Chapter 1

Classification of Nonsimple Graph C^*-algebras

Section (1.1): Graph C^*-algebra results and classification

The classification program for C^*-algebras has for the most part progressed independently for the classes of infinite and finite C^*-algebras. Great strides have been made in this program for each of these classes. In the finite case, Elliott's Theorem classifies all AF-algebras up to stable isomorphism by the ordered K_0-group. In the infinite case, there are a number of results for purely infinite C^*-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite C^*-algebras up to stable isomorphism by the K_0-group together with the K_1-group. For nonsimple purely infinite C^*-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite C^*-algebras containing exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of K-groups, Restorff has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtrated K-theory, and Meyer and Nest have shown that certain purely infinite C∗-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated K-theory. However, in all of these situations the nonsimple C^*-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

Restorff and Ruiz have provided a framework for classifying nonsimple C^*-algebras that are not necessarily AF-algebras or purely infinite C^*-algebras. In particular, they have shown that certain extensions of classifiable C^*-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. This has allowed for the classification of certain nonsimple C^*-algebras in which there are ideals and quotients of mixed type.

We consider the classification of nonsimple graph C^*-algebras. Simple graph C^*-algebras are known to be either AF-algebras or purely infinite algebras, and thus are classified by their K-groups according to either Elliott's Theorem or the Kirchberg-Phillips Theorem. Therefore, we begin by considering nonsimple graph

C^*-algebras with exactly one proper nontrivial ideal. These C^*-algebras will be extensions of simple C^*-algebras that are AF or purely infinite by other simple C^*-algebras that are AF or purely infinite — with mixing of the types allowed. We are able to show that a graph C^*-algebra with exactly one proper nontrivial ideal is classified up to stable isomorphism by the six-term exact sequence in K-theory of the corresponding extension. Additionally, we are able to show that a graph C^*-algebra with a largest proper ideal that is an AF-algebra is also classified up to stable isomorphism by the six-term exact sequence in K-theory of the corresponding extension.

Note that the extensions of graph C^*-algebras constitute a very large class. Every AF-algebra is stably isomorphic to a graph C^*-algebra, and every Kirchberg algebra with free K_1-group is stably isomorphic to a graph C^*-algebra. Thus the extensions we consider comprise a wide variety of extensions of AF-algebras.

We establish some basic facts and notation for graph C^*-algebras and extensions.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s\colon E^1 \to E^0$ identifying the range and source of each edge. A vertex $v \in E^0$ is called a sink if $|s^{-1}(v)| = 0$, and v is called an infinite emitter if $|s^{-1}(v)| = \infty$. A graph E is said to be row-finite if it has no infinite emitters. If v is either a sink or an infinite emitter, then we call v a singular vertex. We write E^0_{sing} for the set of singular vertices. Vertices that are not singular vertices are called regular vertices and we write E^0_{reg} for the set of regular vertices. For any graph E, the vertex matrix is the $E^0 \times E^0$ matrix A_E with $A_e(v,w) := \ |\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}|$. Note that the entries of A_E are elements of $\{0,1,2,\dots\} \cup \{\infty\}$.

If E is a graph, a Cuntz-Krieger E-family is a set of mutually orthogonal projections $\{pv : v \in E^0\}$ and a set of partial isometries $\{se : e \in E^1\}$ with orthogonal ranges which satisfy the Cuntz-Krieger relations:

(i) $s_e^*\, s_e = pr(e)$ for every $e \in E^1$;

(ii) $s_e s_e^* \leq ps(e)$ for every $e \in E^1$;

(iii) $pv \sum_{s(e)=v} s_e s_e^*$ for every $v \in E^0$ that is not a singular vertex.

The graph algebra $C^*(E)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger E-family.

A path in E is a sequence of edges $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \le i < n$, and we say that α has length $|\alpha| = n$. We let E^n denote the set of all paths of length n, and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of finite paths in G. Note that vertices are considered paths of length zero. The maps r, s extend to E^*, and for $v, w \in E^0$ we write $v \ge w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Also for a path $\alpha := \alpha_1 \ldots \alpha_n$ we define $s_\alpha := s_{\alpha_1} \ldots s_{\alpha_n}$, and for a vertex $v \in E^0$ we let $s_v := p_v$. It is a consequence of the Cuntz-Krieger relations that $C^*(E) = \overline{\text{span}}\{s_\alpha s^*\beta : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$.

We say that a path $\alpha := \alpha_1 \ldots \alpha_n$ of length 1 or greater is a cycle if $r(\alpha) = s(\alpha)$, and we call the vertex $s(\alpha) = r(\alpha)$ the base point of the cycle. A cycle is said to be simple if $s(\alpha_i) \ne s(\alpha_1)$ for all $1 < i \le n$. The following is an important condition in the theory of graph C^*-algebras.

Condition (K): No vertex in E is the base point of exactly one simple cycle; that is, every vertex is either the base point of no cycles or at least two simple cycles.

For any graph E a subset $H \subseteq E^0$ is hereditary if whenever $v, w \in E^0$ with $v \in H$ and $v \ge w$, then $w \in H$. A hereditary subset H is saturated if whenever $v \in E^0_{reg}$ with $r(s^{-1}(v)) \subseteq H$, then $v \in H$. For any saturated hereditary subset H, the breaking vertices corresponding to H are the elements of the set

$$B_H \coloneqq \{v \in E^0 : |s^{-1}(v)| = \infty \;\; \text{and} \;\; 0 < |s^{-1}(v) \cap r^{-1}(E^0 \backslash H)| < \infty\}.$$

An admissible pair (H, S) consists of a saturated hereditary subset H and a subset $S \subseteq B_H$. For a fixed graph E we order the collection of admissible pairs for E by defining $(H, S) \le (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. For any admissible pair (H, S) we define

$$I(H, S) := \text{the ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\} \cup \{p^H_{v_0} : v0 \in S\},$$

where $p^H_{v_0}$ is the gap projection defined by

$$p_{v_0}^H := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e)\notin H}} s_e s_e^*.$$

Note that the definition of BH ensures that the sum on the right is finite.

For any graph E there is a canonical gauge action $\gamma: \mathbb{T} \to \mathrm{Aut} C^*(E)$ with the property that for any $z \in \mathbb{T}$ we have $\gamma_z(p_v) = p_v$ for all $v \in E^0$ and $\gamma_z(s_e) = zs_e$ for all $e \in E^1$. We say that an ideal $I \lhd C^*(E)$ is gauge invariant if $\gamma_z(I) \subseteq I$ for all $z \in \mathbb{T}$.

There is a bijective correspondence between the lattice of admissible pairs of E and the lattice of gauge-invariant ideals of $C^*(E)$ given by $(H, S) \mapsto I_{(H,S)}$. When E satisfies Condition (K), all ideals of $C^*(E)$ are gauge invariant and the map $(H, S) \longmapsto I_{(H,S)}$ is onto the lattice of ideals of $C^*(E)$. When $B_H = \emptyset$, we write I_H in place of $I_{(H,\emptyset)}$ and observe that I_H equals the ideal generated by $\{p_v: v \in H\}$. Note that if E is row-finite, then B_H is empty for every saturated hereditary subset H.

All ideals in C^*-algebras will be considered to be closed two-sided ideals. An element a of a C^*-algebra A (respectively, a subset $S \subseteq A$) is said to be full if a (respectively, S) is not contained in any proper ideal of A. A map $\phi: A \to B$ is full if imφ is full in B.

If A and B are C^*-algebras, an extension of A by B consists of a C^*-algebra E and a short exact sequence

$$\text{e:} \qquad 0 \to B \overset{\alpha}{\to} E \overset{\beta}{\to} A \to 0.$$

We say that the extension e is essential if $\alpha(B)$ is an essential ideal of E, and we say that the extension e is unital if E is a unital C^*-algebra. For any extension there exist unique $*$-homomorphisms $\eta_e: E \to \mathcal{M}(B)$ and $\tau_e: A \to \mathcal{Q}(B) := \mathcal{M}(B)/B$ which make the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \overset{\alpha}{\longrightarrow} & E & \overset{\beta}{\longrightarrow} & A & \longrightarrow & 0 \\ & & \| & & \downarrow \eta_e & & \tau_e \downarrow & & \\ 0 & \longrightarrow & B & \overset{i}{\to} & \mathcal{M}(B) & \overset{\pi}{\to} & \mathcal{Q}(B) & \longrightarrow & 0 \end{array}$$

commute. The $*$-homomorphism τ_{e} is called the Busby invariant of the extension, and the extension is essential if and only if τ_{e} is injective. An extension e is full if the associated Busby invariant τ_{e} has the property that $\tau_{\mathrm{e}}(a)$ is full in $\mathcal{Q}(A)$ for every $a \in A\setminus\{0\}$.

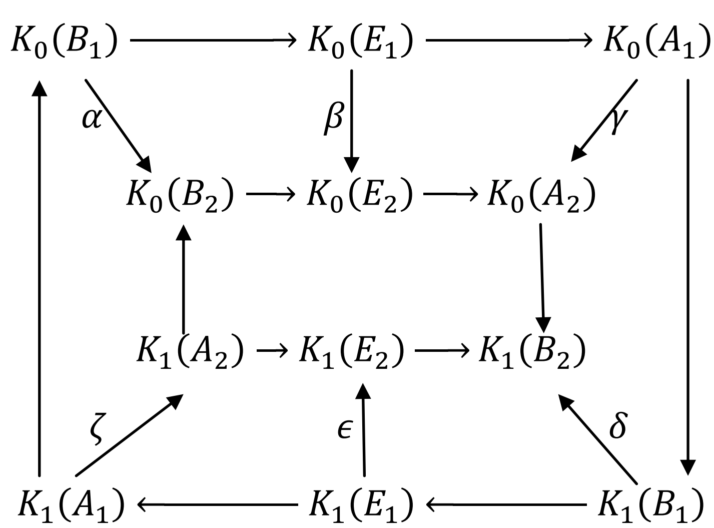
For an extension e, we let $K_{\mathrm{six(e)}}$ denote the cyclic six-term exact sequence of K-groups

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B) \end{array}$$

where $K_0(B), K_0(E)$, and $K_0(A)$ are viewed as (pre-)ordered groups. Given two extensions

$$\begin{array}{ll} \mathrm{e}_1\colon & 0 \longrightarrow B_1 \xrightarrow{\alpha_1} E_1 \xrightarrow{\beta_1} A_1 \longrightarrow 0 \\ \mathrm{e}_2\colon & 0 \longrightarrow B_2 \xrightarrow{\alpha_2} E_2 \xrightarrow{\beta_2} A_2 \longrightarrow 0 \end{array}$$

we say $K_{\mathrm{six(e_1)}}$ is isomorphic to $K_{\mathrm{six(e_2)}}$, written $K_{\mathrm{six(e_1)}} \cong K_{\mathrm{six(e_2)}}$, if there exist isomorphisms $\alpha, \beta, \gamma, \delta, \epsilon$, and ξ making the following diagram commute



and where α, β, and γ are isomorphisms of (pre-)ordered groups.

Lemma (1.1.1) [1]:

If E *is a* graph such that $C^*(E)$ contains a unique proper nontrivial ideal I, then the following six conditions are satisfied:

(i) E satisfies Condition (K),
(ii) E contains exactly three saturated hereditary subsets $\{\emptyset, H, E^0\}$,
(iii) E contains no breaking vertices; i.e., $B_H = \emptyset$,
(iv) I is a gauge invariant ideal and $I_H = I$,
(v) If X is a nonempty hereditary subset of E, then $X \cap H \neq \emptyset$, and
(vi) E has at most one sink, and if v is a sink of E then $v \in H$.

Proof:

Suppose that E does not satisfy Condition (K). Then there exists a saturated hereditary subset $H \subseteq E^0$ such that $E\backslash H$ contains a cycle $\alpha = e_1 \ldots e_n$ with no exits. The set $X = \{s(e_i)\}_{i=1}^{n}$ is a hereditary subset of $E\backslash H$, and I_X is an ideal in $C^*(E\backslash H)$ Morita equivalent to $M_n(C(\mathbb{T})$. Thus I_X, and hence $C^*(E\backslash H)$, contains a countably infinite number of ideals. Since $C^*(E\backslash H) \cong C^*(E)/I_{(H,BH)}$, this implies that $C^*(E)$ has a countably infinite number of ideals. Hence if $C^*(E)$ has a finite number of ideals, E satisfies Condition (K).

Because E satisfies Condition (K), it follows that the ideals of $C^*(E)$ are in one-to-one correspondence with the pairs (H, S) where H is saturated hereditary, and $S \subseteq BH$ is a subset of the breaking vertices of H. Since E contains a unique proper nontrivial ideal, it follows that E contains a unique saturated hereditary subset H not equal to E^0 or $\emptyset$, and that there are no breaking vertices; i.e., $BH = \emptyset$. It must also be the case that $I = IH$. Moreover, since E satisfies Condition (K), shows that all ideals of $C^*(E)$ are gauge-invariant.

In addition, suppose X is a hereditary subset with $X \cap H = \emptyset$. Since H is hereditary, none of the vertices in H can reach X, and thus the saturation X contains no vertices of H, and $X \cap H = \emptyset$. But then X is a saturated hereditary subset of E that does not contain the vertices of H, and hence must be equal to $\emptyset$. Thus if X is a nonempty hereditary subset of E, then $X \cap H \neq \emptyset$.

Finally, suppose v is a sink of E. Consider the hereditary subset $X := \{v\}$. From the previous paragraph it follows that $X \cap H \neq \emptyset$ and hence $v \in H$. In addition, there cannot be a second sink in E, for if v' is a sink, then $X := \{v\}$ and $Y := \{v'\}$ are distinct hereditary sets. Since v cannot reach v', we see that v is not in the saturation Y . Similarly, since v' cannot reach v, we have that v' is not in the saturation X. Thus X and Y are distinct saturated hereditary subsets of E that are proper and nontrivial, which is a contradiction. It follows that there is at most one sink in E.

Definition (1.1.2) [1]:

Let A be a C^*-algebra. A proper ideal $I \lhd A$ is a largest proper ideal of A if whenever $J \lhd A$, then either $J \subseteq I$ or $J = A$.

Observe that a largest proper ideal is always an essential ideal. Also note that if A is a C^*-algebra with a unique proper nontrivial ideal I, then I is a largest proper ideal; and if A is a simple C^*algebra then $\{0\}$ is a largest proper ideal.

Lemma (1.1.3): [1]

Let E be a graph, and suppose that I is a largest proper ideal of $C^*(E)$. Then I is gauge invariant and $I = I_{(H,BH)}$ for some saturated hereditary subset H of E^0. Furthermore, if K is any saturated hereditary subset of E, then either $K \subseteq H$ or $K = E^0$.

Proof:

Let γ denote the canonical gauge action of $\mathbb{T}$ on $C^*(E)$. For any $z \in \mathbb{T}$ we have that $\gamma_z(I)$ is a proper ideal of $C^*(E)$. Since I is a largest proper ideal of $C^*(E)$, it follows that $\gamma_z(I) \subseteq I$. A similar argument shows that $\gamma_{z^{-1}}(I) \subseteq I$. Thus $\gamma_z(I) = I$ and I is gauge invariant. It follows that $I = I_{(H,S)}$ for some saturated hereditary subset H of E^0 and some subset $S \subseteq BH$. Because I is a largest proper ideal, it follows that $S = B_H$, and hence $I = I_{(H,BH)}$. Furthermore, if K is a saturated hereditary subset, then either $I_{(K,BK)} \subseteq I_{(H,BH)}$ or $I_{(K,BK)} = C^*(E)$. Hence either $K \subseteq H$ or $K = E^0$.

Lemma (1.1.4) [1]:

Let E be a graph and suppose that I is a largest proper ideal of $C^*(E)$ with the property that $C^*(E)/I$ is purely infinite. Then $I = I_{(H,BH)}$ for some saturated hereditary subset H of E^0, and there exists a cycle γ in $E \backslash H$ and an edge $f \in E^1$ with $s(f) = s(\gamma)$ and $r(f) \in H$. Furthermore, if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0 \backslash H$.

Proof:

Lemma (1.1.3) shows that $I = I_{(H,BH)}$ for some saturated hereditary subset H of E^0. It follows that $C^*(E)/I(H,BH) \cong C^*(E\backslash H)$ is the subgraph of E with $(E\backslash H)^0 := E^0 \backslash H$ and $(E\backslash H)1 := E1\backslash r - 1(H)$. Since $C^*(E\backslash H)$ is purely infinite, it follows that $E\backslash H$ contains a cycle α. Define $K := \{x \in E^0 : x \ngeq s(\alpha)\}$. Then K is saturated hereditary, $H \subseteq K$, and $K \neq E^0$. Hence $I_{(H,BH)} \subseteq I_{(K,BK)} C^*(E)$, and the fact that $I_{(H,BH)}$ is a largest proper ideal implies that $I_{(H,BH)} = I_{(K,BK)}$ so that $H = K$. Hence for $x \in E^0$ we have $x \geq s(\alpha)$ if and only if $x \in E^0 \backslash H$.

Consider the set $J := \{x \in E^0 : s(\alpha) \geq x\}$. Then J is a hereditary subset and we let $\bar{J}$ denote its saturation. Since $I_{(H,BH)}$ is a largest proper ideal, it follows that either $\bar{J} \subseteq H$ or $\bar{J} = E^0$. Since $s(\alpha) \in \bar{J} \backslash H$, we must have $\bar{J} = E^0$. Choose any element $w \in H$. Since $w \in \bar{J}$ it follows that there exists $v \in J$ with $w \geq v$. But since $w \geq v$ and H is hereditary, it follows that $v \in H$. Hence $v \in J \cap H$, and there is a path from $s(\alpha)$ to a vertex in H. Choose a path $\mu = \mu_1 \mu_2 \ldots \mu_n$ with $s(\mu) = s(\alpha), r(\mu_{n-1})/\notin H$, and $r(\mu_n) \in H$. Since $r(\mu_{n-1})/\notin H$ the previous paragraph shows that there exists a path ν with $s(\nu) = r(\mu_{n-1})$ and $r(\nu = s(\alpha)$. Let $\gamma := \nu \mu_1 \ldots \mu_{n-1}$ and let $f := \mu_n$. Then γ is a cycle in $E\backslash H$ and f is an edge with $s(f) = s(\gamma)$ and $r(f) \in H$. Furthermore, since $s(\alpha)$ is a vertex on the cycle γ, we see that for any $x \in E^0$ we have $x \geq s(\gamma)$ if and only if $x \geq s(\alpha)$. It follows from the previous paragraph that if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0 \backslash H$.

Definition (1.1.5) [1]:

We say that two projections $p, q \in A$ are equivalent, written $p \sim q$, if there exists an element $v \in A$ with $p = vv^*$ and $q = v^* v$. We write $p \precsim q$ to mean that p is equivalent to a subprojection of q; that is, there exists $v \in A$ such that $p = vv^*$

and $v^*v \leq q$. Note that $p \precsim q$ and $q \precsim p$ does not imply that $p \sim q$ (unless A is finite).

If $e \in G^1$ then we see that $pr(e) = s_e^* s_e$ and $s_e s_e^* \leq ps(e)$. Therefore $p_r(e) \precsim p_s(e)$. More generally we see that $v \geq w$ implies $p_w \precsim p_v$.

Lemma (1.1.6) [1]:

Let A be a C^*-algebra with an increasing countable approximate unit $\{p_n\}_{n=1}^{\infty}$ consisting of projections. Then the following are equivalent.

(i) A is stable.

(ii) For every projection $p \in A$ there exists a projection $q \in A$ such that $p \sim q$ and $p \perp q$.

(iii) For all $n \in \mathbb{N}$ there exists $m > n$ such that $p_n \precsim p_m - p_n$.

Lemma (1.1.7) [1]:

Let A be a C^*-algebra. Suppose $p_1, p_2, \ldots, p_n$ are mutually orthogonal projections in A, and $q_1, q_2, \ldots, q_n$ are mutually orthogonal projections in A with $p_i \sim q_i$ for $1 \leq i \leq n$. Then $\sum_{i=1}^{n} p_i \sim \sum_{i=1}^{n} q_i$.

Proof:

Since $p_i \sim q_i$ there exists $v_i \in A$ such that $v_i^* v_i = p_i$ and $v_i v_i^* = q_i$. Thus for $i \neq j$ we have $v_j^* v_i = v_j^* v_j v_j^* v_i v_i^* v_i = v_j^* q_j q_i v_i = 0$ and $v_i v_j^* = v_i v_i^* \, v_i v_j^* v_j v_j^* = v_i p_i p_j v_j^* = 0$. Hence $(\sum_{i=1}^{n} v_i)^* \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} v_i^* v_i = \sum_{i=1}^{n} p_i$ and $\sum_{i=1}^{n} v_i \, (\sum_{i=1}^{n} v_i)^* = \sum_{i=1}^{n} v_i v_i^* = \sum_{i=1}^{n} q_i$. Thus $\sum_{i=1}^{n} p_i \sim \sum_{i=1}^{n} q_i$.

Proposition (1.1.8) [1]:

Let E be a graph with no breaking vertices, and suppose that I is a largest proper ideal of $C^*(E)$ and such that $C^*(E)/I$ is purely infinite and I is AF. Then there exists a projection $p \in C^*(E)$ such that $pC * (E)p$ is a full corner of $C^*(E)$ and $p \, I \, p$ is stable.

Proof:

Lemma (1.1.6) implies that $I = I_{(H,BH)}$ for some saturated hereditary subset H of E_0, and there exists a cycle γ in $E\backslash H$ and an edge $f \in E^1$ with $s(f) = s(\gamma)$ and $r(f) \in H$; and furthermore, if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0\backslash H$. Since E has no breaking vertices, we have that $B_H = \emptyset$ so that $I_{(H,BH)}$ is the ideal generated by $\{p_v : v \in H\}$ and we may write $I_{(H,BH)}$ as IH.

Let $v = s(f) = s(\gamma)$ and let $w = r(f)$. Define $p := p_v + p_w$. Suppose $J \lhd C^*(E)$ and $pC^*(E)p \subseteq J$. Since $v \notin H$ we see that $p_v \notin I$ and hence $p_v \in pC^*(E)p\backslash I \subseteq J \backslash I$. Thus $J \nsubseteq I$ and the fact that I is a largest proper ideal implies that $J = C^*(E)$. Hence $pC^*(E)p$ is a full corner of $C^*(E)$.

In addition, since there are no breaking vertices

$$\begin{aligned} p\, I\, p &= p\, I_H\, p \\ &= p\left(\overline{\text{span}}\{ps_\alpha s_\beta^* p : r(\alpha) = r(\beta) \in H\}\right)p \\ &= \overline{\text{span}}\{ps_\alpha s_\beta^* p : r(\alpha) = r(\beta) \in H\} \\ &= \overline{\text{span}}\left\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{v, w\}\right\}. \end{aligned}$$

Let $S := \{\alpha \in E^* : s(\alpha) = v$ and $r(\alpha) = w\}$. Since S is a countable set we may list the elements of S and write $S = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$. Define $p_0 := p_w$ and $p_n := p_w + \sum_{k=1}^n s_{\alpha_k} s_{\alpha_k}^*$ for $n \in \mathbb{N}$.

We will show that for μ, ν ∈ S we have

$$s_\mu^* s_\nu := \begin{cases} p_r(\mu) & \text{if } \pi = v \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

First suppose that $s_\mu^* s_\nu \neq 0$. Then one of μ and ν must extend the other. Suppose μ extends ν. Then $\mu = \nu\lambda$ for some $\lambda \in E^*$. Thus $s(\lambda) = r(\nu) = w$ and $r(\lambda) = r(\mu) = w$. However, I_H is an AF-algebra, and $C^*(E_H)$ is strongly Morita equivalent to I_H, so $C^*(E_H)$ is an AF-algebra. Thus E_H contains no cycles. Since λ is a path in EH with $s(\lambda) = r(\lambda) = w$, and since E_H contains no cycles, we may conclude that $\lambda = w$. Thus $\mu = \nu$. A similar argument works when ν extends μ. Hence the equation in (1.1) holds. It follows that the elements of the set $\{s_\alpha s_\alpha^* :$

$\alpha \in S\} \cup \{pw\}$ are mutually orthogonal projections, and hence $\{p_n\}_{n=0}^{\infty}$ is an sequence of increasing projections.

Next we shall show that $\{p_n\}_{n=0}^{\infty}$ is an approximate unit for pIp. Given $s_\alpha s_\beta^*$ with $r(\alpha) = r(\beta) \in H$ and $s(\alpha), s(\beta) \in \{v, w\}$, we consider two cases.

Case I: $s(\alpha) = w$. Then for any $\alpha_k \in S$ we see that $(s_{\alpha_k} s_{\alpha_k}^*) s_\alpha s_\beta^*\ s_{\alpha_k} s_{\alpha_k}^* p_w s_\alpha s_\beta^* = 0$. In addition, $p_w(s_\alpha s_\beta^*) = s_\alpha s_\beta^*$. Thus $\lim_{n\to\infty} p_n\, s_\alpha s_\beta^* = s_\alpha s_\beta^*$.

Case II: $s(\alpha) = v$. Then $\alpha = \alpha_j \lambda$ for some $\alpha_j \in S$ and some $\lambda \in E_H^*$ with $s(\lambda) = w$. We have $p_w(s_\alpha s_\beta) = 0$, and also (1) implies that

$$(s_{\alpha_k} s_{\alpha_k}^*) s_\alpha s_\beta^* = s_{\alpha_k} s_{\alpha_k}^* s_{\alpha_j} s_\lambda s_\beta^* = \begin{cases} s_\alpha s_\beta^* & k = j \\ 0 & k \neq j \end{cases} \tag{2}$$

Thus $\lim_{\mathrm{n}\to\infty} p_n s_\alpha s_\beta^* = s_\alpha s_\beta^*$.

The above two cases imply that $\lim_{n\to\infty} p_n\, x = x$ for any $x \in \operatorname{span}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{v, w\}\}$. Furthermore, an $\epsilon/3$-argument shows that $\lim_{n\to\infty} p_n\ x = x$ for any $x \in pI_H p = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{v, w\}\}$. A similar argument shows that $\lim_{n\to\infty} x p_n = x$ for any $x \in pI_H p$. Thus $\{p_n\}_{n=1}^{\infty}$ is an approximate unit for $pI_H p$.

We shall now show that $pI_H p$ is stable. For each $n \in \mathbb{N}$ define

$$\lambda^n := \underbrace{\gamma\gamma \ldots \gamma\, f}_{n \text{ times}}.$$

For any $k, n \in \mathbb{N}$ we have

$$s_{\lambda^n} s_{\lambda^n}^* \ \sim\ s_{\lambda^n}^* s_{\lambda^n} \ =\ p_r(\lambda^n) \ =\ p_w \ =\ s_{\alpha_k}^* s_{\alpha_k} \ \sim\ s_{\alpha_k} s_{\alpha_k}^* \,.$$

For any $n \in \mathbb{N}$ choose q large enough that $|\lambda^q| \geq |\alpha_k|$ for all $1 \leq k \leq n$. Then for all $j \in \mathbb{N}$ we see that $\lambda^{q+j} \in S$ and $\lambda^{q+j} \neq \alpha_k$ for all $1 \leq k \leq n$. Thus for any $1 \leq k \leq n$ we have

$$s_{\alpha_k} s_{\alpha_k}^* \sim s_{\alpha_k}^* s_{\alpha_k} = p_r(\alpha_k) = p_w = p_r(\lambda^{q+k}) = s_{\lambda^{q+k}}^* s_{\lambda^{q+k}} \sim s_{\lambda^{q+k}} s_{\lambda\ \ q+k}^*$$

and

$$p_w = s_{\lambda^q}^* s_{\lambda^q} \sim s_{\lambda^q} s_{\lambda^q}^* .$$

It follows from Lemma (1.1.7) that

$$\mathrm{p_n} = \mathrm{p_w} + \sum_{k=1}^{n} \mathrm{s}_{\alpha_k} \mathrm{s}_{\alpha_k}^* \precsim \sum_{k=0}^{n} \mathrm{s}_{\lambda^{q+k}} \mathrm{s}_{\lambda^{q+k}}^* \precsim \mathrm{p_m} - \mathrm{p_n}$$

where m is chosen large enough that $\lambda^{q+k} \in \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ for all $0 \leq k \leq n$. Lemma (1.1.6) shows that $pI_H p$ is stable.

We apply the methods to classify certain extensions of graph C^*-algebras in terms of their six-term exact sequences of K-groups.

Definition (1.1.9) [1]:

We will be interested in classes $\mathcal{C}$ of separable nuclear unital simple C^*-algebras in the bootstrap category $\mathcal{N}$ satisfying the following properties:

(ii) Any element of $\mathcal{C}$ is either purely infinite or stably finite.
(iii) $\mathcal{C}$ is closed under tensoring with M_n, where M_n is the C^*-algebra of n by n matrices over $\mathcal{C}$.
(iv) If A is in $\mathcal{C}$, then any unital hereditary C^*-subalgebra of A is in $\mathcal{C}$.
(v) For all A and B in $\mathcal{C}$ and for all x in $KK(A, B)$ which induce an isomorphism from $(K_*^+(A), [1_A])$ to $(K_*^+(A), [1_B])$, there exists a $*$-isomorphism $\alpha: A \to B$ such that $KK(\alpha) = x$.

Definition (1.1.10) [1]:

If B is a separable stable C^*-algebra, then we say that B has the Corona factorization property if every full projection in $\mathcal{M}(B)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$.

Lemma (1.1.11) [1]:

Let C_I and C_Q be classes of unital nuclear separable simple C^*-algebras in the bootstrap category $\mathcal{N}$ satisfying the properties of Definition (1.1.9). Let A_1 and A_2 be in C_Q and let B_1 and B_2 be in C_I with $B_1 \otimes \mathbb{K}$ and $B_2 \otimes \mathbb{K}$ satisfying the Corona factorization property. Let

$$
\begin{array}{l}
\mathfrak{e}_1 : \ 0 \longrightarrow B_1 \otimes \mathbb{K} \longrightarrow E_1 \longrightarrow A_1 \longrightarrow 0 \\
\mathfrak{e}_2 : \ 0 \longrightarrow B_2 \otimes \mathbb{K} \longrightarrow E_2 \longrightarrow A_2 \longrightarrow 0
\end{array}
$$

be essential and unital extensions. If $K_{\text{six}}(e_1)K_{\text{six}}(e_2)$, then $E_1 \otimes \mathbb{K}E_2 \otimes \mathbb{K}$.

Proof:

Tensoring the extension $\mathfrak{e}_1$ by $\mathbb{K}$ we obtain a short exact sequence $\mathfrak{e}'_1$ and and vertical maps

$$
\begin{array}{ccccccccc}
\mathfrak{e}_1 : \ 0 & \longrightarrow & B_1 \otimes \mathbb{K} & \longrightarrow & E_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\mathfrak{e}_1 : \ 0 & \longrightarrow & B_1 \otimes \mathbb{K} & \longrightarrow & E_1 & \longrightarrow & A_1 & \longrightarrow & 0
\end{array}
$$

from $\mathfrak{e}_1$ into $\mathfrak{e}'_1$ that are full inclusions. These full inclusions induce isomorphisms of K-groups and hence we have that $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}'_1)$. In addition, since $\mathfrak{e}_1$ is essential, $B_1 \otimes K$ is an essential ideal in E_1, and the Rieffel correspondence between the strongly Morita equivalent C^*-algebras E_1 and $E_1 \otimes K$ implies that $(B_1 \otimes \mathbb{K}) \otimes \mathbb{K}$ is an essential ideal in $E_1 \otimes \mathbb{K}$, so that $\mathfrak{e}'_1$ is an essential extension. Furthermore, since $B_1 \otimes \mathbb{K}$ is stable and e_1 is essential and full, it follows that $\mathfrak{e}'_1$ is full. Moreover, since $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, we may rewrite $\mathfrak{e}'_1$ as

$$
\mathfrak{e}'_1 : \ 0 \longrightarrow \mathrm{B}_1 \otimes \mathbb{K} \longrightarrow \mathrm{E}_1 \otimes \mathbb{K} \longrightarrow \mathrm{A}_1 \otimes \mathbb{K} \longrightarrow 0.
$$

By a similar argument, there is an essential and full extension

$$
\mathfrak{e}'_2 : \ 0 \longrightarrow B_2 \otimes \mathbb{K} \longrightarrow E_2 \otimes \mathbb{K} \longrightarrow A_2 \otimes \mathbb{K} \longrightarrow 0.
$$

such that $K_{\text{six}}(e'_2) \cong K_{\text{six}}(e_2)$. Thus $K_{\text{six}}(e'_1)K_{\text{six}}(e'_2)$, and implies that $E_1 \otimes \mathbb{K} \cong E_2 \otimes \mathbb{K}$.

Lemma (1.1.12) [1]:

Let A be a C^*-algebra and let I be a largest proper ideal of A. If $p \in A$ is a full projection, then the inclusion map $pIp \hookrightarrow I$ and the inclusion map $pAp/pIp \hookrightarrow A/I$ are both full inclusions.

Proof:

Since p is a full projection, we see that A is Morita equivalent to pAp and the Rieffel correspondence between ideals takes the form $J \mapsto pJp$. If J is an ideal of I with $pIp \subseteq J$, then by compressing by p we obtain $pIp \subseteq pJp$. Since the Rieffel correspondence is a bijection, this implies that $I \subseteq J$, and because J is an ideal contained in I, we get that $I = J$. Hence $pIp \hookrightarrow I$ is a full inclusion. Furthermore, because I is a largest proper ideal of A, we know that A/I is simple and thus $pAp/pIp \hookrightarrow A/I$ is a full inclusion.

Theorem (1.1.13) [1]:

If A is a graph C^*-algebra with exactly one proper nontrivial ideal I, then A classified up to stable isomorphism by the six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(EA) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

with all K_0-groups considered as ordered groups. In other words, if A is a graph C^*-algebras with precisely one proper nontrivial ideal I, if A' is a graph C^*-algebras with precisely one proper nontrivial ideal I', and if

$$\begin{array}{lc} \mathfrak{e}_1 : & 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \\ \mathfrak{e}_2 : & 0 \longrightarrow I' \longrightarrow A' \longrightarrow A'/I' \longrightarrow 0 \end{array}$$

are the associated extensions, then $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$ if and only if $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}_2)$.

Theorem (1.1.14) [1]:

If A is a the C^* -algebra of a graph satisfying Condition (K), and if A has a largest proper ideal I such that I is an AF-algebra, then A is classified up to stable isomorphism by the six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

with $K_0(I)$ considered as an ordered group.

In other words, if A is the C^*-algebra of a graph satisfying Condition (K) with a largest proper ideal I that is an AF-algebra, if A' is the C^*-algebra of a graph satisfying Condition (K) with a largest proper ideal I' that is an AF-algebra, and if

$$\begin{aligned} \mathfrak{e}_1 : \quad & 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \\ \mathfrak{e}_2 : \quad & 0 \longrightarrow I' \longrightarrow A' \longrightarrow A'/I' \longrightarrow 0 \end{aligned}$$

are the associated extensions, then $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$ if and only if $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}_2)$.

Examples (1.1.15) [1]:

To illustrate our methods we give a complete classification, up to stable isomorphism, of all C^*-algebras of graphs with two vertices that have precisely one proper nontrivial ideal. Combined with other results, this allows us to give a complete classification of all C^*-algebras of graphs satisfying Condition (K) with exactly two vertices.

If E is a graph with two vertices, and if $C^*(E)$ has exactly one proper ideal, then E must have exactly one proper nonempty saturated hereditary subset with no breaking vertices. This occurs precisely when the vertex matrix of E has the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

where $a, d \in \{0,2,3,\dots,\infty\}$ and $b \in \{1,2,3,\dots,\infty\}$ with the extra conditions

$$a = 0 \Longrightarrow b = \infty \text{ and } b = \infty \Longrightarrow (a = 0 \text{ or } a = \infty),$$

Computing K-groups, we see that in all of these cases the K_1-groups of $C^*(E)$, the unique proper nontrivial ideal I, and the quotient $C^*(E)/I$ all vanish. Thus the six-term exact sequence becomes $0 \to K_0(I) \to K_0(C^*(E)) \to K_0(C^*(E)/I) \to 0$, and to compute the K_0-groups and the induced maps we obtain the following cases.

a	d	b	$K_0(I) \to K_0\big(C^*(E)\big) \to K_0(C^*(E)/I \ \to \mathbb{Z}\oplus)$	Case
0	0	∞	$\mathbb{Z}_{++} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}_{++}$	[11]
0	n	∞	$\mathbb{Z}_{d-1} \to \mathbb{Z}_{d-1}\oplus\mathbb{Z} \to \mathbb{Z}_{++}$	[∞1]
0	∞	∞	$\mathbb{Z}_{\pm} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}_{++}$	[∞1]
n	0	$1, n$	$\mathbb{Z}_{++} \to \text{coker}\left(\begin{bmatrix} b \\ a-1 \end{bmatrix}\right) \to \mathbb{Z}_{a-1}$	[1∞]
n	n	$1, n$	$\mathbb{Z}_{d-1} \to \text{coker}\left(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}\right) \to \mathbb{Z}_{a-1}$	[∞∞]
n	∞	$1, n$	$\mathbb{Z}_{\pm} \to \text{coker}\left(\begin{bmatrix} b \\ a-1 \end{bmatrix}\right) \to \mathbb{Z}_{a-1}$	[∞∞]
∞	0	$1, n, \infty$	$\mathbb{Z}_{++} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}_{\pm}$	[1∞]
∞	n	$1, n, \infty$	$\mathbb{Z}_{d-1} \to \mathbb{Z}_{d-1}\oplus\mathbb{Z} \to \mathbb{Z}_{\pm}$	[∞∞]
∞	∞	$1, n, \infty$	$\mathbb{Z}_{\pm} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}_{\pm}$	[∞∞]

where "n" indicates an integer ≥ 2, "$\mathbb{Z}_{++}$" indicates a copy of $\mathbb{Z}$ ordered with $\mathbb{Z}_+ = N$ and "$\mathbb{Z}_{\pm}$" indicates a copy of $\mathbb{Z}$ ordered with $\mathbb{Z}_+ = \mathbb{Z}$. In addition, in all cases we have written the middle group in such a way that the map from $K_0(I)$ to $K_0(C^*(E))$ is $[x] \mapsto [(x, 0)]$, and the map from $K_0(C^*(E))$ to $K_0(C^*(E)/I)$ is $[(x, y)]7 \to [y]$. Note that in all but the first case, the order structure of the middle K_0-groups is irrelevant and need not be computed.

Section (1.2): Stability of Ideals

Theorem (1.2.1) [1]:

Let E and E' be graphs each with two vertices such that $C^*(E)$ and $C^*(E')$ each have exactly one proper nontrivial ideal, and write the vertex matrix of E as $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and the vertex matrix of E' as $\begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$. Then

$$C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$$

if and only if the following three conditions hold:

(i) $a = a'$

(ii) $d = d'$

(iii) If $a \in \{2, \dots\}$ then

a) If $d \in \{0, \infty\}$ then $[b] = [z][b']$ in $\mathbb{Z}_{a-1}$ for a unit $[z] \in \mathbb{Z}a - 1$

b) If $d \in \{2, \dots\}$ then $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\gcd(a-1,d-1)}$ for a unit $[z_1] \in \mathbb{Z}_{d-1}$ and a unit $[z_2] \in \mathbb{Z}_{a-1}$.

Proof:

Suppose $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$. Then $K_0(I)K_0(I')$ as ordered groups and $K_0(C^*(E)/I)K_0(C^*(E')/I')$ as ordered groups. From a consideration of the invariants in the above table, this implies that $a = a', d = d'$, and the invariants for $C^*(E)$ and $C^*(E')$ both fall into the same case (i.e. the same row) of the table. Thus we need only consider the two cases described in (iii)(a) and (iii)(b).

Case I: $a \in \{2, \dots\}$ and $d \in \{0, \infty\}$.

In this case there are isomorphisms α, β, and γ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} b \\ a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} b' \\ a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \end{array}$$

commutes. Since the only automorphisms on $\mathbb{Z}$ are $\pm Id$, we have that $\alpha(x) = \pm x$. Also, since the only automorphisms on $\mathbb{Z}_{a-1}$ are multiplication by a unit, $\gamma([x]) = [z][x]$ for some unit $[z] \in \mathbb{Z}_{a-1}$. By the commutativity of the left square $\beta([1,0]) = [(\pm 1,0)]$. Also, by the commutativity of the right square, $\beta([0,1]) =$

$([y,z])$ for some $y \in \mathbb{Z}$. It follows from the $\mathbb{Z}$ -linearity of β that $\beta[(r,s)] = [(\pm r + sy, sz)]$, so β is equal to left multiplication by the matrix $\begin{bmatrix} \pm 1 & y \\ 0 & z \end{bmatrix}$. We must have β[(b,a−1)]=[(0,0)], and thus $[(\pm b + (a-1)y, (a-1)z)] = [(0,0)]$ in coker $\left(\begin{bmatrix} b' \\ a-1 \end{bmatrix}\right)$. Hence $\pm b + (a-1)y = b't$ and $(a-1)z = (a-1)t$ for some $t \in \mathbb{Z}$. It follows that $z = t$ and $\pm b + (a-1)y = b'z$, so $b \equiv \pm z \bmod (a-1)$. Since $[\pm z]$ is a unit for $\mathbb{Z}_{a-1}$ it follows that $[b] = [z][b']$ in $\mathbb{Z}_{a-1}$ for a unit $[z] \in \mathbb{Z}_{a-1}$. Thus the condition in (a) holds.

Case II: $a \in \{2, \dots\}$ and $d \in \{2, \dots\}$. In this case there are isomorphisms α, β, and γ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_{d-1} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_{d-1} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \end{array}$$

commutes. Since the only automorphisms on $\mathbb{Z}_{d-1}$ are multiplication by a unit, we have that $\alpha([x]) = [z_1][x]$ for some unit $[z_1] \in \mathbb{Z}_{d-1}$. Likewise, $\gamma([x]) = [z_2][x]$ for some unit $[z_2] \in Za - 1$. By the commutativity of the left square $\beta([1,0]) = [(z_1, 0)]$. Also, by the commutativity of the right square, $\beta([0,1]) = ([y, z_2])$ for some $y \in \mathbb{Z}$. It follows from the $\mathbb{Z}$-linearity of β that $\beta[(r,s)] = [(z_1 r + ys, z_2 s)]$, so β is equal to left multiplication by the matrix $\begin{bmatrix} z_1 & y \\ 0 & z_2 \end{bmatrix}$. Since $\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ a-1 \end{bmatrix}$, we must have $\beta[(b, a-1)] = [(0,0)]$, and thus $[(z_1 b + y(a-1), z_2(a-1))] = [(0,0)]$ in coker$\left(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}\right)$. Hence $z_1 b + y(a-1) = (d-1)s + b't$ and $z_2(a-1) = (a-1)t$ for some $s, t \in \mathbb{Z}$. It follows that $z_2 = t$ and $z_1 b + y(a-1) = (d-1)s + b' z_2$.Writing $(d-1)s - y(a-1) = k\gcd(a-1, d-1)$ we obtain $z_1 b - z_2 b' = k\gcd(a-1, d-1)$ so that $z_1 b \equiv z_2 b' \bmod \gcd(a-1, d-1)$ and $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{gcd(a-1,d-1)}$. Thus the condition in (b) holds.

For the converse, we assume that the conditions in (i)–(iii) hold. Consider the following three cases.

Case I: $a = 0$ or $a = \infty$. In this case, by considering the invariants listed in the above table, we see that we may use the identity maps for the three vertical isomorphisms to obtain a commutative diagram. Thus the six-term exact sequences are isomorphic, and it follows from Theorem (1.1.13) that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

Case II: $a \in \{2, \dots\}$ and $[b] = [z][b']$ in $\mathbb{Z}_{a-1}$ for a unit $[z] \in \mathbb{Z}_{a-1}$. Then $b \cong zb' \bmod (a-1)$. Hence $zb' - b = (a-1)\, y$ for some $y \in \mathbb{Z}$. Consider $\begin{bmatrix} 1 & y \\ 0 & z \end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. It is straightforward to check that this matrix takes $\operatorname{im} \begin{bmatrix} b \\ a-1 \end{bmatrix}$ into $\operatorname{im} \begin{bmatrix} b' \\ a-1 \end{bmatrix}$. Thus multiplication by this matrix induces a map $\beta: \operatorname{coker}(\begin{bmatrix} b \\ a-1 \end{bmatrix}) \to \operatorname{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix})$. In addition, if we let $\alpha = \mathrm{Id}$ and let γ be multiplication by $[z]$, then it is straightforward to verify that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} b \\ a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} b' \\ a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \end{array}$$

commutes. Since α and γ are isomorphisms, an application of the five lemma implies that β is an isomorphism. It follows from Theorem (1.1.13) that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

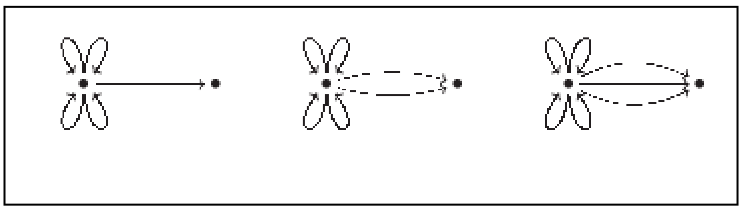
Case III: Suppose that $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\gcd(a-1,d-1)}$ for a unit $[z_1] \in \mathbb{Z}_{d-1}$ and a unit $[z_2] \in \mathbb{Z}_{a-1}$. Then $z_1 b - z_2 b' = k\gcd(a-1, d-1)$ for some $k \in \mathbb{Z}$. Furthermore, we may write $k \gcd(a-1, d-1) = s(d-1) - y(a-1)$ for some $s, y \in \mathbb{Z}$. Consider $\begin{bmatrix} z_1 & y \\ 0 & z_2 \end{bmatrix} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. It is straightforward to check that this matrix takes $\operatorname{im} \begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}$ into $\operatorname{im} \begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}$. Thus multiplication by this matrix induces a map $\beta: \operatorname{coker}(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}) \to \operatorname{coker}(\begin{bmatrix} d-1 \\ b' \end{bmatrix} 0a - 1)$. In addition, if we let α be multiplication by $[z_1]$ and and let γ be multiplication by $[z_2]$, then it is straightforward to verify that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} & \longrightarrow & 0 \end{array}$$

commutes. Since α and γ are isomorphisms, an application of the five lemma implies that β is an isomorphism. It follows from Theorem (1.1.13) that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$

Example (1.2.2) [1]:

Consider the three graphs



which all have graph C^* -algebras with precisely one proper nontrivial ideal. By Theorem (1.2.1) the C^* -algebras of the two first graphs are stably isomorphic to each other, but not to the C^* -algebra of the third graph.

Using the Kirchberg-Phillips Classification Theorem and our results in Theorem (1.2.1) we are able to give a complete classification of the stable isomorphism classes of C^* -algebras of graphs satisfying Condition (K) with exactly two vertices. We state this result in the following theorem. As one can see, there are a variety of cases and possible ideal structures for these stable isomorphism classes.

Theorem (1.2.3) [1]:

Let E and E' be graphs satisfying Condition (K) that each have exactly two vertices. Let AE and AE' be the vertex matrices of E and E', respectively, and order the vertices of each so that $c \leq b$ and $c' \leq b'$. Then $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$ if and only if one of the following five cases occurs.

(i) $A_E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A_E = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ with

$$(b \neq 0 \ \text{ and } c \neq 0) \ \text{ or } \ (a = 0, 0 < b < \infty, c = 0 \ \text{ and } d \geq 2)$$

and

$$(b' \neq 0 \ \text{ and } c' \neq 0) \ \text{ or } \ (a' = 0, 0 < b' < \infty, c' = 0 \ \text{ and } d' \geq 2)$$

and if B_E is the $E^0 \times E^0_{\text{reg}}$ subbmatrix of $A_E^t - I$ and $B_{E'}$ is the $(E')^0 \times (E')^0_{\text{reg}}$ submatrix of $A_{E'}^t - I$ then

$$\operatorname{coker}\left(B_E : \mathbb{Z}^{B^0_{\text{reg}}} \to Z^{E^0}\right) \cong \operatorname{coker}\left(B_{E'} : \mathbb{Z}^{(E')^0_{\text{reg}}} \to Z^{(E')^0}\right)$$

and

$$\ker\left(B_E : \mathbb{Z}^{B^0_{\text{reg}}} \to Z^{E^0}\right) \cong \ker\left(B_{E'} : \mathbb{Z}^{(E')^0_{\text{reg}}} \to Z^{(E')^0}\right)$$

In this case $C^*(E)$ and $C^*(E')$ are purely infinite and simple.

(ii) $A_E = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} 0 & b' \\ 0 & 0 \end{bmatrix}$ with $0 < b < \infty$ and $0 < b' < \infty$. In this case $C^*(E) \cong M_{b+1}(\mathbb{C})$ and $C^*(E') \cong M_{b'+1}(\mathbb{C})$, so that both C^*-algebras are simple and finite-dimensional.

(iii) $A_E = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $b \neq 0$ and $b' \neq 0$,

$$a = 0 \implies b = \infty \ \text{ and } \ b = \infty \implies (a = 0 \ \text{ or } \ a = \infty),$$

and

$$a' = 0 \implies b' = \infty \ \text{ and } \ b' = \infty \implies (a' = 0 \ \text{ or } \ a' = \infty),$$

and the conditions (i)–(iii) of Theorem (1.2.1) hold. In this case $C^*(E)$ and $C^*(E')$ each have exactly one proper nontrivial ideal and have ideal structure of the form

$$\begin{matrix} A \\ | \\ I \\ | \\ \{0\}. \end{matrix}$$

(iv) $A_E = \begin{bmatrix} a & \infty \\ 0 & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & \infty \\ 0 & d' \end{bmatrix}$ with $a \in \{2,3,\ldots\}$ and $a' \in \{2,3,\ldots\}$, and with $a = a'$ and $d = d'$. In this case $C^*(E)$ and $C^*(E')$

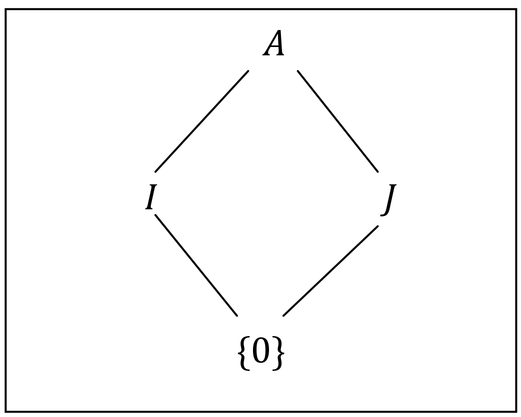
each have exactly two proper nontrivial ideals and have ideal structure of the form

$$\begin{array}{c} A \\ | \\ I \\ | \\ J \\ | \\ \{0\}. \end{array}$$

(v) $A_E = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix}$ with

$$(a = a' \text{ and } d = d') \text{ or } (a = d' \text{ and } d = a').$$

In this case $\mathrm{C}^*(\mathrm{E}) \cong \mathrm{C}^*(\mathrm{E}') \cong \mathrm{I} \oplus \mathrm{J}$, where $\mathrm{I} \coloneqq \begin{cases} \mathcal{O}_{\mathrm{a}} & \text{if a} \geq 2 \\ \mathbb{C} & \text{if a} = 0 \end{cases}$ and $\mathrm{J} \coloneqq \begin{cases} \mathcal{O}_{\mathrm{d}} & \text{if d} \geq 2 \\ \mathbb{C} & \text{if d} = 0 \end{cases}$, and each C^*-algebra has exactly two proper nontrivial ideals and ideal structure of the form



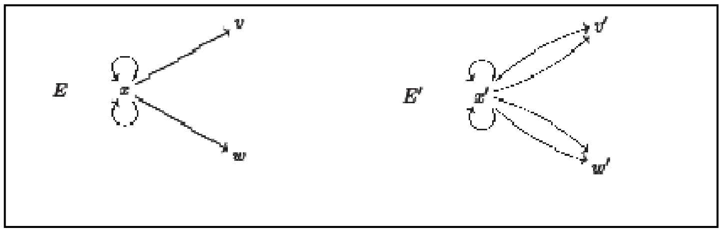
Remark (1.2.4) [1]:

We are not able to classify C^* -algebras of graphs with exactly two vertices that do not satisfy Condition (K). For example if E and E' are graphs with vertex matrices $A_E = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix}$,then $C^*(E)$ and $C^*(E')$ each have uncountably many ideals, and are extensions of $C(\mathbb{T})$ by $C(\mathbb{T} \otimes \mathbb{K})$. Using existing techniques, it is unclear when $C^*(E)$ and $C^*(E')$ will be stably isomorphic.

We conclude this section with an example showing an application of Theorem (1.1.14) to C^* -algebras with multiple proper ideals.

Example (1.2.5) [1]:

Consider the two graphs



The ideal $I := I\{v, w\}$ in $C^*(E)$ is a largest proper ideal that is an AF-algebra, and the six-term exact sequence corresponding to

$$0 \to I \to C^*(E) \to C^*(E)/I \to 0$$

is

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \operatorname{coker}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \to 0$$

where the middle map is $[(x, y)] \longmapsto [(x, y, 0)]$. Likewise, the ideal $I' := I_{\{v\prime, w\prime\}}$ in $C^*(E')$ is a largest proper ideal that is an AF-algebra, and the six-term exact sequence corresponding to

$$0 \to I' \to C^*(E') \to C^*(E')/I' \to 0$$

is

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \operatorname{coker}\left(\begin{bmatrix}2\\2\\1\end{bmatrix}\right) \to 0$$

where the middle map is $[(x, y)] \mapsto [(x, y, 0)]$. If we define $\beta: \operatorname{coker}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \to \operatorname{coker}\left(\begin{bmatrix}2\\2\\1\end{bmatrix}\right)$ by $\beta[(x, y, z)] = [(x + z, y + z, z)]$, then we see that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) & \longrightarrow & 0 \\
 & & \mathrm{Id}\downarrow & & \beta\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix}2\\2\\1\end{bmatrix}\right) & \longrightarrow & 0
\end{array}
$$

commutes. An application of the five lemma shows that β is an isomorphism. It follows from Theorem (1.1.14) that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

In the examples above, both connecting maps in the six-term exact sequences vanish. Since all C^*-algebras considered (and, more generally, all graph C^*-algebras satisfying Condition (K)) have real rank zero, the exponential map $\partial: K_0\left(\frac{A}{I}\right) \to K_1(I)$ is always zero. However, the index map $\partial: K_1\left(\frac{A}{I}\right) \to K_0(I)$ does not necessarily vanish and may carry important information. In forthcoming work, the authors and Carlsen explain how to compute this map for graph C^*-algebras.

In this section we prove that if A is a graph C^* -algebra that is not an AF-algebra, and if A contains a unique proper nontrivial ideal , then I is stable.

Definition (1.2.6) [1]:

If v is a vertex in a graph E we define

$$L(v) \coloneqq \{w \in E^0 \colon \text{there is a path from } w \text{ } to \text{ } v\}.$$

We say that v is left infinite if $L(v)$ contains infinitely many elements.

Definition (1.2.7) [1]:

If $E = (E^0, E^1, r, s)$ is a graph, then a graph trace on E is a function $\mathrm{g}: E^0 \to [0, \infty)$ with the following two properties:

(i) For any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$ we have $\mathrm{g}(v) = \sum_{s(e)=v} \mathrm{g}\big(r(e)\big)$

(ii) For any infinite emitter $v \in G^0$ and any finite set of edges $e_1, \dots, e_n \in s^{-1}(v)$ we have $\mathrm{g}(v) \geq \sum_{i=1}^{n} \mathrm{g}\big(r(e_i)\big)$.

We define the norm of a graph trace g to be the (possibly infinite) quantity $\|g\| := \sum_{v\in E^0} g(v)$, and we say a graph trace g is bounded if $\|g\| < \infty$.

Lemma (1.2.8) [1]:

Let E be a graph such that $C^*(E)$ is simple. If there exists $v \in E^0$ such that v is left infinite, then $C^*(E)$ is stable.

Proposition (1.2.9) [1]:

Let E be a graph such that $C^*(E)$ contains a unique proper nontrivial ideal I, and let $\{E^0, H, \emptyset\}$ be the saturated hereditary subsets of E. Then there are two possibilities:

(i) The ideal I is stable; or

(ii)The graph C^*-algebra $C^*(E)$ is a nonunital AF-algebra, and H is infinite.

Proof:

By Lemma (1.1.1), we see that E contains a unique saturated hereditary subset H not equal to either E^0 or $\emptyset$, and also $I = I_H$. In addition, it follows that I_H is isomorphic to the graph C^*-algebra $C^*({}_HE_\emptyset)$, where ${}_HE_\emptyset$ is the graph described. In particular, if we let

$$F_H := \{\alpha \in E^*: s(\alpha) \notin H, r(\alpha) \in H, \text{ and } r(\alpha_i) \notin H \text{ for } i < |\alpha|\}$$

then

$${}_HE^0_\emptyset := H \cup F_H \text{ and } {}_HE^0_\emptyset := \{e \in E^1: s(e) \in H\} \cup \{\bar{\alpha}: \alpha \in F_H\}$$

where $s(\alpha) = \alpha, r(\alpha) = r(\alpha)$, and the range and source of the other edges is the same as in E. Note that since I is the unique proper nontrivial ideal in $C^*(E)$, we have that $I \cong C^*({}_HE_\emptyset)$ is simple.

Consider three cases.

Case I: H is finite.

Choose a vertex $v \in E^0\backslash H$. By Lemma (1.1.3) v is not a sink in E, and thus there exists an edges $e_1 \in E^1$ with $s(e_1) = v$ and $r(e_1) \notin H$. Continuing inductively, we may produce an infinite path $e_1e_2e_3$... with $r(e_i) \notin H$ for all i. (Note that the vertices of this infinite path need not be distinct.) We shall show that for each i there is a path from $r(e_i)$ to a vertex in H. Fix i, and let

$$X := \{w \in E^0 : \text{ there is a path from } r(e_i) \; to \; w\}.$$

Then X is a nonempty hereditary subset, and by Lemma (1.1.3) it follows that $X \cap H \neq \emptyset$. Thus there is a path from $r(e_i)$ to a vertex in H. Since this is true for all i, it must be the case that F_H is infinite. In the graph ${}_HE_\emptyset$ there is an edge from each element of F_H to an element in H. Since H is finite, this implies that there is a vertex in $H \subseteq {}_HE_\emptyset^0$ that is reached by infinitely many vertices, and hence is left infinite. It follows from Lemma (1.1.7) that $I \cong C^*({}_HE_\emptyset)$ is stable. Thus we are in the situation described in (i).

Case II: H is infinite, and E contains a cycle.

