CHAPTER THREE

CONTROLLER IMPLEMENTATION OF INVERTED PENDULUM SYSTEM

3.1 Inverted Pendulum System

An inverted pendulum is a pendulum that has its center of mass above its pivot point. It is often implemented with the pivot point mounted on a cart that can move horizontally and may be called a cart and pole. Most applications limit the pendulum to one degree of freedom by affixing the pole to an axis of rotation. Whereas a normal pendulum is stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright; this can be done either by applying a torque at the pivot point, by moving the pivot point horizontally as part of a feedback system, changing the rate of rotation of a mass mounted on the pendulum on an axis parallel to the pivot axis and thereby generating a net torque on the pendulum, or by oscillating the pivot point vertically. A simple demonstration of moving the pivot point in a feedback system is achieved by balancing an upturned broomstick on the end of one's finger.

The inverted pendulum is a classic problem in dynamics and control theory and is widely used as a benchmark for testing control algorithms (PID controllers, state space representation, neural networks, fuzzy control, genetic algorithms, etc.). Variations on this problem include multiple links, allowing the motion of the cart to be commanded while maintaining the pendulum, and balancing the cart-pendulum system on a see-saw. The inverted pendulum is related to rocket or missile guidance, where the center of gravity is located behind the center of drag causing aerodynamic instability.

3.2 Inverted Pendulum Classification

There are many series of inverted pendulum systems extended from linear one-stage inverted pendulum, such as linear inverted pendulum, circular inverted pendulum, planar inverted pendulum and configurable inverted pendulum. Inverted pendulum is the system with pendulum plants placed on motion modules. Diverse pendulum plants and motion modules constitute different inverted pendulum series. The following inverted pendulum systems are classified by structure:

3.2.1 Linear inverted pendulum

Linear inverted pendulum has pendulum plant on a linear motion module with one degree of freedom. The cart moves on the sliding shaft horizontally. There are different kinds of linear inverted pendulum systems based on different pendulum plant structure such as the flexible inverted pendulum, which has two carts on the sliding shaft with a spring connected.

3.2.2 Circular inverted pendulum

Circular inverted pendulum system has a pendulum plant on a circular motion module with one degree of freedom. The pendulum is on the arm end rotates around the centre of the circle. Different inverted pendulum system can be setup by varying the stage number in series or parallel.

3.2.3 Planar inverted pendulum

Planar inverted pendulum system has a pendulum plant on the planar motion module with two degree of freedom. There are two classes of planar motion module: XY table and robotic arm. The pendulum can be also divided by stage like one stage or two stages.

3.2.4 Configurable inverted pendulum

Configurable inverted pendulum is a new class of inverted pendulum systems whose pendulum plant is composed of pendulum rod and connection rod. The connection rod can be configured to three modes: level, vertical upper and vertical down. Classified by pendulum stages, there are one-stage, two-stage, three-stage and four-stage inverted pendulum systems. One-stage inverted pendulum is used for basic experiment of control theory while others are mostly used for development of advanced control algorithms. Control complexity increase dramatically as the pendulum stage increase. The feasible maximal stage for inverted pendulum system is four currently [7].

3.3 Inverted Pendulum Properties

Despite the different size and structure, all inverted pendulum systems have the following properties in common:

1) Nonlinearity

Inverted pendulum is a typical nonlinear system. In real control, the system model is usually linearized. Also there is nonlinear control methods applied to inverted pendulum which is becoming a hot topic recently.

2) Uncertainty

Most uncertainties come from model uncertainty, mechanical transmission error and other resistances. In real control, uncertainties are reduced by controlling errors like, tighten the belt or screw to reduce the transmission error, or use ball bearing to reduce the friction.

3) Coupling

There are coupling between each stage of inverted pendulum and the motion module. We will decouple the inverted pendulum near the equilibrium point and ignore some less important coupling variables.

4) Open loop instability

There are two equilibrium states for inverted pendulum systems: vertical upper and vertical down, in which vertical upper is the unstable equilibrium point and vertical down is the stable equilibrium point.

