

Sudan University of Science and Technology

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Bernstein-type Inequality and Lebesgue Classes of Constructible
Functions in Weighted Bergman and Quasi-Banach Spaces

متباينة نوع-بيرنشتاين وعائلات لبيق للدوال القابلة للتركيب في
فضاءات بيرجمان المرجحه وشبه فضاءات باناخ

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Ph.D in Mathematics

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This thesis is dedicated to...

My parents ...

My brothers & sisters ...

My wife & children ...

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In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohamed His servant and messenger.

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Abstract

We show the inequalities of Bernstein-type for derivatives of rational functions, inverse theorems of rational approximation, Kernels of Toeplitz operators, smooth functions and effective essential Hardy space interpolation constrained by weighted Hardy and Bergman norms. The Presburger Sets, P -minimal fields, analytic P -adic cell decomposition, integrals, and the classification of semi-algebraic P -adic sets up to semi algebraic bijection are considered. We characterize the basic sequences and curves with zero derivative in F -spaces and an F -space with trivial dual where the Krein-Milman theorem holds. We discuss the asymptotic sharpness and application of a Bernstein-type inequality for rational functions and interpolation in Hardy, Dirichlet and weighted Bergman spaces. Methods of integration of positive constructible functions against Euler characteristic, dimension, loci of integrability, zero loci, stability under integration for constructible functions on Euclidean space with Lebesgue measure, Lebesgue classes and preparation of real constructible functions are studied. The existence and Lipschitz maps of primitives for continuous functions and the fundamental theorem of calculus with integration in quasi Banach spaces are established.

تم ايضاح المتباينات لنوع بيرنشتاين لاجل الاشتقاقات للدوال النسبية والمبرهنات الانعكاسية للتقريب النسبي والنويات لمؤثرات التبوليتز والدوال الملساء وفضاءات هاردي الاساسية الفعالة بواسطة هاردي المرجح ونظام بيرجمان. اعتبرنا فئات بريسبيرجير والحقول الأصغرية- P وتفكيك الخلية P -أديك التحليلية والتكاملات وتصنيف فئات P -أديك شبه-الجبرية الي الواحد لواحد شبه الجبري. تم تشخيص المتتاليات الأساسية والمنحنيات مع المشتقة الصفرية في فضاءات F -وفضاء F مع الثنائي البديهي حيث مبرهنة كرين-ميلمان تتحقق. درسنا قاطعية التقاربية وتطبيق متباينة نوع-بيرنشتاين لاجل الدوال النسبية والاستكمال في فضاءات هاردي ودرشلت وبيرجمان المرجح. تمت دراسة طرق التكامل للدوال القابلة للبناء الموجب المقابل مميز اويلر والبعد والمحال الهندسية الصفرية والاستقرارية تحت التكامل لاجل دوال البناء علي الفضاء الاقليدي مع قياس لبيق وعائلات لبيق واعادة تجهيز دوال البناء الحقيقة. تم تأسيس وجود ورواسم لبشتز للبدائيات لاجل الدوال المستمرة والمبرهنة الاساسية للحسابان مع التكامل في شبه فضاءات باناخ.

Introduction

Let H_p be the Hardy space of functions f that are analytic in the disk $|z| < 1$ and let $J^\alpha f$ be the derivative of f of order α in the sense of Weyl. It is shown, for example, that if r is a rational function of degree $n \geq 1$ with all its poles in the domain $|z| > 1$, then $\|J^\alpha r\|_{H_p} \leq cn^\alpha \|r\|_{H_p}$, where $p \in (1, \infty]$, $\alpha > 0$, $\sigma = (\alpha + p^{-1})^{-1}$ and $c > 0$ depends only on α and p . Given a finite subset σ of the unit disc $\mathbb{D}$ and a holomorphic function f in $\mathbb{D}$ belonging to a class X , we are looking for a function g in another class Y which satisfies $g|_\sigma = f|_\sigma$ and is of minimal norm in Y . We consider the interpolation constant $c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf_{g|_\sigma = f|_\sigma} \|g\|_Y$. When $Y = H^\infty$, our interpolation problem includes those of Nevanlinna–Pick and Carathéodory–Schur. We show a cell decomposition theorem for Presburger sets and introduce a dimension theory for Z -groups with the Presburger structure. Using the cell decomposition theorem we obtain a full classification of Presburger sets up to definable bijection. We show a conjecture of Denef on parameterized p -adic analytic integrals using an analytic cell decomposition theorem, which we also show. We show that two infinite p -adic semi-algebraic sets are isomorphic (i.e. there exists a semi-algebraic bijection between them) if and only if they have the same dimension. We establish a conjecture of Shapiro that an F -space (complete metric linear space) with the Hahn-Banach Extension Property is locally convex. We show that in certain non-locally convex Orlicz function spaces L_ϕ with trivial dual every compact convex set is locally convex and hence the Krein-Milman theorem holds. We show a Bernstein-type inequality involving the Bergman and the Hardy norms, for rational functions in the unit disc $\mathbb{D}$ having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$. The asymptotic sharpness of this inequality is shown as $n \rightarrow \infty$ and $r \rightarrow 1^-$. Given $n \geq 1$ and $r \in [0, 1)$, we consider the set $\mathcal{R}_{n,r}$ of rational functions having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, where $\mathbb{D}$ is the unit disc of the complex plane. We give an asymptotically sharp Bernstein-type inequality for functions in $\mathcal{R}_{n,r}$ in weighted Bergman spaces with “polynomially” decreasing weights. Following recent work of R. Cluckers and F. Loeser on motivic integration, we develop a direct image formalism for positive constructible functions in the globally subanalytic context. We show a correspondence between zero loci and loci of integrability for constructible functions on Euclidean space, where a function is called constructible if it is a sum of products of globally subanalytic functions and of logarithms of globally subanalytic functions. We call a function constructible if it has a globally subanalytic domain and can be expressed as a sum of products of globally subanalytic functions and logarithms of positively-valued globally subanalytic functions. We show that for a wide class of non-locally convex quasi-Banach spaces X that includes the spaces for $0 < p < 1$, there exists a continuous function

$f: [0,1] \rightarrow X$ failing to have a primitive, thus solving a problem raised by M.M. Popov in 1994. We make a general approach to integrability and its interplay with differentiability in quasi-Banach spaces. This endeavor demands studying first the defects of Bochner and Riemann integration in the setting of p -Banach spaces when $p < 1$. The conclusion will be that the local convexity is a necessary (and sufficient) condition of the space for the integral operator to work in the expected way.

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Chapter 1

Bernstein Type Inequalities and Kernels of Toeplitz Operators with Effective H^∞ Interpolation Constrained

Let φ be a unimodular function on the unit circle $\mathbb{T}$ and let $K_p(\varphi)$ denote the kernel of the Toeplitz operator T_φ in the Hardy space H^p , $p \geq 1$; $K_p(\varphi) \stackrel{\text{def}}{=} \{f \in H^p : T_\varphi f = 0\}$. Suppose $K_p(\varphi) \neq 0$. The problem is to find out how the smoothness of the symbol φ influences the boundary smoothness of functions in $K_p(\varphi)$. If X is a Hilbert space belonging to the families of weighted Hardy and Bergman spaces, we obtain a sharp upper bound for the constant $c(\sigma, X, H^\infty)$ in terms of $n = \text{card} \sigma$ and $r = \max_{\lambda \in \sigma} |\lambda| < 1$. If X is a general Hardy–Sobolev space or a general weighted Bergman space (not necessarily of Hilbert type), we also establish upper and lower bounds for $c(\sigma, X, H^\infty)$ but with some gaps between these bounds. This problem of constrained interpolation is partially motivated by applications in matrix analysis and in operator theory.

Section (1.1): Derivatives of Rational Functions and Inverse Theorems of Rational Approximation

Let X be a quasinormed space of functions that are analytic in the disk $|z| < 1$, and let $R_n(f, X)$ ($f \in X$, $n = 1, 2, \dots$) be the best approximation to f in X by rational fractions of degree at most $n-1$. Dolzhenko [18] showed that if $f \in H_\infty$ and $\Sigma R_n(f, H_\infty) < \infty$ then f belongs to the Hardy-Sobolev space H_1^1 . Under the same conditions on f , Peller [14] showed that f belongs to the Hardy-Besov space B_1^1 . Since $B_1^2 \subsetneq H_1^1$, Peller's result is stronger than Dolzhenko's. Nevertheless (see [18]) both of these inverse theorems on rational approximation are best possible in the following sense. For every nonincreasing sequence of numbers a_n ($n = 1, 2, \dots$) that satisfies the condition $\Sigma a_n = +\infty$, there exists an $f_* \in H_\infty$ such that $R_n(f_*, H_\infty) = O(a_n)$ and $f_* \notin H_1^1$, and consequently $f_* \notin B_1^1$. These results are generalized in the present section. In particular, we obtain the best possible sufficient conditions on the rate of decrease of $R_n(f, H_p)$ ($1 < p \leq \infty$) that guarantee that f belongs to the Hardy-Sobolev space H_σ^α or the Hardy-Besov space B_σ^α ($\alpha > 0$, $\sigma = (\alpha + p^{-1})^{-1}$). In addition, in contrast to [14], [28] and [29], we show the implication $\Sigma(R_n(f, BMOA))^{1/\alpha} < \infty \Rightarrow f \in B_{1/\alpha}^\alpha$ (first obtained by Peller [14] for $0 < \alpha < 1$ and then generalized to the case $\alpha > 1$ in [28], [29] and [32]) without making use of the connection of $R_n(f, BMOA)$ with Hankel operators. The method for solving these problems uses inequalities of Bernstein type, obtained here, for derivatives of rational functions.

The main results of this section were presented without proof in [30], [33].

Let S be a rectifiable curve in the complex plane. We denote by $L_p(S)$, $p \in (0, \infty]$, the set of functions f , measurable on S , for which $\|f\|_{p,S} \leq \infty$, where we set

$$\|f\|_{p,S} = \left(\int_S |f(z)|^p |dz| \right)^{1/p}, \quad p \neq \infty,$$

$$\|f\|_{\infty,S} = \text{ess sup}_{z \in S} |f(z)|, \quad p = \infty.$$

We denote by T , D_+ , and D_- , respectively, the circle $|z| = 1$, the disk $|z| < 1$ and the domain $|z| > 1$; by $A(D_\pm)$ we denote the set of functions that are analytic in $D_\pm$. We denote by H_p , $0 < p \leq \infty$, the Hardy space [1] of functions in $A(D_+)$ for which the quasinorm

$$\|f\|_{H_p} = \lim_{\rho \rightarrow 1-0} \|f(\cdot \rho)\|_p$$

is finite, where we write for short $\|g\|_p = \|g\|_{p,T}$ for $g \in L_p(T)$. The indicated limit exists because of the monotonicity of $\|f(\cdot, \rho)\|_p$ with respect to ρ [2]. If $f \in H_p$ and $z \in T$ we denote by $f(z)$ the nontangential limit of $f(\zeta)$ as $\zeta \rightarrow z$ [2]. It is known that $\|f\|_{H_p} = \|f\|_p$.

Let $f \in A(D_+)$ and let $\hat{f}(k)$ ($k=0,1,\dots$) be the Taylor coefficients of f . If $\alpha \geq 0$, the following functions in $A(D_+)$,

$$f^{(\alpha)}(z) = \sum_{k=[\alpha]}^{\infty} \frac{\Gamma(k - [\alpha] + 1 + \alpha)}{\Gamma(k - [\alpha] + 1)} \hat{f}(k) z^{k - [\alpha]},$$

$$J^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha \hat{f}(k) z^k,$$

where Γ is Euler's gamma function and $[\alpha]$ is the integral part of α , are called the derivatives of f in the Riemann-Liouville and the Weyl senses, respectively. Evidently, if $\alpha = l$ is a positive integer, $f^{(l)}(z)$ is the ordinary derivative, and $J^l f(z) = [(d/dz)z]^l f(z)$. The function $J^\alpha f$ will also be considered for $\alpha < 0$. In this case it is called the integral of f of order $-\alpha$ in the sense of Weyl. It is easy to establish (see also [3]) that when $\alpha > 0$

$$f^{(\alpha)}(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{|\zeta|=\rho} f(\zeta) \left(1 - \frac{z}{\zeta}\right)^{-1-\alpha} \zeta^{-1-[\alpha]} d\zeta, \quad |z| < \rho < 1, \quad (1)$$

where the branch of $(1-\eta)^{-1-\alpha}$ is chosen so that $(1-\eta)^{-1-\alpha} > 0$ for $\eta \in (-\infty, 1)$. We denote by H_p^α ($\alpha \in (-\infty, \infty)$, $p \in (0, \infty]$) the Hardy-Sobolev space, i.e. the set of $f \in A(D_+)$ with finite quasinorm $\|f\|_{H_p^\alpha} = \|J^\alpha f\|_{H_p}$. We denote by $H_{p,q}^\alpha$ ($\alpha \in (-\infty, \infty)$, $p \in (0, \infty]$, $q \in (0, \infty]$), the Hardy-Besov space, i.e. the set of $f \in A(D_+)$ with finite quasinorm

$$\|f\|_{B_{p,q}^\alpha}^{(\beta)} = \left(\int_0^1 (1-\rho)^{(\beta-\alpha)q-1} \|J^\beta f(\cdot, \rho)\|_{H_p}^q d\rho \right)^{1/q}, \quad q \neq \infty,$$

$$\|f\|_{B_{p,\infty}^\alpha}^{(\beta)} = \sup_{0 < \rho < 1} (1-\rho)^{(\beta-\alpha)} \|J^\beta f(\cdot, \rho)\|_{H_p}, \quad q = \infty.$$

Here β is arbitrary, $\beta > \alpha$. The space $B_{p,q}^\alpha$ is independent of β [3] and the quasinorms for different values of β are equivalent. In this connection, we call the quasinorm with $\beta = \alpha + 1$ fundamental, and denote it by $\|f\|_{B_{p,q}^\alpha}$. We abbreviate $B_{p,q}^\alpha$ to B_p^α .

Unlike $J^\alpha f$, the derivative $f^{(\alpha)}$ does not have the semigroup property. In fact, the equality $f^{(\alpha_1+\alpha_2)} = (f^{(\alpha_1)})^{(\alpha_2)}$ is satisfied for every f only in the case when α_1 and α_2 are integers. Lemma (1.1.1), showed below, lets one avoid this inconvenience.

Definition. Let W be a quasinormed space of elements of $A(D_+)$. A sequence $\{\lambda_k\}_0^\infty$ is called a multiplier in W if, for each $f \in W$, we have $\|g\|_W \leq c \|f\|_W$ where $g(z) = \sum_0^\infty \lambda_k \hat{f}(k) z^k$, with $c > 0$ and independent of f .

Lemma (1.1.1)[1]: Let $\alpha, \beta > 0$. Then the sequences $\lambda_k = \Gamma(k + \alpha + \beta)[\Gamma(k + \alpha)(k + 1)^\beta]^{-1}$ and $\mu_k = \lambda_k^{-1}$ ($k = 0, 1, 2, \dots$) are multipliers in the spaces H_p^γ and $B_{p,q}^\gamma$.

Proof. It follows from the definitions of H_p^γ and $B_{p,q}^\gamma$ that we may restrict our attention to $H_p^0 = H_p$. Let m be the smallest integer such that $m \geq p^{-1} + 1$. From the asymptotic series for the gamma function [4] we obtain

$$\lambda_k = b_0 + (k+1)^{-1}b_1 + (k+1)^{-2}b_2 + \dots + (k+1)^{-m}b_m + (k+1)^{-m-1}d_k,$$

where $b_0, b_1, \dots, b_m$ are numbers depending only on α and β , and $\{d_k\}_0^\infty$ is a bounded sequence. Consequently, if $f \in H_p$ and $g(z) = \sum_0^\infty \lambda_k \hat{f}(k) z^k$, then

$$g(z) = \sum_{j=0}^m b_j J^{-j} f(z) + \sum_{k=0}^{\infty} \hat{f}(k)(k+1)^{-m-1} d_k z^k \stackrel{\text{def}}{=} \sum_{j=0}^m b_j f_j(z) + \psi(z). \quad (2)$$

Moreover, we have (see [5]) $\|f_j\|_{H_p} \leq c_1(j)\|f\|_{H_p}$ and (see [3])

$$|\hat{f}(k)| \leq c_2(p)(k+1)^{1/p} \quad (k=0,1,2,\dots).$$

Consequently $|\hat{\psi}(k)| \leq c_3(p)\|f\|_{H_p}(k+1)^{-2}$ and $|\psi|_{H_p} \leq c_4(p)\|f\|_{H_p}$. Thus we obtain $|g|_{H_p} \leq c_5(p)\|f\|_{H_p}$ from (2). We can show in a similar way that the sequence $\{\mu_k\}_0^{\infty}$ is a multiplier in H_p . This completes the proof of Lemma (1.1.1).

Let X and Y be quasinormed spaces. By an embedding $X \subset Y$ we shall always understand a continuous embedding, i.e. if $f \in X$ then $f \in Y$ and $\|f\|_Y \leq c\|f\|_X$, where $c > 0$ is independent of f .

Lemma (1.1.1) lets us extend various embedding theorems that were proved for the Riemann-Liouville derivative to the Weyl derivative. For example, we have [5]

$$H_{p_0}^{\alpha_0} \subset H_{p_1}^{\alpha_1} \quad (0 < p_0 \leq p_1 < \infty, p_0^{-1} - p_1^{-1} = \alpha_0 - \alpha_1). \quad (3)$$

There are the following embeddings between the spaces $B_{p,q}^{\alpha}$ [3]:

$$B_{p_1,q_1}^{\alpha_1} \subset B_{p_0,q_0}^{\alpha_0} \quad (\alpha_1 > \alpha_0, p_1 \geq p_0), \quad (4)$$

$$B_{p,q_1}^{\alpha} \subset B_{p,q_0}^{\alpha} \quad (q_1 < q_0), \quad (5)$$

$$B_{p_1,q}^{\alpha_1} \subset B_{p_0,q}^{\alpha_0} \quad (\alpha_1 - \alpha_0 = p_1^{-1} - p_0^{-1} > 0, p_0 \neq \infty). \quad (6)$$

The following two embeddings [3] reflect the connection between H_p^{α} and B_p^{α} :

$$H_p^{\alpha} \subset B_p^{\alpha} \quad (2 \leq p < \infty), \quad (7)$$

$$B_p^{\alpha} \subset H_p^{\alpha} \quad (0 < p \leq 2). \quad (8)$$

We denote by BMOA the space of analytic functions of bounded mean oscillation [6], i.e. $f \in \text{BMOA}$ if there exists $g \in L_{\infty}(T)$ such that

$$f(z) = \frac{1}{2\pi i} \int_T \frac{g(\zeta) d\zeta}{\zeta - z}, \quad z \in D_+. \quad (9)$$

The norm in BMOA is defined as follows:

$$\|f\|_{\text{BMOA}} = \inf \|g\|_{\infty},$$

where the lower bound is taken over all $g \in L_{\infty}(T)$ for which (9) holds. Evidently

$$H_{\infty} \subset \text{BMOA} \subset H_p \quad (0 < p < \infty). \quad (10)$$

Surveys of inequalities for the derivatives of rational functions are given by Gonchar [7] and Rusak [8]. Here we present only the inequalities that are directly related to the subject of the present section. The first result in this direction was obtained by Dolzhenko [9], who showed that a rational function r of degree $n \geq 1$ with poles only in D_- satisfies

$$\|r\|_{H_1^1} \leq c_1 n \|r\|_{H_{\infty}}, \quad (11)$$

$$\|r\|_{B_2^{\frac{1}{2}}} \leq c_2 n^{\frac{1}{2}} \|r\|_{H_{\infty}}. \quad (12)$$

For any $s \in \mathbb{N}$ and $p \in (0, \infty]$ the following generalization of (11) follows from the results of Sevast'yanov [10]:

$$\|r\|_{H_{\sigma-\varepsilon}^s} \leq c_3(s, p, \varepsilon) n^s \|r\|_{H_p} \quad (\sigma = (s + p^{-1})^{-1}, \varepsilon \in (0, \sigma)). \quad (13)$$

As was observed in [10], one cannot take $\varepsilon = 0$ in the preceding inequality if $\frac{1}{p} \in \mathbb{N}$. To see this, it is enough to consider the function $r(z) = (1 + \delta - z)^{-1}$ as $\delta \rightarrow +0$. We showed in

[11] that for $p = \infty$ and any $s \in \mathbb{N}$ we can take $\varepsilon = 0$ in (13). Inequality (12) was generalized by Danchenko [13], who showed that

$$\|r\|_{B_{t,q}^\alpha} \leq c_3(\alpha, t, q, n) \|r\|_{H_p} \quad (\alpha \in (0, 1], p \in (1, \infty], t \leq (\alpha + p^{-1})^{-1}, q > 0). \quad (14)$$

Another generalization follows from a result of Peller [14] on best rational approximations for the class $B_{1/\alpha}^\alpha$, $\alpha \in (0, 1]$, in the space BMOA. This is

$$\|r\|_{B_{1/\alpha}^\alpha} < c_4(\alpha) n^\alpha \|r\|_{BMOA} \quad (0 < \alpha \leq 1).$$

Our Theorem (1.1.7) generalizes and strengthens the results quoted above.

Lemma (1.1.2)[1]: If $z \in T$ and $l \in \mathbb{N}$ then

$$\frac{(2l-1)!}{2\pi} \int_T |\varrho(z, \zeta)|^{2l} |d\zeta| = z^l \sum_{j=1}^l C_{2l}^{l-j} (-1)^{l-j} B^{-j} [B^j(z) z^{l-1}]^{(2l-1)}.$$

Proof. For z and $\zeta \in T$ we have $|d\zeta| = d\zeta/i\zeta$ and

$$|\varrho(\zeta, z)|^2 = \varrho(\zeta, z) \overline{\varrho(\zeta, z)} = \frac{\zeta^z}{B(\zeta)B(z)} \varrho^2(\zeta, z).$$

Consequently

$$\frac{(2l-1)!}{2\pi} \int_T |\varrho(\zeta, z)|^{2l} |d\zeta| = z^l B^{-l}(z) I_l(z), \quad (15)$$

where

$$I_l(z) = \frac{(2l-1)!}{2\pi i} \int_T \varrho^{2l}(\zeta, z) B^{-l}(\zeta) \zeta^{l-1} d\zeta.$$

Since $I_l(z)$ is continuous in $D_+ \cup T$, it is enough to calculate it for $z \in D_+$. Thus we have

$$I_l(z) = \sum_{j=-1}^l C_{2l}^{l-j} (-B(z))^{l-j} I_{l,j}(z) \quad (z \in D_+), \quad (16)$$

Where

$$I_{l,j}(z) = \frac{(2l-1)!}{2\pi i} \int_T \frac{B^j(\zeta) \zeta^{l-1}}{(\zeta - z)^{2l}} d\zeta.$$

By Cauchy's formula we obtain

$$I_{l,j}(z) = [B^j(z) z^{l-j}]^{(2l-1)} \quad (j = 1, \dots, l). \quad (17)$$

If $-l \leq j \leq 0$, the point $\zeta = 0$ is a zero of order at least 2 of the function

$$B^j(\zeta) \zeta^{l-1} (\zeta - z)^{-2l}.$$

Therefore $I_{l,j}(z) = 0$ ($j = -l, \dots, 0$). By using (15)-(17), we obtain the conclusion of Lemma (1.1.2).

Lemma (1.1.3)[1]: For all $z \in T$ and $s \in \mathbb{N}$

$$|B^{(s)}(z)| \leq 2^s s! \lambda^s(z, 1/s).$$

Proof. We set $b_k(z) = (z - a_k)(1 - \bar{a}_k z)^{-1}$. Then

$$|B^{(s)}(z)| = \sum \frac{s!}{j_0! j_1! \dots j_n!} b_0^{j_0}(z) b_1^{j_1}(z) \dots b_n^{j_n}(z), \quad (18)$$

where the summation is over all collections $j_0, j_1, \dots, j_n$ of nonnegative numbers satisfying the condition $j_0 + j_1 + \dots + j_n = s$. It is evident that for every $z \in T$, $0 \leq k \leq n$ and $1 \leq j \leq s$ we have

$$|b_k^{(j)}(z)| \leq 2j! \left(\left| \frac{1 - |a_k|}{z - a_k} \right|^{1/s} \frac{1}{|z - a_k|} \right)^j. \quad (19)$$

Lemma (1.1.3) follows from (18) and (19).

Lemma (1.1.4)[1]: If $z \in T$ and $\alpha > 0$, then

$$\|\varrho(\cdot, z)\|_{1+\alpha} \leq c(\alpha) \lambda^{\alpha/(\alpha+1)}(z, 1/(\alpha+2)).$$

Proof. It follows immediately from Lemmas (1.1.2) and (1.1.3) that

$$\int_T |\varrho(\zeta, z)|^{2l} |d\zeta| \leq c(l) \lambda^{2l-1} \left(z, \frac{1}{2l-1} \right) \quad (z \in T, l \in N). \quad (20)$$

Let m be the smallest odd number such that $m > \alpha$. We introduce $p = (m+1)(\alpha+1)^{-1}$, $q = (m+1)(m-\alpha)^{-1}$ and $S(z) = \{\zeta \in T : |\arg \zeta - \arg z| \leq \lambda^{-1}(z, m^{-1})\}$. From (20) and Holder's inequality, we obtain

$$\int_{S(z)} |\varrho(\zeta, z)|^{1+\alpha} |d\zeta| \leq \|1\|_{q, S(z)} \|\varrho^{1+\alpha}(\cdot, z)\|_{p, S(z)} \leq c_1(m) \lambda^\alpha \left(z, \frac{1}{m} \right). \quad (21)$$

On the other hand,

$$\int_{T/S(z)} |\varrho(\zeta, z)|^{1+\alpha} |d\zeta| \leq 2^{1+\alpha} \int_{T/S(z)} |\zeta - z|^{-1-\alpha} |d\zeta| \leq c_2(\alpha) \lambda^\alpha \left(z, \frac{1}{m} \right). \quad (22)$$

Since $\lambda(z, \beta)$ does not increase in β for fixed $z \in T$, Lemma (1.1.4) follows from (21) and (22).

Lemma (1.1.5)[1]: if $f \in L_p(T)$, $p \in (1, \infty]$, $\alpha > 0$ and

$$g(z) = \int_T |\varrho(\zeta, z)|^{1+\alpha} |f(\zeta)| |d\zeta|,$$

Then $\|g\|_\sigma \leq c(\alpha, p) n^\alpha \|f\|_p$, where $\sigma = (\alpha + p^{-1})^{-1}$.

Proof. For $p = \infty$ the necessary inequality follows from Lemma (1.1.4) and the relation

$$\int_T \lambda(z, \beta) |dz| \leq c_1(\beta) n \quad (\beta > 0). \quad (23)$$

Now let $p \in (1, \infty]$ and $\alpha = 1 - p^{-1}$. Then $\sigma = 1$ and consequently, by Lemma (1.1.4), Holder's inequality, and (23),

$$\begin{aligned} \|g\|_1 &\leq \int_T \|\varrho^{1+\alpha}(\cdot, z)\|_1 |f(\zeta)| |d\zeta| \leq c_2(\alpha) \int_T \lambda^\alpha \left(\zeta, \frac{1}{\alpha+2} \right) |f(\zeta)| |d\zeta| \\ &\leq c_2(\alpha) \|\lambda\|_1^\alpha \|f\|_p \leq c_3(\alpha) n^\alpha \|f\|_p. \end{aligned}$$

Therefore Lemma (1.1.5) is established in the case under consideration. Now let α be arbitrary. Choose positive numbers γ, τ, l , and s satisfying the conditions $l \in (1, p)$, $l^{-1} + s^{-1} = 1$, $\gamma + \tau = \alpha$ and $l\tau = 1 - l/p$. Then, by Holder's inequality,

$$|g(z)| \leq \|\varrho^{s\gamma+1}(\cdot, z)\|_1^{1/s} \|\varrho^{l\tau+1}(\cdot, z) f^l(\cdot)\|_1^{1/l} \stackrel{\text{def}}{=} \varphi(z) \psi(z) \quad (24)$$

for every $z \in T$. From Lemma (1.1.4) and (23) we have

$$\|\varphi\|_{1/\gamma} \leq c_4(s, \gamma) n^\gamma \quad (25)$$

Using the fact that the lemma has already been established for $\alpha = 1 - p^{-1}$ (in this case $l\tau = 1 - (p/l)^{-1}$), we obtain

$$\|\psi\|_l \leq c_5(l, p) n^{\frac{1}{l} - \frac{1}{p}} \|f\|_p. \quad (26)$$

Thus we obtain the conclusion of Lemma (1.1.5) in the case $p \in (1, \infty)$ and $\alpha > 0$ from (24)-(26) and Holder's inequality.

Lemma (1.1.6)[1]: Let r be a rational function of degree $n \geq 1$ with all its poles in D_- , $\beta > 0$ and $p \in (1, \infty]$.

1) There are continuous functions $\lambda(\varphi)$ and $h(\varphi)$ of period 2π that satisfy the conditions

$$\begin{aligned} \|\lambda\|_{p, [0, 2\pi]} &\leq c(\beta, p) \|r\|_{H_p}, \quad \text{and} \quad \lambda(\varphi) \geq 0 \\ \|h\|_{1, [0, 2\pi]} &\leq n, \quad \text{and} \quad h(\varphi) \geq 1 \end{aligned}$$

$$\left| J^\beta r \left((1-x) e^{i\varphi} \right) \right| \leq \lambda(\varphi) \left(\min(x^{-1}, h(\varphi)) \right)^\beta, \quad x \in (0,1), \varphi \in [0, 2\pi].$$

2) There is a continuous function $g(\varphi)$ of period 2π that satisfies the conditions

$$\|g\|_{1,[0,2\pi]} \leq c(\beta)n \quad \text{and} \quad g(\varphi) > 1,$$

$$\left| J^\beta r \left((1-x) e^{i\varphi} \right) \right| \leq \|r\|_{BMOA} \left(\min(x^{-1}, g(\varphi)) \right)^\beta, \quad x \in (0,1), \varphi \in [0, 2\pi].$$

Proof. It is evident that for $x \in (0,1)$ and $\varphi \in [0, 2\pi]$

$$\left| J^\beta r \left((1-x) e^{i\varphi} \right) \right| \leq \max_{\rho \in [0,1]} \left| J^\beta r \left(\rho e^{i\varphi} \right) \right| \stackrel{\text{def}}{=} G(\varphi). \quad (27)$$

From Lemma (1.1.1) we find that there is a function f , analytic in $D_+ \cup T$, such that $J^\beta r = f^{(\beta)}$ and $\|f\|_{H_p} \leq c_1(p)\|r\|_{H_p}$. Consequently we find from (1) that

$$J^\beta r(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{S(z)} f(\zeta) \left(1 - \frac{z}{\zeta} \right)^{-1-\beta} \zeta^{-1-\lfloor \beta \rfloor} d\zeta \quad (z \in D_+ \setminus \{0\}),$$

where $S(z)$ is the convex curve formed by the circle $|\zeta| = \frac{1}{2}$ and the tangents to it from the point $z/|z|$. Hence we obtain

$$\left| J^\beta r \left((1-x) e^{i\varphi} \right) \right| \leq c_2(\beta) F(\varphi) x^{-\beta}, \quad F(\varphi) = \max_{\zeta \in S(z)} |f(\zeta)|, \quad (28)$$

where $z = (1-x) e^{i\varphi}$. Let us show that the functions

$$\begin{aligned} h(\varphi) &= c_3(\beta, p) n^{1-\beta\gamma} \left[\|r\|_{H_p}^{-1} G(\varphi) \right]^\gamma + 1 \quad (\gamma = (\beta + p^{-1})^{-1}), \\ \lambda(\varphi) &= F(\varphi) + c_4(\beta, p) n^{-\beta(1-\beta\gamma)} \|r\|_{H_p}^{\beta\gamma} G^{1-\beta\gamma}(\varphi) \end{aligned}$$

satisfy the requirements of the lemma for suitable choices of the constants $c_3(\beta, p)$ and $c_4(\beta, p)$. In fact, from (29) together with (Theorem (7.36) of [2]), we obtain

$$\begin{aligned} \|G\|_{\gamma,[0,2\pi]} &\leq c_5(\beta, p) n^\beta \|r\|_{H_p}, \\ \|F\|_{p,[0,2\pi]} &\leq c_6(p) n^\beta \|r\|_{H_p}. \end{aligned}$$

Using (27) and (28), we obtain assertion 1) of Lemma (1.1.6).

For the proof of assertion 2) we observe that

$$J^\beta r(z) = \frac{\Gamma(1+\beta)}{2\pi i} \int_T [f(\zeta) + s(1/\zeta)] \left(1 - \frac{z}{\zeta} \right)^{-1-\beta} \zeta^{-1-\lfloor \beta \rfloor} d\zeta \quad (z \in D_+),$$

where $s \in H_1$ and $s(0) = 0$. Consequently, instead of (28) we must use the inequality

$$\left| J^\beta r \left((1-x) e^{i\varphi} \right) \right| \leq c_7(\beta) \|r\|_{BMOA} x^{-\beta}.$$

To obtain the analog of (27) we have to use (31). Everything else is obtained as in the proof of assertion 1) for $p = \infty$.

Theorem (1.1.7)[1]: Let r be a rational function of degree $n \geq 1$ with all its poles in D_- ; let $\alpha > 0$, $p \in (1, \infty]$, and $\sigma = (\alpha + p^{-1})^{-1}$. Then

$$\|r\|_{H_0^\alpha} \leq c_1(\alpha, p) n^\alpha \|r\|_{H_p}, \quad (29)$$

$$\|r\|_{B_0^\alpha} \leq c_2(\alpha, p) n^\alpha \|r\|_{H_p}, \quad (30)$$

$$\|r\|_{H_{1/\alpha}^\alpha} \leq c_3(\alpha) n^\alpha \|r\|_{BMOA}, \quad (31)$$

$$\|r\|_{B_{1/\alpha}^\alpha} \leq c_4(\alpha) n^\alpha \|r\|_{BMOA}. \quad (32)$$

In the proof of Theorem (1.1.7) we shall consistently use the following notation. Let $a_1, \dots, a_n$, belong to D_+ . We set

$$B(z) = \prod_{k=0}^n \frac{z - a_k}{1 - \bar{a}_k z} \quad (a_0 = 0),$$

$$Q(z, \zeta) = \frac{B(z) - B(\zeta)}{z - \zeta},$$

$$\lambda(z, \beta) = \sum_{k=0}^n \left(\frac{1 - |a_k|}{z - a_k} \right)^\beta \frac{1}{|z - a_k|} \quad (\beta > 0).$$

Proof. Let the poles of the rational function r be located, counting multiplicities, at the points $1/\bar{a}_1, \dots, 1/\bar{a}_n$, where $a_1, \dots, a_n$ belong to D_+ . Then the function $r(\zeta)B^{-k}(\zeta)(1 - \frac{z}{\zeta})^{-1-\alpha\zeta^{-1}[\alpha]}$ ($k \in \mathbb{N}$ and $z \in D_+$) is an analytic function of ζ in D_- and has a zero of order at least 2 at ∞ . Consequently

$$\int r(\zeta)B^{-k}(\zeta) \left(1 - \frac{z}{\zeta}\right)^{-1-\alpha} \zeta^{-1-[\alpha]} d\zeta = 0.$$

Therefore if we expand the function $(1 - B(z)/B(\zeta))^{1+\alpha}$ ($z \in D_+$ and $\zeta \in T$) in a Taylor series in $\frac{B(z)}{B(\zeta)}$, we obtain from (1)

$$r^{(\alpha)}(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_T r(\zeta) \left(1 - \frac{B(z)}{B(\zeta)}\right)^{1+\alpha} \left(1 - \frac{z}{\zeta}\right)^{-1-\alpha} \zeta^{-1-[\alpha]} d\zeta. \quad (33)$$

From (33) and Lemmas (1.1.1) and (1.1.5) we obtain (29). To show (31) it is enough to observe that (33) remains valid if we replace $r(\zeta)$ by $r(\zeta) + h(1/\zeta)$ on the right, where $h \in H_1$ and $h(0) = 0$.

Let h and λ be the functions from Lemma (1.1.6) corresponding to $\beta = \alpha + 1$. Then we obtain (30) from Lemma (1.1.6):

$$\begin{aligned} \|r\|_{B_\alpha}^\sigma &\leq \int_0^{2\pi} \lambda^\sigma(\varphi) \left(\int_0^{1/h(\varphi)} h^{\sigma(\alpha+1)} x^{\sigma-1} dx + \int_{1/h(\varphi)}^1 x^{-\alpha\sigma-1} dx \right) d\varphi \\ &\leq c_1(\alpha, p) \int_0^{2\pi} \lambda^\sigma(\varphi) h^{\alpha\sigma}(\varphi) d\varphi \leq c_2(\alpha, p) n^{\alpha\sigma} \|r\|_{H_p}^\sigma. \end{aligned}$$

Here in obtaining the last inequality we have also applied Holder's inequality. Similarly we obtain (32) from Lemma (1.1.6).

Corollary (1.1.8)[1]: Let $\alpha > 0$, $p \in (1, \infty]$, $\sigma = (\alpha + p^{-1})^{-1}$, $s \in (1, \infty]$, $q \in (1, \infty]$ and

$$A_n(\alpha, p, s, q) = \sup \left(\|r_n\|_{B_{s,q}^\alpha} \|r_n\|_{H_p}^{-1} \right),$$

where the upper bound is taken over all rational functions $r_n \neq 0$ of degree at most n ($n \geq 1$). Then

$$A_n(\alpha, p, \sigma, q) \asymp n^\alpha \quad (q \geq \sigma), \quad (34)$$

$$A_n(\alpha, p, \sigma, q) \asymp n^{q^{-1}-p^{-1}} \quad (q < \sigma), \quad (35)$$

$$A_n(\alpha, p, s, q) = +\infty \quad (s > \sigma, q \in (0, \infty]), \quad (36)$$

$$A_n(\alpha, p, s, q) \asymp n^\alpha \quad (s < \sigma, q \in (0, \infty]). \quad (37)$$

Proof. The upper inequality in (34) follows from (30) and (5). To obtain the lower inequality in (34) it is enough to consider the function $r_n(z) = z^n$. The upper inequality in (35) follows from (30) and (6). To obtain the lower inequality we consider the function

$$r_n(z) = \sum_{k=0}^{n-1} [(1 + \varepsilon)e^{2\pi i k/n} - z]^{-1}$$

for sufficiently small $\varepsilon > 0$. We immediately verify (36) by the example of the function $r_1(z) = (1 + \varepsilon - z)^{-1}$ as $\varepsilon \rightarrow +0$. To obtain the lower inequality in (37) we consider the

function $r_n(z) = z^n$. To obtain the upper inequality in (37) we use Lemma (1.1.6). Let h and λ be the functions of Lemma (1.1.6) corresponding to $\beta = s^{-1} - p^{-1} > \alpha$. Then (with corresponding changes for $q = \infty$) we have

$$\begin{aligned} \|r\|_{B_{s,q}^\alpha}^{(\beta)} &\leq 2^{\frac{1}{q} + \frac{1}{s}} \left(\int_{1/n}^1 \left(\int_0^{2\pi} (\lambda(\varphi) x^{-\beta})^s d\varphi \right)^{\frac{q}{s}} x^{q(\beta-\alpha)-1} dx \right)^{\frac{1}{q}} \\ &\quad + 2^{\frac{1}{q} + \frac{1}{s}} \left(\int_0^{1/n} \left(\int_0^{2\pi} (\lambda(\varphi) h^\beta(\varphi))^2 d\varphi \right)^{\frac{q}{s}} x^{q(\beta-\alpha)-1} dx \right)^{\frac{1}{q}} \\ &\leq c_2(\alpha, p, q, s) n^\alpha \|r_n\|_{H_p}. \end{aligned}$$

Corollary (1.1.8) is showed.

Let the rational function r of degree $n+m$ have no poles on T , but n poles in D_+ and m in D_- . Then $r(z) = r_+(z) + r_-(1/z)$, where r_+ and r_- are rational functions of respective degrees n and m with all their poles in D_- . It is easy to obtain the following corollary of Theorem (1.1.13).

Corollary (1.1.9)[1]: if $\alpha > 0$, $p \in (1, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$ then

$$\|r_+\|_{B_\sigma^\alpha} \leq c(\alpha, p) n^\alpha \|r\|_p, \quad \|r_-\|_{B_\sigma^\alpha} \leq c(\alpha, p) m^\alpha \|r\|_p.$$

In conclusion, we remark that it would be interesting to extend Theorem (1.1.7) to the Smirnov spaces E_p . Some special results in this direction were obtained in [12], [15] and [16].

Let $f \in H_p$ and $n \geq 0$. Let $R_n(f, H_p)$ denote the best approximation to f in H_p by rational fractions of degree at most $n-1$. Following [17], we introduce the approximation space $R_{p,q}^\alpha$ ($\alpha > 0$, $p \in (0, \infty]$, $q \in (0, \infty]$) of functions $f \in H_p$ with finite quasinorm

$$\begin{aligned} \|f\|_{R_{p,q}^\alpha} &= \|f\|_{H_p} + \left(\sum_{k=0}^{\infty} (2^{k\alpha} R_{2^k}(f, H_p))^q \right)^{1/q}, \quad q \neq \infty, \\ \|f\|_{R_{p,\infty}^\alpha} &= \|f\|_{H_p} + \sup_{k=0,1,\dots} 2^{k\alpha} R_{2^k}(f, H_p). \end{aligned}$$

We denote by $R_n(f, BMOA)$ the best approximation to f in BMOA by rational fractions of degree at most $n-1$, and the corresponding approximation space by $R_{*,q}^\alpha$.

Lemma (1.1.10)[1]: [2]. Let $f(x)$ be a nonnegative function defined for $x > 0$, and let $r > 1$ and $s < r-1$. If $f^r(x)x^s$ is integrable on $(0, \infty)$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(y) dy \right)^r x^s dx \leq \left(\frac{r}{r-s-1} \right)^r \int_0^\infty f^r(x) x^s dx.$$

Lemma (1.1.11)[1]: Let $\{\lambda_k\}_{k=-\infty}^\infty$ and $\{h_k\}_{k=-\infty}^\infty$ be sequences of nonnegative numbers satisfying the conditions

$$\frac{h_{k+1}}{h_k} \geq q \quad (k = 0, \pm 1, \pm 2, \dots), \quad \sum_{k=-\infty}^\infty (h_k^{l-m} \lambda_k)^r < \infty,$$

Where $l > m > 0$, $r > 1$ and $q > 1$. If

$$\psi(x) = \sum_{k=-\infty}^\infty \lambda_k (\min(h_k, x^{-1}))^l \quad (x \in [0, \infty)),$$

Then

$$\int_0^\infty \psi^r(x) x^{mr-1} dx \leq c(l, m, q, r) \sum_{k=-\infty}^\infty (h_k^{l-m} \lambda_k)^r.$$

proof. We define a function $\varphi(y)$ on $(0, \infty)$ in the following way. If j is a positive integer and $y \in (q^{j-1}, q^j]$ then $\varphi(y)$ equals $\lambda_k q^{-j}$ if $h_k \in (q^{j-1}, q^j]$ and equals 0 when no h_k belongs to $(q^{j-1}, q^j]$. Since $h_{k+1}/h_k \geq q$ for every k , the interval $(q^{j-1}, q^j]$ contains at most one h_k and consequently $\varphi(y)$ is well defined. It is easy to verify the inequality

$$\psi(x) \leq c_1(q) \int_0^{1/x} \varphi(y) y^l dy + \frac{c_2(q)}{x^l} \int_0^x \varphi\left(\frac{1}{y}\right) \frac{dy}{y^2} \quad (x > 0).$$

Making an appropriate change of variable in the improper integral, we find from Lemma (1.1.10) that

$$\begin{aligned} \int_0^\infty \psi^r(x) x^{mr-1} dx &\leq c_3(r, q) \int_0^\infty \left(\frac{1}{x} \int_0^x \varphi(y) y^l dy \right)^r x^{r(l-m-1)} dx \\ &\quad + c_4(r, q) \int_0^\infty \left(\frac{1}{x} \int_0^x \varphi\left(\frac{1}{y}\right) \frac{dy}{y^2} \right)^r x^{mr-1} dx \\ &\leq c_5(r, m, q, l) \int_0^\infty \varphi^r(x) x^{r(l-m-1)-1} dx. \end{aligned}$$

By the definition of $\varphi(x)$ we obtain

$$\int_0^\infty \varphi^r(x) x^{r(l-m-1)-1} dx \leq c_6(r, m, q, l) \sum_{k=-\infty}^\infty (h_k^{l-m} \lambda_k)^r.$$

Thus the conclusion of Lemma (1.1.11) follows from the preceding two inequalities.

Theorem (1.1.12)[1]: Let $\alpha > 0$, $p \in (0, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$. Then

$$R_{p, \sigma}^\alpha \subset B_\sigma^\alpha, \quad (38)$$

$$R_{p, \min(2, \sigma)}^\alpha \subset H_\sigma^\alpha, \quad (39)$$

$$R_{*, 1/\alpha}^\alpha \subset B_{1/\alpha}^\alpha, \quad (40)$$

$$R_{*, \min(2, 1/\alpha)}^\alpha \subset H_{1/\alpha}^\alpha, \quad (41)$$

proof. is divided into five cases:

1. Embedding (1.1.12) for $\sigma \leq 1$. Following Bernstein's classical method, we represent a function $f \in R_{p, \sigma}^\alpha$ in the form

$$f(z) = a_0 + \sum_{k=0}^\infty u_k(z) \quad (z \in D_+), \quad (42)$$

where u_k is a rational function of degree at most 2^{k+1} , with all its poles in D_- , that satisfies

$$\|u_k\|_{H_p} \leq 3R_{2^k}(f, H_p), \quad (43)$$

and a_0 is a constant such that $|a_0| \leq 2\|f\|_{H_p}$.

Taking account of the restriction $\sigma \leq 1$, we find from (30) and (43) that

$$\|f\|_{B_\sigma^\alpha}^\sigma \leq \|a_0\|_{B_\sigma^\alpha}^\sigma + \sum_{k=0}^\infty \|u_k\|_{B_\sigma^\alpha}^\sigma \leq c(\alpha, p) \|f\|_{R_{p, \sigma}^\alpha}^\sigma.$$

2. Embedding (38) for $\sigma > 1$. We again use (42) and (43), and also suppose that all $u_k \neq 0$. Let λ_k and h_k be the continuous functions of period 2π from Lemma (1.1.6) for u_k and $\beta = \alpha + 1$. We set

$$h_k^*(\varphi) = 2^{k/2} h_0(\varphi) + 2^{(k-1)/2} h_1(\varphi) + \cdots + h_k(\varphi).$$

Then, for every φ , we have

$$\frac{h_{k+1}^*(\varphi)}{h_k^*(\varphi)} \geq \sqrt{2} \quad (\varphi \in [0, 2\pi]), \quad \|h_k^*\|_{1, [0, 2\pi]} \leq 2^{k+2}, \quad (44)$$

$$\|\lambda_k\|_{p, [0, 2\pi]} \leq c_1(\alpha, p) R_{2^k}(f, H_p), \quad (45)$$

$$\left| J^{\alpha+1} u_k \left((1-x) e^{i\varphi} \right) \right| \leq \lambda(\varphi) \left(\min(x^{-1}, h_k^*(\varphi)) \right)^{\alpha+1} \quad (x \in (0,1)).$$

Therefore we find from Lemma (1.1.11) that for every $\varphi \in [0, 2\pi]$

$$\int_0^1 \left| J^{\alpha+1} f \left((1-x) e^{i\varphi} \right) \right|^\sigma x^{\sigma-1} dx \leq c_2(\alpha, p) |a_0|^\sigma + c_3(\alpha, p) \sum_{k=0}^{\infty} \left[\lambda_k(\varphi) (h_k^*(\varphi))^\alpha \right]^\sigma. \quad (46)$$

From Holder's inequality and (44) and (45) we obtain

$$\int_0^{2\pi} \left[\lambda_k(\varphi) (h_k^*(\varphi))^\alpha \right]^\sigma d\varphi \leq c_4(\alpha, p) \left(2^{k\alpha} R_{2^k}(f, H_p) \right)^\sigma.$$

Thus the required embedding follows from (46). If some $u_k \equiv 0$ in (42), we have to make evident modifications in the proof.

3. Embedding (39) for $\sigma \in (0, 2]$. This follows from (38) and (8).

4. Embedding (39) for $\sigma > 2$. This is proved just like (38) for $\sigma > 1$. Here, along with Lemmas (1.1.6) and (1.1.11), we also have to use the Littlewood-Paley theorem [5] according to which

$$\|f\|_{H_p^\sigma}^2 \leq c(\alpha, p) \left\| \int_0^1 \left| J^{\alpha+1} f \left((1-x) e^{i\varphi} \right) \right|^2 x dx \right\|_{\sigma/2, [0, 2\pi]} \quad (\sigma > 2).$$

5. Embeddings (40) and (41). These are proved just like the embeddings (38) and (39) respectively. Theorem (1.1.12) is showed.

Lemma (1.1.13)[1]: If $\alpha > 0$, $0 < q < \infty$, and if the sequence $\{b_k\}_0^\infty$ is nonincreasing and tends to zero, and the series

$$\sum_{k=0}^{\infty} (2^{k\alpha} b_k)^q \quad (47)$$

diverges, then the series

$$\sum_{k=0}^{\infty} (2^{k\alpha} (b_k - b_{k+1}))^q \quad (48)$$

also diverges.

Proof. Suppose that (48) converges. We show that in this case

$$b_k^q \leq c_1(\alpha, p) 2^{-\gamma q k} \sum_{j=k}^{\infty} 2^{j\gamma q} \beta_j^q \quad \left(\gamma = \frac{\alpha}{2}, \beta_j = b_j - b_{j+1} \right) \quad (49)$$

for all $k = 0, 1, \dots$. In fact, since $b_k \downarrow 0$, then $b_k = \beta_k + \beta_{k+1} + \dots$ and since (48) converges we have, for $q \leq 1$,

$$b_k^q \leq \sum_{j=k}^{\infty} \beta_j^q \leq 2^{-\gamma q k} \sum_{j=k}^{\infty} (2^{j\gamma} \beta_j)^q.$$

If $q > 1$, let $q' = q(q-1)^{-1}$ and from Holder's inequality we obtain

$$b_k \leq \left(\sum_{j=k}^{\infty} 2^{-jq'\gamma} \right)^{1/q'} \left(\sum_{j=k}^{\infty} (2^{j\gamma} \beta_j)^q \right)^{1/q} \leq c_2(\alpha, q) 2^{-\gamma k} \left(\sum_{j=k}^{\infty} (2^{j\gamma} \beta_j)^q \right)^{1/q}.$$

Thus we obtain (49) from the preceding two relations. From (49) we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} (2^{k\alpha} b_k)^q &\leq c_3(\alpha, q) \sum_{k=0}^{\infty} 2^{k\gamma q} \sum_{j=k}^{\infty} (2^{j\gamma} \beta_j)^q \\ &= c_3(\alpha, q) \sum_{j=0}^{\infty} \sum_{k=0}^j (2^{(j+k)\gamma} \beta_j)^q \leq c_4(\alpha, q) \sum_{j=0}^{\infty} (2^{j\alpha} (b_j - b_{j+1}))^q. \end{aligned}$$

The last inequality contradicts the divergence of (47). This completes the proof of Lemma (1.1.13).

For use below, we introduce the notation

$$\beta_{n,j} = \pi/j \quad (n \in \mathbb{N}, 2^n - 1 \leq j \leq 2^{n+1} - 2),$$

$$G_{n,j} = \{z : 1 - 2^{-2^n} \leq |z| < 1, \arg z = \beta_{n,j}\}, \quad G_n = \bigcup_{j=2^n-1}^{2^{n+1}-2} G_{n,j}.$$

Lemma (1.1.14)[1]: Let $\alpha > 0$, $p \in (1, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$. Then for every $n \in \mathbb{N}$ there is a rational function φ_n of degree 2^n that satisfies the conditions

- a. $\|\varphi_n\|_{H_p} \leq c_1(\alpha, p),$
- b. $\|\varphi_n^{(\alpha)}\|_{\sigma, G_n} \geq 2^{n\alpha} c_2(\alpha, p),$
- c. $\|\varphi_n^{(\alpha)}\|_{\infty, G_m} \leq 1 (n \neq m).$

Proof. We set

$$\varphi_n(z) = 2^{-n/p} \delta^{1-1/p} \sum_{j=2^n-1}^{2^{n+1}-2} \varphi_{n,j}(z)$$

$$\varphi_{n,j}(z) = (z_{n,j} - z)^{-1}, \quad z_{n,j} = (1 + \delta) e^{i\beta_{n,j}} \quad (\delta > 0).$$

It is easily shown that

$$\lim_{\delta \rightarrow +0} \delta^{1-1/p} \|\varphi_{n,j}\|_{H_p} = \left(\sqrt{\pi} \Gamma\left(\frac{p-1}{2}\right) / \Gamma\left(\frac{p}{2}\right) \right)^{1/p}, \quad (50)$$

$$\lim_{\delta \rightarrow +0} \delta^{1-1/p} \|\varphi_{n,j}^{(\alpha)}\|_{\sigma, G_{n,j}} = \Gamma(1 + \alpha) \left(\sigma - \frac{\sigma}{p} \right)^{1/\sigma}. \quad (51)$$

In (50) the right-hand side is to be taken to be 1 for $p = \infty$; to obtain (51) we need to use the equality

$$\varphi_{n,j}^{(\alpha)}(z) = \Gamma(1 + \alpha) (1 - z z_{n,j}^{-1})^{-1-\alpha} z_{n,j}^{-1-\lfloor \alpha \rfloor},$$

which follows from (1). The functions φ_n and $\varphi_n^{(\alpha)}$ tend uniformly to zero as $\delta \rightarrow +0$, outside an arbitrarily small neighborhood of G_n . Hence it follows from (50) and (51) that φ_n satisfies conditions (a-c) for sufficiently small $\delta > 0$. This completes the proof of Lemma (1.1.14).

Theorem (1.1.15)[1]: Let $\alpha > 0$, $p \in (1, \infty]$, and $\sigma = (\alpha + p^{-1})^{-1}$.

- i. Corresponding to every sequence $\{a_n\}_1^\infty$ that is nonincreasing and tends to zero, and satisfies

$$\sum_{k=0}^\infty (2^k a_{2^k})^{\min(2, \sigma)} = +\infty, \quad (52)$$

there is an $f \in H_p$ such that $R_n(f, H_p) = O(a_n)$ and $f \notin H_\sigma^\alpha$.

- ii. Corresponding to every sequence $\{a_n\}_1^\infty$ that is nonincreasing and tends to zero, and satisfies

$$\sum_{k=0}^\infty (2^k a_{2^k})^\sigma = +\infty, \quad (53)$$

there is an $f \in H_p$ such that $R_n(f, H_p) = O(a_n)$ and $f \notin B_\sigma^\alpha$.

Thus, embeddings (38) and (39) cannot be improved. It follows from (10) that, in the same sense, embeddings (40) and (41) also cannot be improved. Moreover, by a result of Peller [14], [28], there is actually equality in (40). In addition, since (38) admits an inverse for $1 < p < \infty$, see, assertion (i) of Theorem (1.1.15) is of interest only when $p = \infty$. Since the proof is the same for all p , we take $p \in (1, \infty]$ for the sake of completeness of presentation. Assertion (i) for $\alpha = 1$ and $p = \infty$, and (ii) for $\alpha = \frac{1}{2}$ and $p = \infty$ in Theorem (1.1.15), were obtained previously by Dolzhenko [18].

proof. is divided into four cases.

a. Assertion i) for $\sigma \leq 1$. As the required function we take

$$f(z) = \sum_{k=1}^{\infty} p_k \varphi_k(z),$$

where $p_k = a_{2^{k+1}} - a_{2^{k+2}}$ and the φ_k are the rational fractions from Lemma (1.1.14). From condition 1) of Lemma (1.1.14) we obtain

$$R_{2^j}(f, H_p) \leq \left\| \sum_{k=j}^{\infty} p_k \varphi_k \right\|_{H_p} \leq c_1(\alpha, p) a_{2^{j+1}}$$

for every $n \in \mathbb{N}$, and consequently $R_n(f, H_p) = O(a_n)$ as $n \rightarrow \infty$. On the other hand, for arbitrary $n \in \mathbb{N}$ we have from conditions 2) and 3) of Lemma (1.1.14).

$$\|f^{(\alpha)}\|_{\sigma, G_n}^{\sigma} \geq 2^{-\sigma} \|p_n \varphi_n\|_{\sigma, G_n}^{\sigma} - \left\| \sum_{\substack{k=1 \\ k \neq n}}^{\infty} p_k \varphi_k \right\|_{\sigma, G_n}^{\sigma} \geq c_2(\alpha, p) [(2^{n\alpha} p_n)^{\sigma} - a_2^{\sigma} 2^{-n}].$$

Setting $G = \bigcup_1^{\infty} G_n$, we obtain $\|f^{(\alpha)}\|_{\sigma, G} = +\infty$ from Lemma (1.1.13) and (52), and consequently, by Carleson's embedding theorem [19], $f^{(\alpha)} \notin H_{\sigma}$. The proof of this part of the theorem is completed by applying Lemma (1.1.1).

b. Assertion i) for $\sigma > 0$. As the required function we take

$$f(z) = \sum_{k=1}^{\infty} p_k z^{2^k} \quad (p_k = a_{2^{k+1}} - a_{2^{k+2}}). \quad (54)$$

Evidently $R_n(f, H_p) = O(a_n)$. On the other hand, for every $p \in (0, 1)$ we have, by Holder's inequality and Parseval's equality,

$$\|J^{\alpha} f(\rho \cdot)\|_{\sigma, T} \geq (2\pi)^{1/\sigma-1/2} \left(\sum_{k=1}^{\infty} (\rho_k 2^{k\alpha} \rho^{2^k})^2 \right)^{-1/2}.$$

Consequently, we obtain $f \notin H_{\sigma}^{\alpha}$ by letting $\rho \rightarrow 1-0$ and using Lemma (1.1.13) and (52).

c. Assertion ii) for $\sigma \leq 2$. This follows from assertion i) and (8).

d. Assertion ii) for $\sigma > 2$. We show that the function (54) is the required function. In fact, let $\rho_n = 1 - 2^{-n}$, $n \in \mathbb{N}$, and $\rho \in [\rho_n, \rho_{n+1}]$. Then, by Holder's inequality and Parseval's theorem,

$$\int_0^{2\pi} |J^{\alpha+1} f(\rho e^{i\varphi})|^{\sigma} d\varphi \geq (2\pi)^{1-2/\sigma} \left(\int_0^{2\pi} |J^{\alpha+1} f(\rho e^{i\varphi})|^2 d\varphi \right)^{\sigma/2} \geq c_3(\alpha, p) (2^{n(\alpha+1)} p_n)^{\sigma}.$$

By (53) we find from Lemma (1.1.13) that

$$\|f\|_{B_{\sigma}^{\alpha}}^{\sigma} \geq \sum_{n=1}^{\infty} \int_{\rho_n}^{\rho_{n+1}} d\rho \int_0^{2\pi} |J^{\alpha+1} f(\rho e^{i\varphi})|^{\sigma} (1-\rho)^{\sigma-1} d\varphi = +\infty.$$

This completes the proof of Theorem (1.1.15).

We denote by $\omega_k(\delta, f)_p$ ($k \in \mathbb{N}$, $\delta \geq 0$, $f \in L_p(T)$) the k th order modulus of smoothness of f , i.e.

$$\omega_k(\delta, f)_p = \sup_{|h| \leq \delta} \left\| \sum_{\nu=0}^k (-1)^{k-\nu} C_k^{\nu} f(e^{i(\cdot+\nu h)}) \right\|_{p, [0, 2\pi]}.$$

Corollary (1.1.16)[1]: if l is the smallest positive integer such that $l \geq \alpha$, then for every $\delta \in (0, \frac{1}{2}]$

$$\omega_l(\delta, f)_{\sigma} \leq c(\alpha, p) \delta^{\alpha} \left[\sum_{0 \leq m \leq \log_2(1/\delta)} (2^{m\alpha} R_{2^m}(f, H_p))^{\sigma} \right]^{1/\sigma}. \quad (55)$$

To obtain (55) we observe that for $\alpha \notin \mathbb{N}$ we have $l = k$ and it suffices to suppress the terms with $m = 0, 1, \dots, n-1$ on the left-hand side of (61). However, if $\alpha \in \mathbb{N}$, then $l = k-1$ and by Marchaud's inequality (see, [20]) the left-hand side of (61) majorizes $c_1(\alpha, p)(2^{nl} \omega_l(2^{-n}, f)_\sigma)^\sigma$.

In view of Corollary (1.1.9), inequalities (61) and (55) remain valid if we suppose that $f \in L_p(T)$ and $R_{2^m}(f, H_p)$ is replaced by $R_{2^m}(f, L_p(T))$, the best approximation to f in $L_p(T)$ by rational fractions of degree $2^m - 1$.

An inequality of the type of (55) was obtained by Dolzhenko [21] for $\alpha = 1$ and $p = \infty$; by Sevast'yanov [22] for $\alpha \in (0, 1)$ and $p = \infty$; and finally by Brudnyi [23] for $\alpha \geq 1 - p^{-1}$, $p \in [1, \infty]$, and with k instead of l .

For the proof of Theorem (1.1.20) we require the following two lemmas.

Lemma (1.1.17)[1]: Let $p \in (0, \infty]$, $s = \min(1, p)$, $k \in \mathbb{N}$ and $f \in B_{p,s}^0$. Then for every $\delta \in (0, 1)$

$$\omega_k(\delta, f)_p \leq c(k, p) \left(\int_{1-\delta}^1 \|J^k f(\rho)\|_{H_p}^s (1-\rho)^{ks-1} d\rho \right)^{1/s}.$$

Proof. For every $z \in D_+$ we have $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{v=0}^k C_k^v (-1)^v f \left(\left(1 - \frac{v}{k} \delta \right) z \right), \quad f_2(z) = f(z) - f_1(z).$$

From Lemma (1.1.1) and a result of Storozhenko [24] we obtain, since $\|g(\rho)\|_p$ is nondecreasing with respect to ρ ($g \in H_p$),

$$\omega_k(\delta, f_2)_p \leq c_1(k, p) \delta^k \|J^k f_2\|_{H_p} \leq c_2(k, p) \delta^k \|J^k f(\cdot(1-\delta/k))\|_{H_p}. \quad (56)$$

From the properties of finite differences [25] we have, for every $z \in D_+$,

$$\begin{aligned} |f_1(z)| &\leq \left| z^k \int_0^{\delta/k} dt_1 \int_0^{\delta/k} dt_2 \cdots \int_0^{\delta/k} f^{(k)}((1-(t_1+t_2+\cdots+t_k))z) dt_k \right| \\ &\leq \int \cdots \int_{\substack{t_1, t_2, \dots, t_k > 0 \\ t_1+t_2+\cdots+t_k \leq \delta}} |f^{(k)}((1-(t_1+t_2+\cdots+t_k))z)| dt_1 dt_2 \cdots dt_k \\ &= \frac{1}{(k-1)!} \int_0^\delta |f^{(k)}((1-t)z)|^{k-1} dt. \end{aligned} \quad (57)$$

If $p \in [1, \infty]$ we find from (57) that

$$\|f_1\|_p \leq \frac{1}{(k-1)!} \int_0^\delta \|f^{(k)}(\cdot(1-t))\|_{H_p} t^{k-1} dt. \quad (58)$$

Therefore we obtain the necessary inequality for $p \in [1, \infty]$ from (56), (58), and Lemma (1.1.1). For $p \in (0, 1)$ we introduce

$$F(z) = \max_{0 \leq \tau \leq 1} |f^{(k)}(\tau z)|.$$

We find from (57) that

$$\begin{aligned} |f_1(z)|^p &\leq [(k-1)!]^{-p} \left(\sum_{m=0}^\infty \int_{2^{-(m+1)\delta}}^{2^{-m\delta}} F((1-t)z) t^{k-1} dt \right)^p \\ &\leq c_3(k, p) \int_{1-\delta}^1 F^p(\rho z) (1-\rho)^{kp-1} d\rho. \end{aligned} \quad (59)$$

Using the fact that $\|F(\cdot\rho)\|_p \leq c_4(p) \|f^{(k)}(\cdot\rho)\|_p$ for every $p \in (0, 1)$ [2], we obtain the conclusion of Lemma (1.1.17) for $p \in (0, 1)$ from (56), (58) and Lemma (1.1.1). This completes the proof of Lemma (1.1.17).

Remarks (1.1.18)[1]: 1) With a corresponding definition of $\|f\|'_{B_{p,q}^\alpha}$ the conclusion of the lemma remains valid for $q = \infty$.

2) The lemma is well known for $p \in [1, \infty]$ and $q \in [1, \infty]$ (see, for example, [26]).

3) For the proof of Theorem (1.1.20) we need only the necessity for $p = q$.

Lemma (1.1.19)[1]: Let $\alpha > 0$, $p \in (0, \infty]$ and $q \in (0, \infty)$, and let k be the smallest positive integer such that $k > \alpha$. Then a function $f \in H_p$ belongs to class B_p^α if and only if

$$\|f\|'_{B_{p,q}^\alpha} = \|f\|_{H_p} + \left(\sum_{m=1}^{\infty} \left(2^{m\alpha} \omega_k(2^{-m}, f)_p \right)^q \right)^{1/q} < \infty. \quad (60)$$

Here the quasinorm (60) is equivalent to the quasinorm $\|f\|_{B_{p,q}^\alpha}$.

Proof. For $j \in \mathbb{N}$ we introduce $\mu_j = \|J^k f((1-2^{-j})\cdot)\|_{H_p}$. From Lemma (1.1.17) we obtain

$$\omega_k(2^{-m}, f)_p \leq c_1(k, p) \left(\sum_{j=m}^{\infty} (2^{-kj} \mu_j)^s \right)^{1/s}, \quad s = \min(1, p).$$

As in the proof of Lemma (1.1.13), we obtain

$$(2^{m\alpha} \omega_k(2^{-m}, f)_p)^q \leq c_2(\alpha, p, q) 2^{\gamma m q} \sum_{j=m}^{\infty} (2^{-(k-\gamma)j} \mu_j)^q, \quad \gamma = \frac{\alpha}{2}.$$

Consequently $\|f\|'_{B_{p,q}^\alpha} \leq c_3(\alpha, p, q) \|f\|_{B_{p,q}^\alpha}$, since $B_{p,q}^\alpha \subset H_p$ for $\alpha > 0$. The reverse inequality follows from a result of Storozenko [27]:

$$\|f^{(k)}(\rho \cdot)\|_{H_p} \leq c_4(p, k) (1-\rho)^{-k} \omega_k(1-\rho, f)_p \quad \left(\frac{1}{2} \leq \rho < 1\right)$$

and Lemma (1.1.1). This completes the proof of Lemma (1.1.19).

Theorem (1.1.20)[1]: Let $\alpha > 0$, $p \in (1, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$, and let k be the smallest positive integer such that $k > \alpha$.

i. if $f \in H_p$ then for every $n \in \mathbb{N}$

$$\sum_{m=0}^n (2^{m\alpha} \omega_k(2^{-m}, f)_\sigma)^\sigma \leq c(\alpha, p) \sum_{m=0}^n (2^{m\alpha} R_{2^m}(f, H_p))^\sigma. \quad (61)$$

ii. If $f \in BMOA$ then for every $n \in \mathbb{N}$

$$\sum_{m=0}^n (2^{m\alpha} \omega_k(2^{-m}, f)_{1/\alpha})^{1/\alpha} \leq c(\alpha) \sum_{m=0}^n (2^{m\alpha} R_{2^m}(f, BMOA))^{1/\alpha}. \quad (62)$$

Proof. Let $f \in H_p$, $1 < p \leq \infty$, and let r_n be a rational function of degree $2^n - 1$ for which $\|f - r\|_{H_p} \leq 2R_{2^n}(f, H_p)$. From (38) and Lemma (1.1.19) we obtain

$$\|r_n\|'_{B_\sigma^\alpha} \leq c_1(\alpha, p) \|r_n\|_{R_{p,\alpha}^\alpha} \quad (\alpha > 0, \sigma = (\alpha + p^{-1})^{-1}). \quad (63)$$

Evidently, $R_{2^j}(r_n, f) = 0$ for $j \geq n$, and

$$R_{2^j}(r_n, H_p) = R_{2^j}(f - (f - r_n), H_p) \leq R_{2^j}(f, H_p) + \|f - r_n\|_{H_p} \leq 3R_{2^j}(f, H_p)$$

For $j = 0, 1, \dots, n-1$. On the other hand, for every $j \geq \mathbb{N}$,

$$\begin{aligned} \omega_k(2^{-j}, r_n)_\sigma &= \omega_k(2^{-j}, f - (f - r_n))_\sigma \geq -2^{-1/\sigma} \omega_k(2^{-j}, f)_\sigma - \omega_k(2^{-j}, f - r_n)_\sigma \\ &\geq 2^{-1/\sigma} \omega_k(2^{-j}, f)_\sigma - 2^{k+1} R_n(f, H_p). \end{aligned}$$

Consequently, from (63) we obtain

$$\sum_{m=1}^n (2^{m\alpha} \omega_k(2^{-m}, f)_\sigma)^\sigma \leq c_2(\alpha, p) \|f\|_p + c_3(\alpha, p) \sum_{m=0}^n (2^{m\alpha} R_{2^m}(f, H_p))^\sigma.$$

Now if in the preceding inequality we replace $f(z)$ by $f(z) - f(0)$ and use the inequality

$$\|f(z) - f(0)\|_{H_p} \leq c_4(p) R_1(f, H_p),$$

we obtain (61). Inequality (62) is showed similarly. This completes the proof of Theorem (1.1.20).

Section (1.2): Smooth Functions and Bernstein Type Inequalities

In this section we study the relationship between the smoothness of the symbol of a Toeplitz operator and the smoothness of functions belonging to its kernel.

Notation. $\mathbb{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C}: |z| < 1\}$; $\mathbb{T} \stackrel{\text{def}}{=} \partial\mathbb{D}$; m is normalized Lebesgue measure on $\mathbb{T}$; $L^p \stackrel{\text{def}}{=} L^p(\mathbb{T}, m)$, $0 < p \leq \infty$; H^p is the Hardy space [38, 39, 43] of holomorphic functions on $\mathbb{D}$, also treated as a subspace in L^p ; $H_0^p \stackrel{\text{def}}{=} \{f \in H^p: f(0) = 0\}$; $P_+(P_-)$ is the orthogonal projection from L^2 onto $H^2(\overline{H}_0^2)$, extended, if necessary, to L^1 in a natural way.

Let $\varphi \in L^\infty$, and let T_φ be the Toeplitz operator with symbol φ , so that $T_\varphi f \stackrel{\text{def}}{=} P_+(\varphi f)$, $f \in H^1$. By $K_p(\varphi)$, $1 \leq p \leq \infty$, we denote the kernel of T_φ in the Hardy space H^p :

$$K_p(\varphi) \stackrel{\text{def}}{=} \{f \in H^p: T_\varphi f = 0\}.$$

Assume in addition that the function φ is unimodular (i.e., $|\varphi| = 1$ a.e. on $\mathbb{T}$) and possesses certain smoothness properties (say, belongs to the Sobolev space W_r^s or the Besov space B_r^s for some $r, s > 0$); moreover, assume $K_p(\varphi) \neq \{0\}$. What kind of conclusions can one derive concerning the differential properties of functions belonging to $K_p(\varphi)$? In various settings we provide an answer (which often turns out to be unimprovable, in a certain sense) to the above question.

Meanwhile, we note that restricting ourselves to unimodular symbols φ leads in fact to no loss of generality. Indeed, in [50, 51] it was shown that, given $\varphi \in L^\infty$ with $K_2(\varphi) \neq \{0\}$, one can find an h , $h \in H^2$, such that $K_2(\varphi) = K_2(\bar{z}\bar{h}/h)$. We also remark that if θ is an inner function (i.e., a unimodular function lying in H^∞) then the subspace $K_p(\bar{\theta})$ is invariant under the backward shift operator S^* and coincides with the class $K_\theta^p \stackrel{\text{def}}{=} H^p \cap \theta \overline{H}_0^p$. Further, if $\theta \in W_r^s$ or $\theta \in B_r^s$ with $s, r \geq 1$, then θ must be a finite Blaschke product, whereas the functions in K_θ^p are rational fractions.

This section contains a number of statements of the following form:

$$|\varphi| = 1, \quad \varphi \in X \Rightarrow K_p(\varphi) \subset Y,$$

where X and Y are certain spaces of smooth functions on the circle. The results obtained here can also be restated as "Bernstein-type inequalities,"

$$\|f\|_Y \leq \text{const} \|\varphi\|_X^\alpha \|f\|_p, \quad f \in K_p(\varphi), \quad (64)$$

where $\alpha > 0$, $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms (or quasinorms) in X and Y , respectively, while $\|\cdot\|_p$ stands for the L^p norm. As special cases of these inequalities we obtain, first, the estimate

$$\|f^{(n)}\|_p \leq \text{const} \|\theta'\|_\infty^n \|f\|_p, \quad f \in K_\theta^p,$$

due to the author [41] which generalizes the well-known S. N. Bernstein inequality for polynomials (or entire functions), and second, the estimates due to A. A. Pekarskii [1] for derivatives of rational fractions.

We recall that the classical S. N. Bernstein inequality

$$\|Q'\|_p \leq n \|Q\|_p$$

(where Q is a polynomial of degree $\leq n$, i.e., $Q \in K_{z^{n+1}}^p$) arises in the proofs of various inverse theorems of the polynomial approximation theory [36, 44], whereas the above-mentioned A. A. Pekarskii inequalities are used as a tool in the proofs of similar theorems of the rational approximation theory. Similarly, the inequalities of the form (64) provided below can be used to derive certain inverse approximation theorems yielding in turn both the classical (i.e., polynomial) Bernstein-type theorems and the

inverse theorems from [1] pertaining to the rational approximation theory.

Yet another application of (64) enables us to find out how the boundary smoothness of the argument $\arg f$ of an analytic function f affects the smoothness of f itself.

Finally, as by-products, this section contains some results concerning the convergence of Fourier series for inner functions θ (and also for functions lying in K_θ^p) at a fixed point of the circle.

Let $W_r^1 = W_r^1(\mathbb{T})$, $r > 1$, denote the Sobolev space, consisting of those absolutely continuous functions g on $\mathbb{T}$ for which $g' \in L^r$. (Here $g'(\zeta) \stackrel{\text{def}}{=} dg/d\zeta = -ie^{it} dg/dt$, $\zeta = e^{it} \in \mathbb{T}$.)

Theorem (1.2.1)[34]: (Calderon [47]). Let p, q and r satisfy the hypotheses of Theorem (1.2.3), and let b be an absolutely continuous function on the real line $\mathbb{R}$ such that $b' \in L^r(\mathbb{R})$. Then the singular integral operator C_b ("Calderon's commutator") defined by

$$(C_b g)(x) \stackrel{\text{def}}{=} v.p. \int_{-\infty}^{\infty} \frac{b(x) - b(y)}{(x - y)^2} g(y) dy,$$

is a bounded mapping from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. In addition, we have

$$\|C_b\|_{L^p(\mathbb{R}) \rightarrow L^q(\mathbb{R})} \leq c(p, r) \|b'\|_r. \quad (65)$$

A slightly modified version of Theorem (1.2.1) will be needed. In fact, Calderon's proof shows that (65) remains valid if C_b is replaced by $C_b^{(\varepsilon)}$, $\varepsilon > 0$, where

$$(C_b^{(\varepsilon)} g)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{b(x) - b(y)}{(x - y + i\varepsilon)^2} g(y) dy,$$

(Moreover, the constant on the right side of (65) is independent of ε .) The corresponding result for the circle reads as follows.

Theorem (1.2.2)[34]: Under the same assumptions on p, q, r and under the assumption $b \in W_r^1$ the operator $C_{b, \rho}$ ($0 < \rho < 1$) given by

$$(C_{b, \rho} g)(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{b(\zeta) - b(\xi)}{(\zeta - \rho\xi)^2} g(\zeta) d\zeta \quad (\xi \in \mathbb{T}) \quad (66)$$

is a bounded mapping from $L^p = L^p(\mathbb{T}, m)$ to $L^q = L^q(\mathbb{T}, m)$; furthermore,

$$\sup_{0 < \rho < 1} \|C_{b, \rho}\|_{L^p \rightarrow L^q} \leq c(p, r) \|b'\|_r.$$

Theorem (1.2.3)[34]: Let $1 < p, q < +\infty$, $1 < r \leq +\infty$, and $q^{-1} = p^{-1} + r^{-1}$. If $\varphi \in W_r^1$, $|\varphi| \equiv 1$, then

$$K_p(\varphi) \subset W_q^1;$$

moreover, for $f \in K_p(\varphi)$ one has

$$\|f'\|_q \leq c(p, r) \|\varphi'\|_r \|f\|_p,$$

where $c(p, r)$ is a positive constant depending only on p and r .

The proof is based on.

Proof. Since $f \in K_p(\varphi)$, one has $f\varphi \in \overline{H_0^p}$, and so $\int f(\zeta)\varphi(\zeta)(\zeta - z)^{-2} d\zeta = 0$ for any $z \in \mathbb{D}$. Therefore, given $\rho \in (0, 1)$ and $\xi \in \mathbb{T}$, we have

$$f'(\rho\xi) = \frac{1}{2\pi i} \int \frac{f(\zeta) d\zeta}{(\zeta - \rho\xi)^2} = \frac{1}{2\pi i} \int f(\zeta)\varphi(\zeta) \frac{\bar{\varphi}(\zeta) - \bar{\varphi}(\xi)}{(\zeta - \rho\xi)^2} d\zeta = \frac{1}{2\pi i} (C_{\bar{\varphi}, \rho}(f\varphi))(\xi)$$

(this last notation was introduced in (66) above). Applying Theorem (1.2.1) we get

$$\sup_{0 < \rho < 1} \left(\int |f'(\rho\xi)|^q dm(\xi) \right)^{1/q} \leq c(p, r) \|\varphi'\|_r \|f\varphi\|_p = c(p, r) \|\varphi'\|_r \|f\|_p$$

Thus, $f' \in H^q$ and the desired inequality holds true.

