Chapter 2

Uniform Lattices and Decomposition of Group-Valued Measures

With mild extra hypotheses on the ordered topological group, we extend the Boolean decomposition, preserving the uniqueness, to the case where the measure is order bounded instead of being positive. This last result generalizes A. D. Aleksandrov's classical decomposition theorem.

Section (2.1): Uniform Lattices:

Starting point of this section was the observation that there are strong analogies between the theory of topological Riesz spaces and the theory of topological Boolean rings **(**= Boolean rings endowed with an *FN-topology).* So it is of course desirable to unify at least parts of both theories in a more abstract setting. With this aim we here introduce the notion of a uniform lattice, that is a lattice endowed with a uniformity such that the lattice operations $\vee$ and $\wedge$ are uniformly continuous; such a uniformity we call a lattice uniformity. Every topological Riesz space (E,τ) and every topological Boolean ring (R,τ) is a uniform lattice with respect to the uniformity induced by τ; also the order topology of a chain is induced by alattice uniformity. It is surprising and also a justification for the study of uniform lattices that a lot of theorems known for topological Riesz spaces or for topological Boolean rings, in fact, can be proved for this weak structure. Every uniform lattice is, of course, a topological lattice; G. BIRKHOFF has suggested developing a theory of topological lattices [38]; this section is also a contribution to that.

In this section we study uniform lattices with the property (σ) or (F)*.* Let us clearify the meaning of these properties in special cases: The uniformity of a topological Riesz space or of a topological Boolean ring satisfies *(a)* iff its topology is induced by a family of Riesz pseudonorms or of submeasures, respectively, which are σ-subadditive; any uniformity induced by a Fatou topology on a Riesz space satisfies *(F).* We mainly deal with the question, which conditions imply the (sequential) completeness of a lattice uniformity. In particular, we generalize Nakano's completeness theorem 39] for uniform lattices.

We will mention that there are already attempts to unify the notion of Riesz spaces and Boolean rings without topologie, see [40] and [41].

Let L be a lattice and $\Delta := \{(x,x): x \in L\}$ the diagonal of $L \times L$.

Definition (2.1.1)[37]: A uniformity on a lattice is called a lattice uniformity if the lattice operations $\vee$ and $\wedge$ are uniformly continuous. A lattice endowed with a lattice uniformity is called a uniform lattice.

Definition (2.1.2)[37]: A uniformity u on L is a lattice uniformity iff every $U \in u$ contains a$V \in u$ with $V \vee V \subset U$ and $V \vee V \subset U$ iff for every $U \in u$ there is a$V \in u$ u with $V \vee \Delta \subset U$ and $V \vee \Delta \subset U$.

It follows that the supremum of lattice uniformities on L (built in the lattice of all uniformities on L) is also a lattice uniformity; further the discrete and the trivial uniformity on L are lattice uniformities; hence the set of all lattice uniformities on L forms a complete lattice, where the partial ordering is the inclusion. Further one easily proves in the usual manner that every lattice uniformity is the supremum of lattice uniformities, which have a countable base.

Proposition (2.1.3)[37]: Let u be a lattice uniformity on L. Then every $U \in u$ con tains a$V \in u$ such that for all pairs $(a, b) \in V$ the rectangle $[a \wedge b, a \vee b]^2 \subset V$. (Such a V is of course symmetric.)

Proof. For $U \in u$ define $V := \{Ca, b) \in L^2 : [a \wedge b, a \vee b]^2 \subset U\}$. Obviously, $V \subset U$ and $[a \wedge b, a \vee b]^2 \subset V$. We for any $(a, b) \in V$. It remains to prove that $U \in u$: Let W be a symmetric member of u such that $WoW \subset U$. Since $f(x_1, x_2, x_3) = (x_1 \wedge (x_2 \vee x_3)) \vee (x_2 \wedge x_3)$ is uniformly continuous, there is a $W_0 \in u$ such that $(f(x_1, x_2, x_3), f(y_1, y_2, y_3) \in W$ whenever $(x_i, x_i) \in W_0$ for $i = 1, 2, 3$. We prove that $W_0 \subset V$. Let $(a, b) \in W_0$ and $(x, y) \in [a \wedge b, a V b]^2$.*Since* $f(x, a, b) = x$ *and* $f(x, a, a) = a$, one obtains $(x, a) \in W$and just so $(y, a) \in W$. Consequently $(x, Y) \in WoW \subset U$.We have proved that $[a \wedge b, a \vee b]^2 \subset U$ hence $(a, b) \in V$.

A subset C of L is called *convex* if $[a, b] \subset C$ whenever $a, b \in C$ and $a \leq b$. For $A \subset L$ the set

$$c(A) := \{x \in L : \text{there are } a, b \in A \text{ with } a \leq x \leq b\}$$

is the smallest convex subset of L containing A, see [42]; $c(A)$ is the convex hull of A.

Following [42], we call a topology on L locally convex if every point of L has a neighbourhood base consisting of convex sets.

A topological lattice is a lattice endowed with a topology such that the lattice operations ∨ and ∧ are continuous; its topology is called a lattice topology.

Proposition (2.1.4)[37]: If A is a convex subset of a topological lattice L, then the interior A° of A is convex.

Proof. Let $a, b \in A°$ and $x \in L$ with $a \leq x \leq b$. Since $a \wedge x = a \in A°$ and $b \vee x = b \in A°$, there exists by the continuity of the lattice operations a neighbourhood U of x such that $a \wedge y \in A°$ and $b \vee Y \in A°$ for all $y \in U$. But $a \wedge y$, $b \vee y \in A$ and $a \wedge b \leq y \leq b \vee y$ imply $y \in A$, since A is convex. Hence $U \subset A$ and $x \in A°$.

By Proposition (2.1.4), a topological lattice is locally convex iff its topology has a base of convex open sets. Some authors call a topology on a lattice with this last property locally con vex, in contrast to the terminology of [42], which we use here; cf. Example (2.1.11)(e).

Proposition (2.1.5)[37]: Let τ be a topology on L. Suppose that for every $x \in L$ and for every neighbourhood U of x there is a neighbourhood V of x such that $[a, b] \subset U$for all a, $b \in V$ with $a \leq b$;. Then τ is locally convex.

Proof. Let $x \in L$. The convex sets $c(V)$, where V runs through the neighbourhood filter of x, form a neighbourhood base of x.

Proposition (2.1.6)[37]: Let u be a lattice uniformity on L and τ the induced topology. Then (L, τ) is a locally convex topologically lattice. Moreover, every $U \in u$ contains $aV \in u$ such that $V(x)$ is convex for each $x \in L$.

Proof. It is enough the prove the last statement. Let $U \in u$, Choose symmetric sets W_1, $W_2 \in u$ such that $W_2\, 0\, W_2 \subset W_1$, $W_2\, 0\, W_1 \subset U$ and $[a \wedge b, a \vee b]^2 \subset W$ for all $(a, b) \in W_1$. Put $V := \{(x, y): x \in L, y \in c(W_2(x))\}$. Then $W_2 \subset V \in u$ and $V(x) = c(c(W_2(x))$ is convex for all $x \in L$. We show that $V \subset U$. Let $(x, y) \in V$. Then there are $a, b \in W_2(x)$ with $a \leq y \leq b$. Since $(x, a), (x, b) \in W_2$and W_2 is symmetric, we have $(a, b) \in W_2\, 0\, W_2 \subset W_1$, hence $(a, y) \in W_1$, since $a, y \in [a, b]$. Now $(x, a) \in W_2$ and $(a, y) \in W_1$ imply $(x, y) \in W_2\, 0\, W_1 \subset U$

In the following we give some examples of uniform lattices.

Proposition (2.1.7)[37]: Let R be a Boolean ring, τ a group topology on (R, Δ) and u the induced uniformity. Then following conditions are equivalent:

(a) u is a lattice uniformity.

(b) τ is locally convex.

(c) τ is an FN-topology (in the sense of[43]).

Proof. (a) ⇒ (b) is proved in Proposition (2.1.6). (b)⇒(c)⇒(a) is easy and well known.

A subset A of a l-group G is called solid, if $x \in G, Y \in A$, $|x| \leq |y|$ imply $x \in A$ (s. [38]); a group topology on G is called locally solid if 0 has a neighbourhood base consisting of solid sets.

Proposition (2.1.8)[37]: Let τ be a group topology on an l-group and u the right uniformity induced by τ. Then the following conditions are equivalent:

(i) u is a lattice uniformity.

(ii) (G, τ) is a locally convex topological lattice.

(iii) τ is locally solid.

Proof: (i) ⇒ (ii) is proved in Proposition (2.1.6).

(ii)⇒(iii) Let U be a convex 0-neighbourhoodand V a 0-neighbourhood with $-V = V, V \vee V \subset U, V \wedge V \subset U$. First we show that $V \subset W := \{a \in G: [-|a|, |a|] \subset U\}$. Let $a \in V$. Then $-a \in V$, hence $|a| = a \vee (-a) E\ V \vee V\ c\ U$ and $-|a| = a \wedge (-a) E \in V \wedge V \subset U$. Consequently $[-|a|, |a|] \subset U$, since U is convex. It follows that $a \in W$. We have proved that $V \subset W$. Therefore W is a 0-neighbourhood, which is obviously a solid subset of U.

(iii) ⇒ (i) Let U be a solid 0-neighbourhood and $x, x', y \in G$. Since $|(x \vee y) - (x' \vee y)| \leq |x - x'|$ and U is solid, $x - x' \in U$ implies $(x \vee y) - (x' \vee y) \in U$. Analogously $x - x' \in U$ implies $(x \wedge y) - (x' \wedge y) \in U$. It follows with the help of definition (2.1.2) that ∨ and ∧ are uniformly continuous.

Sometimes it is of interest that any order interval of a locally solid l-group is a uniform lattice with respect to the induced uniformity.

Also the uniformities on «$(\leq, +, \Delta)$-lattices» examined in [44] are lattice uniformities; these contain the uniformities induced

(i) by an FN-topology on a Boolean ring,

(ii) by a locally solid group topology on the positive cone of a commutative ℓ-group and

(iii) by an "Integral norm- (see [45]) on the space of all functions from a fixed set X into $[0, +\infty]$".

Proposition (2.1.9)[37]: Let T be a totally ordered set. Then each of the following families of sets

(i) $U_{a,b} := (-\infty, b)^2 \cup (a, +\infty)^2 \ (a, b \in T;\ a < b)$,

(ii) $U_a := (-\infty, b)^2 \cup (a, +\infty)^2 (a \in T)$,

(iii) $V_a := (-\infty, a)^2 \cup \{(a, a)\} \cup (a, +\infty)^2 \ (a \in T)$,

forms a subbase for a totally bounded lattice uniformity U_1, U_2 and $U_3\ on\ T$, which induces on T the interval topology, the half-open interval topology and the discrete topology, respectively.

Proof. (i) Let $a, b \in T$ with $a < b$. Then $U := U_{a,b}$ is symmetric and contains the diagonal of T^2. If $(a, b) = 0$ then $U \circ U \subset U$. If $c \in (a, b)$, then $V \circ V \subset U$ where

$$V := (-\infty, c)^2 \cup (a\ , b)^2 \cup (c, +\infty)^2 = U_{a,c} \cap U_{c,b};$$

in fact, for $(x, y), (y, z) \in V, y \leq c$; implies $x, z < b$ and $y \geq c$: implies $x, z > a$. We have shown that the sets of the form $U_{a,b}\ (a < b)$ form a subbase for a uniformity u. Since the sets $U_{a,b}$ are sublattices of T^2, u is a lattice uniformity. u is totally bounded since $T = (-\infty,\ b) \cup (a, +\infty)$ for $a < b$. The set of the form $(-\infty,\ a)$ or $(a, +\infty)(a \in T)$ form a subbase of the induced topology.

(ii) In case (ii), U_a contains the diagonal of T^2 and is symmetric, $U_a \circ U_a \subset U_a, U_a$ is a sublattice of T^2 and $T = (-\infty,\ a) \cup (a, +\infty)$ for all $a \in T$. Hence $\{U_a: a \in T\}$ is a subbase of a totally bounded lattice uniformity. Obviously, this uniformity induces the half-open interval topology.

(iii) The proof in case (iii) is similar to that in case (ii).

Proposition (2.1.10)[37]: [46]. Let $\mu: L \to G$ be a modular with values in a commutative topological group, i.e. $\mu(a \vee b) + \mu(a \wedge\ b) = \mu(a) + \mu(b)$ for all $a, b \in L$. Then the sets

$$\widehat{U} := \{(a, b) \in L^2: \mu(x) - \mu(y) \in U \text{ for all } x, y \in [a \wedge\ b, a \vee b]\},$$

where U runs through the 0-neighbourhood system of G, form a base for the weakest lattice uniformity on L, which makes μ uniformly continuous.

Not every lattice uniformity is generated by a group valued modular: If there is on L a Hausdorff lattice uniformity generated by a group valued modular, then L is modular by [46, Theorem 1].

Examples (2.1.11)[37]: (a) The sets

$$U_n := \left\{ \bigcup_{i \in F} 2i, 2i + I\} : F \subset \{i \in N : i \geq n\}, F \text{ is finite} \right\} \quad (n \in \mathbb{N})$$

form a 0-neighbourhood base for a ring topology on the algebra of all finite and co- finite subsets of N, which is not an $\boldsymbol{FN}$-topology and therefore not locally convex. Hence ∪,∩ are continuous, but not uniformly continuous (cf. Proposition (2.1.7)).

(b) For any real sequence $x \in (x_n)$ define $\|x\| := \sup_{n \in N} |Xn - n_{n-1}|$ where $X_o := 0$. Then $\| \|$ is a norm on $E := \{x \in \mathbb{R}^N : \|x\| < \infty\}$ and $(E, \| \|)$ is a Banach space (see [39]). On the subspace $C_{00} := \{(x_n) \in \mathbb{R}^N : x_n = 0 \text{ eventually}\}$ the lattice operations ∧,∨ are continuous, but not uniformly continuous with respect to the norm $\| \|$). Hence $(c_{00}, \| \|)$ is a topological group and a topological lattice, which is not locally convex (in the lattice sense; cf. Proposition (2.1.8) and [47]). (Therefore the statements 2.8, 2.9, 2.10 of [48] are not true.)

(c) $p((x_n)) := |\sum_{n=1}^{\infty} x_n|$ defines a seminorm on l_1 The p-topology is locally convex (in the lattice sense), but not locally solid; (l_1, P) is not a topological lattice (cf. Proposition (2.1.8)).

(d) Let $L := \{0,1,a,b\}$ be the free lattice with generators a, b [38]; put $U := \Delta \cup \{(0,b),(b,0)\}. \{V : U \subset V \subset L^2\}$ is a uniformity on L with the property that $[x \wedge y, x \vee y]^2 \subset U$ for all $(x,y) \in U$, but $x \longmapsto x \vee a$ is not continuous in b (cf. Proposition (2.1.3)).

(e) $L := \{-\infty, +\infty\} \cup (\mathbb{N} \times \mathbb{Z})$ becomes a lattice by defining $a \leq b$ iff $a = b$ or $a = -\infty$ or $b = +\infty$ or $a = (n,i), b = (n,j),\ i \leq j$. The sets

$$L, \quad \emptyset, \quad \{(2n,2i),(2n+1,2i)\}, \quad \{(2n-1,2i-1),(2n,2i-I)\},$$
$$\{(k,l) \in \mathbb{N} \times \mathbb{Z} : k \geq 2n,\ \ k > 2n \ \ if\ l\ is\ odd\} \cup \{+\infty\} \qquad (n \in \mathbb{N}, i \in \mathbb{Z})$$

form a base for a locally convex topology, which doesn't admit a base consisting of convex sets.

(f) The same topology on a lattice can be induced by different lattice uniformities u_1, u_2 and by a uniformity u_3, which is not a lattice uniformity: Take on $\mathbb{N}$ the discrete for u_1, the first of the uniformities defined in Proposition (2.1.9) for u_2 and the uniformity with base $\{(k,k) : k \in \mathbb{N}\} \cup \{(2k,2l) : k,l > n\}\ (n \in N) for\ u_3$.

Proof of (b). (i) $x \mapsto x^+$ is continuous on $(c\infty, \|\ \|)$:Let $x = (x_n) \in c\infty$ with $x_i = 0$ for $i > k$, and $\varepsilon > 0$. Choose $\delta > 0$ with $2(k + 1)\,\delta < \varepsilon$. We show that $y \in coo, \|y\| \leq \delta$ imply $\|(x + y)^+ - x^+\| \leq \varepsilon$. Let $y \in (y_n) \in c_{00}$ with $\|y\| \leq \delta$', put $z = (z_n) = (x + y)^+ - x^+$ and $x_0 = y_0 = z_0 = 0$. Then $|z_i| = |(x_i + y_i) \vee 0 - x_i \vee 0| \leq |(x_i + y_i) - x_i| = |y_i|$ for all $i \in \mathbb{N}$ and $z_i = {y_i}^+$ for $i > k$, Since $|y_i| \leq i \cdot \delta \leq \varepsilon/2$ for $i \leq k + 1$, we have $|z_i - z_{i-1}| \leq |y_i| + |y_{i-1}| \leq \varepsilon$ for $i \leq k + 1$. Further $|z_i - z_{i-1}| = |{y_i}^+ - {y_{i-1}}^+| \leq |y_i - y_{i-1}| \leq \delta \leq \varepsilon$ for $i > k + 1$. Consequently $|z| \leq \varepsilon$.

