(y) = \langle \mathbb{K}y, \varphi_1^1 \rangle_{L_2(0,1)\times\mathbb{C}^2} = -\int_0^1 y''(x)dx + 2y'(1) = y'(1) + y'(0),$$

According to Theorem (3.1.16), the system $\{y_\nu\}_{\nu\in\mathbb{Z}}$ is complete in the spaces

$\mathcal{W}_{\mathcal{U},2}^0 = L_2(0,1),$

$\mathcal{W}_{\mathcal{U},2}^1 = \{y \in W_2^1(0,1): y(0) = 0, y(1) = 0\},$

$\mathcal{W}_{\mathcal{U},2}^2 = \{y \in W_2^2(0,1): y(0) = 0, y(1) = 0, y'(1) + y'(0) = 0\}.$

This example is still selfadjoint in the sense of [134], and it has been proved there that $\{y_\nu\}_{\nu\in\mathbb{Z}}$ is an orthonormal basis in

$$H = \{f \in W_2^1(0,1) : f(0) = 0\}, (f,g)_H = \int_0^1 f' \ \overline{g'}\, dx, f, g \in H.$$

Here the corresponding selfadjoint operator A in H is given by

$$D(A) = \{y \in H : y \in W_2^2(0,1), y'(1) + y'(0) = 0\} = \mathcal{W}_{\mathcal{U},2}^2,$$
$$(Ay)(x) = \mathrm{i}y'(x) - \mathrm{i}y'(0), \qquad y \in D(A).$$

It follows that $\{y_\nu\}_{\nu\in\mathbb{Z}}$ is a Riesz basis in $\mathcal{W}_{\mathcal{U},2}^1$.

A completeness or basis result in the Sobolev space $W_2^2(0,1)$ cannot be deduced, neither from the selfadjoint approach in [134] since A is selfadjoint only in H, nor from the nonself-adjoint theory in [147], [148] which does not apply here at all because the boundary conditions depend on the eigenvalue parameter.

Example(3.1.20)[117]: The boundary eigenvalue problem

$$y''' = \lambda y',$$
$$y(1) = 0, \qquad y''(1) = 0, \qquad y''(0) + y'(0) - \lambda y(0) = 0.$$

is the adjoint problem of Example 1.1 in [147], [148]. The eigenvalues and eigenfunctions of this problem are given by

$$\lambda_\nu = -(2\nu - 1)^2 \frac{\pi^2}{4}, \qquad \nu = 1,2,\dots,$$

$$y_\nu(x) = \frac{4}{(2\nu-1)^2\pi^2} \cos\left((2\nu - 1)\frac{\pi}{2}x\right), \ \nu = 1,2,\dots,$$

The operators $\mathbb{K}, \mathbb{H} : L_2(0,1) \to L_2(0,1) \times \mathbb{C}^3$ are defined by $D(\mathbb{K}) = W_2^3(0,1)$, $D(\mathbb{H}) = W_2^1(0,1)$ and

$$\mathbb{K}y = \begin{pmatrix} y''' \\ y(1) \\ y''(1) \\ y''(0) + y'(0) \end{pmatrix}, \quad \mathbb{H}y = \begin{pmatrix} y' \\ 0 \\ 0 \\ y(0) \end{pmatrix},$$

We have

$$\kappa_1 = l_1 = 0 \ , \kappa_2 = l_2 = 2 \ , \kappa_3 = \max\{l_3, n - p + h_3\} = 2 \ , \kappa = 4,$$

and, with $\lambda = -\rho^2$

$$\Delta(\rho) = 2\rho^3(e^{\mathrm{i}\rho} + e^{-\mathrm{i}\rho}).$$

Hence the problem is normal of order 1 and $r(1) = 1$. The operator $\mathbb{A} = \mathbb{H}\mathbb{K}^{-1}$ turns out to be given by

$$(\mathbb{A}f)(x) = \begin{pmatrix} \int_0^x \int_0^t f_1(\tau)d\tau dt + \left(f_3 - 2\int_0^1 f_1(\tau)d\tau\right)x + f_4 - f_3 + 2\int_0^1 f_1(\tau)d\tau \\ 0 \\ 0 \\ f_2 + \frac{1}{2}f_3 - f_4 - \int_0^1 f_1(\tau)d\tau - \int_0^1 \int_0^x \int_0^t f_1(\tau)d\tau\, dtdx \end{pmatrix},$$

where $x \in [0,1], f = (f_1 f_2\ f_3\ f_4)^t \in L_2(0,1) \times \mathbb{C}^3$.A short calculation shows

$$(\mathbb{A}^* g)(x) = \begin{pmatrix} \int_0^x \int_0^t f_1(\tau)d\tau dt + (1-x)\int_0^1 g_1(\tau)d\tau + \int_0^1 \int_0^t g_1(\tau)d\tau\, dt - \frac{1}{2}g_4(1-x^2) \\ g_4 \\ \int_0^1 g_1(\tau)d\tau + \frac{1}{2}g_4 \\ \int_0^1 g_1(\tau)d\tau - g_4 \end{pmatrix}$$

for $x \in [0,1], g = (g_1 g_2\ g_3\ g_4)^t \in L_2(0,1) \times \mathbb{C}^3$. As $r(1) = 1$, the maximal length of a Jordan chain of $\mathbb{A}^*$ corresponding to the eigenvalue 0 is less or equal 2. A basis of $\operatorname{Ker}(\mathbb{A}^*)^2$ is given by $\{\varphi_1^0, \varphi_2^0, \varphi_1^1\}$ where

$$\varphi_1^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \varphi_2^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \varphi_1^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The boundary conditions U_1^0, U_2^0 and U_1^1 read in this case

$$U_1^0(y) = \langle \mathbb{K}\mathrm{y}, \varphi_1^0 \rangle_{L_2(0,1)\times\mathbb{C}^3} = y(1),$$
$$U_2^0(y) = \langle \mathbb{K}\mathrm{y}, \varphi_2^0 \rangle_{L_2(0,1)\times\mathbb{C}^3} = y''(1),$$
$$U_1^1(y) = \langle \mathbb{K}\mathrm{y}, \varphi_1^1 \rangle_{L_2(0,1)\times\mathbb{C}^3}$$
$$= \int_0^1 y'''(x)dx + y''(0) + y'(0) = y''(1) + y'(0),$$

An equivalent normalized system $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$ of boundary conditions is

$$\mathcal{U}_1(y) = y(1), \qquad \mathcal{U}_2(y) = y"(1), \qquad \mathcal{U}_3(y) = y'(o).$$

Hence the system $\{y_\nu\}_{\nu\in\mathbb{N}}$ is complete in the spaces

$\mathcal{W}^0_{\mathcal{U},2} = L_2(0,1)$,
$\mathcal{W}^1_{\mathcal{U},2} = \{y \in W^1_2(0,1){:}\ y(1) = 0\}$.
$\mathcal{W}^2_{\mathcal{U},2} = \{y \in W^2_2(0,1){:}\ y(1) = 0, \quad y'(0) = 0\}$.
$\mathcal{W}^3_{\mathcal{U},2} = \{y \in W^3_2(0,1){:}\ y(1) = 0, \quad y'(0) = 0, y''(1) = 0\}$.

Although this example is adjoint to an example considered in [147], [148], it could not be studied within the theory therein since its boundary conditions depend on λ. Of course, it cannot be viewed within any self-adjoint approach because of the different structure of Example (3.1.20) and its adjoint problem.

Example(3.1.21)[117]: The critical loads for divergence of a clamped-free elastic bar of length l and constant flexural rigidity α exposed to an end load Q are determined by the boundary eigenvalue problem

$$y^{(4)} = \lambda y^{(2)},$$
$$y(0) = 0,\ y'(0) = 0,\ y''(1) = 0,\ y^{(3)}(1) - \lambda(1-\gamma)y'(1) = 0.$$

Here $\lambda = -\frac{Ql^2}{\alpha}$, and $\gamma\varphi_l$, $\gamma \in [0,1]$, is the angle between the end load Q and the x-axis where φ_l denotes the angle between the tangent at the free end of the bar and the x-axis (see Figure 1).

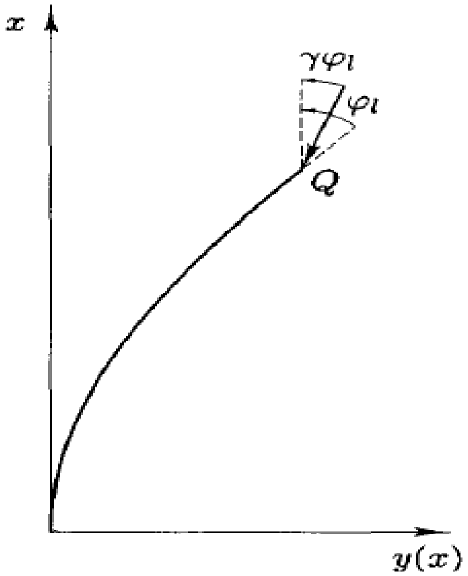

Figure 1

This system is called Petterson-König′s rod (see [132], [135], [129],[141] and [123]), or, if Q is tangential, that is, $\gamma = 1$, Pflüger′s rod. The case $\gamma = 0$ leads to the well-known Euler-loads.

