Chapter 1

Approximation of Ground State and Properties of Convergence

In this chapter we show some approximations of eigenvalues and eigenfunctions, and the first result in full generality in the direction of Bouleau-Hirsch conjecture. Moreover, in multivariate settings, we study the particular case of Sobolev spaces : we show that a convergence for the Sobolev norm $W^{1;p}(\mathbb{R}^d;\mathbb{R}^p)$ toward a non-degenerate limit entails convergence of push-forward measures in the total variation topology.

Sec(1.1): Eigenvalues and Eigenfunctions of Dirichlet Laplacians

Let Ω be a bounded connected open set in $\mathbb{R}^N$, $N \geq 2$, and let $-\Delta_\Omega \geq 0$ be the Dirichlet Laplacian defined in $L^2(\Omega)$. Let $\lambda_\Omega > 0$ be the smallest eigenvalue of $-\Delta_\Omega$, and let $\phi_\Omega > 0$ be its corresponding eigenfunction, normalized by $\|\phi_\Omega\|_2 = 1$. For sufficiently small $\varepsilon > 0$ we let $R(\varepsilon)$ be a connected open subset of Ω satisfying

$$R(\varepsilon) \supseteq \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \varepsilon\}. \tag{1}$$

Let $-\Delta_\varepsilon \geq 0$ be the Dirichlet Laplacian on $R(\varepsilon)$, and let $\lambda_\varepsilon > 0$ and $\phi_\varepsilon > 0$ be its ground state eigenvalue and ground state eigenfunction, respectively, normalized by $\|\phi_\varepsilon\|_2 = 1$. For functions f defined on Ω, we let $S_\varepsilon f$ denote the restriction of f to $R(\varepsilon)$.
For functions g defined on $R(\varepsilon)$, we let $T_\varepsilon\, g$ be the extension of g to Ω satisfying

$$(T_\varepsilon g)(z) = \begin{cases} g(z) & \text{if } z \in R(\varepsilon) \\ 0 & \text{if } z \in \Omega \setminus R(\varepsilon). \end{cases}$$

In this section we show the following.

Theorem(1.1.1)[1]: Suppose that Ω is a bounded simply connected open set in $\mathbb{R}^2$. Let

$$R = \sup\{\text{dist}(z, \partial\Omega) : z \in \Omega\}$$

be the inradius of Ω, let $|\Omega|$ be the area of Ω, and let $\lambda_\Omega < E_2$ be the first and second smallest eigenvalues of $-\Delta_\Omega$. Let $\gamma < \Gamma$ be the first and second smallest eigenvalues of the Dirichlet Laplacian on the unit disc in $\mathbb{R}^2$.

(i) For all $\varepsilon \in (0, 2^{-1}R)$, we have

$$|\lambda_\varepsilon - \lambda_\Omega| \leq C_1 \varepsilon^{1/\varepsilon} \tag{2}$$

where

$$C_1 = (512/3)\pi^{-9/4}\gamma^4 R^{-7} |\Omega|^{9/4} \tag{3}$$

(ii) For all $\varepsilon \in (0, 2^{-1}R)$, we have

$$\|\phi_\Omega - T_\varepsilon \phi_\varepsilon\|_\infty \leq \left[C_2 + C_3(E_2 - \lambda_\Omega)^{-1/2} + C_4(E_2 - \lambda_\Omega)^{-1}\right]\varepsilon^{1/2} \tag{4}$$

Where

$$\begin{aligned} C_2 &= 4096\pi^{-15/4}\gamma^5 R^{-8} |\Omega|^{13/4}, \\ C_3 &= 88301\gamma^7\Gamma\pi^{-21/4}R^{-12}|\Omega|^{19/4} + 9812\gamma^6\Gamma\pi^{-9/2}R^{-21/2}|\Omega|^4, \\ C_4 &= 54146252\gamma^{12}\Gamma^2\pi^{-9}R^{-41/2}|\Omega|^8 + 668473\gamma^{10}\Gamma^2\pi^{-15/2}R^{-35/2}|\Omega|^{13/2} \\ &\quad + 12032501\gamma^{11}\Gamma^2\pi^{-33/4}R^{-19}|\Omega|^{29/4} \end{aligned} \tag{5}$$

Proof : We first show (2). For $\varepsilon \in (0, 2^{-1}R)$, we have, by (28) and (36),

$$\begin{aligned} \left|\left\langle -\Delta_\Omega^{-1}\phi_\varepsilon, \phi_\varepsilon \right\rangle - \lambda_\varepsilon^{-1}\right| &= \left|\left\langle [(-\Delta_\Omega^{-1}) - (-\Delta_\varepsilon^{-1})]\phi_\varepsilon, \phi_\varepsilon \right\rangle\right| \\ &\le \left\|(-\Delta_\Omega^{-1}) - T_\varepsilon(-\Delta_\varepsilon^{-1})S_\varepsilon\right\|_{L^1\to L^\infty} \lambda_\varepsilon^2 \left\|(-\Delta_\varepsilon^{-2})\right\|_{L^1\to L^\infty} \|\phi_\varepsilon\|_1^2 \\ &\le K_1\varepsilon^{1/2} 4R^{-4}\gamma^2 K_2 |\Omega| \\ &= K_3\varepsilon^{1/2}, \end{aligned} \tag{6}$$

where

$$K_3 = \left(\frac{256}{3}\right)\gamma^2\pi^{-3/4}R^{-3}|\Omega|^{9/4}.$$

The minimax principle and (6) now give

$$\begin{aligned} \lambda_\varepsilon^{-1} \le \lambda_\Omega^{-1} &\le \left\langle -\Delta_\Omega^{-1}\phi_\varepsilon, \phi_\varepsilon \right\rangle \\ &\le \lambda_\varepsilon^{-1} + K_3\,\varepsilon^{1/2}, \end{aligned}$$

which implies (2) since

$$\lambda_\varepsilon\lambda_\Omega \le 2R^{-4}\gamma^2.$$

Next we let $E_2 \le E_3 \le \cdots$ be the higher eigenvalues of $-\Delta_\Omega$, with corresponding eigen functions $\psi_2, \psi_3, \cdots$, normalized by $\|\psi_i\|_2 = 1$. Let

$$\phi_\varepsilon = \mu_\varepsilon\phi_\Omega + \sum_{n=2}^{\infty}\sigma_n(\varepsilon)\psi_n = \mu_\varepsilon\phi_\Omega + \tilde{\phi}_\varepsilon. \tag{7}$$

Then

$$\begin{aligned} (\mu_\varepsilon - 1)\phi_\Omega + \tilde{\phi}_\varepsilon &= \phi_\varepsilon - \phi_\Omega \\ &= (\lambda_\varepsilon^2 - \lambda_\Omega^2)(-\Delta_\varepsilon^{-2}\phi_\varepsilon) + \lambda_\Omega^2\,[(-\Delta_\varepsilon^{-2}) - (-\Delta_\Omega^{-2})]\,\phi_\varepsilon \\ &+ \lambda_\Omega^2(-\Delta_\Omega^{-2})[(\mu_\varepsilon - 1)\phi_\Omega + \tilde{\phi}_\varepsilon]. \end{aligned}$$

Hence

$$\tilde{\phi}_\varepsilon - \lambda_\Omega^2(-\Delta_\Omega^{-2})\tilde{\phi}_\varepsilon = (\lambda_\varepsilon^2 - \lambda_\Omega^2)(-\Delta_\varepsilon^{-2}\phi_\varepsilon) + \lambda_\Omega^2[(-\Delta_\varepsilon^{-2}) - (-\Delta_\Omega^{-2})]\phi_\varepsilon \tag{8}$$

Therefore, by (2), (28), (33) and (36), we have

$$\begin{aligned} \left\|\tilde{\phi}_\varepsilon - \lambda_\Omega^2(-\Delta_\Omega^{-2})\tilde{\phi}_\varepsilon\right\|_\infty &= 3\gamma R^{-2}(\lambda_\varepsilon - \lambda_\Omega)\left\|-\Delta_\varepsilon^{-2}\right\|_{L^1\to L^\infty}\|\phi_\varepsilon\|_1 \\ &+2^{-1}\lambda_\Omega^2\Big\|[T_\varepsilon(-\Delta_\varepsilon^{-1})S_\varepsilon - (-\Delta_\Omega^{-1})][T_\varepsilon(-\Delta_\varepsilon^{-1})S_\varepsilon + (-\Delta_\Omega^{-1})] \\ &+[T_\varepsilon(-\Delta_\varepsilon^{-1})S_\varepsilon + (-\Delta_\Omega^{-1})][T_\varepsilon(-\Delta_\varepsilon^{-1})S_\varepsilon - (-\Delta_\Omega^{-1})]\Big\|_{L^\infty\to L^\infty} \\ &\times\lambda_\varepsilon^2\left\|-\Delta_\Omega^{-2}\right\|_{L^1\to L^\infty}\|\phi_\varepsilon\|_1 \\ &\le 3\gamma R^{-2}C_1\varepsilon^{1/2}K_2|\Omega|^{1/2} + 2^{-1}R^{-4}\gamma^2 4K_1^2R^{1/2}\varepsilon^{1/2}4R^{-4}\gamma^2K_2|\Omega|^{1/2} \\ &= K_4\varepsilon^{1/2}, \end{aligned} \tag{9}$$

where

$$K_4 = 4096\gamma^5\pi^{-15/4}R^{-8}|\Omega|^{13/4} + \left(\frac{4096}{9}\right)\gamma^4\pi^{-3}R^{-13/2}|\Omega|^{5/2}. \tag{10}$$

Combining (7), (9) and the inequality

$$0.6197R^{-2} \le \lambda_\Omega < E_2 \le \Gamma R^{-2}$$

from [4], we have, for all $\varepsilon \in (0, 2^{-1}R)$,

$$\begin{aligned} K_1^2\varepsilon &\geq \left\|\tilde{\phi}_\varepsilon - \lambda_\Omega^2(-\Delta_\Omega^{-2}\tilde{\phi}_\varepsilon)\right\|_\infty^2 \geq |\Omega|^{-1}\left\|\tilde{\phi}_\varepsilon - \lambda_\Omega^2(-\Delta_\Omega^{-2}\tilde{\phi}_\varepsilon)\right\|_2^2 \\ &= |\Omega|^{-1}\left\|\sum_{n=2}^{\infty}(1-\lambda_\Omega^2 E_n^{-2})\sigma_n(\varepsilon)\psi_n\right\|_2^2 \\ &\geq |\Omega|^{-1}\Gamma^{-2}R^2(1.2394)(E_2-\lambda_\Omega)\left(\sum_{n=2}^{\infty}\sigma_n(\varepsilon)^2\right) \\ &= (1.2394)\Gamma^{-2}R^2|\Omega|^{-1}(E_2-\lambda_\Omega)\left\|\tilde{\phi}_\varepsilon\right\|_2^2 \\ &\geq (1.2394)\Gamma^{-2}R^2|\Omega|^{-2}(E_2-\lambda_\Omega)\left\|\tilde{\phi}_\varepsilon\right\|_1^2 \end{aligned} \tag{11}$$