Corollary (1.2.4)[34]: Let $1 \leq p < +\infty$. If $\varphi \in Lip1$ ($\stackrel{\text{def}}{=} W_\infty^1$) and $|\varphi| \equiv 1$, then $K_p(\varphi) \subset W_p^1$; moreover, for $f \in K_p(\varphi)$ one has

$$\|f'\|_p \leq c_p \|\varphi'\|_\infty \|f\|_p. \quad (67)$$

Proof. In the case $1 \leq p < +\infty$ it suffices to apply Theorem (1.2.3) with $r = +\infty$. For $p = 1$

inequality (67) (and hence also the inclusion $K_1(\varphi) \subset W_1^1$) can be derived from the proof of Theorem (1.2.3) Indeed, the operator $C_{\bar{\varphi},\rho}$ is a Calderdn-Zygmund operator. Therefore (see, e.g.[40]), being a continuous mapping from L^2 to L^2 , it also acts from $H_{\mathbb{R}}^1 \stackrel{\text{def}}{=} H^1 + \overline{H_0^1}$ to L^1 ; moreover,

$$\sub_{0 < p < 1} \|C_{\bar{\varphi},\rho}\|_{H_{\mathbb{R}}^1 \rightarrow L^1} \leq \text{const} \cdot \sub_{0 < p < 1} \|C_{\bar{\varphi},\rho}\|_{L^2 \rightarrow L^2} \leq \text{const} \|\varphi'\|_{\infty}.$$

Since $f \in K_1(\varphi)$, it follows that $f\varphi \in \overline{H_0^1}$ and

$$\|f'\|_1 = \frac{1}{2\pi} \sub_{0 < p < 1} \|C_{\bar{\varphi},\rho}(f\varphi)\|_1 \leq \text{const} \|\varphi'\|_{\infty} \|f\|_1.$$

Now let Λ^α , $0 < \alpha < +\infty$, denote the Holder class (Zygmund class, if $\alpha \in \mathbb{N}$) on the circle:

$$\Lambda^\alpha \stackrel{\text{def}}{=} \left\{ g \in C(\mathbb{T}) : \sub_{h>0} h^{-\alpha} \|\Delta_h^m g\|_{\infty} < +\infty \right\},$$

where m is any fixed integer with $m > \alpha$ and Δ_h^m stands for the m -th order difference operator with step h . (We recall that the operators Δ_h^k are defined by induction: $(\Delta_h^1 g)(\zeta) = (\Delta_h g)(\zeta) \stackrel{\text{def}}{=} g(e^{ih}\zeta) - g(\zeta)$ and $\Delta_h^k g \stackrel{\text{def}}{=} \Delta_h \Delta_h^{k-1} g$.) Further, we set $\Lambda^0 \stackrel{\text{def}}{=} L^\infty$.

A well-known theorem of Duren, Romberg, and Shields [49] says that the space $\Lambda_A^\alpha \stackrel{\text{def}}{=} P + \Lambda^\alpha$ is the dual of $H^{1/(1+\alpha)}$ with respect to the standard antilinear pairing. From this one can easily deduce the following assertion (which is also known to experts).

Lemma (1.2.5)[34]: Let $s > 0$, $\max(1, s) < p < +\infty$ and $\alpha = s^{-1} - p^{-1}$. Given $\psi \in \Lambda^\alpha$, the Hankel operator H_ψ defined by

$$H_\psi f = P_-(\psi f), \quad f \in H^2,$$

is a bounded mapping (or possesses an extension which is a bounded mapping) from H^s to $\overline{H_0^p}$; furthermore,

$$\|H_\psi\|_{H^s \rightarrow \overline{H_0^p}} \leq \text{const} \|\psi\|_{\Lambda^\alpha},$$

where $\|\cdot\|_{\Lambda^\alpha}$ is a natural norm in Λ^α and const is a constant depending only on α .

Now we point out another consequence of Theorem (1.2.3).

Corollary (1.2.6)[34]: Assume that φ is an absolutely continuous function on $\mathbb{T}$, $|\varphi| = 1$, and $f \in K_1(\varphi)$.

(a) If $1 < q < r \leq +\infty$, $\alpha \geq 0$, $s > 0$, and $q^{-1} = s^{-1} + r^{-1} - \alpha$, then

$$\|f'\|_q \leq c(s, r, \alpha) \|\varphi'\|_r \|\varphi\|_{\Lambda^\alpha} \|f\|_s. \quad (68)$$

(b) If $1 < q < r \leq +\infty$ then

$$\|f'\|_q \leq c(q, r) \|\varphi'\|_r \|\varphi\|_{\Lambda^{1/r}} \|f\|_q \quad (69)$$

(c) If $1 < q < 2$, then

$$\|f'\|_q \leq c_q \|\varphi'\|_2^2 \|f\|_q. \quad (70)$$

Proof. (a) Define the exponent p by $p^{-1} = q^{-1} - r^{-1}$. Then we have $\alpha = s^{-1} - p^{-1}$. On the one hand Theorem (1.2.3) yields $\|f'\|_p \leq \text{const} \|\varphi'\|_r \|f\|_p$. On the other hand, the above Lemma gives

$$\|f\|_p = \|f\varphi\|_p = \|H_\varphi f\|_p \leq \text{const} \|\varphi\|_{\Lambda^\alpha} \|f\|_s.$$

Combining the two inequalities, we arrive at (68).

(b) Apply (68) with $\alpha = \frac{1}{r}$, $s = q$.

(c) Apply (69) with $r = 2$ and note that $W_2^1 \subset \Lambda^{1/2}$.

Remarks (1.2.7)[34]: i. All the above inequalities are sharp in a sense. We consider, for example, inequality (68), which is the most general one. (Theorem (1.2.3) is contained in (68) as a special case where $\alpha = 0$.) Setting $f_a(\zeta) = (1 - \bar{a}\zeta)^{-n}$ and $\varphi_a(\zeta) = (\bar{\zeta} - \bar{a})(1 - a\bar{\zeta} - n$ where $a \in \mathbb{D}$, $n \in \mathbb{N}$, $ns > 1$ we have $f_a \in K_1 \varphi_a$, and a straightforward computation shows that

$$\|f_a'\|_q \asymp \|\varphi_a'\|_r \|\varphi_a\|_{\Lambda^\alpha} \|f_a\|_s \asymp (1 - |a|)^{1/q - n - 1}$$

As $|a| \rightarrow 1 - 0$. (Here the sign $\asymp$ means that the ratio of the two quantities is bounded from below and above by positive constants independent of a .)

ii. It is interesting to compare (67) and (70).

iii. In [41], the author proved a version of inequality (67) (including, in particular, the case $p = +\infty$) in the setting where $\varphi = \bar{\theta}$, θ being an inner function on the upper half-plane $\mathbb{C}_+$, and $f \in K_\theta^p \stackrel{\text{def}}{=} H^p \cap \overline{\theta H^p}$, $H^p = H^p(\mathbb{C}_+)$. If we set $\theta(z) = \exp(i\sigma z)$, $\sigma > 0$, the classical Bernstein inequality for entire functions follows (though not with a sharp constant). Furthermore, it was proved in [41] that, under the assumption $\theta' \in H^\infty(\mathbb{C}_+)$, the higher order derivatives $f^{(n)}$ satisfy

$$\|f^{(n)}\|_p \leq \text{const} \|f\|_p, \quad f \in K_\theta^p, \quad (71)$$

in which case one can take $\text{const} = c(n, p) \|\theta'\|_\infty^n$. The converse was also shown to be true: if (71) holds with a constant independent of f , then $\theta' \in H^\infty(\mathbb{C}_+)$.

Given $f \in L^p$, we set $\hat{f}(k) \stackrel{\text{def}}{=} \int_{\mathbb{T}} f \bar{z}^k dm$ ($k \in \mathbb{Z}$). We denote by $\mathcal{P}_n$ the space of trigonometric polynomials of degree $\leq n$, i.e. $\mathcal{P}_n \stackrel{\text{def}}{=} \{Q \in L^1: \hat{Q}(k) = 0 \text{ and } |k| > n\}$. Finally, let $E_p(f, n) \stackrel{\text{def}}{=} \inf\{\|f - Q\|_p: Q \in \mathcal{P}_n\}$, so that $E_p(f, n)$ is the minimal approximation error for $f \in L^p$ with respect to polynomials of degree $\leq n$. The following simple theorem makes it possible to estimate the Fourier (Taylor) coefficients of a function f , $f \in K_p(\varphi)$, in terms of the quantities $E_{p'}(\varphi, n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem (1.2.8)[34]: Let φ be a unimodular function on $\mathbb{T}$, and let $f \in K_p(\varphi)$, $1 \leq p \leq +\infty$. Then

$$|\hat{f}(n)| \leq \|f\|_p E_{p'}(\varphi, n), \quad n \in \mathbb{Z}_+. \quad (72)$$

Proof. Let $Q \in \mathcal{P}_n$. Since $f\varphi \in \overline{H_0^p}$ and $\bar{z}^n \bar{Q} \in \overline{H^\infty}$, it follows that $\int f\varphi \bar{z}^n \bar{Q} dm = 0$, and so

$$\hat{f}(n) = \int f \bar{z}^n dm = \int f\varphi \bar{z}^n (\bar{\varphi} - \bar{Q}) dm,$$

whence

$$|\hat{f}(n)| \leq \|f\|_p \|\varphi - Q\|_{p'}.$$

Taking the infimum over $Q \in \mathcal{P}_n$, we obtain (72).

Now we recall the definition of the Besov spaces B_{pq}^s ($1 \leq p, q \leq +\infty; s > 0$). Given $f \in L^p$,

$$f \in B_{pq}^s \stackrel{\text{def}}{\iff} \begin{cases} \int_{-\pi}^{\pi} \frac{\|\Delta_t^m f\|_p^q}{|t|^{1+sq}} dt < +\infty, & q < +\infty, \\ \|\Delta_t^m f\|_p = O\left(\frac{1}{n^s}\right), & q = +\infty. \end{cases}$$

where m is some fixed integer with $m > s$. We will also make use of the so-called constructive characterization of Besov classes (see e.g. [44]): given $f \in L^p$,

$$f \in B_{pq}^s \iff \begin{cases} \sum_{n=1}^{\infty} n^{sq-1} E_p(f, n)^q < +\infty, & q < +\infty, \\ E_p(f, n) = O\left(\frac{1}{n^s}\right), & q = +\infty. \end{cases}$$

Moreover, the two norms in B_{pq}^s arising in a natural way in connection with the two definitions above turn out to be equivalent.

As usual, we let $B_p^s \stackrel{\text{def}}{=} B_{pp}^s$. We also note that the spaces Λ^α introduced in the previous section coincide with B_∞^α .

Theorem (1.2.9)[34]: Let $1 \leq p, q \leq +\infty, s > 0$, and let φ be a unimodular function lying in B_{pq}^s ; assume further that $f \in K_{p'}(\varphi)$. The following statement's hold true:

(a) If $q < +\infty$, then

$$\left(\sum_{n=1}^{\infty} n^{sq-1} |\hat{f}(n)|^q\right)^{1/q} \leq \text{const} \|\varphi\|_{B_{pq}^s} \|f\|_{p'}, \quad (73)$$

where const depends only on p, q and s .

(b) If $q = +\infty$ then $|\hat{f}(n)| \leq \text{const} \|\varphi\|_{B_{p\infty}^s} \|f\|_p n^{-s}$.

Proof. (a) In view of (72),

$$|\hat{f}(n)| \leq \|f\|_p E_p(\varphi, n).$$

Raising the two sides to the power q , then multiplying by n^{sq-1} and summing over n , we arrive at (73). (It is here that the constructive characterization of B_{pq}^s is needed.)

The proof of part (b) is similar to the above (in fact, it is even simpler).

Corollary (1.2.10)[34]: Let φ be a unimodular function on $\mathbb{T}$, and let $f \in K_1(\varphi)$. If $1 \leq p \leq 2$ and $1 \leq q < +\infty$, then

$$\|f\|_{l_A^q} \leq \text{const} \|\varphi\|_{B_{pq}^{1/q}} \|f\|_{l_A^p},$$

where

$$\|f\|_{l_A^p} \stackrel{\text{def}}{=} \left(\sum_{n \geq 0} |\hat{f}(n)|^p \right)^{1/p}$$

Proof. It suffices to apply (73) with $s = q^{-1}$ and to use the Hausdorff-Young inequality $\|f\|_{p'} \leq \|f\|_{l_A^p}$.

Corollary (1.2.11)[34]: Let φ and f have the same meaning as above, and let $1 \leq p \leq +\infty$, $2 \leq q \leq +\infty$, $l > 0$. Then

$$\|f^{(l)}\|_q \leq \text{const} \|\varphi\|_{B_{p'q'}^{l+1/q}} \|f\|_p,$$

where $p' = p/(p-1)$, $q' = q/(q-1)$, and $f^{(l)}$ stands for the fractional derivative of order l , defined by

$$f^{(l)}(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} n^l \hat{f}(n) z^n.$$

Proof. We rewrite (73), replacing p with p' and q with q' :

$$\left(\sum_{n=1}^{\infty} n^{sq'-1} |\hat{f}(n)|^{q'} \right)^{1/q'} \leq \text{const} \|\varphi\|_{B_{p'q'}^s} \|f\|_p$$

Set $s = l + \frac{1}{q'}$. Then the left-hand side in the last inequality reduces to $\|f^{(l)}\|_{l_A^{q'}}$, and so, by the Hausdorff-Young theorem, it can be estimated from below by $\|f^{(l)}\|_q$.

Remarks (1.2.12)[34]: i. It might be interesting to compare Corollary (1.2.10) (which becomes interesting for $1 \leq q < p \leq 2$) with the inequality

$$\|f\|_p \leq \text{const} \|\varphi\|_{\Lambda^{1/q-1/p}} \|f\|_q$$

valid for $f \in K_1(\varphi)$ in the case $q > 0$, $\max(1, q) < p < +\infty$ (see the proof of Corollary (1.2.6)).

ii. Corollary (1.2.11), with $l = 1$ and $2 \leq q \leq p \leq +\infty$ yields a result which is close to Theorem (1.2.3) but cannot be reduced to it.

Now let $\varphi = \bar{\theta}$, where θ is an inner function on $\mathbb{D}$. We recall the notation $K_{\theta}^p \stackrel{\text{def}}{=} H^p \cap \overline{\theta H_0^p}$, $1 \leq p \leq +\infty$, and set $K_{*,\theta} \stackrel{\text{def}}{=} K_{\theta}^2 \cap BMOA$. (See [38, Chapter VI] for the definition of the space BMOA and a discussion of its properties.) The BMOA norm $\|\cdot\|_*$ will be introduced as follows:

$$\|f\|_* \stackrel{\text{def}}{=} \sup \left\{ \left| \int f \bar{g} dm \right| : g \in H^2, \|g\|_1 \leq 1 \right\},$$

So that $\|\cdot\|_*$ is equivalent to the standard BMO norm defined in terms of mean oscillations.

The next theorem is a refined version of Theorem (1.2.8) in the case where $\varphi = \bar{\theta}$ and $p = +\infty$.

Theorem (1.2.13)[34]: For $f \in K_{*,\theta}$ and $n \in \mathbb{Z}_+$ one has

$$|\hat{f}(n)| \leq \|f\|_* E_1(\theta, n). \quad (74)$$

Proof. Given $Q \in \mathcal{P}_n$, the proof of Theorem (1.2.8) yields

$$\hat{f}(n) = \int f \bar{\theta} \bar{z}^n (\theta - Q) dm.$$

Since $\bar{\theta} \bar{z}^n (\theta - Q) \in \overline{H^\infty}$, we have $|\hat{f}(n)| \leq \|f\|_* (\theta - Q)$. Taking the infimum over $Q \in \mathcal{P}_n$, we arrive at (74).

Theorem (1.2.9) admits a similar refinement: If $p = 1$ then, given $f \in K_{*,\theta}$, one can replace the inequality (73) by

$$\left(\sum_{n=1}^{\infty} n^{sq-1} |\hat{f}(n)|^q \right)^{1/q} \leq \text{const} \|\theta\|_{B_{1q}^s} \|f\|_*,$$

with the usual interpretation for $q = +\infty$:

$$|\hat{f}(n)| \leq \text{const} \|\theta\|_{B_{1q}^s} \|f\|_* n^{-s}. \quad (75)$$

We cite [45] as a source of some explicit criteria for the membership of an inner function in B_p^s with $sp < 1$; in connection with Blaschke products belonging to $B_{p\infty}^s$, see also [37].

Theorem (1.2.14)[34]: Assume that B is a Blaschke product in $\mathbb{D}$ whose zero sequence $\{a_k\}_{k=1}^{\infty}$ satisfies the "weak Newman condition"

$$\sup_j (1 - |a_j|)^{-1} \sum_{k>j} (1 - |a_k|) < +\infty \quad (76)$$

The following statements hold true:

- (a) For every f , $f \in K_{*,B}$, one has $\hat{f}(n) = O(1/n)$.
- (b) If, moreover, the Frostman condition

$$A(\zeta) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1 - |a_k|}{|\zeta - a_k|} < +\infty \quad (77)$$

holds at some point $\zeta \in \mathbb{T}$, then the series $\sum_{n=0}^{\infty} \hat{f}(n) \zeta^n$ converges for every function $f \in K_{*,B}$.

Proof. It is known [37] that (76) implies $B \in B_{1\infty}^1$. Applying (75) with $\theta = B$ and $s = 1$, we arrive at (a). To show (b), we use a result from [48] saying that (77) is equivalent to the existence of the radial limits $\lim_{r \rightarrow 1-0} f(r\zeta)$ for all $f \in K_{*,B}$. Combining this latter fact, part (a) above, and the Tauberian theorem of Littlewood, we conclude that the Fourier series of any such function f converges at ζ .

Theorem (1.2.15)[34]: Let $\zeta \in \mathbb{T}$, and let $\theta = BS_\mu$ be an inner function. If

$$\sum_k \frac{1 - |a_k|}{|\zeta - a_k|^2} + \int \frac{d\mu(t)}{|t - \zeta|^2} < +\infty, \quad (78)$$

then the series $\sum_{n \geq 0} \hat{\theta}(n) \zeta^n$ converges.

Proof. Let z be a fixed point of the disk, and let $k_z(t) \stackrel{\text{def}}{=} (1 - \overline{\theta(z)}\theta(t))(1 - \bar{z}t)^{-1}$ be the corresponding reproducing kernel in the space K_θ^2 . By Theorem (1.2.8),

$$|\hat{k}_z(n)| \leq \|k_z\|_2 E_2(\theta, n) = \|k_z\|_2 \left(\sum_{k=n+1}^{\infty} |\hat{\theta}(k)|^2 \right)^{1/2} \quad (79)$$

A straightforward verification yields

$$\hat{k}_z(n) = \bar{z}^n (1 - \overline{\theta(z)} \sum_{k=0}^n \hat{\theta}(k) \bar{z}^{-k}); \quad \|k_z\|_2 = \left(\frac{1 - |\theta(z)|^2}{1 - |z|^2} \right)^{1/2}. \quad (80)$$

As is well known (see e.g., [46]), condition (78) ensures that θ has an angular derivative at ζ . This means that the two limits $\lim_{r \rightarrow 1-0} \theta(r\zeta) \stackrel{\text{def}}{=} \theta(\zeta)$ and $\lim_{r \rightarrow 1-0} \theta'(r\zeta) \stackrel{\text{def}}{=} \theta'(\zeta)$ exist, and the former one satisfies $|\theta(\zeta)| = 1$. Moreover,

$$|\theta'(\zeta)| = \lim_{r \rightarrow 1-0} \frac{1 - |\theta(r\zeta)|}{1 - r} = \lim_{r \rightarrow 1-0} \frac{1 - |\theta(r\zeta)|^2}{1 - r^2} = \sum_k \frac{1 - |a_k|^2}{|\zeta - a_k|^2} + 2 \int \frac{d\mu(t)}{|t - \zeta|^2}$$

Substituting (80) in (79), letting $z = r\zeta$ and then making r tend to 1, we get from (79)

$$\left| \theta(\zeta) - \sum_{k=0}^n \hat{\theta}(k) \zeta^k \right| \leq |\theta'(\zeta)|^{1/2} \left(\sum_{k=n+1}^{\infty} |\hat{\theta}(k)|^2 \right)^{1/2},$$

whence the desired conclusion follows immediately.

The results contained on this section are based on the following elementary observation: if $f \in H^1$, then $f \in K_1(\bar{z}\bar{f}/f)$. Indeed,

$$T_{\bar{z}\bar{f}/f}f = P_+(\bar{z}\bar{f}) = 0$$

Due to this fact, we can use the "Bernstein-type inequality"

$$\|f\|_Y \leq \text{const} \|\varphi\|_X^\alpha \|f\|_p, \quad f \in K_p(\varphi) \quad (81)$$

(where X and Y are certain spaces of smooth functions) to derive the following corollary: if $f \in H^p$ and $\bar{f}/f \in X$, then $f \in Y$. To do this, one only needs to apply (81) with $\varphi = \bar{z}\bar{f}/f$ (note that multiplication by $\bar{z}$ preserves membership in a reasonable class X).

Thus, in many cases the smoothness of the function $\bar{f}/f$ (or $\arg f$) on the circle implies the smoothness of f itself, once $f \in H^p$. The next theorem comprises a few assertions to that effect. All of them are readily derived from inequalities of the form (81) that were established above.

Theorem (1.2.16)[34]: Let $f \in H^p$.

(a) If $1 < p < +\infty$, $1 < r \leq +\infty$, and $q^{-1} = p^{-1} + r^{-1} < 1$ (for $r = +\infty$, the values $p = q = 1$ are also admissible), then the implication

$$\bar{f}/f \in W_r^1 \Rightarrow f' \in H^q$$

holds true.

(b) If $1 < p < r$ and $r \geq 2$, then

$$\bar{f}/f \in W_r^1 \Rightarrow f' \in H^p.$$

(c) If $1 \leq p < q < +\infty$ and $\alpha = p^{-1} - q^{-1}$, then

$$\bar{f}/f \in \Lambda^\alpha \Rightarrow f \in H^q.$$

(d) If $1 \leq p \leq +\infty$, $2 \leq q \leq +\infty$, and $l > 0$, then

$$\bar{f}/f \in B_{p,q}^{l+1/q'} \Rightarrow f \in W_q^l, \quad \text{i.e.,} \quad f^{(1)} \in H^q.$$

(We recall that $f^{(l)}(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} n^l \hat{f}(n) z^n$, $p' \stackrel{\text{def}}{=} p/(p-1)$, and $q' \stackrel{\text{def}}{=} q/(q-1)$.)

Finally, we supplement this theorem with the following proposition, which can be derived from Corollary (1.2.10).

Proposition (1.2.17)[34]: Let $1 \leq q < p \leq 2$ and $f \in l_A^p$ (i.e., $f \in H^1$ and $\sum_{n \geq 0} |\hat{f}(n)|^p < +\infty$). If, in addition, $\bar{f}/f \in B_{pq}^{1/q}$, then $f \in l_A^q$.

Here we announce, without providing any proof, one more Bernstein type inequality for the space $K_\theta^p = H^p \cap \theta \bar{H}_0^p$, where θ is an inner function on $\mathbb{D}$.

The proof (which is rather laborious) will be published elsewhere. We remark, however, that the ideas involved are different from those used above.

First we recall the definition of the Sobolev space W_q^l ($p \geq 1$, $l > 0$). Namely, $W_p^l \stackrel{\text{def}}{=} \{f \in L^p: f^{(l)} \in L^p\}$, where $f^{(l)}(\zeta) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} |n|^l \hat{f}(n) \zeta^n$ ($\zeta \in \mathbb{T}$). It is well known that $B_p^l \subset W_q^l$ if $1 \leq p \leq 2$, and $W_q^l \subset B_p^l$ if $p \geq 2$.

The norm $\|\cdot\|_{W_p^l}$ will be defined by $\|f\|_{W_p^l} = |\hat{f}(0)| + \|f^{(l)}\|_p$.

Theorem (1.2.18)[34]: Let $1 \leq p$, $r \leq +\infty$, $s > 0$, and $r' \stackrel{\text{def}}{=} r/(r-1)$. Assume that $\theta' \in H^{rsp}$. Then $K_\theta^{r'p} \subset B_p^s \cap W_p^s$; moreover, for every function f , $f \in K_\theta^{r'p}$, one has

$$\max(\|f\|_{B_p^s}, \|f\|_{W_p^s}) \leq c(p, r, s) \|\theta'\|_{rsp}^s \|f\|_{r'p} \quad (82)$$

Remark (1.2.19)[34]: In the case $rsp \geq 1$, the condition $\theta' \in H^{rsp}$ means that θ is a finite Blaschke product. In this case $K_\theta^{r'p}$ is finite dimensional and consists of rational functions with the same poles as those of θ . Thus, the inclusion $K_\theta^{r'p} \subset B_p^s \cap W_p^s$ becomes obvious; however, inequality (82) is still nontrivial. On the other hand, if $rsp < 1$, then the class of inner functions θ with $\theta' \in H^{rsp}$ is much larger; see [46].

Consider two special cases of inequality (82).

i. Let $r = +\infty$. Then we have $r' = 1$, and so (82) implies

$$\|f^{(s)}\|_p \leq \text{const} \|\theta'\|_\infty^s \|f\|_p, \quad f \in K_\theta^p. \quad (83)$$

In the case where $s \in \mathbb{N}$, this latter inequality was proved by the author [41] via the multiple Calderón commutators.

ii. Let $sp < 1$ and $r = (sp)^{-1}$. Now we have $\|\theta'\|_{r,sp} = \|\theta'\|_1 = n$, where n is the number of zeros (counted with multiplicities) of the Blaschke product θ in the disk $\mathbb{D}$. Thus, (82) reduces to

$$\max(\|f\|_{B_p^s}, \|f\|_{W_p^s}) \leq c(p, s) n^s \|f\|_q, \quad (84)$$

where $q = p/(1 - sp)$ and f is an arbitrary rational function of degree $\leq n$ having all its poles in $\mathbb{C} \setminus \text{clos} \mathbb{D}$. Inequality (84) was obtained by A. A. Pekarskii in [1], where it was used to characterize the classes $B_p^s \cap H^p$ and $W_p^s \cap H^p$ in terms of best rational L^p approximants.

Similarly, following the classical pattern going back to S. N. Bernstein, one can use (82) to derive a number of approximation theorems. We restrict ourselves to stating one of them, arising in the case $r = +\infty$.

Set

$$\mathcal{R}_n \stackrel{\text{def}}{=} \bigcup_{\theta} \{K_\theta^\infty : \|\theta'\|_\infty \leq n\}.$$

Thus, the elements of $\mathcal{R}_n$ are rational functions "of degree not exceeding n ," provided that the "degree" of a rational fraction R is understood as $\|B_R'\|_\infty$, where B_R is the Blaschke product formed from the poles of R in a natural way.

Theorem (1.2.20)[34]: Suppose that $f \in H^p$, $1 \leq p$, $q \leq +\infty$ and $s > 0$. The following are equivalent:

- i. $f \in B_{pq}^s$.
- ii. $\{2^{js} \text{dist}_{L^p}(f, \mathcal{R}_{2^j})\}_{j=0}^\infty \in l^q$.

Sketch of the proof. Since the set $K_{z^n}^\infty$, consisting of polynomials of degree $< n$, is contained in $\mathcal{R}_n$, the "Jackson-type theorem" $i \Rightarrow ii$ is a consequence of the corresponding implication in the classical theorem on polynomial approximation [44].

The proof of the "Bernstein type theorem" $ii \Rightarrow i$ runs exactly as in the classical situation; the only difference is that Bernstein's inequality for polynomials must be replaced by its generalized version (83).

In conclusion, we remark that there is also a "nonanalytic" analog of Theorem (1.2.20), in which case one merely assumes a priori that $f \in L^p$, whereas the approximating rational functions may have poles both in $\mathbb{D}$ and in $\mathbb{C} \setminus \text{clos} \mathbb{D}$ (cf. [42]).

Section (1.3): Weighted Hardy and Bergman Norms

Statement and historical context of the problem. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and let $\text{Hol}(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. We consider here the following problem: given two Banach spaces X and Y of holomorphic functions on the unit disc $\mathbb{D}$, $X, Y \subset \text{Hol}(\mathbb{D})$, and a finite subset σ of $\mathbb{D}$, what is the best possible interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case? The case $X \subset Y$ is of no interest, and so one can suppose that either $Y \subset X$ or X and Y are incomparable. Here and later on, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$.

More precisely, our problem is to compute or estimate the following interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf\{\|g\|_Y : g|_\sigma = f|_\sigma\}.$$

For $r \in [0,1)$ and $n \geq 1$, we also define

$$C_{n,r}(X, Y) = \sup\{c(\sigma, X, Y) : \text{card}\sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma\}.$$

It is explained in [65] why the classical interpolation problems, those of Nevanlinna–Pick and Carathéodory–Schur (see [62, p.231]), on the one hand and Carleson’s free interpolation (1958) (see [62]) on the other hand, are of this nature.

From now on, if $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^n b_{\lambda_j}$$

is the corresponding finite Blaschke product where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$, $\lambda \in \mathbb{D}$. With this notation and supposing that X satisfies the division property

$$[f \in X, \lambda \in \mathbb{D} \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z - \lambda} \in X \right],$$

we have

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf\{\|g\|_Y : g \in Y, g - f \in B_\sigma X\}.$$

A direct relation between the study of the constants $c(\sigma, H^\infty, W)$ and some numerical analysis problems is mentioned in [65, (b)- p.5]. Here, W is the Wiener algebra of absolutely convergent Fourier series. In the same spirit, for general Banach spaces X containing H^∞ , our constants $c(\sigma, X, H^\infty)$ are directly linked with the well known Von-Neumann’s inequality for contractions on Hilbert spaces, which asserts that if A is a contraction on a Hilbert space and $f \in H^\infty$, then the operator $f(A)$ satisfies

$$\|f(A)\| \leq \|f\|_\infty.$$

Using this inequality we get the following interpretation of our interpolation constant $c(\sigma, X, H^\infty)$: it is the best possible constant c such that $\|f(A)\| \leq c \|f\|_X$, $\forall f \in X$. That is to say:

$$c(\sigma, X, H^\infty) = \sup_{\|f\|_X \leq 1} \sup\{\|f(A)\| : A : (\mathbb{C}^n, \|\cdot\|_2) \rightarrow (\mathbb{C}^n, \|\cdot\|_2), \|A\| \leq 1, \sigma(A) \subset \sigma\},$$

where the interior sup is taken over all contractions A on n -dimensional Hilbert spaces $(\mathbb{C}^n, \|\cdot\|_2)$, with a given spectrum $\sigma(A) \subset \sigma$.

An interesting case occurs for f such that $f|_\sigma = (1/z)|_\sigma$ (estimates on condition numbers and the norm of inverses of $n \times n$ matrices) or $f|_\sigma = [1/(\lambda - z)]|_\sigma$ (estimates on the norm of the resolvent of an $n \times n$ matrix), see for instance [67].

Let H^p ($1 \leq p \leq \infty$) be the standard Hardy spaces and let L_a^2 be the Bergman space on $\mathbb{D}$. We obtained in [65] (in which a more general approach to this effective interpolation problem is also given) some estimates on $c(\sigma, X, H^\infty)$ for the cases $X \in \{H^p, L_a^2\}$.

Theorem (1.3.1)[53]: Let $n \geq 1$, $r \in [0,1)$, $p \in [1, +\infty]$ and $|\lambda| \leq r$. Then

$$\frac{1}{32^{\frac{1}{p}} (1-|\lambda|)^{\frac{1}{p}}} \leq c(\sigma_{n,\lambda}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}}, \quad (85)$$

$$\frac{1}{32} \frac{n}{1-|\lambda|} \leq c(\sigma_{n,\lambda}, L_a^2, H^\infty) \leq C_{n,r}(L_a^2, H^\infty) \leq \sqrt{2} 10^{\frac{1}{4}} \frac{n}{1-r}, \quad (86)$$

where

$$\sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}, \quad (n \text{ times}),$$

is the one-point set of multiplicity n corresponding to λ , A_p is a constant depending only on p and the left-hand side inequality in (85) is valid only for $p \in 2\mathbb{Z}_+$. For $p = 2$, we have $A_2 = \sqrt{2}$.

Note that this theorem was partially motivated by a question posed in an applied situation in [58, 59].

Trying to generalize inequalities (85) and (86) for general Banach spaces X (of analytic functions of moderate growth in $\mathbb{D}$), we formulate the following conjecture: $C_{n,r}(X, H^\infty) \leq a \varphi_X(1 - \frac{1-r}{n})$, where a is a constant depending on X only and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X . The aim of this section is to establish this conjecture for some families of weighted Hardy and Bergman spaces.

Here, we extend Theorem (1.3.1) to the case where X is a weighted space

$$l_a^p(\alpha) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p (k+1)^{p\alpha} < \infty \right\}, \quad \alpha \leq 0.$$

First, we study the special case $p = 2$, $\alpha \leq 0$. Then $l_a^p(\alpha)$ are the spaces of the functions $f = \sum_{k \geq 0} \hat{f}(k) z^k$ satisfying

$$\sum_{k \geq 0} |\hat{f}(k)|^2 (k+1)^{2\alpha} < \infty.$$

Notice that $H^2 = l_a^2(1)$. Let $\beta = -2\alpha - 1 > -1$. The scale of weighted Bergman spaces of holomorphic functions

$$X = L_a^2(\beta) = L_a^2\left((1 - |z|^2)^\beta dA\right) = \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\beta dA < \infty \right\},$$

gives the same spaces, with equivalence of the norms:

$$l_a^2(\alpha) = L_a^2(\beta).$$

In the case $\beta = 0$ we have $L_a^2(0) = L_a^2$.

Theorems (1.3.10), (1.3.11) and (1.3.12) were already announced in the note [66]. Let σ be a finite set of $\mathbb{D}$, and let $f \in X$. The technical tools used in the proofs of the upper bounds for the interpolation constants $c(\sigma, X, H^\infty)$ are: a linear interpolation

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

Where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, and $(e_k)_{1 \leq k \leq n}$ is the explicitly known Malmquist basis (see [43]) or Definition 1 below) of the space $K_B = H^2 \ominus B H^2$ where $B = B_\sigma$, a Bernstein-type inequality of Dyakonov (used by induction): $\|f\|_p \leq c_p \|B'\|_\infty \|f\|_p$, for a (rational) function f in the star-invariant subspace $H^p \cap \overline{B z} H^p$ generated by a (finite) Blaschke product B , (Dyakonov [60, 41]); it is used in order to find an upper bound for $\left\| \sum_{k=1}^n \langle f, e_k \rangle e_k \right\|_\infty$ (in terms of $\|f\|_X$), and finally the complex interpolation between Banach spaces, (see [57] or [63]).

The lower bound problem (for $C_{n,r}(X, H^\infty)$) is treated by using the “worst” interpolation n -tuple $\sigma = \sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory–Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ . We note that the spaces

$X = l_a^p(\alpha)$ satisfy the condition $X \circ b_\lambda \subset X$ when $p = 2$, whereas this is not the case for $p \neq 2$. That is why our problem of estimating the interpolation constants is more difficult for $p \neq 2$.

we develop the technical tools mentioned above, which are used later on to establish an upper bound for $c(\sigma, X, H^\infty)$.

In Definitions 1, 2, 3 and in Remark (1.3.2) below, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$ and B_σ is the corresponding Blaschke product.

Definition 1. Malmquist family. For $k \in [1, n]$, we set $f_k = \frac{1}{1 - \lambda_k z}$, and define the family $(e_k)_{1 \leq k \leq n}$, (which is known as Malmquist basis, see [43, p.117]), by

$$e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2}$$

for $k \in [2, n]$; we have $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-\frac{1}{2}}$.

Definition 2. The model space K_{B_σ} . We define K_{B_σ} to be the n -dimensional space:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Definition 3. The orthogonal projection P_{B_σ} on K_{B_σ} . We define P_{B_σ} to be the orthogonal projection of H^2 on its n -dimensional subspace K_{B_σ} .

Remark (1.3.2)[53]: The Malmquist family $(e_k)_{1 \leq k \leq n}$ corresponding to σ is an orthonormal basis of K_{B_σ} . In particular,

$$P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k,$$

where $(\cdot, \cdot)_{H^2}$ means the scalar product on H^2 .

We now recall the following lemma already (partially) established in [65, p. 15] which is useful in the proof of the upper bound in Theorem (1.3.12).

Lemma (1.3.3)[53]: Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc $\mathbb{D}$ and let $(e_k)_{1 \leq k \leq n}$ be the Malmquist family corresponding to σ . Let also $\langle \cdot, \cdot \rangle$ be the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, (if $h \in \text{Hol}(\mathbb{D})$ and $k \in \mathbb{N}$, $\hat{h}(k)$ stands for the k^{th} Taylor coefficient of h). The map $\Phi : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$ given by

$$\Phi : f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

is well defined and has the following properties:

(a) $\Phi|_{H^2} = P_{B_\sigma}$,

(b) Φ is continuous on $\text{Hol}(\mathbb{D})$ with the topology of the uniform convergence on compact sets of $\mathbb{D}$,

(c) if $X = l_a^p(\alpha)$ with $p \in [1, +\infty]$, $\alpha \in (-\infty, 0]$ and $\Psi = Id|_X - \Phi|_X$, then $\text{Im}(\Psi) \subset B_\sigma X$,

(d) if $f \in \text{Hol}(\mathbb{D})$, then

$$|\Phi(f)(\zeta)| = \left| \langle f, P_{B_\sigma} k_\zeta \rangle \right|,$$

for all $\zeta \in \mathbb{D}$, where P_{B_σ} is defined in 3 and $k_\zeta = (1 - \bar{\zeta}z)^{-1}$.

Proof. Points (a), (b) and (c) were already proved in [65]. In order to show (d), we simply need to write that

$$\Phi(f)(\zeta) = \sum_{k=1}^n \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1}^n \overline{e_k(\zeta)} e_k \right\rangle,$$

$\forall f \in \text{Hol}(\mathbb{D})$, $\forall \zeta \in \mathbb{D}$ and to notice that $\sum_{k=1}^n \overline{e_k(\zeta)} e_k = \sum_{k=1}^n (k_\zeta, e_k)_{H^2} e_k = P_{B_\sigma} k_\zeta$.

Bernstein-type inequalities for rational functions are the subject of a number of references and monographs (see, for instance, [55, 56, 60, 41, 61]). We use here a result going back to Dyakonov [60, 41].

Lemma (1.3.4)[53]. Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and let $f \in K_B$. Then

$$\|f'\|_{H^2} \leq 3 \frac{n}{1-r} \|f\|_{H^2}.$$

Lemma (1.3.4) is a partial case ($p=2$) of the following K . Dyakonov's result [41] (which is, in turn, a generalization of Levin's inequality [61] corresponding to the case $p=\infty$): the norm $\|D\|_{K_B^p \rightarrow H^p}$ of the differentiation operator $Df = f'$ on the star-invariant subspace of the Hardy space H^p , $K_B^p := H^p \cap \overline{BzH^p}$, (where the bar denotes complex conjugation) satisfies the following estimate:

$$\|D\|_{K_B^p \rightarrow H^p} \leq c_p \|B'\|_\infty,$$

for every p , $1 \leq p \leq \infty$, where c_p is a positive constant depending only on p , B is a finite Blaschke product and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. In the case $p=2$, Dyakonov's result gives $c_p = \frac{36+2\sqrt{3\pi}}{2\pi}$, which entails an estimate similar to that of Lemma (1.3.4), but with a larger constant ($\frac{13}{2}$ instead of 3). Our lemma is proved in [65].

The sharpness of the inequality stated in Lemma (1.3.4) is discussed in [64]. Here we use it by induction in order to get the following corollary.

Corollary (1.3.5)[53]: Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B$. Then,

$$\|f^{(k)}\|_{H^2} \leq k! 4^k \left(\frac{n}{1-r} \right)^k \|f\|_{H^2},$$

for every $k = 0, 1, \dots$

Proof. Indeed, since $z^{k-1} f^{(k-1)} \in K_{B^k}$, we obtain applying Lemma (1.3.4) with B^k instead of B ,

$$\|z^{k-1} f^{(k)} + (k-1) z^{k-2} f^{(k-1)}\|_{H^2} \leq 3 \frac{kn}{1-r} \|z^{k-1} f^{(k-1)}\|_{H^2} = 3 \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2}.$$

In particular,

$$\|z^{k-1} f^{(k)}\|_{H^2} - \|(k-1) z^{k-2} f^{(k-1)}\|_{H^2} \leq 3 \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2},$$

which gives

$$\|f^{(k)}\|_{H^2} \leq 3 \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2} + (k-1) \|f^{(k-1)}\|_{H^2} \leq 4 \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2}.$$

By induction,

$$\|f^{(k)}\|_{H^2} \leq k! \left(\frac{4n}{1-r} \right)^k \|f\|_{H^2}.$$

Lemma (1.3.6)[53]: Let X_1 and X_2 be two Banach spaces of holomorphic functions in the unit disc $\mathbb{D}$. Let also $\theta \in [0, 1]$ and $(X_1, X_2)_{[\theta]}$ be the corresponding intermediate

Banach space resulting from the classical complex interpolation method applied between X_1 and X_2 , (we use the notation of [57, Chapter 4]). Then,

$$C_{n,r}((X_1, X_2)_{[\theta]}, H^\infty) \leq C_{n,r}(X_1, H^\infty)^{1-\theta} C_{n,r}(X_2, H^\infty)^\theta,$$

for all $n \geq 1$, $r \in [0,1)$.

Proof. For the proof of this lemma, we refer to [65, p.19].

The case $X = l_a^2(\alpha)$, $\alpha \leq 0$. We start with the following result.

Corollary (1.3.7)[53]: Let $N \geq 0$ be an integer. Then,

$$C_{n,r}(l_a^2(-N), H^\infty) \leq A \left(\frac{n}{1-r} \right)^{\frac{2N+1}{2}},$$

for all $r \in [0,1)$, $n \geq 1$, where A depends only on N (of order $N!(4N)^N$, see the proof below).

Proof. Indeed, let $X = l_a^2(-N)$, σ a finite subset of $\mathbb{D}$ and $B = B_\sigma$. If $f \in X$, then using part (c) of Lemma (1.3.3), we get that $\Phi(f)|_\sigma = f|_\sigma$. Now, denoting X^* the dual of X with respect to the Cauchy pairing $\langle \cdot, \cdot \rangle$ (defined in Lemma (1.3.3)).

Applying point (d) of the same lemma, we obtain $X^* = l_a^2(N)$ and

$$|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \|f\|_X K_N \left(\|P_B k_\zeta\|_{H^2}^2 + \|(P_B k_\zeta)^{(N)}\|_{H^2}^2 \right)^{\frac{1}{2}},$$

for all $\zeta \in \mathbb{D}$, where

$$K_N = \max \left\{ N^N, \sup_{k \geq N} \frac{(k+1)^N}{k(k-1)\cdots(k-N+1)} \right\} = \max \left\{ N^N, \frac{(N+1)^N}{N!} \right\} = \begin{cases} N^N, & \text{if } N \geq 3 \\ \frac{(N+1)^N}{N!}, & \text{if } N = 1, 2 \end{cases}.$$

(Indeed, the sequence $\left(\frac{(k+1)^N}{k(k-1)\cdots(k-N+1)} \right)_{k \geq N}$ is decreasing and $\left[N^N > \frac{(N+1)^N}{N!} \right] \Leftrightarrow N \geq 3$. Since $P_B k_\zeta \in K_B$, Corollary (1.3.5) implies

$$|\Phi(f)(\zeta)| \leq \|f\|_X K_N \|P_B k_\zeta\|_{H^2} \left(1 + (N!)^2 \left(4 \frac{n}{1-r} \right)^{2N} \right)^{\frac{1}{2}} \leq A(N) \left(\frac{n}{1-r} \right)^{N+\frac{1}{2}} \|f\|_X,$$

where $A(N) = \sqrt{2} K_N \left(1 + (N!)^2 4^{2N} \right)^{\frac{1}{2}}$, since

$$\|P_B k_\zeta\|_{H^2} = \left\| \sum_{k=1}^n (k_\zeta, e_k)_{H^2} e_k \right\|_{H^2} = \sqrt{\sum_{k=1}^n |e_k(\zeta)|^2} \leq \sqrt{\frac{2n}{1-r}}. \quad (87)$$

An upper bound for $c(\sigma, l_a^p(\alpha), H^\infty)$, $1 \leq p \leq 2$. The purpose of this section is to show the right-hand side inequality of Theorem (1.3.11). We start with a partial case.

Lemma (1.3.8)[53]: Let $N \geq 0$ be an integer. Then

$$C_{n,r}(l_a^1(-N), H^\infty) \leq A_1 \left(\frac{n}{1-r} \right)^{N+\frac{1}{2}},$$

for all $r \in [0,1)$, $n \geq 1$, where A_1 depends only on N (it is of order $N!(4N)^N$, see the proof below).

Proof. In fact, the proof is exactly the same as in Corollary (1.3.7): if σ is a sequence in $\mathbb{D}$ with $\text{card } \sigma \leq n$, and $f \in l_a^1(-N) = X$, then $X^* = l_a^\infty(N)$ (the dual of X with respect to the Cauchy pairing). Using Lemma (1.3.3) we still have $\Phi(f)|_\sigma = f|_\sigma$, and for every $\zeta \in \mathbb{D}$,

$$\begin{aligned}
|\Phi(f)(\zeta)| &\leq \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \|f\|_X K_N \max \left\{ \sup_{0 \leq k \leq N-1} |P_B k_\zeta(k)|, \sup_{k \geq N} \left| (P_B k_\zeta)^{(N)}(k-N) \right| \right\} \\
&\leq \|f\|_X K_N \max \left\{ \|P_B k_\zeta\|_{H^2}, \left\| (P_B k_\zeta)^{(N)} \right\|_{H^2} \right\},
\end{aligned}$$

where K_N is defined in the the proof of Corollary (1.3.7) Since $P_B k_\zeta \in K_B$, Corollary (1.3.5) implies that

$$|\Phi(f)(\zeta)| \leq \|f\|_X K_N \|P_B k_\zeta\|_{H^2} \left(1 + N! 4^N \left(\frac{n}{1-r} \right)^N \right),$$

for all $\zeta \in \mathbb{D}$, which completes the proof using (87) and setting $A_1(N) = 2\sqrt{2}N! 4^N K_N$. An upper bound for $c(\sigma, l_a^p(\alpha), H^\infty)$, $2 \leq p \leq +\infty$. Here, we show the upper bound stated in Theorem (1.3.12). As before, the upper bound $(\frac{n}{1-r})^{\frac{3}{2}-\alpha-\frac{2}{p}}$ is not as sharp as above, we can suppose the constant $(\frac{n}{1-r})^{1-\alpha-\frac{1}{p}}$ should be again a sharp upper (and lower) bound for the quantity $C_{n,r}(l_a^p(\alpha), H^\infty)$, $2 \leq p \leq +\infty$.

First we show the following partial case of Theorem (1.3.12).

Corollary (1.3.9)[53]: Let $N \geq 0$ be an integer. Then,

$$C_{n,r}(l_a^\infty(-N), H^\infty) \leq A_\infty \left(\frac{n}{1-r} \right)^{N+\frac{3}{2}},$$

for all $r \in [0,1)$, $n \geq 1$, where A_∞ depends only on N (it is of order $N!(4N)^N$, see the proof below).

Proof. We use literally the same method as in Corollary (1.3.7) and Lemma (1.3.8). Indeed, if $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$ and $f \in l_a^\infty(-N) = X$, then $X^* = l_a^1(N)$ and applying again Lemma (1.3.3) we get $\Phi(f)_{|\sigma} = f_{|\sigma}$. For every $\zeta \in \mathbb{D}$, we have

$$|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \|f\|_X K_N \left(\|P_B k_\zeta\|_W + \left\| (P_B k_\zeta)^{(N)} \right\|_W \right),$$

where

$$W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_W := \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\},$$

stands for the Wiener algebra, and K_N is defined in the proof of Corollary (1.3.7). Now, applying Hardy's inequality (see [43, p.370],), we obtain

$$\begin{aligned}
|\Phi(f)(\zeta)| &\leq \|f\|_X K_N \left(\pi \left\| (P_B k_\zeta)' \right\|_{H^1} + |(P_B k_\zeta)(0)| + \pi \left\| (P_B k_\zeta)^{(N+1)} \right\|_{H^1} + \left| (P_B k_\zeta)^{(N)}(0) \right| \right) \\
&\leq \|f\|_X K_N \pi \left(\left\| (P_B k_\zeta)' \right\|_{H^2} + \left\| (P_B k_\zeta) \right\|_{H^2} + \left\| (P_B k_\zeta)^{(N+1)} \right\|_{H^2} + \left\| (P_B k_\zeta)^{(N)} \right\|_{H^2} \right),
\end{aligned}$$

for all $\zeta \in \mathbb{D}$. Using Lemma (1.3.4) and Corollary (1.3.5), we get

$$|\Phi(f)(\zeta)| \leq \|f\|_X K_N \pi \|P_B k_\zeta\|_{H^2} \left(\frac{3n}{1-r} + 1 + (N+1)! \left(\frac{4n}{1-r} \right)^{N+1} + N! \right),$$

for all $\zeta \in \mathbb{D}$, which completes the proof using (87).

The case $X = l_a^2(\alpha)$, $\alpha \leq 0$. We start with verifying the sharpness of the upper estimate for the quantity

$$C_{n,r} \left(l_a^2 \left(\frac{1-N}{2} \right), H^\infty \right),$$

(where $N \geq 1$ is an integer), in Theorem (1.3.10). This lower bound problem is treated by estimating our interpolation constant $c(\sigma, X, H^\infty)$ for the one-point interpolation set $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n$, $\lambda \in \mathbb{D}$

$$c(\sigma_{n,\lambda}, X, H^\infty) = \sup \left\{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in X, \|f\|_X \leq 1 \right\},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \left\{ \|f + b_\lambda^n g\|_\infty : g \in X \right\}$. In the proof, we notice that $l_a^2(\alpha)$ is a reproducing kernel Hilbert space on the disc $\mathbb{D}$ (RKHS) and we use the fact that this space has some special properties for particular values of α ($\alpha = \frac{1-N}{2}$, $N = 1, 2, \dots$). Before giving this proof (see below), we show that $l_a^2(\alpha)$ is a RKHS and we focus on the special case $\alpha = \frac{1-N}{2}$, $N = 1, 2, \dots$.

The spaces $l_a^2(\alpha)$ are RKHS. The reproducing kernel of $l_a^2(\alpha)$, by definition, is a $l_a^2(\alpha)$ -valued function $\lambda \mapsto k_\lambda^\alpha$, $\lambda \in \mathbb{D}$, such that $(f, k_\lambda^\alpha) = f(\lambda)$ for every $f \in l_a^2(\alpha)$, where $(\cdot, \cdot)$ means the scalar product $(f, g) = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)} (k+1)^{2\alpha}$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k \frac{1}{(k+1)^{2\alpha}} (k+1)^{2\alpha}$ ($\lambda \in \mathbb{D}$), it follows that

$$k_\lambda^\alpha(z) = \sum_{k \geq 0} \frac{\lambda^{-k} z^k}{(k+1)^{2\alpha}}, z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 \in l_a^2(1)$, we get the Szegő kernel

$$k_\lambda(z) = (1 - \bar{\lambda}z)^{-1},$$

and for the Bergman space $L^2 \in l_a^2(-\frac{1}{2})$, the Bergman kernel $k_\lambda^{-\frac{1}{2}}(z) = (1 - \bar{\lambda}z)^{-2}$.

Now let us explain that more generally if $\alpha = \frac{1-N}{2}$, $N \in \mathbb{N} \setminus \{0\}$, the space $l_a^2(\alpha)$ coincides (topologically) with the RKHS whose reproducing kernel is $(k_\lambda(z))^N = (1 - \bar{\lambda}z)^{-N}$. Following the Aronszajn theory of RKHS (see, for example [54, 62]), given a positive definite function $(\lambda, z) \mapsto k(\lambda, z)$ on $\mathbb{D} \times \mathbb{D}$ (i.e. such that $\sum_{i,j} \bar{a}_i a_j k(\lambda_i, \lambda_j) > 0$ for all finite subsets $(\lambda_i) \subset \mathbb{D}$ and all non-zero families of complex numbers (a_i)) one can define the corresponding Hilbert spaces $H(k)$ as the completion of finite linear combinations $\sum_i \bar{a}_i k(\lambda_i, \cdot)$ endowed with the norm

$$\left\| \sum_i \bar{a}_i k(\lambda_i, \cdot) \right\|^2 = \sum_{i,j} \bar{a}_i a_j k(\lambda_i, \lambda_j).$$

When k is holomorphic with respect to the second variable and antiholomorphic with respect to the first one, we obtain a RKHS of holomorphic functions $H(k)$ embedded into $\text{Hol}(\mathbb{D})$. Now, choosing for k the reproducing kernel of H^2 , $k : (\lambda, z) \mapsto k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$, and $\varphi = z^N$, $N = 1, 2, \dots$, the function $\varphi \circ k$ is also positive definite and the corresponding Hilbert space is

$$H(\varphi \circ k) = l_a^2 \left(\frac{1-N}{2} \right). \quad (88)$$

(Another notation for the space $H(\varphi \circ k)$ is $\varphi(H^2)$ since k is the reproducing kernel of H^2). The equality (88) is a topological identity: the spaces coincide as sets of functions,

and the norms are equivalent. Moreover, the space $H(\varphi \circ k)$ satisfies the following property: for every $f \in H^2$, $\varphi \circ f \in \varphi(H^2)$, and

$$\|\varphi \circ f\|_{H(\varphi \circ k)}^2 \leq \varphi(\|f\|_{H^2}^2), \quad (89)$$

(the Aronszajn-deBranges inequality, see [62, p.320]). The link between spaces of type $l_a^2(\frac{1-N}{2})$ and of type $H(z^N \circ k)$ being established, we give the proof of the left-hand side inequality in Theorem (1.3.10).

Theorem (1.3.10)[53]: Let $n \geq 1$, $r \in [0,1)$, $\alpha \in (-\infty, 0]$ and $|\lambda| \leq r$. Then

$$B \left(\frac{n}{1-|\lambda|} \right)^{\frac{1-2\alpha}{2}} \leq c(\sigma_{n,\lambda}, l_a^2(\alpha), H^\infty) \leq C_{n,r}(l_a^2(\alpha), H^\infty) \leq A \left(\frac{n}{1-r} \right)^{\frac{1-2\alpha}{2}}.$$

Equivalently, if $\beta \in (-1, +\infty)$ then

$$B' \left(\frac{n}{1-|\lambda|} \right)^{\frac{\beta+2}{2}} \leq c(\sigma_{n,\lambda}, L_a^2(\beta), H^\infty) \leq C_{n,r}(L_a^2(\beta), H^\infty) \leq A' \left(\frac{n}{1-r} \right)^{\frac{\beta+2}{2}},$$

where A and B depend only on α , A' and B' depend only on β , and both of the two left-hand side inequalities are valid only for α and β satisfying $1-2\alpha \in \mathbb{N}$ and $\frac{\beta+2}{2} \in \mathbb{N}$.

Proof. There exists an integer N such that $N-1 \leq -\alpha \leq N$. In particular, there exists $0 \leq \theta \leq 1$ such that $\alpha = (1-\theta)(1-N) + \theta(-N)$. Since

$$(l_a^2(1-N), l_a^2(-N))_{|\theta|} = l_a^2(\alpha),$$

(see [57, 63]), this gives, using Lemma (1.3.6) with $X_1 = l_a^2(1-N)$ and $X_2 = l_a^2(-N)$, and Corollary (1.3.7), that

$$C_{n,r}(l_a^2(\alpha), H^\infty) \leq A(N-1)^{1-\theta} A(N)^\theta \left(\frac{n}{1-r} \right)^{\frac{(2N-1)(1-\theta)}{2} + \frac{(2N+1)\theta}{2}}.$$

It remains to use that $\theta = 1 - \alpha - N$ and set $A(\alpha) = A(N-1)^{1-\theta} A(N)^\theta$.

This shows the proof of the right-hand side inequality in Theorem (1.3.10). Now the left-hand side inequality

0) We set $N = 1-2\alpha$, $N = 1, 2, \dots$ and $\varphi(z) = z^N$.

1) Let $b > 0$, $b^2 n^N = 1$. We set

$$Q_n = \sum_{k=0}^{n-1} b_\lambda^k \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}z}, \quad H_n = \varphi \circ Q_n, \quad \Psi = bH_n.$$

Then $\|Q_n\|_2^2 = n$, and hence by (89),

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi(\|Q_n\|_2^2) = b^2 \varphi(n) = 1.$$

Let $b > 0$ such that $b^2 \varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c(\sigma_{n,\lambda}, H_\varphi, H^\infty) = c(\sigma_{n,\mu}, H_\varphi, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$ consider $G \in H^\infty$ such that $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree nN) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}b_\lambda(z)} = (1-|\lambda|^2)^{-\frac{1}{2}} \left(1 + (1-\bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) \\ &= (1-r^2)^{-\frac{1}{2}} \left(1 + (1-r) \sum_{k=1}^{n-1} z^k + rz^n \right) =: (1-r^2)^{-\frac{1}{2}} \psi_1. \end{aligned}$$

Then, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1-r^2)^{-\frac{1}{2}} \psi_1 \right)$. Furthermore,

$$\varphi \circ \psi_1 = \psi_1^N(z).$$

Now, it is clear that ψ is a polynomial of degree Nn such that

$$\psi(1) = \sum_{j=0}^{Nn} \hat{\psi}(j) = b\varphi \left((1-r^2)^{-\frac{1}{2}} (1+r)n \right) = b \left(\sqrt{\frac{1+r}{1-r}} n \right)^N > 0.$$

4) Next, we show that there exists $c = c(N) > 0$ (for example, $c = K/[2^{2N}(N-1)!]$, K being a numerical constant) such that

$$\sum_{j=0}^m (\psi) := \sum_{j=0}^m \hat{\psi}(j) \geq c \sum_{j=0}^{Nn} \hat{\psi}(j) = c\psi(1),$$

where $m \geq 1$ is such that $2m = n$ if n is even and $2m-1 = n$ if n is odd.

Indeed, setting

$$S_n = \sum_{j=0}^n z^j,$$

we have

$$\sum_{j=0}^m (\psi_1^N) = \sum_{j=0}^m \left(\left(1 + (1+r) \sum_{k=1}^{n-1} z^k + rz^n \right)^N \right) \geq \sum_{j=0}^m (S_{n-1}^N).$$

Next, we obtain

$$\begin{aligned} \sum_{j=0}^m (S_{n-1}^N) &= \sum_{j=0}^m \left(\left(\frac{1-z^n}{1-z} \right)^N \right) = \sum_{j=0}^m \left(\frac{1}{(1-z)^N} \right) = \frac{1}{(N-1)!} \sum_{j=0}^m \left(\frac{d^{N-1}}{dz^{N-1}} \frac{1}{1-z} \right) \\ &= \sum_{j=0}^m C_{N+j-1}^j \geq \sum_{j=0}^m \frac{(j+1)^{N-1}}{(N-1)!} \geq K \frac{m^N}{(N-1)!}, \end{aligned}$$

where $K > 0$ is a numerical constant. Finally,

$$\begin{aligned} \sum_{j=0}^m (\psi_1^N) &\geq K \frac{m^N}{(N-1)!} \geq K \frac{(n/2)^N}{(N-1)!} = \frac{K}{2^N (N-1)!} \cdot \frac{((1+r)n)^N}{(1+r)^N} \\ &= \frac{K}{2^N (1+r)^N (N-1)!} \cdot (\psi_1(1))^N, \end{aligned}$$

which gives our estimate.

5) Let $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k -th Fejer kernel. We have $\|g\|_\infty \|F_n\|_{L^1} \geq \|g * F_n\|_\infty$ for every $g \in L^\infty(\mathbb{T})$, and taking the infimum over all $g \in H^\infty$ satisfying $\hat{g}(k) = \hat{\psi}(k)$, $\forall k \in [0, n-1]$, we obtain

$$\|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty,$$

where $*$ stands for the usual convolution product. Now using part 4),

$$\begin{aligned} \|\Psi\|_{H^\infty/b_\lambda^n H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \frac{1}{2} |(\psi * F_n)(1)| \\ &\geq \frac{1}{2} \sum_{j=0}^m \hat{\psi}(j) \geq \frac{c}{2} \psi(1) = \frac{c}{2} b \left(\sqrt{\frac{1+r}{1-r}} n \right)^N \geq B \left(\frac{n}{1-r} \right)^{\frac{N}{2}}. \end{aligned}$$

6) In order to conclude, it remains to use (88).

The case $X = l_a^p(\alpha)$, $1 \leq p \leq \infty$.

Theorem (1.3.11)[53]: Let $r \in [0,1)$, $n \geq 1$, $p \in [1,2]$, and let $\alpha \leq 0$. We have

$$Bn^{1-\alpha-\frac{1}{p}} \leq C_{n,r}(l_a^p(\alpha), H^\infty) \leq A \left(\frac{n}{1-r} \right)^{\frac{1-2\alpha}{2}},$$

Where $A = A(\alpha, p)$ and $B = B(\alpha, p)$ are constants depending only on α and p . It is very likely that the bounds stated in Theorem (1.3.11) are not sharp. The sharp one should be probably $(\frac{n}{1-r})^{1-\alpha-\frac{1}{p}}$. In the same way, for $2 \leq p \leq \infty$, we give the following theorem, in which we feel again that the upper bound $(\frac{n}{1-r})^{\frac{3-\alpha-2}{2}}$ is not sharp. As before, the sharp one is probably $(\frac{n}{1-r})^{1-\alpha-\frac{1}{p}}$.

Proof. Step 1. We start by showing the result for $p=1$ and for all $\alpha \leq 0$. We use the same reasoning as in Theorem (1.3.10) except that we replace $l_a^2(\alpha)$ by $l_a^1(\alpha)$.

Step 2. We now show the result for $p \in [1,2]$ and for all $\alpha \leq 0$: the scheme of this step is completely the same as in Step 1, but we use this time the complex interpolation between $l_a^1(\alpha)$ and $l_a^2(\alpha)$ (the classical Riesz-Thorin Theorem [57, 63]). Applying Lemma (1.3.6) with $X_1 = l_a^1(\alpha)$ and $X_2 = l_a^2(\alpha)$, it suffices to use Theorem (1.3.10) and Theorem (1.3.11) for the special case $p=1$ (already showed in Step 1), to complete the proof of the right-hand side inequality.

Now we give the proof of the left-hand side inequality (the lower bound). We first notice that

$$r \mapsto C_{n,r}(X, H^\infty)$$

increases. As a consequence, if $X = l_a^p(\alpha)$, $1 \leq p \leq \infty$, then

$$C_{n,r}(l_a^p(\alpha), H^\infty) \geq C_{n,0}(l_a^p(\alpha), H^\infty) = c(\sigma_{n,0}, l_a^p(\alpha), H^\infty),$$

where $\sigma_{n,0} = \underbrace{\{0, 0, \dots, 0\}}_n$. Now let $f = \frac{1}{n^{1/p}} \sum_{k=0}^{n-1} (k+1)^{-\alpha} z^k$. Then $\|f\|_X = 1$, and

$$\begin{aligned} c(\sigma_{n,0}, l_a^p(\alpha), H^\infty) &\geq \|f\|_{H^\infty / z^n H^\infty} \geq \frac{1}{2} \|f * F_n\|_\infty \\ &\geq \frac{1}{2} |(f * F_n)(1)| \geq \frac{1}{2} \sum_{j=0}^m \hat{f}(j), \end{aligned}$$

where $*$ and F_n are defined in part 5) of the proof of Theorem (1.3.10) (lower bound) and where $m \geq 1$ is such that $2m = n$ if n is even and $2m-1 = n$ if n is odd as in part 4) of the proof of the same Theorem. Now, since

$$\sum_{j=0}^m \hat{f}(j) = \frac{1}{n^{1/p}} \sum_{k=0}^m (k+1)^{-\alpha},$$

we get the result.

Theorem (1.3.12)[53]: Let $r \in [0,1)$, $n \geq 1$, $p \in [2, +\infty]$, and let $\alpha \leq 0$. We have

$$B'n^{1-\alpha-\frac{1}{p}} \leq C_{n,r}(l_a^p(\alpha), H^\infty) \leq A' \left(\frac{n}{1-r} \right)^{\frac{3-\alpha-2}{2}},$$

where A' and B' depend only on α and p .

Proof. The proof repeats the scheme from Theorem (1.3.11). (the two steps) excepted that this time, we replace (in both steps) the space $X = l_a^1(\alpha)$ by $X = l_a^\infty(\alpha)$.

Chapter 2

Presburger Sets with Analytic P -Adic and Classification of Semi-Algebraic P -Adic Sets

We exhibit a tight connection between the definable sets in an arbitrary p -minimal field and Presburger sets in its value group. We give a negative result about expansions of Presburger structures and show uniform elimination of imaginaries for Presburger structures within the Presburger language. The cell decomposition theorem describes piecewise the valuation of analytic functions (and more generally of subanalytic functions), the pieces being geometrically simple sets, called cells. We also classify subanalytic sets up to subanalytic bijection.

Section (2.1): P -Minimal Fields

In this section we classify the Presburger sets up to definable bijection (2.1.11), using as only classifying invariant the (logical) algebraic dimension. In order to show this classification, we first formulate a cell decomposition theorem for Presburger groups (2.1.4) and a rectilinearisation theorem for the definable sets (2.1.9). Also a rectilinearisation theorem depending on parameters is shown (2.1.10).

Expansions of Presburger groups have recently been studied intensively. One could say that on the one hand one looks for (concrete) expansions which remain decidable and have bounded complexity, and on the other hand different kinds of minimality conditions (like coset-minimality, etc.) are used to characterize general classes of expansions (see e.g., [69], [80]). We examine expansions of Presburger groups satisfying natural kinds of minimality conditions.

In [75], D. Haskell and D. Macpherson defined the notion of p -minimal fields, as a p -adic counterpart of o -minimal fields. A p -minimal field always is a p -adically closed field, and its value group is a $\mathbb{Z}$ -group. Interactions between definable sets in a given p -adically closed field and Presburger sets in its value group have been studied in the context of p -adic integration for several p -minimal structures (see [73], [74]). In Theorem (2.1.17), we exhibit a close connection between definable sets in arbitrary p -minimal fields and Presburger sets in the corresponding value groups.

In the last, we use the cell decomposition theorem in an elementary way to obtain uniform elimination of imaginaries for $\mathbb{Z}$ -groups without introducing extra sorts.

In this section G always denotes a $\mathbb{Z}$ -group, i.e., a group which is elementary equivalent to the integers $\mathbb{Z}$ in the Presburger language $\mathcal{L}_{\text{Pres}} = \langle +, \leq, \{\equiv (\text{mod } n)\}_{n>0}, 0, 1 \rangle$ where $\equiv (\text{mod } n)$ is the equivalence relation in two variables modulo the integer $n > 0$. We call $(G, \mathcal{L}_{\text{Pres}})$ a Presburger structure and we write H for the nonnegative elements in G . By a Presburger set, function, etc., we mean a $\mathcal{L}_{\text{Pres}}$ -definable set, function, etc., and by definable we always mean definable with parameters (otherwise we say $\emptyset$ -definable, S -definable, etc.). We call a set $X \subset G^m$ bounded if there is a tuple $z \in H^m$ such that $-z_i \leq x_i \leq z_i$ for each $x \in X$ and $i = 1, \dots, m$. For $k \leq m$ we write $\pi_k : G^m \rightarrow G^k$ for the projection on the first k coordinates and for $X \subset G^{k+n}$ and $x \in \pi_k(X)$ we write X_x for the fiber $\{y \in G^n \mid (x, y) \in X\}$. We recall that the theory $\text{Th}(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ has definable Skolem functions, quantifier elimination in $\mathcal{L}_{\text{Pres}}$ and is decidable [81].

We show a cell decomposition theorem for Presburger structures, by first showing it in dimension 1 and subsequently using a compactness argument. An elementary arithmetical proof can also be given, using techniques like in the proof in [71], but our

proof has the advantage that it goes through in other contexts as well. As always, G denotes a Z -group.

Definition (2.1.1)[68]: We call a function $f : X \subset G^m \rightarrow G$ linear if there is a constant $\gamma \in G$ and integers a_i , $0 \leq c_i < n_i$ for $i = 1, \dots, m$ such that $x_i - c_i \equiv 0 \pmod{n_i}$ and

$$f(x) = \sum_{i=1}^m a_i \left(\frac{x_i - c_i}{n_i} \right) + \gamma.$$

for all $x = (x_1, \dots, x_m) \in X$. We call f piecewise linear if there is a finite partition $\mathcal{P}$ of X such that all restrictions $f|_A, A \in \mathcal{P}$ are linear. We speak analogously of linear and piecewise linear maps $g : X \rightarrow G^n$.

The following definition fixes the notion of (Presburger) cells.

Definition (2.1.2)[68]: A cell of type (0) (also called a (0)-cell) is a point $\{a\} \subset G$. A (1)-cell is a set with infinite cardinality of the form

$$\{x \in G \mid \alpha \sqcap_1 x \sqcap_2 \beta, x \equiv c \pmod{n}\}, \quad (1)$$

with α, β in $\{a\} \subset G$, integers $0 \leq c < n$ and $\sqcap_i$ either $\leq$ or no condition. Let $i_j \in \{0, 1\}$ for $j = 1, \dots, m$ and $x = (x_1, \dots, x_m)$. A $(i_1, \dots, i_m, 1)$ -cell is a set A of the form

$$A = \{(x, t) \in G^{m+1} \mid x \in D, \alpha(x) \sqcap_1 t \sqcap_2 \beta(x), t \equiv c \pmod{n}\}, \quad (2)$$

with $D = \pi_m(A)$ a $(i_1, \dots, i_m)$ -cell, $\alpha, \beta : D \rightarrow G$ linear functions, $\sqcap_i$ either $\leq$ or no condition and integers $0 \leq c < n$ such that the cardinality of the fibers $A_x = \{t \in G \mid (x, t) \in A\}$ can not be bounded uniformly in $x \in D$ by an integer. A $(i_1, \dots, i_m, 0)$ -cell is a set of the form

$$\{(x, t) \in G^{m+1} \mid x \in D, \alpha(x) = t\},$$

with $\alpha : D \rightarrow G$ a linear function and $D \subset G^m$ a $(i_1, \dots, i_m)$ -cell.

Remarks (2.1.3)[68]: (i) Although we consider in Definition (2.1.2) a condition on the cardinality of fibers, the type of a cell does not alter if one takes elementary extensions.

(ii) To an infinite $(i_1, \dots, i_m)$ -cell $A \subset G^m$ we can associate (as in [82]) a projection $\pi_A : G^m \rightarrow G^k$ such that the restriction of π_A to A gives a bijection from A onto a $(1, \dots, 1)$ -cell $A' \subset G^k$. Also, a $(i_1, \dots, i_m)$ -cell is finite if and only if $i_1 = \dots = i_m = 0$, and then it is a singleton.

(iii) Let A be a $(i_1, \dots, i_m, 1)$ -cell as in Eq. (2), then it is clear that a linear function $f : A \rightarrow G$ can be written as

$$f(x, t) = a \left(\frac{t - c}{n} \right) + \gamma(x), \quad (x, t) \in A, \quad (3)$$

with a an integer, $\gamma : D \rightarrow G$ a linear function and c, n, D as in Eq. (2).

Theorem (2.1.4)[68]: Let $X \subset G^m$ and $f : X \rightarrow G$ be $\mathcal{L}_{Pres}$ -definable. Then there exists a finite partition $\mathcal{P}$ of X into cells, such that the restriction $f|_A : A \rightarrow G$ is linear for each cell $A \in \mathcal{P}$. Moreover, if X and f are S -definable, then also the parts A can be taken S -definable.

Proof. by induction on m . If $X \subset G$, $f : X \rightarrow G$ are $\mathcal{L}_{Pres}$ -definable, then Theorem (2.1.4) follows easily by using quantifier elimination and elementary properties of linear congruences. Alternatively, the more general [80] can be used to show this one dimensional version (see also Proposition (2.1.13) below). Let $X \subset G^{m+1}$ and $f : X \rightarrow G$ be $\mathcal{L}_{Pres}$ -definable, $m > 0$. We write $(\sqcap_1, \sqcap_2) \in \{\leq, \emptyset\}^2$ to say that $\sqcap_1$, resp. $\sqcap_2$, represents either the symbol $\leq$ or no condition. Let $\mathcal{S}$ be the set $\mathbb{Z} \times \{(n, c) \in \mathbb{Z}^2 \mid 0 \leq c < n\} \times \{\leq, \emptyset\}^2$. For any $d = (a, n, c, \sqcap_1, \sqcap_2) \in \mathcal{S}$ and $\xi = (\xi_1, \xi_2, \xi_3) \in G^3$ we define a Presburger function

$$F_{(d,\xi)} : \{t \in G \mid \xi_1 \sqcup_1 t \sqcup_2 \xi_2, t \equiv c \pmod{n}\} \rightarrow G : t \mapsto a\left(\frac{t-c}{n}\right) + \xi_3.$$

The domain $\text{Dom}(F_{(d,\xi)})$ of such a function $F_{(d,\xi)}$ is either empty, a (1)-cell or a finite union of (0)-cells. For fixed $k > 0$ and $d \in \mathcal{S}^k$, let $\varphi_{(d,k)}(x, \xi)$ be a Presburger formula in the free variables $x = (x_1, \dots, x_m)$ and $\xi = (\xi_1, \dots, \xi_k)$, with $\xi_i = (\xi_{i1}, \xi_{i2}, \xi_{i3})$, such that $G \models \varphi_{(d,k)}(x, \xi)$ if and only if the following are true:

- (i) $x \in \pi_m(X)$,
- (ii) the collection of the domains $\text{Dom}(F_{(d_i, \xi_i)})$ for $i = 1, \dots, k$ forms a partition of the fiber $X_x \subset G$,
- (iii) $F_{(d_i, \xi_i)}(t) = f(x, t)$ for each $t \in \text{Dom}(F_{(d_i, \xi_i)})$ and $i = 1, \dots, k$.

Now we define for each k and $d \in \mathcal{S}^k$ the set

$$B_{(d,k)} = \{x \in G^m \mid \exists \xi \varphi_{(d,k)}(x, \xi)\}.$$

Each set $B_{(d,k)}$ is $\mathcal{L}_{\text{Pres}}$ -definable and the (countable) collection $\{B_{(d,k)}\}_{k,d}$ covers $\pi_m(X)$ since each $x \in \pi_m(X)$ is in some $B_{(d,k)}$ by the induction basis. We can do this construction in any elementary extension of G , so by logical compactness we must have that finitely many sets of the form $B_{(d,k)}$ already cover $\pi_m(X)$. Consequently, we can take Presburger sets $D_1, \dots, D_s$ such that $\{D_i\}$ forms a partition of $\pi_m(X)$ and each D_i is contained in a set $B_{(d,k)}$ for some k and k -tuple d . For each $i = 1, \dots, s$, fix a k and k -tuple d with $D_i \in B_{(d,k)}$, then we can define the Presburger set

$$\Gamma_i = \{(x, \xi) \in D_i \times G^{3k} \mid \varphi_{(d,k)}(x, \xi)\}$$

satisfying $\pi_m(\Gamma_i) = D_i$ by construction. Since the theory $\text{Th}(G, \mathcal{L}_{\text{Pres}})$ has definable Skolem functions, we can choose definably for each $x \in D_i$ tuples $\xi \in G^{3k}$ such that $(x, \xi) \in \Gamma_i$. Combining it all, it follows that there exists a finite partition $\mathcal{S}$ of X consisting of Presburger sets of the form

$$A = \{(x, t) \in G^{m+1} \mid x \in C, \alpha(x) \sqcup_1 t \sqcup_2 \beta(x), t \equiv c \pmod{n}\},$$

such that $f|_A$ maps $(x, t) \in A$ to $a\left(\frac{t-c}{n}\right) + \gamma(x)$; with $\alpha, \beta, \gamma : C \rightarrow G$ and $C \subset G^m$ $\mathcal{L}_{\text{Pres}}$ -definable, $\sqcup_i$ either $\leq$ or no condition, integers $a, 0 \leq c < n$ and $\pi_m(A) = C$. The theorem now follows after applying the induction hypothesis to C and $\alpha, \beta, \gamma : C \rightarrow G$ and partitioning further.

Any Presburger structure satisfies the exchange property for algebraic closure. This is a corollary of a more general result in [69] but can also be shown using the cell decomposition theorem elementarily. In particular this yields an algebraic dimension function on the Presburger sets in the following (standard) way.

Definition (2.1.5)[68]: Let $X \subset G^m$ be A -definable for some finite set A by a formula $\varphi(x, a)$ where $a = (a_1, \dots, a_s)$ enumerates A , then the (algebraic) dimension of X , written $\dim(X)$, is the greatest integer k such that in some elementary extension $\bar{G}$ of G there exists $x = (x_1, \dots, x_m) \in \bar{G}^m$ with $\bar{G} \models \varphi(x, a)$ and $\text{rk}(x_1, \dots, x_m, a_1, \dots, a_s) - \text{rk}(a_1, \dots, a_s) = k$, where $\text{rk}(B)$ of a set $B \subset \bar{G}$ is the cardinality of a maximal algebraically independent subset of B (in the sense of model theory, see [77]).

This dimension function is independent of the choice of a set of defining parameters A and the following properties of algebraic dimension are standard.

Proposition (2.1.6)[68]:

- (i) For Presburger sets $X, Y \subset G^m$ we have $\dim(X \cup Y) = \max(\dim X, \dim Y)$.

(ii) Let $f : X \rightarrow G^m$ be $\mathcal{L}_{Pres}$ -definable, then $\dim(X) \geq \dim(f(x))$.

The dimension of a cell C is directly related to the type of C (see (2.1.7)). Also, if we have a Presburger set X and a finite partition $\mathcal{P}$ of X into cells, the dimension of X is directly related to the types of the cells in $\mathcal{P}$ (see (2.1.8)).