The operators $\mathbb{K}, \mathbb{H} : L_2(0,1) \to L_2(0,1) \times \mathbb{C}^4$ are given by

$D(\mathbb{K}) = W^4_2(0,1)$, $D(\mathbb{H}) = W^2_2(0,1)$ and

$$\mathbb{K}y = \begin{pmatrix} y^{(4)} \\ y(0) \\ y'(0) \\ y''(1) \\ y^{(3)}(1) \end{pmatrix}, \qquad \mathbb{H}y = \begin{pmatrix} y^{(2)} \\ 0 \\ 0 \\ 0 \\ (1-\gamma)y'(1) \end{pmatrix}$$

The characteristic determinant is easily seen to be

$$\Delta(\rho) = \rho^5(-2i\gamma - i(1-\gamma)e^{i\rho} - i(1-\gamma)e^{-i\rho})$$

where $\lambda = -\rho^2$. Since

$$\kappa_1 = 0, \ \kappa_2 = 1, \ \kappa_3 = 2, \ \kappa_4 = 3, \ \kappa = 6,$$

the problem is normal of order 1 and $r(1) = 1$. The operator $\mathbb{A} = \mathbb{H}\mathbb{K}^{-1}$is given by

$$(\mathbb{A}f)(x) = \begin{pmatrix} \int_0^x \int_0^t f_1(\tau)d\tau dt + \left(-\int_0^1 f_1(\tau)d\tau + f_5\right)x - \int_0^1 \int_0^t f_1(\tau)d\tau dt + \int_0^1 f_1(\tau)d\tau + f_4 - f_5 \\ 0 \\ 0 \\ 0 \\ \left(\int_0^1 \int_0^x \int_0^t f_1(\tau)d\tau\, dt dx - \int_0^1 \int_0^t f_1(\tau)d\tau dt - \frac{1}{2}\int_0^1 f_1(\tau)d\tau + f_3 + f_4 - \frac{1}{2} f_5\right)(1-\gamma) \end{pmatrix}$$

for $x \in [0,1], f = (f_1 f_2\ f_3\ f_4 f_5)^t \in L_2(0,1) \times \mathbb{C}^4$.After some calculation we find

$$(\mathbb{A}^* g)(x) = \begin{pmatrix} -\int_x^1 \int_0^t f_1(\tau)d\tau dt + \frac{x^2}{2} g_5(1-\gamma) + (1-x)\int_0^1 g_1(\tau)d\tau + \int_0^1 (1-\tau)g_1(\tau)d\tau \\ 0 \\ g_5(1-\gamma) \\ \int_0^1 g_1(\tau)d\tau + g_5(1-\gamma) \\ -\int_0^1 g_5(1-\tau)g_1(\tau)d\tau - \frac{1}{2} g_5(1-\gamma) \end{pmatrix}$$

For $x \in [0,1]$, $g = (g_1 g_2\ g_3\ g_4 g_5)^t \in L_2(0,1) \times \mathbb{C}^4$. Again, as $r(1) = 1$, the maximal length of a Jordan chain of $\mathbb{A}^*$ corresponding to the eigenvalue 0 is less or equal 2. A basis of $\mathrm{Ker}\,(\mathbb{A}^*)^2$ according to Definition (3.1.4) is given by $\{\varphi_1^0, \varphi_2^0, \varphi_3^0, \varphi_1^1\}$ where

$$\varphi_1^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_2^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \varphi_3^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \varphi_1^1 = \begin{pmatrix} 1-\gamma \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus we obtain the boundary conditions

$$\begin{aligned}
\mathcal{U}_1(y) &= U_1^0(y) = \langle \mathbb{K}y, \varphi_1^0 \rangle_{L_2(0,1)\times\mathbb{C}^3} = y(0), \\
\mathcal{U}_2(y) &= U_2^0(y) = \langle \mathbb{K}y, \varphi_2^0 \rangle_{L_2(0,1)\times\mathbb{C}^3} = y'(0), \\
\mathcal{U}_3(y) &= U_3^0(y) = \langle \mathbb{K}y, \varphi_3^0 \rangle_{L_2(0,1)\times\mathbb{C}^3} = y''(1), \\
\mathcal{U}_4(y) &= U_1^1(y) = \langle \mathbb{K}y, \varphi_1^1 \rangle_{L_2(0,1)\times\mathbb{C}^3} \\
&= (1-\gamma)\int_0^1 y^{(4)}(x)dx - y^{(3)}(1) = -\gamma y^{(3)}(1) - (1-\gamma)y^{(3)}(0),
\end{aligned}$$

According to Theorem (3.1.16), the system $\{y_\nu^s\}$ of eigenfunctions and associated functions of the boundary eigenvalue problem for the critical divergence loads of the Petterson-König′s rod is complete in the spaces

$$\begin{aligned}
\mathcal{W}_{\mathcal{U},2}^0 &= L_2(0,1), \\
\mathcal{W}_{\mathcal{U},2}^1 &= \{y \in W_2^1(0,1) : y(0) = 0\}, \\
\mathcal{W}_{\mathcal{U},2}^2 &= \{y \in W_2^2(0,1) : y(0) = 0, y'(0) = 0\},
\end{aligned}$$

$\mathcal{W}^3_{\mathcal{U},2} = \{y \in W^3_2(0,1): y(0) = 0, y'(0) = 0, y''(1) = 0\}.$

$\mathcal{W}^4_{\mathcal{U},2} = \{y \in W^4_2(0,1): y(0) = 0, y'(0) = 0, y''(1) = 0, \gamma y^{(3)}(1) + (1-\gamma) y^{(3)}(0) = 0\}.$

Since the eigenvalues and eigenfunctions of Example (3.1.21) can be computed exactly, it would be interesting to see if the convergence of Ritz-Galerkin methods, which are used for these problems (see[135],[129]), is improved by choosing starting functions in the space $\mathcal{W}^4_{\mathcal{U},2}$, which satisfy the additional boundary condition $\gamma y^{(3)}(1) + (1-\gamma) y^{(3)}(0) = 0$.

Sec(3.2): Some Non-Selfadjont Block Operator Matrices

In the reference [160] selfadjoint operators $\tilde{A}$ in a Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \times \hat{\mathcal{H}}$, given by a block operator matrix

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \tag{29}$$

Were considered under the assumption that the spectra of A and D are separated and B, D, are bounded operators. In [161] and in [164] these investigations were extended to the case that all entries are unbounded.

In this section the results of [161] are generalized to a non-selfadjoint situation. We consider an operator $\tilde{A}$ given by a block operator matrix (29) such that A and $-D$ are m-sectorial (see [162]) and their numerical ranges have a positive distance from the imaginary axis, and B is bounded. We show that the imaginary axis belongs to the resolvent set $\rho(\tilde{A})$. If the spaces $\mathcal{H}$, $\hat{\mathcal{H}}$ are not trivial, then the spectrum $\sigma(\tilde{A})$ consists of some part $\sigma_-(\tilde{A})$ in the left half plane and another part $\sigma_+(\tilde{A})$ in the right half plane (if A or D are unbounded, then $\sigma_+(\tilde{A})$ or $\sigma_-(\tilde{A})$ may be empty, which implies that ∞ belongs to the extended spectrum of $\tilde{A}$).

It is shown below that, if the operator D is bounded, then the spectral subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ corresponding to $\sigma_+(\tilde{A})$ and $\sigma_-(\tilde{A})$ are supported on $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively; that is, e.g., $\mathcal{L}_+$ admits a representation

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{H} \right\}$$

with some bounded linear operator K_+ from $\mathcal{H}$ into $\hat{\mathcal{H}}$. The invariance of $\mathcal{L}_+$ and $\mathcal{L}_-$ under $\tilde{A}$ implies that K_+ and the corresponding operator K_- for $\mathcal{L}_-$ satisfy certain Riccati equations.

As in [160], this result is used in order to show a half range completeness statement : If $\sigma_+(\tilde{A})$ is discrete, under certain assumptions, the first components of a system of root vectors of $\tilde{A}$ corresponding to $\sigma_+(\tilde{A})$ (" half "of the spectrum of $\tilde{A}$) form a complete system in $\mathcal{H}$. An example of an eigenvalues problem of Sturm-Liouville type where this result can be applied is given at the end of this section.

In order to locate the spectrum $\sigma(\tilde{A})$ of the block operator matrix $\tilde{A}$, we introduce the notion of the quadratic numerical range of a general block operator matrix

$$\tilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with closed operators A, D and bounded operators B, C . This quadratic numerical range is the set of all eigenvalues of the matrices

$$\begin{pmatrix} \dfrac{(Ax, x)}{\|x\|^2} & \dfrac{(B\hat{x}, x)}{\|\hat{x}\|\|x\|} \\ \dfrac{(Cx, \hat{x})}{\|x\|\|\hat{x}\|} & \dfrac{(D\hat{x}, \hat{x})}{\|\hat{x}\|^2} \end{pmatrix},$$

where $x \in \mathcal{D}(A)$, $\hat{x} \in \mathcal{D}(D)$, x, $\hat{x} \neq 0$. It has some properties analogous to those of the numerical range. We mention that the quadratic numerical range of a block operator matrix $\tilde{A}$ turns out to be especially useful in the particular case of a selfadjoint operator $\tilde{A}$. This question will be considered elsewhere.