Thus (8), (2), (28), (36) and (11) give, for $\varepsilon \in (0, 2^{-1}R)$,

$$\begin{aligned} \left\|\tilde{\phi}_\varepsilon\right\|_\infty &\leq (\lambda_\varepsilon^2 - \lambda_\Omega^2)\left\|-\Delta_\varepsilon^{-2}\right\|_{L^1\to L^\infty}\left\|\phi_\varepsilon\right\|_1 \\ &+ \lambda_\Omega^2\left\|T_\varepsilon(-\Delta_\varepsilon^{-2})S_\varepsilon - (-\Delta_\Omega^{-2})\right\|_{L^1\to L^\infty}\left\|\tilde{\phi}_\varepsilon\right\|_1 + \lambda_\Omega^2\left\|-\Delta_\Omega^{-2}\right\|_{L^1\to L^\infty}\left\|\tilde{\phi}_\varepsilon\right\|_1 \\ &\leq 3\gamma R^{-2}C_1\,\varepsilon^{1l2}K_2|\Omega|^{1/2} \\ &+ 3\gamma^2 R^{-4}K_2K_4(1.2394)^{-1/2}\Gamma R^{-1}|\Omega|(E_2-\lambda_\Omega)^{-1/2}\varepsilon^{1/2} \\ &= [K_5 + K_6(E_2-\lambda_\Omega)^{-1/2}]\varepsilon^{1/2} \end{aligned} \tag{12}$$

where

$$K_5 = 4096\gamma^5\pi^{-15/4}R^{-8}|\Omega|^{13/4}$$

and

$$\begin{aligned} K_6 &= 98304(1.2394)^{-1/2}\gamma^7\pi^{-21/4}R^{-12}|\Omega|^{19/4} \\ &+\left(\frac{32768}{3}\right)(1.2394)^{-1/2}\gamma^6\Gamma\pi^{-9/2}R^{-21/2}|\Omega|^4. \end{aligned}$$

Since $\left\|\phi_\Omega\right\|_2 = 1 = \left\|\phi_\varepsilon\right\|_2$, (7) and (11) give, for all $\varepsilon \in (0,\ 2^{-1}R)$,

$$\begin{aligned} |1-\mu_\varepsilon| &\leq (1.2394)^{-1}\Gamma^2 R^{-2}|\Omega|K_4^2(E_2-\lambda_\Omega)^{-1}\varepsilon \\ &= K_7(E_2-\lambda_\Omega)^{-1}\varepsilon, \end{aligned} \tag{13}$$

Where

$$\begin{aligned} K_7 &= (1.2394)^{-1}4096^2\gamma^{10}\Gamma^2\pi^{-15/2}R^{-18}|\Omega|^{15/2} \\ &+(1.2394)^{-1}\left(\frac{4096^2}{81}\right)\gamma^8\Gamma^2\pi^{-6}R^{-15}|\Omega|^6 \\ &+2(1.2394)^{-1}\left(\frac{4096^2}{9}\right)\gamma^9\Gamma^2\pi^{-27/4}R^{-33/2}|\Omega|^{27/4}. \end{aligned}$$

Hence (7), (12) and (13) imply that, for $\varepsilon \in (0,\ 2^{-1}R)$,

$$\begin{aligned} \left\|\phi_\Omega - T_\varepsilon\phi_\varepsilon\right\|_\infty &\leq (1-\mu_\varepsilon)\left\|\phi_\Omega\right\|_\infty + \left\|\tilde{\phi}_\varepsilon\right\|_\infty \text{ S} \\ &\leq K_7(E_2-\lambda_\Omega)^{-1}\varepsilon\lambda_\Omega^2\left\|-\Delta_\Omega^{-2}\right\|_{L^1\to L^\infty}|\Omega|^{1/2} + [K_5 + K_6(E_2-\lambda_\Omega)^{-1/2}]\varepsilon^{1/2} \\ &\leq [K_7(E_2-\lambda_\Omega)^{-1}2^{-1/2}R^{1/2}\lambda^2R^{-4}K_2|\Omega|^{1/2} + K_5 + K_6(E_2-\lambda_\Omega)^{-1/2}]\varepsilon^{1/2} \end{aligned}$$

which gives (4).

Note that (i) In [2], Davies showed that if Ω is a bounded simply connected open set in $\mathbb{R}^2$, then for any $\beta \in (0, \frac{1}{2})$, there exists $c = c(\beta) \geq 1$ such that

$$|\lambda_\varepsilon - \lambda_\Omega| \leq c\varepsilon^\beta \tag{14}$$

for all sufficiently small $\varepsilon > 0$. Davies showed that the estimate (14) also holds for higher eigenvalues of $-\Delta_\Omega$. The estimate (2) is therefore an improvement of Davies' result for the ground state eigenvalue.

(ii) The case when Ω is the cardioid in $\mathbb{R}^2$ shows that the term $\varepsilon^{1/2}$ on the right-hand side of (4) cannot be improved.

(iii) The eigenvalue gap $E_2 - \lambda_\Omega$ in (4) depends on the geometry of Ω in a very complicated manner. We refer to [3,4,5] and references therein for recent estimates of this quantity.

(iv) A weaker version of (4) was proved in [6] for the case when Ω is the Koch snowflake in $\mathbb{R}^2$.

Theorem(1.1.2)[1]: Let Ω be a bounded connected open set in $\mathbb{R}^N, N \geq 2$, satisfying the following two assumptions.

(AI) There exist $c_1 \geq 1$ and $\mu \geq N$ such that for all $t > 0$ and all $x, y \in \Omega$,

$$K_\Omega(t,x,y) \leq c_1 t^{-\mu/2} \phi_\Omega(x)\phi_\Omega(y), \tag{15}$$

where $K_\Omega(t,x,y)$ is the heat kernel of $e^{\Delta_\Omega t}$.

(AII) There exist $c_2 \geq 1$ and $\alpha > 0$ such that

$$\phi_\Omega(x) \leq c_2 \text{dist}(x, \partial\Omega)^\alpha \quad (x \in \Omega). \tag{16}$$

Then there exist $c_2 \geq 1$, $\delta > 0$ and $\theta \geq \alpha^2/(2\mu)$ such that:

$$|\lambda_\varepsilon - \lambda_\Omega| \leq c\varepsilon^\theta \tag{17}$$

and

$$\|T_\varepsilon\phi_\varepsilon - \phi_\Omega\|_\infty \leq c\varepsilon^\theta \tag{18}$$

provided that $0 < \varepsilon < \delta$.

Proof: Let $0 < \lambda_\Omega < E_2 \leq E_3 \leq \cdots$ be the eigenvalues of $-\Delta_\Omega$, with corresponding eigenfunctions $\phi_\Omega, \psi_2, \psi_3, \cdots$ normalized by $\|\psi_i\|_2 = 1$. Let

$$T_\varepsilon\phi_\varepsilon = \mu_\varepsilon\phi_\Omega + \sum_{n=2}^{\infty} \sigma_n(\varepsilon)\psi_n = \mu_\varepsilon\phi_\Omega + \tilde{\phi}_\varepsilon. \tag{19}$$

We first show that there exist $c \geq 1$ and $b > 0$ such that if $0 < \varepsilon < b$, then

$$|\lambda_\varepsilon - \lambda_\Omega| \leq C\varepsilon^\theta, \tag{20}$$

where θ is as in (43). Lemma (1.1.11) implies that for $t > \varepsilon^\nu$, we have

$$\begin{aligned}
\left|\left\langle T_\varepsilon e^{\Delta_{R(\varepsilon)}t} S_\varepsilon\phi_\Omega, \phi_\Omega\right\rangle - e^{-\lambda_\Omega t}\right| &= \left|\left\langle (T_\varepsilon e^{\Delta_{R(\varepsilon)}t} S_\varepsilon - e^{\Delta_\Omega t})\phi_\Omega, \phi_\Omega\right\rangle\right| \\
&\leq \left\| e^{\Delta_\Omega t} - T_\varepsilon e^{\Delta_{R(\varepsilon)}t} S_\varepsilon \right\|_{L^\infty \to L^\infty} \|\phi_\Omega\|_\infty \|\phi_\Omega\|_1 \\
&\leq C\varepsilon^\theta \|\phi_\Omega\|_2^2 \\
&= C\varepsilon^\theta
\end{aligned}$$

Thus, by the minimax principle,

$$e^{-\lambda_\Omega} - C\varepsilon^\theta \leq \left\langle e^{\Delta_{R(\varepsilon)}t} S_\varepsilon\phi_\Omega, S_\varepsilon\phi_\Omega\right\rangle \leq e^{-\lambda_\varepsilon} \leq e^{-\lambda_\Omega},$$

which implies (20). Next we have

$$\begin{aligned}(\mu_\varepsilon-1)\phi_\Omega+\tilde{\phi}_\varepsilon &= T_\varepsilon\phi_\varepsilon-\phi_\Omega\\ &= e^{\lambda_\varepsilon}\left(T_\varepsilon e^{\Delta_{R(\varepsilon)}}\phi_\varepsilon\right)-e^{\lambda_\Omega}\left(e^{\Delta_\Omega}\phi_\Omega\right)\\ &= \left(e^{\lambda_\varepsilon}-e^{\lambda_\Omega}\right)\left(T_\varepsilon e^{\Delta_{R(\varepsilon)}}\phi_\varepsilon\right)+e^{\lambda_\Omega}\left(T_\varepsilon e^{\Delta_{R(\varepsilon)}}S_\varepsilon-e^{\Delta_\Omega}\right)T_\varepsilon\phi_\varepsilon\\ &\quad+e^{\lambda_\Omega}e^{\Delta_\Omega}\left[(\mu_\varepsilon-1)\phi_\Omega+\tilde{\phi}_\varepsilon\right].\end{aligned}$$

Hence

$$\tilde{\phi}_\varepsilon-e^{\lambda_\Omega}e^{\Delta_\Omega}\tilde{\phi}_\varepsilon = \left(e^{\lambda_\varepsilon}-e^{\lambda_\Omega}\right)T_\varepsilon e^{\Delta_{R(\varepsilon)}}\phi_\varepsilon+e^{\lambda_\Omega}\left(T_\varepsilon e^{\Delta_{R(\varepsilon)}}S_\varepsilon-e^{\Delta_\Omega}\right)T_\varepsilon\phi_\varepsilon \tag{21}$$

By (20), (21) and Lemma (1.1.11), we have

$$\begin{aligned}C\varepsilon^{2\theta} &\geq \left\|\tilde{\phi}_\varepsilon-e^{\lambda_\Omega}e^{\Delta_\Omega}\tilde{\phi}_\varepsilon\right\|_\infty^2\geq C^{-1}\left\|\tilde{\phi}_\varepsilon-e^{\lambda_\Omega}e^{\Delta_\Omega}\tilde{\phi}_\varepsilon\right\|_2^2\\ &= C^{-1}\left\|\sum_{n=2}^{\infty}(1-e^{-(E_n-\lambda_\Omega)}\sigma_n(\varepsilon)\psi_n\right\|_2^2 = C^{-1}\sum_{n=2}^{\infty}(1-e^{-(E_n-\lambda_\Omega)})^2\sigma_n(\varepsilon)^2\\ &\geq C^{-1}\sum_{n=2}^{\infty}\sigma_n(\varepsilon)^2=C^{-1}\left\|\tilde{\phi}_\varepsilon\right\|_2^2,\end{aligned} \tag{22}$$

where θ is as in (43). Therefore (21) and (22) give

$$\begin{aligned}\left\|\tilde{\phi}_\varepsilon\right\|_\infty &\leq \left\|(e^{\lambda_\varepsilon}-e^{\lambda_\Omega})e^{\Delta_{R(\varepsilon)}}\psi_\varepsilon\right\|_\infty+e^{\lambda_\Omega}\left\|T_\varepsilon e^{\Delta_{R(\varepsilon)}}S_\varepsilon-e^{\Delta_\Omega}\right\|_{L^\infty\to L^\infty}e^{\lambda_\varepsilon}\left\|e^{\Delta_{R(\varepsilon)}}\phi_\varepsilon\right\|_\infty\\ &\quad+e^{\lambda_\Omega}\left\|e^{\Delta_\Omega}\tilde{\phi}_\varepsilon\right\|_\infty\leq C\varepsilon^\theta.\end{aligned} \tag{23}$$

Since $\left\|\phi_\varepsilon\right\|_2=1$, (19) and (23) imply that

$$\left|1-\mu_\varepsilon\right|\leq C\varepsilon^\theta,$$

which implies (18) with θ given by (43).

