Chapter 4

The Strong No Loop Conjecture

With at most two simple modules and $r^4 = 0$, in particular, when Λ is a finite-dimensional algebra over an algebraically closed field with at most two simple modules and $r^3 = 0$. We show the conjecture in case Λ is mild, which means that Λ has only finitely many two-sided ideals and each proper factor algebra Λ/J is representation finite. In fact, it is sufficient that a "small neighborhood" of the support of the projective cover of S is mild.

Section (4.1): Special Biserial Algebras:

Let A be a finite-dimensional algebra over a field given by a quiver with relations. Let S be a simple A-module with a non-split self-extension; that is, the quiver has a loop at the corresponding vertex. The strong no loop conjecture claims that S is of infinite projective dimension; see [118, 119]. This conjecture remains open except for monomial algebras; see, for example, [120, 119, 121, 122]. Under certain hypotheses on the loop, Green, Solberg and Zacharia have shown that $\mathrm{Ext}_A^i(S,S)$ does not vanish for every $i \geq 1$; see [123].

In this Section, we shall first present a short proof of this result, because not only is the original proof rather complex, but also our idea possibly works for other cases. Next we observe that this result reduces the conjecture to the case where some power of the loop is a component of a polynomial relation. This reduction works particularly well when A is special biserial, due to a combinatorial description of the syzygies of string modules; see Proposition (4.1.9). Our main result says that $\mathrm{Ext}_A^i(S,S)$ does not vanish for every $i \geq 1$ if the convex support of S is special biserial. We shall also prove that if S has an almost split self-extension, then the block of A supporting S is a local Nakayama algebra; in particular, $\mathrm{Ext}_A^i(S,S)$ does not vanish for every $i \geq 1$. In the course of its proof, we easily get a characterization of Nakayama algebras, strengthening the one stated in [118]. Contrary to what will be seen in this Section, Happel's example stated in [123] shows the existence of a simple module S with a loop but $\mathrm{Ext}^i(S,S) = 0$ for infinitely many i.

Our motivation for studying special biserial algebras comes from the following two aspects. First of all, since their representations are completely understood, they form naturally a testing class for various well-known conjectures in the representation theory of algebras. Secondly, these algebras play an important role in the modular representation theory of finite groups; see [124], tracing back to the classification of the indecomposable Harish-Chandra modules of the Lorentz group by Gelfand and Ponomarev; see [125].

Let k stands for a field and Q for a finite quiver. Let kQ be the path algebra of Q over k, and Q^+ the ideal generated by the arrows. Note that we shall compose the paths of Q from the left to the right. If I is an ideal such that $(Q^+)^m \subseteq I \subseteq (Q^+)^2$ for some $m \geq 2$, then the pair (Q, I) is called a bound quiver. We shall always assume that I is such an ideal. Let $p_1, \dots, P_r$ be pairwise distinct paths of Q from a vertex a to a vertex b, and let $\lambda_1, \dots, \lambda_r \in k$ be nonzero scalars. We call

$$\rho = \lambda_1 p_1 + \cdots + \lambda_r p_r$$

a relation on Q if $\rho \in I$ while $\sum_{i \in \Omega} \lambda_i p_i \notin I$ for all $\Omega \subset \{1, \dots, r\}$. In this case, $p_1, \dots, p_r$ are called the components of ρ and a the start-point. Moreover, ρ is called monomial, binomial, or polynomial if $r = 1, r = 2$, or $r \geq 2$, respectively.

The quotient $A = kQ/I$ is called the algebra of the bound quiver (Q, I). If $x \in kQ$, we shall denote by $\tilde{x}$ the class $x + I \in A$. An A-module means a right module of finite k-dimension except otherwise stated explicitly. The radical, the top, and the n-th syzygy of an A-module M will be denoted by rad M, top M and $\Omega^n(M)$, respectively. We call $x \in M$ a top element if $x \notin \operatorname{rad} M$. For a vertex a of Q, we shall denote by $S(a)$ and $P(a)$ the simple A-module and the indecomposable projective A-module associated to a, respectively.

The elements of a direct sum of A-modules are written as column matrices. Let $e_1, \dots, e_r$ be idempotents of A. Then an A-linear map $\phi : e_1 A \oplus \cdots \oplus e_r A \to M$ is left multiplication by a matrix $(x_1, \dots, x_r)$ with $x_i \in Me_i$. In this case, we say that ϕ is represented by $(x_1, \dots, x_r)$. In particular, if $f_1, \dots, f_s$ are also idempotents of A, then an A-linear map from $e_1 A \oplus \cdots \oplus e_r A$ to $f_1 A \oplus \cdots \oplus f_s A$ is represented by an $(s \times r)$-matrix whose (i, j)-entry is an element of $f_i A e_j$. For convenience of reference, we state the following well-known result.

Lemma (4.1.1)[117]: Let A be the algebra of a finite bound quiver with $e_1, \dots, e_s$ some primitive idempotents. Let M be a non-zero A-module with $x_i \in Me_i$ for $1 \leq i \leq s$. If the classes of $x_1, \dots, x_s$ in top M are linearly independent over k, then there exist primitive idempotents $e_{s+1}, \dots, e_r \in A$ and $x_{s+1} \in Me_{s+1}, \dots, x_r \in Me_r$ such that

$$(x_1, \ldots, x_s, x_{s+1}, \ldots, x_r) : e_1A \oplus \cdots \oplus e_sA \oplus e_{s+1}A \oplus \cdots \oplus e_rA \to M$$

is a projective cover of M.

We now give the promised alternative proof of Proposition 4.2 in [123].

Proposition (4.1.2)[117]: (Green-Solberg-Zacharia). Let A be the algebra of a bound quiver (Q, I), containing a loop α at a vertex a. If for some $n \geq 2, \alpha^n$ lies in I but not in $IQ^+ + Q^+I$, then for all $i \geq 1, \mathrm{Ext}_A^i\big(S(a), S(a)\big)$ does not vanish.

Proof. Suppose that $\alpha^n \in I$ but $\alpha^n \notin IQ^+ + Q^+I$. In particular, $\alpha^n \notin I$. Let $\alpha_i : \alpha \to c_i, i = 1, \ldots, t$, be the arrows starting at a with $\alpha_1 = \alpha$. Then $S(a)$ admits a minimal projective resolution

$$\cdots \longrightarrow P_m \xrightarrow{\phi_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\phi_1} P_0 \longrightarrow S(a) \longrightarrow 0,$$

where $P_0 = P(a), P_1 = P(c_1) \oplus \cdots \oplus P(c_t)$, and ϕ_1 is represented by $(\bar{\alpha}_1, \ldots, \bar{\alpha}_t)$. Assume, for $m \geq 1$, that $P_m \cong P(b_1) \oplus P(b_2) \oplus \cdots \oplus P(b_s)$ with $s \geq 1$ and $P_{m-1} \cong P(a_1) \oplus P(a_2) \oplus \cdots \oplus P(a_r)$ with $r \geq 1$, where the a_i, b_j are vertices with $a_1 = b_1 = a$, while ϕ_m is isomorphic to the map represented by a matrix of the following form:

$$\begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} & \cdots & \bar{x}_{1s} \\ 0 & \bar{x}_{22} & \cdots & \bar{x}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \bar{x}_{r2} & \cdots & \bar{x}_{rs} \end{pmatrix},$$

where x_{ij} is a linear combination of non-trivial paths from a_i to b_j with x_{11} being α or α^{-1}. We shall show that ϕ_{m+1} is isomorphic to a map represented by a matrix of this form. By Lemma (4.1.1), it suffices to prove that $\ker\, \phi_m$ contains a top element of the form $(\bar{y}_{11}, 0, \ldots, 0)^T$ with $y_{11} = \alpha$ or α^{n-1}. Indeed, if $x_{11} = \alpha^{n-1}$, then $(\alpha, 0, \ldots, 0)^T$ clearly lies in $\ker\, \phi_m$ but not in its radical since $\ker\, \phi_m \subseteq \mathrm{rad}\, P_m$. If $x_{11} = \alpha$, then $Z = (\bar{\alpha}^{n-1}, 0, \ldots, 0)^T \in \ker\, \phi_m$. Assume on the contrary that Z lies in the radical of $\ker\, \phi_m$. Then $Z = Y\bar{u}$ with $Y \in \ker\, \phi_m$ and $u \in Q^+$. Since $Y \in \mathrm{rad}\, P_m$, we may write $Y = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s)^T$ with $y_i \in Q^+$. Now $v = \alpha^{n-1} - y_1u, y_2u, \ldots, y_su \in I$ since $Z = Y\bar{u}$, and $w = \alpha y_1 + x_{12}y_2 + \cdots + x_{1s}y_2 \in I$ since $Y \in \ker\, \phi_m$. Therefore, $\alpha^n = \alpha v + \alpha y_1 u = \alpha v + wu - (x_{12}y_2u + \cdots + x_{1s}y_su) \in Q^+I + IQ^+$, a contradiction. Thus Z is indeed a top element of $\ker\, \phi_m$. By induction, we have shown that $P(a)$ is a direct summand of P_m for all $m \geq 1$. This completes the proof of the proposition.

We now deduce some useful consequences from the above result.

Corollary (4.1.3)[117]: Let A be the algebra of a bound quiver (Q, I), containing a loop a at a vertex a. If no power of α is a component of a polynomial relation, then for all $i \geq 1, \operatorname{Ext}_A^i(S(a), S(a))$ does not vanish.

Proof. Let $n \geq 2$ be minimal such that $\alpha^n \in I$, and assume that α^{n-1} is not a component of any polynomial relation. Note that for a relation ρ on Q and $x, y \in kQ, x\rho y$ is either zero or a sum of relations on Q. Therefore, if $\alpha^n \in Q^+ I + I Q^+$, then

$$\alpha^n = (\beta_1\rho_1 + \cdots + \beta_r\rho_r) + (\rho_{r+1}\beta_{r+1} + \cdots + \rho_s\beta_s),$$

where the ρ_i are relations on Q and the β_i are arrows. Hence, α^{n-1} is a component of at least one of the ρ_i, say ρ_1. By the minimality of n, ρ_1 is a polynomial relation, a contradiction. The proof is now completed by applying Proposition (4.1.2).

The above result implies immediately the following.

Corollary (4.1.4)[117]: Let A be the algebra of a bound quiver (Q, I), containing a loop a at a vertex a. If $\alpha^2 \in I$, then $\operatorname{Ext}_A^i(S(a), S(a))$ does not vanish for every $i \geq 1$.

We shall now study simple modules with an almost split self-extension, that is, invariant under the Auslander-Reiten translation $\tau = \mathrm{DTr}$; see [118]. For this purpose, we need the following result, which is interesting in its own right.

Proposition (4.1.5)[117]: Let A be the algebra of a bound quiver (Q, I), and let a, b be vertices of Q. Then $\tau S(a) \cong S(b)$ if and only if Q contains an arrow $\alpha : a \to b$ which is the only arrow starting at a and the only one ending at b.

Proof. If Q satisfies the stated property, then we have a non-split exact sequence:

$$0 \longrightarrow S(b) \longrightarrow M(\alpha) \longrightarrow S(a) \longrightarrow 0.$$

Since α is the only arrow starting at a and the only one ending at b, one verifies easily that the sequence is almost split. Conversely assume that $S(b) \cong \tau S(a)$. Let $\alpha_i : a \to b_i, i = 1, \ldots, r$, be the arrows starting at a; and $\beta_j : a_j \to b, j = 1, \ldots, s$, be those ending at b. Then $S(a)$ has a minimal projective presentation

$$\delta: P(b_1) \oplus \cdots \oplus P(b_r) \longrightarrow P(a) \longrightarrow S(a) \longrightarrow 0.$$

Applying the duality $\operatorname{Hom}_A(-, A_A)$ to δ, we get a minimal projective presentation

$$\delta^* : Q(a) \longrightarrow Q(b_1) \oplus \cdots \oplus Q(b_r) \longrightarrow \operatorname{Tr} S(a) \longrightarrow 0$$

of the transpose of $S(a)$, where $T(c)$ and $Q(c)$ denote, respectively, the simple and the indecomposable projective left A-module associated to a vertex c. On the other hand, $T(b)$ has a minimal projective presentation

$$\eta : Q(a_1) \oplus \cdots \oplus Q(a_s) \longrightarrow Q(b) \longrightarrow T(b) \longrightarrow 0.$$

Noting $\mathrm{Tr}\, S(a) \cong \mathrm{D}\, S(b) \cong T(b)$, we have $\eta \cong \delta^*$. In particular, $r = 1$ with $b_1 = b$ and $s = 1$ with $a = a_1$. The proof of the proposition is completed.

We call a module homogeneous if it admits an almost split self-extension.

Corollary (4.1.6)[117]: Let A be the algebra of a connected bound quiver (Q, I). Then A admits a homogeneous simple module if and only if Q consists of one loop. In this case, for any indecomposable non-projective A-module M, $\mathrm{Ext}^i_A(M, M)$ does not vanish for every $i \geq 1$.

Proof. By Proposition (4.1.5), Q has a vertex a such that $\tau S(a) \cong S(a)$ if and only if Q contains a loop $\alpha : a \to a$ which is the only arrow starting at a and the only one ending at a. This is equivalent to Q consisting of one loop since Q is connected. The rest of the statement is well known. This completes the proof of the corollary.

An Artin algebra is called a Nakayama algebra if its indecomposable modules are all uniserial. The algebra of a connected bound quiver is a Nakayama algebra if and only if the quiver is a single path (maybe trivial) or an oriented cycle. The following is another characterization of this class of algebras; compare [118].

Theorem (4.1.7)[117]: Let A be the algebra of a connected bound quiver (Q, I). Then A is a Nakayama algebra if and only if there exists a τ-orbit $\mathcal{O}$ consisting of simple A-modules. In this case, $\mathcal{O}$ contains all simple A-modules.

Proof. If Q is a path $a_1 \to \cdots \to a_n$ with $n \geq 1$ and the ai pairwise distinct, by Proposition (4.1.5), $\tau S(a_i) = S(a_{i+1})$ for $1 \leq i < n$. Since $S(a_n)$ is projective and $S(a_1)$ is injective, the $S(a_i)$ form a τ-orbit. If Q is an oriented cycle $a_1 \to \cdots \to a_n \to a_1$ with $n \geq 1$ and the a_i pairwise distinct, then $\tau S(a_i) = S(a_{i+1})$ for $1 \leq i < n$, and $\tau S(a_n) = S(a_1)$. Hence the $S(a_i)$ form a τ-orbit.

Conversely, let $\mathcal{O} = \{S(a_1), \ldots, S(a_n)\}$ be a τ-orbit with the a_i pairwise distinct vertices of Q. Consider first the case $n = 1$. If $S(a_i)$ is projective, then Q consists of the vertex a_1. Otherwise, $\tau S(a_1) = S(a_i)$, and hence Q consists of one loop by Corollary (4.1.6). Assume now that $n > 1$ and that $\tau S(a_i) = S(a_{i+1})$ for $1 \le i < n$. For each $1 \le i < n$, by Proposition (4.1.5), Q contains an arrow $a_i : a_i \to a_{i+1}$, the only one starting at ai and the only one ending at a_{i+1}. If $S(a_n)$ is projective, then $S(a_1)$ is injective. Hence Q contains no arrow starting at a_n or ending at a_1. Thus Q consists of the path $a_1 \to \cdots \to a_{n-1} \to a_n$. Otherwise, $\tau S(a_n) = S(a_1)$. Then Q contains an arrow $\alpha_n : a_n \to a_1$, the only one starting at a_n and the only one ending at a_1. Thus Q consists of the oriented cycle $a_1 \to \cdots \to a_n \to a_n$. The proof of the theorem is completed.

The objective of this section is to establish the conjecture for special biserial algebras. Recall that a finite-dimensional k-algebra is called special biserial if it is isomorphic to the algebra of a bound quiver (Q, I) satisfying (1) each vertex is the start-point of at most two arrows and the end-point of at most two arrows; and (2) for each arrow β, there exists at most one arrow α such that $\alpha\beta \notin I$ and at most one arrow γ such that $\beta\gamma \notin I$. We call such a bound quiver special biserial.

Let (Q, I) be a special biserial bound quiver. Then a relation on Q is either monomial or binomial; see [126]. Moreover, if a vertex is the start-point of a binomial relation, then such a relation is unique up to a scalar. Thus we may assume, without changing its algebra, that every binomial relation of (Q, I) is a multiple of a binomial relation of the form $p - q$. In this case, we call (p, q) a binomial pair.

For a vertex a, the trivial path at a is denoted by ε_a, and for a path p, its start-point and end-point are denoted by $s(p)$ and $e(p)$, respectively. For an arrow α, we introduce a new arrow α^{-1}, the inverse of α with $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A reduced walk w in Q is either a trivial path or $w = c_1 c_2 \dots c_n$ with $n \ge 1$, where c_i is either an arrow or the inverse of an arrow such that $s(c_{i+1}) = e(c_i)$ and $c_{i+1} \neq c_i^{-1}$ for all $1 \le i < n$. In the latter case, a path $p = \alpha_1 \cdots \alpha_r$ with α_i arrows is contained in w if there exists i with $1 \le i \le n$ such that either $c_{i+j} = \alpha_{j+1}$ for all $0 \le j < r$ or $c_{i+j} = \alpha_{r-j}^{-1}$ for all $0 \le j < r$. A reduced walk w is a string if no path contained in w is a component of a relation on Q. A string p is called serial if it is a path and $s(p)$ is not the start-point of any binomial relation. Finally, we say that a string w starts or ends in a deep if there is no arrow γ such that $\gamma^{-1}w$ or $w\gamma$ is a string, respectively.

Let $A = kQ/I$. To each string w, one associates a string module $M(w)$, compare [127, 128], as follows: if $w = \varepsilon_a$ for a vertex a, then $M(w) = S(a)$ with $\{a\}$ a k-basis. If $w = c_1 c_2 \cdots c_n$ with c_i an arrow or the inverse of an arrow, then $M(w)$ has as a k-basis the ordered family $\{a_0, a_1, \ldots, a_n\}$, where $a_0 = s(c_1)$ and $a_i = e(c_i)$ for $1 \leq i \leq n$. Its multiplication is such that for an arrow α, one has $a_i \bar{\alpha} = a_{i+1}$ if $c_{i+1} = \alpha$ with $0 \leq i < n$, and $a_i \bar{\alpha} = a_{i-1}$ if $c_i = \alpha^{-1}$ with $1 \leq i \leq n$, and $a_i \bar{\alpha} = 0$ otherwise. Here a_i with $0 \leq i < n$ is a top element if and only if c_i is the inverse of an arrow whenever $i \geq 1$, and c_{i+1} is an arrow whenever $i < n$.

Furthermore, for a vertex a of Q, we shall fix a k-basis of $P(a)$ with a multi-plication. If ε_a is serial, then there exist paths p, q starting at a such that $p^{-1}q$ is a string starting and ending in a deep. In this case, $P(a) = M(p^{-1}q)$. Oth-erwise, there exists a binomial relation $\alpha_1 \alpha_2 \cdots \alpha_r - \beta_1 \beta_2 \cdots \beta_s$, where α_i, β_j are arrows with $s(\alpha_1) = s(\beta_1) = a$. Then $P(a)$ has as a k-basis the ordered family $\{a_1, \ldots, a_r, b_1, \ldots, b_s\}$, where $a_i = s(\alpha_i)$ and $b_j = e(\beta_j)$. Its multiplication is such that for each arrow α, one has $b_j \bar{\alpha} = b_{j+1}$ if $\alpha = \beta_{j+1}$ with $1 \leq j < s$ and $b_j \bar{\alpha} = 0$ otherwise; moreover,

$$a_i, \bar{\alpha} = \begin{cases} b_1, & i = 1 \text{ and } \alpha = \beta_1, \\ a_{i+1}, & 1 \leq i < r \text{ and } \alpha = \alpha_i, \\ b_s, & i = r \text{ and } \alpha = \alpha_r, \\ 0, & \text{otherwise.} \end{cases}$$

Here a_1 is the only top element. In all cases, we call the fixed k-basis of $M(w)$ or $P(a)$ its canonical basis. Clearly, the classes of the top elements of the canonical basis form a basis for the top in each case.

In order to describe the syzygies of string modules, we need the notion of syzygy strings of a string, defined in the following.

Definition (4.1.8)[117]: Let (Q, I) be a special biserial bound quiver. Let w be a string, and write

$$w = p_1^{-1} q_1 \cdots p_r^{-1} q_r,$$

where $r \geq 1$, the p_i, q_i are paths with p_1, q_r the only ones that may be trivial.

(a) In the case that p_1 is non-serial, we let α_1 be an arrow and u_1, v_1 some paths such that $(p_1 \alpha_1 u_1, q_1 v_1)$ is a binomial pair. Assume now that p_1 is serial. We first define v_1 to be a path such that $q_1 v_1$ is a string ending in a deep. Moreover, if $p_1^{-1} q_1$ does not start in a deep, then let α_1 be an arrow and u_1 a path such that $u_1^{-1}\alpha_1^{-1}p_1^{-1}q_1$ is a string starting in a deep, and we define neither α_1 nor u_1 otherwise.

(b) Now let i be an integer with $1 < i < r$. If p_i is non-serial, then let u_i, v_i be the paths such that $(p_i u_i, q_i v_i)$ is a binomial pair. Otherwise, let u_i, v_i be paths such that piui and $q_i v_i$ are strings ending in a deep.

(c) In the case that p_r is non-serial, let α_r be an arrow and u_r, v_r some paths such that $(p_r u_r, q_r \alpha_r v_r)$ is a binomial pair. Assume now that p_r is serial. First let u_r be a path such that $p_r u_r$ is a string ending in a deep. Moreover, if $p_r^{-1}q_r$ does not end in a deep, then let α_r be an arrow and v_r some path such that $p_r^{-1}q_r\alpha_r v_r$ is a string ending in a deep; we define neither α_r nor v_r otherwise.

For all $1 \leq i < r$, let $w_i = v_i^{-1}u_{i+1}$, which is clearly a string. Moreover, let $w_0 = u_1$ if u_1 is defined, and $w_i = v_i^{-1}$ if v_r is defined. Denote by W the set of defined strings w_i with $0 \leq i \leq r$. We say that w_s, w_t with $0 \leq s < t \leq r$ are connected if p_i is non-serial for all $s < i \leq t$. This relation of connectedness generates an equivalence relation on W. It is easy to see that if $\{w_i, w_{i+1}, \ldots, w_j\}$ with $0 \leq i \leq j \leq r$ is an equivalence class of W, then $w_i w_{i+1} \cdots w_j$ is a string, called a syzygy string of w. Note that the number of the syzygy strings of w is equal to the number of the equivalence classes of W.

The following is the promised combinatorial description of the first syzygy of a string module.

Proposition (4.1.9)[117]: Let $A = kQ/I$ with (Q, I) special biserial. Let w be a string and $\Omega(w)$ the set of the syzygy strings of w. If $M(w)$ is the string module associated to w, then $\Omega\big(M(w)\big) = \oplus_{\omega\in\Omega(w)} M(\omega)$.

Proof. We keep all the notation introduced in Definition (4.1.8). Let $a_i = s(p_i), i = 1, \ldots, r$, be the top elements of the canonical basis of $M(w)$. Then a projective cover of $M(w)$ is given by the map ϕ: $P(a_1) \oplus \cdots \oplus P(a_r) \to M(w)$ such that the top element of the canonical basis of $P(a_i)$ maps to that of $M(p_i^{-1}q_i)$ for all $1 \leq i \leq r$. It is easy to see that $M(w)$ is projective if and only if $\Omega(w)$ is empty. Assume that $\Omega(w)$ is non-empty. Since the result is obvious for the case where $r = 1$, we may assume furthermore that $r > 1$.

In order to describe the kernel of ϕ, we need more notation. If u_1 is defined and it is non-trivial whenever p_1 is non-serial, then let b_1 be the element of the canonical basis of $P(a_1)$ corresponding to $s(u_1)$ and let d_0 be the top element of the canonical basis of $M(w_0)$. Otherwise, we define neither b_1 nor d_0. In any case, let c_1 be the element of the canonical basis of $P(a_1)$ corresponding to $e(q_1)$, and let d_1 be the top element of $M(w_1)$. For each $1 < i < r$, let b_i and c_i be the elements of the canonical basis of $P(a_i)$ corresponding to $e(p_i)$ and $e(q_i)$, respectively, and let d_i be the top element of the canonical basis of $M(w_i)$. Finally, if v_r is defined and it is non-trivial whenever q_r is non-serial, then let c_r be the element of the canonical basis of $P(a_r)$ corresponding to $e(a_r)$, and let d_r be the top element of the canonical basis of $M(w_r)$. Otherwise, we define neither c_r nor d_r. In any case, let b_r be the element of the canonical basis of $P(a_r)$ corresponding to $e(p_r)$.

Write $\hat{d}_i = (0, \ldots, 0, d_i, 0, \ldots, 0)^T \in \bigoplus_{\omega\in\Omega(w)} M(\omega)$ if d_i is defined. Then the classes of the defined d_i with $0 \le i \le r$ form a k-basis of the top of $\bigoplus_{\omega\in\Omega(w)} M(\omega)$. Let $\psi: \bigoplus_{\omega\in\Omega(w)} M(\omega) \to P(a_1) \oplus \cdots \oplus P(a_r)$ be the map such that

$$\psi(\hat{d}_i) = \begin{cases} (b_1, 0, \ldots, 0)^T, & i = 0 \text{ and } d_0 \text{ is defined,} \\ (0, \ldots, 0, -c_i, b_{i+1}, 0, \ldots, 0)^T, & 1 \le i < r, \\ (0, \ldots, 0, c_r)^T, & i = r \text{ and } d_r \text{ is defined.} \end{cases}$$

One easily verifies that ψ is a monomorphism such that $\psi\phi = 0$. By calculating the dimensions of the modules, we deduce finally that ψ is the kernel of ϕ. This completes the proof of the proposition.

By Corollary (4.1.3), we need consider only special biserial bound quivers with a loop such that at least one of its powers is a component of a binomial relation. For convenience, we make the following two definitions.

Definition (4.1.10)[117]: Let (Q, I) be a special biserial bound quiver, containing a loop α at a vertex a and a binomial pair $(\alpha^{n+1}, \alpha_1 \cdots \alpha_m)$ with $n \ge 1$ and $m \ge 2$. We shall consider strings of the following forms:

(a) The trivial string ε_a.

(b) The string

$$q_{r+1} q_r^{-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_r^{-1} p_{r+1},$$

where $r \geq 1$ is odd, the p_i, q_i are paths that are non-trivial for $1 \leq i \leq r$ such that $(p_i q_i, p_{i+1} q_{i+1})$ is a binomial pair for each odd i with $1 \leq i \leq r$, and there exists a binomial pair $(p_r q_r, p_{r+1} \beta_{r+1} q_{r+1})$ with β_{r+1} an arrow.