Let $\alpha = \alpha_1 \ldots \alpha_n$ be a cycle in E. Since H is hereditary, the vertices of α must either all lie outside of H or all lie inside of H. If the vertices all lie in H, then the graph ${}_HE_\emptyset$ contains a cycle, and since $C^*({}_HE_\emptyset)$ is simple, the dichotomy for simple graph C^*-algebras implies that $C^*({}_HE_\emptyset)$ is purely infinite. Since H is infinite, it follows that ${}_HE_\emptyset^0$ is infinite and $C^*({}_HE_\emptyset)$ is nonunital. Because $C^*({}_HE_\emptyset)$ is a simple, separable, purely infinite, and nonunital C^*-algebra, Zhang's Theorem implies that $I \cong C^*({}_HE_\emptyset)$ is stable. Thus we are in the situation described in (i).

If the vertices of α all lie outside H, then the set

$$X := \{w \in E^0 : \text{there is a path from } r(\alpha_n) \text{ to } w\}$$

is a nonempty hereditary set. It follows from Lemma (1.1.3) that $X \cap H \neq \emptyset$. Thus there exists a vertex $v \in H$ and a path β from $r(\alpha_n)$ to v with $r(\beta_i) \notin H$ for $i < |\beta|$. Consequently there are infinitely many paths in F_H that end at (viz. $\beta, \alpha\beta, \alpha\alpha\beta, \alpha\alpha\alpha\beta, \ldots$). Hence there are infinitely many vertices in ${}_HE_\emptyset$ that can reach v, and v is a left infinite vertex in ${}_HE_\emptyset$. It follows from Lemma (1.2.8) that $I \cong C^*({}_HE_\emptyset)$ is stable. Thus we are in the situation described in (i).

Case III: H is infinite, and E does not contain a cycle.

Since E does not contain a cycle, it follows that $C^*(E)$ is an AF-algebra. In addition, since H is infinite it follows that E^0 is infinite and $C^*(E)$ is nonunital. Thus we are in the situation described in (ii).

Corollary (1.2.10) [1]:

If E is a graph with a finite number of vertices and such that $C^*(E)$ contains a unique proper nontrivial ideal I, then I is stable. Furthermore, if $\{E^0, H, \emptyset\}$ are the saturated hereditary subsets of E, then $C^*(E_H)$ is a unital C^*-algebra and $I \cong C^*(E_H) \otimes \mathcal{K}$.

Proof:

Since E^0 is finite it is the case that $C^*(E)$ is unital, and it follows from Proposition (1.2.9) that I is stable. Furthermore, since $I = I_H$ it follows that I is Morita equivalent to $C^*(E_H)$. Since I and $C^*(E_H)$ are separable, it follows that I and $C^*(E_H)$ are stably isomorphic. Thus $I \cong \ I \ \otimes \mathcal{K}\ C^*(E_H) \otimes \mathcal{K}$. Finally, since $E_H^0 = H \subseteq E^0$ is finite, $C^*(E_H)$ is unital.

Chapter 2

C^*-algebras With Finite and Infinite Subquotients

Section (2.1): KK Factor and Classification

Just like a finite group, any C^*-algebra $\mathfrak{A}$ with finitely many ideals has a decomposition series

$$0 = \mathcal{J}_0 \lhd \mathcal{J}_1 \lhd \cdots \lhd \mathcal{J}_n = \mathfrak{A}$$

in such a way that all subquotients are simple. As in the group case, the simple subquotients are unique up to permutation of isomorphism classes, but far from determine the isomorphism class of $\mathfrak{A}$.

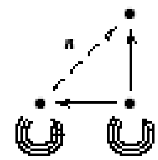
If we further assert that all simple subquotients are classifiable by algebraic invariants such as K-theory we are naturally led to the pertinent question of which algebraic invariants, if any, classify all of $\mathfrak{A}$. This question has previously been studied, leading to a complete solution when all subquotients are AF, and a partial solution when all are purely infinite, but in the case when some are of one type and some of another, only sporadic results have been found. It is the purpose of this section to provide a general framework in which classification of C^*-algebras can be proved for a large class of C^*-algebras of mixed type.

We are able to do so by combining several recent important developments in classification theory, notably:

(i) Kirchberg's isomorphism result ;
(ii) The Corona factorization property;
(iii) The universal coefficient theorem of Meyer and Nest;

A graph C^*-algebra has the property that all simple subquotients are either AF or purely infinite, and examples of mixed type occur even for very small graphs. For instance, consider the graphs $\{E_n\}_{n\in\mathbb{N}}$ given below

$$\begin{bmatrix} 0 & 0 & 0 \\ n & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$



For any $n > 0$, $C^*(E_n)$ decomposes into a linear lattice with simple subquotients

$$\mathbb{K}, \mathcal{O}_3 \otimes \mathbb{K}, \mathcal{O}_3.$$

The results presented have show that $C^*(E_n) \otimes \mathbb{K} \simeq C^*(E_{n+4}) \otimes \mathbb{K}$ and that there are three stable isomorphism classes

$$[C^*(E_1)] = [C^*(E_2)], [C^*(E_3)], [C^*(E_4)].$$

The technical focal point in this work is the general question of when one can deduce from the fact that $\mathfrak{A}$ and $\mathfrak{B}$ in the extension

$$0 \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{A} \longrightarrow 0$$

are classifiable by K-theory, that the same is true for $\mathfrak{E}$. We shall fix a finite (not necessarily Hausdorff) T_0 topological space X with a non-trivial open subset $U \subseteq X$ and require that $\mathfrak{E}$ is a C^*-algebra over X–associating ideals in $\mathfrak{E}$ with open subsets of X– in such a way that $\mathfrak{E}$ is the C^*-algebra corresponding to U. Assuming then that $\mathfrak{A}$ and $\mathfrak{B}$ are KK-strongly classifiable by their filtered and ordered K-theories over $X\backslash U$ and U, respectively, we supply conditions on the extension securing that also $\mathfrak{E}$ is classifiable by filtered and ordered K-theory. Our key technical result to this effect, Theorem (2.1.16) below, provides stable isomorphism in this context under, among other things, the provision of fullness of the extension and KK-liftability of morphisms of filtered K-theory. KK-liftability follows in many cases by the UCT of Meyer and Nest as generalized by Bentmann and Köhler.

Although we are confident that Theorem (2.1.16) will apply in other settings as well, we restrict ourselves to demonstrating how the results lead to classification of certain graph C^*-algebras up to stable isomorphism.

Let X be a topological space and let $\mathbb{O}(X)$ be the set of open subsets of X, partially ordered by set inclusion $\subseteq$. A subset Y of X is called locally closed if

$Y = U \backslash V$ where $U, V \in \mathbb{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of X will be denoted by $\mathbb{LC}(X)$. The set of all connected, non-empty locally closed subsets of X will be denoted by $\mathbb{LC}(X)^*$.

The partially ordered set $(\mathbb{O}(X), \subseteq)$ is a complete lattice, that is, any subset $\mathbb{S}$ of $\mathbb{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset S of $\mathbb{O}(X)$,

$$\bigwedge_{U \in S} U = \left(\bigcap_{U \in S} U\right)^{\circ} \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U.$$

For a C^*-algebra $\mathfrak{A}$, let $\mathbb{I}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathbb{I}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset S of $\mathbb{I}(\mathfrak{A})$,

$$\bigwedge_{\mathcal{J} \in S} \mathcal{J} = \bigcap_{\mathcal{J} \in S} \mathcal{J} \quad \text{and} \quad \bigvee_{\mathcal{J} \in S} \mathcal{J} = \overline{\sum_{\mathcal{J} \in S} \mathcal{J}}.$$

Definition (2.1.1) [2]:

Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathrm{Prim}(\mathfrak{A})$ denote the primitive ideal space of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let X be a topological space. A C^*-algebra over X is a pair $(\mathfrak{A}, \psi)$ consisting of a C^*-algebra $\mathfrak{A}$ and a continuous map $\psi : \mathrm{Prim}(\mathfrak{A}) \to X$. A C^*-algebra over X, $(\mathfrak{A}, \psi)$, is separable if $\mathfrak{A}$ is a separable C^*-algebra. We say that $(\mathfrak{A}, \psi)$ is tight if ψ is a homeomorphism.

We always identify $\mathbb{O}(\mathrm{Prim}(\mathfrak{A}))$ and $\mathbb{I}(\mathfrak{A})$ using the lattice isomorphism

$$U \longmapsto \bigcap_{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) \backslash U} \mathfrak{p}.$$

Let $(\mathfrak{A}, \psi)$ be a C^*-algebra over X. Then we get a map $\psi^* : \mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$ defined by

$$U \longmapsto \{\mathfrak{p} \in \mathrm{Prim}(\mathfrak{A}) : \psi(\mathfrak{p}) \in U\}.$$

Using the isomorphism $\mathbb{O}(\mathrm{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$, we get a map from $\mathbb{O}(X)$ to $\mathbb{I}(\mathfrak{A})$ by

$$U \longmapsto \bigcap \{\mathfrak{p} \in \text{Prim}(\mathfrak{A}) : \psi(\mathfrak{p}) \notin U\}.$$

Denote this ideal by $\mathfrak{A}(U)$. For $Y = U \backslash V \in \mathbb{LC}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By a Lemma, $\mathfrak{A}(Y)$ does not depend on U and V.

We have the following examples:

Example (2.1.2) [2]:

For any C^*-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \text{id}_{\text{Prim}(\mathfrak{A})})$ is a tight C^*-algebra over Prim($\mathfrak{A}$). For each $U \in \mathbb{O}(\text{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap \mathfrak{p} \in \text{Prim}(\mathfrak{A}) \backslash U \, \mathfrak{p}$.

Example (2.1.3) [2]:

Let $X_n = \{1,2,\ldots,n\}$ partially ordered with . Equip X_n with the Alexandrov topology, so the non-empty open subsets are

$$[a, n] = \{x \in X : a \leq x \leq n\}$$

for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathfrak{A}, \phi)$ be a C^*-algebra over X_n. We will use the following notation.

$$\mathfrak{A}[k] = \mathfrak{A}(\{k\}), \;\; \mathfrak{A}[a, b] = \mathfrak{A}([a, b]), \;\; \text{and} \;\; \mathfrak{A}(i, j] = \mathfrak{A}[i + 1, j].$$

Using the above notation we have ideals $\mathfrak{A}[a, n]$ such that

$$[0] \trianglelefteq \mathfrak{A}[n] \trianglelefteq \mathfrak{A}[n-1, n] \trianglelefteq \cdots \trianglelefteq \mathfrak{A}[2, n] \trianglelefteq [1, n] = \mathfrak{A}.$$

Definition (2.1.4) [2]:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras over X. A homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is X-equivariant if $\phi(\mathfrak{A}(U)) \subseteq \mathfrak{B}(U)$ for all $U \in \mathbb{O}(X)$. Hence, for every $Y = U \backslash V$, ϕ induces a homomorphism $\phi Y : \mathfrak{A}(Y) \to \mathfrak{B}(Y)$. Let $\mathfrak{C}^*\text{-alg}(X)$ be the category whose objects are C^*-algebras over X and whose morphisms are X-equivariant homomorphisms.

Remark (2.1.5) [2]:

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are tight C^*-algebras over X_n. Then it is clear that a $*$-homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism if and only if ϕ is an X_n-equivariant isomorphism.

Remark (2.1.6) [2]:

Let $e_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be an extension for $i = 1,2$. Note that $\mathfrak{E}_i$ can be considered as a C^*-algebra over $X_2 = \{1,2\}$ by sending ϕ to the zero ideal, $\{2\}$ to the image of $\mathfrak{B}_i$ in $\mathfrak{E}_i$, and $\{1,2\}$ to $\mathfrak{E}_i$. Hence, there exists a one-to-one correspondence between X_2-equivariant homomorphisms $\phi : \mathfrak{E}_1 \to \mathfrak{E}_2$ and homomorphisms from e_1 and e_2.

Definition (2.1.7) [2]:

Let X be a T_0 topological space and let $\mathfrak{A}$ be a C^*-algebra over X. For open subsets U_1, U_2, U_3 of X with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \backslash U_1$, $Y_2 = U_3 \backslash U_1$, $Y_3 = U_3 \backslash U_2 \in \mathbb{LC}(X)$. Then we have a six-term exact sequence

$$\begin{array}{ccccc} K_0\big(\mathfrak{A}(Y_1)\big) & \xrightarrow{\iota_*} & K_0\big(\mathfrak{A}(Y_2)\big) & \xrightarrow{\pi_*} & K_0\big(\mathfrak{A}(Y_3)\big) \\ \partial_* \uparrow & & & & \uparrow \partial_* \\ K_0\big(\mathfrak{A}(Y_3)\big) & \underset{\pi_*}{\longleftarrow} & K_0\big(\mathfrak{A}(Y_2)\big) & \underset{\iota_*}{\longleftarrow} & K_0\big(\mathfrak{A}(Y_1)\big) \end{array} \qquad (1)$$

The filtered K-theory $\mathrm{FK}_X(\mathfrak{A})$ of $\mathfrak{A}$ is the collection of all K-groups thus occurring and the natural transformations $\{\iota_*, \pi_*, \partial_*\}$. The filtered, ordered K-theory $\mathrm{FK}_X^+(\mathfrak{A})$ of $\mathfrak{A}$ is $\mathrm{FK}_X(\mathfrak{A})$ of $\mathfrak{A}$ together with $K_0(\mathfrak{A}(Y\ ))_+ +$ for all $Y \in \mathbb{LC}(X)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras over X. An isomorphism $\alpha : \mathrm{FK}_X(\mathfrak{A}) \to \mathrm{FK}_X(\mathfrak{B})$ is a collection of group isomorphisms

$$\alpha_{Y,*} : K_*\big(\mathfrak{A}(Y)\big) \to K_*\big(\mathfrak{B}(Y)\big)$$

for each $Y \in \mathbb{LC}(X)$ preserving all natural transformations. An isomorphism $\alpha : \mathrm{FK}_X(\mathfrak{A}) \to \mathrm{FK}_X(\mathfrak{B})$ is an isomorphism $\alpha : \mathrm{FK}_\mathrm{X}^+(\mathfrak{A}) \to \mathrm{FK}_\mathrm{X}^+(\mathfrak{B})$ which satisfies that $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathbb{LC}(X)$.

If $Y \in \mathbb{LC}(X)$ such that $Y = Y_1 \coprod Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \in \mathbb{O}(Y) \subseteq \mathbb{LC}(X)$, then $\mathfrak{A}(Y) \cong \mathfrak{A}(Y_1) \times \mathfrak{A}(Y_2)$ for any C^*-algebra over X. Moreover, there is a natural isomorphism from $K_*(\mathfrak{A}(Y))$ to $K_*\mathfrak{A}(Y_1)) \oplus K_*(\mathfrak{A}(Y_2))$ which is a positive isomorphism from $K_0(\mathfrak{A}(Y))$ to $K_0(\mathfrak{A}(Y_1)) \oplus K_0(\mathfrak{A}(Y_2))$. If X is finite, any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathbb{LC}(X)$ by the subset $\mathbb{LC}(X)^*$.

Let a be an element of a C^*-algebra $\mathfrak{A}$. We say that a is norm-full in $\mathfrak{A}$ if a is not contained in any norm-closed proper ideal of $\mathfrak{A}$. The word "full" is also widely used, but since we will often work in multiplier algebras, we emphasize that it is the norm topology we are using, rather than the strict topology. We say that a sub-C^*-algebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is norm-full if the norm-closed ideal generated by $\mathfrak{B}$ is $\mathfrak{A}$.

Definition (2.1.8) [2]:

An extension e is said to be full if the associated Busby invariant τ_e has the property that $\tau_e(a)$ is a norm-full element of $\mathcal{Q}(\mathfrak{B}) = \mathcal{M}(\mathfrak{B})/\mathfrak{B}$ for every $a \in \mathfrak{A}\backslash\{0\}$.

Let X and Y be T_0 topological spaces. For every continuous function $f : X \to Y$ we have a functor

$$f : \mathfrak{C}^* - \text{alg}(X) \to \mathfrak{C}^* - \text{alg}(Y), \ \ (A, \psi) \mapsto (A, f \circ \psi)$$

(i) Define $g_X^1 : X \to X_1$ by $g_X^1(x) = 1$. Then g_X^1 is continuous. Note that the induced functor $g_X^1 : \mathfrak{C}^* - \text{alg}(X) \to \mathfrak{C}^* - \text{alg}(X_1)$ is the forgetful functor.

(ii) Let U be an open subset of X. Define $g_{U,X}^2 : X \to X_2$ by $g_{U,X}^2(x) = 1$ if $x \notin U$ and $g_{U,X}^2(x) = 2$ if $x \in U$. Then $g_{U,X}^2$ is continuous. Thus the induced functor

$$g_{U,X}^2 : \mathfrak{C}^* - \text{alg}(X) \to \mathfrak{C}^* - \text{alg}(X_2)$$

is just specifying the extension $0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0$.

(iii) We can generalize (ii) to finitely many ideals. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X$ be open subsets of X. Define $g_{U_1, U_2, \ldots, U_n}^n : X \to X_n(x) = n - k + 1$ if

$x \in U_k \backslash U_{k-1}$. Then $\mathrm{g}^n_{U_1, U_2, \ldots, U_n}$ is continuous. Therefore, any C^*-algebra with ideals $0 \trianglelefteq \mathcal{J}_1 \trianglelefteq \mathcal{J}_2 \trianglelefteq \cdots \trianglelefteq \mathcal{J}_n = \mathfrak{A}$ can be made into a C^*-algebra over X_n.

(iv) For all $Y \in \mathbb{LC}(X)$, $r_X^Y : \mathfrak{C}^* - \mathrm{alg}(X) \to \mathfrak{C}^* - \mathrm{alg}(Y)$ is the restriction functor.

Let $\mathfrak{K}\mathfrak{K}(X)$ be the category whose objects are separable C^*-algebras over X and the set of morphisms is $KK(X; \mathfrak{A}, \mathfrak{B})$. By these functors induce functors from $\mathfrak{K}\mathfrak{K}(X)$ to $\mathfrak{K}\mathfrak{K}(Z)$, where $Z = Y, X_1, X_n$.

The proof of the following lemma is straightforward and is left to the reader.

Lemma (2.1.9) [2]:

Let U be an open set of X and $Y = X \backslash U$. Then

$$r_{X_2}^{[2]} \circ \mathrm{g}^2_{U,X} = \mathrm{g}^1_U \circ r_X^U \quad \text{and} \quad r_{X_2}^{[1]} \circ \mathrm{g}^2_{U,X} = \mathrm{g}^1_Y \circ r_X^Y \tag{2}$$

from $\mathfrak{C}^* - \mathrm{alg}(X)$ to $\mathfrak{C}^* - \mathrm{alg}(X_1)$. Consequently, the induced functors from $\mathfrak{K}\mathfrak{K}(X)$ to $\mathfrak{K}\mathfrak{K}(X_1)$ will be equal.

We have the following theorem.

Theorem (2.1.10) [2]:

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear C^*-algebras over X_2. Suppose $e_i: 0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is an essential extension for $i = 1,2$. If $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$, then

$$r_{X_2}^{[1]}(x) \times [\tau_{e_2}] = [\tau_{e_1}] \times r_{X_2}^{[2]}(x)$$

in $KK^1(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$.

Proof:

Let $\mathfrak{C}^*$-$\mathrm{alg}_{\mathrm{nuc}}(X_2)$ be the category whose objects are nuclear, separable C^*-algebras over X_2 and morphisms are X_2-equivariant homomorphisms. Let $\mathfrak{K}\mathfrak{K}_{\mathrm{nuc}}(X_2)$ be the category whose objects are nuclear, separable C^*-algebras over X_2 and the morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ are the elements of $KK(X_2; \mathfrak{A}, \mathfrak{B})$. Let $\mathfrak{K}\mathfrak{K}_{\mathrm{nuc}}$

be the category whose objects are nuclear, separable C^*-algebras and the morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ are the elements of $KK(\mathfrak{A}, \mathfrak{B})$.

Let $\mathfrak{A}$ be in $\mathfrak{C}^*\text{-alg}_{\text{nuc}}(X_2)$ and let $\pi_{\mathfrak{A}}$ be the natural projection from $\mathfrak{A}$ to $\mathfrak{A}[1]$. Let $i_{\mathfrak{A}} : S\mathfrak{A} \to C_{\pi\mathfrak{A}}$ and $j_{\mathfrak{A}} : \mathfrak{A}[2] \to C_{\pi_{\mathfrak{A}}}$ be the natural embeddings, where

$$C_{\pi_{\mathfrak{A}}} = \{(a, f) \in \mathfrak{A} \oplus C_0((0,1], \mathfrak{A}[1]) : \pi_{\mathfrak{A}}(a) = f(1)\}$$

is the mapping cone of $\pi_{\mathfrak{A}}$. Recall that $KK(j_{\mathfrak{A}})$ is invertible in $KK(\mathfrak{A}[2], C_{\pi_{\mathfrak{A}}})$. Then the isomorphism from $KK^1(\mathfrak{A}[1], \mathfrak{B}[2])$ to $KK(S\mathfrak{A}[1], \mathfrak{B}[2])$ sends the class induced by

$$0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$$

to $KK(i_{\mathfrak{A}}) \times KK(j_{\mathfrak{A}})^{-1}$. Using this isomorphism from $KK^1(\mathfrak{A}[1], \mathfrak{B}[2])$ to $KK(S\mathfrak{A}[1], \mathfrak{B}[2])$, the equation

$$r_{X_2}^{[1]}(x) \times [\tau_{e_2}] = [\tau_{e_1}] \times r_{X_2}^{[2]}(x)$$

in $KK^1(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$ becomes

$$Sr_{X_2}^{[1]}(x) \times KK(i_{\mathfrak{A}_2}) \times KK(j_{\mathfrak{A}_2})^{-1} = KK(i_{\mathfrak{A}_1}) \times KK(j_{\mathfrak{A}_1})^{-1} \times r_{X_2}^{[2]}(x)$$

in $KK(S\mathfrak{A}[1], \mathfrak{B}[2])$, where S is the natural isomorphism from $KK(\mathfrak{A}, \mathfrak{B})$ to $KK(S\mathfrak{A}, S\mathfrak{B})$.

Let $m = 1$ or 2. Define $F_m : \mathfrak{C}^* - \text{alg}_{\text{unc}} \to \mathfrak{K}\mathfrak{K}_{\text{nuc}}$ by

$$F_m(\mathfrak{A}) = \begin{cases} \mathfrak{A}[2], & m = 2, \\ S\mathfrak{A}[1] & m = 1, \end{cases} \quad \text{and} \quad F_m(\phi) = \begin{cases} \phi\{2\}, & m = 2. \\ S\phi\{1\}, & m = 1. \end{cases}$$

We claim that $\eta : F_1 \to F_2$ given by $\eta_{\mathfrak{A}} = KK(i_{\mathfrak{A}}) \times KK(j_{\mathfrak{A}})^{-1}$ is a natural transformation between the functors F_1 and F_2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be in $\mathfrak{C}^* - \text{alg}_{\text{unc}}(X_2)$ and let ϕ be an X_2-equivariant homomorphism. By the definition of the mapping cone sequence, there exists a homomorphism $\psi : C_{\pi\mathfrak{A}} \to C_{\pi\mathfrak{B}}$ such that the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S\mathfrak{A}[1] & \overset{i_{\mathfrak{A}}}{\rightarrow} & C_{\pi_{\mathfrak{A}}} & \longrightarrow & \mathfrak{A} & \longrightarrow & 0 \\ & & \downarrow S\phi_{\{1\}} & & \downarrow \psi & & \downarrow \phi & & \\ 0 & \longrightarrow & S\mathfrak{B}[1] & \underset{i_{\mathfrak{B}}}{\rightarrow} & C_{\pi_{\mathfrak{B}}} & \longrightarrow & \mathfrak{B} & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{A}[2] & \overset{j_{\mathfrak{A}}}{\rightarrow} & C_{\pi_{\mathfrak{A}}} & \longrightarrow & C\mathfrak{A} & \longrightarrow & 0 \\ & & \downarrow \phi_{\{2\}} & & \downarrow \psi & & \downarrow C\phi & & \\ 0 & \longrightarrow & \mathfrak{B}[2] & \underset{j_{\mathfrak{B}}}{\rightarrow} & C_{\pi_{\mathfrak{B}}} & \longrightarrow & C\mathfrak{B} & \longrightarrow & 0 \end{array}$$

are commutative. Thus,

$$\begin{aligned} F_1(\phi) \times KK(i_{\mathfrak{B}}) \times KK(j_{\mathfrak{B}})^{-1} &= KK\big(S\phi_{\{1\}}\big) \times KK(i_{\mathfrak{B}}) \times KK(j_{\mathfrak{B}})^{-1} \\ &= KK(i_{\mathfrak{A}}) \times KK(\psi) \times KK(j_{\mathfrak{B}})^{-1} \\ &= KK(i_{\mathfrak{A}}) \times KK(j_{\mathfrak{A}})^{-1} \times KK\big(\phi_{\{2\}}\big) \\ &= KK(i_{\mathfrak{A}}) \times KK(j_{\mathfrak{B}})^{-1} \times F_2(\phi). \end{aligned}$$

Hence, $\eta : F_1 \to F_2$ is a natural transformation.

Since F_1, F_2 are stable, split exact, and homotopy invariant functors, by the universal property of KK, we have that F_m induces a functor $\bar{F}_m : \mathfrak{KK}_{\mathrm{nuc}}(X_2) \to \mathfrak{KK}_{\mathrm{nuc}}$ and η induces a natural transformation $\eta : F_1 \to F_2$. In particular, for each $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$, we have that

$$Sr_{X_2}^{\{1\}}(x) \times KK\big(i_{\mathfrak{A}_2}\big) \times KK\big(j_{\mathfrak{A}_2}\big)^{-1} = KK\big(i_{\mathfrak{A}_1}\big) \times KK\big(j_{\mathfrak{A}_1}\big)^{-1} \times r_{X_2}^{\{2\}}(x).$$

By the comments made in the second paragraph of the proof, we have that

$$r_{X_2}^{\{1\}}(x) \times \big[\tau_{e_2}\big] \times \big[\tau_{e_1}\big] \times r_{X_2}^{\{2\}}(x)$$

in $KK^{-1}(\mathfrak{A}_1[1], \mathfrak{A}_2[2])$.

In this section we prove general classification results for several classes of extension algebras.

Definition (2.1.11) [2]:

For a T_0 topological space X, we will consider classes C_X of separable, nuclear C^*-algebras in $\mathcal{N}$ such that

(i) Any element in C_X is a C^*-algebra over X; and

(ii) If $\mathfrak{A}$ and $\mathfrak{B}$ are in C_X and there exists an invertible element α in $KK(X;\mathfrak{A},\mathfrak{B})$ which induces an isomorphism from $\mathrm{FK}_X^+(\mathfrak{A})$ to $\mathrm{FK}_X^+(\mathfrak{B})$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $KK(\phi) = \mathrm{g}_X^1(\alpha)$.

We concentrate on the following two classes satisfying (i)-(ii) above:

Example (2.1.12) [2]:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, nuclear, stable, $\mathcal{O}_\infty$-absorbing tight C^*-algebras over X. Let α be an invertible element in $KK(X;\mathfrak{A},\mathfrak{B})$. By Kirchberg, there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $KK(X;\phi) = \alpha$. Hence, $KK(\phi) = \mathrm{g}_X^1\big(KK(X;\phi)\big) = \mathrm{g}_X^1(\alpha)$. Thus, if C_X is the class of all stable, separable, nuclear C^*-algebras over X which are $\mathcal{O}_\infty$-absorbing in $\mathcal{N}$, then C_X satisfies the properties of Definition (2.1.11).

Example (2.1.13) [2]:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be stable AF algebras. Let α be an invertible element in $KK(X_1;\mathfrak{A},\mathfrak{B}) = KK(\mathfrak{A},\mathfrak{B})$ which induces an isomorphism from $\mathrm{FK}_{X_1}^+(\mathfrak{A}) = (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ to $\mathrm{FK}_{X_1}^+(\mathfrak{B}) = (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+)$. Then by the classification of AF algebras, there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $K_0(\phi) = K_0(\alpha)$. Since $KK(\mathfrak{A},\mathfrak{B}) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))$, we have that $KK(\phi) = \alpha$. Thus, if C_{X_1} is the class of all stable, AF algebras, then C_{X_1} satisfies the properties of Definition (2.1.11).

Remark (2.1.14) [2]:

The condition

ii$'$. If $\mathfrak{A}$ and $\mathfrak{B}$ are in C_X and there exists an isomorphism β from $\mathrm{FK}_X^+(\mathfrak{A})$ to $\mathrm{FK}_X^+(\mathfrak{B})$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi^* = \beta$

is more closely suited to our purposes, and (i),(ii') is true in general in Example (2.1.13), but not always in Example (2.1.12). In fact, there exists a space X with four points such that (ii') fails in Example (2.1.12).

Lemma (2.1.15) [2]:

For $i = 1,2$, let $e_i: 0 \to \mathcal{I}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be a non-unital full extension. Suppose $\mathcal{I}_i$ is a stable C^*-algebra satisfying the Corona factorization property. If there exist an isomorphism $\phi_0 : \mathcal{I}_1 \to \mathcal{I}_2$ and an isomorphism $\phi_2 : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $KK(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KK(\phi_0)$, then $\mathfrak{E}_1$ is isomorphic to $\mathfrak{E}_2$.

Proof:

Note that $e_1 \cong e_2 \cdot \phi_0$ and $e_2 \cong \phi_2 \cdot e_2$, where $e_1 \cdot \phi_0$ is the push-out of e_1 along ϕ_0 and $\phi_2 \cdot e_2$ is the pull-back of e_2 along ϕ_2. Since $[\tau_{e_1} \cdot \phi_0] = [\tau_{e_1}] \times KK(\phi_0) = KK(\phi_2) \times [\tau_{e_2}] = [\tau_{\phi_2 \cdot e_2}]$ in $KK^1(\mathfrak{A}_1, \mathcal{I}_2)$, we have that $[\tau_{e_1} \cdot \phi_0] = [\tau_{\phi_2 \cdot e_2}]$. Since $\tau_{e_1 \cdot \phi_0}$ and $\tau_{\phi_2 \cdot e_2}$ are non-unital full extensions and $\mathcal{I}_2$ satisfies the Corona factorization property, there exists a unitary u in $\mathcal{M}(\mathcal{I}_2)$ such that $\mathrm{Ad}(\pi(u)) \circ \tau_{e_1} \cdot \phi_0 = \tau_{\phi_2} \cdot e_2$. Hence, $(\mathrm{Ad}(u), \mathrm{Ad}(u), \mathrm{id}_{\mathfrak{A}_1})$ is an isomorphism between $e_1 \cdot \phi_0$ and $\phi_2 \cdot e_2$. Thus, $\mathfrak{E}_1$ is isomorphic to $\mathfrak{E}_2$.

We will apply the theorem below to a certain class of C^*-algebras arising from graphs. See Proposition (2.2.9), Corollary (2.2.10), Proposition (2.2.11), and Theorem (2.2.13).

Theorem (2.1.16) [2]:

Let X be a finite topological space and let $U \in \mathbb{O}(X)$. Set $Y = X \backslash U \in \mathbb{LC}(X)$. For $i = 1,2$, let $\mathfrak{A}_i$ be a C^*-algebra over X such that $\mathfrak{A}_i$ is a stable, separable, nuclear C^*-algebra and every simple subquotient of $\mathfrak{A}_i$ is in $\mathcal{N}$.

Let $\mathcal{C}_{I,U}$ and $\mathcal{C}_{Q,Y}$ be classes of C^*-algebras that satisfy the properties of Definition (2.1.11). Suppose $\mathfrak{A}_i(U)$ is a C^*-algebra in $\mathcal{C}_{I,U}$ and satisfying the Corona factorization property, and $\mathfrak{A}_i(Y)$ is a C^*-algebra in $\mathcal{C}_{Q,Y}$. Suppose for $i = 1,2$

$$e_i : 0 \to \mathfrak{A}_i(U) \to \mathfrak{A}_i \to \mathfrak{A}_i(Y) \to 0$$

are full extensions. If there exists an isomorphism $\alpha : \mathrm{FK}_X^+(\mathfrak{A}_1) \to \mathrm{FK}_X^+(\mathfrak{A}_2)$ such that $\alpha_U : \mathrm{FK}_U^+(\mathfrak{A}_1(U)) \to \mathrm{FK}_U^+(\mathfrak{A}_2(U))$ and $\alpha_Y : \mathrm{FK}_Y^+(\mathfrak{A}_1(Y)) \to \mathrm{FK}_Y^+(\mathfrak{A}_2(Y))$ are isomorphisms, and α lifts to an invertible element in $KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$, then $\mathfrak{A}_1 \cong \mathfrak{A}_2$.

Proof:

Suppose there exists an isomorphism $\alpha : \mathrm{FK}_X(\mathfrak{A}_1) \to \mathrm{FK}_X(\mathfrak{A}_2)$ such that $\alpha_U : \mathrm{FK}_U^+(\mathfrak{A}_1(U)) \to \mathrm{FK}_U^+(\mathfrak{A}_2(U))$ and $\alpha_Y : \mathrm{FK}_Y^+(\mathfrak{A}_1(Y)) \to \mathrm{FK}_Y^+(\mathfrak{A}_2(Y))$ are isomorphisms, and α lifts to an invertible element in $KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$. Let $x \in KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$ be this lifting. Then $r_X^U(x)$ is an invertible element in $KK(U; \mathfrak{A}_1(U), \mathfrak{A}_2(U))$ and $r_X^Y(x)$ is an invertible element in $KK(Y; \mathfrak{A}_1(Y), \mathfrak{A}_2(Y))$. Since $\mathfrak{A}_1(U)$ and $\mathfrak{A}_2(U)$ are in $\mathcal{C}_{I,U}$ and $\mathfrak{A}_1(Y)$ and $\mathfrak{A}_2(Y)$ are in $\mathcal{C}_{Q,Y}$, there exists an isomorphism $\phi_0 : \mathfrak{A}_1(U) \to \mathfrak{A}_2(U)$ which induces $r_X^Y(x)$ and there exists an isomorphism $\phi_2 : \mathfrak{A}_1(Y) \to \mathfrak{A}_2(Y)$ which induces $r_X^Y(x)$. By Theorem (2.1.10) and by Lemma (2.1.9)

$$KK(\phi_2) \times [\tau_{e_2}] = \left(\mathrm{g}_Y^1 \circ r_X^Y(x)\right) \times [\tau_{e_2}] = \left(r_{X_2}^{\{1\}} \circ \mathrm{g}_{U,X}^2(x)\right) \times [\tau_{e_2}]$$

$$= [\tau_{e_1}] \times \left(r_{X_2}^{\{2\}} \circ \mathrm{g}_{U,X}^2\right)(x) = [\tau_{e_1}] \times \left(\mathrm{g}_U^1 \circ r_X^U(x)\right) = [\tau_{e_1}] \times KK(\phi_0)$$

in $KK^1(\mathfrak{A}_1(Y), \mathfrak{A}_2(U))$. Since e_1 and e_2 are full non-unital extensions and $\mathfrak{A}_i(U)$ has the Corona factorization property, by Lemma (2.1.15) we have that $\mathfrak{A}_1 \cong \mathfrak{A}_2$.

Section (2.2): Extensions and Applications to Graph C^*-algebras

In this section, we prove that certain extensions arising from graph C^*algebras are necessarily full, allowing one to use the results.

Let $\mathcal{J}$ be an ideal of a C^*-algebra $\mathfrak{B}$. Set

$$\mathcal{M}(\mathfrak{B};\mathcal{J}) = \{x \in \mathcal{M}(\mathfrak{B}) : x\mathfrak{B} \subseteq \mathcal{J}\}.$$

It is easy to check that $\mathcal{M}(\mathfrak{B};\mathcal{J})$ is a (norm-closed, two-sided) ideal of $\mathcal{M}(\mathfrak{B})$.

Definition (2.2.1) [2]:

Let $\{f_n\}$ be an approximate identity consisting of projections for K, where $f_0 = 0$ and $f_n - f_{n-1}$ is a projection of dimension one. Let $\mathfrak{A}$ be a unital C^*-algebra and set $e_n = 1_{\mathfrak{A}} \otimes f_n$. Note that $\{e_n\}$ is an approximate identity of $\mathfrak{A} \otimes K$ consisting of projections.

We call an element $X \in \mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$ diagonal with respect to $\{e_n\}$ if there exists a strictly increasing sequence $\{\alpha(n)\}$ of integers with $\alpha(0) = 0$ such that

$$X\big(e_{\alpha(n)} - e_{\alpha(n-1)}\big) - \big(e_{\alpha(n)} - e_{\alpha(n-1)}\big)X = 0$$

for all $n \in \mathbb{N}$. We write $X = \operatorname{diag}(x_1, x_2, \dots)$, where

$$x_n = X\Big(\big(e_{\alpha(n)} - e_{\alpha(n-1)}\big)\Big),$$

Conversely, if $\{x_n\}$ is a bounded sequence with $x_n \in M_{k_n}(\mathfrak{A})$, then upon identifying $M_{k_n}(\mathfrak{A})$ with

$$\big(e_{\alpha(n)} - e_{\alpha(n-1)}\big)(\mathfrak{A} \otimes \mathbb{K})\big(e_{\alpha(n)} - e_{\alpha(n-1)}\big)$$

for an appropriate $\alpha(n)$, we have that $X = \operatorname{diag}(x_1, x_2, \dots)$ for some $X \in \mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$.

Let $\epsilon > 0$. Define $f_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_\epsilon(t) = \begin{cases} 0, & \text{if } t \leq \epsilon. \\ \epsilon^{-1}(t-\epsilon), & \text{if } \epsilon \leq t \leq 2\epsilon. \\ 1, & \text{if } t \geq \epsilon. \end{cases} \qquad (3)$$

Theorem (2.2.2) [2]:

Let $\mathfrak{B}_0$ be a unital, $\mathcal{O}_\infty$-absorbing C^*-algebra. Suppose $\mathfrak{B} = \mathfrak{B}_0 \otimes \mathbb{K}$ has a largest proper non-trivial ideal I. If $x \in \mathcal{M}(\mathfrak{B})$ such that x is not an element of $\mathcal{M}(\mathfrak{B}; \mathcal{J}) + \mathfrak{B}$, then $I(x) = \mathcal{M}(B)$, where $I(x)$ is the norm-closed ideal of $\mathcal{M}(\mathfrak{B})$ generated by x.

Consequently, $\mathcal{M}(\mathfrak{B}; \mathcal{J}) + \mathfrak{B}$ is the largest proper ideal of $\mathcal{M}(\mathfrak{B})$ and $(\mathcal{M}(\mathfrak{B}; \mathcal{J}) + \mathfrak{B})/\mathfrak{B}$ is the largest proper ideal of $\mathcal{Q}(\mathfrak{B})$.

Proof:

We first show that if J is an ideal of $\mathcal{M}(\mathfrak{B})$ such that $\mathcal{J} + \mathfrak{B} = \mathcal{M}(\mathfrak{B})$, then $\mathcal{J} = \mathcal{M}(\mathfrak{B})$. Since $1_{\mathcal{M}}(\mathfrak{B})$ is a projection and $\mathcal{J} + \mathfrak{B} = \mathcal{M}(\mathfrak{B})$, there exist $y \in \mathcal{J}$ and $b \in \mathfrak{B}$ such that $1_{\mathcal{M}(\mathfrak{B})} = y + b$. Since $\mathfrak{B}$ is stable, there exists an isometry $S \in \mathcal{M}(B)$ such that $\|S^* bS\| < 1$. Hence,

$$\left\|1_{\mathcal{M}(\mathfrak{B})} - S^* yS\right\| = \|S^* bS\| < 1.$$

Hence, $S^* yS$ is invertible in $\mathcal{M}(\mathfrak{B})$ which implies that $\mathcal{J} = \mathcal{M}(\mathfrak{B})$. Note that the claim also proves that $\mathcal{M}(\mathfrak{B}; \mathcal{J}) + \mathfrak{B}$ is a proper ideal of $\mathcal{M}(\mathfrak{B})$ since $1_{\mathcal{M}}(\mathfrak{B})$ is not an element of $\mathcal{M}(\mathfrak{B}; \mathcal{J})$.

Suppose $x \in \mathcal{M}(\mathfrak{B})$ such that x is not an element of $\mathcal{M}(\mathfrak{B}; \mathcal{J}) + \mathfrak{B}$. Then there exists a diagonal element $y \in \mathcal{M}(\mathfrak{B})$ with respect to $\{e_n\}$ such that $I(x) + \mathfrak{B} = I(y) + \mathfrak{B}$, where diagonal with respect to $\{e_n\}$ we mean there exists a strictly increasing sequence of integers $\{\alpha(n)\}$ with $\alpha(0) = 0$ such that $y = \sum_{k=1}^{\infty} y_k$, where $y_k \in (e_{\alpha(k)} - e_{\alpha(k-1)})\mathfrak{B}(e_{\alpha(k)} - e_{\alpha(k-1)})$ and the sum converges in the strict topology.

Note that y is not an element of $\mathcal{M}(\mathfrak{B}; I) + \mathfrak{B}$ since $I(x) + \mathfrak{B} = I(y) + \mathfrak{B}$ and x is not an element of $\mathcal{M}(\mathfrak{B}; I) + \mathfrak{B}$. Note that we have an exact sequence

$$0 \to (\mathcal{M}(\mathfrak{B}; \mathcal{J}) + \mathfrak{B})/\mathfrak{B} \to \mathcal{Q}(\mathfrak{B}) \to \mathcal{Q}(\mathfrak{B}/\mathcal{J}) \to 0.$$

Since $\mathfrak{B}/\mathcal{J}$ is a σ -unital, purely infinite simple C^*-algebra, $\mathcal{Q}(\mathfrak{B}/\mathcal{J})$ is simple. Since the image of y in $\mathcal{Q}(\mathfrak{B}/\mathcal{J})$ is non-zero and $\mathcal{Q}(\mathfrak{B}/\mathcal{J}))$ is simple, there exists

$\delta > 0$ such that for all $m \in \mathbb{N}$, there exists $m' \geq m$ such that $\sum_{k=m}^{m'} f_\delta(y_k)$ is not an element of $(e_{\alpha(m')} - e_{\alpha(m-1)})\mathcal{J}(e_{\alpha(m')} - e_{\alpha(m-1)})$.

Let $m \in \mathbb{N}$. By the above paragraph, there exists $m' \geq m$ such that $\sum_{k=m}^{m'} f_\delta(y_k)$ is not an element of $(e_{\alpha(m')} - e_{\alpha(m-1)})\mathcal{J}(e_{\alpha(m')} - e_{\alpha(m-1)})$. Therefore, $\sum_{k=m}^{m'} f_\delta(y_k)$ is norm-full in $(e_{\alpha(m')} - e_{\alpha(m-1)})\mathfrak{B}(e_{\alpha(m')} - e_{\alpha(m-1)})$. And there exists $z \in (e_{\alpha(m')} - e_{\alpha(m-1)})\mathfrak{B}(e_{\alpha(m')} - e_{\alpha(m-1)})$ such that

$$\left(e_{\alpha(m')} - e_{\alpha(m-1)}\right) = z\left(\sum_{k=m}^{m'} f_\delta(y_k)\right)z^*.$$

Therefore,

$$1_{\mathfrak{B}_0} \leq \left(e_{\alpha(m')} - e_{\alpha(m-1)}\right) = z\left(\sum_{k=m}^{m'} f_\delta(y_k)\right)z^*.$$

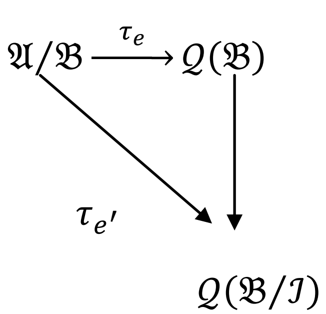
We can find $I(y) = \mathcal{M}(\mathfrak{B})$ which implies that $I(x) + \mathfrak{B} = I(y) + \mathfrak{B} = \mathcal{M}(\mathfrak{B})$. By the above claim, we have that $I(x) = \mathcal{M}(\mathfrak{B})$.

Corollary (2.2.3) [2]:

Let $\mathfrak{B}_0$ be a unital, $\mathcal{O}_\infty$-absorbing C^*-algebra. Suppose $\mathfrak{B} = \mathfrak{B}_0 \otimes \mathbb{K}$ has a largest proper non-trivial ideal $\mathcal{J}$. Suppose $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ such that $e : 0 \to \mathfrak{B}/\mathcal{J} \to \mathfrak{A}/\mathcal{J} \to \mathfrak{A}/\mathfrak{B} \to 0$ is an essential extension. Then the extension $e : 0 \to \mathfrak{B} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{B} \to 0$ is a full extension.

Proof:

Note that the canonical projection from $\mathfrak{A}$ to $\mathfrak{A}/\mathcal{J}$ is an X_2-equivariant homomorphism. Therefore, by the diagram



is commutative.

Since $\mathfrak{B}/\mathcal{J}$ is a stable purely infinite simple C^*-algebra, then we have $\mathcal{Q}(\mathfrak{B}/\mathcal{J})$ is simple. Since e' is an essential extension, we have that e' is a full extension. Let $a \in \mathfrak{A}/\mathfrak{B}$ be a non-zero element. Then the ideal generated by $\tau_{e'}(a)$ in $\mathcal{Q}(\mathfrak{B}/\mathcal{J})$ is $\mathcal{Q}(\mathfrak{B}/\mathcal{J})$.

Since

$$0 \to (\mathcal{M}(\mathfrak{B};\mathcal{J}) + \mathfrak{B})/\mathfrak{B} \to \mathcal{Q}(\mathfrak{B}) \to \mathcal{Q}(\mathfrak{B}/\mathcal{J}) \to 0$$

is an exact sequence, by the above commutative diagram, $\tau_e(a)$ is not an element of $(\mathcal{M}(\mathfrak{B};\mathcal{J}) + \mathfrak{B})/\mathfrak{B}$. Hence, by Theorem (2.2.2), $\tau_e(a)$ is norm-full in $\mathcal{Q}(\mathfrak{B})$.

Proposition (2.2.4) [2]:

Let $\mathfrak{A}$ be a C^*-algebra and let $\mathcal{J}$ and $\mathfrak{D}$ be ideals of $\mathfrak{A}$ with $\mathcal{J} \subseteq \mathfrak{D}$. Suppose $\mathfrak{D}/\mathcal{J}$ is an essential ideal of $\mathfrak{A}/\mathcal{J}$. Then $e_1 : 0 \to \mathcal{J} \to \mathfrak{A} \to \mathfrak{A}/\mathcal{J} \to 0$ is a full extension if and only if $e_2 :\ 0 \to \mathcal{J} \to \mathfrak{D} \to \mathfrak{D}/\mathcal{J} \to 0$ is a full extension.

Proof:

Note that the natural embedding $\iota_{\mathfrak{D}} : \mathfrak{D} \to \mathfrak{A}$ is an X_2-equivariant homomorphism with $\iota_{\mathcal{J}} = \mathrm{id}_{\mathcal{J}}$. Hence, the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathfrak{D} & \longrightarrow & \mathfrak{D}/\mathcal{J} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \iota_{\mathfrak{D}/\mathcal{J}} & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{A}/\mathcal{J} & \longrightarrow & 0 \end{array}$$

is commutative. Therefore, the diagram

$$\begin{array}{ccc} \mathfrak{D}/\mathcal{J} & \xrightarrow{\tau_{e_2}} & \mathcal{Q}(\mathcal{J}) \\ \iota_{\mathfrak{D}/\mathcal{J}} \downarrow & & \| \\ \mathfrak{A}/\mathcal{J} & \xrightarrow[\tau_{e_1}]{} & \mathcal{Q}(\mathcal{J}) \end{array}$$

is commutative.

Suppose e_1 is a full extension. Then it is clear from the above diagram that e_2 is a full extension. Suppose that e_2 is a full extension. Let $a \in \mathfrak{A}/\mathcal{J}$ be a non-zero element. Let $\mathcal{J}$ be the ideal generated by a in $\mathfrak{A}/\mathcal{J}$. Since $\mathfrak{D}/\mathcal{J}$ is an essential ideal of $\mathfrak{A}/\mathcal{J}$, there exists a non-zero $b \in \mathfrak{D}/\mathcal{J}$ such that $\iota_{\mathfrak{D}/\mathcal{J}(b)} \in I$. Hence, $\tau_{e_2}(b) = (\tau_{e_1} \circ \iota_{\mathfrak{D}/\mathcal{J}})(b)$ is norm-full in $\mathcal{Q}(\mathcal{J})$. Since $(\tau_{e_1} \circ \iota_{\mathfrak{D}/\mathcal{J}})(b)$ is in the ideal generated by $\tau_{e_1}(a)$ in $\mathcal{Q}(\mathcal{J})$, we have that $\tau_{e_1}(a)$ is norm-full in $\mathcal{Q}(\mathcal{J})$. Thus, e_1 is a full extension.

Proposition (2.2.5) [2]:

Let $\mathfrak{A}$ be a graph C^*-algebra satisfying Condition (K). Suppose $\mathcal{J}_1 \trianglelefteq \mathcal{J}_2 \trianglelefteq \mathfrak{A}$ such that $\mathcal{J}_1$ is an AF algebra, $\mathcal{J}_1$ is the largest proper ideal of $\mathcal{J}_2, \mathcal{J}_2/\mathcal{J}_1$ is purely infinite. Then $e : 0 \to \mathcal{J}_1 \otimes \mathbb{K} \to \mathcal{J}_2 \otimes \mathbb{K} \to \mathcal{J}_2/\mathcal{J}_1 \otimes \mathbb{K} \to 0$ is a full extension.

Proof:

We can show that, $\mathfrak{A} \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$, where E is a graph satisfying Condition (K) and has no breaking vertices. Hence, $\mathcal{J}_2 \otimes \mathbb{K}$ is isomorphic to a $C^*(E_1) \otimes \mathbb{K}$ where E_1 has no breaking vertices and satisfies Condition (K). Note that $C^*(E_1)$ has a largest proper ideal $\mathfrak{D}_1$, that $C^*(E_1)/\mathfrak{D}_1$ is purely infinite, $\mathfrak{D}_1$ is an AF algebra, and $\mathfrak{D}_1 \otimes \mathbb{K} \cong \mathcal{J}_1 \otimes \mathbb{K}$. Thus, there exists a projection $p \in C^*(E_1)$ such that $pC^*(E_1)p \otimes \mathbb{K} \cong C(E_1) \otimes \mathbb{K}$ and $p\mathfrak{D}_1 p$ is stable. Since $pC^*(E_1)\, p/p\mathfrak{D}_1 p$ is a unital simple C^*-algebra and

$$0 \to p\mathfrak{D}_1 p \to pC^*(E_1)p \to pC^*(E_1)\, p/p\mathfrak{D}_1 p \to 0$$

is a unital essential extension, the extension is full. Since $p\mathfrak{D}_1 p$ is stable, implies that the extension

$$0 \to p\mathfrak{D}_1 p \otimes \mathbb{K} \to pC^*(E_1)p \otimes \mathbb{K} \to (pC^*(E_1)\, p/p\mathfrak{D}_1 p) \otimes \mathbb{K} \to 0$$

is full. The proposition now follows since the isomorphism between $pC^*(E_1)p \otimes \mathbb{K}$ and $C^*(E_1) \otimes \mathbb{K}$ maps the ideal $p\mathfrak{D}_1 p \otimes \mathbb{K}$ onto the ideal $\mathfrak{D}_1 \otimes \mathbb{K}$ by Brown's.

Corollary (2.2.6) [2]:

Let $\mathfrak{A}$ be a graph C^*-algebra satisfying Condition (K). Suppose that $\mathcal{I}$ is an AF algebra such that $\mathcal{I}$ is an ideal of $\mathfrak{A}$, for all ideals $\mathcal{J}$ of $\mathfrak{A}$ we have that $\mathcal{J} \subseteq \mathcal{I}$ or $\mathcal{I} \subseteq \mathcal{J}$, and $\mathfrak{A}/\mathcal{I}$ is $\mathcal{O}_\infty$-absorbing. Then $e : 0 \to \mathcal{I} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A}/\mathcal{I} \otimes \mathbb{K} \to 0$ is a full extension.

Proof:

Let $\{\mathfrak{C}_n : n \in \mathbb{N}\}$ be the set of all minimal ideals of $\mathfrak{A}/\mathcal{I}$ and let $\mathfrak{A}_n$ be an ideal of $\mathfrak{A}$ such that $\mathcal{I} \subseteq \mathfrak{A}_n$ and $\mathfrak{A}_n/\mathcal{I} = \mathfrak{C}_n$.

Let $\mathcal{J}$ be an ideal of $\mathfrak{A}_n$. Then $\mathcal{J}$ is an ideal of $\mathfrak{A}$. Hence, $\mathcal{J} \subseteq \mathcal{I}$ or $\mathcal{I} \subseteq \mathcal{J}$. Suppose $\mathcal{I} \subseteq \mathcal{J}$ but $\mathcal{I} = \mathcal{J}$. Then, $\mathcal{J}/\mathcal{I} = \mathfrak{A}_n/\mathfrak{A} = \mathfrak{C}_n$ since $\mathfrak{C}_n$ is simple. Hence, $\mathcal{I}$ is the largest proper ideal of $\mathfrak{A}_n$, $\mathcal{I}$ is an AF algebra, and $\mathfrak{A}_n/\mathcal{I}$ is purely infinite. Therefore, by Proposition (2.2.5), $0 \to \mathcal{I} \otimes \mathbb{K} \to \mathfrak{A}_n \otimes \mathbb{K} \to \mathfrak{C}_n \otimes \mathbb{K} \to 0$ is a full extension.

Let $\mathfrak{D} = \overline{\sum_{n=1}^{\infty} \mathfrak{A}_n}$. Then $\mathfrak{D}$ is an ideal of $\mathfrak{A}$ such that $\mathfrak{D}/\mathcal{I}$ is an essential ideal of $\mathfrak{A}/\mathcal{I}$. Since $\mathfrak{C}_i \cap \mathfrak{C}_j = \{0\}$ for $i \neq j$, we have that $0 \to \mathcal{I} \otimes \mathbb{K} \to \mathfrak{D} \otimes \mathbb{K} \to \mathfrak{D}/\mathcal{I} \otimes K \to 0$ is a full extension. The corollary now follows from Proposition (2.2.4).

Recall our definition of X_n from Example (2.1.3) above. We now classify a certain class of graph C^*-algebras that are tight C^*-algebras over X_n.

Proposition (2.2.7) [2]:

Suppose $\mathfrak{A}$ is a C^*-algebra with finitely many ideals. If every simple subquotient of $\mathfrak{A} \otimes \mathbb{K}$ satisfies the Corona factorization property, then $\mathfrak{A} \otimes \mathbb{K}$ satisfies the Corona factorization property. Consequently, any graph C^*-algebra with finitely many ideals has the Corona factorization property.

Proof:

We will prove the result of the proposition by induction. If $\mathfrak{A}$ is simple, then by our assumption, $\mathfrak{A} \otimes \mathbb{K}$ has the Corona factorization property. Suppose that the proposition is true for any C^*-algebra $\mathfrak{B}$ with at most n ideals such that any simple subquotient of $\mathfrak{B} \otimes \mathbb{K}$ satisfies the Corona factorization property.

Let $\mathfrak{A}$ be a C^*-algebra with $n+1$ ideals such that every simple subquotient of $\mathfrak{A} \otimes \mathbb{K}$ satisfies the Corona factorization property. Let $\mathcal{J}$ be a proper non-trivial ideal of $\mathfrak{A} \otimes \mathbb{K}$. Then $\mathcal{J}$ and $\mathfrak{A}/\mathcal{J}$ are C^*-algebras with at most n ideals such that every simple subquotient of $\mathcal{J} \otimes \mathbb{K}$ and $\mathfrak{A}/\mathcal{J} \otimes \mathbb{K}$ satisfies the Corona factorization property. Hence, $\mathcal{J} \otimes \mathbb{K}$ and $\mathfrak{A}/\mathcal{J} \otimes \mathbb{K}$ satisfy the Corona factorization property. Therefore, $\mathfrak{A} \otimes \mathbb{K}$ satisfies the Corona factorization property.

Theorem (2.2.8) [2]:

(See Meyer and Nest.) For the topological space X_n, if $\mathfrak{A}$ and $\mathfrak{B}$ are separable, nuclear C^*-algebras over X_n such that $\mathfrak{A}[k]$ and $\mathfrak{B}[k]$ are in $\mathcal{N}$, then any isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}) \to \mathrm{FK}_{X_n}(\mathfrak{B})$ lifts to an invertible element in $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

Proposition (2.2.9) [2]:

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear C^*-algebras over X_n. Suppose $\mathfrak{A}_i[1]$ is an AF algebra and $\mathfrak{A}_i[2,n]$ is a tight stable $\mathcal{O}_\infty$-absorbing C^*-algebra over $[2,n]$, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i[1,2]$. Then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}_1) \to \mathrm{FK}_{X_n}(\mathfrak{B}_1)$ such that $\alpha\{1\}$ is an order isomorphism.

Proof:

Since $\mathfrak{A}_i[2,n]$ is a tight C^*-algebra over $[2,n]$, there exists a norm-full projection p in $\mathfrak{A}_i[2,n]$. By Brown's, $p\mathfrak{A}_i[2,n]p \otimes \mathbb{K} \cong \mathfrak{A}_i[2,n] \otimes \mathbb{K}$. Since $\mathfrak{A}_i[2,n]$ is an $\mathcal{O}_\infty$-absorbing C^*-algebra, $p\mathfrak{A}_i[2,n]p$ is an $\mathcal{O}_\infty$-absorbing C^*-algebra. By Corollary (2.2.3),

$$0 \to \mathfrak{A}_i \to [2,i] \otimes \mathbb{K} \to \mathfrak{A}_i \otimes \mathbb{K} \to \mathfrak{A}_i[1] \otimes \mathbb{K} \to 0$$

is a full extension for $n \geq 3$. Suppose $n = 2$. Then $\mathfrak{A}_i[2,2]$ is a purely infinite simple C^*-algebra, hence, $Q(\mathfrak{A}_i[2,2] \otimes \mathbb{K})$ is a simple C^*-algebra. Thus,

$$0 \to \mathfrak{A}_i[2,2] \otimes \mathbb{K} \to \mathfrak{A}_i \otimes \mathbb{K} \to \mathfrak{A}_i[1] \otimes \mathbb{K} \to 0$$

is a full extension. By Proposition (2.2.7), $\mathfrak{A}_i[2,n]$ has the Corona factorization property. The theorem now follows from Theorem (2.2.8) and Theorem (2.1.16).

Corollary (2.2.10) [2]:

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be graph C^*-algebras satisfying Condition (K) and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-algebras over X_2. Suppose $\mathfrak{A}_i[2]$ is $\mathcal{O}_\infty$-absorbing, $\mathfrak{A}_i[2]$ is the smallest non-zero ideal of $\mathfrak{A}_i$, and $\mathfrak{A}_i[1]$ is an AF algebra. Then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}_1) \to \mathrm{FK}_{X_n}(\mathfrak{A}_2)$ such that $\alpha\{1\}$ is an order isomorphism.