Note that (i) Eigenvalue estimates of the form (17) have been obtained in [2] for bounded connected open sets Ω in $\mathbb{R}^N$ with nonsmooth boundary under different assumptions.

(ii) Assumptions (AI) and (AII) hold for bounded connected John sets Ω in $\mathbb{R}^N$ that satisfy the following twisting external cone condition.

(AIII) There exist $r_0,\zeta\in(0,1)$ such that for all $x_0\in\partial\Omega$ and all $0<r<r_0$, we have

$$\text{vol}(B(x_0,r)\setminus\Omega)\geq\zeta\ \text{vol}\,(B(x_0,r)), \tag{24}$$

where $B(x_0, r)$ is the ball with centre x_0 and radius r, and vol(E) denotes the volume of E (a proof of this can be found in [7] and [8]).

For the rest of this section, we shall write

$$d(x) = \text{dist}\,(x,\partial\Omega)\qquad(x\in\Omega)$$

and

$$\Omega(\varepsilon) = \{x\in\Omega : d(x)>\varepsilon\}.$$

We shall write |Ω| for the area of an open connected set Ω in $\mathbb{R}^2$, and write R for the inradius of Ω. We shall need the following definitions and results.

Proposition(1.1.3)[1]: Let D be a connected open subset of Ω. Then for all $z\in D$, we have

$$E^Z[\tau_\Omega-\tau_D] = E^z[E^{X(\tau_D)}[\tau_\Omega]] \tag{25}$$

where $\{X(t)\}_{t\geq0}$ denotes the Brownian motion, τ_Ω and τ_D are the exit times from Ω and D respectively, and E^z is the expectation with respect to the Brownian motion measure.

If D is a simply connected open set in $\mathbb{R}^2$ and $z \in D$, then we define the density of the hyperbolic metric at z by $\sigma_D(z) = |G'(z)|$, where $G: D \to \mathbb{D}$ is any conformal map from D onto the unit disc $\mathbb{D}$ with $G(z) = 0$. If $z_1, z_2 \in D$, then we define the hyperbolic distance between z_1 and z_2 by

$$\rho_D(z_1, z_2) = \inf_{\Gamma} \left\{ \int_0^1 \sigma_D(\Gamma(t)) |\Gamma'(t)| dt \right\},$$

where the infimum is taken over all rectifiable paths Γ contained in D with $\Gamma(0) = z_1$ and $\Gamma(1) = z_2$. We shall need the following.

Proposition(1.1.4)[1]: Let D be a bounded simply connected open set in $\mathbb{R}^2$, and let $G: D \to \mathbb{D}$ be a conformal map onto $\mathbb{D}$. Then

$$\rho_D(z) = |G'(z)| / (1 - |G(z)|^2) \qquad (z \in D). \tag{26}$$

If $D \subseteq \mathbb{R}^2$ is a bounded connected open set, then we write $G_D(z_1, z_2)$ for the integral kernel of $-\Delta_D^{-1}$. We shall put $G_D(z_1, z_2) = 0$ if either z_1 or z_2 is not in D.

Proposition (1.1.5)[1]: Let D be a bounded simply connected open set in $\mathbb{R}^2$. Then for all $z_0, z \in D$, we have

$$G_D(z_0, z) = (2\pi)^{-1} \log [\coth (\rho_D(z_0, z))]. \tag{27}$$

Lemma (1.1.6)[1]: For all ε in (0, 1), we have

$$\left\| \left(T_\varepsilon \left(-\Delta_\varepsilon^{-1} \right) S_\varepsilon f \right) - \left(-\Delta_\Omega^{-1} f \right) \right\|_\infty \le K_1 \varepsilon^{1/2} \| f \|_\infty \tag{28}$$

for all $f \in L^\infty(\Omega)$, where

$$K_1 = \tfrac{8}{3} \pi^{-3/4} |\Omega|^{3/4} \tag{29}$$

Proof. Using (25) and writing $\omega = u + iv$., we have

$$\begin{aligned} \left\| T_\varepsilon \left(-\Delta_\varepsilon^{-1} \right) S_\varepsilon - \left(-\Delta_\Omega^{-1} \right) \right\|_{L^\infty(\Omega) \to L^\infty(\Omega)} &\le \sup_{\omega_1 \in \Omega} \int_\Omega G_\Omega(\omega_1, \omega_2) - G_{R(\varepsilon)}(\omega_1, \omega_2) du_2 dv_2 \\ &= \sup_{\omega_1 \in \Omega} E^{\omega_1}(\tau_\Omega) - E^{\omega_1}(\tau_{R(\varepsilon)}) \\ &= \sup_{\omega_1 \in \Omega} E^{\omega_1}[E^{X(\tau_{R(\varepsilon)})}(\tau_\Omega)]. \end{aligned} \tag{30}$$

We now estimate the last expression in (30). By (26) and Koebe's distortion theorem (see [13]) , we have

$$4^{-1} d(z)^{-1} \le \sigma_\Omega(z) \le d(z)^{-1} \qquad (z \in \Omega). \tag{31}$$

Let $z \in \partial R(\varepsilon)$. For $\omega \in \Omega$, (31) implies that

$$\begin{aligned} \rho_\Omega(z, \omega) &\ge 4^{-1} \int_0^{|z-\omega|} \left(d(z) + r \right)^{-1} dr \\ &= \log\left[(1 + |z - \omega| d(z)^{-1})^{1/4} \right] \end{aligned}$$

So (27) implies that

$$\begin{aligned} G_\Omega(z, \omega) &\le (2\pi)^{-1} \log\{\coth[(1 + |z - \omega| d(z)^{-1})^{1/4}]\} \\ &= (2\pi)^{-1} \{\log[(d(z) + |z - \omega|)^{1/2} + d(z)^{1/2}] \\ &\quad - \log[(d(z) + |z - \omega|)^{1/2} - d(z)^{1/2}]\}. \end{aligned} \tag{32}$$

Let $Q > 0$ be defined by

$$\pi Q^2 = |\Omega|.$$

Then, writing $\omega = u + iv$, (32) and symmetric rearrangement imply that

$$\begin{aligned} E^z[\tau_\Omega] &= \int_\Omega G_\Omega(z,\omega)\ dudv \\ &\le (2\pi)^{-1}\int_\Omega \log[(d(z)+|z-\omega|)^{1/2}+d(z)^{1/2}] \\ &\qquad -\log[(d(z)+|z-\omega|-d(z)^{1/2}]dudv \\ &\le \int_0^Q\{\log[(d(z)+r)^{1/2}+d(z)^{1/2}]-\log[(d(z)+r)^{1/2}-d(z)^{1/2}]\}rdr \\ &= d(z)^2\int_{d(z)/Q}^{\infty}\{\log[(1+s)^{1/2}+s^{1/2}]-\log[(1+s)^{1/2}-s^{1l2}]s^{-3}\}ds \\ &\le 4d(z)^2\int_{d(z)/Q}^{\infty} s^{-5/2}ds \\ &= \frac{8}{3}\mu^{-3/4}|\Omega|^{3/4}d(z)^{1/2} \\ &\le \frac{8}{3}\mu^{-3/4}|\Omega|^{3/4}\varepsilon^{1/2}, \end{aligned} \tag{33}$$

where we have used the inequality

$$\log[(1+s)^{1/2}+s^{1/2}] \le 2s^{1/2} \qquad (s>0).$$

Combining (30) and (33), we obtain

$$\begin{aligned} \left\|T_\varepsilon(-\Delta_\varepsilon^{-1})S_\varepsilon-(-\Delta_\Omega^{-1})\right\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le \sup_{\omega_1\in\Omega} E^{\omega_1}\left[\frac{8}{3}\mu^{-3/4}|\Omega|^{3/4}\varepsilon^{1/2}\right] \\ &= \frac{8}{3}\pi^{-3/4}|\Omega|^{3/4}\varepsilon^{1/2}, \end{aligned}$$

which proves the lemma.

Proposition(1.1.7)[1]: Writing $\omega = u + iv$, we have for all $z \in \Omega$

$$\int_\Omega G_\Omega(z,\omega)^2\ dudv \le K_2, \tag{34}$$

where

$$K_2 = 8\pi^{-3/2}R\,|\Omega|^{1/2}. \tag{35}$$

Lemma(1.1.8)[1]: We have

$$\left\|-\Delta_\Omega^{-2}\right\|_{L^1\to L^\infty(\Omega)} \le K_2, \tag{36}$$

where K_2 is as in (35).

Proof: Let $I(z_1,z_2)$ be the integral kernel of $-\Delta_\Omega^{-2}$. Then, by (34),

$$\begin{aligned} I(z_1,z_2) &= \int_\Omega G_\Omega(z_1,\omega)\,G_\Omega(\omega,z_2)dudv \\ &\le \left\{\int_\Omega G_\Omega(z_1,\omega)^2\ dudv\right\}^{1/2}\left\{\int_\Omega G_\Omega(\omega,z_2)^2\ dudv\right\}^{1/2} \\ &\le K_2 \end{aligned}$$

Hence

$$\left\|-\Delta_\Omega^{-2}\right\|_{L^1\to L^\infty(\Omega)} \le \|I\|_\infty \le K_2.$$

We assume that Ω is a bounded connected open set in $\mathbb{R}^N$ satisfying the assumptions (AI) and (AII). We let $K(\varepsilon)$ be an $(N-1)$-dimensional C^∞ surface in $\Omega(\varepsilon)\setminus\Omega(2\varepsilon)$ such that for any $x \in \partial\Omega(\varepsilon)$, and $y \in \partial\Omega(2\varepsilon)$, and any path $\ell(x, y)$ joining x and y, we have

$$\ell(x, y) \cap K(\varepsilon) \neq \varnothing.$$

We let

$$U(\varepsilon) = \{x \in \Omega : \text{there exist } y \in \Omega(2\varepsilon) \text{ and a path } \ell(x, y) \text{ joining } x \text{ to } y \text{ with } \ell(x, y) \cap K(\varepsilon) = \varnothing\}.$$

Lemma(1.1.9)[1]: For all $t > 0$ and $x \in U(\varepsilon)$, we have

$$P^x[\tau_{U(\varepsilon)} < t < \tau_\Omega] \le \int_{K(\varepsilon)\times[0,t)} P^y[\tau_\Omega \ge t - s]\, dm_x(y, s), \tag{37}$$

where P^x is the Brownian motion measure and m_x is the Borel measure on $K(\varepsilon) \in [0,t)$ satisfying

$$m_x(B \times (s_1, s_2)) = P^x[s_1 < \tau_{U(\varepsilon)} < s_2, X(\tau_{U(\varepsilon)}) \in B]$$

for all Borel sets $B \subseteq K(\varepsilon)$ and $0 \le s_1 < s_2 < t$.

