Chapter 2
The Extension Property and Boundary Regularity

In this chapter we compare different ways of measuring the “distance” between two sufficiently close Reifenberg flat domains. These results are pivotal to the quantitative stability analysis of the spectrum of the Neumann Laplacian. We show that given an exponent $\alpha \in (0,1)$, there exists an $\varepsilon > 0$ such that the solution of the system is locally Hölder continuous provided that Ω is (ε, r_0)-Reifenberg flat. The proof is based on Alt-Caffarelli-Friedman's monotonicity formula and Morrey-Campanato theorem.

Sec(2.1): Reifenberg-Flat Domains

We establishing extension and geometric properties for a class of domains whose boundaries satisfy a fairly weak regularity requirement introduced by Reifenberg [64]. We show that any domain that is sufficiently flat in the sense of Reifenberg enjoys the so-called extension property and we discuss applications that are relevant for the analysis of PDEs defined in these domains. We also compare different ways of measuring the “distance” between two sufficiently close Reifenberg-flat domains X and Y, in particular we discuss the relations between the Hausdorff distances $d_H(X,Y)$, $d_H(\mathbb{R}^N \backslash X, \mathbb{R}^N \backslash Y)$ and $d_H(\partial X, \partial Y)$ and the measure of the symmetric difference $|X \Delta Y|$.

Although we are confident the results can find different applications, the original motivation was the quantitative stability analysis of the spectrum of the Laplace operator with Neumann boundary conditions defined in Reifenberg-flat domains, see [65].

We mention that Reifenberg-flat domains are in particular NTA domains in the sense of Jerison and Kenig [83].

We denote by d_H the classical Hausdorff distance between two sets X and Y ,

$$d_H(X,Y) := \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\} \qquad (1).$$

Definition(2.1.1)[63]: Let ε, r_0 be two real numbers satisfying $0 < \varepsilon < 1/2$ and $r_0 > 0$. An (ε, r_0)-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^N$ is a nonempty open set satisfying the following two conditions:

(i) for every $x \in \partial\Omega$ and for every $r \leq r_0$, there is a hyperplane $P(x,r)$ containing x which satisfies

$$\frac{1}{r} d_H(\partial\Omega \cap B(x,r), P(x,r) \cap B(x,r)) \leq \varepsilon. \qquad (2)$$

(ii) For every $x \in \partial\Omega$, one of the connected component of

$$B(x,r_0) \cap \{x : \operatorname{dist}(x, P(x,r_0)) \geq 2\varepsilon\, r_0\}$$

is contained in Ω and the other one is contained in $\mathbb{R}^N \backslash \Omega$.

Condition (i) states that the boundary of Ω is an (ε, r_0)-Reifenberg flat set. A Reifenberg flat set enjoys local separability properties (see[84]), however we observe that condition (ii) in the definition is not in general implied by condition (i), as the example of $\Omega = \mathbb{R}^N \backslash \partial B(0,1)$ shows (here $\partial B(0,1)$ denotes the boundary of the unit ball). However, a consequence of the analysis in David [85] is that (i) implies (ii) under some further topological assumption, for instance the implication holds if Ω and $\partial\Omega$ are both connected. Note furthermore that a straightforward consequence of the definition is that, if $\varepsilon_1 < \varepsilon_2$, then any (ε_1, r_0)-Reifenberg flat domain is also an (ε_2, r_0)-Reifenberg flat domain.

Note that we only impose the separability requirement (ii) at scale $r_0$but it simply follows from the definition that it also holds at any scale $r \le r_0$ (see [20] or Lemma (2.1.5) below).

We providing a complete and detailed proof of the fact that Reifenberg flat domains are extension domains. This fact is relevant for the study of elliptic problems and (see e.g. [76]). However, to the best of our knowledge, an explicit proof was so far missing. We recall that the so called extension problem can be formulated as follows: given an open set Ω, we denote by $W^{1,p}$ the classical Sobolev space and we wonder whether or not one can define a bounded linear operator (the so-called extension operator)

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

such that $E(u) \equiv u$ on Ω. If $\partial\Omega$ is Lipschitz, Calderon [86] established the existence of an extension operator in the case when $1 < p < \infty$, while Stein [87] considered the cases $p = 1, \infty$. Jones [88] proved the existence of extension operators for a new class of domains, the so-called (ε, δ)-Jones flat domains. We show that sufficiently flat Reifenberg domains are indeed Jones flat domains. The main result concerning the extension problem is as follows.

Theorem(2.1.2)[63]: Any $(\frac{1}{600}, r_0)$-Reifenberg flat domain is a $\left(\frac{1}{450}, \frac{r_0}{7}\right)$-Jones flat domain.

Proof: We assume $\varepsilon \le 1/600$, we fix an (ε, r_0)-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^N$ and we proceed according to the following steps.

⋄ **Step 1.** We first introduce some notations (see Figure 1 for a representation). For any $x_0 \in \partial\Omega$ and $\rho \le r_0$, we denote as usual by $P(x_0, \rho)$ the hyperplane provided by the definition of Reifenberg flatness, and by $\vec{v}_\rho$ its normal. By Lemma (2.1.5), we can choose the orientation of $\vec{v}_\rho$ in such a way that

$$B^+(x_0, \rho) := \{z + t\vec{v}_\rho : z \in P(x_0, \rho), t \ge 2\varepsilon\rho\} \cap B(x_0, \rho) \subseteq \Omega$$

and

$$B^-(x_0, \rho) := \{z - t\vec{v}_\rho : z \in P(x_0, \rho), t \ge 2\varepsilon\rho\} \cap B(x_0, \rho) \subseteq \Omega^c.$$

Also, we define the hyperplanes $P^+(x_0, \rho)$ and $P^-(x_0, \rho)$ by setting

$$P^+(x_0, \rho) := \{z + 2\varepsilon\rho\vec{v}_\rho : z \in P(x_0, \rho)\}$$

and

$$P^-(x_0, \rho) := \{z - 2\varepsilon\rho\vec{v}_\rho : z \in P(x_0, \rho)\}$$

and we denote by $Y(x_0, \rho)$ the point

$$Y(x_0, \rho) := x_0 + \rho\vec{v}_\rho.$$

Finally, for any $x \in \Omega$, we denote by $x_0 \in \partial\Omega$ the point such that $d(x, \Omega^c) = d(x, x_0)$ (if there is more than one such x_0, we arbitrarily fix one).

⋄ **Step 2.** We provide a preliminary construction: more precisely, given

(i) $x \in \Omega$ such that $d(x, \Omega^c) \leq 2r_0 /7$ and

(ii) r satisfying $d(x, \Omega^c)/2 \leq r \leq r_0/7$,

the curve $\gamma_{x,r}$ is defined as follows.

(I) If $d(x, \Omega^c)/2 \leq r \leq 2d(x, \Omega^c)$, then $\gamma_{x,r}$ is simply the segment $[x, Y(x_0, r)]$.

(II) If $2d(x, \Omega^c) < r \leq r_0/7$, we denote by $k_0 \geq 1$ the biggest natural number k satisfying $2^{-k}r \geq d(x, \Omega^c)$ and we set

$$\gamma_{x,r} := [x, Y(x_0, 2^{-k_0}r)] \cup \bigcup_{k=0}^{k_0-1} [Y\,(x_0, 2^{-k}r), Y\,(x_0, 2^{-(k+1)}\,r)]\,.$$

⋄ **Step 3**. We show that in both cases (I) and (II) we have

$$\mathcal{H}^1(\gamma_{x,r}) \leq 4r. \tag{3}$$

To handle case (I) we just observe that, since by assumption $d(x, \Omega^c) = d(x, x_0) \leq 2r$, then, by recalling $d(x_0, Y\,(x_0, r)) = r$, property (3) follows.

To handle case (II), we first observe that, since $d(x, x_0) \leq 2^{-k_0}r$, then both x and $Y(x_0, 2^{-k_0}r)$ belong to the closure of $B(x_0, 2^{-k_0}r)$. Also, by construction both $Y(x_0, 2^{-k}r)$ and $Y\left(x_0, 2^{-(k+1)}r\right)$ belong to the closure of $B(x_0, 2^{-k}r)$ and by combining these observations we conclude that

$$\begin{aligned} \mathcal{H}^1(\gamma_{x,r}) &\leq d\left(x, Y\,(x_0, 2^{-k_0}r)\right) + \sum_{k=0}^{k_0-1} d(Y\,(x_0, 2^{-k}r), Y\,(x_0, 2^{-(k+1)}r)) \\ &\leq 2 \cdot 2^{-k_0}r + \sum_{k=0}^{k_0-1} 2 \cdot 2^{-k}r \leq 2r \sum_{k \in \mathbb{N}} 2^{-k} = 4r. \end{aligned} \tag{4}$$

Step 4. We show that for every $z \in \gamma_{x,r}$

$$d(z, \Omega^c) \geq \frac{29}{240} d(z, x). \tag{5}$$

We start by handling case (I): we work in the ball $B(x_0, 4r)$ and we recall the definition of $B^+(x_0, 4r)$ and of $B^-(x_0, 4r)$, given at Step 1. Since by assumption $\varepsilon \leq 1/32$, we have

$$16\,\varepsilon r \leq \frac{r}{2} \leq d(x, \Omega^c) \leq d(x, B^-(x_0, 4r))$$

and hence $x \in B^+(x_0, 4r)$. Let β denotes the angle between ν_r and ν_{4r}, then by Lemma (2.1.4) applied with $M = 4$ we get that provided $\varepsilon \leq 1/9$, then $4\varepsilon r \leq r\cos\beta$, so that $Y(x_0, r) \in B^+(x_0, 4r)$. By recalling that $x \in B^+(x_0, 4r)$, we conclude that $[x, Y(x_0, r)] \subseteq B^+(x_0, 4r)$.

We are now ready to establish (5), so we fix $z \in [x, Y\,(x_0, r)]$. To provide a bound from above on $d(z, x)$, we simply observe that, since both x and $Y(x_0, r)$ belong to the closure of $B(x_0, 2r)$, then so does z and hence

$$d(z, x) \leq 4r. \tag{6}$$

Next, we provide a bound from below on $d(z, \Omega^c)$: since $z \in B^+(x_0, 4r) \subseteq \Omega$, then

$$d(z,\Omega^c) \ \geq \ d\big(z,\partial B^+(x_0,4r)\big)$$
$$= \min\{d(z,P^+(x_0,4r)), d(z,\partial B(x_0,4r))\}. \qquad (7)$$

First, we recall that $z \in B(x_0,2r)$ and we provide a bound on the distance from z to the spherical part of $\partial B^+(x_0,4r)$:

$$d(z,\partial B(x_0,4r)) = 4r - d(z,x_0) \geq 4r - 2r = 2r.$$

Next, we observe that

$$d(z,P^+(x_0,4r)) = d(z,P(x_0,4r)) - 8\varepsilon r$$

and, since $z \in [x, Y(x_0,r)]$, then

$$d\big(z,P(x_0,4r)\big) \geq \min\{d(x,P(x_0,4r), d(Y(x_0,r),P(x_0,4r))\}.$$

Note that $d\big(Y\ (x_0,r),P(x_0,4r)\big) = r\cos\beta$ and, using Lemma (2.1.4), we conclude that

$$d(Y\ (x_0,r),P(x_0,4r)) \geq r/2$$

because $\varepsilon \leq 1/10$. Also, since $B^-(x_0,4r) \subseteq \Omega^c$, then

$$r/2 \leq d(x,\Omega^c) \leq \ d(x,B^-(x_0,4r)) = d(x,P(x_0,4r)) + 2\varepsilon r$$

By recalling (7) and the inequality $\varepsilon \leq 1/600$ and by combining all the previous observations we conclude that

$$d(z,\Omega^c) \ \geq d\big(z,\partial B^+(x_0,4r)\big) \geq \min\{d(z,P^+(x_0,4r),2r\}$$
$$= \min\{d(z,P(x_0,4r) - 8\varepsilon r, 2r\}$$
$$\geq \min\big\{\min\big\{d\big(x,P(x_0,4r)\big), d(Y\ (x_0,r),P(x_0,4r))\big\} - 8\varepsilon r, 2r\big\}$$
$$\geq \min\left\{\min\left\{\frac{r}{2} - 2\varepsilon r, \frac{r}{2}\right\} - 8\varepsilon r, 2r\right\} \ = \frac{r}{2} - 10\varepsilon r \geq \frac{29}{60} r. \qquad (8)$$

Finally, by comparing (8) and (6) we obtain (5).

*** Step 5**. We now establish (5) in case (II).

If $z \in [x, Y(x_0, 2^{-k_0} r]$, then we can repeat the argument we used in Step 4 by replacing r with $2^{-k_0} r$, which satisfies

$$d(x,\Omega^c) \leq 2^{-k_0} r \leq 2d(x,\Omega^c).$$

Hence, we are left to consider the case when $z \in [Y\ (x_0, 2^{-k} r), Y(x_0, 2^{-(k+1)} r)]$ for some natural number $k \leq k_0 - 1$. We set $\rho := \ 2^{-k} r$ and we work in the ball $B(x_0, 2\rho)$. We denote by α the angle between $v_{2\rho}$ and ν_ρ, and by β the angle between $v_{2\rho}$ and $\nu_{\rho/2}$. Due to Lemma (2.1.4) applied with $M = 2$ and $M = 4$, we know that, if $\varepsilon \leq \frac{1}{13}$, then

$$\rho\cos\alpha \geq 4\varepsilon\rho \qquad\qquad \frac{1}{2}\rho\cos\beta \geq 4\varepsilon\rho,$$

so that both $Y(x_0,\rho)$ and $Y(x_0,\rho/2)$ belong to $B^+(x_0,2\rho)$. Hence, given

$$z \in [Y\ (x_0,\rho), Y(x_0,\rho/2)] \subseteq B^+(x_0,2\rho) \subseteq \Omega,$$

we have $d(z,\Omega^c) \ \geq \ d(z,\partial B^+(x_0,2\rho))$. The distance from z to the spherical part of $\partial B^+(x_0,2\rho)$ is bounded from below by ρ, while the distance from z to $P^+(x_0,2\rho)$ is bounded from below by $\frac{1}{2}\rho - 4\varepsilon\rho \geq \frac{1}{4}\rho$ provided that $\varepsilon \leq \frac{1}{16}$. Hence, $d(z,\Omega^c) \geq \frac{\rho}{4}$. To provide an upper bound on $d(z,x)$ we observe that, since $d(x,x_0) = d(x,\Omega^c) \leq 2^{-k} r$, then both z and x belong to the closure of $B(x_0,\rho)$. Hence, $d(x,z) \leq 2\rho$ and (5) holds.

* **Step 6.** We are finally ready to show that Ω is a Jones flat domain. Given $x, y \in \Omega$ satisfying $d(x,y) \leq r_0/7$, there are two possible cases:

(i) if either $d(x,\Omega^c) \geq 2d(x,y)$ or $d(y,\Omega^c) \geq 2d(x,y)$, then we set $\gamma := [x,y]$. To see that γ satisfies (17), let us assume that $d(x,\Omega^c) \geq 2d(x,y)$ (the other case is completely analogous), then $y \in B(x, d(x,y)) \subseteq \Omega$ and $[x,y] \subseteq \Omega$. Also, since

$$\sup_{z\in[x,y]} \frac{d(z,x)d(z,y)}{d(x,y)} = \frac{1}{4} d(x,y), \tag{9}$$

then for any $z \in \gamma$,

$$d(z,\Omega^c) \geq d(x,\Omega^c) - d(z,x) \geq d(x,y) \geq 4d(z,x)d(z,y)/d(x,y).$$

Hence, γ satisfies (17) provided that $\delta = 4$.

(ii) we are left to consider the case when both $d(x,\Omega^c) < 2d(x,y)$ and $d(y,\Omega^c) < 2d(x,y)$. Denote by $x_0 \in \partial\Omega$ a point such that $d(x,\Omega^c) = d(x,x_0)$ and $y_0 \in \partial\Omega$ a point such that $d(y,y_0) = d(y,\Omega^c)$ and set $r := d(x,y) \leq r_0/7$. We define

$$\gamma := \gamma_{x,r} \cup \gamma_{y,r} \cup [Y(x_0,r), Y(y_0,r)]. \tag{10}$$

Step 7 is devoted to showing that γ satisfies (16) and (17).

* **Step 7.** First, we establish (16): we observe that

$$d\big(Y(x_0,r),Y(y_0,r)\big)$$
$$\leq d(Y(x_0,r),x_0) + d(x_0,x) + d(x,y) + d(y,y_0) + d(y_0,Y(y_0,r)) \leq 7r$$

and hence by using (13)

$$\mathcal{H}^1(\gamma) \leq \mathcal{H}^1(\gamma_{x,r}) + d\big(Y(x_0,r),Y(y_0,r)\big) + \mathcal{H}^1(\gamma_{y,r}) \leq 15r$$

which proves (16).

Next, we establish (17): we denote by d_γ the geodesic distance on the curve γ and we observe that

$$\frac{d(z,y)}{15d(x,y)} \leq \frac{d_\gamma(z,y)}{d_\gamma(x,y)} \leq 1. \tag{11}$$

Hence, if $z \in \gamma_{x,r}$, then by using (5) we obtain

$$d(z,\Omega^c) \geq \frac{29}{240} d(z,x) \geq \frac{29}{240\cdot 15}\left(\frac{d(z,x)d(z,y)}{d(x,y)}\right)$$

and we next observe $29/240 \cdot 15 \geq 5/60 \cdot 15 = 1/180$. Since the same argument works in the case when $z \in \gamma_{y,r}$, then we are left to establish (17) in the case when z lies on the segment $[Y(x_0,r), Y(y_0,r)]$.

We first observe that

$$d\big(x_0,Y(y_0,r)\big) \leq d(x_0,x) + d(x,y) + d(y,y_0) + d\big(y_0,Y(y_0,r)\big) \leq 6r \tag{12}$$

and hence $[Y(x_0,r), Y(y_0,r)] \subseteq B(x_0,7r)$. Next, we note that $7r \leq r_0$ and we use (8) to get

$$\frac{29}{60} r \leq d(Y(x_0,r),\Omega^c) \leq d(Y(x_0,r), P^-(x_0,7r)), \tag{13}$$

hence since ε is so small that $28\varepsilon r \leq 29r/60$, then we have $d(Y(x_0, r), P^-(x_0, 7r)) \geq 28\varepsilon r$, which means that $Y(x_0, r) \in B^+(x_0, 7r)$. By repeating the same argument we get $Y(x_0, r) \in B^+(x_0, 7r)$ and hence $[Y(x_0, r), Y(y_0, r)] \subseteq B^+(x_0, 7r)$. We fix $z \in [Y(x_0, r), Y(y_0, r)]$ and we observe that

$$\begin{aligned} d(z, x) &\leq d(z, Y(x_0, r)) + d(Y(x_0, r), x_0) + d(x_0, x) \\ &\leq d\big(Y(y_0, r), Y(x_0, r)\big) + d(Y(x_0, r), x_0) + d(x_0, x) \\ &\leq 7r + r + 2r = 10r. \end{aligned} \tag{14}$$

Also,

$$d(z, \Omega^c) \geq d\big(z, \partial B^+(x_0, 7r)\big) \geq \min\{d(z, \partial B(x_0, 7r));\ d(z, P^+(x_0, 7r))\} \tag{15}$$

and by using (12) we get

$$d(z, \partial B(x_0, 7r)) \geq r.$$

Also, we have

$$d(z, P^+(x_0, 7r)) \geq \min\{d(Y(x_0, r), P^+(x_0, 7r)), d(Y(y_0, r), P^+(x_0, 7r))\}$$

and by recalling (13) we get that

$$d\big(Y(x_0, r), P^+(x_0, 7r)\big) = d\big(Y(x_0, r), P^-(x_0, 7r)\big) - 28\varepsilon r \geq \frac{29}{60}r - 28\varepsilon r \geq \frac{r}{3}.$$

Since $Y(y_0, r)$ satisfies the same estimate, then by recalling (11), (14) and (15) we get

$$d(z, \Omega^c) \geq \frac{r}{3} \geq \frac{1}{3 \cdot 10} d(z, x) \geq \frac{1}{3 \cdot 10 \cdot 15} \frac{d(z, x) d(z, y)}{d(x, y)},$$

which concludes the proof because $3 \cdot 10 \cdot 15 = 450$.