(c) The string

$$q_s^{-1} q_{s-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1} p_s^{-1} \cdots p_r^{-1} p_{r+1},$$

where s, r are odd with $1 \leq s \leq r$, the p_i, q_j are paths that are non-trivial for $1 \leq i \leq r$ and $1 \leq j < s$ such that $(p_i q_i, p_{i+1} q_{i+1})$ is a binomial pair for each odd i with $1 \leq i < s$, and $p_s q_s$ is a serial string ending in a deep.

Definition (4.1.11)[117]: Let (Q, I) be as in Definition (4.1.10). We shall consider strings of the following forms:

(a) The string

$$p_{r+1} p_r^{-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_r^{-1} q_{r+1},$$

where $r \geq 0$ is even, the p_i, q_i are paths that are non-trivial for $1 \leq i \leq r$ such that $(q_i p_i, q_{i+1} p_{i+1})$ is a binomial pair for each even $0 \leq i < r$ and there exists a binomial pair $(q_r p_r, \beta_{r+1} p_{r+1})$ with β_{r+1} an arrow, $q_0 = \alpha^n$ and $p_0 = \alpha$.

(b) The string

$$p_s^{-1} p_{s-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1} q_s^{-1} \cdots q_r^{-1} q_{r+1},$$

where s, r are even with $2 \leq s \leq r$, the p_i, q_j are paths that are non-trivial for $1 \leq i < s$ and $1 \leq j \leq r$, such that $(q_i p_i, q_{i+1} p_{i+1})$ is a binomial pair for each even i with $2 \leq i < s$, and $q_s p_s$ is a serial string ending in a deep.

We shall now apply Proposition (4.1.9) to describe the first syzygy of string modules associated to previously defined strings.

Lemma (4.1.12)[117]: Let $A = kQ/I$ with (Q, I) being as in Definition (4.1.10). If M is a string module associated to a string as stated in Definition (4.1.10), then $\Omega(M)$ has as a direct summand a string module associated to a string as stated in Definition (4.1.11).

Proof. Let $M = M(w)$ with w a string. First, if $w = \varepsilon_a$, then $M = S(a)$. Thus $\Omega(M)$ is the string module associated to the string $(\alpha_2 \cdots \alpha_m)\alpha^{-n}$, which is of the form stated in Definition (4.1.11)(a) with $r = 0$ and q_1 trivial. Second, we consider the case

$$w = q_{r+1} q_r^{-1} \cdots q_2 q_1^{-1} \alpha p_1^{-1} p_2 \cdots p_r^{-1} p_{r+1},$$

where $r \geq 1$ is odd, and the p_i, q_i are non-trivial paths such that $(p_i\, q_i, p_{i+1}q_{i+1})$ is a binomial pair for each odd i with $1 \leq i < r$, and there exists a binomial pair $(p_r q_r, p_{r+1}\beta_{r+1}q_{r+1})$ with β_{r+1} an arrow. Note that $q_1 = \alpha_1 \cdots \alpha_t$ with $1 \leq t < m$.

Suppose first that q_{r+1} is trivial and that there exists no even i with $2 \leq i < r$ such that q_i is serial. If $r = 1$, then let u_1 be the path (maybe trivial) such that $\alpha_{t+1} \cdots \alpha_m = a_{t+1}u_1$. Note that we have binomial pairs $(q_i\alpha_{t+1}u_1, \alpha\alpha^n)$ and $(p_1q_1, p_2\beta_2)$. By Proposition (4.1.9), $\Omega(M)$ is the string module associated to the string $u_1\alpha^{-n}q_1$, which is of the form stated in Definition (4.1.11)(a). If $r \geq 3$, let $u_1 = \alpha_{t+1} \cdots \alpha_m$, and for each even i with $2 \leq i \leq r-3$, let u_i, u_{i+1} be non-trivial paths such that $(q_iu_i, q_{i+1}u_{i+1})$ is a binomial pair, and finally, let u_r, u_{r-1} be paths with u_{r-1} non-trivial and δ_1 an arrow such that $(q_{r-1}, u_{r-1}, q_r\delta_r u_r)$ is a binomial pair. Since none of the paths contained in w is a serial string, $\Omega(M)$ is the string module associated to the string $u_r u_{r-1}^{-1} \cdots u_3 u_2^{-1} u_1 \alpha^{-n} q_1 q_2^{-1} q_3 \cdots q_{r-1}^{-1} q_r$, which is of the form stated in Definition (4.1.11)(a) for $r-1$.

Suppose now that q_{r+1} is non-trivial and that there exists no even i with $2 \leq i \leq r+1$ such that q_j is serial. Let $u_1 = \alpha_{t+1} \cdots \alpha_m$ and let, for each even i with $2 \leq i \leq r-1, u_i, u_{i+1}$ be non-trivial paths such that $(q_iu_i, q_{i+1}u_{i+1})$ is a binomial pair, and finally, let u_{r+2}, u_{r+1} be non-trivial paths and δ_{r+2} an arrow such that $(q_{r+1}u_{r+1}, \delta_{r+2}u_{r+2})$ is a binomial pair. For the same reason, $\Omega(M)$ is the string module associated to the string $u_{r+1}u_{r+1}^{-1} \cdots u_2^{-1}u_1\alpha^{-n}q_1q_2^{-1} \cdots q_r q_{r+1}^{-1}$, which is of the form stated in Definition (4.1.11)(a for $r+1$ with q_{r+2} trivial.

Suppose that neither of the above two situations occurs. Then there exists some minimal even integer s with $2 \leq s \leq r+1$ such that q_s is non-trivial and serial. Let $u_1 = \alpha_{t+1} \cdots \alpha_m$ and let, for each even i with $2 \leq i < s, u_i, u_{i+1}$ be non-trivial paths such that $(q_iu_i, q_{i+1}u_{i+1})$ is a binomial pair, and finally, let u_s be a path (maybe trivial) such that q_su_s is a string ending in a deep. Since the p_i, q_j with $1 \leq i \leq r$ and $1 \leq j < s$ are non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s^{-1}u_{s-1} \cdots u_2^{-1}u_1\alpha^{-n}q_1q_2^{-1} \cdots q_{s-1}q_s^{-1} \cdots q_r q_{r+1}^{-1}$, which is of the form stated in Definition (4.1.11)(b) for s and $r-1$ if $s \leq r-1$ and p_{r+1} is trivial, and for s and $r+1$ otherwise.

We conclude the proof with the final case:

$$w = q_s^{-1}q_{s-1}\cdots q_2q_1^{-1}\alpha p_1^{-1}p_2\cdots p_{s-1}p_s^{-1}\cdots p_r^{-1}p_{r+1},$$

where s, r are odd with $1 \le s \le r$, and the p_i, q_j are non-trivial paths for $1 \le i \le r$ and $1 \le j < s$ such that $(p_iq_i, p_{i+1}q_{i+1})$ is a binomial pair for each odd i with $1 \le i < s$, and p_sq_s is a serial string ending in a deep.

Suppose first that q_i is non-serial for each odd i with $1 \le i \le s$. Let u_s, u_{s-1} be paths with u_{s-1} non-trivial and δ_s an arrow such that $(q_s\delta_su_s, q_{s-1}u_{s-1})$ is a binomial pair, where $q_0 = \alpha$ and $u_0 = \alpha^{-n}$ in case $s = 1$; and let $u_1 = \alpha_{t+1}\cdots\alpha_m$ if $s > 1$. Finally for each odd i with $3 \le i < s$, let u_i, u_{i-1} be non-trivial paths such that $(q_{i-1}u_{i-1}, q_iu_i)$ is a binomial pair. Since p_s is serial while the p_i, q_j with $1 \le i < s$ and $1 \le j \le s$ are non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_su_{s-1}^{-1}\cdots u_1\alpha^{-n}q_1q_2^{-1}\cdots q_{s-1}^{-1}q_s$, which is of the form stated in Definition (4.1.11)(a) for $s-1$. Otherwise $s \ge 3$ and there exists a minimal even integer d with $2 \le d \le s-1$ such that q_d is serial. Let u_d be a path (maybe trivial) such that q_du_d is a string ending in a deep. Being non-trivial, $q_1 = \alpha_1\cdots\alpha_t$ for some $1 \le t < m$. Let $u_1 = \alpha_{t+1}\cdots\alpha_m$, and for each even i with $2 \le i < d$, let u_i, u_{i+1} be non-trivial paths such that $(q_iu_i, q_{i+1}u_{i+1})$ is a binomial pair. Since q_d and p_s are serial while the others between them are non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_d^{-1}u_{d-1}\cdots u_2^{-1}u_1q_1q_2^{-1}\cdots q_{s-1}^{-1}q_s$, which is of the form stated in Definition (4.1.11)(b) for d and $s-1$. This completes the proof.

Lemma (4.1.13)[117]: Let $A = kQ/I$ with (Q, I) being as in Definition (4.1.10). If M is a string module associated to a string as stated in Definition (4.1.11), then $\Omega(M)$ has as a direct summand a string module associated to a string as stated in Definition (4.1.10).

Proof. Let $M = M(w)$ with w a string. Let us begin with the case

$$w = p_{r+1}p_r^{-1}\cdots p_2^{-1}p_1\alpha^{-n}q_1q_2^{-1}\cdots q_r^{-1}q_{r+1},$$

where $r \ge 0$ is even, the p_i, q_i are non-trivial for $1 \le i \le r$ such that for each even i with $0 \le i < r$, $(q_ip_i, q_{i+1}p_{i+1})$ is a binomial pair with $q_0 = \alpha^n$ and $p_0 = \alpha$, and there exists a binomial pair $(q_rp_r, q_{r+1}\beta_{r+1}p_{r+1})$ with β_{r+1} some arrow.

Suppose first that p_{r+1} is trivial and that there exists no odd i with $1 \le i < r$ such that p_i is serial. If $r = 0$, then $q_i = \alpha_1\cdots\alpha_{m-1}$ by hypothesis. Thus $\Omega(M) = M(\varepsilon_a)$. If $r \ge 2$, then $p_1 = \alpha_1\cdots\alpha_m$ with $1 < t \le m$. For each odd i with $1 \le i \le r-3$, let u_i, u_{i+1} be non-trivial paths such that $(p_i u_i, p_{i+1}u_{i+1})$ is a binomial pair, and let δ_r be an arrow and u_r, u_{r-1} some paths with u_{r-1} non-trivial such that $(q_{r-1}, u_{r-1}, q_r\delta_r u_r)$ is a binomial pair. Since none of the paths contained in w is a serial string, $\Omega(M)$ is the string module associated to the string $u_r u_{r-1}^{-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{r-1}^{-1} p_r$, which is of the form stated in Definition (4.1.10)(a) for $r-1$.

Suppose secondly that p_{r+1} is non-trivial and that there exists no odd i with $1 \leq i \leq r+1$ such that p_i is serial. For each odd i with $1 \leq i \leq r-1$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1}u_{i+1})$ is a binomial pair. Moreover, let δ_{r+2} be an arrow and u_{r+2}, u_{r+1} some non-trivial paths such that $(q_{r+1}u_{r+1}, \delta_{r+2}u_{r+2})$ is a binomial pair. For the same reason, $\Omega(M)$ is the string module associated to the string $u_{r+2}u_{r+1}^{-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_r p_{r+1}^{-1}$, which is of the form stated in Definition (4.1.10)(a) for $r+1$ with p_{r+2} trivial.

Suppose now that neither of the above two situations occurs. Then there exists a minimal odd integer s with $1 \leq s \leq r+1$ such that p_s is non-trivial and serial. Let us be a path (maybe trivial) such that $p_s u_s$ is a string ending in a deep, and for each odd i with $1 \leq i < s$, let u_i, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1}u_{i+1})$ is a binomial pair. Since the p_i, q_j with $1 \leq i < s$ and $1 \leq j \leq r$ are all non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s^{-1} u_{s-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1} p_s^{-1} \cdots p_r p_{r+1}^{-1}$, which is of the form stated in Definition (4.1.10)(b) for s and $r-1$ if p_{r+1} is trivial with $s \leq r-1$; and otherwise, for s and $r+1$ with p_{r+2} trivial. We shall conclude the proof with the case:

$$w = p_s^{-1} p_{s-1} \cdots p_2^{-1} p_1 \alpha^{-n} q_1 q_2^{-1} \cdots q_{s-1} q_s^{-1} \cdots q_r^{-1} q_{r+1},$$

where s, r are even with $2 \leq s \leq r$, and the p_i, q_j are non-trivial paths for $1 \leq i < s$ and $1 \leq j \leq r$ such that for each even i with $0 \leq i < s, (q_i p_i, q_{i+1}p_{i+1})$ is a binomial pair with $q_0 = \alpha^n, p_0 = \alpha$, whereas $q_s p_s$ is a serial string ending in a deep.

Suppose first that p_i is non-serial for every odd i with $1 \leq i < s$. Let δ_s be an arrow and u_s, u_{s-1} some paths with u_{s-1} non-trivial such that $(p_s \delta_s u_s, p_{s-1}u_{s-1})$ is a binomial pair; and for each even i with $2 \leq i < s$, let u_i and u_{i-1} be non-trivial paths such that $(q_{i-1}u_{i-1}, q_i u_i)$ is a binomial pair. Since q_s is serial while the p_i, q_j with $1 \leq i \leq s$ and $1 \leq j < s$ are all non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_s u_{s-1}^{-1} \cdots u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1}^{-1} p_s$, which is of the form stated in Definition (4.1.10)(a) for $s-1$.

Otherwise, there exists a minimal odd integer d with $1 \leq d \leq s-1$ such that pad is serial. Let u_d be a path (maybe trivial) such that $q_d u_d$ is a string ending in a deep, and for each odd i with $1 \leq i < d$, let u_s, u_{i+1} be non-trivial paths such that $(q_i u_i, q_{i+1} u_{i+1})$ is a binomial pair. Since p_d and q_s are serial while the others between them are all non-serial, $\Omega(M)$ has as a direct summand the string module associated to the string $u_d^{-1} u_{d-1} \cdots u_2 u_1^{-1} \alpha p_1^{-1} p_2 \cdots p_{s-1}^{-1} p_s$, which is of the form stated in Definition (4.1.10)(b) for d and $s-1$. This completes the proof of the lemma.

Let A be the algebra of a bound quiver (Q, I). The convex support of an A-module M is the algebra of the bound quiver (Q_M, I_M), where Q_M is the convex hull in Q of the vertices a with $\mathrm{Hom}_A(P(a), M) \neq 0$, and $I_M = I \cap (kQ_M)$.

Theorem (4.1.14)[117]: Let A be the algebra of a bound quiver (Q, I), and let S be a simple A-module with a non-split self-extension. If the convex support of S is special biserial, then for all $i \geq 1, \mathrm{Ext}_A^i(S, S)$ does not vanish.

Proof. Let B be the convex support of S in A. It is well known that $\mathrm{Ext}_A^i(S,S) = \mathrm{Ext}_B^i(S,S)$ for all $i \geq 1$. Hence we may assume, without loss of generality, that (Q, I) is special biserial. Let α be a loop at the vertex a such that $S = S(a)$. If no power of α is a component of a binomial relation on Q, then the theorem follows from Corollary (4.1.3). Assume now that there exists a binomial pair $(\alpha^{n+1}, \alpha_1 \cdots \alpha_m)$ with $n \geq 1$ and $m \geq 2$. Note that $\Omega^0(S) = M(\varepsilon_a)$. Assume, for $i \geq 0$, that $\Omega^i(S)$ has as a direct summand a string module associated to a string stated in Definitions (4.1.10) or (4.1.11), and in particular, $S(a)$ is a summand of the top of $\Omega^i(S)$. By Lemmas (4.1.12) and (4.1.13), the same holds true for $\Omega^{i+1}(S)$. This completes the proof of the theorem.

We conclude with an even stronger version of the strong no loop conjecture.

Conjecture (4.1.15)[117]: If S is a simple module over an Artin algebra A with $\mathrm{Ext}_A^1(S,S)$ not vanishing, then $\mathrm{Ext}_A^i(S,S)$ does not vanish for infinitely many i.

Section (4.2): Algebras with Two Simples and Radical Cube Zero:

Let Λ be an artinian ring. Many important problems remain to be solved in connection with the homological properties of Λ-modules. We mention the finitistic dimension conjecture (see [130], [131]) and the Cartan determinant conjecture (see [132]). Both these problems deal with studying homological dimensions of Λ-modules. In this Section we will consider no-loop conjectures (see [133], [134]). The (weak) no-loop conjecture says that if $\mathrm{Ext}_{\Lambda}^{1}(S,S) \neq 0$ for a simple Λ-module S, then the global dimension of Λ is infinite. The strong no-loop conjecture says that if $\mathrm{Ext}_{\Lambda}^{1}(S,S) \neq 0$ then the projective dimension of S is infinite.

The weak no-loop conjecture was proven in [133] for a large class of finite-dimensional algebras over a field k, including all finite-dimensional algebras, if k is algebraically closed. The strong no-loop conjecture seems to be more difficult, in fact it has only been established for very special classes of algebras.

For a Λ-module U, let $\mathrm{pd}_{\Lambda} U$ denote the projective dimension of U. Let gldim Λ denote the global dimension of Λ. Let $\mathfrak{r}$ denote the Jacobson radical of Λ. The strong no-loop conjecture holds if $\mathfrak{r}^2 0$. For in this case if $\mathrm{Ext}_{\Lambda}^{1}(S,S) \neq 0$ then S is a summand of its own syzygy and so $\mathrm{pd}_{\Lambda} S = \infty$. If Λ has only one simple module up to isomorphism then all non-projective modules have infinite projective dimension. So in this case $\mathrm{pd}_{\Lambda} S = \infty$ if $\mathrm{Ext}_{\Lambda}^{1}(S,S) \neq 0$. For algebras with two simple modules up to isomorphism and for radical cube zero algebras the situation is much more complicated.

The artinian ring Λ is a filtered ring using its radical filtration. Let $\mathrm{gr}\Lambda = \oplus_i (\mathrm{gr}\Lambda)_i$ denote the corresponding graded ring of Λ. We say that Λ is graded by its radical if the canonical isomorphisms $\mathfrak{r}^i/\mathfrak{r}^{i+1} \to (\mathrm{gr}\Lambda)_i$ lift to an isomorphism of rings $\Lambda \cong \mathrm{gr}\Lambda$. The following is our main result.

Now we only have to note that any basic finite-dimensional algebra over an algebraically closed field with radical cube zero is graded by its radical and we obtain the following

Corollary (4.2.1)[129]: Let Λ be a finite-dimensional k-algebra over an algebraically closed field with $\mathfrak{r}^3 = 0$ and with at most two simple modules up to isomorphism. If S is a simple module of finite projective dimension then $\operatorname{Ext}_\Lambda^1(S, S) = 0$.

Definitions and some basic results. Let Λ be an artinian ring graded by its radical. That is, Λ is a graded artinian ring

$$\Lambda = \bigoplus_{i=0}^{L} \Lambda_i$$

where Λ_0 is semisimple and $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for $i, j \in \{0, \ldots, L\}$ with $i + j \leq L$. Unless otherwise stated, from now on, all modules will be graded left Λ- modules of finite length and all homomorphisms between Λ-modules will be graded of degree 0. For a Λ-module $M = \bigoplus_i M_i$ we denote by $M[j]$ the shifted Λ-module given by $M[j]_i := M_{i-j}$.

Note that the finiteness of the projective dimension of a simple Λ-module S is independent of whether we use graded projective resolutions or not. Moreover the extension group $\operatorname{Ext}_\Lambda^1(S, T[1])$of two simple Λ-modules S and T generated in the same degree may be identified with the group of extensions of S by T we get by forgetting the grading. So for the questions we are interested in we have not lost any generality by considering graded modules over a graded ring.

Let $S_1, \ldots, S_n$ be a complete set of representatives of simple Λ-modules generated in degree 0. Let $P_1, \ldots, P_n$ be a corresponding set of representatives of indecomposable projective Λ-modules. That is, $P_i / \mathfrak{r} P_i \cong S_i$ for $i = 1, \ldots, n$.

Let $[S_i]$ denote the element of $\mathbb{Z}[t]^n$ given by $[S_i]_j = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta. To every P_i we associate the element $[P_i]$ in $\mathbb{Z}[t]^n$ given by

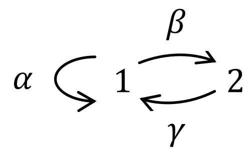
$$[P_i] = \sum_{r=0}^{L} \sum_{j=1}^{n} C_{ij}^{(r)} [S_j] t^r$$

where $C_{ij}^{(r)}$ is the largest integer m such that $\mathfrak{r}^r P_i / \mathfrak{r}^{r+1} P_i \cong S_j[r]^m \oplus U$. Let $C = C(\Lambda)$ be the graded Cartan matrix of Λ (see [135]). That is, $\mathcal{C}$ is an n by n matrix with coefficients in $\mathbb{Z}[t]$ given by

$$C = \sum_{r=0}^{L} C^{(r)} t^r$$

where $C^{(r)}$ is a matrix with coefficients in $\mathbb{Z}$ and where $C_{ij}^{(r)}$ was defined above. In other words, the ith column of C is $[P_i]$. Note that $C^{(0)}$ is the identity matrix.

Example (4.2.2)[129]: Let Λ be the algebra with quiver



and relations $\beta\gamma, \alpha^2 - \gamma\beta, \beta\alpha\gamma$. Then Λ is graded by its radical. A basis of the projective P_1 at vertex 1 is $e_1, \alpha, \beta, \beta\alpha, \alpha^2, \alpha^3$. Thus

$$[P_1] = \begin{pmatrix} 1 + t + t^2 + t^3 \\ t + t^2 \end{pmatrix}$$

A basis of the projective P_2 at vertex 2 is $e_2, \gamma, \alpha\gamma$. Thus

$$[P_2] = \begin{pmatrix} t + t^2 \\ 1 \end{pmatrix}$$

Hence the graded Cartan matrix of Λ is given by

$$C = \begin{pmatrix} 1 + t + t^2 + t^3 & t + t^2 \\ t + t^2 & 1 \end{pmatrix}$$

If N and M are two n by n matrices of integers we write $M \geq N$ if all entries of $M - N$ are non-negative.

Lemma (4.2.3)[129]: The matrices $C^{(r)}$ satisfy the following inequalities:

(i) $C^{(i)} \geq 0$ for all $i = 0, \dots, L$,

(ii) $C^{(l)} C^{(m)} \geq C^{(l+m)}$ for all $l, m \in \{0, \dots, L\}$ with $l + m \leq L$.

Proof. Part (i) is obvious. By the Wedderburn-Artin theorem we have an isomorphism

$$\Lambda_0 \cong \bigoplus_{i=1}^{n} M_{n_i}(D_i)$$

where $M_{n_i}(D_i)$ is the full matrix ring over a division ring D_i. We view this isomorphism as an identification and let e_i denote the identity matrix of the matrix ring $M_{n_i}(D_i)$. Let $l, m \in$

$\{0, \dots, L\}$ with $l+m \le L$. We have $\Lambda_m \Lambda_l = \Lambda_{l+m}$ and so we get a surjective Λ_0-Λ_0-homomorphism $e_j \Lambda_m \otimes_{\Lambda_0} \Lambda_l e_i \to e_j \Lambda_{l+m} e_i$ induced by multiplication. Now

$$e_j \Lambda_m \otimes_{\Lambda_0} \Lambda_l e_i \cong \bigoplus_{r=1}^{n} e_j \Lambda_m e_r \otimes_{M_{n_r}(D_r)} e_r \Lambda_l e_i .$$

The number of indecomposable left summands of $e_j \Lambda_m e_r \otimes_{M_{n_r}(D_r)} e_r \Lambda_l e_i$ is $C_{ir}^{(l)} C_{rj}^{(m)} n_i$. The number of indecomposable left summands of $e_j \Lambda_{l+m} e_i$ is $C_{rj}^{(l+m)} n_i$. Therefore $C_{rj}^{(l+m)} \le \left(C^{(l)} C^{(m)}\right)_{ij}$. This concludes the proof of the lemma.

The following lemma is well known; see for example [136].

Lemma (4.2.4)[129]: We have $C_{ij}^{(1)} > 0$ if and only if $\mathrm{Ext}_\Lambda^1(S_i, S_j[1]) \ne 0$.

Let $Q = Q(\Lambda)$ be the quiver given by the matrix $C^{(1)}$. That is, Q is an oriented graph with vertices $1, \dots, n$ and $C_{ij}^{(1)}$ arrows from vertex i to vertex j. Thus, by the previous lemma, $\mathrm{Ext}_\Lambda^1(S_i, S_i[1])$ is non-zero for some simple S_i if and only if Q has a loop at vertex i.

Let $\Delta = \sum_{i=0}^{n \cdot L} \Delta_i t^i =: \det C$ denote the graded Cartan determinant of Λ. Let M_{ij} be the ijth cofactor of the matrix C. That is, M_{ij} is $(-1)_{i+j}$ times the determinant of the matrix obtained by removing the ith column and the jth row from C. Then

$$C^{-1} = \frac{1}{\Delta} \left(M_{ji}\right)_{ij}$$

is a matrix over the field of rational functions $\mathbb{Q}(t)$. For non-zero polynomials $a_1, \dots, a_n \in \mathbb{Z}[t]$ let $\gcd(a_1, \dots, a_n)$ denote their greatest common factor. We let the coefficient of the lowest degree term of $\gcd(a_1, \dots, a_n)$ be positive.

Lemma (4.2.5)[129]: Let Δ, C and Q be as above. Then

(i) $\Delta_0 = 1$.

(ii) Δ_1 is the number of loops of Q.

(iii) $\gcd\left(M_{1j}, \dots, M_{nj}\right) \big| \Delta$ for all $j = 1, \dots, n$.

(iv) If $\mathrm{gldim}\, \Lambda < \infty$ then $\Delta = 1$.

(v) If $\mathrm{pd}_\Lambda S_j < \infty$ then $\Delta = \gcd\left(M_{1j}, \dots, M_{nj}\right)$.

Proof. We have $\Delta_0 = \det C^{(0)}$. Now $C^{(0)}$ is the identity matrix and so (i) follows. The constant terms in the polynomials off the diagonal of C are all zero. Hence Δ_1 is the trace of $C^{(1)}$. This proves (ii). We have $\Delta = \sum_{i=1}^{n} C_{ij} M_{ij}$, which proves (iii).

Let $\mathrm{pd}_\Lambda S_j < \infty$. We have a graded projective resolution

$$0 \to Q_m \to \cdots \to Q_1 \to Q_0 \to S_j \to 0$$

of S_j. Thus $[S_j] = \sum_{i=0}^{m} (-1)^i [Q_i] = \sum_{i=1}^{n} f_{ij}(t)[P_i]$ for polynomials $f_{ij}(t) \in \mathbb{Z}[t]$. Here

$$f_{ij} = \frac{M_{ij}}{\Delta}$$

for $i = 1, \dots, n$ and so $\Delta = \gcd(M_{1j}, \dots, M_{nj})$ by (iii). This proves (v).