Let $\mathcal{H}$ and $\widehat{\mathcal{H}}$ be Hilbert spaces. We consider an operator $\tilde{A}$ in the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \times \widehat{\mathcal{H}}$ given by a block operator matrix

$$\tilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{30}$$

where A and D are densely defined closed operators in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, and the operators $B \in L(\widehat{\mathcal{H}}, \mathcal{H})$, $C \in L(\mathcal{H}, \widehat{\mathcal{H}})$ are bounded.

For the operator $\tilde{A}$ we call the set

$$W_{\tilde{A}}^2 := \left\{ \lambda \in \mathbb{C} : \det \begin{pmatrix} \dfrac{(Ax,x)}{\|x\|^2} - \lambda & \dfrac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\ \dfrac{(Cx,\hat{x})}{\|x\|\|\hat{x}\|} & \dfrac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} - \lambda \end{pmatrix} = 0, \right.$$
$$\left. x \in \mathcal{D}(A),\ \hat{x} \in \mathcal{D}(D),\ x, \hat{x} \neq 0 \right\} \tag{31}$$

the quadratic numerical range (with respect to the block operator representation (30)).

Thus, for each element $\tilde{x} = (x, \hat{x})^t \in \widetilde{\mathcal{H}}$ such that $x \in \mathcal{D}(A), \hat{x} \in \mathcal{D}(D), x, \hat{x} \neq 0$, two complex numbers λ_1 , λ_2 are defined as the solutions of the quadratic equation in (31), which are the eigenvalues of the matrix

$$\begin{pmatrix} \dfrac{(Ax,x)}{\|x\|^2} & \dfrac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\ \dfrac{(Cx,\hat{x})}{\|x\|\|\hat{x}\|} & \dfrac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} \end{pmatrix},$$

and $W_{\tilde{A}}^2$ is the set of all these solutions or of all these eigenvalues.

It is easy to see that , for a bounded operator $\tilde{A}$, the quadratic numerical range is a bounded subset of $\mathbb{C}$. It is also not difficult to see that the quadratic numerical range consists of at most two connected sets. If $B = 0$ or $C = 0$, then $W_{\tilde{A}}^2 = W_A \cup W_D$, where for a closed operator T in a Hilbert space, W_T denotes its numerical range,

$$W_T := \left\{ \frac{(T\,x,x)}{\|x\|^2} : x \in \mathcal{D}(T), x \neq 0 \right\}.$$

Let $r(\tilde{A})$ be the set of points of regular type of $\tilde{A}$, that is, $\lambda \in r(\tilde{A})$ is equivalent to

$$\|(\tilde{A} - \lambda)\tilde{x}\| \geq \gamma_\lambda \|\tilde{x}\|, \quad \tilde{x} \in \mathcal{D}(\tilde{A}),$$

for some $\gamma_\lambda > 0$.

Theorem(3.2.1)[159]: For the quadratic numerical range $W_{\tilde{A}}^2$ the following inclusions hold:

$$\sigma_p(\tilde{A}) \subset W_{\tilde{A}}^2, \quad \mathbb{C} \setminus r(\tilde{A}) \subset \overline{W_{\tilde{A}}^2}.$$

Proof. Let $\lambda \in \sigma_p(\tilde{A})$. Then there exists a nontrivial vector $(x, \hat{x})^t \in \widetilde{\mathcal{H}}$, $x \in \mathcal{D}(A), \hat{x} \in \mathcal{D}(D)$, such that

$$(A - \lambda)x + B\hat{x} = 0,$$
$$Cx + (D - \lambda)\hat{x} = 0. \tag{32}$$

Suppose first that $x, \hat{x} \neq 0$. Taking the inner product of the first (second, respectively) equation in (32) with x ($\hat{x}$, respectively), we find that the system

$$\begin{pmatrix} (Ax,x) - \lambda\|x\|^2 & (B\hat{x},x) \\ (Cx,\hat{x}) & (D\hat{x},\hat{x}) - \lambda\|\hat{x}\|^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

has the solution $\xi_1 = 1, \xi_2 = 1$, hence

$$\det \begin{pmatrix} (Ax,x) - \lambda\|x\|^2 & (B\hat{x},x) \\ (Cx,\hat{x}) & (D\hat{x},\hat{x}) - \lambda\|\hat{x}\|^2 \end{pmatrix} = 0.$$

But this equation is equivalent to the equation in the definition (31) of $W_{\tilde{A}}^2$, and hence $\lambda \in W_{\tilde{A}}^2$. Now let $\hat{x} = 0$. Then $(A-\lambda)x = 0, Cx = 0$, and hence

$$\det \begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} - \lambda & \frac{(B\hat{x}',x)}{\|\hat{x}'\|\|x\|} \\ \frac{(Cx,\hat{x}')}{\|x\|\|\hat{x}'\|} & \frac{(D\hat{x}',\hat{x}')}{\|\hat{x}'\|^2} - \lambda \end{pmatrix} = \det \begin{pmatrix} 0 & \frac{(B\hat{x}',x)}{\|\hat{x}'\|\|x\|} \\ 0 & \frac{(D\hat{x}',\hat{x}')}{\|\hat{x}'\|^2} - \lambda \end{pmatrix} = 0$$

for all $\hat{x}' \in \mathcal{D}(D), \hat{x}' \neq 0$, which implies $\lambda \in W_{\tilde{A}}^2$. The case $x = 0$ is analogous.

More generally, if $\lambda_0 \notin r(\tilde{A})$, then there exists a sequence $(\tilde{x}_n)_1^\infty \subset \widetilde{\mathcal{H}}$, $\tilde{x}_n = (x_n, \hat{x}_n)^t$, $x_n \in \mathcal{D}(A)$, $\hat{x}_n \in \mathcal{D}(D)$, $\|\tilde{x}_n\| = 1$, such that $\|(\tilde{A} - \lambda_0)\tilde{x}_n\| \to 0$ for $n \to \infty$, that is,

$$\begin{aligned} (A - \lambda_0)x_n + B\hat{x}_n &= f_n, \\ C x_n + (D - \lambda_0)\hat{x}_n &= \hat{f}_n, \end{aligned}$$

where $(f_n)_1^\infty \subset \mathcal{H}, (\hat{f}_n)_1^\infty \subset \widehat{\mathcal{H}}$, $\|f_n\| \to 0$, $\|\hat{f}_n\| \to 0$ for $n \to \infty$.

Suppose first that $\lim_{n\to\infty} \inf \|x_n\| > 0$, $\lim_{n\to\infty} \inf \|\hat{x}_n\| > 0$. Without loss of generality we can assume $\|x_n\| > 0, \|\hat{x}_n\| > 0$ for $n = 1, 2, \ldots$. Then

$$\begin{aligned} \frac{(Ax_n,x_n)}{\|x_n\|^2} - \lambda_0 + \frac{(B\hat{x}_n,x_n)}{\|x_n\|^2} &= \frac{(f_n,x_n)}{\|x_n\|^2}, \\ \frac{(Cx_n,\hat{x}_n)}{\|\hat{x}_n\|^2} + \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda_0 &= \frac{(\hat{f}_n,\hat{x}_n)}{\|\hat{x}_n\|^2}, \end{aligned}$$

We introduce the polynomials

$$\begin{aligned} d_n(\lambda) := \det &\begin{pmatrix} \frac{(Ax_n,x_n)}{\|x_n\|^2} - \lambda & \frac{(B\hat{x}_n,x_n)}{\|x_n\|^2} \\ \frac{(Cx_n,\hat{x}_n)}{\|\hat{x}_n\|^2} & \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda \end{pmatrix} \\ = \det &\begin{pmatrix} \frac{(Ax_n,x_n)}{\|x_n\|^2} - \lambda & \frac{(B\hat{x}_n,x_n)}{\|\hat{x}_n\|\|x_n\|} \\ \frac{(Cx_n,\hat{x}_n)}{\|x_n\|\|\hat{x}_n\|} & \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda \end{pmatrix}. \end{aligned}$$

Then $f_n, \hat{f}_n \to 0$ for $n \to \infty$ implies

$$d_n(\lambda_0) := \det \begin{pmatrix} \frac{(f_n,x_n)}{\|x_n\|^2} & \frac{(B\hat{x}_n,x_n)}{\|x_n\|^2} \\ \frac{(\hat{f}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} & \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda_0 \end{pmatrix} \to 0, n \to \infty.$$

For each n, d_n is a monic quadratic polynomial in λ. If λ_n^1, λ_n^2 are the zeros of d_n, then λ_n^1, $\lambda_n^2 \in W_{\tilde{A}}^2$ and $d_n(\lambda) = (\lambda - \lambda_n^1)(\lambda - \lambda_n^2)$. As $d_n(\lambda_0) \to 0$ for $n \to \infty$, we have $\lambda_n^1 \to \lambda_0$ or $\lambda_n^2 \to \lambda_0$ for $n \to \infty$ and thus $\lambda_0 \in \overline{W_{\tilde{A}}^2}$.