Proposition (2.2.11) [2]:

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be graph C^*-algebras satisfying Condition (K). Suppose $\mathfrak{A}_i$ is a C^*-algebra over X_n such that $\mathfrak{A}_i[n]$ is an AF algebra, and for every ideal $\mathcal{J}$ of $\mathfrak{A}_i$ we have that $\mathcal{J} \subseteq \mathfrak{A}_i[n]$ or $\mathfrak{A}_i[n] \subseteq \mathcal{J}$, and $\mathfrak{A}_i[1, n-1]$ is a tight, $\mathcal{O}_\infty$-absorbing C^*-algebra over $[1, n-1]$. Then $\mathfrak{A}_1 \otimes \mathbb{K} \cong \mathfrak{A}_2 \otimes \mathbb{K}$ if and only if there exists an isomorphism $\alpha : \mathrm{FK}_{X_n}(\mathfrak{A}_1) \to \mathrm{FK}_{X_n}(\mathfrak{A}_2)$ such that $\alpha\{n\}$ is an order isomorphism.

Proof:

By Corollary (2.2.6), $0 \to \mathfrak{A}_i[n] \otimes \mathbb{K} \to \mathfrak{A}_i \otimes \mathbb{K} \to \mathfrak{A}_i[1, n-1] \otimes \mathbb{K} \to 0$ is a full extension. Then $\mathfrak{A}_i[n] \otimes \mathbb{K}$ satisfies the Corona factorization property. The result now follows from Theorem (2.2.8) and Theorem (2.1.16).

Definition (2.2.12) [2]:

For each $n \in \mathbb{N}$, we define a class $\mathfrak{C}_n$ of graph C^*-algebras as follows: $\mathfrak{A}$ is in $\mathfrak{C}_n$ if

(i) $\mathfrak{A}$ is a graph C^*-algebra;
(ii) $\mathfrak{A}$ is a tight C^*-algebra over X_n; and
(iii) There exists $U \in \mathbb{O}(X_n)$ such that either $\mathfrak{A}(U)$ is an AF algebra and $\mathfrak{A}(X_n \backslash U)$ is $\mathcal{O}_\infty$-absorbing or $\mathfrak{A}(U)$ is $\mathcal{O}_\infty$-absorbing and $\mathfrak{A}(X \backslash U)$ is an AF algebra.

Note that if $C^*(E)$ is an element in $\mathfrak{C}_n$, then E satisfies Condition (K).

Theorem (2.2.13) [2]:

Let E_1 and E_2 be graphs such that $C^*(E_1)$ and $C^*(E_2)$ are in $\mathfrak{C}_n$ for some $n \in \mathbb{N}$. Then the following are equivalent:

(i) $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$;

(ii)There exists an isomorphism $\alpha : \mathrm{FK}^+_{X_n}(C^*(E_1)) \to \mathrm{FK}^+_{X_n}(C^*(E_2))$

Proof:

Suppose there exists an isomorphism $\alpha : \mathrm{FK}^+_{X_n}(C^*(E_1)) \to \mathrm{FK}^+_{X_n}(C^*(E_2))$. Note that by Cuntz, if $\mathfrak{A}$ is an $\mathcal{O}_\infty$-absorbing C^*-algebra with a norm-full projection, then $K_0(\mathfrak{A}) = K_0(\mathfrak{A})_+$. Since $K_0(\mathfrak{B}) = K_0(\mathfrak{B})_+$ for any AF algebra, there is no order isomorphism from the K_0-group of an AF algebra to the K_0-group of an $\mathcal{O}_\infty$-absorbing C^*-algebra with a norm-full projection.

With the above observation, one of the following four cases must happen:

(i) $C^*(E_1)$ and $C^*(E_2)$ are AF algebras;

(ii)$C^*(E_1)$ and $C^*(E_2)$ are $\mathcal{O}_\infty$-absorbing;

(iii) There exists $2 \leq k \leq n$ such that $C^*(E_i)[k,n]$ is an AF algebra and $C^*(E_i)[1,k-1]$ is $\mathcal{O}_\infty$-absorbing for $i = 1,2$;

(iv) There exists $2 \leq k \leq n$ such that $C^*(E_i)[k\ ,n]$ is $\mathcal{O}_\infty$-absorbing and $C^*(E_i)[1,k-1]$ is an AF algebra for $i = 1,2$.

Case (i) follows from the classification of AF algebras. Case (ii) also follows. Case (iii) follows from Proposition (2.2.11) and Case (iv) follows from Proposition (2.2.9).

Examples (2.2.14) [2]:

Case (I):

Fix a prime p and consider the class of graph C^*-algebras given by adjacency matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ z & p+1 & 0 \\ y & x & p+1 \end{bmatrix}$$

for $y, z > 0$. Theorem (2.2.13) applies directly as the resulting graph C^*-algebra has a finite linear ideal lattice $0 \lhd \mathcal{J}_1 \lhd \mathcal{J}_2 \lhd \mathfrak{A}$ with subquotients $\mathcal{J}_1 = \mathbb{K}, \mathcal{J}_2/\mathcal{J}_1 = \mathcal{O}_{p+1} \otimes \mathbb{K}$, and $\mathfrak{A}/\mathcal{J}_2 = \mathcal{O}_{p+1}$. All K_1-groups in the filtered K-theory vanish, and the K_0-groups and the natural transformations

$$\begin{array}{ccccc}
K_0(\mathcal{J}_1) & \longrightarrow & K_0(\mathcal{J}_2) & \longrightarrow & K_0(\mathcal{J}_2/\mathcal{J}_1) \\
\| & & \downarrow & & \downarrow \\
K_0(\mathcal{J}_1) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{A}/\mathcal{J}_1) \\
 & & \downarrow & & \downarrow \\
 & & K_0(\mathfrak{A}/\mathcal{J}_2) & = & K_0(\mathfrak{A}/\mathcal{J}_2)
\end{array}$$

may be computed as

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \operatorname{cok}\begin{bmatrix} z \\ p \end{bmatrix} & \longrightarrow & \operatorname{cok}[p] \\
\| & & \downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \operatorname{cok}\begin{bmatrix} z & y \\ p & x \\ 0 & p \end{bmatrix} & \longrightarrow & \operatorname{cok}\begin{bmatrix} p & x \\ 0 & p \end{bmatrix} \\
 & & \downarrow & & \downarrow \\
 & & \operatorname{cok}[p] & = & \operatorname{cok}[p]
\end{array}$$

with all maps induced by the canonical maps from $\mathbb{Z}^r$ into $\mathbb{Z}^s$ for suitably chosen r and s.

Checking when two such filtered K-theories are isomorphic is not easy. Of course it would be necessary that

$$p|x \Leftrightarrow p|x', \ \ p|z \Leftrightarrow p|z'$$

but depending upon the invertibility of x and z in $\mathbb{Z}/p$ we get varying conditions on y. The work explains how to reduce this task to checking isomorphism only in the part of the invariant enclosed in dashed lines. With this, it is easy to conclude:

Example (2.2.15) [2]:

With graphs E and E' given by matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ z & p+1 & 0 \\ y & x & p+1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ z' & p+1 & 0 \\ y' & x' & p+1 \end{bmatrix}.$$

respectively, we have $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$ precisely when

(i) $p|x \Leftrightarrow p|x'$, and
(ii)$p|z \Leftrightarrow p|z'$, and
(iii) a. $p|y \Leftrightarrow p|y'$ when $p|x$ and $p|z$,
b. $p|[y - x\,z/p] \Leftrightarrow p|[y' - x'z'/p]$ when $p \nmid x$ and $p|z$.

Case (II):

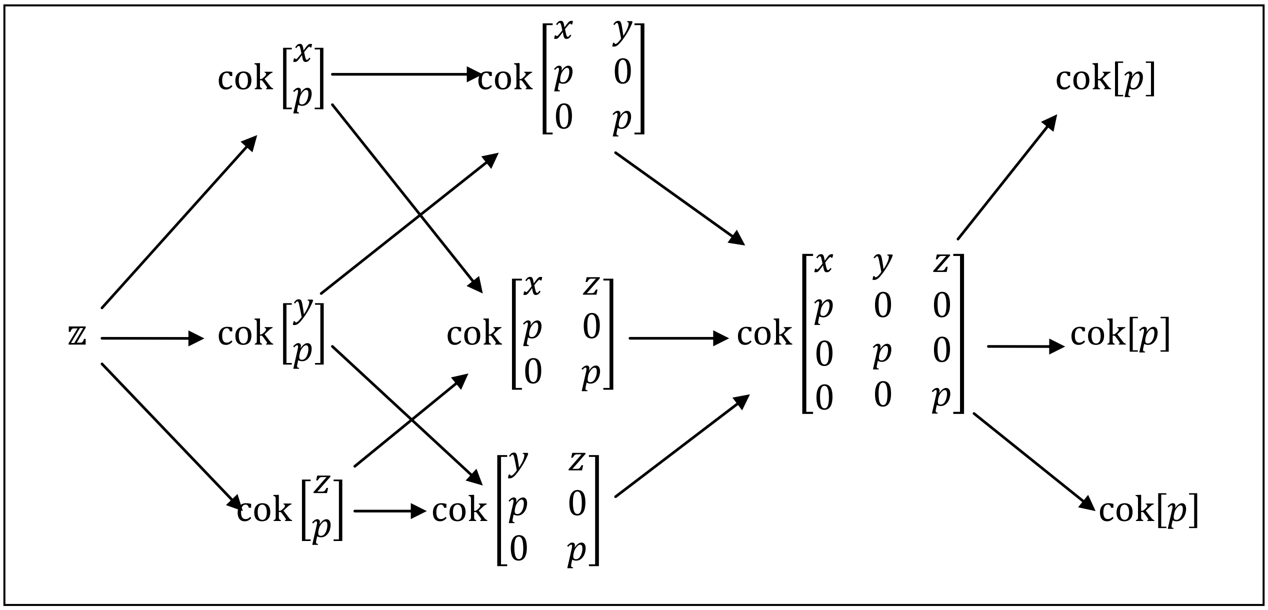
We now consider graphs given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & p+1 & 0 & 0 \\ y & 0 & p+1 & 0 \\ z & 0 & 0 & p+1 \end{bmatrix}$$

with $x, y, z > 0$. The resulting ideal lattice is not linear; in fact we have an extension

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{A} \longrightarrow \mathcal{O}_{p+1} \otimes \mathcal{O}_{p+1} \longrightarrow \mathcal{O}_{p+1} \longrightarrow 0 \qquad (4)$$

showing that the ideal lattice is precisely of the type demonstrated by Meyer and Nest to not generally allow a UCT for filtered K-theory. But since our C^*-algebras have real rank zero, we may appeal to see that isomorphisms of the filtered K-theory, which in this case has the form



lift to invertible elements of KK, so since the ideal $\mathcal{J}_1 \cong \mathbb{K}$ is a least ideal with $\mathfrak{A}/\mathcal{J}_1$ absorbing $\mathcal{O}_\infty$ we may apply Theorem (2.1.16). Further, as explained it suffices to check the existence of isomorphisms on the part of the invariant enclosed in dashed lines, and then it is straightforward to determine when the filtered K-theory for two such matrices are the same; indeed this amounts to

$$p|x \Leftrightarrow p|x', \;\; p|y \Leftrightarrow p|y', \;\; p|z \Leftrightarrow p|z'$$

Taking into account the homeomorphism of $\mathrm{Prim}(\mathfrak{A})$ we arrive at

Example (2.2.16) [2]:

With graphs E and E' given by matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & p+1 & 0 & 0 \\ y & 0 & p+1 & 0 \\ z & 0 & 0 & p+1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ x' & p+1 & 0 & 0 \\ y' & 0 & p+1 & 0 \\ z' & 0 & 0 & p+1 \end{bmatrix}.$$

respectively, we have $C^*(E) \otimes \mathbb{K} C^*(E') \otimes \mathbb{K}$ if and only if the number of entries in (x, y, z) which are multiples of p agrees with the number of entries in (x', y', z') which are multiples of p.

Chapter 3

C^*-algebras with Closed Unitary and Similarity Orbits of Normal Operators

Section (3.1): Normal Operators and Closed Unitary

Significant research has been performed in determining when two normal operators in a unital C^*-algebra are approximately unitarily equivalent. For example the Weyl-von Neumann-Berg Theorem determines when two normal operators in the bounded linear maps on a complex, separable, infinite dimensional Hilbert space are approximately unitarily equivalent and a famous work due to Brown, Douglas, and Fillmore can be used to determine when two normal operators in the Calkin algebra are approximately unitarily equivalent. More recently completely determines when two normal operators in a von Neumann algebra of an arbitrary single type are approximately unitarily equivalent.

Given a normal operator N in a unital C^*-algebra $\mathfrak{U}$, the Continuous Functional Calculus for Normal Operators provides a unital, injective $*$-homomorphism from the continuous functions on the spectrum of N into $\mathfrak{U}$ sending the identity function to N. It is easy to see that two normal operators are approximately unitarily equivalent in $\mathfrak{U}$ if and only if the corresponding unital, injective $*$-homomorphism are approximately unitarily equivalent. Thus it is of interest to determine when two unital, injective $*$-homomorphisms from an abelian C^*-algebra to a fixed unital C^*-algebra are approximately unitarily equivalent. In particular, when $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, several preliminary results were developed and a complete classification was given.

Theorem (3.1.1) [3]:

Let X be a compact metric space, let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra, and let $\varphi, \psi : C(X) \to \mathfrak{U}$ be two unital, injective $*$-homomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if $[[\varphi]] = [[\psi]]$ in $KL(C(X), \mathfrak{U})$.

As a specific case, if $X \subseteq C$ is compact it is a corollary of the Universal Coefficient Theorem for C^*-algebras, the definition of $KL(C(X), \mathfrak{U})$, and the fact that $K_*(C(X))$ is a free abelian group that

$$KL(C(X), \mathfrak{U}) = KK(C(X), \mathfrak{U}) = \mathrm{Hom}\left(K_*\big(C(X)\big), K_*(\mathfrak{U})\right)$$

where $\mathrm{Hom}(K_*(C(X)), K_*(\mathfrak{U}))$ is the set of all homomorphisms from $K_*(C(X))$ to $K_*(\mathfrak{U})$. Thus implies that for a unital, simple, purely infinite C^*-algebra $\mathfrak{U}$ and a compact subset X of $\mathbb{C}$, two unital, injective $*$-homomorphisms $\varphi, \psi : C(X) \to \mathfrak{U}$ are approximately unitarily equivalent if and only if $\varphi^* = \psi^*$ where φ^* and ψ^* are the group homomorphisms from $K_*(C(X))$ to $K_*(A)$ induced by φ and ψ respectively. Thus a complete classification of when two normal operator with the same spectrum in a unital, simple, purely infinite C^*-algebra is obtained.

The proof of Dadarlat's result greatly varies from the traditional proof of when two normal operators on a complex, infinite dimensional, separable Hilbert space are approximately unitarily equivalent. We shall use previously known techniques based on to provide a simple proof of the classification of when two normal operators are approximately unitarily equivalent in a unital, simple, purely infinite C^*-algebra with trivial K_1-group. Although this proof is less powerful than, the techniques used enables the study of additional operator theoretic problems on these C^*-algebras.

One particularly interesting problem is the study of the distance between unitary orbits of operators. Significant progress has been made in determining the distance between two unitary orbits of bounded operators on a complex, infinite dimensional Hilbert space. In terms of determining the distance between unitary orbits of normal operators inside other C^*-algebras, makes significant progress for the Calkin algebra (which is a unital, simple, purely infinite C^*-algebra) and makes significant progress for semifinite factors.

For the discussions, $\mathfrak{U}$ will denote a unital C^*-algebra, $\mathcal{U}(\mathfrak{U})$ will denote the unitary group of $\mathfrak{U}, \mathfrak{U}^{-1}$ will denote the group of invertible elements of $\mathfrak{U}$, and $\mathfrak{U}_0^{-1}$ will denote the connected component of the identity in $\mathfrak{U}^{-1}$. For a fixed unital C^*-algebra $\mathfrak{U}$ and an operator $A \in \mathfrak{U}$, let $\sigma(A)$ denote the spectrum of A in $\mathfrak{U}$, let

$$\mathcal{U}(A) := \{UAU^* \in \mathfrak{U} \mid U \in \mathcal{U}(\mathfrak{U})\}. \tag{1}$$

and let

$$S(A) := \{VAV^{-1} \in \mathfrak{U} \mid V \in \mathfrak{U}^{-1}\}. \quad (2)$$

The set $\mathcal{U}(A)$ is called the unitary orbit of A in $\mathfrak{U}$ and $S(A)$ is called the similarity orbit of A in $\mathfrak{U}$.

Notice if $B \in \mathfrak{U}$ then $B \in \mathcal{U}(A)$ if and only if $A \in \mathcal{U}(B)$ and $B \in S(A)$ if and only if $A \in S(B)$. We will denote $B \in \mathcal{U}(A)$ by $A \sim_u B$ and we will denote $B \in S(A)$ by $A \sim B$. Clearly $\sim_u$ and $\sim$ are equivalence relations.

We will use $\overline{\mathcal{U}(A)}$ and $\overline{S(A)}$ to denote the norm closures in $\mathfrak{U}$ of the unitary and similarity orbits of A respectively. Note if $B \in \overline{\mathcal{U}(A)}$ then $A \in \overline{\mathcal{U}(B)}$ and $B \in \overline{S(A)}$. If $B \in \overline{\mathcal{U}(A)}$ we will say that A and B are approximately unitarily equivalent in $\mathfrak{U}$ and will write $A \sim_{au} B$. Clearly $\sim_{au}$ is an equivalence relation. Furthermore if A is a normal operator and $A \sim_{au} B$ then B is a normal operator. If $B \in \overline{S(A)}$ then it is not necessary that $A \in \overline{S(B)}$ and B need not be normal if A is normal. However if $B \in \overline{S(A)}$ and $C \in \overline{S(B)}$ then $C \in \overline{S(A)}$.

It is an easy application of the semicontinuity of the spectrum to show that if $A, B \in A$ are such that $B \in \overline{S(A)}$ then $\sigma(A) \subseteq \sigma(B)$ and $\sigma(A)$ intersects every connected component of $\sigma(B)$. Thus $\sigma(A) = \sigma(B)$ whenever $A, B \in A$ are approximately unitarily equivalent.

Definition (3.1.2) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra and let $N \in \mathfrak{U}$ be a normal operator. By the Continuous Functional Calculus for Normal Operators, there exists a canonical unital, injective $*$-homomorphism $\varphi N : C(\sigma(N)) \to \mathfrak{U}$ such that $\varphi N(z) = N$. As φN is unital and injective, this induces a group homomorphism $\Gamma(N): K_1(C(\sigma\,(N))) \to K_1(\mathfrak{U})$. The group homomorphism $\Gamma(N)$ is called the index function of N. To simplify notation, we will write $\Gamma(N)(\lambda)$ to denote $[\lambda I_{\mathfrak{U}} - N]_1$ in $\mathfrak{U}$.

In the case that $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, $K_1(\mathfrak{U})$ is canonically isomorphic to $\mathfrak{U}^{-1}/\mathfrak{U}_0^{-1}$. Thus if $N \in \mathfrak{U}$ is a normal operator such that $\Gamma(N)$ is trivial then $\lambda I_{\mathfrak{U}} - N \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(N)$. Furthermore if $N \in A$ is a normal operator and $\lambda \notin \sigma(N)$ then $\Gamma(N)(\lambda)$ describes the connected component of $\lambda I_{\mathfrak{U}} - N$ in $\mathfrak{U}^{-1}$.

The reason for examining the index function in the context of approximately unitarily equivalent normal operators is seen by the following necessary condition.

Lemma (3.1.3) [3]:

Let $\mathfrak{U}$ be a unital and let $N_1, N_2 \in \mathfrak{U}$ be normal operators such that $N_1 \in \overline{S(N_2)}$. Then

(i) If $\lambda I_{\mathfrak{U}} - N \in \mathfrak{U}_0^{-1}$ for some $\lambda \notin \sigma(N_1)$ then $\lambda I_{\mathfrak{U}} - N_1 \in \mathfrak{U}_0^{-1}$, and

(ii) If $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra then $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin \sigma(N_1)$.

Proof:

Suppose $N_1 \in \overline{S(N_2)}$ and $\lambda \notin \sigma(N_1)$. Then $\sigma(N_2) \subseteq \sigma(N_1)$ and there exists a sequence of invertible elements $V_n \in \mathfrak{U}$ such that

$$\lim_{n\to\infty} \|N_1 - V_n N_2 V_n^{-1}\| = 0.$$

Thus it is clear that

$$\lim_{n\to\infty} \|(\lambda I_{\mathfrak{U}} - N_1) - V_n(\lambda I_{\mathfrak{U}} - N_2)V_n^{-1}\| = 0.$$

Therefore, if $\lambda I_{\mathfrak{U}} - N_2 \in \mathfrak{U}_0^{-1}$ then $V(\lambda I_{\mathfrak{U}} - N_2)V_n^{-1} \in \mathfrak{U}_0^{-1}$ for all $n \in \mathbb{N}$ and thus first result trivially follows.

In the case $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, the above implies that $\lambda I_{\mathfrak{U}} - N_1$ and $V_n(\lambda I_{\mathfrak{U}} - N_2)V_n^{-1}$ are in the same connected component of $\mathfrak{U}^{-1}$ for sufficiently large n. Therefore

$$\begin{aligned}[\lambda I_{\mathfrak{U}} - N_1]_1 &= [V_1(\lambda I_{\mathfrak{U}} - N_2)V_0^{-1}]_1 \\ &= [V_n]_1[\lambda I_{\mathfrak{U}} - N_2]_1[V_n^{-1}]_1 \\ &= [\lambda I_{\mathfrak{U}} - N_2]_1.\end{aligned}$$

Hence $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$.

The main tools for our alternate proof are the K-theory of unital, simple, purely infinite C^*-algebras along with the following result due to Lin.

Theorem (3.1.4) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N \in \mathfrak{U}$ be a normal operator. Then N can be approximated by normal operators with finite spectra if and only if $\Gamma(N)$ is trivial.

Using Lin's result and the following trivial technical detail, we can easily provide a simple proof for unital, simple, purely infinite C^*-algebras with trivial K_0-group and normal operators with trivial index function.

Lemma (3.1.5) [3]:

Let $\mathfrak{U}$ be a C^*-algebra, let $N \in \mathfrak{U}$ be a normal operator, let U be an open subset of $\mathbb{C}$ such that $U \cap \sigma(N) = \phi$, and let $(N_n)_{n\geq 1}$ be a sequence of normal operators from $\mathfrak{U}$ such that $N = \lim_{n\to\infty} N_n$. Then there exists a $k \in \mathbb{N}$ such that $\sigma(N_n) \cap U = \phi$ for all $n \geq k$.

Proposition (3.1.6) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra such that $\mathrm{K}_0(\mathfrak{U})$ is trivial. Let $N_1, N_2 \in \mathfrak{U}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.

Proof:

By previous discussions it is clear that $\sigma(N_1) = \sigma(N_2)$ if $N_1 \sim_{au} N_2$. Suppose $\sigma(N_1) = \sigma(N_2)$. Since $K_0(\mathfrak{U}) = \{0\}$, all non-trivial projections are Murray-von Neumann equivalent. Thus any two normal operators with the same finite spectrum are unitarily equivalent.

By the assumption that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial, N_1 and N_2 can be approximated by normal operators with finite spectrum. By small perturbations using Lemma (3.1.5) and the semicontinuity of the spectrum, we can assume that N_1 and N_2 can be approximated arbitrarily well by normal operators with the same finite spectrum. Thus the result follows.

Note the condition '$\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial' holds when $\mathfrak{U}_0^{-1} = \mathfrak{U}^{-1}$ or equivalently when $K_1(\mathfrak{U})$ is trivial.

If $\mathcal{O}_2$ is the Cuntz algebra generated by two isometries, $K_0(\mathcal{O}_2)$ and $K_1(\mathcal{O}_2)$ are trivial. Thus Proposition (3.1.6) completely classifies when two normal operators in $\mathcal{O}_2$ are approximately unitarily equivalent.

Corollary (3.1.7) [3]:

Let $N, M \in \mathcal{O}_2$ be normal operators. Then $N \sim_{au} M$ if and only if $\sigma(N) = \sigma(M)$.

Note that the proof of Proposition (3.1.6) is easily modified to a more general setting. To see this, we recall the following definitions.

Definition (3.1.8) [3]:

Let $\mathcal{U}$ be a unital C^* -algebra. We say that $\mathfrak{U}$ has the finite normal property (property (FN)) if every normal operator in $\mathfrak{U}$ is the limit of normal operators from $\mathfrak{U}$ with finite spectrum. We say that $\mathfrak{U}$ has the weak finite normal property (property weak (FN)) if every normal operator $N \in \mathfrak{U}$ such that $\lambda I_{\mathfrak{U}} - N \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(N)$ is the limit of normal operators from $\mathfrak{U}$ with finite spectrum.

Corollary (3.1.9) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra such that $\mathfrak{U}$ has property weak (FN) and any two non-zero projections in $\mathfrak{U}$ are Murray–von Neumann equivalent. If $N_1, N_2 \in \mathfrak{U}$ are two normal operators such that $\lambda I_{\mathfrak{U}} - N_q \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1,2\}$ then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.

Corollary (3.1.10) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra such that $\mathfrak{U}$ has property (FN) and any two non-zero projections in $\mathfrak{U}$ are Murray-von Neumann equivalent. If $N_1, N_2 \in \mathfrak{U}$ are two normal operators then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.

Corollary (3.1.11) [3]:

Let M be a type (III) factor with separable predual and let $N_1, N_2 \in M$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.

Lemma (3.1.12) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators. Suppose that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial, $\sigma(N_1) = \sigma(N_2)$, and $\sigma(N_1)$ is connected. Then $N_1 \sim_{au} N_2$.

Proof:

We shall begin with the case that $\sigma(N_1) = \sigma(N_2) = [0,1]$ and then modify the proof for the general case.

Suppose $\sigma(N_1) = [0,1] = \sigma\,(N_2)$. Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{\mathrm{n}} < \epsilon$. By (or the fact that unital, simple, purely infinite C^*-algebras have real rank zero), by Lemma (3.1.5), by the semicontinuity of the spectrum, and by perturbing eigenvalues, there exists two collections of non-zero, pairwise orthogonal projections

$$\left\{P_j^{(1)}\right\}_{j=0}^{n} \quad \text{and} \quad \left\{P_j^{(2)}\right\}_{j=0}^{n}$$

in $\mathfrak{U}$ such that

$$\sum_{j=0}^{n} P_j^{(q)} = I_{\mathfrak{U}} \quad \text{and} \quad \left\| N_q - \sum_{j=0}^{n} \frac{j}{n} P_j^{(q)} \right\| < 2\epsilon$$

for all $q \in \{1\ 2\}$. The idea of the proof is to apply a 'back and forth' argument to produce a unitary that intertwines the approximations of N_1 and N_2.

Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, $P_0^{(1)}$ is Murray-von Neumann equivalent to a proper subprojection of $P_0^{(2)}$. Thus we can write $P_0^{(2)} = Q_0^{(2)} + R_0^{(2)}$ where $Q_0^{(2)}$ and $R_0^{(2)}$ are non-zero orthogonal projections in $\mathfrak{U}$ such that $Q_0^{(2)}$ and $P_0^{(1)}$ are Murray-von Neumann equivalent. Furthermore $R_0^{(2)}$ is Murray-von Neumann equivalent to a proper subprojection of $P_1^{(1)}$. Thus we can write $P_1^{(1)} = Q_1^{(1)} + R_1^{(1)}$ where $Q_1^{(1)}$ and $R_1^{(1)}$ are non-zero orthogonal projections in $\mathfrak{U}$ such that $Q_1^{(1)}$ and $R_0^{(2)}$ are Murray-von Neumann equivalent.

For notional purposes, let $Q_0^{(1)} := 0, R_0^{(1)} := P_0^{(1)}, Q_n^{(2)} := P_n^{(2)}$, and $R_n^{(2)} := 0$. By repeating this procedure (using $R_1^{(1)}$ in place of $P_0^{(1)}$), we obtain sets of non-zero, pairwise orthogonal projections

$$\left\{Q_j^{(1)}, R_j^{(1)}\right\}_{j=1}^{n} \quad \text{and} \quad \left\{Q_j^{(2)}, R_j^{(2)}\right\}_{j=0}^{n-1}$$

such that $P_0^{(q)} = Q_j^{(q)} + R_j^{(q)}$ for all $j \in \{0, \dots, n\}$ and $q \in \{1,2\}, R_j^{(2)}$ is Murray-von Neumann equivalent to $Q_{j+1}^{(1)}$ for all $j \in \{0, \dots, n-1\}$, and $R_j^{(1)}$ is Murray-von Neumann equivalent to $Q_j^{(2)}$ for all $j \in \{0, \dots, n-1\}$. Since

$$I_{\mathfrak{U}} = \sum_{j=0}^{n} Q_j^{(1)} + R_j^{(1)} = \sum_{j=0}^{n} Q_j^{(2)} + R_j^{(2)}. \tag{3}$$

we note that

$$\left[R_n^{(1)}\right]_0 = [I_{\mathfrak{U}}]_0 - \sum_{j=0}^{n} \left[Q_j^{(1)}\right]_0 - \sum_{j=0}^{n-1} \left[R_j^{(1)}\right]_0$$

$$= [I_{\mathfrak{U}}]_0 - \sum_{j=0}^{n} \left[R_{j-1}^{(2)}\right]_0 - \sum_{j=0}^{n-1} \left[R_j^{(2)}\right]_0$$

$$= \left[Q_n^{(2)}\right]_0.$$

Hence $R_n^{(1)}$ and $Q_n^{(2)}$ are Murray-von Neumann equivalent.

Let $\{V_j\}_{j=0}^{n} \cup \{W_j\}_{j=0}^{n-1}$ be partial isometries in $\mathfrak{U}$ such that $V_j^* V_j = R_j^{(1)}$ and $V_j V_j^* = Q_j^{(2)}$ for all $j \in \{0, \dots, n\}$, and $W_j^* W_j = Q_{j+1}^{(1)}$ and $W_j W_j^* = R_j^{(2)}$ for all $j \in \{0, \dots, n-1\}$. Hence (3) implies that

$$U = \sum_{j=0}^{n} V_j + \sum_{j=0}^{n-1} W_j$$

is a unitary operator in $\mathfrak{U}$. Moreover

$$U^*\left(\sum_{j=0}^{n}\frac{j}{n}P_j^{(2)}\right)U = U^*\left(\sum_{j=0}^{n}\frac{j}{n}Q_j^{(2)} + \sum_{j=0}^{n}\frac{j}{n}R_j^{(2)}\right)U$$

$$= \sum_{j=0}^{n}\frac{j}{n}R_j^{(1)} + \sum_{j=0}^{n-1}\frac{j}{n}Q_{j+1}^{(1)}.$$

Hence, since

$$\sum_{j=0}^{n}\frac{j}{n}P_j^{(1)} = \sum_{j=0}^{n}\frac{j}{n}Q_j^{(1)} + \sum_{j=0}^{n}\frac{j}{n}R_j^{(1)},$$

we obtain that

$$\|N_1 - U^*N_2U\| \leq 5\epsilon. \tag{4}$$

Since $\epsilon > 0$ was arbitrary, $N_1 \sim_{au} N_2$.

To complete the general case, we will use a technique similar to that used before. To begin, let N_1 and N_2 be as in the statement of the lemma. Fix $\epsilon > 0$ and for each $(n, m) \in \mathbb{Z}^2$ let

$$B_{n,m} := \left(\epsilon n - \frac{\epsilon}{2}, \epsilon n + \frac{\epsilon}{2}\right] + i\left(\epsilon m - \frac{\epsilon}{2}, \epsilon m + \frac{\epsilon}{2}\right] \subseteq \mathbb{C}. \tag{5}$$

Thus the sets $B_{n,m}$ partition the complex plane into a grid with side-lengths ϵ.

For each $(n, m) \in \mathbb{Z}^2$ we label the box $B_{n,m}$ relevant if $\sigma(N_1) \cap B_{n,m} = \phi$ and we will say two boxes are adjacent if their union is connected. Since $\sigma(N_1)$ is connected, the union of the relevant boxes is connected.

We can approximate N_1 and N_2 within by normal operators M_1 and M_2 in $\mathfrak{A}$ with finite spectrum. By Lemma (3.1.5), by the semicontinuity of the spectrum, and by perturbing eigenvalues, we can assume that $\sigma(M_q)$ is precisely the centres of the relevant boxes and $\|N_q - M_q\| \leq 2\epsilon$ for all $q \in \{1,2\}$.

We claim that there exists a unitary $U \in \mathfrak{U}$ such that $\|M_1 - U^* M_2 U\| \le \sqrt{2}\epsilon$. Consider a tree $\mathcal{T}$ in $\mathbb{C}$ whose vertices are the centres of the relevant boxes and whose edges are straight lines that connect vertices in adjacent relevant boxes. Consider a leaf of $\mathcal{T}$. We can identify this leaf with the spectral projections of M_1 and M_2 corresponding to the eigenvalue defined by the vertex. We can then apply the 'back and forth' technique illustrated above to embed the spectral projection of M_1 under the corresponding spectral projection of M_2 and the remaining spectral projection of M_2 under a spectral projection of M_1 corresponding to the adjacent vertex of the leaf (which is within $\sqrt{2}$). By considering $\mathcal{T}$ with the above leaf removed, we then have a smaller tree. By continually repeating this 'back and forth'-crossing technique, we are eventually left with the trivial tree. As before, K-theory implies the remaining projections are Murray-von Neumann equivalent. It is then possible to use the partial isometries from the 'back and forth' construction to create a unitary with the desired properties.

Our next goal is to remove the condition '$\sigma(N_1)$ is connected' from Lemma (3.1.12). Unfortunately, two normal operators having equal spectrum is not enough to guarantee that the normal operators are approximately unitarily equivalent (even in the case that $K_1(\mathfrak{U})$ is trivial). The technicality is the same as why two projections in $\mathcal{B}(\mathcal{H})$ are not always approximately unitarily equivalent. To see this, we note the following lemmas.

Lemma (3.1.13) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra and let $P, Q \in \mathfrak{U}$ be projections. If there exists an element $V \in \mathfrak{U}$ such that

$$\|Q - VPV^{-1}\| < \frac{1}{2}$$

then P and Q are Murray-von Neumann equivalent.

Proof:

Let $P_0 := VPV^{-1} \in \mathfrak{U}$ and let $Z := P_0 Q + (I_{\mathfrak{U}} - P)(I_{\mathfrak{U}} - Q) \in \mathfrak{U}$. Hence P_0 is an idempotent and it is clear that

$$\|Z - I_{\mathfrak{U}}\| = \left\|\left(P_0 Q + (I_{\mathfrak{U}} - P_0)(I_{\mathfrak{U}} - Q)\right) - \left(Q + (I_{\mathfrak{U}} - Q)\right)\right\|$$

$$\leq \|(P_0 - I_{\mathfrak{U}})Q\| + \|((I_{\mathfrak{U}} - P_0) - I_{\mathfrak{U}})(I_{\mathfrak{U}} - Q)\|$$

$$= \|(P_0 - Q)Q\| + \|((I_{\mathfrak{U}} - P_0) - (I_{\mathfrak{U}} - Q))(I_{\mathfrak{U}} - Q)\|$$

$$\leq \|P_0 - Q\| + \|Q - P_0\| < 1.$$

Hence $Z \in \mathfrak{U}^{-1}$. Therefore, if U is the partial isometry in the polar decomposition of $Z, Z = U|Z|$ and U is a unitary element of $\mathfrak{U}$.

We claim that $UQU^* = P_0$. To see this, we notice that $U = Z|Z|^{-1}, ZQ = P_0Q = P_0Z$, and

$$Z^*Z = QP_0Q + (I_{\mathfrak{U}} - Q)(I_{\mathfrak{U}} - P_0)(I_{\mathfrak{U}} - Q).$$

Thus $QZ^*Z = QP_0Q = Z^*ZQ$ so Q commutes with Z^*Z. Hence Q commutes with $C^*(Z^*Z)$ and thus Q commutes with $|Z|^{-1}$. Thus

$$UQU^* = Z[Z]^{-1}Q[Z]^{-1}Z^*$$

$$= ZQ[Z]^{-2}Z^*$$

$$= P_0Z[Z]^{-2}Z^* = P_0$$

as claimed.

Therefore $Q = (U^*V)P(U^*V)^{-1}$ where $U^*V \in \mathfrak{U}^{-1}$. It is standard to verify that if W is the partial isometry in the polar decomposition of U^*V then W is a unitary such that $Q = WP W^*$. Therefore $P \sim_u Q$ and thus P and Q are Murray-von Neumann equivalent.

Lemma (3.1.14) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let P and Q be projections in $\mathfrak{U}$. Then $P \sim_u Q$ if and only if $P \sim_u Q$ if and only if $Q \in \overline{S(P)}$ only if P and Q are Murray-von Neumann equivalent. If $P \neq I_{\mathfrak{U}}$ and $Q = I_{\mathfrak{U}}$, then $P \sim_u Q$ whenever P and Q are Murray-von Neumann equivalent.

Proof:

The above shows that if $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra with $K_0(\mathfrak{U})$ being non-trivial, there exists two projections $P, Q \in \mathfrak{U}$ with $\sigma(P) =$

$\sigma(Q) = \{0\}$ that are not approximately unitarily equivalent. Thus knowledge of the spectrum is not enough to complete our classification.

To avoid the above technicality, we will describe an additional condition for two normal operators to be approximately unitarily equivalent in a unital C^*-algebra. The construction of this conditions makes use of the analytical functional calculus.

Lemma (3.1.15) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra, let $A, B \in \mathfrak{U}$, and let $f : \mathbb{C} \to \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(A) \cup \sigma(B)$. If $A \in \overline{S(B)}$ then $f(A) \in \overline{S(f\ (B))}$. Similarly if $A \sim_{au} B$ then $f(A) \sim_{au} f\ (B)$.

Proof:

Let $(V_n)_{n\geq 1}$ be a sequence of invertible elements in $\mathfrak{U}$ such that

$$\lim_{n\to\infty} \|A - V_n B V_n^{-1}\| = 0.$$

Let γ be any compact, rectifiable curve inside U such that $\big(\sigma(A) \cup \sigma(B)\big) \cap \gamma = \phi$, $\mathrm{lnd}_\gamma(z) \in \{0,1\}$ for all $z \in \mathbb{C}\backslash\gamma$, $\mathrm{lnd}_\gamma(z) = 1$ for all $z \in \sigma(A) \cup \sigma(B)$, and $\{z \in \mathbb{C} |\ \mathrm{lnd}_\gamma(z) \neq 0\} \subseteq U$. Then

$$f(A) - V_n f(B) V_n^{-1} = \frac{1}{2\pi i} \int_\gamma f(z)((zI_{\mathfrak{U}} - A)^{-1} - V_n(zI_{\mathfrak{U}} - B)^{-1} V_n^{-1})\, dz$$

$$= \frac{1}{2\pi} \int_\gamma f(z)((zI_{\mathfrak{U}} - A)^{-1} - (zI_{\mathfrak{U}} - V_n B V_n^{-1})^{-1})\, dz$$

$$= \frac{1}{2\pi} \int_\gamma f(z)(zI_{\mathfrak{U}} - A)^{-1}(A - V_n B V_n^{-1})(zI_{\mathfrak{U}} - V_n B V_n^{-1})^{-1}\, dz$$

Hence $\|f(A) - V_n f(B) V_n^{-1}\|$ is at most

$$\frac{length\ (\gamma)\|A - V_n B V_n^{-1}\|}{2\pi} \sup_{z\in\gamma} |f(z)| \|(zI_{\mathfrak{U}} - A^{-1})\| \|(zI_{\mathfrak{U}} - V_n B V_n^{-1})^{-1}\|.$$

Provided $\|A - V_n B V_n^{-1}\|\|(zI_{\mathfrak{U}} - A)^{-1}\| < 1$ for all $\in \gamma$, the second resolvent equation can be used to show that

$$\|(zI_{\mathfrak{U}} - V_n B V_n^{-1})^{-1}\| \leq \frac{\|(zI_{\mathfrak{U}} - A)^{-1}\|}{1 - \|A - V_n B V_n^{-1}\|\|(zI_{\mathfrak{U}} - A)^{-1}\|}$$

for all $z \in \gamma$. Since $\lim_{n\to\infty}\|A - V_n B V_n^{-1}\| = 0$, γ is compact, and the resolvent function of an operator is continuous on the resolvent, $\|f(A) - V_n f(B) V_n^{-1}\|$ is at most

$$\frac{length\,(\gamma)\|A - V_n B V_n^{-1}\|}{2\pi} \sup_{z\in\gamma}|f(z)| \frac{\|(zI_{\mathfrak{U}} - A)^{-1}\|^2}{1 - \|A - V_n B V_n^{-1}\|\|(zI_{\mathfrak{U}} - A)^{-1}\|}$$

for sufficiently large n. Since the resolvent function is a continuous function on the resolvent of an operator and γ is compact, the above supremum is finite and tends to

$$\sup_{z\in\gamma}|f(z)|\|(zI_{\mathfrak{U}} - A)^{-1}\|^2$$

as $n \to \infty$. Thus, as

$$\lim_{n\to\infty}\|A - V_n B V_n^{-1}\| = 0$$

and $length(\gamma)$ is finite, $f(A) \in \overline{S(f(B))}$.

The proof that $A \sim_{au} B$ implies $f(A) \sim_{au} f(B)$ follows directly by replacing the invertible elements V_n with unitary operators.

If $\mathfrak{U}$ in Lemma (3.1.15) were a unital, simple, purely infinite C^*-algebra, if A and B were normal operators, and if f took values in $\{0,1\}$ with $f(A)$ and $f(B)$ being non-trivial, then Lemma (3.1.14) would imply that the projections $f(A)$ and $f(B)$ are Murray-von Neumann equivalent in $\mathfrak{U}$. Thus, to simplify notation, we make the following definition.

Definition (3.1.16) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra and let $N_1, N_2 \in A$ be normal operators. We say that N_1 and N_2 have equivalent common spectral projections if for every function $f : \mathbb{C} \to \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N_1) \cup \sigma(N_2)$ with

$f\ (U) \subseteq \{0,1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray-von Neumann equivalent.

If $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra and $\sigma(N_1) = \sigma(N_2)$, it is elementary to show that N_1 and N_2 have equivalent spectral projections if and only if they induce the same group homomorphisms from $K_0(\sigma(N_1))$ to $K_0(\mathfrak{U})$ via the Continuous Functional Calculus of Normal Operators.

Finally, we have for planar compact sets in the case that $K_1(\mathfrak{U})$ is trivial.

Theorem (3.1.17) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators. Suppose

(i) $\sigma(N_1) = \sigma(N_2)$,
(ii) $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial, and
(iii) N_1 and N_2 have equivalent common spectral projections.

Then $N_1 \sim_{au} N_2$.

Proof:

Fix $\epsilon > 0$ and consider the ϵ-grid used in Lemma (3.1.12). We label the box $B_{n,m}$ relevant if $B_{n,m} \cap \sigma(N_1) = \phi$. Let K be the union of the relevant boxes. Since $\sigma(N_1)$ is compact, K has finitely many connected components. Let $L_1, \dots, L_k$ be the connected components of K.

By construction $\operatorname{dist}(L_i, L_j) \geq \epsilon$ for all $\neq j$. Therefore, if f_i is the characteristic function of L_i, the third assumptions of the theorem implies $f_i(N_1)$ and $f_i(N_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \dots, k\}$.

Note the second assumption of the theorem implies that there exists normal operators M_1 and M_2 in $\mathfrak{U}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q \in \{1,2\}$. By an application of Lemma (3.1.5), by the semicontinuity of the spectrum, and by small perturbations, we can assume that M_q has spectrum contained in K and $\sigma(M_q) \cap B_{n,m} \neq \phi$ for all relevant boxes $B_{n,m}$ and $q \in \{1,2\}$. Furthermore, since each f_i extends to a continuous function on an open neighbourhood of K, we can assume that $\left\|f_i\left(N_q\right) - f_i\left(M_q\right)\right\| < \frac{1}{2}$ for all $i \in \{1, \ldots, k\}$ and $q \in \{1,2\}$ by properties of the continuous functional calculus. Therefore, for each $i \in \{1, \ldots, k\}$ and $q \in \{1,2\}$, $f_i\left(N_q\right)$ and $f_i\left(M_q\right)$ can be assumed to be Murray-von Neumann equivalent by Lemma (3.1.13). Since $f_i(N_1)$ and $f_i(N_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$, $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$. By perturbing the spectrum of M_1 and M_2 inside each L_i, we can assume that $\sigma(M_q)$ is precisely the centres of the relevant boxes for all $q \in \{1,2\}$, $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$, and $\left\|N_q - M_q\right\| < 2\epsilon$ for all $q \in \{1,2\}$.

Next we apply the 'back and forth' argument of Lemma (3.1.12) to the spectrum of M_1 and M_2 in each L_i separately. This process can be applied to each L_i separately as in Lemma (3.1.12) due to the fact that $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent so the final step of the construction (that is, $R_n^{(1)}$ and $Q_n^{(2)}$ are Murray-von Neumann equivalent) can be completed. Thus, for each $i \in \{1, \ldots, k\}$, the 'back and forth' process produces a partial isometry $V_i \in \mathfrak{U}$ such that $V_i^* V_i = f_i(M_1), V_i V_i^* = f_i(M_2)$, and $\|M_1 f_i(M_1) - V_i^* M_2 f_i(M_2) V_i\| \le \sqrt{2}\epsilon$. Therefore, if $U := \sum_{i=1}^{k} V_i$ then $U \in A$ is a unitary as

$$\sum_{i=1}^{k} f_i(M_1) = I_{\mathfrak{U}} = \sum_{i=1}^{k} f_i(M_2)$$

are sums of orthogonal projections. Moreover, a trivial computation shows

$$\|M_1 - U^* M_2 U\| \le \sqrt{2}\epsilon$$

so

$$\|N_1 - U^* N_2 U\| \le \left(4 + \sqrt{2}\right)\epsilon$$

completing the proof.

Corollary (3.1.18) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra such that $K_1(A)$ is trivial and let $N_1, N_2 \in \mathfrak{U}$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if

(i) $\sigma(N_1) = \sigma(N_2)$ and
(ii) N_1 and N_2 have equivalent common spectral projections.

Proof:

One direction is follows from Theorem (3.1.17) and the fact that $K_1(\mathfrak{U})$ is trivial implies $\mathfrak{U}^{-1} = \mathfrak{U}_0^{-1}$. The other direction follows from Lemmas (3.1.15) and (3.1.14).

Section (3.2): Distamce between Unitary and Closed Similarity Orbits of Normal Opertors

In this section we will make use of the techniques given to provide some bounds for the distance between the unitary orbits of two normal operator in unital, simple, purely infinite C^*-algebras. In particular, Corollary (3.2.7) can be used to deduce Theorem (3.1.17). These results along with others will provide information about the distance between unitary orbits of normal operators with non-trivial index function.

We begin with the following definition that is common in the discussion of the distance between unitary orbits.

Definition (3.2.1) [3]:

Let X and Y be subsets of $\mathbb{C}$. The Hausdorff distance between X and Y, denoted $d_H(X,Y)$, is

$$d_H(X,Y) := \max\left\{\sup_{x\in X} \operatorname{dist}(x,Y), \sup_{y\in Y} \operatorname{dist}(y,X)\right\}.$$

Davidson developed the following notation for the Calkin algebra that will be of particular use to us.

Definition (3.2.2) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra. For normal operators $N_1, N_2 \in \mathfrak{U}$ let $\rho(N_1, N_2)$ denote the maximum of $d_H(\sigma(N_1), \sigma(N_2))$ and

$$\sup\left\{\operatorname{dist}\big(\lambda, \sigma(N_1)\big) + \operatorname{dist}\big(\lambda, \sigma(N_2)\big) \mid \lambda \notin \sigma(N_1) \cup \sigma(N_2), \Gamma(N_1)(\lambda) \neq \Gamma(N_2)(\lambda)\right\}.$$

We begin by noting the following.

Proposition (3.2.3) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators. Then

$$\operatorname{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) \geq d_H\big(\sigma(N_1), \sigma(N_2)\big).$$

If $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra then

$$\operatorname{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq \rho(N_1, N_2).$$

For our discussions of the distance between unitary orbits of normal operators in unital, simple, purely infinite C^*-algebras, we shall begin with the case our normal operators have trivial index function so that $\rho(N_1, N_2) = d_H(\sigma(N_1), \sigma(N_2))$ and we may apply the techniques given. We first turn our attention to the Cuntz algebra $\mathcal{O}_2$. As $K_0(\mathcal{O}_2)$ and $K_1(\mathcal{O}_2)$ are trivial, we are led to the following generalization whose proof is identical to the one given below.

Proposition (3.2.4) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra such that $\mathfrak{U}$ has property weak (FN), any two non-zero projections in $\mathfrak{U}$ are Murray-von Neumann equivalent, and every non-zero projection in $\mathfrak{U}$ is properly infinite. Let $N_1, N_2 \in \mathfrak{U}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Then

$$\operatorname{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof:

One inequality follows from Proposition (3.2.3). Let $\epsilon > 0$. Since $\mathfrak{U}$ has weak (FN), the conditions on N_1 and N_2 imply that there exists two normal operators $M_1, M_2 \in \mathfrak{U}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q \in \{1,2\}$. By Lemma (3.2.4), by the semicontinuity of the spectrum, and by applying small perturbations, we may assume that $\sigma(M_q) \subseteq \sigma(N_q)$ and $\sigma(M_q)$ is an -net for $\sigma(N_q)$ for all $q \in \{1,2\}$.

Let X be the set of all ordered pairs $(\lambda, \mu) \in \sigma(M_1) \times \sigma(M_2)$ such that either

$$|\lambda - \mu| = \operatorname{dist}(\lambda, \sigma(M_2)) \quad \text{or} \quad |\lambda - \mu| = \operatorname{dist}(\lambda, \sigma(M_1)).$$

For each $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$, let $n_\lambda := |\{(\lambda, \xi) \in X\}|$ and $m_\mu := |\{(\zeta, \mu) \in X\}|$. Clearly $n_\lambda \geq 1$ for all $\lambda \in \sigma(M_1), m_\mu \geq 1$ for all $\mu \in \sigma(M_2)$, and $\sum_{\lambda \in \sigma(M_1)} n_\lambda = \sum_{\mu \in \sigma(M_2)} m_\mu$.

Since every projection in $\mathfrak{U}$ is properly infinite, we can write

$$M_1 = \sum_{\lambda\in\sigma(M_1)} \sum_{k=1}^{n_\lambda} \lambda P_{\lambda,k} \quad \text{and} \quad \sum_{\mu\in\sigma(2)} \sum_{k=1}^{m_\mu} \mu Q_{\mu,k}$$

where $\left\{\{P_{\lambda,k}\}_{k=1}^{n_\lambda}\right\}_{\lambda\in\sigma(M_1)}$ and $\left\{\{Q_{\mu,k}\}_{k=1}^{m_\mu}\right\}_{\mu\in\sigma(M_2)}$ are sets of non-zero orthogonal projections in $\mathfrak{U}$ each of which sums to the identity. Since all projections in $\mathfrak{U}$ are Murray-von Neumann equivalent, using X we can pair off the projections in these finite sums to obtain a unitary $U \in \mathfrak{U}$ (that is a sum of partial isometries) such that

$$\|M_1 - UM_2U^*\| \le \sup\{|\lambda - \mu|| \ (\lambda,\mu) \in X\} = d_H\big(\sigma(M_1), \sigma(M_2)\big).$$

Hence

$$\mathrm{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) \le 2\epsilon + d_H\big(\sigma(M_1), \sigma(M_2)\big).$$

Since $\sigma(M_1)$ is an ϵ-net for $\sigma(N_1)$, and $\sigma(M_2)$ is an ϵ-net for $\sigma(N_2)$,

$$d_H\big(\sigma(M_1), \sigma(M_2)\big) \le d_H\big(\sigma(N_1), \sigma(N_2)\big) + \epsilon$$

completing the proof.

Unfortunately Proposition (3.2.4) does not completely generalize to unital, simple, purely infinite C^*-algebras with non-trivial K_0-group. The following uses the ideas given to obtain a preliminary result.

Lemma (3.2.5) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2, \in \mathfrak{U}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. If $\sigma(N_1)$ is connected then

$$\mathrm{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) = d_H\big(\sigma(N_1), \sigma(N_2)\big).$$

Proof:

One inequality follows from Proposition (3.2.3). The proof of the other inequality is a more complicated 'back and forth' argument. Fix $\epsilon > 0$ and let $B_{n,m}$ be as in Lemma (3.1.12). For each $q \in \{1,2\}$, we will say that $B_{n.m}$ is N_q-relevant if $B_{n,m} \cap \sigma(N_q) \neq \phi$. There exists normal operators $M_1, M_2 \in \mathfrak{U}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q = \{1,2\}$. By Lemma (3.1.5), by the semicontinuity of the spectrum, and by a small perturbation, we can assume that $\sigma(M_q)$ is precisely the centres of the N_q-relevant boxes and $\|N_q - M_Q\| < \epsilon$. For each $q \in \{1,2\}$ and $\lambda \in \sigma(M_q)$ let $P_\lambda^{(q)}$ be the non-zero spectral projection of M_q corresponding to λ.

To begin our 'back and forth' argument, we will construct a bipartite graph, $\mathcal{G}$, using $\sigma(M_1)$ and $\sigma(M_2)$ as vertices (where we have two vertices for λ if $\lambda \in \sigma(M_1) \cap \sigma(M_2)$). The process for constructing the edges in G is as follows: for each $i, j \in \{1,2\}$ with $i \neq j$ and each $\lambda \in \sigma(M_i)$, for every $\mu \in \sigma(M_j)$ such that

$$|\lambda - \mu| \leq 2\sqrt{2}\epsilon + d_H\big(\sigma(N_1), \sigma(N_2)\big)$$

add edges to $\mathcal{G}$ from μ to λ and the centre of any N_i-relevant box adjacent (including diagonally adjacent) to the N_i-relevant box λ describes.

Clearly $\mathcal{G}$ is a bipartite graph and, by construction, if $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$ are connected by an edge of $\mathcal{G}$ then $|\lambda - \mu| \leq 2\sqrt{2}\epsilon + d_H\big(\sigma(N_1), \sigma(N_2)\big)$. We claim that $\mathcal{G}$ is connected. To see this, we note that since $\mathcal{G}$ is bipartite and every vertex is the endpoint of at least one edge, it suffices to show that for each pair $\lambda, \mu \in \sigma(M_1)$ there exists a path from λ to μ. Fix a pair $\lambda, \mu \in \sigma(M_1)$. Since $\sigma(N_1)$ is connected, the union of the N_1-relevant boxes is connected so there exists a finite sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$ where $\lambda_{\ell-1}$ and λ are centres of adjacent N_1-relevant boxes for all $\ell \in \{1, \dots, k\}$. However $\lambda_{\ell-1}$ and λ are connected in $\mathcal{G}$ (via an element of $\sigma(M_2)$) by construction. Hence the claim follows.

Now that $\mathcal{G}$ is constructed, we will progressively remove vertices and edges from $\mathcal{G}$ and modify the non-zero projections $\left\{P_\lambda^{(q)}{}_{\lambda \in \sigma(M_j)}\right\}_{q \in \{1,2\}}$ in a specific manner to construct partial isometries in $\mathfrak{U}$ that will enable us to create a unitary $U \in \mathfrak{U}$ such that

$$\|M_1 - U^* M_2 U\| \leq 2\sqrt{2}\epsilon + d_H\big(\sigma(N_1), \sigma(N_2)\big).$$

Since $\mathcal{G}$ is a connected graph, there exists a $j \in \{1,2\}$ and a vertex $\lambda \in \sigma(M_j)$ in $\mathcal{G}$ whose removal (along with all edges with λ as an endpoint) does not disconnect G. Choose any vertex μ in $\mathcal{G}$ connected to λ by an edge. By the construction of

$\mathcal{G}|\lambda-\mu| \le 2\sqrt{2}\epsilon + d_H(\sigma(N_1),\sigma(N_2))$ and $\mu \in \sigma(M_i)$ where $i \in \{1,2\}\setminus\{j\}$. Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra and $P_\mu^{(i)}$ is non-zero, there exists non-zero projections $Q_\mu^{(i)}$ and $R_\mu^{(i)}$ in $\mathfrak{U}$ such that $P_\lambda^{(j)}$ and $Q_\mu^{(i)}$ are Murray-von Neumann equivalent and $P_\mu^{(i)} = Q_\mu^{(i)} + R_\mu^{(i)}$. To complete our recursive step, remove λ from $\mathcal{G}$ (so $\mathcal{G}$ will still be a connected, bipartite graph), remove $P_\lambda^{(j)}$ from our list of projections, and replace $P_\mu^{(i)}$ with $R_\mu^{(i)}$ in our list of projections.

Continue the recursive process in the above paragraph until two vertices are left in $\mathcal{G}$ that must be connected by an edge. Since $\mathcal{G}$ is bipartite, one of these two remaining vertices is a non-zero subprojection of a spectral projection of M_1 and the other is a non-zero subprojection of a spectral projection of M_2. These two projections are Murray-von Neumann equivalent by the same K-theory argument used in Lemma (3.1.12).

By the same arguments as Lemma (3.1.12), the Murray-von Neumann equivalence of the projections created in the above process allows us to create partial isometries and thus, by taking a sum, a unitary $U \in \mathfrak{U}$ with the claimed property. Hence

$$\|N_1 - U^*N_2U\| \le \left(4+2\sqrt{2}\right)\epsilon + d_H\left(\sigma(N_1),\sigma(N_2)\right).$$

As $\epsilon > 0$ was arbitrary, the result follows.

The above proof can be modified to show the following results.

Corollary (3.2.6) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Suppose for each $q \in \{1,2\}$ that $\sigma(N_q) = \bigcup_{i=1}^{n} K_1^{(q)}$ is a disjoint union of compact sets with $K_i^{(1)}$ connected for all $i \in \{1,\dots,n\}$. Let $\chi_i^{(q)}$ be the characteristic function of $K_i^{(q)}$ for all $q \in \{1,2\}$ and $i \in \{1,\dots,n\}$. If $\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent for all $i \in \{1,\dots,n\}$ then

$$\operatorname{dist}\left(\mathcal{U}(N_1),\mathcal{U}(N_2)\right) \le \max_{i\in\{1,\dots,n\}} d_H\left(K_i^{(1)},K_i^{(2)}\right).$$

Proof:

Fix $\epsilon > 0$. The condition that '$\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent' allows the arguments of Lemma (3.2.5) to be applied on each pair $(K_i^{(1)}, K_i^{(2)}$ to produce a partial isometry $V_i \in \mathfrak{U}$ such that $V_i^* V_i = \chi_i^{(2)}(N_2)$, and

$$\left\| N_1 \chi_i^{(1)}(N_1) - V_i^* N_2 \chi_i^{(2)}(N_2) V_i \right\| < \epsilon + d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

If $U := \sum_{i=1}^{k} V_i \in \mathfrak{U}$ then U is a unitary operator such that

$$\|N_1 - U^* N_2 U\| < \epsilon + \max_{i \in \{1,\ldots,n\}} d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

Hence the result follows.