If gldim $\Lambda < \infty$, then again by the graded projective resolution of the simples we see that C^{-1} is a matrix with entries in $\mathbb{Z}[t]$ and consequently Δ is a unit in $\mathbb{Z}[t]$. Hence $\Delta = 1$ by (i). This proves (iv).

Example (4.2.6)[129]: Let Λ be as in Example (4.2.3). Then

$$C^{-1} = \frac{1}{\Delta}\left(M_{ji}\right)_{ij} = \frac{1}{\Delta}\begin{pmatrix} 1 & -t - t^2 \\ -t - t^2 & 1 + t + t^2 + t^3 \end{pmatrix}.$$

Moreover $\gcd(M_{11}, M_{21}) = \gcd(1, -t, t^2) = 1$ and $\gcd(M_{12}, M_{22}) = \gcd(-t - t^2, 1 + t + t^2 + t^3) = 1 + t$. We see that $\Delta = 1 + t - t^3 - t^4$. Consequently $\mathrm{pd}_\Lambda S_1 = \infty$ and $\mathrm{pd}_\Lambda S_2 = \infty$ by Lemma (4.2.5)(v).

Theorem (4.2.7)[129]: Let Λ be an artinian ring graded by its radical with $\mathfrak{r}^4 = 0$ and with at most two simple modules up to isomorphism. If S is a simple module of finite projective dimension then $\mathrm{Ext}^1_\Lambda(S, S) = 0$.

Proof. Let Λ be an artinian ring graded by its radical with at most two simple modules up to isomorphism such that $\mathfrak{r}^4 = 0$. That is, $\Lambda = \bigoplus_{i=0}^{3} \Lambda_i$ where Λ_0 is semisimple and $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for $i, j \in \{0, \dots, 3\}$ with $i + j \le 3$. We may also assume that $\Lambda_2 \neq 0$ and that Λ has exactly two simple modules S_1, S_2 up to isomorphism. We assume that $\mathrm{Ext}^1_\Lambda(S_1, S_1[1]) \neq 0$ and that $\mathrm{pd}_\Lambda S_1 < 1$. We will obtain a contradiction, which proves the theorem.

Let

$$C = \sum_{i=0}^{3} C^{(i)} t^i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the graded Cartan matrix of Λ. We have $\Delta = ad - bc$ and

$$C^{-1} = \frac{1}{\Delta}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

where Δ is the graded Cartan determinant of Λ. Thus by Lemma (4.2.6)(v) we have $\gcd(M_{11}, M_{21}) = \gcd(d, -c) = ad - bc$. So there exist polynomials λ and μ such that $d = \lambda\Delta, c = \mu\Delta$ and $a\lambda - b\mu = 1$. Let $a = \sum_i a_i t^i$ and similarly for b, c and d. Then $a_0 = 1 = d_0$ and $b_0 = 0 = c_0$. If $b_1 = 0$ then $b = 0$ by Lemma (4.2.3)(ii). Similarly if $c_1 = 0$ then $c = 0$. In either case $\mathrm{pd}_\Lambda S_1 = \infty$ (see [131]) so we may assume that $b_1, c_1 > 0$. By Lemma (4.2.5) we see that $a_1 > 0$. Since $\mathrm{pd}_\Lambda S_1 < \infty$ at least one of the projectives has radical length less than 4. Hence we have two cases to consider, either $c_3 = 0 = a_3$ or $d_3 = 0 = b_3$.

We consider first the case $d_3 = 0 = b_3$. We see that deg $\Delta \leq 2$, where degΔ denotes the degree of the polynomial Δ. If deg $\Delta = 2$ then $d = \Delta$, which is a contradiction since the linear term of Δ is $a_1 + d_1$ and $a_1 > 0$. Thus deg $\Delta = 1$ and $\Delta = 1 + (a_1 + d_1)t$. Consequently, $\lambda = 1 - a_1 t$ since $a\lambda - b\mu = 1$. Thus $d = \lambda\Delta = 1 + d_1 t + (-a_1^2 - a_1 d_1)t^2$. But this is a contradiction since $d_2 \geq 0$. This concludes the proof in the case where $d_3 = 0 = b_3$.

We now consider the case $c_3 = 0 = a_3$. As before, $\Delta = 1 + (a_1 + d_1)t$. Thus $\mu = c_1 t$ and $\lambda = 1 - a_1 t + \lambda_2 t^2$ for some integer λ_2. Since $d = \lambda\Delta$ we get $d_2 = \lambda_2 - a_1^2 - a_1 d_1$ and $d_3 = (a_1 + d_1)\lambda_2$. Similarly,

$$c_2 = c_1(a_1 + d_1).$$

Since $a\lambda - b\mu = 1$ we see that $\lambda_2 = b_1 c_1 + a_1^2 - a_2$. Thus

$$d_2 b_1 c_1 - a_2 - a_1 d_1, \qquad d_3 = a_1 b_1 c_1 + a_1^3 - a_1 a_2 + b_1 c_1 d_1 + a_1^2 d_1 - a_2 d_1.$$

Moreover, again by $a\lambda - b\mu = 1$, we get

$$b_2 = a_1 b_1 + \frac{a_1^3 - 2a_1 a_2}{c_1}, \qquad b_3 = a_2 b_1 + \frac{a_1^2 a_2 - a_2^2}{c_1}.$$

By Lemma (4.2.4)(ii) we have $C^{(2)}C^{(1)} \geq C^{(3)}$ and so $d_3 \leq c_1 b_2 + d_1 d_2$. Thus we get $a_1 a_2 + a_1^2 d_1 + a_1 d_1^2 \leq 0$ and so $a_2 = 0$ and $d_1 = 0$. Again by Lemma (4.2.3)(ii) we have $C^{(2)}C^{(1)} \geq C^{(3)}$ and so $d_3 \leq c_2 b_1 + d_2 d_1$. But then $a_1^3 \leq 0$, which is a contradiction since $a_1 > 0$. This completes the proof of the theorem.

Section (4.3): Strong No Loop Conjecture is True for Mild Algebras:

Let Λ be a finite dimensional associative algebra over a fixed algebraically closed field k of arbitrary characteristic. We consider only Λ-right modules of finite dimension.

The strong no loop conjecture says that a simple Λ-module S of finite projective dimension satisfies $\mathrm{Ext}_{\Lambda}^{1}(S, S) = 0$. To prove this conjecture for a given algebra we can switch to the Morita-equivalent basic algebra and therefore assume that $\Lambda = \mathbf{k}Q/I$ for some quiver Q and some ideal I generated by linear combinations of paths of length at least two. Then $S = S_x$ is the simple corresponding to a point x in Q and the conjecture means that there is no loop at x provided the projective dimension $\mathrm{pdim}_{\Lambda} S_x$ is finite.

The conjecture is known in [138, 139, 140, 141, 142].

We prove the conjecture for another class of algebras including all representation finite algebras. To state our result precisely we introduce for any point x in Q its neighborhood $\Lambda(x) = e \Lambda e$. Here e is the sum of all primitive idempotents $e_z \in \Lambda$ such that z belongs to the support of the projective $P_x := e_x \Lambda$ or such that there is an arrow $z \to x$ in Q or a configuration $y^{'} \longleftarrow x \rightleftarrows y \longleftarrow z$ with 4 different points $x, y, y^{'}$ and z.

Recall that an algebra Λ is called distributive if it has a distributive lattice of two-sided ideals and mild if it is distributive and any proper quotient Λ/J is representation-finite.

Our main result reads as follows:

Theorem (4.3.1)[137]: Let $\Lambda = \mathbf{k}Q/I$ be a finite dimensional algebra over an algebraically closed field k. Let x be a point in Q such that the corresponding simple Λ- module S_x has finite projective dimension. If $\Lambda(x)$ is mild, then there is no loop at x.

Of course, it follows immediately that the strong no loop conjecture holds for all mild algebras, in particular for all representation-finite algebras.

Corollary (4.3.2)[137]: Let Λ be a mild algebra over an algebraically closed field. Let S be a simple Λ-module. If the projective dimension of S is finite, then $\mathrm{Ext}_{\Lambda}^{1}(S, S) = 0$.

In order to prove the theorem we do not look at projective resolutions. Instead we refine a little bit the K-theoretic arguments of Lenzing [143], also used by Igusa in his proof of the strong no loop conjecture for monomial algebras [138, Corollary 6.2], to obtain the following result:

Proposition (4.3.3)[137]: Let $\Lambda = \mathrm{k}Q/I$ be a finite dimensional algebra, x a point in Q and α an oriented cycle at x. If P_x has an α-filtration of finite projective dimension, then α is not a loop.

Here an α-filtration $\mathcal{F}$ of P_x is a filtration

$$P_x = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

by submodules with

$$\alpha M_i \subset M_{i+1}\ \forall i = 0 \dots n-1.$$

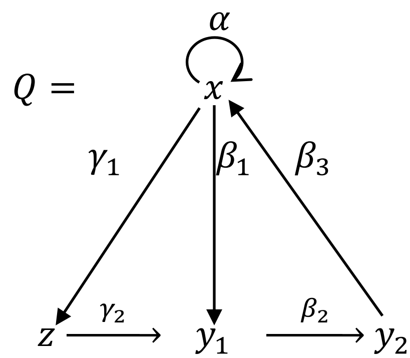
The filtration $\mathcal{F}$ has finite projective dimension if $\mathrm{pdim}_\Lambda M_i < \infty$ holds for all $i = 1 \dots n-1$.

This proposition is shown by Lenzing in [143] for the special filtration $M_i = \alpha^i\Lambda$, but his proof remains valid for all α-filtrations.

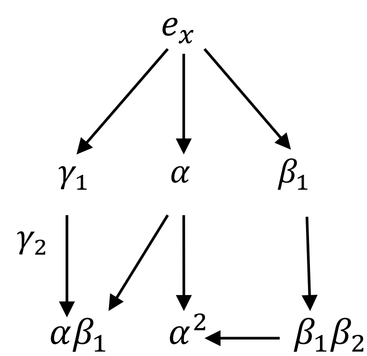
Our strategy to prove Theorem (4.3.1) is then as follows: We consider the point x with $\mathrm{pdim}_\Lambda S_x < \infty$ and its mild neighborhood $A := \Lambda(x)$. We assume in addition that there is a loop α in x. Then we deduce a contradiction either by showing that $\mathrm{pdim}_\Lambda S_x = \infty$ or by constructing a certain α-filtration $\mathcal{F}$ of P_x having finite projective dimension in mod-Λ and implying that α is not a loop by Proposition (4.3.3). Since $\alpha(x)$ contains the support of P_x, this filtrations coincide for P_x as a Λ-module and as a $\Lambda(x)$-module. Thus we are dealing with a mild algebra, and we use in an essential way the deep structure theorems about such algebras given in [144] and [145] to obtain the wanted α-filtrations. In particular, we show that we always work in the ray-category attached to $\Lambda(x)$. This makes it much easier to use cleaving diagrams. But still the construction of the appropriate α-filtrations depends on the study of several cases and it remains a difficult technical problem. The α-filtrations are always built in such a way that they have finite projective dimension in mod- Λ provided $\mathrm{pdim}_\Lambda S_x < \infty$.

To illustrate the method by two examples we define $\langle w_1, \dots, w_k\rangle$ as the submodule of P_x generated by elements $w_1, \dots, w_k \in P_x$.

Example (4.3.4)[137]: Let Λ be an algebra such that $\Lambda(x)$ is given by the quiver



and a relation ideal I such that the projective module P_x is described by the following graph:



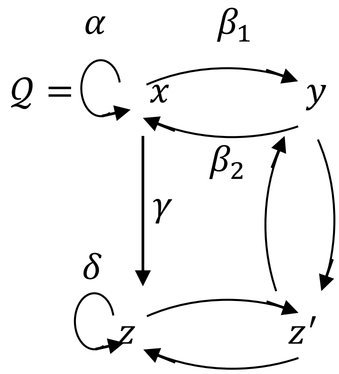
Notice that the picture means that there are relations $\alpha^2 - \lambda_1\beta_1\beta_2\beta_3, \alpha\beta_1 - \lambda_2\gamma_1\gamma_2 \in I$ for some $\lambda_i \in \mathrm{k}\backslash\{0\}$. From the obvious exact sequences

$$0 \to \operatorname{rad} P_x \to P_x \to S_x \to 0$$
$$0 \to \langle\beta_1, \gamma_1\rangle \to \operatorname{rad} P_x \to S_x \to 0$$
$$0 \to \langle\alpha^2, \gamma_1\rangle \to \langle\alpha, \gamma_1\rangle \to S_x \to 0$$

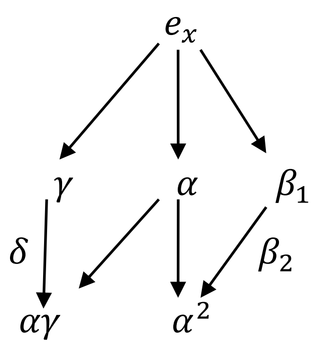
we see that $\mathrm{pdim}_\Lambda S_x < \infty$ leads to $\mathrm{pdim}_\Lambda \operatorname{rad} P_x < \infty$ and $\mathrm{pdim}_\Lambda\langle\beta_1, \gamma_1\rangle < \infty$. Since $\langle\beta_1, \gamma_1\rangle = \langle\beta_1\rangle \oplus \langle\gamma_1\rangle$ and $\langle\alpha^2, \gamma_1\rangle = \langle\alpha^2\rangle \oplus \langle\gamma_1\rangle$ in this example, both $\mathrm{pdim}_\Lambda\langle\gamma_1\rangle$ and $\mathrm{pdim}_\Lambda\langle\alpha, \gamma_1\rangle$ are finite. Then the following α-filtration $\mathcal{F}: P_x \supset \langle\alpha, \gamma_1\rangle \supset \langle\alpha^2\rangle \supset 0$ has finite projective dimension in mod- Λ.

In the next example we see that this method may not work if the neighborhood $\Lambda(x)$ is not mild, even if the support of P_x is mild.

Example (4.3.5)[137]: Let $\Lambda(x) = \mathbf{k}Q/I$ be given by the quiver



and by a relation ideal I such that P_x is represented by



Here we get stuck because the uniserial module with basis $\{\gamma, \alpha\gamma\}$ allows only the composition series as an α-filtration. Since we do not know $\mathrm{pdim}_\Lambda S_x$, which depends on Λ and not only on $\Lambda(x)$, our method does not apply.

We recall some well-known facts from [144], [146].

Let $A := \Lambda(x) = \mathbf{k}Q_A/I_A$ be a basic distributive $\mathbf{k}$-algebra. Then every space e_xAe_y is a cyclic module over e_xAe_x or e_yAe_y and we can associate to A its ray-category $\vec{A}$. Its objects are the points of Q_A. The morphisms in $\vec{A}$ are called rays and $\vec{A}\,(x,y)$ consists of the orbits $\vec{\mu}$ in e_xAe_y under the obvious action of the groups of units in e_xAe_x and e_yAe_y. The composition of two morphisms $\vec{\mu}$ and $\vec{\nu}$ is either the orbit of the composition $\mu\nu$, in case this is independent of the choice of representatives in $\vec{\mu}$ and $\vec{\nu}$, or else 0. We call a non-zero morphism $\eta \in \vec{A}$ long if it is non-irreducible and satisfies $\nu\eta = 0 = \eta\nu'$ for all nonisomorphisms $\nu v' \in \vec{A}$. One crucial fact about ray-categories frequently used in this paper is that A is mild iff $\vec{A}$ is so [146].

The ray-category is a finite category characterized by some nice properties. For instance, given $\lambda\mu\kappa = \lambda\nu\kappa \neq 0$ in $\vec{A}$, $\mu = \nu$ holds. We shall refer to this property as the cancellation law.

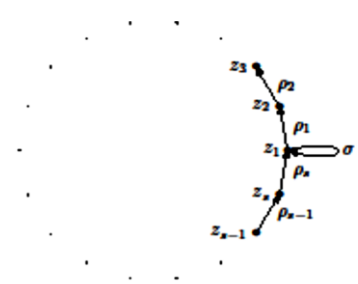
Given $\vec{A}$, we construct in a natural way its linearization $\mathbf{k}(\vec{A})$ and obtain a finite dimensional algebra

$$\vec{A} = \bigoplus_{x,y \in Q_A} K(\vec{A})(x,y),$$

the standard form of A. In general, A and $\vec{A}$ are not isomorphic, but they are if either A is minimal representation-infinite [145] or representation-finite with char $\mathbf{k} \neq 2$ [146].

Similar to A, the ray-category $\vec{A}$ admits a description by quiver and relations. Namely, there is a canonical full functor $\vec{\ }: \mathcal{P}\, Q_A \to \vec{A}$ from the path category of Q_A to $\vec{A}$. Two paths in Q_A are interlaced if they belong to the transitive closure of the relation given by $v \sim w$ iff $v = pv'q, w = pw'q$ and $\vec{v'} = \vec{w'} \neq 0$, where p and q are not both identities.

A contour of $\vec{A}$ is a pair (v, w) of non-interlaced paths with $\vec{v} = \vec{w} \neq 0$. Note that these contours are called essential contours in [144]. Throughout this Section we will need a special kind of contours called penny farthings. A penny-farthing P in $\vec{A}$ is a contour $(\sigma^2, \rho_1 \dots \rho_s)$ such that the full subquiver Q_P of Q_A that supports the arrows of P has the following shape:



Moreover, we ask the full subcategory $A_P \subset A$ living on Q_P to be defined by Q_P and one of the following two systems of relations

$$0 = \sigma^2 - \rho_1 \dots \rho_s = \rho_s\rho_1 = \rho_{i+1} \dots \rho_s\sigma\rho_1 \dots \rho_{f(i)}, \quad (1)$$

$$0 = \sigma^2 - \rho_1 \dots \rho_s = \rho_s\rho_1 - \rho_s\sigma\rho_1 = \rho_{i+1} \dots \rho_s\sigma\rho_1 \dots \rho_{f(i)}, \quad (2)$$

where $f:\{1,2,\ldots.,s-1\}\to\{1,2,\ldots,s\}$ is some non-decreasing function (see [144]. For penny-farthings of type (1) A_P is standard, for that of type (2) A_P is not standard in case the characteristic is two.

A functor $F: D \to \vec{A}$ between ray categories is cleaving ([146]) iff it satisfies the following two conditions and their duals:

(a) $F(\mu)=0$ iff $\mu=0$.

(b) If $\eta\in D(y,z)$ is irreducible and $F(\mu): F(y)\to F(z')$ factors through $F(\eta)$ then μ factors already through η.

The key fact about cleaving functors is that $\vec{A}$ is not representation finite if D is not. In D will always be given by its quiver Q_D, that has no oriented cycles and some relations. Two paths between the same points give always the same morphism, and zero relations are indicated by a dotted line. As in [146], the cleaving functor is then defined by drawing the quiver of D with relations and by writing the morphism $F(\mu)$ in $\vec{A}$ close to each arrow μ.

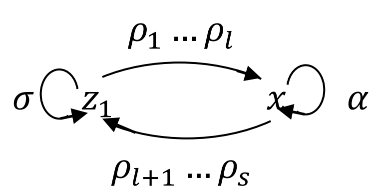
By abuse of notation, we denote the irreducible rays of $\vec{A}$ and the corresponding arrows of Q_A by the same letter.

Using the above notations let $P=(\sigma^2,\rho_1\ldots\rho_s)$ be a penny-farthing in $\vec{A}$. We shall show now that $x=z_1$. Therefore $\sigma=\alpha$ and P is the only penny-farthing in $\vec{A}$ by [146].

Lemma (4.3.6)[137]: If there is a penny-farthing $P=(\sigma^2,\rho_1\ldots\rho_s)$ in $\vec{A}$, then $z_1=x$.

Proof. We consider two cases:

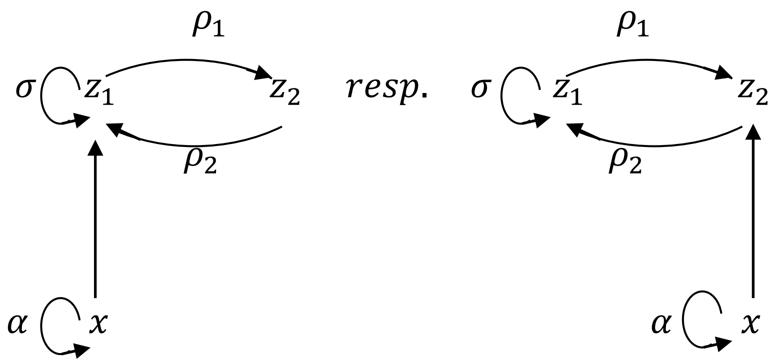
(a) $x\in Q_P$: Hence Q_P has the following shape:



But this can be the quiver of a penny-farthing only for $z_1=x$.

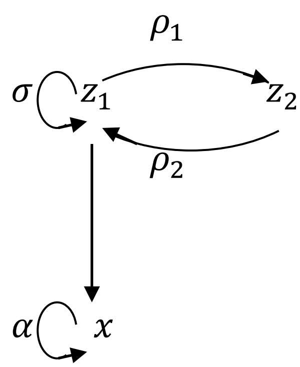
(b) $x\notin Q_P$: Since A is the neighborhood of x, only the following cases are possible:

(a) $e_x A e_z \neq 0$: Since $x \notin Q_P$ we can apply the dual of [147] or [146] to $\vec{A}$ and we see that the following quivers occur as subquivers of Q_A:



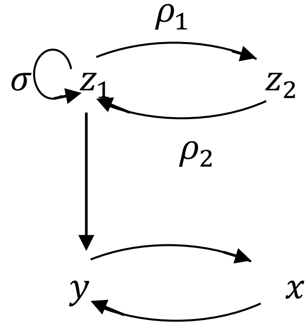
Moreover, there can be only one arrow starting in x. This is a contradiction to the actual setting.

(b) $\exists z_1 \rightarrow x$: By applying [147] or the dual of [146] we deduce that the following quiver occurs as a subquiver of Q_A:



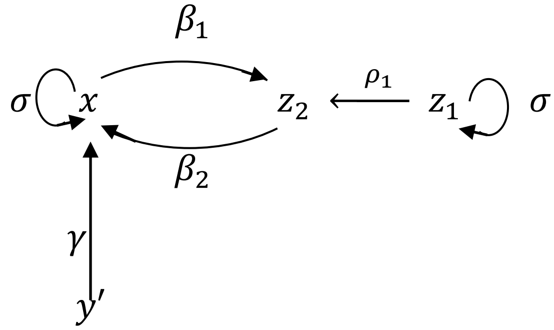
and there can be only one arrow ending in x contradicting the present case.

(c) $\exists y' \longleftarrow x \rightleftarrows y \longleftarrow z_1$: If $y \notin Q_P$, then

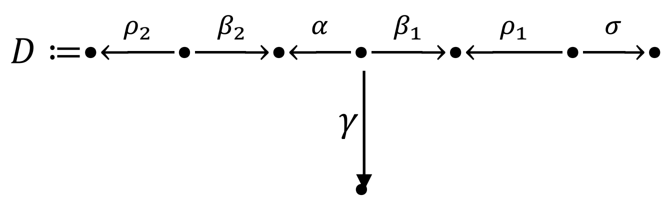


is a subquiver of Q_A leading to the same contradiction as in b).

If $\in Q_P$, then $y = z_2$ and the quiver



is a subquiver of Q_A. Since $\notin Q_P$, all morphisms occurring in the following diagram



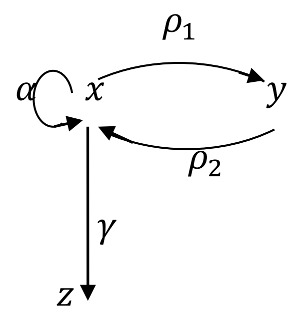
are irreducible and pairwise distinct. Therefore D is a cleaving diagram in $\vec{A}$. Moreover, some long morphism $\eta = v\sigma^3 v'$ does not occur in D; hence D is still cleaving in $\vec{A}/\eta$ by [145]. Since D is of representation-infinite Euclidean type $\tilde{E}_7$, $\vec{A}/\eta$ is representation-infinite contradicting the mildness of A.

Now, we show that, provided the existence of a penny-farthing in $\vec{A}$, there exists an α-filtration of P_x having finite projective dimension.

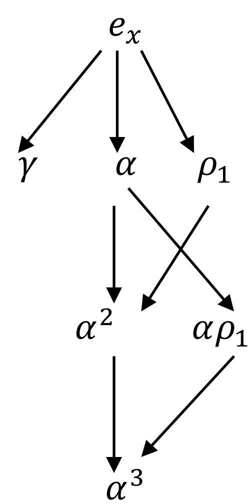
Lemma (4.3.7)[137]: Let $A = \Lambda(x)$ be mild and standard. If there is a penny-farthing in $\vec{A}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If there is a penny-farthing P in $\vec{A}$, then $P = (\alpha^2, \rho_1 \dots \rho_s)$ is the only penny-farthing in $\vec{A}$ by the last lemma. Since A is standard and mild, there are three cases for the graph of P_x which can occur by [147] or the dual of [146].

(i) There exists an arrow $\gamma: x \to z, \gamma \neq \rho_1$. Then $s = 2$, the quiver



is a subquiver of Q_A, and P_x is represented by the following graph:



Let M be a quotient of P_x defined by the following exact sequence:

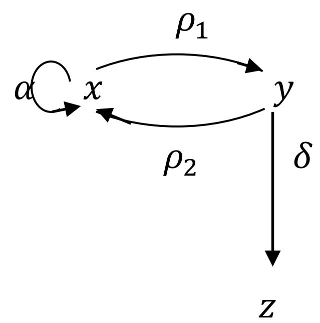
$$0 \to \langle\gamma\rangle \oplus \langle\rho_1, \alpha\rho_1\rangle \to P_x \to M \to 0.$$

Then M has S_x as the only composition factor. Hence $\mathrm{pdim}_\Lambda M < \infty$ and $\mathrm{pdim}_\Lambda\langle\rho_1, \alpha\rho_1\rangle < \infty$. Now, we consider the exact sequence

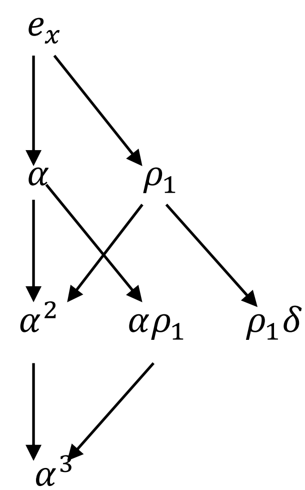
$$0 \to \langle\alpha^3\rangle \to \langle\rho_1, \alpha\rho_1\rangle \to \langle\rho_1\rangle/\langle\alpha^3\rangle \oplus \langle\alpha\rho_1\rangle/\langle\alpha^3\rangle \to 0.$$

But $\langle\alpha^3\rangle \cong S_x$ and $\mathrm{pdim}_\Lambda S_x < \infty$, hence $\langle\alpha\rho_1\rangle/\langle\alpha^3\rangle \cong S_y$Sy has finite projective dimension in mod-Λ. Finally, the Λ-filtration $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \langle\alpha^3\rangle \supset 0$ has finite projective dimension since all filtration modules $\neq P_x$ have S_x and S_y as the only composition factors.