Let $\lim_{n\to\infty} \inf \|\hat{x}_n\| = 0$, without loss of generality $\hat{x}_n \longrightarrow 0$ for $n \to \infty$, $\|x_n\| > 0$ for $n = 1, 2, \ldots$. If we define $\lambda_n := \frac{(Ax_n, x_n)}{\|x_n\|^2}$ and choose $\hat{x}'_n \in \hat{\mathcal{H}}$, $\hat{x}'_n \neq 0$, such that $(C x_n, \hat{x}'_n) = 0$ for $n = 1, 2, \ldots$, then

$$\det\begin{pmatrix} \frac{(Ax_n, x_n)}{\|x_n\|^2} - \lambda_n & \frac{(B\hat{x}'_n, x_n)}{\|\hat{x}'_n\|\|x_n\|} \\ \frac{(Cx_n, \hat{x}'_n)}{\|x_n\|\|\hat{x}'_n\|} & \frac{(D\hat{x}'_n, \hat{x}'_n)}{\|\hat{x}'_n\|^2} - \lambda_n \end{pmatrix} = \det\begin{pmatrix} 0 & \frac{(B\hat{x}'_n, x_n)}{\|\hat{x}'_n\|\|x_n\|} \\ 0 & \frac{(D\hat{x}'_n, \hat{x}'_n)}{\|\hat{x}'_n\|^2} - \lambda_n \end{pmatrix} = 0,$$

that is, $\lambda_n \in W_{\tilde{A}}^2$. As $\hat{x}_n \to 0$ and $f_n \to 0$ for $n \to \infty$, the relation

$$((A - \lambda_0)x_n, x_n) - (B\hat{x}_n, x_n) = (f_n, x_n)$$

implies $\lambda_n \to \lambda_0$ .The case $\lim_{n\to\infty} \inf \|x_n\| = 0$ is analogous. This proves the theorem.

Later we will need the following well-known result connected with the numerical range of a closed operator. Here we assume that $\arg z \in (-\pi, \pi]$ for complex numbers $z \in \mathbb{C}$.

Lemma(3.2.2)[159]: Let T be a closed operator in a Hilbert space. Assume that its numerical range W_T is contained in a sector $\Delta_T = \{z \in \mathbb{C} : |\arg z| \leq \theta_T\}$ for some θ_T, $0 < \theta_T < \frac{\pi}{2}$. Then there exists a constant $C > 0$ such that

$$\|(T - z)^{-1}\| \leq \frac{C}{|z|}, \quad \Re(z) \leq 0, \; z \neq 0.$$

Proof. Let $z \in \mathbb{C}$ with $\Re(z) \leq 0$, $z \neq 0$. If $\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2} + \theta$, then

$$\|(T - z)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, W_T)} \leq \frac{1}{|z| \cos\left(\frac{\pi}{2} + \theta_T - \arg z\right)} \leq \frac{1}{|z| \cos \theta_T}.$$

The case $-\frac{\pi}{2} - \theta_T \leq \arg z \leq -\frac{\pi}{2}$ is analogous. If $|\arg z| \geq \frac{\pi}{2} + \theta_T$, then

$$\|(T - z)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, W_T)} \leq \frac{1}{|z|}.$$

In the sequel we consider a block operator matrix (30) of the particular form

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \tag{33}$$

that is, we assume $C = B^*$. Additionally to the pervious assumptions , we suppose:

(I) The operator A is boundedly invertible and its numerical range W_A is contained in the set

$$\{z \in \mathbb{C} : |\arg z| \leq \theta_A, \; \Re(z) \geq \alpha\}$$

for some θ_A, $0 < \theta_A < \frac{\pi}{2}$, and $\alpha > 0$.

(II) The operator D is boundedly invertible and its numerical range W_D is contained in the set

$$\{z \in \mathbb{C} : |\arg z| \geq \pi - \theta_D, \; \Re(z) \leq -\delta\}$$

for some θ_D, $0 < \theta_D < \frac{\pi}{2}$, and $\delta > 0$.

This means that the operators A and $-D$ are m-sectorial (see[162]) and that their numerical ranges have a positive distance from the imaginary axis. We mention that the role of the imaginary axis can be taken over by any other vertical line in the complex plane.

Lemma(3.2.3)[159]: Let a, b, c and d be complex numbers such that $\Re(a) > 0, \Re(d) < 0$ and $bc \geq 0$. Then the matrix

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

has eigenvalues λ_1, λ_2 such that:

(i) $\Re(\lambda_1) \geq \Re(a)$, $\Re(\lambda_2) \leq \Re(d)$;

(ii) $\min\{\Im(a), \Im(d)\} \leq \Im(\lambda_1), \Im(\lambda_2) \leq \max\{\Im(a), \Im(d)\}$;

(iii) $\lambda_1, -\lambda_2 \in \{z \in \mathbb{C} : |\arg z| \leq \max\{|\arg a|, \pi - |\arg d|\}$.

Proof. We can suppose that $\Im(a) \geq 0$ (otherwise we consider the matrix $\mathcal{A}^*$) and that

$$\arg a \geq \pi - |\arg d| \tag{34}$$

(otherwise in the following considerations we start from d instead of a). The assumption (34) implies

$$\left|\frac{\Im(a-d)}{\Re(a-d)}\right| \leq \tan(\arg a). \tag{35}$$

The eigenvalues λ_1, λ_2 satisfy the equation

$$(a-\lambda)(d-\lambda) - t = 0, \quad t = bc \geq 0.$$

We consider them as functions of t:

$$\lambda_{1,2}(t) = \frac{a+d}{2} \pm \left(\frac{(a-d)^2}{4} + t\right)^{\frac{1}{2}}.$$

If we decompose $\lambda_{1,2}(t)$ and $\frac{a+d}{2}$ in real and imaginary parts, $\lambda_{1,2}(t) = x(t) + iy(t)$ and $\frac{a+d}{2} = \beta + i\gamma$, then we find

$$(x(t)-\beta)^2 - (y(t)-\gamma)^2 = \frac{1}{4}\Re(a-d)^2 + t, \tag{36}$$

$$(x(t)-\beta)(y(t)-\gamma) = \frac{1}{8}\Im(a-d)^2. \tag{37}$$

The last relation shows that the eigenvalues $\lambda_1(t)$, $\lambda_2(t)$ lie on a hyperbola with centre $\beta + i\gamma = \frac{a+d}{2}$, and with the asymptotes $\Im(z) = \gamma$ and $\Re(z) = \beta$ parallel to the real and imaginary axis, the right hand branch passing through a and the left hand branch through d. From the identity (36) it follows that for $0 \leq t \leq \infty$ the eigenvalues $\lambda_1(t)$ fill the part of the right hand branch which extends from a to $+\infty + i\gamma$, and the eigenvalues $\lambda_2(t)$ fill the part of the left hand branch from d to $-\infty + i\gamma$. This implies (i) and (ii). In order to prove (iii), it is sufficient to show that the derivatives of the hyperbola at d and at a are in modulus less than $\tan(\arg a)$. E.g. for the derivative at d it follows from (100)

$$\frac{\dot{y}(0)}{\dot{x}(0)} = -\frac{y(0)-\gamma}{x(0)-\beta} = -\frac{\Im(d) - \frac{1}{2}\Im(a+d)}{\Re(d) - \frac{1}{2}\Re(a+d)} = -\frac{\Im(d-a)}{\Re(d-a)},$$

which is in modulus less than $\tan(\arg a)$ by (35). The lemma is proved.