Corollary (3.2.7) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. If N_1 and N_2 have equivalent common spectral projections then

$$\operatorname{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) = d_H\big(\sigma(N_1), \sigma(N_2)\big).$$

Proof:

Let $\epsilon > 0$ and let M_1 and M_2 be the normal operators as constructed in Lemma (3.2.5). Notice we can apply the same technique as in Theorem (3.1.17) to assume for each $q \in \{1,2\}$ that $\chi_K(N_q)$ and $\chi_K(M_q)$ are Murray-von Neumann equivalent whenever K is a connected component of the union of the N_q-relevant boxes.

Construct the bipartite graph $\mathcal{G}$ as in the proof of Lemma (3.2.5). The only caveat remaining in the proof of Lemma (3.2.5) is that we required $\mathcal{G}$ to be connected. Let $\mathcal{G}_0$ be a connected component of $\mathcal{G}$. If K is the union of the N_1- and N_2-relevant boxes with vertices in $\mathcal{G}_0$ then the distance from K to any other N_q-relevant box is at least . Hence the characteristic function χ_K of K is a continuous function on $\sigma(N_1)$ and $\sigma(N_2)$. Since N_1 and N_2 have equivalent common spectral projections, $\chi_K(N_1)$ and $\chi_K(N_2)$ are Murray-von Neumann equivalent and thus, by our additional assumptions on M_1 and M_2, $\chi_K(M_1)$ and $\chi_K(M_2)$ are Murray-von Neumann equivalent. Hence we can apply the proof of Lemma (3.2.5) to each of the finite number of connected component of $\mathcal{G}$ separately and combine the resulting partial isometries as in Corollary (3.2.6) to obtain a unitary U such that

$$\|N_1 - U^*N_2U\| \le \left(4 + 2\sqrt{2}\right)\epsilon + d_H\left(\sigma(N_1), \sigma(N_2)\right).$$

Hence the result follows.

To illustrate the necessity of these assumptions, we note the following example.

Example (3.2.8) [3]:

Let P and Q be non-trivial projections in $\mathcal{O}_3$ with $[P]_0 \neq [Q]_0$ then $\sigma(P) = \sigma(Q)$ yet $\operatorname{dist}(\mathcal{U}(P), \mathcal{U}(Q)) \ge 1$ or else P and Q would be Murray-von Neumann equivalent.

In particular we have the following quantitative version of the above example.

Proposition (3.2.9) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra, let $N_1, N_2 \in \mathfrak{U}$ be normal operators, and let $f : \mathbb{C} \to \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(N_1) \cup \sigma(N_2)$ with $f\,(U) \subseteq \{0,1\}$. Let γ be a compact, rectifiable curve inside U with $(\sigma(N_1) \cup \sigma(N_2)) \cap \gamma = \phi$, $\operatorname{lnd}_\gamma(z) \in \{0,1\}$ for all $z \in \mathbb{C}\backslash\gamma$, $\operatorname{lnd}_\gamma(z) = 1$ for all $z \in \sigma(N_1) \cup \sigma(N_2)$, and $\{z \in C \mid \operatorname{lnd}_\gamma(z) = 0\} \subseteq U$. If $f(N_1)$ and $f(N_2)$ are not Murray-von Neumann equivalent then

$$\operatorname{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \ge \frac{2\pi}{l_0 \sup_{z\in\gamma} \|(zI_{\mathfrak{U}} - N_1)^{-1}\| \|(zI_{\mathfrak{U}} - N_2)^{-1}\|}$$

where $l_0(\gamma)$ is the length of γ in the regions where $f(z) = 1$.

Proof:

By the proof of Lemma (3.1.15), we know that $\|f(N_1) - Uf(N_2)U^*\|$ is at most

$$\frac{l_0(\gamma)\|N_1 - UN_2U^*\|}{2\pi} \sup_{z\in\gamma} \|(zI_{\mathfrak{U}} - N_1)^{-1}\| \|(zI_{\mathfrak{U}} - N_2)^{-1}\|$$

for all unitaries U in $\mathfrak{U}$. Since $f(N_1)$ and $f(N_2)$ are not Murray-von Neumann equivalent, $f(N_1)$ and $Uf(N_2)U^*$ are not Murray-von Neumann equivalent so

$$1 \leq \|f(N_1) - Uf(N_2)U^*\|$$

Hence the result follows.

Next we desire to examine the distance between unitary orbits of normal operators with nontrivial index function. Unfortunately, as this problem is not complete even for the Calkin algebra and due to the technical restraints illustrated above, a complete description of the distance between unitary orbits will not be given.

We will need a notion of direct sums inside unital, simple, purely infinite C^*-algebras. This leads us to the following construction.

Lemma (3.2.10) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra, let $V \in \mathfrak{U}$ be a non-unitary isometry, and let $P := VV^*$. Then there exists a unital embedding of the 2^∞-UHF C^*-algebra $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ into $(I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$ such that $[Q]_0 = 0$ in $\mathfrak{U}$ for every projection $Q \in \mathfrak{B}$.

Proof:

Let $P_0 := I_{\mathfrak{U}} - P$. Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, there exists a projection $P_1 \in \mathfrak{U}$ such that P_0 and P_1 are Murray-von Neumann equivalent and $0 < P_1 < P_0$. Let $P_2 := P_0 - P_1$ which is a non-trivial projection. Note $[P_0]_0 = 0$ in $\mathfrak{U}$. Hence

$$[P_1]_0 = [P_0]_0 = 0 = [P_1 + P_2]_0 = [P_1]_0 + [P_2]_0 = [P_2]_0.$$

Thus P_1 and P_2 are Murray-von Neumann equivalent in $\mathfrak{U}$. Thus, since $P_1, P_2 \leq P_0, P_1$ and P_2 are Murray-von Neumann equivalent in $P_0\mathfrak{U}P_0$.

For $q \in \{1,2\}$ let $V_q \in P_0\mathfrak{U}P_0$ be an isometry such that $VqV_q^* = P_q$. Then it is not difficult to see for each $\ell \in \mathbb{N}$ that

$$\mathfrak{B}_\ell := * - \text{alg}\left(\left\{V_{i_1}V_{i_2}\dots V_{i_\ell}V_{j_\ell}^* \mid i_1, i_2, \dots, i_\ell, j_1, j_2, \dots, j_\ell \in \{1,2\}\right\}\right)$$

is a C^*-subalgebra of $P_0\mathfrak{U}P_0$ containing P_0 that is isomorphic to $\mathcal{M}_{2^\ell}(\mathbb{C})$. Moreover, it is clear that $\mathfrak{B}_\ell \subseteq \mathfrak{B}_{\ell+1}$ for all $\ell \in \mathbb{N}$ and

$$\left\{V_{i_1}V_{i_2}\dots V_{i_\ell}V_{j_\ell}^* \dots V_{j_2}^*V_{j_1}^* \mid i_1, i_2, \dots, i_\ell, j_1, j_2, \dots, j_\ell \in \{1,2\}\right\}$$

are matrix units for $\mathfrak{B}_\ell$ in such a way that $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1}\mathfrak{B}_\ell}$ is the 2^∞-UHF C^*-algebra. Notice every rank one projection in $\mathfrak{B}$ is Murray-von Neumann equivalent in $\mathfrak{B}$ (and thus in $P_0\mathfrak{U}P_0$) to the rank one matrix unit $(V_1)^\ell(V_1^*)^\ell$ which is Murray-von Neumann equivalent in $\mathfrak{U}$ to P_0.

Therefore $[Q]_0 = [P_0]_0 = 0$ in $\mathfrak{U}$ for every rank one projection $Q \in \mathfrak{B}$. Hence $[Q]_0 = 0$ in $\mathfrak{U}$ for every non-zero projection $Q \in \mathfrak{B}$. However, if $Q \in \mathfrak{B}$ is a non-zero projection, it is easy to see that there exists an $\ell \in \mathbb{N}$ and a non-zero projection $Q_0 \in \mathfrak{B}$ such that $\|Q - Q_0\| < \frac{1}{2}$. Hence Q and Q_0 are Murray-von Neumann equivalent in $\mathfrak{U}$ by Lemma (3.1.13). Thus $[Q]_0 = [Q_0]_0 = 0$ as desired.

We will need the following two well-known results.

Lemma (3.2.11) [3]:

Let $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1}\mathcal{M}_{2^\ell}(\mathbb{C})}$ be the 2^∞-UHF C^*-algebra. If $X \subseteq \mathbb{C}$ is compact, there exists a normal operator $N \in \mathfrak{B}$ such that $\sigma(N) = X$.

Lemma (3.2.12) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra, let $V \in \mathfrak{U}$ be an isometry, and let $U \in \mathfrak{U}$ be a unitary. Then $[U]_1 = [VUV^* + (I_\mathfrak{U} - VV^*)]_1$.

Using the above lemmas we obtain the following extension of Corollary (3.2.7) to a normal operators with non-trivial index functions provided certain assumptions apply.

Lemma (3.2.13) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N, M \in \mathfrak{U}$ be normal operators such that

(i) $\sigma(M) \subseteq \sigma(N)$,
(ii)$\Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(N)$, and
(iii) N and M have equivalent common spectral projections.

Then

$$\operatorname{dist}\big(\mathcal{U}(N), \mathcal{U}(M)\big) = d_H\big(\sigma(N), \sigma(M)\big).$$

Proof:

One inequality follows from Proposition (3.2.3). Since $\mathfrak{A}$ is a unital, simple, purely infinite C^*-algebra, there exists a non-unitary isometry $V \in \mathfrak{A}$. Let $P := VV^*$, let $\mathcal{C} := (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C^*-algebra in $\mathcal{C}$ given by Lemma (3.2.10). By Lemma (3.2.11) there exists normal operators $N_0, M_0 \in \mathfrak{B}$ such that $\sigma(N_0) = \sigma(N)$ and $\sigma(M_0) = \sigma(M)$.

Let $N' \coloneqq VMV^* + N_0$ and let $M' := VMV^* + M_0$ which are clearly normal operators as V is an isometry. We will demonstrate that $N' \in \overline{\mathcal{U}(N)}$ and $M' \in \overline{\mathcal{U}(M)}$. Notice that $\sigma(N') = \sigma(M\,) \cup \sigma(N_0) = \sigma(N)$ as V is an isometry. Furthermore if $f : \mathbb{C} \to \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0,1\}$ then

$$f(N') = f(VMV') + f(N_0) = Vf(M)V^* + f(N_0).$$

If $f(M) = 0$ then $f(N) = 0$ as $f(M)$ and $f(N)$ are Murray-von Neumann equivalent. This implies f is zero on $\sigma(N)$ and thus $f(N') = f(N_0) = 0 = f(N)$. If $f(M) \neq 0$ then $f(N') \neq 0$ and

$$[f(N')]_0 = [Vf(M)V^*]_0 + [f(N_0)]_0 = [f(M)]_0 + [f(N)]_0$$

as $f(N_0) \in \mathfrak{B}$ and as every projection in $\mathfrak{B}$ is trivial in the K_0-group of $\mathfrak{A}$ by Lemma (3.2.10). In any case $f(N')$ and $f(N)$ are Murray-von Neumann equivalent. Furthermore, since $\mathfrak{B}_0^{-1} = \mathfrak{B}^{-1}$ as $\mathfrak{B}$ is a UHF C^*-algebra, we notice for any $\lambda \notin \sigma(N)$ that $\lambda I_{\mathfrak{A}} - N'$ is in the same component of $\mathfrak{A}^{-1}$ as

$$V(\lambda I_{\mathfrak{A}} - M\,)V^* + \ (\lambda I_{\mathfrak{A}} - P)$$

which is in the same connected component of $\mathfrak{A}^{-1}$as $\lambda I_{\mathfrak{A}} - M$ by Lemma (3.2.12). Therefore, since $\Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(N)$ by assumption, we obtain that $\Gamma(N) = \Gamma(N)$. Therefore N and N' are approximately unitarily equivalent in $\mathfrak{U}$. Similarly M and $M'$are approximately unitarily equivalent in $\mathfrak{U}$.

Hence it is easy to see for any unitary $U \in \mathfrak{C}$ that

$$\operatorname{dist}\big(\mathcal{U}(N), \mathcal{U}(M)\big) \leq \|(P+U)N'(P+U)^* - M'\| = \|UN_0U^* - M_0\|.$$

However, since $\mathcal{C}$ is a unital, simple, purely infinite C^*-algebra and $N_0, M_0 \in \mathcal{C}$ are in the unital inclusion of the UHF C^*-algebra $\mathfrak{B}$ in $\mathfrak{C}$, it is easy to see that $\Gamma(N_0)$ and $\Gamma(M_0)$ are trivial (when viewed as elements of $\mathfrak{C}$). Since any two non-zero projections in $\mathfrak{B} \subseteq \mathfrak{C}$ are Murray-von Neumann equivalent, the hypotheses of Corollary (3.2.7) are satisfied for N_0 and M_0 in $\mathcal{C}$. Hence for any $\epsilon > 0$ there exists a unitary $U \in \mathfrak{C}$ such that

$$\|UN_0U^* - M_0\| \leq \epsilon + d_H\big(\sigma(N_0), \sigma(M_0)\big) = \epsilon + d_H\big(\sigma(N), \sigma(M)\big).$$

Hence

$$\big(\mathcal{U}(N), \mathcal{U}(M)\big) \leq d_H\big(\sigma(N), \sigma(M)\big).$$

as desired.

Lemma (3.2.14) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $X \subseteq \mathbb{C}$ be a compact subset. Suppose X is a union of finitely many compact, connected components $\{K_i\}_{i=1}^{n}$ and $\mathbb{C}\backslash X$ is the union of finitely many connected components $\{\Omega_j\}_{j=0}^{m}$ where Ω_0 is the unbounded component. Let $\{g_i\}_{i=1}^{n} \subseteq K_0(\mathfrak{U})$ be such that $\sum_{i=1}^{n} g_i = [I_{\mathfrak{U}}]_0$ and let $\{h_j\}_{j=1}^{m} \subseteq K_1(\mathfrak{U})$. Then there exists a normal operator $N \in \mathfrak{U}$ such that $\sigma(N) = X, [\chi_{K_i}(N)]_0 = g_i$ for all $i \in \{1, \dots, n\}$ (where χ_{K_i} is the characteristic function of K_i), and $[\lambda I_{\mathfrak{U}} - N]_1 = h_j$ whenever $\lambda \in U_j$ for all $j \in \{1, \dots, m\}$. That is, for any element $\gamma \in \operatorname{Hom}(K_*(C(X)), K_*(\mathfrak{U})) \simeq KK(C(X), \mathfrak{U})$ there exists a normal operator in $\mathfrak{U}$ whose continuous functional calculus realizes γ.

Proof:

We may assume without loss of generality that if $1 \leq j_1 < j_2 \leq m$ then Ω_{j_1} is contained in the unbounded component of $\mathbb{C}\backslash\Omega_{j_2}$. Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, $K_1(\mathfrak{U})$ is canonically isomorphic to $\mathfrak{U}^{-1}/\mathfrak{U}_0^{-1}$. Choose a unitary $U_1 \in \mathfrak{U}$ such that $[U_1]_1 = h_1$. By the Continuous Functional Calculus for Normal Operators there exists a normal operator $T_1 \in \mathfrak{U}$ such that $\sigma(T_1)$ is a simple closed curve contained in X such that $[\lambda I_{\mathfrak{U}} - T_1]_1 = h_1$ for all $\lambda \in \Omega_1$. If Ω_2 is contained the unbounded component of $\mathbb{C}\backslash\sigma(T_1)$, we can repeat the above procedure to obtain a normal operator $T_2 \in \mathfrak{U}$ such that $\sigma(T_2)$ is a simple closed curve contained in X and in the unbounded component of $\mathbb{C}\backslash\sigma(T_1)$ such that $[\lambda I_{\mathfrak{U}} - T_2]_1 = h_2$ for all $\lambda \in \Omega_2$. If Ω_2 is contained the bounded component of $\mathbb{C}\backslash\sigma(T_1)$, we can repeat the above procedure to obtain a normal operator $T_2 \in \mathfrak{U}$ such that $\sigma(T_2)$ is a simple closed curve contained in X and in the bounded component of $\mathbb{C}\backslash\sigma(T_1)$ such that $[\lambda I_{\mathfrak{U}} - T_2]_1 = h_2 - h_1$ for all $\lambda \in \Omega_2$. Due to the ordering of $\{\Omega_j\}_{j=1}^{m}$, we can find normal operators $\{T_j\}_{j=1}^{m}$ such that each $\sigma(T_j)$ is a simple closed curve contained in X with the property that if $\mathcal{J}_j \subseteq \{1, \dots, m\}$ is the set of all indices $\ell \in \{1, \dots, m\}$ such that Ω_j is contained in the bounded component of $\mathbb{C}\backslash\sigma(T_1)$ then $\sum_{\ell \in \mathcal{J}_j}\left[\lambda I_{\mathfrak{U}} - T_j\right]_1 = h_j$ for all $\lambda \in \Omega_j$ and $j \in \{1, \dots, m\}$. Hence

$$\sum_{j=1}^{m}\left[\lambda I_{\mathfrak{U}} - T_j\right]_1 = h_j$$

for all $\lambda \in \Omega_i$ and all $\ell \in \{1, \dots, m\}$.

Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, implies there exists m isometries $\{V_j\}_{j=1}^{m}$ such that $Q := \sum_{j=1}^{m} V_j V_j^* < I_{\mathfrak{U}}$. Imply that there exists orthogonal projections $\{Q_j\}_{j=1}^{n-1}$ such that $\sum_{i=1}^{n-1} Q_i < I_{\mathfrak{U}} - Q$ and $[Q_i]_0 + \sum_{j=1}^{m}\left[\chi_{K_i}(T_j)\right]_0 = \mathrm{g}_i$ for all $i \in \{1, \dots, n-1\}$ (where χ_{K_i} is the characteristic function of K_i). Let

$$Q_n := I_{\mathfrak{U}} - Q - \sum_{i=1}^{n-1} Q_i.$$

For each $i \in \{1, \dots, n\}$ choose $\mu_i \in K$ and let

$$M := \sum_{j=1}^{m} V_j T_j V_j^* + \sum_{i=1}^{n} \mu_i Q_i.$$

Clearly M is a normal operator with $\sigma(M) \subseteq X$. Suppose $\lambda \in \Omega_{j_0}$ for some $j_0 \in \{1, \dots, m\}$. Then

$$\lambda I_{\mathfrak{U}} - M = \sum_{j=1}^{m} V_j\big(\lambda I_{\mathfrak{U}} - T_j\big)V_j^* + \sum_{i=1}^{n} (\lambda - \mu_i) Q_i.$$

Since clearly $[Q + \sum_{i=1}^{n} (\lambda - \mu_i) Q_i]_1 = 0$, by writing $\lambda I_{\mathfrak{U}} - M$ as a product of unitaries and by applying Lemma (3.2.12) we clearly obtain that

$$[\lambda I_{\mathfrak{U}} - M]_1 = \sum_{j=1}^{m} \big[\lambda I_{\mathfrak{U}} - T_j\big]_1 = h_j.$$

Furthermore

$$\chi_{K_0}(M) = \sum_{j=1}^{m} V_j \chi_{K_{i_0}}\big(T_j\big)V_j^* + \sum_{i=1}^{n} \chi_{K_{i_0}}(\mu_i) Q_i$$

for all $i_0 \in \{1, \dots, n\}$. Hence

$$\Big[\chi_{K_{i_0}}(M)\Big]_0 = \sum_{j=1}^{m} \Big[\chi_{K_{i_0}}\big(T_j\big)\Big]_0 + [Q_0]_0 = \mathrm{g}_{i_0}$$

for all $i_0 \in \{1, \dots, n-1\}$. Since $\sum_{i=1}^{n} \big[\chi_{K_i}(M)\big]_0 = [I_{\mathfrak{U}}]_0$, by our assumption that $\sum_{i=1}^{n} \mathrm{g}_i = [I_{\mathfrak{U}}]_0$ we clearly obtain $[\chi_{K_n}(M)]_0 = \mathrm{g}_n$. Thus M satisfies the conclusions of the lemma except for the fact that $\sigma(M)$ may be strictly contained in X.

Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, there exists a non-unitary isometry $V \in \mathfrak{U}$. Let $P := VV^*$, let $\mathcal{C} := (I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C^*-algebra in $\mathcal{C}$ given by Lemma (3.2.10). By Lemma

(3.2.11) there exists normal operator $N_0 \in \mathfrak{B}$ such that $\sigma(N_0) = X$. Let $N := VMV^* + N_0 \in \mathfrak{U}$. Then it is clear that N is a normal operator with $\sigma(N) = X$. Furthermore the proof of Lemma (3.2.13) implies that N has the desired properties.

Before generalizing Lemma (3.2.13), we note we may use Lemmas (3.2.13) and (3.2.14) to prove the following corollary.

Corollary (3.2.15) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $X \subseteq \mathbb{C}$ be compact. For each bounded, connected component Ω of $\mathbb{C}\backslash X$ let $h_\Omega \in K_1(\mathfrak{U})$. Let I be the set of closed subsets K of X such that the characteristic function χ_K of K is a continuous function on X. Suppose there exists $\{g_K\}_{K\in I} \subseteq K_0(\mathfrak{U})$ such that $g_X = [I_\mathfrak{U}]$ and $g_{K_1} + g_{K_2} = g_{K_1} \cup g_{K_2}$ whenever $K_1, K_2 \in I$ are disjoint. Then there exists a normal operator $N \in \mathfrak{U}$ such that $[\chi_K(N)]_0 = g_K$ for all $K \in I$ and $[\lambda I_\mathfrak{U} - N]_1 = h_\Omega$ whenever $\lambda \in \Omega$ and Ω is a bounded component of $\mathbb{C}\backslash X$. That is, for any element $\gamma \in \mathrm{Hom}(K_*(C(X)), K_*(\mathfrak{U})) \simeq KK(C(X), \mathfrak{U})$ there exists a normal operator in $\mathfrak{U}$ whose continuous functional calculus realizes γ.

Proof:

For each $n \in \mathbb{N}$ let

$$X_n := \left\{z \in \mathbb{C} | dist(z, X) \leq \frac{1}{2^n}\right\}.$$

Note X_n satisfies the conditions of the compact subset in Lemma (3.2.14) and if K is a connected component of X_n then $K \cap X \in I$. Thus Lemma (3.2.14) implies there exists normal elements $\{M_n\}_{n\geq 1} \subseteq \mathfrak{U}$ such that $\sigma(M_n) = X_n$, if K is a connected component of X_n then $[\chi_K(M_n)] = g_K$, and if $\lambda \in (\mathbb{C}\backslash X) \cap \Omega$ where $\Omega \subseteq \mathbb{C}\backslash X$ is a bounded, connected component then $[\lambda I_\mathfrak{U} - M]_1 = h_\Omega$.

Let $N_1 := M_1$. Since $\sigma(M_2) \subseteq \sigma(N_1)$, since N_2 and N_1 have equivalent common projections by the assumptions on the set $\{g_K\}_{K\in I}$, and since $\Gamma(M_2)(\lambda) = \Gamma(N_1)(\lambda)$ whenever $\lambda \notin \sigma(N)$, Lemma (3.2.13) implies there exists a unitary $U_2 \in \mathfrak{U}$ such that $\|N_1 - U_2 M_2 U_2^*\| \leq \frac{1}{2}$. Let $N_2 := U_2 M_2 U_2^*$. By repeating this process there exists a sequence $(N_n)_{n\geq 1} \subseteq \mathfrak{U}$ such that each N_n is a normal operator with the same conditions as M_n listed in the above paragraph and such that $\|N_n - N_{n+1}\| \leq \frac{1}{2^n}$. Hence $(N_n)_{n\geq 1}$ is a Cauchy sequence and thus converges to a normal operator $N \in \mathfrak{U}$. Clearly $\sigma(N) = X$ by the semicontinuity of the spectrum and by Lemma (3.1.5). Furthermore N has the desired properties by Lemma (3.1.13) and since the connected components of $\mathfrak{U}^{-1}$ are open and completely determine the K_1-group element.

Theorem (3.2.16) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators such that

(i) $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin \sigma(N_1) \cup \sigma(N_2)$, and
(ii) N_1 and N_2 have equivalent common spectral projections.

Then

$$\operatorname{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) = d_H\big(\sigma(N_1), \sigma(N_2)\big).$$

Proof:

Let $\epsilon > 0$. For each q $\in \{1,2\}$ Lemma (3.2,14) implies there exists a normal operator M_q such that

$$\sigma(M_q) = \left\{z \in \mathbb{C} \middle|\, \operatorname{dist}\left(z, \sigma(N_q)\right) \leq \epsilon\right\}.$$

$\Gamma(M_q)(\lambda) = \Gamma(N_q)(\lambda)$ for all $\lambda \notin \sigma(M_q)$, and M_q and N_q have equivalent common spectral projections. Hence Lemma (3.2.13) implies that $\operatorname{dist}(\mathcal{U}(N_q)\mathcal{U}(M_q)) \leq \epsilon$ for all $q \in \{1,2\}$. We claim there exists a normal operator $M \in \mathfrak{U}$ such that $\sigma(M) = \sigma(M_1) \cap \sigma(M_2)$, M and M_q have equivalent common spectral projections for all $q \in \{1,2\}$, and $\Gamma(M)(\lambda) = \Gamma(M_q)(\lambda)$ for all $\lambda \notin \sigma(M_q)$ and $q \in \{1,2\}$. The claim will follow from Lemma (3.2.14) provided $\sigma(M_1) \cap \sigma(M_2)$ is non-empty, we can choose the correct K_1-elements for the bounded, connected components of $\mathbb{C}\backslash\sigma(M)$, and we can construct the correct K_0-elements for the connected components of $\sigma(M)$. Since N_1 and N_2 have equivalent common spectral projections, it is clear that $\sigma(M_1) \cap \sigma(M_2)$ is non-empty.

If Ω is a bounded, connected component of the complement of $\mathbb{C}\backslash\sigma(M)$ then either Ω intersects both or exactly one of $\mathbb{C}\backslash\sigma(M_1)$ and $\mathbb{C}\backslash\sigma(M_2)$. If Ω intersects both $\mathbb{C}\backslash\sigma(M_1)$ and $\mathbb{C}\backslash\sigma(M_2)$, the condition that $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin \sigma(N_1) \cup \sigma(N_2)$ implies we can select a single element of $K_1(\mathfrak{U})$ for $\Gamma(M)(\lambda)$ to take for all $\lambda \in \Omega$ such that $\Gamma(M)(\lambda) = \Gamma(M_q)(\lambda)$ for all $\lambda \in \Omega\backslash\sigma(M_q)$ for $q \in \{1,2\}$. If Ω intersects $\mathbb{C}\backslash\sigma(M_q)$ but not the other complement, we define $\Gamma(M)(\lambda) = \Gamma(M_q)(\lambda)$ for all $\lambda \in \Omega \subseteq C\backslash\sigma(M_q)$.

To construct M such that M and M_q have equivalent common spectral projections for all $q \in \{1,2\}$, we need to define the K_0-elements that should be taken by the spectral projections of the finite number of connected components of $\sigma(M)$ in such a way that if K is a connected component of $\sigma(M_q)$, the sum of K_0-element of the spectral projections of $\sigma(M)$ corresponding to components contained in K is the same as the K_0-element of the spectral projection of M_q corresponding to K. Since, by construction, M_1 and M_2 have equivalent common spectral projections and $\sigma(M_1) \cup \sigma(M_2)$ has a finite number of connected components, we may assume for the purposes of this argument that $\sigma(M_1) \cup \sigma(M_2)$ is connected. Construct a connected, bipartite graph $\mathcal{G}$ whose vertices correspond to the connected components of $\sigma(M_1)$ and $\sigma(M_2)$ and where we connect two vertices with n edges provided the intersection of the corresponding connected components has n connected components. Thus we can view the edges of $\mathcal{G}$ as the connected components of $\sigma(M_1) \cap \sigma(M_2)$. Thinking of each vertex being labelled with the K_0-element of the spectral projection of the corresponding connected component, it suffices to label the edges of $\mathcal{G}$ with K_0-elements in such a way that the K_0-element at any vertex is the sum of the K_0-elements of the adjacent edges. This can be done by selecting a subgraph $\mathcal{T}$ of $\mathcal{G}$ that is a tree, selecting a root for $\mathcal{T}$, labelling all edges not in $\mathcal{T}$ to have the trivial K_0-element, starting at the vertices farthest from the root (which must be leaves) and labelling the one adjacent edge to each vertex to be the correct K_0-element, and by recursively labelling the remaining edges of the vertices farthest from the root that have a unlabelled edges to be such that the K_0-element of the vertex is the sum of the K_0-elements of the adjacent vertices. This process is well-defined (that is, we will always have an edge remaining to label so we can have the correct K_0-element at each vertex we consider), will terminate, and give such a labelling since M_1 and

M_2 have equivalent common spectral projections so the same K-theory using in Lemma (3.1.12) will imply the last step (which is labelling a single edge between the root and another vertex) is correct. Hence the claim is complete.

Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, there exists a non-unitary isometry $V \in \mathfrak{U}$. Let $P := VV^*$, let $\mathcal{C} := (I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C^*-algebra in C given by Lemma (3.2.10). By Lemma (3.2.11) there exists normal operators $M_{q,0} \in \mathfrak{B}$ such that $\sigma(M_{q,0}) = \sigma(M_q)$ for all $q \in \{1,2\}$. For each $q \in \{1,2\}$ let $M_q' := VMV^* + M_{q,0}$. The proof of Lemma (3.1.13) then demonstrates that $M_q \in \overline{\mathcal{U}(M_q)}$ for all $q \in \{1,2\}$,

$$\operatorname{dist}\big(\mathcal{U}(M_1), \mathcal{U}(M_2)\big) \leq \inf_{U \in \mathcal{U}(\mathcal{C})} \left\| U M_{1,0} U^* - M_{2,0} \right\|,$$

and thus

$$\operatorname{dist}\big(\mathcal{U}(M_1), \mathcal{U}(M_2)\big) = d_H\big(\sigma(M_1), \sigma(M_2)\big) \leq 2\epsilon + d_H\big(\sigma(N_1), \sigma(N_2)\big)$$

by Corollary (3.2.7). Hence $\operatorname{dist}\left(\mathcal{U}(N_q), \mathcal{U}(M_q)\right) \leq \epsilon$ for $q \in \{1,2\}$ implies that

$$\operatorname{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) \leq d_H\big(\sigma(N_1), \sigma(N_2)\big) + 4\epsilon.$$

As $\epsilon > 0$, the result follows.

We have the following results.

Proposition (3.2.17) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra with trivial K_0-group. If $N_1, N_2 \in \mathfrak{U}$ are normal operators then

$$\operatorname{dist}\big(\mathcal{U}(N_1), \mathcal{U}(N_2)\big) \leq 2\rho(N_1, N_2)$$

where $\rho(N_1, N_2)$ is as defined in Definition (3.2.2).

Proof:

Since $\mathfrak{U}$ is a unital, simple, purely infinite C^*-algebra, there exists a non-unitary isometry $V \in \mathfrak{U}$. Let $P := VV^*$, let $\mathfrak{C} := (I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C^*-algebra in $\mathcal{C}$ given by Lemma (3.2.10).

Let

$$X := \sigma(N_1) \cup \sigma(N_2) \cup \{\lambda \in \mathbb{C} | \lambda \notin \sigma(N_1) \cup \sigma(N_2) \cdot \Gamma(N_1)(\lambda) \neq \Gamma(N_2)(\lambda)\}.$$

By Lemma (3.2.11) there exists a normal operator $N' \in \mathfrak{B}$ such that $\sigma(N) = X$. Therefore, if

$$M := VN_1V^* + N'$$

then M is a normal operator in $\mathfrak{U}$ such that $\sigma(M) = X$ and $\Gamma(M)(\lambda) = \Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin X$ (alternatively we could have used Lemma (3.2.14) to construct M). Therefore it suffices to show for any $q \in \{1,2\}$ that

$$\text{dist}\left(\mathcal{U}(N_q), \mathcal{U}(M)\right) \leq \rho(N_1, N_2).$$

By the definition of ρ we see that

$$\rho(N_q, M) = d_H\left(\sigma(N_q), \sigma(M)\right) \leq \rho(N_1, N_2).$$

Furthermore, by applying Lemma (3.2.11), there exists normal operators $N_0, M_0 \in \mathfrak{B}$ such that $\sigma(N_0) = \sigma(N_q)$ and $\sigma(M_0) = \sigma(M)$. As in the proof of Lemma (3.213), we see that $VN_qV^* + N_0 \in \overline{\mathcal{U}(N_q)}$ and $VN_qV^* + M_0 \in \overline{\mathcal{U}(M)}$. Hence it is easy to see that for any unitary $U \in \mathcal{C}$ that

$$\text{dist}\left(\mathcal{U}(N_q), \mathcal{U}(M)\right) \leq \left\|(P + U)\left(VN_qV^* + N_0\right)(P + U)^* - \left(VN_qV^* + M_0\right)\right\|$$

$$= \|UN_0U^* - M_0\|.$$

Thus, as in the proof of Lemma (3.2.13), for any $\epsilon > 0$ there exists a $U \in \mathfrak{C}$ such that

$$\|UN_0U^* - M_0\| \leq \epsilon + d_H\left(\sigma(N_1), \sigma(M)\right) \leq \epsilon + \rho(N_1, N_2).$$

Hence the result follows.

Proposition (3.2.18) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra. If $N_1, N_2 \in \mathfrak{U}$ are normal operators with equivalent common spectral projections then

$$\text{dist}\big(\mathcal{U}(N_1),\mathcal{U}(N_2)\big) \le 2\rho(N_1,N_2).$$

Theorem (3.2.19) [3]:

Let N and M be normal operators in the Calkin algebra. Then $N \in \overline{S(M)}$ if and only if

(i) $\sigma_e(M) \subseteq \sigma_e(N)$,
(ii) Each component of $\sigma_e(N)$ intersects $\sigma_e(M)$,
(iii) The Fredholm index of $\lambda I - M$ and $\lambda I - N$ agree for all $\lambda \notin \sigma_e(N)$, and
(iv) If $\lambda \in \sigma_e(N)$ is not isolated in $\sigma_e(N)$, the component of λ in $\sigma_e(N)$ contains some nonisolated point of $\sigma_e(M)$.

Theorem (3.2.20) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N, M \in \mathfrak{U}$ be normal operators. Then $N \in \overline{S(M)}$ if and only if

(i) $\sigma(M) \subseteq \sigma(N)$,
(ii) Each component of $\sigma(N)$ intersects $\sigma(M)$,
(iii) $\Gamma(N)(\lambda) = \Gamma(M)(\lambda)$ for all $\lambda \notin \sigma(N)$,
(iv) If $\lambda \in \sigma(N)$ is not isolated in $\sigma(N)$, the component of λ in $\sigma(N)$ contains some non-isolated point of $\sigma(M)$, and
(v) N and M have equivalent common spectral projections.

Proof:

Let N and M satisfy the five conditions of Theorem (3.2.20). By applying Lemma (3.2.26) recursively a finite number of times, we can find a normal operator M such that $M \in \overline{S(M)}, \sigma(M')$ is $\sigma(M)$ unioned with a finite number of connected components of $\sigma(N)$, and N and M' satisfy the five conditions of Theorem (3.2.20).

Fix $\epsilon > 0$. Since $\sigma(N)$ is compact, $\sigma(N)$ has a finite ϵ-net. Thus the normal operator M in the above paragraph can be selected with the additional requirement that $\text{dist}(\lambda, \sigma(M')) \le \epsilon$ for all $\lambda \in \sigma(N)$. By Lemma (3.2.13) $\text{dist}(\mathcal{U}(N), \mathcal{U}(M')) \le \epsilon$ so $\text{dist}(N, S(M)) \le \epsilon$ as desired.

Theorem (3.2.21) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra with the following properties:

(i) $\mathfrak{U}$ has property weak (FN),
(ii) Every non-zero projection in $\mathfrak{U}$ is properly infinite, and
(iii) Any two non-zero projections in $\mathfrak{U}$ are Murray-von Neumann equivalent.

(For example, $\mathcal{O}_2$ and every type (III) factor with separable predual).

Let $N, M \in \mathfrak{U}$ be normal operators such that $\lambda I_{\mathfrak{U}} - M \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Then $N \in \overline{S(M)}$ if and only if

(i) $\sigma(M) \subseteq \sigma(N)$,
(ii) Each component of $\sigma(N)$ intersects $\sigma(M)$,
(iii) $\lambda I_{\mathfrak{U}} - N \in \mathfrak{U}$ for all $\lambda \notin \sigma(N)$, and
(iv) If $\lambda \in \sigma(N)$ is not isolated in $\sigma(N)$, the component of λ in $\sigma(N)$ contains some non-isolated point of $\sigma(M)$.

To see that the fourth conclusion is necessary, let K_λ be the connected component of $\sigma(N)$ containing λ. We note that if K_λ is not isolated in $\sigma(N)$ (that is, every open neighbourhood of K_λ intersects a different connected component of $\sigma(N)$) then the first two conditions imply that $\sigma(M) \cap K_\lambda$ contains a cluster point of $\sigma(M)$. Otherwise if K_λ is isolated in $\sigma(N)$, the characteristic function χK_λ of K_λ can be extended to an analytic function on a neighbourhood of $\sigma(N)$. Thus Lemma (3.1.15) implies $\chi K_\lambda(N) \in \overline{S(\chi K_\lambda(M))}$. If $\sigma(M) \cap K_\lambda$ does not contain a cluster point of $\sigma(M)$ then $\chi K_\lambda(M)$ must have finite spectrum. Hence there exists a non-zero polynomial p such that $p(\chi K_\lambda(M)) = 0$. Clearly this implies $p(T) = 0$ for all $T \in \overline{S(\chi K_\lambda(M))}$ so $p(\chi K_\lambda(N)) = 0$. Since K_λ is a connected, compact subset of $\sigma(N)$ that is not a singleton, this is impossible. Hence the fourth condition is necessary. An alternative proof of the necessity of the fourth condition may be obtained by considering the separable C^*-algebra generated by N, M, and a countable number of invertible elements, by taking an infinite direct sum of a faithful representation of this C^*-algebra on a separable Hilbert space.

We have the following results.

Corollary (3.2.22) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{U}$ be normal operators. If $N_1 \in \overline{S(N_2)}$ and $N_2 \in \overline{S(N_2)}$ then $N_1 \sim_{au} N_2$.

Lemma (3.2.23) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra, let $P \in \mathfrak{A}$ be a non-trivial projection, let $Z \in (I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$, and let $X \in \mathfrak{U}$ be such that $PX(I_{\mathfrak{U}} - P) = X$. If $\lambda \notin \sigma(I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)(Z)$ then

$$\lambda P + X + Z \sim \lambda P + Z.$$

Proof:

Note that if $Y := X(\lambda(I_{\mathfrak{U}} - P) - Z)^{-1}$ then

$$T := I_{\mathfrak{U}} + Y$$

is invertible with

$$T^{-1} = I_{\mathfrak{U}} - Y.$$

A trivial computation shows

$$T(\lambda P + X + Z)T^{-1} = \lambda P + Z.$$

Corollary (3.2.24) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra, let $n \in \mathbb{N}$, let $\lambda_1, \dots, \lambda_n$ be distinct complex scalars, let $\{P_j\}_{j=1}^{n} \subseteq \mathfrak{U}$ be a set of non-trivial orthogonal projections with $\sum_{j=1}^{n} P_j = I_{\mathfrak{U}}$, and let $\{A_{i,j}\}_{i,j=1}^{n} \subseteq \mathfrak{U}$ be such that $A_{i.j} = 0$ if $i \geq j$ and $P_i A_{i,j} P_j = A_{i,j}$ for all $i < j$. Then

$$\sum_{j=1}^{n} \lambda_j P_j + \sum_{j=1}^{n} A_{i,j} \sim \sum_{j=1}^{n} \lambda_j P_j.$$

Proof:

By applying Lemma (3.2.23) with $P := P_1, Z := \sum_{j=1}^{n} \lambda_j P_j + \sum_{i,j=2}^{n} A_{i,j}$ (it is elementary to show that $\sigma(I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)(Z) = \{\lambda_2, \ldots, \lambda_n\}$ so $\lambda_1 \notin \sigma(Z)$ by assumption), and $X := \sum_{j=1}^{n} A_{1j}$, we obtain that

$$\sum_{j=1}^{n} \lambda_j P_j + \sum_{j=1}^{n} A_{i,j} \sim \sum_{j=1}^{n} \lambda_j P_j + \sum_{j=1}^{n} A_{i,j}.$$

The result then proceeds by recursion by considering the unital C^*-algebra $(I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$.

Lemma (3.2.25) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra, let $M \in \mathfrak{U}$ be a normal operator, let $V \in \mathfrak{U}$ be a non-unitary isometry, let $P := VV^*$, and let $\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ be the unital copy of the 2^∞-UHF C^*-algebra in $\mathfrak{C}$ given by Lemma (3.2.10). Suppose μ is a cluster point of $\sigma(M)$ and $Q \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$ is a nilpotent matrix for some $\ell \in \mathbb{N}$. Then $VMV^* + \mu(I_{\mathfrak{U}} - P) + Q \in \overline{S(M)}$.

Proof:

Since $Q \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$ is a nilpotent matrix, Q is unitarily equivalent to a strictly upper triangular matrix. Thus we can assume Q is strictly upper triangular. By our assumptions on μ there exists a sequence $(\mu_j)_{j \geq 1}$ of distinct scalars contained in $\sigma(M)$ that converges to μ. For each $q \in \mathbb{N}$ let

$$T_q := \operatorname{diag}(\mu_q, \mu_{q+1}, \ldots, \mu_{q+2^\ell-1}) \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$$

be the diagonal matrix with $\mu_q, \ldots, \mu_{q+2^\ell-1}$ along the diagonal.

Let $M_q := VMV^* + T_q \in \mathfrak{U}$. As in the proof of Lemma (3.2.13), it is easy to see that M_q is approximately unitarily equivalent to M for each $q \in \mathbb{N}$. Hence

$$M \sim_{au} M_q \sim VMV^* + (T_q + Q)$$

by Lemma (3.2.24). Since $\lim_{q \to \infty} T_q + Q = \mu(I_{\mathfrak{U}} - P) + Q$, the result follows.

Subsequently we have our next stepping-stone.

Lemma (3.2.26) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra. Let $N, M \in \mathfrak{U}$ be normal operators and write $\sigma(N) = K_1 \cup K_2$ where K_1 and K_2 are disjoint compact sets with K_1connected. Suppose

(i) $\sigma(M) = K_1 \cup 2$ where $K_1' \subseteq K_1$,
(ii) $\Gamma(N)(\lambda) = \Gamma(M)(\lambda)$ for all $\lambda \notin \sigma(N)$, and
(iii) N and M have equivalent common spectral projections.

If K_1' contains a cluster point of $\sigma(M)$ then $N \in \overline{S(M)}$.

Proof:

If $K_1 1$ is a singleton, $K_1' = K$ as K_1' is non-empty. Thus $\sigma(M) = \sigma(N)$ so Theorem (3.1.17) implies N and M are approximately unitarily equivalent.

Otherwise K_1' is not a singleton. Fix a non-unitary isometry $\mathrm{V} \in \mathfrak{U}$ and $\epsilon > 0$. Let $P := VV^*$ and let $\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathcal{M}_{2^\prime}(\mathbb{C})}$ be the unital copy of the 2^∞-UHF C^*-algebra in $(I_\mathfrak{U} - P)\mathfrak{U}(I_\mathfrak{U} - P)$ given by Lemma (3.2.10). There exists a normal operator $T \in \mathfrak{B}$ with

$$\sigma(T) = \{z \in \mathbb{C} |\ |z| \leq \epsilon\}$$

such that T is a norm limit of nilpotent matrices from $\bigcup_{\ell \geq 1} \mathcal{M}_{2^\prime}(\mathbb{C}) \subseteq \mathfrak{B} \subseteq \mathfrak{U}$. Let $\mu \in K_1'$ be any cluster point of $\sigma(M)$. Lemma (3.2.25) implies that

$$VMV^* + \mu(I_\mathfrak{U} - P) + Q \in \overline{S(M)}$$

for every nilpotent matrix $Q \in \bigcup_{\ell \geq 1} \mathcal{M}_{2^\prime}(\mathbb{C}) \subseteq \mathfrak{B}$. Since T is a norm limit of nilpotent matrices from $\bigcup_{\ell \geq 1} \mathcal{M}_{2^\prime}(\mathbb{C})$, we obtain that

$$VMV^* + \mu(I_\mathfrak{U} - P) + T \in \overline{S(M)}$$

Let $M_1 := VMV^* + \mu(I_\mathfrak{U} - P) + T$. As in the proof of Lemma (3.2.13), it is easy to see that M_1 is a normal operator such that $\Gamma(M_1)(\lambda) = \Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(M_1) \cup \sigma(N)$ and M_1 and N have equivalent common spectral projections.

Since K_1 is connected and $\sigma(M_1)$ contains an open neighbourhood around $\mu \in K_1$, we can repeat the above argument a finite number of times to obtain a normal operator $M_0 \in \overline{S(M)}$ such that $\sigma(M_0) = K_1'' \cup K_2$ where K_1'' is connected, $K_1 \subseteq K_1''$,

$$K_1'' \subseteq \{z \in \mathbb{C} | \operatorname{dist}(z, K_1) \leq \epsilon\},$$

$\Gamma(M_0)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(M_1) \cup \sigma\,(N)$, and M_0 and N have equivalent common spectral projections. Therefore Lemma (3.2.13) implies

$$\operatorname{dist}\big(\mathcal{U}(N), \mathcal{U}(M_0)\big) = d_H\big(\sigma(N), \sigma(M_0)\big) \leq \epsilon$$

so $\operatorname{dist}(N, S(M)) \leq \epsilon$. Thus, as $\epsilon > 0$ was arbitrary, the result follows.

Definition (3.2.27) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra. An operator $A \in \mathfrak{U}$ is said to be a scalar matrix in $\mathfrak{U}$ if there exists a finite dimensional C^*-algebra $\mathfrak{B}$ and a unital, injective $*$ homomorphism $\pi : \mathfrak{B} \to \mathfrak{U}$ such that $A \in \pi(\mathfrak{B})$.

Proposition (3.2.28) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra with the three properties listed in Theorem (3.2.21). If $N \in \mathfrak{U}$ is a normal operator with the closed unit disk as spectrum then N is a norm limit of nilpotent scalar matrices from $\mathfrak{U}$.

Using the ideas contained in the proof of Lemma (3.2.25), it is possible to prove the following.

Lemma (3.2.29) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra such that

(i) There exists a unital, injective $*$-homomorphism $\pi : \mathfrak{U} \oplus \mathfrak{U} \to \mathfrak{U}$, and

(ii) If $N_1, N_2 \in \mathfrak{U}$ are normal operators with $\lambda I_{\mathfrak{U}} - N_q \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1,2\}$, $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.

Let $M \in \mathfrak{U}$ be a normal operator with $\lambda I_{\mathfrak{U}} - M \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(M)$, let $\mu \in \sigma(M)$ be a cluster point of $\sigma(M)$, and let $Q \in \mathfrak{U}$ be a nilpotent scalar matrix. Then $\pi(M \oplus (\mu I + Q)) \in \overline{S(M)}$.

By using similar ideas to the proof of Theorem (3.2.20) and by using the following lemma, the proof of Theorem (3.2.21) is also complete.

Lemma (3.2.30) [3]:

Let $\mathfrak{U}$ be a unital C^*-algebra with the three properties listed in Theorem (3.2.21). Let $N, M \in \mathfrak{U}$ be normal operators with $\lambda I_{\mathfrak{U}} - N \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(N)$ and $\lambda I_{\mathfrak{U}} - M \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Let $\{K_\lambda\}_\Lambda$ be the connected components of $\sigma(N)$. Suppose

$$\sigma(M) = \left(\bigcup_{\lambda \in \Lambda \setminus \{\lambda_0\}} K_\lambda \right) \cup K_0$$

where $K_0 \subseteq K_{\lambda_0}$. If K_0 contains a cluster point of $\sigma(M)$ then $N \in \overline{S(M)}$.

With the proofs of Theorems (3.2.20) and (3.2.21) complete, we will use said theorems to classify when a normal operator is a limit of nilpotents in these C^*-algebras.

Corollary (3.2.31) [3]:

Let $\mathfrak{U}$ be a unital, simple, purely infinite C^*-algebra. A normal operator $N \in \mathfrak{U}$ is a norm limits of nilpotent operators from $\mathfrak{U}$ if and only if $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\Gamma(N)$ is trivial.

Proof:

The requirements that $\sigma(N)$ is connected and contains zero was shown. The condition that $\Gamma(N)$ is trivial.

Suppose $N \in \mathfrak{U}$ is a normal operator such that $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\Gamma(N)$ is trivial. Let $\epsilon > 0$ and fix a non-unitary isometry $V \in \mathfrak{U}$. Let $P := VV^*$ and let $\mathfrak{B} := \bigcup_{\ell \geq 1} \overline{\mathcal{M}_{2^\ell}(\mathbb{C})}$ be the unital copy of the 2^∞-UHF C^*-algebra in $(I_{\mathfrak{U}} - P)\mathfrak{U}(I_{\mathfrak{U}} - P)$ given by Lemma (3.2.10). There exists a normal operator $T \in \mathfrak{B}$ with

$$\sigma(T) = \{z \in \mathbb{C} |\ |z| \leq \epsilon\}$$

such that T is a norm limit of nilpotent matrices from $\bigcup_{\ell\geq 1}\mathcal{M}_{2^{\ell}}(\mathbb{C})\subseteq\mathfrak{B}\subseteq\mathfrak{U}$.

Let $M := VNV^* + T \in \mathfrak{U}$. Clearly M is a normal operator such that $\sigma(M) = \sigma(N) \cup \sigma(T)$, M and N have equivalent common spectral projections, and $\Gamma(M)$ is trivial as in the proof of Lemma (3.2.13). Therefore Corollary (3.2.7) implies that

$$\operatorname{dist}(\mathcal{U}(N), \mathcal{U}(M)) \leq \epsilon.$$

However, we note that $\Gamma(T)$ is trivial when we view T as a normal element in $\mathfrak{U}$. Moreover, as $\sigma(N)$ is connected and contains zero, $\sigma(M)$ is connected and contains $\sigma(T)$. Thus Theorem (3.2.20) (where conditions (iv) and (v) are easily satisfied) implies that $M \in \overline{S(T)}$ so

$$\operatorname{dist}(N, S(T)) \leq \epsilon.$$

However, as T is a norm limit of nilpotent operators from $\mathfrak{B} \subseteq \mathfrak{U}$, the above inequality implies N is within 2ϵ of a nilpotent operator from $\mathfrak{U}$. Thus the proof is complete.

Corollary (3.2.32) [3]:

Let $\mathfrak{U}$ be a unital, separable C^*-algebra with the three properties listed in Theorem (3.2.21). A normal operator $N \in \mathfrak{U}$ is a norm limits of nilpotent operators from $\mathfrak{U}$ if and only if $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{U}} - N \in \mathfrak{U}_0^{-1}$ for all $\lambda \notin \sigma(N)$.

Proof:

The proof of this result follows the proof of Corollary (3.2.31) by using direct sums instead of non-unitary isometries (as in Lemma (3.2.27)), Proposition (3.2.4) instead of Corollary (3.2.7), Theorem (3.2.21) instead of Theorem (3.2.20), and Proposition (3.2.28).

To conclude this paper we will briefly discuss closed similarity orbits of normal operators in von Neumann algebras. We recall that completely classifies when two normal operators are approximately unitarily equivalent in von Neumann algebras. Furthermore Theorem (3.2.21) completely determines when one normal operator is in the closed similarity orbit of another normal operator in type (III) factors with separable predual. Thus it is natural to ask whether a generalization of Theorem (3.2.21) to type II factors may be obtained.

Unfortunately the existence of a faithful, normal, tracial state on type (II_1) factors inhibits when a normal operator can be in the closed similarity orbit of another normal operator. Indeed suppose $\mathfrak{M}$ is a type (II_1) factor and let τ be the faithful, normal, tracial state on $\mathfrak{M}$. If $N, M \in \mathfrak{M}$ are such that $N \in \overline{S(M)}$, it is trivial to verify that $\tau(p(N)) = \tau(p(M))$ for all polynomials p in one variable. In particular if $N, M \in \mathfrak{M}$ are self-adjoint and $N \in \overline{S(M)}$ we obtain that $\tau(f(N)) = \tau(f(M))$ for all continuous functions on $\sigma(N) \cup \sigma(N)$ and, as τ is faithful and normal, this implies that N and M must have the same spectral distribution. Therefore, if $N, M \in \mathfrak{M}$ are self-adjoint operators, $\sigma(M) = [0, \frac{1}{2}]$, and $\sigma(N) = [0,1]$, then, unlike in $\mathfrak{B}(\mathcal{H}), N \notin \overline{S(M)}$. Combining the above arguments we have the following result.

Proposition (3.2.33) [3]:

Let $\mathfrak{M}$ be a type (II_1) factor. If $N, M \in \mathfrak{M}$ are self-adjoint operators and $N \in \overline{S(M)}$, then $N \sim_{au} M$.

Chapter 4

Von Neumann algebras and Ultraproducts

Section (4.1): Ultraproduct of von Neumann algebras

The purpose of this section is to study several notions of ultraproducts and central sequence algebras of von Neumann algebras which are not necessarily of finite type. Since it does not seem to be well-known that there are various notions of ultraproducts, let us start from an overview of the history.

The notions of central sequences and ultraproducts play a central role in the study of operator algebras and their automorphisms. The importance of central sequences was already recognized as early as in Murray–von Neumann's work on rings of operators. After establishing the uniqueness of the hyperfinite type II_1 factor R, they tried to prove the existence of non-isomorphic type II_1 factors. In $(R,\tau)=\bigotimes_N(M_2(\mathbb{C}),\frac{1}{2}\mathrm{Tr})$ (Tr denotes the usual trace on $M_2(\mathbb{C})$), consider a sequence

$$u_n = 1^{\otimes n}\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes 1\otimes\cdots, n\in\mathbb{N}. \quad (1)$$

$\{u_n\}_{n=1}^{\infty}$ satisfies

(i) $\sup_n\|u_n\|<\infty$.
(ii) $u_na-au_n\to 0$ strongly for any $a\in R$.
(iii) $\tau(u_n)=0$ for all $n\in\mathbb{N}$ and $\|u_n\|_2\nrightarrow 0$.

A sequence of operators $\{x_n\}_{n=1}^{\infty}$ in a finite von Neumann algebra M is called a central sequence if it satisfies (i) and (ii), and it is called nontrivial if in addition it satisfies (iii). A type II_1 factor with non-trivial central sequence is said to have property Gamma. Using the so-called 14ε argument, they showed that the group von Neumann algebra $L(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ on two generators does not have property Gamma while R does, whence $R\ncong L(\mathbb{F}_2)$. Central sequences were then used to show the existence of uncountably many type II_1 (type II_∞) factors. Variants of the property Gamma, such as property L of Pukanszky were also studied to provide examples of type II_I factors without non-trivial central sequences. On the other hand, the study of the quotient of a finite $(A)W^*$-algebra by its maximal ideals gave rise to the concept of tracial ultraproducts. The study of such quotient algebras was carried out by Wright. He showed that the quotient of an $(A)W^*$-algebra of type II with a trace by its maximal ideal is an $(A)W^*$-factor of type II, and quotient of finite $(A)W^*$-algebra of type I by its maximal ideals are generically $(A)W^*$-factors of type II_1. Sakai showed that the quotient of a finite W^*-algebra M by a maximal ideal $I_\omega = \{x \in M; (x^*x)^\#(\omega) = 0\}$ is a finite W^*-factor. Here, $\#: M \to Z(M) \cong C(\Omega)$ is the center valued trace and ω is a point in the Gelfand spectrum Ω of the center $Z(M)$. When $M = \bigoplus_N M_n(\mathbb{C})$, we have $\Omega = \beta\mathbb{N}$ and M/I_ω is what is now called the tracial ultraproduct of $\{M_n(\mathbb{C})\}_{n=1}^{\infty}$. More generally, the tracial ultraproduct $(M_n, \tau_n)^\omega$ of a sequence of finite von Neumann algebras with faithful tracial states $\{M_n, \tau_n\}_{n=1}^{\infty}$ along a free ultrafilter $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ is defined as the quotient algebra $(M_n, \tau_n)^\omega := \ell_\infty(\mathbb{N}, M_n)/I_\omega(\mathbb{N}, M_n)$, where $\ell_\infty(\mathbb{N}, M_n)$ is the C^*-algebra of all bounded sequences of $\prod_{\mathbb{N}} M_n$, and $I_\omega(\mathbb{N}, M_n)$ is the ideal of $\ell_\infty(\mathbb{N}, M_n)$ consisting of those sequences $(x_n)_n$ which satisfy $\tau_n(x_n^* x_n) \to 0$ along ω. For the case of constant sequence $M_n = M, \tau_n = \tau, (M_n, \tau_n)^\omega$ is written as M^ω and called the ultrapower of M. Few years later after Sakai's work, McDuff revealed the importance of the tracial ultrapower and central sequences. Viewing M as a subalgebra of M^ω by diagonal embedding, central sequences form a von Neumann subalgebra $M_\omega = M' \cap M^\omega$. Among other things, she proved that for a type II_1 factor M, M_ω is either abelian or of type II_1, and the latter case occurs if and only if M absorbs R tensorially: $M \cong M \mathbin{\overline{\otimes}} R$ (such a factor M is now called McDuff).