(ii) In the second case there exists a point $z \notin Q_P$ such that $A(x,z) \neq 0$. Then $s = 2$, the quiver

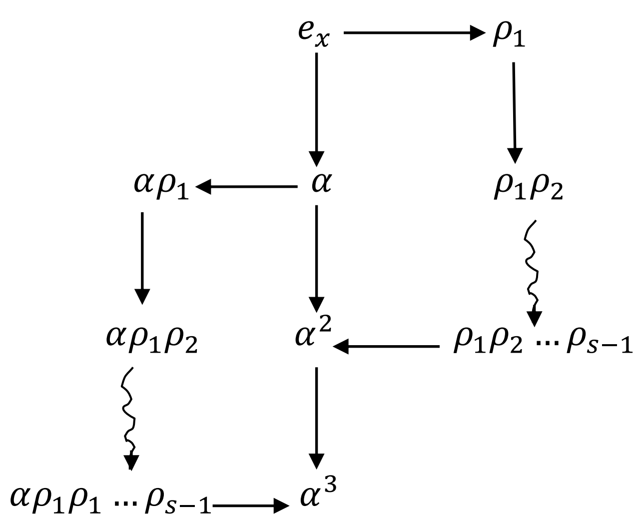


is a subquiver of Q_A, and P_x is represented by:



With similar considerations as in I) we obtain that the same filtration fits.

(iii) In the last possible case we have $A(x,z) = 0$ for all points $z \notin Q_P$. Hence P_x is represented by:



As a Λ-Module, $M := P_x/\langle\alpha^2\rangle$ has finite projective dimension since $\langle\alpha^2\rangle$ has S_x as the only composition factor. Let K be the kernel of the epimorphism $M \to \langle\alpha^2\rangle, e_x \longmapsto \alpha^2$, then $K = \langle\rho_1\rangle/\langle\alpha^2\rangle \oplus \langle\alpha\rho_1\rangle/\langle\alpha^3\rangle$ has finite projective dimension. Moreover, $\mathrm{pdim}_\Lambda\langle\rho_1\rangle, \mathrm{pdim}_\Lambda\langle\alpha\rho_1\rangle < \infty$. Since

$$0 \to \langle\alpha\rho_1\rangle \to \langle\alpha\rangle \overset{\lambda_\alpha}{\to} \langle\alpha^2\rangle \to 0$$

is exact, $\mathrm{pdim}_\Lambda\langle\alpha\rangle < \infty$. Thus the same filtration as in the first two cases fits again.

Lemma (4.3.8)[137]: With above notations let $A = \Lambda(x)$ be mild and non-standard. There exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If A is non-standard, then A is representation finite by [145], $\mathrm{char}\,\mathbf{k} = 2$ and there is a penny-farthing in $\vec{A}$ by [146]. Since Lemma (4.3.6) remains valid, the penny-farthing $(\alpha^2, \rho_1 \dots \rho_s), \rho_i: z_i \to z_{i+1}, z_1 = z_{s+1} = x$, is unique. By [146] the difference between A and $\overline{A}$ in the composition of the arrows shows up in the graphs of the projectives to $z_2, \dots, z_s$ only. Thus the graph of P_x remains the same in all three cases of the proof of Lemma (4.3.7) and the filtrations constructed there still do the job.

If there is no penny-farthing in $\vec{A}$, then $A = \overline{A}$ is standard by Gabriel, Roiter [146] and Bongartz [145]. By a result of Liu, Morin [141], deduced from a proposition of Green, Solberg, Zacharia [140], a power of α is a summand of a polynomial relation in $I = I_\Lambda$. Otherwise $\mathrm{pdim}_\Lambda S_x$ would be infinite contradicting the choice of x. Furthermore, α is a summand of a polynomial relation in I_A by definition of A. But I_A is generated by paths and differences of paths in Q_A. Hence we can assume without loss of generality that there is a relation $\alpha^t - \beta_1\beta_2 \dots \beta_r$ in I_A for some $t \in \mathbb{N}$ and arrows $\beta_1, \beta_2 \dots, \beta_r$. Among all relations of this type we choose one with minimal t. Hence $(\alpha^t, \beta_1\beta_2 \dots \beta_r)$ is a contour in $\vec{A}$ with $t, r \geq 2$. Let $y = e(\beta_1)$ be the ending point of β_1 and $\tilde{\beta} = \beta_2 \dots \beta_r$.

By the structure theorem for non-deep contours in [144] the contour $(\alpha^t, \beta_1\beta_2 \dots \beta_r)$ is deep, i.e. we have $\alpha^{t+1} = 0$ in A. Since A is mild, the cardinality of the set x^+ of all arrows starting in x is bounded by three. Before we consider the cases $|x^+| = 2$ and $|x^+| = 3$ separately we shall prove some useful general facts.

The following trivial fact about standard algebras will be essential hereafter.

Lemma (4.3.9)[137]: Let $A = \overline{A}$ be a standard $\mathbf{k}$-algebra. Consider rays $v_i, w_j \in \vec{A} \backslash \{0\}$ for $i = 1 \dots n$ and $j = 1 \dots m$ such that $v_l \neq v_k$ and $w_l \neq w_k$ for $l \neq k$. If there are $\lambda_i, \mu_j \in k \backslash \{0\}$ such that $\sum_{i=1}^{n} \lambda_i v_i = \sum_{j=1}^{m} \mu_j w_j$, then $n = m$ and there exists a permutation $\pi \in S(n)$ such that $v_i = w_{\pi(i)}$ and $\lambda_i = \mu_{\pi(i)}$ for $i = 1 \dots n$.

Proof. Since the set of non-zero rays in $\vec{A}$ forms a basis of A, it is linearly independent and the claim follows.

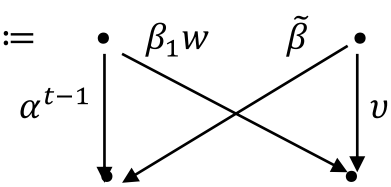
We denote by $\mathcal{L}$ the set of all long morphisms in $\vec{A}$. And by μ we denote some long morphism $v\alpha^t v'$ which exists since $\alpha^t \neq 0$.

Lemma (4.3.10)[137]: Using the above notations we have:

$$\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$$

Proof. We assume to the contrary that $\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle \neq 0$. Then, by Lemma (4.3.9), there are rays $v, w \in \vec{A}$ such that $\beta_1 v = \alpha\beta_1 w \neq 0$. We claim that

$D :=$



is a cleaving diagram in $\vec{A}$. It is of representation-infinite, Euclidean type $\tilde{A}_3$. Since all morphisms occurring in D are not long, the long morphism $\mu = v\alpha^t v'$ does not occur in D and D is still cleaving in $\vec{A}/\mu$ by [145]. Thus $\vec{A}/\mu$ is representation-infinite contradicting the mildness of A.

Now we show in detail, using [145], that D is cleaving. First of all we assume that there is a ray ρ with $\rho\tilde{\beta} = \alpha^{t-1}$. Then we get $0 \neq \alpha^t = \alpha\rho\tilde{\beta} = \beta_1\tilde{\beta}$, whence $\alpha\rho = \beta_1$ by the cancellation law. This contradicts the fact that β_1 is an arrow. In a similar way it can be shown that $\rho\alpha^{t-1} = \tilde{\beta}, \rho v = \beta_1 w$ and $\rho\beta_1 w = v$ are impossible.

The following four cases are left to exclude.

(i) $\alpha^{t-1}\rho = \beta_1 w$: Left multiplication with α gives us $\alpha^t\rho = \alpha\beta_1 w \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}\rho_1 \dots \rho_k, \beta_1 w_1 \dots w_l)$ in $\vec{A}$. Here $\rho = \rho_1 \dots \rho_k$ resp. $w =$

$w_1 \dots w_l$ is a product of irreducible rays (arrows). Since the arrow β_1 is in the contour, the cycle $\beta_1\tilde{\beta}$ and the loop α belong to the contour. Hence it can only be a penny-farthing by the structure theorem for non-deep contours [144]. But this case is excluded in the current section.

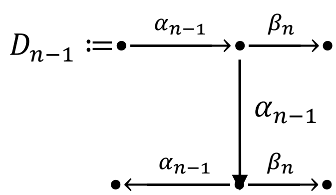
(ii) $\tilde{\beta}\rho = v$: We argue as before and deduce $\beta_1\tilde{\beta}\rho = \beta_1 v = \alpha^t\rho = \alpha\beta_1 w \neq 0$. Hence there is a non-deep contour $(\alpha^t\rho_1 \dots \rho_k, \beta_1 w_1 \dots w_l)$ leading again to a contradiction.

(iii) $\beta_1 w\rho = \alpha^{t-1}$: Since $t - 1 < t$ we have a contradiction to the minimality of t.

(iv) $v\rho = \tilde{\beta}$: Then $\beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t = \alpha\beta_1 v\rho = 0$. Using the cancellation law we get $\alpha^{t-1} = \beta_1 v\rho$ a contradiction as before.

Lemma (4.3.11)[137]: If $t \geq 3$ and $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\alpha^2\beta_1 = 0$.

Proof. If $\alpha^2\beta_1 \neq 0$, then



is a cleaving diagram of Euclidian type $\widetilde{D}_5$ in $\vec{A}$. It is cleaving since:

(i) $\alpha^t = \beta_1\rho \neq 0$ contradicts the choice of $t \geq 3$.

(ii) $\alpha\beta_1 = \beta_1\rho \neq 0$ contradicts Lemma (4.3.10).

It is also cleaving in $\vec{A}/\eta$ for $\eta \in \mathcal{L}\backslash\{\alpha^3, \alpha^2\beta_1\} \neq \emptyset$ contradicting the mildness of A.

Lemma (4.3.12)[137]: If $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle$, then $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

Proof. Let $\alpha^2 u + \beta_1 v = \alpha\beta_1 w \neq 0$ be an element in $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle$. By Lemma (4.3.9) we can assume that u, v, w are rays and the following two cases might occur:

(i) $\beta_1 v = \alpha\beta_1 w \neq 0$: This is a contradiction since $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

(ii) $\alpha^2 u = \alpha\beta_1 w \neq 0$: This is impossible because $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.

Lemma (4.3.13)[137]: If $x^+ = \{\alpha, \beta_1\}$ and $\mathcal{L} \subseteq \{\alpha^3, \alpha^2\beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. We treat two cases:

(i) $\alpha\beta_1 = 0$: Then for $\langle\alpha^k\rangle$ with $k \geq 1$ only S_x is possible as a composition factor; hence $\mathrm{pdim}_\Lambda\langle\alpha^k\rangle < \infty$. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \langle\alpha^3\rangle \supset 0$ is the wanted α-filtration.

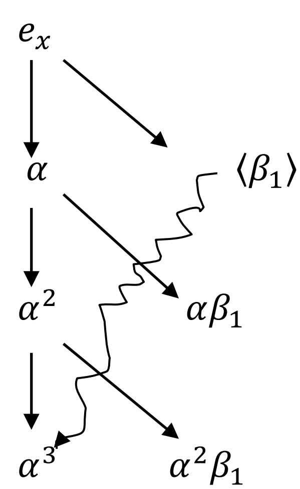
(ii) $\alpha\beta_1 \neq 0$: Since α^3 and $\alpha^2\beta_1$ are the only morphisms in $\vec{A}$ which can be long, we have $t = 3$, $0 \neq \alpha^3 \in \mathcal{L}$, $\langle\alpha\beta_1\rangle = \mathbf{k}\alpha\beta_1 \cong S_y$ and $\langle\alpha^2\beta_1\rangle \in \{\mathbf{k}\alpha^2\beta_1, 0\}$.

Now we show that $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$. If there are rays $v = v_1 \dots v_s$, $w \in \vec{A}$ with irreducible $v_i, i = 1 \dots, s$ such that $\alpha^2 v = \alpha\beta_1 w \neq 0$, then $s > 0$ because $s = 0$ would contradict the irreducibility of α. Therefore $v_1 = \alpha$ or $v_1 = \beta_1$.

(a) If $v_1 = \alpha$, then $v' = v_2 \dots v_s = id$ since α^3 is long and $0 \neq \alpha^2 v = \alpha^3 v'$. Hence $0 \neq \alpha^3 = \alpha^2 v = \alpha\beta_1 w$ and $\alpha^3 = \beta_1 w$ contradicts the minimality of t.

(b) If $v_1 = \beta_1$, then $0 \neq \alpha^2 v = \alpha^2\beta_1 v' = \alpha\beta_1 w$; hence $0 \neq \alpha\beta_1 v' = \beta_1 w \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

Since $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = \langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle$, we deduce $\langle\beta_1, \alpha^2\beta_1\rangle = \langle\beta_1, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle$ by Lemma (4.3.12). Therefore the graph of P_x has the following shape:



Here $\langle\beta_1\rangle$ stands for the graph of the submodule $\langle\beta_1\rangle$ which is not known explicitly. Consider the module M defined by the following exact sequence:

$$0 \to \langle\beta_1, \alpha^2, \alpha\beta_1\rangle \to P_x \to M \to 0$$

Then $\mathrm{pdim}_\Lambda M < \infty$ since M is filtered by S_x and $\mathrm{pdim}_\Lambda(\langle \beta_1, \alpha^2 \rangle \oplus \langle \alpha\beta_1 \rangle) = \mathrm{pdim}_\Lambda \langle \beta_1, \alpha^2, \alpha\beta_1 \rangle < \infty$. Thus $\mathrm{pdim}_\Lambda(\langle \alpha\beta_1 \rangle \cong S_x)$ is finite too and the wanted α-filtration is $P_x \supset \langle \alpha \rangle \supset \langle \alpha^2 \rangle \supset \langle \alpha^3 \rangle \supset 0$.

Lemma (4.3.14)[137]: If $x^+ = \{\alpha, \beta_1\}, t \geq 3$ and $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\alpha^2\rho = 0$ for all rays $\rho \notin \{e_x, \alpha, \dots, \alpha^{t-2}\}$. Moreover, $\langle \alpha^2 \rangle \cap \langle \alpha\beta_1 \rangle = 0$.

Proof. Let $\rho \in \vec{A}$ with $\alpha^2\rho \neq 0$ be written as a composition of irreducible rays $\rho = \rho_1 \dots \rho_s$. Then the following two cases are possible:

(i) $\rho = \alpha^s$: Since $0 \neq \alpha^2\rho = \alpha^{2+s}$ and $\alpha^{t+1} = 0$ we have $s \leq t-2$ and $\rho = \alpha^s \in \{e_x, \alpha, \dots, \alpha^{t-2}\}$.

(ii) There exists a minimal $1 \leq i \leq s$ such that $\rho_i \neq \alpha$. Since $x^+ = \{\alpha, \beta_1\}$, we have $\rho_i = \beta_1$ and $0 \neq \alpha^2\rho = \alpha^{2+i-1}\beta_1\rho_{i+1} \dots \rho_s = 0$ by Lemma (4.3.11).

If $0 \neq \alpha^2 v = \alpha\beta_1 w$, then $v = \alpha^2$ with $0 \leq s \leq t-2$. Hence $0 = \alpha^2 v = \alpha^{s+2} = \alpha\beta_1 w$ and $\alpha^{s+1} = \beta_1 w$ by cancellation law. This contradicts the minimality of t.

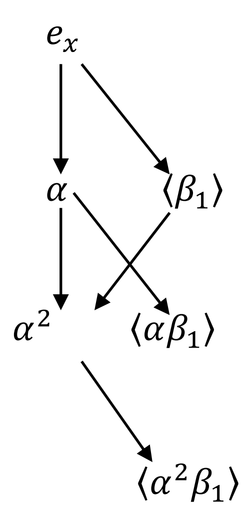
Corollary (4.3.15)[137]: If $x^+ = \{\alpha, \beta_1\}, t \geq 3$ and $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\langle \alpha^2, \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$.

Proof. The claim is trivial using Lemmas (4.3.10), (4.3.12) and (4.3.14).

Proposition (4.3.16)[137]: If $x^+ = \{\alpha, \beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If $\mathcal{L} \subseteq \{\alpha^3, \alpha^2\beta_1\}$, then the claim is the statement of Lemma (4.3.13). If $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then we consider the value of t:

(i) $t = 2$: Then the graph of P_x has the following shape:



Let a subquotient M of P_x be defined by the following exact sequence:

$$0 \to \langle \beta_1, \alpha\beta_1 \rangle \to P_x \to M \to 0$$

Then M and $\langle \beta_1, \alpha\beta_1 \rangle$ have finite projective dimension in mod-Λ. By Lemma (4.3.10) we have $\langle \beta_1, \alpha\beta_1 \rangle = \langle \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle$; hence $\mathrm{pdim}_\Lambda \langle \beta_1 \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha\beta_1 \rangle$ are both finite.

Let K be the kernel of the epimorphism $\lambda_\alpha: \langle \beta_1 \rangle \to \langle \alpha\beta_1 \rangle, \lambda_\alpha(\rho) = \alpha\rho$. Then $\mathrm{pdim}_\Lambda K < \infty$ and for the α-filtration $\mathcal{F}$ we take the following: $P_x \supset \langle \alpha\beta_1 \rangle \supset \langle \beta_1 \rangle \oplus \langle, \alpha\beta_1 \rangle \supset \langle \alpha\beta_1 \rangle \oplus K \supset K \supset 0$.

(ii) $t \geq 3$: Consider the following exact sequences:

$$0 \to, \langle, \alpha, \beta_1 \rangle \to P_x \to S_x \to 0$$

$$0 \to \langle \alpha^2, \beta_1, \alpha\beta_1 \rangle \to \langle \alpha, \beta_1 \rangle \to S_x \to 0$$

Hence $\mathrm{pdim}_\Lambda \langle \alpha, \beta_1 \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha^2, \beta_1, \alpha\beta_1 \rangle$ are finite. By Corollary (4.3.15) $\langle \alpha^2, \beta_1, \alpha\beta_1 \rangle = \langle \alpha^2, \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle$, that means $\mathrm{pdim}_\Lambda \langle \alpha\beta_1 \rangle$ is finite too. With Lemma (4.3.14) it is easily seen that for $2 \leq k \leq t$ the module $\langle \alpha^k \rangle$ is a uniserial module with S_x as the only composition factor. Hence $\mathrm{pdim}_\Lambda \langle \alpha^k \rangle$ is finite for $2 \leq k \leq t$. Thereby we have the wanted α-filtration

$$P_x \supset \langle \alpha, \beta_1 \rangle \supset \langle \alpha^2 \rangle \oplus \langle \alpha\beta_1 \rangle \supset \langle \alpha^3 \rangle \supset \langle \alpha^4 \rangle \supset \ldots \supset \langle \alpha^t \rangle \supset 0.$$

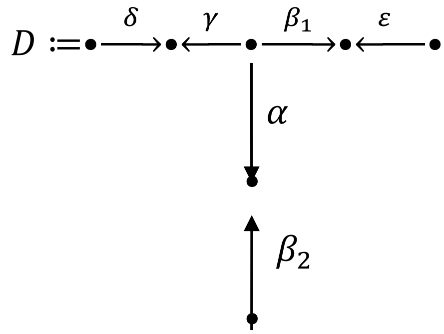
With previous notations $x^+ = \{\alpha, \beta_1, \gamma\}, (\alpha^2, \beta_1\beta_2 \ldots \beta_r)$ is a contour in $\overrightarrow{A}, t \geq 2, \alpha^{t+1} = 0, \tilde{\beta} := \beta_2 \ldots \beta_r$ and $\mu = \nu\alpha^t\nu'$ is a long morphism in $\overrightarrow{A}$.

The α-filtrations will be constructed depending on the set $\mathcal{L}$ of long morphisms in $\overrightarrow{A}$. The case $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$ is treated in Lemma (4.3.24), the case $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$ in (4.3.25) and the remaining case in (4.3.26).

But first, we derive some technical results.

Lemma (4.3.17)[137]: If $r = 2$ and $\delta: z' \to z$ is an arrow in Q_A ending in $z = e(\gamma)$, then $\delta = \gamma$.

Proof. Assume to the contrary that $\gamma \neq \delta: z' \to z$, then there is no arrow $\beta_1 \neq \varepsilon: y' \to y$ in Q_Λ. If there is such an arrow, then by the definition of a neighborhood ε belongs to Q_A. This arrow induces an irreducible ray $\beta_1 \neq \varepsilon: y' \to y$ in $\overrightarrow{A}$ and



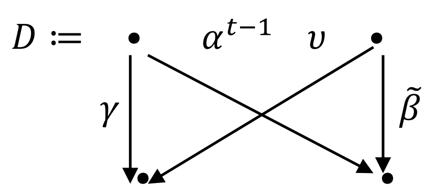
is a cleaving diagram in $\overrightarrow{A}/\mu$ of Euclidian type $\tilde{E}_6$.

In a similar way an arrow $\alpha, \beta_2 \neq \varepsilon: x' \to x$ in Q_Λ leads to a cleaving diagram of type $\widetilde{D}_5$ in $\overrightarrow{A}/\mu$. Hence the full subcategory B of Λ supported by the points x, y is a convex subcategory of Λ. Therefore the projective dimensions of S_x is finite in mod-B since it is finite in mod-Λ. But in B we have $x^+ = \{\alpha, \beta_1\}$, whence we can apply Proposition (4.3.16) together with (4.3.3) to get the contradiction that α is not a loop.

Lemma (4.3.18)[137]: If $\alpha\gamma \neq 0$, then $\beta_1 v \neq \alpha\gamma \neq \gamma w$ for all rays $v, w \in \overrightarrow{A}$.

Proof.

(i) Assume that there exists a ray $v \in \overrightarrow{A}$ such that $\beta_1 v = \alpha\gamma \neq 0$. Then



is a cleaving diagram of Euclidian type e $\tilde{A}_3$ in $\overrightarrow{A}/\mu$.

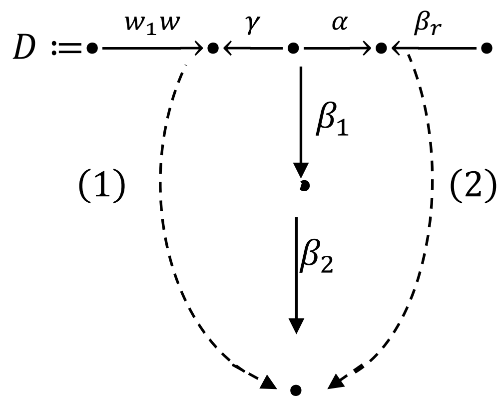
(a) For $\gamma\rho = \alpha^{t-1}$ or $v\rho = \tilde{\beta}$ we have $\alpha\gamma\rho = \beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t \neq 0$. Thus $\alpha^{t-1} = \alpha\rho$ contradicts the choice of t.

(b) If $\alpha^{t-1}\rho = \gamma$ or $\tilde{\beta}\rho = v$, then $\alpha^t\rho = \beta_1\tilde{\beta}\rho = \beta_1 v = \alpha\gamma \neq 0$. Then $\alpha^t\rho = \gamma$ contradicts the irreducibility of γ.

(ii) Assume that there exists a ray $w = w_1 \dots w_s : z \rightsquigarrow z \in \overrightarrow{A}$ with irreducible w_i such that $\gamma w = \alpha\gamma \neq 0$.

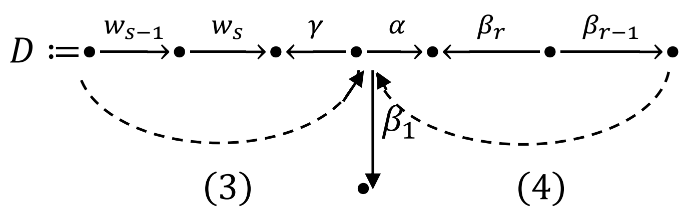
$r = 2$: Since w_s is an irreducible ray ending in $z, w_s = \gamma$ by Lemma (4.3.17). Thus we get a contradiction $\gamma w_s \dots w_{s-1} = \alpha$.

$r \geq 3$: We look at the value of s. If $s = 1$, then $w = w_1$ is a loop and



is a cleaving diagram in $\overrightarrow{A}/\mu$.

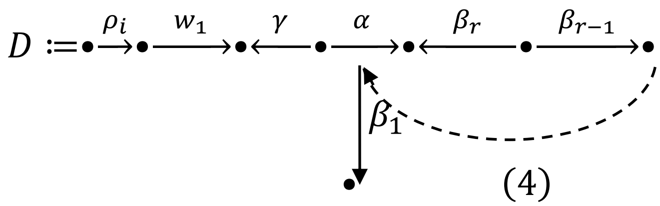
If $s \geq 2$, then



is cleaving in $\overrightarrow{A}/\mu$.

We still have to show that not any morphisms indicated by the dotted lines make the diagrams commute.

(a) : $\gamma\rho = \beta_1\beta_2$, with $\rho = \rho_1 \dots \rho_l$. If $\rho = w_1^l = w^l$, then $\beta_1\beta_2 = \gamma\rho = \gamma w^l = \alpha\gamma w^{l-1}$ and $\beta_1\beta_2 \dots \beta_r = \alpha^t = \alpha\gamma w^{l-1}\beta_3 \dots \beta_r \neq 0$. Therefore $\alpha^{t-1} = \gamma w^{l-1}\beta_3 \dots \beta_r$ is a contradiction. If $\rho \neq w_1^l$, then one of the irreducible rays $\rho_i \neq w_1$ starts in z and



is cleaving in $\overrightarrow{A}/\mu$.

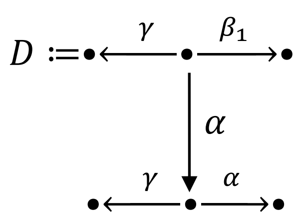
(b) : If $\alpha\rho = \beta_1\beta_2$, then $\alpha\rho\beta_3 \dots \beta_r = \beta_1\beta_2 \dots \beta_r = \alpha^t \neq 0$ and $\alpha^{t-1} = \rho\beta_3 \dots \beta_r$ contradicts the minimality of t.

(c) : If $\rho\gamma = w_{s-1}w_s$, then $\gamma w_1 \dots w_{s-2}\rho\gamma = \gamma w = \alpha\gamma \neq 0$ and $\alpha = \gamma w_1 \dots w_{s-2}\rho$ contradicts the irreducibility of α.