Proof: For all $n \in \mathbb{N}$, let $\{B_{i,n}\}_{i=1}^{m(n)}$ be a collection of subsurfaces of $K(\varepsilon)$ such that each $B_{i,n}$ has piecewise C^∞ boundary and

$$|y_1 - y_2| \le 2^{-n} \qquad (y_1, y_2 \in B_{i,n}).$$

Then we have

$$\begin{aligned}
&P^x[\tau_{U(\varepsilon)} < t < \tau_\Omega] \\
&= \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} P^x[\tau_{U(\varepsilon)} \in [2^{-n}(k-1)t,\ 2^{-n}kt), X(\tau_{U(\varepsilon)}) \in B_{i,n}, \tau_\Omega > t] \\
&= \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} E^x[\tau_{U(\varepsilon)} \in [2^{-n}(k-1)t,\ 2^{-n}kt), X(\tau_{U(\varepsilon)}) \in B_{i,n}; 1_{[\tau_\Omega > t]}] \\
&\le \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} E^x[\tau_{U(\varepsilon)} \in [2^{-n}(k-1)t,\ 2^{-n}kt), X(\tau_{U(\varepsilon)}) \in B_{i,n}; 1_{[\tau_\Omega - \tau_{U(\varepsilon)} > t - kt/2^n]}] \\
&= \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} E^x[\tau_{U(\varepsilon)} \in [2^{-n}(k-1)t,\ 2^{-n}kt), X(\tau_{U(\varepsilon)}) \in B_{i,n}; E^{X(\tau_{U(\varepsilon)})}[1_{[\tau_\Omega > t - kt/2^n]}]] \\
&= \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} E^x[\tau_{U(\varepsilon)} \in \left[2^{-n}(k-1)t,\ 2^{-n}kt\right), X(\tau_{U(\varepsilon)}) \in B_{i,n}; P^{X(\tau_{U(\varepsilon)})}\left[\tau_\Omega > t\frac{kt}{2^n}\right]\right]
\end{aligned} \tag{38}$$

Since

$$\begin{aligned}
&\sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} P^x\left[\tau_{U(\varepsilon)} \in \left[\frac{(k-1)t}{2^n}, \frac{kt}{2^n}\right), X(\tau_{U(\varepsilon)}) \in B_{i,n}\right]\left(\inf_{\substack{y\in B_{i,n} \\ 2^{-n}(k-1)t \le s \le 2^{-n}kt}} P^y\left[\tau_\Omega > t - s\right]\right) \\
&\quad \le \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} E^x\left[\tau_{U(\varepsilon)} \in \left[\frac{(k-1)t}{2^n}, \frac{kt}{2^n}\right), X(\tau_{U(\varepsilon)}) \in B_{i,n}; P^{X(\tau_{U(\varepsilon)})}\left[\tau_\Omega > t - \frac{kt}{2^n}\right]\right] \\
&\quad \le \sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} P^x\left[\tau_{U(\varepsilon)} \in \left[\frac{(k-1)t}{2^n}, \frac{kt}{2^n}\right), X(\tau_{U(\varepsilon)}) \in B_{i,n}\right]\left(\inf_{\substack{y\in B_{i,n} \\ 2^{-n}(k-1)t \le s \le 2^{-n}kt}} P^y\left[\tau_\Omega > t - s\right]\right)
\end{aligned}$$

and since the function

$$(y, s) \to P^y\left[\tau_\Omega > t - s\right]$$

is a bounded continuous function on $K(\varepsilon) \times [0, t)$, we have

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}\sum_{i=1}^{m(n)} E^x\left[\tau_{U(\varepsilon)}\in\left[\frac{(k-1)t}{2^n},\ \frac{kt}{2^n}\right), X(\tau_{U(\varepsilon)})\in B_{i,n}; P^{X(\tau_{U(\varepsilon)})}\left[\tau_\Omega > t-\frac{kt}{2^n}\right]\right]$$
$$= \int_{K(\varepsilon)\times[0,t)} P^y\left[\tau_\Omega > t-s\right]dm_x(y,s). \qquad (39)$$
The lemma follows from (38) and (39).

Proposition(1.1.10)[1]: Let D be a connected open set in $\mathbb{R}^N$, $N\ge 2$, and let $K_D(t, x, y)$ be the heat kernel corresponding to $e^{\Delta_D t}$. Then there exists $c\ge 1$ such that for all $t > 0$ and all $x, y\in D$,
$$\left[\frac{\partial}{\partial t}K_D(t,x,y)\right] \le ct^{-(N/2)-1}\left(1+\frac{|x-y|^2}{t}\right)^{1+N/2} e^{-|x-y|^2/(4t)} \qquad (40)$$
For functions f defined on Ω, we shall let $A_\varepsilon f$ be the restriction of f to $U(\varepsilon)$. For functions g defined on $U(\varepsilon)$, we shall write $B_\varepsilon g$ for the extension of g to Ω, satisfying
$$(B_\varepsilon g)(x) = \begin{cases} g(x) & \text{if } x\in U(\varepsilon),\\ 0 & \text{if } x\in \Omega\setminus U(\varepsilon).\end{cases}$$
Lemma(1.1.11)[1]: There exists $c\ge 1$ such that if
$$0<\beta<\alpha\mu^{-1} \text{ and } 0<\nu<\min\left(\frac{\alpha\beta}{\mu},\frac{2\beta}{N+4}\right), \qquad (41)$$
then for all sufficiently small $\varepsilon > 0$, we have
$$\left\|e^{\Delta_\Omega t}f - B_\varepsilon e^{\Delta_{U(\varepsilon)}t}A_\varepsilon f\right\|_\infty \le c\varepsilon^\theta (t>\varepsilon^\nu, f\in L^\infty(\Omega)), \qquad (42)$$
where
$$\theta = \min(\beta-2^{-1}\nu(N+4), \alpha-\beta\mu, \alpha\beta-\mu\nu). \qquad (43)$$
Proof: For all $x\in U(\varepsilon)$ and $0<r<t$, we have
$$\left\| e^{\Delta_\Omega t}f(x) - B_\varepsilon e^{\Delta_{U(\varepsilon)}t}A_\varepsilon f(x)\right\| = \left|E^x[\tau_\Omega>t; f(X_t)] - E^x[\tau_{U(\varepsilon)}>t; f(X_t)]\right|$$
$$= |E^x[\tau_{U(\varepsilon)}<t<\tau_\Omega; f(X_t)]|$$
$$\le \|f\|_\infty P^x\left[\tau_\Omega<t<\tau_\Omega\right]$$
$$\le \|f\|_\infty \int_{K(\varepsilon)\times[0,t)} P^y\left[\tau_\Omega>t-s\right]dm_x(y,s)$$
$$= \|f\|_\infty\left\{\left(\int_{K(\varepsilon)\times[0,r)}+\int_{K(\varepsilon)\times[r,t)}\right)P^y\left[\tau_\Omega>t-s\right]\, dm_x(y,s)\right\} \qquad (44)$$
First suppose that $x\in\Omega(\varepsilon^\beta)$. If $t>\varepsilon^\nu$, then we have, by Proposition (1.1.10),
$$\int_{K(\varepsilon)\times[t-\varepsilon^\beta,t)} P^y\left[\tau_\Omega>t-s\right]dm_x(y,s) \le P^x\left[\tau_{U(\varepsilon)}\in[t-\varepsilon^\beta,t)\right]$$
$$= \int_{U(\varepsilon)} K_{U(\varepsilon)}(t-\varepsilon^\beta,x,y) - K_{U(\varepsilon)}(t,x,y)dy$$
$$\le c\varepsilon^\beta t^{-(N/2)-2} \le c\varepsilon^{\beta-2^{-1}\nu(n+4)}. \qquad (45)$$
Also, by assumptions (AI) and (AII), we have

$$\int_{K(\varepsilon)\times[0,t-\varepsilon^\beta)} P^y[\tau_\Omega > t-s]dm_x(y,s)$$
$$\leq \int_{K(\varepsilon)\times[0,t-\varepsilon^\beta)} \int_\Omega K_\Omega(\varepsilon^\beta, y, z)\,dz dm_x(y,s)$$
$$\leq \int_{K(\varepsilon)\times[0,t-\varepsilon^\beta)} c\varepsilon^{-\beta\mu}\varepsilon^\alpha \int_\Omega d(z)^\alpha dz dm_x(y,s)$$
$$= c\varepsilon^{\alpha-\beta\mu} \quad (46)$$

Next suppose that $x \in \Omega \setminus \Omega(\varepsilon^\beta)$. Then for $t > \varepsilon^\nu$, assumptions (AI) and (AII) imply that

$$P^x[\tau_\Omega > t] = \int_\Omega K_\Omega(t,x,y)dy \leq Ct^{-\mu}\varepsilon^{\alpha\beta} \leq C\varepsilon^{\alpha\beta-\mu\nu}. \quad (47)$$

The lemma now follows from (44) to (47).

Finally, we note that if we put

$$\beta = 3\alpha/4\mu \text{ and } \nu = \alpha^2/6\mu^2,$$

Then β and ν satisfy the condition (41) and, since $2 \leq N \leq \mu$ and $0 < \alpha \leq 1$, one can check that θ given by (43) then satisfies

$$\theta \geq \frac{\alpha^2}{2\mu}.$$

Sec(1.2): Dirichlet Structures

We set a new property, namely the strong energy image density S.E.I.D., which is a kind of quantitative version of E.I.D.. More precisely, in univariate cases and for general Dirichlet forms, we show that if X_n converge toward X in the $\mathbb{D}$ domain of a Dirichlet structure, then $X_{n*}\left(1_{\{\Gamma[X]>0\}}d\mathrm{P}\right)$ converges in total variation toward $X_*(1_{\{\Gamma[X]>0\}}d\mathrm{P})$.

We redemonstrate the fact that convergence for the Dirichlet topology is preserved by Lipschitz mappings in the univariate case, which has been established in [69]. Besides, in the multivariate case, we show that the distribution of $X \in \mathbb{D}^p$ whose square field operator matrix is almost surely definite, is a Rajchman measure.