The definition of the central sequence algebra M_ω is generalized for arbitrary von Neumann algebras by Connes. It is defined as $M_\omega := \mathcal{M}_\omega(\mathbb{N}, M)/I_\omega(\mathbb{N}, M)$, where $\mathcal{M}_\omega(\mathbb{N}, M)$ is the set of all $(x_n)_n \in \ell_\infty(\mathbb{N}, M)$ satisfying $\|x_n\psi - \psi x_n\| \to 0$ along ω for all $\psi \in M_*$ (here in $I_\omega(\mathbb{N}, M)$, convergence is with respect to strong* topology). M^ω is called the asymptotic centralizer of M. On the other hand, the generalization of $M\omega$ is more involved. If M is not of finite type, then $I_\omega(\mathbb{N}, M)$ is not an ideal of $\ell_\infty(\mathbb{N}, M)$. Therefore one has to modify the definition of M^ω for infinite type von Neumann algebras. The right definition of M^ω was given by Ocneanu in order to generalize Connes' automorphism analysis approach for general injective von Neumann algebras. It is defined as $M^\omega := \mathcal{M}^\omega(N\mathbb{N}, M)/ I_\omega(\mathbb{N}, M)$, where $\mathcal{M}^\omega(\mathbb{N}, M)$ is the two-sided normalizer of $I_\omega(\mathbb{N}, M)$. That is,

$\mathcal{M}^{\omega}(\mathbb{N},M)$ consists of those $(x_n)_n \in \ell_\infty(\mathbb{N},M)$ which satisfy $(x_n)_n I_\omega(\mathbb{N},M) \subset I_\omega(\mathbb{N},M)$ and $I_\omega(\mathbb{N},M)(x_n)_n \subset I_\omega(\mathbb{N},M)$. We call M^ω the Ocneanu ultrapower of M. As same as tracial ultraproducts, any projection p (resp. unitary u) in M^ω is represented by a sequence of projections $(p_n)_n$ (resp. unitaries $(u_n)_n$) of M. A decade before Ocneanu's definition of M^ω, another generalization of $M' \cap M^\omega$ for a general factor M with separable predual was proposed by Golodets. It is defined as follows: let φ be a normal faithful state on M. Consider the GNS representation of M associated with φ, so that $\varphi = \langle \cdot\, \xi_\varphi, \xi_\varphi \rangle$ with a cyclic and separating vector ξ_φ on a Hilbert space H. Consider the following (non-normal) state $\bar{\varphi}$ on $\ell_\infty = \ell_\infty(\mathbb{N},M)$:

$$\bar{\varphi}((x_n)_n) := \lim_{n\to\omega} \varphi(x_n)\,, (x_n)_n \in \ell_\infty(\mathbb{N},M). \tag{2}$$

Let $\pi_{\mathrm{Gol}}\colon \ell_\infty \to \mathbb{B}(H_{\mathrm{Gol}})$ be the GNS representation of $\bar{\varphi}$ with a cyclic vector $\bar{\xi}$ satisfying $\bar{\varphi} = \langle \cdot\, \bar{\xi}, \bar{\xi} \rangle$. Let e_ω be the projection of H_{Gol} onto $\overline{\pi_{\mathrm{Gol}}(\ell_\infty)'\bar{\xi}}$. Define

$$\mathscr{R} := e_\omega \pi_{\mathrm{Gol}}(\ell_\infty)'' e_\omega \subset \mathbb{B}(e_\omega H_{\mathrm{Gol}}).$$

Let $\bar{M}_d$ be the subspace of $\ell^\infty(\mathbb{N},M)$ consisting of constant sequences $(x,x,\ldots)_n, x \in M$. Then the asymptotic algebra C_M^ω of M is defined by

$$C_M^\omega := \mathscr{R} \cap \pi_{\mathrm{Gol}}(\bar{M}_d)' \subset \mathbb{B}(e_\omega H_{\mathrm{Gol}}).$$

Moreover, φ induces a normal faithful state $\tilde{\varphi}$ on R, whence a state $\dot{\varphi} := \tilde{\varphi}|_{C_M^\omega}$ on C_M^ω. He then proved the following interesting property: let

$$\bar{N} := \bar{x} \in \ell^\infty(\mathbb{N},M); \pi_{\mathrm{Gol}}(\bar{x})e_\omega, \pi_{\mathrm{Gol}}(\bar{x}^*)e_\omega \in \mathscr{R}$$

Then $\mathscr{R} = \pi_{\mathrm{Gol}}(\bar{N})e_\omega$, and we have

$$\sigma_t^{\tilde{\varphi}}(\pi_{\mathrm{Gol}}(\bar{x})e_\omega) = \pi_{\mathrm{Gol}}\left(\left(\sigma_t^{\varphi}(x_n)\right)\right)e_\omega, \bar{x} = (x_n)_n \in \bar{N}, t \in \mathbb{R}. \tag{3}$$

Based on the above, he proved that both (the isomorphism class of) C_M^ω and $\dot{\varphi}$ were independent of the choice of φ, and its point spectra characterized Araki's property $L'_\lambda\colon M \mathbin{\overline{\otimes}} R_\lambda \cong M$. Moreover, Golodets and Nessonov proved that its centralizer $(C_M^\omega)_\varphi$ is isomorphic to M_ω. It seems that these works have not been widely recognized, possibly because most of their works were written in Russian. It is not clear from his definition if $\mathscr{R}$ or C_M^{ω} is related to Ocneanu's constructions. We show that Golodets' construction is equivalent to Ocneanu's one.

On the other hand, the development of non-commutative integration theory for von Neumann algebras suggests to seek for a notion of "ultraproduct $\widetilde{\mathrm{M}}^{\omega}$" of M so that the Banach space ultraproduct $(\mathrm{L^p(M)})_{\omega}$ of non-commutative $\mathrm{L^p}$-space for M is isometrically isomorphic to $\mathrm{L^p}(\widetilde{\mathrm{M}}^{\omega})(1 \leq \mathrm{p} < \infty)$. In that viewpoint, it is not the Ocneanu ultraproduct M^{ω} that plays the role. For example, if one uses the Ocneanu ultraproduct, $\mathbb{B}(\mathrm{H})^{\omega} = \mathbb{B}(\mathrm{H})$ holds, while $\mathrm{L}^1(\mathbb{B}(\mathrm{H}))_{\omega} = (\mathbb{B}(\mathrm{H})_*)_{\omega}$ is much larger than $\mathbb{B}(\mathrm{H})_*$ if $\dim(\mathrm{H}) = \infty$. The right definition of the ultraproduct $\widetilde{\mathrm{M}}^{\omega}$ in this context was given by Groh and Raynaud. More precisely, Groh showed that the ultraproduct of the predual M_* of a von Neumann algebra M can be regarded as the predual of some huge von Neumann algebra $\widetilde{\mathrm{M}}^{\omega}$: consider the Banach space ultrapower $(\mathrm{M}_*)_{\omega}(\mathrm{resp.}\,(\mathrm{M})_{\omega})$ of the predual M_* (resp. M), and define a map $\mathrm{J_G}\colon (\mathrm{M}_*)_{\omega} \to ((\mathrm{M})_{\omega})^*$ by

$$x, J_G(\hat{\psi}) := \lim_{n\to\omega} \psi_n(x_n), \qquad x = (x_n)_{\omega} \in (M)_{\omega}, \hat{\psi} = (\psi_n)_{\omega} \in (M_*)_{\omega}. \qquad (4)$$

Then it holds that J_G is an isometric embedding and its range $J_G((M_*)_{\omega})$ is a translation-invariant subspace of $((M)_{\omega})^*$, whence there exists a central projection $z \in ((M)_{\omega})^{**}$ such that $J_G((M_*)_{\omega}) = ((M)_{\omega})^* z$. Therefore $J_G((M_*)_{\omega})$ can be regarded as the predual of the W^*-algebra $((M)_{\omega})^{**}z$. Then almost two decades later, a more handy construction was given by Raynaud: fix a representation π of M on a Hilbert space H so that each $\varphi \in M_*^+$ is represented as a vector functional. Consider the Banach space ultrapower $(M)_{\omega}$ and regard it as a C^*-subalgebra of $\mathbb{B}(H)_{\omega}$. Define $J_R\colon \mathbb{B}(H)_{\omega} \to \mathbb{B}(H_{\omega})$ (H_{ω} is the ultrapower Hilbert space of H) by

$$J_R(x)\xi := (x_n \xi_n)_{\omega}, \qquad x = (x_n)_{\omega} \in \mathbb{B}(H)_{\omega}, \xi = (\xi_n)_{\omega} \in H_{\omega}. \qquad (5)$$

Then it holds that $(M_*)_{\omega}$ is isometrically isomorphic to the predual of the von Neumann algebra $\widetilde{M}^{\omega}$ generated by $J_R((M)_{\omega})$. We write $\widehat{M}_{\omega}$ as $\prod^{\omega} M$ (where we choose the standard representation) and call it the Groh–Raynaud ultrapower of M. Raynaud also showed that $\prod^{\omega} M$ has such nice behaviors as $L^p(M)_{\omega} \cong L^p(\prod^{\omega} M)$ completely isometrically, and $\prod^{\omega} M = (\prod^{\omega} M)'$. The Groh–Raynaud ultrapower was effectively used in e.g., Junge's work on Fubini's Theorem. On the other hand the Groh–Raynaud ultrapower has drawbacks too. In general, even if M has separable predual, $\prod^{\omega} M$ is not even σ-finite (there is no faithful normal state), while M^{ω} is always σ-finite when M is. Moreover, the center of $\prod^{\omega} M$ can be much larger than M^{ω}: for example, Raynaud showed that $\prod^{\mathcal{U}} \mathbb{B}(H)(\dim(H) = \infty)$, is not semifinite for a free ultrafilter $\mathcal{U}$ on a suitable index set I. It seems that there has been no attempts to consider the relationships among the Ocneanu ultraproducts, the Groh–Raynaud ultraproducts and Golodets' asymptotic algebras.

We show that all these ultraproducts are closely related, and the study of one helps that of the other in an essential way. Using the connection, we show some interesting phenomena of the Ocneanu ultraproducts of type III factors which do not appear in the tracial case.

Question (4.1.1) [4]:

Does $M_{\omega} = \mathbb{C}$ imply $M' \cap M^{\omega} = \mathbb{C}$?

We give (Theorem (4.3.3)) an affirmative answer to the question for separable predual case. Moreover, we show that for a σ-finite type III_0 factor M, $M_{\omega} = M \cap M^{\omega}$ holds (Proposition (4.3.4)).

We consider the following questions:

Question (4.1.2) [4]:

Let $(\varphi_n)_n$ be a sequence of normal faithful states on a σ-finite factor M.

(i) Are M^{ω} and $\prod^{\omega} M$ factor too? If so, what are their types?
(ii) Does $(M, \varphi_n)^{\omega}$ depend on the choice of $(\varphi_n)_n$?
(iii) Is $(M, \varphi_n)^{\omega}$ (semi-) finite if M is (semi-) finite?
(iv) Is $(M, \varphi_n)^{\omega}$ of type III if M is of type III?

For (i), if M is of finite type, it is well-known that $M\omega$ is also a finite type factor. Also, it is known that M^{ω} is a type I_{∞} (resp. type II_{∞}) factor if so is M (Proposition (4.3.5)). However, the situation for the Groh–Raynaud ultrapower is different: we show that $\prod^{\omega} R$ is not semifinite (and not a factor), where R is the hyperfinite type II_1 factor (Theorem (4.3.8)). Type III case is more interesting: we show that if M is a σ-finite type $III_{\lambda} (0 < \lambda \leq 1)$ factor, then both M^{ω} and $\prod^{\omega} M$ are type III_{λ} factors (Theorem (4.3.13)). On the other hand, if M is of type III_0, then M^{ω} is not a factor (Theorem (4.3.25)). Moreover, $\prod^{\omega} M$ has a semifinite component and is not a factor (Remark (4.3.21)). As for (ii), we show that if M is of type III_{λ} $(0 < \lambda \leq 1)$, then $(M, \varphi_n)^{\omega} \cong M^{\omega}$ and therefore $(M, \varphi_n)^{\omega}$ does not depend on $(\varphi_n)_n$ (Theorem (4.3.13)). However, regarding (iii), (iv), there exists $(\varphi_n)_n$ such that $(R, \varphi_n)^{\omega}$ is not semifinite (Proposition (4.3.7)). Also, if M is of type III_0, then there exists $(\varphi_n)_n$ such that $(M, \varphi_n)^{\omega} \cong (M_{\varphi_n})^{\omega}$ is of finite type (Theorem (4.3.18)). Finally, let us remark that our ultraproduct analysis has been used for the recent study of QWEP von Neumann algebras and Effros–Maréchal topology on the space of von Neumann algebras.

First we fix a notation and recall basics facts about ultraproducts. Throughout the paper, ω denotes the fixed free ultrafilter on ℕ (in fact, many of the results and proofs are the same for free ultrafilters on any set). For a von Neumann algebra M on a Hilbert space $\mathrm{H}, \mathrm{Z}(\mathrm{M})$ denotes the center of M, and $\mathrm{S_n(M)}$ (resp. $\mathrm{S_{nf}(M)}$) denotes the space of normal (resp. normal faithful) states on M. As usual, we define two seminorms $\|\cdot\|_{\varphi}, \|\cdot\|_{\varphi}^{\#}$, for $\varphi \in \mathrm{S_n(M)}$ by

$$\|x\|_{\varphi} := \varphi(x^* x)^{\frac{1}{2}}, \qquad \|x\|_{\varphi}^{\#} := \varphi(x^* x + x x^*)^{\frac{1}{2}}, \qquad x \in M. \tag{6}$$

If M is σ-finite and φ is faithful, $\|\cdot\|_{\varphi}$ (resp. $\|\cdot\|_{\varphi}^{\#}$) defines the strong (resp. strong*) topology on the unit ball of M. The support projection of a normal state φ is written as $\mathrm{supp}(\varphi)$. For a projection $\mathrm{p} \in \mathrm{M}, z_{\mathrm{M}}(\mathrm{p})$ denotes the central support of p in $\mathrm{M}. \mathcal{U}(\mathrm{M})$ is the group of unitaries in $\mathrm{M}. \mathrm{vN(H)}$ denotes the space of all von Neumann algebras acting on H. Ball(M) is the closed unit ball of M. For a self-adjoint operator A on H, dom(A) is the domain of definition of $\mathrm{A}, \sigma(\mathrm{A})$ (resp. $\sigma_{\mathrm{p}}(\mathrm{A})$) denotes the spectra (resp. point spectra) of A. The range (resp. the domain) of A is written as ran(A) (resp.dom(A)). $\mathrm{G(A)} = \{(\xi, \mathrm{A}\xi); \xi \in \mathrm{dom(A)}\}$ is the graph of A. We denote the sequence of elements of a set like $\{\mathrm{a_n}\}_{\mathrm{n}=1}^{\infty}$. However, we also use the notation $(\mathrm{a_n})_{\mathrm{n}}$ when we think of the sequence as an element in an algebra such as $\ell_{\infty}(\mathbb{N}, \mathrm{B}\mathbb{B}(\mathrm{H}))$. For a unit vector $\xi \in \mathrm{H}$, the corresponding vector state is denoted as ωξ.

Let $(E_n)_n$ be a sequence of Banach spaces, and let $\ell_{\infty}(\mathbb{N}, E_n)$ be the Banach space of all sequences $(x_n)_n^{\bullet} \in \prod_{n=1}^{\infty} E_n$ with $\sup_{n \geq 1} \|x_n\| < \infty$ with the norm $\|(a_n)_n\| = \sup_{n \geq 1} \|a_n\|, (a_n)_n \in \ell^{\infty} \mathbb{N}, E_n)$. The Banach space ultraproduct

$(E_n)_\omega$ is defined as the quotient $\ell^\infty(\mathbb{N}, E_n)_n/\mathcal{J}_\omega$, where $\mathcal{J}_\omega$ is the closed subspace of all $(x_n)_n \in \ell^\infty(\mathbb{N}, E_n)$ which satisfy $\lim_{n\to\omega}\|x_n\| = 0$. An element of $(E_n)_\omega$ represented by $(x_n)_n \in \infty(\mathbb{N}, E)$ is written as $(x_n)_\omega$. One has $\|(x_n)_\omega\| = \lim_{n\to\omega}\|x_n\|$, $(x_n)_\omega \in (E_n)_\omega$. If $(H_n)_n$ is a sequence of Hilbert spaces, then $(H_n)_\omega$ is again a Hilbert space with the inner product given by

$$\langle(\xi_n)_\omega, (\eta_n)_\omega\rangle = \lim_{n\to\omega}\langle\xi_n, \eta_n\rangle, \qquad (\xi_n)_\omega, (\eta_n)_\omega \in (H_n)_\omega. \tag{7}$$

For a sequence $(A_n)_n$ of C^*-algebras, $(A_n)_\omega$ is again a C^*-algebra when equipped with the pointwise multiplication and involution of sequences. However, the Banach space ultraproduct of von Neumann algebras is not a von Neumann algebra in general.

We make a brief summary of modular theory needed for our purpose. In particular we omit the modular theory for weights/Hilbert algebras, which will be used only for Proposition (4.2.24) and Theorem (4.3.25). Let M be a σ-finite von Neumann algebra, and let $\varphi \in S_{\mathrm{nf}}(M)$. Using GNS representation $(M, \pi_\varphi, H, \xi_\varphi), \varphi$ is represented as a vector state ω_{ξ_φ} and $\xi_\varphi \in H$ is a cyclic and separating vector for M (we identify $x \in M$ with $\pi_\varphi(x)$). Then the following operator S_φ^0,

$$\mathrm{dom}(S_\varphi^0) := M\xi_\varphi, \qquad S_\varphi^0 x\xi_\varphi := x^*\xi_\varphi, \qquad x \in M, \tag{8}$$

is a densely defined anti-linear operator on H. Since S_φ^0 is closable, we may consider the polar decomposition $S_\varphi = J_\varphi\Delta_\varphi^{\frac{1}{2}}$ of its closure. It can be shown that J_φ is an anti-linear involution and Δ_φ is a positive, invertible self-adjoint operator on H. Furthermore, $J_\varphi\xi_\varphi = \Delta_\varphi\xi_\varphi = \xi_\varphi$ and $J_\varphi\Delta_\varphi J_\varphi = \Delta_\varphi^{-1}$ hold. J_φ (resp. Δ_φ) is called the modular conjugation operator (resp. modular operator) of φ. Tomita's fundamental Theorem states that

$$J_\varphi M J_\varphi = M', \qquad \Delta_\varphi^{it} M \Delta_\varphi^{-it} = M \text{ for all } t \in \mathbb{R}. \tag{9}$$

Therefore

$$\sigma_t^\varphi(x) := \Delta_\varphi^{it}\, x\Delta_\varphi^{-it}, \qquad x \in M, t \in \mathbb{R}, \tag{10}$$

defines a one parameter automorphism group of M, called the modular automorphism group of φ.

Next we recall Arveson–Connes' spectral theory for automorphism groups. Since we apply the theory only to modular automorphism group, we present the case of oneparameter automorphism group only. In the sequel we identify the dual group $\widehat{\mathbb{R}}$ of the additive group $\mathbb{R}$ with itself. For $f \in L^1(\mathbb{R})$, we define the Fourier transform $\hat{f}$ by

$$\hat{f}(\lambda) := \int_{\mathbb{R}} e^{it\lambda} f(t)\ dt, \qquad \lambda \in \widehat{\mathbb{R}} = \mathbb{R}. \tag{11}$$

We also define $\sigma_f^{\varphi}(x) := \int_{\mathbb{R}} f(t)\sigma_t^{\varphi}(x)dt(x \in M)$.

(i). For $x \in M, \mathrm{S}_{\mathrm{p}_{\sigma^{\varphi}}}(x)$ is defined by

$$\left\{\lambda \in \widehat{\mathbb{R}}; f(\lambda) = 0 \text{ for all } f \in L^1(\mathbb{R}) \text{ with } \sigma_f^{\varphi}(x) = 0\right\}.$$

(ii). The Arveson spectrum of σ^{φ}, denoted by $\mathrm{Sp}(\sigma^{\varphi})$ is the set

$$\left\{\lambda \in \widehat{\mathbb{R}}; f(\lambda) = 0 \text{ for all } f \in L^1(\mathbb{R}) \text{ with } \sigma_f^{\varphi} = 0\right\}.$$

It is shown that $\mathrm{Sp}(\sigma^{\varphi}) = \log(\sigma(\Delta_{\varphi})\backslash\{0\})$.

(iii). For a subset E of R, the spectral subspace of σ^{φ} corresponding to E is given by

$$M(\sigma^{\varphi}, E) := \{x \in M; \mathrm{Sp}_{\sigma^{\varphi}}(x) \subset E\}$$

The fixed point subalgebra $M(\sigma^{\varphi}, \{0\})$ is called the centralizer of φ, and is written as $M\phi$. It is known that $M_{\varphi} = \{x \in M; \varphi(xy) = \varphi(yx), y \in M\}$, and it is always a finite von Neumann algebra with a normal faithful trace $\varphi|_{M_{\varphi}}$. The spectral subspaces have the following properties:

(i) $(M(\sigma^{\varphi}, E)^* = M(\sigma^{\varphi}, -E)$.
(ii)$M(\sigma^{\varphi}, E)M(\sigma^{\varphi}, F) \subset M(\sigma^{\varphi}, \overline{E + F})$.

(iii) $\lambda \in \mathrm{Sp}(\sigma^{\varphi})$ if and only if $M(\sigma^{\varphi}, E) \neq \{0\}$ for any closed neighborhood E of λ.

(iv). The Connes spectrum of σ^{φ}, denoted by $\Gamma(\sigma^{\varphi})$, is given by

$$\Gamma(\sigma^{\varphi}) = \bigcap_{e \in \mathrm{Proj}(M_{\varphi})} \mathrm{Sp}\,(\sigma^{\varphi_e}).$$

Here, for $e \in \mathrm{Proj}(M_{\varphi}), \sigma^{\varphi_e}$ is the restricted action of σ^{φ} to the reduced algebra Me, which coincides with the modular automorphism group of $\varphi_{|_{M_e}}$. It holds that

$$\Gamma(\sigma^{\varphi}) = \bigcap_{0 \neq e \in \mathrm{Proj}\left(\mathcal{Z}(M_{\varphi})\right)} \mathrm{Sp}(\sigma^{\varphi}),$$

whence $\Gamma(\sigma^{\varphi}) = \mathrm{Sp}(\sigma^{\varphi})$ if M_{φ} is a factor.

(v). Let M be a σ-finite factor. The Connes S-invariant is defined by

$$S(M) = \bigcap_{\varphi \in S_{\mathrm{nf}}(M)} \varphi\left(\Delta_{\varphi}\right).$$

It is shown that $S(M)\backslash\{0\}$ is a closed multiplicative subgroup of $\mathbb{R}_{+}^{*} = (0, \infty)$, and $\Gamma(\sigma^{\varphi}) = \log(S(M)\backslash\{0\})$. A σ-finite type III factor M is called of

(i) type III_0 if $S(M) = \{0,1\}$.
(ii) type III_{λ} if $S(M) = \{\lambda^n;\ n \in \mathbb{Z}\} \cup \{0\} (0 < \lambda < 1)$.
(iii) type III_1 if $S(M) = [0, \infty)$.

For general factors, one needs to use normal faithful semifinite weights to define the S-invariant. However, the above classification of type III factors will not be affected by this change.

We first introduce the Ocneanu ultraproduct of a family of von Neumann algebras along ω, with respect to a sequence of their states. This is a slight generalization of the construction of Ocneanu for a single algebra with a single state, and of the construction for tracial states; both generalize classical notions studied by Sakai [44] and McDuff.

Specifically, let $(M_n)_n$ be a sequence of σ-finite von Neumann algebras, and let φ_n be a normal faithful state on M_n for each $n \in \mathbb{N}$. With a slight abuse of notation, put

$$\ell^\infty(\mathbb{N}, M_n) \coloneqq \left\{(x_n)_n \in \prod_{n\in\mathbb{N}} M_n\,; \sup_{n\in\mathbb{N}}\|x_n\| \leq \infty\right\}, \tag{12}$$

$$I_\omega(M_n, \varphi_n) \coloneqq \left\{(x_n)_n \in \ell^\infty(\mathbb{N}, M_n); \lim_{n\to\omega}\|x_n\|_{\varphi_n}^\# = 0\right\}, \tag{13}$$

and also, with the abbreviated notation I_ω for $I_\omega(M_n, \varphi_n)$, let

$$\mathcal{M}^\omega(M_n, \varphi_n) := (x_n)_n \in \ell^\infty(\mathbb{N}, M_n); (x_n)_n I_\omega \subset I_\omega, \text{and } I_\omega(x_n)_n \subset I_\omega. \tag{14}$$

It is then apparent that $\mathcal{M}^\omega(M_n, \varphi_n)$ is a C^*-algebra (with pointwise operations and supremum norm) in which $I_\omega(M_n, \varphi_n)$ is a closed ideal. We then define

$$(M_n, \varphi_n)^\omega := \mathcal{M}^\omega(M_n, \varphi_n)/I_\omega(M_n, \varphi_n) \tag{15}$$

Proposition (4.1.3) [4]:

With the above notation, $(M_n, \varphi_n)^\omega$ is a W^*-algebra.

We remark that Proposition (4.1.17) below gives an alternative proof of Proposition (4.3.1):

We denote the image of $(x_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$ in $(M_n, \varphi_n)^\omega$ as $(x_n)^\omega$. Then we have the following.

Proposition (4.1.4) [4]:

The following defines a normal faithful state $(\varphi_n)^\omega$ on $(M_n, \varphi_n)^\omega$:

$$(\varphi_n)^\omega((x_n)^\omega) := \lim_{n\to\omega} \varphi_n(x_n)\,, \qquad (x_n)^\omega \in (M_n, \varphi_n)^\omega. \tag{16}$$

The special case considered by Ocneanu is the following: all M_n are equal to a fixed von Neumann algebra M, and all φ_n are equal to a fixed normal faithful state φ on M. In this case, we denote $(M_n, \varphi_n)^\omega$ by M^ω, since the latter algebra does not depend on φ (in fact, $I_\omega(M_n, \varphi_n)$ determines the same set of bounded sequences for different state φ); we also denote $(\varphi_n)^\omega$ by φ^ω.

In this section, we define Groh–Raynaud's ultraproduct of a sequence of von Neumann algebras, which is in a rather direct way related to the ultraproduct of C^*-algebras and Hilbert spaces.

Let $(H_n)_n$ be a sequence of Hilbert spaces, and let $H_\omega := (H_n)_\omega$. Let $(\mathbb{B}(H_n))_\omega$ be the Banach space ultraproduct of $(\mathbb{B}(H_n))_n$.

Definition (4.1.5) [4]:

Define $\pi_\omega : (\mathbb{B}(H_n))_\omega \to \mathbb{B}(H_\omega)$ by

$$\pi_\omega((a_n)_\omega)(\xi_n)_\omega := (a_n\xi_n)_\omega, \qquad (a_n)_n \in \ell^\infty(\mathbb{N}, \mathbb{B}(H_n)), \ (\xi_n)_\omega \in H_\omega$$

It is easy to check that $\pi_\omega((a_n)_\omega)$ is a well-defined *-homomorphism, and since

$$\|\pi_\omega((a_n)_\omega)\| = \lim_{n\to\omega}\|a_n\| = \|(a_n)_\omega\|, (a_n)_\omega \in (\mathbb{B}(H_n))_\omega, \qquad (17)$$

π_ω is injective.

Lemma (4.1.6) [4]:

$\pi_\omega((\mathbb{B}(H_n))_\omega)$ is strongly dense in $\mathbb{B}(H_\omega)$.

Proof:

Let $\xi = (\xi_n)_\omega \in H_\omega$ and let $p_n \in \mathbb{B}(H_n)$ be the projection onto $\text{span}(\xi_n) \subset H_n$. Then $p := \pi_\omega((p_n)_\omega)$ is the projection onto $\text{span}(\xi) \subset H_\omega$, as for any $\eta = (\eta_n)_\omega \in H_\omega$ and $\zeta = (\zeta_n)_\omega \in H_\omega$, we have:

$$\begin{aligned}\langle p\eta, \zeta\rangle &= \lim_{n\to\omega}\langle (p\eta)_n, \zeta_n\rangle = \lim_{n\to\omega}\langle p_n\eta_n, \zeta_n\rangle \\ &= \lim_{n\to\omega}\langle \eta_n, \xi_n\rangle\langle \xi_n, \zeta_n\rangle = \langle \eta, \xi\rangle\langle \xi, \zeta\rangle \\ &= \langle\langle \eta, \xi\rangle\xi, \zeta\rangle. \qquad (18)\end{aligned}$$

This shows that any rank one projection in $\mathbb{B}(H_\omega)$ is contained in the subalgebra $\pi_\omega(\mathbb{B}(H_n)_\omega)$. Therefore $\pi_\omega((\mathbb{B}(H_n))_\omega)$ generates $\mathbb{B}(H_\omega)$ as a von Neumann algebra.

Definition (4.1.7) [4]:

Let $(M_n)_n$ be a sequence of W^*-algebras.

(i) Let $M_n \subset \mathbb{B}(H_n)$ be a fixed faithful representation of M_n on a Hilbert space H_n. The abstract ultraproduct of the sequence $(M_n, H_n)_n$ is defined as the strong operator closure of $\pi_\omega((M_n)_\omega)$ in $\mathbb{B}(H_\omega)$, and is denoted as $\prod^\omega (M_n, H_n)$.

(ii) The Groh–Raynaud ultraproduct of $(M_n)_n$, denoted simply as $\prod^\omega M_n$ is defined as $\prod^\omega M_n := \prod^\omega (M_n, H_n)$, where we choose the standard representation of M_n.

From Lemma (4.1.6), it follows that

$$\prod^{\omega}(\mathbb{B}(H_n), H_n) = \mathbb{B}(H_\omega).$$

However, note that the Groh–Raynaud ultraproduct $\prod^\omega \mathbb{B}(H_n)$ is not equal to $\mathbb{B}(H_\omega)$.

Remark (4.1.8) [4]:

Let H be a separable infinite-dimensional Hilbert space. We remark that although $\pi_\omega(\mathbb{B}(H)_\omega)$ is strongly dense in $\mathbb{B}(H_\omega), \pi_\omega$ is not surjective.

To see this, using the weak compactness of the unit ball of H_ω, define $P \in \mathbb{B}(H_\omega)$ by

$$P(\xi_n)_\omega := (\xi)_\omega, \qquad \xi := \text{weak} - \lim_{n\to\omega} \xi_n\,, (\xi_n)_\omega \in H_\omega.$$

P is well-defined and is bounded, because for each n ∈ ℕ we have

$$\begin{aligned} |P(\xi_n)_\omega, P(\xi_n)_\omega| = |\xi\,, \xi| &= \lim_{k\to\omega} \lim_{n\to\omega} |\xi_k, \xi_n| \\ &\le \|(\xi_n)_\omega\|^2. \end{aligned} \tag{19}$$

Therefore $P \in \mathbb{B}(H_\omega)$. It is easy to see that $P^2 = P$ holds. We show that $P \notin \pi(\mathbb{B}(H)_\omega)$. Assume by contradiction that there is a bounded sequence $(p_n)_n \in \ell^\infty(\mathbb{N}, \mathbb{B}(H))$ such that $\pi_\omega((p_n)_\omega) = P$ holds. This means that if a bounded sequence $(\xi_n)_n$ in H converges weakly to $\xi \in H$, then $\|p_n\xi_n - \xi\| \to 0 (n \to \omega)$. In particular, $p_n \to 1 (n \to \omega)$ strongly.

Step 1. We first show that $P = P^*$, hence P is a projection (onto the closed subspace H of H_ω). Let $(\xi_n)_n, (\eta_n)_n \in \infty(\mathbb{N}, H)$ and let $\xi = \text{weak} - \lim_{n\to\omega} \xi_n, \eta = \text{weak} - \lim_{n\to\omega} \eta_n$. We have

$$\langle P(\xi_n)_\omega, (\eta_n)_\omega\rangle = \lim_{n\to\omega}\langle \xi, \eta_n\rangle = \langle \xi , \eta\rangle$$

$$= \lim_{n\to\omega}\langle \xi_n, \eta\rangle = \langle (\xi_n)_\omega, P(\eta_n)\omega\rangle, \tag{20}$$

whence $P = P^*$ holds.

Step 2. There exists a sequence $(\eta_n)_n$ of unit vectors in H such that $\{n \in \mathbb{N}; \eta_n \in \text{ran}(p_n)\} \in \omega$ and $\eta_n \to 0 (n \to \omega)$ weakly.

To see this, fix an orthonormal base (en)n of H. Since $\lim_{n\to\omega}\|p_n e_1 - e_1\| = 0$, we have

$$\left\{n \in \mathbb{N}; \|p_n e_1 - e_1\| < \frac{1}{3}\right\} \subset \left\{n \in \mathbb{N};\ \|p_n - e_1\| \geq \frac{1}{2}\right\} =: I_1 \in \omega. \tag{21}$$

Define $l_n := \max\{1 \leq\ j\ \leq\ n;\ \|p_n e_j\| \geq \frac{1}{2}\}$ for each $n \in I_1$. We then define $(\eta_n)_n$ by

$$\eta_n := \begin{cases} \dfrac{p_n e_{l_n}}{\|p_n e_{l_n}\|} & (n \in I_1), \\ e_1 & (n \notin I_1). \end{cases}$$

Next, suppose $i \geq 1$ and $\varepsilon > 0$ are given. Let

$$I_2 := n\ \in \mathbb{N};\ \|p_n e_i\ -\ e_i\| <\ \varepsilon/2 \in\ \omega.$$

Since $\lim_{n\to\omega} p_n = 1$ strongly, the set I_3 defined by

$$I_3 :=\ \{n\ \in \mathbb{N};\ l_n\ >\ i\}$$

belongs to ω as well. Then for each $n \in I \coloneqq I_1 \cap I_2 \cap I_3 \in \omega$, we have

$$\|\eta_n, e_i\| \leq \frac{1}{\|p_n e_{l_n}\|}\{|e_{l_n}, p_n e_i\ -\ e_i|\ +\ |e_{l_n}, e_i|\} \tag{22}$$

$$\leq 2p_n e_i\ -\ e_i <\ \varepsilon.$$

This shows that $\lim_{n\to\omega} \eta_n, e_i = 0$. Since $i \in \mathbb{N}$ is arbitrary, we obtain the claim.

Step 3. We get a contradiction.

Since $P = \pi((p_n)_\omega)$ is a projection, we may choose pn to be a projection for all $n \in \mathbb{N}$. By Step 2, there exists a sequence of unit vectors $(\eta_n)_n$ such that $J := \{n \in \mathbb{N}; \eta_n \in \operatorname{ran}(p_n)\} \in \omega$ and $weak - \lim_{n\to\omega} \eta_n = 0$. Then, by definition, we have $P(\eta_n)_\omega = 0$. However, for $n \in J$, we have $\|p_n\eta_n\| = \|\eta_n\| = 1$, hence $p_n\eta_n$ does not tend to 0 along ω. This is a contradiction. Hence P is not in the range of π.

As we have seen, there are two notions of ultraproducts for von Neumann algebras. The following theorem explains the relation between the Ocneanu ultraproduct and the Groh–Raynaud ultraproduct:

Theorem (4.1.9) [4]:

Let $(\mathrm{M_n})_\mathrm{n}$ be a sequence of σ-finite von Neumann algebras and let a normal faithful state φ_n on $\mathrm{M_n}$ be given for each $\mathrm{n} \in \mathbb{N}$. Assume that each M_n acts standardly on $\mathrm{H_n} = \mathrm{L}^2(\mathrm{M_n}, \varphi_\mathrm{n})$, so that $\prod^\omega \mathrm{M_n} \subset \mathbb{B}((\mathrm{H_n})_\omega)$. Also let $\mathrm{M}^\omega = (\mathrm{M_n}, \varphi_\mathrm{n})^\omega, \varphi^\omega = (\varphi_\mathrm{n})^\omega$, and define $\mathrm{w}\colon \mathrm{L}^2(\mathrm{M}^\omega, \varphi^\omega) \to (\mathrm{H_n})_\omega$ by

$$w(x_n)^\omega \xi_{\varphi^\omega} := (x_n \xi_{\varphi^n})_\omega, (x_n)^\omega \in M^\omega. \tag{23}$$

Then w is an isometry, and $w^*(\prod^\omega M_n)w = M^\omega$.

To ease notation, let $N = \prod^\omega M_n$ in the sequel. That w is indeed an isometry is seen by direct calculation. To show the identity $w^*Nw = M^\omega$, we need to study the following subsets of $\prod_{n\in\mathbb{N}} M_n$ (for which we use the indicated short notation):

$$\ell^\infty := \ell^\infty(\mathbb{N}, M_n),$$

$$\mathcal{L}_\omega := \left\{(x_n)_n \in \ell^\infty; \lim_{n\to\omega} \varphi_n(x_n^* x_n) = 0\right\}, \mathcal{L}_\omega^* := (x_n^*)_n; (x_n)_n \in \mathcal{L}_\omega,$$

$$\mathcal{M}^\omega := \mathcal{M}^\omega(M_n, \varphi_n), \qquad I_\omega := I_\omega(M_n, \varphi_n).$$

Proof:

First, observe that for $(x_n)_n \in \mathcal{M}^\omega$ and $(y_n)^\omega \in \mathcal{M}^\omega$, we have

$$\pi_\omega((x_n)_\omega)w(y_n)^\omega \xi_{\varphi^\omega} = (x_n y_n \xi_{\varphi_n})_\omega = w(x_n y_n)^\omega \xi_{\varphi^\omega} = w(x_n)^\omega (y_n)^\omega \xi_{\varphi^\omega},$$

so $\pi_\omega((x_n)_\omega)w = w(x_n)^\omega$. Hence $\mathcal{M}^\omega \subset w^*Nw$. To prove $w^*Nw \subset \mathcal{M}^\omega$, it is enough toshow that $w^*\pi_\omega((x_n)_\omega)w \in \mathcal{M}^\omega$ for $(x_n)_n \in \ell^\infty$. Let $(x_n)_n \in \ell^\infty$. By Proposition (4.1.16), we have that $(x_n)_n \in \mathcal{M}^\omega + \mathcal{L}_\omega + \mathcal{L}_\omega^*$. Furthermore, by the above, $w^*\pi_\omega((x_n)_\omega)w \in \mathcal{M}^\omega$if $(x_n)_n \in \mathcal{M}^\omega$. Therefore it suffices to show that $w^*\pi_\omega((x_n)_\omega)w \in \mathcal{M}^\omega$ when $(x_n)_n \in \mathcal{L}_\omega$. But if $(x_n)_n \in \mathcal{L}_\omega$ and $(y_n)_n \in \mathcal{M}^\omega$, we have $(x_n y_n)_n \in \mathcal{L}_\omega$ by Lemma (4.1.13)(i), and so

$$\pi_\omega((x_n)_\omega)w(y_n)^\omega \xi_{\varphi^\omega} = (x_n y_n \xi_{\varphi_n})_\omega = 0,$$

so $w^*\pi_\omega((x_n)_\omega)w = w^* \cdot 0 = 0 \in \mathcal{M}^\omega$.

In this section, we will show (Proposition (4.1.30)) that $ww^* = q$, where $q = pJ_\omega pJ_\omega$, and $w\mathcal{M}^\omega w^* = q(\prod^\omega M_n)q$. Here, J_ω is the ultraproduct of $(J_{\varphi_n})n$. The following result will be used later.

Lemma (4.1.10) [4]:

$\mathcal{L}_\omega$ is a closed left ideal of ℓ^∞, and $I_\omega = \mathcal{L}_\omega \cap \mathcal{L}_\omega^*$.

Proof:

It is easy to see that $\mathcal{L}_\omega$ is a closed subspace of ℓ^∞. Let $(x_n)_n \in \mathcal{L}_\omega$ and $(a_n)_n \in \ell^\infty$. Then we have $\varphi(x_n^* a_n^* a_n x_n) \le \|a_n\|^2 \varphi(x_n^* x_n) \xrightarrow{n\to\omega} 0$. Therefore $(a_n x_n)_n \in \mathcal{L}_\omega$ and $\mathcal{L}_\omega$ is a closed left ideal of ℓ^∞. The last claim is obvious.

Before going further, we prove a result about hereditary C^*-subalgebras. Recall the following

Theorem (4.1.11) [4]:

Let A be a C^*-algebra. If L is a closed left ideal in A, then $L \cap L^*$ is a hereditary C^*-subalgebra of A. The map $L \to B(L) := L \cap L^*$ is a bijection from the set of closed left ideals of A onto the set of hereditary C^*-subalgebras of A. The inverse of the map is given by $B \mapsto L(B)$, where B is ahereditary C^*-subalgebra of A and

$$L(B) := \{a \in A; a^*a \in B\}.$$

Lemma (4.1.12) [4]:

Let A be a C^*-algebra, and let L be a closed left ideal of A. Let $B = L \cap L^*$ be the corresponding hereditary C^*-subalgebra of A, and let M be the two-sided multiplier of B:

$$M := \{a \in A; aB \subset B, Ba \subset B\}.$$

Then we have

(i) $LM \subset L, ML^* \subset L^*$.
(ii) $M \cap (L + L^*) = B$.

Proof:

It is easy to see that M is a C^*-subalgebra of A.

(i) Let $a \in L$ and $x \in M$. Then $a^*a \in L^*L \subset L \cap L^* = B$. Therefore $x^*a^*ax \in B$. By Theorem (4.1.11), $L = L(B)$ implies that $ax \in L$. Therefore $LM \subset L$. Taking the adjoint,we obtain $ML^* \subset L^*$.

(ii) We show the claim in two steps.

Step (1): $M \cap L = M \cap L^* = B$.

Since M and B are self-adjoint, it suffices to show that $\mathrm{M} \cap \mathrm{L} = \mathrm{B}$. Since B is a C^*-algebra, it is clear that $\mathrm{B} \subset \mathrm{M} \cap \mathrm{L}$. Conversely, suppose $\mathrm{x} \in \mathrm{M} \cap \mathrm{L}$. Then $\mathrm{x}^* \in \mathrm{L}^*$ holds, and hence $\mathrm{x}^*\mathrm{x} \in \mathrm{L}^*\mathrm{L} \subset \mathrm{L} \cap \mathrm{L}^* = \mathrm{B}$. On the other hand, as $\mathrm{x} \in \mathrm{M}$, we have $\mathrm{x}(\mathrm{x} * \mathrm{x})\mathrm{x} * \in \mathrm{B}$, which implies that $\mathrm{xx}^* = \{\mathrm{x}(\mathrm{x}^*\mathrm{x})\mathrm{x}^*\}^{\frac{1}{2}} \in \mathrm{B}$. Then by Theorem (4.1.11), again, $\mathrm{x}^* \in \mathrm{L} = \mathrm{L}(\mathrm{B}) \Leftrightarrow \mathrm{x} \in \mathrm{L}^*$ holds. Hence $\mathrm{x} \in (\mathrm{L} \cap \mathrm{L}^*) = \mathrm{B}$.

Step (2): $M \cap (L + L^*) = B$.

By Step (1), it suffices to show that $M \cap (L + L^*) = (M \cap L) + (M \cap L^*)$. It is clear that $(M \cap L) + (M \cap L^*) \subset M \cap (L + L^*)$. Conversely, suppose $x \in M \cap (L + L^*)$. Thenthere is $y \in L, z \in L^*$ such that $x = y + z$ holds. We show that $y, z \in M$. Let $b \in B$. Then $yb \in L$. Furthermore, $yb = xb - zb$ is in L^*, because $b \in B, x \in M$ implies that $xb \in B = L \cap L^*$ and $z \in L^*$. Therefore $yB \subset B$. On the other hand, $by \in L \cap L *$ (since $y \in L, b \in L *$) holds. Therefore $By \subset B$. This shows that $y \in M$. Similarly, we have $z \in M$. Therefore $M \cap (L + L^*) = (M \cap L) + (M \cap L^*)$ holds. This finishes the proof.

Corollary (4.1.13) [4]:

We have

(i) $\mathcal{L}_\omega \mathcal{M}^\omega \subset \mathcal{L}_\omega, \mathcal{M}^\omega \mathcal{L}_\omega^* \subset \mathcal{L}_\omega^*$.
(ii) $\mathcal{M}^\omega \cap (\mathcal{L}_\omega + \mathcal{L}_\omega^*) = I_\omega$.

Proof:

By Lemma (4.1.10), we can apply Lemma (4.1.12) to $A = \ell^\infty, L = \mathcal{L}_\omega, M = \mathcal{M}^\omega, B = I_\omega$.

Now, let $\xi_\omega := (\xi_{\varphi_n})_\omega \in (H_n)_\omega$ and let

$$\varphi_\omega(x) := \langle x\xi_\omega, \xi_\omega \rangle, \qquad x \in N.$$

Then φ_ω is a normal state on N.

Definition (4.1.14) [4]:

We denote by p the support projection of φ_ω, which is the projection onto $\overline{(\prod^\omega M_n)' \xi_\omega}$.

For simplicity, we shall mostly write $\pi_\omega(x)$ as just x in the following (for $x \in (M_n)_\omega$).

Lemma (4.1.15) [4]:

For al l $x \in N$, there is $(x_n)_n \in \infty$ such that

(i) $x\xi_\omega = (x_n)_\omega \xi_\omega$ and $x^*\xi = (x_n^*)_\omega \xi_\omega$.
(ii) $x - p^\perp x p^\perp = (x_n)_\omega - p^\perp (x_n)_\omega p^\perp$.

Proof:

Consider the following subset of $(H_n)_\omega \oplus (H_n)_\omega$:

$$E := \left\{ (x_n)_\omega \xi_\omega, (x_n^*)_\omega \xi_\omega; (x_n)_n \in \ell^\infty, \sup_{n \geq 1} \|x_n\| \leq 1 \right\}.$$

We claim that E is a closed subset of $(H_n)_\omega \oplus (H_n)_\omega$. Indeed, let (η, ζ) be in the closure of E, and choose a sequence $\{(x_n^k)_n\}_{k=1}^\infty \subset \ell^\infty$ such that $\sup_{n\geq 1}\|x_n^k\| \leq 1$ for all $k \in \mathbb{N}$, and such that

$$\left\|(x_n^k)_\omega \xi_\omega - \eta\right\| \leq 2^{-k-1} \text{ and } \left\|(x_n^k)_\omega^* \xi_\omega - \zeta\right\| \leq 2^{-k-1},$$

for all $\kappa \in \mathbb{N}$. Then in particular we have, for all $\kappa \in \mathbb{N}$:

$$\left\|(x_n^{k+1})_\omega \xi_\omega - (x_n^k)_\omega \xi_\omega\right\| \leq 2^{-k} \text{ and } \left\|(x_n^{k+1})_\omega^* \xi_\omega - (x_n^k)_\omega^* \xi_\omega\right\| \leq 2^{-k},$$

so that if we define

$$F_k := \left\{n \in \mathbb{N};\ \left\|x_n^{k+1}\xi_{\varphi_n} - x_n^k \xi_{\varphi_n}\right\| \leq 2^{-k} \text{and } \left\|(x_n^{k+1})^* \xi_{\varphi_n} - (x_n^k)^* \xi_{\varphi_n}\right\| \leq 2^{-k}\right\},$$

then we have $F_k \in \omega$ for all $k \in \mathbb{N}$. Hence with

$$G_k := \{k, k+1, \dots\} \bigcap_{j=1}^{k} F_j ,$$

we have $G_k \in \omega$ for all $k \in \mathbb{N}$ because ω is free, and $(G_k)_k$ is a decreasing sequence with empty intersection. In particular,

$$\mathbb{N} = (\mathbb{N} \setminus G_1) \sqcup \coprod_{j=1}^{\infty} (G_j \setminus G_{j+1})$$

(disjoint union). Now, define a sequence $(x_n)_n \in \ell^\infty$ by

$$x_n := \begin{cases} x_n^1 \ (n \in \mathbb{N} \setminus G_1), \\ x_n^j \ (n \in G_j \setminus G_{j+1}). \end{cases}$$

Then $\sup_{n\geq 1}\|x_n\| \leq 1$. Fix $k \in \mathbb{N}$. If $n \in G_k$, then as $G_k = \sqcup_{j=k}^\infty (G_j \setminus G_{j+1})$, we may choose $j \geq k$ such that $n \in G_j \setminus G_{j+1}$, so that $x_n = x_n^j$ and as $n \in G_j \subset G_m \subset F_m$ for every $m \leq j$, we therefore have

$$\left\|x_n \xi_{\varphi_n} - x_n^k \xi_{\varphi_n}\right\| = \left\|x_n^j \xi_{\varphi_n} - x_n^k \xi_{\varphi_n}\right\|$$

$$\leq \sum_{m=k}^{j-1} \left\| x_n^{m+1}\, \xi_{\varphi_n} - x_n^{m}\, \xi_{\varphi_n} \right\|$$

$$\leq \sum_{m=k}^{j-1} 2^{-m} \leq 2^{-k+1},$$

for every $n \in G_k$. It follows that $\left\|(x_n)_\omega \xi_\omega - (x_n^k)_\omega \xi_\omega\right\| \leq 2^{-k+1}$, so that

$$\|(x_n)_\omega \xi_\omega - \eta\| \leq \|\, (x_n)_\omega \xi_\omega - (x_n^k)_\omega \xi \,\| + \|(x_n^k)_\omega \xi_\omega - \eta\| \leq 2^{-k+1} + 2^{-k+1}.$$

As $k \in \mathbb{N}$ may be chosen to be arbitrarily big, we conclude that $(x_n)_\omega \xi_\omega = \eta$. The proof that $(x_n^*)_\omega \xi_\omega = \zeta$ is similar. Hence E is closed, as claimed.

We are now ready to prove (i). It clearly suffices to consider $x \in \mathbb{N}$ with $\|x\| \leq 1$. By the definition of the Groh–Raynaud ultraproduct, and Kaplansky's Theorem, we may choose a net $\{(x_n^\alpha)_n\}_\alpha \subset \ell^\infty$ such that $\sup_{n\geq 1} x_n^\alpha \leq 1$ for every α and such that $\lim_\alpha (x_n^\alpha)_\omega = x$ in the strong $*$-topology on N. But then $(x\xi_\omega, x^*\xi_\omega)$ is in the closure of E, hence in E by the previous paragraph, and (i) follows.

Finally, (ii) follows from (i): with x and $(x_n)_n$ from there, we have for all $y \in N'$:

$$xy\xi_\omega = yx\xi_\omega = y(x_n)_\omega \xi_\omega = (x_n)_\omega y\xi_\omega,$$

so $xp = (x_n)_\omega p$, and similarly $x^*p = (x_n^*)_\omega p$. Conjugating the latter identity, $px = p(x_n)_\omega$ holds. Now (ii) follows easily.

Proposition (4.1.16) [4]:

There is a vector space isomorphism

$$\rho: \ell^\infty / I_\omega \to pNp \oplus pNp^\perp \oplus p^\perp Np$$

such that $\rho^{-1}(pNp) = \mathcal{M}^\omega / I_\omega, \rho^{-1}(pNp^\perp) = \mathcal{L}_\omega / I_\omega$, and $\rho^{-1}(p^\perp N\, p) = \mathcal{L}_\omega^* / I_\omega$. In particular, we have

$$\ell^\infty = \mathcal{M}^\omega + \mathcal{L}_\omega + \mathcal{L}_\omega^*.$$

Proof:

Observe first that for $(x_n)_n \in \ell^\infty$, we have

$$(x_n)_n \in \mathcal{L}_\omega \Leftrightarrow \|(x_n)_\omega \xi_\omega\| = 0 \Leftrightarrow (x_n)_\omega \in N\, p^\perp.$$

Hence $(x_n)_n \in \mathcal{L}_\omega^* \Leftrightarrow (x_n)_\omega \in p^\perp N$, so by Lemma (4.1.10),

$$(x_n)_n \in I_\omega = \mathcal{L}_\omega \cap \mathcal{L}_\omega^* \;\Leftrightarrow\; (x_n)_\omega \in p^\perp N\, p^\perp.$$

Hence by letting

$$\rho((x_n)_n/I_\omega) := (x_n)_\omega - p^\perp (x_n)_\omega p^\perp,$$

we obtain a well-defined injective linear map from ℓ^∞/I_ω into $pNp \oplus pNp^\perp \oplus p^\perp N\, p$, and it is in fact surjective by Lemma (4.1.15).

By definition and the above, we have for all $(x_n)_n \in \ell^\infty$:

$$\begin{aligned}\rho((x_n)_n/I_\omega) \in pNp^\perp &\Leftrightarrow (x_n)_\omega - p^\perp (x_n)_\omega p^\perp = p(x_n)_\omega p^\perp \\ &\Leftrightarrow (x_n)_\omega \in Np^\perp \\ &\Leftrightarrow (x_n)_n \in \mathcal{L}_\omega,\end{aligned}$$

and from this,

$$\rho((x_n)_n/I_\omega) \in p^\perp Np \;\Leftrightarrow (x_n^*)_n \in \mathcal{L}_\omega \;\Leftrightarrow\; (x_n)_n \in \mathcal{L}_\omega^*.$$

Therefore we have

$$\rho^{-1}(pNp^\perp) = \mathcal{L}_\omega/I_\omega, \qquad \rho^{-1}(p^\perp Np) = \mathcal{L}_\omega^*/I_\omega. \tag{24}$$

Finally, if $\rho((x_n)_n/I_\omega) \in pNp$, and $(y_n)_n \in I_\omega$, we have $(y_n)_\omega \in p^\perp Np^\perp$, and so

$$\begin{aligned}\rho((x_n y_n)_n/I_\omega) &= (x_n)_\omega (y_n)_\omega - p^\perp (x_n)_\omega (y_n)_\omega p^\perp \\ &= ((x_n)_\omega - p^\perp (x_n)_\omega p^\perp)(y_n)_\omega \\ &= \rho((x_n)_n/I_\omega)(y_n)_\omega = 0,\end{aligned}$$

and therefore $(x_n y_n)_n \in I_\omega$. Similarly $(y_n x_n)_n \in I_\omega$, so $(x_n)_n \in \mathcal{M}^\omega$. This shows $\rho^{-1}(pNp) \subset \mathcal{M}^\omega/I_\omega$. On the other hand, by Eq. (24) and by Corollary (4.1.13)(ii), we have

$$\mathcal{M}^\omega/I_\omega \cap \rho^{-1}(pNp^\perp \oplus\ p^\perp Np) = [\mathcal{M}^\omega \cap (\mathcal{L}_\omega + \mathcal{L}_\omega^*)]/I_\omega$$

$$= \{0\},$$

whence we have $\rho^{-1}(pNp)\ =\ \mathcal{M}^\omega/I_\omega$.

In particular, $\ell^\infty/I_\omega = \mathcal{M}^\omega/I_\omega + \mathcal{L}_\omega/I_\omega + \mathcal{L}_\omega^*/I_\omega$, and the last claim is then obvious.

Proposition (4.1.17) [4]:

Let $(x_n)_n \in \ell^\infty$. Then $(x_n)_n \in \mathcal{M}^\omega$ if and only if $p(x_n)_\omega = (x_n)_\omega p$ holds. Moreover, $\rho|_{\mathcal{M}^\omega}: \mathcal{M}^\omega \to pNp, (x_n)_n/I_\omega \to (x_n)_\omega p$ is a^*-isomorphism. Therefore, the Ocneanu ultraproduct is isomorphic to a reduction of the Groh–Raynaud ultraproduct by the support projection p of $\varphi_\omega \in N_*$.

Proof:

By Proposition (4.1.16), $(x_n)_n/I_\omega \in \mathcal{M}^\omega/I_\omega$ holds if and only if $\rho((x_n)_n/I_\omega)\ = p(x_n)_\omega p + p(x_n)_\omega p^\perp + p^\perp(x_n)_\omega p \in pNp$, if and only if $p(x_n)_\omega p^\perp = p^\perp(x_n)_\omega p = 0$. The last condition is equivalent to $(x_n)_\omega p = p(x_n)_\omega$. Since $\rho|_{\mathcal{M}^\omega}: \mathcal{M}^\omega \to pNp$ is linear andbijective, to prove the last assertion it is enough to show that $\rho|M\omega$ is $a\,*$-homomorphism.Let $(a_n)_n, (b_n)_n \in \mathcal{M}^\omega$. Then as $(a_n)_\omega, (b_n)_\omega$ commute with p, we have

$$\rho((a_n b_n)_n/I_\omega) = (a_n b_n)_\omega p = (a_n)_\omega p (b_n)_\omega p,$$

$$= \rho((a_n)_n/I_\omega)\rho((b_n)_n/I_\omega),$$

$$\rho((a_n^*)_n/I_\omega) = (a_n^*)_\omega p\ = (p(a_n)_\omega)^* = ((a_n)_\omega p)^*$$

$$= \rho((a_n)_n/I_\omega)^*$$

whence $\rho|_{\mathcal{M}^\omega}$ is a^*-isomorphism.

Corollary (4.1.18) [4]:

For any $(a_n)_n \in \ell^\infty$, there exist $(b_n)_n \in \mathcal{M}^\omega, (c_n)_n \in \mathcal{L}_\omega$, and $(dx_n)_n \in L_\omega^*$ such that

(i) $a_n = b_n + c_n + d_n \; for \; n \in \mathbb{N}$.
(ii)$\|(b_n)^\omega\| \lim_{n\to\omega} \|a_n\|$.

Proof:

Since $(x_n)_n \in \ell^\infty$, by Proposition (4.1.16), there exist $(b_n)_n \in \mathcal{M}^\omega, (c_n)_n \in \mathcal{L}_\omega$, and $(d_n)_n \in \mathcal{L}_\omega^*$ such that $a_n = b_n + c_n + d_n$. $(b_n)_n$ is unique modulo I_ω, and since $\rho|_{\mathcal{M}^\omega}: \mathcal{M}^\omega \to pNp$ is a^*-isomorphism (Proposition (4.1.17)), we have

$$\|(b_n)^\omega\| = \|\rho^{-1} p(a_n)_\omega p\| = \|p(a_n)_\omega p\| \le \lim_{n\to\omega} \|a_n\| .$$

Our next step is to show (Theorem (4.1.20) below) that the Groh–Raynaud ultraproduct of a sequence of standard von Neumann algebras is again standard, in such a way that the standard form of the ultraproduct algebra is obtained as an ultraproduct of the standard forms of the sequence.

Definition (4.1.19) [4]:

Let (M, H, J, P) be a quadruple, where M is a von Neumann algebra, H is a Hilbert space on which M acts, J is an antilinear isometry on H with $J^2 = 1$, and $P \subset H$ is a closed convex cone which is self-dual, i.e., $P = P^0$, where

$$P^0 := \{\xi \in H; \langle \xi , \eta \rangle \ge 0, \eta \in P\}.$$

Then (M, H, J, P) is called a standard form if the following conditions are satisfied:

(i) $J\,MJ = M'$.
(ii) $J\xi = \xi, \xi \in P$.
(iii) $xJxJ(P) \subset P, x \in M$.
(iv) $JxJ = x^*, x \in Ƶ(M)$.

Theorem (4.1.20) [4]:

Let $(M_n, H_n, J_n, P_n)_n$ be a sequence of standard forms. Let $H_\omega := (H_n)_\omega$, let J_ω be defined on H_ω^* by

$$J_\omega(\xi_n)_\omega := (J_n\xi_n)_\omega, (\xi_n)_\omega \in H_\omega,$$

and let

$$P_\omega := \{(\xi_n)_\omega \in H_\omega; \xi_n \in P_n \; for \; all \; n \in \mathbb{N}\}.$$

Then the quadruple

$$\left(\prod^{\omega} M_n, H_\omega, J_\omega, P_\omega\right)$$

is again a standard form.