(d) : If $\rho\alpha = \beta_{r-1}\beta_r$, then $\beta_1\beta_2 \dots \beta_{r-2}\rho\alpha = \beta_1\beta_2 \dots \beta_r = \alpha^t \neq 0$ and $\alpha^{t-1} = \beta_1\beta_2 \dots \beta_{r-2}\rho$ contradicts the minimality of t.

Lemma (4.3.19)[137]: If $t \geq 3$, then $\alpha\gamma = 0$.

Proof. Assume that $\alpha\gamma \neq 0$, then



is a cleaving diagram of Euclidian type in $\overrightarrow{A}/\mu$. It is cleaving since:

(i) $\gamma\rho = \alpha\gamma$ or $\beta_1\rho = \alpha\gamma$ contradicts Lemma (4.3.18),

(ii) $\gamma\rho = \alpha^2$ or $\beta_1\rho = \alpha^2$ contradicts the minimality of $t \geq 3$.

Lemma (4.3.20)[137]:

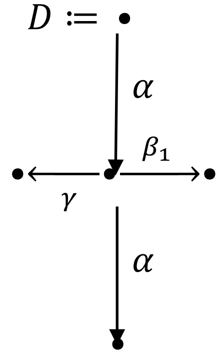
(a) If $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then $\alpha\beta_1 = 0$ or $\alpha\gamma = 0$.

(b) If $\alpha^2\beta_1 \neq 0$, then $\gamma w \neq \alpha\beta_1$ for all $w \in \overrightarrow{A}$.

Proof.

(a) If $\alpha\beta_1 \neq 0$ and $\alpha\gamma \neq 0$, then

$D :=$



is a cleaving diagramof Euclidian type $\widetilde{D}_4$ in $\overrightarrow{A}$. It is still cleaving in $\overrightarrow{A}/\eta$ for $\eta \in \mathcal{L}\backslash\{\alpha^2, \alpha\beta_1, \alpha\gamma\} \neq \emptyset$.

(b) Since $\alpha^2\beta_1 \neq 0$, we have $\alpha\gamma = 0$ by a). But $\gamma w = \alpha\beta_1$ leads to the contradiction $0 \neq \alpha^2\beta_1 = \alpha\gamma w = 0$.

Lemma (4.3.21)[137]: If $t = 2$ or $\mathcal{L} \nsubseteq \{\alpha^t, \alpha^2\beta_1\}$, then:

(a) $\alpha^2\beta_1 = 0 = \alpha^2\gamma, \alpha^2\rho = 0$ for all rays $\rho \notin \{e_x, \alpha, \dots, \alpha^{t-2}\}$.

(b) $\langle\beta_1\rangle \cap \langle\alpha\gamma\rangle = 0$.

(c) If $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$, then $\langle\gamma\rangle \cap \langle\alpha^2\rangle = 0$.

(d) $\langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ or $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$.

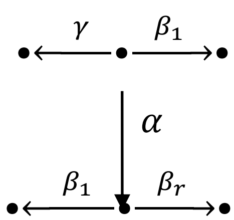
(e) $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$ or $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$.

(f) $\langle\alpha\beta_1\rangle \cap \langle\alpha^2\rangle = 0$ and $\langle\alpha\gamma\rangle \cap \langle\alpha^2\rangle = 0$.

Proof.

(a) Consider the case $t = 2$.

(i) If $\alpha^2\beta_1 \neq 0$, then $\beta_r\beta_1 \neq 0$ and

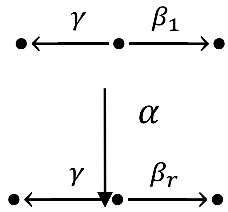


is a cleaving diagram of Euclidian type $\widetilde{D}_5$ in $\overrightarrow{A}/\mu$. The diagram is cleaving because:

- $\beta_1\rho = \alpha\beta_1 \neq 0$ is a contradiction of Lemma (4.3.10),

- $\gamma\rho = \alpha\beta_1 \neq 0$ contradicts Lemma (4.3.20) b).

(ii) If $\alpha^2\gamma \neq 0$, then $\beta_r\gamma \neq 0$ and

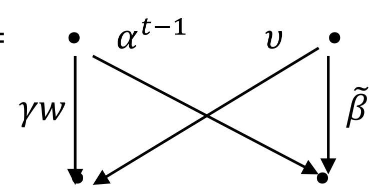


is a cleaving diagram in $\overrightarrow{A}/\mu$. It is cleaving since $\beta_1\rho = \alpha\gamma$ resp. $\gamma\rho = \alpha\gamma$ contradicts Lemma (4.3.18).

In the case $t \geq 3, \alpha^2\gamma = 0$ by Lemma (4.3.19). If $t = 3$, then $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$ by assumption. If $t > 3$, then $\mu = \alpha^t v' \in \mathcal{L}\backslash\{\alpha^3, \alpha^2\beta_1\}$. Hence $\alpha^2\beta_1 = 0$ by Lemma (4.3.11) in both cases.

(b) If v, w are rays in $\overrightarrow{A}$ such that $\beta_1 v = \alpha\gamma w \neq 0$, then the diagram

$D :=$



is a cleaving diagram in $\overrightarrow{A}/\mu$.

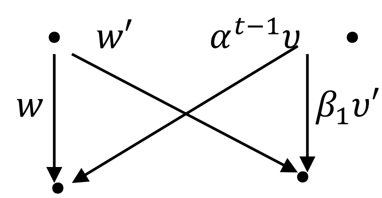
(i) If $\gamma w\rho = \alpha^{t-1}$ or $v\rho = \tilde{\beta}$, then $\beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t = \alpha\gamma w\rho \neq 0$. Hence $\gamma w\rho = \alpha^t$ contradicts the minimality of t.

(ii) If $\alpha^{t-1}\rho = \gamma w$ or $\tilde{\beta}\rho = v$, then $0 \neq \beta_1 v = \beta_1\tilde{\beta}\rho = \alpha\gamma w = \alpha^t\rho = 0$ by a).

(c) Let v, w be rays such that $\gamma v = \alpha^2 w \neq 0$. By a) we have $w = \alpha^k$ with $0 \leq k \leq t-2$, that means $\gamma v = \alpha^{2+k}$. Since t is minimal, we have t=2+k and $0 \neq \gamma v = \alpha^t = \beta_1\tilde{\beta} \in \langle\gamma\rangle \cap \langle\beta_1\rangle = 0$.

(d) Let v, w, v', w' be rays in $\overrightarrow{A}$ such that $\gamma w = \alpha^t v \neq 0$ and $\gamma w' = \alpha\beta_1 v' \neq 0$. Then

$D :=$

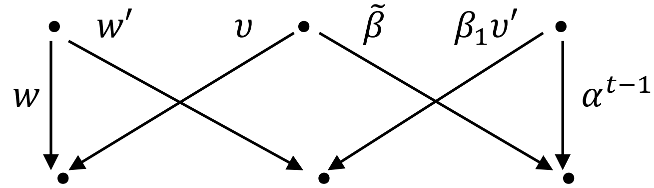


is a cleaving diagram in $\overrightarrow{A}/\mu$.

(i) If $w\rho = w'$ or $\alpha^{t-1}\upsilon\rho = \beta_1\upsilon'$, then $\gamma w\rho = \gamma w' = \alpha^t\upsilon\rho = \alpha\beta_1\upsilon' \neq 0$. Hence there is a nondeep contour $(\alpha^{t-1}\upsilon_1\ldots\upsilon_1\ldots\upsilon_k\rho_1 \ldots \rho_l, \beta_1\upsilon_1'\ldots\upsilon_s')$ in $\overrightarrow{A}$ which can only be a penny-farthing by the structure theorem for non-deep contours. But this case is excluded in the current section.

(ii) If $w' = \rho = w$ or $\beta_1\upsilon'\rho = \alpha^{t-1}\upsilon$, then $\gamma w'\rho = \gamma w = \alpha\beta_1\upsilon'\rho = \alpha^t\upsilon \neq 0$. Again, we have a non-deep contour $(\alpha^{t-1}\upsilon_1 \ldots \upsilon_k, \beta_1\upsilon_1' \ldots \upsilon_l'\rho_1 \ldots \rho_s)$ which leads to a contradiction as before.

(e) Let $\upsilon, w, \upsilon', w'$ be rays such that $\beta_1\upsilon = \gamma w \neq 0$ and $\alpha\beta_1\upsilon' = \gamma w' \neq 0$. Then



is a cleaving diagram in $\overrightarrow{A}/\mu$.

(i) If $w\rho = w'$, we get the contradiction $0 \neq \gamma w\rho = \gamma w' = \beta_1\upsilon = \alpha\beta_1\upsilon' \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

(ii) If $w'\rho = w$, then $0 \neq \gamma w'\rho = \gamma w = \alpha\beta_1\upsilon'\rho = \beta_1\upsilon \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

(iii) If $\upsilon\rho = \tilde{\beta}$, then $0 \neq \beta_1\upsilon\rho = \beta_1\tilde{\beta} = \gamma w\rho = \alpha^t \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

(iv) If $\tilde{\beta}\rho = \upsilon$, then $\beta_1\tilde{\beta}\rho = \beta_1\upsilon = \alpha^t\rho = \gamma w \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

(v) If $\alpha^{t-1}\rho = \beta_1\upsilon'$, then $0 \neq \alpha^t\rho = \alpha\beta_1\upsilon' = \gamma w' \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

The case $\beta_1\upsilon'\rho = \alpha^{t-1}$ contradicts the minimality of t.

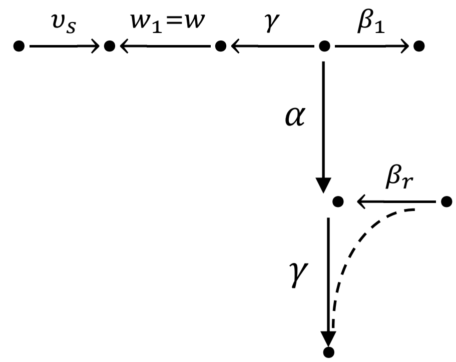
(f) If υ, w are rays in $\overrightarrow{A}$ such that $\alpha\beta_1\upsilon = \alpha^2 w \neq 0$ resp. $\alpha\gamma\upsilon = \alpha^2 w \neq 0$, then $w = \alpha^k$ with $0 \leq k \leq t-2$ and $\beta_1\upsilon = \alpha^{1+k}$ resp. $\gamma\upsilon = \alpha^{1+k}$. Since t is minimal, we get the contradiction $t = 1 + k < t$.

Lemma (4.3.22)[137]: If $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then $\langle\gamma\rangle \cap \langle\alpha\gamma\rangle = 0$.

Proof. In the case $t \geq 3$, the claim is trivial since $\alpha\gamma = 0$ by (4.3.19).

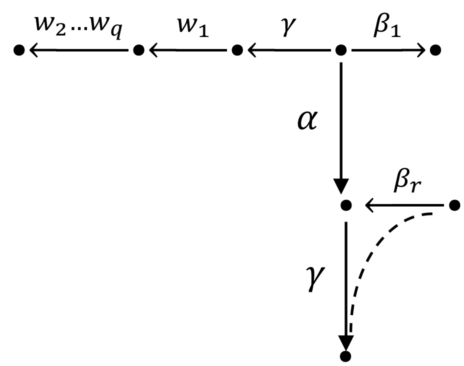
Consider the case $t = 2$. Assume that there exist rays υ, w in $\overrightarrow{A}$ such that $\gamma\upsilon w \neq 0$. First of all, we deduce that $w \neq id$ by Lemma (4.3.18) and $\upsilon \neq id$ since is an arrow. Therefore we can write $\upsilon = \upsilon_1 \ldots \upsilon_s, w = w_1 \ldots, w_q$ with irreducible rays $\upsilon_i, w_j \in \overrightarrow{A}$. Consider the value of q:

(a) If $q = 1$, then the diagram



is a cleaving diagram of Euclidian type $\tilde{E}_7$ in $\overrightarrow{A}/\mu$ (see [146]).

(b) If $q \geq 2$, then the diagram



is cleaving in $\overrightarrow{A}/\mu$.

The diagrams are cleaving because:

(i) $\alpha\rho = \gamma w \neq 0$: Then $0 \neq \alpha\gamma w = \alpha^2\rho = 0$ by Lemma (4.3.21) a).

(ii) $\gamma\rho = \alpha\gamma \neq 0$ contradicts Lemma (4.3.18).

(iii) $\beta_1\rho = \gamma w \neq 0$: Then $0 \neq \alpha\gamma w = \alpha\beta_1\rho = 0$ since $\alpha\beta_1 = 0$ by Lemma (4.3.20).

(iv) $\rho\upsilon_s = \gamma w \neq 0$: Then $\alpha\rho\upsilon_s = \alpha\gamma w \neq 0$. If $\rho = \beta_1\rho'$, then $0 = \beta_1\rho'^{\upsilon_s} = \alpha\gamma w \neq 0$. If $\rho = \gamma\rho'$, then $\alpha\gamma\rho'^{\upsilon_s} = \alpha\beta w$ and $w_1 = w = \rho'\upsilon_s$. Hence $\rho' = id$ and $\upsilon_s = w_1$.

Therefore $0 \neq \gamma v = \gamma v_1 \dots v_{s-1} w_1 = \alpha\gamma w_1$ and $\gamma v_1 \dots v_{s-1} = \alpha\gamma$ contradicting Lemma (4.3.18). If $\rho = \alpha\rho'$, then $0 \neq \alpha\gamma w = \alpha^2 \rho'^{v_s} = 0$ by Lemma (4.3.21) a).

(v) $\beta_1 \rho = \alpha\gamma \neq 0$ contradicts Lemma (4.3.18).

Lemma (4.3.23)[137]: Let $\mathcal{L} \nsubseteq \{\alpha^t, \alpha^2\beta_1\}$ and $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$.

(a) If $\langle\alpha\gamma\rangle = 0 = \langle\gamma\rangle \cap \langle\alpha\beta_1\rangle$, then $\langle\beta_1, \gamma, \alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.

(b) If $\langle\alpha\gamma\rangle = 0 = \langle\gamma\rangle \cap \langle\beta_1\rangle$, then $\langle\beta_1, \alpha^2\rangle \cap \langle\gamma, \alpha\beta_1\rangle = 0$.

(c) If $\langle\alpha\beta_1\rangle = 0$, then $\langle\beta_1, \gamma, \alpha^2\rangle \cap \langle\alpha\gamma\rangle = 0$.

Proof. We only prove b); the other cases are proven analogously. Let $v, v', w, w' \in A$ be such that $\beta_1 v + \alpha^2 v' = \gamma w + \alpha\beta_1 w' \neq 0$. That means we have rays $v_i, w_j \in \overrightarrow{A}$, numbers $\lambda_i, \mu_j \in k$ and integers $s_1, s_2 \geq 0, n_1, n_2 \geq 1$ such that

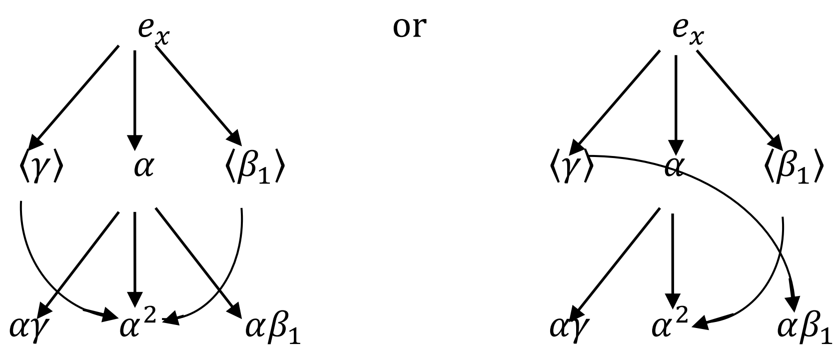
$$\sum_{i=1}^{s_1} \lambda_i \beta_1 v_i + \sum_{i=s_1+1}^{n_1} \lambda_i \alpha^2 v_i = \sum_{j=1}^{s_2} \mu_j \gamma w_j + \sum_{j=s_2+1}^{n_2} \mu_j \alpha\beta_1 w_j$$

and $\beta_1 v_i \neq \beta_1 v_j, \alpha^2 v_i \neq \alpha^2 v_j, \gamma w_i \neq \gamma w_j, \alpha\beta_1 w_i \neq \alpha\beta_1 w_j$ for $i \neq j$. Without loss of generality we can assume that all λ_i, μ_j are non-zero, that $\beta_1 v_i \neq \alpha^2 v_j$ for $i = 1 \dots s_1, j = s_{1+1} \dots n_1$ and $\gamma w_i \neq \alpha\beta_1 w_j$ for $i = 1 \dots s_2, j = s_2 + 1 \dots n_2$. Then by Lemma (4.3.9) we have $n_1 = n_2$ and there exists a permutation π such that $\beta_1 v_i = \gamma w_{\pi(i)} \in \langle\beta_1\rangle \cap \langle\gamma\rangle = 0$ or $\beta_1 v_i = \alpha\beta_1 w_{\pi(i)} \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$ by Lemma (4.3.10). Hence $s_1 = 0$. Moreover, by Lemma (4.3.21) we have $\alpha^2 v_i = \gamma w_{\pi(i)} \in \langle\alpha^2\rangle \cap \langle\gamma\rangle = 0$ or $\alpha^2 v_i = \alpha\beta_1 w_{\pi(i)} \in \langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$; this is possible for $n_1 - s_1 = 0$ only. Hence $n_1 = 0$, contradicting the choice of n_1.

Lemma (4.3.24)[137]: If $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. Since $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}, \mu = \alpha^2$ is long and $t = 2$. Now it is easily seen that $\langle\alpha^2\rangle = \mathbf{k}\alpha^2 \cong S_x, \langle\alpha\gamma\rangle = \mathbf{k}\alpha\gamma, \langle\alpha\beta_1\rangle = \mathbf{k}\alpha\beta_1$ and $\langle\alpha\rangle$ has a $\mathbf{k}$ basis $\{\alpha, \alpha^2, \alpha\beta_1, \alpha\gamma\}$. Using Lemma (4.3.10) and (4.3.18) we conclude $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$ and $\langle\gamma\rangle \cap \langle\alpha\gamma\rangle = 0 = \langle\beta_1\rangle \cap \langle\alpha\gamma\rangle$.

By Lemma (4.3.21) d) $\langle\gamma\rangle \cap \langle\alpha^2\rangle = 0$ or $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$. Thus the graph of P_x has one of the following shapes:



In the first case we consider the following exact sequence:

$$0 \to \langle\alpha^2\rangle \to \langle\alpha,\beta_1,\gamma\rangle \to \langle\alpha,\beta_1,\gamma\rangle/\langle\alpha^2\rangle \to 0$$

Since $\langle\alpha\rangle$ has $\mathbf{k}$ basis $\{\alpha,\alpha^2,\alpha\beta_1,\alpha\gamma\}$ and $\mathcal{L} \subseteq \{\alpha^2,\alpha\beta_1,\alpha\gamma\}$ we have $\langle\alpha,\beta_1,\gamma\rangle/\langle\alpha^2\rangle = \langle\alpha\rangle/\langle\alpha^2\rangle \oplus \langle\beta_1,\gamma\rangle/\langle\alpha^2\rangle$. Hence $\mathrm{pdim}_\Lambda\langle\alpha\rangle < \infty$ and $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset 0$ is the wanted filtration.

In the second case we have $\langle\alpha,\beta_1,\gamma\rangle/\langle\alpha^2\rangle = \langle\alpha,\gamma\rangle/\langle\alpha^2\rangle \oplus \langle\beta_1\rangle/\langle\alpha^2\rangle$. Thus $\mathrm{pdim}_\Lambda\langle\alpha,\gamma\rangle < \infty$. Now we consider

$$0 \to \langle\beta_1,\gamma,\alpha\gamma\rangle \to \langle\alpha,\beta_1,\gamma\rangle \to S_x \to 0.$$

Since $\langle\beta_1,\gamma,\alpha\gamma\rangle = \langle\beta_1,\gamma\rangle \oplus \langle\alpha\gamma\rangle$, we have $\mathrm{pdim}_\Lambda\langle\alpha\gamma\rangle < \infty$ and $P_x \supset \langle\alpha,\gamma\rangle \supset \langle\alpha^2,\alpha\gamma\rangle \supset 0$ is a suitable filtration.

Lemma (4.3.25)[137]: If $\mathcal{L} \subseteq \{\alpha^t,\alpha^2\beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

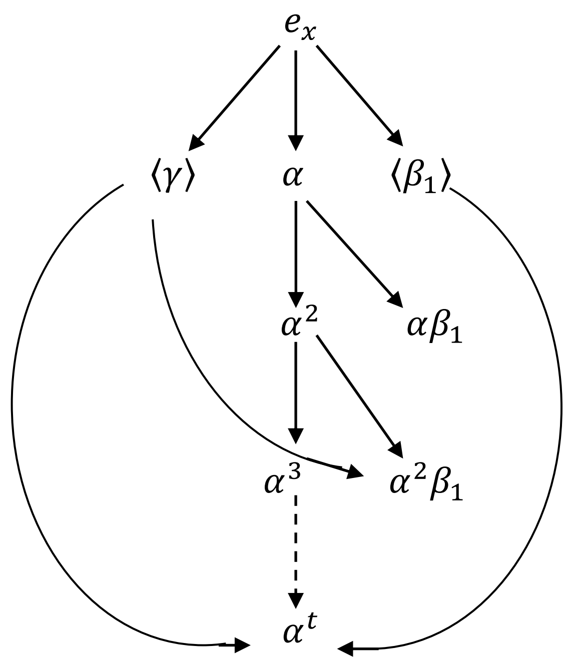
Proof. If $t = 2$, then $\alpha^2\beta_1 = 0$ by Lemma (4.3.21) a). Hence $\mathcal{L} \subseteq \{\alpha^2\}$ and the filtration exists by Lemma (4.3.24).

If $t \geq 3$, then $\alpha\gamma = 0$ by Lemma (4.3.19). From the assumption $\mathcal{L} \subseteq \{\alpha^t,\alpha^2\beta_1\}$ it is easily seen that $\langle\alpha\beta_1\rangle = \mathbf{k}\alpha\beta_1$ and $\langle\alpha^2\beta_1\rangle = \mathbf{k}\alpha^2\beta_1$.

(i) If $\alpha^2\beta_1 = 0$, then α^t is the only long morphism in $\overrightarrow{A}$; hence $\alpha\beta_1 = 0$ and $\langle\alpha^k\rangle, k \geq 1$, is uniserial of finite projective dimension. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \ldots \supset \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

(ii) If $\alpha^2\beta_1 \neq 0$, then $\langle\alpha\beta_1\rangle = \mathbf{k}\alpha\beta_1 \cong S_y \cong \langle\alpha^2\beta_1\rangle$. By (4.3.10) and (4.3.20) b)

$\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\gamma\rangle \cap \langle\alpha\beta_1\rangle$. Therefore the graph of P_x has the following shape:



Moreover, $\langle\alpha\beta_1\rangle \cong S_y$ is a direct summand of the module $\langle\alpha^2, \beta_1, \gamma, \alpha\beta_1\rangle$, which has finite projective dimension. Since the modules $\langle\alpha\rangle, \langle\alpha^2\rangle, \dots, \langle\alpha^t\rangle$ have S_x and S_y as the only composition factors, they are of finite projective dimension. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \dots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

Proposition (4.3.26)[137]: If $x^+ = \{\alpha, \beta_1, \gamma\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. By lemmata (4.3.24) and (4.3.25) we can assume that $\mathcal{L} \not\subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$. Then $\mathrm{pdim}_\Lambda\langle\alpha^k\rangle < \infty$ for $2 \le k \le t$ since $\langle\alpha^k\rangle$ has only S_x as a composition factor by (4.3.21)(a). Moreover, $\mathrm{pdim}_\Lambda\langle\alpha, \beta_1, \gamma\rangle < \infty$ since it is the left hand term of the following exact sequence:

$$0 \to \langle\alpha, \beta_1, \gamma\rangle \to P_x \to S_x \to 0.$$

By Lemma (4.3.20)(a) only the following two cases are possible:

(i) $\alpha\beta_1 = 0$: Consider the following exact sequence:

$$0 \to \langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

Then $\mathrm{pdim}_\Lambda\langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle < \infty$. By (4.3.23)(c) we have $\langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle = \langle\beta_1, \gamma, \alpha^2\rangle \oplus \langle\alpha\gamma\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha\gamma\rangle < \infty$. Therefore $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\alpha\gamma\rangle \supset \langle\alpha^3\rangle \supset \dots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

(ii) $\alpha\gamma = 0$: Then $\mathrm{pdim}_\Lambda\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle < \infty$ since we have the exact sequence

$$0 \to \langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

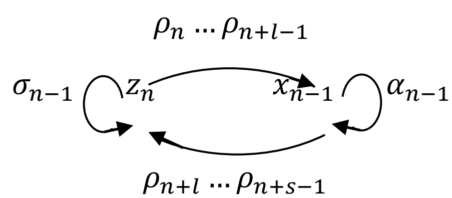
If $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$, then by (4.3.23)(a) we have $\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \gamma, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha\beta_1\rangle < \infty$. Therefore $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\alpha\beta_1\rangle \supset \langle\alpha^3\rangle \supset \ldots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

By Lemma (4.3.21)(e) it remains to consider the case $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$: Then $\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \alpha^2\rangle \oplus \langle\gamma, \alpha\beta_1\rangle$ by (4.3.23)(b). Thus $\mathrm{pdim}_\Lambda\langle\gamma, \alpha\beta_1\rangle < \infty$. Now $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\gamma, \alpha\beta_1\rangle \supset \langle\alpha^3\rangle \supset \ldots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

Corollary (4.3.27)[250]: If there are penny-farthings $P_{n-1} = ((\sigma_{n-1})^2, \rho_n \ldots \rho_{n+s-1})$ in $\vec{A}_{n-1}$, then $z_n = x_{n-1}$.

Proof. We consider two cases:

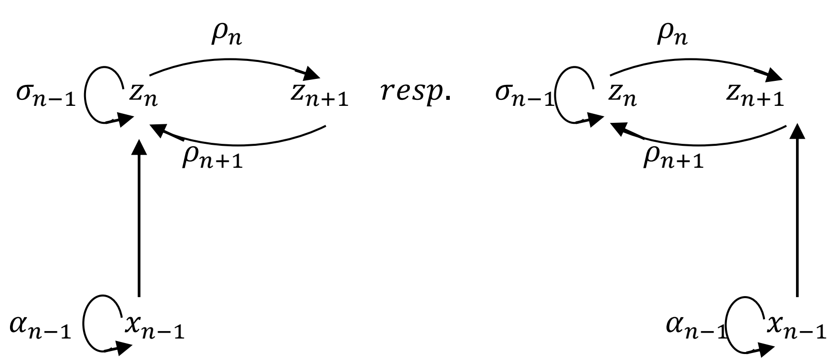
(a) $x_{n-1} \in (Q_{n-1})_{p_{n-1}}$: Hence $(Q_{n-1})_{p_{n-1}}$ has the following shape:



But these can be the quivers of penny-farthings only for $z_n = x_{n-1}$.