Theorem(3.2.4)[159]: Suppose the assumptions (I) and (II) are satisfied and define

$$\Delta := \{z \in \mathbb{C} : |\arg z| \leq \max\{\theta_A, \theta_D\}\}.$$

Then

$$\sigma(\tilde{A}) \subset \{z \in (-\Delta) : \Re(z) \leq -\delta\} \cup \{z \in \Delta : \Re(z) \geq \alpha\} =: \tilde{\Delta}.$$

Proof. First we show that $\overline{W_{\tilde{A}}^2} \subset \tilde{\Delta}$. To this end consider for $x \in \mathcal{D}(A),\ \hat{x} \in \mathcal{D}(D),\ x, \hat{x} \neq 0$, the matrix

$$\begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} & \frac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\ \frac{(B^*x,\hat{x})}{\|x\|\|\hat{x}\|} & \frac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} \end{pmatrix}.$$

According to the assumptions (I) and (II), it has all the properties of the matrix $\mathcal{A}$ in Lemma (3.2.3). Hence its eigenvalues are in $\tilde{\Delta}$ which implies $W_{\tilde{A}}^2 \subset \tilde{\Delta}$ and hence $\overline{W_{\tilde{A}}^2} \subset \tilde{\Delta}$. According to Theorem (3.2.1) we have $\mathbb{C}\backslash r(\tilde{A}) \subset \overline{W_{\tilde{A}}^2}$ and consequently $\mathbb{C}\backslash\tilde{\Delta} \subset r(\tilde{A})$. On the other hand, $\mathbb{C}\backslash\tilde{\Delta}$ consists of only one component, hence the theorem will be proved if we show that at least one point λ_0 of this component belongs to $\rho(\tilde{A})$. To this end we choose λ_0 on the imaginary axis sufficiently large in modulus. According to Lemma (3.2.2) we can choose λ_0 such that

$$\left\| B(D - \lambda_0)^{-1} B^*(A - \lambda_0)^{-1} \right\| < 1.$$

Then $\lambda_0 \in \rho(\tilde{A})$ as

$$(A - \lambda_0 - B(D - \lambda_0)^{-1}B^*)^{-1} = (A - \lambda_0)^{-1} (I - B(D - \lambda_0)^{-1} B^*(A - \lambda_0)^{-1})^{-1}$$

exists and is a bounded everywhere defined operator. The theorem is proved.

In the sequel we consider a block operator matrix $\tilde{A}$ as in (33) which satisfies the assumptions (I), (II) and for which D is bounded. Note that in this case the assumption (II) is fulfilled if there exists a $\delta > 0$ such that

$$\Re(D) \leq -\delta.$$

From Theorem (3.2.4) it follows that $\sigma(\tilde{A})$ splits into the two disjoint subsets

$$\sigma_-(\tilde{A}) := \sigma(\tilde{A}) \cap \{z \in (-\Delta) : \Re(z) \leq -\delta\},$$
$$\sigma_+(\tilde{A}) := \sigma(\tilde{A}) \cap \{z \in \Delta : \Re(z) \geq \alpha\}.$$

Here, as A and hence also $\tilde{A}$ can be unbounded, $\sigma_+(\tilde{A})$ can be empty. Since D is a bounded operator, $\sigma_-(\tilde{A})$ is bounded. Let

$$P_-(\tilde{A}) := -\frac{1}{2\pi i} \int_{\Gamma_-} (\tilde{A} - z)^{-1} \, dz$$

be the corresponding Riesz projection. Here Γ_- is a positively oriented Jordan contour in $\{z \in \mathbb{C} : \Re(z) < 0\}$ surrounding $\sigma_-(\tilde{A})$. If we define the projection

$$P_+(\tilde{A}) := I - P_-(\tilde{A}),$$

then we have a decomposition $\widetilde{\mathcal{H}} = \mathcal{L}_- \dot{+} \mathcal{L}_+$ into the spectral subspaces

$$\mathcal{L}_- := P_-(\tilde{A})\widetilde{\mathcal{H}}, \quad \mathcal{L}_+ := P_+(\tilde{A})\widetilde{\mathcal{H}},$$

and

$$\sigma(\tilde{A}|\mathcal{L}_-) = \sigma_-(\tilde{A}), \quad \sigma(\tilde{A}|\mathcal{L}_+) = \sigma_+(\tilde{A}).$$

Here $\tilde{A}|\mathcal{L}_+$ is, in fact, the restriction of $\tilde{A}$ to $\mathcal{D}(\tilde{A}) \cap \mathcal{L}_+$ and has its values in $\mathcal{L}_+$. If $\mathcal{H} \neq \{0\}$, then $\mathcal{L}_+ \neq \{0\}$, even if $\sigma_+(\tilde{A}) = \emptyset$.

Lemma(3.2.5)[159]: We have

$$-\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (\tilde{A} - z)^{-1} \, dz = \frac{1}{2}(P_-(\tilde{A}) - P_+(\tilde{A})).$$

Proof. Let $r_0 > 0$ be such that $\sigma_-(\tilde{A}) \subset \{z \in \mathbb{C} : \Re(z) < 0, |z| < r_0\}$. Then we have

$$-\frac{1}{2\pi i}\int_{-ir}^{ir}\left(\tilde{A}-z\right)^{-1} dz = P_-\left(\tilde{A}\right) + \frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(\tilde{A}-re^{it}\right)^{-1} ire^{it}\, dt, r \geq r_0.$$

The lemma is proved if we show that

$$\frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(\tilde{A}-re^{it}\right)^{-1} ire^{it}\, dt \longrightarrow -\frac{1}{2}I, \qquad r \longrightarrow \infty, \tag{38}$$

strongly in $\tilde{\mathcal{H}}$ as $I = P_-\left(\tilde{A}\right) + P_+(\tilde{A})$. From

$$\tilde{A}-z = \begin{pmatrix} I & B(D-z)^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A-z-B(D-z)^{-1}B^* & 0 \\ 0 & D-z \end{pmatrix}\begin{pmatrix} I & 0 \\ (D-z)^{-1}B^* & I \end{pmatrix}, \tag{39}$$

it follows

$$\left(\tilde{A}-z\right)^{-1} = \begin{pmatrix} M(z) & -M(z)B(D-z)^{-1} \\ -(D-z)^{-1}B^*M(z) & (D-z)^{-1} + (D-z)^{-1}B^*M(z)B(D-z)^{-1} \end{pmatrix} \tag{40}$$

for $z \in \rho(\tilde{A})$ where

$$M\,(z) := \left(A-z - B(D-z)^{-1}\,B^*\right)^{-1}.$$

For the proof of (38) we first consider the left upper corner of the matrix $\left(\tilde{A}-re^{it}\right)^{-1} re^{it} + I$ and show that

$$\left\|\,(M(re^{it})re^{it} + I)\,x\,\right\| \to 0, \quad r \to \infty, \tag{41}$$

uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ for $x \in \mathcal{H}$. Using Lemma (3.2.2) and

$$\|(D-z)^{-1}\| \leq \frac{2}{|z|}, \qquad |z| > 2\|D\|, \tag{42}$$

we find

$$\begin{aligned} M(re^{it})re^{it} + I &= (A-re^{it} - B(D-re^{it})^{-1}\,B^*)^{-1}re^{it} + I \\ &= (I-(A-re^{it})^{-1}B(D-re^{it})^{-1}B^*)^{-1}(A-re^{it})^{-1}re^{it} + I \\ &= (A-re^{it})^{-1}\,re^{it} + I + O\left(\frac{1}{r}\right). \end{aligned}$$

uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. Hence there exists a $\tilde{C} > 0$ such that

$$\begin{aligned} \left\|(M(re^{it})re^{it} + I)x\right\| &\leq \left\|((A-re^{it})^{-1}re^{it} + I)x\right\| + \frac{\tilde{C}}{r}\|x\| \\ &= \left\|(A-re^{it})^{-1}Ax\right\| + \frac{\tilde{C}}{r}\|x\| \leq \frac{C}{r}\|Ax\| + \frac{\tilde{C}}{r}\|x\| \end{aligned}$$

uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ by Lemma(3.2.2) which proves (41).From (41) it follows

$$\left\|M(re^{it})re^{it}x\right\| \leq K. \tag{43}$$

with some constant $K > 0$ uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ for $x \in \mathcal{H}$. For the off-diagonal elements of $\left(\tilde{A}-re^{it}\right)^{-1}re^{it} + I$ we have

$$\begin{aligned} &\left\|M(re^{it})B(D-re^{it})^{-1}re^{it}\hat{x}\right\| \\ &\quad \leq \left\|(M(re^{it})re^{it} + I)B(D-re^{it})^{-1}\hat{x}\right\| + \left\|B(D-re^{it})^{-1}\hat{x}\right\| \longrightarrow 0, r \longrightarrow \infty \end{aligned}$$

uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ for $\hat{x} \in \widehat{\mathcal{H}}$ by (41) and (42). Furthermore,

$$\left\|(D - re^{it})^{-1} B^* M(re^{it})\, re^{it} x\right\| \to 0, \quad r \to \infty,$$

uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ for $x \in \mathcal{H}$ by (42) and (43). For the right lower corner of $\left(\tilde{A} - re^{it}\right)^{-1} re^{it} + I$ we have

$$\left\|((D - re^{it})^{-1} re^{it} + (D - re^{it})^{-1} B^* M(re^{it}) B (D - re^{it})^{-1} re^{it} + I)\hat{x}\right\|,$$
$$\leq \frac{1}{r} 2\|D\|\|\hat{x}\| + \frac{1}{r}\|B^*\|\left\|M(re^{it}) re^{it} B(D - re^{it})^{-1}\hat{x}\right\| \to 0, r \to \infty,$$

uniformly in $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ for $\hat{x} \in \widehat{\mathcal{H}}$. Summarizing, we obtain that all the components in the matrix representation of $\left(\tilde{A} - re^{it}\right)^{-1} re^{it} + I$ tend strongly to 0 uniformly for $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and hence

$$\left(\frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\tilde{A} - re^{it}\right)^{-1} i re^{it} dt + \frac{1}{2} I\right)\tilde{x} = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\left(\tilde{A} - re^{it}\right)^{-1} re^{it}\tilde{x} + \tilde{x}\right) d \longrightarrow 0,$$

$r \to \infty$, for $\tilde{x} \in \widetilde{\mathcal{H}}$.The lemma is proved.