(ii) It follows from (i) and the equalities $x \vee y = (x - y)^+ + y$ and $(x \vee y)^+ + (x \wedge y) = x + y$ that $\vee$ and $\wedge$ are continuous on c_{00}.

(iii) The $\|\ \|$-topology is not locally solid on c_{00}: Suppose that V is a solid 0-neighbourhood contained in $\{x \in c_{00} : \|x\| < 1\}$. Let $k \in \mathbb{N}$ with $\{x \in c_{00} : \|x\| \leq 1/k\} \subset V$. Define $x = (x_n)$ by $x_i = x_{2k-i} = i / k$ for $i = 1, \ldots, k$ and $x_i = 0$ for $i \geq 2k$ and $e_k = (y_n)$ by $y_k = 1$ and $y_n = 0$ for $n \neq k$, Then $0 \leq e_k \leq x \in V$, hence $e_k \in V$. That contradicts $\|e_k\| = 1$.

(iv) It follows from (iii) and Proposition (2.1.8) (or [47]) that $\vee$, $\wedge$ are not uniformly continuous.

In a natural way to every uniform lattice is assigned a Hausdorff uniform lattice by factorisation. The fact that the Hausdorff property can be important, is shown by the following well-known proposition.

Proposition (2.1.12)[37]: Let (L, τ) be a Hausdorff topological lattice. (a) If $(x_\alpha)_{\alpha \in A}$ and $(y_\alpha)_{\alpha \in A}$ are nets in L converging to *x* and *y*, respectively, and if $x_\alpha \leq y_\alpha$ or all $\alpha \in A$, then $x \leq y$.

(b) If (x_α) is a decreasing or increasing net in *L* converging to *x*, then $x = \inf x_\alpha$, or $x = \sup x_\alpha$ respectively.

(c) The intervals $[a, +\infty), (-\infty, a], [a, b]$ are τ-closed for all $a, b \in L$.

For any uniformity u on a set X denote by $N(u)$ the intersection $\bigcap_{U \in u} U$ i.e. the closure of the diagonal in $(X, U)^2$; then $N(u)$ is an equivalence relation on X; *a* and *b* are equivalent $(i.e. (a, b) \in N(u))$ iff $\overline{\{a\}} = \overline{\{b\}}$. Viceversa, if *N* is an equivalence relation on *X*, then $\{U : N \subset U \subset X^2\}$ is *a* uniformity on *X*.

The proof of the following Propositions (2.1.13) and (2.1.14) is obvious.

Proposition (2.1.13)[37]: For $N \subset L^2$ the following conditions are equivalent:

(a) N is a congruence relation with respect to $\vee$ and $\wedge$ on L.

(b) N contains $\Delta, N^{-1}, N \circ N, \; N \vee N, \; N \wedge N$.

(c) $\{U: N \subset U \subset L^2\}$ is a lattice uniformity on L.

(d) There is a lattice uniformity u on L such that $N = N(u)$.

Proposition (2.1.14)[37]: Let N be a congruence relation with respect to $\vee$ and $\wedge$ on L, denote by $\hat{a} = N(a)$ the equivalence class generated by a, write $a \sim b$ instead of $(a, b) \in N$.

(a) Then $\hat{L} = \{\hat{a}: a \in L\}$ is a lattice with respect to the operation defined by $\left(\hat{a} \vee \hat{b}\right) = (a \vee b)\widehat{}, \hat{a} \wedge \hat{b} = (a \wedge b)^{\wedge}$.

(b) If $a, b \in L$ and $a \sim b$, then $x \sim y$ for every $x, y \in [a \wedge b, a \vee b]$.

(c) $\hat{a} \leq \hat{b}$ iff there are $x, y \in L$ with $x \sim a, \; y \sim b$ and $x \leq y$ iff there are $x, y \in L$ with $x \sim a, y \sim b, \; x \leq y, x \leq a, b \leq y$.

Proposition (2.1.15)[37]: Let u be a lattice uniformity on L and N a congruence rellation on L with $N \subset N(u)$. Put $\hat{a} := N(a)$ for $a \in L, \hat{L} = \{a: \hat{a} \in L\}, \hat{U} := \{(\hat{a}, \hat{b}): (a, b) \in U\}$ for $U \subset L^2$.

(a) Then for any closed subset U of $(L, u)^2$ we have $(a, b) \in U$ iff $\left(\hat{a}, \hat{b}\right) \in \hat{U}$.

(b) $\hat{u} := \{\hat{U}: U \in U\}$ is a lattice uniformity on L; $\hat{u}$ is Hausdorff iff $N(u)=N$.

(c) A net (x_α) in L is u-Cauchy iff $(\hat{x}_\alpha)$ is $\hat{u}$-Cauchy.

(d) A net (x_α) converges to x in (L, u) iff $(\hat{x}_\alpha)$ converges to $\hat{x}$ in $\left(\hat{L}, \hat{u}\right)$.

(e) (L, u) is complete or sequentially complete iff $(\hat{L}, \hat{u})$ is complete or sequentially complete, respectively.

(f) If $(x_\alpha)_{\alpha \in A}$ is a net in L such that $(\hat{x}_\alpha)$ is increasing (or decreasing), then there is an increasing net (or decreasing net, respectively) $(z_\beta)_{\beta \in B}$ such that $(\hat{z}_\beta)$ a subnet of $(\hat{x}_\alpha)$ and A and B have the same cardinality.

Proof. (a)The implication $\Longrightarrow$ holds by the definition of $\hat{U}$. To prove $\Longleftarrow$, let U be a closed subset of (L, u) and $(\hat{a}, \hat{b}) \in \hat{U}$. Then there are elements $x \in \hat{a}$ and $y \in \hat{b}$ such that $(x, y) \in$

U. It follows $(a,b) \in N \circ U \circ N \subset U$, since U is closed and $N \subset N(u)$.Using (a) and the fact that *u* has a base of closed sets, one easily gets (b) to (e).

(f) We consider only the case that $(\hat{x}_\alpha)$ is increasing and *A* is infinite. Let *B* be the set of all finite subset *F* of *A*, which contain a greatest element *g(F),* i.e. $g(F) \in F$ and $\alpha \leq g(F)$ for all $\alpha \in F$. Put $z_\alpha := \sup_{\alpha \in F} x_\alpha$ for $F \in B$. Then $(z_F)_{F \in B}$ is an increasing net, $(\hat{z}_F)$ is a subnet of $(\hat{x}_\alpha)$ since $\hat{z}_F = \hat{x}_{g(F)}$, and *A* and *B* have the same cardinality.

Definition (2.1.16)[37]: Let u be a lattice uniformity on L. We say that (L,u) or that u satisfies the monotone completeness property (briefly (MCP)), if every monotone Cauchy net of (L,u) converges in (L,u) (cj. [47]). We call *u* exhaustive if every monotone net in *L* is *u*-Cauchy.

Corollary (2.1.17)[37]: Under the same assumptions and notations of Proposition (2.1.15) holds:

(a) For a fixed cardinal number x, every increasing (or decreasing) Cauchy net $(x_\alpha)_{\alpha \in A}$ with $|A| \leq x$ converges in (L,u) iff every increasing Cauchy net (or decreasing Cauchy net, respectively) $(\hat{x}_\alpha)_{\alpha \in A}$ with $|A| \leq x$ converges in (L,u). In particular, u satisfies (MCP) iff u satisfies (MCP).

(b) *u* is exhaustive iff it is exhaustive.

Lemma (2.1.18)[37]: Let $(x_\alpha)_{\alpha \in A}$ be a monotone net in a uniform lattice (L,u). Suppose that (x_α) has a subnet, which is Cauchy or converges to $x \in L$. Then also (x_α) is Cauchy or convrges to *x*, respectively.

Proof. Let $U \in u$. For *U* choose $V \in u$ according to Proposition (2.1.3) By assumption, there is an $\alpha_0 \in A$ such that for every $\beta \geq \alpha_0$ there is a $\gamma \in A$ with $\gamma \geq \beta$ and $(x_{\alpha_0}, x_\beta) \in V$; if $\gamma \geq \beta \geq \alpha_0$ then $x_{\alpha_0} \leq x_\beta \leq x_\gamma$ or $x_{\alpha_0} \geq x_\beta \geq x_\gamma$ hence $(x_{\alpha_0}, x_\gamma) \in V$ implies $(x_{\alpha_0}, x_\gamma) \in V \subset U$. We have proved that (x_α) is Cauchy.

Now suppose that a subnet of (x_α) converges to x. Then there is an $\alpha_0 \in A$ such that for every $\beta \geq \alpha_0$ there is a $\gamma \geq \beta$ with $(x_{\alpha_0}, x_\gamma) \in V$ and $(x_{\alpha_0}, x_\beta) \in V$. As before, for $\beta \geq \alpha_0$ we have $(x_{\alpha_0}, x_\beta) \in V$, hence $(x_{\alpha_0}, x_\beta) \in V \circ V \subset U \circ U$.

Proposition (2.1.19)[37]: Every Hausdorff uniform lattice (L, u) is a sub lattice and a dense subspace of a Hausdorff uniform lattice (L, u), which is complete as uniform space.

Proof. Let $(\tilde{L}, \tilde{u})$ be the Hausdorff uniform completion of (L, u). Then $\vee$ and $\wedge$ defined on L^2 have a uniformly continuous extension to $\tilde{L}^2$. So $\tilde{L}$ becomes a lattice and $\tilde{u}$ a lattice uniformity. Obviously, L is a sublattice on $\tilde{L}$[49].

Proposition (2.1.20)[37]: Let (L, u) be a uniform lattice and *q > 1*. Then u is generated by a system $(d_\alpha)_{\alpha \in A}$ of pseudometrics such that for all $x, y, z \in L$ and $\alpha \in A$ $d_\alpha(x \vee z, y \vee z) \leq d_\alpha(x, y)$ and $d_\alpha(x \wedge z, y \wedge z) \sim q \cdot d_\alpha(x, y)$. If u has a countable base, one can choose A with $|A| = 1$.

In general, one cannot replace in this proposition *q >1* by *q = 1*, s. [50]; on the other hand, that is possible if *L* is distributive [50] or if *u* is generated by a group valued modular [46].

Lemma (2.1.21)[37]: Let *d* be a pseudometric on L and $q \geq 1$ such that $d(x \vee z, y \vee z) \leq q \cdot d(x, y)$ for all $x, y, z \in L$.

Then for all $x_1, x_2, x_3, \ldots \in L$ and $n, m \in N$ with $n < m$ we have:

$$(a)\ d\left(\sup_{1 \leq i \leq n} x_i\,,\ \sup_{1 \leq i \leq m} x_i\right) \leq q. \sum_{i=n}^{m-1} d(x_i, x_{i+1})\,;$$

$$(b) d\left(\sup_{i \geq n} x_i\,, \sup_{i \geq m} x_i\right) \leq q. \sum_{i=n}^{m-1} d(x_i, x_{i+1}).$$

if $\sup_{i \geq m} x_i$ (and therefore $\sup_{i \geq n} x_i$) exists.

Proof. (a) Put $a_k = \sup_{1 \leq i \leq k} x_i$Then

$$d(a_n, a_m) \leq \sum_{i=n}^{m-1} d(a_i, a_{i+1}) = \sum_{i=n}^{m-1} d(a_i \vee x_i, a_{i+1} \vee x_{i+1}) \leq q \cdot \sum_{i=n}^{m-1} d(x_i, x_{i+1}).$$

(b) If $s_m := \sup_{i \geq m} x_i$ exists, then $s_k := \sup_{i \geq k} x_i$ exists for *k<m* and

$$d(s_n, s_m) \leq \sum_{i=n}^{m-1} d(s_i, s_{i+1}) = \sum_{i=n}^{m-1} d(s_{i+1} \vee x_i, s_{i+1} \vee x_{i+1}) \leq q \cdot \sum_{i=n}^{m-1} d(x_i, x_{i+1}).$$

Replacing «sup, inf» by «inf, sup» one obtaines statements dual to Proposition (2.1.20) and Lemma (2.1.21).

In this section let u be a lattice uniformity on L.

If $(x_\alpha)_{\alpha\in A}$ is a net in a uniform space (X, v) and $x \in X$, then $x_\alpha \to x(v)$ or equivalently $x = v - \lim_{\alpha\in A} x_\alpha$, means that (x_α) converges to x in (X, v). The following essentially known lemma is here the key of Theorem (2.1.25) and its corollaries.

Lemma (2.1.22)[37]: Let $(x_\alpha)_{\alpha\in A}$, $(y_\beta)_{\beta\in B}$ and $(z_{\alpha\beta})_{\alpha\in A,\beta\in B}$ be nets in a uniform space (X, v) such that $y_\beta = v - \lim_{\alpha} z_{\alpha\beta}$ for $\beta \in B, x_\alpha = v - \lim_{\beta} z_{\alpha\beta}$ for $\alpha \in A$ and $(z_{\alpha\beta})_{\alpha\in A}$ is Cauchy uniform in $\beta \in b$

(a) Then (y_β) is Cauchy.

(b) If $x \in X$ and $y_\beta \to x(u)$ then $x_\alpha \to x(v)$.

Proof. Let V be a closed symmetric member of v. Choose $\alpha_0 \in A$ such that $(z_{\alpha_1\beta}, z_{\alpha_2\beta}) \in V$ for all $\beta \in B$ and $\alpha_{1,}\alpha_2 \geq \alpha_0$. Then $(y_\beta, z_{\alpha_1\beta}) = v - \lim_{\alpha\geq\alpha_0} (z_{\alpha_1\beta}, z_{\alpha_1\beta}) \in V$ for all $\beta \in B$ and $\alpha_1 \geq \alpha_0$. Now choose $\beta_0 \in B$ such that $(z_{\alpha_0\beta}, z_{\alpha_0\beta_0}) \in V$ for $\beta \geq \beta_0$ Then, for $\beta \geq \beta_0$, the pairs $\left(y_\beta, z_{\alpha_0\beta}\right)$, $\left(z_{\alpha_0\beta}, z_{\alpha_0\beta_0}\right)$, $\left(z_{\alpha_0\beta_0}, y_{\beta_0}\right)$ belong to V, hence $(y_\beta, y_{\beta_0}) \in V^3$ for $\beta \geq \beta_0$ This shows that (y_β) is Cauchy.

Suppose now that $y_\beta \to x(v)$. Let $\alpha \geq \alpha_0$. We have just seen that $(y_\beta, (z_{\alpha\beta}) \in V$ for all $\beta \in B$. Hence $(x, x_\alpha) = v - \lim_{\alpha\geq\alpha_0} (y_\beta, z_{\alpha\beta}) \in V$. This shows that $(x_\alpha) \to x(v)$.

Proposition (2.1.23)[37]: Let $(x_\alpha)_{\alpha\in A}$ be a Cauchy net in (L, u) and B an upwards directed subset of L such that $x_\alpha = u - \lim_{b\in B} x_\alpha \wedge b$ for every $\alpha \in A$. Assume that $y_b \in L$ and $y_b = u - \lim_{\alpha\in A} x_\alpha \wedge b$ for every $b \in B$.

(a) Then $(y_b)_{b\in B}$ is Cauchy.

(b) If $x \in L$ and $y_\beta \to x(u)$ then $x_\alpha \to x(u)$.

Proof. Apply Lemma (2.1.22) with $z_{\alpha\beta} = x_\alpha \wedge b$. We have only to check that $(z_{\alpha\beta})_{\alpha\in A}$ is Cauchy uniform in $b \in B$: Let $U \in u$ and $V \in u$ with $V \wedge\Delta \subset U$. If $(x_{\alpha_1}, x_{\alpha_2}) \in V$, then $(z_{\alpha_1}, z_{\alpha_2 b}) = (x_{\alpha_1}, x_{\alpha_2}) \wedge (b, b) \in V \wedge\Delta \subset U$ for all $b \in B$.

Corollary (2.1.24)[37]: Let B be a dense upwards directed subset of (L,u). If (x_α) is a Cauchy net in (L,u) and $x \in L$ such that $x_\alpha \wedge b \to x \wedge b(u)$ for all $b \in B$, then $x_\alpha \to x(u)$.

Proof. Apply Proposition (2.1.23) with $y_b := x \wedge b$; hereby observe that $y_b \to x(u)$ as one easily sees using Proposition (2.1.3).

We denote by $|M|$ the cardinality of any set M.

Theorem (2.1.25)[37]: Let $(x_\alpha)_{\alpha\in A}$ be a Cauchy net in (L,u) such that $((x_\alpha \vee a) \wedge b)_{\alpha\in A}$ converges in (L,u) for every $a, b \in L$. Assume that every monotone Cauchy net $(y_\beta)_{\beta\in B}$ with $|B| \leq |A|$ is convergent. Then (x_α) is convergent.

Proof. Because of Propositions (2.1.14), (2.1.15) and Corollary (2.1.17) we may assume that u is Hausdorff. We further may assume that A is infinite.