As direct consequence of Theorem (2.1.2) we get that one can define extension operators for $(1/600, r_0)$-Reifenberg flat domains.

Some relevant features of this result are the following: first, we provide an explicit and universal threshold on the coefficient ε for the extension property to hold (namely, $\varepsilon \leq 1/600$). Second, $1/600$ is fairly big compared to the usual threshold needed to apply Reifenberg's topological disk theorem (for e.g. the threshold is 10^{-15} in [68], see also [89] for an interesting alternative proof).

As a consequence of the extension extension property, we obtain that the classical Rellich-Kondrachov Theorem applies to Reifenberg flat domains, that the Neumann Laplacian has a discrete spectrum and that the eigenfunctions are bounded. Also, by combining Theorem (2.1.2) with the works by Chua [90, 91,92] and Christ [93] we get that one can define extension operators for weighted Sobolev spaces and Sobolev spaces of fractional order.

We conclude the section by establishing results unrelated to the extension problem, namely we study the relation between different ways of measuring the "distance" between sets of $\mathbb{R}^N$. In particular, for two general open sets X and Y , neither the Hausdorff distance $d_H(X, Y)$ nor the Hausdorff distance between the complements $d_H(\mathbb{R}^N \backslash X, \mathbb{R}^N \backslash Y)$ is, in general, controlled by the Lebesgue measure of the symmetric difference $|X \Delta Y|$. However, we show that they are indeed controlled provided that X, Y are Reifenberg flat and close enough, in a suitable sense.This result will be as well applied in [65] to the stability analysis of the spectrum of the Laplace operator with Neumann boundary conditions.

We denote by $C(a_1, \ldots, a_h)$ a constant only depending on the variables $a_1, \ldots, a_h$. Its precise value can vary from line to line.

Now, we show that any sufficiently flat Reifenberg domain is Jones flat, in the sense of [88]. The extension property follows then as a corollary of the analysis in [88].

First, we provide the precise definition of Jones-flatness.

Definition(2.1.3)[63]: An open and bounded set Ω is a (δ, R_0)-Jones flat domain if for any $x, y \in \Omega$ such that $d(x,y) \leq R_0$ there is a rectifiable curve γ which connects x and y and satisfies

$$\mathcal{H}^1(\gamma) \leq \delta^{-1} d(x,y) \tag{16}$$

and

$$d(z, \Omega^c) \geq \delta \frac{d(z,x)d(z,y)}{d(x,y)}, \text{ for all } z \in \gamma. \tag{17}$$

To investigate the relation between Jones flatness and Reifenberg flatness we need two preliminary lemmas.

Lemma(2.1.4)[63]: Let $\Omega \subseteq \mathbb{R}^N$ be an $(\varepsilon, r_0)-$Reifenberg flat domain. Given $x \in \partial\Omega$ and $r \leq r_0$, we term ν_r the unit normal vector to the hyperplane $P(x,r)$ provided by the definition of Reifenberg-flatness. Given $M \geq 1$, for every $r \leq r_0/M$ we have

$$|\langle \nu_r, \nu_{Mr} \rangle| \geq 1 - (M+1)\varepsilon. \tag{18}$$

Proof: We assume with no loss of generality that x is the origin. For simplicity, in the proof we denote by B_r the ball $B(0,r)$ and by P_r the hyperplane $P(0,r)$. From the definition of Reifenberg flatness we infer that

$$\begin{aligned} d_H(P_{Mr} \cap B_r, P_r \cap B_r) &\leq d_H(P_{Mr} \cap B_r, \partial\Omega \cap B_r) + d_H(\partial\Omega \cap B_r, P_r \cap B_r) \\ &\leq Mr\varepsilon + r\varepsilon \leq (M+1)r\varepsilon. \end{aligned}$$

Since P_{Mr} and P_r are linear spaces we deduce that

$$d_H(P_{Mr} \cap B_1, P_r \cap B_1) \leq (M+1)\varepsilon. \tag{19}$$

We term π_r and π_{Mr} the orthogonal projections onto P_r and P_{Mr}, respectively, and we fix an arbitrary point $y \in P_r \cap B_1$. Inequality (19) states that there is $z \in \bar{P}_{Mr} \cap \bar{B}_1$ satisfying

$$d(z,y) \leq (M+1)\varepsilon.$$

In particular, since $1 = |\nu_{Mr}| = \inf_{z \in P_{Mr}} d(\nu_{Mr}, z)$, we get

$$d(\nu_{Mr}, y) \geq d(\nu_{Mr}, z) - d(z,y) \geq 1 - (M+1)\varepsilon.$$

By taking the infimum for $y \in P_r \cap B_1$we obtain

$$|\nu_{Mr} - \pi_r(\nu_{Mr})| \geq 1 - (M+1)\varepsilon,$$

and the proof is concluded by recalling that $|\langle \nu_{mr}, \nu_r \rangle| = d(\nu_{Mr}, \pi_r(\nu_{Mr}))$.

The following lemma discusses an observation due to Kenig and Toro [20]. Note that the difference between Lemma (2.1.5) and part (ii) in the definition of Reifenberg flatness is that in (ii) we only require the separation property at scale r_0.

Lemma(2.1.5)[63]: Let $\Omega \subseteq \mathbb{R}^N$ be an $(\varepsilon, r_0)-$Reifenberg flat domain. For every $x \in \partial\Omega$ and $r \in]0, r_0]$, one of the connected components of

$$B(x,r) \cap \{x : \text{dist}(x, P(x,r)) \geq 2\varepsilon r\}$$

is contained in Ω and the other one is contained in $\mathbb{R}^N \setminus \Omega$. Here $P(x,r)$ is the same hyperplane as in part (i) of the definition of Reifenberg-flatness.

Proof: We fix $\rho \in]0, r_0]$ and we assume that the separation property holds at scale ρ, namely that one of the connected components of

$$B(x,\rho) \cap \{x : \text{dist}(x, P(x,\rho)) \geq 2\varepsilon\rho\}$$

is contained in Ω and the other one is contained in $\mathbb{R}^N \setminus \Omega$. We now show that the same separation property holds at scale r for every $r \in]\rho/M, \rho]$ provided that $M \leq (1-\varepsilon)/3\varepsilon$. By iteration this implies that the separation property holds at any scale $r \in]0, r_0]$.

Let us fix $r \in]\frac{\rho}{M}, \rho]$ and denote by $B^+(x,r)$ one of the connected components of

$$B(x,r) \cap \{x : \text{dist}(x, P(x,r)) \geq 2\varepsilon r\}$$

and by $B^-(x,r)$ the other one. Also, we term Y^+ and Y^- the points of intersection of the line passing through x and perpendicular to $P(x,r)$ with the boundary of the ball $B(x,r)$.

By recalling (18) and the inequality $r \geq \rho/M$, we get that the distance of $Y^\pm$ from the hyperpane $P(x,\rho)$ satisfies the following inequality:

$$d(Y^\pm, P(x,\rho)) \geq r|\langle \nu_\rho, \nu_{\rho/M}\rangle| \geq r[1-(M+1)\varepsilon] \geq \rho\frac{1-(M+1)\varepsilon}{M}.$$

Since by assumption $M \leq (1-\varepsilon)/3\varepsilon$, this implies that $d(Y^\pm, P(x,\rho)) \geq 2\varepsilon\rho$ and hence that one among Y^+ and Y^- belongs to $B^+(x,\rho)$ and the other one to $B^-(x,\rho)$. Since by assumption the separation property holds at scale ρ, this implies that one of them belongs to Ω and the other one to Ω^c.

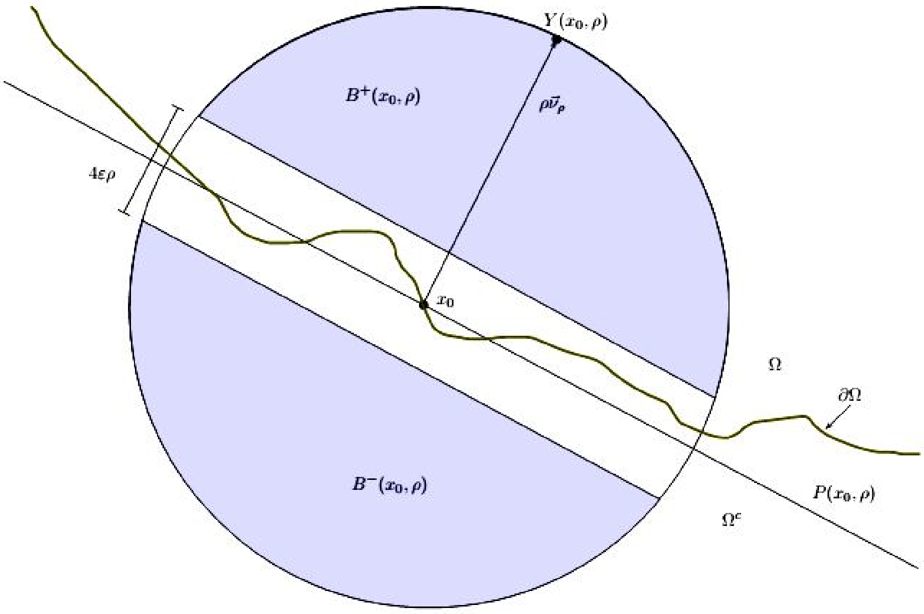

Figure 1. notations for the proof of Theorem (2.1.2)

To conclude, note that part (i) in the definition of Reifenberg flatness implies that

$$B^\pm(x,r) \cap \partial\Omega = \emptyset$$

and hence both $B^+(x,r)$ and $B^-(x,r)$ are entirely contained in either Ω or Ω^c. By recalling that one among Y^+ and Y^- belongs to Ω and the other one to Ω^c, we conclude the proof of the lemma.

We combine the analysis in [88] with Theorem (2.1.2) to show that domains that are sufficiently flat in the sense of Reifenberg satisfy the extension property. We also discuss some direct consequences. Note that we always assume that Ω is connected, as Jones did in [88]. We show that the connectedness assumption can be actually removed in the case of Reifenberg flat domains. Note also that, before providing the precise extension result, we have to introduce a preliminary lemma comparing different notions of “radius” of a given domain Ω.

We term outer radius of a nonempty set $\Omega \subseteq \mathbb{R}^N$ the quantity

$$\mathrm{Rad}(\Omega) := \inf_{x\in\Omega} \sup_{y\in\Omega} d(x,y), \tag{20}$$

and we term inner radius the quantity

$$\mathrm{rad}(\Omega) := \sup_{x\in\Omega} \sup\{r > 0 : B(x,r) \subset \Omega\}. \tag{21}$$

The inner radius is the radius of the biggest ball that could fit inside Ω, whereas the outer radius, as seen below, is the radius of the smallest ball, centered in $\bar{\Omega}$, that contains Ω. Also, we recall that $\mathrm{Diam}(\Omega)$ denotes the diameter of Ω, namely

$$\mathrm{Diam}(\Omega) := \sup_{x,y\in\Omega} d(x,y).$$

We collect some consequences of the definition in the following lemma.

Lemma(2.1.6)[63]: Let Ω be a nonempty subset of $\mathbb{R}^N$, then the following properties hold:

(i) We have the formula

$$\mathrm{Rad}(\Omega) = \inf_{x\in\Omega} \inf\,\{r > 0 : \Omega \subset B(x,r)\}. \tag{22}$$

Also, if $\mathrm{Rad}(\Omega) < +\infty$, then there is a point $x \in \bar{\Omega}$ such that $\Omega \subseteq B(x, \mathrm{Rad}(\Omega))$.

(ii) $\mathrm{rad}(\Omega) \leq \mathrm{Rad}(\Omega) \leq \mathrm{Diam}(\Omega)$.

(iii) If Ω is an (ε, r_0)-Reifenberg flat domain for some $r_0 > 0$ and some ε satisfying $0 < \varepsilon < 1/2$, then $r_0/4 \leq \mathrm{rad}(\Omega) \leq \mathrm{Rad}(\Omega) \leq \mathrm{Diam}(\Omega)$.

Proof. To establish property (i), we first observe that, if Ω is not bounded, then $\mathrm{Rad}(\Omega) = +\infty$ and formula (22) is trivially satisfied. Also, the assumption $\mathrm{Rad}(\Omega) < +\infty$ implies that the closure $\bar{\Omega}$ is compact. Hence, if $\mathrm{Rad}(\Omega) < +\infty$, then

$$\mathrm{Rad}(\Omega) = \min_{x\in\Omega} \sup_{y\in\Omega} d(x,y) \tag{23}$$

and if we term $x_0 \in \bar{\Omega}$ any point that realizes the minimum in (23) we have $\Omega \subset \bar{B}(x_0, \mathrm{Rad}(\Omega))$. This establishes the inequality

$$\mathrm{Rad}(\Omega) \geq \inf_{x\in\Omega} \inf\,\{r > 0 : \Omega \subset B(x,r)\}.$$

To establish the reverse inequality we observe that if $x \in \Omega$ is any arbitrary point and $r > 0$ is such that $\Omega \subset B(x,r)$, then $\sup_{y\in\Omega} d(x,y) \leq r$. By taking the infimum in x and r we conclude. This ends the proof of property (i).

To establish (ii), we focus on the case when $\mathrm{Rad}(\Omega) < +\infty$, because otherwise Ω is unbounded and (ii) trivially holds. Hence, by relying on (i) we infer that $\Omega \subseteq B := B(x_0, \mathrm{Rad}(\Omega))$ for some point $x_0 \in \Omega$. Given $x \in \Omega$ and $r > 0$ satisfying $B(x,r) \subset \Omega$, we have $B(x,r) \subset B(x_0, \mathrm{Rad}(\Omega))$. Hence, $d(x,x_0) + r \leq \mathrm{Rad}(\Omega)$ and hence $r \leq \mathrm{Rad}(\Omega)$. By taking the supremum in r and x we get finally $\mathrm{rad}(\Omega) \leq \mathrm{Rad}(\Omega)$. The inequality $\mathrm{Rad}(\Omega) \leq \mathrm{Diam}(\Omega)$ directly follows from the two definitions.

Given (ii), establishing property (iii) amounts to show that

$$\mathrm{rad}(\Omega) \geq r_0/4. \tag{24}$$

We can assume with no loss of generality that $\partial\Omega \neq \emptyset$, otherwise $\Omega = \mathbb{R}^N$ and (24) trivially holds in this case (we recall that the case $\Omega = \emptyset$ is ruled out by the definition of Reifenberg flat domain).

Hence, we fix $y \in \partial\Omega$, denote by $P(y, r_0)$ the hyperplane in the definition and let $\vec{\nu}$ be its normal vector. We choose the orientation of $\vec{\nu}$ in such a way that

$$\{z + t\nu : z \in P(y, r_0), t \geq 2\varepsilon r\} \cap B(y, r_0) \subseteq \Omega. \tag{25}$$

Since $d_H(P(y, r_0) \cap B(y, r_0), \partial\Omega \cap B(y, r_0)) \leq \varepsilon r$, then from (25) we infer that actually

$$\{z + t\nu : z \in P(y, r_0), t \geq \varepsilon r\} \cap B(y, r_0) \subseteq \Omega.$$

By recalling $\varepsilon < 1/2$, we infer that there is $x \in \Omega$ such that $B(x, r_0/4) \subset \Omega$ and this establishes (24).

The following extension property of Reifenberg flat domains is established by combining Theorem (2.1.2) above with Jones'analysis ([88]).

Corollary(2.1.7)[63]: Let $\Omega \subseteq \mathbb{R}^N$ be a connected, (ε, r_0)-Reifenberg flat domain. If $\varepsilon \leq \frac{1}{600}$, then, for every $p \in [1, +\infty]$, there is an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ satisfying

$$\|E(u)\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p}(\Omega)},$$

where the constant C only depends on N , p, and r_0.

The results of both Christ [93] and Chua [90,91,92] apply to Jones flat domains, hence by relying on Theorem (2.1.2) we infer that they apply to $(\frac{1}{600}, r_0)$-Reifenberg flat domains as well.

As a consequence of Corollary (2.1.7) we get that the classical Rellich-Kondrachov Theorem holds in Reifenberg flat domains.

Proposition(2.1.8)[63]: Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, connected (ε, r_0)-Reifenberg flat domain and assume $0 < \varepsilon \leq 1/600$.

If $1 \leq p < N$, set $p^* := \frac{Np}{N-p}$. Then the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the space $L^{p^*}(\Omega)$ and is compactly embedded in $L^q(\Omega)$ for every $1 \leq q < p^*$.

If $p \geq N$, then the Sobolev space $W^{1,N}(\Omega)$ is continuously embedded in the space $L^\infty(\Omega)$ and is compactly embedded $L^q(\Omega)$ for every $q \in [1, +\infty[$.

Also, the norm of the above embedding operators only depends on N, r_0, q, p and $\mathrm{Rad}(\Omega)$.

As an example of application of Proposition (2.1.8), we establish a uniform bound on the L^∞ norm of Neumann eigenfunctions defined in Reifenberg flat domains. We use this bound in the companion reference [65]. We recall that we term "Neumann eigenfunction" an eigenfunction for the Laplace operator subject to homogeneous Neumann conditions on the boundary of the domain.

Proposition(2.1.9)[63]: Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, connected, (ε, r_0)-Reifenberg flat domain and let u be a Neumann eigenfunction associated to the eigenvalue μ. If $\varepsilon \leq \frac{1}{600}$, then u is bounded and

$$\|u\|_{L^\infty(\Omega)} \leq C\left(1+\sqrt{\mu}\right)^{\gamma(N)} \|u\|_{L^2(\Omega)}, \tag{26}$$

where $\gamma(N) = \max\left\{\frac{N}{2}, \frac{2}{N-1}\right\}$ and $C = C(N, r_0, \mathrm{Rad}(\Omega))$.