Lemma (2.1.7)[68]: Let $C \subset G^m$ be a $(i_1, \dots, i_m)$ -cell, then $\dim(C) = i_1 + \dots + i_m$.

Proof. For a (0)- and a (1)-cell this is clear. Possibly after projecting, we may suppose that $C \subset G^m$ is a $(1, \dots, 1)$ -cell. The Lemma follows now from the definition of the type of a cell using induction on m and a compactness argument.

Corollary (2.1.8)[68]: For any Presburger set $X \subset G^m$ and any finite partition $\mathcal{P}$ of X into cells we have

$$\begin{aligned} \dim(X) &= \max\{i_1 + \dots + i_m \mid C \in \mathcal{P}, C \text{ is a } (i_1 + \dots + i_m)\text{-cell}\} \\ &= \max\{i_1 + \dots + i_m \mid X \text{ contains a } (i_1 + \dots + i_m)\text{-cell}\}. \end{aligned} \quad (4)$$

Proof. The first equality is a consequence of Lemma (2.1.7) and Proposition (2.1.6). To show (4) we take a $(i_1, \dots, i_m)$ -cell $C \subset X$ such that $i_1 + \dots + i_m$ is maximal. By the cell decomposition we can obtain a partition $\mathcal{P}$ of X into cells such that $C \in \mathcal{P}$. Now use the previous equality to finish the proof.

The cell decomposition theorem provides us with the technical tools to classify the $\emptyset$ -definable Presburger sets up to $\mathcal{L}_{Pres}$ -definable bijection. The key step to this classification is a rectilinearisation theorem, which also has a parametric formulation. We recall that G denotes a $\mathbb{Z}$ -group and $H = \{x \in G \mid x \geq 0\}$, we also write $H^0 = \{0\}$. Also notice that a set A is $\emptyset$ -definable if and only if A is $\mathbb{Z}$ -definable, to be precise, definable over $\mathbb{Z} \cdot 1 \subset G$.

Theorem (2.1.9)[68]: Let X be a $\emptyset$ -definable Presburger set, then there exists a finite partition $\mathcal{P}$ of X into $\emptyset$ -definable Presburger sets, such that for each $A \in \mathcal{P}$ there is an integer $l \geq 0$ and a $\emptyset$ -definable linear bijection $f : A \rightarrow H^l$.

Proof. We give a proof by induction on $\dim X$. If $\dim X = 0$ then X is finite and the theorem follows, so we choose a Presburger set X with $\dim X = 0$, $m \geq 0$. Any $\mathcal{L}_{Pres}$ -definable object occurring in this proof will be $\emptyset$ -definable: we will alternately apply $\emptyset$ -definable linear bijections and partition further. By the cell decomposition theorem and possibly after projecting (see remark (2.1.3) Definition (2.1.2)), we may suppose that X is a $(1, \dots, 1)$ -cell contained in G^{m+1} , so we can write

$$X = \{(x, t) \in G^{m+1} \mid x \in D, \alpha(x) \sqcap_1 t \sqcap_2 \beta(x), t \equiv c \pmod{n}\},$$

with $x = (x_1, \dots, x_m)$, $\pi_m(X) = D \subset G^m$ a $(1, \dots, 1)$ -cell, integers $0 \leq c < n$, $\alpha, \beta : D \rightarrow G$ $\emptyset$ -definable linear functions and $\sqcap_i$ either $\leq$ or no condition. By induction we may suppose that $D = H^m$. If both $\sqcap_1$ and $\sqcap_2$ are no condition, the theorem follows easily, so we may suppose that one of the $\sqcap_i$, say $\sqcap_1$, is $\leq$. Moreover, after a linear transformation $(x, t) \mapsto (x, \frac{t-c}{n})$ we may assume that $c = 0$ and $n = 1$, then we can apply the following linear bijection

$$f : X \rightarrow A : (x, t) \mapsto (x_1, \dots, x_m, t - \alpha(x)),$$

onto

$$A = \{(x, t) \in H^{m+1} \mid t \sqcap_2 \beta(x) - \alpha(x)\}.$$

Because $\beta(x) - \alpha(x)$ is a linear function from H^m to G there are integers k_i such that

$$A = \{(x, t) \in H^{m+1} \mid t \sqcap_2 k_0 - \sum_{i=1}^m k_i x_i\}. \quad (5)$$

Moreover, since $\pi_m(A) = H^m$, all integers k_i must be nonnegative. We proceed by induction on $k_1 \geq 0$. If $k_1 = 0$ then $A = H \times \{(x_2, \dots, x_m, t) \in H^m \mid t \leq k_0 + \sum_{i=2}^m k_i x_i\}$ and the theorem follows by induction on the dimension. Now suppose $k_1 > 0$, then we partition A into two parts

$$A_1 = \{(x, t) \in H^{m+1} \mid t \leq x_1 - 1\},$$

$$A_2 = \{(x, t) \in H^{m+1} \mid x_1 \leq t \leq k_0 + \sum_{i=1}^m k_i x_i\},$$

where $\pi_m(A_2) = H^m$ and $\pi_m(A_1) = \{x \in H^m \mid 1 \leq x_1\}$. We apply the linear bijection

$$A_2 \rightarrow B : (x, t) \mapsto (x_1, \dots, x_m, t - x_1)$$

with

$$B = \{(x, t) \in H^{m+1} \mid t \leq k_0 + (k_1 - 1)x_1 + \sum_{i=2}^m k_i x_i\}$$

and the theorem for B follows by induction on k_1 . We conclude the proof by the following linear bijection:

$$A_1 \rightarrow H^{m+1} : (x, t) \mapsto (x_1, -1 - t, x_2, \dots, x_m, t).$$

Theorem (2.1.10)[68]: Let $X \subset G^{m+n}$ be a $\emptyset$ -definable Presburger set, then there exists a finite partition $\mathcal{P}$ of X into $\emptyset$ -definable Presburger sets, such that for each $A \in \mathcal{P}$ there is a set $B \subset G^{m+n}$ with $\pi_m(A) = \pi_m(B)$ and a $\emptyset$ -definable family $\{f_\lambda\}_{\lambda \in \pi_m(A)}$ of linear bijections $f_\lambda : A_\lambda \subset G^n \rightarrow B_\lambda \subset G^n$ with B_λ a set of the form $H^l \times \Lambda_\lambda$ where Λ_λ is a bounded λ -definable set and the integer l only depends on $A \in \mathcal{P}$.

Proof. We give a proof by induction on n , following the lines of the proof of Theorem (2.1.9). So we assume that X is a cell

$$X = \{(\lambda, x, t) \in G^{m+(n+1)} \mid (\lambda, x) \in D, \alpha(\lambda, x) \sqcap_1 t \sqcap_2 \beta(\lambda, x), t \equiv c \pmod{n}\},$$

with $\lambda = (\lambda_1, \dots, \lambda_m)$, $x = (x_1, \dots, x_n)$, $D \subset G^{m+n}$ a cell, integers $0 \leq c < n$, $\alpha, \beta : D \rightarrow G$ $\emptyset$ -definable linear functions and $\sqcap_i$ either $\leq$ or no condition. By subsequently applying the induction hypothesis to D , partitioning further and applying linear bijections (similar as to obtain Eq. (5) in the proof of Theorem (2.1.9), keeping the parameters λ fixed now), we may assume that X has the form

$$X = \{(\lambda, x, t) \in G^{m+n+1} \mid (\lambda, x) \in D', 0 \leq t \leq \gamma(\lambda, x)\},$$

with $\pi_{m+n}(X) = D' \subset G^{m+n}$ a Presburger set such that for each $\lambda \in \pi_m(D')$ $D'_\lambda = H^l \times \Gamma_\lambda$ where Γ_λ is a λ -definable bounded set, l a fixed positive integer and $\gamma : D' \rightarrow G$ a $\emptyset$ -definable linear function. If $l = 0$, X_λ is a bounded set for each λ and the theorem follows immediately. Let thus $l \geq 0$, i.e., the projection of X on the x_1 -coordinate is H , then the function γ can be written as $(\lambda, x) \mapsto k_1 x_1 + \gamma'(\lambda, x_2, \dots, x_m)$ with k_1 an integer, necessarily nonnegative because the projection of X on the x_1 -coordinate is H and γ' is a linear function. The reader can finish the proof by induction on $k_1 \geq 0$, similar as in the proof of Theorem (2.1.9).

Theorem (2.1.11)[68]: Let X be a $\emptyset$ -definable Presburger set with $\dim X = m > 0$, then there exists a $\emptyset$ -definable Presburger bijection $f : X \rightarrow G^m$. In other words, there exists a $\emptyset$ -definable Presburger bijection between two infinite $\emptyset$ -definable Presburger sets X, Y if and only if $\dim X = \dim Y$.

Proof. Let X be $\emptyset$ -definable and infinite. We use induction on $\dim X = m$. We say for short that two Presburger sets X, Y are isomorphic if there exists a $\emptyset$ -definable

Presburger bijection between them and write $X \cong Y$. If $m = 1$, then Theorem (2.1.9) yields a partition $\mathcal{P}$ of X such that each part is either a point or isomorphic to H . Consider the bijections

$$\begin{aligned} f_1: H \rightarrow G : & \begin{cases} 2x & \mapsto x, \\ 2x+1 & \mapsto -x, \end{cases} \\ f_2: H \cup \{-1\} \rightarrow H : & x \mapsto x+1, \\ f_3: (\{0\} \times H) \cup (\{1\} \times H) \rightarrow H : & \begin{cases} (0, x) & \mapsto 2x, \\ (1, x) & \mapsto 2x+1; \end{cases} \end{aligned}$$

the bijections f_1, f_2 , applied repeatedly to (isomorphic copies of) parts in $\mathcal{P}$ yield a definable bijection from X onto H and thus $G \cong X$ by applying f_1 (in the obvious way). Now let $\dim X = m > 1$. Using Theorem (2.1.9) we find a partition $\mathcal{P}$ of X such that each part is isomorphic to H^l and thus to G^l since $H \cong G$ by f_1 . Since $\dim X = m$, at least one part is isomorphic to G^m . Take $A, B \in \mathcal{P}$ with $A \cong G^m$ and $B \cong G^l$, then it suffices to show that $A \cup B \cong G^m$. If $l = 0$ this is clear and if $l > 0$ then $A \cup B \cong G \times (A' \cup B')$ for some disjoint and $\emptyset$ -definable sets A', B' with $A' \cong G^{m-1}$ and $B' \cong G^{l-1}$. The induction hypothesis applied to $A' \cup B'$ finishes the proof.

We define the notion of Presburger minimality ($\mathcal{L}_{Pres}$ -minimality) for expansions of Presburger structures $(G, \mathcal{L}_{Pres})$. This notion of $\mathcal{L}_{Pres}$ -minimality is a concrete instance of the general notion of $\mathcal{L}$ -minimality as in [78] and has already been studied in [80].

Definition (2.1.12)[68]: Let G be a $\mathbb{Z}$ -group and $\mathcal{L}$ an expansion of the language $\mathcal{L}_{Pres}$, then we say that $(G, \mathcal{L})$ is $\mathcal{L}_{Pres}$ -minimal if every $\mathcal{L}$ -definable subset of G is already $\mathcal{L}_{Pres}$ -definable (allowing parameters as always). We say that $\text{Th}(G, \mathcal{L})$ is $\mathcal{L}_{Pres}$ -minimal if every model of this theory is $\mathcal{L}_{Pres}$ -minimal.

Comparing this notion with the terminology of [80], a structure $(G, \mathcal{L})$ is $\mathcal{L}_{Pres}$ -minimal if and only if it is a discrete *coset*-minimal group without definable proper convex subgroups (see [80]). [80] says that a definable function in one variable between such groups is piecewise linear. We reformulate this result with our terminology.

Proposition (2.1.13)[68]: [80] Let $(G, \mathcal{L})$ be $\mathcal{L}_{Pres}$ -minimal, then any definable function $f: G \rightarrow G$ is piecewise linear.

Proposition (2.1.13) allows us to repeat without any change the compactness argument of the proof of the cell decomposition theorem for any model of a $\mathcal{L}_{Pres}$ -minimal theory. This leads to the following description of $\mathcal{L}_{Pres}$ -minimal theories.

Theorem (2.1.14)[68]: Let $(G, \mathcal{L})$ be an expansion of a Presburger structure $(G, \mathcal{L}_{Pres})$, then the following are equivalent:

- (i) $\text{Th}(G, \mathcal{L})$ is $\mathcal{L}_{Pres}$ -minimal;
- (ii) $(G, \mathcal{L})$ is a definitional expansion of $(G, \mathcal{L}_{Pres})$; precisely, any $\mathcal{L}$ -definable set $X \subset G^m$ is already $\mathcal{L}_{Pres}$ -definable.

Thus, the theory $\text{Th}(G, \mathcal{L}_{Pres})$ does not admit any proper $\mathcal{L}_{Pres}$ -minimal expansion.

Proof. Any Presburger minimal theory has definable Skolem functions. For if $X \subset G^{m+1}$ is a definable set in some model G , we can choose definably for any $x \in \pi_m(X)$ the smallest nonnegative element in X_x , if there is any, and the largest negative element otherwise. This implies the definability of Skolem functions by induction. Now replace in the

statement of the cell decomposition Theorem (theorem (2.1.4)) the word $\mathcal{L}_{\text{Pres}}$ - definable by $\mathcal{L}$ - definable. Then repeat the case $m=1$ of the proof of Theorem (2.1.4), using now the $\mathcal{L}_{\text{Pres}}$ - minimality and Proposition (2.1.13). Using the same compactness argument as in the proof of Theorem (2.1.4) we find that any $\mathcal{L}$ - definable set $X \subset G^m$ is a finite union of Presburger cells, thus a fortiori, X is $\mathcal{L}_{\text{Pres}}$ - definable.

We let K be a p -adically closed field with value group G . Recall that a p -adically closed field is a field K which is elementary equivalent to a finite field extension of the field $\mathbb{Q}_p$ of p -adic numbers; in particular, the value group G is a $\mathbb{Z}$ -group and K has quantifier elimination in the Macintyre language $\mathcal{L}_{\text{Mac}} = \langle +, -, \cdot, 0, 1, \{P_n\}_{n \geq 1} \rangle$ where P_n denotes the set of n -th powers in $K^\times$. We write $v: K \rightarrow G \cup \{\infty\}$ for the valuation map and for any $m > 0$ we write $\bar{v}$ for the map $\bar{v}: (K^\times)^m \rightarrow G^m: x \mapsto (v(x_1), \dots, v(x_m))$. We give a definition of p -minimality, extending the original definition of [75] slightly.

Definition (2.1.15)[68]: Let K be a p -adically closed field and let $(K, \mathcal{L})$ be an expansion of $(K, \mathcal{L}_{\text{Mac}})$. We say that the structure $(K, \mathcal{L})$ is p -minimal if any $\mathcal{L}$ - definable subset of K is already $\mathcal{L}_{\text{Mac}}$ - definable (allowing parameters). The theory $\text{Th}(K, \mathcal{L})$ is called p -minimal if every model of this theory is p -minimal.

Lemma (2.1.16)[68]: Let K be a p -adically closed field with value group G , then for any $\mathcal{L}_{\text{Pres}}$ - definable set $S \subset G^m$ the set $\bar{v}^{-1}(S) = \{(x_1, \dots, x_m) \in (K^\times)^m \mid \bar{v}(x) \in S\}$ is $\mathcal{L}_{\text{Mac}}$ - definable.

Proof. Let $S \subset G^m$ be $\mathcal{L}_{\text{Pres}}$ - definable. By Theorem (2.1.4) we may suppose that S is a Presburger cell. The Lemma follows now inductively from the fact that conditions imposed on $(x_1, \dots, x_{m-1}, t) \in (K^\times)^m$ of the form $\pm v(t) \leq \frac{1}{e} (\sum_{i=1}^{m-1} a_i v(x_i)) + d$ or $v(t) \equiv c \pmod{n}$ are $\mathcal{L}_{\text{Mac}}$ - definable for any integers a_i , $e \neq 0$, $0 \leq c < n$ and $d \in G$ (see e.g., [72]).

Theorem (2.1.17)[68]: Let $(K, \mathcal{L})$ be a p -minimal field with p -minimal theory and let G be the value group of K . Then for any $\mathcal{L}$ - definable set $X \subset (K^\times)^m$ the set

$$\bar{v}(X) = \{v(x_1), \dots, v(x_m) \in (G)^m \mid (x_1, \dots, x_m) \in X\} \subset G^m$$

is $\mathcal{L}_{\text{Pres}}$ - definable.

Proof. Put $S_m = \{\bar{v}(X) \subset G^m \mid X \subset (K^\times)^m, X \text{ is } \mathcal{L} \text{-definable}\}$, then it is easy to see that the collection $(S_m)_{m \geq 0}$ determines a structure on G (i.e., the collection $\bigcup_m S_m$ is precisely the collection of $\mathcal{L}'$ - definable sets for some language $\mathcal{L}'$). We first argue that this structure is in fact $\mathcal{L}_{\text{Pres}}$ - minimal. Choose a $\mathcal{L}$ - definable set $X \subset K^\times$, then, by p -minimality, X is $\mathcal{L}_{\text{Mac}}$ - definable. We can thus apply the p -adic semi-algebraic cell decomposition ([72], in the formulation of [70, Lemma 4]) to the set X to obtain that X is a finite union of p -adic cells, i.e., sets of the form

$$\{x \in K \mid v(a_1) \sqcap_1 v(x - c) \sqcap_2 v(a_2), x - c \in \lambda P_n\} \subset K^\times,$$

with $a_1, a_2, c, \lambda \in K$ and $\sqcap_i$ either $\leq$, $<$ or no condition. The image under v of such a cell is either a finite union of (0)-cells or a (1)-cell and thus a $\mathcal{L}_{\text{Pres}}$ - definable subset of G . By consequence, the structure $(S_m)_{m \geq 0}$ is $\mathcal{L}_{\text{Pres}}$ - minimal. By the Presburger minimality of $(S_m)_{m \geq 0}$, the p -minimality of $\text{Th}(K, \mathcal{L})$, and Lemma (2.1.16) to interpret G into K , we can repeat the compactness argument of the proof of the cell decomposition theorem (2.1.4) for the structure $(S_m)_{m \geq 0}$ on G to find that each $A \in \bigcup_m S_m$ is a finite union of Presburger cells. This shows the theorem.

As a last application of the cell decomposition theorem we show uniform elimination of imaginaries for Presburger structures. We say that a structure $(M, \mathcal{L})$ has uniform elimination of imaginaries if for any $\emptyset$ -definable equivalence relation on M^k there exists a $\emptyset$ -definable function $F: M^k \rightarrow M^r$ for some r such that two tuples $x, y \in M^k$ are equivalent if and only if $F(x) = F(y)$.

Theorem (2.1.18)[68]: The theory $\text{Th}(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ has uniform elimination of imaginaries, precisely, any Presburger structure $(G, \mathcal{L}_{\text{Pres}})$ eliminates imaginaries uniformly.

Proof. Since $\text{Th}(\mathbb{Z}, \mathcal{L}_{\text{Pres}})$ has definable Skolem functions, we only have to show the following statement for an arbitrary $\mathbb{Z}$ -group G (see e.g., [77]). For any $\emptyset$ -definable Presburger set $X \subset G^{m+1}$ there exists a $\emptyset$ -definable Presburger function $F: G^m \rightarrow G^n$ for some n , such that $F(x) = F(x')$ if and only if $X_x = X_{x'}$, (if $x \notin \pi_m(X)$ then we put $X_x = \emptyset$). So let $X \subset G^{m+1}$ be a $\emptyset$ -definable Presburger set. Apply the cell decomposition theorem to obtain a partition $\mathcal{P}$ of X into cells. For each cell $A \in \mathcal{P}$ of the form $A = \{(x, t) \in G^{m+1} \mid x \in D, \alpha(x) \sqcup_{1A} t \sqcup_{2A} \beta(x), t \equiv c \pmod{n}\}$, (as in Eq. 2) and each $\xi = (\xi_1, \xi_2) \in G^2$ we define a set

$$C_A(\xi) = \{t \in G \mid \xi_1 \sqcup_{1A} t \sqcup_{2A} \xi_2, t \equiv c \pmod{n}\}.$$

Notice that for each $x \in \pi_m(X)$ we have at least one partition of X , into sets of the form $C_A(\xi)$ with $A \in \mathcal{P}$ and $\xi \in G^2$. For $x, y \in G$ we write $x \triangleleft y$ if and only if one of the following conditions is satisfied

- (i) $0 \leq x < y$,
- (ii) $0 < x < -y$,
- (iii) $0 < -x < y$,
- (iv) $0 < -x < -y$.

This gives a new ordering $0 \triangleleft 1 \triangleleft -1 \triangleleft 2 \triangleleft -2 \triangleleft \dots$ on G with zero as its smallest element. For each $k > 0$ we also write $\triangleleft$ for the lexicographical order on G^k built up with $\triangleleft$. The order $\triangleleft$ is $\mathcal{L}_{\text{Pres}}$ -definable and each Presburger set has a unique $\triangleleft$ -smallest element. For each $x \in G^m$ and each $I \in \mathcal{P}$ with cardinality $|I| = s \geq 0$ we let $y_I(x) = (\xi_A)_{A \in I}$, $\xi_A \in G^2$, be the $\triangleleft$ -smallest tuple in G^{2s} such that $\bigcup_{A \in I} C_A(\xi_A) = X_x$ if there exists at least one such tuple and we put $y_I(x) = (0, \dots, 0) \in G^{2s}$ otherwise. One can reconstruct the set X_x given all tuples $y_I, I \subset \mathcal{P}$. Let F be the function mapping $x \in \pi_m(X)$ to $y = (y_I(x))_{I \subset \mathcal{P}}$. Since the lexicographical order $\triangleleft$ is $\mathcal{L}_{\text{Pres}}$ -definable it is clear that F is $\mathcal{L}_{\text{Pres}}$ -definable and that $F(x) = F(x')$ if and only if $X_x = X_{x'}$ for each $x, x' \in G^m$.

Section (2.2): Cell Decomposition and Integrals

Let p denote a fixed prime number, $\mathbb{Z}_p$ the ring of p -adic integers, $\mathbb{Q}_p$ the field of p -adic numbers, $|\cdot|$ the p -adic norm, and $v(\cdot)$ the p -adic valuation.

Let $f = (f_1, \dots, f_r)$ be an r -tuple of restricted power series over $\mathbb{Z}_p$ in the variables $(\lambda, x) = (\lambda_1, \dots, \lambda_s, x_1, \dots, x_m)$, i.e., the f_i are power series converging on $\mathbb{Z}_p^{s+m}$. To f we associate a parametrized p -adic integral

$$I(\lambda) = \int_{\mathbb{Z}_p^m} |f(\lambda, x)| |dx|, \quad (6)$$

where $|dx|$ is the Haar measure on $\mathbb{Z}_p^m$ normalized so that $\mathbb{Z}_p^m$ has measure 1.

A subanalytic constructible function on a subanalytic set X is by definition a $\mathbb{Q}$ -linear combination of products of functions of the form $\nu(h)$ and $|h'|$, where $h: X \rightarrow \mathbb{Q}_p^\times$ and $h': X \rightarrow \mathbb{Q}_p$ are subanalytic functions.

We show the following conjecture of Denef [73]:

Theorem (2.2.1)[83]: The function I is a subanalytic constructible function on $\mathbb{Z}_p^s$.

In the case that the functions f_i are polynomials, the map I has been studied by Igusa for $r=1$, by Lichtin for $r=2$, and by Denef for arbitrary r (see [91, 92, 93], [94], and [73]). In the more general case that the $|f(\lambda, x)|$ in (6) is replaced by an arbitrary subanalytic constructible function, the conclusion of Theorem (2.2.1) still holds (see Theorem (2.2.14) below), where now I is identically zero if the integrated function is not integrable for some λ .

The rationality of the analytic p -adic Serre–Poincaré series was conjectured in [101] and [102] and proven by Denef and van den Dries in [74]; the rationality can immediately be obtained as a corollary of integration Theorem (2.2.1). This is because it is well known how to express the Poincaré series as a p -adic integral (see [73]).

A second key result of the present section is a cell decomposition theorem for subanalytic sets and subanalytic functions (Theorem (2.2.10)), in perfect analogy to the semialgebraic cell decomposition theorem of [86] and [72]. Roughly speaking, p -adic cell decomposition theorems describe the norm of given functions after partitioning the domain of the functions in finitely many basic sets, called cells. Cell decompositions are very useful to study parameterized p -adic integrals (see below and [73]) and to show the rationality of Igusa’s local zeta functions and of several Poincaré series (see [86]). Many of the applications of cell decomposition (in for example [73] and [71]) cannot, up to now, be proven with other techniques.

The proof of the analytic cell decomposition is based on several results by van den Dries, Haskell, and Macpherson [76] on the geometry of subanalytic p -adic sets and subanalytic functions; we state some of these results in this section.

We also extensively use the theory of p -adic subanalytic sets, developed by Denef and van den Dries in [74] in analogy to the theory of real subanalytic sets; in particular, we use the dimension theory of [74]. We apply cell decomposition to obtain the following classification:

Theorem (2.2.2)[83]: Let $X \subset \mathbb{Q}_p^m$ and $Y \subset \mathbb{Q}_p^n$ be infinite subanalytic sets. Then there exists a subanalytic bijection $X \rightarrow Y$ if and only if $\dim(X) = \dim(Y)$.

This classification of subanalytic sets is similar to the classification of semialgebraic sets in [70]. Note that in particular there exists a semialgebraic bijection between $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$; this is the main result of [85].

The theory of p -adic integration has also served as an inspiring example for the theory of motivic integration and there are many connections to it (see e.g. [89] and [88]).

Many of the results of [74] and [76] are formulated for $\mathbb{Q}_p$ and not for finite field extensions of $\mathbb{Q}_p$; nevertheless, all results referred to in this section also hold for finite field extensions of $\mathbb{Q}_p$ (see [74]). All results of this section also hold in finite field extensions of $\mathbb{Q}_p$.

Let p denote a fixed prime number, $\mathbb{Q}_p$ the field of p -adic numbers and K a fixed finite field extension of $\mathbb{Q}_p$ with valuation ring R . For $x \in K^\times$ let $v(x) \in \mathbb{Z}$ denote the p -adic valuation of x and $|x| = q^{-v(x)}$ the p -adic norm, with q the cardinality of the residue class field. We write $P_n = \{y^n \mid y \in K^\times\}$, and μP_n denotes $\{\mu x \mid x \in P_n\}$ for $\mu \in K$.

For $x = (x_1, \dots, x_m)$ let $K\{x\}$ be the ring of restricted power series over K in the variables x ; it is the ring of power series $\sum a_i x^i$ in $K[[x]]$ such that $|a_i|$ tends to 0 as $|i| \rightarrow \infty$. (Here, we use the multi-index notation where $i = (i_1, \dots, i_m)$, $|i| = i_1 + \dots + i_m$ and $x^i = (x_1^{i_1}, \dots, x_m^{i_m})$.) For $x_0 \in R^m$ and $f = \sum a_i x^i$ in $K\{x\}$ the series $\sum a_i x_0^i$ converges to a limit in K , thus, one can associate to f a restricted analytic function given by

$$f : K^m \rightarrow K : x \mapsto \begin{cases} \sum_i a_i x^i & \text{if } x \in R^m, \\ 0 & \text{else.} \end{cases}$$

We extend the notion of D -functions of [74] to our setting:

Definition (2.2.3)[83]: A D -function is a function $K^m \rightarrow K$ for some $m \geq 0$, obtained by repeated application of the following rules:

- (i) for each $f \in K\{x_1, \dots, x_m\}$, the associated restricted analytic function $x \mapsto f(x)$ is a D -function;
- (ii) for each polynomial $f \in K\{x_1, \dots, x_m\}$, the polynomial map $x \mapsto f(x)$ is a D -function;
- (iii) the function $x \mapsto x^{-1}$, where $0^{-1} = 0$ by convention, is a D -function;
- (iv) for each D -function f in n variables and each D -functions $g_1, \dots, g_n$ in m variables, the function $f(g_1, \dots, g_n)$ is a D -function.

A (globally) subanalytic subset of K^m is a subset of the form

$$X = \bigcup_{i=1}^r \bigcap_{j=1}^s X_{ij}$$

where each X_{ij} is of the form $\{x \in K^m \mid f_{ij}(x) = 0\}$ or $\{x \in K^m \mid f_{ij}(x) \in P_{n_{ij}}\}$, where the functions f_{ij} are D -functions and $n_{ij} > 0$. We call a function $g : A \subset K^m \rightarrow K^n$ subanalytic if its graph is a subanalytic set. We refer to [74], [73] and [76] for the theory of subanalytic p -adic geometry and to [96] for the theory of rigid subanalytic sets.

We will use the framework of model theory. We let $\mathcal{L}_{an}$ be the first order language consisting of the symbols

$$+, -, \cdot, ^{-1}, \{P_n\}_{n>0},$$

together with an extra function symbol f for each restricted analytic function associated to restricted power series in $\bigcup_m K\{x_1, \dots, x_m\}$. We consider K as an $\mathcal{L}_{an}$ -structure using the natural interpretations of the symbols of $\mathcal{L}_{an}$.

We mention the following fundamental result in the theory of subanalytic sets.

Theorem (2.2.4)[83]: ([74]). The collection of subanalytic sets is closed under taking complements, finite unions, finite intersections, and images under subanalytic maps.

A semialgebraic subset of K^m is a subset of the same form as X above but with the f_{ij} polynomials over K , and a function is semialgebraic if its graph is a semialgebraic set. It is well known that also the collection of semialgebraic sets is closed under taking complements, finite unions and intersections, and images under semialgebraic maps (see [98], [72]).

To state cell decomposition one needs basic sets called (subanalytic) cells, which we define inductively. For $m, l > 0$ write $\pi_m : K^{m+l} \rightarrow K^m$ for the linear projection on the first m variables and, for $A \subset K^{m+l}$ and $x \in \pi_m(A)$, write A_x for the fiber $\{t \in K^l \mid (x, t) \in A\}$.

Definition (2.2.5)[83]: A cell $A \subset K$ is a (nonempty) set of the form

$$\{t \in K \mid |\alpha| \square_1 |t - \gamma| \square_2 |\beta|, t - \gamma \in \mu P_n\}, \quad (7)$$

with constants $n > 0$, $\mu, \gamma \in K$, $\alpha, \beta \in K^\times$, and $\square_i$ either $<$ or no condition. If $\mu = 0$ we call A a 0-cell and we call A a 1-cell otherwise.

A (subanalytic) cell $A \subset K^{m+1}$, $m \geq 1$, is a set of the form

$$\{(x, t) \in K^{m+1} \mid x \in D, |\alpha(x)| \square_1 |t - \gamma(x)| \square_2 |\beta(x)|, t - \gamma(x) \in \mu P_n\}, \quad (8)$$

with $(x, t) = (x_1, \dots, x_m, t)$, $n > 0$, $\mu \in K$, $D = \pi_m(A)$ a cell, subanalytic functions $\alpha, \beta : K^m \rightarrow K^\times$, $\gamma : K^m \rightarrow K$, and $\square_i$ either $<$ or no condition. We call γ the center and μP_n the coset of the cell A . If D is a cell of type $(i_1, \dots, i_m)$ with $i_i \in \{0, 1\}$, we call A an $(i_1, \dots, i_m, 0)$ -cell if $\mu = 0$ and we call A an $(i_1, \dots, i_m, 1)$ -cell otherwise. If at each stage of this inductive definition all occurring functions α, β , and γ are analytic on the respective projections $\pi_i(A)$, $i = 1, \dots, m-1$, we call A an analytic cell.

Let K_1 be an $\mathcal{L}_{an}$ -elementary extension of K and let R_1 be its valuation ring. In view of Theorem (2.2.4), we can call a set $X \subset K_1^m$ subanalytic if it is $\mathcal{L}_{an}$ -definable (with parameters from K_1) and analogously for subanalytic functions, cells, and so on. Expressions of the form $|x| < |y|$ for $x, y \in K_1$ are abbreviations for the corresponding $\mathcal{L}_{an}$ -formula's expressing $|x| < |y|$ for $x, y \in K$, as in Lemma 2.1 of [72]. Cells in K_1^m are defined just as in K^m by replacing K by K_1 everywhere in the definition. By a D -function $K_1^m \rightarrow K_1$ we mean a function given by an $\mathcal{L}_{an}$ -term (with parameters from K_1) in m variables. Similarly, one can speak of semialgebraic subsets of K_1^m (with parameters from K_1).

Theorem (2.2.6)[83]: ([76]). Each subanalytic subset of K_1 is semialgebraic.

The following two lemmas treat the one-dimensional part of Theorem (2.2.10).

Lemma (2.2.7)[83]: Let $f : R_1 \rightarrow K_1$ be a subanalytic function. Then there exists a finite partition of R_1 into semialgebraic sets A such that for each A there exist polynomials p and q such that

$$|f(x)| = |p(x)/q(x)|^{1/e}, \quad \text{for each } x \in A,$$

where q has no zeros in A and $e > 0$ is an integer.

Proof. By [87], there exists a finite partition of R_1 into subanalytic sets B such that

$$|f(x)| = |g_B(x)/h_B(x)|^{1/e}, \quad \text{for each } x \in B,$$

Where g_B and h_B are D -functions, $h_B(x) \neq 0$ on B and $e > 0$. (In [87] this is proven for subanalytic functions $\mathbb{Z}_p^m \rightarrow \mathbb{Z}_p$ using quantifier elimination in an elementary way; its proof can be repeated for our situation $R_1 \rightarrow K_1$ or otherwise one can instantiate parameters in the result of [87] to deduce this as a corollary.) By Theorem B of [76], the sets B are semialgebraic.

In [76] it is proven that the norm of any D -function in one variable is piecewise equal to the norm of a rational function, the pieces being semialgebraic sets. More precisely, by

Proposition (4.1), Corollary (3.4) and Lemma (2.10) of [76], there exists for each function g_B a finite partition of R_1 into semialgebraic sets C such that on each C

$$|g_B(x)| = |g_{BC}(x)/h_{BC}(x)|, \text{ for each } x \in C,$$

where g_{BC} and h_{BC} are polynomials over K_1 and $h_{BC}(x) \neq 0$ on C . The same holds for each function h_B . Taking an appropriate partition using intersections of these sets C and B the lemma follows.

Lemma (2.2.8)[83]: Let $X \subset R_1$ be a subanalytic set and $f : X \rightarrow K_1$ a subanalytic function. Then there exists a finite partition $\mathcal{P}$ of X into cells, such that for each cell $A \in \mathcal{P}$ with center $\gamma \in K_1$ and coset μP_n

$$|f(t)| = |\delta| |(t - \gamma)^a \mu^{-a}|^{\frac{1}{n}} \text{ for each } t \in A,$$

with $\delta \in K_1$ and a an integer. We use the convention that $a = 0$ and $0^0 = 1$ when $\mu = 0$.

Proof. We extend f to a function $R_1 \rightarrow K_1$ by putting $f(x) = 0$ if $x \notin X$. By [76], the set X is semialgebraic. Apply Lemma (2.2.7) to f to obtain a partition $\mathcal{P}$. Intersecting each set in $\mathcal{P}$ with X , we obtain a partition $\mathcal{P}'$ of X . Now apply the semialgebraic cell decomposition (in the formulation of [70]) to the sets in $\mathcal{P}'$ and the respective polynomials occurring in the application of Lemma (2.2.7). If we refine the obtained partition such that for each cell $A \subset C$ with coset μP_n the number n is a multiple of e (for the occurring fractional powers $1/e$), then the lemma follows.

We will use the previous lemma and a model-theoretical compactness argument to show the following variant of Theorem (2.2.10).

Theorem (2.2.9)[83]: Let K_1 be an arbitrary $\mathcal{L}_m$ -elementary extension of K with valuation ring R_1 . Let $X \subset K_1^{m+1}$ be subanalytic and $f_j : X \rightarrow K_1$ subanalytic functions for $j = 1, \dots, r$. Then there exists a finite partition of X into subanalytic cells A with center $\gamma : K_1^m \rightarrow K_1$ and coset μP_n such that for each $(x, t) \in A$

$$|f_j(x, t)| = |\delta_j(x)| |(t - \gamma(x))^{a_j} \mu^{-a_j}|^{\frac{1}{n}},$$

with $(x, t) = (x_1, \dots, x_m, t)$, integers a_j , and $\delta_j : K_1^m \rightarrow K_1$ subanalytic functions, $j = 1, \dots, r$. Here we use the convention that $a_j = 0$ and $0^0 = 1$ when $\mu = 0$.

Proof. The proof goes by induction on $m \geq 0$. It is enough to show the theorem for $r = 0$ (the theorem then follows after a straightforward further partitioning; see for example [72]).

When $m = 0$, the usual change of variables $t' = 1/t$ reduces the description of what happens outside R_1 to what happens on R_1 , and an application of Lemma (2.2.8) gives the desired result.

Let $X \subset K_1^{m+1}$ and $f : X \rightarrow K_1$ be subanalytic, $m > 0$. We write $(x, t) = (x_1, \dots, x_m, t)$ and know by the previous discussion that for each fixed $x \in K_1^m$ we can decompose the fiber X_x and the function $t \mapsto f(x, t)$ on this fiber. We will measure the complexity of given decompositions on which $|f(x, \cdot)|$ has a nice description and see that this must be uniformly bounded when x varies.

To do this, we define a countable set $\mathcal{S} = \{\mu P_n \mid \mu \in K, n > 0\} \times \mathbb{Z} \times \{<, 0\}^2$ and $\mathcal{S}' = (K_1^\times)^2 \times K_1^2$. To each $d = (\mu P_n, a, \square_1, \square_2)$ in $\mathcal{S}$ and $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{S}'$ we associate a set $\text{Dom}_{(d, \xi)}$ as follows:

$$\text{Dom}_{(d,\xi)} = \{t \in K_1 \mid |\xi_1|_1 |t - \xi_3|_2 | \xi_2|, t - \xi_3 \in \mu P_n\}.$$

The set $\text{Dom}_{(d,\xi)}$ is either empty or a cell and is independent of ξ_4 and a . For fixed $k > 0$ and tuple $d = (d_1, \dots, d_k) \in S^k$, let $\varphi_{(d,k)}(x, \xi)$ be an $\mathcal{L}_{an}$ -formula in the free variables $x = (x_1, \dots, x_k)$ and $\xi = (\xi_1, \dots, \xi_k)$, with $\xi_i = (\xi_{i1}, \xi_{i2}, \xi_{i3}, \xi_{i4})$, such that $(x, \xi) \in K_1^{m+4k}$ satisfies $\varphi_{(d,k)}$ if and only if the following are true:

- (i) $x \in \pi_m(X)$ and $\xi \in (S')^k$,
- (ii) the collection of the sets $\text{Dom}_{(d_i, \xi_i)}$ for $i = 1, \dots, k$ forms a partition of the fiber $X_x = \{t \in K_1 \mid (x, t) \in X\}$,
- (iii) $|\xi_{i4}| \left| (t - \xi_{i3})^{a_i} \mu_i^{-a_i} \right|^{\frac{1}{n_i}} = |f(x, t)|$ for each $t \in \text{Dom}_{(d_i, \xi_i)}$ and each $i = 1, \dots, k$.

Now we define for each $k > 0$ and $d \in S^k$ the set

$$B_d = \{x \in K_1^m \mid \exists \xi \quad \varphi_d(x, \xi)\}.$$

Each set B_d is subanalytic and the (countable) collection $\{B_d\}_{k,d}$ covers $\pi_m(X)$, because each $x \in \pi_m(X)$ is in some B_d by the induction. Since K_1 is an arbitrary elementary extension of K , finitely many sets of the form B_d must already cover $\pi_m(X)$ by model-theoretical compactness. Consequently, we can take subanalytic sets $D_1, \dots, D_s$ such that $\{D_i\}$ forms a partition of $\pi_m(X)$ and each D_i is contained in a set B_d for some $k > 0$ and k -tuple d . For each $i = 1, \dots, s$, fix such a d with $D_i \subset B_d$, and let Γ_i be the subanalytic set

$$\Gamma_i = \{(x, \xi) \in D_i \times (S')^k \mid \varphi_d(x, \xi)\}.$$

Then $\pi_m(\Gamma_i) = D_i$ by construction (π_m is the projection on the x -coordinates). By [74] on definable Skolem functions, there is a subanalytic function $D_i \rightarrow K_1^{4k}$ associating to x a tuple $\xi(x) \in (S')^k$ such that $(x, \xi(x)) \in \Gamma_i$ for each $x \in D_i$. The theorem now follows by partitioning further with respect to the x -variables and using the induction hypothesis.

Theorem (2.2.10)[83]: Let $X \subset K^{m+1}$ be a subanalytic set, $m \geq 0$, and $f_j : X \rightarrow K$ subanalytic functions for $j = 1, \dots, r$. Then there exists a finite partition of X into cells A with center $\gamma : K^m \rightarrow K$ and coset μP_n such that for each $(x, t) \in A$

$$|f_j(x, t)| = |\delta_j(x)| \left| (t - \gamma(x))^{a_j} \mu^{-a_j} \right|^{\frac{1}{n}}, \quad \text{for each } j = 1, \dots, r, \quad (9)$$

with $(x, t) = (x_1, \dots, x_m, t)$, integers a_j , and $\delta_j : K^m \rightarrow K$ subanalytic functions. If $\mu = 0$, we use the conventions $a_j = 0$ and $0^0 = 1$. Moreover, the cells A can be taken to be analytic cells such that the δ_j are analytic on $\pi_m(A)$.

For the proof of Theorem (2.2.10) we use techniques from model theory, namely a compactness argument. (For general notions of model theory we refer to [77].)

Proof. We only have to show that we can partition X using analytic cells A in such a way that the functions δ_j are analytic on $\pi_m(A)$. In [74] one proves that any subanalytic function is piecewise analytic. Theorem (2.2.10) then follows from Theorem (2.2.10) by partitioning further using this fact.

For $X \subset K^m$ subanalytic and nonempty, the dimension $\dim(X)$ of X is defined as the largest integer n such that there is a K -linear map $\pi : K^m \rightarrow K^n$ and a nonempty $U \subset \pi(X)$, open in K^n (for alternative definitions, see [74]).

Theorem (2.2.11)[83]: For any subanalytic set $X \subset K^m$ and subanalytic functions $f_i : X \rightarrow K$, $i = 1, \dots, r$, there is a semialgebraic set Y , a subanalytic bijection $F : X \rightarrow Y$ and

there are semialgebraic maps $g_i : Y \rightarrow K$ such that

$$|g_i(F(x))| = |f_i(x)| \quad \text{for each } x \in X.$$

Proof. We will give a proof by induction on m . Suppose that $X \subset K^{m+1}$ is subanalytic and that $f_i : X \rightarrow K$ are subanalytic functions, $m \geq 0$. Apply cell decomposition to X and the functions f_i to obtain a finite partition $\mathcal{P}$ of X . For $A \in \mathcal{P}$ and $(x, t) \in A$, suppose that

$$|f_j(x, t)| = |\delta_j(x)| \left| (t - \gamma(x))^{a_j} \mu^{-a_j} \right|^{\frac{1}{n}}, \quad i = 1, \dots, r, \text{ and suppose that } A \text{ is a cell of the form}$$

$$\{(x, t) \in K^{m+1} \mid x \in D, |\alpha(x)| \square_1 |t - \gamma(x)| \square_2 |\beta(x)|, t - \gamma(x) \in \mu P_n\},$$

as in (8). After the translation $(x, t) \mapsto (x, t - \gamma(x))$ we may suppose that γ is zero on D . Apply the induction hypotheses to the sets D and the subanalytic functions α, β , and δ_i . Repeating this process for every $A \in \mathcal{P}$, and noting that there is a semialgebraic function $h : P_n \rightarrow K$ such that $|h(t)| = |t|^{1/n}$, the proposition follows after taking appropriate disjoint unions inside K^m of the occurring semialgebraic sets.

We show the following generalization of Theorem (2.2.2).

Theorem (2.2.12)[83]: Let $X \subset K^m$ and $Y \subset K^n$ be infinite subanalytic sets. Then there exists a subanalytic bijection $X \rightarrow Y$ if and only if $\dim(X) = \dim(Y)$.

Proof. By Theorem (2.2.11) there are subanalytic bijections $X \rightarrow X'$ and $Y \rightarrow Y'$ with X' and Y' semialgebraic, but then there exists a semialgebraic bijection $X' \rightarrow Y'$ if and only if $\dim(X') = \dim(Y')$ by [70]. Since the dimension of a subanalytic set is invariant under subanalytic bijections (see [74]), the theorem follows.

We show that certain algebras of functions from $\mathbb{Q}_p^m$ to the rational numbers $\mathbb{Q}$ are closed under p -adic integration. These functions are called subanalytic constructible functions, and they come up naturally when one calculates parametrized p -adic integrals such as (6).

For $x = (x_1, \dots, x_m)$ an m -tuple of variables, we will write $|dx|$ to denote the Haar measure on K^m , so normalized that R^m has measure 1.

Definition (2.2.13)[83]: For each subanalytic set X , we let $\mathcal{C}(X)$ be the $\mathbb{Q}$ -algebra generated by the functions $X \rightarrow \mathbb{Q}$ of the form $x \mapsto \nu(h(x))$ and $x \mapsto |h'(x)|$, where $h : X \rightarrow K^\times$ and $h' : X \rightarrow K$ are subanalytic functions. We call $f \in \mathcal{C}(X)$ a subanalytic constructible function on X .

To any function f in $\mathcal{C}(K^{m+n})$, $m, n \geq 0$, we associate a function $I_m(f) : K^m \rightarrow \mathbb{Q}$ by putting

$$I_m(f)(\lambda) = \int_{K^n} f(\lambda, x) |dx|, \quad (10)$$

if the function $x \mapsto f(\lambda, x)$ is absolutely integrable for all $\lambda \in K^m$, and we put $I_m(f)(\lambda) = 0$ otherwise.

Theorem (2.2.14)[83]: (Basic Theorem on p -Adic Analytic Integrals). For any function $f \in \mathcal{C}(K^{m+n})$, the function $I_m(f)$ is in $\mathcal{C}(K^m)$.

Proof. By induction it is enough to show that for a function f in $\mathcal{C}(K^{m+1})$ in the variables $(\lambda_1, \dots, \lambda_m, t)$ the function $I_m(f)$ is in $\mathcal{C}(K^m)$. Suppose that f is a $\mathbb{Q}$ -linear combination of products of functions $|f_i|$ and $\nu(g_j)$, $i = 1, \dots, r$, $j = 1, \dots, s$, where f_i and g_j are subanalytic functions $K^{m+1} \rightarrow K$ and $g_j(\lambda, t) \neq 0$. Applying cell decomposition to K^{m+1} and the functions f_i and g_j , we obtain a partition $\mathcal{P}$ of K^{m+1} into cells such that $I_m(f)(\lambda)$ is a

sum of integrals over $A_\lambda = \{t \mid (\lambda, t) \in A\}$ for each cell $A \in \mathcal{P}$, where the integrands on these pieces A_λ have a very simple form. More precisely, on each piece A_λ the integrand is a $\mathbb{Q}$ -linear combination of functions of the form

$$\delta(\lambda) \left| (t - \gamma)^a \mu^{-a} \right|^{\frac{1}{n}} \nu(t - \gamma(\lambda))^l, \quad (11)$$

where A is a cell with center $\gamma: K^m \rightarrow K$ and coset μP_n , and with integers a and $0 \leq l$, and a function δ in $\mathcal{C}(K^m)$. We may suppose that $\delta(\lambda) \neq 0$ for some $\lambda \in \pi_m(A)$. Regroup all such terms where the same exponents a and l appear, possibly by replacing the functions $\delta(\lambda)$ by other functions in $\mathcal{C}(K^m)$. The integrability of such an integrand then only depends on the integers a , l , and n occurring in each of the terms as in (11) and on the symbols $\square_1$ and μ occurring in the description of the cell A . By consequence, we may suppose that the integrand is a single term of the form as in (11) and that this term is absolutely integrable over A . It suffices to show that the integral

$$\delta(\lambda) \int_{t \in A_\lambda} \left| (t - \gamma(\lambda))^a \mu^{-a} \right|^{\frac{1}{n}} \nu(t - \gamma(\lambda))^l |dt| \quad (12)$$

is in $\mathcal{C}(K^m)$. Write $u = t - \gamma(\lambda)$; since A is a cell with center γ and coset μP_n , the set A is of the form

$$A = \{(\lambda, u) \in K^{m+1} \mid \lambda \in D, |\alpha(\lambda)| \square_1 |u| \square_2 |\beta(\lambda)|, u \in \mu P_n\},$$

with $\square_i$ either $<$ or no condition, D a cell, and $\alpha, \beta: K^m \rightarrow K^\times$ subanalytic functions. Taking into account that the integral (12) is, by supposition, integrable, only a few possibilities can occur (with respect to the integers a , l , and n , the conditions $\square_i$, and μ being zero or nonzero). If $\mu = 0$, the set A_λ is a point for each $\lambda \in D$, thus the statement is clear. Suppose $\mu \neq 0$. In case that both $\square_1$ and $\square_2$ represent no condition, the integrand has to be zero by the supposition of integrability, and the above integral trivially is in $\mathcal{C}(K^m)$. We suppose from now on that $\square_1$ is $<$; the other cases can be treated similarly. The integral (12) can be rewritten as

$$\begin{aligned} \delta(\lambda) \cdot \int_{u \in A_\lambda} \left| \mu^a \mu^{-a} \right|^{\frac{1}{n}} \nu(u)^l |du| &= \delta(\lambda) \sum_k \left(q^{-ak} \left| \mu^{-a} \right|^{\frac{1}{n}} k^l \cdot \text{Measure}\{u \in A_\lambda \mid \nu(u) = k\} \right) \\ &= \epsilon \delta(\lambda) \sum_k \left(q^{-ak} \left| \mu^{-a} \right|^{\frac{1}{n}} k^l q^{-k} \right) \end{aligned}$$

for $\epsilon = q^s$. $\text{Measure}\{u \in A_\lambda \mid \nu(u) = s\}$ (where s is any number such that $\emptyset = \{u \in A_\lambda \mid \nu(u) = s\}$), and where the summation is over those integers $k \equiv \nu(\mu) \pmod{n}$ satisfying

$$|\alpha(\lambda)| < q^{-k} \square_2 |\beta(\lambda)|.$$

We may suppose that on A , the residue classes

$$\nu(\alpha(\lambda)) \pmod{n} \text{ and } \nu(\beta(\lambda)) \pmod{n}$$

are fixed (possibly after refining the partition $\mathcal{P}$). Then this sum is equal to a $\mathbb{Q}$ -linear combination of products of the functions δ , $|\alpha|$, $|\beta|$, $\nu(\alpha)$ and $\nu(\beta)$. For example, if $a/n = -1$, $\square_1$ and $\square_2$ are necessarily $<$ and one obtains a polynomial in $\nu(\alpha)$ and $\nu(\beta)$ of degree $\leq l + 1$, multiplied with δ . For more examples of calculations of sums of this kind, see [71]. Thus, the integral (12) is in $\mathcal{C}(K^m)$ as was to be shown.

As a corollary we will formulate another version of the basic integration theorem, conjectured in [8].

Definition (2.2.15)[83]: A set $A \subset \mathbb{Z}^n \times K^m$ is called simple if

$$\{(\lambda, x) \in K^{n+m} \mid (\nu(\lambda_1), \dots, \nu(\lambda_n), x) \in A \ \& \ \prod_{i=1, \dots, n} \lambda_i \neq 0\}$$

is a subanalytic set. A function $h : A \subset \mathbb{Z}^n \times \mathbb{Q}_p^m \rightarrow \mathbb{Z}$ is called simple if its graph is simple. For a simple set X we let $\mathcal{C}_{\text{simple}}(X)$ be the $\mathbb{Q}$ -algebra generated by all simple functions on X and all functions of the form q^h , where h is a simple function on X .

For a function f in $\mathcal{C}_{\text{simple}}(\mathbb{Z}^{k+l} \times K^{m+n})$, k, l, m, n integers ≥ 0 , we define $I_{k,m}(f) : \mathbb{Z}^k \times K^m \rightarrow \mathbb{Q}$ as

$$I_{k,m}(f)(z, \lambda) = \sum_{z' \in \mathbb{Z}^l} \int_{K^n} f(z, z', \lambda, x) |dx|$$

if the function $(z', x) \mapsto f(z, z', \lambda, x)$ is absolutely integrable for all $(z, \lambda) \in \mathbb{Z}^k \times K^m$ with respect to the Haar measure on K^n and the discrete measure on $\mathbb{Z}^l$, and we define $I_{k,m}(f)(z, \lambda) = 0$ otherwise.

Theorem (2.2.16)[83]: For each f in $\mathcal{C}_{\text{simple}}(\mathbb{Z}^{k+l} \times K^{m+n})$, the function $I_{k,m}(f)$ is in $\mathcal{C}_{\text{simple}}(\mathbb{Z}^k \times K^l)$.

Proof. It is enough to show that for a function f in $\mathcal{C}_{\text{simple}}(\mathbb{Z}^k \times K^m)$ in the variables $(z_1, \dots, z_k, x_1, \dots, x_m)$ the function obtained by eliminating x_m by integration, resp. eliminating z_k by summation, is in the respective algebra $\mathcal{C}_{\text{simple}}$.

We first focus on integration with respect to x_m . To $f : \mathbb{Z}^k \times K^m \rightarrow \mathbb{Q}$ we can associate a function $g : K^{k+m} \rightarrow \mathbb{Q}$ by replacing the variables z running over $\mathbb{Z}^k$ by variables λ running over K^k in such a way that $g(\lambda, x) = f(\nu(\lambda_1), \dots, \nu(\lambda_k), x)$ for each $\lambda \in (K^\times)^k$ and $g(\lambda, x) = 0$ if one of the λ_i is zero. By the definitions it is immediate that g is in $\mathcal{C}(K^{k+m})$ and the integral of f with respect to x_m corresponds to the integral of the function g with respect to x_m . If we eliminate x_m by integration from g , then we get the function $I_{k+m-1}(g)$ which is in $\mathcal{C}(K^{k+m-1})$ by Theorem (2.2.14). This function only depends on $(\nu(\lambda_1), \dots, \nu(\lambda_k), x_1, \dots, x_{m-1})$ and thus corresponds to a function in $\mathcal{C}_{\text{simple}}(K^{k+m-1})$, as one can check.

If we want to eliminate z_k by summation, we associate to f the subanalytic constructible function $g' : K^{k+m} \rightarrow \mathbb{Q}$ determined by

$$g'(\lambda, x) = |\lambda_k|^{-1} \frac{p}{p-1} f(\nu(\lambda_1), \dots, \nu(\lambda_k), x)$$

if $\prod_{i=1, \dots, n} \lambda_i \neq 0$ and $g'(\lambda, x) = 0$ if $\prod_{i=1, \dots, n} \lambda_i = 0$. Integrating with respect to λ_k then corresponds to summing over z_k , and the same argument as above can be applied to complete the proof.

Section (2.3): Semi-Algebraic Bijection

In real semi-algebraic geometry (as opposed to p -adic semi-algebraic geometry) the following classification is well-known [82]:

There exists a real semi-algebraic bijection between two real semi-algebraic sets if and only if they have the same dimension and Euler characteristic.

More generally L. van den Dries [82] gave such a classification for o -minimal expansions of the real field, using the dimension and Euler characteristic as defined for o -minimal structures. Since the semi-algebraic Euler characteristic χ is in fact the canonical map from the real semi-algebraic sets onto the Grothendieck ring (see [85]) of

$\mathbb{R}$ (which is $\mathbb{Z}$), we see that the isomorphism class of a real semi-Algebraic set only depends on its image in the Grothendieck ring and its dimension.

we treat the p -Adic analogue of this classification. The Grothendieck ring of $\mathbb{Q}_p$ is recently proved to be trivial by D. Haskell and the author [85], so the analogue of the real case is a classification of the p -adic semi-algebraic sets up to semi-algebraic bijection using only the dimension. We give such a classification for the p -adic semi-algebraic sets and for finite field extensions of $\mathbb{Q}_p$, using explicit isomorphisms of [85] and the p -adic Cell Decomposition Theorem of J. Denef [72]. The most difficult part in giving this classification is to show that for any semi-algebraic set X there is a finite partition into semi-algebraic sets, such that each part is isomorphic to a Cartesian product of one dimensional sets, in other words semi-algebraic sets have a rectilinearization. Since all arguments hold also for finite field extensions of $\mathbb{Q}_p$, we work in this more general setting.

Let p denote a fixed prime number, $\mathbb{Q}_p$ the field of p -adic numbers and K a fixed finite field extension of $\mathbb{Q}_p$. For $x \in K$ let $v(x) \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of x . Let $R = \{x \in K \mid v(x) \geq 0\}$ be the valuation ring, $K^\times = K \setminus \{0\}$ and for $n \in \mathbb{N}_0$ let P_n be the set $\{x \in K^\times \mid \exists y \in K \ y^n = x\}$. We call a subset of K^n semi-algebraic if it is a Boolean combination (i.e. obtained by taking finite unions, complements and intersections) of sets of the form $\{x \in K^m \mid f(x) \in P_n\}$, with $f(x) \in K[X_1, \dots, X_m]$. The collection of semi-algebraic sets is closed under taking projections $K^m \rightarrow K^{m-1}$, even more: it consists precisely of Boolean combinations of projections of affine p -Adic varieties. Further we have that sets of the form $\{x \in K^m \mid v(f(x)) \leq v(g(x))\}$ with $f(x), g(x) \in K[X_1, \dots, X_m]$ are semi-Algebraic (see [72] and [98]). A function $f: A \rightarrow B$ is semi-Algebraic if its graph is a semi-Algebraic set; if further f is a bijection, we call f an isomorphism and we write $A \cong B$.

Let π be a fixed element of R with $v(\pi) = 1$, thus π is a uniformizing parameter for R . For a semi-algebraic set $X \subset K$ and $K > 0$ we write

$$X^{(k)} = \{x \in X \mid x \neq 0 \text{ and } v(\pi^{-v(x)}x - 1) \geq k, x \neq 0\},$$

which is semi-algebraic (see [72]); $X^{(k)}$ consists of those points $x \in X$ which have a p -adic expansion $x = \sum_{i=s}^{\infty} a_i \pi^i$ with $a_s = 1$ and $a_i = 0$ for $i = s+1, \dots, s+k-1$. By a finite partition of a semi-algebraic set we mean a partition into finitely many semi-algebraic sets. Let $X \subset K^n$, $Y \subset K^m$ be semi-algebraic. Choose disjoint semi-algebraic sets $X', Y' \subset K^k$ for some k , such that $X \cong X'$ and $Y \cong Y'$, then we define the disjoint union of X and Y up to isomorphism as $X' \cup Y'$. In the introduction of [85] it is shown that we can take $k = \max(m, n)$, i.e. we can realize the disjoint union without going into higher dimensional affine spaces.

We recall some well-known facts.

Lemma (2.3.1)[70]: Let $f(t)$ be a polynomial over R in one variable t , and let $\alpha \in R$, $e \in \mathbb{N}$. Suppose that $f(\alpha) \equiv 0 \pmod{\pi^{2e+1}}$ and $v(f'(\alpha)) \leq e$, where f' denotes the derivative of f . Then there exists a unique $\bar{\alpha} \in R$ such that $f(\bar{\alpha}) = 0$ and $f(\bar{\alpha}) \equiv 0 \pmod{\pi^{2e+1}}$.

Corollary (2.3.2)[70]: Let $n > 1$ be a natural number. For each $k > v(n)$, and $k' = k + v(n)$ the function

$$K^{(k)} \rightarrow P_n^{(k')}: x \rightarrow x^n$$

is an isomorphism.

The next theorem gives some concrete isomorphisms between one dimensional sets.

Proposition (2.3.3)[70]:[85].

- (i) The union of two disjoint copies of $R \setminus \{0\}$ is isomorphic to $R \setminus \{0\}$.
- (ii) For each $k > 0$ the union of two disjoint copies of $R^{(k)}$ is isomorphic to $R^{(k)}$.
- (iii) $R \cong R \setminus \{0\}$.

We deduce an easy corollary, also consisting of concrete isomorphisms.

Corollary (2.3.4)[70]: For each k we have isomorphisms

- (i) $R^{(k)} \cong R \setminus \{0\}$,
- (ii) $R \setminus \{0\} \cong K$.

Proof. (i) There is a finite partition $R \setminus \{0\} = \bigcup_{\alpha} \alpha R^{(k)}$ with $\nu(\alpha) = 0$, say with s parts. Then $R \setminus \{0\}$ is a fortiori isomorphic to the union of s disjoint copies of $R^{(k)}$, which is by Proposition (2.3.3)(ii) isomorphic to $R^{(k)}$.

(ii) The map

$$(\{0\} \times R) \cup (\{1\} \times R \setminus \{0\}) \rightarrow K : \begin{cases} (0, x) \mapsto x, \\ (1, x) \mapsto 1/(\pi x), \end{cases}$$

is a well-defined isomorphism. It follows that K is isomorphic to the disjoint union of R and $R \setminus \{0\}$. Now use (i) and (iii) of Proposition (2.3.3).

Give K^m the topology induced by the norm $|x| = \max(|x_i|_p)$ with $|x_i|_p = p^{-\nu(x_i)}$ for $x = (x_1, \dots, x_m) \in K^m$. P. Scowcroft and L. van den Dries [104] proved there exists no isomorphism from an open set $A \subset K^m$ onto an open set $B \subset K^n$ with $n \neq m$, so we can define the dimension of semi-algebraic sets as follows.

Definition (2.3.5)[70]: [104]. The dimension of a semi-algebraic set $X \neq \emptyset$ is the greatest natural number n such that we have a nonempty semi-algebraic subset $A \subset X$ and an isomorphism from A to a nonempty semi-algebraic open subset of K^n . We put $\dim(\emptyset) = -1$.

P. Scowcroft and L. van den Dries [104] proved many good properties of this dimension, for example that it is invariant under isomorphisms.

Proposition (2.3.6)[70]: [104]. Let A and B be semi-algebraic sets, then the following is true:

- (i) If $A \cong B$ then $\dim(A) = \dim(B)$,
- (ii) $\dim(A \cup B) = \max(\dim(A), \dim(B))$.
- (iii) $\dim(A) = 0$ if and only if A is finite and nonempty.

We will show the converse of (i) for infinite semi-algebraic sets.

Lemma (2.3.7)[70]: For any semi-algebraic set X of dimension $m \in \mathbb{N}_0$ there exists a semi-algebraic injection $X \rightarrow K^m$.

Proof. By [104] there is a finite partition of X such that each part A is isomorphic to a semi-algebraic open $A' \subset K^k$ for some $k \leq m$. Now realize the disjoint union of the sets A' without going into higher embedding dimension (see the introduction).

We formulate the p -adic Cell Decomposition Theorem by J. Denef [72, 86], which is the analogue of the real semi-algebraic Cell Decomposition Theorem.

Theorem (2.3.8)[70]: [72,86]. Let $x = (x_1, \dots, x_m)$ and $\hat{x} = (x_1, \dots, x_{m-1})$, $m > 0$. Let $f_i(\hat{x}, x_m)$, $i = 1, \dots, r$, be polynomials in x_m with coefficients which are semi-algebraic functions from K^{m-1} to K . Let $n \in \mathbb{N}_0$ be fixed. Then there exists a finite partition of K^m into sets A of the form

$$A = \{x \in K^m \mid \hat{x} \in D \text{ and } \nu(a_1(\hat{x})) \square_1 \nu(x_m - c(\hat{x})) \square_2 \nu(a_2(\hat{x}))\},$$

such that

$$f_i(x) = u_i(x)^n h_i(\hat{x})(x_m - c(\hat{x}))^{\nu_i}, \text{ for each } x \in A, i = 1, \dots, r,$$

with $u_i(x)$ a unit in R for each x , $D \subset K^{m-1}$ semi-algebraic, $\nu_i \in \mathbb{N}_0$, h_i , a_1 , a_2 , c semi-algebraic functions from K^{m-1} to K and $\square_1$, $\square_2$ either $\leq$, $<$, or no condition.

The next Lemma is also due to J. Denef [86].

Lemma (2.3.9)[70]: [86]. Let $b : K^m \rightarrow K$ be a semi-algebraic function. Then there exists a finite partition of K^m such that for each part A we have $e > 0$ and polynomials $f_1, f_2 \in R[X_1, \dots, X_m]$ such that

$$\nu(b(x)) = \frac{1}{e} \nu\left(\frac{f_1(x)}{f_2(x)}\right), \text{ for each } x \in A,$$

with $f_2(x) \neq 0$ for each $x \in A$.

We give an application of the Cell Decomposition Theorem and Lemma (2.3.9), inspired by similar applications in [86]. For details of the proof we refer to the proof of [86], By λP_n with $\lambda = 0$ we mean $\{0\}$.

Lemma (2.3.10)[70]: Let $X \subset K^m$ be semi-algebraic and $b_j : K^m \rightarrow K$ semi-algebraic functions for $j = 1, \dots, r$. Then there exists a finite partition of X s.t. each part A has the form

$$A = \{x \in K^m \mid \hat{x} \in D, \nu(a_1(\hat{x})) \square_1 \nu(x_m - c(\hat{x})) \square_2 \nu(a_2(\hat{x})), x_m - c(\hat{x}) \in \lambda P_n\},$$

and such that for each $x \in A$ we have

$$\nu(b_j(x)) = \frac{1}{e_j} \nu((x_m - c(\hat{x}))^{\mu_j} d_j(\hat{x})),$$

with $\hat{x} = (x_1, \dots, x_{m-1})$, $D \subset K^{m-1}$ semi-algebraic, $e_j > 0$, $\mu_j \in \mathbb{Z}$, $\lambda \in K$, c , a_1 , d_j semi-algebraic functions from K^{m-1} to K and $\square_i$ either $<$, $\leq$ or no condition.

Proof. By Lemma (2.3.9) we have a finite partition of X such that for each part A_0 we have $e_j > 0$ and polynomials $g_j, g'_j \in R[X_1, \dots, X_m]$ with

$$\nu(b_j(x)) = \frac{1}{e_j} \nu\left(\frac{g_j(x)}{g'_j(x)}\right), \text{ for each } x \in A_0, j = 1, \dots, r.$$

Let f_i be the polynomials which appear in a description of A_0 as a Boolean combination of sets of the form $\{x \in K^m \mid f(x) \in P_n\}$.. Apply now the Cell Decomposition Theorem as in the proof of [86], to the polynomials f_i , g_j and g'_j to obtain the lemma.

The proof of the next proposition is an application of both the Cell Decomposition Theorem and some hidden Presburger arithmetic in the value group of K ; it is the technical heart of this section. If $l = 0$ then $\prod_{i=1}^l R^{(k)}$ denotes the set $\{0\}$.

Definition (2.3.11)[70]: We say that a semi-algebraic function $f : B \rightarrow K$ satisfies condition (13) (with constants e , μ_i , β) if we have constants $e \in \mathbb{N}_0$, $\mu_i \in \mathbb{Z}$, $\beta \in K$ such that each $x = (x_i) \in B$ satisfies

$$\nu(f(x)) = \frac{1}{e} \nu(\beta \prod_i x_i^{\mu_i}). \quad (13)$$

Proposition (2.3.12)[70]: Let X be a semi-algebraic set and $b_j : X \rightarrow K$ semi-algebraic functions for $j = 1, \dots, r$. Then there exists a finite partition of X such that for each part A we have constants $l \in \mathbb{N}$, $k \in \mathbb{N}_0$, $\mu_{ij} \in \mathbb{Z}$, $\beta_j \in K$, and an isomorphism

$$f : \prod_{i=1}^l R^{(k)} \rightarrow A ,$$

such that for each $x = (x_1, \dots, x_l) \in \prod_{i=1}^l R^{(k)}$ we have

$$\nu(b_j \circ f(x)) = \nu(\beta_j \prod_{i=1}^l x_i^{\mu_{ij}}) .$$

Proof. We work by induction on $m = \dim(X)$. Let $\dim(X) = 1$ and $b_j : X \rightarrow K$ semi-algebraic functions, $j = 1, \dots, r$. By Lemma (2.3.7) we may suppose that $X \subset K$. We reduce first to the case that X and b_j have the special form (14) (see below). By Lemma (2.3.10) there is a partition such that each part A is either a point or of the form

$$A = \{x \in K \mid \nu(a_1) \sqcup_1 \nu(x - c) \sqcup_2 \nu(a_2), x - c \in \lambda P_n\} ,$$

and such that for each $x \in A$ we have $\nu(b_j(x)) = \frac{1}{e_j} \nu(\beta_j(x - c)^{\mu_j})$, with $a_1, c, \lambda, \beta_j \in K$, $e_j > 0$ and $\mu_j \in \mathbb{Z}$. We may assume that $\lambda \neq 0$, $a_1 \neq 0 \neq a_2$, $\sqcup_i$ is either $\leq$ or no condition and since the translation

$$\{x \in K \mid \nu(a_1) \sqcup_1 \nu(x) \sqcup_2 \nu(a_2), x \in \lambda P_n\} \rightarrow A : x \mapsto x + c$$

is an isomorphism, we may also assume that $c = 0$. If both $\sqcup_1$ and $\sqcup_2$ are no condition we can partition A into parts $\{x \in A \mid 0 \leq \nu(x)\}$ and $\{x \in A \mid \nu(x) \leq -1\}$. It follows that if $\sqcup_1$ is no condition we may suppose that $\sqcup_2$ is $\leq$, then we can apply the isomorphism

$$\{x \in K \mid \nu(\frac{1}{a_2}) \leq \nu(x), x \in \frac{1}{\lambda} P_n\} \rightarrow A : x \mapsto \frac{1}{x} ,$$

and replace μ_j by $-\mu_j$. This shows we can reduce to the case that X has the form

$$X = \{x \in K \mid \nu(a_1) \leq \nu(x) \sqcup_2 \nu(a_2), x \in \lambda P_n\} , \quad (14)$$

with $a_1 \neq 0 \neq a_2$, $\lambda \neq 0$, $\sqcup_2$ either $\leq$ or no condition and $\nu(b_j(x)) = \frac{1}{e_j} \nu(\beta_j x^{\mu_j})$ for each $x \in X$.

Case 1: $\sqcup_2$ is $\leq$ (in equation (14)).

By Hensel's Lemma we can partition X into finitely many parts of the form $y + \pi^s R$ for some fixed $s > \nu(a_2)$ and with $\nu(a_1) \leq \nu(y) \leq \nu(a_2)$ for each y . For each such part there is a finite partition $y + \pi^s R = \bigcup_{\gamma \in \Gamma} A_\gamma \cup \{y\}$, with $A_\gamma = y + \pi^s \gamma R^{(1)}$ and $\nu(\gamma) = 0$ for each γ . The functions $f_\gamma : R^{(1)} \rightarrow A_\gamma : x \mapsto y + \pi^s \gamma x$ are isomorphisms which satisfy $\nu(b_j \circ f_\gamma(x)) = \frac{1}{e_j} \nu(\beta_j y^{\mu_j})$ for all $x \in R^{(1)}$. This last expression is independent of x , so there exists $\beta'_j \in K$ such that $\nu(b_j \circ f_\gamma(x)) = \nu(\beta'_j)$ for all $x \in R^{(1)}$. This shows Case 1.

Case 2: $\sqcup_2$ is no condition (in equation (14)).

The map

$$f_1 : R \cap \lambda' P_n \rightarrow X : x \mapsto a_1 x ,$$

with $\lambda' = \lambda/a_1$ is an isomorphism. Let n' be a common multiple of $e_1, \dots, e_r$ and n . Choose $k > \nu(n')$ and put $k' = k + \nu(n')$. Let $R \cap \lambda' P_n = \bigcup_\gamma B_\gamma$ be a finite partition, with $B_\gamma = \gamma(R \cap P_{n'}^{(k')})$ and $0 \leq \nu(\gamma) < n'$. Now we have that the map $f_\gamma : R^{(k)} \rightarrow B_\gamma : x \mapsto \gamma x^{n'}$ is an isomorphism by corollary (2.3.2). Let g_γ be the semi-algebraic function $f_1 \circ f_\gamma$, which is an isomorphism from $R^{(k)}$ onto a semi-algebraic set $A_\gamma \subset X$. The sets A_γ form a finite partition of X . Put $\mu'_j = \mu_j n' / e_j$, then we have for each $x \in R^{(k)}$ that

$$\begin{aligned} \nu(b_j \circ g_\gamma(x)) &= \frac{1}{e_j} \nu(\beta_j(a_1 \gamma x^{n'})^{\mu_j}) = \frac{1}{e_j} \nu(\beta_j(a_1 \gamma)^{\mu_j}) + \nu(x^{\mu_j}) \\ &= \nu(\beta'_j) + \nu(x^{\mu_j}) = \nu(\beta'_j x^{\mu_j}), \end{aligned}$$

with β'_j in K such that $\nu(\beta'_j) = \frac{1}{e_j} \nu(\beta_j(a_1 \gamma)^{\mu_j})$. This shows case 2.

Now let $\dim(X) = m > 1$ and let $b_j : X \rightarrow K$ be semi-algebraic functions, $j = 1, \dots, r$. By Lemma (2.3.7) we may suppose that $X \subset K^m$.

Claim. We can partition X such that for each part A we have an isomorphism of the form $f : D_1 \times D_{m-1} \rightarrow A$, with $D_1 \subset K$ and $D_{m-1} \subset K^{m-1}$ semi-algebraic, such that the functions $b_j \circ f$ satisfy condition (13), i.e. there are constants $e_j \in \mathbb{N}_0$, $\mu_{ij} \in \mathbb{Z}$, $\beta_j \in K$ such that each $x = (x_i) \in D_1 \times D_{m-1}$ satisfies

$$\nu(b_j \circ f(x)) = \frac{1}{e_j} \nu(\beta_j \prod_i x_i^{\mu_{ij}}).$$

If the claim is true, we can apply the induction hypotheses once to D_1 and the functions $x \mapsto x_1^{\mu_{1j}}$ and once to D_{m-1} and the functions $(x_2, \dots, x_m) \mapsto \beta_j \prod_{i=2}^m x_i^{\mu_{ij}}$ for $j = 1, \dots, r$. It follows easily that we can partition X such that for each part A there is an isomorphism $f : \prod_i R^{(k)} \rightarrow A$ such that all $f \circ b_j$ satisfy condition (13) with constants e'_j , μ'_{ij} and β'_j . Now we can proceed as in Case 2 for $m = 1$ to make all $e'_j \in \mathbb{N}_0$ occurring in condition (13) equal to 1. The proposition follows now immediately.

Proof of the claim. First we show we can reduce to the case described in equation (15) below. Using Lemma (2.3.10) and its notation, we find a finite partition of X such that each part A has the form

$$A = \{x \in K^m \mid \hat{x} \in D, \nu(a_1(\hat{x})) \sqcap_1 \nu(x_m - c(\hat{x})) \sqcap_2 \nu(a_2(\hat{x})), x_m - c(\hat{x}) \in \lambda P_n\},$$

and such that for each $x \in A$ we have $\nu(b_j(x)) = \frac{1}{e_j} \nu((x_m - c(\hat{x}))^{\mu_{mj}} d_j(\hat{x}))$, with $\mu_{mj} \in \mathbb{Z}$.

Similar as for $m = 1$, we may suppose that $c(\hat{x}) = 0$ for all $\hat{x}$. Apply now the induction hypotheses to the set $D \subset K^{m-1}$ and the functions a_1, a_2, d_j . We find a finite partition of A such that for each part A' we have an isomorphism $f : B \rightarrow A'$, where B is a set of the form

$$B = \{x \in K^m \mid \hat{x} \in D', \nu(\alpha_1 \prod_{i=1}^l x_i^{\eta_i}) \sqcap_1 \nu(x_m) \sqcap_2 \nu(\alpha_2 \prod_{i=1}^l x_i^{\epsilon_i}), x_m \in \lambda P_n\},$$

with $D' = \prod_{i=1}^l R^{(k)}$, $l \leq m-1$, such that each $b_j \circ f$ satisfies condition (13). We will alternately partition further and apply isomorphisms to the parts which compositions with b_j will always satisfy condition (13). By the induction hypotheses we may suppose that $\lambda \neq 0$ and $\dim(D') = m-1$, i.e. $D' = \prod_{i=1}^{m-1} R^{(k)}$. Analogously as for $m = 1$ we may suppose that $\alpha_1 \neq 0 \neq \alpha_2$, $\sqcap_2$ is either $\leq$ or no condition and $\sqcap_1$ is the symbol $\leq$ (possibly after partitioning or applying $x \mapsto (x_1, \dots, x_{m-1}, 1/x_m)$).

Choose $\bar{k} > \nu(n)$ and put $k' = \bar{k} + \nu(n)$. We may suppose that $k' > k$, so we have a finite partition $B = \bigcup_\gamma B_\gamma$ with $\gamma = (\gamma_1, \dots, \gamma_m) \in K^m$, $0 \leq \nu(\gamma_i) < n$ and $B_\gamma = \{x \in B \mid x_i \in \gamma_i P_n^{(k')}\}$. Now we have isomorphisms

$$f_\gamma : C_\gamma \rightarrow B_\gamma : x \mapsto (\gamma_1 x_1^n, \dots, \gamma_m x_m^n),$$

with

$$C_\gamma = \{x \in (\prod_{i=1}^{m-1} R^{(\bar{k})}) \times K^{(\bar{k})} \mid \nu(\alpha'_1 \prod_{i=1}^{m-1} x_i^{\eta_i}) \leq \nu(x_m) \sqsubset_2 \nu(\alpha'_2 \prod_{i=1}^{m-1} x_i^{\varepsilon_i})\},$$

for appropriate choice of $\alpha'_i \in K$. Put $\nu_i = \varepsilon_i - \eta_i$, $\beta = \alpha'_2 / \alpha'_1$, then we have the isomorphism

$$\{x \in \prod_{i=1}^m R^{(\bar{k})} \mid \nu(x_m) \sqsubset_2 \nu(\beta \prod_{i=1}^{m-1} x_i^{\nu_i})\} \rightarrow C_\gamma : x \mapsto (x_1, \dots, x_{m-1}, \alpha'_1 x_m \prod_{i=1}^{m-1} x_i^{\eta_i}).$$

If $\sqsubset_2$ is no condition, the claim is trivial. It follows that we can reduce to the case that we have an isomorphism

$$f : E = \{x \in \prod_{i=1}^m R^{(\bar{k})} \mid \nu(x_m) \leq \nu(\beta \prod_{i=1}^{m-1} x_i^{\nu_i})\} \rightarrow X \quad (15)$$

with $\beta \neq 0$, $\bar{k} > 0$, and $\nu_i \in \mathbb{Z}$, such that each $b_j \circ f$ satisfies condition (13).