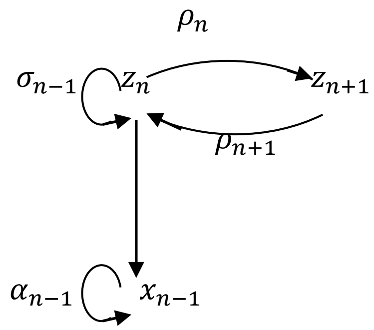
(b) $x_{n-1} \notin (Q_{n-1})_{p_{n-1}}$: Since A_{n-1} are the neighborhoods of x_{n-1}, only the following cases are possible:

(i) $e_{x_{n-1}} A_{n-1} e_{z_{n-1}} \neq 0$: Since $x_{n-1} \notin (Q_{n-1})_{p_{n-1}}$ we can apply the dual of [147] or [146] to $\overrightarrow{A}_{n-1}$ and we see that the following quivers occurs as subquivers of $(Q_{n-1})_{A_{n-1}}$:



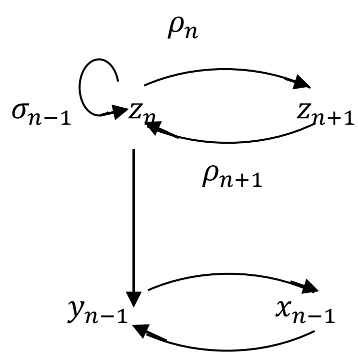
Moreover, there can be only one arrow starting in x_{n-1}. This is a contradiction to the actual setting.

(ii) $\exists z_n \rightarrow x_{n-1}$: By applying [147] or the dual of [146] we deduce that the following quivers occurs as subquivers of $(\mathcal{Q}_{n-1})_{A_{n-1}}$:



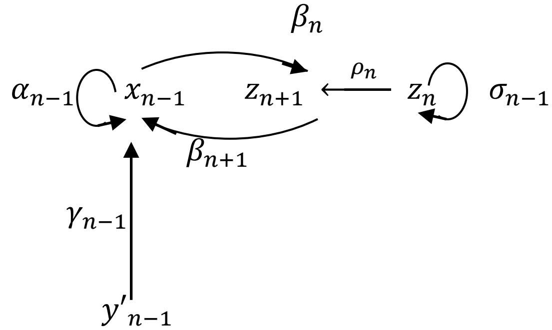
and there can be only one arrow ending in x_{n-1} contradicting the present case.

(iii) $\exists y'_{n-1} \longleftarrow x_{n-1} \rightleftarrows y_{n-1} \longleftarrow z_n$: If $y_{n-1} \notin (\mathcal{Q}_{n-1})_{p_{n-1}}$, then

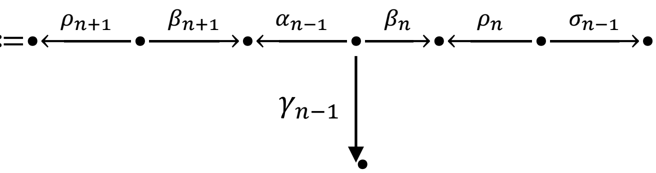


are subquivers of $(\mathcal{Q}_{n-1})_{A_{n-1}}$ leading to the same contradiction as in b).

If $y_{n-1} \in (\mathcal{Q}_{n-1})_{p_{n-1}}$, then $y_{n-1} = z_{n+1}$ and the quivers



are subquivers of $(Q_{n-1})_{A_{n-1}}$. Since $x_{n-1} \notin (Q_{n-1})_{p_{n-1}}$, all morphisms occurring in the following diagram

$D_{n-1} :=$ 

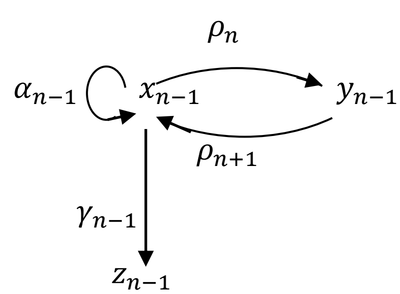
are irreducible and pairwise distinct. Therefore D_{n-1} is a cleaving diagram in $\overrightarrow{A}_{n-1}$. Moreover, some long morphism $\eta_{n-1} = \nu_{n-1}(\sigma_{n-1})^3\nu'_{n-1}$ does not occur in D_{n-1}; hence D_{n-1} is still cleaving in $\overrightarrow{A}_{n-1}/\eta_{n-1}$ by [145]. Since D_{n-1} is of representation-infinite Euclidean type $\left(\tilde{E}_{n-1}\right)_7$, $\overrightarrow{A}_{n-1}/\eta_{n-1}$ is representation-infinite contradicting the mildness of A_{n-1}.

We show that, provided the existence of penny-farthings in $\overrightarrow{A}_{n-1}$, there exist α_{n-1}-filtrations of $P_{x_{n-1}}$ having finite projective dimension.

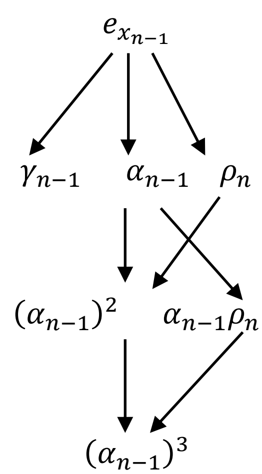
Corollary (4.3.28)[250]: Let $A_{n-1} = \Lambda(x_{n-1})$ be mild and standard. If there are penny-farthings in $\overrightarrow{A}_{n-1}$, then there exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimension.

Proof. If there are penny-farthings P_{n-1} in $\overrightarrow{A}_{n-1}$, then $P_{n-1} = ((\alpha_{n-1})^2, \rho_n \dots \rho_{n+s-1})$ are the only penny-farthings in $\overrightarrow{A}_{n-1}$ by the last lemma. Since A_{n-1} is standard and mild, there are three cases for the graph of $P_{x_{n-1}}$ which can occur by [147] or the dual of [146].

(i) There exist arrows $\gamma_{n-1}: x_{n-1} \to z_{n-1}, \gamma_{n-1} \neq \rho_n$. Then $s = 2$, the quivers



are subquivers of $(Q_{n-1})_{A_{n-1}}$, and $P_{x_{n-1}}$ is represented by the following graph:



Let M be a quotient of $P_{x_{n-1}}$ defined by the following exact sequence:

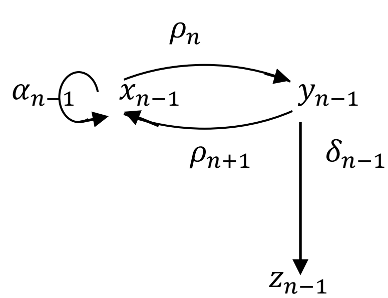
$$0 \to \langle \gamma_{n-1} \rangle \oplus \langle \rho_n, \alpha_{n-1}\rho_n \rangle \to P_{x_{n-1}} \to M \to 0.$$

Then M has $S_{x_{n-1}}$ as the only composition factor. Hence $\operatorname{pdim}_\Lambda M < \infty$ and $\operatorname{pdim}_\Lambda \langle \rho_n, \alpha_{n-1}\rho_n \rangle < \infty$. Now, we consider the exact sequence

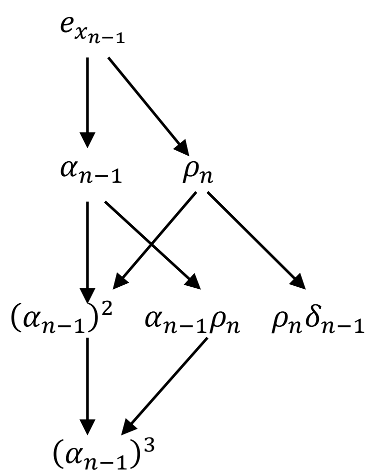
$$0 \to \langle (\alpha_{n-1})^3 \rangle \to \langle \rho_n, \alpha_{n-1}\rho_n \rangle \to \langle \rho_n \rangle / \langle (\alpha_{n-1})^3 \rangle \oplus \langle \alpha_{n-1}\rho_n \rangle / \langle (\alpha_{n-1})^3 \rangle \to 0.$$

But $\langle (\alpha_{n-1})^3 \rangle \cong S_{x_{n-1}}$ and $\operatorname{pdim}_\Lambda S_{x_{n-1}} < \infty$, hence $\langle \alpha_{n-1}\rho_n \rangle / \langle (\alpha_{n-1})^3 \rangle \cong S_{y_{n-1}}$ has finite projective dimension in mod-Λ. Finally, the Λ-filtrations $P_{x_{n-1}} \supset \langle \alpha_{n-1} \rangle \supset \langle (\alpha_{n-1})^2 \rangle \supset \langle (\alpha_{n-1})^3 \rangle \supset 0$ have finite projective dimension since all filtration modules $\neq P_{x_{n-1}}$ have $S_{x_{n-1}}$ and $S_{y_{n-1}}$ as the only composition factors.

(ii) In the second case there exist the points $z_{n-1} \notin (Q_{n-1})_{p_{n-1}}$ such that $A_{n-1}(x_{n-1}, z_{n-1}) \neq 0$. Then $s = 2$, the quivers

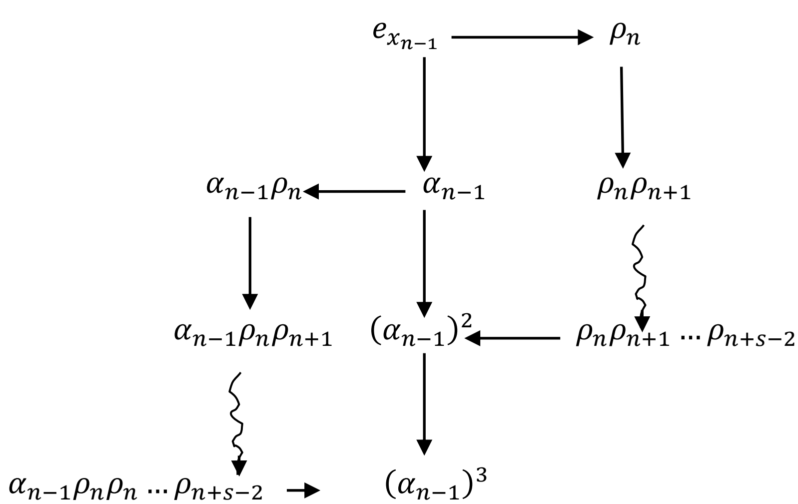


are subquivers of $(Q_{n-1})_{y_{n-1}}$, and $P_{x_{n-1}}$ is represented by:



With similar considerations as in I) we obtain that the same filtrations fits.

(iii) In the last possible case we have $A_{n-1}(x_{n-1}, z_{n-1}) = 0$ for all points $z_{n-1} \notin (Q_{n-1})_{P_{n-1}}$. Hence $P_{x_{n-1}}$ is represented by:



As a Λ-Module, $M := P_{x_{n-1}}/\langle(\alpha_{n-1})^2\rangle$ has finite projective dimension since $\langle(\alpha_{n-1})^2\rangle$ has $S_{x_{n-1}}$ as the only composition factor. Let K be the kernel of the epimorphism $M \to \langle(\alpha_{n-1})^2\rangle, e_{x_{n-1}} \mapsto (\alpha_{n-1})^2$, then $K = \langle\rho_n\rangle/\langle(\alpha_{n-1})^2\rangle \oplus \langle\alpha_{n-1}\rho_n\rangle/\langle(\alpha_{n-1})^3\rangle$ has finite projective dimension. Moreover, $\mathrm{pdim}_\Lambda\langle\rho_n\rangle, \mathrm{pdim}_\Lambda\langle\alpha_{n-1}\rho_n\rangle < \infty$. Since

$$0 \to \langle \alpha_{n-1}\rho_n \rangle \to \langle \alpha_{n-1} \rangle \xrightarrow{\lambda_{\alpha_{n-1}}} \langle (\alpha_{n-1})^2 \rangle \to 0$$

is exact, $\mathrm{pdim}_\Lambda \langle \alpha_{n-1} \rangle < \infty$. Thus the same filtrations as in the first two cases fits again.

Corollary (4.3.29)[250]: With above notations let $A_{n-1} = \Lambda(x_{n-1})$ be mild and non-standard. There exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimension.

Proof. If A_{n-1} is non-standard, then A_{n-1} is representation finite by [145], $\mathrm{char}\,\mathbf{k} = 2$ and there are penny-farthings in $\vec{A}_{n-1}$ by [146]. Since Corollary (4.3.27) remains valid, the penny-farthings $((\alpha_{n-1})^2, \rho_n \dots \rho_{n+s-1})$, $\rho_{n+i-1}: z_{n+i-1} \to z_{n+i}, z_n = z_{n+s} = x_{n-1}$, are unique. By [146] the difference between A_{n-1} and $\overline{A}_{n-1}$ in the composition of the arrows shows up in the graphs of the projectives to $z_{n+1}, \dots, z_{n+s-1}$ only. Thus the graph of $P_{x_{n-1}}$ remains the same in all three cases of the proof of Corollary (4.3.28) and the filtrations constructed there still do the job.

Corollary (4.3.30)[250]: Let $A_{n-1} = \bar{A}_{n-1}$be a standard $\mathbf{k}$-algebra. Consider rays $(v_{n-1})_i, (w_{n-1})_j \in \overrightarrow{A}_{n-1} \backslash \{0\}$ for $i = 1 \dots n$ and $j = 1 \dots m$ such that $(v_{n-1})_l \neq (v_{n-1})_k$ and $(w_{n-1})_l \neq (w_{n-1})_k$ for $l \neq k$. If there are $(\lambda_{n-1})_i, (\mu_{n-1})_j \in k \backslash \{0\}$ such that $\sum_{i=1}^{n} (\lambda_{n-1})_i (v_{n-1})_i = \sum_{j=1}^{m} (\mu_{n-1})_j (w_{n-1})_j$, then $n = m$ and there exist permutations $\pi \in S_{n-1}(n)$ such that $(v_{n-1})_i = (w_{n-1})_{\pi(i)}$ and $(\lambda_{n-1})_i = (\mu_{n-1})_{\pi(i)}$ for $i = 1 \dots n$.

Proof. Since the set of non-zero rays in $\overrightarrow{A}_{n-1}$ forms a basis of A_{n-1}, it is linearly independent and the claim follows.

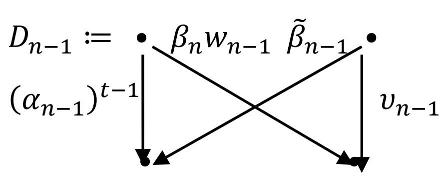
We denote by $\mathcal{L}$ the set of all long morphisms in $\overrightarrow{A}_{n-1}$. By μ_{n-1} we denote some long morphisms $\nu_{n-1}(\alpha_{n-1})^t \nu'_{n-1}$ which exist since $(\alpha_{n-1})^t \neq 0$.

Corollary (4.3.31)[250]: Using the above notations we have:

$$\langle \beta_n \rangle \cap \langle \alpha_{n-1}\beta_n \rangle = 0$$

Proof. We assume to the contrary that $\langle \beta_n \rangle \cap \langle \alpha_{n-1}\beta_n \rangle \neq 0$. Then, by Corollary (4.3.30), there are rays $v_{n-1}, w_{n-1} \in \vec{A}_{n-1}$ such that $\beta_n v_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$. We claim that

$D_{n-1} :=$



are cleaving diagrams in $\overrightarrow{A}_{n-1}$. It is of representation-infinite, Euclidean type $\left(\tilde{A}_{n-1}\right)_3$. Since all morphisms occurring in D_{n-1} are not long, the long morphisms $\mu_{n-1} = \nu_{n-1}(\alpha_{n-1})^t \nu'_{n-1}$ do not occurs in D_{n-1} and D_{n-1} still cleaving in $\overrightarrow{A}_{n-1}/\mu_{n-1}$ by [145]. Thus $\overrightarrow{A}_{n-1}/\mu_{n-1}$ is representation-infinite contradicting the mildness of A_{n-1}.

Now we show (using [145)], that D_{n-1} is cleaving. First of all we assume that there is a ray ρ_{n-1} with $\rho_{n-1}\tilde{\beta}_{n-1} = (\alpha_{n-1})^{t-1}$. Then we get $0 \neq (\alpha_{n-1})^t = \alpha_{n-1}\rho_{n-1}\tilde{\beta}_{n-1} = \beta_n\tilde{\beta}_{n-1}$, whence $\alpha_{n-1}\rho_{n-1} = \beta_n$ by the cancellation law. This contradicts the fact that β_n is an arrow. In a similar way it can be shown that $\rho_{n-1}(\alpha_{n-1})^{t-1} = \tilde{\beta}_{n-1}, \rho_{n-1}\upsilon_{n-1} = \beta_n w_{n-1}$ and $\rho_{n-1}\beta_n w_{n-1} = \upsilon_{n-1}$ are impossible.

The following four cases are left to exclude.

(i) $(\alpha_{n-1})^{t-1}\rho_{n-1} = \beta_n w_{n-1}$: Left multiplication with α_{n-1} gives us $(\alpha_{n-1})^t\rho_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$. Hence there is a non-deep contour $((\alpha_{n-1})^{t-1}\rho_n \dots \rho_{n+k-1}, \beta_n w_n \dots w_{n+l-1})$ in $\overrightarrow{A}_{n-1}$. Here $\rho_{n-1} = \rho_n \dots \rho_{n+k-1}$ resp. $w_{n-1} = w_n \dots w_{n+l-1}$ is a product of irreducible rays (arrows). Since the arrow β_n is in the contour, the cycle $\beta_n\tilde{\beta}_{n-1}$ and the loop α_{n-1} belong to the contour. Hence they can only be penny-farthings by the structure theorem for non-deep contours [144]. But this case is excluded in the current section.

(ii) $\tilde{\beta}_{n-1}\rho_{n-1} = \upsilon_{n-1}$: We argue as before and deduce $\beta_n\tilde{\beta}_{n-1}\rho_{n-1} = \beta_n\upsilon_{n-1} = (\alpha_{n-1})^t\rho_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$. Hence there is a non-deep contour $((\alpha_{n-1})^t\rho_n \dots \rho_{n+k-1}, \beta_n w_n \dots w_{n+l-1})$ leading again to a contradiction.

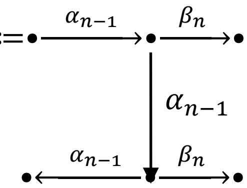
(iii) $\beta_n w_{n-1}\rho_{n-1} = (\alpha_{n-1})^{t-1}$: Since $t - 1 < t$ we have a contradiction to the minimality of t.

(iv) $\upsilon_{n-1}\rho_{n-1} = \tilde{\beta}_{n-1}$:Then$\beta_n\upsilon_{n-1}\rho_{n-1} = \beta_n\tilde{\beta}_{n-1} = (\alpha_{n-1})^t = \alpha_{n-1}\beta_n\upsilon_{n-1}\rho_{n-1} \neq 0$. Using the cancellation law we get $(\alpha_{n-1})^{t-1} = \beta_n\upsilon_{n-1}\rho_{n-1}$ a contradiction as before.

Corollary (4.3.32)[250]: If $t \geq 3$ and $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$, then $(\alpha_{n-1})^2\beta_n = 0$.

Proof. If $(\alpha_{n-1})^2\beta_n \neq 0$, then

$D_{n-1} :=$



are cleaving diagrams of Euclidian type $\left(\widetilde{D}_{n-1}\right)_5$ in $\overrightarrow{A}_{n-1}$. It is cleaving since:

(i) $(\alpha_{n-1})^t = \beta_n \rho_{n-1} \neq 0$ contradicts the choice of $\epsilon \geq 0$.

(ii) $\alpha_{n-1}\beta_n = \beta_n \rho_{n-1} \neq 0$ contradicts Corollary (4.3.31).

It is also cleaving in $\overrightarrow{A}_{n-1}/\eta_{n-1}$ for $\eta_{n-1} \in \mathcal{L} \backslash \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\} \neq \emptyset$ contradicting the mildness of A_{n-1}.

Corollary (4.3.33)[250]: If $\langle(\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0 = \langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle$, then $\langle(\alpha_{n-1})^2, \beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

Proof. Let $(\alpha_{n-1})^2 u_{n-1} + \beta_n v_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$ be an element in $\langle(\alpha_{n-1})^2, \beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle$. By Corollary (4.3.30) we can assume that $u_{n-1}, v_{n-1}, w_{n-1}$ are rays and the following two cases might occur:

(i) $\beta_n v_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$: This is a contradiction since $\langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

(ii) $(\alpha_{n-1})^2 u_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$: This is impossible because $\langle(\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

Corollary (4.3.34)[250]: If $(x_{n-1})^+ = \{\alpha_{n-1}, \beta_n\}$ and $\mathcal{L} \subseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$, then there exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimensions.

Proof. We treat two cases:

(i) $\alpha_{n-1}\beta_n = 0$: Then for $\langle(\alpha_{n-1})^k\rangle$ with $k \geq 1$ only $S_{x_{n-1}}$ is possible as a composition factor; hence $\mathrm{pdim}_\Lambda \langle(\alpha_{n-1})^k\rangle < \infty$. Thus $P_{x_{n-1}} \supset \langle\alpha_{n-1}\rangle \supset \langle(\alpha_{n-1})^2\rangle \supset \langle(\alpha_{n-1})^3\rangle \supset 0$ are the wanted α_{n-1}-filtrations.

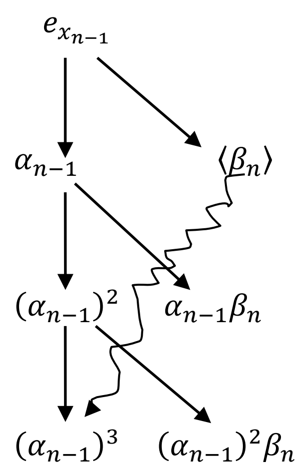
(ii) $\alpha_{n-1}\beta_n \neq 0$: Since $(\alpha_{n-1})^3$ and $(\alpha_{n-1})^2\beta_n$ are the only morphisms in $\overrightarrow{A}_{n-1}$ which can be long, we have $t = 3$, $0 \neq (\alpha_{n-1})^3 \in \mathcal{L}$, $\langle\alpha_{n-1}\beta_n\rangle = \mathbf{k}\alpha_{n-1}\beta_n \cong S_{y_{n-1}}$ and $\langle(\alpha_{n-1})^2\beta_n\rangle \in \{\mathbf{k}(\alpha_{n-1})^2\beta_n, 0\}$.

Now we show that $\langle(\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$. If there are rays $v_{n-1} = v_n \dots v_{n+s-1}$, $w_{n-1} \in \overrightarrow{A}_{n-1}$ with irreducible $v_{n+i-1}, i = 1 \dots, s$ such that $(\alpha_{n-1})^2 v_{n-1} = \alpha_{n-1}\beta_n w_{n-1} \neq 0$, then $s > 0$ because $s = 0$ would contradict the irreducibility of α_{n-1}. Therefore $v_n = \alpha_{n-1}$ or $v_n = \beta_n$.

(a) If $v_n = \alpha_{n-1}$, then $v'_{n-1} = v_{n+1} \dots v_{n+s-1} = id$ since $(\alpha_{n-1})^3$ is long and $0 \neq (\alpha_{n-1})^2 v_{n-1} = (\alpha_{n-1})^3 v'_{n-1}$. Hence $0 \neq (\alpha_{n-1})^3 = (\alpha_{n-1})^2 v_{n-1} = \alpha_{n-1}\beta_n w_{n-1}$ and $(\alpha_{n-1})^3 = \beta_n w_{n-1}$ contradicts the minimality of t.

(b) If $v_n = \beta_n$, then $0 \neq (\alpha_{n-1})^2 v_{n-1} = (\alpha_{n-1})^2 \beta_n v'_{n-1} = \alpha_{n-1}\beta_n w_{n-1}$; hence $0 \neq \alpha_{n-1}\beta_n v'_{n-1} = \beta_n w_{n-1} \in \langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

Since $\langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = \langle(\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle$, we deduce $\langle\beta_n, (\alpha_{n-1})^2\beta_n, \alpha_{n-1}\beta_n\rangle = \langle\beta_n, (\alpha_{n-1})^2\rangle \oplus \langle\alpha_{n-1}\beta_n\rangle$ by Corollary (4.3.33). Therefore the graph of $P_{x_{n-1}}$ has the following shape:



Here $\langle\beta_n\rangle$ stands for the graph of the submodule $\langle\beta_n\rangle$ which is not known explicitly. Consider the module M defined by the following exact sequence:

$$0 \to \langle\beta_n, (\alpha_{n-1})^2, \alpha_{n-1}\beta_n\rangle \to P_{x_{n-1}} \to M \to 0$$

Then $\mathrm{pdim}_\Lambda M < \infty$ since M is filtered by $S_{x_{n-1}}$ and $\mathrm{pdim}_\Lambda(\langle\beta_n, (\alpha_{n-1})^2\rangle \oplus \langle\alpha_{n-1}\beta_n\rangle) = \mathrm{pdim}_\Lambda\langle\beta_n, (\alpha_{n-1})^2, \alpha_{n-1}\beta_n\rangle < \infty$. Thus $\mathrm{pdim}_\Lambda\big(\langle\alpha_{n-1}\beta_n\rangle \cong S_{x_{n-1}}\big)$ is finite too and the wanted α_{n-1}-filtrations are $P_{x_{n-1}} \supset \langle\alpha_{n-1}\rangle \supset \langle(\alpha_{n-1})^2\rangle \supset \langle(\alpha_{n-1})^3\rangle \supset 0$.

Corollary (4.3.35)[250]: If $x_{n-1}^{+} = \{\alpha_{n-1}, \beta_n\}, t \geq 3$ and $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$, then $(\alpha_{n-1})^2\rho_{n-1} = 0$ for all rays $\rho_{n-1} \notin \{e_{x_{n-1}}, \alpha_{n-1}, \dots, (\alpha_{n-1})^{t-2}\}$. Moreover, $\langle(\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

Proof. Let $\rho_{n-1} \in \overrightarrow{A}_{n-1}$ with $(\alpha_{n-1})^2\rho_{n-1} \neq 0$ be written as a composition of irreducible rays $\rho_{n-1} = \rho_n \dots \rho_{n+s-1}$. Then the following two cases are possible:

(i) $\rho_{n-1} = (\alpha_{n-1})^s$: Since $0 \neq (\alpha_{n-1})^2\rho_{n-1} = (\alpha_{n-1})^{2+s}$ and $(\alpha_{n-1})^{t+1} = 0$ we have $s \leq t-2$ and $\rho_{n-1} = (\alpha_{n-1})^s \in \{e_{x_{n-1}}, \alpha_{n-1}, \dots, (\alpha_{n-1})^{t-2}\}$.