As $\tilde{A}$ is not supposed to be selfadjoint, the spectral subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ need not be orthogonal. However, the block operator matrix $\tilde{A}^*$ adjoint to $\tilde{A}$ given by

$$\tilde{A}^* = \begin{pmatrix} A^* & B \\ B^* & D^* \end{pmatrix}$$

fulfills the assumptions of the present part as well. Hence $P_-(\tilde{A}^*)$ and $P_+(\tilde{A}^*)$ are defined, and $\tilde{A}^*$ has the spectral subspaces

$$\mathcal{L}_-^* := P_-(\tilde{A}^*)\,\widetilde{\mathcal{H}}, \ \mathcal{L}_+^* := P_+(\tilde{A}^*)\widetilde{\mathcal{H}}.$$

Lemma(3.2.6)[159]: $\mathcal{L}_-^\perp = \mathcal{L}_+^*$, $\mathcal{L}_+^\perp = \mathcal{L}_-^*$.

Proof. We have

$$\mathcal{L}_-^\perp = R\left(P_-(\tilde{A})\right)^\perp = \ker\left(P_-(\tilde{A})^*\right) = \ker\left(P_-(\tilde{A}^*)\right)$$
$$= \ker\left(I - P_+(\tilde{A}^*)\right) = R\left(P_+(\tilde{A}^*)\right) = \mathcal{L}_+^*$$

as $\sigma_-(\tilde{A}^*) = \overline{\sigma_-(\tilde{A})}$ and hence $P_-(\tilde{A})^* = P_-(\tilde{A}^*)$ (see [162]).The proof of the second statement is similar.

Now we are ready to show that the spectral subspaces $\mathcal{L}_-$and $\mathcal{L}_+$ of $\tilde{A}$ can be represented by means of angular operators. For $1 \leq p \leq \infty$, we denote by $\mathcal{S}_p$ the von Neumann−Schatten classes of linear operators in $\mathcal{H}$ (see [154]); in particular, $\mathcal{S}_\infty$ is the class of all compact operators and $\mathcal{S}_1$ is the class of all nuclear or trace class operators.

Theorem(3.2.7)[159]: There exist bounded linear operators $K_- \in L(\widehat{\mathcal{H}}, \mathcal{H})$ and $K_+ \in L(\mathcal{H}, \widehat{\mathcal{H}})$ such that:

(i) The spectral subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ have the representations

$$\mathcal{L}_- = \left\{\begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} : \hat{x} \in \widehat{\mathcal{H}}\right\}, \quad \mathcal{L}_+ = \left\{\begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{H}\right\}.$$

(ii) The operator K_- has the property $R(K_-) \subset \mathcal{D}(A)$ and K_-, K_+ satisfy the Riccati equations

$$K_- B^* K_- - B - AK_- + K_- D = 0 \text{ on } \widehat{\mathcal{H}},$$

$$K_+B\,K_+ - B^* - DK_+ + K_+A \;=\; 0 \;\text{ on }\; \mathcal{D}(A),$$

(iii) The restriction $\tilde{A}|\mathcal{L}_-$ is unitarily equivalent to the operator $D + B^*K_-$ in the Hilbert space $(\widehat{\mathcal{H}}, [\,\cdot\,,\cdot\,]_\wedge)$ where

$$[\hat{x},\hat{y}]_\wedge := ((I + K_-^*\,K_-)\hat{x},\hat{y}),\;\; \hat{x},\hat{y} \in \widehat{\mathcal{H}}.$$

There is a $\hat{\gamma} > 0$ such that

$$\Re[(D + B^*K_-)\hat{x},\hat{x}]_\wedge \;\le\; -\hat{\gamma}[\hat{x},\hat{x}]_\wedge,\;\; \hat{x} \in \widehat{\mathcal{H}}.$$

(iv) The restriction $\tilde{A}|\mathcal{L}_+$ is unitarily equivalent to the operator $A + BK_+$ in the Hilbert space $(\mathcal{H}, [\,\cdot\,,\cdot\,])$ where

$$[x,y] \;:=\; ((I + K_+^*\;K_+)x,y)\,,\quad x,y \in \mathcal{H}.$$

There is a $\gamma > 0$ such that

$$\Re[(A + BK_+)x,x] \;>\; \gamma[x,x],\quad x \in \mathcal{D}(A).$$

If for one (and hence for all) $z \in \rho(A)$ the resolvent $(A - z)^{-1}$ of A belongs to some class $\mathcal{S}_p$, $1 \le p \le \infty$, then the operators K_- and K_+ belong to the same class $\mathcal{S}_p$.

Proof. From the representation (40) and Lemma (3.2.5) it follows that

$$\Re\left(P_+(\tilde{A}) - P_-(\tilde{A})\right)_{11} \;=\; \Re\left(\frac{1}{\pi i}\int_{-i\infty}^{i\infty}(A - z - B(D - z)^{-1}B^*)^{-1}dz\right)$$
$$=\; \Re\left(\frac{1}{\pi}\int_{-\infty}^{\infty}(A - i\eta - B(D - i\eta)^{-1}B^*)^{-1}d\eta\right) \gg 0$$

since for $-\delta < R(z) < \alpha$,

$$\Re\big((A\; - z - B(D - z)^{-1}\,B^*)x,x\big) \ge \tilde{\alpha}(x,x),\quad x \in \mathcal{D}(A), \tag{44}$$

with some $\tilde{\alpha} > 0$ by the assumptions(I) and (II). Here and in the following we use the notation $(X_{ij}\;)_{i,j=1}^2 := X$ for the components of a block operator matrix X in $\widetilde{\mathcal{H}} = \mathcal{H} \times \widehat{\mathcal{H}}$. On the other hand, $\Re\left(P_+(\tilde{A}) + P_-(\tilde{A})\right)_{11} = I$ and hence

$$\Re\big(P_+(\tilde{A})\big)_{11} \gg \tfrac{1}{2}. \tag{45}$$

(i) Let $x \in \mathcal{H}$ be such that $\binom{x}{0} \in \mathcal{L}_-$. Then $P_+(\tilde{A})\binom{x}{0} = \left(I - P_-(\tilde{A})\right)\binom{x}{0} = 0$ and thus $\left(\Re\big(P_+(\tilde{A})\big)_{11}\,x,x\right) = \;0$. By (45) this implies $x \;= 0$.

Now consider a sequence$\left(\binom{x_n}{\hat{x}_n}\right)_1^\infty \subset \mathcal{L}_-$ with $x_n \in \mathcal{H}$, $\hat{x}_n \in \widehat{\mathcal{H}}$, $\|x_n\| = 1, n = 1,2,\ldots,$ and $\|\hat{x}_n\| \to 0$ for $n \to \infty$. By Lemma (3.2.6),

$$0 = \left(P_+\,(\tilde{A}^*)\binom{x_n}{0},\binom{x_n}{\hat{x}_n}\right) = \left(\big(P_+(\tilde{A}^*)\big)_{11}\,x_n,x_n\right) + \left(\big(P_+(\tilde{A}^*)\big)_{21}\,x_n,\hat{x}_n\right).$$

The last term tends to 0 for $n \to \infty$ and hence

$$\left(\big(P_+(\tilde{A}^*)\big)_{11}\,x_n\,,x_n\right) \;\to 0,\quad n \to \infty.$$

By means of the inequality (45) for $\tilde{A}^*$, we find $\|x_n\| \to 0$ for $n\; \to \infty$. This proves that $\mathcal{L}_-$ can be represented as

$$\mathcal{L}_- \;=\; \left\{\binom{K_-\hat{x}}{\hat{x}} :\; \hat{x} \in \widehat{\mathcal{H}}'\right\},$$

where K_- is a bounded linear operator from some closed subspace $\widehat{\mathcal{H}}'$ of $\widehat{\mathcal{H}}$ into $\mathcal{H}$. It remains to be proved that $\widehat{\mathcal{H}}' = \widehat{\mathcal{H}}$. To this end let $\hat{z} \in \widehat{\mathcal{H}}'^{\perp}$. Then $\hat{z} = 0$ as

$$\begin{pmatrix} 0 \\ \hat{z} \end{pmatrix} \in \mathcal{L}_-^{\perp} = \left\{ \begin{pmatrix} x \\ -K_-^* x \end{pmatrix} : x \in \mathcal{D}(K_-^*) \right\}.$$

The proof of the assertion for $\mathcal{L}_+$ is analogous. Here we have to use that

$$\Re(D - z - B^*(A-z)^{-1} B) \ll 0$$

for $-\delta < R(z) < \alpha$ instead of (44) and the representation

$$\tilde{A} - z = \begin{pmatrix} I & 0 \\ B^*(A-z)^{-1} & I \end{pmatrix} \begin{pmatrix} A - z & 0 \\ 0 & D - z - B^*(A-z)^{-1}B \end{pmatrix} \begin{pmatrix} I & (A-z)^{-1}B \\ 0 & I \end{pmatrix}$$

for $z \in \rho(A)$ instead of (39) in order to obtain

$$\Re\big(P_-(\tilde{A})\big)_{22} \gg \frac{1}{2}. \tag{46}$$

(ii) Since $P_-(\tilde{A})$ is a Riesz projection, $\mathcal{L}_- = R(P_-(\tilde{A})) \subset \mathcal{D}(\tilde{A})$. On the other hand,

$$\mathcal{D}(\tilde{A}) = \left\{ \begin{pmatrix} x \\ \hat{x} \end{pmatrix} : x \in \mathcal{D}(A), \hat{x} \in \widehat{\mathcal{H}} \right\},$$

and hence the representation of $\mathcal{L}_-$ according to (i) yields $R(K_-) \subset \mathcal{D}(A)$.