Proof. By using classical techniques coming from the regularity theory for elliptic operators, Ross [94] established (26) in the case of Lipschitz domains. However, in [94] the only reason why one needs the regularity assumption on the domain Ω is to use the Sobolev inequality

$$\|u\|_{L^{2^*}(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right), \qquad C = C\left(N, r_0, \mathrm{Rad}(\Omega)\right) \tag{27}$$

as the starting point for a bootstrap argument. Since Proposition (2.1.8) states that (27) holds if Ω is a bounded Reifenberg-flat domain, then the proof in [94] can be extended to the case of Reifenberg flat domains.

We have always assumed that the domain Ω is connected. We now show that the results we have established can be extended to general (i.e., not necessarily connected) Reifenberg-flat domains. Although extension of the result of Jones [88] to non-connected domains were already widely known, we decided to provide here a self-contained proof. In this way, we obtain results on the structure of Reifenberg flat domains that may be of independent interest.

We first show that any sufficiently flat Reifenberg flat domain is finitely connected and we establish a quantitative bound on the Hausdorff distance between two connected components.

Proposition(2.1.10)[63]: Let $\Omega \subseteq \mathbb{R}^N$ be a bounded, (ε, r_0)-Reifenberg flat domain and we assume $\varepsilon \leq 20^{-N}$. Then Ω has a finite number of nonempty, open and disjoint connected components $U_1, \ldots, U_n$, where

$$n \leq \frac{20^N|\Omega|}{\omega_N r_0^N}. \tag{28}$$

Moreover, if $i \neq j$, then for every $z \in \partial U_i$ we have

$$d(z, U_j) > r_0/70. \tag{29}$$

Proof. We proceed according to the following steps.

*** Step 1** We recall that any nonempty open set $\Omega \subseteq \mathbb{R}^N$ can be decomposed as

$$\Omega := \bigcup_{i\in I} U_i, \tag{30}$$

where the connected components U_i satisfy

(i) for every $i \in I$, U_i is a nonempty, open, arcwise connected set which is also closed in Ω. Hence, in particular, $\partial U_i \subseteq \partial\Omega$.

(ii) $U_i \cap U_j = \emptyset$ if $i \neq j$.

Indeed, for any $x \in \Omega$ we can define $U_x := \{ y \in \Omega :$ there is a continuous curve $\gamma: [0,1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y \}$ and observe that any U_x is a nonempty, open, arcwise connected set which is also closed in Ω. Also, given two points $x, y \in \mathbb{R}^N$, we have either $U_x = U_y$ or $U_x \cap U_y = \emptyset$.

⋄ **Step 2** Let Ω as in the statement of the proposition, and let the family $\{U_i\}_{i\in I}$ be as in (30). We fix $i \in I$ and we prove that $|U_i| \geq C\,(r_0, N)$. This straightforwardly implies that $\#I \leq C\,(|\Omega|, r_0, N\,)$. Since U_i is bounded, then $\partial U_i \neq \emptyset$: hence, we can fix a point $\tilde{x} \in \partial U_i$, and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in U_i$ and $x_n \to \tilde{x}$ as $n \to +\infty$. We recall that $\partial U_i \subseteq \partial\Omega$ and we infer that, for any $n \in \mathbb{N}$, the following chain of inequalities holds:

$$d(x_n, \partial U_i) = d(x_n, U_i^c) \leq d(x_n, \Omega^c) = d(x_n, \partial\Omega) \leq d(x_n, \partial U_i),$$

which implies $d(x_n\,, \Omega^c) = d(x_n, \partial U_i)$. We fix n sufficiently large such that $d(x_n, \tilde{x}) \leq \frac{r_0}{7}$, so that

$$d(x_n, \Omega^c) = d(x_n, \partial U_i) \leq r_0/7.$$

We term $\Gamma := \gamma_{x_n, r_0/7}$ the polygonal curve constructed as in Step 2 of the proof of Theorem (2.1.2) and we observe that, if $\varepsilon \leq 1/32$, then (5) holds and $\Gamma \subseteq \Omega$ and hence, by definition of $U_i, \Gamma \subseteq U_i$. We use the same notation as in Step 1 of the proof of Theorem (2.1.2) and we recall that Γ connects x_n to some point $Y(x_0, r_0/7)$, defined with some $x_0 \in \partial\Omega$. Hence, in particular, $Y(x_0, r_0/7) \in U_i$ and this implies that $B^+(x_0, r_0/7) \subseteq U_i$ because $B^+(x_0, r_0/7)$ is connected. This finally yields

$$|U_i| \geq \left|B^+\left(x_0, \frac{r_0}{7}\right)\right| \geq \omega_N \left(\frac{r_0}{14}(1 - 2\varepsilon)\right)^N \geq \omega_N \left(\frac{9r_0}{140}\right)^N \geq \omega_N \left(\frac{r_0}{20}\right)^N,$$

because $\varepsilon \leq 1/20$. We deduce that

$$\#I \leq \frac{20^N |\Omega|}{\omega_N r_0^N}.$$

⋄ **Step 3** We establish the separation property (29).

We set $r_1 := r_0/70$ and we argue by contradiction, assuming that there are $z \in \partial U_i$, $y \in \partial U_j$ such that

$$d(z, U_j\,) = d(z, \partial U_j) = d(z, y) \leq r_1.$$

Let $\{z_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in U_i and U_j converging to z and y, respectively. We fix n sufficiently large such that

$$d(z_n, \partial U_i) \leq d(z_n, z\,) \leq r_1 \leq r_0/14$$

and we term $\bar{z}$ be a point in ∂U_i satisfying $d(z_n, \bar{z}) = d(z_n, \partial U_i)$ (if there is more than one such $\bar{z}$, we arbitrarily fix one). By arguing as in Step 2, we infer that $B^+\left(\bar{z}, \frac{r_0}{14}\right) \subseteq U_i$. Next, we do the same for U_j, namely we fix m sufficiently large that

$$d(y_m, \partial U_j) \leq d(y_m, y\,) \leq r_1 \leq r_0/7,$$

we let $\bar{y}$ be a point in ∂U_j satisfying $d(y_m, \bar{y}) = d(y_m, \partial U_j)$ and, by arguing as in Step 2, we get that $B^+(\bar{y}, r_0/7) \subseteq U_j$. Also, we note that

$$d(\bar{z}, \bar{y}) \leq d(\bar{z}, z_n) + d(z_n, z\,) + d(z, y) + d(y, y_m) + d(y_m, \bar{y}) \leq 5r_1.$$

Since $r_1 = r_0/70$, then $B^+(\bar{z}, r_0/14) \subseteq B(\bar{z}, r_0/14) \subseteq B(\bar{y}, r_0/7)$. We observe that
$$B^+(\bar{z}, r_0/14) \cap B^-(\bar{y}, r_0/7) = \emptyset \tag{31}$$
since by construction $B^+(\bar{z}, r_0/14) \subseteq \Omega$ and $B^-(\bar{y}, r_0/7) \subseteq \Omega^c$. Also, by recalling that
$$B^+(\bar{z}, r_0/14) \subseteq U_i \text{ , } B^+(\bar{y}, r_0/7) \subseteq U_j \text{ and } U_i \cap U_j = \emptyset \text{ ,}$$
we have that
$$B^+(\bar{z}, r_0/14) \cap B^+(\bar{y}, r_0/7) = \emptyset \tag{32}$$
By combining (31) and (32) we get
$$B^+(\bar{z}, r_0/14) \subseteq B(\bar{y}, r_0/7) \backslash B^+(\bar{y}, r_0/7) \cup B^-(\bar{y}, r_0/7). \tag{33}$$
We now use the inequality
$$\omega_N \geq \omega_{N-1} \frac{1}{2^{N-1}}, \tag{34}$$
which will be proven later. By relying on (34) and by recalling that $\varepsilon \leq 20^{-N} \leq 1/20$ we obtain
$$|B^+(\bar{z}, r_0/14)| \geq \omega_N \left(\frac{r_0}{28}(1-2\varepsilon)\right)^N \geq 2\omega_{N-1}\left(\frac{9r_0}{560}\right)^N$$
and
$$|B(\bar{y}, r_0/7) \backslash B^+(\bar{y}, r_0/7) \cup B^-(\bar{y}, r_0/7)| \leq 4\varepsilon\omega_{N-1}\left(\frac{r_0}{7}\right)^N \leq 2\omega_{N-1}\left(\frac{2r_0}{140}\right)^N,$$
which contradicts (33) since $2/140 < 9/560$.
To finish the proof we are thus left to establish (34). To do this, we use the relation
$$\omega_N = \omega_{N-1} \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^{N-1} dx \,.$$
This implies that, for any $\lambda \in (0,1)$, we have
$$\omega_N \geq \omega_{N-1} 2 \int_0^{\lambda} \left(\sqrt{1-x^2}\right)^{N-1} dx \geq \omega_{N-1} 2\lambda \left(\sqrt{1-\lambda^2}\right)^{N-1}$$
By choosing $\lambda = \sqrt{3/2}$ we obtain the inequality
$$\omega_N \geq \omega_{N-1} \frac{\sqrt{3}}{2^{N-1}} \geq \omega_{N-1} \frac{1}{2^{N-1}},$$
and this concludes the proof.
Corollary(2.1.11)[63]: Let $N \geq 2$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded, (ε, r_0)-Reifenberg flat domain with $\varepsilon \leq \min(20^{-N}, 1/600)$. Then for every $p \in [1, +\infty]$ there is an extension operator
$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N) \tag{35}$$
whose norm is bounded by a constant which only depends on N, p, and r_0.
Proof. We employ the same notation as in the statement of Proposition (2.1.10) and we fix a connected component U_i. By recalling that $\partial U_i \subseteq \partial\Omega$ and the separation property (29), we infer that U_i is itself a $(\varepsilon, r_0/140)$-Reifenberg flat domain. Since by definition U_i is connected, we can apply Proposition (2.1.7) which says that, for every $p \in [1, +\infty]$, there is an extension operator
$$E_i : W^{1,p}(U_i) \to W^{1,p}(\mathbb{R}^N)$$
whose norm is bounded by a constant which only depends on N, p and r_0.

In order to "glue together" the extension operators $E_1, \dots, E_n$ we proceed as follows. Given $i = 1, \dots, n$, we set $\delta := r_0/280$ and we introduce the notation

$$U_i^\delta := \{x \in \mathbb{R}^N : d(x, U_i) < \delta\}.$$

Note that the separation property (29) implies that $U_i^{2\delta} \cap U_j^{2\delta} = \emptyset$ if $i \neq j$.

We now construct suitable cut-off functions $\varphi_i, i = 1, \dots, n$. Let $\ell : [0, +\infty[\to [0,1]$ be the auxiliary function defined by setting

$$\ell(t) := f(x) = \begin{cases} 1 & \text{if } t \leq \delta \\ 1 + \dfrac{\delta - t}{\delta} & \text{if } \delta \leq t \leq 2\delta \\ 0 & \text{if } t \geq 2\delta \end{cases}$$

We set

$\varphi_i(x) := \ell(d(x, U_i))$ and we recall that the function $x \mapsto d(x, U_i)$ is 1-Lipschitz and that $\delta = r_0/280$. Hence, the function φ_i satisfies the following properties:

$$0 \leq \varphi_i(x) \leq 1, |\nabla\varphi_i(x)| \leq C(r_0)\ \forall x \in \mathbb{R}^N, \varphi_i \equiv 1 \text{ on } U_i, \varphi_i \equiv 0 \text{ on } \mathbb{R}^N \setminus U_i^{2\delta} \quad (36)$$

We then define $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ by setting

$$E(u) := \sum_{i=1}^n E_i(u)(x)\varphi_i(x).$$

We recall that the sets $U_1, \dots, U_n$ are all pairwise disjoint, we focus on the case $p < +\infty$ and we get

$$\|E(u)\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left|\sum_{i=1}^n E_i(u)(x)\varphi_i(x)\, dx\right|^p\right)^{\frac{1}{p}} \leq \sum_{i=1}^n \left(\int_{U_i^{2\delta}} |E_i(u)(x)\varphi_i(x)|^p\, dx\right)^{1/p}$$

$$\leq \sum_{i=1}^n \|E_i(u)\|_{L^p(\mathbb{R}^N)} \leq \sum_{i=1}^n C(N,p,r_0)\|(u)\|_{W^{1,p}(U_i)} \leq C(N,p,r_0)\|(u)\|_{W^{1,p}(\Omega)}$$

Also, by using the bound on $|\nabla\varphi_i|$ provided by (36), we get

$$\|\nabla E(u)\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left|\sum_{i=1}^n \big(\nabla E_i(u)(x)\nabla\varphi_i(x) + E_i(u)(x)\nabla\varphi_i(x)\big)\, dx\right|^p\right)^{\frac{1}{p}}$$

$$\leq \sum_{i=1}^n \left(\int_{U_i^{2\delta}} |\nabla E_i(u)(x)\varphi_i(x)|^p\, dx\right)^{\frac{1}{p}} + \sum_{i=1}^n \left(\int_{U_i^{2\delta}} |E_i(u)(x)\nabla\varphi_i(x)|^p\, dx\right)^{1/p}$$

$$\leq \sum_{i=1}^n \|\nabla E_i(u)\|_{L^p(\mathbb{R}^N)} + C(r_0)\sum_{i=1}^n \|E_i(u)\|_{L^p(\mathbb{R}^N)} \leq C(N,p,r_0)\|(u)\|_{W^{1,p}(\Omega)}$$

The proof in the case $p = \infty$ is a direct consequence of the bounds on the norm of E_i and on the uniform norms of φ_i and $\nabla\varphi_i$. This concludes the proof of the corollary.

We end this section by comparing different ways of measuring the "distance" between Reifenberg-flat domains.

Comparing the Hausdorff distances $d_H(X,Y)$, $d_H(X^c, Y^c)$ and $d_H(\partial X, \partial Y)$, where X and Y are subsets of $\mathbb{R}^N$.

First, we exhibit two examples showing that, in general, neither $d_H(X, Y)$ controls $d_H(X^c, Y^c)$ nor $d_H(X^c, Y^c)$ controls $d_H(X, Y)$. We term $B := B(1, \vec{0})$ the unit ball and we consider the two perturbations A and C as represented in Figure 2.

Next, we exhibit an example showing that, in general, $d_H(\partial X, \partial Y)$ controls neither $d_H(X, Y)$ nor $d_H(X^c, Y^c)$. Let $X := B(R, \vec{0})$ and $Y := B(R + \varepsilon, \vec{0}) \backslash B(R, \vec{0})$, then

$$\varepsilon = d_H(\partial X, \partial Y) << d_H(X, Y) = d_H(X^c, Y^c) = R.$$

$\varepsilon \simeq d_H(A^c, B^c) << d_H(A, B) \simeq 1$ $\quad$ $\varepsilon \simeq d_H(C, B) << d_H(C^c, B^c) \simeq 1$

Figure 2.

Also, note that the examples represented in Figure 2 show that, in general, neither $d_H(X, Y)$ nor $d_H(X^c, Y^c)$ controls $d_H(\partial X, \partial Y)$. Indeed, $d_H(\partial A, \partial B) \simeq 1$ and $d_H(\partial C, \partial B) \simeq 1$.

However, if X and Y are two sufficiently close Reifenberg flat domains, then we have the following result.

Lemma(2.1.12)[63]: Let X and Y be two (ε, r_0)-Reifenberg flat domains satisfying $d_H(\partial X, \partial Y) \leq 2r_0$. Then

$$d_H(\partial X, \partial Y) \leq \frac{4}{1 - 2\varepsilon} \min\{d_H(X, Y), d_H(X^c, Y^c)\}. \quad (37)$$

Proof. Just to fix the ideas, assume that $d_H(\partial X, \partial Y) = \sup_{x \in \partial X} d(x, \partial Y)$.

Since by assumption $d_H(\partial X, \partial Y) < +\infty$, then for every $h > 0$ there is $x_h \in \partial X$ such that

$$d_H(\partial X, \partial Y) - h \leq d_h := d(x_h, \partial Y) \leq d_H(\partial X, \partial Y).$$

Note that $\partial Y \cap B\left(x_h, \frac{d_h}{2}\right) = \emptyset$ and hence either (i) $B\left(x_h, \frac{d_h}{2}\right) \subseteq Y$ or (ii) $B\left(x_h, \frac{d_h}{2}\right) \subseteq Y^c$.

First, consider case (i): let $P(x_h, \frac{d_h}{2})$ be the hyper plane prescribed by the definition of Reifenberg flatness, then by Lemma (2.1.5) we can choose the orientation of the normal vector ν in such a way that

$$B^-(x_h, d_h/2) := \{z + t\nu : z \in P(x_h, d_h/2), t \geq \varepsilon d_h\} \cap B(x_h, d_h/2) \subseteq X^c$$

and

$$B^+(x_h, d_h/2) := \{z - t\nu : z \in P(x_h, d_h/2), t \geq \varepsilon d_h\} \cap B(x_h, d_h/2) \subseteq X.$$

Fix the point

$$\bar{z} := x_h + \frac{(1 + 2\varepsilon)d_h}{4} \nu,$$

then we have

$$B\left(\bar{z}, \frac{(1 - 2\varepsilon)d_h}{4}\right) \subseteq B^-(x_h, d_h/2) \subseteq X^c \cap Y$$

and hence

$$d_H(X^c, Y^c) \geq \sup_{z \in X^c} d(z, Y^c) \geq d(\bar{z}, Y^c) \geq \frac{(1 - 2\varepsilon)d_h}{4}$$

and

$$d_H(X,Y) \geq \sup_{z\in Y} d(z,X) \geq d(\bar{z},X) \geq \frac{(1-2\varepsilon)d_h}{4}.$$

Since case (ii) can be tackled in an entirely similar way, by the arbitrariness of h we deduce that

$$d_H(\partial X,\partial Y) \leq \frac{4}{1-2\varepsilon} d_H(X,Y). \tag{38}$$

The proof of (37) is concluded by making the following observations:

(i) if X is an (ε,r_0)-Reifenberg flat domain, then X^c is also an (ε,r_0)-Reifenberg flat domain.

(ii) $\partial X = \partial X^c$ and $\partial Y = \partial Y^c$.

Hence, by replacing in (38) X with X^c and Y with Y^c we obtain (37).

Comparing the Hausdorff distances $d_H(X,Y)$ and $d_H(X^c,Y^c)$ with the Lebesgue measure of the symmetric difference, $|X\Delta Y|$. As usual, X and Y are subsets of $\mathbb{R}^N$. The results we state are applied in [65] to the stability analysis of the spectrum of the Laplace operator with Neumann boundary conditions.

First, we observe that the examples illustrated in Figure 2 show that, in general, $|X\Delta Y|$ controls neither $d_H(X,Y)$ nor $d_H(X^c,Y^c)$. Indeed, $|A\Delta B| \simeq \varepsilon$ and $|C\Delta B| \simeq \varepsilon$. However, if X and Y are two sufficiently close Reifenberg-flat domains, then the following result hold.