Let $a \in L$. Then $(z_\alpha) := (x_\alpha \vee a)$ is Cauchy, since (x_α) is Cauchy. By assumption $(z_\alpha \wedge b)_{\alpha\in A}$ converges to an element $y_b \in L$ for every $b \in L$. Put $B := \{\sup_{\alpha\in F} z_\alpha : F \text{ is a finite subset of } A\}$. Then $(y_b)_{b\in B}$ is an increasing Cauchy net by Proposition (2.1.12)(a) and Proposition (2.1.23)(a).

Since $|\mathrm{B}| \leq |\mathrm{A}|$, the net $(\mathrm{y_b})_{\mathrm{b\in B}}$ converges by assumption and therefore $(\mathrm{z}_\alpha)_{\alpha\in\mathrm{A}}$ converges by Proposition (2.1.23)(b).

We have proved that $(x_\alpha \vee a)_{\alpha\in A}$ converges to an element $y_b^{'} \in L$ for evry $a \in L$. Put $B^{'} := \{\inf_{\alpha\in F} z_\alpha : F \text{ is a finite subset of } A\}$ Then $(y_b^{'})_{b\in B'}$ is a decreasing Cauchy net by Proposition (2.1.12)(a) and the statement dual to Proposition (2.1.23)2.2(a) (which one obtains replacing in Proposition (2.1.23) $\wedge$ by $\vee$ and «upwards» by «downwards»). Since $|B'| \leq |A|$, the net $(y_b^{'})_{b\in B'}$ converges by as sumption. Therefore (x_α) converges by the statement dual to Proposition (2.1.23)(b).

Corollary (2.1.26)[37]: (a) (L,u) is sequentially complete iff for every $a, b \in L$ with $a < b$ the interval $[a,b]$ is sequentially u-compleie (i.e. sequentially complete with re spect to the uniformity induced by u on $[a,b]$) and every monotone Cauchy sequence converges in (L,u).

(b) (L,u) is complete iff for every $a, b \in L$ with $a < b$ the interval $[a,b]$ is u-complete and u satisfies [51].

Follows immediately from Theorem (2.1.24) observing that $(x \vee a) \wedge b \in [a \wedge b, b]$ for every $a, b, x \in L$.

Corollary (2.1.26)(b) generalizes [47].

Definition (2.1.27)[37]: We say that u satisfies the property (L), if every increasing Cauchy net in (L, u) has an upper bound and every decreasing Cauchy net in (L, u) has a lower bound. (cf. Proposition (2.1.29)).

Corollary (2.1.28)[37]: (a) If $[a, b]$ is u-complete for all $a, b \in L$ with $a < b$ and u satisfies (L), then (L, u) is complete.

(b) If (L, u) is Hausdorff, then (L, u) is complete iff $[a, b]$ is u-complete for all $a, b \in L$ with $a < b$ and u satisfies (L).

Proof. (a) Because cf. Corollary (2.1.26)(b) we have only to show that u satisfies (MPC). If $(x_\alpha)_{\alpha\in A}$ is an increasing Cauchy net, then (x_α) has by (L) an upper bound $z \in L$. For $\alpha_0 \in A$, the net $(x_\alpha)_{\alpha\geq\alpha_0}$ is an increasing Cauchy net, then (x_α) has by (L) an upper bound $z \in L$ is Cauchy net in $[(x_{\alpha_0}, z]$ and is therefore convergent. Hence $(x_\alpha)_{\alpha\in A}$ is convergent. Analogously one treats the decreasing case.

(b) If (L, u) is Hausdorff and complete, then (L) holds by Proposition (2.1.12)(b). The rest follows from (a).

In view of the following Proposition 2.7(a) generalizes [47].

Proposition (2.1.29)[37]: Let E be a Riesz space endowed with a locally solid linear topology T.

(a) If $(x_\alpha)_{\alpha\in A}$ is an increasing Cauchy net in E^+, then $\{x_\alpha : \alpha \in A\}$ is bounded (in the linear topological space sense).

(b) If τ is Levi (in the sense of [47] or of [39]), then the uniformity in duced by τ satisfies (L).

Proof. (a) Let $(x_\alpha)_{\alpha\in A}$ be an increasing Cauchy net in E^+ and U, V solid 0-neighbourhoods with $V + V \subset U$. Choose $\gamma \in A$ such that $x_\beta - x_\gamma \in V$ for all $\beta \geq \gamma$ and $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1$ and $\varepsilon x_\gamma \in V$. We show that $\{\varepsilon x_\alpha : \alpha \in A\}\, c\, U$. Let $\alpha \in A$. Choose $\beta \in A$ with $\beta \geq \alpha$ and $\beta \geq \gamma$. Then $0 \leq \varepsilon x_\alpha \leq \varepsilon x_\beta = \varepsilon(x_\beta - x_\gamma) + \varepsilon x_\gamma \in \varepsilon V + V \subset V + V \subset U$ and therefore $\varepsilon x_\alpha \in U$.

(b) Follows from (a).

The main results of this section are the Theorems (2.1.33), (2.1.44) and the Corollaries (2.1.39), (2.1.43); there we give sufficient conditions for a uniform lattice to be (sequentially) complete or a Baire space. In these theorems, the condition (σ) defined in Definition (2.1.30) is important.

In this section we assume that u is a lattice uniformity on L and write

$$a \sim b \text{ iff } (a, b) \in N(u).$$

Definition (2.1.30)[37]: u is said to satisfy the property (σ) iffor every $U \in u$. there is a sequence $(U_n)_{n\in N}$ in u with the following property: if $(a_n)_{n\in N}$ is a monotone sequence in L order-converging to a with $(a_i, a_j) \in U_n$; for $i, j \geq n$, then $(a_1, a) \in U$.

The meaning of this property in special cases is given in Proposition (2.1.45) and Proposition (2.1.46).

Proposition (2.1.31)[37]: (a) The supremum of lattice uniformities on L satisfying (σ) also satisfies (σ).

(b) If u satisfies (σ), then there is a net $(u_\alpha)_{\alpha\in A}$ of lattice uniformities on L with countable base satisfying (σ) such that $u_\alpha \uparrow u$ (i.e. (u_α) is increasing and $u = \sup u_\alpha$.).

Proof. (a) Suppose that $(u_\alpha)_{\alpha\in A}$ is a family of lattice uniformities with (σ) and $u = \sup_{\alpha\in A} u_\alpha$.: Let B be a finite subset of A and $U_\alpha \in u_\alpha$ $(\alpha \in B)$. If we choose for U; a sequence $(U_{\alpha,n})_{n\in N}$ in u_α ; according to Definition (2.1.30), then $(\bigcap_{\alpha\in B} U_{\alpha,n})_{n\in N}$ is a sequence for $\bigcap_{\alpha\in B} U_\alpha$ according to Definition (2.1.30). Therefore u satisfies (σ) .

(b) Let $U \in u$. Put $U_0 \coloneqq U_{0,n} := U$ for $n \in \mathbb{N}$; we define U_k and $U_{k,n}$ $(k, n \in \mathbb{N};\ n > k)$ by induction as follows: Choose a symmetric $U_k \in u$ with $U_{k,n} \subset U_{k-1,n}$ and $U_k \wedge U_k, U_k \vee U_k, U_k \circ U_k \subset U_{k-1}$ and $U_{k,n} \in U$ $(n > k)$ with $U_{k,n} \subset U_{k-1,n}$ and with the following property: If $(a_i, a_j) \in U_{k,n+k}$ $(i, j \geq n)$, further $a_n \uparrow a$ or $a_n \downarrow a$, then $(a_1, a) \in U_k$. Since $U_{n+k} \subset U_{n+k-1,n+k,} \subset U_{k,n+k}$, $(a_i, a_j) \in U_{n+k}$ $(i, j \geq n)$ implies that $(a_1, a) \in U_k$ if $a_n \uparrow a$ or $a_n \downarrow a$. Therefore $(U_k)_{k\in N}$ is a base of a lattice uniformity with (σ) and $U_k \subset U$ $(k \in \mathbb{N})$. It follows that u is the supremum of all lattice uniformities coarser than u, satisfying (σ), with countable base.

Proposition (2.1.32)[37]: If u has a countable base, then the following conditions are equivalent:

(i) u satisfies (σ).

(ii) Every monotone Cauchy sequence (a_n) which order-converges to a, converges to a in (L, u).

(iii) u has a base $(U_n)_{n \in N}$ such that every sequence (a_n) with $(a_n, a_{n+1}) \in U$; $(n \in \mathbb{N})$ holds: If $s = \limsup a_n$ exists, then $a_n \to s(u)$; if $t = \liminf a_n$ exists, then $a_n \to t(u)$; if both $\limsup a_n$ and $\liminf a_n$ exist, then $\limsup a_n \sim \liminf a_n$.

Assume that d is a pseudometric, which generates u, and that $q, Q \geq 1$ such that

$$d(a \vee c, b \vee c) \leq q \cdot d(a, b) \quad \text{and} \quad d(a \wedge c, b \wedge c) \leq Q \cdot d(a, b)$$

for all $a, b, c \in L$ (cf. Proposition (2.1.20)). Then the following condition (4) is equivalent to (1):

(iv) For any sequence a_n in L and $a \in L$,

(a) $a = \sup a_n$ implies $d(a_1, a) \leq q. \sum_{n=1}^{\infty} d(a_n, a_{n+1})$ and

(b) $a = \inf a_n$ implies $d(a_1, a) \leq Q. \sum_{n=1}^{\infty} d(a_n, a_{n+1})$

Proof. (i) $\Longrightarrow$(ii): Let (a_n) be a Cauchy sequence in (L, u) and $a \in L$ such that $a_n \uparrow a$ or $a_n \downarrow a$. Let $U \in u$ such that $[a \wedge b, a \vee b]^2 \subset U$ for all $(a, b) \in u$ (cf. Proposition (2.1.3)).

Choose (U_n) for U according to Definition (2.1.30) and a subsequence (b_n) of (a_n) such that $(b_i, b_j) \in U$; for $i, j \geq n$. Also (b_n) order-converges to a. Using (i) it follows that $(b_1, a) \in U$, hence $(a_n, a) \in U$ if $n \geq k$ and $a_k = b_1$.

(ii) $\Longrightarrow$ (iv): (a) Suppose that $a, a_n \in La = \sup_{n \in N} a_n$. Put $s_n = \sup_{1 \leq i \leq n} a_i$. Then by Lemma (2.1.21),

$$d(a_1, a) \leq d(a_1, s_n) + d(s_n, a) \leq q. \sum_{i=1}^{\infty} d(a_i, a_{i+1}) + d(s_n, a)$$

It remains to show that $\sum_{i=1}^{\infty} d(a_i, a_{i+1})) < \infty$ implies $d(s_n, s) \to 0$. Since by Lemma (2.1.21) $d(s_n, s_m) \leq q. \sum_{i=n}^{m-1} d(a_i, a_{i+1})$ for $n < m$, (s_n) then is Cauchy. Further $s_n \uparrow a$. Hence $d(s_n, a) \to 0$ by (b).

(b) follows from (a) by duality.

(iv) $\Longrightarrow$(iii): Observe first that by Proposition (2.1.20) there exists a pseudometric d according to the assumption stated between (iii) and (iv).

Put $U_n := \{(a,b) \in L^2 : d(a,b) \leq 2^{-n}\}$. Let (a_n) be a sequence in L with $(a_n, a_{n+1}) \in U$; $(n \in \mathbb{N})$. Suppose that $s_k = \sup_{i \geq k} a_i$ exist for $k \in \mathbb{N}$ and $s_k \downarrow s$. Then, by (iv)(b),

$$d(s_n, s) \leq Q \,.\sum_{i \geq n} d(s_i, s_{i+1}) = Q \,.\sum_{i \geq n} d(s_{i+1} \vee a_i, s_{i+1} \vee a_{i+1})$$

$$\leq Q \,.\sum_{i \geq n} q \cdot d(a_i, a_{i+1}) \leq Q \cdot q \cdot 2^{-n+1},$$

hence $s_n \to s(u)$. Since $d(a_n, s_n) \leq q \,.\sum_{i \geq n} d(a_i, a_{i+1}) \leq q \cdot 2^{-n+1}$ by (iv)(a) and therefore $d(a_n, s_n) \to 0$, it follows that also $a_n \to s(u)$.

If $t = \liminf a_n$ exists, then analogously one gets that $a_n \to t(u)$. If both $S = \limsup a_n$and If $t = \liminf a_n$, then $a_n \to s(u)$, $a_n \to t(u)$ and therefore $s \sim t$.

(iii) ⟹(i): Let $U\ E\ u$ and U be closed in $(L,u)^2$ Choose (U_n) according to (iii) and $a, a_n \in L$ for $n \in \mathbb{N}$ such that $(a_i, a_j) \in U_n \cap U$ for $i, j \geq n$: and $a_n \uparrow a$ or $a_n \downarrow a$. Using (iii) it follows that $a_n \to a(u)$ hence $(a_1, a) = u - \lim(a_1, a_n) \in \bar{U} = U$.

(i) ⟹ (iii) ⟹ (ii) ⟹ (i) of Proposition (2.1.32) and the Lemma (2.1.34) below also can be proved easily without using Proposition (2.1.20).

For topological Riesz spaces the property (ii) of Proposition (2.1.32) is called in [47] the pseudo σ-Lebesgue property.

Theorem (2.1.33)[37]: Assume that u has a countable base and every monotone Cauchy sequence in (L,u) is order-convergent. Then the following conditions are equivalent:

(a) u satisfies (σ),

(b) Every Cauchy sequence (a_n) in (L,u), for which $\limsup a_n$ and $\liminf a_n$ exist and $\limsup a_n \sim \liminf a_n$, converges to $\limsup a_n$.

(c) (i) (L,u) is complete and

(ii) every sequence (a_n) converging in (L,u) to an element a has a subsequence (b_n) such that $\limsup b_n$ and $\liminf b_n$ exist and $\limsup b_n \sim a \sim \liminf b_n$.

Proof. (a) ⟹(c) Choose V_n according to Lemma (2.1.34) and U_n according to Proposition (2.1.32)(c). Let (a_n) be a Cauchy sequence in (L,u). Then (a_n) has a subsequence (b_n) such that $(b_n, b_{n+1}) \in U_n \cap V_n$ $(n \in \mathbb{N})$. By the lemma, $s \coloneqq \limsup b_n$ and $t := \liminf b_n$ exist and so by Proposition (2.1.32) (a) $\Longrightarrow$ (c) $s \sim t$ and $b_n \to s(u)$, hence $a_n \to s(u)$. This proves (i). To prove (ii), suppose that (a_n) converges to a. Since (a_n) also converges to s, it follows that $s \sim a$. We have just shown that $s \sim t$, hence $s \sim a \sim t$.

(c) $\Longrightarrow$ (b) Given a sequence (a_n) as stated in (b). Then (a_n) converges to an element $a \in L$ by (c)(i). Let (b_n) be a subsequence of (a_n) according to (c)(ii). Then $lim\ sup\ b_n \sim a$ and

$$\liminf a_n \leq \liminf b_n \leq \limsup b_n \leq \limsup a_n \sim \liminf a_n \ ,$$

hence $\limsup a_n \sim \limsup b_n \sim a$. If follows that (a_n) converges to $\limsup a_n$.

(b)$\Longrightarrow$(a) follows from Proposition (2.1.32) (b)$\Longrightarrow$(a).

The completeness assumption of Theorem (2.1.33) is of course satisfied if $(L, \leq)$ is σ-complete or if L is a topological Riesz space with the σ -Levi property, see [47].

In special cases Theorem (2.1.33) (a)$\Longrightarrow$(c) is well known, s. e.g. [47], [43], [45], [44], [52]. In a special case (c)(ii) means that every convergent sequence has a subsequence converging pointwise almost everywhere.

In Theorem (2.1.33), neither (c)(i) nor (c)(ii) imply (a): The power set $\mathbf{P}(\mathbb{N})$ of $\mathbb{N}$ with the **FN**-topology induced by a {0,1}-valued real content, which is not a Dirac measure, is complete; but (σ) is not satisfied $[0, 1[\cup \{2\}$ with the usual uniformity (induced by the metric $(x, y) \longmapsto |x - y|$) satisfies (c)(ii), but not (σ).

The examples [47] and [44] show that in Theorem (2.1.33) without the countability assumption (a) implies neither (c)(i) nor (c)(ii).

Lemma (2.1.34)[37]: Under the assumptions of Theorem (2.1.33) there is a base $(V_n)_{n\in\mathbb{N}}$ of u such that $lim\ sup\ b_n$ and $\liminf b_n$; exists for every sequence (b_n) with $(b_n, b_{n+1}) \in V_n$ $(n \in \mathbb{N})$.