5) Limitations

The inverted pendulum system performance is limited by mechanisms like motion module travel distance, motor torque, etc. To make it convenient and reduce the cost, the structure size and the motor power of inverted pendulum are required to be small. The effect of travel distance to inverted pendulum swing up is especially evident: short travel distance easily gets the cart exceed the limit switch.

3.4 Applications of Inverted Pendulum

Among the some considerable applications of inverted pendulum, some of these applications will be described below:

(1) Simulation of dynamics of robotic arm

The inverted pendulum problem resembles the control systems that exist in robotic arms. The dynamics of inverted pendulum simulates the dynamics of robotic arm in the condition when the center of pressure lies below the centre of gravity for the arm so that the system is also unstable. Robotic arm behaves very much like inverted pendulum under this condition.

(2) Model of human standing still

The ability to maintain stability while standing straight is of great importance for the daily activities of people. The Central Nervous System (CNS) registers the pose and changes in the pose of the human body, and activates muscles in order to maintain balance. The inverted pendulum is widely accepted as an adequate model of a human standing still (quiet standing).

3.5 Inverted Pendulum Control

The inverted pendulum is an ubiquitous example of nonlinear control systems analysis and design. It is difficult, if not impossible, to design stabilization control algorithms and identify the corresponding stability regions for the nonlinear system, since our interest is to control the system in a neighborhood of an equilibrium state, it is reasonable to consider the linearization of the system at the equilibrium. A simple demonstration of moving the pivot point in a feedback system is achieved by balancing an upturned broomstick on the end of one's hand.

The problem of balancing a broomstick on a person's hand is illustrated in Figure 3.1. The only equilibrium condition is at $\theta(t) = 0$ and $d\theta/dt = 0$. The problem of balancing a broomstick on one's hand is not unlike the problem of controlling the attitude of a missile during the initial stages of launch. This problem is the classic and intriguing problem of the inverted pendulum mounted on a cart, as shown in Figure 3.2.

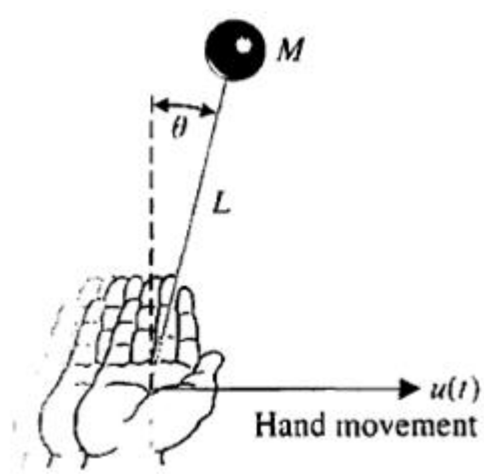

Figure 3.1: An inverted pendulum balanced on a Person's hand by moving the hand to reduce $\theta(t)$

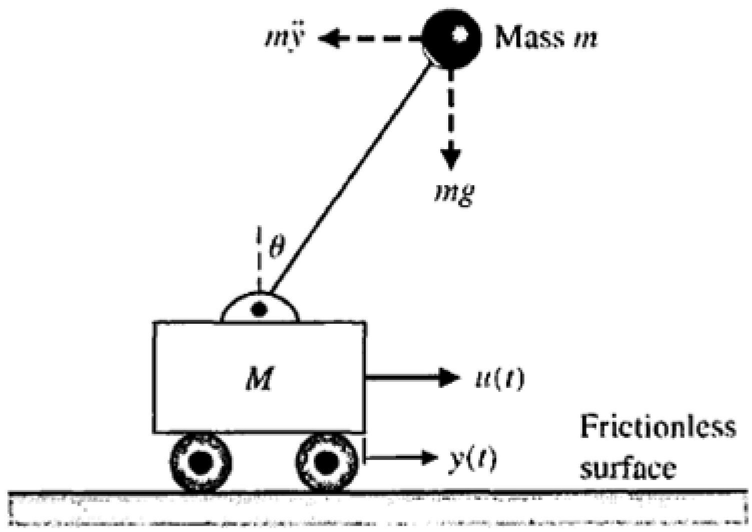

Figure 3.2: A cart and an inverted pendulum.