Lemma(2.1.13)[63]: Let X and Y be two (ε,r_0)-Reifenberg flat domains in $\mathbb{R}^N$.Then the following implications hold:

(I) if $d_H(X,Y) \leq 4r_0$, then

$$d_H(X,Y) \leq \frac{8}{(1-2\varepsilon)}\left(\frac{|X\Delta Y|}{\omega_N}\right)^{1/N}. \tag{39}$$

(II) If $d_H(X^c,Y^c) \leq 4r_0$, then

$$d_H(X^c,Y^c) \leq \frac{8}{(1-2\varepsilon)}\left(\frac{|X\Delta Y|}{\omega_N}\right)^{1/N} \tag{40}$$

In both the previous expressions, ω_N denotes the measure of the unit ball in $\mathbb{R}^N$.

Proof. The argument relies on ideas similar to those used in the proof of Lemma (2.1.12).

We first establish (39). Just to fix the ideas, assume that $d_H(X,Y) = \sup_{x\in X} d(x,Y)$ and note that by assumption $d_H(X,Y) < +\infty$. Hence, for every $h > 0$ there is $x_h \in X$ such that

$$d_H(X,Y) - h \leq d_h := d(x_h,Y) \leq d_H(X,Y)$$

Note that, by the very definition of $d(x_h,Y)$, we have $B(x_h,d_h) \subseteq Y^c$. We now separately consider two cases: if $B(x_h,d_h/2) \subseteq X$, then

$$B(x_h,d_h/2) \subseteq X\cap Y^c \subseteq |X\Delta Y|$$

and hence

$$\omega_N\left(\frac{d_h}{2}\right)^N \leq |X\Delta Y|,$$

and by the arbitrariness of h this implies (39).

Hence, we are left to consider the case when there is $x_0 \in B(x_h, d_h/2) \cap \partial X$. We make the following observations: first,

$$B(x_0, d_h/4) \subseteq B(x_h, d_h) \subseteq Y^c. \qquad (41)$$

Second, since $\frac{d_h}{4} \leq d_H(X, Y)/4 \leq r_0$, then we can apply the definition of Reifenberg flatness in the ball $B(x_0, d_h/4)$. Let $P(x_0, d_h/4)$ be the hyperplane provided by property (i) in the definition, and let ν_0 denote the normal vector. By relying on Lemma (2.1.5) we infer that we can choose the orientation of ν_0 in such a way that

$$B\left(x_0 + \frac{(1+2\varepsilon)d_h}{8}\,\nu_0, \frac{(1-2\varepsilon)d_h}{8}\right) \subseteq X \cap B\left(x_0, \frac{d_h}{4}\right) \qquad (42)$$

By combining (41) and (42) we infer that

$$\omega_N\left(\frac{(1-2\varepsilon)d_h}{8}\right)^N \leq |X \cap Y^c| \leq |X \Delta Y|$$

and by the arbitrariness of h this completes the proof of (39).

Estimate (40) follows from (39) by relying on the following two observations:

(i) $X\Delta Y = (X^c \cap Y) \cup (X \cap Y^c) = X^c \Delta Y^c$.

(ii) if X is an (ε, r_0)-Reifenberg flat domain, then X^c is also an (ε, r_0)-Reifenberg flat domain.

Hence, by replacing in (39) X with X^c and Y with Y^c we get (40).

Sec(2.2): The Poisson Equation In Reifenberg-Flat Domains

The goal of the present section is to show a boundary regularity result for the Poisson equation with homogeneous Dirichlet boundary conditions in non smooth domains.

We consider the case of Reifenberg flat domains as given by the following definition.

Definition(2.2.1)[96]: Let ε, r_0 be two real numbers satisfying $0 < \varepsilon < 1/2$ and $r_0 > 0$. An (ε, r_0)-Reifenberg-flat domain $\Omega \in \mathbb{R}^N$ is an open, bounded, and connected set satisfying the two following conditions:

(i) for every $x \in \partial\Omega$ and for any $r \leq r_0$, there exists a hyperplane $P(x, r)$ containing x which satisfies

$$\frac{1}{r} d_H(\partial\Omega \cap B(x,r), P(x,r) \cap B(x,r)) \leq \varepsilon.$$

(ii) for every $x \in \partial\Omega$, one of the connected component of

$$B(x, r_0) \cap \{x; d(x, P(x, r_0)) \geq 2\varepsilon r_0\}$$

is contained in Ω and the other one is contained in Ω^C.

We consider the following problem in the (x, r_0)-Reifenberg flat domain $\Omega \subset \square^N$ for some $f \in L^q(\Omega)$,

$$(\text{P1})\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Reifenberg flat domains are less smooth than Lipschitz domains and it is well known that we cannot expect more regularity than Hölder for boundary regularity of the Poisson equation in Lipschitz domains (see [97], [65] or [100]).

Historically, Reifenberg flat domains came into consideration because of their relationship with the regularity of the Poisson kernel and the harmonic measure, as shown in a series of famous and deep references by Kenig and Toro (see [20, 70, 71, 73]). In particular, they are Non Tangentially Accessible (in short NTA) domains as described in [83].
Notice that the Poisson kernel is defined as related to the solution of the equation $-\Delta u = 0$ in Ω with $u = f$ on $\partial\Omega$. In this section we consider the equation (P1) which is of different nature. However, the regularity result is again based on the monotonicity formula of Alt, Caffarelli and Friedman [106], which is known to be one of the key estimate in the study harmonic measure as well.

More recently, regularity of elliptic PDEs in Reifenberg flat domains has been studied by Byun and Wang in [107, 108, 76, 77, 78]. One of their main result regarding to equation of the type of (P1) is the existence of a global $W^{1,p}(\Omega)$ bound on the solution. This fact will be used in Corollary (2.2.4) below.

The case of domains of $\mathbb{R}^n$ has been investigated by Caffarelli and Peral [112]. See also [100] for the case of Lipschitz domains. Some other type of elliptic problems in Reifenberg-flat domains can be found in [80, 81, 74, 16, 65].

The present section is the first step towards a general boundary regularity theory for elliptic PDEs in divergence form on Reifenberg flat domains, that might be pursued in some future work. We have the following.

Theorem(2.2.2)[96]: Let $p, q, p_0 \geq 1$ be some exponents satisfying $1/p + 1/q = 1/p_0$ and $p_0 > N/2$. Let $\alpha > 0$ be any given exponent such that

$$\alpha < \frac{p_0 - N/2}{p_0}$$

Then one can find an $\varepsilon = \varepsilon(N, \alpha)$ such that the following holds. Let $\Omega \in \mathbb{R}^N$ be an (ε, r_0)-Reifenberg flat domain for some $r_0 > 0$, and let u be a solution for the problem (P1) in Ω with $u \in L^p(\Omega)$ and $f \in L^q(\Omega)$. Then

$$u \in C^{0,\alpha}(B(x, r_0/12) \cap \overline{\Omega}) \quad \forall x \in \overline{\Omega}.$$

Moreover $\|u\|_{C^{0,\alpha}(B(x,\ r_0/12)\cap\overline{\Omega})} \leq C(N, r_0, \alpha, p_0, \|u\|_p, \|f\|_q)$.

Proof. Considering u as a function of $W^{1,2}(\mathbb{R}^N)$ by setting 0 outside Ω, and applying Proposition (2.2.11) with $\beta = 2\alpha$, we obtain that

$$\int_{B(x,r)} |\nabla u|^2 dx \leq C r^{N-2+\beta} \|u\|_p \|f\|_q \quad \forall x \in \overline{\Omega}, \forall r \in (0, r_0/6), \quad (43)$$

with $C = C(N, r_0, \beta, p_0)$. Recalling now the classical Poincaré inequality in a ball $B(x, r)$

$$\int_{B(x,r)} |u - u_{x,r}| dx \leq C(N) r^{1+\frac{N}{2}} \left(\int_{B(x,r)} |\nabla u|^2 dx \right)^{\frac{1}{2}},$$

we get

$$\int_{B(x,r)} |u - u_{x,r}|dx \leq Cr^{N+\frac{\beta}{2}} \quad \forall x \in \overline{\Omega}, \forall\, r \in (0, r_0/6), \tag{44}$$

with $C = C(N, r_0, \beta, p_0, \|u\|_p \|f\|_q)$. But this implies that

$$u \in \mathcal{L}^{1,N+\frac{\beta}{2}}(B(x, r_0/12) \cap \overline{\Omega}) \quad \forall x \in \overline{\Omega}.$$

Moreover $\frac{\beta}{2} < \frac{p_0 - \frac{N}{2}}{p_0} \leq 1$ and hence Theorem (2.2.12) says that

$$u \in C^{0,\alpha}(B(x, r_0/12) \cap \overline{\Omega})$$

with $\alpha = \frac{\beta}{2}$, and the norm is controlled by $C = C(N, r_0, \beta, p_0, \|u\|_p \|f\|_q)$.

Observe that in the statement of Theorem (2.2.2), some a priori L^p integrability on u is needed to get some Hölder regularity. In what follows we shall see at least two situations where we know that $u \in L^p$ for some $p > 2$, and consequently state two Corollaries where the integrability hypothesis is given on f only, without any a priori requirement on u.

First, notice that when $f \in L^q(\Omega)$ for $2 \leq q \leq +\infty$, then the application $v \mapsto \int_\Omega vf\, dx$ is a bounded linear form on $W_0^{1,2}(\Omega)$, endowed with the scalar product $\int_\Omega \nabla u \cdot \nabla v\, dx$. Therefore, using Riesz representation theorem we deduce the existence of a unique weak solution $u \in W_0^{1,2}(\Omega)$ for the problem (P1). Moreover, the Sobolev inequality says that $u \in L^{2^*}(\Omega)$, with $2^* = \dfrac{2N}{N-2}$. Some simple computations shows that in this situation, u and f verify the statement of Theorem (2.2.2) provided that $2 \leq N \leq 5$, which leads to the following corollary.

Corollary (2.2.3)[96]: Assume that $2 \leq N \leq 5$ and let $q \in I_N$ be given where

$$I_N = \begin{cases} [2, +\infty) & if \quad N = 2 \\ \left(\frac{2N}{6-N}, +\infty\right) & for \quad 3 \leq N \leq 5. \end{cases}$$

Then for any $\alpha > 0$ verifying

$$\alpha < 1 - \frac{N}{2}\left(\frac{N-2}{2N} + \frac{1}{q}\right),$$

we can find an $\varepsilon = \varepsilon(N, \alpha)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^N$ be an (ε, r_0)-Reifenberg flat domain for some $r_0 > 0$, let $f \in L^q(\Omega)$ and let $u \in W_0^{1,2}(\Omega)$ be the unique solution for the problem (P1) in Ω. Then

$$u \in C^{0,\alpha}(B(x, r_0/12) \cap \overline{\Omega}) \quad \forall x \in \overline{\Omega}.$$

Moreover $\|u\|_{C^{0,\alpha}(B(x, r_0/12)\cap\overline{\Omega})} \leq C(N, r_0, \alpha, q, \|\nabla u\|_2, \|f\|_q)$.

Proof. Since $u \in W_0^{1,2}(\Omega)$, the Sobolev embedding says that $u \in L^p(\Omega)$, with $p = 2^* = \frac{2N}{N-2}$. And by assumption $f \in L^q$ for some $q \in I_N$ (notice that $q \geq 2$, which guarantees existence and uniqueness of the weak solution). We now try to apply Theorem (2.2.2) with those p and q. Let p_0 be defined by

$$1/p + 1/q = 1/p_0 .$$

Then a simple computation yields that $p_0 > N/2$, provided that

$$q > \frac{2N}{6-N}.$$

This fixes the range of dimension $N \leq 5$ and notice that in this case $\frac{2N}{6-N} \geq 2$ except for $N = 2$, which justifies the definition of I_N. We then conclude by applying Theorem (2.2.2).

In the proof of Corollary (2.2.3) we brutally used the Sobolev embedding on $W_0^{1,2}(\Omega)$ to obtain an L^p integrability on u. But under some natural hypothesis we can get more using a theorem by Byun and Wang [76]. Precisely, if $f = \mathrm{div} F$ for some $F \in L^2$,then f lies in the dual space of $H_0^1(\Omega)$ which guarantees the existence and uniqueness of a weak solution u for (P1), again by the Reisz representation theorem. The theorem of Byun and Wang [76] implies moreover that if $F \in L^r(\Omega)$ then $\nabla u \in L^r(\Omega)$ as well. But then the Sobolev inequality says that $u \in L^{r^*}$which allows us to apply Theorem (2.2.2) for a larger range of dimensions and exponents. Of course this analysis is interesting only for $r \leq N$ because if $r > N$ we directly get some Hölder estimates by the classical Sobolev embedding. This leads to the second corollary.

Corollary(2.2.4)[96]: Let $\frac{N}{3} < r \leq N$ and $q > \frac{rN}{3r-N}$ be given, so that moreover $r \geq 2$. Then for any $\alpha > 0$ satisfying

$$\alpha < 1 - \frac{N}{2}\left(\frac{1}{r^*} + \frac{1}{q}\right), \quad \text{with} \quad r^* := \frac{rN}{N-r},$$

we can find an $\varepsilon = \varepsilon(N, \alpha, |\Omega|, r)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^N$ be an (ε, r_0)-Reifenberg flat domain for some $r_0 > 0$, let $f \in L^q(\Omega)$ and assume that $f = -\mathrm{div} F$ for some $F \in L^r(\Omega)$. Let $u \in W_0^{1,2}(\Omega)$ be the unique weak solution for the problem (P1) in Ω. Then

$$u \in C^{0,\alpha}(B(x, r_0/12) \cap \overline{\Omega}) \; \forall x \in \overline{\Omega}.$$

Moreover $\|u\|_{C^{0,\alpha}(B(x,r_0/12)\cap\overline{\Omega})} \leq C(N, r_0, \alpha, r, q, |\Omega|, \|f\|_q, \|F\|_r)$.

Proof. First we apply [76] which provides the existence of a threshold $\varepsilon_0 = \varepsilon(N, |\Omega|, r)$ such that for any solution $u \in W_0^{1,2}(\Omega)$ of (P1) with $f = -\mathrm{div} F$ and $F \in L^r(\Omega)$., we have that $\nabla u \in L^r(\Omega)$, provided that Ω is (ε_0, r_0)-Reifenberg flat. But then the Sobolev inequality implies that $u \in L^{r^*}$ with

$$r^* := \frac{rN}{N-r} \quad \text{if} \quad r < N,$$

and $r^* = +\infty$ otherwise.

In order to apply Theorem (2.2.2) we define p_0 such that

$$1/r^* + 1/q = 1/p_0 ,$$

and we only need to check that $p_0 > N/2$. This implies the following condition on r and q:

$$r > N/3 \text{ and } q > \frac{rN}{3r - N},$$

as required in the statement of the Corollary. We finally conclude by applying Theorem (2.2.2).

Lemma(2.2.5)[96]: Let u be a solution for the problem (P1). Then for every $x_0 \in \partial\Omega$ and a.e. $r > 0$ we have

$$\int_{B(x_0,r)\cap\Omega} |\nabla u|^2 |x|^{2-N} dx \leq r^{2-N} \int_{\partial B(x_0,r)\cap\Omega} u \frac{\partial u}{\partial \nu} dS + \frac{N-2}{2} r^{1-N} \int_{\partial B(x_0,r)\cap\Omega} u^2 dS + \int_{B(x_0,r)\cap\Omega} uf|x|^{2-N} dx. \quad (45)$$

Proof. Although (56) below can be formally obtained through an integration by parts, the rigorous proof is a bit technical. In the sequel we use the notation $\Omega_r^+ := B(x_0, r) \cap \Omega$ and $S_r^+ := \partial B(x_0, r) \cap \Omega$. We find it convenient to define, for a given $\varepsilon > 0$, the regularized norm

$$|x|_\varepsilon := \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2 + \varepsilon},$$

so that $|x|_\varepsilon$ is a C^∞ function. A direct computation shows that

$$\Delta(|x|_\varepsilon^{2-N}) = (2 - N)N \frac{\varepsilon}{|x|_\varepsilon^{N+2}} \leq 0,$$

in other words $|x|_\varepsilon^{2-N}$ is superharmonic, and hopefully enough this goes in the right direction regarding to the next inequalities.

We use one more regularization thus we let $u_n \in C_c^\infty(\Omega)$ be a sequence of functions converging in $W^{1,2}(\mathbb{R}^N)$ to u. We now proceed as in the proof of Alt, Caffarelli and Friedman monotonicity formula [106]: by using the equality

$$\Delta(u_n^2) = 2|\nabla u_n|^2 + 2u_n \Delta u_n \quad (46)$$

we deduce that

$$2\int_{\Omega_r^+} |\nabla u_n|^2 |x|_\varepsilon^{2-N} = \int_{\Omega_r^+} \nabla(u_n^2)|x|_\varepsilon^{2-N} - 2\int_{\Omega_r^+} (u_n \nabla u_n)|x|_\varepsilon^{2-N}. \quad (47)$$

Since $\Delta(|x|_\varepsilon^{2-N}) \leq 0$, the Gauss-Green Formula yields

$$\int_{\Omega_r^+} \nabla(u_n^2)|x|_\varepsilon^{2-N} dx = \int_{\Omega_r^+} u_n^2 \nabla(|x|_\varepsilon^{2-N}) dx + I_{n,\varepsilon}(r) \leq I_{n,\varepsilon}(r), \quad (48)$$

where

$$I_{n,\varepsilon}(r) = (r^2 + \varepsilon)^{\frac{2-N}{2}} \int_{\partial\Omega_r^+} 2u_n \frac{\partial u_n}{\partial \nu} dS + (N-2)\frac{r}{(r^2 + \varepsilon)^{\frac{N}{2}}} \int_{\partial\Omega_r^+} u_n^2 dS.$$

In other words, (47) reads

$$2\int_{\Omega_r^+} |\nabla u_n|^2 |x|^{2-N} dx \leq I_{n,\varepsilon}(r) - 2\int_{\Omega_r^+} (u_n \nabla u_n)|x|_\varepsilon^{2-N} dx. \quad (49)$$

We now want to pass to the limit, first as $n \to +\infty$, and then as $\varepsilon \to 0^+$. To tackle some technical problems, we first integrate over $r \in [r, r + \delta]$ and divide by δ, thus obtaining

$$\frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+}|\nabla u_n|^2|x|_\varepsilon^{2-N}dx\right)d\rho \;\le\; A_n - R_n, \tag{50}$$

where

$$A_n \;=\frac{1}{\delta}\int_r^{r+\delta} I_{n,\varepsilon}(\rho)d\rho$$

and

$$R_n \;= 2\frac{1}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+}(u_n\Delta u_n)|x|_\varepsilon^{2-N}dx\right)d\rho.$$

First, we investigate the limit of A_n as $n\to+\infty$: by applying the coarea formula, we rewrite A_n as

$$A_n \;=\frac{1}{\delta}\left(2\int_{\Omega_{r+\delta}^+\setminus\,\Omega_r^+}(|x|^2+\varepsilon)^{\frac{2-N}{2}}\,u_n\nabla u_n\cdot x\,dx+(N-2)\int_{\Omega_{r+\delta}^+\setminus\,\Omega_r^+}\frac{|x|}{(|x|^2+\varepsilon)^{\frac{N}{2}}}u_n^2dx\right).$$