Suppose we are in the case described in (15). If $\nu_i \leq 0$ for $i = 1, \dots, m-1$ then we have a finite partition $E = \bigcup_s E^=s$, with $s \in \{0, 1, \dots, \nu(\beta)\}$ and $E^=s = \{x \in E \mid \nu(x_m) = s\}$. Also, $E^=s = \{(x_1, \dots, x_{m-1}) \mid \exists x_m (x_1, \dots, x_m) \in E^=s\} \times \{x_m \in R^{(\bar{k})} \mid \nu(x_m) = s\}$ and the claim follows.

Suppose now that $\nu_1 > 0$ in (15). First we show the proposition when $\nu_1 = 1$, using some implicit Presburger arithmetic on the value group. We can partition E into parts E_1 and E_2 , with

$$\begin{aligned} E_1 &= \{x \in E \mid \nu(x_m) < \nu(\beta \prod_{i=2}^{m-1} x_i^{\nu_i})\}, \\ E_2 &= \{x \in E \mid \nu(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}) \leq \nu(x_m)\} \\ &= \{x \in \prod_{i=1}^m R^{(\bar{k})} \mid \nu(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}) \leq \nu(x_m) \leq \nu(\beta x_1 \prod_{i=2}^{m-1} x_i^{\nu_i})\}. \end{aligned}$$

Since $\nu(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}) \leq \nu(x_1 \beta \prod_{i=2}^{m-1} x_i^{\nu_i})$ for $x \in E_1$, it follows that

$$E_1 = R^{(\bar{k})} \times \{(x_2, \dots, x_m) \in \prod_{i=2}^m R^{(\bar{k})} \mid \nu(x_m) < \nu(\beta \prod_{i=2}^{m-1} x_i^{\nu_i})\},$$

and the restrictions $b_j \circ f \mid E_1$ satisfy condition (13).

As for E_2 , let D_{m-1} be the set

$$D_{m-1} = \{(x_2, \dots, x_m) \in \prod_{i=2}^m R^{(\bar{k})} \mid \nu(\beta \prod_{i=2}^{m-1} x_i^{\nu_i}) \leq \nu(x_m)\}.$$

We may suppose that $\beta \in K^{(\bar{k})}$, then the map

$$R^{(\bar{k})} \times D_{m-1} \rightarrow E_2 : x \mapsto (\frac{x_1 x_m}{\beta \prod_{i=2}^{m-1} x_i^{\nu_i}}, x_2, \dots, x_m),$$

can be checked by elementary Presburger arithmetic to be an isomorphism. This shows the claim when $\nu_1 = 1$.

Suppose now that X is of the form described in (15) and $\nu_1 > 1$. We show we can reduce to the case $\nu_1 = 1$ by partitioning and applying appropriate power maps. Choose $\tilde{k} > \nu(\nu_1)$ and put $\tilde{k}' = \tilde{k} + \nu(\nu_1)$. We may suppose that $\tilde{k} \geq \bar{k}$, so we have a finite partition $E = \bigcup_\alpha E_\alpha$, with $\alpha = (\alpha_1, \dots, \alpha_m) \in K^m$, $\nu(\alpha_1) = 0$, $0 \leq \nu(\alpha_i) < \nu_1$ for $i = 2, \dots, m$ and

$$E_\alpha = \{x \in E \mid x_1 \in \alpha_1 R^{(\bar{k})}, x_i \in \alpha_i P_{\nu_1}^{(\tilde{k}')} \text{ for } i = 2, \dots, m\}.$$

By corollary (2.3.2) we have isomorphisms

$$f_\alpha : C_\alpha \rightarrow E_\alpha : x \mapsto (\alpha_1 x_1, \alpha_2 x_2^{\nu_1}, \dots, \alpha_m x_m^{\nu_1}),$$

with $C_\alpha = \{x \in \prod_{i=1}^m R^{(\tilde{k})} \mid v(x_m) \leq v(\beta' x_1 \prod_{i=2}^{m-1} x_i^{v_i})\}$, where $\beta' \in K^\times$ depends on α . This reduces the problem to the case described in (15) with $v_1=1$ and thus the proposition is showed.

Theorem (2.3.13)[70]: Let X be a semi-algebraic set, then either X is finite or there exists a semi-algebraic bijection $X \rightarrow K^k$ with $k \in \mathbb{N}_0$ the dimension of X .

Proof. We give a proof by induction on $\dim(X) = m$. Let $\dim(X) = 1$. Use Proposition (2.3.12) to partition X such that each part is isomorphic to either $R^{(k)}$ or a point. By combining the isomorphisms of Proposition (2.3.3) and Corollary (2.3.4), it follows that $X \cong K$.

Now suppose $\dim(X) = m > 1$. Proposition (2.3.3) together with the case $m = 1$ implies that we can finitely partition X such that each part is isomorphic to K^l , for some $l \in \{0, \dots, m\}$, with $K^0 = \{0\}$. By proposition (2.3.3) at least one part must be isomorphic to K^m . Suppose that A and B are disjoint parts, such that $A \cong K^l$ and $B \cong K^m$, with $l \in \{0, \dots, m\}$. It is enough to show that $A \cup B \cong K^m$. First suppose that $l = 0$, so A is a singleton $\{a\}$. Since $m > 1$ there exists an injective semi-algebraic function $i : R \rightarrow A \cup B$ such that $i(R \setminus \{0\}) \subset B$ and $i(0) = a$. It follows that $A \cup B \cong B \cong K^m$ since $R \cong R \setminus \{0\}$ (Proposition 1). If $1 \leq l$ we have $A \cup B \cong K \times (A' \cup B')$, for some disjoint sets $A' \cong K^{l-1}$ and $B' \cong K^{m-1}$. By induction we find $A' \cup B' \cong K^{m-1}$ and thus $A \cup B \cong K^m$. This shows Theorem (2.3.13).

We obtain as a corollary of Theorem (2.3.13) the following classification of the p -adic semi-algebraic sets.

Corollary (2.3.14)[70]: Two infinite semi-algebraic sets are isomorphic if and only if they have the same dimension.

Chapter 3

Basic Sequences and F -Space with Trivial Dual and Curves

This complements the example constructed by Roberts of a compact convex set without extreme points in L_p ($0 < p < 1$) and answers a question raised by Shapiro. We substantiate a conjecture of Rolewicz that every F -Space X with trivial dual admits a non-constant curve $g : [0,1] \rightarrow X$ with zero derivative.

Section (3.1): F -Spaces and Their Applications

The aim of this section is to establish a conjecture of Shapiro [115] that an F -space (complete metric linear space) with the Hahn-Banach Extension Property is locally convex. This result was proved by Shapiro for F -Spaces with Schauder bases; other similar results have been obtained by Ribe [113]. The method used in this section is to establish the existence of basic sequences in most F -spaces.

It was originally stated by Banach that every B -Space contains a basic sequence, and proofs have been given by Bessaga and Pelczynski [106], [107], Gelbaum [109] and Day [108]. In [106] Bessaga and Pelczynski give a general method of construction in locally convex F -Spaces, but we shall show in this Section that this construction can be modified to apply in any F -space (X, τ) on which there is a weaker vector topology ρ such that τ has a base of ρ -closed neighbourhoods. The basic result of the section is Theorem (3.1.6), and this is a natural generalization of a locally convex version due to Bessaga and Mazur and given (essentially) in Pelczynski [111], [112].

We study the problem of existence of a basic sequence in an arbitrary F -Space, and show that in fact repeated applications of Theorem (3.1.6) give a basic sequence in any F -Space with a non-minimal topology. Since the only example we know of a minimal F -Space is the space ω of all sequences (which has a basis) it seems likely that every F -Space contains a basic sequence.

We show that if (X, τ) is an F -Space and $\rho \leq \tau$ is a topology defining the same closed linear subspaces as τ , then ρ and τ define the same bounded sets—a result familiar in locally convex theory. The Shapiro conjecture follows immediately. The final theorem is a generalisation of the Eberlein-Smulian theorem employing techniques developed by Pelczynski [112].

An F -semi-norm η on a vector space X is a non-negative real-valued function defined on X such that

- (i) $\eta(x + y) \leq \eta(x) + \eta(y)$.
- (ii) $\eta(tx) \leq \eta(x) \quad |t| \leq 1$,
- (iii) $\lim_{t \rightarrow 0} \eta(tx) = 0 \quad x \in X$

If in addition $\eta(x) = 0$ implies that $x = 0$ then we call η an F -norm. Any vector topology on X may be defined by a collection of F -semi-norms; any metrisable topology may be defined by one F -norm. From this point, unless specifically stated, all vector topologies are assumed to be Hausdorff.

Now suppose (X, ρ) is a topological vector space and τ is a vector topology on X ; we shall say that τ is ρ -polar if τ has a base of neighbourhoods which are ρ -closed.

Proposition (3.1.1)[105]: If τ is ρ -polar then τ may be defined by a collection of F -semi-norms $(\eta_\alpha : \alpha \in A)$ of the form

$$\eta_\alpha(x) = \sup\{\lambda(x) : \lambda \in \Lambda_\alpha\}$$

where each Λ_α is a collection of ρ -continuous F -semi-norms. If τ is metrisable then τ may be defined by one such F -norm.

Proof. Let $(\gamma_\alpha : \alpha \in A)$ be a collection of F -semi-norms defining τ such that every τ -neighbourhood of 0 contains a set $\{x : \gamma_\alpha(x) \leq \varepsilon\}$ for some $\alpha \in A$ and $\varepsilon > 0$; let Δ be the collection of all ρ -continuous F -semi-norms. We define Λ_α to be the collection of F -semi-norms of the form

$$\lambda_\delta^\alpha(x) = \inf(\delta(y) + \gamma_\alpha(z) : y + z = x).$$

(Thus $\Lambda_\alpha = \{\lambda_\delta^\alpha : \delta \in \Delta\}$.) As $\lambda_\delta^\alpha \leq \delta$ each λ_δ^α is ρ -continuous and an F -semi-norm ($\lambda_\delta^\alpha \leq \delta$ implies condition (iii) in particular). Now define

$$\eta_\alpha(x) = \sup(\lambda_\delta^\alpha(x) : \delta \in \Delta).$$

Clearly $\eta_\alpha \leq \gamma_\alpha$ and so is an F -semi-norm. Now if U is a τ -neighbourhood of 0 we may find α_1 and $\varepsilon > 0$ such that if $x_0 \in \overline{\{x : \gamma_{\alpha_1}(x) \leq \varepsilon\}}$ (closure in ρ) then $x_0 \in U$. Suppose now $x_0 \in \{x : \eta_{\alpha_1}(x) < \varepsilon\}$; then it is easy to show that $x_0 \in \overline{\{x : \gamma_{\alpha_1}(x) \leq \varepsilon\}}$ and so $(\eta_\alpha : \alpha \in A)$ defines τ .

If τ is metrisable, A may be taken to be a singleton and therefore τ may be defined by a single F -norm of the required type.

Proposition (3.1.2)[105]: Suppose (X, τ) is an F -Space (complete metric linear space) and suppose $\rho < \tau$ is a vector topology on X . Then

(i) If the net $x_a \rightarrow 0(\rho)$ but $x_a \not\rightarrow 0(\tau)$, then there are vector topologies α, β such that

- (a) $\rho \leq \alpha < \beta \leq \tau$;
- (b) β is metrisable and α -polar;
- (c) $x_a \rightarrow 0(\alpha)$ but $x_a \not\rightarrow 0(\beta)$.

(ii) If U is a τ -neighbourhood of 0 but not a ρ -neighbourhood then there are vector topologies α, β satisfying (a), (b) and (c)' U is a β -neighbourhood of 0 but not an α -neighbourhood of 0.

(iii) If τ is locally bounded then there is a topology α such that $\alpha < \tau$ but τ is α -polar.

Proof. (i) Let α be the largest vector topology such that $\rho \leq \alpha \leq \tau$ and $x_a \rightarrow 0(\alpha)$ (it is easy to see that there is such a topology). Let β be the vector topology with a base of neighbourhoods consisting of the α -closures of τ -neighbourhoods of 0. Since $\alpha \leq \tau$ it follows that $\alpha \leq \beta \leq \tau$. If $\alpha = \beta$ then the identity map $i : (X, \alpha) \rightarrow (X, \tau)$ is almost continuous and so by the Closed Graph Theorem (cf. Kelley [110]) $\alpha = \tau$ contrary to hypothesis on the net (x_a) . Therefore $\alpha < \beta$; clearly also since τ is metrisable so is β , and $x_a \not\rightarrow 0(\beta)$.

(ii) By an application of Zorn's Lemma it may be shown that there is a maximal vector topology α such that $\rho \leq \alpha \leq \tau$ and U is not an α -neighbourhood (we do not assert that α is the largest such topology). Then proceed as in (i).

(iii) Follows from (ii) by considering a single bounded neighbourhood ($\beta = \tau$).

Two vector topologies on X will be called compatible if they define the same closed subspaces.

Proposition (3.1.3)[105]: Let τ and ρ be compatible topologies on X ; they define the same continuous linear functionals.

Proof. f is τ - or ρ -continuous according as its null space is τ - or ρ -closed.

A sequence (x_n) in a topological vector space X is called a basis if every $x \in X$ has a unique expansion in the form

$$x = \sum_{i=1}^{\infty} t_i x_i .$$

In this case we may define linear functionals f_n such that

$$f_n(x) = t_n$$

and linear operators S_n by

$$S_n(x) = \sum_{i=1}^n t_i x_i = \sum_{i=1}^n f_i(x) x_i .$$

If X is an F -space then it is well known (cf. [115], [117]) that each f_n , is necessarily continuous and the family $\{S_n\}$ is equicontinuous.

Suppose now that X is metrisable but not necessarily complete; we shall call a sequence (x_n) in X a basic sequence if it is a basis for its closed linear span in the completion of X . We shall call (x_n) a semi-basic sequence if we simply have $x_n \notin \overline{\text{lin}\{x_{n+1}, x_{n+2}, \dots\}}$ for every n .

We now give a useful and elementary criterion for a sequence (x_n) to be basic or semi-basic. Let (x_n) be linearly independent and let E be the linear span of (x_n) ; then for $x \in E$

$$x = \sum_{i=1}^{\infty} t_i x_i$$

uniquely where (t_i) is finitely non-zero. Define

$$f_n(x) = t_n$$

and

$$S_n x = \sum_{i=1}^n f_i(x) x_i ,$$

where $S_n : E \rightarrow E$ is linear.

Lemma (3.1.4)[105]: (i) (x_n) is semi-basic if and only if each S_n is continuous or equivalently each f_n is continuous.

(ii) (x_n) is basic if and only if the family $\{S_n\}$ is equicontinuous.

Proof. (i) If $\{x_n\}$ is semi-basic, let N_k be the null space of f_k ; then N_k is a maximal linear subspace of E . Then $N_1 = \text{lin}\{x_i : i \geq 2\}$ and since $x_1 \notin \overline{N_1}$, N_1 is closed and f_1 is continuous; while if $k \geq 2$,

$$N_k = \text{lin}\{x_i : i \neq k\} = \text{lin}\{x_i : i < k\} + \text{lin}\{x_i : i > k\} .$$

Hence

$$\overline{N_k} = \overline{\text{lin}\{x_i : i < k\}} + \overline{\text{lin}\{x_i : i > k\}} ,$$

since the former space is finite-dimensional. Suppose $x_k \in \overline{N_k}$; then

$$x_k = \sum_{i=1}^{k-1} t_i x_i + y ,$$

where $y \in \overline{\text{lin}\{x_i : i > k\}}$. Since $x_k \notin \overline{\text{lin}\{x_i : i > k\}}$ we conclude that there is a first index l such that $t_l \neq 0$. Then we obtain $x_l \in \overline{\text{lin}\{x_{l+1}, x_{l+2}, \dots\}}$ and a contradiction. Hence $x_k \notin \overline{N_k}$ and by the maximality of N_k , N_k is closed and f_k is continuous.

The converse is trivial.

(ii) (Cf. Shapiro [117], Proposition C.)

It follows from the definition of basic sequence that if (x_n) is basic then the family $\{S_n\}$ is equicontinuous (consider (x_n) as a basis of its closed linear span in the completion of X). Conversely, $S_n(x) \rightarrow x$ for $x \in E$ and if the family is equicontinuous $S_n(x) \rightarrow x$ for $x \in \bar{E}$ (closure in the completion of X), and it easily follows that (x_n) is a basis for $\bar{E}$.

Lemma (3.1.5)[105]: Let E be a finite-dimensional space and suppose V is a closed balanced subset of E . If V intersects every one-dimensional subspace of E in a bounded set then V is bounded.

Proof. We may suppose E is normed; suppose $x_n \in V$ and $\|x_n\| \rightarrow \infty$. Then by selecting a subsequence we may suppose $\|x_n\|^{-1} x_n \rightarrow z$ where $\|z\| = 1$. Then for any N there is an m such that for $n \geq m$, $\|x_n\| \geq N$ and

$$\|x_n\|^{-1} x_n \in \|x_n\|^{-1} V \subset N^{-1} V.$$

Therefore $z \in N^{-1} V$ for all N and hence $V \supset \text{lin}\{z\}$.

Theorem (3.1.6)[105]: Suppose (X, τ) is a metric linear space and ρ is a vector topology on X such that τ is ρ -polar. Suppose (x_a) is a net such that $x_a \rightarrow 0(\rho)$ but $x_a \not\rightarrow 0(\tau)$; suppose $z_1 \neq 0 \in X$. Then there is a sequence $(a(k) : k \geq 2)$ such that

$$a(k+1) > a(k)$$

for all $k \geq 2$ and the sequence $(z_n)_{n=1}^\infty$ is a basic sequence where $z_n = x_{a(n)}$ $n \geq 2$.

Proof. We may suppose (Proposition (3.1.1)) that (X, τ) is normed by an F -norm $\|\cdot\|$ such that

$$\|x\| = \sup(\lambda(x) : \lambda \in \Lambda),$$

where Λ is a collection of ρ -continuous F -norms. Let $\theta > 0$ be chosen such that

$$(i) \|z_1\| \geq 4\theta.$$

$$(ii) \text{ For all } a, \exists a' \geq a \text{ such that } \|x_{a'}\| \geq 4\theta.$$

Let $V = \{x : \|x\| \leq \theta\}$; then $\text{Klin } V \cap \text{lin}\{z_1\}$ is compact (since $\|z_1\| \geq 4\theta$). We shall construct the sequence $(a(n) : n \geq 2)$ by induction so that if

$$E_n = \text{lin}(z_1, x_{a(2)}, \dots, x_{a(n)})$$

then $E_n \cap V$ is compact.

Suppose $\{a(2), \dots, a(n)\}$ have been chosen (this set can be empty at the first step, the selection of $a(2)$) and let $E_n = \text{lin}(z_1, x_{a(2)}, \dots, x_{a(n)})$. By the inductive hypothesis $V \cap E_n$ is compact.

For $1 \leq k \leq 2^{n+3}$ let

$$W_k^n = \{x : \|x\| = k \cdot 2^{-(n+3)} \theta\} \cap E_n.$$

Each W_k^n is compact and so we may choose finite subsets U_k^n so that for $w \in W_k^n$ there exists $u \in U_k^n$ with

$$\|w - u\| \leq 2^{-(n+3)} \theta.$$

Let $U^n = \bigcup_{k=1}^{2^{n+3}} U_k^n$, and for $u \in U^n$ choose $\lambda_u \in \Lambda$ so that

$$\lambda_u(u) \geq \|u\| - 2^{-(n+3)}\theta. \quad (1)$$

Then choose $b > a(n)$ so that if $c \geq b$ then

$$\lambda_u(x_c) \leq 2^{-(n+3)}\theta \quad (2)$$

for $u \in U^n$ (possible since U^n is finite and $x_a \rightarrow 0(\rho)$).

Choose a subnet $x_d : d \in D$ of $(x_c : c \geq b)$ such that $\|x_d\| \geq 4\theta$, and suppose for every such x_d the set $V \cap \text{lin}(E_n, x_d)$ is unbounded. By Lemma (3.1.5), for every d there exists $t_d x_d + u_d \neq 0$ where $u_d \in E_n$ such that the linear span of $(t_d x_d + u_d)$ is contained in V . Clearly $u_d \neq 0$ and so we may normalize in such a way that $\|u_d\| = \theta$ (since $V \cap E_n$ is compact). Then

$$\|t_d x_d\| \leq \|t_d x_d + u_d\| + \|u_d\| \leq 2\theta$$

so that $|t_d| \leq 1$. Hence since $x_d \rightarrow 0(\rho)$, $t_d x_d \rightarrow 0$ in (ρ) . By selection again of a subnet we may suppose $u_d \rightarrow u$ in E_n (since $V \cap E_n$ is compact) and $\|u\| = \theta$.

Then for any $t \in R$

$$\|tu\| \leq \liminf_{d \rightarrow \infty} \|t(t_d x_d + u_d)\| \leq \theta$$

so that $\text{lin}\{u\} \subset V \cap E_n$, a contradiction.

Hence we may choose $a(n+1) \geq b$ such that $\|x_{a(n+1)}\| \geq 4\theta$ and $V \cap E_{n+1}$ is compact. This completes the construction of $a(n)$; now let $z_n = x_{a(n)} n \geq 2$. It remains to establish that by using (1) and (2) (z_n) is a basic sequence.

For convenience we shall replace $\|\cdot\|$ by an equivalent F-norm $\|\cdot\|^*$ given by

$$\|x\|^* = \min(\|x\|, \theta).$$

We next show that if $t_1, \dots, t_{n+1}$ is a scalar sequence

$$\left\| \sum_{i=1}^{n+1} t_i z_i \right\|^* \geq \left\| \sum_{i=1}^n t_i z_i \right\|^* - 2^{-(n+1)}\theta \quad (3)$$

Choose the greatest integer k such that

$$\left\| \sum_{i=1}^n t_i z_i \right\|^* \geq k \cdot 2^{-(n+3)}\theta.$$

Then $0 \leq k \leq 2^{(n+3)}$; if $k = 0$ there is nothing to show. If $k \geq 1$ then we may choose a scalar s with $|s| \leq 1$ such that

$$\left\| \sum_{i=1}^n s t_i z_i \right\| = k \cdot 2^{-(n+3)}\theta.$$

Then choose $u \in U_k^n$ so that

$$\left\| u - \sum_{i=1}^n s t_i z_i \right\| \leq 2^{-(n+3)}\theta.$$

If $|s t_{n+1}| \leq 1$ then

$$\|u + s t_{n+1} z_{n+1}\| \geq \lambda_u(u) - \lambda_u(z_{n+1}) \geq (k-2) \cdot 2^{-(n+3)}\theta$$

by (1) and (2). If $|s t_{n+1}| \geq 1$ then

$$\|u + s t_{n+1} z_{n+1}\| \geq \|z_{n+1}\| - \|u\| \geq 3\theta \geq (k-2) \cdot 2^{-(n+3)}\theta.$$

Hence

$$\begin{aligned}
\left\| s \sum_{i=1}^{n+1} t_i z_i \right\| &\geq (k-2)2^{-(n+3)}\theta - 2^{-(n+3)}\theta \\
&= (k-3)2^{-(n+3)}\theta \\
&\geq \left\| \sum_{i=1}^n t_i z_i \right\|^* - 2^{-(n+1)}\theta.
\end{aligned}$$

Hence since $|s| \leq 1$

$$\left\| \sum_{i=1}^{n+1} t_i z_i \right\| \geq \left\| \sum_{i=1}^n t_i z_i \right\|^* - 2^{-(n+1)}\theta$$

and (3) follows.

From (3) it is clear that (z_n) is linearly independent for if $\left\| \sum_{i=1}^n t_i z_i \right\| \geq \theta$ then $\left\| \sum_{i=1}^{n+1} t_i z_i \right\| \geq \frac{1}{2}\theta$; thus if $\sum_{i=1}^{n+1} t_i z_i = 0$, then for every s , $\left\| s \sum_{i=1}^n t_i z_i \right\| \leq \theta$ and so since $V \cap E_n$ is compact, $\sum_{i=1}^n t_i z_i = 0$. Let E be the linear span of $\{z_n\}$ and define S_k by

$$S_k \left(\sum_{i=1}^{\infty} t_i z_i \right) = \sum_{i=1}^k t_i z_i$$

where (t_i) is finitely non-zero. Then by (3)

$$\|S_{n+k}x\|^* \geq \|S_nx\|^* - 2^{-n}\theta \quad (k \geq 0)$$

and therefore for $x \in E$ and $n \geq 1$

$$\|x\|^* \geq \|S_nx\|^* - 2^{-n}\theta.$$

Suppose $\|x_m\| \rightarrow 0$ but $\|S_k x_m\| \not\rightarrow 0$; then since $V \cap E_k$ is compact we may, by selecting a subsequence and multiplying by a bounded sequence of scalars, suppose that $\|S_k x_m\| = \theta$. Thus $\|x_m\| \geq \frac{1}{2}\theta > 0$, and we have a contradiction.

Thus each S_k is continuous.

To establish equicontinuity of $\{S_m : m \geq 1\}$ we must show that if $p(m)$ is any sequence and $x_m \rightarrow 0$ then $S_{p(m)}x_m \rightarrow 0$. Suppose not; then we may suppose

$$\|S_{p(m)}x_m\|^* \geq \gamma > 0$$

for all m . Then

$$\|x_m\|^* \geq \gamma - 2^{-p(m)}\theta$$

and as $\|x_m\|^* \rightarrow 0$ we conclude that $p(m)$ is bounded. But then we may select a constant subsequence and this contradicts the continuity of each S_n . Thus by Lemma

(3.1.4) we have established the theorem.

Corollary (3.1.7)[105]: Under the assumptions of Theorem (3.1.6) suppose μ is a pseudo-metrisable topology on X such that $\mu \leq \rho$. Then (z_n) may be chosen so that $z_n \rightarrow 0(\mu)$.

An examination of the proof of Theorem (3.1.6) reveals that we can insist that $\eta(z_n) \rightarrow 0$ for any single ρ -continuous F -semi-norms.

Corollary (3.1.8)[105]: Suppose that (X, τ) is an F -Space and that ρ is a vector topology on X with $\rho < \tau$. Suppose $x_a \rightarrow 0(\rho)$ but $x_a \not\rightarrow 0(\tau)$, and that $z_1 \in X$. Then there

is a sequence $a(k)$ so that $a(k+1) > a(k)$ $k \geq 2$ and such that the sequence (z_n) is a semi-basic sequence where $z_n = x_{a(n)} n \geq 2$.

Proof. Proposition (3.1.2) combined with Theorem (3.1.6) establishes that we may choose (z_n) to be a basic sequence in a weaker topology than τ . This clearly implies that (z_n) is at least a semi-basic sequence in (X, τ) .

We consider the question of whether an F -Space need possess a basic sequence. We shall call a topological vector space (E, τ) minimal if for every Hausdorff vector topology $\rho \leq \tau$ we have $\rho = \tau$. It is well known that ω is minimal if we restrict to locally convex topologies.

Proposition (3.1.9)[105]: ω is a minimal F -Space.

Proof. Suppose ρ is a weaker vector topology on ω and $x_a \rightarrow 0(\rho)$ but $\|x_a\| \geq \theta$ (where $\|\cdot\|$ is an F -norm determining the topology of ω). Then there is a sequence (z_n) , with $\|z_n\| \geq \theta$, which is a basic sequence for some weaker Hausdorff vector topology on ω (Proof of (3.1.8)). Let E be the closed linear span of (z_n) in the original topology, then $E \cong \omega$. However, the dual functional of (z_n) induce on E a weaker Hausdorff locally convex topology. It follows that $z_n \rightarrow 0$ contrary to assumption.

We do not know any other examples of minimal F -spaces; their existence is crucial to the problem of basic sequences in view of the following theorem.

We first show a stability theorem for basic sequences similar to a locally convex version given by Weill [118] (cf. also Shapiro [116]). A sequence in a topological vector space is regular if it is bounded away from zero.

Lemma (3.1.10)[105]: Suppose X is an F -space and (x_n) is a regular basic sequence. Suppose $\sum \|u_n\| < \infty$, and let $y_n = x_n + u_n$. If whenever

$$\sum_{n=1}^{\infty} t_n y_n = 0$$

then $t_n = 0$, then (y_n) is also a basic sequence.

Proof. Define a map $S : l_{\infty} \rightarrow X$ by

$$S(t) = \sum_{n=1}^{\infty} t_n u_n.$$

Since $\sum \|u_n\| < \infty$, S is well defined and S is continuous by the Banach-Steinhaus Theorem. Now suppose $(t^{(n)})$ is a sequence in l_{∞} such that

$$\sup \|t^{(n)}\|_{\infty} < \infty$$

and

$$\lim_{n \rightarrow \infty} t_k^{(n)} = 0 \text{ for each } k.$$

Then it is easy to verify that $\|S(t^{(n)})\| \rightarrow 0$.

Let E be the closed linear span of $\{x_n\}$ and suppose $f_n \in E'$ is the bi-orthogonal sequence. For $x \in E$, $\lim_{n \rightarrow \infty} f_n(x) = 0$, since (x_n) is regular. We define $R : E \rightarrow c_0$ by $R(x) = (f_n(x))$; R is continuous by the Closed Graph Theorem. Hence the map $T : E \rightarrow X$ defined by $T = I + SR$ is also continuous. Since T takes the form

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n.$$

T is injective. Now suppose $(z_n) \subset E$ is a sequence such that $\|T(z_n)\| \rightarrow 0$; suppose $\|z_n\| > \varepsilon > 0$. We suppose at first

$$\sup_n \|R(z_n)\|_\infty < \infty.$$

Then by selecting a subsequence we may suppose $R(z_n) \rightarrow t$ co-ordinatewise in l_∞ and hence

$$S(R(z_n)) \rightarrow S(t) \text{ in } X.$$

Now

$$z_n = T(z_n) - S(R(z_n)) \rightarrow -S(t).$$

Therefore $S(t) \in E$ and

$$R(z_n) + RS(t) \rightarrow 0 \text{ in } l_\infty.$$

i.e.

$$t + RS(t) = 0$$

$$S(t) + SRS(t) = 0$$

$$T(S(t)) = 0$$

$$S(t) = 0$$

and so

$$\lim_{n \rightarrow \infty} z_n = 0$$

contrary to assumption. It follows that no subsequence of $(\|Rz_n\|_\infty)$ is bounded.

If, on the contrary, $\|Rz_n\|_\infty \rightarrow \infty$, then we may consider $(\|Rz_n\|_\infty^{-1} z_n)$ and obtain a similar contradiction. We establish that for such a sequence $\|Rz_n\|_\infty^{-1} z_n \rightarrow 0$ and hence $\|Rz_n\|_\infty^{-1} Rz_n \rightarrow 0$ in l_∞ which is a contradiction. Hence T is an isomorphism on to its image, and as $Tx_n = y_n$, (y_n) is a basic sequence.

Theorem (3.1.11)[105]: Every non-minimal F -space contains a basic sequence.

Proof. Let U_n be a base of neighbourhoods of 0 in (X, τ) ; We may assume, without loss of generality, that U_1 is not a neighbourhood of 0 in some weaker vector topology. By Proposition (3.1.2) there are vector topologies α, β in X such that $\alpha < \beta \leq \tau$, β is metrisable and α -polar and U_1 is a β -neighbourhood. Then by Theorem (3.1.6) there is a basic sequence $(w_k^{(1)})$ in (X, β) . Then let E_1 be the τ -closed linear hull of the sequence $(w_k^{(1)})$ and let F_1 be the linear span; let $\gamma_1 = \beta$. Then by induction we construct sequences $(h_k^{(n)})$, E_n , F_n , γ_n such that $F_n = \text{lin}\{w_k^{(n)} : k = 1, 2, \dots\}$, E_n is the τ -closure of F_n and γ_n is a metrisable vector topology on E_n such that $(w_k^{(n)} : k = 1, 2, \dots)$ is a basis of (E_n, γ_n) . Furthermore

(i) $(w_k^{(n)})$ is block basic with respect to $(w_k^{(n-1)})$ for $n \geq 2$, i.e. $w_k^{(n)}$ takes the form

$$w_k^{(n)} = \sum_{p_{k-1}+1}^{p_k} c_i w_i^{(n-1)},$$

where $p_0 = 0 < p_1 < p_2 \dots$. Thus $F_n \subset F_{n-1}$ for $n \geq 2$ and $F_n \subset F_{n-1}$ $n \geq 2$.

(ii) The topology γ_n on E_n is finer than γ_{n-1} restricted to E_n for $n \geq 2$, and coarser than τ .

(iii) $U_n \cap E_n$ is a γ_n -neighbourhood of 0.

We now describe the inductive construction; suppose $(w_k^{(n)})$, E_n , F_n and γ_n have

been chosen. If $U_{n+1} \cap E_n$ is a γ_n -neighbourhood of 0 then let $\gamma_{n+1} = \gamma_n$ and $w_k^{(n+1)} = w_k^{(n)}$ for all k . Otherwise by Proposition (3.1.2) we may find topologies α and γ_{n+1} on E_n such that $\gamma_n \leq \alpha < \gamma_{n+1} \leq \tau$, γ_{n+1} is α -polar and metrisable and $U_{n+1} \cap E_n$ is a γ_{n+1} -neighbourhood of 0 but not an α -neighbourhood.

Since F_n is τ -dense in E_n , F_n is also γ_{n+1} -dense and hence $\alpha < \gamma_{n+1}$ on F_n . Thus by Corollary (3.1.7) we may determine a γ_{n+1} -regular basic sequence (z_k) in F_n such that $z_k \rightarrow 0(\gamma_n)$. Thus

$$z_k = \sum_{i=1}^{q(k)} c_{k,i} w_i^{(n)},$$

where $\lim_{k \rightarrow \infty} c_{k,i} = 0$ for each i (since the co-ordinate functionals for $(w_i^{(n)})$ are γ_n -continuous). It follows easily that we may find a subsequence (y_k) and a block basic sequence $(w_k^{(n+1)})$ such that $\sum_k \|y_k - w_k^{(n+1)}\|_{n+1} < \infty$ where $\|\cdot\|_{n+1}$ is an F -norm determining γ_{n+1} . If

$$\sum_{k=1}^{\infty} t_k w_k^{(n+1)} = 0 \quad (\gamma_{n+1})$$

then

$$\sum_{k=1}^{\infty} t_k w_k^{(n+1)} = 0 \quad (\gamma_n)$$

and thus since the co-ordinate functionals for $w_i^{(n)}$ are γ_n -continuous $t_k \rightarrow 0$ for all k . Thus $(w_k^{(n+1)})$ is a γ_{n+1} -basic sequence, and we proceed by letting $F_{n+1} = \text{lin}\{w_k^{(n+1)}\}$, $E_{n+1} = \bar{F}_{n+1}$ (in τ). This completes the inductive construction.

Finally take the "diagonal sequence"

$$v_n = w_n^{(n)}.$$

Then for each n , $(v_k : k \geq n)$ is block basic with respect to $(w_k^{(n)})$. In particular (v_k) is block basic with respect to $(w_k^{(1)})$ and hence there are γ_1 -continuous linear functionals (f_k) defined on $\text{lin}\{v_k\}$ such that $f_i(v_j) = \delta_{ij}$. These are then also τ -continuous and extend to the closed linear span H of $\{v_k\}$. Now suppose $x \in H$; we show

$$\sum_{i=1}^{\infty} f_i(x) v_i = x.$$

For any n , $(v_k : k \geq n)$ is a basic sequence in (E_n, γ_n) ; let

$$R_n(x) = x - \sum_{i=1}^{n-1} f_i(x) v_i.$$

Then $R_n(x)$ is in the τ -closure of $\text{lin}\{v_k : k \geq n\}$, as this space is easily seen to be $\bigcap_{i=1}^{n-1} f_i^{-1}(0)$.

Thus $R_n(x)$ is in E_n and in the γ_n -closure of $\text{lin}\{v_k : k \geq n\}$.

Therefore

$$R_n(x) = \sum_{i=n}^{\infty} f_i(x) v_i \quad (\gamma_n)$$

and so for some N and all $m \geq N$,

$$R_n(x) - \sum_{i=n}^m f_i(x) v_i \in U_n,$$

and

$$x - \sum_{i=1}^m f_i(x) v_i \in U_n.$$

Thus $x = \sum_{i=1}^{\infty} f_i(x) v_i$ for $x \in H$, and (v_i) is a basic sequence.

If E is a minimal F -space, then E may still possess a basic sequence (see Proposition (3.1.9)).

Theorem (3.1.12)[105]: Let (X, τ) be an F -space; the following are equivalent:

- (i) X contains no basic sequence.
- (ii) Every closed subspace of X with a separating dual is finite-dimensional.

Proof. Clearly (ii) $\Rightarrow$ (i) so we have to show (i) $\Rightarrow$ (ii). If E is a subspace of X with a separating dual, then the weak topology σ on E is weaker than τ . If E is infinite-dimensional, then by Theorem (3.1.11) $\sigma = \tau$. But in this case $E \cong \omega$, and so has a basis. Therefore, E is finite-dimensional.

We now can apply basic sequences or rather semi-basic sequences to derive many results familiar in locally convex theory.

Theorem (3.1.13)[105]:

(i) Let (X, τ) be an F -space and suppose $\rho \leq \tau$ is a vector topology on X compatible with τ . Then every ρ -bounded set is τ -bounded.

(ii) Suppose X is a vector space and $\rho \leq \tau$ are two vector topologies on X such that ρ and τ are compatible and τ is ρ -polar. Then any ρ -bounded set is τ -bounded.

Proof. (i) It is enough to show that if $x_n \rightarrow 0(\rho)$ and c_n is a sequence of scalars such that $c_n \rightarrow 0$ then $c_n x_n \rightarrow 0(\tau)$. Suppose $x_n \rightarrow 0(\rho)$; then choose $x_0 \neq 0$. For $c_n \rightarrow 0$, $c_n \neq 0$,

$$c_n(x_n + x_0) \rightarrow 0(\rho).$$

Suppose $c_n(x_n + x_0) \not\rightarrow 0(\tau)$; then by Corollary (3.1.8), there is a semi-basic sequence (z_n) with $z_1 = x_0$ and

$$z_n = c_{m_n}(x_{m_n} + x_0) \quad (n \geq 2),$$

where (m_n) is an increasing sequence of integers. Then

$$c_{m_n}^{-1} z_n \rightarrow x_0(\rho)$$

and hence x_0 is in the ρ -closure of $\text{lin}\{z_n : n \geq 2\}$. Thus x_0 is also in the τ -closure of $\text{lin}\{z_n : n \geq 2\}$, contradicting the fact that (z_n) is a semi-basic sequence. Thus since $c_n x_0 \rightarrow 0$, $c_n x_n \rightarrow 0(\tau)$.

The proof of (ii) is somewhat similar; let η be a ρ -lower-semi-continuous τ -continuous F -semi-norm and let $N = \{x : \eta(x) = 0\}$. Then X/N metrisable under η and may be given the quotient topology $\hat{\rho}$ of ρ (N is ρ -closed). Every η -closed subspace of X/N is $\hat{\rho}$ -closed and so an argument similar to (i) may be employed.

Corollary (3.1.14)[105]: Suppose (X, τ) is an F -space and $\rho \leq \tau$ is a metrisable vector topology compatible with τ . Then $\rho = \tau$.

Corollary (3.1.15)[105]: Let (X, τ) be an F -space with the Hahn-Banach Extension Property. Then X is locally convex.

Proof. Let σ be the weak topology on N ; then $\sigma \leq \tau$ and σ and τ are compatible by the HBEP. For suppose Y is a τ -closed subspace and $x \notin Y$; then by HBEP there is a continuous linear functional ϕ such that $\phi(Y) = 0$ and $\phi(x) = 1$. Let μ be the associated

Mackey topology; then (see Shapiro [115]) $\sigma \leq \mu \leq \tau$ and μ is metrisable. Hence by Corollary (3.1.14) $\mu = \tau$ and τ is locally convex.

Corollary (3.1.16)[105]: Suppose (X, τ) is an F -space and $\rho \leq \tau$ is a vector topology compatible with τ . Then τ is ρ -polar.

Proof. Let γ be the topology induced by the ρ -closures of τ -neighbourhoods of 0; then $\rho \leq \gamma \leq \tau$ and γ is metrisable. Hence by (3.1.14), $\gamma = \tau$.

Theorem (3.1.17)[105]: Let (X, τ) be an F -space and let (x_n) be a basis of X in a compatible topology $\rho \leq \tau$. Then (x_n) is a basis of X .

Proof. By the previous corollary we may assume that τ is defined by a ρ -lower-semi-continuous F -norm $\|\cdot\|$ (see Proposition (3.1.1)). Each $x \in X$ may be expanded in the form

$$x = \sum_{i=1}^{\infty} f_i(x) x_i(\rho)$$

(the linear functionals f_n are not necessarily ρ -continuous). Now for each $x \in X$, the sequence $\left(\sum_{i=1}^n f_i(x) x_i \right)$ is ρ - and therefore τ -bounded (Theorem (3.1.13)) and so we may define

$$\|x\|^* = \sup_n \left\| \sum_{i=1}^n f_i(x) x_i \right\|.$$

Then $\lim_{t \rightarrow 0} \|tx\|^* = 0$ since $\lim_{t \rightarrow 0} ty = 0$ uniformly for y in a bounded set; hence $\|\cdot\|^*$ is an F -norm on X . Clearly also $\|x\|^* \geq \|x\|$ by the ρ -lower-semicontinuity of $\|\cdot\|$.

It remains to establish that $(X, \|\cdot\|^*)$ is complete and then by the Closed Graph Theorem it will follow that $\|\cdot\|^*$ and $\|\cdot\|$ are equivalent. Let (y_n) be a $\|\cdot\|^*$ -Cauchy sequence; then since $\|y_n - y_m\| \leq \|y_n - y_m\|^*$ for all m, n , (y_n) is τ -convergent to y say. Furthermore, it can be seen that the sequences

$$\left(\sum_{i=1}^m f_i(y_n) x_i \right)$$

are τ -convergent uniformly in m ; clearly $\lim_{n \rightarrow \infty} f_i(y_n) = t_i$ exists and

$$\lim_{n \rightarrow \infty} \sum_{i=1}^m f_i(y_n) x_i = \sum_{i=1}^m t_i x_i$$

uniformly in m for the topology τ . Thus working in the weaker topology ρ

$$\lim_{m \rightarrow \infty} \sum_{i=1}^m t_i x_i = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{i=1}^m f_i(y_n) x_i = y.$$

(The limits are interchangeable by uniform convergence.) Therefore it follows that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^m f_i(y_n) x_i = \sum_{i=1}^m f_i(y) x_i(\tau)$$

uniformly in m and that $\|y - y_n\|^* \rightarrow 0$. Hence $\|\cdot\|$ and $\|\cdot\|^*$ are equivalent, and by an application of Lemma (3.1.4), (x_n) is a basic sequence in $(X, \|\cdot\|)$. By the compatibility of ρ , (x_n) is a basis of X .

Shapiro [117] proves that the Weak Basis Theorem fails in any non-locally convex locally bounded F -space. With regard to this theorem we show that a weaker version of

the Weak Basis Theorem holds always.

Proposition (3.1.18)[105]: Let (x_n) be a weak basis of (X, τ) , where (X, τ) is an F -space with a separating dual. Then the associated linear functionals $\{f_n\}$ are continuous.

Proof. Let σ be the weak topology and μ the (metrisable) Mackey topology. Then (X, μ) is barrelled, for if C is a μ -barrel then C is τ -closed and by the Baire Category Theorem we may show C has τ -interior. It follows easily that C is a τ -neighbourhood of 0 and thus a μ -neighbourhood [115].

Now let $\|\cdot\|_n$ be a sequence of semi-norms defining μ and let

$$\|x\|_n^* = \sup_m \left\| \sum_{i=1}^m f_i(x) x_i \right\|_n$$

(finite, since μ and σ have the same bounded sets). Let μ^* be the topology induced by the sequence $\|\cdot\|_n^*$ and let $\hat{X}$ be the μ^* -completion of X . Consider the identity map

$i : (X, \mu) \rightarrow (\hat{X}, \mu^*)$. Suppose $z_n \in X$, $z_n \rightarrow z(\mu)$ and $z_n \rightarrow z'(\mu^*)$. Then $\left\{ \sum_{i=1}^m f_i(z_n) x_i \right\}_{n=1}^\infty$ is

uniformly μ -Cauchy for $m = 1, 2, \dots$; thus in the topology $\sigma \leq \mu$

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{i=1}^m f_i(z_n) x_i = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^m f_i(z_n) x_i$$

and we conclude

$$\lim_{n \rightarrow \infty} f_i(z_n) = t_i \text{ exists for each } i$$

and

$$\lim_{n \rightarrow \infty} z_n = z = \sum_{i=1}^\infty t_i x_i \text{ in } \sigma.$$

Thus $f_i(z) = t_i$ and therefore

$$\lim_{n \rightarrow \infty} \sum_{i=1}^m f_i(z_n - z) x_i = 0 \text{ } \mu\text{-uniformly in } m.$$

Hence $z_n \rightarrow z$ in (X, μ^*) and i has Closed Graph. By the Closed Graph Theorem

[114], since $(\hat{X}, \mu^*)$ is complete and metric, $\mu \geq \mu^*$ and it follows easily that each f_n is μ and hence τ -continuous.

The idea of the next theorem is due to Pelczynski [112].

Theorem (3.1.19)[105]: Let (X, τ) be an F -space and suppose $\sigma \leq \tau$ is a compatible vector topology. Let K be a subset of X ; then the following are equivalent

- (i) K is ρ -compact,
- (ii) K is ρ -sequentially compact,
- (iii) K is ρ -countably compact.

Proof. (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) are well known. Let $\|\cdot\|$ be an F -norm determining τ ; by Corollary (3.1.16) we may suppose $\|\cdot\|$ is ρ -lower-semi-continuous.

(iii) $\Rightarrow$ (i). It is easy to see that K is ρ -precompact; we show that K is also ρ -complete. Let $(\hat{X}, \hat{\rho})$ be the ρ -completion of X and let $Y \subset \hat{X}$ be the vector space of all $y \in \hat{X}$ such that there is a ρ -bounded net $x_\alpha \in X$ such that $x_\alpha \rightarrow y$. By Theorem (3.1.13) a ρ -bounded net is τ -bounded. Let $B_\lambda = \{x \in X : \|x\| \geq \lambda\}$; then for $y \in Y$ we define

$$\|y\|^* = \inf\{\lambda : y \in \bar{B}_\lambda, \text{ closure in } \hat{\rho}\}.$$

Let $y \in Y$ and suppose x_α is a τ -bounded net converging to y in $\hat{\rho}$; then

$$\|y\|^* \leq \sup_\alpha \|x_\alpha\| < \infty$$

and

$$\lim_{t \rightarrow 0} \|ty\|^* \leq \limsup_{t \rightarrow 0} \|tx_\alpha\| = 0$$

since the net $\{x_\alpha\}$ is bounded (cf. Theorem (3.1.17)). It follows without difficulty that $\|\cdot\|^*$ is an F -semi-norm on Y , and that $\|\cdot\|^*$ is $\hat{\rho}$ -lower-semi-continuous; also from the definition, $\|x\| = \|x\|^*$ for $x \in X$, since each B_λ is ρ -closed. Next if $y \in Y$ and $\|y\|^* = 0$ then for each $\lambda > 0$ and V a neighbourhood of 0 in $(\hat{X}, \hat{\rho})$ we may find $x_{\lambda, V} \in X$ such that $x_{\lambda, V} - y \in V$ and $\|x_{\lambda, V}\| \leq \lambda$. The $\{(\lambda, V) : \lambda > 0, V \text{ a } \hat{\rho}\text{-neighbourhood of } 0\}$ is directed in the obvious way $[(\lambda, V) \geq (\lambda', V') \text{ if and only if } \lambda \leq \lambda' \text{ and } V \subset V']$; then the net $x_{\lambda, V}$ converges to 0 in (X, τ) and $x_{\lambda, V} \rightarrow 0$ in (X, ρ) . However $x_{\lambda, V} \rightarrow y$ in $(\hat{X}, \hat{\rho})$ and so $y = 0$. Thus Y is a metrisable vector space under $\|\cdot\|^*$ and $\|\cdot\|^*$ is $\hat{\rho}$ -lower-semi-continuous.

Now suppose $x_\alpha \in K$ is a ρ -Cauchy net; then $x_\alpha \rightarrow y$ in $(\hat{X}, \hat{\rho})$ and $y \in Y$. Suppose at first $\|x_\alpha - y\|^* \rightarrow 0$; then by the completeness of (X, τ) $y \in X$, and there is a sequence $(\alpha(n))$ such that $x_{\alpha(n)} \rightarrow y(\tau)$. Thus y is the sole ρ -cluster point of $\{x_{\alpha(n)}\}$ in X ; since K is countably compact, $y \in K$, and $x_\alpha \rightarrow y$ in (K, ρ) .

Now suppose $\|x_\alpha - y\|^* \not\rightarrow 0$ and that $y \notin X$; since $y \neq 0$ we may suppose $x_\alpha \notin V$ for all α , where V is a ρ -neighbourhood of 0. Then by Theorem (3.1.6) there is a basic sequence (z_n) in $(Y, \|\cdot\|^*)$ such that:

- (i) $z_1 = y$.
- (ii) $z_n = w_n - y$, $n \geq 2$ where $w_n = x_{\alpha(n)}$ for some increasing sequence.
- (iii) $\inf \|z_n\|^* > 0$.

Let Z be the closed linear span of $\{z_n\}_{n=1}^\infty$ and let W be the closed linear span of $\{w_n\}_{n=2}^\infty$. Since $z_1 \notin X$ and $W \subset X$, W is a closed subspace of co-dimension one in Z . Let ϕ be the continuous linear functional on $(Z, \|\cdot\|^*)$ such that $\phi(z_1) = 1$ and $\phi(W) = 0$; we define $A : Z \rightarrow Z$ by $Az = z - \phi(z)z_1$. Then for $n \geq 2$

$$Az_n = Aw_n - Az_1 = w_n.$$

Similarly define $B : Z \rightarrow Z$ by

$$B\left(\sum_{i=1}^\infty t_i z_i\right) = \sum_{i=2}^\infty t_i z_i.$$

Then

$$Bw_n = B(z_1 + z_n) = z_n.$$

It follows that $BAz_n = z_n$, $n \geq 2$ and hence that A is an isomorphism of $\overline{\text{lin}}\{z_n : n \geq 2\}$ on to its image. In particular $(w_n : n \geq 2)$ is a basic sequence in $(X, \|\cdot\|)$. However $w_n \in K$ for $n \geq 2$, and so (w_n) possesses a ρ -cluster point. Now suppose w_0 is a ρ -cluster point; then w_0 is in the τ -closed linear span of (w_n) by compatibility. It follows that

$$w_0 = \sum_{i=2}^\infty \psi_i(w_0)w_i.$$

where ψ_i is the dual sequence of τ -continuous linear functionals on W . Each ψ_i is also ρ -continuous by compatibility and hence

$$\psi_i(w_0) = 0 \quad i \geq 2.$$

Therefore $w_0 = 0$. This contradicts the original choice of $x_\alpha \notin V$, where V is a ρ -neighbourhood of 0. Thus we have a contradiction.

Finally suppose $\|x_\alpha - y\|^* \not\rightarrow 0$ and $y \in X$; determine the basic sequence $(z_n : n \geq 2)$ satisfying (ii)-(iii). In this case if w_0 is a ρ -cluster point of $(w_n : n \geq 2)$ then $w_0 - y$ is a ρ -cluster point of $(z_n : n \geq 2)$. Since $w_0 - y \in X$ and $z_n \in X$ we conclude that $w_0 - y$ is in the τ -closed linear span of $\{z_n : n \geq 2\}$ by compatibility and it follows as usual that $w_0 - y = 0$. Hence $y \in K$. We conclude that any ρ -Cauchy net converges in K and so K is complete and therefore compact.

(iii) $\Rightarrow$ (ii). Let (x_n) be a sequence in K and let x_0 be a ρ -cluster point. Then there is a net (z_α) in K such that each z_α is some x_n and $z_\alpha \rightarrow x_0(\rho)$. If $z_\alpha \rightarrow x_0$ in τ then there is nothing to show, as it will follow that some subsequence of (x_n) converges to x_0 . Otherwise we may find a basic sequence (u_n) of the form $u_n = z_{\alpha(n)} \rightarrow x_0$. Let w be a ρ -cluster point of $(z_{\alpha(n)})$ in K ; then clearly $w - x_0 \in \overline{\text{lin}\{u_n\}}$ and since τ and ρ are compatible it follows as in (iii) $\Rightarrow$ (i) that $w - x_0 = 0$. Hence x_0 is the sole cluster point of $(z_{\alpha(n)})$ and so $z_{\alpha(n)} \rightarrow x_0$. However $z_{\alpha(n)}$ is simply a subsequence of (x_n) ($\alpha(n) \rightarrow \infty$ since the $z_{\alpha(n)}$ are distinct).

[Added In Proof: The problem of determining conditions under which the Hahn-Banach Extension Property is equivalent to local convexity was originally posed by Duren, Romberg and Shields [119].]

Section (3.2): The Krein-Milman Theorem

In [124] Roberts answered a long outstanding question by constructing an example of a compact convex subset of a non-locally convex F -space without extreme points; thus the Krein-Milman theorem fails in general without local convexity. Later in [123], Roberts showed that such examples can be constructed in the spaces L_p ($0 < p < 1$) (or more generally Orlicz spaces L_ϕ where ϕ is sub-additive and $x^{-1}\phi(x) \rightarrow 0$ as $x \rightarrow \infty$).

The basic ingredient of Roberts's construction is the notion of a needle point. If E is an F -space with associated F -norm $|\cdot|$, then $x \in E$ is a needle point if given any $\varepsilon > 0$, there exist $u_1, \dots, u_n \in E$ such that $|u_i| < \varepsilon$ ($i = 1, 2, \dots, n$) and

$$(i) \quad x = (1/n)(u_1 + \dots + u_n),$$

$$(ii) \quad \text{if } a_1 + \dots + a_n = 1 \text{ and } a_i \geq 0 \text{ (} i = 1, 2, \dots, n \text{) then there exists } t, 0 \leq t \leq 1 \text{ such that}$$

$$\left| tx - \sum_{i=1}^n a_i u_i \right| < \varepsilon.$$

Roberts [123] showed that if E contains a non-zero needle point then E contains a compact convex subset which is not locally convex. Also if every element of E is a needle point then E contains a compact convex set with no extreme points; in this case E is called a needle-point space.

Following the work of Roberts, the question was asked (Shapiro [127]) whether every F -space with trivial dual contains a compact convex set without extreme points. We shall show that this is not the case and that there exist F -spaces with trivial dual in

which every compact convex set is locally convex. In particular every compact convex set is affinely embeddable in a locally convex space [125] and obeys the Krein-Milman theorem. Our example is an Orlicz function space L_ϕ .

We start by defining an element x of an F -space E to be approachable if there is a bounded subset B of E such that whenever $\varepsilon > 0$ there exist $u_1, \dots, u_n \in E$ with $|u_i| < \varepsilon$ ($i = 1, 2, \dots, n$) and

$$(i) \quad |x - (1/n)(u_1 + \dots + u_n)| < \varepsilon$$

$$(ii) \quad \text{if } |a_1| + \dots + |a_n| \leq 1 \text{ then } \sum_{i=1}^n a_i u_i \in B.$$

Theorem (3.2.1)[120]: Suppose E is an F -space in which 0 is the only approachable point. Then every compact convex subset of E is affinely embeddable in a locally convex space.

Proof. Suppose $K \subset E$ is a compact convex set and let $K_1 = \overline{\text{co}}(K \cup (-K))$. Then K_1 is also compact. We show $0 \in K_1$ has a base of convex neighborhoods in K_1 . For $\varepsilon > 0$, let $V_\varepsilon = \{x : |x| < \varepsilon\}$. Suppose $x \in K_1$ and $x \in \overline{\text{co}}(K_1 \cap V_\varepsilon)$ for every $\varepsilon > 0$. Then x is approachable (take $B = K_1$ in the definition) and hence $x = 0$. Now by compactness for any $\delta > 0$ there exists $\varepsilon > 0$ so that

$$\overline{\text{co}}(K_1 \cap V_\varepsilon) \subset V_\delta.$$

Now the finest vector topology on the linear span F of K_1 (i.e. $F = \bigcup (nK_1 : n \in \mathbb{N})$), which agrees with the given topology on K_1 has a base of neighborhoods of the form

$$\bigcup_{n=1}^{\infty} \sum_{m=1}^n (mK_1 \cap V_{\varepsilon_m})$$

where ε_m is a sequence of positive numbers [129]. By the above result this is locally convex, and the theorem is showed.

We remark that the second half of this proof was used in [121] in the introduction; an alternative approach would be to show that every point of K_1 has a base of convex neighborhoods (this follows easily from the same fact for 0) and then use Roberts's deeper results in [125].

Lemma (3.2.2)[120]: Suppose E and F are F -spaces and $T : E \rightarrow F$ is a continuous linear operator. If $x \in E$ is approachable, then Tx is approachable in F .

The proof is immediate.

We now recall that an Orlicz function ϕ is an increasing function defined on $[0, \infty)$ which is continuous at 0, satisfies $\phi(0) = 0$ and $\phi(x) > 0$ for some $x > 0$. The function ϕ is said to satisfy the Δ_2 -condition if for some constant K , we have $\phi(2x) \leq K\phi(x)$ ($0 \leq x < \infty$). If ϕ satisfies the Δ_2 -condition then the Orlicz space $L_\phi(0, 1)$ is defined to be the set of measurable functions f such that

$$\int_0^1 \phi(|f(t)|) dt < \infty.$$

L_ϕ is an F -space (after the usual identification of functions differing on a set of measure zero) with a base of neighborhoods $V(\varepsilon)$ where $f \in V(\varepsilon)$ if and only if

$$\int_0^1 \phi(|f(t)|) dt < \varepsilon.$$

Theorem (3.2.3)[120]: Suppose ϕ is an Orlicz function satisfying the Δ_2 -condition and

$$\phi(x) = x, \quad 0 \leq x \leq 1, \quad (4)$$

$$\text{there exist } c_n \ (n \in N) \text{ such that } c_n \geq 0 \text{ for all } n, \sum c_n < \infty \quad (5)$$

and if

$$G(x) = \sum_{n=1}^{\infty} c_n \frac{n}{x} \phi\left(\frac{x}{n}\right), \quad (x > 0)$$

then $G(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Then 0 is the only approachable point in $L_\phi(0,1)$.

Proof. Given any $f \in L_\phi$, with $f \neq 0$, there exists a continuous linear operator $T : L_\phi \rightarrow L_\phi$ with $Tf = 1$ (where (3.2.1) denotes the constantly one function). Hence it suffices to show that (3.2.1) is not approachable.

Suppose on the contrary (3.2.1) is approachable. In this case there is a constant M so that whenever $\delta > 0$ there exist $n = n(\delta)$ and $u_1, \dots, u_{2n}, h \in L_\phi$ with

$$1 = \frac{1}{2n}(u_1 + \dots + u_{2n}) + h, \quad (6)$$

$$\int_0^1 \phi(|u_i(t)|) dt \leq \delta, \quad (7)$$

$$\int_0^1 \phi(|h(t)|) dt \leq \delta, \quad (8)$$

$$\int_0^1 \phi\left(\left|\sum_{i=1}^{2n} a_i u_i(t)\right|\right) dt \leq M, \quad (9)$$

whenever $|a_1| + |a_2| + \dots + |a_{2n}| \leq 1$.

Now let

$$B = \sup_{0 < x \leq 2} \frac{\phi(x)}{x},$$

$$C = \sum_{n=1}^{\infty} c_n,$$

so that both B and C are finite. Now choose $\varepsilon < 1/10$ so that if $x \geq \varepsilon^{-1}$

$$G(x) \geq C(8\varepsilon^2 M + B).$$

Then we may choose $u_1, \dots, u_{2n}, h$ as above with $\delta = \varepsilon^2$. Let $u_1^*, \dots, u_{2n}^*$ be the pointwise decreasing re-arrangement of $|u_1|, \dots, |u_{2n}|$. Clearly each u_i^* is measurable and belongs to L_ϕ . Next let

$$w_i(t) = \min\left(u_i^*(t), \frac{2n}{i}\right), \quad 1 \leq i \leq 2n.$$

We shall show first that

$$\frac{1}{2n} \sum_{i=1}^n \int_{w_i(t) \geq \varepsilon^{-1}} w_i(t) dt \geq \frac{1}{2}. \quad (10)$$

Let λ denote Lebesgue measure on $(0,1)$ and let $N(t)$ for each t be the largest k so that $u_k^*(t) \geq 1$ (and $N(t) = 0$ if $u_k^*(t) < 1$ for all k). Then

$$\int_0^1 N(t) dt = \sum_{i=1}^{2n} \lambda(|u_i| \geq 1) \leq \sum_{i=1}^{2n} \int_0^1 \phi(|u_i(t)|) dt \leq 2n\varepsilon^2.$$

Hence $\lambda(t : N(t) \geq 2n\varepsilon) \leq \varepsilon^2$.

Similarly

$$\lambda(t : |h(t)| \geq \varepsilon) \leq \varepsilon^2.$$

Now let $A = \{t : |h(t)| < \varepsilon, N(t) < 2n\varepsilon\}$; then $\lambda(A) \geq 1 - 2\varepsilon^2$. For $t \in A$

$$\sum_{i=1}^{2n} |u_i(t)| \geq 2n(1-\varepsilon)$$

and hence

$$\sum_{i=1}^{2n} |w_i(t)| \geq 2n(1-\varepsilon).$$

Now

$$\begin{aligned} \int_{|u_i| \leq \varepsilon^{-1}} |u_i| dt &= \int_{|u_i| \leq \varepsilon} |u_i| dt + \int_{\varepsilon < |u_i| \leq \varepsilon^{-1}} |u_i| dt \\ &\leq \varepsilon + \varepsilon^{-1} \lambda(|u_i| > \varepsilon) \\ &\leq \varepsilon + \varepsilon^{-2} \int_0^1 \phi(|u_i|) dt \leq 2\varepsilon. \end{aligned}$$

Hence

$$\frac{1}{2n} \sum_{i=1}^{2n} \int_{u_i^* \leq \varepsilon^{-1}} u_i^*(t) dt \leq 2\varepsilon.$$

If $t \in A$ and $w_i(t) \leq \varepsilon^{-1}$ then $u_i^*(t) \leq \varepsilon^{-1}$. For otherwise $2n/i \leq \varepsilon^{-1}$ so that $i \geq 2n\varepsilon > N(t)$ and hence $u_i^*(t) < 1 \leq 2n/i$. Hence

$$\frac{1}{2n} \sum_{i=1}^{2n} \int_{A \cap (w_i > \varepsilon^{-1})} w_i(t) dt \leq 2\varepsilon.$$

However

$$\frac{1}{2n} \sum_{i=1}^{2n} \int_A w_i(t) dt \geq (1-\varepsilon)\lambda(A) \geq 1-3\varepsilon.$$

Thus

$$\frac{1}{2n} \sum_{i=1}^{2n} \int_{A \cap (w_i > \varepsilon^{-1})} w_i(t) dt \geq 1-5\varepsilon \geq \frac{1}{2}.$$

Since for $t \in A$, $w_i(t) \leq 1 \leq \varepsilon^{-1}$ for $i \geq n(> N(t))$, we see that (10) holds.

We now fix r with $1 \leq r \leq n$. We define two sets of random variables $(X_1, \dots, X_{2n})$, $(Y_1, \dots, Y_{2n})$ on some probability space (Ω, P) where Ω is a finite set. The random variables $(Y_1, \dots, Y_{2n})$ are mutually independent and independent of $(X_1, \dots, X_{2n})$ with common distribution given by $P(Y_i = +1) = P(Y_i = -1) = \frac{1}{2}$. The random variables $(X_1, \dots, X_{2n})$ are not mutually independent. Their distribution may be described as follows: select an r -subset γ at random from the collection of r -subsets of $\{1, 2, \dots, 2n\}$; then let $X_i = 1$ if $i \in \gamma$ and $X_i = 0$ otherwise.

Then for every $\omega \in \Omega$, $\sum_{i=1}^{2n} |X_i(\omega)Y_i(\omega)| = r$ and hence

$$\int_0^1 \phi\left(\frac{1}{r} \left| \sum_{i=1}^{2n} X_i(\omega)Y_i(\omega)u_i(t) \right| \right) dt \leq M. \quad (11)$$

Let $s = [2n/r]$, and let γ be any fixed s -subset of $\{1, 2, \dots, 2n\}$. For $j \in \gamma$, let $E_j = \{\omega : X_j(\omega) = 1, X_i(\omega) = 0 \text{ if } i \in \gamma \setminus \{j\}\}$. Then if $r > 1$,

$$\begin{aligned} P(E_j) &= \binom{2n-s}{r-1} / \binom{2n}{r} \\ &= \frac{r}{2n} \cdot \frac{2n-s}{2n-1} \cdot \frac{2n-s-1}{2n-2} \cdots \frac{2n-r-s+2}{2n-r+1} \\ &\geq \frac{r}{2n} \exp\left[-(s-1)\left(\frac{1}{2n-s} + \frac{1}{2n-s-1} + \cdots + \frac{1}{2n-r-s+2}\right)\right] \end{aligned}$$

$$\geq \frac{r}{2n} \exp\left(-\frac{rs}{n}\right) \geq \frac{r}{2ne^2}$$

and this also holds for $r=1$.

Now by symmetry for fixed $t \in (0,1)$

$$P\left\{E_j \cap \left(\omega: \left|\sum_{i=1}^{2n} X_i(\omega) Y_i(\omega) u_i(t)\right| > |u_i(t)|\right)\right\} \geq \frac{r}{4ne^2}.$$

Thus

$$\int_{E_j} \phi\left(\frac{1}{r} \left|\sum_{i=1}^{2n} X_i(\omega) Y_i(\omega) u_i(t)\right|\right) dP(\omega) \geq \frac{r}{4ne^2} \phi\left(\frac{|u_i(t)|}{r}\right).$$

As the events $(E_j, j \in \gamma)$ are disjoint, we conclude

$$\int_{\Omega} \phi\left(\frac{1}{r} \left|\sum_{i=1}^{2n} X_i Y_i u_i(t)\right|\right) dP(\omega) \geq \frac{r}{4ne^2} \sum_{j \in \gamma} \phi\left(\frac{|u_j(t)|}{r}\right).$$

Choosing γ to maximize the right-hand side, we have

$$\int_{\Omega} \phi\left(\frac{1}{r} \left|\sum_{i=1}^{2n} X_i Y_i u_i(t)\right|\right) dP(\omega) \geq \frac{r}{4ne^2} \sum_{j=1}^s \phi\left(\frac{u_j^*(t)}{r}\right).$$

Thus by (11) and Fubini's theorem, we have

$$\int_0^1 \frac{r}{2n} \sum_{j=1}^s \phi\left(\frac{u_j^*(t)}{r}\right) dt \leq 2e^2 M. \quad (12)$$

Now summing over $r=1,2,\dots,n$ we have

$$\frac{1}{2n} \int_0^1 \sum_{r=1}^n \sum_{j=1}^{\lfloor 2n/r \rfloor} c_r r \phi\left(\frac{u_j^*(t)}{r}\right) dt \leq 2e^2 CM.$$

Interchanging the order of summation and discarding terms with $rj > n$ we have

$$\frac{1}{2n} \int_0^1 \sum_{j=1}^n \sum_{r=1}^{\lfloor n/j \rfloor} c_r r \phi\left(\frac{u_j^*(t)}{r}\right) dt \leq 2e^2 CM. \quad (13)$$

If $x \leq 2n/j$, we have

$$\begin{aligned} \sum_{r=1}^{\lfloor n/j \rfloor} c_r r \phi\left(\frac{x}{r}\right) &= x \left[G(x) - \sum_{r > \lfloor n/j \rfloor} c_r \frac{r}{x} \phi\left(\frac{x}{r}\right) \right] \\ &\geq x [G(x) - BC]. \end{aligned}$$

Thus

$$\sum_{r=1}^{\lfloor n/j \rfloor} c_r r \phi\left(\frac{w_j(t)}{r}\right) \geq w_j(t) [G(w_j(t)) - BC]. \quad (14)$$

From (13) since $w_j \leq u_j^*$ we have

$$\frac{1}{2n} \sum_{j=1}^n \int_{w_j \geq \varepsilon^{-1}} \sum_{r=1}^{\lfloor n/j \rfloor} c_r r \phi\left(\frac{w_j(t)}{r}\right) dt \leq 2e^2 CM$$

and hence, recalling the choice of ε and (14),

$$\frac{1}{2n} \sum_{j=1}^n \int_{w_j \geq \varepsilon^{-1}} 8e^2 CM w_j(t) dt \leq 2e^2 CM$$

or

$$\frac{1}{2n} \sum_{j=1}^n \int_{w_j \geq \varepsilon^{-1}} w_j(t) dt \leq \frac{1}{4}$$

which contradicts (10) and completes the proof.

We are now in a position to construct the example.

Example. There exists a locally bounded Orlicz space $L_\phi(0,1)$ with trivial dual in which the only approachable point is $\{0\}$.

We shall construct ϕ to satisfy (4), (5), the Δ_2 -condition and

$$\liminf_{x \rightarrow \infty} \frac{\phi(x)}{x} = 0, \quad (15)$$

$$\text{for some } \beta > 0, \ x^{-\beta} \phi(x) \text{ is non-decreasing.} \quad (16)$$

Then (15) will imply that $L_\phi^* = \{0\}$ (Rolewicz [126], Turpin [128]) and (16) will imply that L_ϕ is locally bounded (Rolewicz [126], Turpin [128]).

Let $(t_n : n = 0, 1, 2, \dots)$ be an increasing sequence of positive numbers such that $t_{n+1} > t_n + 4n + 2$ ($n \geq 0$). Define a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \sigma(t) &= 0, & t &\leq t_0; \\ \sigma(t) &= (1-\beta)(n - (t - t_n)), & t_n &\leq t \leq t_n + 2n; \\ \sigma(t) &= (1-\beta)(t - t_n - 3n), & t_n + 2n &\leq t \leq t_n + 4n + 1; \\ \sigma(t) &= (1-\beta)(n + 1), & t_n + 4n + 1 &\leq t \leq t_{n+1}. \end{aligned}$$

Suppose $0 < \alpha < \frac{1}{4}(1-\beta)$ and define

$$\theta(t) = \max_{n=0,1,2,\dots} (\sigma(t - n \log 2) - \alpha n \log 2).$$

Then if $t_n \leq t \leq t_n + 4n + 1$, there exists m with $m \log 2 < 4n + 2$ and

$$\sigma(t - m \log 2) = n(1-\beta).$$

Hence

$$\sigma(t) \geq n(1-\beta) - \alpha(4n + 2).$$

If $t_n + 4n + 1 \leq t \leq t_{n+1}$, $\theta(t) \geq \sigma(t) = (1-\beta)(n + 1)$, so that $\lim_{t \rightarrow \infty} \theta(t) = \infty$.

Now we define

$$\begin{aligned} \phi(x) &= x \exp(\sigma(\log x)), & 0 < x < \infty, \\ \phi(0) &= 0. \end{aligned}$$

Then $\phi(x) = x$ for $0 \leq x \leq 1$, and satisfies the Δ_2 -condition. Also $\log x^{-\beta} \phi(x) = \sigma(\log x) + (1-\beta) \log x$ is non-decreasing, so that (16) holds. For (15) observe that $\log(\phi(x)/x) = \sigma(\log x)$ and $\sigma(t_n + 2n) = -n(1-\beta)$.

Finally we show that (5) holds:

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{-n\alpha} \frac{2^n}{x} \phi\left(\frac{x}{2^n}\right) &= \sum_{n=0}^{\infty} 2^{-n\alpha} \exp(\sigma(\log x - n \log 2)) \\ &\geq \exp \theta(\log x) \rightarrow \infty \text{ as } x \rightarrow \infty. \end{aligned}$$

Of course by Theorem (3.2.1) the space L_ϕ we have constructed has the property that every compact convex subset is locally convex.

There are a number of obvious questions arising from this example. We do not know if a condition like (5) is necessary for the conclusion of Theorem (3.2.3). In particular if we simply have

$$\liminf_{x \rightarrow \infty} x^{-1} \phi(x) = 0 \text{ and } \limsup_{x \rightarrow \infty} x^{-1} \phi(x) = \infty,$$

then can L_ϕ contain a non-zero needle point? In [127] Shapiro asks whether the Krein-Milman theorem holds in certain quotients of H_p ($0 < p < 1$). This example perhaps suggests that the failure of the Krein-Milman theorem and the existence of needle points is a rarer phenomenon than previously suspected.

Section (3.3): Zero Derivative in F -Spaces

Let X be an F -space (complete metric linear space) and suppose $g:[0,1] \rightarrow X$ is a continuous map. Suppose that g has zero derivative on $[0,1]$, i.e.

$$g'(t) = \lim_{h \rightarrow 0} \frac{1}{h} (g(t+h) - g(t)) = 0$$

for $0 \leq t \leq 1$ (we take the left and right derivatives at the end points). Then, if X is locally convex or even if it merely possesses a separating family of continuous linear functionals, we can conclude that g is constant by using the Mean Value Theorem. If however $X^* = \{0\}$ then it may happen that g is not constant; for example, let $X = L_p(0,1)$ ($0 \leq p < 1$) and $g(t) = 1_{[0,t]}$ ($0 \leq t \leq 1$) (the characteristic function of $[0,t]$). This example is due to Rolewicz [133], [134].

The aim of this section is to substantiate a conjecture of Rolewicz [134] that every F -space X with trivial dual admits a non-constant curve $g:[0,1] \rightarrow X$ with zero derivative. In fact we shall show, given any two points $x_0, x_1 \in X$, there exists a map $g:[0,1] \rightarrow X$ with $g(0) = x_0$, $g(1) = x_1$ and

$$\lim_{|t-s| \rightarrow 0} \frac{g(t) - g(s)}{t - s} = 0 \text{ uniformly for } 0 \leq s, t \leq 1.$$

To establish this result we shall need to study X -valued martingales. Let $\mathcal{B}$ be the σ -algebra of Borel subsets of $[0,1]$ and let $\mathcal{F}_n$ ($n \geq 0$) be an increasing family of finite sub-algebras of $\mathcal{B}$. Then a sequence of functions $u_n:[0,1] \rightarrow X$ is an X -valued F_n -martingale if each u_n is F_n -measurable and for $n \geq m$ we have $\mathcal{E}(u_n | \mathcal{F}_m) = u_m$. Here the definition of conditional expectation is the standard one with respect to Lebesgue measure λ and there are no integration problems since each u_n is finitely-valued.

It is easy to show that every F -space X with trivial dual contains a non-constant martingale $\{u_n | \mathcal{F}_n\}$ which converges to zero uniformly. However we shall need to consider dyadic martingales. Let $D_{n,k} = [(k-1)/2^n, k/2^n)$ ($1 \leq k \leq 2^n, 0 \leq n < \infty$). Then, for $n \geq 0$, let $\mathcal{B}_n$ be the sub-algebra of $\mathcal{B}$ generated by the sets $\{D_{n,k} : 1 \leq k \leq 2^n\}$. A dyadic martingale is simply a $\mathcal{B}_n$ -martingale. The main point of the argument will be to show that we can find non-zero dyadic martingales which converge uniformly to zero.

We note here a connection with the recent work of Roberts [124], [123] on the existence of compact convex sets without extreme points. Indeed, in a needlepoint space (see [123]) it would be easy to show that there are non-zero dyadic martingales which converge uniformly to zero. However there are F -spaces with trivial dual which contain no needlepoints [120].

As usual an F -norm on a (real) vector space X is a map $x \rightarrow \|x\|$ such that

$$\|x\| > 0 \text{ if } x \neq 0, \quad (17)$$

$$\|x + y\| \leq \|x\| + \|y\| \quad (x, y \in X), \quad (18)$$

$$\|tx\| \leq \|x\| \quad (|t| \leq 1), \quad (19)$$

$$\lim_{t \rightarrow 0} \|tx\| = 0 \quad (x \in X). \quad (20)$$

The F -norm is said to be strictly concave if, for each $x \in X$ with $x \neq 0$, the map $t \rightarrow \|tx\|$ is strictly concave on $[0, \infty)$, i.e.

if $0 \leq s < t < \infty$ and $0 < a, b < 1$ with $a + b = 1$ then, if $x \neq 0$,

$$\|(as + bt)x\| > a\|sx\| + b\|tx\|. \quad (21)$$

Every F -space can be equipped with an (equivalent) F -norm which is strictly concave. This follows from the results of Bessaga, Petczynski and Rolewicz [131]. We may give X an F -norm $\|\cdot\|_0$ so that the map $t \rightarrow \|tx\|_0$ is concave and strictly increasing for each $x \neq 0$. Now define $\|x\| = \|x\|_0^{1/2}$.

Suppose N is a positive integer. We consider the space $\mathbb{R}^N$ with the natural co-ordinatewise partial ordering (i.e. $x \geq y$ if and only if $x_i \geq y_i$ for $1 \leq i \leq N$). We shall denote by $(e_k : 1 \leq k \leq N)$ the natural basis elements of $\mathbb{R}^N$. We shall use the idea of $\mathbb{R}^N$ -valued submartingales and supermartingales; these have obvious meaning with respect to the ordering denned above. In addition, standard scalar convergence theorems can be applied co-ordinatewise to produce the same theorems for $\mathbb{R}^N$.