We study multivariate cases for the $W^{1,p}(\Omega)$ Sobolev structures (Ω open subset or $\mathbb{R}^d$) and establish a weaker form of S.E.I.D. but with rather precise estimates. Indeed, we need some additional assumptions in order to ensure that the Jacobian of the mappings is integrable, which enables us to show an integration by parts formula.

Let us mention that the proof provides with a generalization of a recent result from H. Brezis and H.M.Nguyen (c.f.[43]). Besides, these results are extended to the case of the Ornstein-Uhlenbeck form in the Wiener space giving a generalization (by a purely algebraic method and avoiding every coarea type arguments) of classical results from Bouleau-Hirsch [37]. Every results are heavily inspired by [39], where functional calculus and completeness of $\mathbb{D}$ are combined, in order to show E.I.D.. As explained of this section, the proof has a common structure with [39]. In [39], the preponderant argument is the completeness of $\mathbb{D}$ whereas in these proofs we use the infinitesimal generator A (inducing the Dirichlet form $\varepsilon[\cdot,\cdot]$). Although these two facts are totally equivalent (c.f.[36]), we highlight that using the generator is a more powerful approach of E.I.D..

We focus on the univariate case for a general structure. Next, we end by studying the particular case of Sobolev spaces Writing $W^{1,p}(\Omega)$.

Originally introduced by Beurling and Deny (c.f. [34]), a Dirichlet form is a symmetric non-negative bilinear form $\varepsilon[\cdot,\cdot]$ acting on a dense subdomain $D(\varepsilon)$ of an Hilbert space H, such that $D(\varepsilon)$ endowed with the norm $\sqrt{<X,X>_H+\varepsilon[X,X]}$ is complete.
In the sequel we only focus on the particular case of local Dirichlet forms admitting square field operators. In order to avoid unessential difficulties, we restrict our attention to the case of probability spaces $(\Omega,\mathrm{F},\mathrm{P})$ instead of measured spaces . The (Ω,F,m) next definition is central.

Definition(1.2.1)[33]: Following the terminology of [37], in this section, a Dirichlet structure will X denote a term $(\Omega,\mathrm{F},\mathrm{P},\mathrm{D},\Gamma)$ such that:

(a) $(\Omega,\mathrm{F},\mathrm{P})$ is a probability space.

(b) D is a dense subdomain of $L^2(\mathrm{P})$.

(c) $\Gamma[\cdot,\cdot]:\mathrm{D}\times\mathrm{D}\to L^1(\mathrm{P})$ is bilinear, symmetric, non-negative.

(d) For all $m\geq 1$, for all $X=(X_1,\ldots,X_m)\in\mathrm{D}^m$, and for all $F\in C^1(\mathbb{R}^m,\mathbb{R})$ and K-Lipschitz:

$(i)\ F(X)\in\mathrm{D}$,

$(ii)\ \Gamma[F(X),\ F(X)]=\sum_{i=1}^{m}\sum_{j=1}^{m}\partial_i F(X)\partial j F(X)\Gamma[X_i,X_j].$

(e) Setting $\varepsilon[X,X]=\mathrm{E}(\Gamma[X,X])$, the domain D endowed with the norm:

$$\|X\|_{\mathrm{D}}=\sqrt{\mathrm{E}(X^2)+\varepsilon[X,X]},$$

is complete. Thus, ε is a Dirichlet form with domain D on the Hilbert space $L^2(\mathrm{P})$.

Let us recall briefly, that there exists an operator A defined on a $D(A)$ dense subdomain of $L^2(\mathrm{P})$ such that:

(I) $U\in D(A)$ if and only if there exists $C_U>0$ satisfying :

$$\forall X\in\mathrm{D},\ \left|\mathrm{E}\{\Gamma[X,U]\}\right|\leq C_U\sqrt{\mathrm{E}(X^2)},$$

(II) for all $U\in\mathrm{D}(A)$, for all $X\in\mathrm{D}$ and all $Z\in\mathrm{D}\cap L^\infty(\mathrm{P})$, we have :

$$\mathrm{E}\{\Gamma[X,\ U]Z\}\ =\ -\ \mathrm{E}\{XZA[U]\}+\mathrm{E}\{X\,\Gamma[Z,U]\},$$

(III) $\mathrm{D}(A)$ is dense in D for the norm $\|\cdot\|_{\mathrm{D}}$.

Let us enumerate the notations adopted in the present section :

(i) For $X\in\mathrm{D}$, we set $\Gamma[X]=\Gamma[X,X]$ and $\varepsilon[X]=\varepsilon[X,X]$,

(ii) for $X=(X_1,\ldots,X_m)\in\mathrm{D}^m$:

$$\Gamma[X]\ =\ \Gamma\left[X^t,X\right]\ =\ \begin{pmatrix}\Gamma[X_1,X_1] & \Gamma[X_1,X_2] & \ldots & \Gamma[X_1,X_n]\\ \Gamma[X_2,X_1] & \Gamma[X_2,X_2] & \ldots & \Gamma[X_2,X_n]\\ \vdots & \vdots & \ldots & \vdots\\ \Gamma[X_n,X_1] & \Gamma[X_n,X_2] & \ldots & \Gamma[X_n,X_n]\end{pmatrix},$$

(iii) for $\phi\in C^1(\mathbb{R}^d,\mathbb{R})$, we set $\vec{\nabla}\phi(x)=(\partial_1\phi(x),\ldots,\ \partial_d\phi(x))$,

(iv) in a topological space (E,T), $x_n\xrightarrow[n\to\infty]{T}x$ naturally means that x_n converges toward x in the topology T ,

(v) for a random variable X taking values in $\mathbb{R}^p$, L_X is the distribution of X and $\hat{\mathsf{L}}_X(\xi)$ its characteristic function,

(vi) for a Radon measure μ, we set $\|\mu\|_{TV} = \sup_{\{\|\phi\|_{C^0}\le 1\}} <\phi,\mu>$ the total variation o μ

(vii) finally, in the spaces $\mathbb{R}^p$, $\|\cdot\|$ will be the Euclidean norm.

The following definition is preponderant in this section.

Definition(1.2.2)[33]: Let $S = (\Omega, \mathcal{F}, \mathrm{P}, \mathrm{D}, \Gamma)$ be a Dirichlet structure. We say that S satisfies the energy image density criterion, if and only if, for all $p \ge 1$, for all $X = (X_1,\dots,X_p) \in \mathrm{D}^p$:

$$X_*\left(1_{\{\det \Gamma[X]>0\}} d\mathrm{P}\right) << d\lambda_p\Big).$$

Conjecture. (Bouleau-Hirsch)

Every Dirichlet structure (in the sense of (1.2.1) satisfies the criterion E.I.D..

As already mentioned, we refer to [39, 40, 41] for examples and sufficient conditions entailing E.I.D.. The most illustrative example of this kind of structure is the Sobolev space $\mathrm{H}^1(\Omega,\lambda_d)$ where Ω is a bounded open subset of $\mathbb{R}^d$, and λ_d the d-dimensional Lebesgue measure In this case:

(i) $(\Omega,\mathcal{F},\mathrm{P}) = \left(\Omega, \mathcal{B}(\Omega), \frac{d\lambda_d}{\lambda_d(\Omega)}\right)$,

(ii) $\mathrm{D} = \mathrm{H}^1(\Omega)$,

(iii) $\Gamma[\phi] = \vec{\nabla}\phi \cdot {}^t\vec{\nabla}\phi$,

(iv) $\varepsilon[\phi] = \dfrac{1}{\lambda_d(\Omega)}\displaystyle\int_\Omega \vec{\nabla}\phi\, {}^t\vec{\nabla}\phi \; d\lambda_d$

Theorem(1.2.3)[33]: Let $S = (\Omega, \mathcal{F}, \mathrm{P}, \mathrm{D}, \Gamma)$ be a Dirichlet structure. Let $(X_n)_{n\in\mathbb{N}}$ and X be in the domain D. We assume that $X_n \xrightarrow[n\to\infty]{\mathrm{D}} X$, then for any $Z \in L^1(\mathrm{P})$ supported in $\{\Gamma[X] > 0\}$:

$$\sup_{\|\phi\|_{C^0}\le 1} \mathrm{E}\{(\phi(X_n) - \phi(X))Z\} \xrightarrow[n\to\infty]{} 0. \tag{48}$$

For instance, if almost surely $\Gamma[\mathrm{X}] > 0$, then $\left\|\mathrm{L}_{X_n} - \mathrm{L}_X\right\|_{TV} \xrightarrow[n\to\infty]{} 0.$

Proof: Let M be a positive constant and let us fix a sequence ϕ_n in E_M such that:

$$\left|\sup_{\phi\in E_M} \mathrm{E}\{(\phi(X_n) - \phi(X))Z\} - \mathrm{E}\{(\phi_n(X_n) - \phi_n(X))Z\}\right| \le \frac{1}{n}.$$

Up to extracting a subsequence, we assume that $(\phi_n(X_n) - \phi_n(X))$ converges weakly in $L^\infty(\mathrm{P})$ toward $Y \in L^\infty(\mathrm{P})$. In particular, for all $U \in \mathrm{D}(A)$ and all $\mathrm{W} \in \mathrm{D} \cap L^\infty(\mathrm{P})$:

$$\mathrm{E}\{(\phi_n(X_n) - \phi_n(X))\,\Gamma[X,U]\,W\} \xrightarrow[n\to\infty]{} \mathrm{E}\{Y\,\Gamma[X,U]\,W\}.$$

Using Lemma (1.2.9), we deduce $\mathrm{E}\{Y\,\Gamma[X,U]\,W\} = 0$: Besides, $\mathsf{D}(A)$ is dense in D for the norm $\|\cdot\|_{\mathsf{D}}$ and $\mathsf{D} \cap L^\infty(\mathsf{P})$ is dense in $L^1(\mathrm{P})$ so that $Y\,\Gamma[X] = 0$ a.s. and hence $YZ = 0$ a.s., and in consequence,

$$\lim_{n\to\infty} \sup_{\phi\in E_M} \mathrm{E}\{(\phi(X_n) - \phi(X))Z\} = \mathrm{E}\{YZ\} = 0.$$

In the general case, let us notice that:

$$\left|\sup_{\phi\in E_M} \mathrm{E}\left\{(\phi(X_n)-\phi(X))1_{\{\Gamma[X]>0\}}Z\right\} - \sup_{\|\phi\|_{C^0}\leq 1} \mathrm{E}\left\{(\phi(X_n)-\phi(X))1_{\{\Gamma[X]>0\}}Z\right\}\right|$$
$$\leq \mathrm{E}\left\{\left(1_{\{|X_n|>M\}}+1_{\{|X|>M\}}\right)Y\right\}\xrightarrow[M\to\infty]{}0.$$

Corollary(1.2.4)[33]: $S=(\Omega,\mathcal{F},\mathrm{P},\mathbb{D},\Gamma)$ be a Dirichlet structure and let $(X_n)_{n\in\mathbb{N}}$ and X be in $\mathbb{D}$. If $X_n \xrightarrow[n\to\infty]{\mathbb{D}} X$ then for any Lipschitz map $F, F(X_n)\xrightarrow[n\to\infty]{\mathbb{D}} F(X)$.