Proof. Let d be a pseudometric on L generating u such that $d(a \vee c, b \vee c) \leq 2d(a, b)$ and $d(a \wedge c, b \wedge c) \leq 2d(a, b)$ for all $a, b, c \in L$; s. Proposition (2.1.20). Let (b_n) be a sequence in L with $d(b_n, b_{n+1}) \leq 2^{-n}$. Then we have by Lemma (2.1.21)(a)

$$d\left(\sup_{k\leq i\leq n} b_i \, , \sup_{k\leq i\leq n} b_i\right) \leq 2 \cdot \sum_{i=n}^{m-1} d(b_i, b_{i+1}) \leq 2 \cdot \sum_{i\geq n} 2^{-i} = 2^{-n+2}$$

for $m \geq n \geq k$. Therefore $(\sup_{k \leq i \leq n+k} b_i)_{n \in \mathbb{N}}$ is an increasing Cauchy sequence and so by assumption there is an $s_k \in L$ with $s_k = \sup_{k \leq i} b_i$. Since by Lemma (2.1.21) (b) $d(s_n, s_m) \leq 2 \cdot \sum_{i=n}^{m-1} d(b_i, b_{i+1}) \leq 2 \cdot \sum_{i \geq n} 2^{-i} = 2^{-n+2}$ for $m \geq n$, the sequence $(s_n)_{n \in \mathbb{N}}$ is

Cauchy and decreasing, hence $\limsup b_n = \inf s_n$ exists by assumption. Analogously one gets that $\liminf b_n$ exists.

It follows that the sets $V_n := \{(a, b) \in L^2 : d(a, b) \leq 2^{-n}\}$ $(n \in \mathbb{N})$ have the desired property.

Theorem (2.1.35)[37]: If u is metrizable, then the following conditions are equivalent:

(i) (L, u) is complete.

(ii) Every monotone Cauchy sequence is convergent in (L, u).

(iii)u satisfies (σ) and every monotone Cauchy sequence in (L, u) is order-convergent.

(iv)u satisfies (σ). and every Cauchy sequence in (L, u) has an order-convergent subsequence.

Proof. (iv)⟹(iii) and (i)⟹(ii) are obvious. (iii)⟹(i) follows from Theorem (2.1.33) (a) ⟹ (c)(i).

(ii) ⟹ (iii) Let (a_n) be a monotone Cauchy sequence in (L, u). By (ii), (a_n) con verges to an element a in (L, u). Then a is the order-limit of (a_n) by Proposition (2.1.12)(b). It follows with Proposition (2.1.32) (ii) ⟹ (i) that u satisfies (σ).

(iii) ⟹ (iv): Let (a_n) be a Cauchy sequence in (L, u). Since (iii) implies (i), there is an element $a \in L$ with $a_n \to a(u)$. By Theorem (2.1.33) (a) ⟹ (c) (ii) there is a subsequence (b_n) of (a_n) such that $s := \limsup b_n$ and $t := \liminf b_n$ exist and $s \sim t$. Since u is Hausdorff, $s \sim t$ implies $s = t$, i.e. (b_n) is order-convergent.

Corollary (2.1.36)[37]: If u has a countable base, then (L, u) is complete iff every monotone Ca1tchy sequence converges in (L, u) iff (L, u) satisfies (MCP).

This follows immediately from Theorem (2.1.35)(ii)⟹(i), Proposition (2.1.15)(e) and Corollary (2.1.17)(a).

For Boolean rings endowed with a submeasure, Corollary (2.1.36) is due to Drewnowski, for commutative 1-groups endowed with a Riesz pseudonorm to Wilhelm, s. [52]; cf. also [47].

Now we examine property (σ) for u also in the case that u doesn't have a countable base. To obtain criteria for sequential completeness of (L, u) we use the following lemma.

Lemma (2.1.37)[37]: Assume that u is Hausdorff and satisfies (σ), that $(L, \leq)$ is σ-complete and (a_n) a Cauchy sequence in (L, u):

$$Put\ X := \left\{\inf_{n\in M} a_n : M \text{ is an infinite subset of } N\right\} \text{and } Y:$$

$$= \{\sup_{n\in M} a_n : M \text{ is an infine subset of } N\}.$$

(a) Then $x \leq y$ for every $x \in X$ and $y \in Y$.

(b) If $a \in L$, then $a_n \to a(u)$ iff $x \leq a \leq y$ for every $x \in X$ and $y \in X$.

(c) The following conditions are equivalent:

(i) (a_n) converges in (L, u).

(ii) sup X exists in L.

(iii) inf Y exists in L.

(iv) sup X and inf Y exist in L and sup X = inf Y.

(d) Denite by $\mathfrak{C}$ the set of all countable subset of X directed by inclusion and put $z_C :=$ sup C for $C \in \mathfrak{C}$. Then $(z_C)_{C\in\mathfrak{C}}$ is an increasing Cauchy net; if (z_C) converges in (L, u), then so does (a_n).

Proof. (a), (b) $\Longrightarrow$: By Proposition (2.1.19) there is a uniform lattice $(\tilde{L}, \tilde{u})$, which is the comple tion of (L, u). Let a be the limit of (a_n). in $(\tilde{L}, \tilde{u})$. Let $x \in X$ and M be a countable subset of $\mathbb{N}$ such that $x = \inf_{n\in M} a_n$ (inf built in L). Since also the subsequence $(a_n)_{n\in M}$ of $(a_n)_{n\in N}$ converges to a, one gets $x \leq a$ by Proposition (2.1.12). By duality holds $y \geq a$ for all $y \in Y$. It follows that $x \leq a \leq y$ and therefore $x \leq y$ for every $x \in X$ and $y \in Y$. (Up to now we have not used that *u* satisfies (σ).)

(b) $\Longleftarrow$ By Proposition (2.1.31) there is a net $(u_\alpha)_{\alpha\in A}$ of lattice uniformities on L with countable base satisfying (σ) such that $u_\alpha \uparrow u$. Let $a \in L$ such that $x \leq a \leq y$ for all $x \in X$ and $y \in Y$. For fixed $\alpha \in A$ there is by Proposition (2.1.32) (ii)$\Longrightarrow$ (iii) a subsequence (b_n)

of (a_n) such that, with $s := \limsup b_n$ and $t := \liminf b_n$, we have $(s,t) \in N(u_\alpha)$ and $b_n \to s(u_\alpha)$. Since $t \le a \le s$ and $(s,t) \in N(u_\alpha)$, we have $(a,s) \in N(u_\alpha)$. It follows that $b_n \to a(u_\alpha)$. and there fore $a_n \to a(u_\alpha)$. since (a_n) is u_α-Cauchy and (b_n) is a subsequence of b_n We have proved that $a_n \to a(u_\alpha)$. for every $\alpha \in A$; hence $a_n \to a(u_\alpha)$.

(c) (i) ⟹ (iv): Suppose that (a_n) converges to a in (L,u). Then a is an upper bound of X by (b) ⟹ If b is an upper bound in L of X with $b \le a$, then $x \le b \le a$, y for all $x \in X$ and $y \in Y$, hence $a_n \to b(u)$ by (b) ⟸. It follows that *a* = *b*, since u is Hausdorff. We have proved that *a* = sup X. By duality one gets *a* = inf Y.

(ii) ⟹(i) If *a* = sup X exists in *L*, then $x \le a \le y$ for all $x \in X$ and $y \in Y$ by (a). Consequently $a_n \to a(u_\alpha)$ by (b) ⟸(iii)⟹(i) one gets by duality. Obviously, (iv) implies (ii) and (iii).

(d) As in the proof of (b) ⟸, we choose for fixed $\alpha \in A$ a subsequence $(b_n)$of (a_n) such that $(s,t) \in N(u_\alpha)$ for s = lim sup b_n; and $t =$ lim inf b_n. Then $D := \{\inf_{i \ge n} b_i : n \in \mathbb{N}\} \in \mathfrak{C}$ and $t = z_D$ For every $C \in \mathfrak{C}$ with $D \subset C$ we get $t = z_D \le z_C \le s$ with the help of *(a)*, hence $(z_D, z_C) \in N(u_\alpha)$. It follows that (z_C) is u_α-Cauchy for every $\alpha \in A$ and therefore *u*-Cauchy, too.

Suppose now that (z_C) has a limit a in (L,u). Then $a = \sup_{c\in\mathfrak{C}} z_C$ by Proposition (2.1.15) (b), i.e. *a* = sup X; hence (a_n) converges by (c) (ii) ⟹(i).

Denote by *c* the cardinality of $\mathbb{R}$. In Lemma (2.1.37), the cardinalities $|X|, |Y|$ and $|\mathfrak{C}|$ are not greater than *c*. Since $a = \sup_{c\in\mathfrak{C}} z_C$ implies *a* = sup X, one immediately gets from Lemma (2.1.37) the following theorem.

Theorem (2.1.38)[37]: Assume that u satisfies (σ), is Hausdorff and that c is a σ- complete. Assume further that every increasing Cauchy net $(x_\alpha)_{\alpha\in A}$ in *(L, u)* with $|A| \le c$; has a supremum. Then *(L, u)* is sequentially complete.

$(L, \le)$ said to be *c*-complete, if every subset of *L* with a cardinality not greater than *c* has a supremum and an infimum.

The following corollary generalizes [47].

Corollary (2.1.39)[37]:.Assume that u satisfies (σ), is Hausdorff and that $(L, \le)$ is *c*-complete. Then *(L, u)* is sequentially complete.

Corollary (2.1.40)[37]: If every increasing Cauchy net $(x_\alpha)_{\alpha\in A}$ with $|A| \leq c$ is conver gent in $(L,u), u$ satisfies (σ) and $(L,\leq)$ is σ -complete, then (L,u) is sequentially complete.

Proof. In case of u being Hausdorff the assertion immediately follows from Lemma (2.1.37)(d) or from Theorem (2.1.38) and Proposition (2.1.12)(b). One can reduce Corollary (2.1.40) to the Hausdorff case with the help of Proposition (2.1.15)(d), Corollary (2.1.17) and the following facts Proposition (2.1.41) and Proposition (2.1.42).

Proposition (2.1.41)[37]: Let N be a congruence relation with respect to $\vee$ and $\wedge$ on *L*. Then the following conditions are equivalent:

(a) There is a lattice uniformity *u* con *L* satisfying (σ) such that $N = N(u)$.

(b) $\{U: N \subset U \subset L^2\}$ is a lattice uniformity on *L* satisfying (σ).

(c) *N* is a σ-sublattice of L^2 (in the sense of [39]).

(d) With the notation of Proposition (2.1.14), $a \mapsto \hat{a}$ defines a sequentially order-continuous map on *L* to $\hat{L}$ (in the sense of [39]).

Proof. (b)⟹(a) is obvious.

(a)⟹(d) If (a_n) is a sequence in *L* and $a \in L$ with $a_n \uparrow a$, then $\hat{a}_n \uparrow$ and $\hat{a}_n \leq \hat{a}$ $(n \in N)$. Let $b \in L$ such that $\hat{a}_n \leq \hat{b}$; $(n \in N)$. Then $\hat{a}_n \vee \hat{b} = \hat{b}$ i.e. $(a_n \vee b, b) \in N(u)$. Since *u* satisfies (σ) and $a_n \vee b \uparrow a \vee b$, we have $(a_1 \vee b, a \vee b) \in N(u)$, hence $\hat{b} = \hat{a}_1 \vee \hat{b} = \hat{a} \vee \hat{b}$ and $\hat{a} \leq \hat{b}$. This shows that $\hat{a}_n \uparrow \hat{a}$. Similarly, $\hat{b}_n \uparrow \hat{b}$ implies $\hat{a}_n \downarrow \hat{a}$.

(d) ⟹(c) First observe that N is a sublattice of L^2 by Proposition (2.1.13). Let (a_n, b_n) be a sequence in *N* and $a, b \in L$ such that $(a_n, b_n) \uparrow (a, b)$. By (d), $\hat{a}_n \uparrow \hat{a}$ and $\hat{b}_n \uparrow \hat{b}$. But $(a_n, b_n) \in N$ means that $\hat{a}_n = \hat{b}_n$. Hence $\hat{a} = \hat{b}$, i.e. $(a, b) \in N$. Similarly, $N \ni (a_n, b_n) \downarrow (a, b)$ implies $(a, b) \in N$.

(c) ⟹ (b) The given filter is a lattice uniformity by Proposition (2.1.13). If (a_n) is a monotone sequence with order-limit a such that $(a_i, a_j) \in N$ for all $i, j \in N$, then (a_1, a_n) is a mono tone sequence in *N* with order-limit (a_1, a), hence $(a_1, a), \in N$.

Proposition (2.1.42)[37]: Let L be σ-compleie, N a σ -sublattice of L^2, which is a congruence relation with respect to $\vee$ and $\wedge$, and $N \subset N(u)$. Using the notation of Lemma (2.1.21) we have:

(a) $\hat{L}$ is σ -complete.

(b) If u satisfies (σ), then also u satisfies (σ),

Proof. (a) Let (a_n) be a sequence in L such that $(\hat{a}_n)$ is increasing. Put $s_n = \sup_{1\le i\le n} a_i$ ($n \in \mathbb{N}$) and a$= \sup_{n\in\mathbb{N}} s_n$. By Proposition (2.1.41) (c) $\Longrightarrow$ (d), $\hat{s}_n \uparrow \hat{a}$, hence $\hat{a}_n \uparrow \hat{a}$, since $\hat{s}_n = \sup_{1\le i\le n} \hat{a}_i = \hat{a}_n$ Analogously, every decreasing sequence in $\hat{L}$ has an infimum.

(b) Let $U \in u$. Choose for U a sequence (U_n) according to Definition (2.1.30). Let $a, a_n \in L$ for $n \in\in$ such that $(\hat{a}_i, \hat{a}_j) \in \widehat{U}_n$, for $i, i \ge n$ and $\hat{a}_n \uparrow \hat{a}$ or $\hat{a}_n \downarrow \hat{a}$. We may assume that $a_n \uparrow a$ or $a_n \downarrow a$ (s, the proof of (a)) and that U, U_n; are closed. Since U_n; are closed, $(\hat{a}_i, \hat{a}_j) \in \widehat{U}_n$; implies $(a_i, a_j) \in U_n$, see Proposition (2.1.15)(a). Hence $(a_1, a) \in U$ by the choice of (U_n) It follows that $(\hat{a}_1, \hat{a}) \in \widehat{U}$

Combining Corollary (2.1.26)(a) with Lemma (2.1.37), Theorem (2.1.38), Corollary (2.1.39) or Corollary (2.1.40) one can obtain other criteria for sequential completeness of (L, u). We carry this out in one case combining Corollary (2.1.26)(a) and Corollary (2.1.40).

Corollary (2.1.43)[37]: Let $(L, \le)$ be Dedekind σ-complete. Suppose that u satisfies (σ) and that every monotone Cauchy net $(x_\alpha)_{\alpha\,\in\, A}$ with $|A| \le c$; is convergent in *(L, u).* Then *(L, u)* is sequentially complete.

Proof. By Corollary (2.1.40), the order-intervals $[a, b]$ are sequentially u-complete for all $a, b \in L$. Therefore (L, u) is sequentially complete by Corollary (2.1.26)(a).

Corollary (2.1.43) generalizes [47]. Corollary (2.1.39) and Corollary (2.1.43) show in particular that in the assumptions of [47] one can replace the σ-Fatou property by the property (σ). For a topological Riesz space the property (σ) is strictly weaker than the σ -Fatou property, see example (2.1.48).

Theorem (2.1.44)[37]: If L is a s-complete lattice and u. satisfies (σ), then (L, u) is a Baire space.

Proof. By Proposition (2.1.31)(b) there are lattice uniformities u_α; with countable base satisfying (σ) such that $u = \sup_{\alpha\,\in\, A} u_\alpha$. For every countable subset B of A the uniformity $u_B = \sup_{\alpha\,\in\, B} u_\alpha$ has a countable base and satisfies (σ) by Proposition (2.1.31)(a). Therefore

(L, u_B) is complete by Theorem (2.1.33)(a) ⟹(c)(i). It follows by [53] that (L, u) is a Baire space.

Theorem (2.1.44) generalizes [53].

We now examine the meaning of the property (σ) for locally solid commutative l-groups and topological Boolean rings.

A Riesz pseudonorm $\| \|$ on a commutative l-group G is a function $\| \|: G \to [0, +\infty]$ such that $\|0\| = 0$, $\|x + y\| \leq \|x\| + \|Y\|$, $|x| \leq |y|$ Iimplies $\|x\| \leq \|Y\|$ for all $x, y \in G$; a Riesz pseudonorm $\| \|: G \to [0, +\infty]$ is σ-subaddiiioe if $x = \sum_{n=0}^{\infty} x_n = \sup_{n \in \mathbb{N}} \sum_{n=0}^{\infty} x_n$ implies $\|x\| = \sum_{n=1}^{\infty} \|x_n\|$ for all $x, x_n \in G^+ := \{z \in G: z \geq 0\}$, see [51].

Proposition (2.1.45)[37]: Let G be a commutative l-group.

(a) The uniformity induced by a Riesz pseudonorm $\| \|$ on G satisfies (σ) iff $\| \|$ is σ -subadditive.

(b) The following conditions for a locally solid group topology τ on G are equivalent:

(i) The uniformity induced by τ satisfies (σ).

For every 0-neighbourhoo U there is a sequence (U_n) of 0-neighbourhoods with the following property: If $a_n \in U_n$; for all $n \in N$ and $a \in G$ with $a = \sum_{n=1}^{\infty} a_n$, then $a \in U$.

(ii) τ is generated by a family of σ-subadditive Riesz pseudonorms.

Proof. (b) (iii) ⟹(ii) is obvious.