Since u_n converges to u in $W^{1,2}(\mathbb{R}^N)$ when $n\to+\infty$, then by using again the coarea formula we get that

$$A_n \;\to\frac{1}{\delta}\int_r^{r+\delta} I_\varepsilon(\rho)d\rho \qquad n\to+\infty$$

where

$$I_\varepsilon(\rho)\;=\;(\rho^2+\varepsilon)^{\frac{2-N}{2}}\int_{\partial\Omega_\rho^+}2u\frac{\partial u}{\partial\nu}dS+(N-2)\frac{\rho}{(\rho^2+\varepsilon)^{\frac{N}{2}}}\int_{\partial\Omega_\rho^+}u^2dS\,.$$

Next, we investigate the limit of R_n as $n\to+\infty$. By using Fubini's Theorem, we can rewrite R_n as

$$R_n \;=\int_\Omega(u_n\Delta u_n)G(x)dx\,,$$

where

$$G(x)\;=\;|x|_\varepsilon^{2-N}\frac{1}{\delta}\int_r^{r+\delta}\mathbf{1}_{\Omega_\rho^+}(x)d\rho.$$

Since

$$\frac{1}{\delta}\int_r^{r+\delta}\mathbf{1}_{\Omega_\rho^+}(x)d\rho=\begin{cases}1 & \text{if } x\in\Omega_r^+\\ \dfrac{r+\delta-|x|}{\delta} & \text{if } x\in\Omega_{r+\delta}^+\setminus\,\Omega_r^+\,,\\ 0 & \text{if } x\notin\Omega_{r+\delta}^+\end{cases}$$

then G is Lipschitz continuous and hence by recalling $u_n\in C_c^\infty(\Omega)$ we get

$$\left|\int_\Omega(u_n\Delta u_n-u\Delta u)\,G\,dx\right|\le\left|\int_\Omega(\Delta u_n-\Delta u)\,u_nG\,dx\right|+\left|\int_\Omega(u_n-u)\,\Delta uG\,dx\right|$$

$$=\left|\int_\Omega(\nabla u_n-\nabla u)(u_n\nabla G+\nabla u_nG)\,dx\right|+\left|\int_\Omega\nabla u\big((\nabla u_n-\nabla u)G+(u_n-u)\,\nabla G\big)\,dx\right|$$

$$\le\;\|\nabla u_n\;-\;\nabla u\|_{L^2(\Omega)}\big(\|u_n\|_{L^2(\Omega)}\|\nabla G\|_{L^\infty(\Omega)}\;+\|\nabla u_n\|_{L^2(\Omega)}\|G\|_{L^\infty(\Omega)}\big)$$

$$+\| \nabla u\|_{L^2(\Omega)}\left(\|\nabla u_n - \nabla u\|_{L^2(\Omega)}\|G\|_{L^\infty(\Omega)} + \|u_n - u\|_{L^2(\Omega)}\|\nabla G\|_{L^\infty(\Omega)}\right)$$

and hence the expression at the first line converges to 0 as $n \to +\infty$.

By combining the previous observations and by recalling that u satisfies the equation in the problem (P1) we infer that by passing to the limit $n \to \infty$ in (50) we get

$$\frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+}|\nabla u|^2|x|_\varepsilon^{2-N}dx\right)dr \le \frac{1}{\delta}\int_r^{r+\delta} I_\varepsilon(\rho)d\rho + \frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+} uf|x|_\varepsilon^{2-N}dx\right)d\rho.$$

Finally, dividing by 2, by passing to the limit $\delta \to 0^+$, and then $\varepsilon \to 0^+$ we obtain (56) below.

Next, we will need the following Lemma of Gronwall type.

Lemma(2.2.6)[96]: Let $\gamma > 0$, $r_0 > 0$, $\psi : (0, r_0) \to \mathbb{R}$ be a continuous function and $\varphi: (0, r_0) \to \mathbb{R}$ be an absolutely continuous function that satisfies the following inequality for a.e. $r \in (0, r_0)$,

$$\varphi(r) \le \gamma r \varphi'(r) + \psi(r). \tag{51}$$

Then

$$r \mapsto \frac{\varphi(r)}{r^{1/\gamma}} + \frac{1}{\gamma}\int_0^r \frac{\psi(s)}{s^{1+1/\gamma}}ds$$

is a nondecreasing function on $(0, r_0)$.

Proof. We can assume that

$$\int_0^{r_0} \frac{\psi(s)}{s^{1+1/\gamma}} ds < +\infty,$$

otherwise the Lemma is trivial. Under the hypothesis, the function

$$F(r) := \frac{\varphi(r)}{r^{1/\gamma}} + \frac{1}{\gamma}\int_0^r \frac{\psi(s)}{s^{1+1/\gamma}}ds$$

is differentiable a.e. and absolutely continuous. A computation gives

$$F'(r) = \frac{\varphi'(r)r^{1/\gamma} - \varphi(r)\,1/\gamma\, r^{1/\gamma-1}}{r^{2/\gamma}} + 1/\gamma\,\frac{\psi(r)}{r^{1+1/\gamma}}$$
$$= \frac{\varphi'(r) - \varphi(r)\,1/\gamma\, r^{-1}}{r^{1/\gamma}} + 1/\gamma\,\frac{\psi(r)}{r^{1+1/\gamma}}$$

thus (51) yields

$$r\gamma F'(r) = \frac{r\gamma\varphi'(r) - \varphi(r) + \psi(r)}{r^{1/\gamma}} \ge 0,$$

which implies that F is nondecresing.

We now show the monotonicity Lemma, which is inspired by Alt, Caffarelli and Friedman [106]. The following statement and its proof, is an easy variant of [65], where the same estimate is performed on Dirichlet eigenfunctions of the Laplace operator. We decided to write the full details in order to enlighten the role of the second member f in the inequalities.

Lemma(2.2.7)[96]: Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let u be a solution for the problem (P1). Given $x_0 \in \overline{\Omega}$ and a radius $r > 0$, we denote by $\Omega_r^+ := B(x_0, r) \cap \Omega$, by $S_r^+ := \partial B(x_0, r) \cap \Omega$ and by $\sigma(r)$ the first Dirichlet eigenvalue of the Laplace operator on the spherical domain S_r^+. If there are constants $r_0 > 0$ and $\sigma^* \in]0, N-1[$ such that

$$\inf_{0<r<r_0} (r^2\sigma(r)) \geq \sigma^*, \tag{52}$$

then the function

$$(r \longmapsto \left(\frac{1}{r^\beta}\int_{\Omega_r^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}}dx\right) + \beta\int_0^r \frac{\psi(s)}{s^{1+\beta}}ds \tag{53}$$

is non decreasing on $]0, r_0[$, where $\beta \in]0,2[$ is given by

$$\beta = \sqrt{(N-2)^2 + 4\sigma^*} - (N-2)$$

and

$$\psi(s) := \int_{\Omega_s^+} |uf||x-x_0|^{2-N}dx.$$

We also have the bound

$$\int_{\Omega_{r_0/2}^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}}dx \leq C(N, r_0, \beta)\|\nabla u\|_{L^2(\Omega)}^2 + \psi(r_0). \tag{54}$$

Note that: Of course the Lemma is interesting only when

$$\int_0^r \frac{\psi(s)}{s^{1+\beta}}ds < +\infty. \tag{55}$$

This will be satisfied if u and f are in some L^p spaces with suitable exponents, as will be shown in Lemma (2.2.8).

Proof. We assume without lose of generality that $x_0 = 0$ and to simplify notation we denote by B_r the ball $B(x, r)$.

By Lemma (2.2.5) we know that for a.e. $r > 0$,

$$\int_{\Omega_r^+} |\nabla u|^2|x|^{2-N}dx \leq r^{2-N}\int_{S_r^+} 2u\frac{\partial u}{\partial \nu}dS + \frac{(N-2)}{2}r^{1-N}\int_{S_r^+} u^2 dS$$
$$+ \int_{\Omega_r^+} uf|x|^{2-N}dx. \tag{56}$$

Let us define

$$\psi(r) := \int_{\Omega_r^+} |uf||x|^{2-N}dx. \tag{57}$$

and assume that (55) holds (otherwise there is nothing to prove).

Next, we point out that the definition of σ^* implies that

$$\int_{S_r^+} u^2 dS \leq \frac{1}{\sigma^*}r^2\int_{S_r^+} |\nabla_{\mathcal{T}} u|^2 dS \quad r \in]0, r_0[, \tag{58}$$

where $\nabla_{\mathcal{T}}$ denotes the tangential gradient on the sphere. Also, let $\alpha > 0$ be a parameter that will be fixed later, then by combining Cauchy-Schwarz inequality, (58) and the inequality $ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$, we get

$$\left|\int_{S_r^+} u\frac{\partial u}{\partial \nu}dS\right| \leq \left(\int_{S_r^+} u^2 dS\right)^{\frac{1}{2}}\left(\int_{S_r^+}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)^{\frac{1}{2}}$$

$$\leq r/\sqrt{\sigma^*}\left(\int_{S_r^+}|\nabla_{\mathcal{T}} u|^2 dS\right)^{\frac{1}{2}}\left(\int_{S_r^+}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)^{\frac{1}{2}}$$

$$\leq r/\sqrt{\sigma^*}\left(\frac{\alpha}{2}\int_{S_r^+}|\nabla_{\mathcal{T}} u|^2 dS + \frac{1}{2\alpha}\int_{S_r^+}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right) \tag{59}$$

Hence,

$$r^{2-N}\int_{S_r^+} 2u\frac{\partial u}{\partial \nu}dS + (N-2)r^{1-N}\int_{S_r^+} u^2 dS$$

$$\leq r^{2-N}\frac{2r}{\sqrt{\sigma^*}}\left[\frac{\alpha}{2}\int_{S_r^+}|\nabla_{\mathcal{T}} u|^2 dS + \frac{1}{2\alpha}\int_{S_r^+}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right] + (N-2)r^{1-N}\frac{1}{\sigma^*}r^2\int_{S_r^+}|\nabla_{\mathcal{T}} u|^2 dS$$

$$\leq r^{1-N}\left[\left(\frac{\alpha}{\sqrt{\sigma^*}} + \frac{N-2}{\sigma^*}\right)\int_{S_r^+}|\nabla_{\mathcal{T}} u|^2 dS + \frac{1}{\alpha\sqrt{\sigma^*}}\int_{S_r^+}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right]. \tag{60}$$

Next, we choose $\alpha > 0$ in such a way that

$$\frac{\alpha}{\sqrt{\sigma^*}} + \frac{N-2}{\sigma^*} = \frac{1}{\alpha\sqrt{\sigma^*}}.$$

namely

$$\alpha = \frac{1}{2\sqrt{\sigma^*}}\left[\sqrt{(N-2)^2 + 4\sigma^*} - (N-2)\right].$$

Hence, by combining (56), (58) and (60) we finally get

$$\int_{\Omega_r^+}|\nabla u|^2|x|^{2-N}dx \leq r^{3-N}\gamma(N,\sigma^*)\int_{S_r^+}|\nabla u|^2 dS + \psi(r), \tag{61}$$

where

$$\gamma(N,\sigma^*) = \left[\sqrt{(N-2)^2 + 4\sigma^*} - (N-2)\right]^{-1}.$$

Let us set

$$\varphi(r) := \int_{\Omega_r^+}|\nabla u|^2|x|^{2-N}dx$$

and observe that

$$\varphi'(r) = r^{2-N}\int_{S_r^+}|\nabla u|^2 \qquad a.e.\, r \in]0, r_0[\,,$$

hence (61) implies that

$$\varphi(r) \leq \gamma r\varphi'(r) + \psi(r), \tag{62}$$

with $\gamma = \gamma(N,\sigma^*)$.

But now Lemma (2.2.6) exactly says that the function

$$r \longmapsto \frac{1}{r^\beta}\int_{\Omega_r^+}\frac{|\nabla u|^2}{|x-x_0|^{N-2}}dx + \beta\int_0^r \frac{\psi(s)}{s^{1+\beta}}ds \tag{63}$$

is non decreasing on $(0, r_0)$, where $\beta \in (0,2)$ is given by

$$\beta = \frac{1}{\gamma} = \sqrt{(N-2)^2 + 4\sigma^*} - (N-2),$$

which proves the monotonicity result.

To finish the proof of the Lemma it remains to establish (54). For this purpose, we start by finding a radius $r_1 \in (r_0/2, r_0)$ such that $\int_{S_{r_1}^+} |\nabla u|^2 \, ds$ is less than average, which means

$$\int_{S_{r_1}^+} |\nabla u|^2 \, ds \le \frac{2}{r_0} \int_{\Omega_{r_0}^+} |\nabla u|^2 dx \le \frac{2}{r_0} \|\nabla u\|_{L^2(\Omega)}^2 .$$

By combining (61) with the fact that $r_0/2 \le r_1 \le r_0$ we infer that

$$\int_{\Omega_{r_1}^+} |\nabla u|^2 |x|^{2-N} \, dx \le C(N, r_0, \beta) \|\nabla u\|_{L^2(\Omega)}^2 + \psi(r_1)$$

$$\le C(N, r_0, \beta) \|\nabla u\|_{L^2(\Omega)}^2 + \psi(r_0). \tag{64}$$

It follows that

$$\int_{\Omega_{r_0/2}^+} \frac{|\nabla u|^2}{|x|^{N-2}} dx \le \int_{\Omega_{r_1}^+} \frac{|\nabla u|^2}{|x|^{N-2}} \le C(N, r_0, \beta) \|\nabla u\|_{L^2(\Omega)}^2 + \psi(r_0),$$

and (54) is proved.

In order to apply Lemma (2.2.7), the first thing to check is that (55) holds. The purpose of the following Lemma is to show that it is the case when u and f are in suitable L^p spaces.

Lemma(2.2.8)[96]: Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain and let $p_0 > \frac{N}{2}$. Then or any $g \in L^{p_0}(\Omega)$, denoting

$$\psi(r) := \int_{B(0,r)} |g| \, |x|^{2-N} dx ,$$

we have

$$|\psi(r)| \le C(N, p_0) \|g\|_{p_0} r^{\frac{2p_0 - N}{p_0}} . \tag{65}$$

As a consequence

$$\int_0^{r_0} \frac{\psi(s)}{s^{1+\beta}} ds < +\infty$$

for any $\beta > 0$ satisfying

$$\beta < \frac{2p_0 - N}{p_0}. \tag{66}$$

Proof. First, we observe that $|x|^{2-N} \in L^m$ for any m that satisfies

$$(N-2)m < N \Rightarrow m < \frac{N}{N-2}.$$

Moreover a computation gives that under this condition,

$$\| |x|^{2-N} \|_{L^m(B(0,r))}^m = \int_{B(0,r)} |x|^{(2-N)m} = C(N, m) r^{N - m(N-2)}. \tag{67}$$

Let us define m as being the conjugate exponent of p_0, namely

$$\frac{1}{m} := 1 - \frac{1}{p_0}.$$

Then the hypothesis $p_0 > \frac{N}{2}$ implies that $m < \frac{N}{N-2}$, and by use of Hölder inequality and by (67) we can estimate

$$|\psi(r)| \le \|g\|_{p_0} \||x|^{2-N}\|_{L^m(B(0,r))} \le C(N,m)\|g\|_{p_0} r^{\frac{N-m(N-2)}{m}},$$

or equivalently in terms of p_0,

$$|\psi(r)| \le C(N,p_0)\|g\|_{p_0} r^{\frac{2p_0-N}{p_0}}. \tag{68}$$

The conclusion of the Lemma follows directly from (68).

We come now to the first application of Lemma (2.2.7), which is a decay estimate for the energy at interior points, for arbitrary domains.

Proposition(2.2.9)[96]: Let $p, q, p_0 > 1$ be some exponents satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{p_0}$ and $p_0 > \frac{N}{2}$. *Let* $\beta > 0$ be any given exponent such that

$$\beta < \frac{2p_0 - N}{p_0}. \tag{69}$$

Let $\Omega \subseteq \mathbb{R}^N$ be any domain, $x_0 \in \Omega$, and let u be a solution for the problem (P1) in Ω with $u \in L^p(\Omega)$ and $f \in L^q(\Omega)$. Then

$$\int_{B(x_0,r)\cap\Omega} |\nabla u|^2 dx \le C r^{N-2+\beta} \|u\|_p \|f\|_q \quad \forall\, r \in (0, \mathrm{dist}(x_0, \partial\Omega)/2), \tag{70}$$

with $C = C(N, \beta, p_0, \mathrm{dist}(x_0, \partial\,\Omega))$.

Proof. We assume that $x_0 = 0$ and we define $r_1 := \mathrm{dist}(x_0, \partial\,\Omega)$ in such a way that $B(x, r)$ is totally contained inside Ω for $r \le r_1$ and under the notation of Lemma (2.2.7), $\Omega_r^+ = B(0, r)$ and by $S_r^+ = \partial B(0, r)$ for any $r \le r_1$. Under the hypothesis and in virtue of Lemma (2.2.8) that we apply with $g = uf$, we know that (55) holds for any $\beta > 0$ that satisfies

$$\beta < \frac{2p_0 - N}{p_0}. \tag{71}$$

Notice that $\frac{2p_0 - N}{p_0} > 0$, because $p_0 > N/2$.

In the sequel we chose any exponent $\beta > 0$ satisfying (70), so that (55) holds and moreover Lemma (2.2.8) says that

$$\begin{aligned}\int_0^{r_1} \frac{\psi(s)}{s^{1+\beta}}\, ds &\le C(N,p_0)\|u\|_p \|f\|_q \int_0^{r_1} s^{\frac{2p_0-N}{p_0}-1-\beta}\, ds \\ &\le C(N,p_0,\beta)\|u\|_p\|f\|_q . \end{aligned} \tag{72}$$

We are now ready to show (70), β still being a fixed exponent satisfying (71).

We recall that the first eigenvalue of the spherical Dirichlet Laplacian on the unit sphere is equal to $N - 1$, thus hypothesis (52) in the present context reads

$$\inf_{0<r<r_1} \left(r^2\sigma(r)\right) = N - 1, \tag{73}$$

so that (52) holds for any $\sigma^* < N - 1$. Let us choose σ^* exactly equal to the one that satisfies

$$\beta = \sqrt{(N-2)^2 + 4\sigma^*} - (N-2).$$

one easily verifies that $\sigma^* < N - 1$. because of (71).