(ii) There exists a minimal $1 \leq i \leq s$ such that $\rho_{n+i-1} \neq \alpha_{n-1}$. Since $(x_{n-1})^+ = \{\alpha_{n-1}, \beta_n\}$, we have $\rho_{n+i-1} = \beta_n$ and $0 \neq (\alpha_{n-1})^2\rho_{n-1} = (\alpha_{n-1})^{1+i}\beta_n\rho_{n+i} \dots \rho_{n+s-1} = 0$ by Corollary (4.3.32).

If $0 \neq (\alpha_{n-1})^2 v_{n-1} = \alpha_{n-1}\beta_n w_{n-1}$, then $v_{n-1} = (\alpha_{n-1})^s$ with $0 \leq s \leq t-2$. Hence $0 = (\alpha_{n-1})^2 v_{n-1} = (\alpha_{n-1})^{s+2} = \alpha_{n-1}\beta_n w_{n-1}$ and $(\alpha_{n-1})^{s+1} = \beta_n w_{n-1}$ by cancellation law. This contradicts the minimality of t.

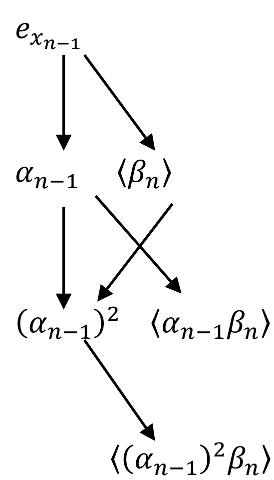
Corollary (4.3.36)[250]: If $x_{n-1}^{+} = \{\alpha_{n-1}, \beta_n\}, t \geq 3$ and $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$, then $\langle(\alpha_{n-1})^2, \beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

Proof. The claim is trivial using Corollarys (4.3.10), (4.3.12) and (4.3.14).

Corollary (4.3.37)[250]: If $x_{n-1}^{+} = \{\alpha_{n-1}, \beta_n\}$, then there exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimension.

Proof. If $\mathcal{L} \subseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$, then the claim is the statement of Corollary (4.3.24). If $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$, then we consider the value of t:

(i) $t = 2$: Then the graph of $P_{x_{n-1}}$ has the following shape:



Let a subquotient M of $P_{x_{n-1}}$ be defined by the following exact sequence:

$$0 \to \langle \beta_n, \alpha_{n-1}\beta_n \rangle \to P_{x_{n-1}} \to M \to 0$$

Then M and $\langle \beta_n, \alpha_{n-1}\beta_n \rangle$ have finite projective dimension in mod-Λ. By Corollary (4.3.31) we have $\langle \beta_n, \alpha_{n-1}\beta_n \rangle = \langle \beta_n \rangle \oplus \langle \alpha_{n-1}\beta_n \rangle$; hence $\mathrm{pdim}_\Lambda \langle \beta_n \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha_{n-1}\beta_n \rangle$ are both finite.

Let K be the kernel of the epimorphism $\lambda_{(\alpha_{n-1})}: \langle \beta_n \rangle \to \langle \alpha_{n-1}\beta_n \rangle, \lambda_{(\alpha_{n-1})}(\rho_{n-1}) = \alpha_{n-1}\rho_{n-1}$. Then $\mathrm{pdim}_\Lambda K < \infty$ and for the α_{n-1}-filtrations $\mathcal{F}$ we take the following: $P_{x_{n-1}} \supset \langle \alpha_{n-1}\beta_n \rangle \supset \langle \beta_n \rangle \oplus \langle \alpha_{n-1}\beta_n \rangle \supset \langle \alpha_{n-1}\beta_n \rangle \oplus K \supset K \supset 0.$

(ii) $t \geq 3$: Consider the following exact sequences:

$$0 \to, \langle \alpha_{n-1}, \beta_n \rangle \to P_{x_{n-1}} \to S_{x_{n-1}} \to 0$$

$$0 \to \langle (\alpha_{n-1})^2, \beta_n, \alpha_{n-1}\beta_n \rangle \to \langle \alpha_{n-1}, \beta_n \rangle \to S_{x_{n-1}} \to 0$$

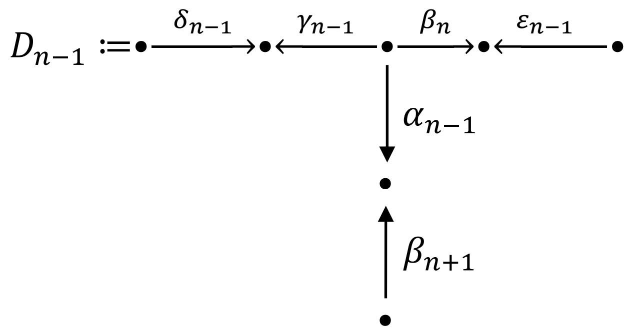
Hence $\mathrm{pdim}_\Lambda \langle \alpha_{n-1}, \beta_n \rangle$ and $\mathrm{pdim}_\Lambda \langle (\alpha_{n-1})^2, \beta_n, \alpha_{n-1}\beta_n \rangle$ are finite. By Corollary (4.3.36) $\langle (\alpha_{n-1})^2, \beta_n, \alpha_{n-1}\beta_n \rangle = \langle (\alpha_{n-1})^2, \beta_n \rangle \oplus \langle \alpha_{n-1}\beta_n \rangle$, that means $\mathrm{pdim}_\Lambda \langle \alpha_{n-1}\beta_n \rangle$ is finite too. With Corollary (4.3.35) it is easily seen that for $2 \leq k \leq t$ the module $\langle (\alpha_{n-1})^k \rangle$ is a uniserial module with $S_{x_{n-1}}$ as the only composition factor. Hence $\mathrm{pdim}_\Lambda \langle (\alpha_{n-1})^k \rangle$ is finite for $2 \leq k \leq t$. There by we have the wanted α_{n-1}-filtrations

$$P_{x_{n-1}} \supset \langle \alpha_{n-1}, \beta_n \rangle \supset \langle (\alpha_{n-1})^2 \rangle \oplus \langle \alpha_{n-1}\beta_n \rangle \supset \langle (\alpha_{n-1})^3 \rangle \supset \langle (\alpha_{n-1})^4 \rangle \supset \ldots \supset \langle (\alpha_{n-1})^t \rangle \supset 0.$$

With previous notations $(x_{n-1})^+ = \{\alpha_{n-1}, \beta_n, \gamma_{n-1}\}, ((\alpha_{n-1})^t, \beta_n\beta_{n+1} \ldots \beta_{n+r-1})$ is a contour in $\vec{A}_{n-1}, t \geq 2, (\alpha_{n-1})^{t+1} = 0, \tilde{\beta}_{n-1} := \beta_{n+1} \ldots \beta_{n+r-1}$ and $\mu_{n-1} = \nu_{n-1}(\alpha_{n-1})^t \nu'{}'_{n-1}$ is a long morphism in $\vec{A}_{n-1}$.

Corollary (4.3.38)[250]: If $r = 2$ and $\delta_{n-1}: z'_{n-1} \to z_{n-1}$ is an arrow in $(Q_{n-1})_{A_{n-1}}$ ending in $z_{n-1} = e_{n-1}(\gamma_{n-1})$, then $\delta_{n-1} = \gamma_{n-1}$.

Proof. Assume to the contrary that $\gamma_{n-1} \neq \delta_{n-1}: z'_{n-1} \to z_{n-1}$, then there is no arrow $\beta_n \neq \varepsilon_{n-1}: y'_{n-1} \to y_{n-1}$ in $(Q_{n-1})_\Lambda$. If there is such an arrow, then by the definition of a neighborhood ε_{n-1} belongs to $(Q_{n-1})_{A_{n-1}}$. This arrow induces an irreducible ray $\beta_n \neq \varepsilon_{n-1}: y'_{n-1} \to y_{n-1}$ in $\vec{A}_{n-1}$ and

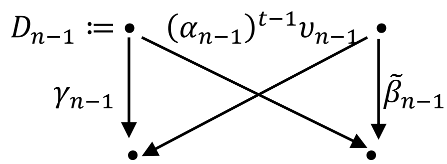


is a cleaving diagram in $\overrightarrow{A}_{n-1}/\mu_{n-1}$ of Euclidian type $\left(\tilde{E}_{n-1}\right)_6$.

Corollary (4.3.39)[250]: If $\alpha_{n-1}\gamma_{n-1} \neq 0$, then $\beta_n v_{n-1} \neq \alpha_{n-1}\gamma_{n-1} \neq \gamma_{n-1}w_{n-1}$ for all rays $v_{n-1}, w_{n-1} \in \overrightarrow{A}_{n-1}$.

Proof.

(i) Assume that there exists a ray $v_{n-1} \in \overrightarrow{A}_{n-1}$ such that $\beta_n v_{n-1} = \alpha_{n-1}\gamma_{n-1} \neq 0$. Then



is a cleaving diagram of Euclidian type e $\left(\tilde{A}_{n-1}\right)_3$ in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

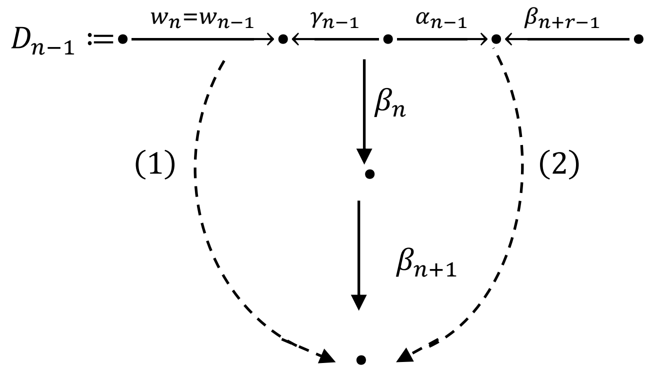
(a) For $\gamma_{n-1}\rho_{n-1} = (\alpha_{n-1})^{t-1}$ or $v_{n-1}\rho_{n-1} = \tilde{\beta}_{n-1}$ we have $\alpha_{n-1}\gamma_{n-1}\rho_{n-1} = \beta_n v_{n-1}\rho_{n-1} = \beta_n\tilde{\beta}_{n-1} = (\alpha_{n-1})^t \neq 0$. Thus $(\alpha_{n-1})^{t-1} = \alpha_{n-1}\rho_{n-1}$ contradicts the choice of t.

(b) If $(\alpha_{n-1})^{t-1}\rho_{n-1} = \gamma_{n-1}$ or $\tilde{\beta}_{n-1}\rho_{n-1} = v_{n-1}$, then $(\alpha_{n-1})^t\rho_{n-1} = \beta_n\tilde{\beta}_{n-1}\rho_{n-1} = \beta_n v_{n-1} = \alpha_{n-1}\gamma_{n-1} \neq 0$. Then $(\alpha_{n-1})^t\rho_{n-1} = \gamma_{n-1}$ contradicts the irreducibility of γ_{n-1}.

(ii) Assume that there exist rays $w_{n-1} = w_n \ldots w_{n+s-1}: z_{n-1} \rightsquigarrow z_{n-1} \in \overrightarrow{A}_{n-1}$ with irreducible w_{n+i-1} such that $\gamma_{n-1}w_{n-1} = \alpha_{n-1}\gamma_{n-1} \neq 0$.

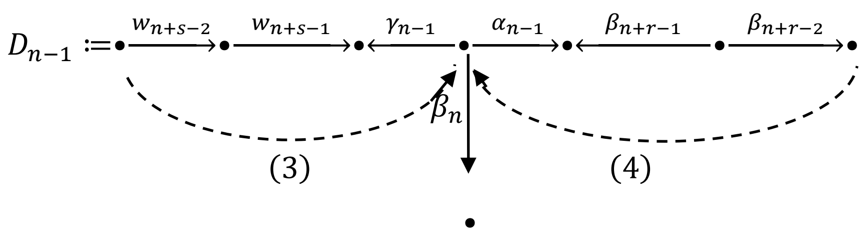
$r = 2$: Since w_{n+s-1} is an irreducible ray ending in $z_{n-1}, w_{n+s-1} = \gamma_{n-1}$ by Corollary (4.3.38). Thus we get a contradiction $\gamma_{n-1}w_{n+s-1} \ldots w_{n+s-2} = \alpha_{n-1}$.

$r \geq 3$: We look at the value of s. If $s = 1$, then $w_{n-1} = w_n$ is a loop and



is a cleaving diagram in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

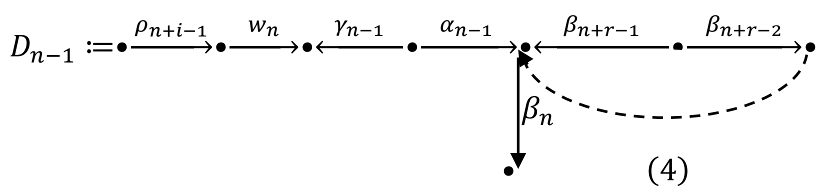
If $s \geq 2$, then



is cleaving in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

We still have to show that not any morphisms indicated by the dotted lines make the diagrams commute.

(a) : $\gamma_{n-1}\rho_{n-1} = \beta_n\beta_{n+1}$, with $\rho_{n-1} = \rho_n \dots \rho_{n+l-1}$. If $\rho_{n-1} = (w_n)^l = w_n^l$, then $\beta_n\beta_{n+1} = \gamma_{n-1}\rho_{n-1} = \gamma_{n-1}w_{n-1}^l = \alpha_{n-1}\gamma_{n-1}w_{n-1}^l$and $\beta_n\beta_{n+1} \dots \beta_{n+r-1} = (\alpha_{n-1})^t = \alpha_{n-1}\gamma_{n-1}(w_{n-1})^{l-1}$ $\beta_{n+2} \dots \beta_{n+r-1} \neq 0$. Therefore $(\alpha_{n-1})^{t-1} = \gamma_{n-1}w_{n-1}^{l-1}\beta_{n+2} \dots \beta_{n+r-1}$ is a contradiction. If $\rho_{n-1} \neq (w_n)^l$, then one of the irreducible rays $\rho_{n+i-1} \neq w_{n+i-1}$ starts in z_{n-1} and



is cleaving in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

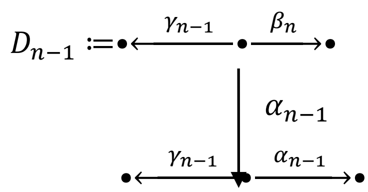
(b) : If $\alpha_{n-1}\rho_{n-1} = \beta_n\beta_{n+1}$, then $\alpha_{n-1}\rho_{n-1}\beta_{n+2} \dots \beta_{n+r-1} = \beta_n\beta_{n+1} \dots \beta_{n+r-1} = (\alpha_{n-1})^t \neq 0$ and $(\alpha_{n-1})^{t-1} = \rho_{n-1}\beta_{n+2} \dots \beta_{n+r-1}$ contradicts the minimality of t.

(c) : If $\rho_{n-1}\gamma_{n-1} = w_{n+s-2}w_{n+s-1}$, then $\gamma_{n-1}w_n \dots w_{n+s-3}\rho_{n-1}\gamma_{n-1} = \gamma_{n-1}w_{n-1} = \alpha_{n-1}\gamma_{n-1} \neq 0$ and $\alpha_{n-1} = \gamma_{n-1}w_n \dots w_{n+s-3}\rho_{n-1}$ contradicts the irreducibility of α_{n-1}.

(d) : If $\rho_{n-1}\alpha_{n-1} = \beta_{n+r-2}\beta_{n+r-1}$, then $\beta_n\beta_{n+1} \dots \beta_{n+r-3}\rho\alpha_{n-1} = \beta_n\beta_{n+1} \dots \beta_{n+r-1} = (\alpha_{n-1})^t \neq 0$ and $(\alpha_{n-1})^{t-1} = \beta_n\beta_{n+1} \dots \beta_{n+r-3}\rho_{n-1}$ contradicts the minimality of t.

Corollary (4.3.40)[250]: If $t \geq 3$, then $\alpha_{n-1}\gamma_{n-1} = 0$.

Proof. Assume that $\alpha_{n-1}\gamma_{n-1} \neq 0$, then



is a cleaving diagram of Euclidian type in $\overrightarrow{A}_{n-1}/\mu_{n-1}$. It is cleaving since:

(i) $\gamma_{n-1}\rho_{n-1} = \alpha_{n-1}\gamma_{n-1}$ or $\beta_n\rho_{n-1} = \alpha_{n-1}\gamma_{n-1}$ contradicts Corollary (4.3.30),

(ii) $\gamma_{n-1}\rho_{n-1} = (\alpha_{n-1})^2$ or $\beta_n\rho_{n-1} = (\alpha_{n-1})^2$ contradicts the minimality of $t \geq 3$.

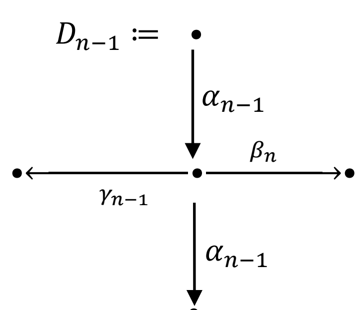
Corollary (4.3.41)[250]:

(a) If $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$, then $\alpha_{n-1}\beta_n = 0$ or $\alpha_{n-1}\gamma_{n-1} = 0$.

(b) If $(\alpha_{n-1})^2\beta_n \neq 0$, then $\gamma_{n-1}w_{n-1} \neq \alpha_{n-1}\beta_n$ for all $w_{n-1} \in \overrightarrow{A}_{n-1}$.

Proof.

(a) If $\alpha_{n-1}\beta_n \neq 0$ and $\alpha_{n-1}\gamma_{n-1} \neq 0$, then



is a cleaving diagram of Euclidian type $\left(\widetilde{D}_{n-1}\right)_4$ in $\overrightarrow{A}_{n-1}$. It is still cleaving in $\overrightarrow{A}_{n-1}/\eta_{n-1}$ for $\eta_{n-1} \in \mathcal{L}\backslash\{(\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\} \neq \emptyset$.

(b) Since $(\alpha_{n-1})^2\beta_n \neq 0$, we have $\alpha_{n-1}\gamma_{n-1} = 0$ by a). But $\gamma_{n-1}w_{n-1} = \alpha_{n-1}\beta_n$ leads to the contradiction $0 \neq (\alpha_{n-1})^2\beta_n = \alpha_{n-1}\gamma_{n-1}w_{n-1} = 0$.

Corollary (4.3.42)[250]: If $t = 2$ or $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^t, (\alpha_{n-1})^2\beta_n\}$, then:

(a) $(\alpha_{n-1})^2\beta_n = 0 = (\alpha_{n-1})^2\gamma_{n-1}, (\alpha_{n-1})^2\rho_{n-1} = 0$ for all rays $\rho_{n-1} \notin \left\{e_{x_{n-1}}, (\alpha_{n-1}), \dots, (\alpha_{n-1})^{t-2}\right\}$.

(b) $\langle\beta_n\rangle \cap \langle\alpha_{n-1}\gamma_{n-1}\rangle = 0$.

(c) If $\langle\gamma_{n-1}\rangle \cap \langle\beta_n\rangle = 0$, then $\langle\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^2\rangle = 0$.

(d) $\langle\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^t\rangle = 0$ or $\langle\gamma_{n-1}\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

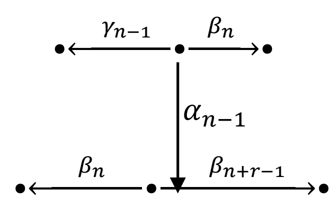
(e) $\langle\gamma_{n-1}\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$ or $\langle\gamma_{n-1}\rangle \cap \langle\beta_n\rangle = 0$.

(f) $\langle\alpha_{n-1}\beta_n\rangle \cap \langle(\alpha_{n-1})^2\rangle = 0$ and $\langle\alpha_{n-1}\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^2\rangle = 0$.

Proof.

(a) Consider the case $t = 2$.

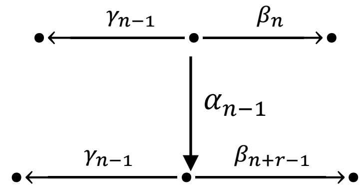
(i) If $(\alpha_{n-1})^2\beta_n \neq 0$, then $\beta_{n+r-1}\beta_n \neq 0$ and



is a cleaving diagram of Euclidian type $\left(\widetilde{D}_{n-1}\right)_5$ in $\overrightarrow{A}_{n-1}/\mu_{n-1}$. The diagram is cleaving because:

- $\beta_n\rho_{n-1} = \alpha_{n-1}\beta_n \neq 0$ is a contradiction of Corollary (4.3.31),
- $\gamma_{n-1}\rho_{n-1} = \alpha_{n-1}\beta_n \neq 0$ contradicts Corollary (4.3.41)(b).

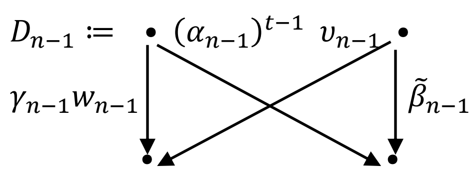
(ii) If $(\alpha_{n-1})^2\gamma_{n-1} \neq 0$, then $\beta_{n+r-1}\gamma_{n-1} \neq 0$ and



is a cleaving diagram in $\overrightarrow{A}_{n-1}/\mu_{n-1}$. It is cleaving since $\beta_n\rho_{n-1} = \alpha_{n-1}\gamma_{n-1}$ resp. $\gamma_{n-1}\rho_{n-1} = \alpha_{n-1}\gamma_{n-1}$ contradicts Corollary (4.3.40).

In the case $t \geq 3, \alpha_{n-1}^2\gamma_{n-1} = 0$ by Corollary (4.3.41). If $t = 3$, then $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$ by assumption. If $t > 3$, then $\mu_{n-1} = \nu_{n-1}(\alpha_{n-1})^t\nu'_{n-1} \in \mathcal{L}\backslash\{(\alpha_{n-1})^3, (\alpha_{n-1})^2\beta_n\}$. Hence $(\alpha_{n-1})^2\beta_n = 0$ by Corollary (4.3.32) in both cases.

(b) If v_{n-1}, w_{n-1} are rays in $\overrightarrow{A}_{n-1}$ such that $\beta_n v_{n-1} = \alpha_{n-1}\gamma_{n-1}w_{n-1} \neq 0$, then the diagram



is a cleaving diagram in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

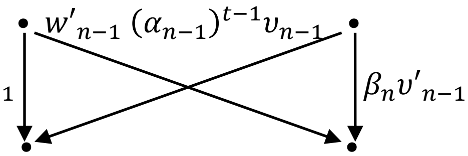
(iii) If $\gamma_{n-1}w_{n-1}\rho_{n-1} = (\alpha_{n-1})^{t-1}$ or $v_{n-1}\rho_{n-1} = \tilde{\beta}_{n-1}$, then $\beta_n v_{n-1}\rho_{n-1} = \beta_n\tilde{\beta}_{n-1} = (\alpha_{n-1})^t = \alpha_{n-1}\gamma_{n-1}w_{n-1}\rho_{n-1} \neq 0$. Hence $\gamma_{n-1}w_{n-1}\rho_{n-1} = (\alpha_{n-1})^{t-1}$ contradicts the minimality of t.

(iv) If $(\alpha_{n-1})^{t-1}\rho_{n-1} = \gamma_{n-1}w_{n-1}$ or $\tilde{\beta}_{n-1}\rho_{n-1} = v_{n-1}$, then $0 \neq \beta_n v_{n-1} = \beta_n\tilde{\beta}_{n-1}\rho_{n-1} = \alpha_{n-1}\gamma_{n-1}w_{n-1} = (\alpha_{n-1})^t\rho_{n-1} = 0$ by a).

(c) Let v_{n-1}, w_{n-1} be rays such that $\gamma_{n-1}v_{n-1} = (\alpha_{n-1})^2 w_{n-1} \neq 0$. By a) we have $w_{n-1} = (\alpha_{n-1})^k$ with $0 \leq k \leq t-2$, that means $\gamma_{n-1}v_{n-1} = (\alpha_{n-1})^{2+k}$. Since t is minimal, we have $t = 2 + k$ and $0 \neq \gamma_{n-1}v_{n-1} = (\alpha_{n-1})^t = \beta_n\tilde{\beta}_{n-1} \in \langle\gamma_{n-1}\rangle \cap \langle\beta_n\rangle = 0$.

(d) Let $v_{n-1}, w_{n-1}, v'_{n-1}, w'_{n-1}$ be rays in $\overrightarrow{A}_{n-1}$ such that $\gamma_{n-1}w_{n-1} = (\alpha_{n-1})^t v_{n-1} \neq 0$ and $\gamma_{n-1}w'_{n-1} = \alpha_{n-1}\beta_n v'_{n-1} \neq 0$. Then

$D_{n-1} :=$

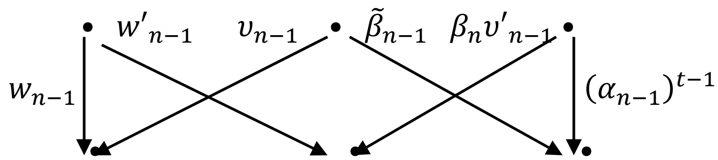


is a cleaving diagram in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

(v) If $w_{n-1}\rho_{n-1} = w'_{n-1}$ or $(\alpha_{n-1})^{t-1}v_{n-1}\rho_{n-1} = \beta_n v'_{n-1}$, then $\gamma_{n-1}w_{n-1}\rho_{n-1} = \gamma_{n-1}w'_{n-1} = (\alpha_{n-1})^t v_{n-1}\rho_{n-1} = \alpha_{n-1}\beta_n v'_{n-1} \neq 0$. Hence there is a non-deep contour $((\alpha_{n-1})^{t-1}v_n \ldots v_{n+k-1}\rho_n \ldots \rho_{n+l-1}, \beta_n v'_n \ldots v'_{n+s-1})$ in $\overrightarrow{A}_{n-1}$ which can only be a penny-farthing by the structure theorem for non-deep contours. But this case is excluded in the current section.

(vi) If $w'_{n-1}\rho_{n-1} = w_{n-1}$ or $\beta_n v'_{n-1}\rho_{n-1} = (\alpha_{n-1})^{t-1}v_{n-1}$, then $\gamma_{n-1}w'_{n-1}\rho_{n-1} = \gamma_{n-1}w_{n-1} = \alpha_{n-1}\beta_n v'_{n-1}\rho_{n-1} = (\alpha_{n-1})^t v_{n-1} \neq 0$. Again, we have a non-deep contour $((\alpha_{n-1})^{t-1}v_n \ldots v_{n+k-1}, \beta_n v'_n \ldots v'_{n+l-1}\rho_n \ldots \rho_{n+s-1})$ which leads to a contradiction as before.