For $1 \leq i \leq N$, let F_i be a continuous map $F_i : [0, \infty) \rightarrow [0, \infty)$ which is strictly increasing, strictly concave and satisfies $F_i(0) = 0$, $F_i(1) = 1$. Then F_i is also subadditive since

$$F_i(s) \geq \frac{s}{s+t} F_i(s+t) \quad (s, t > 0).$$

Hence we may define an absolute F -norm on $\mathbb{R}^N$ by

$$\|x\| = \sum_{i=1}^N F_i(|x_i|) \quad x \in \mathbb{R}^N. \quad (22)$$

Now, for $x \in \mathbb{R}^N$, define

$$\sigma(x) = \inf \{ \max(\|y\|, \|z\|) : x = \frac{1}{2}(y + z) \}. \quad (23)$$

We shall need the following properties of σ .

Lemma (3.3.1)[130]: (a) If $x \in \mathbb{R}^N$ and $x \geq 0$ then there exist $y, z \in \mathbb{R}^N$ with $y \geq 0$, $z \geq 0$, $x = \frac{1}{2}(y + z)$ and $\|y\| \leq \sigma(x)$, $\|z\| \leq \sigma(x)$.

(b) For $x, y \in \mathbb{R}^N$,

$$|\sigma(x) - \sigma(y)| \leq \|x - y\|, \quad (24)$$

$$\sigma(x) \leq \|x\|. \quad (25)$$

(c) If $x \geq 0$ and $\sigma(x) = \|x\| = 1$ then, for some k , we have $x = e_k$.

Proof. (a) is an easy consequence of a compactness argument. For (b) (24), observe that if $x = \frac{1}{2}(z + z')$ then

$$y = \frac{1}{2}[(z + y - x) + (z' + y - x)],$$

so that $\sigma(y) \leq \sigma(x) + \|y - x\|$ and so (24) follows. (25) is an immediate consequence of the definition of σ .

Suppose $x \geq 0$, $\|x\| = 1$, $x_i > 0$ and $x_j > 0$ where $i \neq j$. We show $\sigma(x) < 1$.

Since F_i is concave, it has left and right derivatives at x_i , α_1 and α_2 , say, with $0 \leq \alpha_2 \leq \alpha_1$. Similiarly F_j has left and right derivatives at x_j , β_1 and β_2 with $0 \leq \beta_2 \leq \beta_1$. For small $t > 0$,

$$\|x + t(\beta_1 e_i - \alpha_2 e_j)\| < \|x\|,$$

$$\|x - t(\beta_1 e_i - \alpha_2 e_j)\| < \|x\| + t(-\alpha_1 \beta_1 + \beta_2 \alpha_2) \leq \|x\|.$$

Hence $\sigma(x) < 1$.

We conclude that if $\sigma(x) = 1$ then $x = e_k$ for some k , $1 \leq k \leq N$.

Now let $\pi(x) = x_1 + \dots + x_n \quad (x \in \mathbb{R}^N)$.

Theorem (3.3.2)[130]: Suppose $a \in \mathbb{R}^N$, $a \geq 0$ and $\pi(a)=1$. Then there are disjoint Borel subsets $E_1, \dots, E_N$ of $[0,1)$ with $\lambda(E_i)=a_i$ ($1 \leq i \leq N$) and a scalar valued dyadic supermartingale θ_n ($0 \leq n < \infty$) such that

$$0 \leq \theta_n(t) \leq 1 \quad (0 \leq t < 1, 0 \leq n < \infty), \quad (26)$$

$$\lim_{n \rightarrow \infty} \theta_n(t) = 0 \text{ a.e.} \quad (27)$$

and if

$$u_n = \mathcal{E} \left(\sum_{i=1}^N 1_{E_i} e_i \mid \mathcal{B}_n \right) \quad (0 \leq n < \infty) \quad (28)$$

then

$$u_n(t) \geq \theta_n(t)a \quad (0 \leq t < 1, 0 \leq n < \infty), \quad (29)$$

$$\|u_n(t) - \theta_n(t)a\| \leq 1 \quad (0 \leq t < 1, 0 \leq n < \infty). \quad (30)$$

Proof. To start observe

$$\|a\| = \sum_{i=1}^N F_i(a_i) \geq \pi(a) = 1.$$

Define $\alpha_0(t) \equiv \alpha_0$ for $0 \leq t < 1$, where $0 < \alpha_0 \leq 1$ and $\|\alpha_0 a\| = 1$; then let $w_0(t) = \alpha_0 a$, $0 \leq t < 1$. We then define inductively sequences $(w_n : n \geq 0)$, $(w_n^* : n \geq 1)$, $(\alpha_n : n \geq 0)$ of functions on $[0,1)$, where

$$w_n \ (n \geq 0) \text{ and } w_n^* \ (n \geq 1) \text{ are } \mathbb{R}^N \text{-valued and } \mathcal{B}_n \text{-measurable,} \quad (31)$$

$$\alpha_n \ (n \geq 0) \text{ is } \mathbb{R} \text{-valued and } \mathcal{B}_n \text{-measurable,} \quad (32)$$

$$w_n(t) \geq 0 \quad (0 \leq t < 1, n \geq 0),$$

$$w_n^*(t) \geq 0 \quad (0 \leq t < 1, n \geq 0), \quad (33)$$

$$\alpha_n(t) \geq 0 \quad (0 \leq t < 1, n \geq 0),$$

$$\mathcal{E}(w_n^* \mid \mathcal{B}_n) = w_n \quad (n \geq 0), \quad (34)$$

$$w_n(t) = w_n^*(t) + \alpha_n(t)a \quad (0 \leq t < 1, n \geq 0), \quad (35)$$

$$\|w_n(t)\| = 1 \quad (0 \leq t < 1, n \geq 0), \quad (36)$$

$$\|w_{n+1}^*(t)\| \leq \sigma(w_n(t)) \quad (0 \leq t < 1, n \geq 0). \quad (37)$$

Indeed suppose w_j , w_j^* and α_j have been chosen for $j \leq n$. Then

$$w_n(t) = b_{n,k} \quad (t \in D_{n+1,k}).$$

where $\|b_{n,k}\| = 1$, and $b_{n,k} \geq 0$. Choose $y_{2k-1}, y_{2k} \geq 0$ so that $\max(\|y_{2k-1}\|, \|y_{2k}\|) = \sigma(b_{n,k})$ and $b_{n,k} = \frac{1}{2}(y_{2k-1} + y_{2k})$ (see Lemma (3.3.1) (a)). Now define

$$w_{n+1}^*(t) = y_k \quad (t \in D_{n+1,k}).$$

Then (34) and (37) are clear. Since

$$\|w_{n+1}^*(t)\| \leq 1 \quad (0 \leq t < 1),$$

we can determine α_{n+1} to be $\mathcal{B}_{n+1}$ -measurable so that $\alpha_{n+1} \geq 0$ and

$$\|w_{n+1}^*(t) + \alpha_{n+1}(t)a\| = 1 \quad (0 \leq t < 1).$$

Now define

$$w_{n+1}(t) = w_{n+1}^*(t) + \alpha_{n+1}(t)a \quad (0 \leq t < 1)$$

and clearly (36) holds.

Observe that

$$\mathcal{E}(w_{n+1} \mid \mathcal{B}_n) = w_n + \mathcal{E}(\alpha_{n+1} \mid \mathcal{B}_n)a$$

and if $m > n$

$$\mathcal{G}(w_n | \mathcal{B}_n) = w_n + \left(\sum_{k=n+1}^m \mathcal{G}(\alpha_k | \mathcal{B}_n) \right) a. \quad (38)$$

Hence w_n is a submartingale and it is clearly bounded. Thus $\lim_{n \rightarrow \infty} w_n(t) = w_\infty(t)$ exists almost everywhere, and $\|w_\infty(t)\| = 1$ a.e.

The real-valued submartingale $(\pi \circ w_n : n \geq 0)$ is uniformly bounded and converges to $\pi \circ w_\infty$ a.e. Hence

$$\begin{aligned} \int_0^1 \pi(w_\infty(t)) dt &= \lim_{n \rightarrow \infty} \int_0^1 \pi(w_n(t)) dt \\ &= \int_0^1 \pi(w_n(t)) dt + \sum_{k=1}^{\infty} \int_0^1 \alpha_k(t) dt \end{aligned}$$

by (38) since $\pi(a) = 1$. Hence

$$\int_0^1 \sum_{k=1}^{\infty} \alpha_k(t) dt < \infty$$

and so (a.e.) $\sum \alpha_k(t) < \infty$. Thus $\alpha_k(t) \rightarrow 0$ a.e. and $\|w_{n+1}(t) - w_{n+1}^*(t)\| \rightarrow 0$ a.e. Hence $\|w_{n+1}^*(t)\| \rightarrow 1$ and $\sigma(w_n(t)) \rightarrow 1$ a.e. By Lemma (3.3.1)(b), σ is continuous and so (a.e.)

$$\sigma(w_\infty(t)) = \|w_\infty(t)\| = 1.$$

As $w_\infty(t) \geq 0$, we conclude that

$$w_\infty(t) = \sum_{i=1}^N 1_{E_i} e_i \quad \text{a.e.,}$$

where $E_1, \dots, E_N$ are disjoint Borel sets with $E_1 \cup \dots \cup E_N = [0, 1)$.

Now define $u_n = \mathcal{G}(w_\infty | \mathcal{B}_n)$. Then, since $\{w_n\}$ is uniformly bounded and $w_n \rightarrow w_\infty$ a.e.,

$$\begin{aligned} u_n &= \lim_{m \rightarrow \infty} \mathcal{G}(w_m | \mathcal{B}_n) \\ &= w_n + \left(\sum_{k=n+1}^{\infty} \mathcal{G}(\alpha_k | \mathcal{B}_n) \right) a = w_n + \theta_n a, \end{aligned}$$

where $\theta_n \geq 0$ is $\mathcal{B}_n$ -measurable. Since (w_n) is a submartingale, (θ_n) is a supermartingale. As $u_n - w_n \rightarrow 0$ a.e., we have $\theta_n \rightarrow 0$ a.e. As $\pi(w_\infty) \leq 1$ a.e., $\pi(u_n) \leq 1$ a.e. and so $\theta_n \leq 0$ a.e. Also $\|u_n - \theta_n a\| = \|w_n\| = 1$. Finally observe

$$u_0 = (\alpha_0 + \theta_0)a = \sum_{i=1}^N \lambda(E_i) e_i.$$

Hence

$$\pi(u_0) = \sum_{i=1}^N \lambda(E_i) = 1 = \alpha_0 + \theta_0$$

Thus $\lambda(E_i) = a_i$ ($1 \leq i \leq N$), and the proof is complete.

In fact we shall not use Theorem (3.3.2); instead we use its "finite" version.

Theorem (3.3.3)[130]: Under the same hypotheses as Theorem (3.3.2), given $\varepsilon > 0$, there is a finite dyadic martingale $(v_0, v_1, \dots, v_m)$ with

$$v_0(t) = a \quad (0 \leq t < 1), \quad (39)$$

$$\|v_m(t)\| \leq 1 + \varepsilon \quad (0 \leq t < 1). \quad (40)$$

For $1 \leq n \leq m-1$, there is a positive $\mathcal{B}_n$ -measurable function ϕ_n with $\phi_n \leq 1$ and

$$\|v_n(t) - \phi_n(t)a\| \leq 1 + \varepsilon \quad (0 \leq t < 1). \quad (41)$$

Proof. Suppose $0 < \delta_0 < \frac{1}{2}$ is chosen so that $\|2\delta_0 a\| < \frac{1}{2}\varepsilon$ and $\|(1-\delta_0)^{-1}\| < 1 + \frac{1}{2}\varepsilon$ whenever $\|x\| < 1$.

Let u_n, θ_n be chosen as in Theorem (3.3.2) and select m so that

$$\int_0^1 \theta_m(t) dt = \delta \leq \delta_0.$$

Define

$$v_m = (1-\delta)^{-1}(u_m - \theta_m a)$$

and

$$v_n = \mathcal{E}(v_m \mid \mathcal{B}_n) \quad (0 \leq n \leq m).$$

Then $\|v_m(t)\| \leq 1 + \varepsilon$ and

$$\begin{aligned} v_m &= (1-\delta)^{-1}(u_m - \mathcal{E}(\theta_m \mid \mathcal{B}_n)a) \\ &= (1-\delta)^{-1}(u_n - \theta_n a) + (1-\delta)^{-1}(\theta_n - \mathcal{E}(\theta_m \mid \mathcal{B}_n))a. \end{aligned}$$

Define

$$\phi_n = \theta_n - \mathcal{E}(\theta_m \mid \mathcal{B}_n) \quad (0 \leq n \leq m).$$

Then $0 \leq \phi_n \leq \theta_n \leq 1$ and

$$v_n - \phi_n a = (1-\delta)^{-1}(u_n - \theta_n a) + \delta(1-\delta)^{-1}\phi_n a$$

and so

$$\|v_n - \phi_n a\| \leq 1 + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = 1 + \varepsilon.$$

We now turn to the general infinite-dimensional problem.

Lemma (3.3.4)[130]: Suppose X is an F -space with a strictly concave F -norm. Suppose $x_0 \neq 0$ and that $x_0 \in \text{co}\{x : \|x\| \leq \delta\}$. Then there is a finite dyadic martingale u_n ($0 \leq n \leq m$) with $u_0(t) \equiv x_0$, and

$$\|u_m(t)\| \leq 2\delta \quad (0 \leq t < 1), \quad (42)$$

$$\|u_n(t)\| \leq \|x_0\| + 2\delta \quad (0 \leq t < 1, 0 \leq n \leq m). \quad (43)$$

Proof. There exist $y_1, \dots, y_N \in X$ with $y_i \neq 0$ ($1 \leq i \leq N$), $\|y_i\| \leq \delta$ and $x_0 = a_1 y_1 + \dots + a_N y_N$, where $a_i \geq 0$ and $a_1 + a_2 + \dots + a_N = 1$.

For $0 \leq t < \infty$, define

$$F_i(t) = \|ty_i\| / \|y_i\|.$$

Then F_i is strictly concave. Define the absolute norm on $\mathbb{R}^N$ by

$$\|b\| = \sum_{i=1}^N F_i(|b_i|).$$

Now, by Theorem (3.3.3), there is a finite $\mathbb{R}^N$ -valued dyadic martingale $(v_n : 0 \leq n \leq m)$ with (taking $\varepsilon = 1$)

$$v_0 \equiv a = (a_1, \dots, a_N) \quad (0 \leq t < 1),$$

$$\|v_m(t)\| \leq 2 \quad (0 \leq t < 1)$$

and

$$\|v_n(t) - \phi_n(t)a\| \leq 2 \quad (0 \leq t < 1, 0 \leq n < m),$$

where $0 \leq \phi_n(t) \leq 1$. Define $T : \mathbb{R}^N \rightarrow X$ by

$$Tb = \sum_{i=1}^N b_i y_i.$$

Then

$$\|Tb\| \leq \sum_{i=1}^N \|b_i y_i\| \leq \sum_{i=1}^N \|y_i\| F_i(|b_i|) \leq \delta \|b\|.$$

Now let $u_n = T v_n$. Then $u_0(t) \equiv x_0$ and $\|u_m(t)\| \leq 2\delta$. Also

$$\|u_n(t)\| \leq \|\phi_n(t)x_0\| + 2\delta \leq \|x_0\| + 2\delta.$$

Theorem (3.3.5)[130]: Suppose X is an F -space with trivial dual, and that $x_0 \in X$. Then there is a dyadic martingale $(u_n : n \geq 0)$ with $u_0(t) \equiv x_0$ and

$$\max_{0 \leq n < \infty} \|u_n(t)\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (44)$$

Proof. As explained in the introduction we may suppose that the F -norm on X is strictly concave (passing to an equivalent F -norm does not affect (44)). The hypotheses guarantee that the convex hull of any neighborhood of zero is X . The construction is inductive, based on Lemma (3.3.4). To start the construction we may find a finite martingale $(u_n : 0 \leq n \leq N_1)$ so that $u_0(t) \equiv x_0$, $\|u_{N_1}(t)\| \leq \frac{1}{2}\|x_0\|$ and $\|u_n(t)\| \leq 2\|x_0\|$ ($1 \leq n \leq N_1$), by applying Lemma (3.3.4) with $\delta = \frac{1}{4}\|x_0\|$ if $x_0 \neq 0$ (the case $x_0 = 0$ is trivial).

Suppose now we have defined $(u_n : 1 \leq n \leq N_k)$ so that

$$\|u_{N_j}(t)\| \leq \left(\frac{1}{2}\right)^j \|x_0\| \quad (1 \leq j \leq k), \quad (45)$$

$$\|u_n(t)\| \leq 2\left(\frac{1}{2}\right)^j \|x_0\| \quad (N_j \leq n \leq N_{j+1}, 1 \leq j \leq k). \quad (46)$$

We shall show how to extend to a finite dyadic martingale $(u_n : 1 \leq n \leq N_{k+1})$ so that (45) and (46) hold for $j \leq k+1$ and $j \leq k$ respectively.

We have

$$u_{N_k}(t) = y_l \quad (t \in D_{N_k, l}).$$

For each y_l , there is a finite martingale $(v_n^l : 0 \leq n \leq M)$ with

$$v_0^l(t) = y_l \quad (0 \leq t \leq 1),$$

$$\|v_M^l(t)\| \leq \left(\frac{1}{2}\right)^{k+1} \|x_0\| \quad (0 \leq t \leq 1),$$

$$\|v_n^l(t)\| \leq \|y_l\| + \left(\frac{1}{2}\right)^{k+1} \|x_0\|$$

$$\leq \left(\frac{1}{2}\right)^{k-1} \|x_0\| \quad (0 \leq t \leq 1, 0 \leq n \leq M).$$

Here M may be taken independent of l by simply extending the martingale where necessary by adding further terms equal to the last term of the sequence.

Now let $N_{k+1} = N_k + M$ and define

$$u_{N_{k+1}} = v_1^l(2^{N_k}t - l + 1) \quad (t \in D_{N_{k+1}, l}).$$

It is now easy to verify that conditions (45) and (46) hold where applicable. Continuing in this way we clearly have (44) for the (infinite) martingale (u_n) .

The step from Theorem (3.3.5) is a very simple one if X is a quasi-Banach space or more generally is exponentially galbed (see Turpin [135]). In such space there is a natural correspondence between curves with uniform zero derivative and dyadic martingales converging uniformly to 0. In a general F -space a little more subtlety is required in the proof of the main theorem.

Theorem (3.3.6)[130]: Suppose X is an F -space with trivial dual and that $x_0, x_1 \in X$. Then there is a curve $g : [0, 1] \rightarrow X$ with $g(0) = x_0$, $g(1) = x_1$ and

$$\lim_{|t-s| \rightarrow 0} \frac{g(t) - g(s)}{t - s} = 0 \quad \text{uniformly for } 0 \leq s, t \leq 1. \quad (47)$$

In particular $g'(t) = 0$ for $0 \leq t \leq 1$.

Proof. It suffices to suppose $x_0 = 0$. Then there is a dyadic martingale $(u_n : n \geq 0)$ with

$$u_0(t) = x_1 \quad (0 \leq t < 1),$$

$$\max_{0 \leq t < 1} \|u_n(t)\| = \varepsilon_n \rightarrow 0.$$

Choose $N_0 = 0$. Since each u_n has finite range it is possible to choose a strictly increasing sequence of positive integers $(N_k : k \geq 1)$ so that

$$\|2^{N_j - N_k}(u_k(t) - u_{k-1}(t))\| \leq 2^{j-k} \varepsilon_j \quad (48)$$

for $0 \leq j \leq k-1$, $0 \leq t < 1$. Each $t \in [0,1)$ has a unique binary expansion

$$t = \sum_{j=1}^{\infty} \tau_j 2^{-j},$$

where each τ_j is zero or one and $\tau_j = 0$ infinitely often. Now define

$$v_k(t) = u_k \left(\sum_{j=1}^k \tau_j 2^{-j} \right).$$

(Recall that u_k is constant on the interval $\sum_{j=1}^k \tau_{N_j} 2^{-j} \leq t < \sum_{j=1}^k \tau_{N_j} 2^{-j} + 2^{-k}$.) Then we observe that v_k is a $\mathcal{B}_{N_k}$ -martingale, with

$$\max_{0 \leq t < 1} \|v_k(t)\| = \varepsilon_k,$$

$$\mathcal{E}(v_k | \mathcal{B}_0) = \int_0^1 v_k(t) dt = \int_0^1 u_k(t) dt = x_1.$$

In fact we observe that

$$\mathcal{E}(v_k | \mathcal{B}_{N_{k-1}}) = v_{k-1}. \quad (49)$$

For $k \geq 1$ and $0 \leq t \leq 1$, we define

$$g_k(t) = \int_0^t v_k(s) ds$$

(the integrand is simple). Then each g_k is continuous and from (49) we have

$$g_k(t) = g_{k-1}(t) \quad \text{if} \quad 2^{N_{k-1}} t \in \mathbb{Z}.$$

Now suppose that $0 < t < 1$ and that $2l \leq 2^{N_k} t < 2l+1$, where l is an integer. Then

$$g_k(t) - g_{k-1}(t) = \int_{2l/2^{N_k}}^t (v_k(s) - v_{k-1}(s)) ds$$

$$= (t - 2l(2^{-N_k})) (v_k(t) - v_{k-1}(t)). \quad (50)$$

Equally, if $2l+1 \leq 2^{N_k} t < 2l+2$,

$$g_k(t) - g_{k-1}(t) = ((2l+2)2^{-N_k} - t) (v_k(t) - v_{k-1}(t)). \quad (51)$$

Combining these results, we have

$$\|g_k(t) - g_{k-1}(t)\| \leq \max_{0 \leq t < 1} \|2^{-N_k}(v_k(t) - v_{k-1}(t))\|$$

$$= \max_{0 \leq t < 1} \|2^{-N_k}(u_k(t) - u_{k-1}(t))\| \leq 2^{-k} \varepsilon_0.$$

Hence (g_k) converges uniformly to a continuous function g on $[0,1]$, and $g(0) = 0$, $g(1) = x_1$.

Now suppose $0 \leq s < t \leq 1$. Then there is a least integer n so that for some integer l we have $2^n s \leq l < l+1 \leq 2^n t$. Clearly $2^{n-1}t - 2^{n-1}s < 2$ and $2^n t - 2^n s \geq 1$. Hence $2^{-n} \leq t - s < 4 \cdot 2^{-n}$ and $n \geq \log_2 1/(t-s)$.

Now suppose $N_{k-1} \leq n < N_k$, where $1 \leq k < \infty$. Suppose l_1 is the least integer not less than $2^n s$ and l_2 is the greatest integer not greater than $2^n t$. Then

$$2^n (g_{k-1}(t) - g_{k-1}(l_2 2^{-n})) = (2^n t - l_2) v_{k-1}(l_2 2^{-n}),$$

$$\begin{aligned} 2^n(g_{k-1}(l_1 2^{-n}) - g_{k-1}(s)) &= (l_1 - 2^n s) v_{k-1}(l_1 2^{-n}), \\ 2^n(g_{k-1}(i 2^{-n}) - g_{k-1}((i-1) 2^{-n})) &= v_{k-1}((i-1) 2^{-n}). \end{aligned}$$

Hence

$$\|2^n(g_{k-1}(l_2 2^{-n}) - g_{k-1}(l_1 2^{-n}))\| \leq (l_2 - l_1) \varepsilon_{k-1}$$

and

$$\|2^n(g_{k-1}(t) - g_{k-1}(s))\| \leq (l_2 - l_1 + 2) \varepsilon_{k-1}.$$

However $l_2 - l_1 \leq 2^n(t - s) < 4$ so that $l_2 - l_1 + 2 \leq 5$. Hence

$$\|2^n(g_{k-1}(t) - g_{k-1}(s))\| \leq 5 \varepsilon_{k-1}. \quad (52)$$

Now

$$2^n(g_k(t) - g_{k-1}(t)) = 2^{n-N_k} \rho(v_k(t) - v_{k-1}(t)),$$

where $0 \leq \rho \leq 1$, by (50) and (51). Hence

$$\|2^n(g_k(t) - g_{k-1}(t))\| \leq \varepsilon_k + \varepsilon_{k-1}. \quad (53)$$

A similar inequality holds for s .

If $r > k$

$$2^n(g_r(t) - g_{r-1}(t)) = 2^{n-N_r} \rho(v_r(t) - v_{r-1}(t)),$$

where $0 \leq \rho \leq 1$, and so

$$\begin{aligned} \|2^n(g_r(t) - g_{r-1}(t))\| &= \|2^{N_k - N_r} \rho(v_r(t) - v_{r-1}(t))\| \\ &\leq \max_{0 \leq t < 1} \|2^{N_k - N_r} (u_r(t) - u_{r-1}(t))\| \leq 2^{k-r} \varepsilon_k \end{aligned}$$

by (48). Hence

$$\|2^n(g(t) - g_k(t))\| \leq \left(\sum_{r>k} 2^{k-r} \right) \varepsilon_k = \varepsilon_k. \quad (54)$$

A similar inequality holds for s .

Combining (52), (53) and (54) and the similar results for s we obtain

$$\|2^n(g(t) - g(s))\| \leq 7 \varepsilon_{k-1} + 4 \varepsilon_k$$

and hence

$$\left\| \frac{g(t) - g(s)}{t - s} \right\| \leq 7 \varepsilon_{k-1} + 4 \varepsilon_k,$$

Where $N_k > \log_2 1/(t - s)$. Hence g has the properties specified in the theorem.

Every F -space X has a unique maximal linear subspace with trivial dual; this subspace is closed. Let us call this maximal subspace the *core* of X . If $\text{core}(X) = \{0\}$, it does not necessarily follow that X has a separating dual; for a detailed investigation of related ideas see Ribe [132]. We conclude with a simple corollary.

Corollary (3.3.7)[130]: Suppose X is an F -space and $x \in X$. In order that there exists a curve $g : [0, 1] \rightarrow X$ with $g(0) = 0$, $g(1) = x$ and $g'(t) = 0$ for $0 \leq t \leq 1$ if is necessary and sufficient that $x \in \text{core}(X)$.

Proof. If $x \in \text{core}(X)$ the existence of g is given by Theorem (3.3.6). Suppose conversely such a g exists and let Y be the closed linear span of $\{g(t) : 0 \leq t \leq 1\}$. Suppose ϕ is a continuous linear functional on Y . Then $(\phi \circ g)'(t) = 0$ ($0 \leq t \leq 1$) and hence by the Mean Value Theorem $\phi(g(t)) = 0$ ($0 \leq t \leq 1$). Thus $\phi = 0$ and so $Y \subset \text{core}(X)$; in particular $x \in \text{core}(X)$.

Chapter 4

Asymptotic Sharpness and Applications of Bernstein-Type Inequalities

A Bernstein-Type inequality in the standard Hardy space H^2 of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, for rational functions in $\mathbb{D}$ having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0 < r < 1$, is considered. The asymptotic sharpness is shown as $n \rightarrow \infty$, for every $r \in [0, 1)$. We apply our Bernstein-Type inequality to an effective Nevanlinna-Pick interpolation problem in the standard Dirichlet space, constrained by the H^2 -norm. We show that this result can not be extended to weighted Bergman spaces with “super-polynomially” decreasing weights.

Section (4.1): A Bernstein-Type Inequality for Rational Functions in H^2

First we recall the classical Bernstein inequality for polynomials: we denote by $\mathcal{P}_n$ the class of all polynomials with complex coefficients, of degree n : $P = \sum_{k=0}^n a_k z^k$. Let

$$\|P\|_2 = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{T}} |P(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} = \left(\sum_{k=0}^n |a_k|^2 \right)^{\frac{1}{2}}.$$

The classical inequality

$$\|P'\|_2 \leq n \|P\|_2 \quad (1)$$

is known as Bernstein's inequality. A great number of refinements and generalizations of (1) have been obtained. See [148, Part III] for an extensive study of that subject. The constant n in (1) is obviously sharp (take $P = z^n$).

Now let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc $\mathbb{D}$, the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional space defined by

$$K_{B_\sigma} = \mathcal{L}in(k_{\lambda_i} : i = 1, \dots, n),$$

where σ is a family of distinct elements of $\mathbb{D}$, and where $k_\lambda = \frac{1}{1 - \bar{\lambda}z}$ is the Szegő kernel associated to λ . An obvious modification allows to generalize the definition of K_{B_σ} in the case where the sequence σ admits multiplicities.

Notice that using the scalar product $(\cdot, \cdot)_{H^2}$ on H^2 , an equivalent description of this space is:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2,$$

where H^2 stands for the standard Hardy space of the unit disc $\mathbb{D}$,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_2^2 = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\},$$

m being the Lebesgue normalized measure on $\mathbb{T}$. We notice that the case $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ gives $K_{B_\sigma} = \mathcal{P}_{n-1}$. The issue of this section is to generalize classical Bernstein inequality (1) to the spaces K_{B_σ} . Notice that every rational functions with poles outside of $\mathbb{D}$ lies in a space K_{B_σ} . It has already been proved in [65] that if $r = \max_j |\lambda_j|$, and $f \in K_{B_\sigma}$, then

$$\|f'\|_2 \leq \frac{5}{2} \frac{n}{1-r} \|f\|_2. \quad (2)$$

In fact, Bernstein-type inequalities for rational functions were the subject of a number of references (see, for instance, [61], [140], [141], [136], [137], [138], [55] and [139]). Perhaps, the stronger and closer to ours of all known results are due to K.

Dyakonov [60]. In particular, it is proved in [60] that the norm $\|D\|_{K_B \rightarrow H^2}$ of the differentiation operator $Df = f'$ on a space K_B satisfies the following double inequality

$$a\|B'\|_\infty \leq \|D\|_{K_B \rightarrow H^2} \leq A\|B'\|_\infty,$$

where $a = \frac{1}{36c}$, $A = \frac{36+c}{2\pi}$ and $c = 2\sqrt{3\pi}$ (as one can check easily (c is not precised in [60])). It implies an inequality of type (2) (with a constant about $\frac{13}{2}$ instead of $\frac{5}{2}$).

Our goal is to find an inequality for $\sup \|D\|_{K_B \rightarrow H^2} = C_{n,r}$ (sup is over all B with given $n = \deg B$ and $r = \max_{\lambda \in \sigma} |\lambda|$), which is asymptotically sharp as $n \rightarrow \infty$. Our result is that there exists a limit $\lim_{n \rightarrow \infty} \frac{C_{n,r}}{n} = \frac{1+r}{1-r}$ for every r , $0 \leq r < 1$. Our method is different from [60] and is based on an elementary Hilbert space construction for an orthonormal basis in K_B .

Theorem (4.1.1)[64]: Let $n \geq 1$, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc $\mathbb{D}$, and B_σ the finite Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$. Let also K_{B_σ} be the n -dimensional subspace of H^2 defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Let D be the operator of differentiation on $(K_{B_\sigma}, \|\cdot\|_2)$:

$$D : (K_{B_\sigma}, \|\cdot\|_2) \rightarrow (H^2, \|\cdot\|_2) \quad f \mapsto f',$$

where $\|f\|_2 = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{T}} |f(\zeta)|^2 d\zeta \right)^{\frac{1}{2}}$. For $r \in [0,1)$ and $n \geq 1$, we set

$$C_{n,r} = \sup \left\{ \|D\|_{K_B \rightarrow H^2} : 1 \leq \text{card } \sigma \leq n, |\lambda| \leq r \ \forall \lambda \in \sigma \right\}.$$

(i) If $n = 1$ and $\sigma = \{\lambda\}$, we have

$$\|D\|_{K_B \rightarrow H^2} = |\lambda| \left(\frac{1}{1-|\lambda|^2} \right)^{\frac{1}{2}}.$$

If $n \geq 2$,

$$a(n,r) \frac{n}{1-r} \leq C_{n,r} \leq A(n,r) \frac{n}{1-r},$$

where

$$a(n,r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right)^{\frac{1}{2}},$$

and

$$A(n,r) \leq 1 + r + \frac{1}{\sqrt{n}}.$$

(ii) Moreover, the sequence

$$\left(\frac{1}{n} C_{n,r} \right)_{n \geq 1},$$

is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{n,r} = \frac{1+r}{1-r},$$

for all $r \in [0,1)$.

Proof. We first show (i). We suppose that $n = 1$. In this case, $K_B = \mathbb{C}e_1$, where

$$e_1 = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{(1-\bar{\lambda}z)}, |\lambda| \leq r,$$

(e_1 being of norm 1 in H^2). Calculating,

$$e'_1 = \frac{\bar{\lambda}(1-|\lambda|^2)^{\frac{1}{2}}}{(1-\bar{\lambda}z)^2},$$

and

$$\begin{aligned} \|e'_1\|_2 &= |\lambda|(1-|\lambda|^2)^{\frac{1}{2}} \left\| \frac{1}{(1-\bar{\lambda}z)^2} \right\|_2 \\ &= |\lambda|(1-|\lambda|^2)^{\frac{1}{2}} \left(\sum_{k \geq 0} (k+1)|\lambda|^{2k} \right)^{\frac{1}{2}} = |\lambda|(1-|\lambda|^2)^{\frac{1}{2}} \frac{1}{(1-|\lambda|^2)} = |\lambda| \left(\frac{1}{1-|\lambda|^2} \right)^{\frac{1}{2}}, \end{aligned}$$

we get

$$\|D_{|K_{B_\sigma}}\| = |\lambda| \left(\frac{1}{1-|\lambda|^2} \right)^{\frac{1}{2}}.$$

Now, we suppose that $n \geq 1$. First, we show the left-hand side inequality. Let

$$e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}.$$

Then $e_n \in K_{b_r^n}$ and $\|e_n\|_2 = 1$, (see [43], Malmquist-Walsh Lemma). Moreover,

$$\begin{aligned} e'_n &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2} \\ &= -\frac{r}{(1-r^2)^{\frac{1}{2}}} b'_r b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2}, \end{aligned}$$

since $b'_r = \frac{r^2-1}{(1-rz)^2}$. Then,

$$e'_n = b'_r \left[-\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$

and

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(w)| |b'_r(w)| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right|^2 dm(w) \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(w)| |b'_r(w)| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right|^2 dm(w), \end{aligned}$$

which gives, using the variables $u = b_r(w)$,

$$\|e'_n\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{T}} |b'_r(b_r(w))| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 dm(u).$$

But $1-rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\begin{aligned} \|e'_n\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} \left| \frac{(1-ru)^2}{1-r^2} \right| \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right|^2 dm(u) \\ &= \frac{1}{(1-r^2)^2} \frac{1}{2\pi} \int_{\mathbb{T}} |(1-ru)(-ru + (n-1)(1-ru))|^2 dm(u). \end{aligned}$$

Without loss of generality we can replace r by $-r$, which gives

$$\|e'_n\|_2 = \frac{1}{1-r^2} \|\varphi_n\|_2,$$

where $\varphi_n = (1+rz)(rz + (n-1)(1+rz))$. Expanding, we get

$$\begin{aligned}\varphi_n &= (1+rz)(nrz + (n-1)) = nrz + (n-1) + nr^2z^2 + (n-1)rz \\ &= (n-1) + (nr + (n-1)r)z + nr^2z^2,\end{aligned}$$

and

$$\begin{aligned}\|e'_n\|_2^2 &= \frac{1}{(1-r^2)^2} ((n-1)^2 + (2n-1)^2 r^2 + n^2 r^4) \\ &= \frac{n^2}{(1-r^2)^2} \left(1 + 4r^2 + r^4 - \frac{2}{n} - \frac{4r^2}{n} + \frac{1}{n^2} + \frac{r^2}{n^2} \right) \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 4r^2 + r^4 - \frac{2}{n} - \frac{4r^2}{n} + \frac{1+r^2}{n^2} \right) \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 4r^2 + r^4 - 5r^4 + 5r^4 - \frac{4r^4}{n} + \frac{4r^4}{n} - \frac{4r^2}{n} - \frac{2}{n} + \frac{1+r^2}{n^2} \right) \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(4r^2(1-r^2) - \frac{4r^2}{n}(1-r^2) + \frac{1+r^2}{n^2} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} \right) \\ &= \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(4r^2(1-r^2)(1-\frac{1}{n}) + \frac{1+r^2}{n^2} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} \right) \\ &\geq \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \begin{cases} 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{n} & \text{if } n > 0 \\ \frac{1}{4} + 1 + 5r^4 - \frac{4r^4}{n} - \frac{2}{2} & \text{if } n = 0 \end{cases} \\ &\geq \left(\frac{n}{1-r} \right)^2 \left(\frac{1}{1+r} \right)^2 \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right),\end{aligned}$$

and

$$a(n, r) \geq \frac{1}{1+r} \left(1 + 5r^4 - \frac{4r^4}{n} - \min\left(\frac{3}{4}, \frac{2}{n}\right) \right)^{\frac{1}{2}},$$

which completes the proof of the left hand side inequality .

We show now the right hand side one. Let σ be a sequence in $\mathbb{D}$ such that $1 \leq \text{card} \sigma \leq n$, $|\lambda| \leq r \ \forall \lambda \in \sigma$. Using [65], Proposition 4.1, we have

$$\begin{aligned}\|D\|_{K_{B_\sigma} \rightarrow H^2} &\leq \frac{1}{1-r} + \frac{1+r}{1-r} (n-1) + \frac{1}{1-r} \sqrt{n-2} \\ &= \frac{1}{1-r} (1 + (1+r)(n-1) + \sqrt{n-2}) = \frac{1}{1-r} (n(1+r) - r + \sqrt{n-2}) \\ &= \frac{n}{1-r} \left(1 + r - \frac{r}{n} + \sqrt{\frac{1}{n} - \frac{2}{n^2}} \right) \leq \frac{n}{1-r} \left(1 + r + \sqrt{\frac{1}{n}} \right),\end{aligned}$$

which gives the result.

Now, we show (ii). Step 1. We first show the right-hand side inequality:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} C_{n,r} \leq \frac{1+r}{1-r},$$

which becomes obvious since

$$\|D\|_{K_{B_\sigma} \rightarrow H^2} \leq \frac{n}{1-r} \left(1 + r + \sqrt{\frac{1}{n}} \right).$$

Step 2. We now show the left-hand side inequality:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} C_{n,r} \geq \frac{1+r}{1-r}.$$

More precisely, we show that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \|D\|_{K_{b_r^n} \rightarrow H^2} \geq \frac{1+r}{1-r}.$$

Let $f \in K_{b_r^n}$. Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1)(f, e_k)_{H^2} \frac{b'_r}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1)(f, e_k)_{H^2} e_k \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1)(f, e_k)_{H^2} b_r^{k-1}, \end{aligned}$$

which gives

$$f' = -b'_r \left[\frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{z-r} \sum_{k=2}^n (k-1)(f, e_k)_{H^2} b_r^{k-1} \right]. \quad (3)$$

Now using the change of variables $v = b_r(u)$, we get

$$\begin{aligned} \|f'\|_2^2 &= \int_{\mathbb{T}} |b'_r(u)| |b'_r(u)| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1)(f, e_k)_{H^2} b_r^{k-1} \right|^2 du \\ &= \int_{\mathbb{T}} |b'_r(b_r(u))| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v)-r} \sum_{k=2}^n (k-1)(f, e_k)_{H^2} v^{k-1} \right|^2 dv. \end{aligned}$$

But

$$b_r - r = \frac{r-z-r(1-rz)}{1-rz} = \frac{z(r^2-1)}{1-rz},$$

and

$$b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2},$$

which gives

$$\begin{aligned} \|f'\|_2^2 &= \frac{1}{1-r^2} \int_{\mathbb{T}} |(1-rv)^2| \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^n (k-1)(f, e_k)_{H^2} v^{k-1} \right|^2 dv \\ &= \frac{1}{(1-r^2)^2} \int_{\mathbb{T}} |(1-rv)^2| \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1-rv) \sum_{k=2}^n (k-1)(f, e_k)_{H^2} v^{k-2} \right|^2 dv, \end{aligned}$$

and

$$\|f'\|_2^2 = \frac{1}{(1-r^2)^2} \int_{\mathbb{T}} \left| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right|^2 dv. \quad (4)$$

In particular,

$$\begin{aligned} \frac{1}{n \|f\|_2} \left[\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2 + \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \right] &\geq \frac{1-r^2}{n} \frac{\|f'\|_2}{\|f\|_2} \\ &\geq \frac{1}{n \|f\|_2} \left[\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2 - \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \right]. \end{aligned} \quad (5)$$

Now, we notice that on one hand

$$\left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 \leq r(1+r) \left(\sum_{k=0}^{n-1} |(f, e_{k+1})_{H^2}|^2 \right)^{\frac{1}{2}} \leq r(1+r) \|f\|_2, \quad (6)$$

and on the other hand,

$$\begin{aligned} (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k &= (1-2rv + r^2 v^2) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \\ &= \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k - 2r \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^{k+1} + r^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^{k+2} \\ &= \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k - 2r \sum_{k=0}^{n-1} k(f, e_{k+1})_{H^2} v^k + r^2 \sum_{k=0}^n (k-1)(f, e_k)_{H^2} v^k \\ &= (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} \left[(k+1)(f, e_{k+2})_{H^2} - 2rk(f, e_{k+1})_{H^2} + r^2(k-1)(f, e_k)_{H^2} \right] v^k \\ &\quad - 2r \left[(f, e_2)_{H^2} v + (n-1)(f, e_n)_{H^2} v^{n-1} \right] + r^2 \left[(n-2)(f, e_{n-1})_{H^2} v^{n-1} + (n-1)(f, e_n)_{H^2} v^n \right], \end{aligned}$$

which gives

$$\begin{aligned} (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k &= (f, e_2)_{H^2} + 2[(f, e_3)_{H^2} - r(f, e_2)_{H^2}] v + \\ &\quad + \sum_{k=2}^{n-2} \left[(k+1)(f, e_{k+2})_{H^2} - 2rk(f, e_{k+1})_{H^2} + r^2(k-1)(f, e_k)_{H^2} \right] v^k + \\ &\quad + [r^2(n-2)(f, e_{n-1})_{H^2} - 2r(n-1)(f, e_n)_{H^2}] v^{n-1} + r^2(n-1)(f, e_n)_{H^2} v^n. \end{aligned} \quad (7)$$

Now, let $s = (s_n)_n$ be a sequence of even integers such that

$$\lim_{n \rightarrow \infty} s_n = \infty \text{ and } s_n = o(n) \text{ as } n \rightarrow \infty.$$

Then we consider the following function f in $K_{b_r^n}$:

$$f = e_n - e_{n-1} + e_{n-2} - e_{n-3} + \dots + (-1)^k e_{n-k} + \dots + e_{n-s} - e_{n-s-1} + e_{n-s-2} = \sum_{k=0}^{s+2} (-1)^k e_{n-k}.$$

Using (6) on one hand, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| r(1-rv) \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_2 = 0, \quad (8)$$

and applying (7) on the other hand, we obtain

$$\begin{aligned} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 &= |(f, e_2)_{H^2}|^2 + 4|(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \\ &\quad + |r^2(n-2)(f, e_{n-1})_{H^2} - 2r(n-1)(f, e_n)_{H^2}|^2 + r^4(n-1)^2 |(f, e_n)_{H^2}|^2 + \\ &\quad + \sum_{k=2}^{n-2} \left| (k+1)(f, e_{k+2})_{H^2} - 2rk(f, e_{k+1})_{H^2} + r^2(k-1)(f, e_k)_{H^2} \right|^2, \end{aligned}$$

which gives

$$\begin{aligned} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 &= |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 + \\ &\quad + \sum_{l=2}^{n-2} \left| (n-l+1)(f, e_{n-l+2})_{H^2} - 2r(n-l)(f, e_{n-l+1})_{H^2} + r^2(n-l-1)(f, e_{n-l})_{H^2} \right|^2, \end{aligned}$$

setting the change of index $l = n - k$ in the last sum. This finally gives

$$\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 = |r^2(n-2) + 2r(n-1)|^2 + r^4(n-1)^2 +$$

$$+ \sum_{l=2}^{s+1} \left| (n-l+1) + 2r(n-l) + r^2(n-l-1) \right|^2 + \left| (n-s-1) + 2r(n-s-2) \right|^2 + \left| n-s-2 \right|^2.$$

And

$$\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \left| r^2(n-2) + 2r(n-1) \right|^2 + r^4(n-1)^2 + \\ + s \left| (n-s) + 2r(n-s-1) + r^2(n-s-2) \right|^2 + \left| (n-s-1) + 2r(n-s-2) \right|^2 + \left| n-s-2 \right|^2.$$

In particular,

$$\left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq s \left| (n-s) + 2r(n-s-1) + r^2(n-s-2) \right|^2. \quad (9)$$

Passing after to the limit as $n \rightarrow \infty$ in (5), we obtain (using (8))

$$\frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2 \geq \liminf_{n \rightarrow \infty} \frac{1-r}{n} \frac{\|f'\|_2}{\|f\|_2} \quad (10) \\ \geq \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2.$$

This gives

$$\liminf_{n \rightarrow \infty} \frac{1-r}{n} \frac{\|f'\|_2}{\|f\|_2} = \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2. \quad (11)$$

Now, since

$$\|f\|_2^2 = s_n + 3,$$

using (9) we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ \geq \liminf_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} (\|f\|_2^2 - 3) \left| (n-s) + 2r(n-s-1) + r^2(n-s-2) \right|^2.$$

Since

$$\lim_{n \rightarrow \infty} \frac{3}{n^2 s_n^2} \left| (n-s) + 2r(n-s-1) + r^2(n-s-2) \right|^2 = 0,$$

we get

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2 \|f\|_2^2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2^2 \geq \\ \geq \liminf_{n \rightarrow \infty} \frac{3}{n^2 s_n^2} s_n^2 \left| (n-s_n) + 2r(n-s_n-1) + r^2(n-s_n-2) \right|^2 = \\ = \lim_{n \rightarrow \infty} \frac{1}{n^2} \left| (n-s_n) + 2r(n-s_n-1) + r^2(n-s_n-2) \right|^2 = \lim_{n \rightarrow \infty} \frac{1}{n^2} \left| n + 2rn + r^2n \right|^2 = (1+r)^4.$$

We can now conclude that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \|D\|_{K_{b_r^n} \rightarrow H^2} \geq \liminf_{n \rightarrow \infty} \frac{1-r}{n} \frac{\|f'\|_2}{\|f\|_2} = \\ = \frac{1}{1+r} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_2} \left\| (1-rv)^2 \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_2 \geq \frac{(1+r)^2}{1+r} = 1+r.$$

Step 3. Conclusion. Using both Step 1 and Step 2, we get

$$\limsup_{n \rightarrow \infty} \frac{1-r}{n} C_{n,r} = \liminf_{n \rightarrow \infty} \frac{1-r}{n} C_{n,r} = 1+r,$$

which means that the sequence $(\frac{1}{n}C_{n,r})_{n \geq 1}$ is convergent and

$$\lim_{n \rightarrow \infty} \frac{1}{n}C_{n,r} = \frac{1+r}{1-r}.$$

(a) Bernstein-type inequalities for K_B appeared as early as in 1991. There, the boundedness of $D : (K_B, \|\cdot\|_{H^p}) \rightarrow (H^p, \|\cdot\|_{H^p})$ was covered for the full range $1 \leq p \leq \infty$. In [60], the chief concern of K. Dyakonov was compactness (plus a new, simpler, proof of boundedness). Now, using both [140], (or equivalently M. Levin's inequality [61]) and complex interpolation, we could recover the result of K. Dyakonov for H^p spaces, $2 \leq p \leq \infty$ and our method could give a better numerical constant c_p in the inequality

$$\|f'\|_{H^p} \leq c_p \|B'\|_{\infty} \|f\|_{H^p}.$$

The case $1 \leq p \leq 2$ can be treated using the partial result of K. Dyakonov ($p=1$) and still complex interpolation.

(b) In the same spirit, it is also possible to generalize the above Bernstein-type inequality to the same class of rational functions f in $\mathbb{D}$, replacing the Hardy space H^2 by Besov spaces $B_{2,2}^s$, $s \in \mathbb{R}$, of all holomorphic functions $f = \sum_{k \geq 0} \hat{f}(k) z^k$ in $\mathbb{D}$ satisfying

$$\|f\|_{B_{2,2}^s} := \left(\sum_{k \geq 0} (k+1)^{2s} |\hat{f}(k)|^2 \right)^{1/2} < \infty.$$

The same spaces are also known as Dirichlet-Bergman spaces. (In particular, the classical Bergman space corresponds to $p = -\frac{1}{2}$ and the classical Dirichlet space corresponds to $p = \frac{1}{2}$). Using the above approach, one can show the sharpness of the growth order $\frac{n}{1-r}$ in the corresponding Bernstein-type inequality

$$\|f'\|_{B_{2,2}^s} \leq c_s \frac{n}{1-r} \|f\|_{B_{2,2}^s}, \quad (12)$$

(at least for integers values of s).

(c) One can also show an inequality

$$\|f\|_{B_{2,2}^s} \leq c'_s \left(\frac{n}{1-r} \right)^s \|f\|_{H^2}, \quad (13)$$

for $s \geq 0$ and the same class of functions (essentially, this inequality can be found in [142]), and show the sharpness of the growth order $(\frac{n}{1-r})^s$ (at least for integers values of s). An application of this inequality lies in constrained H^∞ interpolation in weighted Hardy and Bergman spaces, see [65] and [63] for details.

Notice that already E. M. Dyn'kin (in [144]), and A. A. Pekarskii (in [1], [146] and [147]), studied Bernstein-type inequalities for rational functions in Besov and Sobolev spaces. In particular, they applied such inequalities to inverse theorems of rational approximation. Our approach is different and more constructive. We are able to obtain uniform bounds depending on the geometry of poles of order n , which allows us to obtain estimates which are asymptotically sharp.

Also, in [143] of K. Dyakonov, there are Bernstein-type inequalities involving Besov and Sobolev spaces that contain, as special cases, the earlier version from , Pekarskii's inequalities for rational functions, and much more. K. Dyakonov used those Bernstein-type inequalities to "interpolate", in a sense, between the polynomial and rational inverse approximation theorems (in response to a question raised by E. M. Dyn'kin). Finally, he has recently studied the "reverse Bernstein inequality" in K_B ; this is done in [143].

(d) The above comments can lead to wonder what happens if we replace Besov

spaces $B_{2,2}^s$ by other Banach spaces, for example by W , the Wiener algebra of absolutely convergent Taylor series. In this case, we obtain

$$\|f\|_W \leq c(n, r) \|f\|_{H^2} \quad (14)$$

where $c(n, r) \leq c \left(\frac{n^2}{1-r} \right)^{\frac{1}{2}}$ and c is a numerical constant. We suspect that $\left(\frac{n^2}{1-r} \right)^{\frac{1}{2}}$ is the right growth order of $c(n, r)$. An application of this inequality to an estimate of the norm of the resolvent of an $n \times n$ power-bounded matrix T on a Banach space is given in [67]. Inequality (14), above, is deeply linked with the inequality

$$\|f'\|_{H^1} \leq \gamma n \|f\|_{H^\infty}, \quad (15)$$

through Hardy's inequality :

$$\|f\|_W \leq \pi \|f'\|_{H^1} + |f(0)|,$$

for all $f \in W$, (see [62]).

Inequality (15) is (shown and) used by R. J. LeVeque and L. N. Trefethen in [145] with $\gamma = 2$, and later by M. N. Spijker in [149] with $\gamma = 1$ (an improvement) so as to apply it to the Kreiss Matrix Theorem in which the power boundedness of $n \times n$ matrices is related to a resolvent condition on these matrices.

Section (4.2): A Bernstein Type Inequality to Rational Interpolation in the Dirichlet Space

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and let $\text{Hol}(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. Let also X and Y be two Banach spaces of holomorphic functions on the unit disc $\mathbb{D}$, $X, Y \subset \text{Hol}(\mathbb{D})$. Here and later on, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. We suppose that $n \geq 1$ is an integer, $r \in [0, 1)$ and we consider the two following problems.

Problem 1. Let $\mathcal{P}_n$ be the complex space of analytic polynomials of degree less or equal than n , and

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q} : q \in \mathcal{P}_n, d^\circ p < d^\circ q, q(\zeta) = 0 \Rightarrow |\zeta| \geq \frac{1}{r} \right\},$$

(where $d^\circ p$ means the degree of any $p \in \mathcal{P}_n$) be the set of all rational functions in $\mathbb{D}$ of degree less or equal than $n \geq 1$, having at most n poles all outside of $\frac{1}{r}\mathbb{D}$. Notice that for $n = 1$, we get $\mathcal{R}_{n,0} = \mathcal{P}_{n-1}$. Our first problem is to search for the “best possible” constant $\mathcal{C}_{n,r}(X, Y)$ such that

$$\|f'\|_X \leq \mathcal{C}_{n,r}(X, Y) \|f\|_Y$$

for all $f \in \mathcal{R}_{n,r}$.

Problem 2. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a finite subset of $\mathbb{D}$. What is the best possible interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case? The case $X \subset Y$ is of no interest, and so one can suppose that either $Y \subset X$ or X and Y are incomparable. More precisely, our second problem is to compute or estimate the following interpolation constant

$$I(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \}.$$

We also define

$$\mathcal{I}_{n,r}(X, Y) = \sup \{ I(\sigma, X, Y) : \text{card } \sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma \}.$$

Bernstein-type inequalities for rational functions are applied

1.1. in matrix analysis and in operator theory (see “Kreiss Matrix Theorem” [145, 149] or [152, 67] for resolvent estimates of power bounded matrices),

1.2. to “inverse theorems of rational approximation” using the classical Bernstein decomposition (see [151, 14, 1]),

1.3. to effective H^∞ interpolation problems (see [65] and our Theorem (4.2.6) below in Subsection d), and more generally to our Problem 1.

We can give three main motivations for Problem 2.

2.1. It is explained in [65] (the case $Y = H^\infty$) why the classical interpolation problems, those of Nevanlinna-Pick (1908) and Carathéodory-Schur (1916) (see [62] for these two problems), on the one hand and Carleson’s free interpolation problem (1958) (see [63]) on the other hand, are of the nature of our interpolation problem.

2.2. It is also explained in [65] why this constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

2.3. It has already been proved in [65] that for $X = H^2$ and $Y = H^\infty$,

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq \mathcal{I}_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}. \quad (16)$$

The above estimate (16) answers a question of L. Baratchart (private communication), which is part of a more complicated question arising in an applied situation in [58] and [59]: given a set $\sigma \in \mathbb{D}$, how to estimate $I(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only?

Now let us define some Banach spaces X and Y of holomorphic functions in $\mathbb{D}$ which we will consider throughout this section. From now on, if $f \in \text{Hol}(\mathbb{D})$ and $k \in \mathbb{N}$, $\hat{f}(k)$ stands for the k^{th} Taylor coefficient of f .

1. The standard Hardy space $H^2 = H^2(\mathbb{D})$,

$$H^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{H^2}^2 = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty \right\}$$

where m stands for the normalized Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. An equivalent description of the space H^2 is

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^2} = \left(\sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

2. The standard Bergman space $L_a^2 = L_a^2(\mathbb{D})$,

$$L_a^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{L_a^2}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty \right\},$$

where A is the standard area measure, also defined by

$$L_a^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{L_a^2} = \left(\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty \right\}.$$

3. The analytic Besov space $B_{2,2}^{\frac{1}{2}}$ (also known as the standard Dirichlet space) defined by

$$B_{2,2}^{\frac{1}{2}} = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{B_{2,2}^{\frac{1}{2}}} = \left(\sum_{k \geq 0} (k+1) |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Then if $f \in B_{2,2}^{\frac{1}{2}}$, we have the following equality

$$\|f\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|f\|_{L_a^2}^2 + \|f\|_{H^2}^2, \quad (17)$$

which establishes a link between the spaces $B_{2,2}^{\frac{1}{2}}$ and L_a^2 .

Here and later on, the letter c denotes a positive constant that may change from one step to the next. For two positive functions a and b , we say that a is dominated by b , denoted by $a = O(b)$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a = O(b)$ and $b = O(a)$ hold.

Problem 1. Our first result (Theorem (4.2.5), below) is a partial case ($p = q = 2, s = \frac{1}{2}$) of the following K. Dyakonov's result [142]: if $p \in [1, \infty)$, $s \in (0, +\infty)$, $q \in [1, +\infty]$, then there exists a constant $c_{p,s} > 0$ such that

$$C_{n,r}(B_{p,p}^{s-1}, H^q) \leq c_{p,s} \sup \|B'\|_{H^\gamma}^s, \quad (18)$$

where γ is such that $\frac{s}{\gamma} + \frac{1}{q} = \frac{1}{p}$, and the supremum is taken over all finite Blaschke products B of order n with n zeros outside of $\frac{1}{r}\mathbb{D}$. Here $B_{p,p}^s$ stands for the Hardy-Besov space which consists of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{B_{p,p}^s} = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbb{D}} (1-|w|)^{(n-s)p-1} |f^{(n)}(w)|^p dA(w) < \infty.$$

For the (tiny) partial case considered here, our proof is different and the constant $c_{2,\frac{1}{2}}$, is asymptotically sharp as r tends to 1^- and n tends to $+\infty$.

Problem 2. Looking at 2.3, we replace the algebra H^∞ by the Dirichlet space $B_{2,2}^{\frac{1}{2}}$. We show that the “gap” between $X = H^2$ and $Y = H^\infty$ (see (16)) is asymptotically the same as the one which exists between $X = H^2$ and $Y = B_{2,2}^{\frac{1}{2}}$. In other words,

$$\mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \asymp \mathcal{I}_{n,r}(H^2, H^\infty) \asymp \sqrt{\frac{n}{1-r}}. \quad (19)$$

We first give some definitions introducing the main tools used in the proofs of Theorem (4.2.5) and Theorem (4.2.6). After that, we show these theorems.

From now on, if $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^n b_{\lambda_j}$$

is the corresponding finite Blaschke product where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$, $\lambda \in \mathbb{D}$. In Definitions (4.2.1), (4.2.2), (4.2.3) and in Remark (4.2.4) below, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$ and B_σ is the corresponding Blaschke product.

Definition (4.2.1)[150]: For $k \in [1, n]$, we set $f_k = \frac{1}{1-\bar{\lambda}_k z}$, and define the family $(e_k)_{1 \leq k \leq n}$, (which is known as Malmquist basis, see [43]), by

$$e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2}, \quad (20)$$

for $k \in [2, n]$; we have $\|f_k\|_2 = (1-|\lambda_k|^2)^{-\frac{1}{2}}$.

Definition (4.2.2)[150]: The model space K_{B_σ} . We define K_{B_σ} to be the n -dimensional space:

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2. \quad (21)$$

Definition (4.2.3)[150]: The orthogonal projection P_{B_σ} on K_{B_σ} . We define P_{B_σ} to be the orthogonal projection of H^2 on its n -dimensional subspace K_{B_σ} .

Remark (4.2.4)[150]: The Malmquist family $(e_k)_{1 \leq k \leq n}$ corresponding to σ is an orthonormal basis of K_{B_σ} . In particular,

$$P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k, \quad (22)$$

Where $(\cdot, \cdot)_{H^2}$ means the scalar product on H^2 .

Theorem (4.2.5)[150]: (i) Let $n \geq 1$ and $r \in [0, 1)$. We have

$$\tilde{a}(n, r) \sqrt{\frac{n}{1-r}} \leq C_{n,r}(L_a^2, H^2) \leq \tilde{A}(n, r) \sqrt{\frac{n}{1-r}}, \quad (23)$$

where

$$\tilde{a}(n, r) \geq \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}} \text{ and } \tilde{A}(n, r) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}.$$

(ii) Moreover, the sequence

$$\left(\frac{C_{n,r}(L_a^2, H^2)}{\sqrt{n}} \right)_{n \geq 1}$$

is convergent and there exists a limit

$$\lim_{n \rightarrow \infty} \frac{C_{n,r}(L_a^2, H^2)}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}}. \quad (24)$$

for all $r \in [0, 1)$.

Notice that it has already been proved in [64] that there exists a limit

$$\lim_{n \rightarrow \infty} \frac{C_{n,r}(H^2, H^2)}{n} = \frac{1+r}{1-r}, \quad (25)$$

for every r , $0 \leq r < 1$.

Proof. (i). 1) We first show the the right-hand side inequality of (23). Using both Cauchy-Schwarz inequality and the fact that $f'(k) = (k+1)f(k+1)$ for all $k \geq 0$, we get

$$\begin{aligned} \|f'\|_{L_a^2}^2 &= \sum_{k \geq 0} \frac{|f'(k)|^2}{k+1} = \sum_{k \geq 0} \frac{(k+1)^2 |f(k+1)|^2}{k+1} = \sum_{k \geq 1} k |f(k)|^2 \\ &\leq \left(\sum_{k \geq 1} k^2 |f(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k \geq 1} |f(k)|^2 \right)^{\frac{1}{2}} = \|f'\|_{H^2} \|f\|_{H^2} \leq C_{n,r}(H^2, H^2) \|f\|_{H^2}^2, \end{aligned}$$

and hence,

$$\|f\|_{L_a^2} \leq \sqrt{C_{n,r}(H^2, H^2)} \|f\|_{H^2},$$

which means

$$C_{n,r}(L_a^2, H^2) \leq \sqrt{C_{n,r}(H^2, H^2)}.$$

Then it remains to use [64]:

$$C_{n,r}(H^2, H^2) \leq \left(1 + r + \frac{1}{\sqrt{n}}\right) \frac{n}{1-r},$$

for all $n \geq 1$ and $r \in [0, 1)$.

2) The proof of the left-hand side inequality of (23) repeats the one of [64, (i)] (for the left-hand side inequality) excepted that this time, we replace the Hardy norm $\|\cdot\|_{H^2}$ by the

Bergman one $\|\cdot\|_{L_a^2}$. Indeed, we use the same test function $e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1}$ (the n^{th} vector of the Malmquist family associated with the one-point set $\sigma_{n,r} = \underbrace{\{r, r, \dots, r\}}_n$ see Definition

(4.2.5)) and show by the same changing of variable $\circ b_r$ (in the integral on the unit disc $\mathbb{D}$ which defines the L_a^2 -norm) that

$$\|e'_n\|_{L_a^2}^2 = \frac{n}{1-r} \left(1 - \frac{1-r}{n}\right),$$

which gives

$$C_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{n}{1-r}} \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}}.$$

Here are the details of the proof. We have $e_n \in K_{b_r^n}$ and $\|e_n\|_{H^2}$, (see [43], Malmquist-Walsh Lemma). Moreover,

$$e'_n = \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2} = -\frac{r}{(1-r^2)^{\frac{1}{2}}} b'_r b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b'_r b_r^{n-2},$$

since $b'_r = \frac{r^2-1}{(1-rz)^2}$. Then,

$$e'_n = b'_r \left[-\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],$$

and

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} |b'_r(w)|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r(w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r(w))^{n-2} \right|^2 dm(w) \\ &= \frac{1}{2\pi} \int_{\mathbb{D}} |b'_r(w)|^2 |(b_r(w))^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right|^2 dm(w) \end{aligned}$$

which gives, using the variables $u = b_r(w)$,

$$\|e'_n\|_{L_a^2}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 dm(u).$$

But $1-rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b'_r \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-r^2} (1-ru) \right|^2 dm(u) \\ &= \frac{1}{(1-r^2)} \frac{1}{2\pi} \int_{\mathbb{D}} |u^{n-2}|^2 |(-ru + (n-1)(1-ru))|^2 dm(u), \end{aligned}$$

which gives

$$\|e'_n\|_{L_a^2}^2 = \frac{1}{(1-r^2)^{\frac{1}{2}}} \|\varphi_n\|_2,$$

where $\varphi_n = z^{n-2}(-rz + (n-1)(1-rz))$. Expanding, we get

$$\varphi_n = z^{n-2}(-rz + n-1+rz - nrz) = z^{n-2}(-nrz + n-1) = (n-1)z^{n-2} - nrz^{n-1},$$

and

$$\begin{aligned} \|e'_n\|_{L_a^2}^2 &= \frac{1}{(1-r^2)} \left(\frac{(n-1)^2}{n-1} + \frac{n^2}{n} r^2 \right) = \frac{1}{(1-r^2)} (n(1+r) - 1) \\ &= \frac{n}{(1-r)(1+r)} \left((1+r) - \frac{1}{n} \right) = \frac{n}{(1-r)} \left(1 - \frac{1-r}{n} \right), \end{aligned}$$

which gives

$$C_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{n}{1-r}} \left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}}.$$

Proof of (ii). This is again the same proof as [64, (ii)] (the three steps). More precisely in Step 2, we use the same test function

$$f = \sum_{k=0}^{s+2} (-1)^k e_{n-k},$$

(where $s = (s_n)$ is defined in [64]), and the same changing of variable ϕ_{b_r} in the integral on $\mathbb{D}$. Here are the details of the proof.

Step 1. We first show the right-hand-side inequality:

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} C_{n,r}(L_a^2, H^2) \leq \sqrt{\frac{1+r}{1-r}},$$

which becomes obvious since

$$\frac{1}{\sqrt{n}} C_{n,r}(L_a^2, H^2) \leq \frac{1}{\sqrt{n}} \sqrt{C_{n,r}(H^2, H^2)}.$$

and

$$\frac{1}{\sqrt{n}} \sqrt{C_{n,r}(H^2, H^2)} \rightarrow \sqrt{\frac{1+r}{1-r}},$$

as n tends to infinity, see [152].

Step 2. We now show the left-hand-side inequality:

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} C_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{1+r}{1-r}}.$$

More precisely, we show that

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|D\|_{\left(K_{b_r^n} \| \cdot \|_{L_a^2}\right) \rightarrow H^2} \geq \sqrt{\frac{1+r}{1-r}}.$$

Let $f \in K_{b_r^n}$. Then,

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{r}{(1-rz)} e_1 + \sum_{k=2}^n (k-1) (f, e_k)_{H^2} \frac{b'_r}{b_r} e_k + r \sum_{k=2}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k \\ &= r \sum_{k=1}^n (f, e_k)_{H^2} \frac{1}{(1-rz)} e_k + \frac{1-r^2}{(1-rz)(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} e_k \\ &= \frac{r(1-r^2)^{\frac{1}{2}}}{(1-rz)^2} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{3}{2}}}{(1-rz)^2(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \\ &= -b'_r \left[\frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{(z-r)} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right]. \end{aligned}$$

Now using the change of variables $v = b_r(u)$, we get

$$\begin{aligned} \|f'_n\|_{L_a^2}^2 &= \int_{\mathbb{D}} |b'_r(u)|^2 \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} b_r^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{u-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} b_r^{k-1} \right|^2 du \\ &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{b_r(v)-r} \sum_{k=2}^n (k-1) (f, e_k)_{H^2} v^{k-1} \right|^2 dv. \end{aligned}$$

Now, $b_r - r = \frac{r-z-r(1-rz)}{1-rz} = \frac{z(r^2-1)}{1-rz}$, which gives

$$\begin{aligned}
\|f_n'\|_{L_a^2}^2 &= \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^n (k-1)(f, e_k)_{H^2} v^{k-1} \right|^2 dv \\
&= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=1}^n (f, e_k)_{H^2} v^{k-1} - (1-rv) \sum_{k=2}^n (k-1)(f, e_k)_{H^2} v^{k-2} \right|^2 dv \\
&= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k - (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right|^2 dv.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{\|f\|_{H^2} \sqrt{n(1+r)}} \left[\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} + \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right] \\
&\geq \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \frac{1}{\|f\|_{H^2} \sqrt{n(1+r)}} \left[\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} - \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \right].
\end{aligned} \tag{26}$$

Now,

$$\begin{aligned}
(1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k &= \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k - r \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^{k+1} \\
&= \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k - r \sum_{k=0}^{n-1} k(f, e_{k+1})_{H^2} v^k \\
&= (f, e_2)_{H^2} + 2(f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k+1)(f, e_{k+2})_{H^2} - rk(f, e_{k+1})_{H^2}] v^k + \\
&\quad + r[(f, e_2)_{H^2} v + (n-1)(f, e_n)_{H^2} v^{n-1}] \\
&= (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r(f, e_2)_{H^2}] v + \sum_{k=2}^{n-2} [(k+1)(f, e_{k+2})_{H^2} - rk(f, e_{k+1})_{H^2}] v^k + \\
&\quad - r(n-1)(f, e_n)_{H^2} v^{n-1},
\end{aligned}$$

which gives

$$\begin{aligned}
\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 &= |(f, e_2)_{H^2}|^2 + \frac{1}{2} |(f, e_3)_{H^2} - r(f, e_2)_{H^2}|^2 + \\
&\quad + \frac{1}{n} r^4 (n-1)^2 |(f, e_n)_{H^2}|^2 + \sum_{k=2}^{n-2} \left| (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right|^2.
\end{aligned} \tag{27}$$

On the other hand,

$$\left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} \leq r \left(\sum_{k=0}^{n-1} \frac{1}{k+1} |(f, e_{k+1})_{H^2}|^2 \right)^{\frac{1}{2}} \leq r \|f\|_{H^2}, \tag{28}$$

Now, let $s = (s_n)$ be a sequence of even integers such that

$$\lim_{n \rightarrow \infty} s_n = \infty \text{ and } s_n = o(n) \text{ as } n \rightarrow \infty.$$

Then we consider the following function f in $K_{b_r^n}$:

$$f = \sum_{k=0}^{s+2} (-1)^k e_{n-k}.$$

Applying (27) with such an f , we get

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 = r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{n-2} (n-l+1) \left| (f, e_{n-l+2})_{H^2} - \frac{r(n-l)}{n-l+1} (f, e_{n-l+1})_{H^2} \right|^2$$

setting the change of index $l = n - k$ in the last sum. This finally gives

$$\begin{aligned} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 &= r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left| 1 + \frac{r(n-l)}{n-l+1} \right|^2 = \\ &= r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left[1 + r \left(1 - \frac{1}{n-l+1} \right) \right]^2, \end{aligned}$$

and

$$\begin{aligned} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 &\geq \\ &\geq r^4 \frac{(n-1)^2}{n} + (s+1-2+1)(n-(s+1)+1) \left[1 + r \left(1 - \frac{1}{n-(s+1)+1} \right) \right]^2 \\ &= r^4 \frac{(n-1)^2}{n} + s(n-s) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2. \end{aligned}$$

In particular,

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq s(n-s) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2.$$

Now, since $\|f\|_{H^2}^2 = s_n + 3$, we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 &\geq \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n \|f\|_{H^2}^2} \|f\|_{H^2}^2 (n - \|f\|_{H^2}^2) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 = \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{s_n}{n} \right) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 = (1+r)^2. \end{aligned}$$

On the other hand, applying (28) with this f , we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} \|f\|_{H^2}} \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0$$

Thus, we can conclude passing after to the limit as n tends to $+\infty$ in (26), that

$$\liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} = \frac{1}{\sqrt{1+r}} \liminf_{n \rightarrow \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1)(f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq \frac{1+r}{\sqrt{1+r}} = \sqrt{1+r},$$

and

$$\liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \|D\|_{K_{b_r^n} \rightarrow H^2} \geq \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \sqrt{1+r}.$$

Step 3. Conclusion. Using both Step 1 and Step 2, we get

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} C_{n,r}(L_a^2, H^2) = \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} C_{n,r}(L_a^2, H^2) = 1+r,$$

which means that the sequence $\left(\frac{1}{\sqrt{2}} C_{n,r}(L_a^2, H^2) \right)_{n \geq 1}$ is convergent and

$$\lim_{n \rightarrow \infty} \sqrt{\frac{1}{n}} C_{n,r}(L_a^2, H^2) = \sqrt{\frac{1+r}{1-r}}.$$

Theorem (4.2.6)[150]: Let $n \geq 1$, and $r \in [0,1)$. Then,

$$\mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \leq \left[(C_{n,r}(L_a^2, H^2))^2 + 1 \right]^{\frac{1}{2}}. \quad (29)$$

Let $\lambda \in \mathbb{D}$ and the corresponding one-point interpolation set $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n$. We have,

$$I(\sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}}) \geq \sqrt{\frac{n}{1-|\lambda|}} \left[\frac{(1+|\lambda|)^2 - \frac{2}{n} - \frac{2|\lambda|}{n}}{2(1+|\lambda|)} \right]^{\frac{1}{2}}. \quad (30)$$

In particular,

$$\left[\frac{1+r}{2} \left(1 - \frac{1}{n} \right) \right]^{\frac{1}{2}} \sqrt{\frac{n}{1-r}} \leq \mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \leq \left(1+r + \frac{1}{\sqrt{n}} + \frac{1-r}{n} \right)^{\frac{1}{2}} \sqrt{\frac{n}{1-r}}, \quad (31)$$

$$\sqrt{\frac{1+r}{2}} \leq \liminf_{n \rightarrow \infty} \frac{\mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}})}{\sqrt{n}} \leq \limsup_{n \rightarrow \infty} \frac{\mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}})}{\sqrt{n}} \leq \sqrt{\frac{1+r}{1-r}}, \quad (32)$$

and

$$\frac{\sqrt{2}}{2} \leq \liminf_{r \rightarrow 1^-} \liminf_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \leq \limsup_{r \rightarrow 1^-} \limsup_{n \rightarrow \infty} \sqrt{\frac{1-r}{n}} \mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \leq \sqrt{2}. \quad (33)$$

Proof. Proofs of inequality (29) and of the right-hand side inequality of (31). Let σ be a sequence in $\mathbb{D}$, and $B = B_\sigma$ the finite Blaschke product corresponding to σ . If $f \in H^2$, we use the same function g as in [65] which satisfies $g|_\sigma = f|_\sigma$. More precisely, let $g = P_B f \in K_B$ (see Definitions (4.2.2), (4.2.3) and Remark (4.2.4) above for the definitions of K_B and K_B). Then $g - f \in BH^2$ and using the definition of $\mathcal{C}_{n,r}(L_a^2, H^2)$

$$\|g'\|_{L_a^2}^2 \leq (\mathcal{C}_{n,r}(L_a^2, H^2))^2 \|g\|_{H^2}^2.$$

Now applying the identity (17) to g we get

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 \leq [(\mathcal{C}_{n,r}(L_a^2, H^2))^2 + 1] \|g\|_{H^2}^2.$$

Using the fact that $\|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2}$, we finally get

$$\|g\|_{B_{2,2}^{\frac{1}{2}}} \leq [(\mathcal{C}_{n,r}(L_a^2, H^2))^2 + 1]^{\frac{1}{2}} \|f\|_{H^2},$$

and as a result,

$$I(\sigma, H^2, B_{2,2}^{\frac{1}{2}}) \leq [(\mathcal{C}_{n,r}(L_a^2, H^2))^2 + 1]^{\frac{1}{2}}.$$

It remains to apply the right-hand side inequality of (23) in Theorem (4.2.5) to show the right-hand side one of (31).

Proof of inequality (30). 1) We use the same test function

$$f = \sum_{k=0}^{n-1} (1-|\lambda|)^{\frac{1}{2}} b_\lambda^k (1-\bar{\lambda}z)^{-1},$$

as the one used in the proof of [65] (the lower bound[65]). f being the sum of n elements of H^2 which are an orthonormal family known as Malmquist's basis (associated with $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n$, see Remark (4.2.4) above or [43]), we have $\|f\|_{H^2}^2 = n$.

2) Since the spaces H^2 and $B_{2,2}^{\frac{1}{2}}$ are rotation invariant, we have $I(\sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}}) = I(\sigma_{n,\mu}, H^2, B_{2,2}^{\frac{1}{2}})$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|f\|_{B_{2,2}^{\frac{1}{2}}/b_\lambda^n B_{2,2}^{\frac{1}{2}}}$ consider g such that $f - g \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $f \circ b_\lambda - g \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we notice that

$$\begin{aligned}\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 &= \|(g \circ b_\lambda)'\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|b_\lambda \cdot (g' \circ b_\lambda)\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \\ &= \int_{\mathbb{D}} |b_\lambda(u)|^2 |g'(b_\lambda(u))|^2 du + \|g \circ b_\lambda\|_{H^2}^2 = \int_{\mathbb{D}} |g'(w)|^2 dw + \|g \circ b_\lambda\|_{H^2}^2,\end{aligned}$$

using the changing of variable $w = b_\lambda(u)$. We get

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g'\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|g\|_{B_{2,2}^{\frac{1}{2}}}^2 + \|g \circ b_\lambda\|_{H^2}^2 - \|g\|_{H^2}^2$$

and

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g\|_{H^2}^2 + \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 \geq \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2.$$

Now, we notice that

$$\begin{aligned}f \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}b_\lambda(z)} = (1-|\lambda|^2)^{-\frac{1}{2}} \left(1 + (1-\bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1-r^2)^{-\frac{1}{2}} \left(1 + (1+r) \sum_{k=1}^{n-1} z^k + r z^n \right).\end{aligned}$$

4) Next,

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 = \sum_{k \geq 1} k |g \circ b_\lambda(k)|^2 \geq \sum_{k=1}^{n-1} k |g \circ b_\lambda(k)|^2 = \sum_{k=1}^{n-1} k |f \circ b_\lambda(k)|^2,$$

since $g \circ b_\lambda(k) = f \circ b_\lambda(k)$, $\forall k \in [0, n-1]$. This gives

$$\begin{aligned}\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 &\geq \frac{1}{1-r^2} \left((1+r)^2 \sum_{k=1}^{n-1} k \right) = \\ &= \frac{(1+r)^2}{1-r^2} \frac{n(n-1)}{2} = \frac{1+r}{1-r} \frac{n(n-1)}{2} = \frac{1+r}{1-r} \frac{(n-1)}{2} \|f\|_{H^2}^2,\end{aligned}$$

for all $n \geq 2$ since $\|f\|_{H^2}^2 = n$. Finally,

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 \geq \frac{n}{1-r} \frac{1+r}{2} \left(1 - \frac{1}{n} \right) \|f\|_{H^2}^2.$$

In particular,

$$\mathcal{I}_{n,r}(H^2, B_{2,2}^{\frac{1}{2}}) \geq \sqrt{\frac{n}{1-r}} \left[\frac{1+r}{2} \left(1 - \frac{1}{n} \right) \right]^{\frac{1}{2}}.$$

Extension of Theorem (4.2.5) to spaces $B_{2,2}^s$, $s \geq 0$. Using the techniques developped in the proof of our Theorem (4.2.5) (combined with complex interpolation (between Banach spaces) and a reasoning by induction), it is possible both to precise the sharp numerical constant $c_{2,s}$ in K. Dyakonov's result (18) (mentioned above in paragraph d. of the Introduction) and to show the asymptotic sharpness (at least for $s \in \mathbb{N} \cup \frac{1}{2}\mathbb{N}$) of the right-hand side inequality of (18). In the same spirit, we would obtain that there exists a limit:

$$\lim_{n \rightarrow \infty} \frac{C_{n,r}(B_{2,2}^{s-1}, H^2)}{n^s} = \left(\frac{1+r}{1-r} \right)^s. \quad (34)$$

Our Theorem (4.2.5) corresponds to the case $s = \frac{1}{2}$.