As a consequence, we are in position to apply Lemma (2.2.7) which ensures that, if u is a solution for the problem (P1), then the function in (63) is non decreasing. In particular, by monotonicity we know that for every $r \le r_1$,

$$\frac{1}{r^{N-2+\beta}}\int_{\Omega_r^+}|\nabla u|^2\,dx \le \left(\frac{1}{r^{\beta}}\int_{\Omega_r^+}\frac{|\nabla u|^2}{|x-x_0|^{N-2}}\,dx\right)+\beta\int_0^r\frac{\psi(s)}{s^{1+\beta}}\,ds$$

$$\le \left(\frac{1}{\left(\frac{r_1}{2}\right)^{\beta}}\int_{\Omega_{\frac{r_1}{2}}^+}\frac{|\nabla u|^2}{|x-x_0|^{N-2}}\,dx\right)+\beta\int_0^{r_1/2}\frac{\psi(s)}{s^{1+\beta}}\,ds. \qquad (74)$$

and we conclude that for every $r \le r_1/2$,

$$\int_{\Omega_r^+}|\nabla u|^2dx \le Kr^{N-2+\beta},$$

with

$$K=\left(\frac{1}{(r_1/2)^{\beta}}\int_{\Omega_{r_1/2}^+}\frac{|\nabla u|^2}{|x-x_0|^{N-2}}dx\right)+\beta\int_0^{r_1/2}\frac{\psi(s)}{s^{1+\beta}}ds,$$

Let us now provide an estimate on K. To estimate the first term in K we use (54) to write

$$\int_{\Omega_{r_1/2}^+}\frac{|\nabla u|^2}{|x-x_0|^{N-2}}dx \le C(N,r_1,\beta)\|\nabla u\|_{L^2(\Omega)}^2+\psi(r_1) \qquad (75)$$

Then we use (65) to estimate

$$\psi(r_1) \le C(N,p_0,r_1)\|u\|_p\|f\|_q\,,$$

and from the equation satisfied by u we get

$$\|\nabla u\|_{L^2(\Omega)}^2=\int_\Omega uf\,dx \le \|u\|_p\|f\|_q$$

so that in total we have

$$\int_{\Omega_{r_1/2}^+}\frac{|\nabla u|^2}{|x-|^{N-2}}dx \le C(N,r_0,\beta,p_0)\|u\|_p\|f\|_q.$$

Finally, the last estimate together with (72) yields

$$K \le C(N,r_1,\beta,p_0)\|u\|_p\|f\|_q,$$

and this ends the proof of the Proposition.

We now use Lemma (2.2.7) again to provide an estimate on the energy at boundary points, this time for Reifenberg flat domains.

Proposition(2.2.10)[96]: Let $p,q,p_0>1$ be some exponents satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{p_0}$ and $p_0>\frac{N}{2}$. Let $\beta>0$ be any given exponent such that

$$\beta<\frac{2p_0-N}{p_0}. \qquad (76)$$

Then one can find an $\varepsilon=\varepsilon(N,\beta)$ such that the following holds. Let $\Omega\subseteq\mathbb{R}^N$ be any (ε,r_0)-Reifenberg flat domain for some $r_0>0$, let $x_0\in\partial\Omega$ and let u be a solution for the problem (P1) in Ω with $u\in L^p(\Omega)$ and $f\in L^q(\Omega)$. Then

$$\int_{B(x_0,r)\cap\Omega} |\nabla u|^2 dx \le C r^{N-2+\beta} \|u\|_p \|f\|_q \forall\, r \in \left(0, \frac{r_0}{2}\right), \tag{77}$$

with $C = C(N, r_0, \beta, p_0)$.

Proof. As before we assume that $x_0 = 0$ and we denote by $\Omega_r^+ := B(0,r) \cap \Omega$ and by $S_r^+ := \partial B(0,r) \cap \Omega$. To obtain the decay estimate on $\int_{\Omega_r^+} |\nabla u|^2 dx$ we will follow the proof of Proposition (2.2.9) the main difference is that for boundary points, (73) does not hold. This is where Reifenberg flatness will play a role.

Let $\beta > 0$, be an exponent satisfying (76), so that invoquing Lemma (2.2.8) we have

$$\int_0^{r_0} \frac{\psi(s)}{s^{1+\beta}} ds \le C(N, p_0, \beta\,) \|u\|_p \|f\|_q < +\infty. \tag{78}$$

Next, we recall that the first eigenvalue of the spherical Dirichlet Laplacian on a half sphere is equal to $N-1$ (as for the total sphere). For $t \in (-1,1)$, let S_t be the spherical cap $S_t := \partial B(0,1) \cap \{x_N > t\}$ so that $t = 0$ corresponds to a half sphere. Let $\lambda_1(S_t)$ be the first Dirichlet eigenvalue in S_t . In particular, $t \longmapsto \lambda_1(S_t)$ is continuous and monotone in t.

Therefore, since and $\lambda_1(S_t) \to 0$ as $t \downarrow -1$, there is $t^*(\beta) < 0$ such that

$$\beta = \sqrt{(N-2)^2 + 4\lambda_1(t^*)} - (N-2)$$

By applying the definition of Reifenerg flat domain, we infer that, if $\varepsilon < t^*(\eta)/2$, then $\partial B(x_0, r) \cap \Omega$ is contained in a spherical cap homothetic to S_{t^*} for every $r \le r_0$. Since the eigenvalues scale of by factor r^2 when the domain expands of a factor $1/r$, by the monotonicity property of the eigenvalues with respect to domains inclusion, we have

$$\inf_{r<r_0} r^2 \lambda_1(\partial B(x_0, r) \cap\ \Omega) \ge \lambda_1(S_{t^*}\,) = \frac{\beta}{2}\left(\frac{\beta}{2} + N - 2\right). \tag{79}$$

As a consequence, we are in position to apply the monotonicity Lemma (Lemma 2.2.7) which ensures that, if u is a solution for the problem (P1) and $x_0 \in \partial\Omega$, then the function in (63) is non decreasing.

We then conclude as in the proof of Proposition (2.2.9), i.e. by monotonicity we know that for every $r \le r_0 < 1$,

$$\begin{aligned} \frac{1}{r^{N-2+\beta}} \int_{\Omega_r^+} |\nabla u|^2\, dx &\le \left(\frac{1}{r^\beta} \int_{\Omega_r^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}}\, dx\right) + \beta \int_0^r \frac{\psi(s)}{s^{1+\beta}}\, ds \\ &\le \left(\frac{1}{\left(\frac{r_0}{2}\right)^\beta} \int_{\Omega_{\frac{r_0}{2}}^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}}\, dx\right) + \beta \int_0^{r_0/2} \frac{\psi(s)}{s^{1+\beta}}\, ds. \end{aligned} \tag{80}$$

hence for every $r \le r_0/2$,

$$\int_{\Omega_r^+} |\nabla u|^2 dx \le K r^{N-2+\beta},$$

with

$$K = \left(\frac{1}{(r_0/2)^\beta} \int_{\Omega_{r_0/2}^+} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} dx\right) + \beta \int_0^{r_0/2} \frac{\psi(s)}{s^{1+\beta}} ds,$$

Then we estimate K exactly as in the end of the proof of Proposition (2.2.9), using (54), (65) and (78) to bound

$$K \leq C(N, r_0, \beta, p_0)\|u\|_p \|f\|_q ,$$

and this ends the proof of the Proposition. Gathering together Proposition (2.2.9) and Proposition (2.2.10) we deduce the following global result.

Proposition(2.2.11)[96]: Let $p, q, p_0 > 1$ be some exponents satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{p_0}$ and $p_0 > \frac{N}{2}$. Let $\beta > 0$ be any given exponent such that

$$\beta < \frac{2p_0 - N}{p_0}. \tag{81}$$

Then one can find an $\varepsilon = \varepsilon(N, \beta)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^N$ be any (ε, r_0)-Reifenberg flat domain for some $r_0 > 0$, and let u be a solution for the problem (P1) in Ω with $u \in L^p(\Omega)$ and $f \in L^q(\Omega)$. Then

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2 dx \leq C r^{N-2+\beta} \|u\|_p \|f\|_q \forall\, x \in \overline{\Omega}, \forall\, r \in \left(0, \frac{r_0}{6}\right), \tag{82}$$

with $C = C(N, r_0, \beta, p_0)$.

Proof. By Proposition (2.2.9) and Proposition (2.2.10), we already know that (82) holds true for every $x \in \partial\Omega$, or for points x such that $\operatorname{dist}(x, \partial\Omega) \geq r_0/3$. It remains to consider balls centered at points $x \in \Omega$ verifying

$$\operatorname{dist}(x, \partial\Omega) \leq r_0/3.$$

Let x be such a point. Then Proposition (2.2.9) directly says that (82) holds for every radius r such that $0 < r \leq \operatorname{dist}(x, \partial\Omega)/2$, and it remains to extend this for the radii r in the range

$$\operatorname{dist}(x, \partial\Omega)/2 \leq r \leq r_0/6. \tag{83}$$

For this purpose, let $y \in \partial\Omega$ be such that

$$\operatorname{dist}(x, \partial\Omega) = \|x - y\| \leq r_0/3.$$

Denoting $d(x) := \operatorname{dist}(x, \partial\Omega)$ we observe that for the r that satisfies (83) we have

$$B(x, r) \subseteq B(y, r + d(x)) \subseteq B(y, 3r). \tag{84}$$

Then since $y \in \partial\Omega$, Proposition (2.2.10) says that

$$\int_{B(x,r)\cap\Omega} |\nabla u|^2 dx \leq C r^{N-2+\beta} \|u\|_p \|f\|_q \quad \forall\, r \in (0, r_0/2), \tag{85}$$

so that (82) follows, up to change C with $3^{N-2+\beta}C$.

The classical results on Campanato Spaces can be found for instance in [116]. We define the space

$$\mathcal{L}^{p,\lambda}(\Omega) := \left\{u \in L^p(\Omega) ;\ \sup_{x,\rho}\left(\rho^{-\lambda}\int_{B(x,r)\cap\Omega} \left|u - u_{x,r}\right|^p dx\right) < +\infty)\right\}$$

where the supremum is taken over all $x \in \Omega$ and all $\rho \leq \operatorname{diam}(\Omega)$, and where $u_{x,r}$ means the average of u on the ball $B(x, r)$. A proof of the next result can be found in [116].

Theorem(2.2.12)[96]: (Campanato). If $N < \lambda \leq N + p$ then

$$\mathcal{L}^{p,\lambda}(\Omega) \simeq C^{0,\alpha}(\overline{\Omega}), \quad \text{with } \alpha = \frac{\lambda - N}{p}.$$

Corollary(2.2.13)[206]: Let $p, \frac{(1+\varepsilon)p}{p-(\varepsilon+1)}, \varepsilon \geq 0$ be some exponents. Let $\varepsilon < 1$ be any given exponent such that

$$\frac{2\varepsilon^2}{(1-\varepsilon)} > N \tag{86}$$

Then one can find an $\varepsilon = \varepsilon(N)$ such that the following holds. Let $\Omega \in \mathbb{R}^N$ be an $(\varepsilon, r+\varepsilon)$-Reifenberg flat domain for some $r+\varepsilon > 0$, and let the sequence u_i be solutions for the problem (P1) in Ω with $u_i \in L^p(\Omega)$ and $f \in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$. Then

$$u_i \in C^{0,1-\varepsilon}(B(x,(r+\varepsilon)/12) \cap \overline{\Omega}) \quad \forall x \in \overline{\Omega}.$$

Moreover $\|u_i\|_{C^{0,1-\varepsilon}(B(x,\ (r+\varepsilon)/12)\cap\overline{\Omega})} \leq C(N, r, \varepsilon, \|u_i\|_p, \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}})$.

Proof . Considering the sequence u_i as functions of $W^{1,2}(\mathbb{R}^N)$ by setting 0 outside Ω, and applying Corollary(2.2.22) below with $\varepsilon = 0$, we obtain that

$$\int_{B(x,r)} |\nabla u_i|^2 dx \leq C r^{N-\varepsilon} \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} \quad \forall x \in \overline{\Omega}, \quad \forall r \in (0,(r+\varepsilon)/6), \tag{87}$$

with $C = C(N, r, \varepsilon)$. Recalling now the classical Poincaré inequality in a ball $B(x, r)$

$$\int_{B(x,r)} |u_i - (u_i)_{x,r}| dx \leq C(N) r^{1+\frac{N}{2}} \left(\int_{B(x,r)} |\nabla u_i|^2 dx \right)^{\frac{1}{2}},$$

we get

$$\int_{B(x,r)} |u_i - (u_i)_{x,r}| dx \leq C r^{N+1-\frac{\varepsilon}{2}} \quad \forall x \in \overline{\Omega}, \forall r \in (0,(r+\varepsilon)/6), \tag{88}$$

with $C = C\left(N, r, \varepsilon, \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}\right)$. But this implies that

$$u_i \in \mathcal{L}^{1,N+1-\frac{\varepsilon}{2}}(B(x,(r+\varepsilon)/12) \cap \overline{\Omega}) \quad \forall x \in \overline{\Omega}.$$

Moreover $0 \leq \frac{2N}{\varepsilon N + 2\varepsilon^2} < 1$ and hence Theorem (2.2.12) says that

$$u_i \in C^{0,1-\varepsilon}(B(x,(r+\varepsilon)/12) \cap \overline{\Omega})$$

with $\varepsilon = 0$, and the norm is controlled by $C = C\left(N, r, \varepsilon, \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}\right)$.

Corollary(2.2.14)[206]: Assume that $2 \leq N \leq 5$ and let $\frac{(1+\varepsilon)p}{p-(\varepsilon+1)} \in I_N$ be given where

$$I_N = \begin{cases} [2,+\infty) & if \quad N = 2 \\ \left(\frac{2N}{6-N}, +\infty\right) & for \quad 3 \leq N \leq 5. \end{cases}$$

Then for any $\varepsilon < 1$ verifying

$$\varepsilon = \frac{-(6p - Np + 2N) \pm \sqrt{(6p - Np + 2N)^2 - 16p(2p + 2N - 3Np)}}{8p}$$

we can find an $\varepsilon = \varepsilon(N, p)$ such that the following holds. Let $\Omega \in \mathbb{R}^N$ be an $(\varepsilon, r+\varepsilon)$-

Reifenberg-flat domain for some $r+\varepsilon>0$, let $f\in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$ and let $u_i\in W_0^{1,2}(\Omega)$ be the unique solution for the problem (P1) in Ω . Then

$$u_i\in C^{0,1-\varepsilon}(B(x,(r+\varepsilon)/12)\cap\overline{\Omega})\quad\forall x\in\overline{\Omega}.$$

Moreover $\|u_i\|_{C^{0,1-\varepsilon}(B(x,(r+\varepsilon)/12)\cap\overline{\Omega})}\le C(N,r,\varepsilon,p,\|\nabla u_i\|_2,\|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}})$.

Proof. Since $u_i\in W_0^{1,2}(\Omega)$, the Sobolev embedding says that $u_i\in L^p(\Omega)$, with $p=2^*=\frac{2N}{N-2}$.

And by assumption $f\in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}$ for some $\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}\in I_N$ (notice that $\varepsilon\ge\frac{p-2}{p+2}$, which guarantees existence and uniqueness of the weak solution). We now try to apply Corollary (2.2.13) with those p and $\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}$. Let $1+\varepsilon$ be defined .

Then a simple computation yields that $\varepsilon>0$, provided that

$$\varepsilon>\frac{3Np-6p-2N}{-Np+6p-2N}$$

This fixes the range of dimension $N\le 5$ and notice that in this case $N\ge 3$ except for $N=2$, which justifies the definition of I_N. We then conclude by applying Corollary (2.2.13).

Corollary(2.2.15)[206]:

Let $\frac{N-6}{3}<\varepsilon\le N-2$, and $\varepsilon=\frac{-(-2Np+9p+3N)\pm\sqrt{(-2Np+9p+3N)^2-4(3p+N)(-3Np+6p+2N)}}{6p+2N}$ be given, so that moreover $\varepsilon\ge 0$. Then for any $\varepsilon<1$ satisfying

$$\varepsilon=\frac{-(2r^*p-Np+Nr^*)\pm\sqrt{(2r^*p-Np+Nr^*)^2-8(r^*p)(-Np-Nr^*p+Nr^*)}}{4r^*p},$$

$$\text{with}\quad r^*:=\frac{(2+\varepsilon)N}{N-(2+\varepsilon)},$$

we can find an $\varepsilon=\varepsilon(N,|\Omega|,p,r^*)$ such that the following holds. Let $\Omega\subseteq\mathbb{R}^N$ be an $(\varepsilon, r+\varepsilon)$-Reifenberg-flat domain for some $r+\varepsilon>0$, let $f\in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$ and assume that $f=-\mathrm{div}F$ for some $F\in L^{2+\varepsilon}(\Omega)$. Let $u_i\in W_0^{1,2}(\Omega)$ be the unique weak solution for the problem (P1) in Ω. Then

$$u_i\in C^{0,1-\varepsilon}(B(x,(r+\varepsilon)/12)\cap\overline{\Omega})\ \forall x\in\overline{\Omega}.$$

Moreover $\|u_i\|_{C^{0,1-\varepsilon}(B(x,(r+\varepsilon)/12)\cap\overline{\Omega})}\le C(N,r,\varepsilon,p,|\Omega|,\|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}},\|F\|_{2+\varepsilon})$.

Proof. First we apply [76] which provides the existence of a threshold $\varepsilon_0=\varepsilon(N,|\Omega|)$ such that for any solution $u_i\in W_0^{1,2}(\Omega)$ of (P1) with $f=-\mathrm{div}F$ and $F\in L^{2+\varepsilon}(\Omega)$, we have that $\nabla u_i\in L^{2+\varepsilon}(\Omega)$, provided that Ω is $(\varepsilon_0,r+\varepsilon)$-Reifenberg flat. But then the Sobolev inequality implies that $u_i\in L^{r^*}$ with

$$r^*:=\frac{(2+\varepsilon)N}{N-(2+\varepsilon)}\quad\text{if}\ \ \varepsilon<N-2,$$

and $r^* = +\infty$ otherwise.

In order to apply Corollary(2.2.13) we define $1+\varepsilon$ such that

$$\varepsilon = \frac{-p+r^*}{p-r^*},$$

and we only need to check that $\varepsilon > 0$. This implies the following condition on $2+\varepsilon$ and $\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}$:

$$\varepsilon > \frac{N-6}{3}$$

$$\text{and} \quad \varepsilon = \frac{-(9p-2Np+3N) \pm \sqrt{(9p-2Np+3N)^2 - 4(3p+N)(-3Np+6p+2N)}}{6p+2N},$$

as required in the statement of the Corollary. We finally conclude by applying Corollary (2.2.13).

Corollary(2.2.16)[206]: Let u_i be a solution for the problem (P1). Then for every $x_0 \in \partial\Omega$ and a.e. $r > 0$ we have

$$\int_{B(x_0,r)\cap\Omega} |\nabla u_i|^2 |x|^{2-N} dx \le r^{2-N} \int_{\partial B(x_0,r)\cap\Omega} u_i \frac{\partial u_i}{\partial \nu_i} dS + \frac{N-2}{2} r^{1-N} \int_{\partial B(x_0,r)\cap\Omega} {u_i}^2 dS + \int_{B(x_0,r)\cap\Omega} u_i f |x|^{2-N} dx. \quad (89)$$

Proof. Although see (100) below can be formally obtained through an integration by parts, the rigorous proof is a bit technical. In the sequel we use the notation $\Omega_r^+ := B(x_0,r)\cap\Omega$, and $S_r^+ := \partial B(x_0,r)\cap\Omega$.We find it convenient to define, for a given $\varepsilon > 0$, the regularized norm

$$|x|_\varepsilon := \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2 + \varepsilon},$$

so that $|x|_\varepsilon$ is a C^∞ function. A direct computation shows that

$$\Delta(|x|_\varepsilon^{2-N}) = (2-N)N \frac{\varepsilon}{|x|_\varepsilon^{N+2}} \le 0,$$

in other words $|x|_\varepsilon^{2-N}$ is superharmonic, and hopefully enough this goes in the right direction regarding to the next inequalities.