Proof: Let us be placed under the assumptions of Corollary, let f be a Borel representation of F' and let K be the Lipschitz constant of *F*. First, as the mapping *F* is *K*-Lipschitz, it is straightforward that:

$$F(X_n)\xrightarrow[n\to+\infty]{L^2(\mathrm{P})} F(X).$$

For real valued variables of $\mathbb{D}$, the E.I.D. criterion enables Lipschitz functional calculus. Thus, we have:

$$\begin{aligned}\Gamma[F(X_n)-F(X)] &= f(X_n)^2\Gamma[X_n]+f(X)^2\Gamma(X)\\ &- 2f(X_n)f(X)\Gamma\left[X_n,X\right],\\ &= (F(X)-F(X_n))^2\Gamma\left[X\right]+R_n,\end{aligned}$$

with

$$\mathrm{E}\{|R_n|\} \leq K^2\,\mathrm{E}\{|\Gamma[X_n]-\Gamma[X]|\}+2K^2\sqrt{\mathrm{E}[X_n-X]}\sqrt{\mathrm{E}[U]}\xrightarrow[n\to\infty]{}0.$$

Moreover,

$$\begin{aligned}\mathrm{E}\{(f(X)-f(X_n))^2\Gamma[X]\} &= \mathrm{E}\{(f(X_n)^2-f(X)^2)\Gamma[X]\}\\ &+ 2\mathrm{E}\{f(X)\Gamma[X](f(X)-f(X_n))\}\\ &= A_n+B_n.\end{aligned}$$

By Theorem (1.2.3), we know that for every bounded Borelian mapping $\phi:\mathbb{R}\to\mathbb{R}$ and every Z in $L^1(\mathrm{P})$ supported in $\{\Gamma[X]\neq 0\}$,

$$\mathrm{E}\{(\phi(X_n)-\phi(X))Z\}\xrightarrow[n\to\infty]{}0$$

In particular:

(i) For $Z=\Gamma[X]$ and $\phi=f^2$, we obtain that $\lim_{n\to\infty}A_n=0.$

(ii) For $Z=f(X)\Gamma[X]$ and $\phi=f$, we obtain that $\lim_{n\to\infty}B_n=0.$

In conclusion:

$$\lim_{n\to\infty}\mathrm{E}\{\Gamma[F(X_n)-F(X)]\} = 0$$

and the corollary is proven.

Now we come to following Theorem , it is the first result in full generality in the direction of showing the conjecture of Bouleau-Hirsch. In order to exploit the uniformity provided by Theorem (1.2.3), we will use linear combinations of the variables $(X_1,\cdots,X_p)$ and $(X_n^{(1)},\cdots,X_n^{(p)})$.

Theorem(1.2.5)[33]: Let $S=(\Omega,\mathcal{F},\mathrm{P},\mathbb{D},\Gamma)$ a Dirichlet structure and let $X_n=(X_n^{(1)},\cdots,X_n^{(p)})$ and $X=(X^{(1)},\cdots,X^{(p)})$ be in $\mathbb{D}^p$. Besides, we assume that almost surely $\det\Gamma\left[X\right]>0$ and that $X_n\xrightarrow[n\to\infty]{\mathbb{D}^p}X$. Then:

$$\sup_{\zeta\in\mathbb{R}^p}\left\|\hat{L}_{X_n}(\zeta)-\hat{L}_X(\zeta)\right\|\xrightarrow[\|\zeta\|\to\infty]{}0. \tag{49}$$

Particularly, we get that $\hat{L}_X(\zeta)\xrightarrow[\|\zeta\|\to\infty]{}0,$ that is to say L_X is a Rajchman measure.

Proof: Let us be placed under the assumptions of the Theorem , and let ζ_n be in $\mathbb{R}^p$ such that:

$$\left|\sup_{\zeta\in\mathbb{R}^p}\mathrm{E}\left\{e^{i<X_n,\zeta>}-e^{i<X,\zeta>}\right\}-\mathrm{E}\left\{e^{i<X_n,\zeta_n>}-e^{i<X,\zeta_n>}\right\}\right|\le\frac{1}{n}. \tag{50}$$

Now, we rewrite the right term:

$$\mathrm{E}\left\{e^{i<X_n,\zeta_n>}-e^{i<X,\zeta_n>}\right\}=\mathrm{E}\left\{e^{i\|\zeta_n\|<X_n,\frac{\zeta_n}{\|\zeta_n\|}>}-e^{i\|\zeta_n\|<X,\frac{\zeta_n}{\|\zeta_n\|}>}\right\}. \tag{51}$$

By compactness of the Euclidean p -sphere, up to extracting a subsequence, we may assume that $\frac{\zeta_n}{\|\zeta_n\|}\xrightarrow[n\to\infty]{}\zeta$, where $\|\zeta\|=1$. We then deduce that:

$$\left\langle X_n,\frac{\zeta_n}{\|\zeta_n\|}\right\rangle\xrightarrow[n\to\infty]{\mathrm{D}}\left\langle X,\zeta\right\rangle.$$

Besides $\Gamma\left[<X,\zeta>\right]=\zeta\ \Gamma[X]^t\ \zeta>0$. We are under the assumptions of Theorem (1.2. 3) by choosing $\phi_n(x)=e^{i\|\zeta_n\|x}$, and thanks to (51) we get $\lim_{n\to\infty}\mathrm{E}\left\{e^{i<X_n,\zeta_n>}-e^{i<X,\zeta_n>}\right\}=0.$ Using inequality (50):

$$\sup_{\zeta\in\mathbb{R}^p}\left|\hat{L}_{X_n}(\zeta)-\hat{L}_X(\zeta)\right|\xrightarrow[n\to\infty]{}0$$

In order to show that L_X is Rajchman, it is enough to make a convolution. More precisely, let $Y_n=X+\frac{1}{n}(U_1,\cdots,U_p)$ where U_i are i.i.d. with common law $1_{[0,1]}(x)\ dx$. In fact, the variables U_i may be thought as the coordinates of the usual Dirichlet structure:

$$([0,1],\mathrm{B}([0,1]),1_{[0,1]}(x)\,dx,\mathrm{H}^1([0,1]),\Gamma[\phi]=((\phi')^2)^p.$$

In addition, $Y_n\xrightarrow[n\to\infty]{\mathrm{D}}X$ where D is the domain of the product structure:

$$S\times([0,1],\mathrm{B}([0,1]),1_{[0,1]}(x)\,dx,\mathrm{H}^1([0,1]),\Gamma[\phi]=((\phi')^2)^p.$$

Being absolutely continuous, $\hat{L}_{Y_n}(\zeta)\xrightarrow[\|\zeta\|\to\infty]{}0.$ We may use the first part of the Theorem which ensures that:

$$\sup_{\zeta\in\mathbb{R}^p}\left|\hat{L}_{Y_n}(\zeta)-\hat{L}_X(\zeta)\right|\xrightarrow[n\to\infty]{}0$$

to complete the proof.

Theorem(1.2.6)[33]: Let $d\ge p$ be two integers, and Ω be an open subset of $\mathbb{R}^d$. (F_n) be a sequence in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^p)$ converging to F in $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^p)$ and K in $L^1(\Omega,\mathbb{R})$ a measurable map supported in $\{\det\Gamma[F]\neq 0\}$, where $\Gamma[F]=\vec{\nabla}F.{}^t\vec{\nabla}F$. Then

$$(F_n)_*(Kd\lambda)\xrightarrow[n\to+\infty]{VT}F_*(Kd\lambda)$$

Proof: Let $\mu_n=(F_n)_*(K\ d\lambda),\ \mu=F_*(K\ d\lambda)$. We fix ω in $C_C^1(\Omega,\mathbb{R})$ supported on some bounded open set Ω', and we write:

$$\int_{\mathbb{R}^p} \varphi \, d\mu - \int_{\mathbb{R}^p} \varphi \, d\mu_n = I_1 + I_2 + I_3$$

with

$$\begin{cases} I_1 = \int_{\Omega'} (\varphi o F)\omega \mathrm{J}_I(F) \, d\lambda - \int_{\Omega'} (\varphi o F_n)\omega \mathrm{J}_I(F_n) \, d\lambda \\ I_2 = \int_{\Omega'} (\varphi o F_n)\omega(\mathrm{J}_I(F_n) - \mathrm{J}_I(F)) \, d\lambda \\ I_2 = \int_{\Omega} (\varphi o F - \varphi o F_n)(K - \omega \mathrm{J}_I(F)) \, d\lambda. \end{cases}$$

Since (F_n) is bounded in $\mathrm{W}^{1,p}(\Omega')$, using Lemma (1.2.14) with $r=s=p$ we can bound I_1 by $C\|F_n - F\|_{L^p(\Omega')}$ where C does not depend on n. Besides, integrals I_2 and I_3 are respectively less than $2\|\mathrm{J}_I(F_n) - \mathrm{J}_I(F)\|_{L^1(\Omega')}$ and This leads to $\|\mu_n - \mu\|_{TV} \le C\|F_n - F\|_{L^p(\Omega')} + 2\|\mathrm{J}_I(F_n) - \mathrm{J}_I(F)\|_{L^1(\Omega')} + \|\omega \mathrm{J}_I(F) - K\|_{L^1(\Omega)}$ Letting n tend to $+\infty$, we get

$$\limsup_{n \to +\infty} \|\mu_n - \mu\|_{TV} \le \|\omega \mathrm{J}_I(F) - K\|_{L^1(\Omega)}.$$

Since $\frac{K}{\mathrm{J}_I(F)} \in L^1(\Omega, |\mathrm{J}_I(F)| d\lambda)$ and since $C_C^1(\Omega, \mathbb{R})$ is dense in $L^1(\Omega, |\mathrm{J}_I(F)| d\lambda)$, we may choose ω such that $\|\omega \mathrm{J}_I(F) - K\|_{L^1(\Omega)}$ is as small as wished. We deduce:

$$\limsup_{n \to +\infty} \|\mu_n - \mu\|_{TV} = 0.$$

Note that: The conclusion of Theorem (1.2.6) would remain true replacing the assumption $F_n \xrightarrow[n \to +\infty]{\mathrm{W}^{1,p}(\Omega)} F$ by

$$\begin{cases} F_n \xrightarrow[n \to +\infty]{\text{Prob}} F \\ \sup_n \left\| \vec{\nabla}_I F_n \right\|_{L^{p-1}(\Omega)} < +\infty \; . \\ J_I(F_n) \xrightarrow[n \to +\infty]{L^1(\Omega)} J_I(F) \end{cases}$$

This is a consequence of the following estimate : for ψ in $C^0(\mathbb{R}^p, \mathbb{R})$ supported in $[-M, M]^p$ such that $\|\psi\|_{C^0} \le 1$,

$$\left| \int_{\Omega} (\psi o F_n) \mathrm{J}_I(F_n)\omega \, d\lambda - \int_{\Omega} (\psi o F) \mathrm{J}_I(F)\omega \, d\lambda \right|$$

$$\le \int_{\Omega} \left(\|F_n(x) - F(x)\| \wedge 2M \right) \left| \det(\vec{\nabla}_I \omega, \vec{\nabla}_I f_2, \dots, \vec{\nabla}_I f_p) \right| d\lambda.$$

The right term tends to 0 under above assumptions, which next allows to conclude as in Theorem (1.2.6) proof.