Extension of Theorem (4.2.6) to spaces $B_{2,2}^s$, $s \geq 0$. The proof of the upper bound in our Theorem B can be extended so as to give an upper (asymptotic) estimate of the interpolation constant $\mathcal{I}_{n,r}(H^2, B_{2,2}^s)$, $s \geq 0$. More precisely, applying K. Dyakonov's result (18) (mentioned above in paragraph d. of the Introduction) we get

$$\mathcal{I}_{n,r}(H^2, B_{2,2}^s) \leq \tilde{c}_s \left(\frac{n}{1-r} \right)^s, \text{ with } \tilde{c}_s \asymp c_{2,s}, \quad (35)$$

where $c_{2,s}$ is defined in (18) and precised in (34). Looking at the above comment 1, $\tilde{c}_s \asymp (1+r)^s$ for sufficiently large values of n . Our Theorem (4.2.6) corresponds again to the case $s = \frac{1}{2}$. In this Theorem B, we show the sharpness of the right-hand side inequality in (35) for $s = \frac{1}{2}$. However, for the general case $s \geq 0$, the asymptotic sharpness of $(\frac{n}{1-r})^s$ as $r \rightarrow 1^-$ and $n \rightarrow \infty$ is less obvious. Indeed, the key of the proof (for the sharpness) is based on the property that the Dirichlet norm (the one of $B_{2,2}^{1/2}$) is “nearly” invariant composing by an elementary Blaschke factor b_λ , as this is the case for the H^∞ norm. A conjecture given by N. K. Nikolski is the following:

$$\mathcal{I}_{n,r}(H^2, B_{2,2}^s) \asymp \begin{cases} \frac{n^s}{\sqrt{1-r}} & \text{if } s \geq \frac{1}{2} \\ (\frac{n}{1-r})^s & \text{if } s \leq \frac{1}{2} \end{cases}, \quad (36)$$

and is due to the position of the spaces $B_{2,2}^s$, $s \geq 0$ with respect to the algebra H^∞ .

Section (4.3): Rational Functions in Weighted Bergman Spaces

Estimates of the norms of derivatives for polynomials and rational functions (in different functional spaces) is a classical topic of complex analysis (see surveys by A.A. Gonchar [7], V.N. Rusak [8], and P. Borwein and T. Erdélyi [140]). Such inequalities have applications in many domains of analysis; to mention just some of them: 1) matrix analysis and in operator theory (see “Kreiss Matrix Theorem” [145, 149] or [152, 67] for resolvent estimates of power bounded matrices), 2) inverse theorems of rational approximation (see [151, 14, 1]), 3) effective Nevanlinna–Pick interpolation problems (see [65, 53]).

Here, we present Bernstein-type inequalities for rational functions f of degree n with poles in $\{z : |z| > 1\}$, involving Hardy norms and weighted Bergman norms. Let $\mathcal{P}_n$ be the complex space of polynomials of degree less or equal to $n \geq 1$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ its closure. Given $r \in [0, 1)$, we define

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q} : p, q \in \mathcal{P}_n, d^\circ p < d^\circ q, q(\zeta) \neq 0, |\zeta| < \frac{1}{r} \right\},$$

(where $d^\circ p$ denotes the degree of $p \in \mathcal{P}_n$), the set of all rational functions in $\mathbb{D}$ of degree less or equal than $n \geq 1$, having at most n poles all outside of $\frac{1}{r}\mathbb{D}$. Notice that for $r = 0$, we get $\mathcal{R}_{n,0} = \mathcal{P}_{n-1}$.

We denote by $\text{Hol}(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. From now on, if $f \in \text{Hol}(\mathbb{D})$ then for every $\rho \in (0, 1)$ we define

$$f_\rho : \xi \mapsto f(\rho\xi), \quad \xi \in \frac{1}{\rho}\mathbb{D}.$$

We consider the two following scales of Banach spaces $X \subset \text{Hol}(\mathbb{D})$.

a. The Hardy spaces $H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$:

$$H^p = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{H^p}^p = \sup_{0 \leq \rho < 1} \int_{\mathbb{T}} |f_\rho(\xi)|^p dm(\xi) < \infty \right\},$$

where m stands for the normalized Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. As usual, we denote by H^∞ the space of all bounded analytic functions in $\mathbb{D}$.

b. The radial weighted Bergman spaces $L_a^p(w)$, $1 \leq p < \infty$ (where "a" means analytic),

$$L_a^p(w) = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{L_a^p(w)}^p = \int_0^1 \rho w(\rho) \int_{\mathbb{T}} |f_\rho(\zeta)|^p dm(\zeta) d\rho < \infty \right\},$$

where the weight w satisfies $w \geq 0$ and $\int_0^1 w(\rho) d\rho < \infty$. For the classical power weights $w(\rho) = w_\beta(\rho) = (1-\rho)^\beta$, $\beta > -1$, we have $L_a^p(w_\beta) = L_a^p((1-|z|)^\beta dA(z))$, A being the normalized area measure on $\mathbb{D}$.

For general properties of these spaces we refer to [156, 157].

From now on, for two positive functions a and b , we say that a is dominated by b , denoted by $a \lesssim b$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$.

By Bernstein-type inequalities for rational functions one usually understands the inequalities of the form

$$\|f'\|_X \leq \phi_{X,Y}(n) \|f\|_Y, \quad f \in \mathcal{R}_n, \quad (37)$$

where $\mathcal{R}_n$ is the set of all proper rational functions of degree at most n with the poles in $\{|z| > 1\}$, X and Y are some normed spaces of functions analytic in the unit disc, and ϕ is some increasing (often polynomially growing) function. Thus, for a given pair of the function spaces X and Y , the question is to determine the dependence on n for the norm of the differentiation operator $(\mathcal{R}_n, \|\cdot\|_X)$ to Y . Bernstein-type inequalities of E.P. Dolzhenko [18] and A.A. Pekarskii [1] are of this form; e.g., it is shown in [18] that

$$\|f\|_{H_1^1} \leq c_1 n \|f\|_\infty, \quad \|f\|_{B_{2,2}^{1/2}} \leq c_2 n^{1/2} \|f\|_\infty, \quad f \in \mathcal{R}_n,$$

where H_1^1 is the Hardy–Sobolev space, and $B_{2,2}^{1/2}$ is the Besov (or Dirichlet) space. Let us also mention that this problem is a part of a more general one given by (see [155]).

Looking at (37), we notice that for some choices of X and Y , we have $\phi_{X,Y}(n) = +\infty$ for every $n = 1, 2, \dots$. Indeed, it may happen for instance when the poles of our function f are allowed to be arbitrary close to the torus $\mathbb{T}$: we can observe this phenomenon for example in the special case $X = Y = H^p$, $1 \leq p \leq +\infty$ but also when $X = Y = L_a^p(w)$, $1 \leq p \leq +\infty$. This observation leads us to come back on the problem in (37) and to state it more generally: that is replacing $\mathcal{R}_n$ by $\mathcal{R}_{n,r}$ (for any fixed $r \in [0, 1)$) and $\phi_{X,Y}(n)$ by $\phi_{X,Y}(n, r)$ so that to focus on this phenomenon of “natural dependence on the parameter r ”. For most of the classical cases already studied by others (for instance E. P. Dolzhenko [18], A. A. Pekarskii [1], V.V. Peller [14]) the spaces X and Y are such that $\sup_{r \in (0,1)} \phi_{X,Y}(n, r) < +\infty$: in this case we can set $\phi_{X,Y}(n) = \sup_{r \in (0,1)} \phi_{X,Y}(n, r)$. As a consequence, if $\sup_{r \in (0,1)} \phi_{X,Y}(n, r) = +\infty$, it may be of interest to search (as a continuation of the investigations of the second author [64, 150]) for the “best possible” $\phi_{X,Y}(n, r)$ in an asymptotically sense, that is to say as $n \rightarrow \infty$ and $r \rightarrow 1^-$. This question has already been answered for the case $X = Y = H^p$, $1 \leq p \leq +\infty$ by K. M. Dyakonov [34] see (38) below. In this section, we answer the same question for the case $X = Y = L_a^p(w)$, $1 \leq p \leq +\infty$. Let us give a general formulation of our problem for the special case $X = Y$ for which we set $C_{n,r}(X) = \phi_{X,Y}(n, r)$: given a Banach space X of holomorphic functions in $\mathbb{D}$, we are searching for the best possible constant $C_{n,r}(X)$ such that

$$\|f'\|_X \leq C_{n,r}(X) \|f\|_X, \quad f \in \mathcal{R}_{n,r}.$$

For the case where $X = H^p$ is a Hardy space, an estimate which gives a correct order of growth for $C_{n,r}(X)$ was obtained by K.M. Dyakonov [34] (as a very special case of more general results): for any $p \in [1, \infty]$ there exist positive constants A_p and B_p such that

$$A_p \frac{n}{1-r} \leq C_{n,r}(H^p) \leq B_p \frac{n}{1-r} \quad (38)$$

for all $n \geq 1$ and $r \in [0, 1)$. More precisely, the upper estimate for $p \in (1, +\infty)$ is treated in [34], the case $p = 1$, in [34], and the case $p = +\infty$ (known much earlier) is given in [140]. The below estimate follows trivially when applying the differentiation operator to the test function $f(z) = (1 - rz)^{-n}$.

For the case $p = 1$ an asymptotically sharp result was obtained later in [64]: for any $r \in (0, 1)$ there exists the limit

$$\lim_{n \rightarrow \infty} \frac{C_{n,r}(H^2)}{n} = \frac{1+r}{1-r}.$$

Related results about Bernstein-type inequalities in a more general setting of the so-called model or star invariant subspaces may be found in [142], [60], and [55, 154].

We obtain estimates for the derivatives of rational functions with respect to weighted Bergman norms. It turns out that there is an essential difference between slowly (polynomially) decreasing weights and fast (superpolynomially) decreasing weights. In the first case we have a two-sided estimate analogous to (38), while in the second case only the above estimate remains true. Let us give the precise definitions. Recall that w is always an integrable nonnegative function on $(0, 1)$.

Definition (4.3.1)[153]: The weight w is said to be γ -polynomially decreasing if there exists $\gamma > 0$ such that

$$\rho \mapsto (1 - \rho)^{-\gamma} w(\rho),$$

is increasing on $[r_0, 1)$ for some $0 \leq r_0 < 1$. We say that w is polynomially decreasing if it is γ -polynomially decreasing for some $\gamma > 0$.

Definition (4.3.2)[153]: The weight w is said to be super-polynomially decreasing if for any $\gamma > 0$ there exists $r(\gamma) \in (0, 1)$ such that the function

$$\rho \mapsto (1 - \rho)^{-\gamma} w(\rho),$$

decreases on the interval $[r(\gamma), 1)$.

Typical example of the weights from the first class are given by $w(r) = (1 - r)^\beta$, $\beta > -1$, or $w(r) = (1 - r)^\beta (|\log(1 - r)| + 1)^\gamma$, $\beta > -1$, $\gamma \in \mathbb{R}$. The weights $w(r) = \exp(-c(1 - r)^{-\gamma})$, $c > 0$, $\gamma > 0$ are super-polynomially decreasing.

Our first result may be considered as an analogue of Dyakonov's theorem for the radial weighted Bergman spaces.

Theorem (4.3.3)[153]: Let $1 \leq p < \infty$ and let w be an integrable nonnegative function on $[0, 1)$. Then there exists a positive constant K depending only on p (but not on the weight w) such that

$$C_{n,r}(L_a^p(w)) \leq K \frac{n}{1-r} \quad (39)$$

for all $r \in [0, 1)$ and $n \geq 1$. Moreover, if we fix $r \in (0, 1)$ and let n tend to infinity, then we have

$$\frac{\tilde{K}r}{1-r} \leq \liminf_{n \rightarrow \infty} \frac{C_{n,r}(L_a^p(w))}{n} \leq \limsup_{n \rightarrow \infty} \frac{C_{n,r}(L_a^p(w))}{n} \leq \frac{K}{1-r}, \quad (40)$$

where $\tilde{K}$ is, as K , a positive constant depending only on p .

Proof. First, we notice that for any $0 \leq \alpha < 1$,

$$\|f\|_{L_a^p(w)}^p \asymp \iint_{u \in C_\alpha} \rho |f(\rho \xi)|^p w(\rho) dm(\xi) d\rho \quad (41)$$

for all $f \in L_a^p(w)$, where $C_\alpha = \{z : \alpha < |z| < 1\}$. Let $f \in \mathcal{R}_{n,r}$ with $r \in [0,1)$ and $n \geq 1$. Using (41) with $\alpha = \frac{1}{2}$ we get

$$\|f'\|_{L_a^p(w)}^p \asymp \iint_{\rho \xi \in C_{1/2}} |f'(\rho \xi)|^p w(\rho) dm(\xi) d\rho = \int_{\frac{1}{2}}^1 \rho w(\rho) \frac{1}{\rho^p} \left(\|f_\rho\|_{H^p}^p \right) d\rho.$$

Now using the fact that $f_\rho \in \mathcal{R}_{n,\rho r} \subset \mathcal{R}_{n,r}$ for every $\rho \in (0,1)$, we get

$$\begin{aligned} \int_{\frac{1}{2}}^1 \rho w(\rho) \frac{1}{\rho^p} \left(\|f_\rho\|_{H^p}^p \right) d\rho &\leq (2C_{n,r}(H^p))^p \int_{\frac{1}{2}}^1 \rho w(\rho) \left(\|f_\rho\|_{H^p} \right)^p d\rho \\ &\asymp (C_{n,r}(H^p))^p \|f\|_{L_a^p(w)}^p. \end{aligned}$$

In particular, using the right-hand side inequality of (38), we get

$$C_{n,r}(L_a^p(w)) \leq K_p \frac{n}{1-r}$$

for all $p \in [1, \infty)$, and $\beta \in (-1, \infty)$, where K_p is a constant depending on p only.

Now, let us show (40). Let

$$f_n(z) = \frac{1}{(1-rz)^n} \in \mathcal{R}_{n,r},$$

and $D = \{z \in \mathbb{D} : |1-rz| \leq 2|1-r|\}$. We claim that

$$\|f_n\|_{L_a^p(w)}^p \sim \int_D |f_n(z)|^p w(z) dA(z), \quad n \rightarrow \infty,$$

and, analogously,

$$\|f_n'\|_{L_a^p(w)}^p \sim \int_D |f_n'(z)|^p w(z) dA(z), \quad n \rightarrow \infty.$$

Indeed, by a very rough estimate

$$\int_{\mathbb{D} \setminus D} |f_n(z)|^p w(z) dA(z) \leq \frac{C_1}{2^{pn}(1-r)^{pn}},$$

where $C_1 > 0$ depends only on w . On the other hand, if we put $\tilde{D} = \{z \in \mathbb{D} : |1-rz| \leq \frac{3}{2}|1-r|\}$, then

$$\int_D |f_n(z)|^p w(z) dA(z) \geq \frac{1}{(3/2)^{pn}(1-r)^{pn}} \int_{\tilde{D}} w(z) dA(z).$$

Since r (thus D and $\tilde{D}$) are fixed we see that

$$\frac{1}{2^{pn}(1-r)^{pn}} = o\left(\frac{1}{(3/2)^{pn}(1-r)^{pn}} \int_{\tilde{D}} w(z) dA(z)\right), \quad n \rightarrow \infty.$$

Thus,

$$\frac{\|f_n'\|_{L_a^p(w)}^p}{\|f_n\|_{L_a^p(w)}^p} \sim \int_D |f_n'(z)|^p w(z) dA(z) / \int_D |f_n(z)|^p w(z) dA(z).$$

Obviously,

$$\begin{aligned} \int_D |f_n'(z)|^p w(z) dA(z) &= \int_D \frac{n^p r^p}{|1-rz|^{pn+p}} w(z) dA(z) \\ &\geq \frac{n^p r^p}{2^p (1-r)^p} \int_D \frac{1}{|1-rz|^{pn}} w(z) dA(z) \geq \frac{n^p r^p}{2^p (1-r)^p} \int_D |f_n(z)|^p w(z) dA(z). \end{aligned}$$

Thus,

$$\liminf_{n \rightarrow \infty} \frac{\|f'_n\|_{L^p_a(w)}}{\|f_n\|_{L^p_a(w)}} \geq \frac{r}{2(1-r)}.$$

Lemma (4.3.4)[153]: Let $r \in [0,1)$ and $t \geq 1$. We set

$$I(t, r) = \int_{\mathbb{T}} |1 - r\xi|^{-t} dm(\xi) \text{ and } \varphi_r(t) = \int_{\mathbb{T}} |1 + r\xi|^t dm(\xi).$$

Then,

$$I(t, r) = \frac{1}{(1-t^2)^{t-1}} \varphi_r(t-2)$$

for every $t \geq 2$, and $t \mapsto \varphi_r(t)$ is an increasing function on $[0, +\infty)$ for every $r \in [0,1)$. Moreover, both

$$r \mapsto \varphi_r(t-2) \text{ and } r \mapsto I(t, r),$$

are increasing on $[0,1)$, for all $t \geq 0$.

Proof. Indeed, supposing that $t \geq 2$, we can write

$$I(t, r) = \frac{1}{1-r^2} \int_{\mathbb{T}} |b'_r(\xi)| \frac{1}{|1-r\xi|^{t-2}} dm(\xi),$$

(where $b_r(z) = \frac{r-z}{1-rz}$). Using the fact that $b_r \circ b_r(z) = z$ and changing the variable in the above integral we get

$$\begin{aligned} I(t, r) &= \frac{1}{1-r^2} \int_{\mathbb{T}} |b'_r(\xi)| \frac{1}{|1-rb_r \circ b_r(\xi)|^{t-2}} dm(\xi) \\ &= \frac{1}{1-r^2} \int_{\mathbb{T}} |b'_r(\xi)| \frac{1}{|1-rb_r(\xi)|^{t-2}} dm(\xi) = \frac{1}{(1-t^2)^{t-1}} \varphi_r(t-2), \end{aligned}$$

since $1-rb_r(z) = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$. Now,

$$\begin{aligned} \varphi_r(t) &= \int_0^{2\pi} \exp\left(\frac{t}{2} \ln(1+r^2-2r \cos s)\right) ds, \\ \varphi'_r(t) &= \frac{1}{4} \int_0^{2\pi} \ln(1+r^2+2r \cos s) \exp\left(\frac{t}{2} \ln(1+r^2+2r \cos s)\right) ds, \end{aligned}$$

and

$$\varphi''_r(t) = \frac{1}{4} \int_0^{2\pi} [\ln(1+r^2-2r \cos s)]^2 \exp\left(\frac{t}{2} \ln(1+r^2-2r \cos s)\right) ds \geq 0,$$

for every $t \geq 0$, $r \in [0,1)$. Thus, φ_r is a convex function on $[0, \infty)$ and φ'_r is increasing on $[0, \infty)$ for all $r \in [0,1)$. Moreover,

$$\varphi'_r(0) = \frac{1}{4} \int_0^{2\pi} \ln(1+r^2-2r \cos s) ds = 0.$$

Thus,

$$\varphi'_r(t) = \varphi'_r(0) = 0, \quad \forall t \in [0, \infty), \quad r \in [0,1),$$

and so φ_r is increasing on $[0, \infty)$. The fact that

$$r \mapsto I(t, r),$$

is increasing on $[0,1)$ for all $t \geq 0$ is obvious since

$$I(t, r) = \left\| \frac{1}{(1-rz)^{t/2}} \right\|_{H^2}^2 = \sum_{k \geq 0} a_k^2(t) r^{2k},$$

where $a_k(t)$ is the k th Taylor coefficient of $(1-z)^{-t/2}$. The same reasoning gives that $r \mapsto \varphi_r(t)$ is increasing on $[0,1)$.

Lemma (4.3.5)[153]: If for some $r_0 \in [0,1)$ and $\gamma > 0$ the function $\frac{w(\rho)}{(1-\rho^2)^\gamma}$ is increasing on $[r_0,1)$, then

$$\int_r^1 \rho w(\rho) I(t, r\rho) d\rho \asymp \int_{r_0}^1 \rho w(\rho) I(t, r\rho) d\rho,$$

for all t such that $t \geq \gamma + 3$ and for all $r \geq r_0$, with constants independent on t .

Proof. Clearly,

$$\int_{r_0}^1 \rho w(\rho) I(t, r\rho) d\rho \geq \int_r^1 \rho w(\rho) I(t, r\rho) d\rho, \quad r \in [r_0, 1).$$

Moreover,

$$\int_{r_0}^1 \rho w(\rho) I(t, r\rho) d\rho = \int_r^1 \rho w(\rho) I(t, r\rho) d\rho + \int_{r_0}^r \rho w(\rho) I(t, r\rho) d\rho,$$

and applying Lemma (4.3.4),

$$\begin{aligned} \int_{r_0}^r \rho w(\rho) I(t, r\rho) d\rho &= \int_{r_0}^r \frac{\rho w(\rho)}{(1-\rho^2)^\gamma} \frac{(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho \\ &\leq \frac{w(r)}{(1-r^2)^\gamma} \int_{r_0}^r \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho \leq \frac{w(r)}{(1-r^2)^\gamma} \varphi_{r^2}(t) \int_{r_0}^1 \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho, \end{aligned}$$

because $u \rightarrow \varphi_u(t)$ is increasing for all $t > 0$. For the same reason,

$$\begin{aligned} \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho &= \int_r^1 \frac{w(\rho)}{(1-\rho^2)^\gamma} \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho \\ &\geq \frac{w(r)}{(1-r^2)^\gamma} \varphi_{r^2}(t) \int_r^1 \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho. \end{aligned}$$

Now note that

$$\int_{r_0}^r \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho \lesssim \int_r^1 \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho, \quad r \in [r_0, 1),$$

with constants independent on $t \geq \gamma + 3$. Indeed, this estimate holds for $t = \gamma + 3$, and, hence, by monotonicity of the function $\rho \mapsto (1-(\rho r)^2)^{-1}$, for all $t \geq \gamma + 3$.

Thus, using Lemma (4.3.4) and the fact that the function $(1-\rho)^{-\gamma} w(\rho)$ is increasing on $[r_0, 1)$, we obtain

$$\begin{aligned} \int_{r_0}^r \rho w(\rho) I(t, r\rho) d\rho &\leq \frac{w(r)}{(1-r^2)^\gamma} \varphi_{r^2}(t-2) \int_{r_0}^r \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho \\ &\leq k_1 \frac{w(r)}{(1-r^2)^\gamma} \varphi_{r^2}(t-2) \int_r^1 \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho \leq k_2 \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho, \end{aligned}$$

(where k_1, k_2 are positive constants which do not depend on t), which completes the proof.

The next theorem shows that for the polynomially decreasing weights the quantity $\mathcal{C}_{n,r}(L_a^p(w))$ admits a below estimate of the same form.

Theorem (4.3.6)[153]: If w is γ -polynomially decreasing, then there exists a positive constant K' depending only on w and p such that

$$K' \frac{n}{1-r} \leq \mathcal{C}_{n,r}(L_a^p(w)) \leq K \frac{n}{1-r}, \quad (42)$$

where K is defined in (39) and where the left-hand side inequality of (42) holds for all $r \in [0,1)$ and $n \geq \frac{\gamma+3}{p} + 1$. In particular, (42) holds for the classical weights $w(\rho) = w_\beta(\rho) = (1-\rho)^\beta$, $\beta > -1$.

The polynomial decrease is essential and provides a sharp bound for the validity of the

uniform estimate (42) for all possible values of n and r . Namely, if the weight is super-polynomially decreasing, then (42) will fail along some sequence of radii.

Proof. We need to show only the lower bound, the upper bound is already showed in Theorem (4.3.3). Let us show the minoration with the test function $f(z) = \frac{1}{(1-rz)^n}$.

Using (41) with $\alpha = r_0$, we need to show that

$$\begin{aligned} \frac{\|f\|_{L_a^p(w)}^p}{n^p r^p} &= \int_{r_0}^1 \rho w(\rho) I(pn+p, r\rho) d\rho \\ &\geq \frac{C}{(1-r)^p} \int_{r_0}^1 \rho w(\rho) I(pn, r\rho) d\rho = \frac{C}{(1-r)^p} \|f\|_{L_a^p(w)}^p. \end{aligned}$$

Since $r \in [r_0, 1)$ and $n \geq \frac{\gamma+3}{p}$, by Lemma (4.3.5) applied with $t = pn+p$ and $t = pn$ this means that

$$\int_r^1 \rho w(\rho) I(pn+p, r\rho) d\rho \geq \frac{C}{(1-r)^p} \int_r^1 \rho w(\rho) I(pn, r\rho) d\rho.$$

By Lemma (4.3.4), this is equivalent to the estimate

$$\int_r^1 \rho w(\rho) \frac{\varphi_{rp}(pn+p-2)}{(1-(rp)^2)^{pn+p-1}} d\rho \geq \frac{C}{(1-r)^p} \int_r^1 \rho w(\rho) \frac{\varphi_{rp}(pn-2)}{(1-(rp)^2)^{pn-1}} d\rho$$

The last statement is obvious since

$$\begin{aligned} \int_r^1 \rho w(\rho) \frac{\varphi_{rp}(pn+p-2)}{(1-(rp)^2)^{pn+p-1}} d\rho &\geq \frac{1}{(1-r^2)^p} \int_r^1 \rho w(\rho) \frac{\varphi_{rp}(pn+p-2)}{(1-(rp)^2)^{pn-1}} d\rho \\ &\geq \frac{1}{(1-r^2)^p} \int_r^1 \rho w(\rho) \frac{\varphi_{rp}(pn-2)}{(1-(rp)^2)^{pn-1}} d\rho, \end{aligned}$$

where the last inequality is due to the fact that $u \rightarrow \varphi_u(t)$ is increasing for all $0 \leq u < 1$.

Lemma (4.3.7)[153]: Let $n \geq 1$, $r, s \in [0, 1)$ and $p \in [1, +\infty]$. We set

$$M_{p,s}(n, r) = \sup \left\{ |f(\xi)| : \xi \in \mathbb{D}, f \in \mathcal{R}_{n,r}, \|f\|_{L_a^p(s\mathbb{D})} \leq 1 \right\}.$$

Then

$$M_{p, \frac{2}{3}}(n, r) \leq d \frac{c^n}{(1-r)^{n+b}}, \quad (43)$$

where $d > 0$, $b > 0$, $c > 1$ are some absolute positive constants (may be, depending on p).

Remark (4.3.8)[153]: Lemma (4.3.7) is valid not only for $s = \frac{2}{3}$, but for every $s \in (0, 1)$, with constants $d > 0$, $b > 0$, $c > 1$ depending both on s and p .

Proof. For every $f \in \mathcal{R}_{n,r}$ and $\xi \in \mathbb{D}$, we have

$$\left| f\left(\frac{1}{2}\xi\right) \right| = \left| f_{\frac{2}{3}}\left(\frac{3}{4}\xi\right) \right| = \left| \int_{\mathbb{D}} f_{\frac{2}{3}}(u) \left(\overline{k_{\frac{3}{4}\xi}(u)}\right)^2 dA(u) \right|,$$

where $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$ is the standard Cauchy kernel associated with $\lambda \in \mathbb{D}$, and A is the normalized area measure on $\mathbb{D}$. Applying Holder's inequality we obtain

$$\left| f\left(\frac{1}{2}\xi\right) \right| \leq \|f_{\frac{2}{3}}\|_{L_a^p} \left\| \left(k_{\frac{3}{4}\xi}\right)^2 \right\|_{L_a^{p'}} = \left(\frac{3}{2}\right)^{1/p} \|f\|_{L_a^p(\frac{2}{3}\mathbb{D})} \left\| \left(k_{\frac{3}{4}\xi}\right)^2 \right\|_{L_a^{p'}}, \quad \xi \in \mathbb{T},$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Now, note that

$$\left\| \left(k_{\frac{3}{4}\xi}\right)^2 \right\|_{L_a^{p'}} \leq \left\| \left(k_{\frac{3}{4}\xi}\right)^2 \right\|_{H^\infty} = \left(\frac{1}{1-\frac{3}{4}}\right)^2 = 16.$$

Finally, supposing $\|f\|_{L_a^p(\frac{2}{3}\mathbb{D})} \leq 1$, we obtain

$$\|f\|_{L_a^p(\frac{2}{3}\mathbb{D})} \leq \|f\|_{H^\infty(\frac{1}{2}\mathbb{D})} \leq 16 \left(\frac{3}{2}\right)^{1/p} \leq 24,$$

which gives

$$M_{p, \frac{2}{3}}(n, r) \leq 24M_{2, \frac{1}{2}}(n, r). \quad (44)$$

It remains to obtain a suitable upper bound for $M_{2, \frac{1}{2}}(n, r)$. Let us show that

$$M_{2, \frac{1}{2}}(n, r) \leq 2\sqrt{n} \left(\frac{2}{1-r}\right)^{n+\frac{1}{2}}. \quad (45)$$

For every $f \in \mathcal{R}_{n, r}$, we have $f_{\frac{1}{2}} \in \mathcal{R}_{n, \frac{1}{2}r} \subset \mathcal{R}_{n, r}$. If $\{1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n\}$ is the set of the poles of f (thus, $|\lambda_j| \leq r$, $j = 1, \dots, n$), then $f \in K_{B_\sigma}$ with $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset r\mathbb{D}$, whereas the set $\{2/\bar{\lambda}_1, \dots, 2/\bar{\lambda}_n\}$ is the set of the poles of the function $f_{\frac{1}{2}}$ and $f_{\frac{1}{2}} \in K_{B_{\sigma'}}$ with $\sigma' = \{\frac{1}{2}\lambda_1, \dots, \frac{1}{2}\lambda_n\} \subset \frac{r}{2}\mathbb{D}$. Hence, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that

$$f_{\frac{1}{2}} = \sum_{k=1}^n \alpha_k e_k, \quad (46)$$

on $\mathbb{D}$, where $(e_k)_{k=1}^n$ is the Malmquist basis associated with the set σ' . Since both $f_{\frac{1}{2}}$ and $\sum_{k=1}^n \alpha_k e_k$ are meromorphic in $\mathbb{C}$ the equality (46) is in fact valid everywhere in $\mathbb{C}$. Thus,

$$f(\xi) = \sum_{k=1}^n \alpha_k \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{(1 - \frac{1}{4}|\lambda_k|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}_k \xi}, \quad \xi \in \mathbb{D},$$

and by the Cauchy–Schwarz inequality,

$$|f(\xi)| \leq \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \left| \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{(1 - \frac{1}{4}|\lambda_k|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}_k \xi} \right|^2 \right)^{\frac{1}{2}}. \quad (47)$$

for any $\xi \in \mathbb{D}$. Now, if $\lambda \in r\mathbb{D}$ and $\xi \in \mathbb{D}$,

$$\frac{\frac{\lambda}{2} - 2\xi}{1 - \bar{\lambda}\xi} = \frac{2(\frac{\lambda}{4} - \xi)}{1 - \frac{\bar{\lambda}}{4}\xi} \frac{1 - \frac{\bar{\lambda}}{4}\xi}{1 - \bar{\lambda}\xi} = 2b_{\frac{\lambda}{4}}(\xi) \left(1 + \frac{3\bar{\lambda}}{4(1 - \bar{\lambda}\xi)} \right),$$

which gives

$$\left| \frac{\frac{\lambda}{2} - 2\xi}{1 - \bar{\lambda}\xi} \right| \leq 2 \left(1 + \frac{3r}{4} \frac{1}{1-r} \right) = \frac{4-r}{2(1-r)} \leq \frac{2}{1-r}.$$

We get

$$\sum_{k=1}^n \left| \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{(1 - \frac{1}{4}|\lambda_k|^2)^{\frac{1}{2}}}{1 - \bar{\lambda}_k \xi} \right|^2 \leq \frac{1}{(1-r)^2} \sum_{k=1}^n 2^{2(k-1)} \left(\frac{1}{1-r} \right)^{2(k-1)} \leq \frac{1}{4} \left(\frac{2}{1-r} \right)^{2n+1}. \quad (48)$$

Now we first notice that

$$\left(\sum_{k=1}^n |\alpha_k|^2 \right)^{1/2} = \|f_{\frac{1}{2}}\|_{H^2}.$$

For any function $\varphi(z) = \sum_{k \geq 0} \varphi(k) z^k$ in H^2 , one has

$$\|\varphi\|_{H^2}^2 = \sum_{k \geq 0} \frac{|\varphi(k)|^2}{\sqrt{k+1}} \sqrt{k+1} |\varphi(k)| \leq \|\varphi\|_{L_a^2} \|\varphi\|_{B_{2,2}^{1/2}},$$

We now use the upper bound of [150, Theorem A, (4)]: for $\varphi \in \mathcal{R}_{n, \rho}$ one has

$$\begin{aligned}\|\varphi\|_{B_{2,2}^{1/2}}^2 &= \|\varphi'\|_{L_a^2}^2 + \|\varphi\|_{H^2}^2 \\ &\leq (2+r) \frac{n}{1-r} \|\varphi\|_{H^2}^2 + \|\varphi\|_{H^2}^2 \leq \frac{4n}{1-r} \|\varphi\|_{H^2}^2,\end{aligned}$$

which gives

$$\|\varphi\|_{H^2} \leq 2\sqrt{\frac{n}{1-r}} \|\varphi\|_{L_a^2}.$$

In particular, with $\varphi = f_{\frac{1}{2}}$ we get $\varphi \in \mathcal{R}_{n,\frac{1}{2}r}$ and

$$\|f_{\frac{1}{2}}\|_{H^2} \leq 2\sqrt{\frac{n}{1-r/2}} \|f_{\frac{1}{2}}\|_{L_a^2} \leq 2\sqrt{2n} \|f_{\frac{1}{2}}\|_{L_a^2}. \quad (49)$$

We conclude from (47), (48) and (49) that for any $\xi \in \mathbb{D}$,

$$|f(\xi)| \leq \|f_{\frac{1}{2}}\|_{H^2} \left(\frac{1}{4} \frac{2^{2n+1}}{(1-r)^{2n+1}} \right)^{\frac{1}{2}} \leq \frac{1}{2} \left(\frac{2}{1-r} \right)^{n+\frac{1}{2}} 2\sqrt{2n} \|f_{\frac{1}{2}}\|_{L_a^2},$$

that is,

$$|f(\xi)| \leq \sqrt{2n} \left(\frac{2}{1-r} \right)^{n+\frac{1}{2}} \|f\|_{L_a^2(\frac{1}{2}\mathbb{D})}, \quad \xi \in \mathbb{D}.$$

Taking the supremum over $\xi \in \mathbb{D}$ and $f \in \mathcal{R}_{n,r}$ we obtain (45).

Combining (44) and (45) and choosing $d = 48$, $b = \frac{1}{2}$ and $c > 0$ such that $2^n \sqrt{n} \leq c^n$ for any $n \geq 1$, we complete the proof and obtain (43).

Theorem (4.3.9)[153]: Suppose that w is super-polynomially decreasing. Then there exists a sequence $r_n \rightarrow 1-$ such that for any p ,

$$\frac{C_{r_n,n}(L_a^p(w))}{n} \leq o\left(\frac{1}{1-r_n}\right), \quad n \rightarrow \infty.$$

For the proof of Theorem (4.3.9) we will need a definition from the theory of model subspaces of the Hardy space. For a finite subset σ of $\mathbb{D}$ with $\text{card } \sigma = n$, consider the finite Blaschke product

$$B_\sigma = \prod_{\lambda \in \sigma} b_\lambda,$$

where $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$, $\lambda \in \mathbb{D}$. Define the model space K_{B_σ} by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Consider the family $(e_k)_{1 \leq k \leq n}$ in K_{B_σ} (known as Malmquist basis, see [43]),

$$e_1(z) = \frac{(1 - |\lambda_1|)}{1 - \lambda_1 z} \text{ and } e_k(z) = \left(\prod_{j=1}^{k-1} b_{\lambda_j}(z) \right) \frac{(1 - |\lambda_k|)}{1 - \lambda_k z}, \quad k \in [2, n],$$

The family $(e_k)_{1 \leq k \leq n}$ associated with σ is an orthonormal basis of the n -dimensional space K_{B_σ} .

In what follows we denote by $L_a^p(w, s\mathbb{D})$ and by $H^p(s\mathbb{D})$, $s > 0$, the weighted Bergman space and the Hardy space in the disc $s\mathbb{D} = \{z : |z| < s\}$, respectively. If $w \equiv 1$, we write simply $L_a^p(s\mathbb{D})$ and we write L_a^p if $s = 1$.

Proof. Take $r \in (0,1)$ and $R \in (0,r)$ and let us represent the norm $\|f'\|_{L_a^p(w)}^p$ of a function $f \in \mathcal{R}_{n,r}$ as $I_1 + I_2$,

$$I_1 = \int_0^R \|(f_\rho)'\|_p^p w(\rho) d\rho, \quad I_2 = \int_R^1 \|(f_\rho)'\|_p^p w(\rho) d\rho.$$

Here and everywhere below in this proof, C_i , $i=1,\dots,5$, are positive constants, depending, may be, only on p and w (but not on n and r). By (38), we have for the first integral

$$I_1 = C_1 \left(\frac{n}{1-R} \right)^p \int_0^R \|f_\rho\|_p^p w(\rho) d\rho \leq C_2 \left(\frac{n}{1-R} \right)^p \|f\|_{L_a^p(w)}^p.$$

Note that $f_\rho \in \mathcal{R}_{n,\rho r} \subset \mathcal{R}_{n,r}$, and, thus, $\|f_\rho\|_\infty \leq M_{p,2/3}(n,r) \|f_\rho\|_{L_a^p(\frac{2}{3}\mathbb{D})}$. Applying (38) once again together with an obvious inequality $\|f_\rho\|_p \leq \|f_\rho\|_\infty$, we get

$$\begin{aligned} I_2 &\leq C_3 \left(\frac{n}{1-r} \right)^p \int_R^1 \|f_\rho\|_\infty^p w(\rho) d\rho \\ &\leq C_3 \|f\|_{L_a^p(\frac{2}{3}\mathbb{D})}^p \left(\frac{n}{1-r} \right)^p \int_R^1 M_{p,2/3}^p(n,r) w(\rho) d\rho \\ &\leq C_3 \|f\|_{L_a^p(\frac{2}{3}\mathbb{D})}^p \left(\frac{n}{1-r} \right)^p \frac{c^{pn}}{(1-r)^{pn+pb}} w(R), \end{aligned}$$

where the last inequality follows from Lemma (4.3.7). Note that

$$\|f\|_{L_a^p(\frac{2}{3}\mathbb{D})}^p \leq (w(2/3))^{-1} \|f\|_{L_a^p(w)}^p.$$

Hence,

$$I_2 \leq C_4 \left(\frac{n}{1-r} \right)^p \frac{c^{pn}}{(1-r)^{pn+pb}} w(R) \|f\|_{L_a^p(w)}^p.$$

Now, choose a positive increasing sequence $(\gamma_n)_{n \in \mathbb{N}}$ such that $n = o(\gamma_n)$, as $n \rightarrow +\infty$. For any n we fix r_n° such that the function $w(r)(1-r)^{-\gamma_n}$ decreases on $[r_n^\circ, 1)$. Now for a fixed n take r, R so that $r_n^\circ < R < r < 1$ and

$$1-R = (1-r)^{1/2}, \quad 1-r_n^\circ = (1-r)^{1/4}$$

We have

$$w(R) \leq w(r_n^\circ) \frac{(1-R)^{\gamma_n}}{(1-r_n^\circ)^{\gamma_n}} = w(r_n^\circ) (1-r)^{\gamma_n/4}.$$

Hence, using the fact that w is bounded on $[r_1^\circ, 1)$, we obtain

$$I_2 \leq C_4 \left(\frac{n}{1-r} \right)^p \|f\|_{L_a^p(w)}^p \cdot c^{pn} \frac{(1-r)^{\gamma_n/4}}{(1-r)^{pn+pb}}.$$

Let us show that for sufficiently large n ,

$$c^{pn} \frac{(1-r)^{\gamma_n/4}}{(1-r)^{pn+pb}} \rightarrow 0, \quad r \rightarrow 1-.$$

Indeed, choosing r so that $c < (1-r)^{-1}$, we get

$$c^{pn} \frac{(1-r)^{\gamma_n/4}}{(1-r)^{pn+pb}} \leq (1-r)^{\frac{\gamma_n}{4} - pn - pb} \rightarrow 0, \quad r \rightarrow 1-.$$

since $n = o(\gamma_n)$, $n \rightarrow \infty$. Hence, there exists a sequence (r_n) , $r_n \rightarrow 1-$, such that

$$\frac{I_2^{1/p}}{n \|f\|_{L_a^p(w)}} = o\left(\frac{1}{1-r_n}\right), \quad n \rightarrow \infty.$$

The corresponding estimate for I_1 is obvious since $1-R_n = (1-r_n)^{1/2}$.

Corollary (4.3.10)[221]: Let $\epsilon > 0$ and let w be an integrable nonnegative function on $[0,1)$. Then there exists a positive constant K depending only on $1+\epsilon$ (but not on the weight w) such that

$$\mathcal{C}_{1+\epsilon, 1-\epsilon}(L_a^{1+\epsilon}(w)) \leq \frac{1+\epsilon}{\epsilon} K \quad (50)$$

for all $\epsilon > 0$. Moreover, if we fix $\epsilon > 0$ and let $1+\epsilon$ tend to infinity, then we have

$$\frac{\tilde{K}(1-\epsilon)}{\epsilon} \leq \liminf_{\epsilon \rightarrow \infty} \frac{\mathcal{C}_{1+\epsilon, 1-\epsilon}(L_a^{1+\epsilon}(w))}{1+\epsilon} \leq \limsup_{\epsilon \rightarrow \infty} \frac{\mathcal{C}_{1+\epsilon, 1-\epsilon}(L_a^{1+\epsilon}(w))}{1+\epsilon} \leq \frac{K}{\epsilon}, \quad (51)$$

where $\tilde{K}$ is, as K , a positive constant depending only on $1+\epsilon$.

The next theorem (see [1]) shows that for the polynomially decreasing weights of quadratic factor the quantity $\mathcal{C}_{1+\epsilon, 1-\epsilon}(L_a^{1+\epsilon}(w))$ admits a below estimate of the same form.

Proof. First, we notice that for any $0 \leq r_0 < 1$,

$$\|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon} \asymp \iint_{u \in C_{r_0}} (1-\epsilon) |f((1-\epsilon)\xi)|^{1+\epsilon} w(1-\epsilon) dm(\xi) d(1-\epsilon) \quad (52)$$

for all $f \in L_a^{1+\epsilon}(w)$, where $C_{r_0} = \{z : r_0 < |z| < 1\}$. Let $f \in \mathcal{R}_{1+\epsilon, 1-\epsilon}$ with $\epsilon > 0$. Using (52) with $r_0 = \frac{1}{2}$ we get

$$\begin{aligned} \|f'\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon} &\asymp \iint_{(1-\epsilon)\xi \in C_{1/2}} |f'((1-\epsilon)\xi)|^{1+\epsilon} w(1-\epsilon) dm(\xi) d(1-\epsilon) \\ &= \int_{\frac{1}{2}}^1 w(1-\epsilon) \frac{1}{(1-\epsilon)^\epsilon} \left(\|f_{1-\epsilon}\|_{H^{1+\epsilon}}^{1+\epsilon} \right) d(1-\epsilon). \end{aligned}$$

Now using the fact that $f_{1-\epsilon} \in \mathcal{R}_{1+\epsilon, (1-\epsilon)^2} \subset \mathcal{R}_{1+\epsilon, 1-\epsilon}$ for every $\epsilon > 0$, we get

$$\begin{aligned} \int_{\frac{1}{2}}^1 w(1-\epsilon) \frac{1}{(1-\epsilon)^\epsilon} \left(\|f_{1-\epsilon}\|_{H^{1+\epsilon}}^{1+\epsilon} \right) d(1-\epsilon) &\leq (2\mathcal{C}_{1+\epsilon, 1-\epsilon}(H^{1+\epsilon}))^{1+\epsilon} \int_{\frac{1}{2}}^1 (1-\epsilon) w(1-\epsilon) \left(\|f_{1-\epsilon}\|_{H^{1+\epsilon}} \right)^{1+\epsilon} d(1-\epsilon) \\ &\asymp (\mathcal{C}_{1+\epsilon, 1-\epsilon}(H^{1+\epsilon}))^{1+\epsilon} \|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}. \end{aligned}$$

In particular, using the right-hand side inequality of (38), we get

$$\mathcal{C}_{1+\epsilon, 1-\epsilon}(L_a^{1+\epsilon}(w)) \leq \frac{1+\epsilon}{\epsilon} K_{1+\epsilon}$$

for all $\epsilon > 0$, where $K_{1+\epsilon}$ is a constant depending on $1+\epsilon$ only.

Now, let us show (51). Let

$$f_{1+\epsilon}(z) = \frac{1}{(1-(1-\epsilon)z)^{1+\epsilon}} \in \mathcal{R}_{1+\epsilon, 1-\epsilon},$$

and $D = \{z \in \mathbb{D} : |1-(1-\epsilon)z| \leq 2|\epsilon|\}$. We claim that

$$\|f_{1+\epsilon}\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon} \sim \int_D |f_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z), \quad \epsilon \rightarrow \infty,$$

and, analogously,

$$\|f'_{1+\epsilon}\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon} \sim \int_D |f'_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z), \quad \epsilon \rightarrow \infty.$$

Indeed, by a very rough estimate

$$\int_{\mathbb{D} \setminus D} |f_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z) \leq \frac{C_1}{2^{(1+\epsilon)^2} (\epsilon)^{(1+\epsilon)^2}},$$

where $C_1 > 0$ depends only on w . On the other hand, if we put $\tilde{D} = \{z \in \mathbb{D} : |1-(1-\epsilon)z| \leq \frac{3}{2}|\epsilon|\}$, then

$$\int_D |f_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z) \geq \frac{1}{(3/2)^{(1+\epsilon)^2} (\epsilon)^{(1+\epsilon)^2}} \int_{\tilde{D}} w(z) dA(z).$$

Since $1-\epsilon$ (thus D and $\tilde{D}$) are fixed we see that

$$\frac{1}{2^{(1+\epsilon)^2} (\epsilon)^{(1+\epsilon)^2}} = o\left(\frac{1}{(3/2)^{(1+\epsilon)^2} (\epsilon)^{(1+\epsilon)^2}} \int_{\tilde{D}} w(z) dA(z)\right), \quad \epsilon \rightarrow \infty.$$

Thus,

$$\frac{\|f'_{1+\epsilon}\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}}{\|f_{1+\epsilon}\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}} \sim \int_D |f'_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z) / \int_D |f_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z).$$

Obviously,

$$\begin{aligned} \int_D |f'_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z) &= \int_D \frac{(1-\epsilon^2)^{1+\epsilon}}{|1-(1-\epsilon)z|^{(2+\epsilon)(1+\epsilon)}} w(z) dA(z) \\ &\geq \frac{(1-\epsilon^2)^{1+\epsilon}}{2^{1+\epsilon}(\epsilon)^{1+\epsilon}} \int_D \frac{1}{|1-(1-\epsilon)z|^{(1+\epsilon)^2}} w(z) dA(z) \\ &\geq \frac{(1-\epsilon^2)^{1+\epsilon}}{2^{1+\epsilon}(\epsilon)^{1+\epsilon}} \int_D |f_{1+\epsilon}(z)|^{1+\epsilon} w(z) dA(z). \end{aligned}$$

Thus,

$$\liminf_{\epsilon \rightarrow \infty} \frac{\|f'_{1+\epsilon}\|_{L_a^{1+\epsilon}(w)}}{\|f_{1+\epsilon}\|_{L_a^{1+\epsilon}(w)}} \geq \frac{1-\epsilon}{2\epsilon}.$$

Corollary (4.3.11)[221]. Let $\epsilon > 0$ and $t \geq 1$. We set

$$I(t, 1-\epsilon) = \int_{\mathbb{T}} |1-(1-\epsilon)\xi|^t dm(\xi) \text{ and } \varphi_{1-\epsilon}(t) = \int_{\mathbb{T}} |1+(1-\epsilon)\xi|^t dm(\xi).$$

Then,

$$I(t, 1-\epsilon) = \frac{1}{((2-\epsilon)\epsilon)^{t-1}} \varphi_{1-\epsilon}(t-2)$$

for every $t \geq 2$, and $t \mapsto \varphi_{1-\epsilon}(t)$ is an increasing function on $[0, +\infty)$ for every $\epsilon > 0$. Moreover, both

$$(1-\epsilon) \mapsto \varphi_{1-\epsilon}(t-2) \text{ and } (1-\epsilon) \mapsto I(t, 1-\epsilon),$$

are increasing on $[0, 1)$, for all $t \geq 0$.

Proof. Indeed, supposing that $t \geq 2$, we can write

$$I(t, 1-\epsilon) = \frac{1}{(2-\epsilon)\epsilon} \int_{\mathbb{T}} |b'_{1-\epsilon}(\xi)| \frac{1}{|1-(1-\epsilon)\xi|^{t-2}} dm(\xi),$$

(where $b_{1-\epsilon}(z) = \frac{1-\epsilon-z}{1-(1-\epsilon)z}$). Using the fact that $b_{1-\epsilon} \circ b_{1-\epsilon}(z) = z$ and changing the variable in the above integral we get

$$\begin{aligned} I(t, 1-\epsilon) &= \frac{1}{(2-\epsilon)\epsilon} \int_{\mathbb{T}} |b'_{1-\epsilon}(\xi)| \frac{1}{|1-(1-\epsilon)b_{1-\epsilon} \circ b_{1-\epsilon}(\xi)|^{t-2}} dm(\xi) \\ &= \frac{1}{(2-\epsilon)\epsilon} \int_{\mathbb{T}} |b'_{1-\epsilon}(\xi)| \frac{1}{|1-(1-\epsilon)b_{1-\epsilon}(\xi)|^{t-2}} dm(\xi) = \frac{1}{((2-\epsilon)\epsilon)^{t-1}} \varphi_{1-\epsilon}(t-2), \end{aligned}$$

since $1-(1-\epsilon)b_{1-\epsilon}(z) = \frac{1-(1-\epsilon)z-(1-\epsilon)(1-\epsilon-z)}{1-(1-\epsilon)z} = \frac{(2-\epsilon)\epsilon}{1-(1-\epsilon)z}$. Now,

$$\varphi_{1-\epsilon}(t) = \int_0^{2\pi} \exp\left(\frac{t}{2} \ln(1+(1-\epsilon)^2 - 2(1-\epsilon)\cos s)\right) ds,$$

$$\varphi'_{1-\epsilon}(t) = \frac{1}{4} \int_0^{2\pi} \ln(1+(1-\epsilon)^2 + 2(1-\epsilon)\cos s) \exp\left(\frac{t}{2} \ln(1+(1-\epsilon)^2 + 2(1-\epsilon)\cos s)\right) ds,$$

and

$$\varphi''_{1-\epsilon}(t) = \frac{1}{4} \int_0^{2\pi} [\ln(1+(1-\epsilon)^2 - 2(1-\epsilon)\cos s)]^2 \exp\left(\frac{t}{2} \ln(1+(1-\epsilon)^2 - 2(1-\epsilon)\cos s)\right) ds \geq 0,$$

for every $t \geq 0$, $\epsilon > 0$. Thus, $\varphi_{1-\epsilon}$ is a convex function on $[0, \infty)$ and $\varphi'_{1-\epsilon}$ is increasing on $[0, \infty)$ for all $\epsilon > 0$. Moreover,

$$\varphi'_{1-\epsilon}(0) = \frac{1}{4} \int_0^{2\pi} \ln(1+(1-\epsilon)^2 - 2(1-\epsilon)\cos s) ds = 0.$$

Thus,

$$\varphi'_{1-\epsilon}(t) \geq \varphi'_{1-\epsilon}(0) = 0, \quad \forall t \in [0, \infty), \quad \epsilon > 0,$$

and so $\varphi_{1-\epsilon}$ is increasing on $[0, \infty)$. The fact that

$$1-\epsilon \mapsto I(t, 1-\epsilon),$$

is increasing on $[0, 1)$ for all $t \geq 0$ is obvious since

$$I(t, 1-\epsilon) = \left\| \frac{1}{(1-(1-\epsilon)z)^{t/2}} \right\|_{H^2}^2 = \sum_{k \geq 0} a_k^2(t) (1-\epsilon)^{2k},$$

where $a_k(t)$ is the k th Taylor coefficient of $(1-z)^{-t/2}$. The same reasoning gives that $(1-\epsilon) \mapsto \varphi_{1-\epsilon}(t)$ is increasing on $[0, 1)$.

Corollary (4.3.12)[221]. If for some $r_0 \in [0, 1)$ and $\gamma^2 > 1$ the function $\frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}}$ is increasing on $[r_0, 1)$, then

$$\int_{1-\epsilon}^1 (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon) \asymp \int_{r_0}^1 (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon),$$

for all t such that $t \geq \gamma^2 + 2$ and for all $\epsilon \leq 1-r_0$, with constants independent on t .

Proof. Clearly,

$$\int_{r_0}^1 (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon) \geq \int_{1-\epsilon}^1 (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon), \quad (1-\epsilon) \in [r_0, 1).$$

Moreover,

$$\begin{aligned} \int_{r_0}^1 (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon) &= \int_{1-\epsilon}^1 (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon) \\ &\quad + \int_{r_0}^{1-\epsilon} (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon), \end{aligned}$$

and applying Lemma (4.3.4),

$$\begin{aligned} \int_{r_0}^{1-\epsilon} (1-\epsilon)w(1-\epsilon)I(t, (1-\epsilon)^2)d(1-\epsilon) &= \int_{r_0}^{1-\epsilon} \frac{(1-\epsilon)w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \frac{((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} \varphi_{(1-\epsilon)^2}(t) d(1-\epsilon) \\ &\leq \frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \int_{r_0}^{1-\epsilon} \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} \varphi_{(1-\epsilon)^2}(t) d(1-\epsilon) \\ &\leq \frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \varphi_{(1-\epsilon)^2}(t) \int_{r_0}^1 \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} d(1-\epsilon), \end{aligned}$$

because $u \rightarrow \varphi_u(t)$ is increasing for all $t > 0$. For the same reason,

$$\begin{aligned} \int_{1-\epsilon}^1 (1-\epsilon)w(1-\epsilon) \frac{1}{(1-(1-\epsilon)^4)^{t-1}} \varphi_{(1-\epsilon)^2}(t) d(1-\epsilon) &= \\ &= \int_{1-\epsilon}^1 \frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} \varphi_{(1-\epsilon)^2}(t) d(1-\epsilon) \\ &\geq \frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \varphi_{(1-\epsilon)^2}(t) \int_{1-\epsilon}^1 \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} d(1-\epsilon). \end{aligned}$$

Now note that

$$\int_{r_0}^{1-\epsilon} \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} d(1-\epsilon) \lesssim \int_{1-\epsilon}^1 \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} d(1-\epsilon), \quad (1-\epsilon) \in [r_0, 1),$$

with constants independent on $t \geq \gamma^2 + 2$. Indeed, this estimate holds for $t = \gamma^2 + 2$, and, hence, by monotonicity of the function $(1-\epsilon) \mapsto (1-(1-\epsilon)^4)^{-1}$, for all $t \geq \gamma^2 + 2$.

Thus, using Lemma (4.3.4) and the fact that the function $(\epsilon)^{1-\gamma^2} w(1-\epsilon)$ is increasing on $[r_0, 1)$, we obtain

$$\begin{aligned} \int_{r_0}^{1-\epsilon} (1-\epsilon) w(1-\epsilon) I(t, (1-\epsilon)^2) d(1-\epsilon) &\leq \frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \varphi_{(1-\epsilon)^2}(t-2) \int_{r_0}^{1-\epsilon} \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} d(1-\epsilon) \\ &\leq k_1 \frac{w(1-\epsilon)}{((2-\epsilon)\epsilon)^{\gamma^2-1}} \varphi_{(1-\epsilon)^2}(t-2) \int_{1-\epsilon}^1 \frac{(1-\epsilon)((2-\epsilon)\epsilon)^{\gamma^2-1}}{(1-(1-\epsilon)^4)^{t-1}} d(1-\epsilon) \\ &\leq k_2 \int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) \frac{1}{(1-(1-\epsilon)^4)^{t-1}} \varphi_{(1-\epsilon)^2}(t) d(1-\epsilon), \end{aligned}$$

(where k_1, k_2 are positive constants which do not depend on t), which completes the proof.

Corollary (4.3.13)[221]. If w is (γ^2-1) -polynomially decreasing, then there exists a positive constant K' depending only on w and $1+\epsilon$ such that

$$\frac{1+\epsilon}{\epsilon} K' \leq \mathcal{C}_{1+\epsilon, 1-\epsilon}(L_a^{1+\epsilon}(w)) \leq \frac{1+\epsilon}{\epsilon} K, \quad (53)$$

where K is defined in (50) and where the left-hand side inequality of (53) holds for all $\epsilon > 0$ and $\epsilon(1+\epsilon) \geq \gamma^2 + 2$. In particular, (53) holds for the classical weights $w(1-\epsilon) = w_{\epsilon-1}(1-\epsilon) = (\epsilon)^{\epsilon-1}(1-\epsilon)$.

The polynomial decrease is essential and provides a sharp bound for the validity of the uniform estimate (53) for all possible values of $1+\epsilon$ and $1-\epsilon$. Namely, if the weight is super-polynomially decreasing, then (53) will fail along some sequence of radii.

Proof. We need to show only the lower bound, the upper bound is already showed in Theorem (4.3.3). Let us show the minoration with the test function $f(z) = \frac{1}{(1-(1-\epsilon)z)^{1+\epsilon}}$.

Using (52) with r_0 , we need to show that

$$\begin{aligned} \frac{\|f'\|_{L_a^{(1+\epsilon)}(w)}^{(1+\epsilon)}}{(1-\epsilon^2)^{(1+\epsilon)}} &= \int_{r_0}^1 (1-\epsilon) w(1-\epsilon) I((1+\epsilon)(2+\epsilon), (1-\epsilon)^2) d(1-\epsilon) \\ &\geq \frac{C}{(\epsilon)^{1+\epsilon}} \int_{r_0}^1 (1-\epsilon) w(1-\epsilon) I((1+\epsilon)^2, (1-\epsilon)^2) d(1-\epsilon) = \frac{C}{(\epsilon)^{1+\epsilon}} \|f\|_{L_a^{(1+\epsilon)}(w)}^{(1+\epsilon)}. \end{aligned}$$

Since $(1-\epsilon) \in [r_0, 1)$ and $(1+\epsilon)^2 \geq \gamma^2 + 2$, by Lemma (4.3.5) applied with $\epsilon > 0$ this means that

$$\int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) I((1+\epsilon)(2+\epsilon), (1-\epsilon)^2) d(1-\epsilon) \geq \frac{C}{(\epsilon)^{(1+\epsilon)}} \int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) I((1+\epsilon)^2, (1-\epsilon)^2) d(1-\epsilon)$$

By Lemma (4.3.4), this is equivalent to the estimate

$$\int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) \frac{\varphi_{1-\epsilon^2}((3+\epsilon)\epsilon)}{(1-(1-\epsilon)^4)^{(1+\epsilon)^2+\epsilon}} d(1-\epsilon) \geq \frac{C}{(\epsilon)^{(1+\epsilon)}} \int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) \frac{\varphi_{(1-\epsilon)^2}((1+\epsilon)^2-2)}{(1-(1-\epsilon)^4)^{(2+\epsilon)\epsilon}} d(1-\epsilon)$$

The last statement is obvious since

$$\begin{aligned} \int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) \frac{\varphi_{(1-\epsilon)^2}((3+\epsilon)\epsilon)}{(1-(1-\epsilon)^4)^{(1+\epsilon)^2+\epsilon}} d(1-\epsilon) &\geq \\ &\geq \frac{1}{((2-\epsilon)\epsilon)^{1+\epsilon}} \int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) \frac{\varphi_{1-\epsilon^2}((3+\epsilon)\epsilon)}{(1-(1-\epsilon)^4)^{(2+\epsilon)\epsilon}} d(1-\epsilon) \\ &\geq \frac{1}{((2-\epsilon)\epsilon)^{1+\epsilon}} \int_{1-\epsilon}^1 (1-\epsilon) w(1-\epsilon) \frac{\varphi_{(1-\epsilon)^2}((1+\epsilon)^2-2)}{(1-(1-\epsilon)^4)^{(2+\epsilon)\epsilon}} d(1-\epsilon), \end{aligned}$$

where the last inequality is due to the fact that $u \rightarrow \varphi_u(t)$ is increasing for all $0 \leq u < 1$.

Corollary (4.3.14)[221]. Suppose that w is super-polynomially decreasing. Then there exists a sequence $(1-\epsilon)_{1+\epsilon} \rightarrow 1-$ such that for any $1+\epsilon$,

$$\frac{C_{(1-\epsilon)_{1+\epsilon}, 1+\epsilon}(L_a^{1+\epsilon}(w))}{1+\epsilon} \leq o\left(\frac{1}{1-(1-\epsilon)_{1+\epsilon}}\right), \quad \epsilon \rightarrow \infty.$$

Proof. Take $\epsilon > 0$ and $R \in (0, 1-\epsilon)$ and let us represent the norm $\|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}$ of a function $f \in \mathcal{R}_{1+\epsilon, 1-\epsilon}$ as $I_1 + I_2$,

$$I_1 = \int_0^R \|(f_{1-\epsilon})'\|_{1+\epsilon}^{1+\epsilon} w(1-\epsilon) d(1-\epsilon), \quad I_2 = \int_R^1 \|(f_{1-\epsilon})'\|_{1+\epsilon}^{1+\epsilon} w(1-\epsilon) d(1+\epsilon).$$

Here and everywhere below in this proof, C_i , $i=1, \dots, 5$, are positive constants, depending, may be, only on $1+\epsilon$ and w (but not on $1+\epsilon$ and $1-\epsilon$). By (38), we have for the first integral

$$I_1 = C_1 \left(\frac{1+\epsilon}{1-R}\right)^{1+\epsilon} \int_0^R \|(f_{1-\epsilon})'\|_{1+\epsilon}^{1+\epsilon} w(1-\epsilon) d(1-\epsilon) \leq C_2 \left(\frac{1+\epsilon}{1-R}\right)^{1+\epsilon} \|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}.$$

Note that $f_{1-\epsilon} \in \mathcal{R}_{1+\epsilon, (1-\epsilon)^2} \subset \mathcal{R}_{1+\epsilon, 1-\epsilon}$, and, thus, $\|f_{1-\epsilon}\|_\infty \leq M_{1+\epsilon, 2/3}(1+\epsilon, 1-\epsilon) \|f\|_{L_a^{1+\epsilon}(2/3)}^{1+\epsilon}$. Applying (38) once again together with an obvious inequality $\|f_{1-\epsilon}\|_{1+\epsilon} \leq \|f\|_{1-\epsilon}$, we get

$$\begin{aligned} I_2 &\leq C_3 \left(\frac{1+\epsilon}{\epsilon}\right)^{1+\epsilon} \int_R^1 \|f_{1-\epsilon}\|_\infty^{1+\epsilon} w(1-\epsilon) d(1-\epsilon) \\ &\leq C_3 \|f\|_{L_a^{1+\epsilon}(2/3)}^{1+\epsilon} \left(\frac{1+\epsilon}{\epsilon}\right)^{1+\epsilon} \int_R^1 M_{1+\epsilon, 2/3}^{1+\epsilon}(1+\epsilon, 1-\epsilon) w(1-\epsilon) d(1-\epsilon) \\ &\leq C_3 \|f\|_{L_a^{1+\epsilon}(2/3)}^{1+\epsilon} \left(\frac{1+\epsilon}{\epsilon}\right)^{1+\epsilon} \frac{c^{(1+\epsilon)^2}}{\epsilon^{(1+\epsilon)^2+(1+\epsilon)b}} w(R), \end{aligned}$$

where the last inequality follows from Lemma (4.3.7). Note that

$$\|f\|_{L_a^{1+\epsilon}(2/3)}^{1+\epsilon} \leq (w(2/3))^{-1} \|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}.$$

Hence,

$$I_2 \leq C_4 \left(\frac{1+\epsilon}{\epsilon}\right)^{1+\epsilon} \frac{c^{(1+\epsilon)^2}}{(\epsilon)^{(1+\epsilon)^2+(1+\epsilon)b}} w(R) \|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon}.$$

Now, choose a positive increasing sequence $((\gamma^2-1)_{1+\epsilon})_{(1+\epsilon) \in \mathbb{N}}$ such that $1+\epsilon = o((\gamma^2-1)_{1+\epsilon})$, as $\epsilon \rightarrow +\infty$. For any $1+\epsilon$ we fix $(1-\epsilon)_{1+\epsilon}^\circ$ such that the function $w(1-\epsilon)(\epsilon)^{(1-\gamma^2)_{1+\epsilon}}$ decreases on $[(1-\epsilon)_{1+\epsilon}^\circ, 1)$. Now for a fixed $1+\epsilon$ take $1-\epsilon$, R so that $(1-\epsilon)_{1+\epsilon}^\circ < R < 1-\epsilon < 1$ and

$$1-R = (\epsilon)^{\frac{1}{2}}, \quad 1-(1-\epsilon)_{1+\epsilon}^\circ = (\epsilon)^{\frac{1}{4}}$$

We have

$$w(R) \leq w((1-\epsilon)_{1+\epsilon}^\circ) \frac{(1-R)^{(\gamma^2-1)_{1+\epsilon}}}{(1-(1-\epsilon)_{1+\epsilon}^\circ)^{(\gamma^2-1)_{1+\epsilon}}} = w((1-\epsilon)_{1+\epsilon}^\circ) (\epsilon)^{(\gamma^2-1)_{1+\epsilon}/4}.$$

Hence, using the fact that w is bounded on $[(1-\epsilon)_{1+\epsilon}^\circ, 1)$, we obtain

$$I_2 \leq C_4 \left(\frac{1+\epsilon}{\epsilon}\right)^{1+\epsilon} \|f\|_{L_a^{1+\epsilon}(w)}^{1+\epsilon} \cdot c^{(1+\epsilon)^2} \frac{(\epsilon)^{(\gamma^2-1)_{1+\epsilon}/4}}{(\epsilon)^{(1+\epsilon)^2+(1+\epsilon)b}}.$$

Let us show that for sufficiently large $1+\epsilon$,

$$c^{(1+\epsilon)^2} \frac{(\epsilon)^{(\gamma^2-1)_{1+\epsilon}/4}}{(\epsilon)^{(1+\epsilon)^2+(1+\epsilon)b}} \rightarrow 0, \quad 1-\epsilon \rightarrow 1-.$$

Indeed, choosing $1-\epsilon$ so that $c < (\epsilon)^{-1}$, we get

$$c^{(1+\epsilon)^2} \frac{(\epsilon)^{(\gamma^2-1)_{1+\epsilon}/4}}{(\epsilon)^{(1+\epsilon)^2+(1+\epsilon)b}} \leq (\epsilon)^{\frac{(\gamma^2-1)_{1+\epsilon}-2(1+\epsilon)^2-(1+\epsilon)b}{4}} \rightarrow 0, \quad 1-\epsilon \rightarrow 1-.$$

since $1+\epsilon = o((\gamma^2-1)_{1+\epsilon})$, $\epsilon \rightarrow \infty$. Hence, there exists a sequence $((1-\epsilon)_{1+\epsilon})$, $(1-\epsilon)_{1+\epsilon} \rightarrow 1-$, such that

$$\frac{I_2^{1/1+\epsilon}}{(1+\epsilon)\|f\|_{L_a^{1+\epsilon}(w)}} = o\left(\frac{1}{1-(1-\epsilon)_{1+\epsilon}}\right), \quad \epsilon \rightarrow \infty.$$

The corresponding estimate for I_1 is obvious since $1-R_{1+\epsilon} = (1-(1-\epsilon)_{1+\epsilon})^{1/2}$.

Corollary (4.3.15)[221]. Lemma (4.3.7) is valid not only for $s = \frac{2}{3}$, but for every $s \in (0,1)$, with constants $d > 0$, $b > 0$, $c > 1$ depending both on s and $1+\epsilon$.

Proof. For every $f \in \mathcal{R}_{1+\epsilon,1-\epsilon}$ and $\xi \in \mathbb{D}$, we have

$$\left|f\left(\frac{1}{2}\xi\right)\right| = \left|f_{\frac{2}{3}}\left(\frac{3}{4}\xi\right)\right| = \left|\int_{\mathbb{D}} f_{\frac{2}{3}}(u) \left(\overline{k_{\frac{3}{4}\xi}(u)}\right)^2 dA(u)\right|,$$

where $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$ is the standard Cauchy kernel associated with $\lambda \in \mathbb{D}$, and A is the normalized area measure on $\mathbb{D}$. Applying Holder's inequality we obtain

$$\left|f\left(\frac{1}{2}\xi\right)\right| \leq \|f_{\frac{2}{3}}\|_{L_a^{1+\epsilon}} \left\|\left(k_{\frac{3}{4}\xi}\right)^2\right\|_{L_a^{(1+\epsilon)'}} = \left(\frac{3}{2}\right)^{1/(1+\epsilon)} \|f\|_{L_a^{1+\epsilon}(\frac{2}{3}\mathbb{D})} \left\|\left(k_{\frac{3}{4}\xi}\right)^2\right\|_{L_a^{(1+\epsilon)'}}^2, \quad \xi \in \mathbb{T},$$

where $\epsilon > 0$. Now, note that

$$\left\|\left(k_{\frac{3}{4}\xi}\right)^2\right\|_{L_a^{(1+\epsilon)'}} \leq \left\|\left(k_{\frac{3}{4}\xi}\right)^2\right\|_{H^\infty} = \left(\frac{1}{1-\frac{3}{4}}\right)^2 = 16.$$

Finally, supposing $\|f\|_{L_a^{1+\epsilon}(\frac{2}{3}\mathbb{D})} \leq 1$, we obtain

$$\|f\|_{L_a^{1+\epsilon}(\frac{1}{2}\mathbb{D})} \leq \|f\|_{H^\infty(\frac{1}{2}\mathbb{D})} \leq 16 \left(\frac{3}{2}\right)^{1/(1+\epsilon)} \leq 24,$$

which gives

$$M_{1+\epsilon, \frac{2}{3}}(1+\epsilon, 1-\epsilon) \leq 24M_{2, \frac{1}{2}}(1+\epsilon, 1-\epsilon). \quad (54)$$

It remains to obtain a suitable upper bound for $M_{2, \frac{1}{2}}(1+\epsilon, 1-\epsilon)$. Let us show that

$$M_{2, \frac{1}{2}}(1+\epsilon, 1-\epsilon) \leq 2\sqrt{1+\epsilon} \left(\frac{2}{\epsilon}\right)^{\epsilon+\frac{3}{2}}. \quad (55)$$

For every $f \in \mathcal{R}_{1+\epsilon,1-\epsilon}$, we have $f_{\frac{1}{2}} \in \mathcal{R}_{1+\epsilon, \frac{1}{2}(1-\epsilon)} \subset \mathcal{R}_{1+\epsilon,1-\epsilon}$. If $\{1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_{1+\epsilon}\}$ is the set of the poles of f (thus, $|\lambda_j| \leq 1-\epsilon$, $j=1, \dots, 1+\epsilon$), then $f \in K_{B_{(1+\epsilon)}}$ with $1+\epsilon = \{\lambda_1, \dots, \lambda_{1+\epsilon}\} \subset (1-\epsilon)\mathbb{D}$, whereas the set $\{2/\bar{\lambda}_1, \dots, 2/\bar{\lambda}_{1+\epsilon}\}$ is the set of the poles of the function $f_{\frac{1}{2}}$ and $f_{\frac{1}{2}} \in K_{B_{(1+\epsilon)'}}$ with $(1+\epsilon)' = \{\frac{1}{2}\lambda_1, \dots, \frac{1}{2}\lambda_{1+\epsilon}\} \subset \frac{1-\epsilon}{2}\mathbb{D}$. Hence, there exist $r_0, \dots, r_{0_{1+\epsilon}} \in \mathbb{C}$ such that

$$f_{\frac{1}{2}} = \sum_{k=1}^{1+\epsilon} r_{0_k} e_k, \quad (56)$$

on $\mathbb{D}$, where $(e_k)_{k=1}^{1+\epsilon}$ is the Malmquist basis associated with the set $(1+\epsilon)'$. Since both $f_{\frac{1}{2}}$ and $\sum_{k=1}^{1+\epsilon} r_{0_k} e_k$ are meromorphic in $\mathbb{C}$ the equality (56) is in fact valid everywhere in $\mathbb{C}$. Thus,

$$f(\xi) = \sum_{k=1}^{1+\epsilon} r_{0_k} \left(\prod_{j=1}^{k-1} \frac{\lambda_j - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{(1 - \frac{1}{4}|\lambda_k|^2)^{1/2}}{1 - \bar{\lambda}_k \xi}, \quad \xi \in \mathbb{D},$$

and by the Cauchy-Schwarz inequality,

$$|f(\xi)| \leq \left(\sum_{k=1}^{1+\epsilon} |r_{0_k}|^2 \right)^{1/2} \left(\sum_{k=1}^{1+\epsilon} \left| \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{(1 - \frac{1}{4} |\lambda_k|^2)^{1/2}}{1 - \bar{\lambda}_j \xi} \right|^2 \right)^{1/2}. \quad (57)$$

for any $\xi \in \mathbb{D}$. Now, if $\lambda \in (1-\epsilon)\mathbb{D}$ and $\xi \in \mathbb{D}$,

$$\frac{\frac{\lambda}{2} - 2\xi}{1 - \bar{\lambda}\xi} = \frac{2(\frac{\lambda}{4} - \xi)}{1 - \frac{\bar{\lambda}}{4}\xi} \frac{1 - \frac{\bar{\lambda}}{4}\xi}{1 - \bar{\lambda}\xi} = 2b_{\frac{\lambda}{4}}(\xi) \left(1 + \frac{3\bar{\lambda}}{4(1 - \bar{\lambda}\xi)} \right),$$

which gives

$$\left| \frac{\frac{\lambda}{2} - 2\xi}{1 - \bar{\lambda}\xi} \right| \leq 2 \left(1 + \frac{3(1-\epsilon)}{4} \frac{1}{\epsilon} \right) = \frac{3+\epsilon}{2(\epsilon)} \leq \frac{2}{\epsilon}.$$

We get

$$\sum_{k=1}^{1+\epsilon} \left| \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{(1 - \frac{1}{4} |\lambda_k|^2)^{1/2}}{1 - \bar{\lambda}_j \xi} \right|^2 \leq \frac{1}{\epsilon^2} \sum_{k=1}^{1+\epsilon} 2^{2(k-1)} \left(\frac{1}{\epsilon} \right)^{2(k-1)} \leq \frac{1}{4} \left(\frac{2}{\epsilon} \right)^{3+2\epsilon}. \quad (58)$$

Now we first notice that

$$\left(\sum_{k=1}^{1+\epsilon} |r_{0_k}|^2 \right)^{1/2} = \|f_{\frac{1}{2}}\|_{H^2}.$$

For any function $\varphi(z) = \sum_{k \geq 0} \varphi(k) z^k$ in H^2 , one has

$$\|\varphi\|_{H^2}^2 = \sum_{k \geq 0} \frac{|\varphi(k)|^2}{\sqrt{k+1}} \sqrt{k+1} |\varphi(k)| \leq \|\varphi\|_{L_a^2} \|\varphi\|_{B_{2,2}^{1/2}},$$

We now use the upper bound of [21]: for $\varphi \in \mathcal{R}_{1+\epsilon, 1-\epsilon}$ one has

$$\begin{aligned} \|\varphi\|_{B_{2,2}^{1/2}}^2 &= \|\varphi'\|_{L_a^2}^2 + \|\varphi\|_{H^2}^2 \\ &\leq (3-\epsilon) \frac{1+\epsilon}{\epsilon} \|\varphi\|_{H^2}^2 + \|\varphi\|_{H^2}^2 \leq \frac{4(1+\epsilon)}{\epsilon} \|\varphi\|_{H^2}^2, \end{aligned}$$

which gives

$$\|\varphi\|_{H^2} \leq 2 \sqrt{\frac{1+\epsilon}{\epsilon}} \|\varphi\|_{L_a^2}.$$

In particular, with $\varphi = f_{\frac{1}{2}}$ we get $\varphi \in \mathcal{R}_{1+\epsilon, \frac{1}{2}(1-\epsilon)}$ and

$$\|f_{\frac{1}{2}}\|_{H^2} \leq 2 \sqrt{\frac{1+\epsilon}{1-(1-\epsilon)/2}} \|f_{\frac{1}{2}}\|_{L_a^2} \leq 2\sqrt{2} \|f_{\frac{1}{2}}\|_{L_a^2}. \quad (59)$$

We conclude from (57), (58) and (59) that for any $\xi \in \mathbb{D}$,

$$|f(\xi)| \leq \|f_{\frac{1}{2}}\|_{H^2} \left(\frac{1}{4} \frac{2^{3+2\epsilon}}{\epsilon} \right)^{\frac{1}{2}} \leq \frac{1}{2} \left(\frac{2}{\epsilon} \right)^{\epsilon + \frac{3}{2}} 2\sqrt{2(1+\epsilon)} \|f_{\frac{1}{2}}\|_{L_a^2},$$

that is,

$$|f(\xi)| \leq \sqrt{2(1+\epsilon)} \left(\frac{2}{\epsilon} \right)^{\epsilon + \frac{3}{2}} \|f\|_{L_a^2(\frac{1}{2}\mathbb{D})}, \quad \xi \in \mathbb{D}.$$

Taking the supremum over $\xi \in \mathbb{D}$ and $f \in \mathcal{R}_{1+\epsilon, 1-\epsilon}$ we obtain (55).