Conditions (ii) and (iii) can be easily verified. For (i), we have to show the Raynaud Theorem that $(\prod^\omega M_n)' = \prod^\omega M_n'$ (Theorem (4.1.24) below). It might look obvious that (iv) holds. However, we will see that $\mathcal{Z}(\prod^\omega M_n)$ is different from $\prod^\omega \mathcal{Z}(M_n)$ in general. Therefore it is not obvious that the equality $J_\omega x J_\omega = x^*$ holds for $x \in \mathcal{Z}(\prod^\omega M_n)$. However, this can be fixed by showing that condition (iv) is redundant.

Proof:

It is clear that P_ω is a closed convex cone. We prove self-duality as follows: assume that $\xi = (\xi_n)_\omega \in P_\omega^0$. For each $n \in \mathbb{N}$, there are $\eta_n^+, \eta_n^-, \zeta_n^+, \zeta_n^- \in P_n$ such that $\eta_n^+ \perp \eta_n^-, \zeta_n^+ \perp \zeta_n^-$ and $\xi_n = \eta_n^+ - \eta_n^- + i(\zeta_n^+ - \zeta_n^-)$. Then by $\xi \in P_\omega^0$, we have

$$\lim_{n\to\omega}\langle \xi_n, \eta_n^- \rangle = -\lim_{n\to\omega}\|\eta_n^-\|^2 + i\lim_{n\to\omega}\langle \zeta_n^+ - \zeta_n^-, \eta_n^- \rangle$$

$$\geq 0.$$

Therefore $(\eta_n^-)\omega = 0$. We also have

$$\lim_{n\to\omega}\langle \xi_n, \zeta_n^\pm \rangle = \lim_{n\to\omega}\langle \eta_n^+, \zeta_n^\pm \rangle \pm i\left\| \zeta_n^\pm \right\|^2$$

$$\geq 0.$$

Therefore $(\zeta_n^\pm)_\omega = 0$, and $(\xi_n)_\omega = (\eta_n^+)_\omega \in P_\omega$, so $P_\omega^0 = P_\omega$. By Theorem (4.1.24), it follows that (i) in Definition (4.1.19) holds for the quadruple

$(\prod^{\omega} M_n, H_\omega, J_\omega, P_\omega)$, and the properties (ii)–(iii) in Definition (4.1.19) are easily checked. By Lemma (4.1.21), the claim follows.

Lemma (4.1.21) [4]:

Let (M, H, J, P) be a quadruple satisfying, conditions (i)–(iii) in Definition (4.1.19). Then (M, H, J, P) satisfies condition (iv), whence it is a standard form.

We use the following Araki's characterization of the modular conjugation operator.

Proof of Lemma (4.1.21):

The proof is in three steps. Throughout, conditions (i)–(iii). in Definition (4.1.19) are assumed to hold.

Step 1: Assume first that M has a cyclic and separating vector $\xi \in P$. Then by Theorem (4.1.22), J is the modular involution associated with ξ. But then (iv) is immediate from Tomita–Takesaki theory.

Step 2: Assume now the slightly more general situation where we have $\xi \in P$ such that $\overline{MM'\xi} = H$. Let e and e be the projections onto $\overline{M'\xi}$ and $\overline{M\xi}$, respectively. If f is a central projection in M, and $f \geq e$, then we have, for all $x \in M$ and $x' \in M'$:

$$f\, xx'\xi = xfx'\xi = xx\xi\ ,$$

and so $f = 1$; it follows that the central support of e is 1, and similarly, it follows that the central support of e' is 1. Moreover, as

$$JM'\xi = JM'J\xi = M\xi,$$

we have that $JeJ = e'$.

Now, let $f := ee$. Then $JfJ = JeJeJ^2 = e'e = ee' = f$, it follows that $(fMf, f(H), J|_{f(H)}, f(P))$ does also satisfy the conditions (i)–(iii) But as

$$\overline{fMf\xi} = \overline{ee'M\xi} = ee'(H) = f(H),$$

and similarly $\overline{fM'f\xi} = f(H)$, we see that ξ is a separating and cyclic vector for fMf, acting on $f(H)$. Hence by Step 1, we have

$J|_{f(H)}dJ|_{f(H)} = d^*$ for all central element d of fMf. But as e and e have central support 1, the map $c \mapsto fcf$ is a^*-isomorphism from the center of M onto the center of fMf. We now prove that 4. holds in the case under consideration: let $c \in M \cap M'$, then $JcJ - c^* \in M \cap M'$, so as $JfJ = f$, we get from the above:

$$0 = J|_{f(H)}f\,cf\,J|_{f(H)} - (f\,cf)^*$$

$$= (Jfc\,J - fc^*f)_{f(H)}$$

$$= f(J\,cJ - c^*)f|_{f(H)},$$

hence $JcJ = c^*$ holds by the injectivity of $c \mapsto fcf$.

Step 3. We now consider the general case. Let $(\xi_\alpha)_\alpha \subset P\backslash\{0\}$ be a maximal familywith respect to the property that $(\overline{MM'\xi_\alpha})_\alpha$ forms an orthogonal family of subspacesof H. Let qα be a projection onto $\overline{MM'\xi_\alpha}$. The projections $(q_\alpha)_\alpha$ are clearly central, and as

$$JMM'\xi_\alpha = (JMJ)(JM'J)J\xi_\alpha = M'M\xi_\alpha = MM'\xi_\alpha,$$

one has also $Jq_\alpha = q_\alpha J$ for all α. Hence with $p := 1 - \sum_\alpha q_\alpha$, we have $JpJ = p$. Now,assume that $p \neq 0$. As P spans H, we may then choose $\eta \in P$ such that $p\eta \neq 0$. Let $\xi = p\eta$. Then

$$\xi = p\eta = p^2\eta = pJpJ\eta \in P,$$

and as $\xi \perp \overline{MM'\xi_\alpha}$ for all α, it is easy to see that $\overline{MM'\xi} \perp \overline{MM'\xi_\alpha}$ for all α. But this contradicts the maximality of $(\xi_\alpha)_\alpha$, so that $p = 0$ and hence $\sum_\alpha q_\alpha = 1$. Now, each of the quadruples

$$\left(q_\alpha M, q_\alpha(H), J|_{q_\alpha(H)}, q_\alpha(P)\right)$$

satisfies (i)–(iii) and the condition considered in Step 2, since $q_\alpha(H) = \overline{MM'\xi_\alpha}$; hence 4.holds for the above quadruple, i.e.,

$$J|_{q_\alpha(H)}c_\alpha J|_{q_\alpha(H)} = c_\alpha^*|_{q_\alpha(H)}$$

whenever c_α is a central element of $q_\alpha M$.

Now, let $c \in M \cap M'$. Then qαc is a central element of $q_\alpha M$, and so

$$JcJ = J\left(\sum_\alpha q_\alpha c q_\alpha\right)J = \sum_\alpha J q_\alpha c q_\alpha J q_\alpha$$

$$= \sum_\alpha J|_{q_\alpha(H)} c q_\alpha J|_{q_\alpha(H)} q_\alpha = \sum_\alpha c^* q_\alpha$$

$$= c^*.$$

Next we show that the Groh–Raynaud ultraproduct preserves commutant. This result was obtained by Raynaud in the case of a constant sequence of algebras.

Theorem (4.1.22) [4]:

Let ξ be a cyclic and separating vector for a von Neumann algebra M on a Hilbert space H. Then a conjugate-linear involution J is the modular conjugation operator associated with the state $\omega_\xi = \langle \cdot\, \xi, \xi \rangle$ if and only if J satisfies the following conditions.

(i) $JMJ = M'$.
(ii) $J\xi = \xi$.
(iii) $\langle \xi, aJaJ\xi \rangle \geq 0$ for all $a \in M$, and equality holds if and only if $a = 0$.

Lemma (4.1.23) [4]:

Let $(\mathrm{H_n})_\mathrm{n}$ be a sequence of Hilbert spaces, and let $\mathrm{M_n} \in \mathrm{vN}(\mathrm{H_n})$ for each $\mathrm{n} \in \mathbb{N}$. Let $\mathrm{H}_\omega = (\mathrm{H_n})_\omega$, and $\mathrm{M} = \prod^\omega(\mathrm{M_n}, \mathrm{H_n})$ and $\mathrm{N} = \prod^\omega(\mathrm{M_n}, \mathrm{H_n})$. For any $\xi \in \mathrm{H}_\omega$ and $\mathrm{a}' \in \mathrm{M}'$, there exists $\mathrm{a} \in \mathrm{N}$ such that $\mathrm{a}\xi = \mathrm{a}'\xi$ and $\|\mathrm{a}\| \leq \|\mathrm{a}'\|$.

Proof:

Let $\xi = (\xi_n)_\omega \in H_\omega$ and let $a' \in M'$; to prove the lemma, we may and do assume that $\|a'\| = 1$. Let $\eta = a'\xi = (\eta_n)_\omega$ and put

$$\varepsilon_n := \sup\{\langle x\eta_n, \eta_n \rangle - \langle x\xi_n, \xi_n \rangle; x \in M_n, 0 \leq x \leq 1\}, n \in \mathbb{N}.$$

Then $\varepsilon_n \geq 0 (n \in \mathbb{N})$, and by weak-compactness of Ball $(M_n) \cap M_n^+$, there is $(x_n)_n \in \prod_{n\in\mathbb{N}} M_n$ such that $0 \leq x_n \leq 1$ and

$$\varepsilon_n = \langle x_n \eta_n, \eta_n \rangle - \langle x_n \xi_n, \xi_n \rangle,$$

for all $n \in \mathbb{N}$. In particular, $\pi_\omega(x) \in M$, where $x = (x_n)_\omega$. Also, we have

$$\lim_{n\to\omega} \varepsilon_n = \langle x\eta, \eta \rangle - \langle x\xi, \xi \rangle = \langle xa'\xi , a'\xi \rangle - \langle x\xi, \xi \rangle \leq 0,$$

and hence $\lim_{n\to\omega} \varepsilon_n = 0$. Moreover, by the definition of $(\varepsilon_n)_n$, we have

$$\omega_{\eta_n}(x) \leq \omega\xi_n(x) + \varepsilon_n, \qquad x \in M_n, 0 \leq x \leq 1, n \in \mathbb{N},$$

so

$$\omega_{\eta_n}\left(\frac{x}{\|x\|}\right) \leq \omega\xi_n\left(\frac{x}{\|x\|}\right) + \varepsilon_n, \qquad x \in M_n^+ \backslash \{0\}, \qquad n \in \mathbb{N},$$

and hence, there exists $(\eta'_n)_n \in \prod_{n\in\mathbb{N}} H_n$ such that $\|\eta_n - \eta'_n\| \leq \varepsilon_n^{\frac{1}{2}}$ and $\omega\eta'_n \leq \omega\xi_n$ for each $n \in \mathbb{N}$. In particular, $(\eta_n)_\omega = (\eta'_n)_\omega$, since $\lim_{n\to\omega} \varepsilon_n = 0$. We then get $(a'_n)_n \in \prod_{n\in\mathbb{N}} M_n$ such that $\|a'_n\| \leq 1$ and $a'_n \xi_n = \eta'_n (n \in \mathbb{N})$. Let $a := \pi_\omega((a'_n)_\omega)$. Then $a \in N$, and

$$a\xi = (a'_n \xi_n)_\omega = (\eta'_n)_\omega = (\eta_n)_\omega = \eta = a'\xi .$$

Also

$$\|a\| = \lim_{n\to\omega} \|a'_n\| \leq 1 = \|a'\|.$$

Theorem (4.1.24) [4]:

Let $(\mathrm{M_n}, \mathrm{H_n}, \mathrm{J_n}, \mathrm{P_n})_\mathrm{n}$ be as in Theorem (4.1.20). Then one has

$$\left(\prod^{\omega} M_n\right)' = \prod^{\omega} M_n .$$

Proof:

Let $A = (M_n)_\omega$ and $B = (M'_n)_\omega$ and identify these with their images under $\pi\omega$.Then $B \subset A'$ is clear, so it suffices to prove that $A' \subset B''$. Let $a' \in A'$ and

$\xi_1, \dots, \xi_m \in H_\omega (m \in \mathbb{N})$. Let F be the type I_m-factor, acting on $K = \mathbb{C}^m$. Using the matrix picture of $M_n \otimes F$, it is clear that

$$(M_n \otimes F)_\omega = A \otimes F \text{ on } H_\omega \otimes K$$

(as *-algebras) and hence

$$((M_n \otimes F)_\omega)' = A' \otimes \mathbb{C}1,$$

as von Neumann algebras. Thus $a' \otimes 1 \in (\prod^\omega (M_n \otimes F, H_n \otimes K))'$, and so by Lemma (4.1.23), there is

$$a \otimes 1 \in \prod^{\omega} ((M_n \otimes F)', H_n \otimes K) = \prod^{\omega} M_n \otimes \mathbb{C}1$$

with $\|a\| \leq \|a'\|$ and

$$(a \otimes 1)(\xi_1, \dots, \xi_m) = (a' \otimes 1)(\xi_1, \dots, \xi_m),$$

hence $a\xi_j = a'\xi_j (j = 1, \dots, m)$. This means that B meets any so-neighborhood of a'. As $a' \in A'$ was arbitrary, we conclude that $A' \subset B''$, as desired.

Theorem (4.1.25) [4]:

Let $(\mathrm{M_n})_\mathrm{n}$ be a sequence of standard von Neumann algebras. Then $(\prod^\omega M_n)_*$ is Banach space isomorphic to the Banach space ultraproduct $((M_n)_*)_\omega$, in such a way that a normal functional on $\prod^\omega M_n$ is implemented by the ultraproduct vectors corresponding to the isomorphic image in $((M_n)_*)_\omega$.

Proof:

Let $(\varphi_n)_\omega \in ((M_n)_*)_\omega$. As each M_n is standard, we have sequences $(\xi_n)_n, (\eta_n)_n \in \prod_{n\in\mathbb{N}} H_n$ such that

$$\varphi_n(x) = \langle x\xi_n, \eta_n \rangle, \qquad x \in M_n, \qquad n \in \mathbb{N},$$

and $\|\varphi_n\| = \|\xi_n\|^2 = \|\eta_n\|^2 (n \in \mathbb{N})$. In particular, both $(\xi_n)_n$ and (η_n) are bounded. Define $\xi_\omega := (\xi_n)_\omega$ and $\eta_\omega := (\eta_n)_\omega$ in $(Hn)\omega$. Then define $\varphi_\omega \in (\prod^\omega M_n)_*$ by

$$\varphi_\omega(x) = \langle x\xi_\omega, \eta_\omega\rangle, \qquad x \in \prod^\omega M_n,$$

and $\|\varphi_\omega\| = \lim_{n\to\omega}\|\varphi_n\|$. Hence $\Phi: ((M_n)_*)_\omega \to (\prod^\omega M_n)_*$ defined by $\Phi((\varphi_n)_\omega) := \varphi_\omega$ is isometric. Note also that for $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$, we have

$$\begin{aligned}\Phi((\varphi_n)_\omega)\big(\pi_\omega((x_n)_\omega)\big) &= \lim_{n\to\omega}\langle x_n\xi_n, \eta_n\rangle \\ &= \lim_{n\to\omega}\varphi_n(x_n). \qquad (25)\end{aligned}$$

Since $\pi_\omega(\ell^\infty(\mathbb{N}, M_n))$ is strongly dense in $\prod^\omega M_n, \Phi((\varphi_n)_\omega)$ is uniquely determined by Eq. (25) and is independent of the choice of $(\xi_n)_n, (\eta_n)_n$. It is clear that $\Phi(\lambda(\varphi_n)_\omega) = \lambda\Phi((\varphi_n)_\omega)$ for $\lambda \in \mathbb{C}$ and $(\varphi_n)_\omega \in ((M_n)_*)_\omega$. Note also that if $(\phi_n)_\omega, (\psi_n)_\omega \in ((M_n)_*)_\omega$, then by Eq. (48), for $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ we have

$$\begin{aligned}\Phi((\varphi_n + \psi_n)_\omega)\big(\pi_\omega((x_n)_\omega)\big) &= \lim_{n\to\omega}(\varphi_n + \psi_n)(x_n) \\ &= \lim_{n\to\omega}\varphi_n(x_n) + \lim_{n\to\omega}\psi_n(x_n) \\ &= [\Phi(\phi_n)_\omega + \Phi((\psi_n)_\omega)]\big(\pi_\omega((x_n)_\omega)\big).\end{aligned}$$

Therefore by the strong density of $\pi_\omega(\ell^\infty(\mathbb{N}, M_n)), \Phi((\varphi_n + \psi_n)_\omega) = \Phi((\varphi_n)_\omega) + \Phi((\psi_n)_\omega)$ holds. Hence Φ is linear. Surjectivity of Φ follows from Theorem (4.1.20) by reversing the above argument. Therefore Φ is an isometric isomorphism.

In the following, $(M_n)_n$ is a sequence of standard von Neumann algebras, and we identify $(\varphi_n)_\omega \in ((M_n)_*)_\omega$ with its image φ_ω in $(\prod^\omega M_n)_*$.

Corollary (4.1.26) [4]:

Let ϕ be a normal state on $\prod^\omega M_n$. Then there are normal states $\varphi_n \in (M_n)_*$ such that $\varphi = (\varphi_n)_\omega$. If all M_n are σ-finite, then we may choose the states φ_n such that they are also faithful.

Proof**:**

Since $\prod^\omega M_n$ is standard (Theorem (4.1.20)), there exists $\xi_\varphi \in P_\omega$ such that $\varphi = \omega_{\xi_\varphi}$. By definition, ξ_φ has a representative $(\xi_n)_n$ where $\xi_n \in P_n$ for all $n \in$

$\mathbb{N}$. As $1 = \|\varphi\| = \|\xi_\varphi\|^2 = \lim_{n\to\omega}\|\xi_n\|^2$, we may choose each ξ_n to be a unit vector. Then $\varphi_n := \omega_{\xi_\varphi} \in S_n(M_n)$, and $\varphi = (\varphi_n)_\omega$. Now suppose each $M_n(n \in \mathbb{N})$ is σ-finite and take $\psi_n \in S_{\rm nf}(M_n)(n \in \mathbb{N})$. Let

$$\varphi_n := \left(1 - \frac{1}{n}\right)\varphi_n + \frac{1}{n}\psi_n, \qquad n \in \mathbb{N}.$$

Then φ'_n is a normal faithful state on M_n for each $n \in \mathbb{N}$, and $(\varphi'_n)_\omega = (\varphi_n)_\omega = \varphi$.

Recall that

Lemma (4.1.27) [4]:

Let (M, H, J, P) be a standard form, p a projection in M, and $q = pJpJ$. Then $(qMq, q(H), q\,J\,q, q(P))$ is standard, and $pMp \ni pxp \mapsto qxq \in qMq$ is an isomorphism.

Therefore by Proposition (4.1.17), Theorem (4.1.20) and Lemmay (4.1.27), we have

Corollary (4.1.28) [4]:

Let $M^\omega = (M_n, \varphi_n)^\omega, \varphi^\omega = (\varphi_n)^\omega, N = \prod^\omega M_n, q = pJ_\omega pJ_\omega$ and $H_{\varphi^\omega} = L^2(M^\omega, \varphi^\omega)$. Then $(M^\omega, H_{\varphi^\omega}, J_{\varphi^\omega}, P_{\varphi^\omega})$ is isomorphic to $(qNq, qH_\omega, qJ_\omega q, qP_\omega)$ as a standard form.

Corollary (4.1.29) [4]:

Under the same notation as in Theorem (4.1.9), the following hold.

(i) $ww^* = q = pJ_\omega pJ_\omega$.
(ii) $wM^\omega w^* = q(\prod^\omega M_n)q$.

Proof:

Let $\xi_\omega = (\xi_{\varphi_n})_\omega$. Then consider the GNS representation π_{φ^ω} of M^ω with respect to φ^ω. Recall also by Proposition (4.1.17) that $\rho_0 := \rho|_{M^\omega} : M^\omega \ni (x_n)^\omega \mapsto (x_n)_\omega p \in p(\prod^\omega M_n)p$ is a^*-isomorphism, so we have another representation λ of M^ω on $q(L^2(M_n, \varphi_n)_\omega)$ given by $\lambda((x_n)^\omega) := q\rho_0((x_n)^\omega)q =$

$q(x_n)^\omega q, (x_n)^\omega \in M^\omega$. Since ξ_ω is cyclic for $q(\prod^\omega M_n)q$, it is cyclic for $\lambda(M^\omega)$, and for $(x_n)^\omega \in M^\omega$,

$$\begin{aligned}\langle \lambda((x_n)^\omega)\xi_\omega, \xi_\omega\rangle &= \langle q(x_n)_\omega q\xi_\omega, \xi_\omega\rangle \\ &= \lim_{n\to\omega}\langle x_n\xi_{\varphi_n}, \xi_{\varphi_n}\rangle \\ &= \varphi^\omega((x_n)^\omega).\end{aligned}$$

Therefore by the proof of the uniqueness of GNS representation, there is a unitary $\widetilde{w}: L^2(M^\omega, \varphi^\omega) \to q(L^2(M_n, \varphi_n))_\omega$ determined by

$$\widetilde{w}\pi^*_{\varphi^\omega}((x_n)^\omega)\xi_{\varphi^\omega} = \lambda((x_n)^\omega)\xi_\omega, \qquad (x_n)^\omega \in M^\omega,$$

which implements the unitary equivalence of π_{φ^ω} and λ. But by Proposition (4.1.17), for $(x_n)^\omega \in M^\omega, (x_n)_\omega q = q(x_n)_\omega$ holds, whence

$$\begin{aligned}\widetilde{w}\pi_{\varphi^\omega}((x_n)^\omega)\xi_{\varphi^\omega} &= q(x_n)_\omega\xi_\omega = (x_n)_\omega\xi_\omega \\ &= w\pi_{\varphi^\omega}(x_n)^\omega,\end{aligned}$$

and $w = \widetilde{w}$ holds. Therefore $ww^* = q$. (2) By Theorem (4.1.9), it holds that $wM^\omega w *= ww^*(\prod^\omega M_n)ww^* = q(\prod^\omega M_n)q$.

The next corollary shows that every normal faithful state on the Ocneanu ultraproduct is the ultraproduct state for some sequence of normal faithful states.

Corollary (4.1.30) [4]:

Under the same notation as in Theorem (4.1.9), let ψ be a normal faithful state on $M^\omega = (M_n, \varphi_n)^\omega$. Then there exist $\psi_n \in S_{\mathrm{nf}}(M_n)(n \in \mathbb{N})$ such that $(M_n, \psi_n)^\omega = M^\omega$ and $\psi = (\psi_n)^\omega$.

Proof:

Let $N := \prod^\omega M_n$. Define the isometry $w: L^2(M^\omega, \varphi^\omega) \to (L^2(M_n, \phi_n))_\omega$ as in Theorem (4.1.9). Let $\hat{\psi}$ be a normal state on N given by

$$\hat{\psi}(x) := \psi(w^*xw), \qquad x \in N.$$

Note that $w^*(\prod^\omega M_n)_w = (M_n, \phi_n)^\omega$ by Theorem (4.1.9). Then $\mathrm{supp}(\hat{\psi})$ is $p = \mathrm{supp}((\varphi_n)_\omega)$. By Corollary (4.1.26), we may choose normal faithful states ψ_n on each M_n such that $\hat{\psi} = (\psi_n)_\omega$. Now by (the proof of) Proposition (4.1.16), for $(x_n)_n \in \ell^\infty \mathbb{N}, M_n)$ we have

$$(x_n)_n \in I_\omega(M_n, \phi_n) \Leftrightarrow (x_n)_\omega \in p^\perp N p^\perp$$

$$\Leftrightarrow (x_n)_n \in I_\omega(M_n, \psi_n),$$

so $I_\omega(M_n, \phi_n) = I_\omega(M_n, \psi_n)$, which implies that $(M_n, \phi_n)^\omega = (M_n, \psi_n)^\omega$. Recall (Corollary (4.1.29)) also that $\Phi: (M_n, \psi_n)^\omega x \to wxw^* \in qNq$ gives a^*-isomorphism such that $(\psi n)\omega|_{qNq} \circ \Phi = (\psi_n)^\omega$. Therefore for $x \in (M_n, \psi_n)^\omega = (M_n, \phi_n)^\omega$, we have

$$(\psi_n)^\omega(x) \;=\; \hat{\psi}(wxw^*) = \;\psi(w^*wxw^*w) = \;\psi(x).$$

In this section, we describe Golodets' construction of the asymptotic algebra C_M^ω from our viewpoint. Let M be a σ-finite von Neumann algebra, and let $\varphi \in S_{\mathrm{nf}}(M)$. Consider the GNS representation of M associated with ϕ, so that $\varphi = \omega_{\xi_\varphi}$ with a cyclic and separating vector ξ_φ on a Hilbert space H. Consider the following state $\bar{\varphi}$ on $\ell^\infty = \ell^\infty(\mathbb{N}, M)$:

$$\bar{\varphi}((x_n)_n) := \lim_{n\to\omega} \varphi(x_n)\,, \qquad (x_n)_n \in \ell^\infty(\mathbb{N}, M).$$

Let $\pi_{\mathrm{Gol}}: \ell^\infty \to \mathbb{B}(H_{\mathrm{Gol}})$ be the GNS representation of $\bar{\varphi}$ with cyclic vector $\bar{\xi}$ satisfying $\bar{\varphi} = \omega_{\bar{\xi}}$. Let eω be the projection of H_{Gol} onto $\overline{\pi_{\mathrm{Gol}}(\ell^\infty)'\bar{\xi}}$. Define

$$\mathscr{R} := e_\omega \pi_{\mathrm{Gol}}(\ell^\infty)'' e_\omega \subset \mathbb{B}(e_\omega H_{\mathrm{Gol}}).$$

Let $\bar{N}$ be the set of all $\bar{x} = (x_n)_n \in \ell^\infty$ for which $\pi_{\mathrm{Gol}}(\bar{x})e_\omega \in \mathscr{R}$ and $\pi_{\mathrm{Gol}}(\bar{x}^*)e_\omega \in \mathscr{R}$ hold. Then $\bar{N}$ is a $C*$-subalgebra of ℓ^∞. Moreover,

$$I := \left\{x = (x_n)_n \in \bar{N};\; \lim_{n\to\omega} \varphi(x_n^* x_n) = 0\right\}$$

is a closed two-sided ideal in $\bar{N}$, and $\pi_{\mathrm{Gol}}(\bar{N})e_\omega = \mathscr{R} \cong \bar{N}/I$. Let $\bar{M}_d$ be the subspace of $\ell^\infty(\mathbb{N}, M)$ consisting of constant sequences $(x, x, \ldots)_n, x \in M$. The asymptotic algebra C_M^ω of M is defined by

$$C_M^{\omega} := \mathscr{R} \cap \pi_{\mathrm{Gol}}(\bar{M}_d)' \subset \mathbb{B}(e_\omega H_{\mathrm{Gol}}).$$

We show that Golodets' construction is equivalent to Ocneanu's construction.

Lemma (4.1.31) [4]:

Let $\bar{x} = (x_n)_n \in \ell^\infty(\mathbb{N}, M)$, and let $q = pJ_\omega pJ_\omega$. Then $\bar{x} \in \mathcal{M}^\omega$ if and only if $q(x_n)_\omega = (x_n)_\omega q$.

Proof:

If $\bar{x} \in \mathcal{M}^\omega$, then $p(x_n)_\omega = (x_n)_\omega p$ holds by Proposition (4.1.17). Since $p, (x_n)_\omega (\in \prod^\omega M)$ commute with $J_\omega pJ_\omega$, we have $(x_n)_\omega q = q(x_n)_\omega$. Conversely, suppose that $(x_n)_\omega = q(x_n)_\omega$ holds. This implies that

$$(x_n^*)_\omega p^\perp J_\omega pJ_\omega = J_\omega pJ_\omega p^\perp (x_n^*)_\omega. \tag{26}$$

Let $(x_n)_\omega \in I_\omega$. We have to show that $(x_n a_n)_n^* \in \mathcal{L}_\omega$ and $(a_n x_n)_n \in \mathcal{L}_\omega$. By $(a_n x_n)_n \in \ell^\infty$, Eq. (26) and $(a_n^*)_\omega = (a_n^*)_\omega p^\perp$, we have

$$\begin{aligned}(a_n^* x_n^*)_\omega \xi_\omega &= J_\omega pJ_\omega (a_n^* x_n^*)_\omega \xi_\omega = J_\omega pJ_\omega (a_n^*)_\omega p^\perp (x_n^*)_\omega \xi_\omega \\ &= (a_n^*)_\omega J_\omega pJ_\omega p^\perp (x_n^*)_\omega \xi_\omega \\ &= (a_n^*)_\omega (x_n^*)_\omega J_\omega pJ_\omega p^\perp \xi_\omega = 0,\end{aligned}$$

hence $(a_n^* x_n^*)_n \in \mathcal{L}_\omega \cdot (a_n x_n)_n \in \mathcal{L}_\omega$ is proved similarly. Therefore $(x_n a_n)_n, (a_n x_n)_n$ belong to I_ω and $(x_n)_n \in \mathcal{M}^\omega$ holds.

Theorem (4.1.32) [4]:

$\bar{N} = \mathcal{M}^\omega$, $R \cong M^\omega$ and $C_M^\omega \cong M' \cap M^\omega$ hold.

Let H_ω be the ultrapower of H, and let $\pi_\omega : \ell^\infty \to \mathbb{B}(H_\omega)$ be the ultrapower map $(a_n)_n \mapsto (a_n)_\omega$ (we identify $(a_n)_\omega$ with its image in $\mathbb{B}(H_\omega)$ as before), and let $\xi_\omega := (\xi_\varphi)_\omega \in H_\omega$. Let $J = J_\varphi$ be the modular conjugation and J_ω be its ultrapower. For $\bar{x} \in \ell^\infty$, we have

$$\bar{\varphi}(\bar{x}) = \lim_{n \to \omega} \langle x_n \xi_\varphi, \xi_\varphi \rangle = \langle (x_n)_\omega \xi_\omega, \xi_\omega \rangle.$$

Therefore by the uniqueness of GNS representation, we may identify

$$H_{\mathrm{Gol}} = \overline{\pi_\omega \ell^\infty \xi_\omega}, \qquad \bar{\xi} = \xi_\omega, \qquad \pi_{\mathrm{Gol}}(\bar{x}) = \pi_\omega(x)|_{H_{\mathrm{Gol}}} \ (\bar{x} \in \ell^\infty).$$

Recall that we defined a projection p of H_ω onto $\overline{(\prod^\omega M)'\xi_\omega}$ (see Definition (4.1.14)). Then $J_\omega p J_\omega$ is the projection of H_ω onto $\overline{(\prod^\omega M)\xi_\omega}$. We use such abbreviation as $\mathcal{M}^\omega, I_\omega, \mathcal{L}_\omega$ given.

Proof:

Let e'_{Gol} be the projection of H_ω onto H_{Gol}. By Lemma (4.1.15)(ii), we have

$$e'_{\mathrm{Gol}} H_\omega = \overline{\pi_\omega(\ell^\infty)\xi_\omega} = \overline{\pi_\omega(\ell^\infty)''\xi_\omega} = J_\omega p J_\omega H_\omega.$$

Therefore $e'_{\mathrm{Gol}} = J_\omega p J_\omega$. Furthermore, as $\pi_{\mathrm{Gol}} = \pi_\omega|_{H_{\mathrm{Gol}}}$, we have

$$e_\omega H_{\mathrm{Gol}} = \overline{\pi_{\mathrm{Gol}}(\ell^\infty)'\xi_\omega} = \overline{e'_{\mathrm{Gol}}\pi_\omega(\ell^\infty)' e'_{\mathrm{Gol}}\xi_\omega}$$

$$= e'_{\mathrm{Gol}}\pi_\omega(\ell^\infty)'^{\xi_\omega} = J_\omega p J_\omega p H_\omega$$

$$= q H_\omega.$$

Therefore it holds.

$$\mathscr{R} = e_\omega \pi_{\mathrm{Gol}}(\ell^\infty)''^{e_\omega} = q\pi_\omega(\ell^\infty)'' q$$

$$= q\left(\prod^\omega M\right) q \cong M^\omega.$$

We now show that $\bar{N} = \mathcal{M}^\omega$.

Suppose $\bar{x} \in \mathcal{M}^\omega$. Then by Lemma (4.1.31),

$$\pi_{\mathrm{Gol}}(\bar{x}) e_\omega = \pi_\omega(\bar{x}) q = q\pi_\omega(\bar{x}) \in \mathscr{R}.$$

Similarly $\pi_{\mathrm{Gol}}(\bar{x}^*) e_\omega \in \mathscr{R}$ holds, and $\bar{x} \in \bar{N}$. Conversely, suppose $\bar{x} \in \bar{N}$. Then $\pi_{\mathrm{Gol}}(\bar{x}) e_\omega \in \mathscr{R}$, so

$$\pi_{\mathrm{Gol}}(\bar{x}) e_\omega = e_\omega \pi_{\mathrm{Gol}}(\bar{x}),$$

whence $\pi_\omega(\bar{x}) q = q\pi_\omega(\bar{x})$. By Lemma (4.1.31), $\bar{x} \in \mathcal{M}^\omega$ holds. Therefore $\bar{N} = \mathcal{M}^\omega$. Note that by Corollary (4.1.13), this also shows that

$$I = \overline{N} \cap \mathcal{L}_\omega = \mathcal{M}^\omega \cap L_\omega = I_\omega.$$

Finally, as the constant sequence M in M^ω is mapped to $\pi_{\mathrm{Gol}}(\overline{M}_d)$ under the isomorphism $M^\omega q(\prod^\omega M)q$, we see that $C_M^\omega \cong M' \cap M^\omega$. This finishes the proof.

Section (4.2): Theory of Ultraproduct

In this section we show that the ultraproduct action of the modular automorphism group on the Ocneanu ultraproduct is still continuous. This is the key result for all the subsequent analysis. In the case of constant algebras, similar results were obtained by Golodets for his auxiliary algebra $\mathscr{R}$, and by Raynaud for the corner $p(\prod^{\omega} M)p$ which corresponds to M^{ω} (see Proposition (4.1.17). Here, p is the support projection of $(\varphi_n)_{\omega}$ as in Definition (4.1.14)).

We prove next the corresponding result for a general sequence of σ-finite von Neumann algebras with normal faithful states.

Lemma (4.2.1) [4]:

Let $(a_n)_n \in \prod_{n\in\mathbb{N}} \mathbb{B}(H_n)_{sa}$ with spectra satisfying $\sigma(a_n) \subset [0,1]$ and $0,1 \notin \sigma_p(a_n), n \in \mathbb{N}$. Then $a(a_n)_{\omega} \in \mathbb{B}(H_{\omega})$ satisfies $0 \leq a \leq 1$. Moreover, if $K \subset H_{\omega}$ is a closed subspace invariant under a, and with

$$K \cap \operatorname{Ker}(a) = \{0\} = K \cap \operatorname{Ker}(1-a), \tag{27}$$

then for every bounded continuous function g on $(0,1)$, we have:

(i) K is invariant under $(g(a_n)_{\omega}$;

(ii) $(g(a_n))_{\omega}|_K = g(a|_K)$.

From the assumption and Eq. (27), $\sigma(a|_K) \subset [0,1]$ and $0,1 \notin \sigma_p(a|_K)$, so $g(a|_K)$ is well-defined.

Proof:

From the identification $a = \pi_{\omega}((a_n)_{\omega})$, it is straightforward that $0 \leq a \leq 1$, also that

$$f((a_n)_{\omega}) = f(a) \tag{28}$$

for every polynomial f on $[0,1]$, hence (by Weierstrass' Theorem) for every continuous function on $[0,1]$.

Now, let p denote the projection of H_{ω} onto K. As $a(K) \subset K, p$ commutes with a, hence with all spectral projections of a. In particular, K is perpendicular to both

$\mathrm{Ker}(a)$ and $\mathrm{Ker}(1-a)$, so $K \subset 1_{(0,1)}(a)(H_\omega)$, where 1_X denotes the indicator function of $X \subset \mathbb{R}$. Hence $K = \bigvee_{0<\varepsilon<\frac{1}{2}} K_\varepsilon$, where $K_\varepsilon = 1_{(\varepsilon,1-\varepsilon)}(a)(H_\omega) \cap K$.

Fix now $\varepsilon \in (0,\frac{1}{2})$ and fix a continuous function f_ε on $[0,1]$ with

$$f_\varepsilon\left(\left[0,\frac{\varepsilon}{2}\right] \cup \left[1-\frac{\varepsilon}{2},1\right]\right) = \{0\}, \qquad f_\varepsilon(\varepsilon, 1-\ \varepsilon] = \{1\}.$$

Also, fix $\xi = (\xi_n)_\omega \in K_\varepsilon$ and a continuous function g on $(0,1)$. Choose a continuous function h on $[0,1]$ such that $g(t) = h(t)$ whenever $\frac{\varepsilon}{2} \le t \le 1-\frac{\varepsilon}{2}$. Also, let $\xi_n' := f_\varepsilon(a_n)\xi_n, n \in \mathbb{N}$. Then

$$(\xi_n')_\omega = (f_\varepsilon(a_n)\xi_n)_\omega = f_\varepsilon(a)\xi\ = \xi, \tag{29}$$

where we used Eq. (28) in the second last equality, and in the last equality that $\xi \in 1_{(\varepsilon,1-\varepsilon)}(a)(H_\omega)$ and $f_\varepsilon 1_{(\varepsilon,1-\varepsilon)} = 1_{(\varepsilon,1-\varepsilon)}$. Because $g = h$ on the support of f_ε, we have

$$g(a_n)\xi_n' = h(a_n)\xi_n', \qquad n \in \mathbb{N}. \tag{30}$$

Also, as p commutes with $1_{(\varepsilon,1-\varepsilon)}(a)$, and $g = h$ on $(\varepsilon, 1-\varepsilon)$, one has

$$\begin{aligned} g(a|_K)\xi\ &=\ g(a|_K)1_{(\varepsilon,1-\varepsilon)}(a)p\xi \\ &=\ g(a|_K)1_{(\varepsilon,1-\varepsilon)}(a|_K)p\xi \\ &=\ h(a|K)\xi\ , \end{aligned}$$

but as h is continuous on $[0,1]$ (hence bounded) and $\xi \in K$, this entails

$$g(a|_K)\xi = h(a)\xi. \tag{31}$$

Now we get, by Eqs., (30), (29), (28) and (31) respectively:

$$\begin{aligned} g(a_n)_\omega\xi\ &= (g(a_n)\xi_n')_\omega = (h(a_n)\xi_n')_\omega \\ &= \big(h(a_n)\big)_\omega\xi = h(a)\xi = g(a|K)\xi\ . \end{aligned}$$

As $K = 0 < \varepsilon < 12K\varepsilon$, we now get (i) and (ii).

Lemma (4.2.2) [4]:

For each $n \in \mathbb{N}$, let Δn be a positive self-adjoint (possibly unbounded) operator on H_n, such that $0 \notin \sigma_p(\Delta_n)$. Let $a = ((1 + \Delta_n)^{-1})_\omega \in \mathbb{B}(H_\omega)$, and let $K \subset H_\omega$ be a closed subspace which is invariant under a. If Δ is a positive self-adjoint operator on K with $0 \notin \sigma_p(\Delta)$ and with

$$(1 + \Delta)^{-1} = a|_K,$$

then

$$\left(\Delta_n^{it}\right)_\omega|_K = \Delta^{it}, \qquad t \in \mathbb{R}.$$

Proof:

Let $a_n := (1 + \Delta n)^{-1}, n \in \mathbb{N}$. Define

$$g_t(x) := (x^{-1} - 1)^{it}, \qquad x \in (0,1), t \in \mathbb{R}.$$

As $0,1 \notin \sigma_p(a|_K) = \sigma_p((1 + \Delta)^{-1})$, Lemma (4.2.2) gives

$$(g_t(a_n))_\omega|_K = g_t(a|_K), \qquad t \in \mathbb{R}.$$

This shows the claim, as

$$g_t(a_n) = \Delta_n^{it}, \qquad n \in \mathbb{N}, t \in \mathbb{R},$$

and

$$g_t(a|_K) = g_t((1 + \Delta)^{-1}) = \Delta^{it}, \qquad t \in \mathbb{R}.$$

Lemma (4.2.3) [4]:

Let e, f be projections on a real Hilbert space such that $e \wedge f + e^\perp \wedge f = f$. Then $ef = fe$ holds.

Proof:

Note that $e \wedge f = f(e \wedge f)f \leq fef$ and similarly $e^\perp \wedge f \leq fe^\perp f$. Moreover, $fef + f e^\perp f = f$. Hence $e \wedge f + e^\perp \wedge f = f$ implies that $e \wedge f = fef$ and $e^\perp \wedge f = fe^\perp f$. It follows that efe is a positive self-adjoint operator whose square

$(efe)^2 = efefe = fef$ is the projection $e \wedge f$, whence efe itself is the projection $e \wedge f$. It then holds that

$$(ef - fe)^*(ef - fe) = fef - efef - fefe + efe$$

$$= e \wedge f - e(e \wedge f) - (e \wedge f)e + e \wedge f$$

$$= 0,$$

whence $ef = fe$ holds.

Theorem (4.2.4) [4]:

Let $\{(M_n, \phi_n)\}_{n=1}^{\infty}$ be a sequence of von Neumann algebras with faithful normal states. Let $M^\omega = (M_n, \phi_n)^\omega, \varphi^\omega = (\varphi_n)^\omega$. Then

$$\sigma_t^{\varphi^\omega}((x_n)^\omega) = \left(\sigma_t^{\varphi_n}(x_n)\right)^\omega, t \in \mathbb{R}, (x_n)^\omega \in M^\omega.$$

In particular, $t \mapsto (\sigma_t^{\varphi_n})^\omega$ is a continuous flow on $(M_n, \phi_n)^\omega$.

This is not an obvious result as it might look for the first sight. Indeed, it is known that the ultrapower of a continuous action of a topological group on a von Neumann algebra M is often discontinuous on M^ω.

To prove Theorem (4.2.4), we need preparations. Consider a sequence $(H_n = L^2(M_n, \phi_n))_n$ of standard Hilbert spaces, let $H_\omega = (H_n)_\omega$, and as before we identify $(a_n)_\omega$ with $\pi_\omega((a_n)_\omega) \in \mathbb{B}(H_\omega)$ for every $(a_n)_\omega \in (\mathbb{B}(H_n))_\omega$.

Proof:

Consider for each $n \in \mathbb{N}$ the standard representation of M_n on $H_n = L^2(M_n, \phi_n)$, and write $N = \prod^\omega M_n$ for simplicity. Define $H_\omega := (H_n)_\omega$ and $J_\omega := (J_{\varphi_n})_\omega$. Let $\varphi_\omega := (\varphi_\omega)_\omega \in ((M_n)_*)_\omega \cong N_*$ and let $p := \mathrm{supp}(\varphi_\omega) \in \prod^\omega M_n$ (see Theorem (4.1.25)). By Proposition (4.1.30), we have $M^\omega \cong qNq = wM^\omega w^*$ with $q := pJ_\omega pJ_\omega$, and with this identification, we have

$$L^2(M^\omega, \varphi^\omega) \cong qH_\omega, \qquad J_{\varphi^\omega} \cong qJ_\omega q = pJ_\omega p.$$

Let S_{φ_n} (resp. F_{φ_n}) be the closure of the closable (conjugate-linear) operator $S_{\varphi_n}^0$ (resp. $F_{\varphi_n}^0$) on H_n defined by

$$\mathrm{dom}\left(S^0_{\varphi_n}\right) = M_n\xi_{\varphi_n}, \qquad S^0_{\varphi_n}x\xi_{\varphi_n} := x^*\xi_{\varphi_n}(x \in M_n),$$

$$\mathrm{dom}\left(F^0_{\varphi_n}\right) = M'_n\xi_{\varphi_n}, \qquad F^0_{\varphi_n}y\xi_{\varphi_n} := y^*\xi_{\varphi_n}(y \in M'_n),$$

Since F_{φ_n} is the adjoint of S_{φ_n}, they are the adjoint of each other with respect to the real Hilbert space structure of H_n. Therefore, we have the following decomposition as a real Hilbert space:

$$H_n \oplus_{\mathbb{R}} H_n = G\left(S_{\varphi_n}\right) \oplus_{\mathbb{R}} VG\left(F_{\varphi_n}\right), \qquad n \in \mathbb{N},$$

where $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $G(T)$ is the graph of a closed operator T. Taking the ultraproduct (as a real Hilbert space), we obtain

$$H_\omega \oplus_{\mathbb{R}} H_\omega = \left(G\left(S_{\varphi_n}\right)\right)_\omega \oplus_{\mathbb{R}} V_\omega \left(G\left(F_{\varphi_n}\right)\right)_\omega, \tag{32}$$

where $V_\omega = (V)_\omega$. Let $\tilde{\varphi}_\omega := \varphi_\omega|_{qNq} \in (qNq)_*$ be the image of φ^ω under the isomorphism $M^\omega \cong qNq$. Let $x \in qNq$. Then by Lemma (4.1.15)(i) and Proposition (4.1.17), there exists $(x_n)_n \in \mathcal{M}^\omega$ such that $x = (x_n)_\omega q = q(x_n)_\omega$. Therefore it holds that

$$\begin{aligned} S_{\tilde{\varphi}_\omega}x\xi_\omega &= S_{\tilde{\varphi}_\omega}(x_n)_\omega\xi_\omega \\ &= (x_n^*)_\omega\left(\xi_{\varphi_n}\right)_\omega \\ &= (S_{\varphi_n}x_n\xi_{\varphi_n})_\omega, \end{aligned}$$

which shows that $(x\xi_\omega, S_{\tilde{\varphi}_n}x\xi_\omega) \in qH_\omega \oplus_{\mathbb{R}} qH_\omega \cap (G(S_{\varphi_n}))_\omega$. Doing similar computationsfor $F_{\tilde{\varphi}_\omega}$, we obtain

$$G(S_{\tilde{\varphi}_\omega}) \subset \left(G\left(S_{\varphi_n}\right)\right)_\omega \cap (qH_\omega \oplus_{\mathbb{R}} qH_\omega), \tag{33}$$

$$V_\omega G(F_{\tilde{\varphi}_\omega}) \subset V_\omega \left(G\left(F_{\varphi_n}\right)\right)_\omega \cap (qH_\omega \oplus_{\mathbb{R}} qH_\omega). \tag{34}$$

Similarly, using $F_{\tilde{\varphi}_\omega} = (S_{\tilde{\varphi}_\omega})^*$, we have

$$G(S_{\tilde{\varphi}_\omega}) \oplus_{\mathbb{R}} V_\omega G(F_{\tilde{\varphi}_\omega}) = qH_\omega \oplus_{\mathbb{R}} qH_\omega. \tag{35}$$

Let E be the real orthogonal projection of $H_\omega \oplus_{\mathbb{R}} H_\omega$ onto $(G(S_{\varphi_n}))_\omega$, and let $F := q \oplus q$. By Eq. (32), $E^\perp$ is the real orthogonal projection onto $V_\omega G(F_{\tilde{\varphi}_\omega})$. By Eqs. (33), (34) and (35), we have

$$ran(E \wedge F) \supset G\big(S_{\tilde{\varphi}_\omega}\big),$$

$$ran(E^\perp \wedge F) \supset V_\omega G(F_{\tilde{\varphi}_\omega}),$$

$$ran(F) = qH_\omega \oplus_{\mathbb{R}} qH_\omega.$$

Let P, Q be real orthogonal projections from $H_\omega \oplus_{\mathbb{R}} H_\omega$ onto $G(S_{\tilde{\varphi}_\omega})$ and $V_\omega G(F_{\tilde{\varphi}_\omega})$, respectively. Then $P \leq E \wedge F, Q \leq E^\perp \wedge F$. On the other hand, by Eq. (35) we have

$$P + Q = F \geq E \wedge F + E^\perp F.$$

Therefore it follows that $P = E \wedge F, Q = E^\perp \wedge F$ and $E \wedge F + E^\perp \wedge F = F$. Therefore by Lemma (4.2.3), E and F commute. Let E_n be the real orthogonal projection of $H_n \oplus_{\mathbb{R}} H_n$ onto $G(S_{\varphi_\omega})(n \in \mathbb{N})$. Then E is the ultraproduct of $(E_n)_n$, and we know that

$$E_n = \begin{pmatrix} (1 + \Delta_{\varphi_n})^{-1} & J_{\varphi_n}(\Delta_{\varphi_n}^{\frac{1}{2}} + \Delta_{\varphi_n}^{-\frac{1}{2}})^{-1} \\ J_{\varphi_n}(\Delta_{\varphi_n}^{\frac{1}{2}} + \Delta_{\varphi_n}^{-\frac{1}{2}})^{-1} & (1 + \Delta_{\varphi_n})^{-1} \end{pmatrix} \quad (n \in \mathbb{N}).$$

Let $a_n := (1 + \Delta_{\varphi_n})^{-1}, b_n := J_{\varphi_n}(\Delta_{\varphi_n}^{\frac{1}{2}} + \Delta_{\varphi_n}^{-\frac{1}{2}})^{-1} \in \mathbb{B}(H_n)$, and let $a_\omega := (a_n)_\omega, b_n := (b_n)_\omega \in \mathbb{B}(H_\omega)$ (b_n, b_ω are regarded as real linear operators). Then it holds that

$$E = \begin{pmatrix} a_\omega & b_\omega \\ b_\omega & a_\omega \end{pmatrix}.$$

Since $E = (E_n)_\omega$ commutes with $F = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$, a_ω commutes with q and qH_ω is a_ω-invariant. Therefore we see that EF is the projection of $H_\omega \oplus_{\mathbb{R}} H_\omega$ onto $G(S_{\tilde{\varphi}_\omega})$, which is of the following form:

$$EF = E \wedge F = \begin{pmatrix} (1+\Delta_{\tilde{\varphi}_\omega})^{-1}q & J_{\tilde{\varphi}_\omega}(\Delta_{\tilde{\varphi}_\omega}^{\frac{1}{2}} + \Delta_{\tilde{\varphi}_\omega}^{-\frac{1}{2}})^{-1}q \\ J_{\tilde{\varphi}_\omega}(\Delta_{\tilde{\varphi}_\omega}^{\frac{1}{2}} + \Delta_{\tilde{\varphi}_\omega}^{-\frac{1}{2}})^{-1}q & (1+\Delta_{\tilde{\varphi}_\omega})^{-1}q \end{pmatrix} \quad (n \in \mathbb{N})$$

This shows that

$$a\omega|_{qH_\omega} = (1+\Delta_{\tilde{\varphi}_\omega})^{-1}. \qquad (36)$$

Now by Lemma (4.2.4), we have

$$\left(\Delta_{\varphi_n}^{it}\right)^\omega|_{qH_\omega} = \Delta_{\tilde{\varphi}_\omega}^{it}, \qquad t \in \mathbb{R}. \qquad (37)$$

From this equality, we have that $(\sigma_t^{\varphi_n})^\omega = \sigma_t^{\varphi_\omega}$ for all $t \in \mathbb{R}$ because φ^ω corresponds to $\tilde{\varphi}_\omega$ under the identification $M^\omega \cong qNq$.

Example (4.2.5) [4]:

Let $0 < \lambda < 1$ and $R_\lambda = \otimes_{n=1}^\infty (M_2(\mathbb{C}), \tau_\lambda)$ be the Powers factor of type III_λ, where $\tau_\lambda = \mathrm{Tr}(\rho_\lambda \cdot), \rho_\lambda = \mathrm{diag}(\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda})$. The modular automorphism group of $\varphi = \otimes_{n=1}^\infty \tau_\lambda$ is given by $\otimes_{n=1}^\infty \sigma_t^{\tau_\lambda}$, where

$$\sigma_t^{\tau_\lambda}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & \lambda^{it}b \\ \lambda^{-it}c & d \end{pmatrix}, \qquad a,b,c,d \in \mathbb{C}, \qquad t \in \mathbb{R}.$$

Define

$$x_n := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes n} \otimes 1 \otimes 1 \cdots \in R_\lambda, \qquad n \in \mathbb{N}.$$

It is clear that $(x_n) \in \ell^\infty(\mathbb{N}, R_\lambda)$. We see that

$$\left\|\sigma_t^\varphi(x_n) - x_n\right\|_\varphi^2 = \left\|\begin{pmatrix} 0 & \lambda^{it} \\ \lambda^{-it} & 0 \end{pmatrix}^{\otimes n} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes n}\right\|_{\tau_\lambda^{\otimes n}}^2$$

$$= 2 - \tau_\lambda^{\otimes n}\left[\begin{pmatrix} \lambda^{it} & 0 \\ 0 & \lambda^{-it} \end{pmatrix}^{\otimes n} + \begin{pmatrix} \lambda^{-it} & 0 \\ 0 & \lambda^{it} \end{pmatrix}^{\otimes n}\right]$$

$$= 2 - \left(\frac{\lambda^{it+1} + \lambda^{-it}}{1+\lambda}\right)^n - \left(\frac{\lambda^{-it+1} + \lambda^{it}}{1+\lambda}\right)^n.$$

It follows that since

$$\left|\frac{\lambda^{it+1} + \lambda^{-it}}{1+\lambda}\right|^2 = \left|\frac{\lambda + \lambda^{-2it}}{1+\lambda}\right|^2$$

$$= \frac{\lambda^2 + 2\lambda\cos(2t\log\lambda) + 1}{\lambda^2 + 2\lambda + 1}.$$

the second term tends to zero as $n \to \infty$ whenever $|t|$ is small but nonzero, say $0 < |t|\pi/(6|\log\lambda|)$. The same happens for the third term, and we see that $\lim_{n\to\omega}\left\|\sigma_t^{\varphi}(x_n) - x_n\right\|_{\varphi} = \sqrt{2}$ for small enough $|t| \neq 0$. This shows that

$$\lim_{t\to 0}\lim_{n\to\omega}\left\|\sigma_t^{\varphi}(x_n) - x_n\right\|_{\varphi} = \sqrt{2} \neq 0.$$

We state few immediate useful consequences.

Corollary (4.2.6) [4]:

Let $(M_n, \varphi_n)_n$ be a sequence of pairs of σ-finite von Neumann algebras and normal faithful states. Let $(x_n)_n \in \mathcal{M}^{\omega}(M_n, \varphi_n)$ and put $\varphi^{\omega} = (\varphi_n)^{\omega}$.

(i) $\Delta_{\varphi^{\omega}}^{it}(x_n\xi_{\varphi_n})_{\omega} = (\Delta_{\varphi_n}^{it} x_n\xi_{\varphi_n})_{\omega}$ for all $t \in \mathbb{R}$.

(ii) $\Delta_{\varphi^{\omega}}^{\frac{1}{2}}(x_n\xi_{\varphi_n})_{\omega} = (\Delta_{\varphi_n}^{\frac{1}{2}} x_n\xi_{\varphi_n})_{\omega}$.

(iii) If $M_n = M, \varphi_n = \varphi (n \in \mathbb{N})$ for a fixed M and φ, then $\sigma(\Delta_{\varphi^{\omega}}) = \sigma(\Delta_{\varphi})$.

Proof:

By Theorem (4.2.4), we have

$$\Delta_{\varphi^{\omega}}^{it}(x_n)^{\omega}\xi_{\varphi^{\omega}} = \left(\sigma_t^{\varphi^{\omega}}(x_n)\right)^{\omega}\xi_{\varphi^{\omega}} = \left(\sigma_t^{\varphi_n}(x_n)\right)^{\omega}\xi_{\varphi^{\omega}} = \left(\Delta_{\varphi_n}^{it}x_n\xi_{\varphi_n}\right)_{\omega}.$$

Therefore (i) follows. For (ii), by the proof of Theorem (4.2.4), we have $G(S_{\varphi^{\omega}}) = (G(S_{S_{\varphi_n}}))_{\omega} \cap (qH_{\omega} \oplus_{\mathbb{R}} qH_{\omega})$. This implies that

$$\Delta_{\varphi^\omega}^{\frac{1}{2}}(x_n\xi_\varphi)\omega = J_{\varphi^\omega} S_{\varphi^\omega}(x_n\xi_\varphi)_\omega = J_\omega(S_{\varphi_n}x_n\xi_\varphi)_\omega = \left(\Delta_{\varphi_n}^{\frac{1}{2}} x_n\xi_\varphi\right)_\omega .$$

To prove (iii), note that $\Delta_{\varphi^\omega}|_{L^2(M,\varphi)} = \Delta_\varphi$, so $\sigma(\Delta_\varphi) \subset \sigma(\Delta_{\varphi^\omega})$ and $\sigma((1+\Delta_\varphi)^{-1}) \subset \sigma((1+\Delta_{\varphi^\omega})^{-1})$. On the other hand, by Eq. (36), we have

$$\left(\sigma(1+\Delta_{\varphi^\omega})^{-1}\right) \subset \left(\sigma\left((1+\Delta_\varphi)^{-1}\right)^\omega\right) = \sigma\left((1+\Delta_\varphi)^{-1}\right),$$

because $(1+\Delta_\varphi)^{-1}$ is bounded and $\sigma(a^\omega) = \sigma(a)$ holds for a bounded operator a.Therefore $\sigma\left(\left(1+\Delta_{\varphi^\omega}\right)^{-1}\right) = \sigma\left(\left(1+\Delta_\varphi\right)^{-1}\right)$, whence $\sigma(\Delta_{\varphi^\omega}) = \sigma(\Delta_\varphi)$ holds.

Therefore Δ_{φ^ω} behaves like the ultrapower of Δ_φ. Let us remark a subtle difference between the ultrapower of bounded operators and Δ_{φ^ω}. It is easy to see that for a bounded self-adjoint operator $a, \sigma(a^\omega) = \sigma_p(a^\omega)$ holds. However, the analogous result for Δ_{φ^ω} does not hold.

Proposition (4.2.7) [4]:

Let M be a type II_1 factor. There exists $\varphi \in S_{\mathrm{nf}}(M)$ for which $\sigma_p(\Delta_{\varphi^\omega}) \subseteq \sigma(\Delta_{\varphi^\omega})\backslash\{0\}$ holds.