Corollary(1.2.7)[33]: We endow $\mathbb{R}^{\mathbb{N}}$ with the Gaussian probability $\mathrm{P} = \mathrm{N}(0,1)^{\mathbb{N}}$. Let p an integer, (F_n) a sequence in $\mathrm{W}^{1,p}(\mathrm{P})$ converging to F in $\mathrm{W}^{1,p}(\mathrm{P})$. Let I a finite subset of $\mathbb{N}$ with cardinal p and K in $L^1(\mathrm{P})$ supported in $\{\det \Gamma[F] \neq 0\}$ where $\Gamma[F] = \vec{\nabla} F . {}^t\vec{\nabla} F$. Then

$$(F_n)_*(KdP) \xrightarrow[n \to +\infty]{T.V.} F_*(Kdp)$$

Proof: We fix an integer d larger than $\max I$. For y in $\mathbb{R}^{\mathbb{N}}$ and x in $\mathbb{R}^d$ we denote $i_y(x) = (x, y)$. Up to extracting a subsequence of (F_n), we can assume that for almost every y in $\mathbb{R}^{\mathbb{N}}$,

$$F_n \circ i_y \xrightarrow[n\to+\infty]{W^{1,p}(\mathbb{R}^d, N(0,1)^d)} F \circ i_y .$$

Thus, for almost every y in $\mathbb{R}^{\mathbb{N}}$, and every ψ in $C^0(\mathbb{R}^N, \mathbb{R})$ satisfying $\|\psi\|_{C^0} \le 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\psi \circ (F_n \circ i_y)(x) - \psi \circ (F \circ i_y)(x)\right)\left(K \circ i_y(x)\right) e^{-|x|^2}\, dx \\
&\quad = \int_{|x| \le M} \left(\psi \circ (F_n \circ i_y)(x) - \psi \circ (F \circ i_y)(x)\right)\left(K \circ i_y(x)\right) e^{-|x|^2}\, dx \\
&\quad + \int_{|x| > M} \left(\psi \circ (F_n \circ i_y)(x) - \psi \circ (F \circ i_y)(x)\right)\left(K \circ i_y(x)\right) e^{-|x|^2}\, dx \\
&\quad = A_n + B_n .
\end{aligned}$$

Using a Markov inequality, for all $\epsilon > 0$ there exist $M > 0$ such that $\sup_{n,\psi} B_n \le \epsilon$. Besides, A_n tends to 0 as n tends to $+\infty$ uniformly in ψ by Theorem (1.2.6). Integrating on y, we deduce that:

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\psi \circ F_n - \psi \circ F) K \; d\mathrm{P} \\
&\quad = \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^d} (\psi \circ (F_n \circ i_y)(x) - \psi \circ (F \circ i_y)(x))(K \circ i_y(x))\; e^{-|x|^2}\, dx d\mathrm{P}(y)
\end{aligned}$$

tends to 0 uniformly in ψ as n tends to $+\infty$.

In the same way as Corollary (1.2.4), we can deduce from Theorem (1.2.6) that the convergence in $W^{1,p}(\Omega)$ is preserved by Lipschitz mapings. But in this case contrarly to Corollary (1.2.4), a non-degenerescence of the limit is required.

Corollary(1.2.8)[33]: Let $d \ge p$ be two integers, Ω be an open set of $\mathbb{R}^d$ and (F_n), F belong to $W^{1,p}(\Omega, \mathbb{R}^p)$ such that

$$\begin{cases} F_n \xrightarrow[n\to+\infty]{W^{1,p}} F \\ \det \Gamma[F] = \det\left(\vec{\nabla} F . {}^t\vec{\nabla} F\right) > 0 \; a.e. \end{cases}$$

Then for any Lipschitz mapping $\Phi : \mathbb{R}^p \to \mathbb{R}$:

$$\Phi \circ F_n \xrightarrow[n\to+\infty]{W^{1,p}} \Phi \circ F .$$

Proof: Since Φ is Lipchitz it is straightforward that

$$\Phi \circ F_n \xrightarrow[n\to+\infty]{L^p} \Phi \circ F .$$

Next, fixing k in $\{1,...,d\}$, we have

$$\begin{aligned}
\partial_k (\Phi \circ F_n - \Phi \circ F) &= \sum_{i=1}^{n} \partial_k f_i^{(n)} (\partial_i \Phi \circ F_n) - \partial_k f_i (\partial_i \Phi \circ F) \\
&= \sum_{i=1}^{n} \partial_k f_i (\partial_i \Phi \circ F_n - \partial_i \Phi \circ F) + R_n
\end{aligned}$$

where $R_n \xrightarrow[n\to+\infty]{L^p(\Omega)} 0.$

Now we fix i in $\{1,...,d\}$, and we want to show that $\partial_k f_i(\partial_i\Phi \circ F_n - \partial_i\Phi\circ F)$ converges to 0 in $L^p(\Omega)$. Since this sequence is clearly bounded in $L^p(\Omega)$, it is sufficient to show the convergence in $L^p_{\text{loc}}(\Omega)$, and hence we can assume that $\lambda(\Omega)<+\infty$. We will follow the same strategy that in proof of corollary (1.2.4). First, we write that

$$\|\partial_i\Phi\circ F_n - \partial_i\Phi\circ F\|^2_{L^2(\Omega)} = \int_\Omega (\partial_i\Phi\circ F_n)^2\, d\lambda$$
$$-\ 2\int_\Omega (\partial_i\Phi\circ F_n)(\partial_i\Phi\circ F)\, d\lambda + \int_\Omega (\partial_i\Phi\circ F)^2\, d\lambda$$

By Theorem (1.2.6), we know that for any bounded borelian map $\varphi : \mathbb{R}^p \to \mathbb{R}$ and any mapping K in $L^1(\Omega)$,

$$\lim_{n\to+\infty}\int_\Omega (\varphi \circ F_n) K d\lambda = \int_\Omega (\varphi \circ F_n) K d\lambda.$$

In particular:

(i) For $K = 1$ and $\varphi = \partial_i\Phi^2$, we obtain

$$\lim_{n\to+\infty}\int_\Omega (\partial_i\Phi \circ F_n)^2\, d\lambda = \int_\Omega (\partial_i\Phi \circ F)^2\, d\lambda.$$

(ii) For $K = \partial_i\Phi \circ F$ and $\varphi = \partial_i\Phi$, we obtain

$$\lim_{n\to+\infty}\int_\Omega (\partial_i\Phi \circ F_n)(\partial_i\Phi \circ F)\, d\lambda = \int_\Omega (\partial_i\Phi \circ F)^2\, d\lambda.$$

Thus, we deduce that $\partial_i\Phi \circ F_n$ converges toward $\partial_i\Phi \circ F$ in $L^2(\Omega)$,and hence in $L^p(\Omega)$ by Holder inequality since $\partial_i\Phi$ is bounded. Finally, writing that

$$\|\partial_k f_i(\varphi \circ F_n - \varphi \circ F)\|_{L^p(\Omega)} \le 2\left\|\partial_k f_i 1_{\{|\partial_k f_i|>M\}}\right\|_{L^p(\Omega)} + M\|\varphi \circ F_n - \varphi \circ F\|_{L^p(\Omega)}$$

Where M is arbitrarily large, we deduce that $\partial_k f_i(\varphi\circ F_n)$ converges toward $\partial_k f_i(\varphi\circ F)$ in $L^p(\Omega)$.

Let us mention that in the multivariate setting, the integration by parts relies on Schwartz property $(\partial_x\partial_y = \partial_y\partial_x)$, and we failed in finding an analogue in the general context.

Lemma(1.2.9)[33]: For all $M>0$, let $E_M = \left\{\phi\in C_c^0([-M,M]) \,\middle|\, \|\phi\|_{C^0}\le 1\right\}$. Then for any sequence ϕ_n in E_M, for any $U\in \mathbb{D}(A)$ and any $W\in L^\infty(\mathbb{P})\cap\mathbb{D}$:

$$\lim_{n\to\infty} \mathbb{E}\{(\phi_n(X_n)-\phi_n(X))\Gamma[X,U]\,W\} = 0.$$

Proof. Let us be placed under the assumptions of the lemma. Using functional calculus we have:

$$\Gamma[\Phi_n(X_n)-\Phi_n(X),U] = \phi_n(X_n)\Gamma[X_n,U]-\phi_n(X)\Gamma[X,U] \qquad (52)$$

Cauchy-Schwarz inequality entails:

$$\mathbb{E}\{|\Gamma[X_n,U]-\Gamma[X,U]|\} \le \sqrt{\mathbb{E}[X_n-X]}\sqrt{\mathbb{E}[U]}\quad . \qquad (53)$$

Functional calculus (52) and inequality (53) ensure that:

$$\left|\mathbb{E}\{\Gamma[\Phi_n(X_n)-\Phi_n(X),U]\,W\} - \mathbb{E}\{(\phi_n(X_n)-\phi_n(X)),\Gamma[X,U]\,W\}\right|$$
$$\le \|W\|_\infty\sqrt{\mathbb{E}[X_n-X]}\sqrt{\mathbb{E}[U]}. \qquad (54)$$

Moreover, the usual integration by parts leads to:

$$\mathbb{E}\{\Gamma[\Phi_n(X_n)-\Phi_n(X),U]\,W\} = \mathbb{E}\{(\Phi_n(X_n)-\Phi_n(X))(\Gamma[U,W]$$
$$-A[U]W)\} \qquad (55)$$

Finally using both (54) and (55), we get:

$$\begin{aligned}\left|E\{(\phi_n(X_n)-\phi_n(X))\Gamma[X,U]\,W\}\right| &\le E\{|\Phi_n(X_n)-\Phi_n(X)|\,|\Gamma[U,W]-A[U]\,W|\}\\ &\le E\{|X_n-X|\wedge(2M))\,|\Gamma[U,W]-A[U]\,W|\}\\ &+\|W\|_\infty\sqrt{E[X_n-X]}\sqrt{E[U]}. \qquad (56)\end{aligned}$$

Since, $|\Gamma[U,W]-A[U]\,W| \in L^1(P)$, inequality (56) ensures that:

$$\lim_{n\to\infty} E\{(\phi_n(X_n)-\phi_n(X))\,\Gamma[X,U]\,W\} = 0.$$

We introduce the following definition which is a generalization to the multivariate case.

Definition(1.2.10)[33]: Let $S=(\Omega,F,P,D,\Gamma)$ a Dirichlet structure. We will say that S satisfies the "strong" energy image density E.I.D., if for any $X=(X_1,\cdots,X_p)$ and any sequence $X_n=(X_n^{(1)},\cdots,X_n^{(p)})$ in D^p with $\|X_n-X\|_{D^p}\xrightarrow[n\to\infty]{}0$ and for any

$$Z\in L^1(P):\ \sup_{\|\phi\|_{C^0(\mathbb{R}^p,\mathbb{R})}\le 1} E\left\{(\phi(X_n)-\phi(X))1_{\{\det\Gamma[X]>0\}}Z\right\}\xrightarrow[n\to\infty]{}0. \qquad (57)$$

The terminology of E.I.D. is justified by the next proposition.