Combining (54) and (55) and choosing $d = 48$, $b = \frac{1}{2}$ and $c > 0$ such that $2^{1+\epsilon} \sqrt{1+\epsilon} \leq c^{1+\epsilon}$ for any $\epsilon \geq 0$, we complete the proof and obtain (43) (see [1]).

Chapter 5

Integration and Loci of Integrability with Lebesgue Classes

The formalism is generalized to arbitrary first-order logic models and is illustrated by several examples on the p -adics, on the Presburger structure and on o -minimal expansions of groups. Furthermore, within this formalism, we define the Radon transform and show the corresponding inversion formula. We generalize the main result of the authors in Cluckers and Miller about the stability under integration of the class of constructible functions, by relaxing the conditions on integrability. Further, we give an interpolation result for constructible functions by constructible functions with maximal locus of integrability. For any $q > 0$ and constructible functions f and μ on $E \times \mathbb{R}^n$, we show a theorem describing the structure of the set

$$\{(x, p) \in E \times (0, \infty] : f(x, \cdot) \in L^p(|\mu|_x^q)\},$$

where $|\mu|_x^q$ is the positive measure on $\mathbb{R}^n$ whose Radon–Nikodym derivative with respect to the Lebesgue measure is $|\mu(x, \cdot)|^q : y \mapsto |\mu(x, y)|^q$. We also show a closely related preparation theorem for f and μ . These results relate analysis (the study of L^p -spaces) with geometry (the study of zero loci).

Section (5.1): Positive Constructible Functions against Euler Characteristic and Dimension

By a subanalytic set we will always mean a globally subanalytic subset $X \subset \mathbf{R}^n$, meaning that X is subanalytic in the classical sense inside $\mathbf{P}^n(\mathbf{R})$ under the embedding $\mathbf{R}^n = \mathbf{A}^n(\mathbf{R}) \subseteq \mathbf{P}^n(\mathbf{R})$. By a subanalytic function we mean a function whose graph is a (globally) subanalytic set.

By Sub we denote the category of subanalytic subsets $X \subset \mathbf{R}^n$ for all $n > 0$, with subanalytic maps as morphisms. We work with the Euler characteristic $\chi : \text{Sub} \rightarrow \mathbf{Z}$ and the dimension $\dim : \text{Sub} \rightarrow \mathbf{N}$ of subanalytic sets as defined for o -minimal structures in [82].

Note that if $X \in \text{Sub}$, then, by the o -minimal triangulation theorem in [82], the o -minimal Euler characteristic $\chi(X)$ coincides with the Euler characteristic $\chi_{BM}(X)$ of X with respect to the Borel–Moore homology. If $X \in \text{Sub}$ is locally compact, the o -minimal Euler characteristic $\chi(X)$ coincides with the Euler characteristic $\chi_c(X)$ of X with respect to sheaf cohomology of X with compact supports and constant coefficient sheaf.

By [82], the Euler characteristic $\chi : \text{Sub} \rightarrow \mathbf{Z}$ satisfies the following:

$$\chi(\emptyset) = 0,$$

$$\chi(X) = \chi(Y) \text{ if } X \text{ and } Y \text{ are isomorphic in } \text{Sub}$$

and

$$\chi(X \cup Y) = \chi(X) + \chi(Y)$$

whenever $X, Y \in \text{Sub}$ are disjoint. The last equality for χ_{BM} and χ_c follows from the long exact (co)homology sequence. If we take X to be the unit circle in the plane $\mathbf{R}^2$ and Y a point in X , we see that this equality does not hold for the Euler characteristic associated with the topological singular (co)homology.

Thus we can think of $\chi : \text{Sub} \rightarrow \mathbf{Z}$ as a measure with values in the Grothendieck ring $K_0(\text{Sub})$ of the category Sub and, for any $X \in \text{Sub}$ and any function $f : X \rightarrow \mathbf{Z}$ with finite range and the property that $f^{-1}(a) \in \text{Sub}$ for all $a \in \mathbf{Z}$ (constructible functions), one has an obvious definition for

$$\int_X f \chi$$

such that $\chi(X) = \int_X 1_X \chi$ (cf. [171]).

This measure and integration against Euler characteristic is what is considered by Viro [171], Shapira [169,170] and Brocker [159]. However, for the measure $\chi: \text{Sub} \rightarrow \mathbb{Z}$ it is not true that $\chi(X) = \chi(Y)$ if and only if X and Y are isomorphic in Sub . Following the recent work of the first author and Francois Loeser [160–162] on motivic integration, we construct the universal measure μ for the category Sub with values in the Grothendieck semi-ring $SK_0(\text{Sub})$ of Sub such that $\mu(X) = \mu(Y)$ if and only if X and Y are isomorphic in Sub . Furthermore, we develop a direct image formalism for positive constructible functions, i.e., functions $f: X \rightarrow SK_0(\text{Sub})$ with finite range and the property that $f^{-1}(a) \in \text{Sub}$ for all $a \in SK_0(\text{Sub})$. This formalism is generalized to arbitrary first-order logic models and is illustrated by several examples on the p -adics, on the Presburger structure and on o -minimal expansions of groups. Moreover, within this formalism, we define the Radon transform and show the corresponding inversion formula.

We start by pointing out that, instead of Sub , we can work in this section with any o -minimal expansion of a field R using the category Def whose objects are definable sets and whose morphisms are definable maps.

By a semi-group we mean a commutative monoid with a unit element. Likewise, a semi-ring is a set equipped with two semi-group structures: addition and multiplication such that 0 is a unit element for the addition, 1 is the unit element for multiplication, and the two operations are connected by $x(y+z) = xy + xz$ and $0x = 0$. A morphism of semi-rings is a mapping compatible with the unit elements and the operations.

Let $A := \mathbb{Z} \times \mathbb{N}$ be the semi-ring where addition is given by $(a,b) + (a',b') = (a+a', \max(b,b'))$, the additive unit element is $(0,0)$, multiplication is given by $(a,b)(a',b') = (aa', b+b')$, and the multiplicative unit is $(1,0)$. Note that the ring generated by A by inverting additively any element of A is $\mathbb{Z}$ with the usual ring structure.

For $Z \in \text{Sub}$, we define $\mathcal{C}_+(Z)$ as the semi-ring of functions $Z \rightarrow A$ with finite image and whose fibers are subanalytic sets. We call $\mathcal{C}_+(Z)$ the semi-ring of positive constructible functions on Z . In particular, $\mathcal{C}_+(\{0\}) = A$.

If $Z \in \text{Sub}$, then we denote by Sub_Z the category of subanalytic maps $X \rightarrow Z$ for $X \in \text{Sub}$ with morphisms subanalytic maps that make the obvious diagrams commute. We define the Grothendieck semi-group $SK_0(\text{Sub}_Z)$ as the quotient of the free abelian semi-group over symbols $[Y \rightarrow Z]$ with $Y \rightarrow Z$ in Sub_Z by relations

$$[\emptyset \rightarrow Z] = 0, \tag{1}$$

$$[Y \rightarrow Z] = [Y' \rightarrow Z] \tag{2}$$

if $Y \rightarrow Z$ is isomorphic to $Y' \rightarrow Z$ in Sub_Z and

$$[(Y \cup Y') \rightarrow Z] + [(Y \cap Y') \rightarrow Z] = [Y \rightarrow Z] + [Y' \rightarrow Z] \tag{3}$$

for Y and Y' subsets of some $X \rightarrow Z$. There is a natural semi-ring structure on $SK_0(\text{Sub}_Z)$ where the multiplication is induced by taking fiber products over Z .

We write $SK_0(\text{Sub})$ for $SK_0(\text{Sub}_{\{0\}})$ and $[X]$ for $[X \rightarrow \{0\}]$. Note that any element of $SK_0(\text{Sub}_Z)$ can be written as $[X \rightarrow Z]$ for some $X \in \text{Sub}_Z$, because we can take disjoint unions in Sub corresponding to finite sums in $SK_0(\text{Sub}_Z)$.

Proposition (5.1.1)[158]: For $Z \in \text{Sub}$, there is a natural isomorphism of semi-rings

$$T : SK_0(\text{Sub}_Z) \rightarrow \mathcal{C}_+(Z)$$

induced by sending $[X \rightarrow Z]$ in Sub_Z to $Z \rightarrow A : z \mapsto (\chi(X_z), \dim(X_z))$, where X_z is the fiber above z . By consequence, $SK_0(\text{Sub}) = A$.

Proof. This follows immediately from the trivialisation property for definable maps in any o -minimal expansion of a field. See [82].

By means of this result, we may identify $SK_0(\text{Sub}_Z)$ and $\mathcal{C}_+(Z)$.

A general notion of positive measures on a Boolean algebra $\mathcal{S}$ of sets is a map $\mu : \mathcal{S} \rightarrow G$ with G a semi-group satisfying

$$\mu(X \cup Y) = \mu(X) + \mu(Y)$$

and

$$\mu(\emptyset) = 0$$

whenever $X, Y \in \mathcal{S}$ are disjoint. Often, one has a notion of isomorphisms between sets in $\mathcal{S}$ under which the measure should be invariant and which allows one to take disjoint unions of given sets in $\mathcal{S}$ (by taking disjoint isomorphic copies of the sets).

We let $\mu : \text{Sub} \rightarrow A$ be the positive measure which sends X to $(\chi(X), \dim(X))$. This measure is a universal measure on Sub with the property that $\mu(X) = \mu(Y)$ whenever there exists a subanalytic bijection between X and Y and where universal means that any other positive measure with this property factorises through μ .

Note that μ measures, in some sense, the topological size since, by the cell decomposition theorem from [82], $\mu(A) = \mu(B)$ will hold for two subanalytic sets A, B if and only if, for any fixed $n \geq 0$, there exists a finite partition of A , resp. B , into subanalytic C^n -manifolds $\{A_i\}_{i=1}^m$, resp. $\{B_i\}_{i=1}^m$, and subanalytic maps $A_i \rightarrow B_i$ which are isomorphisms of C^n -manifolds.

Now we can define the integral of any positive function $f \in \mathcal{C}_+(Z)$ as

$$\int_Z f \mu := \sum_i f_i \mu(Z_i)$$

where $\{Z_i\}$ is any finite partition of Z into subanalytic sets such that f is constant on each part Z_i with value f_i .

To show that this is independent of the partition $\{Z_i\}$, we just note that there is a unique $[X \rightarrow Z]$ in $SK_0(Z)$ which corresponds to f under T and that $\sum_i f_i \mu(Z_i)$ corresponds to $[X] = (\chi(X), \dim(X))$ in $A = SK_0(\text{Sub})$. This independence follows also from the cell decomposition theorem ([82]).

For $f : X \rightarrow Y$, there is an immediate notion of pushforward $f_! : \mathcal{C}_+(X) \rightarrow \mathcal{C}_+(Y)$, $f_! : SK_0(\text{Sub}_X) \rightarrow SK_0(\text{Sub}_Y)$, which is given by

$$f_!(g)(y) = \int_{f^{-1}(y)} g|_{f^{-1}(y)} \mu$$

for $g \in \mathcal{C}_+(X)$, resp. by

$$f_!([Z \rightarrow X]) = [Z \rightarrow Y],$$

for $Z \rightarrow X$ in Sub_X and where $Z \rightarrow Y$ is given by composition with $X \rightarrow Y$. Note that these pushforwards are compatible with T .

If $Y = \{0\}$, then $SK_0(\text{Sub}_Y) = A$ and we write $\mu([Z \rightarrow X])$ for $f_!([Z \rightarrow X])$ which is the integral of $[Z \rightarrow X]$.

Thus the functoriality condition $(f \circ h)_! = h_! \circ f_!$ can be interpreted as Fubini's Theorem, since

$$\int_X g \mu = \int_Y \left(\int_{f^{-1}(y)} g|_{f^{-1}(y)} \mu \right) \mu$$

for $g \in \mathcal{C}_+(X)$ and $h: Y \rightarrow \{0\}$.

For $f: X \rightarrow Y$ a morphism in Sub , there is an immediate notion of pullback $f^*: \mathcal{C}_+(Y) \rightarrow \mathcal{C}_+(X)$, resp. $f^*: SK_0(\text{Sub}_Y) \rightarrow SK_0(\text{Sub}_X)$, which is given by

$$f^*(g) = g \circ f$$

for $g \in \mathcal{C}_+(X)$, resp. by

$$f^*([Z \rightarrow Y]) = [Z \otimes_Y X \rightarrow X],$$

for $Z \rightarrow Y$ in Sub_Y and where $Z \otimes_Y X \rightarrow X$ is the projection and $Z \otimes_Y X$ is the set-theoretical fiber product. Note that these pullbacks are also compatible with T and satisfy the functoriality property $(f \circ h)^* = h^* \circ f^*$.

Proposition (5.1.2)[158]: Let $f: X \rightarrow Y$ be a morphism in Sub and let g be in $\mathcal{C}_+(X)$ and h in $\mathcal{C}_+(Y)$. Then

$$f_!(gf^*(h)) = f_!(g)h.$$

Proof. This is immediate at the level of SK_0 , since both the multiplication in SK_0 and the pullback are defined by the fiber product.

Let $S \subset X \times Y$, X, Y be subanalytic sets and write $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ for the projections and $q_X = \pi_{X|S}$ and $q_Y = \pi_{Y|S}$. For $g \in \mathcal{C}_+(X)$, we define the Radon transform $\mathcal{R}_S(g) \in \mathcal{C}_+(Y)$ by

$$\mathcal{R}_S(g) = q_{Y!} \circ q_X^*(g) = \pi_{Y!} \circ (\pi_X^*(g) 1_S)$$

where 1_S is the characteristic function on S .

Example:

Consider the case $X = \mathbf{R}^n$, $Y = \text{Gr}(n)$ with $S = \{(p, \Pi): p \in \Pi\}$. Let $Z \subseteq \mathbf{R}^n$ be a subanalytic subset and $\sigma_Z: \text{Gr}(n) \rightarrow A: \Pi \mapsto (\chi(\Pi \cap Z), \dim(\Pi \cap Z))$. Then $\sigma_Z = \mathcal{R}_S(1_Z)$.

Let $S' \subset Y \times X$ be another subanalytic set and put $q'_X = \pi_{X|S'}$ and $q'_Y = \pi_{Y|S'}$. The following proposition is showed just as in [170].

Proposition (5.1.3)[158]: Let $r: S \otimes_Y S' \rightarrow X \times X$ be the projection and suppose that the following hypotheses hold:

(*) there exists $\lambda \in A$ such that $[r^{-1}(x, x')] = \lambda$ for all $x \neq x'$, $x, x' \in X$;

(**) there exists $0 \neq \theta \in A$ such that $[r^{-1}(x, x)] = \theta + \lambda$ for all $x \in X$.

If g is in $\mathcal{C}_+(Y)$, then

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = \theta g + \lambda \int_X g \mu \quad (4)$$

and this is independent of the choice of θ .

Proof. Let h and h' be the projections from $S \otimes_Y S'$ to S and S' , respectively. Then, by definition of fiber product, $q_Y \circ h = q'_Y \circ h'$, and so, by functoriality of pullback and pushforward, we have $h'_! \circ h^* = q_{Y'}^* \circ q_{Y!}$. Thus $\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = q_{X'}^* \circ (q'_Y)^* \circ q_{Y!} \circ q_X^*(g) = q_{X'}^* \circ h'_! \circ h^* \circ q_X^*(g)$.

The last formula is also equal to $p_{2!} \circ r_! \circ r^* \circ p_1^*(g)$, where $p_1, p_2: X \times X \rightarrow X$ are the projections onto the first and second coordinates respectively, since $q_X \circ h = p_1 \circ r$ and

$q'_X \circ h' = p_2 \circ r$. The hypothesis shows that $r_!(1_{S \otimes_Y S'}) = \theta 1_{\Delta_X} + \lambda 1_{X \times X}$, moreover this expression is independent of the choice of θ . By the projection formula, $r_!(r^*(p_1^*(g))) = r_!(1_{S \otimes_Y S'} r^*(p_1^*(g))) = r_!(1_{S \otimes_Y S'}) p_1^*(g) = (\theta 1_{\Delta_X} + \lambda 1_{X \times X}) p_1^*(g)$ holds, hence we obtain $p_{2!}((\theta 1_{\Delta_X} + \lambda 1_{X \times X}) p_1^*(g)) = \theta p_{2!}(1_{\Delta_X} p_1^*(g)) + \lambda p_{2!}(p_1^*(g)) = \theta g + \lambda \int_X g \mu$, as required.

We now show that the inversion formula is independent of the choice of θ . If $\theta + \lambda = \theta' + \lambda$ and $\theta \neq \theta'$, then necessarily $\lambda_2 > \theta_2$, $\lambda_2 > \theta'_2$ and $\theta_1 = \theta'_1$ with $\lambda = (\lambda_1, \lambda_2)$, $\theta = (\theta_1, \theta_2)$ and $\theta' = (\theta'_1, \theta'_2)$. Hence, $\theta g + \lambda \int_X g \mu = \theta' g + \lambda \int_X g \mu$ for all $x \in X$.

Example:

Consider the case $X = \mathbf{R}^n$, $Y = \text{Gr}(n)$ with $S = \{(p, \Pi) : p \in \Pi\}$ and $S' = \{(\Pi, p) : p \in \Pi\}$. Then $[r^{-1}(x, x)] = [\mathbf{P}^{n-1}]$ and $[r^{-1}(x, x')] = [\mathbf{P}^{n-2}]$ for all $x, x' \in \mathbf{R}^n$ with $x \neq x'$. Since $[\mathbf{P}^n] = (\frac{1+(-1)^n}{2}, n)$, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = ((-1)^{n+1}, n-1)g + \left(\frac{1+(-1)^n}{2}, n-2 \right) \int_X g \mu .$$

In particular, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(1_Z) = ((-1)^{n+1}, n-1)1_Z + \left(\frac{1+(-1)^n}{2}, n-2 \right) [Z]$$

for every subanalytic subset Z of $\mathbf{R}^n$.

Let $\mathcal{M}$ be a model of a theory in a language $\mathcal{L}$ with at least two constant symbols c_1 , c_2 satisfying $c_1 \neq c_2$. For Z a definable set, we define the category $\text{Def}_Z(\mathcal{M})$, also written Def_Z for short, whose objects are definable sets X with a definable map $X \rightarrow Z$ and whose morphisms are definable maps that make the obvious diagram commute. We write $\text{Def}(\mathcal{M})$ or Def for $\text{Def}_{\{c_1\}}(\mathcal{M})$. In $\mathcal{M}$, one can pursue the usual operations of set theory like finite unions, intersections, Cartesian products, disjoint unions and fiber products.

We define the Grothendieck semi-group $SK_0(\text{Def}_Z)$ as the quotient of the free abelian semi-group over symbols $[Y \rightarrow Z]$ with $Y \rightarrow Z$ in Def_Z by relations

$$[\phi \rightarrow Z] = 0, \tag{5}$$

$$[Y \rightarrow Z] = [Y' \rightarrow Z] \tag{6}$$

if $Y \rightarrow Z$ is isomorphic to $Y' \rightarrow Z$ in Def_Z and

$$[(Y \cup Y') \rightarrow Z] + [(Y \cap Y') \rightarrow Z] = [Y \rightarrow Z] + [Y' \rightarrow Z] \tag{7}$$

for Y and Y' subsets of some $X \rightarrow Z$. There is a natural semi-ring structure on $SK_0(\text{Def}_Z)$ where the multiplication is induced by taking fiber products over Z . Note that any element of $SK_0(\text{Def}_Z)$ can be written as $[X \rightarrow Z]$ for some $X \rightarrow Z \in \text{Def}_Z$, because we can take disjoint unions in $\mathcal{M}$ corresponding to finite sums in $SK_0(\text{Def}_Z)$.

The map $\text{Def} \rightarrow SK_0(\text{Def})$ sending X to its class $[X]$ is a universal positive measure with the property that two sets have the same measure if there exists a definable bijection between them. For $f : X \rightarrow Y$, there is an immediate notion of pushforward $f_! : SK_0(\text{Sub}_X) \rightarrow SK_0(\text{Sub}_Y)$ given by

$$f_!([Z \rightarrow X]) = [Z \rightarrow Y],$$

for $Z \rightarrow X$ in Def_X and where $Z \rightarrow Y$ is given by composition with $X \rightarrow Y$.

If $Y = \{c_1\}$, then we write $\mu([Z \rightarrow X])$ for $f_!([Z \rightarrow X])$, which we call the integral of $[Z \rightarrow X]$; note that $\mu([Z \rightarrow X])$ is just $[Z]$ in $SK_0(\text{Def})$. Thus the functoriality condition

$(f \circ h)_i = h_i \circ f_i$ can be interpreted as Fubini's Theorem.

There is also an immediate notion of pullback $f^*: SK_0(\text{Sub}_Y) \rightarrow SK_0(\text{Sub}_X)$ given by

$$f^*([Z \rightarrow Y]) = [Z \otimes_Y X \rightarrow X],$$

for $Z \rightarrow Y$ in Def_Y and where $Z \otimes_Y X \rightarrow X$ is the projection and $Z \otimes_Y X$ the set-theoretical fiber product. The pullback is functorial, i.e., $(f \circ h)^* = h^* \circ f^*$.

Proposition (5.1.4)[158]: Let $f: X \rightarrow Y$ be a morphism in Def and let g be in $SK_0(\text{Def}_X)$ and h in $SK_0(\text{Def}_Y)$. Then

$$f_!(gf^*(h)) = f_!(g)h.$$

Proof. Exactly the same proof as for the subanalytic sets above works.

One can also define the Radon transform in this context in exactly the same way as in the subanalytic case. Furthermore, the same argument as in the subanalytic case gives the corresponding inversion formula. However, since, in general, there is no trivialisation theorem, the conditions (*) and (**) in Proposition (2.8.1) have to be replaced by global conditions. Using the embedding $SK_0(\text{Sub}) \rightarrow SK_0(\text{Sub}_U)$ sending $[W]$ to $[W \times U \rightarrow U]$ where $W \times U \rightarrow U$ is the projection, the statement becomes:

Let $r: S \otimes_Y S' \rightarrow X \times X$ be the projection and suppose that the following hypotheses hold:

(*) there exists Z_1 in Def such that in $SK_0(\text{Def}_{X_1})$ we have

$$[B_1 \rightarrow X_1] = [Z_1],$$

(**) there exists Z_2 in Def such that in $SK_0(\text{Def}_{\Delta_X})$ we have

$$[B_2 \rightarrow \Delta_X] = [Z_1] + [Z_2]$$

where $X_1 = X \times X \setminus \Delta_X$, $B_1 = S \otimes_Y S' \setminus r^{-1}(\Delta_X)$, $B_2 = S \otimes_Y S' \cap r^{-1}(\Delta_X)$ and $B_1 \rightarrow X_1$ and $B_2 \rightarrow \Delta_X$ are the restrictions of the projection $r: S \otimes_Y S' \rightarrow X \times X$. If $Z \rightarrow X$ is in Def_X , then

$$\mathcal{R}_{S'} \circ \mathcal{R}_S([Z \rightarrow X]) = [Z_2][Z \rightarrow X] + [Z_1][Z] \quad (8)$$

and this is independent of the choice of Z_2 .

Example (5.1.5)[158]: For K any finite field extension of the field $\mathbb{Q}_p$ of p -adic numbers, one can calculate explicitly the semi-ring of semialgebraic sets $SK_0(K, \text{Sem})$, resp. of globally subanalytic sets $SK_0(K, \text{Sub})$, using work of [70] for semialgebraic sets, resp. using work of [83] for the subanalytic sets. In both cases it is a subset of $\mathbf{N} \times \mathbf{N}$, and the class of a semialgebraic set X , resp. a subanalytic set X , is $(\sharp X, 0)$ if X is finite and $(0, \dim X)$ if X is infinite. This is because there exists a semialgebraic bijection between two infinite semialgebraic sets if and only if they have the same dimension, and similarly for subanalytic sets. However, no trivialisation theorem is known, hence the relative semi-Grothendieck rings $SK_0(K, \text{Sem}_Z)$, resp. $SK_0(K, \text{Sub}_Z)$, for Z semialgebraic, resp. subanalytic, are expected to be much more complicated than maps $Z \rightarrow \mathbf{N} \times \mathbf{N}$ with finite image.

Example (5.1.6)[158]: Consider the Presburger structure on $\mathbf{Z}$ by using the Presburger language

$$\mathcal{L}_{\text{PR}} = \{+, -, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \mathbf{N}, n > 1\},$$

with $\equiv_n$ the equivalence relation modulo n . Again, one can calculate explicitly the semi-

ring $SK_0(\mathbf{Z}, \mathcal{L}_{\text{PR}})$, using work of [68]. It is a subset of $\mathbf{N} \times \mathbf{N}$, and the class of a Presburger set X is $(\#X, 0)$ if X is finite and $(0, \dim X)$ if X is infinite, where the dimension of [68] is used. Again, this is because there exists a Presburger bijection between two infinite Presburger sets if and only if they have the same dimension. Again, no trivialisation theorem is known, hence the relative semi-Grothendieck rings are expected to be more complicated.

Example (5.1.7)[158]: Let $K = (K, 0, 1, +, \cdot, <)$ be an ordered field and consider the structure $\mathcal{M} = (K, 0, 1, +, (\lambda_c)_{c \in K}, <)$, where λ_c is the scalar multiplication by $c \in K$. The category Def in this case is the category of K -semilinear sets with K -semilinear maps.

By [166], the Grothendieck ring $K_0(\text{Def})$ is isomorphic to $E = \mathbf{Z}[x]/(x(x+1))$ and there is a universal Euler characteristic $\epsilon: \text{Def} \rightarrow E$ (see also [164]).

Let D be the set whose elements are of the form $\sum_{i=1}^n y^{k_i} z^{l_i} \in \mathbf{N}[y, z]$ with $k_i \leq l_i$ and, for $i \neq j$, $\neg(y^{k_i} z^{l_i} = y^{k_j} z^{l_j}) \wedge \neg(y^{k_i} z^{l_i} \prec y^{k_j} z^{l_j}) \wedge \neg(y^{k_j} z^{l_j} \prec y^{k_i} z^{l_i})$. Here, $y^{k_i} z^{l_i} \prec y^{k_j} z^{l_j}$ if and only if $k_i < k_j$ and $l_i < l_j$.

The set D can be equipped with a semi-ring structure in the following way: the zero element 0_D is $\sum_{i=1}^0 y^{k_i} z^{l_i}$, the identity element 1_D is $y^0 z^0$, the addition is given by

$$\sum_{i=1}^n y^{k_i} z^{l_i} + D \sum_{i=1}^m y^{k'_i} z^{l'_i} = \sum \max_{\prec} \left\{ y^{k_i} z^{l_i} : y^{k_i} z^{l_i} \text{ a monomial in } \sum_{i=1}^n y^{k_i} z^{l_i} + \sum_{i=1}^m y^{k'_i} z^{l'_i} \right\}$$

and multiplication is given by

$$\sum_{i=1}^n y^{k_i} z^{l_i} \cdot D \sum_{i=1}^m y^{k'_i} z^{l'_i} = \sum \max_{\prec} \left\{ y^{k_i} z^{l_i} : y^{k_i} z^{l_i} \text{ a monomial in } \sum_{i=1}^n y^{k_i} z^{l_i} \cdot \sum_{i=1}^m y^{k'_i} z^{l'_i} \right\}$$

where the symbol $\sum \max_{\prec} S$ mean that we sum up the $\prec$ -maximal elements of the finite set S .

By [166], there is a universal abstract dimension $\delta: \text{Def} \rightarrow D$ and two sets in Def are isomorphic in Def if and only if they have the same universal Euler characteristic and the same universal abstract dimension. Thus, if A is the semi-ring $E \times D$, then the Grothendieck semi-ring $SK_0(\text{Def})$ is isomorphic to A and the map $\mu: \text{Def} \rightarrow A$ given by $\mu(X) = (\epsilon(X), \delta(X))$ is the positive universal measure on Def.

Note that the results that we used above from [166] were proved in the field of real numbers, but the same arguments hold in any arbitrary ordered field K .

Example (5.1.8)[158]: Let $K = (K, 0, 1, +, \cdot, <)$ be a real closed field and consider the structure $\mathcal{M} = (K, 0, 1, +, (\lambda_c)_{c \in K}, B, <)$, where λ_c is the scalar multiplication by $c \in K$ and B is the graph of multiplication on a bounded interval. The category Def in this case is the category of K -semibounded sets with K -semibounded maps.

By [165], all bounded semialgebraic subsets are in Def and, by [168], $\mathcal{M}$ is, up to definability, the only $\mathcal{o}$ -minimal structure properly between $(K, 0, 1, +, (\lambda_c)_{c \in K}, <)$ and $(K, 0, 1, +, \cdot, <)$.

By [166], the Grothendieck ring $K_0(\text{Def})$ is isomorphic to $E = \mathbf{Z}[x]/(x(x+1))$ and there is a universal Euler characteristic $\epsilon: \text{Def} \rightarrow E$ (see also [164]). Furthermore, if D is the semi-ring of Example (5.1.7), then there is a universal abstract dimension $\delta: \text{Def} \rightarrow D$ and two sets in Def are isomorphic in Def if and only if they have the same universal Euler characteristic and the same universal abstract dimension. Thus, if A is the semi-ring $E \times D$, then the Grothendieck semi-ring $SK_0(\text{Def})$ is isomorphic to A and the

map $\mu: \text{Def} \rightarrow A$ given by $\mu(X) = (\epsilon(X), \delta(X))$ is the positive universal measure on Def .

The results that we used above from [166] were proved in the field of real numbers and are based on Peterzil's [167] structure theorem for semibounded sets in the real numbers. However, the same arguments hold in any arbitrary real closed field K using the structure theorem from [163].

Section (5.2): Zero Loci and Stability under Integration for Constructible Functions on Euclidean Space with Lebesgue Measure

We define and study loci of integrability of certain (families of) functions. A recent insight into parameterized integrals is that, for functions f belonging to certain classes of functions on certain product measure spaces $E \times T$, a set of the form

$$\{x \in E \mid T \rightarrow \mathbb{C} : t \mapsto f(x, t) \text{ is measurable and integrable over } T\}, \quad (9)$$

is in fact equal to the zero locus of a function on E belonging to the same class of functions; see [176]. If we call the set in (9) the locus of integrability of f in E , then we can rephrase the recent insight as a link between loci of integrability and zero loci for certain kinds of functions.

We give such a link for the class of constructible functions on Euclidean spaces with the Lebesgue measure; see Theorem (5.2.8). We follow the terminology of [178]: a constructible function is by definition a sum of products of globally subanalytic functions and of logarithms of globally subanalytic functions; see below for more detailed definitions. The advantage of the class of constructible functions is that it is closed under integration. Indeed, in Cluckers and Miller [178] proved that if f is constructible on $\mathbb{R}^n \times \mathbb{R}^m$ such that $y \mapsto f(x, y)$ is integrable over $\mathbb{R}^m$ for each $x \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^m} f(x, y) dy$$

is constructible on $\mathbb{R}^n$, which generalizes results of [180]. We extend this stability result by relaxing the conditions on integrability; see Theorem (5.2.10). Further, we give an interpolation result, Theorem (5.2.9), of constructible functions by constructible functions with maximal locus of integrability.

Recall that a function $f: X \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is called globally subanalytic if its graph is a globally subanalytic set, and a set $A \subset \mathbb{R}^n$ is called globally subanalytic if its image under the natural embedding of $\mathbb{R}^n$ into n -dimensional real projective space, namely $\mathbb{R}^n \rightarrow \mathbb{P}^n(\mathbb{R}) : (x_1, \dots, x_n) \mapsto (1: x_1, \dots, x_n)$, is a subanalytic subset of $\mathbb{P}^n(\mathbb{R})$ in the classical sense; see Definition (5.2.3) below for a self-contained definition.

From now on in this section, we write “subanalytic” instead of “globally subanalytic” (see again Definition (5.2.3)).

Definition (5.2.1)[172]: For each subanalytic set X , let $\mathcal{C}(X)$ be the $\mathbb{R}$ -algebra of real-valued functions on X generated by all subanalytic functions on X and all the functions $x \mapsto \log f(x)$, where $f: X \rightarrow (0, +\infty)$ is subanalytic. Functions in $\mathcal{C}(X)$ are called constructible functions on X and $\mathcal{C}(X)$ is called the algebra of constructible functions on X .

In the whole section, we use the Lebesgue measure on $\mathbb{R}^n$. We introduce the locus of integrability of a function, as follows.

Definition (5.2.2)[172]: For E a set, and for $f: E \times \mathbb{R}^n \rightarrow \mathbb{C}$ a function, define the locus of integrability of f in E as the set

$$\text{Int}(f, E) := \{x \in E \mid f(x, \cdot) \text{ is measurable and integrable over } \mathbb{R}^n\},$$

where $f(x, \cdot)$ is the function sending $y \in \mathbb{R}^n$ to $f(x, y)$, and where the Lebesgue measure is used on $\mathbb{R}^n$.

The main results of this section are the following three theorems, for which we will give relatively short and simple proofs.

Definition (5.2.3)[172]: Call a function $f : X \subset \mathbb{R}^l \rightarrow \mathbb{R}^k$ analytic if it extends to an analytic function on an open neighborhood of X . A restricted analytic function is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the restriction of f to $[-1,1]^n$ is analytic and $f(x) = 0$ on $\mathbb{R}^n \setminus [-1,1]^n$.

Call a set or a function subanalytic if and only if it is definable in the expansion of the real field by all restricted analytic functions. Thus in this section, “subanalytic” is an abbreviation of “globally subanalytic”, and in this meaning, the natural logarithm $\log : (0, +\infty) \rightarrow \mathbb{R}$ is not subanalytic.

For the rest of this section we fix an ordered list of variables $x_1, \dots, x_{n+1}$, where $n \geq 0$, and we write x for $(x_1, \dots, x_n)$ and write y for x_{n+1} , since the variable x_{n+1} will play a special role.

Definition (5.2.4)[172]: Consider subanalytic sets $A \subset \mathbb{R}^{n+1}$ and $B \subset \mathbb{R}^n$ and an analytic subanalytic function $\theta : B \rightarrow \mathbb{R}$. Then A is called a 0-cell over $\mathbb{R}^n$ with base B if A equals the graph of an analytic subanalytic function $c : B \rightarrow \mathbb{R} : x \mapsto y = c(x)$.

Call A a 1-cell over $\mathbb{R}^n$ with base B and with center θ if there are analytic subanalytic functions $a : B \rightarrow \mathbb{R}$ and $b : B \rightarrow \mathbb{R}$, with $a < b$ on B , such that A is of the following form:

$$A = \{(x, y) \in B \times \mathbb{R} : a(x) \sqcap_i y \sqcap_i b(x)\},$$

with $\sqcap_i$ either $<$ or no condition for each $i = 1, 2$, and such that the graph $\Gamma(\theta)$ of θ satisfies either

$$\Gamma(\theta) \subset \bar{A} \setminus A, \quad \text{or,} \quad \Gamma(\theta) \cap \bar{A} = \emptyset,$$

where $\bar{A}$ is the topological closure of A inside $\mathbb{R}^{n+1}$. In any case, A is called a cell over $\mathbb{R}^n$.

Definition (5.2.5)[172]: Let A be a 1-cell over $\mathbb{R}^n$ with base B and with center θ . A basic function with center θ is a function $\varphi : A \rightarrow \mathbb{R}^{N+2}$, for some $N \geq 0$, with bounded image and which is of the form

$$\varphi(x, y) = (a_1(x), \dots, a_N(x), b_1(x)|y - \theta(x)|^{1/p}, b_2(x)|y - \theta(x)|^{-1/p}), \quad (10)$$

where $a_1, \dots, a_N, b_1, b_2$ are analytic subanalytic functions from B to $\mathbb{R}$ and p is a positive integer. A strong function on A with center θ is a function $A \rightarrow \mathbb{R}$ of the form $F \circ \varphi$, where φ is a basic function with center θ and where the function F is given by a single power series that converges on an open neighborhood of the image of φ . Note that strong functions are automatically subanalytic functions.

Theorem (5.2.6)[172]: (Preparation of subanalytic functions [184, 189]). Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $X \subset \mathbb{R}^{n+1}$. Then there exists a finite partition of X into cells over $\mathbb{R}^n$ such that the following holds for any 1-cell A over $\mathbb{R}^n$ in this partition:

There exists a center θ for A such that each $f \in \mathcal{F}$ can be written in the form

$$f(x, y) = g(x)|y - \theta(x)|^r S(x, y)$$

on A , where g is an analytic subanalytic function on the base of A , r is a rational number, and S is a strong function on A with center θ , and such that, moreover, $S > \epsilon$ on A for some $\epsilon > 0$.

The last part in the following corollary is new and simplifies the proofs concerning integration and integrability when compared with [178].

Corollary (5.2.7)[172]: (Preparation of constructible functions). Let $\mathcal{F}$ be a finite set of constructible functions on a subanalytic set $X \subset \mathbb{R}^{n+1}$. Then there exists a finite partition of X into cells over $\mathbb{R}^n$ such that for each 1-cell A over $\mathbb{R}^n$ with base B in this decomposition, there exists a center θ such that, the following holds for each $f \in \mathcal{F}$ and all $(x, y) \in A$, and with $\tilde{y} := y - \theta(x)$:

$$f(x, y) = \sum_{i=1}^M d_i(x) S_i(x, y) |\tilde{y}|^{\alpha_i} (\log|\tilde{y}|)^{\ell_i}, \quad (11)$$

for some $M \geq 0$, functions $d_i \in \mathcal{C}(B)$, rational numbers α_i , integers $\ell_i \geq 0$, and strong functions S_i on A with center θ . Moreover, one can ensure for each i that at least one of the following two conditions holds:

- (1) $S_i(x, y) = 1$ on A ;
- (2) $y \mapsto |\tilde{y}|^{\alpha_i}$ is integrable over A_x for all $x \in B$.

Proof. Let $\mathcal{F}'$ be a finite collection of subanalytic functions such that each $f \in \mathcal{F}$ is a finite sum of products of functions in $\mathcal{F}'$ and of logarithms of functions in $\mathcal{F}'$. Apply Theorem (5.2.6) to $\mathcal{F}'$. Note that $\log(S)$ is a strong function with center θ if S is a strong function with center θ satisfying $S > \epsilon$ for some $\epsilon > 0$. Hence, we are done with the first part of the statement by writing logarithms of products as sums of logarithms, and since the product of strong functions with center θ is a strong function with center θ . Suppose now that, for some occurring term $S_i(x, y) d_i(x) |\tilde{y}|^{\alpha_i}$ on some cell A with center θ and base B , one has that $y \mapsto |\tilde{y}|^{\alpha_i}$ is not integrable over A_x for some (and hence for all) $x \in B$.

Then, by the supposed presence of this nonintegrable term and by partitioning the cells slightly further, we may suppose that exactly one of the following two conditions holds:

- (i) The graph of the center θ lies in $\bar{A}$ and A_x is bounded in $\mathbb{R}$ for each value of $x \in B$.
- (ii) The graph of the center θ is disjoint from $\bar{A}$ and A_x is not contained in a compact subset of $\mathbb{R}$ for any value of $x \in B$.

Since the argument is completely similar in both cases, let us suppose (i) holds. Then, writing the strong function S_i as $F_i \circ \varphi$ with φ a basic function with center θ , as in (10), and F_i a converging power series, and by recalling that the image of φ is bounded, one sees that $b_2(x) = 0$ for all $x \in B$, with notation from (10). Moreover, $\tilde{y}$ is bounded on A_x for each x , and thus, $|\tilde{y}|^{q+\alpha_i}$ is integrable over A_x for all $x \in B$ as soon as $q \in \mathbb{Q}$ is sufficiently large. For any $s > 0$ we can develop finitely many terms of F_i in $|\tilde{y}|^{1/p}$ plus the remaining series in $|\tilde{y}|^{1/p}$, as follows:

$$S_i(x, y) = \left(\sum_{j=0}^{s-1} c_j(x) |\tilde{y}|^{j/p} \right) + \left(\sum_{j \geq s} c_j(x) |\tilde{y}|^{j/p} \right). \quad (12)$$

By pulling out the factor $|\tilde{y}|^{s/p}$ from the last term, by writing out $S_i(x, y) d_i(x) |\tilde{y}|^{\alpha_i}$ using distributivity and (12), and by taking s large enough, the first s such terms will be as in part (1) of the corollary, and the last term will be integrable as in (2). This completes the proof.

Theorem (5.2.8)[172]: Let f be in $\mathcal{C}(E \times \mathbb{R})$ for some subanalytic set E . Then there exists h in $\mathcal{C}(E)$ such that

$$\text{Int}(f, E) = \{x \in E \mid h(x) = 0\}. \quad (13)$$

Conversely, for every h in $\mathcal{C}(E)$ there exists f in $\mathcal{C}(E \times \mathbb{R})$ such that (13) holds.

Theorem (5.2.8) thus gives a correspondence between loci of integrability and zero loci of constructible functions, at least when integration is in dimension 1, that is, over $\mathbb{R}$. One should not misunderstand Theorem (5.2.8): zero loci of constructible functions are much more general than, say, Zariski closed sets, and for example, a zero locus of $h \in \mathcal{C}(E)$ can easily be dense in E . Indeed, the characteristic function of any subanalytic subset of E lies in $\mathcal{C}(E)$. Note that when f in Theorem (5.2.8) is moreover subanalytic, then one can take h to be a subanalytic function as well by the main result of [180]. Theorem (5.2.8) implies Theorem 1.4 of [178]. In Cluckers and Miller [179] we treat a higher dimensional variant of Theorem (5.2.8), also treating L^p -integrability for various p .

Constructible functions allow an interpolation by constructible functions with maximal locus of integrability, as follows.

Theorem (5.2.9)[172]: Let f be in $\mathcal{C}(E \times \mathbb{R})$ for some subanalytic set E . Then there exists $g \in \mathcal{C}(E \times \mathbb{R})$ with

$$\text{Int}(g, E) = E$$

and such that, for all $x \in \text{Int}(f, E)$ and all $y \in \mathbb{R}$, one has

$$g(x, y) = f(x, y).$$

Finally, we can integrate in any dimension m to find the following generalization of the principal result, Theorem 1.3, of [178].

Proofs of Theorems (5.2.8) and (5.2.9). Let f be in $\mathcal{C}(E \times \mathbb{R})$, with $E \subset \mathbb{R}^n$ for some n . Apply Corollary (5.2.7) to the collection of functions consisting only of f . Consider a 1-cell A over $\mathbb{R}^n$ in the obtained partition, with center θ , and write f as in (11). By regrouping the terms and using the notation of (11), we may suppose, for each i , that either $|\tilde{y}|^{\alpha_i}$ is integrable over A_x , or that (α_i, ℓ_i) is different from the (α_j, ℓ_j) for all $j \neq i$. Let I be those indices i such that $|\tilde{y}|^{\alpha_i}$ is not integrable over A_x . Now define Q_A as the set $\{x \in B \mid d_i(x) = 0 \text{ for } i \in I\}$ and define, for $(x, y) \in A$, the constructible function

$$g(x, y) := \sum_{i \notin I} d_i(x) S_i(x, y) |\tilde{y}|^{\alpha_i} (\log |\tilde{y}|)^{\ell_i}.$$

Note that

$$\{x \in B : f(x, \cdot) \text{ is integrable over } A_x\} = Q_A,$$

because of condition (1) in Corollary (5.2.7), and because we have taken the exponent pairs (α_i, ℓ_i) mutually different for nonintegrable terms. Do the above construction for each occurring 1-cell A over $\mathbb{R}^n$. On any 0-cell A' over $\mathbb{R}^n$ in our partition, define $g(x, y)$ as $f(x, y)$. Then g is as desired by Theorem (5.2.9). Now note that a finite union of zero loci of constructible functions h_i equals the zero locus of a single constructible function by taking the product of the h_i . Similarly, a finite intersection of zero loci of constructible functions h_i equals the zero locus of a single constructible function by taking the sum of the squares of the h_i . Now one is done for $\text{Int}(f, E)$. Indeed, $\text{Int}(f, E)$ equals the finite intersection

$$\bigcap_A Q'_A,$$

where A runs over all 1-cells over $\mathbb{R}^n$ in the partition, and where, for any such 1-cell A , Q'_A equals the set $Q_A \cup (E \setminus B)$. Note that $E \setminus B$ is a subanalytic set and each of the Q'_A equals thus the zero locus of a constructible function on E . For the converse statement of Theorem (5.2.8), given h , it suffices to put $f(x, y) = h(x)y$ for all $(x, y) \in E \times \mathbb{R}$.

Theorem (5.2.10)[172]: Let f be in $\mathcal{C}(E \times \mathbb{R}^m)$ for some subanalytic set E and some $m > 0$. Then there exists $g \in \mathcal{C}(E)$ such that, for each $x \in \text{Int}(f, E)$, one has

$$g(x) = \int_{y \in \mathbb{R}^m} f(x, y) dy .$$

The above theorem is proved in Cluckers and Miller [178] under the extra condition that $\text{Int}(f, E)$ equals E (which in turn generalized main results from [180, 185]). Note that integrals of constructible functions are related to what one could call families of periods; see [181–183]. In several special cases, explicit formulas for parameterized integrals of constructible functions are given in [173, 188]. Parameterized integrals of constructible functions are often used for the study of singularities, as in [174, 186, 187]. For context on subanalytic functions we refer the reader to [175, 74].

[176] which contains several p -adic and motivic analogues of this section, where [178] was more closely inspired on p -adic and motivic results of [83, 177]. The results and proofs of this section can be used to replace some of the technical difficulties encountered in Cluckers and Miller [178].

In this section, we recall a basic form of the subanalytic preparation theorem from [184] (see also [189]), we fix some notation, and we give a new preparation result for constructible functions.

Proof. Consider f in $\mathcal{C}(E \times \mathbb{R}^m)$ for some $m > 0$. If $m = 1$, then apply Theorem (5.2.9) to f to find g_0 in $\mathcal{C}(E \times \mathbb{R})$ with $\text{Int}(g_0, E) = E$ and such that $g_0(x, y) = f(x, y)$ for all $x \in \text{Int}(f, E)$ and all $y \in \mathbb{R}$. Now Apply Theorem 1.3 of [178] to g_0 , which states that, if one defines, for $x \in E$,

$$g(x) := \int_{\mathbb{R}} g_0(x, y) dy ,$$

then g lies in $\mathcal{C}(E)$. Then this g is as desired. The result for general m now follows from Fubini's Theorem.

Alternatively to deriving Theorem (5.2.10) for $m = 1$ from Theorem 1.3 of [178], one can also derive the case $m = 1$ from Corollary (5.2.7) by the integration procedure by Lion and Rolin of [185], which is also used and explained in Cluckers and Miller [178]. This self-contained approach for obtaining Theorem (5.2.10) is simpler than the approaches of [178, 180, 185], which moreover only yielded special forms of Theorem (5.2.10).

Section (5.3): Preparation of Real Constructible Functions

The Lebesgue spaces, $L^p(\mu)$ for $p \in (0, \infty]$, are ubiquitous in many areas of mathematical analysis and its applications. Much of the research about the Lebesgue spaces has been conducted in a very general measure-theoretic framework, with the focus being on discovering a host of relationships between the various L^p spaces. A number of the classical theorems are inequalities that explain how various function operations behave with respect to the Lebesgue spaces. For example, for addition there is Minkowski's inequality; for multiplication there is Hölder's inequality; for convolutions there is Young's convolution inequality; for Fourier transforms of periodic functions there is the Hausdorff–Young inequality. Other classical theorems explain the structure of linear maps between the various L^p spaces, such as the duality of Lebesgue spaces with conjugate exponents and the Riesz–Thorin interpolation theorem.

This section explores theorems about the Lebesgue spaces of a rather different sort. We use geometric techniques to study the structure of the Lebesgue classes of parameterized families of functions, along with a related preparation theorem. The

starting point of our investigation is the observation that, although much of the utility of the Lebesgue spaces – and more generally, of the theory of integration as a whole – stems from the generality of the measure-theoretic framework in which it has been developed, it is many times applied to study integrals of very special functions that arise naturally in real analytic geometry. And, if we focus our attention on studying the L^p properties of these very special functions, we should be able to obtain rather strong theorems that cannot be shown, or even reasonably formulated, in a very general measure-theoretic framework. This is because by focusing on special functions, we can supplement the very general tools from mathematical analysis with much more specialized tools from real analytic geometry and $\mathcal{o}$ -minimal structures. Similar approaches have been followed in the context of p -adic and motivic integration; see e.g. [177].

The $\mathcal{o}$ -minimal framework is still a bit too general for our purposes, and we choose to focus on the constructible functions, by which we mean the real-valued functions that have globally subanalytic domains and that can be expressed as sums of products of globally subanalytic functions and logarithms of positively-valued globally subanalytic functions. The study of constructible functions largely originated in the work of Lion and Rolin, [196], where these functions naturally arose in their study of integration of globally subanalytic functions. (In the context of p -adic integration, analogues of constructible functions arose from the work by J. Denef [194].) The integration theory of globally subanalytic and constructible functions was then further developed by Comte, Lion and Rolin in [193] and also in [178] and [172]. Much of the utility of the constructible functions stems from the fact that they are stable under integration – from which it follows that they are the smallest class of functions that is stable under integration and contains the subanalytic functions – and that they have very simple asymptotic behavior (see [178]). In fact, these results have typically lagged behind the motivic and p -adic developments. In this section, the real situation takes the lead over the p -adic and motivic results.

We obtain two main theorems about the constructible functions; see Theorems (5.3.44) and (5.3.2). The first theorem considers a constant $q > 0$ and constructible functions f and μ on $E \times \mathbb{R}^n$, and it describes the structure of the set

$$\mathrm{LC}(f, |\mu|^q, E) := \{(x, p) \in E \times (0, \infty] : f(x, \cdot) \in L^p(|\mu|_x^q)\}, \quad (14)$$

where $|\mu|_x^q$ is the positive measure on $\mathbb{R}^n$ whose Radon–Nikodym derivative with respect to the Lebesgue measure is $|\mu(x, \cdot)|^q : y \mapsto |\mu(x, y)|^q$. The theorem and its corollaries show that the set of all fibers of $\mathrm{LC}(f, |\mu|^q, E)$ over E is a finite set of open subintervals of $(0, \infty]$, and that the set of all fibers of $\mathrm{LC}(f, |\mu|^q, E)$ over $(0, \infty]$ is a finite set of subsets of E , each of which is the zero locus of a constructible function on E . This theorem therefore relates analysis with geometry, in the sense that Lebesgue classes are an object of study in analysis, while zero loci of functions are widely studied in analytic geometry. A similar link between geometry and analysis (but with $\mu=1$ and with focus on L^1 -integrability) is obtained in p -adic and motivic contexts in [191].

The second theorem is a closely related preparation result that expresses f and μ as finite sums of terms of a very simple form that naturally reflect the structure of $\mathrm{LC}(f, |\mu|^q, E)$. This theorem can be most easily appreciated through the historical context in which it was developed, starting with the following simple preparation result for

constructible functions, which is a rather direct consequence of Lion and Rolin's preparation theorem for globally subanalytic functions:

Let $f : E \times \mathbb{R}^n \rightarrow \mathbb{R}$ be constructible, with $E \subset \mathbb{R}^m$, and write $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ for the standard coordinates on $E \times \mathbb{R}^n$. Then f can be piecewise written on subanalytic sets as finite sums $\sum_{k \in K} T_k(x, y)$, where up to performing translations in y by globally subanalytic functions of a triangular form, each term is of the form $T_k(x, y) = g_k(x) \left(\prod_{j=1}^n |y_j|^{r_{k,j}} (\log |y_j|)^{s_{k,j}} \right) u_k(x, y)$ for some constructible function g_k , rational numbers $r_{k,j}$, natural numbers $s_{k,j}$, and globally subanalytic unit u_k which is of the special form as given by the globally subanalytic preparation theorem. (15)

Lion and Rolin [195] used (15) when proving that any parameterized integral of a constructible function is piecewise given by constructible functions, but on pieces that need not be globally subanalytic sets. Comte, Lion and Rolin [193] also used (15) when proving that any parameterized integral of a globally subanalytic function is a constructible function. The authors then subsumed both of these results in [178] by showing that $F(x) = \int_{\mathbb{R}^n} f(x, y) dy$ is a constructible function on E if $f : E \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a constructible function such that $f(x, \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in E$. The key to doing this was to improve (15) by showing that in the special case of $n=1$, if $f(x, \cdot) \in L^1(\mathbb{R})$ for every $x \in E$, then the sums can be constructed in such a way so that each term $T_k(x, y)$ is also integrable in y for every $x \in E$. This alleviated various analytic considerations employed in [195] and [193] to get around the awkward fact that (15) allows the possibility of expressing integrable functions as sums of nonintegrable functions. In [172] improved upon (15) in the special case of $n=1$ by dropping the assumption that $f(x, y)$ be integrable in y for every $x \in E$, and then showing that the set $\text{Int}(f, E) := \{x \in E : f(x, \cdot) \in L^1(\mathbb{R})\}$ is the zero locus of a constructible function on E , and that the sums in (15) can be constructed so that each term $T_k(x, y)$ is integrable in y for every $x \in E$, provided that we only require the equation $f(x, y) = \sum_k T_k(x, y)$ to hold for those values of (x, y) with $x \in \text{Int}(f, E)$.

The preparation theorem of this section strengthens this line of results even further by considering an arbitrary positive integer n , not just $n=1$, and by considering all L^p classes simultaneously, not just L^1 . In order to convey the main idea of the theorem without getting bogged down in technicalities, let us use the Lebesgue measure on $\mathbb{R}^n$ (thus $\mu=1$, where μ is the function from (14)), and let us also only consider the L^p classes for finite values of p . Under these simplifying assumptions, the preparation theorems states that the sums $\sum_{k \in K} T_k(x, y)$ in (15) can be constructed in such a way so that there is a partition $\{K_i\}_i$ of the finite index set K such that for each $x \in E$ and $p \in (0, \infty)$ with $f(x, \cdot) \in L^p(\mathbb{R}^n)$, and for each i , either $T_k(x, \cdot)$ is in L^p for all $k \in K_i$, or else $\sum_{k \in K_i} T_k(x, y) = 0$ for all y . So, for instance, if for some fixed value of p the function $f(x, \cdot)$ happened to be in $L^p(\mathbb{R}^n)$ for every $x \in E$, then the sums in (15) can be constructed so that each term $T_k(x, \cdot)$ is in L^p for every $x \in E$, for we may simply omit the remaining terms in the sum because they collectively sum to zero.

Part of our interest in developing a good integration theory for constructible

functions comes from a desire to study various integral transforms in the constructible setting. And, to summarize, we now have three main tools at our disposal to conduct such studies: the constructible functions are stable under integration, they have simple asymptotic behavior, and they have a multivariate preparation theorem with good analytic properties. We apply these three tools to the field of harmonic analysis in [192] by showing a theorem that bounds the decay rates of parameterized families of oscillatory integrals. This is an adaptation of a classical theorem found in Stein [197] but with different assumptions. The classical theorem bounds a single oscillatory integral with an amplitude function that is smooth and compactly supported and a phase function that is smooth and of finite type. In contrast, we give a uniform bound on a parameterized family of oscillatory integrals with an amplitude function that is constructible and integrable and a phase function that is globally subanalytic and satisfies a certain “hyperplane condition” (which closely relates to the notion of “finite type” in our setting). Thus by restricting our attention to the special classes of constructible and globally subanalytic functions, we obtain a much more global, parameterized version of the classical theorem with significantly weaker analytic assumptions. This application of our preparation theorem was, in fact, the initial stimulus for our work in this section.

This section formulates our main theorem on the structure of diagrams of Lebesgue classes and also a simple version of the related preparation theorem; see Theorems (5.3.44) and (5.3.2). It also gives two key supporting theorems used to show these results; see Theorems (5.3.22) and (5.3.32). The full version of the preparation theorem can be found as Theorem (5.3.48). We begin by fixing some notation to be used throughout the section.

Notation (5.3.1)[190]: Denote the set of natural numbers by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Denote the subset and proper subset relations by $\subset$ and $\subsetneq$, respectively. Write $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ for the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If $f = (f_1, \dots, f_n) : D \rightarrow \mathbb{R}^n$ is a differentiable map with $D \subset \mathbb{R}^{m+n}$, write

$$\frac{\partial f}{\partial y}(x, y) = \left(\frac{\partial f_i}{\partial y_j}(x, y) \right)_{(i,j) \in \{1, \dots, n\}^2}$$

for its Jacobian matrix in y . Define the coordinate projection $\Pi_m : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ by

$$\Pi_m(x, y) = x.$$

For any $D \subset \mathbb{R}^{m+n}$ and $x \in \mathbb{R}^m$, define the fiber of D over x by

$$D_x = \{y \in \mathbb{R}^n : (x, y) \in D\}.$$

For any $d \in \{0, \dots, n\}$ and $\square \in \{<, \leq, >, \geq\}$, define $y_{\square d} = (y_i)_{i \in \square d}$. For example, $y_{\leq d} = (y_1, \dots, y_d)$, and in accordance with our above notation for coordinate projections, the maps $\Pi_d : \mathbb{R}^n \rightarrow \mathbb{R}^d$ and $\Pi_{m+d} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+d}$ are given by $\Pi_d(y) = y_{\leq d}$ and $\Pi_{m+d}(x, y) = (x, y_{\leq d})$. More generally, if $\lambda : \{1, \dots, d\} \rightarrow \{1, \dots, n\}$ is an increasing map, define $\Pi_{m,\lambda} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+d}$ by

$$\Pi_{m,\lambda}(x, y) = (x, y_\lambda),$$

where $y_\lambda = (y_{\lambda(1)}, \dots, y_{\lambda(d)})$.

For any set $D \subset \mathbb{R}^n$, call a function $f : D \rightarrow \mathbb{R}^m$ analytic if it extends to an analytic function on a neighborhood of D in $\mathbb{R}^n$. A restricted analytic function is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the restriction of f to $[-1, 1]^n$ is analytic and $f(x) = 0$ on $\mathbb{R}^n \setminus [-1, 1]^n$. We shall henceforth call a set or function subanalytic if, and only if, it is definable (in the sense of first-order logic) in the expansion of the real field by all restricted analytic

functions. Thus in this section, the word “subanalytic” is an abbreviation for the phrase “globally subanalytic”, and in this meaning, the natural logarithm $\log: (0, \infty) \rightarrow \mathbb{R}$ is not subanalytic. For any subanalytic set D , let $\mathcal{C}(D)$ denote the $\mathbb{R}$ -algebra of functions on D generated by the functions of the form $x \mapsto f(x)$ and $x \mapsto \log g(x)$, where $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow (0, \infty)$ are subanalytic. A function that is a member of $\mathcal{C}(D)$ for some subanalytic set D is called a constructible function.

Consider a Lebesgue measurable set $D \subset \mathbb{R}^{m+n}$ and Lebesgue measurable functions $f: D \rightarrow \mathbb{R}$ and $v: D \rightarrow [0, \infty)$, and put $E = \Pi_m(D)$. Define the diagram of Lebesgue classes of f over E with respect to v to be the set

$$\text{LC}(f, v, E) = \{(x, p) \in E \times (0, \infty] : f(x, \cdot) \in L^p(v_x)\},$$

where v_x is the positive measure on D_x defined by setting

$$v_x(Y) = \int_Y v(x, y) dy \quad (16)$$

for each Lebesgue measurable set $Y \subset D_x$, where the integration in (16) is with respect to the Lebesgue measure on $\mathbb{R}^n$. Thus for each $x \in E$, when $0 < p < \infty$, the function $f(x, \cdot)$ is in $L^p(v_x)$ if and only if

$$\int_{D_x} |f(x, y)|^p v(x, y) dy < \infty,$$

and the function $f(x, \cdot)$ is in $L^\infty(v_x)$ if and only if there exist a constant $M > 0$ and a Lebesgue measurable set $Y \subset D_x$ such that $v_x(Y) = 0$ and $|f(x, y)| \leq M$ for all $y \in D_x \setminus Y$.

The fibers of $\text{LC}(f, v, E)$ over E and over $(0, \infty]$ are both of interest, so we give them special names. For each $x \in E$, define the set of Lebesgue classes of f at x with respect to v to be the set

$$\text{LC}(f, v, x) = \{p \in (0, \infty] : f(x, \cdot) \in L^p(v_x)\}.$$

For each $p \in (0, \infty]$, define the L^p -locus of f in E with respect to v to be the set

$$\text{Int}^p(f, v, E) = \{x \in E : f(x, \cdot) \in L^p(v_x)\}.$$

When $v = 1$ (which is the case of most interest because it means we are simply using the n -dimensional Lebesgue measure on D_x), it is convenient to simply write $\text{LC}(f, E)$, $\text{LC}(f, x)$ and $\text{Int}^p(f, E)$ and to drop the phrase “with respect to v ” in the names of these sets. Also when $v = 1$, we shall write $L^p(D_x)$ rather than $L^p(v_x)$. The set $\text{Int}^1(f, E)$ was studied by the authors in [172] (focusing on the case of $n = 1$), where it was denoted by $\text{Int}(f, E)$ and called the “locus of integrability of f in E .”

We order the set $[0, \infty]$ in the natural way, and we topologize $(0, \infty]$ by letting

$$\{(a, b) : 0 \leq a < b < \infty\} \cup \{\{\infty\}\}$$

be a base for its topology. A convex subset of $(0, \infty]$ is called a subinterval of $(0, \infty]$. The endpoints of a subinterval of $(0, \infty]$ are its supremum and infimum in $[0, \infty]$. Note that the empty set is a subinterval of $(0, \infty]$, and that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

It is elementary to see that $\text{LC}(f, v, x)$ is a subinterval of $(0, \infty]$ for each $x \in E$. Much more can be said when f and v are assumed to be constructible functions or their powers.

Theorem (5.3.44) has been formulated in such a way so as to make it adaptable to a variety of situations. This section contains an extensive list of corollaries that further explain how the theorem elucidates the structure of $\text{LC}(f, |\mu|^q, E)$, and how it can be easily adapted to give analogous theorems about local L^p spaces, complex measures, and

measures defined from differential forms on subanalytic sets, all within the context of constructible functions.

The proof of Theorem (5.3.44) is intimately linked to the proof of a preparation theorem for constructible functions that is stated in full strength in this section, where it is showed. Here we state only a simple version of the preparation theorem that is sufficient for our application to oscillatory integrals in [192]. But first, we need one more definition: a cell over $\mathbb{R}^m$ is a subanalytic set $A \subset \mathbb{R}^{m+n}$ such that for each $i \in \{1, \dots, n\}$, the set $\Pi_{m+i}(A)$ is either the graph of an analytic subanalytic function on $\Pi_{m+i-1}(A)$, or

$$\Pi_{m+i}(A) = \{(x, y_{\leq i}) : (x, y_{< i}) \in \Pi_{m+i-1}(A), a_i(x, y_{< i}) \square_1 y_i \square_2 b_i(x, y_{< i})\} \quad (17)$$

for some analytic subanalytic functions $a_i, b_i : \Pi_{m+i-1}(A) \rightarrow \mathbb{R}$: for which $a_i(x, y_{< i}) < b_i(x, y_{< i})$ on $\Pi_{m+i-1}(A)$, where $\square_1$ and $\square_2$ denote either $<$ or no condition.

Theorem (5.3.2)[190]: Let $p \in (0, \infty)$ and $f \in \mathcal{C}(D)$ for some subanalytic set $D \subset \mathbb{R}^{m+n}$, and assume that $\text{Int}^p(f, \Pi_m(D)) = \Pi_m(D)$. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ whose fibers over $\Pi_m(A)$ are open in $\mathbb{R}^n$, we may write f as a finite sum

$$f(x, y) = \sum_k T_k(x, y)$$

on A , with $\text{Int}^p(T_k, \Pi_m(A)) = \Pi_m(A)$ for each k , as follows: there exists a bounded function $\varphi : A \rightarrow (0, \infty)^M$ of the form

$$\varphi(x, y) = \left(c_i(x) \prod_{j=1}^n |y_j - \theta_j(x, y_{< j})|^{\gamma_{i,j}} \right)_{i \in \{1, \dots, M\}},$$

and for each k ,

$$T_k(x, y) = g_k(x) \left(\prod_{i=1}^n |y_i - \theta_i(x, y_{< i})|^{r_{k,i}} (\log |y_i - \theta_i(x, y_{< i})|)^{s_{k,i}} \right) U_k \circ \varphi(x, y), \quad (18)$$

where the $g_k : \Pi_m(A) \rightarrow \mathbb{R}$ are constructible, the $c_i : \Pi_m(A) \rightarrow (0, \infty)$ and $\theta_i : \Pi_{m+i-1}(A) \rightarrow \mathbb{R}$ are analytic subanalytic functions, the graph of each θ_i is disjoint from $\Pi_{m+i}(A)$, the $\gamma_{i,j}$ and $r_{k,i}$ are rational numbers, the $s_{k,i}$ are natural numbers, and the U_k are positively-valued analytic functions on the closure of the range of φ .

In addition, the fact that $\text{Int}^p(T_k, \Pi_m(A)) = \Pi_m(A)$ only depends on the values of the $r_{k,i}$, and not the values of $s_{k,i}$, in the following sense: we have $\text{Int}(T'_k, \Pi_m(A)) = \Pi_m(A)$ for any function T'_k on A of the form

$$T'_k(x, y) = \prod_{i=1}^n |y_i - \theta_i(x, y_{< i})|^{r_{k,i}} (\log |y_i - \theta_i(x, y_{< i})|)^{s'_{k,i}},$$

where the $r_{k,i}$ are as in (18) and the $s'_{k,i}$ are arbitrary natural numbers.

The key aspect of Theorem (5.3.2) that is of interest, and what makes its proof nontrivial, is that the piecewise sum representation of f can be constructed so that each of its terms $T_k(x, \cdot)$ are in the same L^p class as $f(x, \cdot)$; namely, $\text{Int}^p(T_k, \Pi_m(A)) = \Pi_m(A)$ for each A and T_k , provided that $\text{Int}^p(f, \Pi_m(D)) = \Pi_m(D)$. There is an analog of Theorem (5.3.2) for $p = \infty$, but then one must replace (18) with the more complicated form

$$T_k(x, y) = g_k(x) \left(\prod_{i=1}^n |y_i - \theta_i(x, y_{< i})|^{r_{k,i}} \left(\log \prod_{j=1}^n |y_j - \theta_j(x, y_{< j})|^{\beta_{i,j}} \right)^{s_{k,i}} \right) U_k \circ \varphi(x, y), \quad (19)$$

where the $\beta_{i,j}$ are rational numbers and everything else is as before, and where the fact

that $\text{Int}^\infty(T_k, \Pi_m(A)) = \Pi_m(A)$ now depends on all the values of the $r_{k,i}$, $s_{k,i}$ and $\beta_{i,j}$, not just the values of the $r_{k,i}$ alone.

We shall also show a theorem on the fiberwise vanishing of constructible functions and a theorem on parameterized rectilinearization of subanalytic functions, given below.

This section formulates a version of the subanalytic preparation theorem of Lion and Rolin [195]. We begin with some multi-index notation.

Notation (5.3.3)[190]: For any tuples $y = (y_1, \dots, y_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{R}^n$, define

$$|y| = (|y_1|, \dots, |y_n|),$$

$$\log y = (\log y_1, \dots, \log y_n), \text{ provided that } y_1, \dots, y_n > 0,$$

$$y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \text{ provided that this is defined,}$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$\text{supp}(\alpha) = \{i \in \{1, \dots, n\} : \alpha_i \neq 0\}, \text{ which is called the support of } \alpha.$$

There is a conflict of notation between this use of $|y|$ and $|\alpha|$, but the context will always distinguish the meaning: if α is a tuple of exponents of a tuple of real numbers, then $|\alpha|$ means $\alpha_1 + \cdots + \alpha_n$; if y is a tuple of real numbers not used as exponents, then $|y|$ means $(|y_1|, \dots, |y_n|)$. These notations may be combined, such as with $|y|^\alpha = |y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n}$ and $(\log |y|)^\alpha = (\log |y_1|)^{\alpha_1}, \dots, (\log |y_n|)^{\alpha_n}$.

Definitions (5.3.4)[190]: Consider a subanalytic set $A \subset \mathbb{R}^{m+n}$. We say that A is open over $\mathbb{R}^m$ if A_x is open in $\mathbb{R}^n$ for all $x \in \Pi_m(A)$.

We call a function $\theta = (\theta_1, \dots, \theta_n) : A \rightarrow \mathbb{R}^n$ a center for A over $\mathbb{R}^m$ if A is open over $\mathbb{R}^m$, and if for each $i \in \{1, \dots, n\}$ the component θ_i is an analytic subanalytic function $\theta_i : \Pi_{m+i-1}(A) \rightarrow \mathbb{R}$ with the following two properties.

1. The range of θ_i is contained in either $(-\infty, 0)$, $\{0\}$ or $(0, \infty)$. And, when θ_i is nonzero, the closure of the set $\{y_i / \theta_i(x, y_{<i}) : (x, y) \in A\}$ is a compact subset of $(0, \infty)$.
2. Let $\tilde{y}_i = y_i - \theta_i(x, y_{<i})$. The set $\{\tilde{y}_i : (x, y) \in A\}$ is a subset of either $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ or $(1, \infty)$.

We call $(x, \tilde{y}) := (x, \tilde{y}_1, \dots, \tilde{y}_n)$ the coordinates on A with center θ .

A rational monomial map on A over $\mathbb{R}^m$ with center θ is a bounded function $\varphi : A \rightarrow \mathbb{R}^m$ of the form

$$\varphi(x, y) = (c_1(x)|\tilde{y}|^{\gamma_1}, \dots, c_M(x)|\tilde{y}|^{\gamma_M}), \quad (20)$$

where $c_1, \dots, c_M$ are positively-valued analytic subanalytic functions on $\Pi_m(A)$ and $\gamma_1, \dots, \gamma_M$ are tuples in $\mathbb{Q}^n$. Note that $\varphi(A) \subset (0, \infty)^M$. If $A \subset \mathbb{R}^m \times (0, 1)^n$ and $\theta = 0$, we say that φ is basic.

An analytic function is called a unit if its range is contained in either $(-\infty, 0)$ or $(0, \infty)$. A function $f : A \rightarrow \mathbb{R}$ is called a φ -function if $f = F \circ \varphi$ for some analytic function F whose domain is the closure of the range of φ ; if F is also a unit, then we call f a φ -unit.

A function $f : A \rightarrow \mathbb{R}$ is φ -prepared if

$$f(x, y) = g(x)|\tilde{y}|^\alpha u(x, y)$$

on A for some analytic subanalytic function g , tuple $\alpha \in \mathbb{Q}^n$ and φ -unit u .

Definition (5.3.5)[190]: To any rational monomial map $\varphi: A \rightarrow \mathbb{R}^M$ over $\mathbb{R}^m$ with center θ , we associate a basic rational monomial map over $\mathbb{R}^m$, denoted by φ_θ , as follows. For each $i \in \{1, \dots, n\}$, the set $\{\tilde{y}_i : (x, y) \in A\}$ is contained in either $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ or $(1, \infty)$, so there exist unique $\varepsilon_i, \zeta_i \in \{-1, 1\}$ such that $0 < \varepsilon_i \tilde{y}_i^{\zeta_i} < 1$ for all $(x, y) \in A$. Define an analytic isomorphism $T_\theta: A \rightarrow A_\theta$ by

$$T_\theta(x, y) = (x, \varepsilon_1 \tilde{y}_1^{\zeta_1}, \dots, \varepsilon_n \tilde{y}_n^{\zeta_n}).$$

Define $\varphi_\theta := \varphi \circ T_\theta^{-1}: A_\theta \rightarrow \mathbb{R}^M$.

Notation (5.3.6)[190]: Write $\varphi_\theta(x, y) = (c_1(x)y^{\gamma_1}, \dots, c_M(x)y^{\gamma_M})$ for some $\gamma_1, \dots, \gamma_M \in \mathbb{Q}^n$. For each $i \in \{0, \dots, n\}$, define $\varphi_{\theta, i}$ to be the function on $\Pi_{m+i}(A)$ consisting of the components $c_j(x)y^{\gamma_j}$ of φ_θ such that $\text{supp}(\gamma_j) \subset \{1, \dots, i\}$, and when $i > 0$, such that $i \in \text{supp}(\gamma_j)$. Thus

$$\varphi_\theta(x, y) = (\varphi_{\theta, 0}(x), \varphi_{\theta, 1}(x, y_1), \dots, \varphi_{\theta, n}(x, y_1, \dots, y_n)).$$

For each $i \in \{0, \dots, n\}$ and $\square \in \{<, \leq, >, \geq\}$, define $\varphi_{\theta, \square i} = (\varphi_{\theta, j})_{j, \square i}$ on its appropriate domain. For example, $\varphi_{\theta, \leq i}$ is the function on $\Pi_{m+i}(A)$ given by

$$\varphi_{\theta, \leq i}(x, y_{\leq i}) = (\varphi_{\theta, 0}(x), \varphi_{\theta, 1}(x, y_1), \dots, \varphi_{\theta, i}(x, y_{\leq i})).$$

Definition (5.3.7)[190]: If $C \subset \mathbb{R}^{m+n}$ is a cell over $\mathbb{R}^m$, then there exists a unique increasing map $\lambda: \{1, \dots, d\} \rightarrow \{1, \dots, n\}$ whose image consists of the set of all $i \in \{1, \dots, n\}$ for which $\Pi_{m+i}(C)$ is of the form (17). We call C a λ -cell.

Note that $\Pi_{m, \lambda}$ defines an analytic isomorphism from a λ -cell C onto $\Pi_{m, \lambda}(C)$, and $\Pi_{m, \lambda}(C)$ is a cell over $\mathbb{R}^m$ that is open over $\mathbb{R}^m$.

Definition (5.3.8)[190]: We say that φ is prepared over $\mathbb{R}^m$ if A is a cell over $\mathbb{R}^m$ such that for each $i \in \{1, \dots, n\}$, if we write

$$\Pi_{m+i}(A_\theta) = \{(x, y_{\leq i}) : (x, y_{< i}) \in \Pi_{m+i-1}(A_\theta), a_i(x, y_{< i}) < y_{< i} < b_i(x, y_{< i})\},$$

then the functions a_i , b_i and $b_i - a_i$ are $\varphi_{\theta, < i}$ -prepared, and a_i is either identically zero or is strictly positively-valued.

Proposition (5.3.9)[190]: Suppose that $\mathcal{F}$ is a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$, if A is a λ -cell over $\mathbb{R}^m$ and we write $g: \Pi_{m, \lambda}(A) \rightarrow A$ for the inverse of the projection $\Pi_{m, \lambda}|_A: A \rightarrow \Pi_{m, \lambda}(A)$, then there exists a prepared rational monomial map $\varphi: \Pi_{m, \lambda}(A) \rightarrow \mathbb{R}^M$ over $\mathbb{R}^m$ such that $f \circ g$ is φ -prepared for each $f \in \mathcal{F}$.

Proof. This follows from the subanalytic preparation theorem (see [195] or [189]) by induction on n .

Corollary (5.3.10)[190]: Suppose that $\mathcal{F}$ is a finite set of constructible functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ and $f \in \mathcal{F}$, the restriction of f to A is analytic. Moreover, if each function in $\mathcal{F}$ is subanalytic, then $\mathcal{A}$ can be chosen so that $f(A)$ is contained in either $(-\infty, 0)$, $\{0\}$ or $(0, \infty)$ for each $A \in \mathcal{A}$ and $f \in \mathcal{F}$.

Proof. When $\mathcal{F}$ consists entirely of subanalytic functions, this follows directly from Proposition (5.3.9). In the general constructible case, fix a finite set $\mathcal{F}'$ of subanalytic functions such that each function in $\mathcal{F}$ is a sum of products of functions of the form

$(x, y) \mapsto f(x, y)$ and $(x, y) \mapsto \log g(x, y)$ with $f, g \in \mathcal{F}'$. Now apply the result of the subanalytic case to $\mathcal{F}'$.

Definition (5.3.11)[190]: If $\mathcal{S}$ is a set of subsets of a set X , we say that a partition $\mathcal{A}$ of X is compatible with $\mathcal{S}$ if for each $A \in \mathcal{A}$ and each $S \in \mathcal{S}$, either $A \subset S$ or $A \subset X \setminus S$.

Note that in Proposition (5.3.9) and Corollary (5.3.10), the partition $\mathcal{A}$ can be made to be compatible with any prior given finite set of subanalytic subsets of D .

Throughout this section we use the notation of Theorem (5.3.44).

Corollary (5.3.12)[190]: For each $I \in \mathcal{I}$,

$$\{x \in E : \text{LC}(f, |\mu|^q, x) = I\} = \left\{ x \in E : (g_I(x) = 0) \wedge \left(\bigwedge_{J \in \mathcal{I}_I} g_J(x) \neq 0 \right) \right\}, \quad (21)$$

where $\mathcal{I}_I = \{J \in \mathcal{I} : I \subsetneq J\}$.

Proof. This follows from (49) and from the fact that for each $x \in E$, $\text{LC}(f, \mu, x) = I$ if and only if $I \subset \text{LC}(f, \mu, x)$ and $J \not\subset \text{LC}(f, \mu, x)$ for all $J \in \mathcal{I}_I$.

The final sentence of Theorem (5.3.44) shows that when f is subanalytic, so is the set (21).

Remark (5.3.13)[190]: The set $\text{LC}(f, |\mu|^q, E)$ can be expressed as the disjoint union

$$\bigcup_{I \in \mathcal{I}} (\{x \in E : \text{LC}(f, |\mu|^q, x) = I\} \times I) \quad (22)$$

and as the (not necessarily disjoint) union

$$\bigcup_{I \in \mathcal{I}} (\{x \in E : I \subset \text{LC}(f, |\mu|^q, x)\} \times I). \quad (23)$$

Proof. The fact that $\text{LC}(f, |\mu|^q, E)$ equals (22), and that (22) is contained in (23), are both clear. To see that (23) is contained in (22), note that if (x, p) is such that $I \subset \text{LC}(f, |\mu|^q, x)$ and $p \in I$, then $J = \text{LC}(f, |\mu|^q, x)$ and $p \in J$ for some $J \in \mathcal{I}$ with $I \subset J$.