We use one more regularization thus we let $(u_i)_n \in C_c^\infty(\Omega)$ be a sequence of functions converging in $W^{1,2}(\mathbb{R}^N)$ to u_i. We now proceed as in the proof of Alt , Caffarelli and Friedman monotonicity formula [106]: by using the equality

$$\Delta((u_i)_n^2) = 2|\nabla(u_i)_n|^2 + 2(u_i)_n \Delta(u_i)_n \quad (90)$$

we deduce that

$$2\int_{\Omega_r^+} |\nabla(u_i)_n|^2 |x|_\varepsilon^{2-N} = \int_{\Omega_r^+} \nabla((u_i)_n^2)|x|_\varepsilon^{2-N} - 2\int_{\Omega_r^+} ((u_i)_n \nabla(u_i)_n)|x|_\varepsilon^{2-N}. \quad (91)$$

Since $\Delta(|x|_\varepsilon^{2-N}) \le 0$, the Gauss-Green Formula yields

$$\int_{\Omega_r^+} \nabla((u_i)_n^2)|x|_\varepsilon^{2-N} dx = \int_{\Omega_r^+} (u_i)_n^2 \nabla(|x|_\varepsilon^{2-N}) dx + I_{n,\varepsilon}(r) \le I_{n,\varepsilon}(r), \quad (92)$$

where

$$I_{n,\varepsilon}(r) = (r^2+\varepsilon)^{\frac{2-N}{2}} \int_{\partial\Omega_r^+} 2(u_i)_n \frac{\partial (u_i)_n}{\partial \nu_i} dS + (N-2)\frac{r}{(r^2+\varepsilon)^{\frac{N}{2}}} \int_{\partial\Omega_r^+} (u_i)_n^2 \, dS.$$

In other words, (91) reads

$$2\int_{\Omega_r^+} |\nabla (u_i)_n|^2 |x|^{2-N} dx \;\le\; I_{n,\varepsilon}(r) - 2\int_{\Omega_r^+} ((u_i)_n \nabla (u_i)_n)|x|_\varepsilon^{2-N}\, dx. \tag{93}$$

We now want to pass to the limit, first as $n \to +\infty$, and then as $\varepsilon \to 0^+$. To tackle some technical problems, we first integrate over $r \in [r, r+\delta]$ and divide by δ, thus obtaining

$$\frac{2}{\delta}\int_r^{r+\delta} \left(\int_{\Omega_\rho^+} |\nabla (u_i)_n|^2 |x|_\varepsilon^{2-N} dx \right) d\rho \;\le\; A_n - R_n, \tag{94}$$

where

$$A_n = \frac{1}{\delta}\int_r^{r+\delta} I_{n,\varepsilon}(\rho)\, d\rho$$

and

$$R_n = 2\frac{1}{\delta}\int_r^{r+\delta} \left(\int_{\Omega_\rho^+} ((u_i)_n \, \Delta (u_i)_n)|x|_\varepsilon^{2-N} dx \right) d\rho.$$

First, we investigate the limit of A_n as $n \to +\infty$: by applying the coarea formula, we rewrite A_n as

$$A_n = \frac{1}{\delta}\Bigg(2\int_{\Omega_{r+\delta}^+ \setminus \Omega_r^+} (|x|^2+\varepsilon)^{\frac{2-N}{2}} (u_i)_n \, \nabla (u_i)_n \cdot x \, dx + (N-2)\int_{\Omega_{r+\delta}^+ \setminus \Omega_r^+} \frac{|x|}{(|x|^2+\varepsilon)^{\frac{N}{2}}} (u_i)_n^2 \, dx \Bigg).$$

Since $(u_i)_n$ converges to u_i in $W^{1,2}(\mathbb{R}^N)$ when $n \to +\infty$, then by using again the coarea formula we get that

$$A_n \to \frac{1}{\delta}\int_r^{r+\delta} I_\varepsilon(\rho)\, d\rho \qquad n \to +\infty$$

where

$$I_\varepsilon(\rho) = (\rho^2+\varepsilon)^{\frac{2-N}{2}} \int_{\partial\Omega_\rho^+} 2u_i \frac{\partial u_i}{\partial \nu_i} dS + (N-2)\frac{\rho}{(\rho^2+\varepsilon)^{\frac{N}{2}}} \int_{\partial\Omega_\rho^+} {u_i}^2 \, dS.$$

Next, we investigate the limit of R_n as $n \to +\infty$. By using Fubini's Theorem, we can rewrite R_n as

$$R_n = \int_\Omega ((u_i)_n \, \Delta (u_i)_n) G(x)\, dx,$$

where

$$G(x) = |x|_\varepsilon^{2-N} \frac{1}{\delta}\int_r^{r+\delta} \mathbf{1}_{\Omega_\rho^+}(x)\, d\rho.$$

Since

$$\frac{1}{\delta}\int_r^{r+\delta} \mathbf{1}_{\Omega_\rho^+}(x)d\rho = \begin{cases} 1 & \text{if } x \in \Omega_r^+ \\ \dfrac{r+\delta-|x|}{\delta} & \text{if } x \in \Omega_{r+\delta}^+ \setminus \Omega_r^+ \\ 0 & \text{if } x \notin \Omega_{r+\delta}^+ \end{cases},$$

then G is Lipschitz continuous and hence by recalling $(u_i)_n \in C_c^\infty(\Omega)$ we get

$$\left|\int_\Omega ((u_i)_n \Delta(u_i)_n - u_i\Delta u_i)\, G\, dx\right| \le \left|\int_\Omega (\Delta(u_i)_n - \Delta u_i)\,(u_i)_n\, G\, dx\right|$$
$$+\left|\int_\Omega ((u_i)_n - u_i)\, \Delta u_i\, G\, dx\right| = \left|\int_\Omega (\nabla(u_i)_n - \nabla u_i)((u_i)_n \nabla G + \nabla(u_i)_n\, G)\, dx\right|$$
$$+\left|\int_\Omega \nabla u_i\big((\nabla(u_i)_n - \nabla u_i)G + ((u_i)_n - u_i)\,\nabla G\big)\, dx\right|$$
$$\le \|\nabla(u_i)_n - \nabla u_i\|_{L^2(\Omega)}\big(\|(u_i)_n\|_{L^2(\Omega)}\|G\|_{L^\infty(\Omega)} + \|\nabla(u_i)_n\|_{L^2(\Omega)}\|G\|_{L^\infty(\Omega)}\big)$$
$$+\|u_i\|_{L^2(\Omega)}\big(\|\nabla(u_i)_n - \nabla u_i\|_{L^2(\Omega)}\|G\|_{L^\infty(\Omega)} + \|(u_i)_n - u_i\|_{L^2(\Omega)}\|\nabla G\|_{L^\infty(\Omega)}\big)$$

and hence the expression at the first line converges to 0 as $n \to +\infty$.

By combining the previous observations and by recalling that u_i satisfies the equation in the problem (P1) we infer that by passing to the limit $n \to \infty$ in (94) we get (see [21])

$$\frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+} |\nabla u_i|^2 |x|_\varepsilon^{2-N} dx\right) dr \le \frac{1}{\delta}\int_r^{r+\delta} I_\varepsilon(\rho)d\rho + \frac{2}{\delta}\int_r^{r+\delta}\left(\int_{\Omega_\rho^+} u_i\, f\, |x|_\varepsilon^{2-N} dx\right) d\rho.$$

Finally, dividing by 2, by passing to the limit $\delta \to 0^+$, and then $\varepsilon \to 0^+$ we obtain (100) below.

Corollary(2.2.17)[206]: Let $\gamma > 0$, $r+\varepsilon > 0$, $\psi : (0, r+\varepsilon) \to \mathbb{R}$ be a continuous function and $\varphi : (0, r+\varepsilon) \to \mathbb{R}$ be an absolutely continuous function that satisfies the following inequality for a.e. $r \in (0, r+\varepsilon)$,

$$\varphi(r) \le \gamma\, r\, \varphi'(r) + \psi(r). \tag{95}$$

Then

$$r \mapsto \frac{\varphi(r)}{r^{\frac{1}{\gamma}}} + \frac{1}{\gamma}\int_0^r \frac{\psi(s)}{s^{1+\frac{1}{\gamma}}} ds$$

is a nondecreasing function on $(0, r+\varepsilon)$.

Proof. We can assume that

$$\int_0^{r+\varepsilon} \frac{\psi(s)}{s^{1+\frac{1}{\gamma}}} ds < +\infty,$$

otherwise the Lemma is trivial. Under the hypothesis, the function

$$F(r) := \frac{\varphi(r)}{r^{\frac{1}{\gamma}}} + \frac{1}{\gamma}\int_0^r \frac{\psi(s)}{s^{1+\frac{1}{\gamma}}} ds$$

is differentiable a.e. and absolutely continuous. A computation gives

$$F'(r) = \frac{\varphi'(r) r^{\frac{1}{\gamma}} - \varphi(r) \frac{1}{\gamma} r^{\frac{1}{\gamma}-1}}{r^{\frac{2}{\gamma}}} + \frac{1}{\gamma} \frac{\psi(r)}{r^{1+\frac{1}{\gamma}}}$$

$$= \frac{\varphi'(r) - \varphi(r) \frac{1}{\gamma} r^{-1}}{r^{\frac{1}{\gamma}}} + \frac{1}{\gamma} \frac{\psi(r)}{r^{1+\frac{1}{\gamma}}}$$

thus (95) yields

$$r\,\gamma F'(r) = \frac{r\,\gamma\,\varphi'(r) - \varphi(r) + \psi(r)}{r^{\frac{1}{\gamma}}} \geq 0,$$

which implies that F is nondecresing.

Corollary(2.2.18)[206]: Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let u_i be a sequence of solutions for the problem (P1). Given $x_0 \in \overline{\Omega}$ and a radius $r > 0$, we denote by $\Omega_r^+ := B(x_0, r) \cap \Omega$, by $S_r^+ := \partial B(x_0, r) \cap \Omega$ and by $\sigma(r)$ the first Dirichlet eigenvalue of the Laplace operator on the spherical domain S_r^+. If there are constants $r + \varepsilon > 0$ and $\sigma^* \in (0, N-1)$ such that

$$\inf_{0<r<r+\varepsilon} (r^2 \sigma(r)) \geq \sigma^*, \tag{96}$$

then the function

$$\left(r \mapsto \left(\frac{1}{r^{(2-\varepsilon)}} \int_{\Omega_r^+} \frac{|\nabla u_i|^2}{|x - x_0|^{N-2}} dx\right) + (2-\varepsilon) \int_0^r \frac{\psi(s)}{s^{3-\varepsilon}} ds \right. \tag{97}$$

is non decreasing on $(0, r + \varepsilon)$, where $0 < \varepsilon < 2$ is given by

$$\varepsilon = -\sqrt{(N-2)^2 + 4\sigma^*} + N$$

and

$$\psi(s) := \int_{\Omega_s^+} |u_i\, f| |x - x_0|^{2-N} dx\,.$$

We also have the bound

$$\int_{\Omega_{r+\varepsilon/2}^+} \frac{|\nabla u_i|^2}{|x - x_0|^{N-2}} dx \leq C(N, r, \varepsilon) \|\nabla u_i\|_{L^2(\Omega)}^2 + \psi(r + \varepsilon). \tag{98}$$

Notice that**.** Of course the Lemma is interesting only when

$$\int_0^r \frac{\psi(s)}{s^{3-\varepsilon}} ds < +\infty. \tag{99}$$

This will be satisfied if u_i and f are in some L^p spaces with suitable exponents, as will be shown in Corollary(2.2.19).

Note that it is not known in general whether $\nabla u_i \in L^{2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ and therefore it is not obvious to find a bound for the left hand side of (98).

Proof. We assume without lose of generality that $x_0 = 0$ and to simplify notation we denote by B_r the ball $B(x, r)$.

By Corollary(2.2.16) we know that for a.e. $r > 0$,

$$\int_{\Omega_r^+} |\nabla u_i|^2 |x|^{2-N} dx \le r^{2-N} \int_{S_r^+} 2u_i \frac{\partial u_i}{\partial \nu_i} dS + \frac{(N-2)}{2} r^{1-N} \int_{S_r^+} u_i^2 \, dS + \int_{\Omega_r^+} u_i \, f \, |x|^{2-N} dx. \tag{100}$$

Let us define

$$\psi(r) := \int_{\Omega_r^+} |u_i \, f| |x|^{2-N} dx. \tag{101}$$

and assume that (99) holds (otherwise there is nothing to prove).

Next, we point out that the definition of σ^* mimplies that

$$\int_{S_r^+} u_i^2 \, dS \le \frac{1}{\sigma^*} r^2 \int_{S_r^+} |\nabla_{\mathcal{T}} u_i|^2 dS \quad r \in (0, r+\varepsilon), \tag{102}$$

where $\nabla_{\mathcal{T}}$ denotes the tangential gradient on the sphere. Also, let $\varepsilon < 1$ be a parameter that will be fixed later, then by combining Cauchy-Schwarz inequality, (102) and the inequality $ab \le \frac{(1-\varepsilon)}{2} a^2 + \frac{1}{2(1-\varepsilon)} b^2$, we get

$$\begin{aligned} \left| \int_{S_r^+} u_i \frac{\partial u_i}{\partial \nu_i} dS \right| &\le \left(\int_{S_r^+} u_i^2 \, dS \right)^{\frac{1}{2}} \left(\int_{S_r^+} \left| \frac{\partial u_i}{\partial \nu_i} \right|^2 dS \right)^{\frac{1}{2}} \\ &\le r/\sqrt{\sigma^*} \left(\int_{S_r^+} |\nabla_{\mathcal{T}} u_i|^2 dS \right)^{\frac{1}{2}} \left(\int_{S_r^+} \left| \frac{\partial u_i}{\partial \nu_i} \right|^2 dS \right)^{\frac{1}{2}} \\ &\le r/\sqrt{\sigma^*} \left(\frac{1-\varepsilon}{2} \int_{S_r^+} |\nabla_{\mathcal{T}} u_i|^2 dS + \frac{1}{2(1-\varepsilon)} \int_{S_r^+} \left| \frac{\partial u_i}{\partial \nu_i} \right|^2 dS \right) \end{aligned} \tag{103}$$

Hence,

$$\begin{aligned} & r^{2-N} \int_{S_r^+} 2u_i \frac{\partial u_i}{\partial \nu_i} dS + (N-2) r^{1-N} \int_{S_r^+} u_i^2 \, dS \\ & \le r^{2-N} \frac{2r}{\sqrt{\sigma^*}} \left[\frac{1-\varepsilon}{2} \int_{S_r^+} |\nabla_{\mathcal{T}} u_i|^2 dS + \frac{1}{2(1-\varepsilon)} \int_{S_r^+} \left| \frac{\partial u_i}{\partial \nu_i} \right|^2 dS \right] \\ & + (N-2) r^{1-N} \frac{1}{\sigma^*} r^2 \int_{S_r^+} |\nabla_{\mathcal{T}} u_i|^2 dS \\ & \le r^{1-N} \left[\left(\frac{1-\varepsilon}{\sqrt{\sigma^*}} + \frac{N-2}{\sigma^*} \right) \int_{S_r^+} |\nabla_{\mathcal{T}} u_i|^2 dS + \frac{1}{(1-\varepsilon)\sqrt{\sigma^*}} \int_{S_r^+} \left| \frac{\partial u_i}{\partial \nu_i} \right|^2 dS \right]. \end{aligned} \tag{104}$$

Next, we choose $\varepsilon < 1$ in such a way that

$$\frac{1-\varepsilon}{\sqrt{\sigma^*}} + \frac{N-2}{\sigma^*} = \frac{1}{(1-\varepsilon)\sqrt{\sigma^*}}.$$

namely

$$\varepsilon = -\frac{1}{2\sqrt{\sigma^*}} \left[\sqrt{(N-2)^2 + 4\sigma^*} - (N-2) \right] + 1.$$

Hence, by combining (100), (102) and (104) we finally get

$$\int_{\Omega_r^+} |\nabla u_i|^2 |x|^{2-N} dx \le r^{3-N} \gamma(N, \sigma^*) \int_{S_r^+} |\nabla u_i|^2 dS + \psi(r), \tag{105}$$

where

$$\gamma(N, \sigma^*) = \left[\sqrt{(N-2)^2 + 4\sigma^*} - (N-2)\right]^{-1}.$$

Let us set

$$\varphi(r) := \int_{\Omega_r^+} |\nabla u_i|^2 |x|^{2-N} dx$$

and observe that

$$\varphi'(r) = r^{2-N} \int_{S_r^+} |\nabla u_i|^2 \quad a.e.\, r \in (0, r+\varepsilon),$$

hence (105) implies that

$$\varphi(r) \le \gamma r \varphi'(r) + \psi(r), \tag{106}$$

with $\gamma = \gamma(N, \sigma^*)$.

But now Corollary(2.2.17) exactly says that the function

$$r \longmapsto \frac{1}{r^{(2-\varepsilon)}} \int_{\Omega_r^+} \frac{|\nabla u_i|^2}{|x-x_0|^{N-2}} dx + (2-\varepsilon) \int_0^r \frac{\psi(s)}{s^{3-\varepsilon}} ds \tag{107}$$

is non decreasing on $(0, r+\varepsilon)$, where $0 < \varepsilon < 2$ is given by

$$\varepsilon = 2 - \frac{1}{\gamma} = -\sqrt{(N-2)^2 + 4\sigma^*} + N,$$

which proves the monotonicity result.

To finish the proof of the Lemma it remains to establish (98). For this purpose, we start by finding a radius $r_1 \in (\frac{r+\varepsilon}{2}, r+\varepsilon)$ such that $\int_{S_{r_1}^+} |\nabla u_i|^2 ds$ is less than average, which means

$$\int_{S_{r_1}^+} |\nabla u_i|^2 ds \le \frac{2}{r+\varepsilon} \int_{\Omega_\varepsilon^+} |\nabla u_i|^2 dx \le \frac{2}{r+\varepsilon} \|\nabla u_i\|^2_{L^2(\Omega)}.$$

By combining (105) with the fact that $\frac{r+\varepsilon}{2} \le r_1 \le r+\varepsilon$ we infer that

$$\int_{\Omega_{r_1}^+} |\nabla u_i|^2 |x|^{2-N} dx \le C(N, r, \varepsilon) \|\nabla u_i\|^2_{L^2(\Omega)} + \psi(r_1)$$
$$\le C(N, r, \varepsilon) \|\nabla u_i\|^2_{L^2(\Omega)} + \psi(r+\varepsilon). \tag{108}$$

It follows that

$$\int_{\Omega^+_{\frac{r+\varepsilon}{2}}} \frac{|\nabla u_i|^2}{|x|^{N-2}} dx \le \int_{\Omega_{r_1}^+} \frac{|\nabla u_i|^2}{|x|^{N-2}} \le C(N, r, \varepsilon) \|\nabla u_i\|^2_{L^2(\Omega)} + \psi(r+\varepsilon),$$

and (98) is proved.