Proposition(1.2.11)[33]: Let $S=(\Omega,F,P,D,\Gamma)$ be a Dirichlet structure satisfying S.E. I.D., then it satisfies the criterion E.I.D..

Proof: Let A be a Borel subset of $\mathbb{R}^n$ negligible with respect to the Lebesgue measure and let $X=(X_1,\cdots,X_n)$ be in D^n. Let us be given $\hat{U}=(\hat{U}_1,\cdots,\hat{U}_n)$ n random variables i.i.d. with common law $1_{[0,1]}(x)\,dx$ and defined on an independent probability space $(\hat{\Omega},\hat{F},\hat{P})$. Let us take $Z=1$ and let us apply E.I.D.:

$$\lim_{n\to\infty}\hat{E}\left\{E\left\{\left(\chi_A(X+\frac{1}{n}\hat{U})-\chi_A(X)\right)1_{\{\det\Gamma[X]>0\}}\right\}\right\} = 0 \qquad (58)$$

But Fubini theorem entails:

$$\hat{E}\left\{E\left\{\chi_A(X+\frac{1}{n}\hat{U})\right\}\right\} = E\left\{\hat{E}\left\{\chi_A(X+\frac{1}{n}\hat{U})\right\}\right\} = 0.$$

Finally, (58) ensures that $E\left\{\chi_A(X)1_{\{\det\Gamma[X]>0\}}\right\} = 0$.

We show a weaker version of E.I.D. criterion in the Sobolev spaces $W_1^p(\mathbb{R}^p)$. The result is weaker because we need stronger moments on the variables $\Gamma[X]$. Fortunately, in every practically encountered cases, this assumption is fulfilled. Besides, in the particular setting of standard Sobolev spaces, we are able to show several generalisations of the S.E.I.D. criterion.

Let us be placed under the assumptions of the theorem(1.2.6). If $I=\{i_1,\cdots,i_p\}$ is a subset of $\{1,\ldots,d\}$, for f in $W^{1,p}(\Omega,\mathbb{R})$ we denote $\vec{\nabla}_I f=(\partial_{i_1}f,\ldots,\partial_{i_p}f)$, and for $F=(f_1,\ldots,f_p)$ in $W^{1,p}(\Omega,\mathbb{R}^p)$, we denote $J_I(F)=\det(\vec{\nabla}_I f_1,\ldots,\vec{\nabla}_I f_d)$. With these Notations, we have

$$\det\Gamma[F] = \sum_{\substack{I\subset\{1,\ldots,d\}\\ |I|=p}}|J_I(F)|^2.$$

Thus, up to replacing K by $K1_{\{J_I(F)\neq 0\}}$, we will assume that K is supported in $\{J_I(F)\neq 0\}$ from some subset $I=\{i_1,\cdots,i_p\}$.

Lemma(1.2.12)[33]: For $f_1,...,f_p$ in $W^{1,p}(\Omega,\mathbb{R})$ and ω in $C_c^1(\Omega,\mathbb{R})$:

$$\int_\Omega \omega \ \det(\vec{\nabla}_I f_1,\vec{\nabla}_I f_2,...,\vec{\nabla}_I f_p)\,d\lambda = -\int_\Omega f_I \ \det(\vec{\nabla}_I \omega,\vec{\nabla}_I f_2...,\vec{\nabla}_I f_p)\,d\lambda.$$

Proof: Without loss of generality, we can assume that $I=\{1,...,p\}$. Besides, since $C^2(\Omega,\mathbb{R})$ is dense in $W^{1,p}(\Omega,\mathbb{R})$ and $(f_1,...,f_p)\to\det(\vec{\nabla}_I f_1,\dots,\vec{\nabla}_I f_p)$ is continuous from $W^{1,p}(\Omega,\mathbb{R}^p)$ into $L^1(\Omega,\mathbb{R})$, we can also assume that $f_1,...,f_p$ are C^2 on Ω. Noticing that

$$\omega\ \det(\vec{\nabla}_I f_1,\vec{\nabla}_1 f_2,...,\vec{\nabla}_I f_p) + f_1\det(\vec{\nabla}_I\omega,\vec{\nabla}_I f_2,...,\vec{\nabla}_I f_p) = \det(\vec{\nabla}_I(\omega f_1),\vec{\nabla}_I f_2,...,\vec{\nabla}_I f_p).$$

and that

$$\int_\Omega \det(\vec{\nabla}_I(\omega f_1),\vec{\nabla}_I f_2,...,\vec{\nabla}_I f_p)\,d\lambda = \int_\Omega\left(\sum_{\sigma\in S_p}\varepsilon(\sigma)\partial_{\sigma(1)}f_1\cdots\partial_{\sigma(p)}f_p\right)d\lambda.$$

$$= -\int_\Omega \omega f_1\left(\sum_{\sigma\in S_p}\varepsilon(\sigma)\partial_{\sigma(1)}(\partial_{\sigma(2)}f_2...\partial_{\sigma(p)}f_p)\right)d\lambda.$$

it is sufficient to prove the algebraic relation

$$\sum_{\sigma\in S_p}\varepsilon(\sigma)\,\partial_{\sigma(1)}(\partial_{\sigma(2)}f_2...\partial_{\sigma(p)}f_p) = 0.$$

The left term equals $\sum_{k=1}^p h_k$ where

$$h_k = \sum_{\sigma\in S_p}\varepsilon(\sigma)\,\partial_{\sigma(1)}\partial_{\sigma(k)}f_1\prod_{j\neq k}\partial_{\sigma(j)}f_j.$$

Denoting by τ the transposition $(1,k)$ we have

$$h_k = \sum_{\sigma\in S_p}\varepsilon(\sigma\, o\,\tau)\,\partial_{\sigma\, o\,\tau(1)}\ \partial_{\sigma\, o\,\tau(k)}f_1\prod_{j\neq k}\partial_{\sigma(j)}f_j$$

$$= -\sum_{\sigma\in S_p}\varepsilon(\sigma)\,\partial_{\sigma(k)}\ \partial_{\sigma(1)}f_1\prod_{j\neq k}\partial_{\sigma(j)}f_j = -h_k$$

so that h_k is null, which completes the proof.

Using this integration by parts, we will establish this new one more general, relating integral of $\varphi o F$ and $\partial_1\varphi o F$ against some weights.

Lemma(1.2.13)[33]: For $F=(f_1,...,f_p)$ in $W^{1,p}(\Omega,\mathbb{R}^p)$, φ in $C^1(\mathbb{R}^p,\mathbb{R})$ and ω in $C_c^1(\Omega,\mathbb{R})$

$$\int_\Omega(\partial_1\varphi o F)\,\omega J_I(F)\ d\lambda = -\int_\Omega \varphi o F\ \det(\vec{\nabla}_I\omega,\vec{\nabla}_I f_2,...,\vec{\nabla}_I f_p)\ d\lambda$$

Proof: Let $\Phi=(\varphi,q_2,...,q_p)$, where q_k is the $k-$th canonical projection on $\mathbb{R}^d$. Then $\Phi o F=(\varphi o F, f_2,..., f_p)$, and Lemma (1.2.12) applied to $\varphi o F, f_2,..., f_p$ leads to

$$\int_\Omega \omega J_I(\Phi\, o F)\,d\lambda = -\int_\Omega \varphi o f\ \det(\vec{\nabla}_I\omega,\vec{\nabla}_I f_2,\cdots,\vec{\nabla}_I f_p)\ d\lambda.$$

Noticing that $J_I(\Phi)=\partial_1\varphi$ we have

$$J_I(\Phi o F) = \left(J_I(\Phi) o F\right) J_I\left(F\right) = \left(\partial_1 \varphi o F\right) J_I\left(F\right),$$

which completes the proof.

Lemma(1.2.14)[33]: For $F = \left(f_1, ..., f_p\right)$ and $G = \left(g_1, ..., g_p\right)$ in $W^{1,p}\left(\Omega, \mathbb{R}^p\right)$, for ψ in $C^0(\mathbb{R}^p, \mathbb{R})$ with $\|\psi\|_{C^0} \le 1$, for ω in $C_C^1(\Omega, \mathbb{R})$ and for r, s positive numbers with $\frac{1}{r+p} - \frac{1}{s} = 1$:

$$\left|\int_\Omega (\psi o F) J_I(F)\omega \, d\lambda - \int_\Omega (\psi o G) J_I(G)\omega \, d\lambda\right| \le C \|F - G\|_{L^r(\Omega)}$$

where

$$C = \sup_{1 \le k \le p}\left(\left\|\vec{\nabla}_I f_k\right\|_{L^s(\Omega)} + \left\|\vec{\nabla}_I g_K\right\|_{L^s(\Omega)}\right)^{p-1} \left\|\vec{\nabla}_I \omega\right\|_\infty .$$

Proof: Let us denote $H_k = \left(f_1, ..., f_k, g_{k+1}, ... g_p\right)$ for $k = 0, ..., p$. Since

$$(\psi o F) J_I(F) - (\psi o G) J_I(G) = \sum_{k=0}^{p-1} (\psi o H_{k+1}) J_I(H_{k+1}) - (\psi o H_k) J_I(H_k),$$

up to replacing $\left(F, G\right)$ by $\left(H_k, H_{k+1}\right)$, we may assume that F and G differ from only one coordinate, namely j. Besides, up to a permutation, we may assume $j = 1$, so that $f_k = g_k$ for $k \ne 1$. We set

$$\varphi\left(x_1, ..., x_p\right) = \int_0^{x_1} \psi\left(t, x_2, ..., x_p\right) dt.$$

Then by Lemma (1.2.13),

$$\begin{cases} \int_\Omega (\psi o F)\omega J_I(F) d\lambda = \int_\Omega \varphi o F \det(\vec{\nabla}_I \omega, \vec{\nabla}_I f_2, \cdots, \vec{\nabla}_I f_p) d\lambda \\ \int_\Omega (\psi o G)\omega J_I(G) d\lambda = \int_\Omega \varphi o G \det(\vec{\nabla}_I \omega, \vec{\nabla}_I f_2, \cdots, \vec{\nabla}_I f_p) d\lambda \end{cases},$$

and hence

$$\left|\int_\Omega (\psi o F) J_I(F)\omega \, d\lambda - \int_\Omega (\psi o G) J_I(G)\omega \, d\lambda\right|$$
$$\le \int_\Omega \left|\varphi o F - \varphi o G\right| \left|\det(\vec{\nabla}_I \omega, \vec{\nabla}_I f_2, \cdots, \vec{\nabla}_I f_p)\right| d\lambda.$$

Since $\left|\partial_1 \varphi\right| = \left|\psi\right| \le 1$, we have

$$\left|\varphi o F(x) - \varphi o G(x)\right| = \left|\int_{f_1(x)}^{g_1(x)} \psi(t, f_2(x), ..., f_p(x)) dt\right| \le \left|f_1(x) - g_1(x)\right|$$

and Lemma (1.2.14) is then as a consequence of the Hölder inequality.