The Riccati equations for K_- and K_+ follow from the invariance of the subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ under $\tilde{A}$ and the representations of $\mathcal{L}_-$ and $\mathcal{L}_+$ in (i).

(iii) The operator $U_- \in L(\mathcal{L}_-, (\widehat{\mathcal{H}}, [\,\cdot\,,\cdot\,]_\wedge))$ defined by

$$U_- \begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} := \hat{x}, \qquad \hat{x} \in \widehat{\mathcal{H}},$$

is isometric and bijective. Using the Riccati equation for K_-, we find

$$\tilde{A}|\mathcal{L}_- = U_-^{-1}(D + B^*K_-)U_-$$

and

$$\begin{aligned} \Re[(D + B^*K_-)\hat{x}, \hat{x}]_\wedge &= \Re\big((I + K_-^*K_-)(D + B^*K_-)\hat{x}, \hat{x}\big) \\ &= \Re\Big(\big(D + B^*K_- + K_-^*(K_-D + K_-B^*K_-)\big)\hat{x}, \hat{x}\Big) \\ &= \Re((D + K_-^*AK_- + B^*K_- - K_-^*B)\hat{x}, \hat{x}) \\ &= \Re(D\hat{x}, \hat{x}) - \Re(AK_-\hat{x}, K_-\hat{x}) \le -\hat{\gamma}(\hat{x}, \hat{x}) \end{aligned}$$

for $\hat{x} \in \widehat{\mathcal{H}}$ with some $\hat{\gamma} > 0$ by the assumptions (I) and (II).

(iv) The operator $U_+ \in L(\mathcal{L}_+, (\mathcal{H}, [\,\cdot\,,\cdot\,]))$ defined by

$$U_+ \begin{pmatrix} x \\ K_+x \end{pmatrix} := x, \qquad x \in \mathcal{H},$$

is isometric and bijective. The Riccati equation for K_+ yields

$$\tilde{A}|\mathcal{L}_+ = U_+^{-1}(A + BK_+)U_+$$

and, similar as in the proof of (iii),

$$\Re[(A + BK_+)x, x] = \Re(Ax, x) - \Re(DK_+x, K_+x) \ge \gamma(x, x)$$

for $x \in \mathcal{D}(A)$ with some $\gamma > 0$ by the assumptions (I) and (II).

Finally, the Riccati equations for K_- and K_+ can be written in the form

$$K_-B^*K_- - B - (A - \mu)K_- + K_-(D - \mu) = 0,$$
$$K_+BK_+ - B^* - (D - \mu)K_+ + K_+(A - \mu) = 0,$$

where $\mu \in \mathbb{C}$ is arbitrary. If we choose $\mu \in \rho(A)$ and multiply the first relation from the left hand side and the second equation from the right hand side by $(A - \mu)^{-1}$, it follows that K_- and K_+ are in $\mathcal{S}_p$ if $(A - \mu)^{-1}$ is in $\mathcal{S}_p$ for some p, $1 \le p \le \infty$. The theorem is proved.

If in Theorem(3.2.7) the operators A and D are selfadjoint, then $K_+ = -K_-^*$, which is immediate from the relation $\mathcal{L}_+ = \mathcal{L}_-^{\perp}$. Thus, in the selfa-djoint case the assertions of Theorem (3.2.7) coincide with the respective statements established in [160], which were proved by a different method. Additionally, it was shown therein that in this case K_- and K_+ are strict contractions. To show a generalization of this property, we use the following two results.

Lemma(3.2.8)[159]: The operators $I - K_+ K_-$ and $I - K_- K_+$ are bijective.

Proof. Let $\hat{x} \in \ker(I - K_+ K_-)$. Then $\hat{x} = K_+ K_- \hat{x}$,

$$\begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} K_-\hat{x} \\ K_+ K_-\hat{x} \end{pmatrix} \in \mathcal{L}_- \cap \mathcal{L}_+ = \{0\}$$

and consequently $\hat{x} = 0$. Now let $\hat{z} \in \widehat{\mathcal{H}}$. As $\mathcal{H} = \mathcal{L}_- \dot{+} \mathcal{L}_+$, there exist $x \in \mathcal{H}$ and $\hat{x} \in \widehat{\mathcal{H}}$ such that

$$\begin{pmatrix} 0 \\ \hat{z} \end{pmatrix} = \begin{pmatrix} K_-\hat{x} + x \\ \hat{x} + K_+ x \end{pmatrix}.$$

This implies $\hat{z} = \hat{x} - K_+ K_- \hat{x} \in \Re(I - K_+ K_-)$. The proof for $I - K_- K_+$ is analogous.

Proposition(3.2.9)[159]: The projections $P_-(\tilde{A})$ and $P_+(\tilde{A})$ have the matrix representations

$$P_-(\tilde{A}) = \begin{pmatrix} -K_-(I - K_+ K_-)^{-1} K_+ & K_-(I - K_+ K_-)^{-1} \\ -(I - K_+ K_-)^{-1} K_+ & (I - K_+ K_-)^{-1} \end{pmatrix},$$

$$P_+(\tilde{A}) = \begin{pmatrix} (I - K_- K_+)^{-1} & -(I - K_- K_+)^{-1} K_- \\ K_+(I - K_- K_+)^{-1} & -K_+(I - K_- K_+)^{-1} K_- \end{pmatrix}.$$

Proof. As $P_-(\tilde{A})$ is the projection onto $\mathcal{L}_-$ along $\mathcal{L}_+$, we have

$$P_-(\tilde{A}) = \begin{pmatrix} K_- X & K_- Y \\ X & Y \end{pmatrix}$$

with operators $X \in L(\mathcal{H}, \widehat{\mathcal{H}})$ and $Y \in L(\widehat{\mathcal{H}})$. From $P_-(\tilde{A})|\mathcal{L}_+ = 0$, it follows that

$$0 = \begin{pmatrix} K_- X & K_- Y \\ X & Y \end{pmatrix} \begin{pmatrix} x \\ K_+ x \end{pmatrix} = \begin{pmatrix} K_-(X + Y K_+) x \\ (X + Y K_+) x \end{pmatrix}, x \in \mathcal{H},$$

and hence $X = -Y K_+$. Then $P_-(\tilde{A})|\mathcal{L}_- = I$ reads

$$\begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} -K_- Y K_+ & K_- Y \\ -Y K_+ & Y \end{pmatrix} \begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} K_- Y(-K_+ K_- + I)\hat{x} \\ Y(-K_+ K_- + I)\hat{x} \end{pmatrix}, \hat{x} \in \widehat{\mathcal{H}}.$$

By Lemma (3.2.8), $I - K_+ K_-$ is invertible and hence

$$Y = (I - K_+ K_-)^{-1}, \qquad X = -(I - K_+ K_-)^{-1} K_+.$$

This shows the representation of $P_-(\tilde{A})$. The proof for $P_+(\tilde{A})$ is analogous.

Theorem(3.2.10)[159]: The operators $K_- K_+$ and $K_+ K_-$ are strict contractions in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, that is,

$$\|K_- K_+\| < 1, \quad \|K_+ K_-\| < 1.$$

Proof. From (45) and the preceding proposition it follows that

$$\Re((I - K_- K_+)^{-1}) \gg \frac{1}{2}.$$

Consequently,

$$\begin{aligned} I &\ll (I - K_- K_+)^{-1} + (I - (K_- K_+)^*)^{-1} \\ &= (I - K_- K_+)^{-1}(2I - K_- K_+ - (K_- K_+)^*)(I - (K_- K_+)^*)^{-1}. \end{aligned}$$

This implies

$$(I - K_- K_+)(I - (K_- K_+)^*) \ll 2I - K_- K_+ - (K_- K_+)^*$$

and hence

$$(K_- K_+)(K_- K_+)^* \ll I,$$

which shows $\|K_- K_+\| < 1$. The proof of $\|K_+ K_-\| < 1$ is analogous.