Proof:

Let τ be the unique tracial state on M and consider the standard representation of M. Let $h \in M_+$ be such that $\sigma(h) = [\frac{1}{2}, 2]$ and that the distribution measure μ_h corresponding to h with respect to τ has absolutely continuous spectra in $[\frac{1}{2}, 1]$ and purely atomic spectra in (1,2]. Here, μh is determined by moments

$$\int_{\mathbb{R}} t^p d\mu_h(t) = (\tau h^p) \ (p \in \mathbb{N}).$$

Define $\varphi \in S_{\mathrm{nf}}(M)$ by $\varphi(x) := \tau(hx)/\tau(h), x \in M$. Then $\Delta_\varphi = h(J\,hJ)^{-1}$. Since M is afactor, $\sum_{i=1}^n x_i y_i \mapsto \sum_{i=1}^n x_i \otimes y_i \ (x_i \in M, y_i \in M')$ induces a^*-isomorphism between the *-algebra generated by M and M' and the algebraic tensor product $M \otimes M$. Therefore $C^*(M, M') \cong M \otimes_\alpha M'$ for a C^*-tensor norm $\|\cdot\|_\alpha$, and since

$C^*(h), C^*(J\,hJ)$ are abelian hence nuclear, we have $C^*(h, JhJ) \cong C^*(h) \otimes C^*(JhJ)$. Consequently, it holds that (see Corollary (4.2.6)(iii))

$$\sigma(\Delta_{\varphi^\omega}) = \sigma(\Delta_\varphi) = \left\{\frac{s}{t}; s \in \sigma(h), t \in \sigma(J\,hJ)\right\} = \left[\frac{1}{4} + 4\right].$$

Let $\tilde{h}$ be the image of h under the canonical embedding $M \subset M^\omega$. Let μ_h be thedistribution measure of $\tilde{h}$ with respect to τ^ω. Since $\tau^\omega(\tilde{h}^p) = \tau(h^p)$ holds for all $p \in \mathbb{N}$ and both $h, \tilde{h}$ are bounded, $\mu_h = \mu_{\tilde{h}}$ holds. Now we show that

$$\sigma_p(\Delta_{\varphi^\omega}) \cap \left(\left[\frac{1}{4}, \frac{1}{2}\right] \cup [2,4]\right) = \emptyset.$$

Suppose there were $\lambda \in [2,4] \cap \sigma_p(\Delta_{\varphi^\omega})$. Then by Takesaki's result, there exists $u \in M^\omega$ such that $u\varphi^\omega = \lambda\varphi^\omega u$ holds. By taking the polar decomposition, we may assume that u is a partial isometry. Since $\varphi^\omega = \tau^\omega(\tilde{h}\,\cdot)$, this implies that $u\tilde{h} = \lambda\tilde{h}u$. Moreover, as u^*u and uu^* belong to $(M^\omega)_{\varphi^\omega}$, they commute with h. It follows that

$$u(\tilde{h}u^*u)u^* = (\lambda\tilde{h})uu^*.$$

This shows that both $K = u^*uL^2(M^\omega, \tau^\omega)$ and $L = uu^*L^2(M^\omega, \tau^\omega)$ are $\tilde{h}$-invariant subspaces, and u induces an isometry of K onto L. In particular, $\tilde{h}|_K$ and $(\lambda\tilde{h})|_L$ are unitarily equivalent operators, whence $\sigma(\tilde{h}|_K) = \sigma(\lambda h|_L)$ holds. On the other hand, we know that

$$\sigma(\tilde{h}|_K) \subset \left[\frac{1}{2}, 1\right], \qquad \sigma(\lambda\tilde{h}|_L) \subset \left[\frac{\lambda}{2}, 2\lambda\right].$$

Since $\lambda \in [2,4]$, this shows that $\sigma(\tilde{h}|_K) = \sigma(\lambda\tilde{h}|_L) \subset [\frac{\lambda}{2}, 2]$ However, $\mu_{\tilde{h}}|_K$ restricted to $[1,2]$ is discrete, while $\mu_{\lambda\tilde{h}}|_L$ restricted to $[\frac{\lambda}{2}, 2] \subset [1,2]$ is absolutely continuous,a contradiction. Therefore $\sigma_p(\Delta_{\varphi^\omega}) \cap [2,4] = \emptyset$. $\sigma_p(\Delta_{\varphi^\omega}) \cap [\frac{1}{4}, \frac{1}{2}] = \emptyset$ can be shown similarly. This proves that $\sigma_p(\Delta_{\varphi^\omega}) \subsetneq \sigma(\Delta_{\varphi^\omega})\backslash\{0\}$.

Remark (4.2.8) [4]:

Proposition (4.2.7) states in particular that for $0 < \lambda \in \sigma(\Delta_\varphi)\backslash\sigma_p(\Delta_{\varphi^\omega})$, there is no bounded sequence $(x_n)_n$ of M with $\|x_n\xi_\varphi\| = 1 (n \in \mathbb{N})$ satisfying

$$\lim_{n\to\infty}\left\|\Delta_\varphi^{\frac{1}{2}}x_n\xi_\varphi - \lambda^{\frac{1}{2}}x_n\xi_\varphi\right\| = 0. \tag{38}$$

For if there were such sequence, Corollary (4.2.6) would imply that $(x_n)_n$ defines a nonzero element $(x_n)^\omega \in M^\omega$ satisfying $\Delta_{\varphi^\omega}^{\frac{1}{2}}(x_n)^\omega\xi_{\varphi^\omega} = \lambda^{\frac{1}{2}}(x_n)^\omega\xi_{\varphi^\omega}$, whence $\lambda \in \sigma_p(\Delta_{\varphi^\omega})$. Onthe other hand, as $M(\sigma^\varphi, [log\,\lambda - \frac{1}{n}, \log\lambda + \frac{1}{n}]) \neq \{0\}$ for each $n \in \mathbb{N}$, there existsa (necessarily unbounded) sequence $(x_n)_n \subset M$ with $\|x_n\xi_\varphi\| = 1 (n \in \mathbb{N})$ satisfying Eq. (38).

Next, we show that elements of $\mathcal{M}^\omega$ are characterized by the spectral condition for $(\sigma^{\varphi_n})_n$.

Proposition (4.2.9) [4]:

Let $(M_n, \varphi_n)_n$ be a sequence of σ-finite von Neumann algebras and normal faithful states. Then for $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$, the following conditions are equivalent.

(I) $(x_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$.

(II) For every $\varepsilon > 0$, there exist $a > 0$ and $(y_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$ such that

(i) $\lim_{n\to\omega}\|x_n - y_n\|_\varphi^\# < \varepsilon$,

(ii) $y_n \in M_n(\sigma^{\varphi_n}, [-a, a]), n \in \mathbb{N}$.

In this case, $(y_n)_n$ can be chosen to satisfy $\|(y_n)^\omega\| \leq \|(x_n)^\omega\|$.

We need preparations. Recall two summability kernels on $\mathbb{R}$.

Proof:

(I) $\Rightarrow$ (II): Let $(x_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$ and put $x := (x_n)^\omega$. Also, define

$$x_a := \sigma_{F_a}^{\varphi^\omega}(x) \in (M_n, \varphi_n)^\omega (a > 0).$$

Then we have $\lim_{a\to\infty} \|x_a - x\|_{\varphi^\omega}^{\#} = 0$. Indeed, since $\Phi: t \mapsto \left\|x - \sigma_t^{\varphi^\omega}(x)\right\|_{\varphi^\omega}^{\#}$ is continuou and bounded, we have

$$\|x_a - x\|_{\varphi^\omega}^{\#} = \left\|\int_{\mathbb{R}} F_a(t)\left(\sigma_t^{\varphi^\omega}(x) - x\right) dt\right\|_{\varphi^\omega}^{\#}$$

$$\leq \int_{\mathbb{R}} F_a(t) \left\|x - \sigma_t^{\varphi^\omega}(x)\right\|_{\varphi^\omega}^{\#} dt$$

$$\xrightarrow{a\to\infty} \Phi(0) = 0,$$

whence the claim follows. Therefore there exists $a > 0$ such that $y := \sigma_{F_a}^{\varphi^\omega}(x)$ satisfies $\|y - x\|_{\varphi^\omega} < \varepsilon$. We have $\|y\| \leq \|F_a\|_1 x = x$, and by Lemma (4.2.12), $y = (y_n)^\omega$, where $y_n = \sigma_{F_a}^{\varphi_n}(x_n)(n \in \mathbb{N})$. Therefore $(y_n)_n$ satisfies all conditions in (II). Note that we also have $\|y_n\| \leq \|x_n\|(n \in \mathbb{N})$.

(II) ⇒ (I): Suppose $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ satisfies the conditions in (II). Let $\varepsilon > 0$.

Then by Lemma (4.2.11) and by assumption, there is $(x'_n)_n \in \mathcal{M}^\omega$ such that $\lim_{n\to\omega} \|x_n - x'_n\|_{\varphi_n}^{\#} < \varepsilon$. Let $(y_n)_n \in I_\omega$ with $\sup_{n\geq 1}\|y_n\|$. Then we see that

$$\lim_{n\to\omega} \|(x_n y_n)^*\|_{\varphi_n} \leq \lim_{n\to\omega} \left\{\|y_n^*\| \|x_n^* - (x_n')^*\|_{\varphi_n} + \|y_n^*(x_n')^*\|_{\varphi_n}\right\}$$

$$\leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n\to\omega} \|(x_n y_n)^*\|_{\varphi_n} = 0$. Similarly, we also have $\lim_{n\to\omega} \|y_n x_n\|_{\varphi_n} = 0$. This proves that $(x_n)_n \in \mathcal{M}^\omega$.

As an application of the Groh–Raynaud ultraproduct, we prove that it provides examples of von Neumann algebras for which all normal faithful states are unitarily equivalent. We also prove that this property is only possible for von Neumann algebras with nonseparable preduals (besides $\mathbb{C}$).

Definition (4.2.10) [4]:

The Fejér kernel $F_a : \mathbb{R} \to \mathbb{R} (a > 0)$ is defined by

$$F_a(t) := \begin{cases} \dfrac{1 - \cos(at)}{\pi a t^2} & (t \neq 0), \\ a/2\pi & (t = 0). \end{cases}$$

Its Fourier transform is

$$\widehat{F_a}(\lambda) = \begin{cases} 1 - \dfrac{|\lambda|}{a} & (|\lambda| \leq a), \\ 0 & (|\lambda| > a). \end{cases}$$

It holds that $0 \leq F_a$ and $\|F_a\|_1 = \hat{F}_a(0) = 1$. Moreover, we have

$$\lim_{a\to\infty} \int_{\mathbb{R}} F_a(s)\phi(s)ds = \phi(0), \qquad \lim_{a\to\infty} \|F_a * f - f\|_1 = 0,$$

for all continuous bounded function ϕ on $\mathbb{R}$ and $f \in L^1(\mathbb{R})$. The de la Vallée Poussin $kernel\ D_a : \mathbb{R} \to \mathbb{R}$ is given by

$$D_a(t) = 2F_{2a}(t) - F_a(t) = \begin{cases} \dfrac{\cos(at) - \cos(2at)}{\pi a t^2} & (t \neq 0), \\ 3a/2\pi & (t = 0). \end{cases}$$

Its Fourier transform is

$$\widehat{D_a}(\lambda) = \begin{cases} 1 & (|\lambda| \leq a), \\ 2 - \dfrac{|\lambda|}{a} & (a \leq |\lambda| \leq 2a), \\ 0 & (|\lambda| > 2a). \end{cases}$$

Lemma (4.2.11) [4]:

Let $(x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$. If there exists $a > 0$ such that $x_n \in M_n(\sigma^{\varphi_n}, [-a, a])$ holds for all $n \in \mathbb{N}$, then $(x_n)_n \in \mathcal{M}^\omega(M_n, \varphi_n)$.

Proof:

We show that the map $t \mapsto \sigma_t^{\varphi_n}(x_n)$ is extended to an entire analytic M_n-valued function satisfying

$$\left\|\sigma_z^{\varphi_n}(x_n)\right\| \leq C_{a,z}\|x_n\|, \qquad z \in \mathbb{C},$$

where $C_{a,z}$ is a constant depending only on a, z. Since $x_n \in M_n(\sigma^{\varphi_n}, [-a, a])$ and the de la Vallée Poussin kernel satisfies $\widehat{D}_a = 1$ on $[-a, a]$, we have $x_n = \sigma_{D_a}^{\varphi_n}(x_n)$. Therefore for $t \in R$, we have

$$\sigma_t^{\varphi_n}(x_n) = \int_{\mathbb{R}} D_a(s)\sigma_{t+s}^{\varphi_n}(x_n)\, ds$$

$$= \int_{\mathbb{R}} D_a(s-t)\sigma_s^{\varphi_n}(x_n)\ ds.$$

By the explicit form, $D_a = 2F_{2a} - F_a$ has an analytic continuation to $\mathbb{C}$. We have

$$\int_{\mathbb{R}} |F_a(s+it)|\, ds \leq e^{a|t|}(t \in \mathbb{R}).$$

Therefore for $z \in \mathbb{C}, s \to D_a(s-z)$ is in $L^1(\mathbb{R})$, and $t \to \sigma_t^{\varphi_n}(x_n)$ has an Mn-valued analytic extension:

$$\sigma_z^{\varphi_n}(x_n) = \int_{\mathbb{R}} D_a(s-z)\sigma_s^{\varphi_n}(x_n)\, ds, \qquad z \in \mathbb{C}.$$

Then we have

$$\left\|\sigma_z^{\varphi_n}(x_n)\right\| \leq \int_{\mathbb{R}} |D_a(s-z)|\left\|\sigma_s^{\varphi_n}(x_n)\right\| ds \leq C_{a,z}\|x_n\|,$$

where $C_{a,z} := 2e^{2a|\mathrm{Im}(z)|} + e^{a|\mathrm{Im}\, z|}$. Let $(y_n)_n \in I_\omega$. It follows that

$$\|(x_n y_n)^*\|_{\varphi_n} = \left\|y_n^* J_{\varphi_n}\Delta_{\varphi_n}^{\frac{1}{2}} x_n \xi_{\varphi_n}\right\|$$

$$= \left\| J_{\varphi_n} y_n^* J_{\varphi_n} \sigma^{\varphi_n}_{-\frac{i}{2}}(x_n) \xi_{\varphi_n} \right\|$$

$$\leq \left\| \sigma^{\varphi_n}_{-\frac{i}{2}}(x_n) \right\| \cdot \; \| J_{\varphi_n} y_n^* J_{\varphi_n} \xi_{\varphi_n} \|$$

$$\leq C_{a,-i/2} \|x_n\| \cdot \; \|y_n\|_{\varphi_n}$$

$$\xrightarrow{n \to \omega} 0.$$

Similarly, $\|y_n x_n\|_{\varphi} \to 0 (n \to \omega)$. Hence $(x_n)_n \in \mathcal{M}^{\omega}$.

Lemma (4.2.12) [4]:

Let $f \in L^1(\mathbb{R})$, and $((x_n)_n \in \mathcal{M}^{\omega}(M_n, \varphi_n)$. Then $(\sigma_f^{\varphi_n}(x_n))_n \in \mathcal{M}^{\omega}(M_n, \varphi_n)$ and $\sigma_f^{\varphi^{\omega}}((x_n)^{\omega}) = (\sigma_f^{\varphi_n}(x_n))^{\omega}$ holds.

Proof of Lemma (4.2.14):

We first prove

Claim. $(\sigma^{\varphi_n}_{f * F_a}(x_n))_n \in \mathcal{M}^{\omega}(M_n, \varphi_n)$ and $\sigma^{\varphi^{\omega}}_{f * F_a}((x_n)^{\omega}) = (\sigma^{\varphi_n}_{f * F_a}(x_n))^{\omega}$ holds.

Since $\text{supp}(\widehat{f * F_a}) \subset \text{supp}(\widehat{F_a}) = [-a, a]$, we have $\sigma^{\varphi_n}_{f * F_a}(x_n) \in M_n(\sigma^{\varphi_n}, [-a, a])$ for all $n \in \mathbb{N}$. Therefore by Lemma (4.2.11), we have $(\sigma^{\varphi_n}_{f * F_a}(x_n))_n \in \mathcal{M}^{\omega}(M_n, \varphi_n)$. Next, consider a bounded continuous function $Q_a: (0,1) \to \mathbb{C}$ given by

$$Q_a(t) := \left(\widehat{f * F_a}\right)\left(\log(t^{-1} - 1)\right), \qquad t \in (0,1).$$

Then we have

$$Q_a((1+t)^{-1}) = \left(\widehat{f * F_a}\right)(\log t), \qquad t \in \mathbb{R}.$$

By Lemma (4.2.4) and Theorem (4.2.4), we have

$$Q_a\left(\left(1 + \Delta_{\varphi^{\omega}}\right)^{-1}\right) = \left(Q_a\left(\left(1 + \Delta_{\varphi_n}\right)^{-1}\right)\right)_{\omega} |_{qH\omega}.$$

It then follows that

$$\sigma^{\varphi^\omega}_{f*F_a}((x_n)^\omega)\xi_{\varphi^\omega} = \widehat{f*F_a}(\log\Delta_{\varphi^\omega})(x_n)^\omega\xi_{\varphi^\omega} = Q_a\left(\left(1+\Delta_{\varphi^\omega}\right)^{-1}\right)(x_n)^\omega\xi_{\varphi^\omega}$$

$$= \left(Q_a\left(\left(1+\Delta_{\varphi_n}\right)^{-1}\right)\right)_\omega \left(x_n\xi_\varphi\right)_\omega = \left(\widehat{f*F_a}(\log\Delta_{\varphi_n})x_n\xi_\varphi\right)_\omega$$

$$= \left(\sigma^{\varphi_n}_{f*F_a}(x_n)\xi_{\varphi_n}\right)_\omega = \left(\sigma^{\varphi_n}_{f*F_a}(x_n)\right)^\omega \xi_{\varphi^\omega}\,.$$

Since $\xi\phi\omega$ is separating for $(M_n,\varphi_n)^\omega$, we have $\sigma^{\varphi^\omega}_{f*F_a}((x_n)^\omega) = (\sigma^{\varphi_n}_{f*F_a}(x_n))^\omega$.

Now we prove that $(\sigma^{\varphi_n}_f(x_n))_n \in \mathcal{M}^\omega(M_n,\varphi_n)$ and $\sigma^{\varphi^\omega}_f((x_n)^\omega) = (\sigma^{\varphi_n}_f(x_n))^\omega$ holds. Since $\|f*F_a - f\|_1 \xrightarrow{a\to\infty} 0$, we have

$$\sup_{n\geq 1}\left\|\sigma^{\varphi_n}_f(x_n) - \sigma^{\varphi_n}_{f*F_a}(x_n)\right\| \leq \sup_{n\leq 1}\int_{\mathbb{R}} |f(t)-(f\ *F_a)(t)|\cdot\left\|\sigma^{\varphi_n}_t(x_n)\right\|\,dt$$

$$= \sup_{n\geq 1}\|x_n\|\cdot\ \|f - f*F_a\|_1 \xrightarrow{a\to\infty} 0\,.$$

By the Claim, $(\sigma^{\varphi_n}_{f*F_a}(x_n))_n \in \mathcal{M}^\omega(M_n,\varphi_n)$. Therefore as $\mathcal{M}^\omega(M_n,\varphi_n)$ is norm-closed, we have $(\sigma^{\varphi_n}_f(x_n))_n \in \mathcal{M}^\omega(M_n,\varphi_n)$. Finally, suppose $\varepsilon > 0$ is given. By similar arguments as above, there exists $a > 0$ such that

$$\left\|\sigma^{\varphi_n}_{f*F_a}((x_n)^\omega) - \sigma^{\varphi^\omega}_f((x_n)^\omega)\right\| < \varepsilon,\quad \left\|\left(\sigma^{\varphi_n}_{f*F_a}(x_n)\right)^\omega - \left(\sigma^{\varphi_n}_f(x_n)\right)^\omega\right\| < \varepsilon.$$

Then by the Claim, we see that

$$\begin{aligned}&\left\|\sigma^{\varphi^\omega}_f((x_n)^\omega) - \left(\sigma^{\varphi_n}_f(x_n)\right)^\omega\right\| \\ &\qquad\leq \left\|\sigma^{\varphi^\omega}_f((x_n)^\omega) - \sigma^{\varphi^\omega}_{f*F_a}((x_n)^\omega)\right\| \\ &\qquad+ \left\|\left(\sigma^{\varphi_n}_{f*F_a}(x_n)\right)^\omega - \left(\sigma^{\varphi_n}_f(x_n)\right)^\omega\right\| < 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the lemma is proved.

Remark (4.2.13) [4]:

One might think that this is a direct consequence of $\sigma_t^{\varphi^\omega}((x_n)^\omega) = (\sigma_t^{\varphi_n}(x_n))^\omega$ (Theorem (4.2.4)). However, we must show that

$$\int_{\mathbb{R}} f(t)\left(\sigma_t^{\varphi_n}(x_n)\right)^\omega dt = \left(\int_{\mathbb{R}} f(t)\sigma_t^{\varphi_n}(x_n)dt\right)^\omega,$$

i.e., the order of integration and ultralimit can be changed.

Definition (4.2.14) [4]:

Let M be a σ-finite von Neumann algebra. Then $S_{\mathrm{nf}}(M)$ is said to be

- homogeneous, if for any $\varphi, \psi \in S_{\mathrm{nf}}(M)$ and any $\varepsilon > 0$, there is $u \in \mathcal{U}(M)$ such that $\|u\varphi u^* - \psi\| < \varepsilon$;
- strictly homogeneous, if for any $\phi, \psi \in S_{\mathrm{nf}}(M)$ there is $u \in \mathcal{U}(M)$ such that $u\varphi u^* = \psi$.

We have the following

Theorem (4.2.15) [4]:

Let M be a σ-finite von Neumann algebra. The following are equivalent:

(i) M is a factor of type I_1 or type III_1.
(ii) $S_{\mathrm{nf}}(M)$ is homogeneous.

Lemma (4.2.16) [4]:

Let M be a σ-finite factor not isomorphic to $\mathbb{C}$ with strictly homogeneous state space. Then

(i) M is a type III_1 factor.
(ii)For $\varphi, \psi \in S_{\mathrm{n}}(M)$, there exists a partial isometry $u \in M$ such that

$$u^*u = \mathrm{supp}(\varphi), \qquad uu^* = \mathrm{supp}(\psi), \;\; and \;\; \psi = u\varphi u^*.$$

Proof.:

(i) We have to show that M has state space diameter 0. But since $S_{\mathrm{nf}}(M)$ is norm-dense in $S_{\mathrm{n}}(M)$, this is the consequence of the strict homogeneity of $S_{\mathrm{nf}}(M)$.

(ii) By (i), M is a type III factor. Hence there is a partial isometry $v \in M$ such that $v^*v = \mathrm{supp}(\varphi), vv^* = \mathrm{supp}(\psi)$ holds. Put $\psi' := v^*\psi v$. We see that $\mathrm{supp}(\psi') = v^*\mathrm{supp}(\psi)v = \mathrm{supp}(\varphi)$. Since M is of type III, $M_{\mathrm{supp}(\varphi)} \cong M$ has strictly homogeneousstate space. Therefore regarding $\varphi, \psi' \in S_{\mathrm{nf}}(M_{\mathrm{supp}(\varphi)})$ we may find $w \in M_{\mathrm{supp}(\varphi)}$ with $w^*w = ww^* = \mathrm{supp}(\varphi)$ such that $\psi' = w\phi w^*$. Then $u := vw$ satisfies

$$u^*u = w^*\mathrm{supp}(\varphi)w = \mathrm{supp}(\varphi),$$

$$uu^* = v\,\mathrm{supp}(\varphi)v^* = vv^* = \mathrm{supp}(\psi),$$

$$u\varphi u^* = vw\varphi w^*v^* = v\psi v^* = \psi.$$

By the homogeneity, the Ocneanu ultraproduct of a type III_1 factor does not depend on the choice of a sequence of normal faithful states.

Corollary (4.2.17) [4]:

Let M be a σ-finite factor of type III_1 and $(\psi_n)_n \subset S_{\mathrm{nf}}(M)$. Then $(M,\ \psi_n)^\omega \cong M^\omega$.

Proof:

Let $\psi \in S_{\mathrm{nf}}(M)$ and choose (by the Connes–Størmer transitivity, see Theorem (4.2.15)) a sequence $(u_n)_n$ of unitaries in M such that

$$\|\psi - u_n\psi_n u_n^*\| \leq \frac{1}{n}, \qquad n \in \mathbb{N}.$$

Then $\psi_\omega = (u_n\psi_n u_n^*)_\omega$ and so $M^\omega = (M,\psi)^\omega \cong (M,\psi_n)^\omega$ by Theorem (4.1.9) (see also the remark after Proposition (4.1.4)).

Theorem (4.2.18) [4]:

Let M be a σ-finite factor of type III_1, let $M_n = M (n \in \mathbb{N})$, and let $N = \prod^{\omega} M_n$. Then N is not σ-finite, but for any σ-finite projection $p \in N$, one hasthat pNp has strictly homogeneous state space. In particular, N and M^{ω} are factors of type $III1$.

Proof:

Let φ, ψ be normal states in N. By Corollary (4.1.26), there are sequences of normal states $(\varphi_n)_n, (\psi_n)_n \subset M_*$ such that $\varphi = (\varphi_n)_\omega$ and $\psi = (\psi_n)_\omega$. By Theorem (4.2.15), there is $(u_n)_n \subset \mathcal{U}(M)$ such that $\|u_n \varphi_n u_n^* - \psi_n\| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Now, let $u := (u_n)_\omega \in U(N)$. Then $u\varphi u^* = \psi$. Hence all normal states of N are unitarily equivalent; in particular, N is not σ-finite (there can be no faithful normal states in this situation).

If $p \in N$ is a σ-finite projection, let φ, ψ be normal faithful states on pNp. Then $\tilde{\varphi} := p\phi p$ and $\tilde{\psi} := p\psi p$ define normal states on N with support p. By the above, we maychoose $u \in \mathcal{U}(N)$ such that $u\tilde{\varphi}u^* = \tilde{\psi}$. Then $upu^* = p$ and hence $v := up \in \mathcal{U}(pNp)$. Also $v\phi v^* = u\varphi u^* = \tilde{\psi} = \psi$ on elements of pNp. Hence $S_{\mathrm{nf}}(pNp)$ is strictly homogeneous.

We remark that no von Neumann algebra with separable predual has strictly homogeneous state space:

Lemma (4.2.19) [4]:

Let M be a σ-finite factor not isomorphic to $\mathbb{C}$ with strictly homogeneous state space, and let $\varphi \in S_{\mathrm{nf}}(M)$. Then for any $\lambda \in (0,1)$, there is a projection $p \in M_\varphi$ such that $\varphi(p) = \lambda$ holds.

Proof:

Put $\widetilde{M} := M \otimes M_2(\mathbb{C})$, $\theta := \begin{pmatrix} \lambda\varphi & 0 \\ 0 & (1-\lambda)\varphi \end{pmatrix}$, and $q := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

It holds that $q \in \widetilde{M}_\theta$, and $\theta(q) = \lambda$. Since M is of type III, there is an isomorphism $\Phi: \widetilde{M} \xrightarrow{\sim} M$. Then $\psi := \theta \circ \Phi^{-1}, p' := \Phi(q)$ satisfies $p' \in M_\psi$ and

$\psi(p') = \lambda$. Choose, by strict homogeneity of $S_{\mathrm{nf}}(M), u \in \mathcal{U}(M)$ such that $u\psi u^* = \varphi$. Then $p := up'u^*$ works.

Proposition (4.2.20) [4]:

Let M be a σ-finite factor not isomorphic to $\mathbb{C}$ with strictly homogeneous state space. Then M_* is not separable.

Proof:

Choose $0 < \lambda < 1$. By Lemma (4.2.19), there is a projection $p \in M_\varphi$ such that $\varphi(p) = \lambda$. Put $\psi := \frac{1}{\lambda} p\varphi$. By Lemma (4.2.16)(ii), there is a partial isometry $v \in M$ such that $v^*v = \mathrm{supp}(\varphi) = 1, vv^* = \mathrm{supp}(\psi) = p$, and $\psi = v\varphi v^*$. We see that

$$\begin{aligned} v\varphi &= (v\varphi v^*)v = \psi v = \psi(v\, \cdot) \\ &= \frac{1}{\lambda} p\varphi(v\, \cdot) = \frac{1}{\lambda}\varphi(v \cdot p) = \frac{1}{\lambda}\varphi(pv\, \cdot) \\ &= \frac{1}{\lambda}\varphi v. \end{aligned}$$

Therefore, $\sigma_t^\varphi(v) = \lambda^{it} v$ holds for all $t \in \mathbb{R}$, which is equivalent to $\lambda \in \sigma_p(\Delta_\varphi)$. Since $\lambda \in (0,1)$ is arbitrary, Δ_φ has uncountably many eigenvalues. This shows that $L^2(M, \varphi)$ is not separable, whence M_* is not separable.

Proposition (4.2.21) [4]:

Let M be a σ-finite factor not isomorphic to $\mathbb{C}$ with strictly homogeneous state space. Then for any $\varphi \in S_{\mathrm{nf}}(M), M_\varphi$ is a factor of type II_1.

Proof:

It is clear that M_φ is a finite von Neumann algebra. If M_φ were not a factor, choose a projection $p \in Z(M_\varphi)\backslash\{0,1\}$. We may assume that $0 < s := \varphi(p) \leq \frac{1}{2}$. Then $\varphi(p^\perp) = 1 - s \geq s = \varphi(p)$. Since M is of type III, $(1-p)M(1-p) \cong M$. Hence by Lemma (4.2.19) applied to $\frac{1}{1-s}\varphi|_{M(1-p)}$, there is a projection $q \in M_\varphi$ such that $q \leq 1 - p$ and $\varphi(q) = s$. Since $\frac{1}{s} p\varphi$ and $\frac{1}{s} q\phi$ are normal states on M

with support p and q, respectively. By Lemma (4.2.16), there is a partial isometry $v \in M$ such that $v^*v = p, vv^* = q$ and $v(p\varphi)v^* = q\varphi$. Since p, $q \in M_\varphi$, we have

$$\varphi v = \varphi(v\, \cdot) = \varphi(qv\, \cdot) = \varphi(v \cdot q)$$
$$= q\varphi v = (vp\varphi v^*)v = vp\varphi p = vp\varphi$$
$$= v\varphi,$$

whence $v \in M_\varphi$. This shows that $p \sim q$ in M_φ. However, as $q \leq 1 - p$, we know that $z_{M_\varphi}(q) \perp z_{M_\varphi}(p) = p$. Therefore $p \sim q$ in $M\phi$ cannot be the case. This shows that M_φ is a factor. Then by Lemma (4.2.19), M_φ is a II_1 factor.

Let $\mathcal{W}_{\mathrm{nfs}}(M)$ be the set of all normal faithful semifinite weights on a σ-finite von Neumann algebra M, and let $E: M^\omega \ni (x_n)^\omega \to wot - \lim_{n\to\omega} x_n \in M$ be the canonical normal faithful conditional expectation.

Definition (4.2.22) [4]:

We define $\varphi^\omega \in \mathcal{W}_{\mathrm{nfs}}(M^\omega)$ by

$$\varphi^\omega := \varphi \circ E, \qquad \varphi \in \mathcal{W}_{\mathrm{nfs}}(M).$$

Since both φ and E are normal and faithful, and since φ is semifinite, $\varphi^\omega \in W_{\mathrm{nfs}}(M^\omega)$ holds. Note that this definition is in agreement with the definition of the ultra power state φ^ω when $\varphi \in S_{\mathrm{nf}}(M)$. We then have a following partial generalization of Theorem (4.2.4).

Lemma (4.2.23) [4]:

Let M be a σ-finite von Neumann algebra, and let $\varphi \in \mathcal{W}_{\mathrm{nfs}}(M)$. Then we have

$$\sigma_t^{\varphi^\omega}((x_n)^\omega) = \left(\sigma_t^{\varphi}(x_n)\right)^\omega, \quad (x_n)^\omega \in M^\omega, \quad t \in \mathbb{R}.$$

Proof:

Let $\psi \in S_{\mathrm{nf}}(M)$, and let $ut := (D\varphi^\omega : D\psi^\omega)_t$ $(t \in \mathbb{R})$. Since $\varphi^\omega = \varphi \circ E$ and $\psi^\omega = \psi \circ E$, by Theorem (4.2.4), we have for $x = (x_n)^\omega \in M^\omega$ and $t \in \mathbb{R}$ that

$$\sigma_t^{\varphi^\omega}((x_n)^\omega) = u_t \sigma_t^{\psi^\omega}((x_n)^\omega) u_t^*$$

$$= \left(D(\varphi \circ E): D(\psi \circ E)\right)_t \left(\sigma_t^{\psi}(x_n)\right)^{\omega} \left(D(\varphi \circ E): D(\psi \circ E)\right)_t^*$$

$$= \left((D\varphi: D\psi)_t \sigma_t^{\psi}(x_n)(D\varphi: D\psi)_t^*\right)^{\omega}$$

$$= \left(\sigma_t^{\varphi}(x_n)\right)^{\omega}.$$

This proves the lemma.

Recall that a normal faithful semifinite weight φ on a von Neumann algebra M is called lacunary if 1 is isolated in $\sigma(\Delta_\varphi)$. The next result will be important for the analysis of the Ocneanu ultraproduct of type III_0 factors.

Proposition (4.2.24) [4]:

Let M be a σ-finite von Neumann algebra, and let $\varphi \in \mathcal{W}_{\mathrm{nfs}}(M)$ be lacunary. Then $(M^{\omega})_{\varphi^{\omega}} \cong (M_\varphi)^{\omega}$ holds.

Proof:

We first prove that $(M_\varphi)^{\omega} \subset (M^{\omega})_{\varphi^{\omega}}$. Since φ is lacunary, it is strictly semifinite and therefore there exists a normal faithful φ-preserving conditional expectation $E: M \to M_\varphi$. Therefore we may regard $(M_\varphi)^{\omega} \subset M^{\omega}$. Let $x = (x_n)^{\omega} \in (M_\varphi)^{\omega}$. Then by Lemma (4.2.23), we have $\sigma_t^{\varphi^{\omega}}(x) = (\sigma_t^{\varphi}(x_n))^{\omega} = x (t \in \mathbb{R})$, whence $x \in (M^{\omega})_{\varphi^{\omega}}$ holds. Let $0 < \lambda < 1$ be such that $\sigma(\Delta_\varphi) \cap (\lambda, \lambda^{-1}) = \{1\}$.

Step 1. We next prove $(M^{\omega})_{\varphi^{\omega}} \subset (M_\varphi)^{\omega}$ for the case where $\varphi(1) < \infty$. Let $x = (x_n)^{\omega} \in (M^{\omega})_{\varphi^{\omega}}$ with $\|x\xi_{\varphi^{\omega}}\| = 1$. Then by Corollary (4.2.6)(ii), we have

$$\Delta_{\varphi^{\omega}}^{\frac{1}{2}} x\xi_{\varphi^{\omega}} = x\xi_{\varphi^{\omega}} \iff \lim_{n \to \omega} \left\| \Delta_{\varphi}^{\frac{1}{2}} x_n \xi_\varphi - x_n \xi_\varphi \right\| = 0.$$

Let $p := 1_{\{1\}}(\Delta_\varphi)$ be the spectral projection of Δ_φ corresponding to the eigenvalue 1. Then by assumption, we have

$$\left\| \Delta_{\varphi}^{\frac{1}{2}} x_n \xi_\varphi - x_n \xi_\varphi \right\| = \left\| \Delta_{\varphi}^{\frac{1}{2}} p^{\perp} x_n \xi_\varphi - p^{\perp} x_n \xi_\varphi \right\|$$

$$\geq \min\left(1-\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}-1\right)\left\|p^{\perp}x_n\xi_\varphi\right\|$$

$$= \left(1-\lambda^{\frac{1}{2}}\right)\left\|p^{\perp}x_n\xi_\varphi\right\|.$$

Therefore we have

$$\lim_{n\to\omega}\left\|x_n\xi_\varphi - px_n\xi_\varphi\right\| = 0.$$

Let $f \in L^1(\mathbb{R})_+$ be such that $\mathrm{supp}(\hat{f}) \subset (\log\lambda, -\log\lambda)$ and $\int_{\mathbb{R}} f(t)\,dt = 1$. Let

$$y_n \coloneqq \sigma_f^{\varphi}(x_n) = \int_{\mathbb{R}} f(t)\sigma_t^{\varphi}(x_n)\,dt, \qquad n \geq 1.$$

Since

$$\sigma_f^{\varphi}(x_n)\xi_\varphi = f(\log\Delta_\varphi)x_n\xi_\varphi,$$

we have $\mathrm{Sp}_{\sigma^\varphi}(y_n) \subset \mathrm{Sp}_{\sigma_\varphi}(x_n) \cap (\log\lambda, -\log\lambda) = \{0\}$ and $y_n \in M_\varphi$. It is clear that $\sup_{n\geq 1}\|y_n\| \leq \|f\|_1 \sup_{n\geq 1}\|x_n\| < \infty$. We have

$$px_n\xi_\varphi = \hat{f}\left(\log\Delta_\varphi\right)x_n\xi_\varphi = y_n\xi_\varphi, \qquad n \geq 1.$$

This implies that $\left\|x_n\xi_\varphi - y_n\xi_\varphi\right\| \to (n\to\omega)$. Since $\Delta_{\varphi^\omega}\, x^*\xi_{\varphi^\omega} = x^*\xi_{\varphi^\omega}$ also holds, we have also $\left\|x_n^*\xi_\varphi - y_n^*\xi_\varphi\right\| \to 0$. Since M_φ is a finite von Neumann algebra, $(y_n)_n$ defines an element in $(M_\varphi)^\omega$, and $x = (y_n)^\omega$ holds. Therefore $(M^\omega)_{\varphi^\omega} \subset (M_\varphi)^\omega$.

Step 2. Finally, we prove $(M^\omega)_{\varphi^\omega} \subset (M_\varphi)^\omega$ for a general lacunary $\varphi \in W_{\mathrm{nfs}}(M)$. Take $\lambda > 0$ as in Step 1. Since the restriction of φ to M_φ is a semifinite trace, there exists an increasing net $\{p_i\}_{i\in I}$ of projections in M_φ such that $\{p_i\}_{i\in I}$ converges strongly to 1, and $\varphi(p_i) < \infty$ for all $i \in I$. Let $x \in (M^\omega)_{\varphi^\omega}$. Fix arbitrary $i \in I$. Identifying $p_iM^\omega p_i$ with $(p_iMp_i)^\omega$, we may regard $p_ixp_i \in (p_iMp_i)^\omega$. Furthermore, as $p_i \in M_\varphi$ and $\varphi^\omega(p_i) = \varphi(p_i) < \infty$, the restriction $\varphi_{p_i}^\omega$ of φ^ω to $(p_iMp_i)^\omega$ is a normal faithful positive linear functional, and $p_iM^\omega p_i \cap (M^\omega)_{\varphi^\omega} = ((p_iM\,p_i)^\omega)_{\varphi_{p_i}^\omega}$. It also holds that $\varphi_{p_i}^\omega$ is the ultrapower of φ_{p_i}. Since

$\Delta_{\varphi_{p_i}} = \Delta_\varphi|_{p_i J_\varphi p_i J_\varphi L^2(M,\varphi)}$, we have $\sigma(\Delta_{\varphi_{p_i}}) \cap (\lambda, \lambda^{-1}) \subset \sigma(\Delta_\varphi) \cap (\lambda, \lambda^{-1}) = \{1\}$, and hence φ_{p_i} is lacunary on $p_i M p_i$. Therefore by Step 1, we have $((p_i M p_i)^\omega)_{\varphi^\omega_{p_i}} = ((p_i M p_i)_{\varphi_{p_i}})^\omega$ holds. Therefore $p_i x p_i \in ((p_i M\, p_i)_{\varphi_{p_i}})^\omega \subset (M_\varphi)^\omega$. Since $i \in I$ is arbitrary, and $p_i x p_i \to x$ strongly, we have that $x \in (M_\varphi)^\omega$. Therefore $(M^\omega)_{\varphi^\omega} \subset (M_\varphi)^\omega$.

We reinterpret the main result of Golodets' work on the asymptotic algebra from our viewpoint. Let M be a factor with separable predual, and consider the asymptotic algebra C_M^ω induced by $\varphi \in S_{\text{nf}}(M)$. φ naturally induces a normal faithful state $\tilde{\varphi} = \bar{\varphi}|_{\mathscr{R}}$ on $\mathscr{R} = e_\omega \pi_{\text{Gol}}(\ell^\infty)'' e_\omega$, hence a normal faithful state $\dot{\varphi} = \tilde{\varphi}|_{C_M^\omega}$. The main results of Golodets' work were

(i) to generalize the central sequence algebra $M' \cap M^\omega$ for type III factors and give a characterization of Araki's property $L'_\lambda (0 < \lambda < 1)$: $M \cong M \overline{\otimes} R_\lambda$ if and only if λ is the eigenvalue of $\Delta_{\dot{\varphi}}$.

(ii) to show that the centralizer $(C_M^\omega)_{\dot{\varphi}}$ plays the similar role as Connes' asymptotic centralizer $M\omega$ (see Definition (4.2.29) below), namely M is McDuff if and only if $(C_M^\omega)_{\dot{\varphi}}$ is noncommutative.

Regarding (ii), Golodets and Nessonov later showed that $(C_M^\omega)_{\dot{\varphi}}$ is indeed isomorphic to M_ω for a factor M with separable predual.

We start from the following observation:

Proposition (4.2.25) [4]:

Let M be a σ-finite von Neumann algebra, and let $\varphi, \psi \in S_{\text{nf}}(M)$ such that $\varphi|_{\mathrm{Z}(M)} = \psi|_{\mathrm{Z}(M)}$. Then $\dot{\varphi}_\omega = \psi_\omega$, where $\dot{\varphi}_\omega := \varphi^\omega|_{M' \cap M^\omega}, \dot{\psi}_\omega := \psi^\omega|_{M' \cap M^\omega}$. In particular, if M is a σ-finite factor, then $\dot{\varphi}_\omega$ does not depend on φ.

Proof:

Recall that $M^\omega \ni (x_n)^\omega \mapsto wot - \lim_{n\to\omega} x_n \in M$ defines a normal faithful conditional expectation E. It is easy to see that $E((x_n)^\omega) \in \mathrm{Z}(M)$ if $(x_n)^\omega \in M' \cap M^\omega$. Since $\varphi^\omega = \varphi \circ E, \psi^\omega = \psi \circ E$, and since φ and ψ agree on $\mathrm{Z}(M)$, we have

$$\dot{\varphi}_\omega = \varphi \circ E|_{M' \cap M^\omega} = \psi \circ E|_{M' \cap M^\omega} = \dot{\psi}_\omega.$$

Definition (4.2.26) [4]:

Let M be a σ-finite von Neumann algebra, and let $\varphi \in S_{\mathrm{nf}}(M)$. We call $\dot{\varphi}_\omega = \varphi^\omega|_{M' \cap M^\omega}$ the Golodets state associated with φ.

The next theorem corresponds to Golodets' work (i) mentioned above.

Lemma (4.2.27) [4]:

Let $0 < \lambda < 1$ and let M be a σ-finite factor of type III. The following conditions are equivalent.

(i) $M \cong M \mathbin{\overline{\otimes}} R_\lambda$.

(ii) For any $n \in \mathbb{N}, \varepsilon > 0$ and $\varphi_1, \dots, \varphi_n \in S_{\mathrm{nf}}(M)$, there exists nonzero $x \in M$ such that

$$\left\| \left(\Delta_{\varphi_j}^{\frac{1}{2}} - \lambda^{\frac{1}{2}} \right) x \xi_{\varphi_j} \right\|^2 \leq \varepsilon \sum_{j=1}^{n} \varphi_j(x^* x).$$

Theorem (4.2.28) [4]: (Golodets).

Let M be a σ-finite factor of type III. Then $M \cong M \mathbin{\overline{\otimes}} R_\lambda$ holds if and only if $\lambda \in \sigma_p(\Delta_{\dot{\varphi}^\omega})$ for some (hence any) $\varphi \in S_{\mathrm{nf}}(M)$.

To prove the theorem we use the following characterization of the condition $M \cong M \mathbin{\overline{\otimes}} R_\lambda$.

Proof:

(1) Assume $\lambda \in \sigma_p(\Delta_{\dot{\varphi}^\omega})$, and suppose $\varepsilon > 0, n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in S_{\mathrm{nf}}(M)$ are given. Define $\psi := \sum_{i=1}^{n} \varphi_i \in M_*^+$. By assumption, there exists $y \in M' \cap M^\omega$ with $\|y\|_{\dot{\psi}} = 1$ satisfying

$$\sigma_t^{\dot{\varphi}_\omega}(y) = \lambda^{it} \, y, \qquad t \in \mathbb{R}.$$

Take a representative $(y_n)_n$ of y. By Proposition (4.2.25), $\dot{\varphi}_{j_\omega} = \dot{\psi}_\omega (= \dot{\varphi}_\omega)$ holds for $j = 1,2, \dots, n$. Note that since $[\sigma_t^\varphi(x), y]_\varphi^\# = \left[x, \sigma_{-t}^\varphi(y)\right]_\varphi^\# (x, y \in M), M' \cap M^\omega$

is $\sigma_t^{\varphi^\omega}$ -invariant thanks to Theorem (4.2.4). Therefore we have $\sigma_t^{\dot{\varphi}_{j\omega}} = \sigma_t^{\varphi_j^\omega}|_{M' \cap M^\omega} (t \in \mathbb{R}, 1 \leq j \leq n)$. This implies that

$$\Delta_{\varphi_j^\omega}^{\frac{1}{2}} y \xi_{\varphi_j^\omega} = \lambda^{\frac{1}{2}} y \xi_{\varphi_j^\omega} \ \ (1 \leq j \leq n).$$

This means that (Corollary (4.2.6)(ii))

$$\lim_{k \to \omega} \left\| \Delta_{\varphi_j}^{\frac{1}{2}} y_k \xi_{\varphi_j} - \lambda^{\frac{1}{2}} y_k \xi_{\varphi_j} \right\| = 0 \quad (1 \leq j \leq n).$$

Choose $k \in N$ such that the following inequalities hold:

$$\left\| \Delta_{\varphi_j}^{\frac{1}{2}} y_k \xi_{\varphi_j} - \lambda^{\frac{1}{2}} y_k \xi_{\varphi_j} \right\| \leq \varepsilon(1-\varepsilon) \qquad (1 \leq j \leq n),$$

$$\left| \|y_k\|_\psi^2 - 1 \right| < \varepsilon.$$

It follows that for each $1 \leq j \leq n$ and for $x = y_k$, we have

$$\left\| \Delta_{\varphi_j}^{\frac{1}{2}} x \xi_{\varphi_j} - \lambda^{\frac{1}{2}} x \xi_{\varphi_j} \right\| \leq \varepsilon \|x\|_\psi^2 = \varepsilon \sum_{i=1}^{n} \varphi_i(x^* x).$$

By Lemma (4.2.27), we have $M \cong M \overline{\otimes} R_\lambda$.

Conversely, assume $M \cong M \overline{\otimes} R_\lambda$ holds. Fix $\psi \in S_{\text{nf}}(M)$ and put $N := M \overline{\otimes} R_\lambda$. Let $\varphi_\lambda = \otimes_N \text{Tr}(\rho_\lambda \cdot)$ and let $x_n := \pi^{-1}(1 \otimes u_n) \in M$, where

$$u_n := 1^{\otimes_n} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes 1 \cdots \in R_\lambda, \qquad n \in \mathbb{N},$$

and $\pi: M \overset{\cong}{\to} N$ is a^*-isomorphism. Then it holds that $(x_n)_n \in \mathcal{M}^\omega(M)$. Indeed, it is clear that $\|x_n\| = 1, n \geq 1$, and hence $(x_n)_n \in \ell^\infty(\mathbb{N}, M)$ Let $L_\psi: M \overline{\otimes} R_\lambda \to R_\lambda$ be a left-slice map given as the extension of the map L_ψ^0 defined on the algebraic tensor product $M \odot R_\lambda$ by

$$L_{\psi}^{0}\left(\sum_{i} a_i \otimes b_i\right) := \sum_{i} \psi(a_i) b_i\,, \qquad a_i \in M, \qquad b_i \in R_{\lambda}.$$

L_{ψ} is a normal conditional expectation. Let $(b_n)_n \in I_{\omega}(N)$. Using $u_n \varphi_{\lambda} = \lambda^{-1} \varphi_{\lambda} u_n$, we have

$$\|b_n \pi(x_n)\|_{\psi \otimes \varphi_{\lambda}}^{2} = \psi \otimes \varphi_{\lambda}\big((1 \otimes u_n^*) b_n^* b_n (1 \otimes u_n)\big)$$
$$= \varphi_{\lambda}\left(L_{\psi}\big((1 \otimes u_n^*) b_n^* b_n (1 \otimes u_n)\big)\right)$$
$$= \varphi_{\lambda}\big(u_n^* L_{\psi}(b_n^* b_n) u_n\big) = \lambda^{-1} \varphi_{\lambda}\left(u_n u_n^* L_{\psi}(b_n b_n^*)\right)$$
$$\leq \lambda^{-1} \big\|L_{\psi}(b_n^* b_n)\big\|_{\varphi_{\lambda}} \|u_n u_n^*\|_{\varphi_{\lambda}}$$
$$\leq \lambda^{-1} \big|\varphi_{\lambda}\big(L_{\psi}(b_n^* b_n)^2\big)\big|^{\frac{1}{2}} \left(L_{\psi} \text{ is a conditional expectation}\right),$$

and since $(b_n^* b_n)_{n=1}^{\infty} \in I_{\omega}(N)$, we have

$$\left(\varphi_{\lambda}\big(L_{\psi}(b_n^* b_n)^2\big)\right) = \psi \otimes \varphi_{\lambda}((b_n^* b_n)^2) = \|b_n^* b_n\|_{\psi \otimes \varphi_{\lambda}}^{2}$$
$$\xrightarrow{n \to \omega} 0.$$

Therefore $(b_n \pi(x_n))_{n=1}^{\infty} \in \mathcal{L}_{\omega}(N)$. Since $(b_n \pi(x_n))_{n=1}^{\infty} \in \mathcal{L}_{\omega}^{*}(N)$ automatically, we have $(b_n \pi(xn))_{n=1}^{\infty} \in I_{\omega}(N)$ Similarly, we have $(\pi(x_n) b_n)_n \in \mathcal{L}_{\omega}^{*}(N)$ and thus $(\pi(x_n) b_n)_n \in I_{\omega}(N)$, which shows that $(\pi(x_n))_n \in \mathcal{M}^{\omega}(N)$, and hence $(x_n)_n \in \mathcal{M}^{\omega}(M)$. It is then easy to show that $(x_n)^{\omega} \in M' \cap M^{\omega}$. It also holds that $\sigma_t^{\psi \otimes \varphi_{\lambda}}(\pi(x_n)) = \lambda^{it} \pi(x_n)$ for each $t \in \mathbb{R}$, where φ_{λ} is the Powers state and ψ is a normal faithful state on M. Therefore $\varphi := (\psi \otimes \varphi_{\lambda}) \circ \pi \in S_{\mathrm{nf}}(M)$ satisfies $\sigma_t^{\dot{\varphi}_{\omega}}((x_n)^{\omega}) = \lambda^{it} (x_n)^{\omega}$. Therefore $\lambda \in \sigma_p(\Delta_{\dot{\varphi}_{\omega}})$ holds.

Now recall the definition of Connes' asymptotic centralizer.

Definition (4.2.29) [4]:

The asymptotic centralizer M_{ω} of M is defined as the quotient C^*algebra $\mathcal{M}_{\omega}(\mathbb{N}, M)/I_{\omega}(\mathbb{N}, M)$, where

$$\mathcal{M}_\omega(\mathbb{N},M):=\left\{(x_n)_n\in\ell^\infty(\mathbb{N},M);\ \lim_{n\to\omega}\|x_n\psi-\psi x_n\|=0,\forall\psi\in M_*\right\}.$$

M_ω is a finite von Neumann algebra for any M.

Regarding Golodets' and Golodets–Nessonov's work (2) above, we prove next that $(C_M^\omega)_{\dot{\varphi}^\omega}$ is nothing but M_ω when we identify C_M^ω with $M'\cap M^\omega$. Note that we do not need the factoriality of M or the separability of the predual.

Lemma (4.2.30) [4]:

Let $(M_n,\varphi_n)_n$ be a sequence of pairs of σ-finite von Neumann algebras and normal faithful states. Let $(x_n)_n,(y_n)_n\in\mathcal{M}^\omega(M_n,\varphi_n)$. Then we have

$$\|(x_n)^\omega(\varphi_n)^\omega-(\varphi_n)^\omega(y_n)^\omega\|=\lim_{n\to\omega}\|x_n\varphi_n-\varphi_n y_n\|$$

In particular, $(x_n)^\omega\in((M_n,\varphi_n)^\omega)_{\varphi_\omega}$ holds if and only if $\lim_{n\to\omega}\|x_n\varphi_n-\varphi_n x_n\|=0$ holds.

Proof:

We use abbreviated notation as $\ell^\infty,\mathcal{M}^\omega,L_\omega,I_\omega$. Put $C_1:=\|(x_n)^\omega(\varphi_n)^\omega-(\varphi_n)^\omega(y_n)^\omega\|$ and $C_2:=\lim_{n\to\omega}\|x_n\varphi_n-\varphi_n y_n\|$. Let $\varepsilon>0$, and choose $a\in\mathrm{Ball}((M_n,\varphi_n)^\omega)$ such that

$$|\langle a,(x_n)^\omega(\varphi_n)^\omega-(\varphi_n)^\omega(y_n)^\omega\rangle|>C_1-\varepsilon.$$

Since $(M_n,\varphi_n)^\omega$ is a quotient of $\mathcal{M}^\omega$, we may find $(a_n)_n\in\mathcal{M}^\omega$ with $a=(an)\omega$, such that $\|(a_n)_n\|=\sup_{n\geq 1}\|a_n\|\leq 1$.

Therefore we have

$$C_1-\varepsilon<\lim_{n\to\omega}|\langle a_n,x_n\varphi_n-\varphi_n y_n\rangle|\leq\lim_{n\to\omega}\|x_n\varphi_n-\varphi_n y_n\|.$$

Since $\varepsilon>0$ is arbitrary, we have $C_1\leq C_2$.

To prove $C_1\leq C_2$, let $a_n\in\mathrm{Ball}(M_n)(n\in\mathbb{N})$ be such that

$$|\langle a_n,x_n\varphi_n-\varphi_n y_n\rangle|>\|x_n\varphi_n-\varphi_n y_n\|-\frac{1}{n}. \tag{39}$$

By Proposition (4.1.18), there exist $(b_n)_n \in \mathcal{M}^\omega, (c_n)_n \in \mathcal{L}_\omega$, and $(d_n)_n \in \mathcal{L}_\omega^*$ such that $a_n = b_n + c_n + d_n (n \in \mathbb{N})$ and $\|(b_n)^\omega\| \lim_{n\to\omega}\|a_n\| \leq 1$. It follows that

$$\begin{aligned}\langle a_n, x_n\varphi_n - \varphi_n y_n\rangle \\ &= \langle b_n, x_n\varphi_n - \varphi_n y_n\rangle + \langle c_n x_n \xi_{\varphi_n}, \xi_{\varphi_n}\rangle - \langle c_n \xi_{\varphi_n}, y_n^* \xi_{\varphi_n}\rangle \\ &+ \langle x_n \xi_{\varphi_n}, d_n^* \xi_{\varphi_n}\rangle - \langle \xi_{\varphi_n}, d_n^* y_n^* \xi_{\varphi_n}\rangle. \qquad (40)\end{aligned}$$

Since $(c_n)_n \in \mathcal{L}_\omega$ and $(d_n)_n \in \mathcal{L}_\omega^*$, the third and the fourth terms in the right hand side of Eq. (40) will vanish as $n \to \omega$. Also, as $(x_n)_n \in \mathcal{M}^\omega$, Corollary (4.1.13)(i) implies that $(c_n x_n)_n \in \mathcal{L}_\omega$ and $(d_n^* y_n^*)_n \in \mathcal{L}_\omega$, whence the second and the fifth terms will vanish as $n \to \omega$. Therefore we have

$$\lim_{n\to\omega} |\langle a_n, x_n\varphi_n - \varphi_n y_n\rangle| = \lim_{n\to\omega} |\langle b_n, x_n\varphi_n - \varphi_n y_n\rangle|. \qquad (41)$$

Then by Eqs. (39) and (41), we have

$$\begin{aligned}\lim_{n\to\omega}\|x_n\varphi_n - \varphi_n y_n\| &\leq \lim_{n\to\omega}|b_n, x_n\varphi_n - \varphi_n y_n| \\ &= \langle (b_n)^\omega, (x_n)^\omega(\varphi_n)^\omega - (\varphi_n)^\omega (y_n)^\omega\rangle \\ &\leq \|(x_n)^\omega(\varphi_n)^\omega - (\varphi_n)^\omega(y_n)^\omega\|,\end{aligned}$$

whence $C_2 \leq C_1$. This finishes the proof.

Proposition (4.2.31) [4]:

Let M be a σ-finite von Neumann algebra. Let $\varphi \in S_{\mathrm{nf}}(M)$. Then the centralizer of the Golodets state $\dot{\varphi}^\omega$ is M_ω.

Note that Proposition (4.2.31) gives an alternative proof of the fact that M_ω is a (finite) von Neumann algebra.

Proof:

For $M_\omega \subset (M' \cap M^\omega)_{\dot{\varphi}_\omega}$, let $(x_n)^\omega \in M_\omega$. Then for $(y_n)^\omega \in M^\omega$, we have

$$|(y_n x_n - x_n y_n)| = |[x_n, \varphi](y_n)| \leq \|y_n\| \cdot \|x_n, \varphi\| \xrightarrow{n\to\omega} 0.$$

Hence $(x_n)^\omega \in (M^\omega)_{\varphi^\omega} \cap (M' \cap M^\omega) \subset (M' \cap M^\omega)_{\dot{\varphi}^\omega}$ holds.

For $M_\omega \supset (M' \cap M^\omega)_{\dot{\varphi}^\omega}$, let $(x_n)^\omega \in (M' \cap M^\omega)_{\dot{\varphi}^\omega}$. Since $\sigma_t^{\dot{\varphi}^\omega} = \sigma_t^{\varphi^\omega}|_{M' \cap M^\omega} (t \in \mathbb{R})$ (see the proof of Theorem (4.2.32)), we have $\sigma_t^{\varphi^\omega}((x_n)^\omega) = (x_n)^\omega (t \in \mathbb{R})$. Therefore by Lemma (4.2.30), we have

$$(x_n)^\omega \varphi^\omega = \varphi^\omega (x_n)^\omega \Leftrightarrow \lim_{n \to \omega} \|x_n \varphi - \varphi x_n\| = 0.$$

Then, $(x_n)^\omega \in M_\omega$ holds.

Note that the equivalence $(x_n)^\omega \varphi^\omega = \varphi^\omega (x_n)^\omega \Leftrightarrow \lim_{n \to \omega} \|x_n \varphi - \varphi x_n\| = 0$ can be seen using Corollary (4.2.6)(ii).

Section (4.3):

Let M be a σ-finite von Neumann algebra. Connes defined the asymptotic centralizer M_ω (see Definition (4.2.29)) as a generalization of $M' \cap M^\omega$ for the case of type II_1 factor.It is known that if M is σ-finite, and if $(x_n)^\omega \in M' \cap M^\omega$ satisfies $\lim_{n\to\omega}\|x_n\varphi - \varphi x_n\| = 0$ for one $\varphi \in S_{\mathrm{nf}}(M)$, then $(x_n)^\omega \in M_\omega$. Therefore the existence of a normal faithful tracial state shows that $M' \cap M^\omega = M_\omega$ for a finite von Neumann algebra. The same is true for type II_∞ factors. However, for type III factors, it is often the case that $M_\omega \subsetneq M' \cap M^\omega$.