(ii)⟹(i) Let U be a symmetric 0-neighbourhood and $\widetilde{U} := \{(a, b) \in G^2: a - b \in U_n\}$. Choose $U_n)$; for U according to (ii) and put $\widetilde{U}_n := \{(a, b) \in G^2: a - b \in U\}$. It is easy to check that for every monotone sequence (a_n) in G order-converging to a $(a_i, a_j) \in \widetilde{U}_n, (i, j \in n)$ implies $(a, a_1) \in \widetilde{U}$.

(i) ⟹ (iii). With the method of the proof of Proposition (2.1.31)(b) one can easily see that τ is the supremum of a family $(\tau_\alpha)_{\alpha \in A}$ of locally solid group topologies, which satisfy (σ) and have a countable 0-neighbourhoodbase. Let $\| \|_\alpha^*$; be a group pseudonorm on G generating τ_α. Then $\|x\|_\alpha := \sup \{\|y\|_\alpha: |y| \leq |x|$ defines a Riesz pseudonorm generating τ_α.

Therefore τ is induced by the family $(\| \|_\alpha)_{\alpha \in A}$ As we will prove in (a)⟹, $\| \|_\alpha$ is σ -sub-additive for every $\alpha \in A$.

(a)⟹: Suppose that the uniformity induced by $\| \|$ satisfies (σ) and that $x, x_n \in G^+$ with $x = \sum_{n=1}^{\infty} x_n$. Put $x_0 = 0$ and $s_n = \sum_{n=0}^{\infty} x_n$ Applying Proposition (2.1.32)(a) ⟹ (d) with $d(a,b) := \|a - b\|$ and $q = Q = 1$ one obtains

$$\|x\| = d(x, d_0) \le \sum_{i=0}^{\infty} d(s_i\,, s_{i+1}) = \sum_{i=0}^{\infty} \|x_i\|$$

(a) ⟸follows from (b) (iii)⇒ (i).

Proposition (2.1.46)[37]: Let R be a Boolean ring.

(a) The uniformity induced by a submeasure η on R satisfies (σ) iff η is σ -subadditive.

(b) The following conditions for an FN-topology τ on R are equivalent:

(i) The uniformity induced by τ satisfies (σ).

(ii) For every 0-neighbourhood U there is a sequence (U_n) of 0-neighbour hoods with the following property. If $a_n \in U_n$ for all $n \in \mathbb{N}$ and $a \in R$ with $a = sup_{n \in \mathrm{N}}\, a_n$, then $a \in U$.

(iii) τ is generated by a family of σ -submeasures.

One can prove Proposition (2.1.46) similar to Proposition (2.1.45). The equivalence (ii) ⇔ (iii) of Proposition (2.1.46) (b) is also contained in [52].

In this section let u be a lattice uniformity on L.

Definition (2.1.47)[37]: u is said to satisfy property (F) if one of the following equivalent conditions holds:

(a) Every $U \in u$ contains $aV \in u$ with the following property: If $(x_\alpha,\ y_\alpha)_{\alpha \in A}$ is a net in V and $x,\ y \in L$ such that $x_\alpha \downarrow x$ and $y_\alpha \uparrow y$, then $(x,\ y) \in U$.

(b) Every $U \in u$ contains $aV \in u$ with the following property: If $(x_\alpha,\ y_\alpha)_{\alpha \in A}$ is a net in V and $x,\ y \in L$ such that $x_\alpha \le y_\alpha$ for all $\alpha \in A$, $x_\alpha \downarrow x$ and $y_\alpha \uparrow y$, then $(x,\ y) \in U$.

Proof of (b) ⟹ (a). Let $U_0 \in u$. By Proposition (2.1.3), U_0 contains a $U_1 \in u$ such that $(a, b) \in U_1$ implies $[a \wedge b, a \vee b]^2 \subset U_1$ Choose $U \in u$ with $U^3 \subset U_1$ further V for U according to (b) and a symmetric member V_0 of u such that $V_0 \vee \Delta \subset V, V_0 \wedge \Delta \subset V$ and $V_0 \subset U$. Suppose that $(x_\alpha,\ y_\alpha) \in V_0$ and $x_\alpha \downarrow x$ and $y_\alpha \uparrow y$. If $\beta \le \alpha$ then $(x_\beta,\ y_\alpha \vee x_\beta) = (x_\alpha,\ y_\alpha) \vee (x_\beta,\ y_\beta) \in V_0 \vee \Delta \subset V$ hence $(x_\beta,\ y \vee x_\beta) \in U$ by (b). Analogously one gets $(x \wedge$

$y_\beta,\ y_\beta) \in U$. Moreover $(y_\beta, x_\beta) \in V_0 \subset U$. It follows that $(x \wedge y_\beta, y \vee x_\beta) \in U^3 \subset U_1$ and $(x,\ y) \in U_1 \subset U_0$, since $x \wedge y_\beta \leq x,\ \ y \leq y \vee x_\beta$.

If τ is an FN-topology on a Boolean ring or a locally solid group topology on a com mutative l-group, then the uniformity induced by τ satisfies (F) iff every τ -neighbourhood *U* of 0 contains a τ-neighbourhood *V* of 0 with the following property: If *x* is the supremum of an increasing net in *V*, then $x \in U$; this is satisfied if there is a 0-neighbourhood base consisting of order closed sets. Therefore the uniformity induced by a Fatou topology (see [47]) on a Riesz space satisfies (F). The question, whether viceversa a topological Riesz space, the uniformity of which satisfies (F), has the Fatou property, is related to an unsolved question posed by D. FREMLIN asking whether any weakly Fatou norm (= Riesz norm such that $x_\alpha \uparrow x$ implies $\|x\| \leq 2 \cdot \sup \|x\|$) is equivalent to a Fatou norm (s. [47], [39] fpr the definition of a Fatou norm).

If (σF) denotes the property, which one obtaines replacing in Definition (2.1.47) nets by sequences, then (F) implies (σF) and (σF) implies (σ); on the other hand (σ) doesn't imply (σF) and (σF) doesn't imply (F).

Example (2.1.48)[37]: (cf. [53]). Let (A_n) be a disjoint sequence of infinite subsets of $\mathbb{N}$ and $\mathfrak{U}_n$ the ideal in $\mathrm{P}(\mathbb{N}))$ consisting of all subsets *A* of $\mathbb{N}$ such that $A\ \cap A_i$ is finite for all $i \in \mathbb{N}$ and $A\ \cap A_i = \emptyset$ for $i <\ n$ if $n \geq\ 2$; put $\mathfrak{U}_1 =\ \mathrm{P}(\mathbb{N}))$. Then $\eta(A) :=\ \inf\{1/n\colon\ n\ \in \mathbb{N}, A \in \mathfrak{U}_n\}$ defines a σ -submeasure on $\mathrm{P}(\mathbb{N})$ and

$$\|(x_n)\| :=\inf\{\varepsilon > 0\colon\ \eta(\{n \in \mathbb{N}\colon\ |x_n|\ \geq \varepsilon\}) \geq \varepsilon\}$$

a σ –subadditive Riesz pseudonorm (in the sense of [47]) on l_∞. (l_∞, $\|\ \|$ is a locally solid Riesz space, which has the property (σ), but not the σ -Eatou. property (s. [47]). More precisely, the η-uniformity on $\mathrm{P}(\mathbb{N}))$ and the $\|\ \|$-uniformity on l_∞; satisfy (σ), but not (σF), i.e. the η -topologyand the $\|\ \|$-topology don't have the following property: Every 0-neighbourhood *U* contains a 0-neighbourhood *V* such that $x_n \in V, x_n \uparrow x$ imply $x \in U$.

Now we prepare the proof of Theorem (2.1.52), the main theorem of this section.

Proposition (2.1.49)[37]: (a) the supremum of lattice uniformities on *L* satisfying (*F*) also satisfies (F).

(b) If *u* satisfies (*F*), then there is a net $(u_\alpha)_{\alpha \in A}$ of lattice uniformities on *L* with countable base satisfying (*F*) such that $u_\alpha \uparrow u$.

The proof of Proposition (2.1.49) is obvious. The following fact can be proved in the same way as Proposition (2.1.41).

Proposition (2.1.50)[37]: Let N be a congruence relation with respect to $\vee$ and $\wedge$ on L. Then the following conditions are equivalent.

(i) There is a lattice uniformity u on L satisfying (F) such that $N = N(u)$.

(ii) $\{U: N \subset U \subset L^2\}$ is a lattice uniformity on L satisfying (F).

(iii) N is an order-closed sublattice of L^2 (in the sense of [39].

(iv) With the notation of Proposition (2.1.14), $a \mapsto \hat{a}$ defines an order-continuous map on L to $\hat{L}$ (in the sense of [39]).

Lemma (2.1.51)[37]: Let v and v_α $(\alpha \in A)$ be uniformities on a set X such that $v_\alpha \uparrow v$.

(a) Then $\{N(v_\alpha): \alpha \in A\}$ is a base of a uniformity w on X, which is finer than v. If (X, v) is complete, then (X, w) is complete.

(b) If (X, w) is complete and (X, v_α) is complete for all $\alpha \in A$, then (X, v) is complete.

Proof. (a) The first statement holds obviously. Since w has a base of sets closed in $(X, V)^2$, the completeness of (X, v) implies that (X, w) is complete.

(b) Suppose that the spaces (X, w) and (X, v_α) are complete, $\alpha \in A$. Let $(x_\gamma)_{\gamma \in \Gamma}$ be a Cauchy net in (X, v). Since $v_\alpha \subset V$ for $\alpha \in A$, the net (x_γ) is also v_α-Cauchyand there fore converges in (X, v_α) to an element z_α. For $\alpha_1, \alpha_2, \alpha_3 \in A$ with exl , $\alpha_1, \alpha_2 \geq \alpha$ the net (x_γ) converges because of $v_\alpha \subset v_{\alpha_1}$ as well to z_{α_1} as to z_{α_2} with respect to v_α hence $(z_{\alpha_1}, z_{\alpha_2}) \in N(v_\alpha)$. It follows that (z_α) is w-Cauchy and therefore converges in (X, w) to an element z. Let $\alpha \in A$; since $N(v_\alpha)$ is w-closed, we have $(z_\alpha, z) = w - \lim_{\gamma \geq \alpha}(z_\alpha, z_\gamma) \in N(v_\alpha)$; therefore (x_γ) converges (X, v_α) also to z. This is true for all $\alpha \in A$. Hence z is the v -limit of (x_γ).

Theorem (2.1.52)[37]: Assume that u satisfies (F) and (L) and that there is a $U_0 \in u$ with the following property: If $(x_\alpha, y_\alpha)_{\alpha \in A}$is an order-bounded net in U_0 such that $x_\alpha \leq y_\alpha$ for all e $\alpha \in A, x_\alpha \downarrow$ and $y_\alpha \uparrow$, then there are elements $x, y \in L$ with $x_\alpha \downarrow x$ and $y_\alpha \uparrow y$, Then (L, u) is complete.

Proof. By Corollary (2.1.28)(a), it is enough to prove that for every $a, b \in L$ with $a < b$ the order interval $[a, b]$ is u-complete. Therefore we may assume-replacing $[a, b]$ by L-that L

has a least and a greatest element. In particular, in the assumption for U_0 one can replace order-bounded nets by arbitrary nets.

By Proposition (2.1.49), there is a net $(u_\alpha)_{\alpha\in A}$ of lattice uniformities on *L* with countable base satisfying (*F*) such that $u_\alpha \uparrow u$ and $U_0 \in u_\alpha$; for all $\alpha \in A$. Applying Lemma (2.1.51)(b), it is enough to show that (L, u_α) is complete for every $\alpha \in A$ and that (L, w) is complete, where w de notes the lattice uniformity with base $\{N(u_\alpha): \alpha \in A\}$.

Let $\alpha \in A$. Since $U_0 \in u_\alpha$; every monotone Cauchy net in (L, u_α) order-converges by the assumption for U_0: Satisfying (*F*), the uniformity u_α also satisfies (σ). Hence (L, u_α) is complete by Theorem (2.1.33) (a)$\Longrightarrow$ (c)(i).

Let $(x_\gamma)_{\gamma\in\Gamma}$ be a Cauchy net in (L, w). For fixed $\alpha \in A$, there is a $\gamma_0 \in \Gamma$ such that $(x_\nu, x_\mu) \in N(u_\alpha)$ for $\nu, \mu \geq \gamma_0$. Since $N(u_\alpha)$ is an order-closed sublattice of L^2 by Proposition (2.1.50) and $N(u_\alpha) \subset U_0$, there exists $x = \limsup x_\gamma$ and $(x_\nu, x) \in N(u_\alpha)$ for all $\nu \geq \gamma_0$. This proves that $x_\gamma \to x(w)$.

Corollary (2.1.53)[37]: (L, u) is complete, if *u* satisfies (*F*) and if one of the following three conditions holds:

(a) $(L, \leq)$ is Dedekind complete and (L, u) satisfies (*L*).

(b) $(L, \leq)$ is complete.

(c) $(L, \leq)$ is Dedekind complete and (L, u) satisfies (MCP):

Proof. In case (a), apply Theorem (2.1.52). Case (b) can be reduced to case (a). In case (c), apply Corollary (2.1.26)(b) and this corollary for case (b).

Corollary (2.1.53) generalizes Nakano's completeness theorem [47] and the Theorems 13.1 and 13.10 of [47]. It is of interest that Corollary (2.1.53) in case (b) also contains the well- known (more elementary) theorem that the order topology on a complete chain is compact. To see this one has to observe that the totally bounded lattice uniformity, which induces by Proposition (2.1.9)(i) the order topology on a chain, obviously satisfies (*F*) and that a totally bounded complete uniform space is compact.

Section (2.2): Group-Valued Measures on Orthoalgebras:

The initial impetus for the study of measures on non-Boolean orthostructures such as orthomodular lattices and orthomodular posets came from the logico-probabilistic foundations of quantum mechanics (see [55], [56], [57] and [58]). But the non-existence of a tensor product for orthomodular lattices or orthomodular posets (see [59])—necessary to describe coupled physical entities—has led to the introduction of orthoalgebras, a more general orthostructure a large class of which—called unital orthoalgebras—admits a tensor product (see [60]). Orthoalgebras are apparently one of the simplest and most natural orthostructures that can carry orthogonally additive measures, and today they provide a mathematical basis for the rapidly developing field of non-commutative measure theory (see [61], [62], [63]–[64], [65]–[66]).

In this Section we present a general decomposition theorem for a positive inner regular measure on an orthoalgebra L with values in an ordered Hausdorff topological group G not necessarily commutative. The generality of the context prevents of course the uniqueness of our decomposition. We show that it holds when L is a Boolean algebra, getting what we call the First Decomposition Theorem. Moreover, adding some natural hypotheses on G we can eliminate, in the Boolean case, the positivity restriction and give a unique decomposition of an order bounded inner regular measure on L with values in G. This Second Decomposition Theorem generalizes the first classical Aleksandrov Decomposition Theorem [67], and allows us, upon imposing a mild "topological" condition, to derive a Yosida–Hewitt decomposition of a G-valued order bounded inner regular finitely additive set function (see [68]).

The Section is organized as follows. After the introduction of some pertinent subsets of G^L, we establish our general decomposition theorem and we deduce the First Decomposition Theorem. We define the notion of topological lattice group satisfying the condition (M) following Jameson [69] and formulates the Second Decomposition Theorem. Finally, we present the Yosida–Hewitt decomposition of an order bounded inner regular finitely additive set function.

For the basic theory of orthoalgebras needed in this work, the reader is referred to [63], [65] and [66].

The terminology and notation on partially ordered sets, ordered groups and ordered topological groups collected of [70] will be used implicitly in the remainder of this Section. In order to complete the list of ordered topological groups given there, we indicate the non-commutative examples 4.6(4), 4.6(5) and 4.6(7) of [65].

The unexplained terminology and properties concerning lattice groups— called also l-groups—can be found in [71] or [72].

Throughout this section, $L = (L, \perp, \oplus, 0, 1)$ is an orthoalgebra and $G = (G, \cdot, e, \leq, \tau)$ is an ordered Hausdorff topological group.

A subset K of L is called a *δ-paving in L* if the following conditions are satisfied:

(a) $0 \in K$.

(b) Every finite subset of K has a supremum in $(L, \leq)$ which belongs to K.

(c) Every countable subset of K has an infimum in $(L, \leq)$ which belongs to K.

Examples (2.2.1)[54]: (a) Let Ω be a non-empty set, let $L = 2$ be the Boolean algebra of all subsets of Ω and let K be a δ-ring of subsets of Ω. Then K is a δ-paving in L.

(b) For a non-empty set X, let $(X, \mathcal{F})$ be a space in the sense of Aleksandrov [73] and let L be the Boolean subalgebra of 2^X generated by $\mathcal{F}$. Then $\mathcal{F}$ is a δ-paving in L.

(c) Let τ be the usual topology on $\mathbb{R}$, let $X = [0, 1]$ be a subspace of $(\mathbb{R}, \tau)$, let $L = 2^X$ and let K be the set of all Souslin sets in X. Then K is a δ-paving in L.

(d) Let (X, τ) be a Hausdorff topological space, let $L = 2^X$ and let K be the set of all compact subsets of X. Then K is a δ-paving in L.