The cart must be moved so that mass *(m)* is always in an upright position. The state variables must be expressed in terms of the angular rotation $\theta(t)$ and the position of the cart *y(t)*. The differential equations describing the motion of the system can be obtained by writing the sum of the forces in the horizontal direction and the sum of the moments about the pivot point. By assuming that *M* >> *m* and the angle of rotation *(θ)* is small so that the equations are linear. The sum of the forces in the horizontal direction is:

$$M\ddot{y} + ml\ddot{\Theta} - u(t) = 0 \tag{3.1}$$

Where $u(t)$ equals the force on the cart, (l) is the distance from the mass *(m)* to the pivot point, *(θ)* the angle of the pendulum and *(M)* the cart mass. The sum of the torques about the pivot point is:

$$\text{Ml}\,\ddot{v} + m\,l^2\,\ddot{\Theta} - mlg\theta = 0 \tag{3.2}$$

The state variables for the two second-order equations are chosen as $(x_1, x_2, x_3, x_4) = (y, \dot{y}, \theta, \dot{\theta})$. Then Equations (3.1) and (3.2) are written in terms of the state variables as:

$$\text{M}\,\dot{x_2} + ml\dot{x_4} - u(t) = 0 \tag{3.3}$$

And:

$$\dot{x_2} + l\dot{x_4} - gx_3 = 0 \tag{3.4}$$

Where g is gravity. To obtain the necessary first-order differential equations, we solve for $l\dot{x_4}$ in Equation (3.4) and substitute into Equation (3.3) to obtain:

$$\text{M}\,\dot{x_2} + mgx_3 = u(t) \tag{3.5}$$

Since M >> m. Substituting x_2 from Equation (3.3) into Equation (3.4), we have:

$$\text{Ml}\,\dot{x_4} + Mgx_3 + u(t) = 0 \tag{3.6}$$

Therefore, the four first-order differential equations can be written as:

$$\dot{x_1} = x_2, \quad \dot{x_2} = -\frac{mg}{M}x_3 + \frac{1}{M}\,u(t)$$
$$\dot{x_3} = x_4, \quad \dot{x_4} = \frac{g}{l}x_3 - \frac{1}{Ml}\,u(t) \tag{3.7}$$

Thus, the system matrices are:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix} \quad (3.8)$$

$$B = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/Ml \end{bmatrix} \quad (3.9)$$

Without applying a control force, the angle of pendulum (*θ)* will increase without limit, i.e. the response is unbounded, and also the root of the characteristic equation will be positive. This again confirms an unbounded response and that is to say the system is unstable, that is, the inverted pendulum will fall over unless a suitable control force via full state feedback is used [7].

3.6 The Implementation of Pole Placement

The main objective is the angle stabilization of the inverted pendulum under uncertainties. If the output is the angle of the pendulum relative to the vertical axis (in upright position), Then it will be realize that the system is unstable, the goal is to design a control system that keeps the pendulum in upright position ($\theta=0$), and the cart can be brought back to the reference position ($x = 0$). The system becomes less sensitive to parameter variation and external disturbances by implementation of a suitable controller, inverted pendulum of its highly nonlinear characteristic.

On the other side, modern control theory is based on the description of system equations in terms of n first-order differential equations, and the system can be described by state space equation form. This means that only one controller can successfully stabilize the inverted pendulum system. The main design approach for systems described in state-space form is the use of state feedback, or pole placement. The first step in the pole placement design approach is to choose the locations of the desired closed loop poles. The most frequently used approach is to choose such poles based on experience in the root locus design, placing a dominant pair of closed poles and choosing other poles so that they are far to the left of the dominant closed loop poles. Another approach is based on the quadratic optimal control approach. This approach will determine the desired closed-loop poles such that it balances between the acceptable response and the amount of control energy required. The design of the state feedback controller based on pole placement approach is given in details. This scheme makes the inverted pendulum control design very easy based on pole placement approach.

In the pole-placement or pole-assignment techniques, assuming that all state variables are measurable and are available for feedback. It will be shown that if the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix. The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements. The control is achieved by feeding back the state variables through a regulator with constant gains. Consider the control system presented in the state-variable form as shown in Equation (2.13). The block diagram of the control system shown in Figure 3.3 with the following state feedback control as shown in Equation (2.15).