Example (4.3.1) [4]:

The following example has been known to experts. We add it for convenience. Let $(R_\lambda, \varphi) = \otimes_{n\in\mathbb{N}}(M_2(\mathbb{C}), \mathrm{Tr}(\rho_\lambda\ \cdot))$ be the Powers factor of type $III\lambda(0 < \lambda < 1)$, where $\rho_\lambda = \mathrm{diag}(\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda})$. Let

$$u_n := 1^{\otimes n} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \in R_\lambda, \qquad n \geq 1.$$

Then $(u_n)_n \in \mathcal{M}^\omega(R_\lambda)$ and $(u_n)^\omega \in R'_\lambda \cap R_\lambda^\omega$. On the other hand, we have

$$\varphi u_n = \lambda u_n \varphi, \quad n \in \mathbb{N}.$$

Therefore $\|u_n\varphi - \varphi u_n\| = (1-\lambda) \neq 0 (n \in \mathbb{N})$, and hence $(u_n)^\omega \notin (R_\lambda)_\omega$. Moreover, $R'_\lambda \cap R_\lambda^\omega$ is a type III_λ factor. To see this, $(R_\lambda)_\omega$ is a type II_1 factor. Therefore by Proposition (4.2.31), the centralizer of the Golodets state $\dot{\varphi}_\omega = \varphi_\omega|_{R'_\lambda \cap (R_\lambda)^\omega}$ is a factor, whence by Corollary (4.2.6)(iii), we have

$$\Gamma(\sigma^{\dot{\varphi}_\omega}) = \mathrm{Sp}(\sigma^{\dot{\varphi}_\omega}) = \log\left(\sigma(\Delta_{\dot{\varphi}_\omega})\backslash\{0\}\right)$$

$$\subset \log\left(\sigma(\Delta_{\varphi_\omega})\{0\}\right) = \log\left(\sigma(\Delta_\varphi)\{0\}\right)$$

$$= (\log\lambda)\mathbb{Z}.$$

On the other hand, we have $(\log\lambda)\mathbb{Z} \subset \mathrm{Sp}(\sigma^{\dot{\varphi}^\omega})$. Therefore as $\Gamma(\sigma^{\dot{\varphi}^\omega}) = \log(S(R'_\lambda \cap R_\lambda^\omega)\backslash\{0\})$, we have

$$S(R'_\lambda \cap R_\lambda^\omega) = \{\lambda^n; n \in \mathbb{Z}\} \cup \{0\}.$$

This proves that $R'_\lambda \cap R_\lambda^\omega$ is a type III_λ factor.

In spite of the above example, Ueda asked whether $M_\omega = \mathbb{C}$ implies $M' \cap M^\omega = \mathbb{C}$. We prove that the answer to his question is affirmative when M has separable predual.

Lemma (4.3.2) [4]:

Let M be a von Neumann algebra, φ be a normal faithful state on M with $M_\varphi = \mathbb{C}$. Then M is either $\mathbb{C}$ or a factor of type III_1.

Proof:

Let H be a Hilbert space on which M acts. Since $Z(M) \subset M_\varphi = \mathbb{C}, M$ is a factor. Suppose M is semifinite with a normal faithful semifinite trace τ. Then there exists a positive self-adjoint operator $h \in L^1(M,\tau)$ with $\tau(h) = 1$ such that $\varphi = \tau(h\,\cdot)$ holds. It is well known that this implies $\sigma_t^\varphi\backslash(x) = h^{it}xh^{-it}$ for every $x \in M$ and $t \in \mathbb{R}$. Let A be the abelian von Neumann algebra generated by all spectral projections of h. Then for $x \in M, x \in M_\varphi$ holds if and only if x commutes with h^{it} for all $t \in \mathbb{R}$, which is equivalent to the condition $x \in A'$, hence $M_\varphi = A' \cap M = \mathbb{C}$. Since $A \subset A' \cap M = \mathbb{C}, h$ must be a multiple of 1 and τ is a tracial state. This implies that $\varphi = \tau$, and

$$M_\varphi = M_\tau = M = \mathbb{C}.$$

Suppose next that M is of type $\mathrm{III}_\lambda (\lambda \neq 1)$. Then $(0 < \lambda < 1$ case) and $(\lambda = 0$ case) of Connes, there exists a maximal abelian subalgebra A of M_φ which is maximal abelian in M. This in particular means that M_φ cannot be $\mathbb{C}$. This finishes the proof.

Theorem (4.3.3) [4]:

Let M be a von Neumann algebra with a separable predual for which $M_\omega = C$ holds. Then $M' \cap M^\omega = \mathbb{C}$ holds.

The following lemma is well-known.

Proof:

Put $N := M' \cap M^\omega$. Take an arbitrary $\varphi \in S_{\rm nf}(M)$. Since $\mathrm{Z}(M) \subset M_\omega = \mathbb{C}, M$ is a factor. By Proposition (4.2.25), the Golodets state $\dot{\varphi}_\omega := \varphi^\omega|_N \in S_{\rm nf}(N)$ does not depend on the choice of φ. By Proposition (4.2.31), $N_{\dot{\varphi}^\omega} = \mathbb{C}$. Then by Lemma (4.3.2), N is either $\mathbb{C}$ or a factor of type III_1. Suppose N is a type III1 factor and we shall get a contradiction. Fix $0 < \lambda < 1$. Since N is of type III, there exists an automorphism $\alpha: N \to N \otimes M_2(\mathbb{C})$. Define $\psi \in S_{\rm nf}(N)$ by

$$\psi := [\dot{\varphi}_\omega \otimes \mathrm{Tr}(\rho_\lambda \cdot)] \circ \alpha,$$

where $\rho_\lambda := \mathrm{diag}\left(\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda}\right)$. Let $\varepsilon > 0$ be given. By the Connes–Størmer transitivity (note that the transitivity holds without any assumption on the predual), there exists $u \in \mathcal{U}(N)$ such that

$$\|\dot{\varphi}_\omega - u\psi u^*\| < \varepsilon. \tag{42}$$

Define a 2×2 matrix unit $\{f_{i,j}\}_{i,j=1}^2$ in N by

$$f_{ij} := u^* \alpha^{-1}\big(1 \otimes e_{i,j}\big)u, \qquad 1 \le i, j \le 2,$$

where $\{e_{i,j}\}_{i,j=1}^2$ is the standard matrix unit in $M_2(\mathbb{C})$. For $x \in N$, write $\alpha(x) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, where $x_{ij} \in N$. By a straightforward computation, we have

$$\begin{aligned}
[\psi\alpha^{-1}(1 \otimes e_{12})](x) &= \psi(\alpha^{-1}(1 \otimes e_{12})x) \\
&= [\dot{\varphi}_\omega \otimes \mathrm{Tr}(\rho_\lambda \cdot)]\begin{pmatrix} x_{21} & x_{22} \\ 0 & 0 \end{pmatrix} \\
&= \frac{\lambda}{1+\lambda}\dot{\varphi}_\omega(x_{21}), \\
[\alpha^{-1}(1 \otimes e_{12})\psi](x) &= [\dot{\varphi}_\omega \otimes \mathrm{Tr}(\rho_\lambda \cdot)]\begin{pmatrix} 0 & x_{11} \\ 0 & x_{21} \end{pmatrix} \\
&= \frac{1}{1+\lambda}\dot{\varphi}_\omega(x_{21}).
\end{aligned}$$

Doing similar computations, we have the following equalities:

$$\psi\alpha^{-1}(1 \otimes e_{ii}) = \alpha^{-1}(1 \otimes e_{ii})\psi \qquad (i = 1,2), \tag{43}$$

$$\psi\alpha^{-1}(1 \otimes e_{12}) = \lambda\alpha^{-1}(1 \otimes e_{12})\psi, \tag{44}$$

$$\psi\alpha^{-1}(1 \otimes e_{21}) = \lambda^{-1}\alpha^{-1}(1 \otimes e_{21})\psi. \tag{45}$$

Using Eq. (42) and Eqs. (43)–(45), it follows that

$$\begin{aligned}\|\dot{\varphi}_\omega f_{12} - \lambda f_{12}\dot{\varphi}_\omega\| &= \|\dot{\varphi}_\omega u^*\alpha^{-1}(1 \otimes e_{12})u - \lambda u^*\alpha^{-1}(1 \otimes e_{12})u\dot{\varphi}_\omega\| \\ &= \|u\dot{\varphi}_\omega u^*\alpha^{-1}(1 \otimes e_{12}) - \lambda\alpha^{-1}(1 \otimes e_{12})u\dot{\varphi}_\omega u^*\| \\ &\leq \|(u\dot{\varphi}_\omega u^* - \psi)\alpha^{-1}(1 \otimes e_{12})\| + \|\lambda\alpha^{-1}(1 \otimes e_{12})(\psi - u\dot{\varphi}_\omega u^*)\| \\ &\leq (1 + \lambda)\varepsilon.\end{aligned}$$

Doing similar computations, we obtain

$$\|\dot{\varphi}_\omega f_{ii} - f_{ii}\dot{\varphi}_\omega\| \leq 2\varepsilon \quad (i = 1,2), \tag{46}$$

$$\|\dot{\varphi}_\omega f_{12} - \lambda f_{12}\dot{\varphi}_\omega\| \leq (1 + \lambda)\varepsilon, \tag{47}$$

$$\|\dot{\varphi}_\omega f_{21} - \lambda - 1 f_{21}\dot{\varphi}_\omega\| \leq (1 + \lambda^{-1})\varepsilon. \tag{48}$$

Let $\{a_n\}_{n=1}^{\infty}$ be a $\|\cdot\|_\varphi^\sharp$-dense sequence of the unit ball of M.

Claim 1. For each $n \in \mathbb{N}$ there exist $f_{ij}^{(n)} \in M (i, j = 1,2)$ satisfying the following conditions:

(i) $\left\|f_{ij}^{(n)}\right\| \leq 1 (i, j = 1,2)$.

(ii) $\left\|\varphi f_{ii}^{(n)} - f_{ii}^{(n)}\varphi\right\| \leq \frac{1}{n} (i = 1,2)$.

(iii) $\left\|\varphi f_{12}^{(n)} - \lambda f_{12}^{(n)}\varphi\right\| \leq \frac{1}{n}$.

(iv) $\left\|\varphi f_{21}^{(n)} - \lambda^{-1} f_{21}^{(n)}\varphi\right\| \leq \frac{1}{n}$.

(v) $\left\|\left(f_{ij}^{(n)}\right)^* - f_{ji}^{(n)}\right\|_\varphi^\sharp \leq \frac{1}{n} (i, j = 1,2)$.

(vi) $\left\|f_{ij}^{(n)} a_m - a_m f_{ij}^{(n)}\right\|_\varphi^\sharp \leq \frac{1}{n} (1 \leq m \leq n, i, j = 1,2)$.

(vii) $\left\| f_{ij}^{(n)} f_{kl}^{(n)} - \delta_{j\,k} f_{il}^{(n)} \right\|_{\varphi}^{\#} \leq \frac{1}{n} (i, j, k, l = 1,2).$

(viii) $\left\| f_{11}^{(n)} + f_{22}^{(n)} - 1 \right\|_{\varphi}^{\#} \leq \frac{1}{n}.$

By Eqs. (46)–(48), there exists a matrix unit $(f_{ij})_{i,j=1}^{2} \in M' \cap M^{\omega}$ satisfying the following conditions.

$$\|\dot{\varphi}_{\omega} f_{ii} - f_{ii} \dot{\varphi}_{\omega}\| \leq \frac{1}{2n} (i = 1,2),$$

$$\|\dot{\varphi}_{\omega} f_{12} - \lambda f_{12} \dot{\varphi}_{\omega}\| \leq \frac{1}{2n},$$

$$\|\dot{\varphi}_{\omega} f_{21} - \lambda^{-1} f_{21} \dot{\varphi}_{\omega}\| \leq \frac{1}{2n}.$$

Since $M' \cap M^{\omega}$ is $\sigma_t^{\varphi^{\omega}}$-invariant, by Takesaki's Theorem, there exists a normal faithful conditional expectation $E: M^{\omega} \to M' \cap M^{\omega}$ with $\varphi^{\omega} = \dot{\varphi}_{\omega} \circ E$. Since $f_{ij} \in M' \cap M^{\omega}$, we have $E(f_{ij}) = f_{ij}$. Therefore for every $a \in M^{\omega}$, we have

$$(\varphi^{\omega} f_{ii} - f_{ii} \varphi^{\omega})(a) = \varphi^{\omega}(f_{ii} a - a f_{ii}) = \dot{\varphi}_{\omega} \circ E(f_{ii} a - a f_{ii})$$
$$= \dot{\varphi}_{\omega}(f_{ii} E(a) - E(a) f_{ii}) = (\dot{\varphi}_{\omega} f_{ii} - f_{ii} \dot{\varphi}_{\omega})\big(E(a)\big),$$

and hence $\|\varphi^{\omega} f_{ii} - f_{ii} \varphi^{\omega}\| \leq \|\dot{\varphi}_{\omega} f_{ii} - f_{ii} \dot{\varphi}_{\omega}\|$. Since $\|\varphi^{\omega} f_{ii} - f_{ii} \varphi^{\omega}\| \geq \|\dot{\varphi}_{\omega} f_{ii} - f_{ii} \dot{\varphi}_{\omega}\|$, we have

$$\|\varphi^{\omega} f_{ii} - f_{ii} \varphi^{\omega}\| = \|\dot{\varphi}_{\omega} f_{ii} - f_{ii} \dot{\varphi}_{\omega}\| \leq \frac{1}{2n} \ (i = 1,2).$$

Similarly, we have

$$\|\varphi^{\omega} f_{12} - \lambda f_{12} \varphi^{\omega}\| \leq \frac{1}{2n},$$

$$\|\varphi^{\omega} f_{21} - \lambda f_{21} \varphi^{\omega}\| \leq \frac{1}{2n}.$$

Choose $\left(f_{ij}^{(k)}\right)_k \in \mathcal{M}^{\omega}(i,j=1,2)$ such that $f_{ij} = \left(f_{ij}^{(k)}\right)^{\omega}$. They can be chosen to satisfy $\left\|f_{ij}^{(k)}\right\| \leq 1$. By the definition of fij and the matrix unit property, together with Lemma (4.2.30), we have

(ii)∗ $\lim_{k\to\omega} \left\|\varphi f_{ii}^{(k)} - f_{ii}^{(k)}\varphi\right\| \leq \frac{1}{2n}(i = 1,2)$.

(iii)∗ $\lim_{k\to\omega} \left\|\varphi f_{12}^{(k)} - \lambda f_{12}^{(k)}\varphi\right\| \leq \frac{1}{2n}$.

(iv)∗$\lim_{k\to\omega} \left\|\varphi f_{21}^{(k)} - \lambda^{-1} f_{21}^{(k)}\varphi\right\| \leq \frac{1}{2n}$.

(v)∗ $\lim_{k\to\omega} \left\|(f_{ij}^{(k)}) * - f_{j\,i}^{(k)}\varphi\right\|_{\varphi}^{\sharp} = 0(i,j = 1,2)$.

(vi)∗ $\lim_{k\to\omega} \left\|f_{ij}^{(k)} a_m - a_m f_{ij}^{(k)}\right\|_{\varphi}^{\sharp} = 0(m \geq 1, i,j = 1,2)$.

(vii)∗ $\lim_{k\to\omega} \left\|f_{ij}^{(k)} f_{lm}^{(k)} - \delta_{jl} f_{im}^{(k)}\right\|_{\varphi}^{\sharp} = 0(i,j,l,m = 1,2)$.

(viii)∗ $\lim_{k\to\omega} \left\|f_{11}^{(k)} + f_{22}^{(k)} - 1\right\|_{\varphi}^{\sharp} = 0$.

For fixed n, there are only finitely many conditions. Therefore there exists $k = k(n) \in \mathbb{N}$ such that $f_{ij}^{(k(n))}$ satisfies all the conditions (i)–(viii) in the claim.

Claim 2. If $\left(f_{ij}^{(n)}\right)_{i,j=1}^{2} \in M$ satisfies conditions (i)–(viii) in Claim 1 for all $n \geq 1$, then $\left(f_{ij}^{(n)}\right)_n \in \mathcal{M}^{\omega}$ holds for $i,j = 1,2$.

By (i), $\left(f_{ij}^{(n)}\right)_n \in \ell^{\infty}(\mathbb{N}, M)$ holds. Let $(b_n)_{n=1}^{\infty} \in I_{\omega}$ with $\sup_{n\geq 1}\|b_n\| \leq 1$. Then $\left(f_{ij}^{(n)} b_n\right) \in \mathcal{L}_{\omega}$ holds automatically. On the other hand we have

$$\varphi\left(f_{ij}^{(n)} b_n \left(f_{ij}^{(n)} b_n\right)^*\right) \leq \left|\varphi\left(f_{ij}^{(n)} b_n b_n^* \left\{\left(f_{ij}^{(n)}\right)^* - f_{ji}^{(n)}\right\}\right)\right| + \left|\varphi\left(f_{ij}^{(n)} b_n b_n^* f_{ji}^{(n)}\right)\right|$$

$$\leq \left\| b_n b_n^* \left(f_{ij}^{(n)}\right)^* \right\|_\varphi \left\| \left(f_{ij}^{(n)}\right)^* - f_{ji}^{(n)} \right\|_\varphi + \left| \left(\varphi f_{ij}^{(n)} - c(i,j) f_{ij}^{(n)} \varphi\right) \left(b_n b_n^* f_{ji}^{(n)}\right) \right|$$

$$+ c(i,j) \left| \varphi \left(b_n b_n^* f_{ji}^n f_{ij}^{(n)}\right) \right|$$

$$\leq \|b_n\|^2 \cdot \frac{1}{n} + \frac{1}{2n} \cdot \|b_n\|^2 + c(i,j) \|b_n b_n^*\|_\varphi \left\| f_{j\,i}^{(n)} f_{ij}^{(n)} \right\|_\varphi$$

$$\leq \frac{3}{2n} + c(i,j) \|b_n b_n^*\|_\varphi \xrightarrow{n \to \omega} 0,$$

where

$$c(i,j) := \begin{cases} 1 & (i = j), \\ \lambda & (i = 1, j = 2), \\ \lambda^{-1} & (i = 2, j = 1). \end{cases}$$

This shows that $\left(f_{ij}^{(n)} b_n\right)_n \in \mathcal{L}_\omega^*$, and hence $\left(f_{ij}^{(n)} b_n\right) \in I_\omega = \mathcal{L}_\omega \cap \mathcal{L}_\omega^*$. Similarly, we have $\left(b_n f_{ij}^{(n)}\right)_n \in I_\omega$. This proves that $\left(f_{ij}^{(n)}\right)_n \in \mathcal{M}^\omega$ for $i, j = 1,2$.

Therefore by Claim 1 and Claim 2, we see that $(g_{ij})_{i,j=1}^2$, where $g_{ij} := \left(f_{ij}^{(n)}\right)^\omega$ is a well-defined matrix unit in $M' \cap M^\omega$, and using conditions (i)–(viii) in Claim 1, we have

$$g_{ii} \dot{\varphi}_\omega = \dot{\varphi}_\omega g_{ii} (i = 1,2),$$

$$\dot{\varphi}_\omega g_{12} = \lambda g_{12} \dot{\varphi}_\omega,$$

$$\dot{\varphi}_\omega g_{21} = \lambda^{-1} g_{21} \dot{\varphi}_\omega.$$

In particular, $g_{ii} \in N_{\dot{\varphi}_\omega} (i = 1,2)$ holds and λ is in the point spectrum of $\Delta_{\dot{\varphi}_\omega}$. Then we have

$$\dot{\varphi}_\omega(g_{11}) = \dot{\varphi}_\omega(g_{12} g_{21}) = (g_{21} \dot{\varphi}_\omega)(g_{12})$$

$$= \lambda (\dot{\varphi}_\omega g_{21})(g_{12}) = \lambda \dot{\varphi}_\omega (g_{21} g_{12})$$

$$= \lambda \dot{\varphi}_\omega(g_{22}),$$

and hence $\dot{\varphi}_\omega(g_{11}) = \frac{\lambda}{1+\lambda}$, which is neither 0 nor 1. Therefore $g_{11} \in N_{\dot{\varphi}_\omega}$ is a nontrivial projection. This implies $\dim(N_{\dot{\varphi}_\omega}) \geq 2$, a contradiction. Hence N must be $\mathbb{C}$.

Finally, we remark that there is no difference between M_ω and $M' \cap M^\omega$ when M is of type III_0.

Proposition (4.3.4) [4]:

If M is a σ-finite type III_0 factor, then $M' \cap M^\omega$ is a finite von Neumann algebra and $M' \cap M^\omega = M_\omega$ holds.

Proof:

Let $\varphi \in S_{\mathrm{nf}}(M)$. By Proposition (4.2.25), the Golodets state $\dot{\varphi}_\omega = \varphi^\omega|_{M' \cap M^\omega}$ does not depend on φ. Hence by Corollary (4.2.6)(iii), we have

$$\sigma\left(\Delta_{\dot{\varphi}_\omega}\right) = \bigcap_{\psi \in S_{\mathrm{nf}}(M)} \sigma\left(\Delta_{\psi_\omega}\right) \subset \bigcap_{\psi \in S_{\mathrm{nf}}(M)} \sigma\left(\Delta_{\psi^\omega}\right)$$

$$= \bigcap_{\psi \in S_{\mathrm{nf}}(M)} \sigma(\Delta_\psi) = S(M) = \{0,1\},$$

whence $\sigma(\Delta_{\dot{\varphi}_\omega}) = \{1\}$ because $0 \notin \sigma_p(\Delta_{\dot{\varphi}_\omega})$. This shows that $\dot{\varphi}_\omega$ is a normal faithfultrace on $M' \cap M^\omega$. Since M_ω is the centralizer of $\dot{\varphi}_\omega$ by Proposition (4.2.31), we see that $M' \cap M^\omega \omega = M_\omega$ holds.

We study the factoriality and the Murray–von Neumann–Connes type of the ultraproduct of factors.

The answers to factoriality/type questions for the Ocneanu ultrapower M^ω of a semifinite factor M has been known. In fact, it has been known to experts that for a von Neumann algebra M with separable predual, $(M \mathbin{\overline{\otimes}} \mathbb{B}(H))^\omega \cong M^\omega \mathbin{\overline{\otimes}} \mathbb{B}(H)$ and $(M \mathbin{\overline{\otimes}} \mathbb{B}(H))_\omega \cong M_\omega \otimes \mathbb{C}$ holds, where H is a separable Hilbert space. The proof can be found. On the other hand, it is well-known that M^ω is a type II_1 factor if so is M. This shows the following folklore result:

Proposition (4.3.5) [4]:

Let M be a semifinite factor with separable predual. Then M^{ω} is a factor. If M is of type $I_n(n \in \mathbb{N} \cup \{\infty\}), \mathrm{II}_1$ or II_{∞}, so is M^{ω}.

On the other hand, the situation for the factoriality of the Groh–Raynaud ultraproduct is very different. Based on the local reflexivity principle for Banach spaces and the fact that $\mathbb{B}(H)^{**}$ is not semifinite, Raynaud showed that $\prod^{\mathcal{U}}\mathbb{B}(H)$ is not semifinite (for a free ultrafilter $\mathcal{U}$ on a suitable index set I and infinite-dimensional H). We prove that $\prod^{\omega}R$ is not semifinite, where R is the hyperfinite type II_1 factor. For a fixed $\lambda \in (0,1)$, put $\rho_{\lambda} = \mathrm{diag}\left(\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda}\right) \in M_2(\mathbb{C}) +$, and let $R_{\lambda} = \otimes_N (M_2(\mathbb{C}), \mathrm{Tr}(\rho_{\lambda}\ \cdot))$ be the Powers factor of type III_{λ}. Define $\varphi_n \in S_{\mathrm{nf}}(R)$ by

$$\varphi_n := \bigotimes_{k=1}^{n} \mathrm{Tr}(\rho_{\lambda}\ \cdot) \otimes \bigotimes_{k=n+1}^{\infty} \frac{1}{2}\mathrm{Tr}, \quad n \geq 1.$$

Proposition (4.3.6) [4]:

There exists a normal injective *-homomorphism $\pi: R_{\lambda} \to (R, \varphi_n)^{\omega}$ whose range is a normal faithful conditional expectation $\varepsilon: (R\varphi_n)^{\omega} \to \pi(R_{\lambda})$.

This shows that

Proof:

Put $A_m := \otimes_{k=1}^{m} M_2(\mathbb{C}) \otimes \mathbb{C} \otimes \mathbb{C} \otimes \cdots$ considered as a subalgebra of R_{λ}, and let $\hat{A}_m$ be the same algebra now considered as a subalgebra of R. Moreover, put

$$A \coloneqq \bigcup_{m=1}^{\infty} A_m \subset R_{\lambda}, \qquad \hat{A} \coloneqq \bigcup_{m=1}^{\infty} \hat{A}_m \subset R.$$

For $x \in A$, let $\hat{x}$ denote the corresponding element in $\hat{A}$. Define now a *-monomorphism $\pi_0: A \to \ell^{\infty}(\mathbb{N}, R)$ by $\pi_0(x) = (\hat{x})_n, x \in A$ (constant sequence). Note that for $x \in A_m$ ($m \in \mathbb{N}$ fixed), we have

$$\varphi_n(\hat{x}) = \varphi_{\lambda}(x), \qquad \sigma_t^{\varphi_n}(x) = \widehat{\sigma_t^{\varphi_{\lambda}}(x)}, \qquad n \geq m.$$

Since $\sigma(\Delta_{\mathrm{Tr}(\rho_\lambda\cdot)}) = \sigma(\rho_\lambda)\cdot\sigma(\rho_\lambda^{-1}) = \{\lambda, 1, \lambda^{-1}\}$, we have

$$\sigma\left(\bigotimes_{k=1}^{m}\Delta_{\mathrm{Tr}(\rho_\lambda\cdot)}\right) = \{\lambda^m, \lambda^{m-1}, \dots, \lambda^{-m}\}.$$

Therefore it holds that

$$\hat{A}_m \subset R(\sigma^{\varphi_n}, [m\log\lambda, -m\log\lambda])$$

for all nm. Thus by Lemma (4.2.11), $\pi_0(A_m) \subset \mathcal{M}^\omega(R, \varphi_n)$ holds for all $m \in \mathbb{N}$ and hence also $\pi_0(A) \subset \mathcal{M}^\omega(R, \varphi_n)$ holds. Let $\pi_1 : A \to M^\omega := (R, \varphi_n)^\omega$ be π_0 composed with the quotient map from $\mathcal{M}^\omega(R, \varphi_n)$ onto $M^\omega = \mathcal{M}^\omega(R, \varphi_n)/I_\omega(R, \varphi_n)$. Then it is elementary to check that

$$\varphi^\omega\big(\pi_1(x)\big) = \varphi_\lambda(x), \qquad x \in A,$$

where $\varphi^\omega := (\varphi_n)^\omega$. Using Theorem (4.2.4), we also have

$$\sigma_t^{\varphi^\omega}(\pi_1(x)) = \pi_1\big(\sigma_t^{\varphi_\lambda}(x)\big), \qquad x \in A, t \in \mathbb{R}.$$

Therefore π_1 extends to a normal $*$-monomorphism π of $R_\lambda = \bar{A}^{sot}$ onto a von Neumann subalgebra $\pi(R_\lambda)$ of M^ω, which is invariant under $\sigma_t^{\varphi^\omega}$ $(t \in \mathbb{R})$, whence there is a normal faithful conditional φ^ω-preserving expectation of M^ω onto $\pi(R_\lambda)$.

Let M be a σ-finite type $\mathrm{III}_\lambda (0 < \lambda \le 1)$ factor. We show that $\prod^\omega M$, as well as $(M, \varphi_n)^\omega$, is again a type III_λ factor, and the isomorphism class of $(M, \varphi_n)^\omega$ does not depend on the choice of $(\varphi_n)_n \subset S_{\mathrm{nf}}(M)$. To do this, we first recall the state space diameter of factors. Let M be a von Neumann algebra. Then an equivalence relation $\sim$ on $\mathrm{S_n}(M)$ is defined by $\varphi \sim \psi$ if they are approximately unitarily equivalent, i.e., there is a sequence of unitaries $(u_n)_n \subset \mathcal{U}(M)$ such that $\lim_{n\to\infty}\|\varphi - u_n\psi u_n^*\| = 0$. Denote by $[\varphi]$ the equivalence class in $\mathrm{S_n}(M)$ represented by $\varphi \in \mathrm{S_n}(M)$. Then $\mathrm{S_n}(M)/\sim$ is a metric space by

$$d([\varphi], [\psi]) := \inf_{u \in \mathcal{U}(M)}\|\varphi - u\psi u^*\|, \quad [\varphi][\psi] \in \mathrm{S_n}(M)/\sim.$$

Proposition (4.3.7) [4]:

$(R, \varphi_n)^\omega$ is not semifinite.

Therefore, we have

Theorem (4.3.8) [4]:

$\prod^\omega R$ is not semifinite, and not a factor.

Definition (4.3.9) [4]:

The state space diameter of M, denoted as $d(M)$ is defined by

$$d(M) := \sup_{\varphi,\psi \in \mathrm{S_n(M)}} d([\varphi],[\psi]).$$

It holds that $d(M) \leq 2$, and $d(M) = 2$ if M is not a factor. By the result of Connes Størmer, Connes–Haagerup–Størmer, and Haagerup–Størmer, the explicit form of $d(M)$ is given as follows.

Theorem (4.3.10) [4]:

Let M be a factor. Then the $d(M)$ is

(i) $2\left(1-\frac{1}{n}\right)$ if M is of type $I_n (n \in \mathbb{N} \cup \{\infty\})$.
(ii) 2 if M is of type II.
(iii) $2\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}$ if M is of type $\mathrm{III}_\lambda (0 \leq \lambda \leq 1)$.

Let $(M_n, H_n)_n$ be a sequence of standard von Neumann algebras, and define the Groh–Raynaud ultraproduct $N = \prod^\omega M_n$. We will show the diameter formula $d(N) = \lim_{n\to\omega} d(M_n)$.

Lemma (4.3.11) [4]:

Let $(\varphi_n)_n, (\psi_n)_n \in \prod_{n\in\mathbb{N}} \mathrm{S_n(M)}$ and let $\varphi = (\varphi_n)_n$ and $\psi = (\psi_n)_n$ be the corresponding normal states on N (see Theorem (4.1.25)). Then

$$d([\varphi],[\psi]) = \lim_{n\to\omega} d([\varphi_n],[\psi_n]).$$

Proof:

For each $n \in \mathbb{N}$, choose a unitary $u_n \in M_n$ such that

$$\|\varphi_n - u_n\psi_n u_n^*\| \leq d([\varphi_n],[\psi_n]) + \frac{1}{n},$$

then with $u := (u_n)_\omega \in N$ we have

$$\|\varphi - u\psi u^*\| = \lim_{n\to\omega} \|\varphi_n - u_n\psi_n u_n^*\| \leq \lim_{n\to\omega} d([\varphi_n],[\psi_n]).$$

Hence $d([\varphi],[\psi]) \lim_{n\to\omega} d([\varphi_n],[\psi_n])$.

For the converse inequality, we use that the unitary group of $\pi_\omega((M_n)_\omega)$ is Strongly $*$-dense in the unitary group of N by the Kaplansky density Theorem (cf. Definition (4.1.7)). Hence given $\varepsilon > 0$, we may choose a unitary $u_n \in M_n$ for each $n \in \mathbb{N}$, such that with $u := (u_n)_\omega$, we have

$$\|\varphi - u\psi u^*\| \leq d([\varphi],[\psi]) + \varepsilon.$$

But then

$$\lim_{n\to\omega} d([\varphi_n],[\psi_n]) \leq \lim_{n\to\omega} \|\varphi_n - u_n\psi_n u_n^*\| = \|\varphi - u\psi u^*\| \leq d([\varphi],[\psi]) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $d([\phi],[\psi]) \geq \lim_{n\to\omega} d([\varphi_n],[\psi_n])$.

Lemma (4.3.12) [4]:

With the above notation, $d(N) = \lim_{n\to\omega} d(M_n)$.

Proof:

For all $\varphi, \psi \in \mathrm{S_n}(N)$ we may, by Corollary (4.1.26), choose normal states $(\varphi_n), (\psi_n) \prod_{n\in\mathbb{N}} \mathrm{S_n}(N)$ such that $\varphi = (\varphi_n)_\omega$ and $\psi = (\psi_n)_\omega$. By Lemma (4.3.11),

$$d([\varphi_n],[\psi_n]) = \lim_{n\to\omega} d([\varphi_n],[\psi_n]) \leq \lim_{n\to\omega} d(M_n).$$

Hence $d(N) \leq \lim_{n\to\omega} d(M_n)$.

Conversely, we may for each $n \in \mathbb{N}$ choose $\varphi_n, \psi_n \in \mathrm{S_n}(M_n)$ such that

$$d([\varphi_n],[\psi_n]) \geq d(M_n) - \frac{1}{n}, \ \ n \in \mathbb{N}.$$

Let $\varphi := (\varphi_n)_\omega$ and $\psi := (\psi_n)_\omega$. By Lemma (4.3.11), we get (taking the limit of the inequalities above): $d([\phi],[\psi]) \geq \lim_{n\to\omega} d(M_n)$. Hence $d(N) \geq \lim_{n\to\omega} d(M_n)$.

Theorem (4.3.13) [4]:

Let M be a σ-finite factor of type $\mathrm{III}_\lambda(\lambda \neq 0)$. Then $\prod^\omega M$ is a type III_λ factor. Moreover, for any sequence $(\varphi_n)_n \subset \mathrm{S_{nf}}(M), (M,\varphi_n)^\omega \cong M^\omega$ is also a factor of type III_λ.

Proof:

Let $p := \mathrm{supp}(\varphi_\omega)$, where $\varphi_\omega = (\varphi_n)_\omega \in (M_*)_\omega$. Then by Proposition (4.1.17), we have

$$(M,\varphi_n)^\omega \cong pNp, \quad N := \prod^\omega M.$$

By Theorem (4.3.10), the state space diameter of N is

$$d(N) = \lim_{n\to\omega} d(M) = 2\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}.$$

Hence N is a type III_λ factor, so is its corner pNp. Since all σ-finite projections in a type III factor are equivalent, all $(M,\varphi_n)^\omega$'s are mutually isomorphic.

Remark (4.3.14) [4]:

Let M be a σ-finite factor of type $\mathrm{III}_\lambda(0<\lambda<1)$. Then the factoriality of M^ω can be shown using Theorem (4.2.4).

Proof:

Let $x \in Z(M^\omega)$. Let $\varphi \in S_{\mathrm{nf}}(M)$ be such that $\sigma_T^\varphi = \mathrm{id}$, where $T = -2\pi/\log\lambda$. By Proposition (4.2.24), we have

$$x \in Z(M^\omega) \subset (M^\omega)_{\varphi^\omega} = \left(M_\varphi\right)^\omega.$$

Then by Takesaki's Theorem for periodic state, M_φ is a type II_1 factor and $\sigma(\Delta_\varphi) = \{\lambda^n; n \in \mathbb{Z}\} \cup \{0\}$, whence $(M_\varphi)^\omega$ is also a type II_1 factor by a standard argument. This shows that

$$x \in \left(M_\varphi\right)^\omega \cap (M^\omega)' \subset Z\left(\left(M_\varphi\right)^\omega\right) = \mathbb{C}.$$

Therefore M^ω is a factor, and since $(M^\omega)_{\varphi^\omega}$ is a factor, we have

$$S(M^\omega) = \sigma\left(\Delta_{\varphi^\omega}\right) = \sigma\left(\Delta_\varphi\right) = \{\lambda^n; n \in \mathbb{Z}\} \cup \{0\}.$$

This shows that M^ω is a type III_λ factor.

As we have seen, in the case of type $\mathrm{III}_\lambda(\lambda \neq 0)$ factor, the Ocneanu ultraproduct $(M, \varphi_n)^\omega$ does not depend on the choice of $(\varphi_n)_n$. In this section we see that the situation is different for the case of type III_0 factors. Moreover, we will show that M^ω is not a factor.

Lemma (4.3.15) [4]:

Let α be a continuous action of a locally compact abelian group G on a factor M. Denote by G the Pontrjagin dual of G. Then the family $\mathcal{F}$ of subsets of G of the form $\{\mathrm{Sp}(\alpha^e) + K\}$, where e is a non-zero projection in M^α and K is a compact neighborhood of 0 in G, forms a directed set with intersection $\Gamma(\alpha)$.

Lemma (4.3.16) [4]:

Let σ be a continuous action of $\mathbb{R}$ on a factor M, and assume there is $c > 0$ such that

$$\mathrm{Sp}(\sigma) \cap \{[-2c] \cup [c, 2c]\} = \phi,$$

where we identify $\widehat{\mathbb{R}} = \mathbb{R}$. Then there exists $h \in M^\sigma, -c/2 \leq h \leq c/2$ such that the action σ' of $\mathbb{R}$ defined by

$$\sigma'_t(x) := e^{-ith}\sigma_t(x)e^{ith}, \;\; x \in M, t \in \mathbb{R},$$

satisfies $\mathrm{Sp}(\sigma') \cap (-c, c) = \{0\}$.

Lemma (4.3.17) [4]:

Let M be a σ-finite factor of type III_0. Then for each $n \in \mathbb{N}$, there exists $\varphi_n \in S_{\mathrm{nf}}(M)$ such that $\mathrm{Sp}(\sigma^{\varphi_n}) \cap (-\log n, \log n) = \{0\}$.

Proof:

For $n \in \mathbb{N}$ and $\varepsilon > 0$, define $I_n := [-2\log n, -\log n] \cup [\log n, 2\log n]$ and $K_\varepsilon := [-\varepsilon, \varepsilon]$. Assume that there is $n \geq 2$ such that $\mathrm{Sp}(\sigma^\psi) \cap I_n \neq \phi$ for every $\psi \in S_{\mathrm{nf}}(M)$. Fix $\psi \in S_{\mathrm{nf}}(M)$. Let $e \in \mathrm{Proj}(M_\psi)\backslash\{0\}$. Since $M \cong eMe$, the assumption implies that $\mathrm{Sp}(\sigma^{\psi_e}) \cap I_n = \phi$. Now given finitely many $e_1, \dots, e_N \in \mathrm{Proj}(M_\psi)\backslash\{0\}$ and $\varepsilon_1, \dots, \varepsilon_N > 0$. By Lemma (4.3.15), there is $e \in \mathrm{Proj}(M_\psi)\backslash\{0\}$ and $\varepsilon > 0$ such that

$$\phi \neq I_n \cap \left\{\mathrm{Sp}\left(\sigma^{\psi_e}\right) + K_e\right\} \subset I_n \cap \bigcap_{i=1}^{N} \left\{\mathrm{Sp}\left(\sigma^{\psi_{e_i}}\right) + K_{\varepsilon_i}\right\}.$$

Therefore by the compactness of In and by Lemma (4.3.15), we have

$$\phi \neq I_n \cap \bigcap_{0 \neq e \in M_\psi, \varepsilon > 0} \left\{\mathrm{Sp}\left(\sigma^{\psi_e}\right) + K_e\right\}$$

$$= I_n \cap \Gamma\left(\sigma^\psi\right) = I_n \cap \log(S(M)\backslash\{0\})$$

$$= I_n \cap \{0\} = \phi,$$

which is a contradiction. Therefore for each $n \in \mathbb{N}$, there is $\psi_n \in S_{\mathrm{nf}}(M)$ such that $\mathrm{Sp}(\sigma^{\psi_n}) \cap I_n = \phi$. Then choose $h_n \in M_{\psi_n}, -\frac{1}{2}\log n \leq h_n \leq \frac{1}{2}\log n$ as in Lemma (4.3.19) for ψ_n. Then set $\varphi_n := \psi_n(h_n' \cdot), h_n' := \left(h_n + \frac{1}{2}\log n + 1\right)^{-1}$. Then we have $\varphi_n \in S_{\mathrm{nf}}(M)$ and $\mathrm{Sp}(\sigma^{\varphi_n}) \cap (-\log n, \log n) = \phi$.

Theorem (4.3.18) [4]:

Let M be a σ-finite type III_0 factor. Then there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ of normal faithful states on M such that $(M, \varphi_n)^\omega$ is isomorphic to the finite von Neumann algebra $(M_{\varphi_n}, \tau_n)^\omega$ where $\tau_n := \varphi_n | M_{\varphi_n}$.

Proof:

By Lemma (4.3.17), for each $n \in \mathbb{N}$ there exists $\varphi_n \in S_{\mathrm{nf}}(M)$ such that $\mathrm{Sp}(\sigma^{\varphi_n}) \cap (-\log n, \log n) = \{0\}$. Let $x = (x_n)^\omega \in (M, \varphi_n)^\omega$. By Proposition (4.2.11), x can be approximated strongly by elements of the form $(y_n)^\omega$, where $(y_n)_n$ satisfies $y_n \in M(\sigma^{\varphi_n}, [-a, a])$ for each n for a fixed $a > 0$. Fix one such (y_n) and $a > 0$. Let $n_0 \in \mathbb{N}$ be such that a log n_0. Then by $\mathrm{Sp}(\sigma^{\varphi_n}) \cap (-\log n, \log n) = \{0\}$, for $n > n_0$ we have

$$M(\sigma^{\varphi_n}, [-a, a]) \subset M(\sigma^{\varphi_n}, [-\log n_0, \log n_0]) = M_{\varphi_n},$$

whence $(y_n)^\omega \in (M_{\varphi_n}, \tau_n)^\omega$, where $\tau_n := \varphi_n|(M_n)_{\varphi_n}$. Since $(x_n)^\omega$ is approximated by these elements, $(x_n)^\omega \in (M_{\varphi_n}, \tau_n)^\omega$ holds too. This finishes the proof.

Lemma (4.3.19) [4]:

Let $\mathrm{A} = \mathrm{L}^\infty(\mathrm{X}, \mu)$ be a (possibly non-separable) diffuse abelian von Neumann algebra, where (X, μ) is a probability space without atoms. Let T be an ergodic transformation on (X, μ). Let $\alpha(\mathrm{f})(\omega) := \mathrm{f}(\mathrm{T}^{-1}\omega)$ be the corresponding automorphism of A. Then $\alpha^\omega \in \mathrm{Aut}(\mathrm{A}^\omega)$ is not ergodic.

Proof:

By Lemma (4.3.22), we can find measurable sets $B \subset X (n \in \mathbb{N})$ with $\mu(B_n \,\Delta\, TB_n) \leq \frac{2}{n}$. Then put $p := (1B_n)^\omega \in A^\omega$. By assumption, p is an α^ω-invariant projection in $A^\omega \backslash \{0,1\}$. Hence α^ω is not ergodic.

We next show that for type III_0 factors, discrete decomposition is preserved under the Ocneanu ultrapower.

Let M be a type III_0 factor. There is a normal faithful lacunary weight φ on M such that M_φ is of type II_∞ with diffuse center, and let $\tau := \varphi|_{M_\varphi}$. There is $0 < \lambda_0 < 1$ and $U \in M(\sigma^\varphi, (-\infty, \log \lambda_0])$ such that $\theta = \mathrm{Ad}(U)|_{M_\varphi} \in \mathrm{Aut}(M_\varphi)$ is a centrally ergodic automorphism satisfying $\tau \circ \theta \leq \lambda_0 \tau$. In this setting, we have $M \cong M_\varphi \rtimes_0 \mathbb{Z}$ and $\varphi = \hat{\tau}$ (dual weight of) under this isomorphism. We call this a discrete decomposition of M. Similar decompositions are possible for type

$\mathrm{III}_\lambda (0 < \lambda < 1)$ factors, in which case we have $\tau \circ \theta = \lambda_\tau$ and $U \in M(\sigma^\varphi, \{\log \lambda\})$.

Remark (4.3.20) [4]:

Schmidt showed that if (X, μ) is a standard nonatomic probability space, there exist measurable sets $\{B_n\}_{n=1}^\infty \subset X$ which are non-trivial asymptotically T-invariant sets. That is, it satisfies

$$\lim_{n\to\infty} \mu(TB_n \,\Delta\, B_n) = 0, \quad \liminf_{n\to\infty} \mu(B_n)\mu(1 - B_n) > 0.$$

Therefore $p := (1B_n)^\omega$ is a non-trivial projection in $(A^\omega)^{\alpha^\omega}$, and Lemma (4.3.19) follows. Since we could not check if his proof works for non-separable space (X, μ), we add a proof of Schmidt's result for non-separable space below (Lemma (4.3.22)).

We need a slight modification of Rokhlin's Theorem from due to Kawahigashi Sutherland-Takesaki. We include a proof for reader's convenience.

Lemma (4.3.21) [4]:

Let (Ω, μ) be a non-atomic probability space, $T: X \to X$ be a non-singular ergodic transformation. Then for each $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a measurable subset $E \subset X$ such that

(i) $E, T(E), \dots, T^{n-1}(E)$ are mutually disjoint.
(ii) $\mu\left(X - \bigcup_{j=0}^{n-1} T^j E\right) < \varepsilon$.
(iii) $\mu(E) \leq \frac{1}{n}$.

Proof:

Let $\nu_n := \sum_{j=0}^{n-1} \mu \circ T^j$. Then ν is absolutely continuous with respect to μ. Therefore given $\varepsilon > 0$, there is $\delta = \delta(n, \varepsilon) > 0$ such that

$$\mu(F) < \delta \quad \Rightarrow \quad \nu_n(F) < \varepsilon$$

holds. This implies that $\mu(T^j F) < \varepsilon, 0 \leq j \leq n - 1$. By the Rokhlin tower Theorem, there exists a measurable set $F \subset X$ such that $F, TF, \dots, T^{n-1}F$ are mutually disjoint, and $G := X - \bigcup_{j=0}^{n-1} T^j F$ has $\mu(G) < \delta(n, \varepsilon)$. In particular, we have $\mu(T^j G) < \varepsilon, 0 \le j \le n-1$. Since $\sum_{j=0}^{n-1} \mu(T^j F) \le 1$, we may choose $k \in \{0,1,\dots,n-1\}$ with $\mu(T^k F) \le \frac{1}{n}$. Put $E := T^k F$. Then $E, TE, \dots, T^{n-1}E$ are mutually disjoint, and

$$X - \bigcup_{j=0}^{n-1} T^j E = T^k \left(X - \bigcup_{j=0}^{n-1} T^j E \right) = T^k G.$$

whence $\mu\left(X - \bigcup_{j=0}^{n-1} T^j E\right) < \varepsilon$, and $\mu(E) \le \frac{1}{n}$.

Lemma (4.3.22) [4]:

Let (X, μ) be a non-atomic probability space, $T: X \to X$ be a non-singular ergodic transformation. For each $n \ge 2$, there exists a measurable set $B_n \subset X$ with $\mu(B_n) = \frac{1}{2}$ such that $\mu(TB_n \,\Delta\, B_n) \le \frac{2}{n}$ holds.

Proof:

Put $\varepsilon = \frac{1}{n}$ and choose $E \subset X$ as in Lemma (4.3.21). Since μ has no atoms, there exists a family $\{G(t)\}_{t \in [0,1]}$ of measurable subsets of X with the following properties:

(i) $G(0) = \phi, G(1) = E$.
(ii) $0 \le t \le s \le 1 \Rightarrow G(t) \subset G(s)$.
(iii) $\mu\big(G(t)\big) = t\mu(E), 0 \le t \le 1$.

Put

$$B(t) := \bigcup_{j=0}^{n-1} T^j G(t), \quad 0 \le t \le 1.$$

We see that $B(0) = \phi, B(1) = \bigcup_{j=0}^{n-1} T^j E$, so that $\mu(B(1)) > 1 - \frac{1}{n} \ge \frac{1}{2}$. Since $t \mapsto \mu\big(B(t)\big) = \sum_{j=0}^{n-1} \mu\left(T^j G(t)\right)$ is continuous by the choice of $\{G(t)\}_{t \in [0,1]}$, we can find $t_0 \in [0,1]$ with $\mu(B(t_0)) = \frac{1}{2}$. Then put $B_n := B(t_0)$. Since $E, TE, \dots, T^{n-1}E$ are disjoint, we see that

$$B_n \,\Delta\, TB_n \subset T^n E \cup E \subset G \cup E,$$

where $G := X - \bigcup_{j=0}^{n-1} T^j E$ (the last inclusion is true modulo null sets, since $\mu(T^n E \cap T^j E) = 0$ for $1 \le j \le n-1$). Therefore we have

$$\mu(B_n \,\Delta\, TB_n) \le \mu(G) + \mu(E) \le \frac{2}{n}.$$

Lemma (4.3.23) [4]:

Let N be a von Neumann subalgebra of M, and let $\theta \in \mathrm{Aut}(N)$ be such that $p(\theta^k) = 0$ for all $k = 0$. Suppose N satisfies

(i) $N' \cap M \subset N$.
(ii) There is a normal faithful conditional expectation E from M onto N.
(iii) There is $U \in U(M)$ such that $UxU^* = \theta(x)$ for all $x \in N$.
(iv) M is generated by $\{U\} \cup N$ as a von Neumann algebra.

Then there is a $*$-isomorphism $\Phi: M \to N \rtimes_\theta \mathbb{Z}$ sending U (resp. N) to the canonical implementing unitary of θ (resp. the canonical image of N) in the crossed product.

Proposition (4.3.24) [4]:

Let M be a σ-finite factor of type III_0 with discrete decomposition $M = M_\varphi \rtimes_\theta \mathbb{Z}$ (φ is chosen as above). Then $M^\omega \cong (M_\varphi)^\omega \rtimes_{0^\omega} \mathbb{Z}$.

Recall that for a von Neumann algebra M and $\theta \in \mathrm{Aut}(M), p(\theta)$ is the greatest projection $e \in M^\theta$ for which $\theta|_{M_e}$ is an inner automorphism.

Proof:

We have to verify (i)–(iv) in Lemma (4.329) for $(M_\varphi)^\omega \subset M^\omega$ and θ^ω. Let U be the implementing unitary of θ in the discrete decomposition. Then we have $U \in M(\sigma^\varphi, (-\infty, \log \lambda_0])$ for some $0 < \lambda_0 < 1$, and $\varphi = \hat{\tau} \in \mathcal{W}_{\mathrm{nfs}}(M)$ is lacunary. Also, $\tau \circ \theta \le \lambda_0 \tau$.

Since φ is strictly semifinite, there is a normal faithful conditional expectation $E: M \to M_\varphi$. By Proposition (4.2.24), we have $(M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega$, so that the normal faithful φ^ω-preserving conditional expectation coincides with $E^\omega: M^\omega \to (M_\varphi)^\omega$. So (ii), (iii) are clearly satisfied. Regarding $p((\theta^\omega)^k)$, note that $\tau^\omega := \varphi^\omega|_{(M_\varphi)^\omega}$ is a normal faithful semifinite trace satisfying $\tau^\omega \circ \theta^\omega \leq \lambda_0 \tau^\omega$. This implies, that $p((\theta^\omega)^k) = 0$ for all $k \neq 0$.

Next, we show (iv): M^ω is generated by $(M_\varphi)^\omega$ and U (canonical image of U in M^ω). Let $\{p_i'\}_{i \in I}$ be a net of projections in M_φ such that $\tau(p_i') < \infty (i \in I)$ and $p_i' \nearrow 1$ strongly. Then put $p_i := \bigvee_{n=1}^{\infty} \theta^n (p_i')$. Then it holds that

$$\tau(p_i) \leq \sum_{n=1}^{\infty} \tau\left(\theta^n(p_i')\right) \leq \sum_{n=1} \lambda^n \tau(p_i') < \infty,$$

and $\theta(p_i) \leq p_i, p_i \nearrow 1$ strongly. Now fix one of such finite projection $p = p_i$ in M_φ, and we prove that $p(M^\omega)p$ is generated by pQp, where

$$Q := \text{span}\left((M_\varphi)^\omega \cup \bigcup_{k=1}^{\infty} (M_\varphi)^\omega \cup U^k \bigcup_{k=1}^{\infty} (U^*)^k \left(M_\varphi\right)^\omega \right).$$

By construction, each $x \in M = M_\varphi \rtimes_\theta \mathbb{Z}$ has a formal expansion

$$x \sim x(0) + \sum_{k=1}^{\infty} \{x(k)U^k + (U^*)^k x(-k)\},$$

where $x(k) \in M_\varphi (k \in \mathbb{Z})$ is uniquely determined by

$$x(k) = E((U^*)^k), \quad x \in (-k) = E(U^k x) \ \ (k \geq 0)$$

(the order of U^k and $x(k)$ is a matter of convention). Let $x \in M^\omega$, and put $y := pxp \in (M^\omega)_p$ (here we used $p(M^\omega)_p = (M^\omega)^\omega$. Since $p \in (M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega$, we may consider φ_p^ω as a faithful normal positive functional on $(M^\omega)_p$. Let $\varepsilon > 0$ be given. By Proposition (4.2.9), we may find $a > 0$ and $z = (z_n)^\omega \in$

$(M^\omega)_p$ with $z_n \in M_p(\sigma^{\varphi_p}, [-a, a])$ $(n \in \mathbb{N})$ such that $\|y - z\|_{\varphi_p^\omega} < \varepsilon$, and $\|z\| \leq \|y\|$. Consider the expansion of z_n (in M):

$$z_n \sim z_n(0) + \sum_{k=1}^{\infty} \{z_n(k)U^k + (U^*)^k z_n(-k)\}, \ \ n \in \mathbb{N}.$$

Let $V := Up$. Then by $UpU^* = \theta(p) \leq p$, we have

$$V^2 = UpUp = Up(UpU^*)U = U\theta(p)U$$

$$= U^2 p.$$

Similar computations show that

$$V^k = U^k p, \quad (V^*)^k, \ \ k \geq 1.$$

We see that f or $k \geq 0$,

$$z_n(k) = E(z_n(U^*)^k) = E(z_n p(U^*)^k) = E(z_n(V^*)^k),$$

$$z_n(-k) = E(U^k p z_n) = E(V^k z_n).$$

In particular, we have $z_n(k) \in pM_\varphi\theta^k(p), z_n(-k) \in \theta^k(p)M_\varphi p$ and

$$z_n(k)U^k = z_n(k)V^k, \ \ (U^*)^k z_n(-k) = (V^*)^k z_n(-k).$$

Therefore the expansion of z_n can be rewritten as

$$z_n \sim z_n(0) + \sum_{k=1}^{\infty} \{z_n(k)V^k + (V^*)^k z_n(-k)\}, \ \ n \in \mathbb{N}.$$

Since $U \in M(\sigma^\varphi, (-\infty, \log\lambda_0])$, we have

$$V^k z_n \in M_p(\sigma^{\varphi_p}, (-\infty, k\log\lambda_0 + a]), z_n(V^*)^k \in M_p\big([-k\log\lambda_0 - a, \infty)\big)$$

for each $k \geq 1$. Let $K := [a/(-\log\lambda_0)] + 1 \in \lambda$, and consider the GNS representation of $\left(\varphi_p, \left(M_\varphi\right)_p\right)$. E induces φ_p-preserving conditional expectation $E_p : M_p \to \left(M_\varphi\right)_p$. Then for $k \geq K$, we have

$$k\log\lambda_0 + a < 0 < -k\log\lambda_0 - a.$$

Hence

$$z_n(k)\xi_{\varphi_p} = E_p(z_n(V^*)^k)\xi_{\varphi_p} = 1_{\{1\}}\left(\Delta_{\varphi_p}\right)\left(z_n(V^*)^k\xi_{\varphi_p}\right) = 0,$$

$$z_n(-k)\xi_{\varphi_p} = E_p(V^k z_n)\xi_{\varphi_p} = 1_{\{1\}}(\Delta_{\varphi_n})\left(V^k z_n \xi_{\varphi_p}\right) = 0.$$

Since $\xi_{\varphi_p}$is separating for M_p, we have $z_n(k) = 0, |k| \geq K$. Therefore we have

$$z_n = z_n(0) + \sum_{k=1}^{K-1}\{z_n(k)V^k + (V^*)^k z_n(-k)\},\ \ n \in \mathbb{N}.$$

Now, since $\left(M_\varphi\right)_p$ is a finite von Neumann algebra, each $\left(z_n(k)\right)_n(|k| \leq K-1)$ is in $\mathcal{M}^\omega\left(\mathbb{N}, M_{\varphi_p}\right)$, and we have

$$z = \left(z_n(0)\right)^\omega + \sum_{k=1}^{K-1}\left\{\left(z_n(k)\right)^\omega U^k + (U^*)^k\left(z_n(-k)\right)^\omega\right\} \in pQp.$$

Since $\varepsilon > 0$ is arbitrary, $y = pxp$ can be approximated strongly by elements from pQp. Hence $pM^\omega p = \overline{pQp}^{\mathrm{sot}}$. Since i is arbitrary (recall that $p = p_i$), this implies that x is in $\bar{Q}^{\mathrm{sot}}$ as well. This proves the claim.

Namely, let $x \in \left(\left(M_\varphi\right)^\omega\right)' \cap M^\omega$. Then by the above, x has a formal expansion by $x \sim x(0) + \sum_{k=1}^{\infty}\left(x(k)U^k + (U^*)^k x(-k)\right)\left(x(k) \in \left(M_\varphi\right)^\omega\right)$. Since $ax = xa$ for $a \in \left(M_\varphi\right)^\omega$, this implies that $ax(k) = x(k)\theta^k(a), \theta^k(a)x(-k) = x(-k)a$ for all $a \in \left(M_\varphi\right)^\omega$ and $k \geq 0$. Then by $p((\theta^\omega)^k) = 0(k \neq 0)$ and we have $x(k) = 0, k \neq 0$ and hence $x = x(0) \in \left(M_\varphi\right)^\omega$. This proves that $\left(\left(M_\varphi\right)^\omega\right)' \cap M^\omega \subset \left(M_\varphi\right)^\omega$.

Now we are ready to prove the non-factoriality of the Ocneanu ultrapower for type III_0 factors.

Theorem (4.3.25) [4]:

Let M be a σ-finite factor of type III_0. Then M^ω is not a factor.

Proof:

Now we show that M^ω is not a factor. By Claim 1, M^ω is generated by $\left(M_\varphi\right)^\omega$ and U^ω, which implements θ^ω. Representing the center of M_φ as $L^\infty(X,\mu)$ where (X,μ) is a diffuse probability space, implies that the center of $\left(M_\varphi\right)^\omega$ is $L^\infty(X,\mu)^\omega$. By Lemma (4.3.19), θ^ω is not centrally ergodic. This implies that there is a nontrivial element $x \in (L^\infty(X,\mu)^\omega)^{\theta^\omega}$, whence a nontrivial element in $\mathbb{Z}(M^\omega)$. Therefore M^ω is not a factor.

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