(f) Let (X, τ) be a locally compact Hausdorff space, let $L = 2^X$ and let K be the set of all compact G_δ-sets in X. Then K is a δ-paving in L.

(e) Let H be an infinite-dimensional Hilbert space over $\mathbb{C}$, let A be a von Neumann algebra acting on H, let $P(A)$ be the orthomodular lattice of all projections in A, let L be the orthoalgebra determined by $P(A)$ and let $K = \{e \in L : e \text{ has a finite-dimensional range}\}$. Then K is a δ-paving in L.

For every $a \in L$, let $G_a = (G, \cdot, e)$ and let G^L be the direct product of the family $(G_a)_{a \in L}$. Then G^L ordered by the canonical order $\leq$ induced by $\leq$ is an ordered group.

An element μ of G^L is said to be a *measure on* L (with values in G) if $a, b \in G$ and $a \perp b$ imply $\mu(a \oplus b) = \mu(a) \cdot \mu(b)$. For example, if we denote by e the identity element of the group G^L, then e is a measure on L. Henceforth, the set of all measures on L (with values in G) will be denoted by $a(L, G)$.

Let μ be an element of $a(L, G)$. Then the following properties are immediate:

(i) $\mu(a) \cdot \mu(b) = \mu(b) \cdot \mu(a)$ *whenever* $a, b \in L$ *and* $a \perp b$.

(ii) $\mu(0) = e$.

(iii) *If* $a, b \in G$ *and* $a \leq b$, *then* $\mu(b - a) = \mu(b) \cdot \mu(a)^{-1} = \mu(a)^{-1} \cdot \mu(b)$.

(iv) *If* $n \in \mathbb{N} \backslash \{0, 1\}$ *and* $(a_i)_{0 \leq i \leq n}$ *is a finite orthogonal sequence in* $L \backslash \{0\}$, *then* $\mu(\bigoplus_{i=0}^{n} a_i) = \prod_{i=0}^{n} \mu(a_i)$.

Let μ be an element of G^L such that $\mu(0) = e$. We say that

(a) μ is *positive* if $\mu \geq e$.

(b) μ is *countably additive* if, for every orthogonal sequence $(a_i)_{i\in\mathbb{N}}$ in L such that $\bigoplus_{i\in\mathbb{N}} a_i$ exists in $(L, \leq)$, we have

$$\mu\left(\bigoplus_{i\in\mathbb{N}} a_i\right) = \tau - \lim_{n\to\infty} \prod_{i=0}^{n} \mu(a_i).$$

(c) μ is *inner regular* if there exists a δ-paving K in L with the following property: For every $c \in L$ and every $U \in \mathcal{N}(e)$ there exists $b \in K$ such that $b \leq c$ and $\mu(d) \in U$ whenever $d \in L$ and $d \leq c - b$. Sometimes, for more precision, we say that μ is K-*inner regular*.

(d) *Let μ be a positive element of $a(L, G)$. If $a, b \in L$ and $a \leq b$, then $\mu(a) \leq \mu(b)$.*

If K is a δ-paving in L, we denote by $\mathrm{ra}_K(L, G)$ the set of all K-inner regular measures on L (with values in G).

Lemma (2.2.2)[54]: *Let K be a δ-paving in L and let $\mu_1, \mu_2 \in \mathrm{ra}_K(L, G)$. Then*

(a) $\mu_1^{-1} \in \mathrm{ra}_K(L, G)$.

(b) $\mu_1 \cdot \mu_2 \in \mathrm{ra}_K(L, G)$.

Proof. The statement (a) is trivial. It is also clear that $\mu = \mu_1 \cdot \mu_2$ is an element of $a(L, G)$. Let $c \in L$ and let $U \in \mathcal{N}(e)$. We can choose $V \in \mathcal{N}(e)$ such that $V \cdot V \subseteq U$. Then there exist $b_1, b_2 \in K$ such that $b_i \in c$ and $\mu_i(d) \in V$ whenever $d \in L$ and $d \leq c - b_i$ for $i = 1, 2$. Since K is a δ-paving in L, it follows that $b_1 \vee b_2$ exists in $(L, \leq)$ and belongs to K. Put $b = b_1 \vee b_2$. Let $d \in L$ be such that $d \leq c - b$. Since $b_i \leq b \leq c$ $(i = 1, 2)$, it follows from [65] that $c - b \leq c - b_i$. Then $\mu(d) = \mu_1(d) \cdot \mu_2(d) \in V \cdot V \subseteq U$.

Examples (2.2.3)[54]: (i) Let $A = (A, +, \cdot, 0)$ be the ring with unity 1 of all 2×2 matrices with coefficients in $\mathbb{R}$ and let $*$ be the involution on A given by the transposition. Then $(A, *)$ is a Baer $*$-ring such that the set $P(A)$ of all projections in A consists of 0, 1 and all matrices of the form $\begin{pmatrix} x & y \\ y & 1-x \end{pmatrix}$ with $x, y \in \mathbb{R}$ and $x^2 + y^2 = x$. Since $P(A)$ is an orthomodular lattice (see [74] or [65]), it determines an orthoalgebra L. Let G be the usual ordered additive topological group of real numbers. For $f = \begin{pmatrix} x & y \\ y & 1-x \end{pmatrix}$ in L, put $\mu(f) = x$. Then μ is a positive element of $a(L, G)$.

(ii) Let X be a Polish space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X, let $L = B(X)$ be the σ-complete Boolean subalgebra of 2^X of Borel sets in X and let G be the additive lattice topological group derived from the Banach lattice $(c_0(\mathbb{N}), \leq, \|\cdot\|_1)$. For every $B \in L$, put $\mu(B) = \left(2^{-n}\delta_{x_n}(B)\right)_{n\in\mathbb{N}}$ where δ_{x_n} denotes the Dirac measure on L concentrated at x_n. Then μ is a positive countably additive element of $\mathrm{ra}_K(L, G)$, where K is the set of all compact subsets of X.

(iii) Let $(\mathbb{C}, V, \langle\cdot,\cdot\rangle)$ be the quadratic space of [65] and let $\mathcal{L}_S(V) = \{M : M \text{ is a linear subspace of } V \text{ and } V = M + M^\perp\}$. Then $(\mathcal{L}_S(V), \subseteq, {}^\perp, 0, V)$ is an orthomodular poset (see [75] or [65]). Let L be the orthoalgebra derived from $\mathcal{L}_S(V)$ and let G be the usual ordered additive topological group of real numbers. Let f be a fixed vector of V. Then, for every $M \in L$, there exists a unique decomposition $f = f_M + f_{M^\perp}$ with $f_M \in M$ and $f_{M^\perp} \in M^\perp$. For every $M \in L$, put $\mu(M) = \int_{\mathbb{R}} |f_M(x)|^2 dx$. Then μ is a positive element of $a(L, G)$ which is not countably additive (see [75]).

(iv) Let H be an infinite-dimensional Hilbert space over $\mathbb{C}$, let $A = B(H)$ be the von Neumann algebra acting on H of all bounded linear operators in H, let λ be a positive linear form on A, let L be the orthoalgebra determined by $P(A)$ and let G be the usual ordered multiplicative topological group of non-zero real numbers. For $f \in L$, put $\mu(f) = \exp\big(\lambda(f)\big)$. Then μ is a positive element of $a(L, G)$. Moreover, if λ is normal, then μ is countably additive and K-inner regular, where $K = \{f \in L : f \text{ has a finite-dimensional range}\}$ (see [76]).

(v) Let $L = 2^{\mathbb{R}}$ and let $I = \mathbb{R}_+$. For every $i \in I$, let G_i be the additive group of integers ordered by $P = \mathbb{N}$ and endowed with the discrete topology τ_i on $\mathbb{Z}$, let $[i]$ denote the integer part of i and let $J(i)$ be the order interval $\big[-[i], [i]\big]$ in $(G_i, \leq_{\mathbb{N}})$. Let G be the direct product of the groups G_i, let $\leq$ be the canonical order on G induced by $\leq_{\mathbb{N}}$ and let τ be the product topology of the topologies τ_i. Then $G = (\mathbb{Z}^I, +, 0, \leq, \tau)$ is an ordered Hausdorff topological group. For every $A \in L$, put $\mu(A) = \big(card\big(A \setminus J(i)\big)\big)_{i\in I}$. Then μ is a positive countably additive element of $a(L, G)$.

Let μ be a positive element of a(L, G) and let K be a δ-paving in L. We say that

(a) μ is K-*smooth* if, for every subset D of $K\backslash\{0\}$ such that $D \downarrow 0$ in $(K\backslash\{0\}, \leq)$, we have $\tau - \lim_{D} \mu(d) = e$.

(b) μ is K-*singular* if μ is K-inner regular and, for every positive K-smooth element γ of $\mathrm{ra}_K(L, G)$ such that $\gamma \leq \mu$, we have $\gamma = e$.

For example, e is the only positive element of $\mathrm{ra}_K(L, G)$ which is K-smooth and K-singular. The positive measure of Example (2.2.3)(b) is K-smooth. Also, if K is any δ-ring of subsets of $\mathbb{R}$, then the positive measure of Example (2.2.3)(f) is K-smooth.

It is clear that if K is a δ-paving in L and μ_1, μ_2 are two positive elements of $a(L, G)$ which are K-smooth, then $\mu_1 \cdot \mu_2$ is also K-smooth.

Lemma (2.2.4)[54]: *Assume that K is a δ-paving in L and G is locally order convex. Let γ_1 and γ_2 be two positive elements of $a(L, G)$ such that $\gamma_1 \leq \gamma_2$. If γ_2 is K-inner regular, then so is γ_1.*

Proof. Let $c \in L$ and let $U \in \mathcal{N}(e)$. Since G is locally order convex, it follows from [70] that there exists $V \in \mathcal{N}(e) =$ such that $e \leq x \leq y$ and $y \in V$ imply $x \in U$. By the K-inner regularity of γ_2, there exists $b \in K$ such that $b \leq c$ and $\gamma_2(d) \in V$ whenever $d \in L$ and $d \leq c - b$. Since γ_1 is positive and $\gamma_1 \leq \gamma_2$, it follows that $\gamma_1(d) \in U$ whenever $d \in L$ and $d \leq c - b$.

Lemma (2.2.5)[54]: *Assume that K is a δ-paving in L and G is locally order convex. Let γ_1 and γ_2 be two positive elements of $a(L, G)$ such that $\gamma_1 \leq \gamma_2$. If γ_2 is K-smooth, then so is γ_1.*

Proof. Let D be a subset of $K\backslash\{0\}$ such that $D \downarrow 0$ in $(K\backslash\{0\}, \leq)$ and let $U \in \mathcal{N}(e)$. Then there exists $V \in \mathcal{N}(e)$ such that $e \leq x \leq y$ and $y \in V$ imply $x \in U$. Since γ_2 is K-smooth, there exists $d_0 \in D$ such that $d \in D$ and $d \leq d_0$ imply $\gamma_2(d) \in V$. Then the positivity of γ_1 and the inequality $\gamma_1 \leq \gamma_2$ imply $\gamma_1(d) \in U$ whenever $d \in D$ and $d \leq d_0$. So $\tau - \lim_{D} \gamma_1(d) = e$.

Theorem (2.2.6)[54]: *Assume that L is a Boolean algebra and G is locally order convex. Let K be a δ-paving in L and let μ be a positive K-smooth element of $\mathrm{ra}_K(L, G)$. Then μ is countably additive.*

Proof. It suffices to show that $\tau - \lim_{n\to\infty} \mu(c_n) = e$ for every decreasing sequence $(c_n)_{n\in\mathbb{N}}$ of elements of L such that $\bigwedge_{n\in\mathbb{N}} c_n = 0$. Let $U \in \mathcal{N}(e)$. Choose $V \in \mathcal{N}(e)$ such that $V \cdot V \subseteq U$. Then there exists $W \in \mathcal{N}(e)$ such that $e \leq x \leq y$ and $y \in W$ imply $x \in V$. Moreover, there exists a sequence $(W_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{N}(e)$ such that $W_0 \cdot W_1, \dots, W_n \subseteq W$ for all $n \in \mathbb{N}$. Using the K-inner regularity of μ and the assumption that L is a Boolean algebra, we can construct inductively a sequence $(b_i)_{i\in\mathbb{N}}$ of elements of K such that $b_i \le c_i \wedge b_{i-1}$ and $\mu(c_i \wedge b_{i-1} \wedge b_i') \in W_i$ for all $i \in \mathbb{N}$, where $b - 1 = 1$. We may suppose that $b_i \neq 0$ for all $i \in \mathbb{N}$. Then $D = \{b_i : i \in \mathbb{N}\}$ is a downwards filtering subset of $K\backslash\{0\}$ such that $D \downarrow 0$ in $(K\backslash\{0\}, \le)$. By the K-smoothness of μ, we have $\tau - \lim\limits_{i\to\infty} \mu(b_i) = e$. Since $c_n - b_n = c_n \wedge b_n' = c_n \wedge (\vee_{i=0}^{n}(b_{i-1} \wedge b_i')) = \vee_{i=0}^{n}(c_n \wedge b_{i-1} \wedge b_i') \le \vee_{i=0}^{n}(c_i \wedge b_{i-1} \wedge b_i')$, $(c_i \wedge b_{i-1} \wedge b_i')_{i\in\mathbb{N}}$ is an orthogonal sequence in L and μ is a positive element of $a(L, G)$, it follows from properties (iii)–(v) that $e \le \mu(b_n)^{-1} \cdot \mu(c_n) \le \prod_{i=0}^{n} \mu(a_i \wedge b_{i-1} \wedge b_i') \in W_0 \cdot W_1, \dots, W_n \subseteq W$ for all $n \in \mathbb{N}$, and therefore $\mu(b_n)^{-1} \cdot \mu(c_n) \in V$ for all $n \in \mathbb{N}$.

Choose $n_0 \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n \ge n_0$ imply $\mu(b_n) \in V$. Then $\mu(c_n) = \mu(b_n) \cdot \left(\mu(b_n)^{-1} \cdot \mu(c_n)\right) \in V \cdot V \subseteq U$ whenever $n \ge n_0$.

Theorem (2.2.7)[54]: *Assume that K is a δ-paving in L and G is quasi-order complete, locally order convex and has the property* (oc). *Then, for every positive element μ of $\mathrm{ra}_K(L, G)$, there exist two positive elements ξ and η of $\mathrm{ra}_K(L, G)$ with the following properties*:

(a) $\mu = \xi \cdot \eta$.

(b) ξ *is K-smooth.*

(c) η *is K-singular.*

Proof. Set $\Gamma = \{\gamma \in ra_K(L, G) : \text{is positive}e, K\text{-smooth and } \gamma \le \mu\}$.

Since $e \in \Gamma, \Gamma$ is non-empty. Let Γ_0 be a totally ordered subset of $(\Gamma, \le)$.

Let $c \in L$. Then $D(c) = \{(c) : \gamma \in \Gamma_0\}$ is a majorized, upwards filtering subset of G. Since G is quasi-order complete, $\vee D(c)$ exists in G. Write $\gamma_0(c) = \vee D(c)$ for all $c \in L$. Clearly, γ_0 is a positive element of G^L such that $\gamma_0 \le \mu$ and, in particular, $\gamma_0(0) = e$. Since $(\Gamma_0, \ge)$ is a directed set, $\left(\gamma(c)\right)_{\gamma\in(\Gamma_0,\ge)}$ is an increasing net in G such that $\vee_{\gamma\in\Gamma_0} \gamma(c) = \gamma_0(c)$ for all $c \in L$. Since G has the property (oc), it follows from [70] that

$$\gamma_0(c) = \tau - \lim_{\Gamma_0} \gamma(c) \quad \text{for all } c \in L. \tag{1}$$

We show that γ_0 is a measure on L. Let $a, b \in L$ be such that $a \perp b$. By (1) we have

$$\gamma_0(a \oplus b) = \tau - \lim_{\Gamma_0} \gamma(a \oplus b) = \tau - \lim_{\Gamma_0} \gamma_0(a) \cdot \gamma(b)$$

$$= \tau - \lim_{\Gamma_0} \gamma(a) \cdot \tau - \lim_{\Gamma_0} \gamma(b) = \gamma_0(a) \cdot \gamma_0(b).$$

In particular, $\gamma_0(1) = \gamma_0(c \oplus c') = \gamma_0(c) \cdot \gamma_0(c')$ for all $c \in L$.

We show that

$$\gamma_0(c) = \tau - \lim_{\Gamma_0} \gamma(c) \quad \text{uniformly for } c \in L. \tag{2}$$

Let $U \in \mathcal{N}(e)$. Since G is locally order convex, there exists a symmetric order convex neighbourhood V of e contained in U. By (1), $\gamma_0(1) = \tau - \lim_{\Gamma_0} \gamma_0(1)$. So there exists $\gamma_1 \in \Gamma_0$ such that $\gamma \in \Gamma_0$ and $\gamma \geq \gamma_1$ imply $\gamma(1) \in V \cdot \gamma_0(1)$ and therefore $\gamma_0(1) \cdot \gamma(1)^{-1} \in V$. Let $c \in L$. Since $\gamma(c) \leq \gamma_0(c)$ and $\gamma(c') \leq \gamma_0(c')$ for all $\gamma \in \Gamma_0$, we have $e \leq \gamma_0(c) \cdot \gamma(c)^{-1} = \gamma_0(1) \cdot \gamma_0(c')^{-1} \cdot (c') \cdot \gamma(1)^{-1} \leq \gamma_0(1) \cdot \gamma(1)^{-1}$ for all $\gamma \in \Gamma_0$. Then $\gamma \geq \Gamma_0$ and $\gamma \geq \gamma_1$ imply $\gamma_0(c) \cdot \gamma(c)^{-1} \in V$ and therefore $\gamma(c) \in U \cdot \gamma_0(c)$ for all $c \in L$.