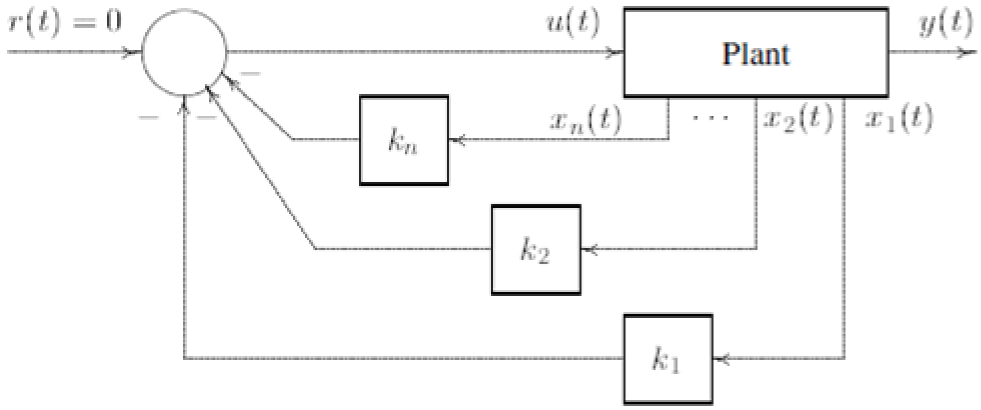

Figure 3.3: Control system design via pole placement

Where

K is a *1 x n* vector of constant feedback gains. The control system input $r(t)$ is assumed to be zero. The purpose of this system is to return all state variables to values of zero when the states have been perturbed. Substituting Equation (2.15) into Equation (2.13), the closed-loop system state-variable representation is:

$$\dot{x}(t) = (A-BK)\, x(t) = A_f \; x(t) \tag{3.9}$$

The closed-loop system characteristic equation is:

$$|sI - A + BK| = 0 \tag{3.10}$$

Assume the system is represented in the phase variable canonical form as follows:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \vdots \\ \dot{x_{n-1}} \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{3.11}$$

Substituting for A and B into Equation (3.12), the closed-loop characteristic equation for the control system is found as:

$$|sI - A + BK| = s^n + (a_{n-1} + k_n)\, s^{n-1} + \ldots (a_1 + k_2)\, s + (a_0 + k_1) = 0 \qquad (3.12)$$

For the specified closed-loop pole locations $-\lambda_1, \ldots\ldots, -\lambda_n$ the desired characteristic equation is:

$$\alpha_c(s) = (s + \lambda_1) \ldots .(s + \lambda_n) = s^n + \alpha_{n-1}\, s^{n-1} \ldots\ldots \alpha_1 s + \alpha_0 = 0 \qquad (3.13)$$

The design objective is to find the gain matrix K such that the characteristic equation for the controlled system is identical to the desired characteristic equation. Thus, the gain vector K is obtained by equating coefficients of Equations (3.13) and (3.12).

$$k_i = \alpha_{i-1} - a_{i-1} \qquad (3.14)$$

If the state model is not in the phase-variable canonical form, we can use the transformation technique to transform the given state model to the phase-variable canonical form. The gain factor is obtained for this model and then transformed back to confirm with the original model. This procedure results in the following formula, known as Ackermann's formula.

$$K = [0 \quad 0 \quad \cdots \quad 0 \quad 1]\, {M_c}^{-1}\, \alpha_c(A) \qquad (3.15)$$

Where α_c(A) is given by:

$$\alpha_c(A) = A^n + \alpha_{n-1} A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \qquad (3.16)$$

The function *[K,A_f] = placepol (A, B, C, p)* is developed for the pole-placement design. A, B, and C are system matrices and p is a row vector containing the desired closed-loop poles. This function returns the gain vector K and the closed-loop system matrix A_f. Also, the MATLAB control system toolbox contains two functions for pole-placement design. Function *K =acker (A, B, p)* is for single input systems, and function *K = place (A, B, p),* which uses a more reliable algorithm, is for multi-input systems.

The condition that must exist to place the closed-loop poles at the desired location is to be able to transform the given state model into phase-variable canonical form [7].