Corollary(2.2.19)[206]: Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain and let $\varepsilon > 0$. Then for any $g \in L^{\frac{N}{2}+\varepsilon}(\Omega)$, denoting

$$\psi(r) := \int_{B(0,r)} |g| |x|^{2-N} dx,$$

we have

$$|\psi(r)| \leq C(N,\varepsilon)\|g\|_{\frac{N}{2}+\varepsilon} \quad r^{2-\frac{N}{1+\varepsilon}}. \tag{109}$$

As a consequence

$$\int_0^{r+\varepsilon} \frac{\psi(s)}{s^{3-\varepsilon}} ds \ < \ +\infty$$

for any $\varepsilon < 2$ satisfying

$$\varepsilon\,(1+\varepsilon) \ > \ N. \tag{110}$$

Proof. First, we observe that $|x|^{2-N} \in L^m$ for any m that satisfies

$$(N-2)m < N \ \Rightarrow \ m \ < \frac{N}{N-2}.$$

Moreover a computation gives that under this condition,

$$\||x|^{2-N}\|^m_{L^m(B(0,r))} = \int_{B(0,r)} |x|^{(2-N)m} = C(N,m)r^{N-m(N-2)}. \tag{111}$$

Let us define m as being the conjugate exponent of $\frac{N}{2}+\varepsilon$, namely

$$m := \frac{N+2\varepsilon}{N+2(\varepsilon-1)}.$$

Then the hypothesis $\varepsilon > 0$, and by use of Hölder inequality and by (111) we can estimate

$$|\psi(r)| \leq \|g\|_{\frac{N}{2}+\varepsilon}\||x|^{2-N}\|_{L^m(B(0,r))} \leq C(N,m)\|g\|_{\frac{N}{2}+\varepsilon} \ r^{\frac{N-m(N-2)}{m}},$$

or equivalently in terms of $\frac{N}{2}+\varepsilon$,

$$|\psi(r)| \leq C(N,\varepsilon)\|g\|_{\frac{N}{2}+\varepsilon} \ r^{\frac{4\varepsilon}{N+2\varepsilon}}. \tag{112}$$

The conclusion of the Lemma follows directly from (112).

Corollary(2.2.20)[206]: Let $p, \frac{(1+\varepsilon)p}{p-(\varepsilon+1)}, \varepsilon > 0$ be some exponents . Let $\varepsilon < 2$ be any given exponent such that

$$\frac{2\varepsilon^2}{2-\varepsilon} \ > \ N. \tag{113}$$

Let $\Omega \subseteq \mathbb{R}^N$ be any domain, $x_0 \in \Omega$, and let u_i be a sequence of solutions for the problem (P1) in Ω with $u_i \in L^p(\Omega)$ and $f \in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$. Then

$$\int_{B(x_0,r)\cap\Omega} |\nabla u_i|^2 dx \leq Cr^{N-\varepsilon}\|u_i\|_p\|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} \quad \forall\, r \in (0, dist(x_0,\partial\Omega)/2)\,, \tag{114}$$

with $C = C(N,\varepsilon,\mathrm{dist}(x_0,\partial\Omega))$.

Proof. We assume that $x_0 = 0$ and we define $r+\varepsilon := \mathrm{dist}(x_0,\partial\,\Omega)$ in such a way that $B(x,r)$ is totally contained inside Ω for $\varepsilon \geq 0$ and under the notation of Corollary (2.2.18), $\Omega_r^+ = B(0,r)$ and by $S_r^+ = \partial B(0,r)$ for any $\varepsilon \geq 0$. Under our hypothesis and in virtue of Corollary (2.2.19) that we apply with $g = u_i\, f$, we know that (99) holds for any $\varepsilon < 2$ that satisfies

$$\frac{2\varepsilon^2}{2-\varepsilon} \ > \ N. \tag{115}$$

Notice that $\varepsilon > 0$.

In the sequel we chose any exponent $\varepsilon < 2$ satisfying (115), so that (99) holds and moreover Corollary (2.2.19) says that

$$\int_0^{r+\varepsilon} \frac{\psi(s)}{s^{3-\varepsilon}}\, ds \le C(N,\varepsilon)\|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} \int_0^{r+\varepsilon} s^{\frac{4\varepsilon}{N+2\varepsilon}-3+\varepsilon}\, ds$$
$$\le C(N,\varepsilon)\|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}. \tag{116}$$

We are now ready to prove (114), ε still being a fixed exponent satisfying (115). We recall that the first eigenvalue of the spherical Dirichlet Laplacian on the unit sphere is equal to $N-1$, thus hypothesis (96) in the present context reads

$$\inf_{0<r<r+\varepsilon}\left(r^2\sigma(r)\right) = N-1, \tag{117}$$

so that (96) holds for any $\sigma^* < N-1$. Let us choose σ^* exactly equal to the one that satisfies

$$\varepsilon = -\sqrt{(N-2)^2 + 4\sigma^*} + N.$$

one easily verifies that $\sigma^* < N-1$. because of (115).

As a consequence, we are in position to apply Corollary (2.2.18) which ensures that, if u_i are solutions for the problem (P1), then the function in (107) is non decreasing. In particular, by monotonicity we know that for every $\varepsilon \ge 0$,

$$\frac{1}{r^{N-\varepsilon}}\int_{\Omega_r^+} |\nabla u_i|^2\, dx \le \left(\frac{1}{r^{2-\varepsilon}}\int_{\Omega_r^+} \frac{|\nabla u_i|^2}{|x-x_0|^{N-2}}\, dx\right) + (2-\varepsilon)\int_0^r \frac{\psi(s)}{s^{3-\varepsilon}}\, ds$$
$$\le \left(\frac{1}{\left(\frac{r+\varepsilon}{2}\right)^{2-\varepsilon}}\int_{\Omega^+_{\frac{r+\varepsilon}{2}}} \frac{|\nabla u_i|^2}{|x-x_0|^{N-2}}\, dx\right)$$
$$+ (2-\varepsilon)\int_0^{\frac{r+\varepsilon}{2}} \frac{\psi(s)}{s^{3-\varepsilon}}\, ds. \tag{118}$$

and we conclude that for every $\varepsilon \ge 0$,

$$\int_{\Omega_r^+} |\nabla u_i|^2 dx \le K r^{N-\varepsilon},$$

with

$$K = \left(\frac{1}{(r+\varepsilon)^{2-\varepsilon}}\int_{\Omega^+_{r+\varepsilon}} \frac{|\nabla u_i|^2}{|x-x_0|^{N-2}} dx\right) + (2-\varepsilon)\int_0^{r+\varepsilon} \frac{\psi(s)}{s^{3-\varepsilon}} ds,$$

Let us now provide an estimate on K. To estimate the first term in K we use (98) to write

$$\int_{\Omega^+_{r+\varepsilon}} \frac{|\nabla u_i|^2}{|x-x_0|^{N-2}} dx \le C(N,r,\varepsilon)\|\nabla u_i\|^2_{L^2(\Omega)} + \psi(2(r+\varepsilon)) \tag{119}$$

Then we use (109) to estimate

$$\psi(2(r+\varepsilon)) \le C(N,\varepsilon,r)\|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}},$$

and from the equation satisfied by u_i we get

$$\|\nabla u_i\|^2_{L^2(\Omega)} = \int_\Omega u_i\, f dx \le \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}$$

so that in total we have

$$\int_{\Omega^+_{r+\varepsilon}} \frac{|\nabla u_i|^2}{|x-|^{N-2}} dx \le C(N,r,\varepsilon)\|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}.$$

Finally, the last estimate together with (116) yields

$$K \le C(N,r,\varepsilon)\|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}},$$

and this ends the proof of the Proposition.

Corollary(2.2.21)[206]: Let $p, \frac{(1+\varepsilon)p}{p-(\varepsilon+1)}, \varepsilon > 0$ be some exponents. Let $\varepsilon < 2$ be any given exponent such that

$$\frac{2\varepsilon^2}{2-\varepsilon} > N. \tag{120}$$

Then one can find an $\varepsilon = \varepsilon(N)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^N$ be any $(\varepsilon, r+\varepsilon)$ Reifenberg flat domain for some $r+\varepsilon > 0$, let $x_0 \in \partial\Omega$ and let u_i be a sequence of solutions for the problem (P1) in Ω with $u_i \in L^p(\Omega)$ and $f \in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$. Then

$$\int_{B(x_0,r)\cap\Omega} |\nabla u_i|^2 dx \le C r^{N-\varepsilon} \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} \quad \forall\, r \in \left(0, \frac{r+\varepsilon}{2}\right), \tag{121}$$

with $C = C(N,r,\varepsilon)$.

Proof. As before we assume that $x_0 = 0$ and we denote by $\Omega^+_r := B(0,r)\cap\Omega$ and by $S^+_r := \partial B(0,r)\cap\Omega$. To obtain the decay estimate on $\int_{\Omega^+_r} |\nabla u_i|^2 dx$ we will follow the proof of Corollary (2.2.20): the main difference is that for boundary points, (117) does not hold. This is where Reifenberg-flatness will play a role.

Let $\varepsilon < 2$, be an exponent satisfying (120), so that invoquing Corollary (2.2.19) we have

$$\int_0^{r+\varepsilon} \frac{\psi(s)}{s^{3-\varepsilon}} ds \le C(N,\varepsilon)\|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} < +\infty. \tag{122}$$

Next, we recall that the first eigenvalue of the spherical Dirichlet Laplacian on a half sphere is equal to $N-1$ (as for the total sphere). For $t \in (-1,1)$, let S_t be the spherical cap $S_t := \partial B(0,1) \cap \{x_N > t\}$ so that $t = 0$ corresponds to a half sphere. Let $\lambda_1(S_t)$ be the first Dirichlet eigenvalue in S_t . In particular, $t \longmapsto \lambda_1(S_t)$ is continuous and monotone in t. Therefore, since and $\lambda_1(S_t) \to 0$ as $t \downarrow -1$, there is $t^*(2-\varepsilon) < 0$ such that

$$\varepsilon = -\sqrt{(N-2)^2 + 4\lambda_1(t^*)} + N$$

By applying the definition of Reifenerg flat domain, we infer that, if $\varepsilon < t^*(\eta)/2$, then $\partial B(x_0,r)\cap\Omega$ is contained in a spherical cap homothetic to S_{t^*} for every $\varepsilon \ge 0$. Since the eigen-values scale of by factor r^2 when the domain expands of a factor $1/r$, by the monotonicity property of the eigenvalues with respect to domains inclusion, we have

$$\inf_{r<r+\varepsilon} r^2\lambda_1(\partial B(x_0,r)\cap\Omega) \ge \lambda_1(S_{t^*}) = (2-\varepsilon)\left(\frac{2N-\varepsilon-2}{4}\right). \tag{123}$$

As a consequence, we are in position to apply the monotonicity Corollary (Corollary (2.2. 18)) which ensures that, if u_i is a sequence of solutions for the problem (P1) and $x_0 \in \partial\Omega$, then the function in (107) is non decreasing. We then conclude as in the proof of Corollary (2.2.20), i.e. by monotonicity we know that for every $0 \le \varepsilon < 1 - r$,

$$\frac{1}{r^{N-\varepsilon}} \int_{\Omega_r^+} |\nabla u_i|^2 \, dx \le \left(\frac{1}{r^{2-\varepsilon}} \int_{\Omega_r^+} \frac{|\nabla u_i|^2}{|x - x_0|^{N-2}} \, dx \right) + (2-\varepsilon) \int_0^r \frac{\psi(s)}{s^{3-\varepsilon}} \, ds$$
$$\le \left(\frac{1}{\left(\frac{r+\varepsilon}{2}\right)^{2-\varepsilon}} \int_{\Omega^+_{\frac{r+\varepsilon}{2}}} \frac{|\nabla u_i|^2}{|x - x_0|^{N-2}} \, dx \right)$$
$$+ (2-\varepsilon) \int_0^{\frac{(r+\varepsilon)}{2}} \frac{\psi(s)}{s^{3-\varepsilon}} \, ds. \tag{124}$$

hence for every $\varepsilon \ge 0$,

$$\int_{\Omega_r^+} |\nabla u_i|^2 dx \le K r^{N-\varepsilon},$$

with

$$K = \left(\frac{1}{(r+\varepsilon)^{2-\varepsilon}} \int_{\Omega^+_{(r+\varepsilon)}} \frac{|\nabla u_i|^2}{|x - x_0|^{N-2}} dx \right) + (2-\varepsilon) \int_0^{(r+\varepsilon)} \frac{\psi(s)}{s^{3-\varepsilon}} ds,$$

Then we estimate K exactly as in the end of the proof of Corollary (2.2.20), using (98), (109) and (122) to bound

$$K \le C(N, r, \varepsilon) \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}},$$

and this ends the proof of the Proposition.

Corollary(2.2.22)[206]: Let $p, \frac{(1+\varepsilon)p}{p-(\varepsilon+1)}, \varepsilon > 0$ be some exponents. Let $\varepsilon < 2$ be any given exponent such that

$$\frac{2\varepsilon^2}{2-\varepsilon} > N. \tag{125}$$

Then one can find an $\varepsilon = \varepsilon(N)$ such that the following holds. Let $\Omega \subseteq \mathbb{R}^N$ be any $(\varepsilon, r + \varepsilon)$-Reifenberg-flat domain for some $r + \varepsilon > 0$, and let u_i be a sequence of solutions for the problem (P1) in Ω with $u_i \in L^p(\Omega)$ and $f \in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$. Then

$$\int_{B(x,r)\cap\Omega} |\nabla u_i|^2 dx \le C r^{N-\varepsilon} \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} \quad \forall\, x \in \overline{\Omega}, \forall\, r \in \left(0, \frac{r+\varepsilon}{6}\right), \tag{126}$$

with $C = C(N, r, \varepsilon)$.

Proof. By Corollary (2.2.20) and Corollary (2.2.21), we already know that (126) holds true for every $x \in \partial\Omega$, or for points x such that $\operatorname{dist}(x, \partial\Omega) \ge (r + \varepsilon)/3$. It remains to consider balls centered at points $x \in \Omega$ verifying

$$\operatorname{dist}(x, \partial\Omega) \le (r + \varepsilon)/3.$$

Let x be such a point. Then Corollary (2.2.20) directly says that (126) holds for every radius r such that $0 < r \le \operatorname{dist}(x, \partial\Omega)/2$, and it remains to extend this for the radii r in the range

$$\operatorname{dist}(x, \partial\Omega)/2 \le r \le (r + \varepsilon)/6. \tag{127}$$

For this purpose, let $y \in \partial\Omega$ be such that

$$\mathrm{dist}(x, \partial\Omega) = \|x - y\| \leq (r + \varepsilon)/3.$$

Denoting $d(x) := \mathrm{dist}(x, \partial\Omega)$ we observe that for the r that satisfies (127) we have

$$B(x, r) \subseteq B(y, r + d(x)) \subseteq B(y, 3r). \tag{128}$$

Then since $y \in \partial\Omega$, Corollary (2.2.21) says that

$$\int_{B(x,r)\cap\Omega} |\nabla u_i|^2 dx \leq C r^{N-\varepsilon} \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}} \quad \forall\, r \in (0, (r + \varepsilon)/2), \tag{129}$$

so that (126) follows, up to change C with $3^{N-\varepsilon}C$.

Corollary(2.2.23)[206]: Let $p, \frac{(1+\varepsilon)p}{p-(\varepsilon+1)}, \varepsilon \geq 0$ be some exponents. Let $0 \leq \varepsilon < 1$ be any given prescribe exponent such that

$$\frac{2\varepsilon^2}{(1 - \varepsilon)} > N \tag{130}$$

Then we can find an $\varepsilon = \varepsilon(N)$ such that for $\Omega \in \mathbb{R}^N$ and $(\varepsilon, r^2 + \varepsilon)$-Reifenberg flat domain for some $\varepsilon > -r^2$, and let the sequence u_i be solutions for (P1) in Ω such that $\sum_i u_i \in L^p(\Omega)$ and $f \in L^{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}(\Omega)$. Then

$$\sum_i u_i \in \tilde{C}^{0,1-\varepsilon}(B(x, (r^2 + \varepsilon)/12) \cap \overline{\Omega}) \qquad \forall x \in \overline{\Omega}.$$

Hence

$$\sum_i \|u_i\|_{\tilde{C}^{0,1-\varepsilon}(B(x,(r^2+\varepsilon)/12)\cap\overline{\Omega})} \leq \tilde{C}\left(N, r^2, \varepsilon, \sum_i \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}\right)$$

Proof. In spite of Corollary (2.2.13) we have

$$\int_{B(x,r^2)} \sum_i |\nabla u_i|^2\, dx \leq \tilde{C}\, r^{2(N-\varepsilon)} \sum_i \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}$$
$$\forall x \in \overline{\Omega}, r^2 \in (0, (r^2 + \varepsilon)/6), \tag{131}$$

with $\tilde{C} = \tilde{C}\,(N, r^2, \varepsilon)$. By the classical Poincaré inequality in a ball $B(x, r^2)$

$$\int_{B(x,r^2)} \sum_i |u_i - (u_i)_{x,r^2}|\, dx \leq \tilde{C}(N)\, r^{2\left(1+\frac{N}{2}\right)} \left(\int_{B(x,r^2)} \sum_i |\nabla u_i|^2\, dx\right)^{\frac{1}{2}}$$
$$\leq \tilde{C}\, r^{2\left(N+1-\frac{\varepsilon}{2}\right)} \quad \forall x \in \overline{\Omega},\ r^2 \in (0, (r^2 + \varepsilon)/6), \tag{132}$$

with $\tilde{C} = \tilde{C}\left(N, r^2, \varepsilon, \sum_i \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}\right)$. which implies that

$$\sum_i u_i \in \mathcal{L}^{1,N+1-\frac{\varepsilon}{2}}(B(x, (r^2 + \varepsilon)/12) \cap \overline{\Omega}) \quad \forall x \in \overline{\Omega}.$$

So $0 \leq \frac{2N}{\varepsilon N + 2\varepsilon^2} < 1$ and hence Theorem (2.2.12) gives that

$$\sum_i u_i \in \tilde{C}^{0,1-\varepsilon}(B(x, (r^2 + \varepsilon)/12) \cap \overline{\Omega})$$

where $\tilde{C} = \tilde{C}\left(N, r^2, \varepsilon, \sum_i \|u_i\|_p \|f\|_{\frac{(1+\varepsilon)p}{p-(\varepsilon+1)}}\right)$.