We show that γ_0 is K-inner regular. Let $c \in L$ and let $U \in \mathcal{N}(e)$. Choose $V \in \mathcal{N}(e)$ such that $V^{-1} \cdot V \subseteq U$. By (2) there exists $\gamma \in \gamma_0$ such that $\gamma(d) \in V \cdot \gamma_0(d)$ for all $d \in L$. Since γ is K-inner regular, there exists $b \in K$ such that $b \leq c$ and $\gamma(d) \in V$ whenever $d \in L$ and $d \leq c - b$. Then $\gamma_0(d) = (\gamma_0(d) \cdot \gamma(d)^{-1}) \cdot \gamma(d) \in V^{-1} \cdot V \subseteq U$ whenever $d \in L$ and $d \leq c - b$.

To show that γ_0 is an upper bound of Γ_0 in $(\Gamma, \leq)$, it remains to show that γ_0 is K-smooth. Let D be a downwards filtering subset of $K \backslash \{0\}$ such that $D \downarrow 0$ in $(K \backslash \{0\}, \leq)$. Let $U \in \mathcal{N}(e)$. Choose an element V of $\mathcal{N}(e)$ such that $V^{-1} \cdot V \subseteq U$. Then, by (2), there exists $\gamma \in \Gamma_0$ such that $\gamma(d) \in V \cdot \gamma_0(d)$ for all $d \in L$. Since γ is K-smooth, there exists $d_0 \in D$ such that $d \in D$ and $d \leq d_0$ imply $\gamma(d) \in V$. Then $\gamma_0(d) = (\gamma_0(d) \cdot \gamma(d)^{-1}) \cdot \gamma(d) \in V^{-1} \cdot V \subseteq U$ whenever $d \in L$ and $d \leq d_0$.

By the Zorn Lemma, the partially ordered set $(\Gamma, \leq)$ contains a maximal element ξ. Then ξ is a positive K-smooth element of $\mathrm{ra}_K(L, G)$ such that $\xi \leq \mu$. Let $\eta = \xi^{-1} \cdot \mu$. Clearly, η is a positive element of G^L and $\mu = \xi \cdot \eta$. By Lemma (2.2.2), η is an element of $\mathrm{ra}_K(L, G)$. It remains to show that η is K-singular. Let γ be a positive K-smooth element of $\mathrm{ra}_K(L, G)$ such that $\gamma \leq \eta$. Then $\xi \cdot \gamma \leq \mu$ and, by Lemma (2.2.2), $\xi \cdot \gamma$ is a positive element of $\mathrm{ra}_K(L, G)$ which is K-smooth. Then $\xi \cdot \gamma \in \Gamma$, and the maximality of ξ implies that $\xi \cdot \gamma = \xi$, and therefore $\gamma = e$.

Remark (2.2.8)[54]: In the same way we can show that $\eta_1 = \mu \cdot \xi^{-1}$ is a positive element of $\mathrm{ra}_K(L, G)$ which is K-singular. Since $\mu = \eta_1 \cdot \xi$, we get a “right” decomposition of μ.

Assume that $\mathrm{H} = (\mathrm{H}, \cdot, \mathrm{e}, \leq)$ is a lattice group. The following properties will be used in proving the subsequent decomposition theorems:

(i) *Let* $z \in H$ *be such that* $z = x \cdot y^{-1}$ *where* x *and* y are positive elements of H. Then $z^{+} \leq x$ and $z^{-} \leq y$.

(ii) Let $x, y \in H$. Then $(x \cdot y^{-1})^{+} \leq |x| \cdot |y|$ and $(x \cdot y^{-1})^{-} \leq |y| \cdot |x|$.

We denote by $\mathrm{ba}(L, G)$ the set of all order bounded elements of $a(L, G)$. For example, if μ is a positive element of $a(L, G)$, then $\mu \in \mathrm{ba}(L, G)$. Clearly, $\mathrm{ba}(L, G)$ is an ordered subgroup of $(a(L, G), \leq)$ and, moreover, if G is order complete, then $\bigvee_{a \in L} \mu(a)$ and $\bigwedge_{a \in L} \mu(a)$ exist in G for all $\mu \in \mathrm{ba}(L, G)$ (see [77]).

Lemma (2.2.9)[54]: Assume that L is a Boolean algebra and G is order complete. Then $(\mathrm{ba}(L, G), \cdot, e, \leq)$ is a lattice group.

Proof. Using [72] it suffices to show that, for every $\mu \in \mathrm{ba}(L, G)$, there exists $\mu \vee e$ in $\mathrm{ba}(L, G)$.

Define $v(a) = \bigvee\{\mu(b) : b \in L \text{ and } b \leq a\}$ for all $a \in L$. Then v is a positive element of G^{L}. We shall show that v is an element of $a(L, G)$. Let $c, d \in L$ be such that $c \perp d$. Using the assumption that L is a Boolean algebra, we can deduce the following property:

$$b \in L \text{ and } b \leq c \oplus d \iff \text{there exist } c_1, d_1 \in L \text{ such that } c_1 \leq c, d_1 \leq d \text{ and } b = c_1 \oplus d_1.$$

From this and the fact that $\mu \in a(L, G)$, we get

$$\begin{aligned} v(c \oplus d) &= \bigvee\{\mu(b) : b \in L \text{ and } b \leq c \oplus d\} \\ &= \bigvee\{\mu(c_1 \oplus d_1) : c_1, d_1 \in L, c_1 \leq c \text{ and } d_1 \leq d\} \\ &= \bigvee\{\mu(c_1) \cdot \mu(d_1) : c_1, d_1 \in L, c_1 \leq c \text{ and } d_1 \leq d\} \\ &= \bigvee\{\mu(c_1) : c_1 \in L \text{ and } c_1 \leq c\} \cdot \bigvee\{\mu(d_1) : d_1 \in L \text{ and } d_1 \leq d\} \\ &= v(c) \cdot v(d). \end{aligned}$$

It remains to show that $v = \mu \vee e$ in $\mathrm{ba}(L, G)$. Clearly, v is an upper bound of the set $\{\mu, e\}$ in $(\mathrm{ba}(L, G), \cdot, e, \leq)$. Let λ be an upper bound of the set $\{\mu, e\}$ in $(\mathrm{ba}(L, G), \cdot, e, \leq)$. Then $\lambda \geq e$ and $\lambda \geq \mu$. Let $a \in L$. Then, for all $b \in L$ such that $b \leq a$, we have, by property (v), $\lambda(a) \geq \lambda(b) \geq \mu(b)$, and therefore $\lambda(a) \geq v(a)$.

Corollary (2.2.10)[54]: Assume that L is a Boolean algebra and G is order complete. If $\mu \in \mathrm{ba}(L, G)$, then μ^{+} and μ^{-} are given respectively by the formulae:

(a) $\mu^+(a) = \bigvee\{\mu(b) : b \in L \text{ and } b \le a\}$

(b) $\mu^-(a) = (\bigwedge\{\mu(b) : b \in L \text{ and } b \le a\})^{-1}$ for all $a \in L$.

Proof. (a) follows from the proof of Lemma (2.2.9) and (b) follows from (a), [71] and [72].

First Decomposition Theorem. *Assume that L is a Boolean algebra, K is a δ-paving in L and G is order complete, locally order convex and has the property* (oc). *Then, for every positive element μ of $\mathrm{ra}_K(L,G)$, there exist exactly two positive elements ξ and η of $\mathrm{ra}_K(L,G)$ with the following properties*:

(a) $\mu = \xi \cdot \eta$.

(b) ξ *is K-smooth.*

(c) η *is K-singular.*

Proof. The existence of the decomposition of μ follows from Theorem (2.2.7).

To show the uniqueness, suppose that there exist four positive elements ξ_1, ξ_2, η_1 and η_2 of $\mathrm{ra}_K(L,G)$ such that ξ_1 and ξ_2 are K-smooth, η_1 and η_2 are K-singular, $\mu = \xi_1 \cdot \eta_1$ and $\mu = \xi_2 \cdot \eta_2$. Then $\xi_2^{-1} \cdot \xi_1 = \eta_2 \cdot \eta_1^{-1}$. From Lemma (2.2.9) it follows that $(\mathrm{ba}(L,G),\cdot,e,\le)$ is a lattice group. Then $\xi_2^{-1} \cdot \xi_1$ belongs to $\mathrm{ba}(L,G)$ and from (vi) it follows that $(\xi_2^{-1} \cdot \xi_1)^+ = (\eta_2 \cdot \eta_1^{-1})^+ \le \eta_2$ and $(\xi_2^{-1} \cdot \xi_1)^- = (\eta_2 \cdot \eta_1^{-1})^- \le \eta_1$. Hence, Lemma (2.2.4) implies that $(\xi_2^{-1} \cdot \xi_1)^+$ and $(\xi_2^{-1} \cdot \xi_1)^-$ are K-inner regular. Moreover, from (vii) it follows that $(\xi_2^{-1} \cdot \xi_1)^+ \le |\xi_2^{-1}| \cdot |\xi_1^{-1}| = \xi_2 \cdot \xi_1$ and $(\xi_2^{-1} \cdot \xi_1)^- \le |\xi_1^{-1}| \cdot |\xi_2^{-1}| = \xi_1 \cdot \xi_2$. Then, by Lemma (2.2.5), $(\xi_2^{-1} \cdot \xi_1)^+$ and $(\xi_2^{-1} \cdot \xi_1)^-$ are K-smooth. Since η_2 and η_1 are K-singular, it follows that $(\xi_2^{-1} \cdot \xi_1)^+ = e = (\xi_2^{-1} \cdot \xi_1)^-$. Then [71] implies that $\xi_2^{-1} \cdot \xi_1 = e$. Hence $\xi_1 = \xi_2$ and therefore $\eta_1 = \eta_2$.

The following result will be used in Section 3:

Lemma (2.2.11)[54]: *Assume that L is a Boolean algebra and G is order complete and locally order convex. Let K be a δ-paving in L. If η_1 and η_2 are two positive elements of $\mathrm{ra}_K(L,G)$ which are K-singular, then $\eta_1 \cdot \eta_2$ is also K-singular.*

Proof. Let γ be a positive K-smooth element of $\mathrm{ra}_K(L,G)$) such that $\gamma \le \eta_1 \cdot \eta_2$. By Lemma (2.2.9) it follows that $(\mathrm{ba}(L,G),\cdot,e,\le)$ is a lattice group. Applying [71] to this group, we can write $\gamma = \gamma_1 \cdot \gamma_2$ where γ_i is a positive element of $a(L,G)$ such that $\gamma_i \le \eta_i$ for $i = 1,2$. Since G is locally order convex, it follows from Lemma (2.2.4) that γ_i is K-inner regular $(i = 1,2)$.

But $\gamma_i \leq \gamma$. Then Lemma (2.2.5) implies that γ_i is K-smooth for $i = 1, 2$. Since η_1 and η_2 are K-singular, we conclude that $\gamma_1 = \gamma_2 = e$, and therefore $\gamma = e$.

In this section we assume that L is a Boolean algebra and K is a δ-paving in L.

If $G = (G, +, 0, \leq, \tau)$ is an ordered Hausdorff topological group, we write $\mathrm{rba}_K(L, G) = \mathrm{ba}(L, G) \cap \mathrm{ra}_K(L, G)$.

Before we formulate the following result appearing in [76], we recall that, by the Iwasawa Theorem (see [72]), every order complete lattice group is commutative:

Lemma (2.2.12)[54]: *Let $G = (G, +, 0, \leq, \tau)$ be a Hausdorff commutative topological lattice group* which is order complete and has the property (oc). *If V is a sublattice of $(G, \leq)$ and D is a non-empty majorized subset of V, then $x = \bigvee D$ exists in G and $x \in \bar{V}$.*

Proof. The existence of $\bigvee D$ in G follows from the order completeness of G. Let $\mathcal{F}(D)$ denote the set of all finite subsets of D. Since $x = \bigvee D = \bigvee\{\bigvee F : F \in \mathcal{F}(D)\}$ and $(\bigvee F)_{F\in(\mathcal{F}(D),\supseteq)}$ is an increasing net in G, the property (oc) implies that $x = \tau - \lim_{\mathcal{F}(D)} \bigvee F$. But V is a sublattice of $(G, \leq)$ and $D \subseteq V$. Then $\bigvee F \in V$ for all $F \in \mathcal{F}(D)$. Hence $x \in V$.

Following Jameson [69], a topological lattice group $(G, +, 0, \leq, \tau)$ satisfies the *condition* (M) if there exists a base for the neighbourhood system $\mathcal{N}(e)$ consisting of sublattices of $(G, \leq)$. For example, let $(V, \leq)$ be a Riesz space with an Archimedean order unit u and let τ denote the metric topology on V induced by $(x, y) \to \varrho_u(x - y)$. Then the additive topological lattice group $(V, +, 0, \leq, \tau)$ satisfies the condition (M).

The following result appears also in [76]:

Proposition (2.2.13)[54]: *Let $G = (G, +, 0, \leq, \tau)$* be a Hausdorff commutative topological lattice group which is order complete, has the property (oc) and satisfies the condition (M). *If $\mu \in \mathrm{rba}_K(L, G)$, then μ^+* and μ^- are K-inner regular.

Proof. Let $c \in L$ and let $U \in \mathcal{N}(0)$. Since *G* satisfies the condition (M), there exists $V \in \mathcal{N}(0)$ such that $\bar{V} \subseteq U$ and V is a sublattice of $(G, \leq)$. From the K-inner regularity of μ, it follows that there exists $b \in K$ such that $b \leq c$ and $\mu(d) \in V$ whenever $d \in L$ and $d \leq c - b$. In particular, $\{\mu(a) : a \in L \text{ and } a \leq d\} \subseteq V$ for all $d \in D$ and $d \leq c - b$.

Since $\mu \in \mathrm{ba}(L, G)$, it follows from Corollary (2.2.10) and Lemma (2.2.12) that $\mu^+(d) = \bigvee\{\mu(a) : a \in L \text{ and } a \in d\} \in \bar{V} \subseteq U$ whenever $d \in L$ and $d \leq c - b$.

The K-inner regularity of μ^- follows from the above result, Lemma (2.2.2) and [71].

Let $G = (G, +, e, \leq, \tau)$ be an ordered Hausdorff topological group which is order complete and let $\mu \in \mathrm{ba}(L, G)$. We say that

(a) μ is K-*smooth* if μ^+ and μ^- are K-smooth.

(b) μ is K-*singular* if μ is K-inner regular and, for every pair γ_1 and γ_2 of positive K-smooth elements of $\mathrm{ra}_K(L, G)$ such that $\gamma_1 \leq \mu^+$ and $\gamma_2 \leq \mu^-$, we have $\gamma_1 = \gamma_2 = e$.

Clearly, if μ is K-smooth, then so is $|\mu|$.

Lemma (2.2.14)[54]: *Let $G = (G, +, 0, \leq, \tau)$ be a Hausdorff commutative topological lattice group which is order complete, has the property* (oc) *and satisfies the condition* (M). *If $\mu \in \mathrm{rba}_K(L, G)$ is K-smooth and K-singular, then $\mu = 0$.*

Proof. This follows immediately from the definitions and Proposition (2.2.13).

The following result generalizes [67]:

Theorem (2.2.15)[54]: *Let $G = (G, +, 0, \leq, \tau)$ be a Hausdorff commutative topological lattice group which is order complete, locally order convex, has the property* (oc) *and satisfies the condition* (M), *and let μ be an element of $\mathrm{rba}_K(L, G)$. If μ is K-smooth, then μ is countably additive.*

Proof. By Lemma (2.2.9) and [71] we can write $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive elements of $a(L, G)$. Moreover, Proposition (2.2.13) implies that μ^+ and μ^- are K-inner regular. Since μ is K-smooth, it follows that so are μ^+ and μ^-. Then, by Theorem (2.2.6), μ^+ and μ^- are countably additive. Therefore μ is countably additive.

Second Decomposition Theorem. Let $G = (G, +, 0, \leq, \tau)$ be a Hausdorff commutative topologica*l* lattice group which is order complete, locally order convex, has the property (oc) and satisfies the condition (M). Then, for every element μ *of* $\mathrm{rba}_K(L, G)$, there exist exactly two elements ξ and η of $\mathrm{rba}_K(L, G)$ with the following properties:

(a) $\mu = \xi + \eta$.

(b) ξ *is K-smooth.*

(c) η *is K-singular.*

Proof. We first show the existence of the decomposition.