If A and D are selfadjoint, then it follows from Theorem (3.2.10) and from the relation $K_+ = -K_-^*$ that

$$\|K_-\| = \|K_+\| < 1.$$

Additionally to the assumptions (I) and (II) we suppose now that A has a compact resolvent. Then $\sigma_+(\tilde{A})$ is discrete and ∞ is the only accumulation point as

$$\sigma_+(\tilde{A}) = \sigma(\tilde{A}|\mathcal{L}_+) = \sigma(A + BK_+)$$

by Theorem (3.2.7) where K_+ is compact.

In the following we denote by P_1 the projection from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$. If $\lambda \in \sigma(\tilde{A})$, then $\mathcal{L}_\lambda(\tilde{A})$ denotes the root subspace of $\tilde{A}$ at λ.

Theorem(3.2.11)[159]: Suppose additionally to the assumptions (I) and (II) that the resolvent $(A - z)^{-1}$ of A belongs to the class $\mathcal{S}_1$ for some $z \in \rho(\tilde{A})$. Then

$$\overline{\bigcup_{\lambda \in \sigma_+(\tilde{A})} P_1 \mathcal{L}_\lambda(\tilde{A})} = \mathcal{H},$$

that is, the first components of the root vectors of $\tilde{A}$ corresponding to the eigenvalues in the right half plane form a complete system in $\mathcal{H}$.

Proof. Let $\lambda_0 \in \sigma_+(\tilde{A})$ and assume that $\{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_h\} \subset \tilde{\mathcal{H}}$ is a Jordan chain of $\tilde{A}$ at λ_0. Then $\tilde{x}_j \in \mathcal{L}_+$ and hence $\tilde{x}_j = \begin{pmatrix} x_j \\ K_+ x_j \end{pmatrix}$ with $x_j \in \mathcal{H}$ for $j = 0, 1, \ldots, h$ by Theorem (3.2.7). Then the relation

$$\begin{pmatrix} A - \lambda_0 & B \\ B^* & D - \lambda_0 \end{pmatrix} \begin{pmatrix} x_j \\ K_+ x_j \end{pmatrix} = \begin{pmatrix} x_{j-1} \\ K_+ x_{j-1} \end{pmatrix}, \quad j = 0, 1, \ldots, h,$$

implies that $\{x_0, x_1, \ldots, x_h\} = \{P_1\tilde{x}_0, P_1\tilde{x}_1, \ldots, P_1\tilde{x}_h\}$ is a Jordan chain of the operator $A + BK_+$ at λ_0. Hence

$$\bigcup_{\lambda \in \sigma_+(\tilde{A})} P_1 \mathcal{L}_\lambda(\tilde{A}) = \bigcup_{\lambda \in \sigma(A + BK_+)} \mathcal{L}_\lambda(A + BK_+).$$

For sufficiently large $\xi > 0$, the operator $A + BK_+ + \xi$ is accretive, that is, $\Re((A + BK_+ + \xi)x, x) \geq 0$ for $x \in \mathcal{D}(A)$. If $z \in \rho(A + BK_+ + \xi), \Re(z) < 0$, the resolvent

$$(A + BK_+ + \xi - z)^{-1} = (A - z)^{-1}(I + (BK_+ + \xi)(A - z)^{-1})^{-1}$$

belongs to $\mathcal{S}_1$ as $(A - z)^{-1} \in \mathcal{S}_1$ according to the assumption. Then the statement follows from [154], applied to the dissipative operator $-i(A + BK_+ + \xi)^{-1}$ as $\lambda \in \sigma(A + BK_+)$ if and only if $-\frac{i}{\lambda+\xi} \in \sigma(-i(A + BK_+ + \xi)^{-1})$ and

$$\mathcal{L}_\lambda(A + BK_+) = \mathcal{L}_{-\frac{i}{\lambda+\xi}}(-i(A + BK_+ + \xi)^{-1}), \quad \lambda \in \sigma(A + BK_+).$$

Example(3.2.12)[159]: We consider the λ-rational boundary eigenvalue problem

$$y'' + \lambda y + \frac{q}{u - \lambda} y = 0,$$
$$y(0) - \beta y'(0) = 0, \quad y(1) = 0, \tag{47}$$

in $L_2(0,1)$ where $q, u \in C[0,1]$, $q > 0$, $\Re(u) < 0$, and $\beta \in \mathbb{C}$, $\Re(\beta) > 0$.

Problems of this type with real-valued u and Dirichlet boundary conditions, that is, $\beta = 0$, were studied in [163] and[160]. If we define $\hat{y} := -\frac{q^{1/2}}{u-\lambda} y$, then the problem (47) is equivalent on $\rho(u)$, which is the complement of the set of all values of the continuous function u, to the λ-linear problem

$$(\tilde{A} - \lambda)\tilde{y} = 0, \quad \tilde{y} \in \mathcal{D}(\tilde{A}),$$

in $L_2(0,1) \times L_2(0,1)$ where $\tilde{A}$ is a block operator matrix of the form (33) given by

$$\tilde{A} = \begin{pmatrix} -\frac{d^2}{dx^2} & q^{1/2} \\ q^{1/2} & u \end{pmatrix}$$

and

$$\mathcal{D}(\tilde{A}) := \{\tilde{y} = (y, \hat{y})^t \in W_2^2(0,1) \times L_2(0,1) : y(0) - \beta y'(0) = 0, \ \ y(1) = 0\}.$$

In the following we show that the operator $\tilde{A}$ fulfills the assumptions of Theorem (3.2.11). As $\Re(u) < 0$, the assumption (II) is fulfilled with $\theta_D := \pi - \mathrm{Min}_{x\in[0,1]}|\arg u(x)| < \frac{\pi}{2}$ and $\delta := -\max_{x\in[0,1]} \Re(u(x))$. The operator A in $L_2(0,1)$ given by

$$Ay := -y'', \quad \mathcal{D}(A) := \{y \in W_2^2(0,1) : y(0) - \beta y'(0) = 0, \ \ y(1) = 0\}$$

is densely defined, closed, and it satisfies the assumption (I). Indeed, A is boundedly invertible, and

$$(Ay, y) = \beta|y'(0)|^2 + \int_0^1 |y'(x)|^2 dx, \qquad y \in \mathcal{D}(A). \tag{48}$$

First we show that $\Re(Ay, y) \geq \alpha$ for $y \in \mathcal{D}(A), \|y\| = 1$, for some $\alpha > 0$. From (48) and the assumption $\Re(\beta) > 0$ it follows that

$$\Re(Ay, y) = \Re(\beta)|y'(0)|^2 + \int_0^1 |y'(x)|^2 dx \geq 0, \qquad y \in \mathcal{D}(A). \tag{49}$$

Now suppose that there exists a sequence $(y_n)_1^\infty \subset \mathcal{D}(A)$, $\|y_n\| = 1$, with $\Re(Ay_n, y_n) \to 0$ for $n \to \infty$. Then $y'_n(0) \to 0$ and $\|y'_n\| \to 0$ for $n \to \infty$ by (49) and hence

$$|y_n(x) - y_n(0)| \leq \int_0^x |y'_n(t)|\, dt \leq \sqrt{x}\left(\int_0^x |y'_n(t)|^2 dt\right)^{\frac{1}{2}} \leq \|y'_n\| \to 0, \qquad n \to \infty,$$

uniformly for $x \in [0,1]$, and hence $\|y_n\| \to 0$ as $y_n(0) = \beta y'(0) \to 0$ for $n \to \infty$, a contradiction. Furthermore, (49) implies

$$\frac{|\Im(Ay, y)|}{\Re(Ay, y)} = \frac{|\Im(\beta)||y'(0)|^2}{\Re(\beta)|y'(0)|^2 + \|y'\|^2} \leq \frac{|\Im(\beta)|}{\Re(\beta)}$$

which proves that the numerical range W_A of A is contained in a set $\{z \in \mathbb{C} : \arg z \leq \theta_A, \Re(z) \geq \alpha\}$ with $\theta_A = |\arg \beta| < \frac{\pi}{2}$.

It remains to be shown that $A^{-1} \in \mathcal{S}_1$. In order to see this, we introduce the selfadjoint operator A_0 in $L_2(0,1)$,

$$A_0 y := -y'', \quad \mathcal{D}(A_0) := \{y \in W_2{}^2(0,1) : y(0) = 0, \ \ y(1) = 0\}.$$

Its eigenvalues are the numbers $\mu_n := \pi^2 n^2$, $n = 1, 2, \ldots$, hence $A_0^{-1} \in \mathcal{S}_1$. As the difference $A^{-1} - A_0^{-1}$ is a one-dimensional operator, it follows that $A^{-1} \in \mathcal{S}_1$.

Thus Theorem (3.2.11) can be applied to the operator $\tilde{A}$ and we get:

Theorem(3.2.13)[159]: The spectrum σ_+ of the eigenvalue problem (47) in the right half plane is discrete, ∞ is its only accumulation point, and σ_+ is contained in the sector

$$\left\{z \in \mathbb{C} : |\arg z| \leq \max\left\{\pi - \min_{x\in[0,1]} |\arg u(x)|, |\arg \beta|\right\}\right\}.$$

The root vectors corresponding to the eigenvalues in σ_+ form a complete system in $L_2(0,1)$.