(e) Let $v_{n-1}, w_{n-1}, v'_{n-1}, w'_{n-1}$ be rays such that $\beta_n v_{n-1} = \gamma_{n-1}w_{n-1} \neq 0$ and $\alpha_{n-1}\beta_n v'_{n-1} = \gamma_{n-1}w'_{n-1} \neq 0$. Then



is a cleaving diagram in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

(i) If $w_{n-1}\rho_{n-1} = w'_{n-1}$, we get the contradiction $0 \neq \gamma_{n-1}w_{n-1}\rho_{n-1} = \gamma_{n-1}w'_{n-1} = \beta_n v_{n-1}\rho_{n-1} = \alpha_{n-1}\beta_n v'_{n-1} \in \langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

(ii) If $w'_{n-1}\rho_{n-1} = w_{n-1}$, then $0 \neq \gamma_{n-1} w'_{n-1}\rho_{n-1} = \gamma_{n-1}w_{n-1} = \alpha_{n-1}\beta_n v'_{n-1}\rho_{n-1} = \beta_n v_{n-1} \in \langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

(iii) If $v_{n-1}\rho_{n-1} = \tilde{\beta}_{n-1}$, then $0 \neq \beta_n v_{n-1}\rho_{n-1} = \beta_n\tilde{\beta}_{n-1} = \gamma_{n-1}w_{n-1}\rho_{n-1} = (\alpha_{n-1})^t \in \langle\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^t\rangle = 0$ by d).

(iv) If $\tilde{\beta}_{n-1}\rho_{n-1} = v_{n-1}$, then $\beta_n\tilde{\beta}_{n-1}\rho_{n-1} = \beta_n v_{n-1} = (\alpha_{n-1})^t\rho_{n-1} = \gamma_{n-1}w_{n-1} \in \langle\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^t\rangle = 0$ by d).

(v) If $(\alpha_{n-1})^{t-1}\rho_{n-1} = \beta_n v'_{n-1}$, then $0 \neq (\alpha_{n-1})^t\rho_{n-1} = \alpha_{n-1}\beta_n v'_{n-1} = \gamma_{n-1}w'_{n-1} \in \langle\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^t\rangle = 0$ by d).

The case $\beta_n v'_{n-1}\rho_{n-1} = (\alpha_{n-1})^{t-1}$ contradicts the minimality of t.

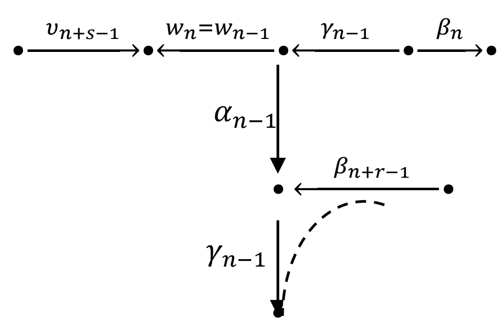
(vi) If v_{n-1}, w_{n-1} are rays in $\overrightarrow{A}_{n-1}$ such that $\alpha_{n-1}\beta_n v_{n-1} = (\alpha_{n-1})^2 w_{n-1} \neq 0$ resp. $\alpha_{n-1}\gamma_{n-1}v_{n-1} = (\alpha_{n-1})^2 w_{n-1} \neq 0$, then $w_{n-1} = (\alpha_{n-1})^k$ with $0 \leq k \leq t-2$ and $\beta_n v_{n-1} = (\alpha_{n-1})^{1+k}$ resp. $\gamma_{n-1}v_{n-1} = (\alpha_{n-1})^{1+k}$. Since t is minimal, we get the contradiction $t = 1 + k < t$.

Corollary (4.3.43)[250]: If $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$, then $\langle\gamma_{n-1}\rangle \cap \langle\alpha_{n-1}\gamma_{n-1}\rangle = 0$.

Proof. In the case $t \geq 3$, the claim is trivial since $\alpha_{n-1}\gamma_{n-1} = 0$.

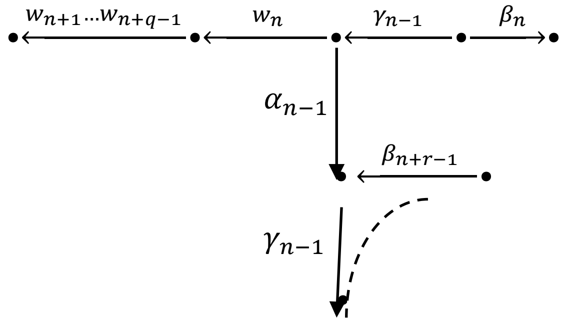
Consider the case $t = 2$. Assume that there exist rays v_{n-1}, w_{n-1} in $\overrightarrow{A}_{n-1}$ such that $\alpha_{n-1}v_{n-1} = \gamma_{n-1}v_{n-1}w_{n-1} \neq 0$. First of all, we deduce that $w_{n-1} \neq id$ by Corollary (4.3.40) and $v_{n-1} \neq id$ since is an arrow. Therefore we can write $v_{n-1} = v_n \dots v_{n+s-1}, w_{n-1} = w_n \dots, w_{n+q-1}$ with irreducible rays $v_{n+i-1}, w_{n+j-1} \in \overrightarrow{A}_{n-1}$. Consider the value of q:

(a) If $q = 1$, then the diagram



is a cleaving diagram of Euclidian type $(\tilde{E}_{n-1})_7$ in $\overrightarrow{A}_{n-1}/\mu_{n-1}$ (see [146]).

(b) If $q \geq 2$, then the diagram



is cleaving in $\overrightarrow{A}_{n-1}/\mu_{n-1}$.

The diagrams are cleaving because:

(i) $\alpha_{n-1}\rho_{n-1} = \gamma_{n-1}w_{n-1} \neq 0$: Then $0 \neq \alpha_{n-1}\gamma_{n-1}w_{n-1} = (\alpha_{n-1})^2\rho_{n-1} = 0$ by Corollary (4.3.42) a).

(ii) $\gamma_{n-1}\rho_{n-1} = \alpha_{n-1}\gamma_{n-1} \neq 0$ contradicts Corollary (4.3.41).

(iii) $\beta_n\rho_{n-1} = \gamma_{n-1}w_{n-1} \neq 0$: Then $0 \neq \alpha_{n-1}\gamma_{n-1}w_{n-1} = \alpha_{n-1}\beta_n\rho_{n-1} = 0$ since $\alpha_{n-1}\beta_n = 0$ by Corollary (4.3.41).

(iv) $\rho_{n-1}v_{n+s-1} = \gamma_{n-1}w_{n-1} \neq 0$: Then $\alpha_{n-1}\rho_{n-1}v_{n+s-1} = \alpha_{n-1}\gamma_{n-1}w_{n-1} \neq 0$. If $\rho_{n-1} = \beta_n\rho'_{n-1}$, then $0 = \alpha_n\beta_n\rho'_{n-1}v_{n+s-1} = \alpha_{n-1}\gamma_{n-1}w_{n-1} \neq 0$. If $\rho_{n-1} = \gamma_{n-1}\rho'_{n-1}$, then $\alpha_{n-1}\gamma_{n-1}\rho'_{n-1}v_{n+s-1} = \alpha_{n-1}\beta_{n-1}w_{n-1}$ and $w_n = w_{n-1} = \rho'_{n-1}v_{n+s-1}$. Hence $\rho'_{n-1} = id$ and $v_{n+s-1} = w_n$. Therefore $0 \neq \gamma_{n-1}v_{n-1} = \gamma_{n-1}v_n \dots v_{n+s-2}w_n = \alpha_{n-1}\gamma_{n-1}w_n$ and $\gamma_{n-1}v_n \dots v_{n+s-2} = \alpha_{n-1}\gamma_{n-1}$ contradicting Corollary (4.3.41). If $\rho_{n-1} = \alpha_{n-1}\rho'_{n-1}$, then $0 \neq \alpha_{n-1}\gamma_{n-1}w_{n-1} = (\alpha_{n-1})^2\rho'_{n-1}v_{n+s-1} = 0$ by Corollary (4.3.42) a).

(v) $\beta_n\rho_{n-1} = \alpha_{n-1}\gamma_{n-1} \neq 0$ contradicts Corollary (4.3.41).

Corollary (4.3.44)[250]: Let $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^t, (\alpha_{n-1})^2\beta_n\}$ and $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$.

(a) If $\langle\alpha_{n-1}\gamma_{n-1}\rangle = 0 = \langle\gamma_{n-1}\rangle \cap \langle\alpha_{n-1}\beta_n\rangle$, then $\langle\beta_n, \gamma_{n-1}, (\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$.

(b) If $\langle\alpha_{n-1}\gamma_{n-1}\rangle = 0 = \langle\gamma_{n-1}\rangle \cap \langle\beta_n\rangle$, then $\langle\beta_n, (\alpha_{n-1})^2\rangle \cap \langle\gamma_{n-1}, \alpha_{n-1}\beta_n\rangle = 0$.

(c) If $\langle\alpha_{n-1}\beta_n\rangle = 0$, then $\langle\beta_n, \gamma_{n-1}, (\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\gamma_{n-1}\rangle = 0$.

Proof. We only show b); the other cases are shown analogously. Let $v_{n-1}, v'_{n-1}, w_{n-1}, w'_{n-1} \in A_{n-1}$ be such that $\beta_n v_{n-1} + (\alpha_{n-1})^2 v'_{n-1} = \gamma_{n-1}w_{n-1} +$

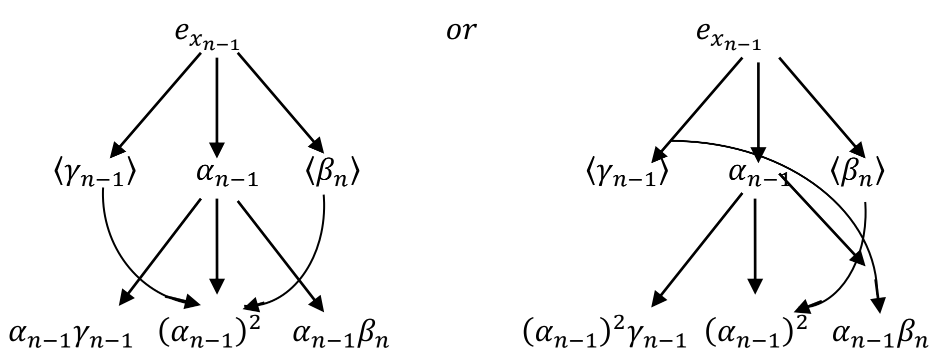
$\alpha_{n-1}\beta_n w'_{n-1} \neq 0$. That means we have rays $(v_{n-1})_i, (w_{n-1})_j \in \overrightarrow{A}_{n-1}$, numbers $(\lambda_{n-1})_i, (\mu_{n-1})_j \in k$ and integers $s_1, s_2 \geq 0, n_1, n_2 \geq 1$ such that

$$\sum_{i=1}^{s_1} (\lambda_{n-1})_i \beta_n (v_{n-1})_i + \sum_{i=s_1+1}^{n_1} (\lambda_{n-1})_i (\alpha_{n-1})^2 (v_{n-1})_i$$
$$= \sum_{j=1}^{s_2} (\mu_{n-1})_j \gamma_{n-1} (w_{n-1})_j + \sum_{j=s_2+1}^{n_2} (\mu_{n-1})_j \alpha_{n-1}\beta_n (w_{n-1})_j$$

and $\beta_n(v_{n-1})_i \neq \beta_n(v_{n-1})_j, (\alpha_{n-1})^2(v_{n-1})_i \neq (\alpha_{n-1})^2(v_{n-1})_j, \gamma_{n-1}(w_{n-1})_i \neq \gamma_{n-1}(w_{n-1})_j$, $\alpha_{n-1}\beta_n(w_{n-1})_i \neq \alpha_{n-1}\beta_n(w_{n-1})_j$ for $i \neq j$. Without loss of generality we can assume that all $(\lambda_{n-1})_i, (\mu_{n-1})_j$ are non-zero, that $\beta_n(v_{n-1})_i \neq (\alpha_{n-1})^2(v_{n-1})_j$ for $i = 1 \ldots s_1, j = s_1 + 1 \ldots n_1$ and $\gamma_{n-1}(w_{n-1})_i \neq \alpha_{n-1}\beta_n(w_{n-1})_j$ for $i = 1 \ldots s_2, j = s_2 + 1 \ldots n_2$. Then by Corollary (4.3.30) we have $n_1 = n_2$ and there exists a permutation π such that $\beta_n(v_{n-1})_i = \gamma_{n-1}(w_{n-1})_{\pi(i)} \in \langle\beta_n\rangle \cap \langle\gamma_{n-1}\rangle = 0$ or $\beta_n(v_{n-1})_i = \alpha_{n-1}\beta_n(w_{n-1})_{\pi(i)} \in \langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$ by Corollary (4.3.31). Hence $s_n = 0$. Moreover, by Corollary (4.3.41) we have $(\alpha_{n-1})^2(v_{n-1})_i = \gamma_{n-1}(w_{n-1})_{\pi(i)} \in \langle(\alpha_{n-1})^2\rangle \cap \langle\gamma_{n-1}\rangle = 0$ or $(\alpha_{n-1})^2(v_{n-1})_i = \alpha_{n-1}\beta_n(w_{n-1})_{\pi(i)} \in \langle(\alpha_{n-1})^2\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$; this is possible for $n_1 - s_1 = 0$ only. Hence $n_1 = 0$, contradicting the choice of n_1.

Corollary (4.3.45)[250]: If $\mathcal{L} \subseteq \{(\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$, then there exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimension.

Proof. Since $\mathcal{L} \subseteq \{(x_{n-1})^2, x_{n-1}\beta_n, x_{n-1}\gamma_{n-1}\}, \mu_{n-1} = (\alpha_{n-1})^2$ is long and $t = 2$. It is easily seen that $\langle(\alpha_{n-1})^2\rangle = \mathbf{k}(\alpha_{n-1})^2 \cong S_{x_{n-1}}, \langle\alpha_{n-1}\gamma_{n-1}\rangle = \mathbf{k}\alpha_{n-1}\gamma_{n-1}, \langle\alpha_{n-1}\beta_n\rangle = \mathbf{k}\alpha_{n-1}\beta_n$ and $\langle\alpha_{n-1}\rangle$ has a $\mathbf{k}$ basis $\{\alpha_{n-1}, (\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$. Using Corollary (4.3.31) we conclude $\langle\beta_n\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$ and $\langle\gamma_{n-1}\rangle \cap \langle\alpha_{n-1}\gamma_{n-1}\rangle = 0 = \langle\beta_n\rangle \cap \langle\alpha_{n-1}\gamma_{n-1}\rangle$.
By Corollary (4.3.42) d) $\langle\gamma_{n-1}\rangle \cap \langle(\alpha_{n-1})^2\rangle = 0$ or $\langle\gamma_{n-1}\rangle \cap \langle\alpha_{n-1}\beta_n\rangle = 0$. Thus the graph of $P_{x_{n-1}}$ has one of the following shapes:



In the first case we consider the following exact sequence:

$$0 \to \langle (\alpha_{n-1})^2 \rangle \to \langle \alpha_{n-1}, \beta_n, \gamma_{n-1} \rangle \to \langle \alpha_{n-1}, \beta_n, \gamma_{n-1} \rangle / \langle (\alpha_{n-1})^2 \rangle \to 0$$

Since $\langle \alpha_{n-1} \rangle$ has $\mathbf{k}$ basis $\{\alpha_{n-1}, (\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$ and $\mathcal{L} \subseteq \{(\alpha_{n-1})^2, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$ we have $\langle \alpha_{n-1}, \beta_n, \gamma_{n-1} \rangle / \langle (\alpha_{n-1})^2 \rangle = \langle \alpha_{n-1} \rangle / \langle (\alpha_{n-1})^2 \rangle \oplus \langle \beta_n, \gamma_{n-1} \rangle / \langle (\alpha_{n-1})^2 \rangle$. Hence $\mathrm{pdim}_\Lambda \langle \alpha_{n-1} \rangle < \infty$ and $P_{x_{n-1}} \supset \langle \alpha_{n-1} \rangle \supset \langle (\alpha_{n-1})^2 \rangle \supset 0$ are the wanted filtrations.

In the second case we have $\langle \alpha_{n-1}, \beta_n, \gamma_{n-1} \rangle / \langle (\alpha_{n-1})^2 \rangle = \langle \alpha_{n-1}, \gamma_{n-1} \rangle / \langle (\alpha_{n-1})^2 \rangle \oplus \langle \beta_n \rangle / \langle (\alpha_{n-1})^2 \rangle$. Thus $\mathrm{pdim}_\Lambda \langle \alpha_{n-1}, \gamma_{n-1} \rangle < \infty$. Now we consider

$$0 \to \langle \beta_n, \gamma_{n-1}, \alpha_{n-1}\gamma_{n-1} \rangle \to \langle \alpha_{n-1}, \beta_n, \gamma_{n-1} \rangle \to S_{x_{n-1}} \to 0.$$

Since $\langle \beta_n, \gamma, \alpha_{n-1}\gamma_{n-1} \rangle = \langle \beta_n, \gamma_{n-1} \rangle \oplus \langle \alpha_{n-1}\gamma_{n-1} \rangle$, we have $\mathrm{pdim}_\Lambda \langle \alpha_{n-1}\gamma_{n-1} \rangle < \infty$ and $P_{x_{n-1}} \supset \langle \alpha_{n-1}, \gamma_{n-1} \rangle \supset \langle (\alpha_{n-1})^2, \alpha_{n-1}\gamma_{n-1} \rangle \supset 0$ are suitable filtrations.

Corollary (4.3.46)[250]: If $\mathcal{L} \subseteq \{(\alpha_{n-1})^t, (\alpha_{n-1})^2\beta_n\}$, then there exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimension.

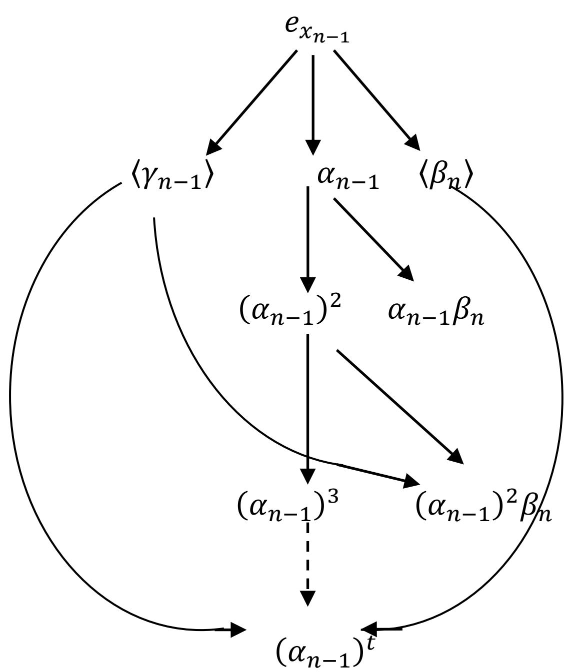
Proof. If $t = 2$, then $(\alpha_{n-1})^2\beta_n = 0$ by Corollary (4.3.41) a). Hence $\mathcal{L} \subseteq \{(\alpha_{n-1})^2\}$ and the filtrations exist by Corollary (4.3.45).

If $t \geq 3$, then $\alpha_{n-1}\gamma_{n-1} = 0$ by Corollary (4.3.42). From the assumption $\mathcal{L} \subseteq \{(\alpha_{n-1})^t, (\alpha_{n-1})^2\beta_n\}$ it is easily seen that $\langle \alpha_{n-1}\beta_n \rangle = \mathbf{k}\alpha_{n-1}\beta_n$ and $\langle (\alpha_{n-1})^2\beta_n \rangle = \mathbf{k}(\alpha_{n-1})^2\beta_n$.

(i) If $(\alpha_{n-1})^2\beta_n = 0$, then $(\alpha_{n-1})^t$ is the only long morphism in $\overrightarrow{A}_{n-1}$; hence $\alpha_{n-1}\beta_n = 0$ and $\langle (\alpha_{n-1})^k \rangle, k \geq 1$, is uniserial of finite projective dimension. Thus $P_{x_{n-1}} \supset \langle \alpha_{n-1} \rangle \supset \langle (\alpha_{n-1})^2 \rangle \supset \ldots \supset \langle (\alpha_{n-1})^t \rangle \supset 0$ are suitable α_{n-1}-filtrations.

(ii) If $(\alpha_{n-1})^2\beta_n \neq 0$, then $\langle \alpha_{n-1}\beta_n \rangle = \mathbf{k}\alpha_{n-1}\beta_n \cong S_{y_{n-1}} \cong \langle (\alpha_{n-1})^2\beta_n \rangle$.

$\langle \beta_n \rangle \cap \langle \alpha_{n-1}\beta_n \rangle = 0 = \langle \gamma_{n-1} \rangle \cap \langle \alpha_{n-1}\beta_n \rangle$. Therefore the graph of $P_{x_{n-1}}$ has the following shape



Moreover, $\langle\alpha_{n-1}\beta_n\rangle \cong S_{y_{n-1}}$ is a direct summand of the module $\langle(\alpha_{n-1})^2, \beta_n, \gamma_{n-1}, \alpha_{n-1}\beta_n\rangle$, which has finite projective dimension. Since the modules $\langle\alpha_{n-1}\rangle, \langle(\alpha_{n-1})^2\rangle, \dots, \langle(\alpha_{n-1})^t\rangle$ have $S_{x_{n-1}}$ and $S_{y_{n-1}}$ as the only composition factors, they are of finite projective dimension. Thus $P_{x_{n-1}} \supset \langle\alpha_{n-1}\rangle \supset \langle(\alpha_{n-1})^2\rangle \supset \dots \langle(\alpha_{n-1})^t\rangle \supset 0$ are suitable α_{n-1}-filtrations.

Corollary (4.3.47)[250]: If $x_{n-1}^+ = \{\alpha_{n-1}, \beta_n, \gamma_{n-1}\}$, then there exist α_{n-1}-filtrations $\mathcal{F}$ of $P_{x_{n-1}}$ having finite projective dimension.

Proof. By Lemmas (4.3.24) and (4.3.25) we can assume that $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^t, (\alpha_{n-1})^2\beta_n\}$ and $\mathcal{L} \nsubseteq \{(\alpha_{n-1})^t, \alpha_{n-1}\beta_n, \alpha_{n-1}\gamma_{n-1}\}$. Then $\mathrm{pdim}_\Lambda\langle(\alpha_{n-1})^k\rangle < \infty$ for $2 \leq k \leq t$ since $\langle(\alpha_{n-1})^k\rangle$ has only $S_{x_{n-1}}$ as a composition factor by Lemma (4.3.21)(a). Moreover, $\mathrm{pdim}_\Lambda\langle\alpha_{n-1}, \beta_n, \gamma_{n-1}\rangle < \infty$ since it is the left hand term of the following exact sequence:

$$0 \to \langle\alpha_{n-1}, \beta_n, \gamma_{n-1}\rangle \to P_{x_{n-1}} \to S_{x_{n-1}} \to 0.$$

By Corollary (4.3.41)(a) only the following two cases are possible:

(i) $\alpha_{n-1}\beta_n = 0$: Consider the following exact sequence:

$$0 \to \langle\beta_n, \gamma_{n-1}, (\alpha_{n-1})^2, \alpha_{n-1}\gamma_{n-1}\rangle \to \langle\alpha_{n-1}, \beta_n, \gamma_{n-1}\rangle \to S_{x_{n-1}} \to 0.$$

Then $\mathrm{pdim}_\Lambda\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2,\alpha_{n-1}\gamma_{n-1}\rangle<\infty$. By Lemma (4.3.23) c) we have $\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2,\alpha_{n-1}\gamma_{n-1}\rangle=\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2\rangle\oplus\langle\alpha_{n-1}\gamma_{n-1}\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha_{n-1}\gamma_{n-1}\rangle<\infty$. Therefore $P_{x_{n-1}}\supset\langle\alpha_{n-1},\beta_1,\gamma_{n-1}\rangle\supset\langle(\alpha_{n-1})^2\rangle\oplus\langle\alpha_{n-1}\gamma_{n-1}\rangle\supset\langle(\alpha_{n-1})^3\rangle\supset\ldots\langle(\alpha_{n-1})^t\rangle\supset 0$ are suitable α_{n-1}-filtrations.

(ii) $\alpha_{n-1}\gamma_{n-1}=0$: Then $\mathrm{pdim}_\Lambda\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2,\alpha_{n-1}\beta_n\rangle<\infty$ since we have the exact sequence

$$0\to\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2,\alpha_{n-1}\beta_n\rangle\to\langle\alpha_{n-1},\beta_n,\gamma_{n-1}\rangle\to S_{x_{n-1}}\to 0.$$

If $\langle\gamma_{n-1}\rangle\cap\langle\alpha_{n-1}\beta_n\rangle=0$, then by Lemma (4.3.23) a) we have $\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2,\alpha_{n-1}\beta_n\rangle=\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2\rangle\oplus\langle\alpha_{n-1}\beta_n\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha_{n-1}\beta_n\rangle<\infty$. Therefore $P_{x_{n-1}}\supset\langle\alpha_{n-1},\beta_n,\gamma_{n-1}\rangle\supset\langle(\alpha_{n-1})^2\rangle\oplus\langle\alpha_{n-1}\beta_n\rangle\supset\langle(\alpha_{n-1})^3\rangle\supset\ldots\langle(\alpha_{n-1})^t\rangle\supset 0$ are suitable α_{n-1}-filtrations.

By Corollary (4.3.42)(e) it remains to consider the case $\langle\gamma_{n-1}\rangle\cap\langle\beta_n\rangle=0$: Then $\langle\beta_n,\gamma_{n-1},(\alpha_{n-1})^2,\alpha_{n-1}\beta_n\rangle=\langle\beta_n,(\alpha_{n-1})^2\rangle\oplus\langle\gamma_{n-1},\alpha_{n-1}\beta_n\rangle$ by Lemma (4.3.23)(b). Thus $\mathrm{pdim}_\Lambda\langle\gamma_{n-1},\alpha_{n-1}\beta_n\rangle<\infty$. Now $P_{x_{n-1}}\supset\langle\alpha_{n-1},\beta_n,\gamma_{n-1}\rangle\supset\langle(\alpha_{n-1})^2\rangle\oplus\langle\gamma_{n-1},\alpha_{n-1}\beta_n\rangle\supset\langle(\alpha_{n-1})^3\rangle\supset\ldots\langle(\alpha_{n-1})^t\rangle\supset 0$ are suitable α_{n-1}-filtrations.