From Lemma (2.2.9), [71] and Proposition (2.2.13), we can write

$\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive elements of $\mathrm{ra}_K(L,G)$. By the First Decomposition Theorem there exist four positive elements ξ_1, ξ_2, η_1 and η_2 of $\mathrm{ra}_K(L,G)$ such that ξ_1 and ξ_2 are K-smooth, η_1 and η_2 are K-singular, $\mu^+ = \xi_1 + \eta_1$ and $\mu^- = \xi_2 + \eta_2$. Since G is commutative, we have $\mu = (\xi_1 - \xi_2) + (\eta_1 - \eta_2)$. Clearly, $\xi_1 - \xi_2, \eta_1 - \eta_2 \in \mathrm{ba}(L,G)$, and it follows from Lemma (2.2.2) that $\xi_1 - \xi_2$ and $\eta_1 - \eta_2$ are K-inner regular. So $\xi_1 - \xi_2, \eta_1 - \eta_2 \in \mathrm{rba}_K(L,G)$. On the other hand, from property (vi) and Lemma (2.2.5) it follows that $\xi_1 - \xi_2$ is K-smooth. Finally, to show that $\eta_1 - \eta_2$ is K-singular, let γ_1 and γ_2 be two positive K-smooth elements of $\mathrm{raK}(L,G)$ such that $\gamma_1 \le (\eta_1 - \eta_2)^+$ and $\gamma_2 \le (\eta_1 - \eta_2)^-$. By (vii) we have $(\eta_1 - \eta_2)^+ \le \eta_1$ and $(\eta_1 - \eta_2)^- \le \eta_2$. Since η_1 and η_2 are K-singular, we get $\gamma_1 = 0$ and $\gamma_2 = 0$.

Now we show the uniqueness of the decomposition.

Suppose that there exist $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathrm{rba}_K(L,G)$ such that ξ_1 and ξ_2 are K-smooth, η_1 and η_2 are K-singular and $\mu = \xi_1 + \eta_1 = \xi_2 + \eta_2$. Then $\xi_1 - \xi_2 = \eta_2 - \eta_1$, and for $i = 1,2$, $|\xi_i|$ and $|\eta_i|$ are positive elements of $\mathrm{raK}(L,G)$ such that $|\xi_i|$ is K-smooth. Since, by (vii), $(\xi_1 - \xi_2)^+, (\xi_1 - \xi_2)^- \le |\xi_1| + |\xi_2|$, it follows from Lemma (2.2.5) that $\xi_1 - \xi_2$ is K-smooth. On the other hand, since ξ_i is K-singular, Proposition (2.2.13) implies that η_i^+ and η_i^- are K-singular, and therefore, by Lemma (2.2.11), it follows that $|\eta_i|$ is K-singular for $i = 1,2$. Again, by Lemma (2.2.11), $|\eta_1| + |\eta_2|$ is K-singular. We show that $\eta_2 - \eta_1$ is K-singular. Let γ_1 and γ_2 be two positive K-smooth elements of $\mathrm{raK}(L,G)$ such that $\gamma_1 \le (\eta_2 - \eta_1)^+$ and $\gamma_2 \le (\eta_2 - \eta_1)^-$. Then, by formula (vii), we have $\gamma_1, \gamma_2 \le |\eta_1| + |\eta_2|$ and therefore $\gamma_1 = 0$ and $\gamma_2 = 0$. So $\xi_1 - \xi_2$ is a K-smooth and K-singular element of $\mathrm{rba}_K(L,G)$. By Lemma (2.2.14), $\xi_1 - \xi_2 = 0$ and therefore $\xi_1 = \xi_2$. This implies that $\eta_1 = \eta_2$.

Remark (2.2.16)[54]: Using the results established in [70] it is easy to verify that the Second Decomposition Theorem generalizes the classical first Aleksandrov Decomposition Theorem [67].

Let X be a non-empty set, let 2^X be the Boolean algebra of all subsets of X and let $\mathcal{F}$ be a δ-paving in 2^X containing X. Then the pair $(X, \mathcal{F})$ is called, following Aleksandrov [73], a *space*. The elements of $\mathcal{F}$ are called *closed sets* in X and their complements are called *open sets* in X. For example, if (X, τ) is a topological space and $\mathcal{F} = \{F \subseteq X : F = f^{-1}(0)$ for some continuousfunction f from X into $\mathbb{R}\}$, then $(X, \mathcal{F})$ is a space.

A space $(X,\mathcal{F})$ is said to be *Lindelöf* if every covering of X by open sets in X contains a countable subcovering. For example, if $(X,\mathcal{F})$ is a Souslin space and $\mathcal{F}=\{F\subseteq X : X\backslash F\in\tau\}$, then $(X,\mathcal{F})$ is a Lindelöf space.

We note that in [67] it is shown that in every normal and non-bicompact space $(X,\mathcal{F})$ there exists an element $\mu\in \mathrm{rba}_{\mathcal{F}}(L,\mathbb{R})$ which is not $\mathcal{F}$-smooth, where L is the Boolean subalgebra of 2^X generated by $\mathcal{F}$. Then, considering this element μ, the Second Decomposition Theorem gives a non-trivial decomposition for μ.

Let L be an orthoalgebra, let $G=(G,\cdot,e,\tau)$ be a Hausdorff topological group and let μ be an element of G^L such that $\mu(0)=e$. We say that μ is *s-bounded* if, for every orthogonal sequence $(a_i)_{i\in\mathbb{N}}$ of elements of L, we have $\tau-\lim\limits_{i\to\infty}\mu(a_i)=e$. For example, the measure given by Example (2.2.3)(3) is s-bounded. We will denote by $\mathrm{sa}(L,G)$ (resp. $\mathrm{ca}(L,G)$) the set of all s-bounded (resp. countably additive) measures on L (with values in G). Clearly, if $\mu_1,\mu_2\in\mathrm{sa}(L,G)$, then $\mu_1^{-1},\mu_1\cdot\mu_2\in\mathrm{sa}(L,G)$.

Lemma (2.2.17)[54]: *Let L be an orthoalgebra and let $G=(G,\cdot,e,\leq,\tau)$ be an ordered Hausdorff topological group which is quasi-order complete and has the property* (oc). *If μ is a positive element of $a(L,G)$, then μ is s-bounded.*

Proof. Let $(a_i)_{i\in\mathbb{N}}$ be an orthogonal sequence in L. For every $n\in\mathbb{N}$, put $x_n=\prod_{i=0}^{n}\mu(a_i)$. Since μ is positive, it follows that $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in G. Moreover, the set $\{x_n : n\in\mathbb{N}\}$ is an upwards filtering subset of G majorized by $\mu(1)$. Since G is quasi-order complete, $x=\bigvee_{n\in\mathbb{N}}x_n$ exists in G. Then, by the property (oc), we get $x=\tau-\lim\limits_{n\to\infty}x_n$. Since $\mu(a_n)=x_n\cdot x_{n-1}^{-1}$ with $x_{-1}=e$, it follows that $\tau-\lim\limits_{n\to\infty}\mu(a_n)=x\cdot x^{-1}=e$.

Corollary (2.2.18)[54]: *Let L be a Boolean algebra and let $G=(G,\cdot,e,\leq,\tau)$ be an ordered Hausdorff topological group which is order complete and has the property* (oc). *Then every element μ of* $\mathrm{ba}(L,G)$ *is s-bounded.*

Proof. This follows immediately from Lemmas (2.2.9) and (2.2.17) and [71].

Lemma (2.2.19)[54]: Let L be a Boolean algebra and let $G=(G,\cdot,e,\leq,\tau)$ be a Hausdorff commutative topological lattice group which is order complete, has the property (oc) and satisfies the condition (M). If $\mu\in\mathrm{ba}(L,G)\cap\mathrm{ca}(L,G)$*, then* μ^+ and μ^- are countably additive.

Proof. It suffices to show the countable additivity of μ^+. Let $(c_n)_{n\in\mathbb{N}}$ be a decreasing sequence of elements of L such that $\bigwedge_{n\in\mathbb{N}}c_n=0$. It remains to show that $\tau-\lim\limits_{n\to\infty}\mu^+(c_n)=0$. Let $U\in$

$\mathcal{N}(0)$. Since G satisfies the condition (M), there exists $V \in \mathcal{N}(0)$ such that $V \subseteq U$ and V is a sublattice of $(G, \leq)$. By [78], there exists $n_0 \in \mathbb{N}$ such that $a \in L$ and $a \leq c_{n_0}$ imply $\mu(a) \in V$. So $\{\mu(a) : a \in L \text{ and } a \leq c_{n_0}\} \subseteq V$. Let $n \in \mathbb{N}$ be such that $n \geq n_0$. Since $\{\mu(a) : a \in L \text{ and } a \leq c_n\} \subseteq \{\mu(a) : a \in L \text{ and } a \leq c_{n_0}\}$ and $\mu \in \text{ba}(L, G)$, it follows from Lemma (2.2.12) that $\mu^+(c_n) = \bigvee\{\mu(a) : a \in L \text{ and } a \leq c_n\} \in \bar{V} \subseteq U$, and therefore $\tau - \lim_{n\to\infty} \mu^+(c_n) = 0$.

Lemma (2.2.20)[54]: Let L be a Boolean algebra, let $G = (G, \cdot, e, \leq, \tau)$ be an ordered Hausdorff topological group which is locally order convex and let μ be a positive element of *ca*(L, G). If γ is a positive element of $a(L, G)$ such that $\gamma \leq \mu$, then γ is countably additive.

Proof. Let $(c_n)_{n\in\mathbb{N}}$ be a decreasing sequence of elements of L such that $\bigwedge_{n\in\mathbb{N}} c_n = 0$. It suffices to show that $\tau - \lim_{n\to\infty} (c_n) = e$. Let $U \in \mathcal{N}(e)$. Since G is locally order convex, there exists $V \in \mathcal{N}(e)$ such that $e \leq x \leq y$ and $y \in V$ imply $x \in U$. But μ is countably additive. Then $\tau - \lim_{n\to\infty} (c_n) = e$. So there exists $n_0 \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n \geq n_0$ imply $\mu(c_n) \in V$. Hence $\gamma(c_n) \in U$ whenever $n \geq n_0$.

Let L be an orthoalgebra, let $G = (G, \cdot, e, \leq, \tau)$ be an ordered Hausdorff topological group and let μ be a positive element of $a(L, G)$.We say that μ is *purely finitely additive* if, for every positive element of ca(L, G) such that $\gamma \leq \mu$, we have $\gamma = e$. For example, let H be a separable infinite-dimensional Hilbert space over $\mathbb{C}$, let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal base of H, let L be the orthoalgebra determined by $P\big(B(H)\big)$ and let LIM denote the Banach limit on $\ell^\infty(\mathbb{N})$. For every $f \in L$, put $\mu(f) = \text{LIM}((\|f\varphi_n\|)_{n\in\mathbb{N}})$. Then it is easy to see that μ is a positive element of $a(L, \mathbb{R})$ which is purely finitely additive.

Now assume that L is a Boolean algebra and G is order complete. Then an element μ of ba(L, G) is said to be *purely finitely additive* if μ^+ and μ^- are purely finitely additive. For example, let $L = 2^{\mathbb{Z}}$, let $M^+(\mathbb{Z}) = \{\mu \in a(L, \mathbb{R}) : \mu \text{ takes only the values } 0 \text{ and } 1\}$ and let $M(\mathbb{Z}) = \{\mu = \mu_1 - \mu_2 : \mu_1, \mu_2 \in M^+(\mathbb{Z})\}$. We note that an element $\mu \in M^+(\mathbb{Z}) \backslash \{0\}$ is countably additive if and only if there exists $k \in \mathbb{Z}$ such that $\mu(\{k\}) = 1$. Let $\mu = \mu_1 - \mu_2$ be an element of $M(\mathbb{Z})$ such that $\mu_1(\{k\}) = \mu_2(\{k\}) = 0$ for all $k \in \mathbb{Z}$. Then μ is purely finitely additive.

Lemma (2.2.21)[54]: Let L *be a Boolean algebra*, *let* $G = (G,\cdot,e,\leq,\tau)$ be an ordered Hausdorff topological group which is order complete and locally order convex and let μ_1 and μ_2 be two positive elements of $a(L,G)$. If μ_1 and μ_2 are purely finitely additive, then so is $\mu_1 \cdot \mu_2$.

Proof. Let γ be a positive element of $\mathrm{ca}(L,G)$ such that $\gamma \leq \mu_1 \cdot \mu_2$. Consider the lattice group $(\mathrm{ba}(L,G),\cdot,e,\leq)$. Then, by [71] we can write $\gamma = \gamma_1 \cdot \gamma_2$ where γ_1 and γ_2 are positive elements of $a(L,G)$ such that $\gamma_1 \leq \mu_1$ and $\gamma_2 \leq \mu_2$. Since $\gamma_1, \gamma_2 \leq \gamma$, it follows from Lemma (2.2.20) that γ_1 and γ_2 are countably additive. But μ_1 and μ_2 are purely finitely additive. Then $\gamma_1 = e$ and $\gamma_2 = e$, and therefore $\gamma = e$.

Corollary (2.2.22)[54]: Let L be a Boolean algebra, let $G = (G,\cdot,e,\leq,\tau)$ be an ordered Hausdorff topological group which is order complete and locally order convex and let μ be an element of $\mathrm{ba}(L,G)$. If μ is purely finitely additive, then so is $|\mu|$.

Proof. This follows immediately from the definition, [71] and Lemma (2.2.21).

We also need the following corrected version of [70]:

Lemma (2.2.23)[54]: *Let* $(X,\mathcal{F})$ *be a Lindelöf space*, *let* L *be a Boolean subalgebra of* 2^X *containing* $\mathcal{F}$ *and let* $G = (G,\cdot,e,\leq,\tau)$ *be a Hausdorff topological group. If* $\mu \in \mathrm{sa}(L,G)\setminus \mathrm{ca}(L,G)$ *then* μ *is* $\mathcal{F}$*-smooth.*

Theorem (2.2.24)[54]: Let $(X,\mathcal{F})$ be a Lindelöf space, let L be a Boolean subalgebra of 2^X containing $\mathcal{F}$, and let $G = (G,+,0,\leq,\tau)$ be a Hausdorff commutative topological lattice group which is order complete, locally order convex, has the property (oc) and satisfies the condition (M). Then for every $\mu \in \mathrm{rba}_{\mathcal{F}}(L,G)$, there exist exactly two elements ξ and η *of* $\mathrm{rba}_{\mathcal{F}}(L,G)$ with the following properties:

(a) $\mu = \xi + \eta$.

(b) ξ *is countably additive.*

(c) η *is purely finitely additive.*

Proof. We first show the existence of the decomposition.

By the Second Decomposition Theorem there exist $\xi, \xi \in \mathrm{rba}_{\mathcal{F}}(L,G)$ such that ξ is $\mathcal{F}$-smooth, η is $\mathcal{F}$-singular and $\mu = \xi + \eta$. By Theorem (2.2.15), ξ is countably additive. It remains to show that η is purely finitely additive. Let γ_1 and γ_2 be two positive elements of $\mathrm{ca}(L,G)$ such that $\gamma_1 \leq \gamma^+$ and $\gamma_2 \leq \gamma^-$. Since $\eta \in \mathrm{rba}_{\mathcal{F}}(L,G)$, Proposition (2.2.13) shows that η^+ and η^- are $\mathcal{F}$-inner regular. Then, by Lemma (2.2.4), γ_1 and γ_2 are $\mathcal{F}$-inner regular. On the other hand, Lemma (2.2.17) implies that γ_1 and γ_2 are s-bounded. Since $(X, \mathcal{F})$ is a Lindelöf space and L is a Boolean subalgebra of 2^X containing $\mathcal{F}$, it follows from Lemma (2.2.23) that γ_1 and γ_2 are $\mathcal{F}$-smooth. But η is $\mathcal{F}$-singular. Hence $\gamma_1 = 0$ and $\gamma_2 = 0$. This implies that η is purely finitely additive.

Now we show the uniqueness of the decomposition.

Suppose that there exist $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathrm{rba}_{\mathcal{F}}(L, G)$ such that ξ_1 and ξ_2 are countably additive, η_1 and $\eta_2$are purely finitely additive and $\mu = \xi_1 + \eta_1 = \xi_2 + \eta_2$. Then $\xi_1 - \xi_2 = \eta_2 - \eta_1$. Since $\xi_1 - \xi_2 \in \mathrm{ba}(L, G) \backslash \mathrm{ca}(L, G)$, Lemma (2.2.19) implies that $(\xi_1 - \xi_2)^+$ and $(\xi_1 - \xi_2)^-$ are countably additive. By (vii) we have $(\xi_1 - \xi_2)^+ = (\eta_2 - \eta_1)^+ \leq |\eta_1| + |\eta_2|$ and $(\xi_1 - \xi_2)^- = (\eta_2 - \eta_1)^- \leq |\eta_1| + |\eta_2|$. Since η_1 and η_2 are purely finitely additive, Corollary (2.2.22) and Lemma (2.2.21) imply that $|\eta_1| + |\eta_2|$ is purely finitely additive as well. Then $(\xi_1 - \xi_2)^+ = 0$ and $(\xi_1 - \xi_2)^- = 0$ and therefore $\xi_1 = \xi_2$. Hence $\eta_1 = \eta_2$.