Observe that (21) and (49) show how to use the functions $\{g_I\}_{I \in \mathcal{I}}$ to define the sets occurring in (22) and (23).

Corollary (5.3.14)[190]: For each $p \in (0, \infty]$ there exists $G_p \in \mathcal{C}(E)$ such that

$$\{x \in E : p \subset \text{LC}(f, |\mu|^q, x)\} = \{x \in E : G_p(x) = 0\}. \quad (24)$$

Proof. Define G_p to be the product of the g_I for all $I \in \mathcal{I}$ with $p \subset I$. Then (24) follows from (49) and from the fact that for each $x \in E$, we have $p \subset \text{LC}(f, |\mu|^q, x)$ if and only if $\text{LC}(f, |\mu|^q, x) = I$ for some $I \in \mathcal{I}$ with $p \subset I$.

For each $p \in (0, \infty]$, taking $P = \{p\}$ in (24) shows that $\text{Int}^p(f, |\mu|^q, E)$ is the zero locus of a constructible function. A very elementary proof of this fact is given in [172] for the special case when $\mu = 1$, $p = 1$ and $n = 1$.

Corollary (5.3.15)[190]: The set $\{\text{Int}^p(f, |\mu|^q, E) : p \in (0, \infty]\}$ is finite.

Proof. Since $\mathcal{I}$ is finite by Theorem (5.3.44), we may fix a finite partition $\mathcal{J}$ of $(0, \infty]$ compatible with $\mathcal{I}$. If $J \in \mathcal{J}$ and $P \in J$, then for each $I \in \mathcal{I}$, $p \in I$ if and only if $J \subset I$; so $\text{Int}^p(f, |\mu|^q, E) = \{x \in E : J \subset \text{LC}(f, |\mu|^q, x)\}$. Therefore

$$\{\text{Int}^p(f, |\mu|^q, E) : p \in (0, \infty]\} = \{\{x \in E : J \subset \text{LC}(f, |\mu|^q, x)\} : J \in \mathcal{J}\},$$

which is finite because $\mathcal{J}$ is finite.

Corollary (5.3.16)[190]: There exists $g \in \mathcal{C}(E)$ such that

$$\{x \in E : f(x, \cdot) \text{ is bounded on } D_x\} = \{x \in E : g(x) = 0\}.$$

Proof. Zero loci of constructible functions are closed under intersections and unions (by taking sums of squares and by taking products, respectively), so we may assume by Corollary (5.3.10) that D is a cell over $\mathbb{R}^m$ and that f is analytic. By projecting into a lower dimensional space, we may further assume that D is open over $\mathbb{R}^m$. Thus $f(x, \cdot)$ is bounded on D_x if and only if it is in $L^\infty(D_x)$, so we are done by applying Corollary (5.3.14) with $P = \{\infty\}$.

Although we will use Theorem (5.3.22) to show Theorem (5.3.44), it is interesting to observe that, conversely, Theorem (5.3.22) also follows from Theorem (5.3.44), as follows.

Corollary (5.3.17)[190]: There exist $g, h \in \mathcal{C}(E)$ such that

$$\{x \in E : f(x, y) = 0 \text{ for all } y \in D_x\} = \{x \in E : g(x) = 0\}$$

and

$$\{x \in E : f(x, y) = 0 \text{ for } |\mu|_x\text{-almost all } y \in D_x\} = \{x \in E : h(x) = 0\}.$$

Proof. Define $F : D \times \mathbb{R} \rightarrow \mathbb{R}$ by $F(x, y, z) = zf(x, y)$. Note that for each $x \in E$, $f(x, y) = 0$ for all $y \in D_x$ if and only if $(y, z) \mapsto F(x, y, z)$ is bounded on $D_x \times \mathbb{R}$, and that $f(x, y) = 0$ for $|\mu|_x$ -almost all $y \in D_x$ if and only if $(y, z) \mapsto F(x, y, z)$ is in $L^\infty(\nu_x)$, where $\nu : D \times \mathbb{R} \rightarrow [0, \infty)$ is defined by $\nu(x, y, z) = |\mu(x, y)|$. So we are done by applying Corollaries (5.3.16) and (5.3.14) (with $P = \{\infty\}$) to F .

The following result generalizes [178, Theorem 1.4].

Corollary (5.3.18)[190]: Let $q > 0$, $p \subset (0, \infty]$, and $F, \nu \in \mathcal{C}(X \times Y \times \mathbb{R}^k)$ for some subanalytic sets X and Y . Suppose that for each $x \in X$, the set $\{y \in Y : P \subset \text{LC}(F, |\nu|^q, (x, y))\}$ is dense in Y . Then there exists a subanalytic set $C \subset X \times Y$ such that $C \times P \subset \text{LC}(F, |\nu|^q, X \times Y)$ and C_x is dense in Y for each $x \in X$.

Proof. Assume that $X \subset \mathbb{R}^m$. We may assume that $Y = \mathbb{R}^n$ because the case of a general subanalytic set Y follows from this special case by arguing as in the second paragraph of the proof of [178, Theorem 1.4]. By Corollary (5.3.14) we may fix $g \in \mathcal{C}(X \times \mathbb{R}^n)$ such that

$$\{(x, y) \in X \times \mathbb{R}^n : P \subset \text{LC}(F, |\nu|^q, (x, y))\} = \{(x, y) \in X \times \mathbb{R}^n : g(x, y) = 0\}. \quad (25)$$

By Corollary (5.3.10) we may fix a partition $\mathcal{A}$ of $X \times \mathbb{R}^n$ into subanalytic cells over $\mathbb{R}^m$ such that g restricts to an analytic function on each $A \in \mathcal{A}$. Let C be the union of the members of $\mathcal{A}$ that are open over $\mathbb{R}^m$. Then C is subanalytic, $\Pi_m(C) = X$, and C_x is open and dense in $\mathbb{R}^n$ for each $x \in X$. If there exists $(a, b) \in C$ such that $g(a, b) \neq 0$, then $\{y \in C_a : g(a, y) = 0\}$ would be a proper analytic subset of the open set C_a , so $\{y \in \mathbb{R}^n : g(a, y) = 0\}$ would not be dense in $\mathbb{R}^n$, contradicting (25) and our assumption on F and $|\nu|^q$. Therefore $g(x, y) = 0$ for all $(x, y) \in C$, which by (25) shows the corollary.

We now show how Theorem (5.3.44) adapts easily to the study of local integrability, complex measures, and measures defined from constructible differential forms on subanalytic sets. We only discuss the analogs of Theorem (5.3.44) itself, but it follows that analogs of the previous list of corollaries of this theorem hold as well, via the same proofs.

Suppose that $Y \subset \mathbb{R}^n$ and $f : Y \rightarrow \mathbb{R}$ are Lebesgue measurable, that ν is a positive measure on Y that is absolutely continuous with respect to the n -dimensional Lebesgue

measure, and that $p \in (0, \infty]$. We say that f is locally in $L^p(\nu)$, written as $f \in L^p_{\text{loc}}(\nu)$, if for each $y \in Y$ there exists a neighborhood U of y in Y such that $f|_U$ is in $L^p(\nu|_U)$. Similarly, we say that f is locally bounded on Y if for each $y \in Y$ there exists a neighborhood U of y in Y such that $f|_U$ is bounded.

For measurable functions $f : D \rightarrow \mathbb{R}$ and $\nu : D \rightarrow [0, \infty)$, where $D \subset \mathbb{R}^{m+n}$ and $E = \Pi_m(D)$, define the sets $\text{LC}_{\text{loc}}(f, \nu, E)$, $\text{LC}_{\text{loc}}(f, \nu, x)$ and $\text{Int}^p_{\text{loc}}(f, \nu, E)$ analogously to how $\text{LC}(f, \nu, E)$, $\text{LC}(f, \nu, x)$ and $\text{Int}^p(f, \nu, E)$ were defined in this Section, but replacing the condition $f(x, \cdot) \in L^p(\nu_x)$ with $f(x, \cdot) \in L^p_{\text{loc}}(\nu_x)$.

Proposition (5.3.19)[190]: The local analog of Theorem (5.3.44) holds, which describes the structure of $\text{LC}_{\text{loc}}(f, |\mu|^q, E)$ rather than $\text{LC}(f, |\mu|^q, E)$.

Proof. By extending f and μ by 0 on $(E \times \mathbb{R}^n) \setminus D$, we may assume that $D = E \times \mathbb{R}^n$. Define functions F and ν on $E \times \mathbb{R}^n \times [-1, 1]^n$ by $F(x, y, z) = f(x, y + z)$ and $\nu(x, y, z) = |\mu(x, y + z)|^q$. The compactness of $[-1, 1]^n$ implies that for each $x \in E$ and $p \in (0, \infty]$, $f(x, \cdot) \in L^p_{\text{loc}}(|\mu|^q_x)$ if and only if $F(x, y, \cdot) \in L^p(\nu_{(x, y)})$ for all $y \in \mathbb{R}^n$. Therefore

$$\text{LC}_{\text{loc}}(f, |\mu|^q, x) = \bigcap_{y \in \mathbb{R}^n} \text{LC}(F, \nu, (x, y)).$$

Theorem (5.3.44) shows that $\{\text{LC}(F, \nu, (x, y)) : (x, y) \in E \times \mathbb{R}^n\}$ is a finite set of subintervals of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, so the set

$$\mathcal{I}_{\text{loc}} := \{\text{LC}_{\text{loc}}(f, |\mu|^q, x) : x \in E\}$$

is of this form as well. Let $I \in \mathcal{I}_{\text{loc}}$. By Corollary (5.3.14) we may fix $g \in \mathcal{C}(E \times \mathbb{R}^n)$ such that

$$\{(x, y) \in E \times \mathbb{R}^n : I \subset \text{LC}(F, \nu, (x, y))\} = \{(x, y) \in E \times \mathbb{R}^n : g(x, y) = 0\}.$$

Thus

$$\{x \in E : I \subset \text{LC}_{\text{loc}}(f, |\mu|^q, x)\} = \{x \in E : g(x, y) = 0 \text{ for all } y \in \mathbb{R}^n\},$$

and this set is the zero locus of a constructible function by Theorem (5.3.22) (or Corollary (5.3.17)).

Suppose that f and ν are complex-valued Lebesgue measurable functions on a measurable set $D \subset \mathbb{R}^{m+n}$ such that $\nu(x, \cdot)$ is Lebesgue integrable on D_x for all $x \in E$, where $E = \Pi_m(D)$. For each $x \in E$, define a complex measure ν_x on D_x by setting

$$\nu_x(Y) = \int_Y \nu(x, y) dy$$

for each Lebesgue measurable set $Y \subset D_x$. The notion of an L^p -class with respect to a complex measure is defined using the absolute variation of the measure, so we define $\text{LC}(f, \nu, E) := \text{LC}(|f|, |\nu|, E)$, $\text{LC}(f, \nu, x) := \text{LC}(|f|, |\nu|, x)$ for each $x \in E$, and $\text{Int}^p(f, \nu, E) := \text{Int}^p(|f|, |\nu|, E)$ for each $p \in (0, \infty]$.

Proposition (5.3.20)[190]: The complex analog of Theorem (5.3.44) holds with $q = 1$, which describes the structure of $\text{LC}(f, \mu, E)$ for complex-valued functions f and μ on a subanalytic set $D \subset \mathbb{R}^{m+n}$ whose real and imaginary parts are constructible, where $\mu(x, \cdot)$ is Lebesgue integrable on D_x for all x in $E = \Pi_m(D)$.

Proof. Apply Theorem (5.3.44) to the constructible functions $|f|^2$ and $|\mu|^2$ with $q = 1/2$. Then note that for any $p \in (0, \infty]$, $|f| \in L^p(|\mu|_x)$ if and only if $|f|^2 \in L^{p/2}(|\mu|_x)$.

For the last result of this section, consider a subanalytic set $D \subset \mathbb{R}^{m+n}$ such that for

each x in $E = \Pi_m(D)$, the fiber D_x is a smooth k -dimensional submanifold of $\mathbb{R}^n$. For each $x \in E$, consider a smooth k -form ω_x on D_x , such that moreover there exist constructible functions $\omega_{i_1, \dots, i_k}(x, y)$ on D with $1 \leq i_1 < \dots < i_k \leq n$ such that

$$\omega_x(y) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega_{i_1, \dots, i_k}(x, y) dy_{i_1} \wedge \dots \wedge dy_{i_k}.$$

For each $x \in E$, write $|\omega_x|$ for the measure on D_x associated to the smooth k -form ω_x . For $f \in \mathcal{C}(D)$, consider

$$\text{LC}(f, \omega_x, x) = \{p \in (0, \infty] : f(x, \cdot) \in L^p(|\omega_x|)\},$$

and

$$\text{LC}(f, \omega, E) = \{(x, p) \in E \times (0, \infty] : f(x, \cdot) \in L^p(|\omega_x|)\},$$

where ω stands for the family $(\omega_x)_{x \in E}$.

Proposition (5.3.21)[190]: With the above notation for D , ω and E , and with $f \in \mathcal{C}(D)$, the analog of Theorem (5.3.44) holds for $\text{LC}(f, \omega, E)$. To adapt the last sentence of Theorem (5.3.44) to $\text{LC}(f, \omega, E)$, the extra assumption that μ be subanalytic should be replaced by the condition that the $\omega_{i_1, \dots, i_k}$ be subanalytic.

Proof. Because D is subanalytic, basic o-minimality implies that there exists a finite family $\mathcal{U}$ of subanalytic subsets of D which covers D and is such that the following hold for each $U \in \mathcal{U}$:

1. for every $x \in \Pi_m(U)$, the fiber U_x is open in D_x ;
2. there exists an increasing function $\lambda^U : \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ such that for each $x \in \Pi_m(U)$, the projection Π_{λ^U} is injective on U_x and has constant rank k .

For each $U \in \mathcal{U}$, let $G^U(x, y) = (x, g^U(x, z))$ be the inverse of $\Pi_{m, \lambda^U} : U \rightarrow \Pi_{m, \lambda^U}(U)$, where $z = (z_1, \dots, z_k)$. Then for each $U \in \mathcal{U}$, the functions $f \circ G^U$ and

$$\omega^U(x, z) := \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega_{i_1, \dots, i_k}(x, g^U(x, z)) \frac{\partial(g_{i_1}^U, \dots, g_{i_k}^U)}{\partial(z_1, \dots, z_k)}(x, z)$$

are both constructible functions on U , and in the case that f and all the $\omega_{i_1, \dots, i_k}$ are subanalytic, the ω^U and $f \circ G^U$ also are. Hence, Theorem (5.3.44) applies $\text{LC}(f \circ G^U, |\omega^U|, \Pi_m(U))$. The proposition now follows relatively easily from this and from the fact that

$$\text{LC}(f|_U, \omega|_U, \Pi_m(U)) = \text{LC}(f \circ G^U, |\omega^U|, \Pi_m(U))$$

for each $U \in \mathcal{U}$.

Theorem (5.3.22)[190]: If $f \in \mathcal{C}(D)$ for a subanalytic set $D \subset \mathbb{R}^{m+n}$ and $E = \Pi_m(D)$, then there exists $g \in \mathcal{C}(E)$ such that

$$\{x \in E : f(x, y) = 0 \text{ for all } y \in D_x\} = \{x \in E : g(x) = 0\}.$$

The parameterized rectilinearization theorem requires some additional terminology to state. For any sets $A \subset \mathbb{R}^{m+n}$ and $B \subset \mathbb{R}^{m+d}$, we call a map $f = (f_1, \dots, f_{m+n}) : B \rightarrow A$ an analytic isomorphism over $\mathbb{R}^m$ if f is a bijection, f and f^{-1} are both analytic, and $f_1(x, z) = x_1, \dots, f_m(x, z) = x_m$, where $z = (z_1, \dots, z_d)$.

For $l \in \{0, \dots, d\}$, we say that a set $B \subset \mathbb{R}^{m+d}$ is l -rectilinear over $\mathbb{R}^m$ if B is a cell over $\mathbb{R}^m$ such that for each $x \in \Pi_m(B)$, the fiber B_x is an open subset of $(0, 1)^d$ of the form

$$B_x = \Pi_l(B_x) \times (0, 1)^{d-l},$$

where the closure of $\Pi_l(B_x)$ is a compact subset of $(0, 1]^l$. When $B \subset \mathbb{R}^{m+d}$ is l -rectilinear

over $\mathbb{R}^m$, we call a function u on B an l -rectilinear unit if it may be written in the form $u = U \circ \psi$, where $\psi: B \rightarrow (0, \infty)^{N+d-l}$ is a bounded function of the form

$$\psi(x, z) = \left(c_1(x) \prod_{j=1}^l z_j^{\gamma_{1,j}}, \dots, c_N(x) \prod_{j=1}^l z_j^{\gamma_{N,j}}, z_{l+1}, \dots, z_d \right) \quad (26)$$

for some positively-valued analytic subanalytic functions c_i and rational numbers $\gamma_{i,j}$, and where U is a positively-valued analytic function on the closure of the range of ψ .

Proof. Let $f \in \mathcal{C}(D)$ for a subanalytic set $D \subset \mathbb{R}^{m+n}$, and put $E = \Pi_m(D)$. Write $V = \{x \in E : f(x, y) = 0 \text{ for all } y \in D_x\}$. We proceed by induction on n .

First suppose that $n = 1$. By Corollary (5.3.10) we may fix a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that the restriction of f to A is analytic for each $A \in \mathcal{A}$. We claim that for each $A \in \mathcal{A}$ there exists $g_A \in \mathcal{C}(\Pi_m(A))$ such that

$$\{x \in \Pi_m(A) : f(x, y) = 0 \text{ for all } y \in A_x\} = \{x \in \Pi_m(A) : g_A(x) = 0\}.$$

The theorem (with $n = 1$) follows from the claim, for then

$$V = \left\{ x \in E : \sum_{A \in \mathcal{A}} (g'_A(x))^2 = 0 \right\},$$

where $g'_A : E \rightarrow \mathbb{R}$ is the extension of g_A by 0 on $E \setminus \Pi_m(A)$. To show the claim, fix $A \in \mathcal{A}$. We may assume that A is open over $\mathbb{R}^m$, else the claim is trivial. Since $f(x, \cdot)$ is analytic on A_x for each $x \in \Pi_m(A)$, and since $f|_A$ is definable in the expansion of the real field by all restricted analytic functions and the exponential function, which is o-minimal (see [198], or [195]), it follows that we may fix a positive integer N such that for each $x \in \Pi_m(A)$, $f(x, y) = 0$ for all $y \in A_x$ if and only if there exist distinct $y_1, \dots, y_N \in A_x$ such that $f(x, y_1) = \dots = f(x, y_N) = 0$. So fix subanalytic functions $\xi_1, \dots, \xi_N : \Pi_m(A) \rightarrow \mathbb{R}$ whose graphs are disjoint subsets of A . Then the claim holds for the function

$$g_A(x) = \sum_{i=1}^N (f(x, \xi_i(x)))^2.$$

This establishes the theorem when $n = 1$.

Now suppose that $n > 1$, and inductively assume the theorem holds with k in place of n for each $k < n$. The set V is defined by the formula

$$(x \in E) \wedge \forall y \in \mathbb{R}^n ((x, y) \in D \rightarrow f(x, y) = 0).$$

Applying the induction hypothesis twice shows that this formula is equivalent to

$$(x \in E) \wedge \forall y_1 \in \mathbb{R} ((x, y_1) \in \Pi_{m+1}(D) \rightarrow h(x, y_1) = 0)$$

for some $h \in \mathcal{C}(\Pi_{m+1}(D))$, which in turn is equivalent to

$$(x \in E) \wedge (g(x) = 0)$$

for some $g \in \mathcal{C}(E)$. Thus $V = \{x \in E : g(x) = 0\}$.

Definition (5.3.23)[190]: Consider $l \in \{0, \dots, n\}$ and a rational monomial map ψ on B over $\mathbb{R}^m$, where $B \subset \mathbb{R}^{m+n}$. We say that ψ is l -rectilinear over $\mathbb{R}^m$ if B is l -rectilinear over $\mathbb{R}^m$ (as defined prior to Theorem (5.3.32)) and if ψ is of the form

$$\psi(x, y) = (c_1(x) y_{\leq l}^{\gamma_1}, \dots, c_N(x) y_{\leq l}^{\gamma_N}, y_{l+1}, \dots, y_n)$$

for some positively-valued analytic subanalytic functions $c_1, \dots, c_N$ on $\Pi_m(B)$ and tuples $\gamma_1, \dots, \gamma_N$ in $\mathbb{Q}^l$. We say that set B , or a rational monomial map ψ on B over $\mathbb{R}^m$, is rectilinear over $\mathbb{R}^m$ to mean that it is l -rectilinear over $\mathbb{R}^m$ for some l .

Definition (5.3.24)[190]: For a subanalytic set $D \subset \mathbb{R}^{m+n}$, an open partition of D over $\mathbb{R}^m$ is a finite family $\mathcal{A}$ of disjoint subanalytic subsets of D that are open over $\mathbb{R}^m$ and are

such that $\dim(D \setminus \bigcup A)_x < n$ for all $x \in \Pi_m(D)$.

The following lemma of one-variable calculus, and its corollary, are apparent.

Lemma (5.3.25)[190]: Let $\alpha \in \mathbb{R}$ and $\beta \geq 0$. Then the function $t \rightarrow t^\alpha (\log t)^\beta$ is

1. integrable on $(0,1)$ if and only if $\alpha > -1$;
2. bounded on $(0,1)$ if and only if $\alpha > 0$ or $\alpha = \beta = 0$.

Corollary (5.3.26)[190]: Suppose that $A \subset \mathbb{R}^n$ is l -rectilinear over $\mathbb{R}^0$, and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in [0, \infty)^n$. Then the function $y \mapsto y^\alpha |\log y|^\beta$ is

1. integrable on A if and only if for all $i \in \{l+1, \dots, n\}$, $\alpha_i > -1$;
2. bounded on A if and only if for all $i \in \{l+1, \dots, n\}$, $\alpha_i > 0$ or $\alpha_i = \beta_i = 0$.

Note that if $A \subset \mathbb{R}^{m+n}$ is l -rectilinear over $\mathbb{R}^m$, then by applying Corollary (5.3.26) to each of the fibers A_x , we see that $y \mapsto y^\alpha |\log y|^\beta$ is integrable on A_x either for all $x \in \Pi_m(A)$ or for no $x \in \Pi_m(A)$, according to whether the condition given in clause 1 of the corollary holds; and likewise for boundedness and clause 2.

Lemma (5.3.27)[190]: Let $A \subset \mathbb{R}^n$ be l -rectilinear over $\mathbb{R}^0$, and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$.

1. If $\{y^\alpha : y \in A\}$ is bounded, then $\alpha_{l+1}, \dots, \alpha_n \geq 0$.
2. Let $\beta \in \mathbb{Q}$ and $B = \{(y, z) \in A \times \mathbb{R} : a(y) < z < 1\}$, where $0 \leq a(y) < 1$ for all $y \in A$. If $\{y^\alpha z^\beta : (y, z) \in B\}$ is bounded, then $\alpha_{l+1}, \dots, \alpha_n \geq 0$.

Proof. Statement 1 is clear. Statement 2 follows from Statement 1 because $\{y^\alpha : y \in A\}$ is in the closure of the set $\{y^\alpha z^\beta : (y, z) \in B\}$, so $\{y^\alpha : y \in A\}$ is bounded if $\{y^\alpha z^\beta : (y, z) \in B\}$ is bounded.

The following lemma is apparent.

Lemma (5.3.28)[190]: Let $\varphi : A \rightarrow \mathbb{R}$ be a basic rational monomial map over $\mathbb{R}^m$, where $A \subset \mathbb{R}^{m+n}$ and $\varphi(x, y) = c(x)y^\alpha$.

1. If A is l -rectilinear over $\mathbb{R}^m$ and $\alpha \in \mathbb{Q}^l \times \mathbb{N}^{n-l}$, then $c(x)y^{\alpha_{\leq l}}$ is bounded on $\Pi_{m+l}(A)$, and φ is a $(c(x)y^{\alpha_{\leq l}}, y_l, \dots, y_n)$ -function.
2. Let $j \in \{1, \dots, n\}$, and put $y' = (y_{< j}, y_{> j})$ and $\alpha' = (\alpha_{< j}, \alpha_{> j})$. If the closure of $\{y_j : (x, y) \in A\}$ is contained in $(0, 1]$, then $c(x)(y')^{\alpha'}$ is bounded on A , and φ is a $(c(x)(y')^{\alpha'}, y_j)$ -function.

The proof of Proposition (5.3.31) will use two types of constructions, called pullback and pushforward constructions, to achieve the desired pullback and pushforward properties.

Definition (5.3.29)[190]: Suppose we are given a basic rational monomial map $\varphi : A \rightarrow \mathbb{R}^M$ over $\mathbb{R}^m$, where $A \subset \mathbb{R}^{m+n}$ is a cell over $\mathbb{R}^m$. A pullback construction for φ consists of a subanalytic map $F : A \rightarrow B$ and a basic rational monomial map $\psi : B \rightarrow \mathbb{R}^N$ over $\mathbb{R}^m$, diagrammed as follows,

$$\begin{array}{ccc} B & \xrightarrow{F} & A \\ \psi \downarrow & & \downarrow \varphi \\ \mathbb{R}^N & & \mathbb{R}^M \end{array},$$

where $B \subset \mathbb{R}^{m+n}$ is a cell over $\mathbb{R}^m$, $F : B \rightarrow F(B)$ is an analytic isomorphism over $\mathbb{R}^m$, $\det \frac{\partial F}{\partial y}$ and the components of F are ψ -prepared, and $\varphi \circ F$ is a ψ -function.

Observe that these properties ensure that if h is any φ -prepared function, then $h \circ F$ is ψ -prepared.

We will use the six types of pullback constructions listed below, where

$$\Pi_{m+j}(A) = \{(x, y_{\leq j}) : (x, y_{< j}) \in \Pi_{m+j-1}(A), a_i(x, y_{< j}) < y_j < b_j(x, y_{< j})\} \quad (27)$$

for each $j \in \{1, \dots, n\}$. When defining F below, we only specify its action on coordinates on which it acts nontrivially.

1. Adjustment: This means that F is the identity map (but ψ may be different from φ).
2. Restriction: This means that F is an inclusion map and $\psi = \varphi|_B$.
3. Power Substitution in y_j : This means that F sends $y_j \mapsto y_j^p$ for some positive integer p , and $\psi = \varphi \circ F$.
4. Blowup in y_j : This means that we are assuming that $\varphi_{\leq j}$ is prepared over $\mathbb{R}^{m+j-1}$, that F sends $y_j \mapsto y_j b_j(x, y_{< j})$, and that ψ is the pullback of $\bar{\varphi}$ by the transformation sending $y_j \mapsto y_j b(x) y_{< j}^\beta$, where $b_j(x, y_{< j}) = b(x) y_{< j}^\beta u(x, y_{< j})$ is the $\varphi_{< j}$ -prepared form of b_j and $\bar{\varphi}$ is the natural extension of φ to $\Pi_m(A) \times (0, \infty)^n$.
5. Flip in y_j : This means we are assuming that φ is prepared over $\mathbb{R}^{m+j-1}$, that the closure of $\{y_j : (x, y) \in A\}$ is contained in $(0, 1]$, that $b_j = 1$, and that φ is of the form

$$\varphi(x, y) = (\varphi_{< j}(x, y_{< j}), y_j, \varphi_{> j}(x, y_{< j}, y_{> j})); \quad (28)$$

F is the transformation sending $y_j \mapsto 1 - y_j$, and ψ is defined by the formula on the right side of (28), but on B rather than on A .

6. Swap in y_i and y_j : This means that F is the transformation sending $(y_i, y_j) \mapsto (y_j, y_i)$ and $\psi = \varphi \circ F$, provided that the resulting set B is still a cell over $\mathbb{R}^m$.

Definition (5.3.30)[190]: Suppose that we are given a basic rational monomial map $\psi: B \rightarrow \mathbb{R}^N$ over $\mathbb{R}^m$ and a subanalytic analytic isomorphism $F: B \rightarrow A$ over $\mathbb{R}^m$, where $A, B \subset \mathbb{R}^{m+n}$. A pushforward construction for ψ and F is a basic rational monomial map $\varphi: A \rightarrow \mathbb{R}^M$ over $\mathbb{R}^m$, diagrammed as follows,

$$\begin{array}{ccc} B & \xrightarrow{F} & A \\ \downarrow \psi & & \downarrow \varphi \\ \mathbb{R}^N & & \mathbb{R}^M, \end{array}$$

where the components of F^{-1} are φ -prepared and $\psi \circ F^{-1}$ is a φ -function.

Observe that these properties ensure that if h is any ψ -prepared function, then $h \circ F^{-1}$ is φ -prepared.

If $F: B \rightarrow A$ is a map from any one of the six types of pullback constructions described above, $\psi': B' \rightarrow \mathbb{R}^{N'}$ is a basic rational monomial map over $\mathbb{R}^m$ with $B' \subset B$, and $A' = F(B')$, then the maps $F|_{B'}: B' \rightarrow A'$ and ψ' have an obvious pushforward construction $\varphi': A' \rightarrow \mathbb{R}^{M'}$, provided that when F is a flip in y_j , the map ψ' is of the form $\psi'(x, y) = (\psi'_{< j}(x, y_{< j}), y_j, \psi'_{> j}(x, y_{< j}, y_{> j}))$.

The main purpose of this section is to show the following proposition.

Proposition (5.3.31)[190]: Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exists a subanalytic analytic isomorphism $F: B \rightarrow A$ over $\mathbb{R}^m$ with $B \subset \mathbb{R}^{m+n}$, and there exist rational monomial maps φ on A and ψ on B over $\mathbb{R}^m$ with the following properties.

1. Pullback property: Each function in $\{f \circ F\}_{f \in \mathcal{F}} \cup \{\det \frac{\partial F}{\partial y}\}$ is ψ -prepared, and ψ is rectilinear over $\mathbb{R}^m$.

2. Pushforward property: The components of F^{-1} are φ -prepared, and $\psi \circ F^{-1}$ is a φ -function.

The purpose of the pushforward property is that it ensures that for each subanalytic function $h: B \rightarrow \mathbb{R}$ that is ψ -prepared, $h \circ F^{-1}$ is φ -prepared. This proposition is essentially Theorem (5.3.32), the only differences being that the theorem does not mention the pushforward property and that the theorem deals with an actual partition of D rather than just an open partition of D over $\mathbb{R}^m$. In the proposition we use open partitions over $\mathbb{R}^n$, rather than actual partitions, because it allows the proof of the proposition to be stated somewhat more simply since we may ignore subsets of D whose fibers over $\mathbb{R}^m$ have dimension less than n , and doing so is of no loss to the study of L^p -spaces on D_x .

Proof. Let $\mathcal{F}$ be a finite set of subanalytic functions on $D \subset \mathbb{R}^{m+n}$. Apply Proposition (5.3.9) to $\mathcal{F}$, and focus on one rational monomial map $\varphi: A \rightarrow \mathbb{R}^M$ over $\mathbb{R}^m$ that this gives for which A is open over $\mathbb{R}^m$. Thus φ is prepared, and each function in $\mathcal{F}$ restricts to a φ -prepared function on A . Let θ be the center of φ . We will first construct finitely many sequences of maps diagrammed as follows,

$$\begin{array}{ccccccc} B = A_k & \xrightarrow{F_k} & A_{k-1} & \longrightarrow \cdots \longrightarrow & A_1 & \xrightarrow{F_1} & A_0 = A_\theta \xrightarrow{T_\theta^{-1}} & A \\ \downarrow \varphi^{[k]} = \psi & & \downarrow \varphi^{[k-1]} & & \downarrow \varphi^{[1]} & & \downarrow \varphi^{[0]} = \varphi_\theta & \downarrow \varphi \\ \mathbb{R}^N = \mathbb{R}^{M_k} & & \mathbb{R}^{M_{k-1}} & & \mathbb{R}^{M_1} & & \mathbb{R}^{M_0} = \mathbb{R}^M & \mathbb{R}^M \end{array} \quad (29)$$

where for each $i \in \{1, \dots, k\}$ the maps F_i and $\varphi^{[i]}$ are a pullback construction for $\varphi^{[i-1]}$ of one of the six types listed above, the map ψ is rectilinear over $\mathbb{R}^m$, and the ranges of the maps $F: B \rightarrow A$ given by $F = T_\theta^{-1} \circ F_1 \circ \cdots \circ F_k$ for all such sequences (29) constructed form an open partition of A over $\mathbb{R}^m$. Doing this shows the pullback property. We will construct (29) to also have the following property.

For each $j \in \{1, \dots, n\}$, at most one map F_i in (29) is a flip in y_j . (30)

Assuming we can construct (29) as such, to show the pushforward property it suffices to define $A' = F(B)$, to inductively define $B_k = B$ and $B_{i-1} = F_i(B_i)$ for each $i \in \{1, \dots, k\}$, and to show that we can construct maps diagrammed as follows,

$$\begin{array}{ccccccc} B = B_k & \xrightarrow{F_k} & B_{k-1} & \longrightarrow \cdots \longrightarrow & B_1 & \xrightarrow{F_1|_{B_1}} & B_0 \xrightarrow{T_\theta^{-1}|_{B_0}} & A' \\ \downarrow \psi^{[k]} = \psi & & \downarrow \psi^{[k-1]} & & \downarrow \psi^{[1]} & & \downarrow \psi^{[0]} & \downarrow \varphi' = \psi^{[0]} \circ T_\theta|_{A'} \\ \mathbb{R}^N = \mathbb{R}^{N_k} & & \mathbb{R}^{N_{k-1}} & & \mathbb{R}^{N_1} & & \mathbb{R}^{N_0} & \mathbb{R}^{M'} = \mathbb{R}^{N_0}, \end{array} \quad (31)$$

where for each $i \in \{1, \dots, k\}$, $\psi^{[i-1]}$ is a pushforward construction for $F_i|_{B_i}: B_i \rightarrow B_{i-1}$ and $\psi^{[i]}$. (Thus the map $\varphi: A \rightarrow \mathbb{R}^M$ in the statement of the theorem is being denoted by $\varphi': A' \rightarrow \mathbb{R}^{M'}$ here in the proof.) These pushforward constructions will be possible because if a map F_i in (29) is a flip in y_j , we can ensure that $\psi^{[i]}$ is of the form (28). Indeed, from among the six types of pullback and pushforward constructions we use, only blowups in one of the variables $y_j, \dots, y_n$ can possibly destroy the form (28). So (30) imply that, in fact, all the maps $\varphi^{[i]}, \dots, \varphi^{[k]}$ and $\psi^{[k]} \rightarrow \psi^{[i]}$ are of the form (28).

So it remains to construct the sequences (29). This is done by an induction, and to simplify notation we will write $\varphi: A \rightarrow \mathbb{R}^M$ instead of the more cumbersome

$\varphi^{[l]}: A_l \rightarrow \mathbb{R}^{M_l}$. (So we are now assuming that φ is basic.) Let $d \in \{1, \dots, n\}$, and inductively assume that $\varphi_{\leq d}$ is l -rectilinear over $\mathbb{R}^m$ for some $l \in \{0, \dots, d-1\}$ and that φ is prepared over $\mathbb{R}^{m+d-1}$. Thus A is a cell over $\mathbb{R}^m$, so we use the notation (27). To complete the construction, it suffices to show that after taking an open partition of A over $\mathbb{R}^m$ and pulling back φ , we may reduce to the case that $\varphi_{\leq d}$ is rectilinear and φ is prepared over $\mathbb{R}^{m+d}$.

By pulling back by a blowup in y_d and then by power substitutions in $y_{l+1}, \dots, y_d$, and using Lemma (5.3.27), we may assume that $b_d = 1$ and that all the powers of $y_{l+1}, \dots, y_d$ occurring in the components of φ are natural numbers, and when $a_d > 0$, that all the powers of $y_{l+1}, \dots, y_{d-1}$ in the monomials occurring outside the units in the $\varphi_{\leq d}$ -prepared forms of a_d and $1 - a_d$ are also natural numbers. There are two cases that can be handled very easily.

Case 1: $a_d = 0$.

In this case, $\Pi_{m+d}(A)$ is l -rectilinear, so we are done after using Lemma (5.3.31.1) to adjust φ .

Case 2: The closure of $\{y_d : (x, y) \in A\}$ is contained in $(0, 1]$.

In this case, use Lemma (5.3.31.2) to adjust φ to assume that φ is of the form (28), and then apply a flip in y_d to reduce to Case 1.

(Note that if we reduce to either of these two cases, we need not require that $b_d = 1$ or that the requisite powers of $y_{l+1}, \dots, y_d$ are natural numbers, because the blowup and power substitutions mentioned just prior to these cases can be applied if needed.) So assume that $a_d > 0$, and write

$$a_d(x, y_{\leq d}) = \hat{a}(x) y_{\leq d}^{\alpha} u(x, y_{\leq d})$$

for some analytic subanalytic function $\hat{a}$, tuple of rational numbers $\alpha = (\alpha_1, \dots, \alpha_{d-1})$, and $\varphi_{\leq d}$ -unit u . We proceed by induction on $|\text{supp}(\alpha_{>l})|$, the cardinality of the set $\text{supp}(\alpha_{>l})$.

Suppose that $\text{supp}(\alpha_{>l})$ is empty, and write $y_{\leq l}^{\alpha}$ instead of $y_{\leq d}^{\alpha}$. Fix a constant C that is greater than the supremum of the range of u . Construct a partition of $\Pi_{m+l}(A)$ into cells over $\mathbb{R}^m$ compatible with the condition $\hat{a}(x) y_{\leq l}^{\alpha} C = 1$. By considering the restriction of φ to $A \cap (B \times \mathbb{R}^{n-l})$ for each cell B from this partition that is open over $\mathbb{R}^m$, we may assume that either $\hat{a}(x) y_{\leq l}^{\alpha} C > 1$ on A or $\hat{a}(x) y_{\leq l}^{\alpha} C < 1$ on A . If $\hat{a}(x) y_{\leq l}^{\alpha} C > 1$ on A , then a_d is bounded below by a positive constant, and we are in Case 2. So assume that $\hat{a}(x) y_{\leq l}^{\alpha} C < 1$ on A . Consider the two sets

$$\{(x, y) \in A : a_d(x, y_{\leq d}) < y_d < \hat{a}(x) y_{\leq l}^{\alpha} C\} \text{ and } \{(x, y) \in A : \hat{a}(x) y_{\leq l}^{\alpha} C < y_d < 1\}.$$

By restricting φ to the first set and then pulling back by a blowup in y_d , we reduce to Case 2. By restricting φ to the second set and then swapping the coordinates y_{l+1} and y_d , we reduce to the case that $\varphi_{\leq d}$ is $(l+1)$ -rectilinear and φ is prepared over $\mathbb{R}^{m+d}$, and we are done. This completes the proof when $\text{supp}(\alpha_{>l})$ is empty.

Now suppose that $\text{supp}(\alpha_{>l})$ is nonempty. By pulling back by a swap, we may assume that $l+1 \in \text{supp}(\alpha_{>l})$. By pulling back by the power substitution $y_d \mapsto y_d^{\alpha_{l+1}}$, we may also assume that $\alpha_{l+1} = 1$. Let y' and α' be the tuples indexed by $\{1, \dots, d-1\} \setminus \{l+1\}$ that are respectively obtained from $y_{\leq d}$ and α by omitting their $(l+1)$ -th components, and

write $y_{<d} = (y', y_{l+1})$; thus $\alpha_{>l} = (1, \alpha_{>l+1})$ and $\alpha'_{>l} = \alpha_{>l+1}$. Fix a constant $C > 1$ that is greater than the supremum of the range of $\hat{a}(x)(y')^{\alpha'} u(x, y', y_{l+1})$; this may be done because $\hat{a}(x)(y')^{\alpha'} y_{l+1}$ is bounded (since it equals $a_d(x, y_{<d})/u(x, y_{<d})$) and y_{l+1} may freely approach 1 independently of the other variables. Thus

$$a_d(x, y', y_{l+1}) = \hat{a}(x)(y')^{\alpha'} y_{l+1} u(x, y', y_{l+1}) < C y_{l+1}$$

on A . Consider the three sets,

$$\{(x, y) \in A : C^{-1} < y_{l+1} < 1\}$$

$$\{(x, y) \in A : 0 < y_{l+1} < C^{-1} \text{ and } a(x, y', y_{l+1}) < y_d < C y_{l+1}\}$$

and

$$\{(x, y) \in A : 0 < y_{l+1} < C^{-1} \text{ and } C y_{l+1} < y_d < 1\}.$$

By restricting φ to the first set, we reduce to the case that $\varphi_{\leq d}$ is $(l+1)$ -rectilinear, and we are done by the induction hypothesis since $|\text{supp}(\alpha_{>l+1})| < |\text{supp}(\alpha_{>l})|$. If we restrict φ to either the second or third set, we may pull back by a blowup in y_{l+1} to assume that $C = 1$. On the second set, we may then pull back by a blowup in y_d , and we are done by the induction hypothesis since $|\text{supp}(\alpha'_{>l})| < |\text{supp}(\alpha_{>l})|$. The third set can also be written as $\{(x, y) \in A : 0 < y_d < 1, 0 < y_{l+1} < y_d\}$, so we may reduce to Case 1 by swapping the coordinates y_{l+1} and y_d .

We use the proposition above to show Theorem (5.3.32) and also Theorem (5.3.44) when f and μ are assumed to be subanalytic.

Theorem (5.3.32)[190]: Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. Then there exists a finite partition $\mathcal{A}$ of D into subanalytic sets such that for each $A \in \mathcal{A}$ there exist $d \in \{0, \dots, n\}$, $l \in \{0, \dots, d\}$ and a subanalytic map $F : B \rightarrow A$ such that F is an analytic isomorphism over $\mathbb{R}^m$, the set $B \subset \mathbb{R}^{m+d}$ is l -rectilinear over $\mathbb{R}^m$, and each function g in the set $\mathcal{G}$ defined by

$$\mathcal{G} = \begin{cases} \{f \circ F\}_{f \in \mathcal{F}}, & \text{if } d < n, \\ \{f \circ F\}_{f \in \mathcal{F}} \cup \{\det \frac{\partial F}{\partial y}\}, & \text{if } d = n, \end{cases}$$

may be written in the form

$$g(x, z) = h(x) \left(\prod_{j=1}^d z_j^{r_j} \right) u(x, z) \quad (32)$$

on B for some analytic subanalytic function h , rational numbers r_j , and l -rectilinear unit u .

Note that if one desires, one can take the $\gamma_{i,j}$ in (26) and the r_j in (32) to all be integers. To do this, simply pull back each map F in Theorem (5.3.32) by a map, $(x, z) \mapsto (x, z_1^{k_1}, \dots, z_d^{k_d})$ for a suitable choice of positive integers $k_1, \dots, k_d$.

Proof. Let $\mathcal{F}$ be a finite set of subanalytic functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. We proceed by induction on n . The base case of $n = 0$ is trivial, so assume that $n > 0$ and that the theorem holds with k in place of n for all $k < n$. Let $\mathcal{A}$ be the open partition of D over $\mathbb{R}^m$ given by applying Proposition (5.3.31) to $\mathcal{F}$, and let $D' = \bigcup \mathcal{A}$. Thus the theorem holds for $\mathcal{F}|_{D'}$. It follows from the induction hypothesis that the theorem also holds for $\mathcal{F}|_{D \setminus D'}$, since $D \setminus D'$ may be partitioned into cells over $\mathbb{R}^m$, and each of these cells projects via an analytic isomorphism into $\mathbb{R}^{m+d}$ for some $d < n$.

Notation (5.3.33)[190]: For any set $E \subset \mathbb{R}^m$, let $\mathcal{O}_E$ denote the ring of all analytic germs

on E , and let $\mathcal{O}_E[y]$ denote the ring of all polynomials in $y = (y_1, \dots, y_n)$ with coefficients in $\mathcal{O}_E$. Each member of $\mathcal{O}_E[y]$ is an equivalence class of functions defined on neighborhoods of $E \times \mathbb{R}^n$ in $\mathbb{R}^{m+n}$, and hence defines a function on $E \times \mathbb{R}^n$. For each $\mathcal{F} \subset \mathcal{O}_E[y]$, define the variety of $\mathcal{F}$ by

$$\mathbb{V}(\mathcal{F}) = \{(x, y) \in E \times \mathbb{R}^n : f(x, y) = 0 \text{ for all } f \in \mathcal{F}\}.$$

For each $x \in \mathbb{R}^m$, the ring $\mathcal{O}_{\{x\}}$ is Noetherian, so $\mathcal{O}_{\{x\}}[y]$ is as well. This implies that when E is compact, the varieties of $\mathcal{O}_E[y]$ form the collection of closed subsets of a Noetherian topological space on $E \times \mathbb{R}^n$; in other words, for any $\mathcal{F} \subset \mathcal{O}_E[y]$ there exists a finite $\mathcal{F}' \subset \mathcal{F}$ such that $\mathbb{V}(\mathcal{F}') = \mathbb{V}(\mathcal{F})$.

Notation (5.3.34)[190]: We partially order $\mathbb{N}^k$ by defining $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ for all $j \in \{1, \dots, k\}$, where $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$. For any $\alpha \in \mathbb{N}^k$ write $[\alpha] = \{\beta \in \mathbb{N}^k : \beta \geq \alpha\}$, and for any $A \subset \mathbb{N}^k$ write $[A] = \bigcup_{\alpha \in A} [\alpha]$ for the upward closure of A . If $A \subset \mathbb{N}^k$ is nonempty, define $\min A$ to be the set of minimal members of A , and define $\min \emptyset = \emptyset$.

Dickson's lemma states that $\min A$ is finite for every $A \subset \mathbb{N}^k$. The following is a parameterized version of Dickson's lemma.

Lemma (5.3.35)[190]: Let $E \subset \mathbb{R}^m$ be compact and $\{f_\alpha\}_{\alpha \in \mathbb{N}^k} \subset \mathcal{O}_E[y]$. Then the set

$$\bigcup_{(x, y) \in E \times \mathbb{R}^n} \min\{\alpha \in \mathbb{N}^k : f_\alpha(x, y) \neq 0\} \quad (33)$$

is finite.

Proof. The proof is by induction on k , with the base case of $k = 0$ being trivial. For the inductive step, use topological Noetherianity to fix $\beta \in \mathbb{N}^k$ such that $\mathbb{V}(\{f_\alpha\}_{\alpha \leq \beta}) = \mathbb{V}(\{f_\alpha\}_{\alpha \in \mathbb{N}^k})$. Then (33) is finite because it is contained in

$$\bigcup_{i=1}^k \bigcup_{j=0}^{\beta_i} \left(\bigcup_{(x, y) \in E \times \mathbb{R}^n} \min\{\alpha \in \mathbb{N}^k : f_\alpha(x, y) \neq 0, \alpha_j = j\} \right), \quad (34)$$

and each of the sets in parenthesis in (34) is finite by the induction hypothesis.

Lemma (5.3.36)[190]: Let $M \subset \mathbb{N}^k$ be finite. Then there exists a finite partition of $[M] \setminus M$ that is compatible with $\{[\alpha]\}_{\alpha \in M}$ and is such that each member of the partition has a unique minimal member.

Proof. Define $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ by $\epsilon_i = \max\{\alpha_i : \alpha \in M\}$ for each $i \in \{1, \dots, k\}$. Let the partition of $[M] \setminus M$ consist of all the singletons $\{\alpha\}$ with $\alpha \in \left(\prod_{i=1}^k [0, \epsilon_i]\right) \cap [M] \setminus M$ and all sets of the form

$$\left\{ \alpha \in \mathbb{N}^k : \left(\bigwedge_{i \in N} \alpha_i > \epsilon_i \right) \wedge \left(\bigwedge_{j \in \{1, \dots, k\} \setminus N} \alpha_j = \beta_j \right) \right\},$$

for each nonempty $N \subset \{1, \dots, k\}$ and $\beta = (\beta_1, \dots, \beta_k)$ in $\left(\prod_{i=1}^k [0, \epsilon_i]\right) \cap [M]$.

Lemma (5.3.37)[190]: Let $E \times \mathbb{R}^m$ be compact, and suppose that f is represented by a convergent power series

$$f(x, y, z) = \sum_{\alpha \in \mathbb{N}^k} f_\alpha(x, y) z^\alpha$$

on $E \times \mathbb{R}^n \times [0, 1]^k$, where $f_\alpha \in \mathcal{O}_E[y]$ for each $\alpha \in \mathbb{N}^k$. Then we may write

$$f(x, y, z) = \sum_{\alpha \in M^{cr}} z^\alpha f_\alpha(x, y) + \sum_{\beta \in M^{nc}} z^\beta f_\beta(x, y, z) \quad (35)$$

on $E \times \mathbb{R}^n \times [0, 1]^k$, where the sets $M^{cr}, M^{nc} \subset \mathbb{N}^k$ are finite and disjoint, each f_β with

$\beta \in M^{cr}$ is represented by a subseries of $\sum_{\alpha \geq \beta} f_\alpha(x, y) z^{\alpha - \beta}$, and for each $(x, y) \in E \times \mathbb{R}^n$ and each $\beta \in M^{nc}$, if $f_\beta(x, y, z) \neq 0$ for some $z \in [0, 1]^k$, then $f_\alpha(x, y) \neq 0$ for some $\alpha \in M^{cr}$ with $\alpha \leq \beta$.

Proof. Let M^{cr} be the set defined in (33), let $\mathcal{S}$ be the partition of $[M^{cr}] \setminus M^{cr}$ given by Lemma (5.3.36), and let M^{nc} be the set of minimal members of the sets in $\mathcal{S}$. For each $\beta \in M^{nc}$, write S_β for the unique member of $\mathcal{S}$ whose minimal member is β , and define $f_\beta(x, y, z) = \sum_{\alpha \in S_\beta} f_\alpha(x, y) z^{\alpha - \beta}$. Then (35) holds. Consider $\beta \in M^{nc}$ and $(x, y) \in E \times \mathbb{R}^n$ such that $f_\beta(x, y, z) \neq 0$ for some $z \in [0, 1]^k$. Then $f_\gamma(x, y) \neq 0$ for some $\gamma \in S_\beta$. Fix $\alpha \in M^{cr}$ such that $f_\alpha(x, y) \neq 0$ and $\alpha \leq \gamma$. Thus $S_\beta \cap [\alpha]$ is nonempty, so $S_\beta \subset [\alpha]$ by the compatibility property of $\mathcal{S}$, and hence $\alpha \leq \beta$.

This section shows the following proposition, which is a preparation result for constructible functions in transformed coordinates on rectilinear sets.

Proposition (5.3.38)[190]: Let $\mathcal{F}$ be a finite set of constructible functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$. There exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exist a subanalytic analytic isomorphism $F = (F_1, \dots, F_{m+n}): B \rightarrow A$ over $\mathbb{R}^m$, rational monomial maps φ on A and ψ on B over $\mathbb{R}^m$, and $l \in \{1, \dots, n\}$ with the following properties.

1. Pullback property: The map ψ is l -rectilinear over $\mathbb{R}^m$, $\det \frac{\partial F}{\partial y}$ is ψ -prepared, and for every $f \in \mathcal{F}$ we may write $f \circ F$ in the form

$$f \circ F(x, y) = \sum_{s \in S} (\log y_{>l})^s \left(\sum_{r \in R_s^{cr}} y_{>l}^r f_{r,s}(x, y_{\leq l}) + \sum_{r \in R_s^{nc}} y_{>l}^r f_{r,s}(x, y) \right) \quad (36)$$

on B , where the sets $S \subset \mathbb{N}^{n-1}$ and $R_s^{cr}, R_s^{nc} \subset \mathbb{Z}^{n-1}$ are finite with $R_s^{cr} \cap R_s^{nc} = \emptyset$ for each s , and each function $f_{r,s}$ may be written as a finite sum

$$\begin{cases} f_{r,s}(x, y_{\leq l}) = \sum_j g_j(x) y_{\leq l}^{\alpha_j} (\log y_{\leq l})^{\beta_j} h_j(x, y_{\leq l}), & \text{if } r \in R_s^{cr}, \\ f_{r,s}(x, y) = \sum_j g_j(x) y_{\leq l}^{\alpha_j} (\log y_{\leq l})^{\beta_j} h_j(x, y), & \text{if } r \in R_s^{nc}, \end{cases} \quad (37)$$

where $g_j \in \mathcal{C}(\Pi_m(A))$, $\alpha_j \in \mathbb{Z}^l$, $\beta_j \in \mathbb{N}^l$, h_j is either a $\psi_{\leq l}$ -function or a ψ -function according to whether r is in R_s^{cr} or R_s^{nc} , and the following holds:

$$\begin{cases} \text{For each } s \in S, r' \in R_s^{nc} \text{ and } (x, y_{\leq l}) \in \Pi_{m+l}(B), \text{ if } f_{r',s}(x, y_{\leq l}, y_{>l}) \neq 0 \\ \text{for some } y_{>l} \in (0, 1)^{n-l}, \text{ then } f_{r,s}(x, y_{\leq l}) \neq 0 \text{ for some } r \in R_s^{cr} \text{ with } r \leq r'. \end{cases} \quad (38)$$

2. Pushforward property: The components of F^{-1} are φ -prepared, and $\psi \circ F^{-1}$ is a φ -function.

The superscripts “cr” and “nc” in the notation R_s^{cr} and R_s^{nc} stand for critical and noncritical. We will use (38) to see that the L^p -classes of $f(x, \cdot)$ are determined by which of the terms $f_{r,s}(x, \cdot)$ with $r \in R_s^{cr}$ are identically zero, so in this sense these are the “critical” terms.

In the degenerate case of $l = n$, (36) and (37) simply mean that

$$f \circ F(x, y) = \sum_j g_j(x) y^{\alpha_j} (\log y)^{\beta_j} h_j(x, y)$$

for some constructible functions g_j , tuples $\alpha_j \in \mathbb{Z}^n$ and $\beta_j \in \mathbb{N}^n$, and ψ -functions h_j . To see this, note that if $f \circ F$ is nonzero and $l = n$, then $S = \mathbb{N}^0 = \{0\}$ and $R_0^{cr}, R_0^{nc} \subset \mathbb{Z}^0 = \{0\}$

with $R_0^{cr} \cap R_0^{nc} = \emptyset$, so $R_0^{cr} = \emptyset$ and $R_0^{nc} = \emptyset$ by (38).

Proof. For each $f \in \mathcal{F}$ write $f(x, y) = \sum_i f_i(x, y) \prod_j \log f_{i,j}(x, y)$ for finitely many subanalytic functions $f_i = D \rightarrow \mathbb{R}$ and $f_{i,j} : D \rightarrow (0, \infty)$. Apply Proposition (5.3.31) to $\bigcup_{f \in \mathcal{F}} \{f_i, f_{i,j}\}_{i,j}$, and focus on one set A in the open partition of D over $\mathbb{R}^m$ that this gives, along with its associated maps $F : B \rightarrow A$, φ on A , and ψ on B , where ψ is l -rectilinear over $\mathbb{R}^m$. Thus $\det \frac{\partial F}{\partial y}$ is ψ -prepared, and we may write

$$f \circ F(x, y) = \sum_i a_i(x) y^{\alpha_i u_i(x, y)} \prod_j \log a_{i,j}(x) y^{\alpha_{i,j} u_{i,j}(x, y)}$$

on B for some analytic subanalytic functions a_i and $a_{i,j}$, tuples α_i and $\alpha_{i,j}$ in $\mathbb{Q}^n$, and ψ -units u_i and $u_{i,j}$. By expanding the logarithms and distributing, we may rewrite this in the form

$$f \circ F(x, y) = \sum_i g_i(x) y^{\alpha_i} (\log y)^{\beta_i} h_i(x, y) \quad (39)$$

for some constructible functions g_i , tuples $\alpha_i \in \mathbb{Q}^n$ and $\beta_i \in \mathbb{N}^n$, and ψ -functions h_i . By pulling back by power substitutions in y , we may assume that $\alpha_i \in \mathbb{Z}^n$ for each α_i in (39). Write $h_i(x, y) = H_i(\psi_{\leq l}(x, y_{\leq l}), y_{>l})$ for some analytic function $H_i(X, y_{>l})$ on the closure of the image of ψ .

We are done if $l = n$, so assume that $l < n$ and work by induction on $n - l$. Since the closure of the range of $\psi_{\leq l}$ is compact, we may fix $\epsilon > 0$ such that each function H_i is given by a single convergent power series in $y_{>l}$ with analytic coefficients in $(X, y_{\leq l})$, say

$$H_i(X, y_{>l}) = \sum_{\gamma \in \mathbb{N}^{n-l}} H_{i,\gamma}(X) y_{>l}^{\gamma}, \quad (40)$$

for all X in the closure of the range of $\psi_{\leq l}$ and all $y_{>l}$ in $[0, \epsilon]^{n-l}$. For each $j \in \{l+1, \dots, n\}$, by restricting ψ to $\{(x, y) \in B : y_j > \epsilon\}$ and swapping the coordinates y_{l+1} and y_j , we may reduce to the case that ψ is $(l+1)$ -rectilinear, in which case we are done by our induction on $n - l$. So it suffices to restrict ψ to $B \cap (\mathbb{R}^{m+l} \times (0, \epsilon)^{n-l})$. After pulling back by the maps sending $y_j \mapsto \epsilon y_j$ for each $j \in \{l+1, \dots, n\}$, and again expanding the logarithms $\log y_j \epsilon = \log y_j + \log \epsilon$ and distributing, we may assume that $\epsilon = 1$. We are now done pulling back ψ . The pushforward property of the proposition we are showing follows from the fact that φ satisfies the pushforward property of Proposition (5.3.31), because we have only applied some very simple pullback constructions to the map ψ originally given by Proposition (5.3.31). It remains to show that we can express $f \circ F$ as a sum in the desired form.

By grouping terms in (39) according to like powers of $\log y_{>l}$, factoring out suitable monomials in y , and absorbing any remaining monomials in $y_{>l}$ with nonnegative powers inside of ψ -functions, we may rewrite (39) in the form

$$f \circ F(x, y) = \sum_{s \in S} (\log y_{>l})^s y^{\delta_s} \sum_{j \in J_s} g_j(x) y_{\leq l}^{\alpha_j} (\log y_{\leq l})^{\beta_j} h_j(x, y) \quad (41)$$

for some finite $S \subset \mathbb{N}^{n-l}$ and finite index sets J_s , constructible functions g_j , tuples $\delta_s \in \mathbb{Z}^n$ and $\alpha_j, \beta_j \in \mathbb{N}^l$, and ψ -functions h_j , which we still write as $h_j = H_j \circ \psi$ with H_j written as a power series (40). For each $s \in S$ write

$$G_s \circ \Psi_s(x, y) = \sum_{j \in J_s} g_j(x) y_{\leq l}^{\alpha_j} (\log y_{\leq l})^{\beta_j} h_j(x, y),$$

where

$$\begin{aligned} \Psi_s(x, y) &= (\psi_{\leq l}(x, y_{\leq l}), \log y_{\leq l}, (g_j(x))_{j \in J_s}, y), \\ G_s(X, Y, Z_s, y) &= \sum_{j \in J_s} Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} H_j(X, y_{> l}), \end{aligned}$$

with $Z_s = (Z_j)_{j \in J_s}$ and $Y = (Y_1, \dots, Y_l)$. By computing

$$\sum_{j \in J_s} \left(Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} \sum_{\gamma \in \mathbb{N}^{n-l}} H_{j,\gamma}(X) y_{> l}^{\gamma} \right) = \sum_{\gamma \in \mathbb{N}^{n-l}} \left(\sum_{j \in J_s} Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} H_{j,\gamma}(X) \right) y_{> l}^{\gamma}, \quad (42)$$

we may write

$$G_s(X, Y, Z_s, y_{> l}) = \sum_{\gamma \in \mathbb{N}^{n-l}} G_{s,\gamma}(X, Y, Z_s, y_{\leq l}) y_{> l}^{\gamma}$$

with each

$$G_{s,\gamma}(X, Y, Z_s, y_{\leq l}) = \sum_{j \in J_s} Z_j y_{\leq l}^{\alpha_j} Y^{\beta_j} H_{j,\gamma}(X).$$

Note that each $G_{s,\gamma}$ is a polynomial in $(Y, Z_s, y_{\leq l})$ with analytic coefficients in X , and X ranges over a compact set. So we may apply Lemma (5.3.40) to get

$$G_s(X, Y, Z_s, y) = \sum_{\gamma \in R_s^{cr}} y_{> l}^{\gamma} G_{s,\gamma}(X, Y, Z_s, y_{\leq l}) + \sum_{\gamma \in R_s^{nc}} y_{> l}^{\gamma} G_{s,\gamma}^{nc}(X, Y, Z_s, y),$$

where R_s^{cr} and R_s^{nc} are disjoint subsets of $\mathbb{N}^{n-l}$, each $G_{s,\gamma}^{nc}$ is an analytic function represented by a subseries of $\sum_{\delta \geq \gamma} y_{> l}^{\delta-\gamma} G_{s,\delta}(X, Y, Z_s, y_{\leq l})$, and for each choice of $(X, Y, Z_s, y_{\leq l})$ and $\gamma' \in R_s^{nc}$, if $G_{s,\gamma'}^{nc}(X, Y, Z_s, y_{\leq l}, y_{> l}) \neq 0$ for some $y_{> l} \in [0, 1]^{n-l}$, then there exists $\gamma \in R_s^{cr}$ such that $G_{s,\gamma}(X, Y, Z_s, y_{\leq l}) \neq 0$ and $\gamma \leq \gamma'$. Write

$$f \circ F(x, y) = \sum_{s \in S} (\log y_{> l})^s y^{\delta_s} \left(\sum_{\gamma \in R_s^{cr}} y_{> l}^{\gamma} G_{s,\gamma} \circ \Psi_{s,\leq l}(x, y_{\leq l}) + \sum_{\gamma \in R_s^{nc}} y_{> l}^{\gamma} G_{s,\gamma}^{nc} \circ \Psi_s(x, y) \right), \quad (43)$$

where $\Psi_{s,\leq l}$ is the map obtained from Ψ_s by omitting its components $y_{> l}$. By distributing each y^{δ_s} and expressing each function $G_{s,\gamma}^{nc}$ as a sum of terms indexed by $j \in J_s$, via a computation analogous to what was done in (42) for G_s (but going from right to left rather than from left to right), we see that (43) expresses $f \circ F$ in the desired form.

We begin by fixing some notation to describe a situation that will be encountered throughout the section.

Notation (5.3.39)[190]: Consider a finite set $\mathcal{F}$ of constructible functions on a subanalytic set $D \subset \mathbb{R}^{m+n}$, and let $\mathcal{A}$ be an open partition of D over $\mathbb{R}^m$ obtained by applying Proposition 6.1 to $\mathcal{F}$. Focus on one $A \in \mathcal{A}$, along with its associated maps $F = (F_1, \dots, F_{m+n}) : B \rightarrow A$, φ on A , and ψ on B , where ψ is l -rectilinear over $\mathbb{R}^m$, as in the statement of the proposition. Write $(x, \tilde{y})$ for the coordinates on A with center θ , where θ is the center of φ . Write

$$\det \frac{\partial F}{\partial y}(x, y) = H(x) y^{\gamma} U(x, y)$$

on B for some analytic subanalytic function H , tuple $\gamma = (\gamma_1, \dots, \gamma_n)$ in $\mathbb{Q}^n$, and φ -unit U . For each $f \in \mathcal{F}$ write Eq. (36) as

$$f \circ F(x, y) = \sum_{(r,s) \in \Delta(f,A)} f_{r,s}(x, y)$$

on B , where

$$\begin{aligned}
\Delta^{\text{cr}}(f, A) &= \{(r, s) : s \in S \text{ and } r \in R_s^{\text{cr}}\}, \\
\Delta^{\text{nc}}(f, A) &= \{(r, s) : s \in S \text{ and } r \in R_s^{\text{nc}}\}, \\
\Delta(f, A) &= \Delta^{\text{cr}}(f, A) \cup \Delta^{\text{nc}}(f, A), \\
f_{r,s}(x, y) &= \begin{cases} y_{>l}^r (\log y_{>l})^s f_{r,s}(x, y_{\leq l}), & \text{if } (r, s) \in \Delta^{\text{cr}}(f, A), \\ y_{>l}^r (\log y_{>l})^s f_{r,s}(x, y), & \text{if } (r, s) \in \Delta^{\text{nc}}(f, A), \end{cases}
\end{aligned}$$

for the sets S , R_s^{cr} and R_s^{nc} and the functions $f_{r,s}$ defined from f and A in Proposition (5.3.38). For each $f \in \mathcal{F}$ and $x \in \Pi_m(A)$, define

$$\begin{aligned}
\Delta^{\text{cr}}(f, A, x) &= \{(r, s) \in \Delta^{\text{cr}}(f, A) : f_{r,s}(x, y_{\leq l}) \neq 0 \text{ for some } y_{\leq l} \in \Pi_l(B_x)\}, \\
\Delta^{\text{nc}}(f, A, x) &= \{(r, s) \in \Delta^{\text{nc}}(f, A) : f_{r,s}(x, y) \neq 0 \text{ for some } y \in B_x\}, \\
\Delta(f, A, x) &= \Delta^{\text{cr}}(f, A, x) \cup \Delta^{\text{nc}}(f, A, x), \\
\Omega(f, A, x) &= \{y_{\leq l} \in \Pi_l(B_x) : f_{r,s}(x, y_{\leq l}) \neq 0 \text{ for all } (r, s) \in \Delta^{\text{cr}}(f, A, x)\}.
\end{aligned}$$

For each $x \in \Pi_m(A)$ and $i \in \{l+1, \dots, n\}$, define

$$\begin{aligned}
\bar{r}_i(f, A, x) &= \inf\{r_i : (r, s) \in \Delta^{\text{cr}}(f, A, x)\}, \\
\bar{s}_i(f, A, x) &= \sup\{s_i : (r, s) \in \Delta^{\text{cr}}(f, A, x) \text{ and } r_i = \bar{r}_i(f, A, x)\},
\end{aligned}$$

under the convention that $\bar{r}_i(f, A, x) = \infty$ and $\bar{s}_i(f, A, x) = 0$ when $\Delta^{\text{cr}}(f, A, x)$ is empty.

Remarks (5.3.40)[190]. Consider the situation described in Notation (5.3.39), and let $f \in \mathcal{F}$.

- i. For each $x \in \Pi_m(A)$, the set $\Omega(f, A, x)$ is dense and open in $\Pi_l(B_x)$.
- ii. For each $x \in \Pi_m(A)$, the $\Delta^{\text{cr}}(f, A, x)$ is empty if and only if $f(x, y) = 0$ for all $y \in A_x$.

Proof. i. This follows from the fact that for each $x \in \Pi_m(A)$ and $(r, s) \in \Delta^{\text{cr}}(f, A, x)$, $f_{r,s}(x, \cdot)$ is a nonzero analytic function on $\Pi_l(B_x)$, and $\Pi_l(B_x)$ is connected and open in $\mathbb{R}^l$.

ii. If $\Delta^{\text{cr}}(f, A, x)$ is empty, then (38) implies that $f(x, \cdot)$ is identically zero on A_x . If $\Delta^{\text{cr}}(f, A, x)$ is nonempty, then the following lemma implies that $f(x, \cdot)$ is not identically zero on A_x .

Lemma (5.3.41)[190]: Consider the situation described in Notation (5.3.39). Fix $f \in \mathcal{F}$,

$i \in \{l+1, \dots, n\}$, $x \in \Pi_m(A)$ with $\Delta^{\text{cr}}(f, A, x) \neq \emptyset$, and $y_{\leq l} \in \Omega(f, A, x)$. For any tuple $y_{>l} = (y_{l+1}, \dots, y_n)$, write $y' = (y_j)_{j \in \{l+1, \dots, n\} \setminus \{i\}}$ and $y_{>l} = (y', y_i)$. Then the limit

$$\lim_{y_i \rightarrow 0} \frac{f \circ F(x, y)}{y_i^{\bar{r}_i(f, A, x)} (\log y_i)^{\bar{s}_i(f, A, x)}} \quad (44)$$

exists for all $y' \in (0, 1)^{n-l-1}$, and the set

$$\{y' \in (0, 1)^{n-l-1} : (44) \text{ is nonzero}\} \quad (45)$$

is dense and open in $(0, 1)^{n-l-1}$.

Proof. Define

$$\begin{aligned}
\Delta_i(f, A, x) &= \{(r, s) \in \Delta(f, A, x) : r_i = \bar{r}_i(f, A, x) \text{ and } s_i = \bar{s}_i(f, A, x)\}, \\
\Delta_i^{\text{cr}}(f, A, x) &= \Delta_i(f, A, x) \cap \Delta^{\text{cr}}(f, A), \\
\Delta_i^{\text{nc}}(f, A, x) &= \Delta_i(f, A, x) \cap \Delta^{\text{nc}}(f, A).
\end{aligned}$$

It follows from (38) that for each $(r, s) \in \Delta(f, A, x)$, either $r_i > \bar{r}_i(f, A, x)$, or $r_i = \bar{r}_i(f, A, x)$ and $s_i \leq \bar{s}_i(f, A, x)$. Therefore the limit (44) exists and equals $g(y')$, where

$g : (0,1)^{n-l-1} \rightarrow \mathbb{R}$ is the analytic function defined by

$$g(y') = \sum_{(r,s) \in \Delta_i^{\text{cr}}(f, A, x)} (y')^{r'} (\log y')^{s'} f_{r,s}(x, y_{\leq l}) + \sum_{(r,s) \in \Delta_i^{\text{nc}}(f, A, x)} (y')^{r'} (\log y')^{s'} f_{r,s}(x, y_{\leq l}, y', 0).$$

So to show that (45) is dense and open in $(0,1)^{n-l-1}$, it suffices to show that g is not identically zero. To do that we will show that $g \circ \eta$ is not identically zero, where $\eta : \Lambda \times (0,1) \rightarrow (0,1)^{n-l-1}$ is defined by

$$\eta(\lambda, t) = (t^{\lambda_j})_{j \in \{l+1, \dots, n\} \setminus \{i\}}$$

for some suitably chosen open set $\Lambda \subset (0, \infty)^{n-l-1}$.

Note that

$$\begin{aligned} g \circ \eta(\lambda, t) &= \sum_{(r,s) \in \Delta_i^{\text{cr}}(f, A, x)} t^{\lambda \cdot r'} \lambda^{s'} (\log t)^{|s'|} f_{r,s}(x, y_{\leq l}) \\ &\quad + \sum_{(r,s) \in \Delta_i^{\text{nc}}(f, A, x)} t^{\lambda \cdot r'} \lambda^{s'} (\log t)^{|s'|} f_{r,s}(x, y_{\leq l}, \eta(\lambda, t), 0). \end{aligned}$$

We may choose Λ so that there exist $\bar{r}' \in \{r' : (r,s) \in \Delta_i^{\text{cr}}(f, A, x)\}$ and $c > 0$ such that for all $(r,s) \in \Delta_i^{\text{cr}}(f, A, x)$ with $r' \neq \bar{r}'$,

$$\lambda \cdot \bar{r}' + c < \lambda \cdot r' \quad \text{for all } \lambda \in \Lambda. \quad (46)$$

By (38), for each $(r,s) \in \Delta_i^{\text{nc}}(f, A, x)$ there exists ρ such that $(\rho, s) \in \Delta_i^{\text{cr}}(f, A, x)$ and $\rho \leq r$ (and necessarily $\rho \neq r$), so $\lambda \cdot \rho' < \lambda \cdot r'$ for all $\lambda \in \Lambda$. Therefore by shrinking Λ and c , we can ensure that (46) also holds for all $(r,s) \in \Delta_i^{\text{nc}}(f, A, x)$. So by defining

$$\bar{s}' = \max\{|s'| : (r,s) \in \Delta_i^{\text{cr}}(f, A, x) \text{ and } r' = \bar{r}'\},$$

$$\Delta_{i,\Lambda}^{\text{cr}}(f, A, x) = \{(r,s) \in \Delta_i^{\text{cr}}(f, A, x) : r' = \bar{r}' \text{ and } |s'| = \bar{s}'\},$$

we see that as t tends to 0, $g \circ \eta(\lambda, t)$ is asymptotic with

$$t^{\lambda \cdot \bar{r}'} (\log t)^{\bar{s}'} \left(\sum_{(r,s) \in \Delta_{i,\Lambda}^{\text{cr}}(f, A, x)} \lambda^{s'} f_{r,s}(x, y_{\leq l}) \right),$$

which is not identically zero because the sum in parentheses is a nonzero polynomial in λ .

To show the next lemma, we need the following inequality:

$$(x_1 + \dots + x_k)^p \leq x_1^p + \dots + x_k^p \quad \text{if } x_1, \dots, x_k \geq 0 \text{ and } 0 < p \leq 1. \quad (47)$$

The inequality (47) can be verified when $k=2$ by considering $f(t) = (x_1 + t)^p$ and $g(t) = x_1^p + t^p$, where $x_1 \geq 0$ and $0 < p \leq 1$, and then showing that $f(0) = g(0)$ and $f'(t) \leq g'(t)$ for all $t > 0$. The general case then follows by induction on k .

Lemma (5.3.42)[190]: Let ν be a positive measure on a set Y , let $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ be finite families of real-valued ν -measurable functions on Y , and let $p, q > 0$. Put $M = \max\{p, q\}$. Then

$$\int_Y \left| \sum_{i \in I} f_i \right|^p \left| \sum_{j \in J} g_j \right|^q d\nu \leq \begin{cases} \sum_{(i,j) \in I \times J} \int_Y |f_i|^p |g_j|^q d\nu & \text{if } M < 1, \\ \left(\sum_{(i,j) \in I \times J} \left(\int_Y |f_i|^p |g_j|^q d\nu \right)^{1/M} \right)^M, & \text{if } M \geq 1. \end{cases}$$

Proof. By symmetry we may assume that $p \geq q$. Then

$$\int_Y \left| \sum_{i \in I} f_i \right|^p \left| \sum_{j \in J} g_j \right|^q d\nu \leq \int_Y \left(\sum_{i \in I} |f_i| \right)^p \left(\sum_{j \in J} |g_j| \right)^q d\nu = \int_Y \left(\left(\sum_{i \in I} |f_i| \right) \left(\sum_{j \in J} |g_j| \right)^{q/p} \right)^p d\nu$$

$$\begin{aligned}
&\leq \int_Y \left(\left(\sum_{i \in I} |f_i| \right) \left(\sum_{j \in J} |g_j|^{q/p} \right) \right)^p d\nu \quad \text{by (7.4),} \\
&= \int_Y \left(\sum_{(i,j) \in I \times J} |f_i| |g_j|^{q/p} \right)^p d\nu, \\
&\leq \begin{cases} \sum_{(i,j) \in I \times J} \int_Y |f_i|^p |g_j|^q d\nu, & \text{if } p < 1, \\ \left(\sum_{(i,j) \in I \times J} \left(\int_Y |f_i|^p |g_j|^q d\nu \right)^{1/p} \right)^p, & \text{if } p \geq 1, \end{cases}
\end{aligned}$$

with the last inequality following from (47) when $p < 1$ and from the triangle inequality for $L^p(\nu)$ when $p \geq 1$.

Lemma (5.3.43)[190]: Consider the situation described in Notation (5.3.39), and suppose that $f, \mu \in \mathcal{F}$, $q > 0$ and $x \in \Pi_m(A)$. Then

$$LC(f|_A, |\mu|^q|_A, x) \cap (0, \infty) = \bigcap_{i=l+1}^n \{p \in (0, \infty) : \bar{r}_i(f, A, x)p + \bar{r}_i(\mu, A, x)q + \gamma_i > -1\}.$$

And, $\infty \in LC(f|_A, |\mu|^q|_A, x)$ if and only if either $\Delta^c(\mu, A, x)$ is empty or else for each $i \in \{l+1, \dots, n\}$, $\bar{r}_i(f, A, x) > 0$ or $\bar{r}_i(f, A, x) = \bar{s}_i(f, A, x) = 0$.

Proof. Let $x \in \Pi_m(A)$. The conclusion is clear from Remark (5.3.42.2) when either $\Delta^c(f, A, x)$ or $\Delta^c(\mu, A, x)$ is empty, for then $LC(f|_A, |\mu|^q|_A, x) = (0, \infty]$ and either $\bar{r}_i(f, A, x) = \infty$ for all $i \in \{l+1, \dots, n\}$ (when $\Delta^c(f, A, x)$ is empty), or $\bar{r}_i(\mu, A, x) = \infty$ for all $i \in \{l+1, \dots, n\}$ (when $\Delta^c(\mu, A, x)$ is empty). So we assume that $\Delta^c(f, A, x)$ and $\Delta^c(\mu, A, x)$ are both nonempty. Let $p \in (0, \infty)$.

Suppose that

$$\bar{r}_i(f, A, x)p + \bar{r}_i(\mu, A, x)q + \gamma_i > -1 \quad (48)$$

for all $i \in \{l+1, \dots, n\}$. Then

$$r_i p + r'_i q + \gamma_i > -1$$

for all $i \in \{l+1, \dots, n\}$, $(r, s) \in \Delta(f, A, x)$ and $(r', s') \in \Delta(\mu, A, x)$. By applying Lemma (5.3.42) to the sums $f \circ F = \sum_{(r,s)} f_{r,s}$ and $\mu \circ F = \sum_{(r,s)} \mu_{r,s}$ using the measure defined from the Jacobian of F in y , and then by applying Corollary (5.3.26), we see that $p \in LC(f|_A, |\mu|^q|_A, x)$.

Conversely, suppose that $p \in LC(f|_A, |\mu|^q|_A, x)$, and let $i \in \{l+1, \dots, n\}$. Fubini's theorem and Remark (5.3.42.1) imply that there exist $y_{\leq i} \in \Omega(f, A, x) \cap \Omega(\mu, A, x)$ and y' in the set (45) such that

$$y_i \mapsto |f \circ F(x, y)|^p |\mu \circ F(x, y)|^q \det \frac{\partial F}{\partial y}(x, y)$$

is integrable on $(0, 1)$. So (48) holds by Lemmas (5.3.25) and (5.3.41).

The L^∞ case is similar. Indeed, suppose that $\bar{r}_i(f, A, x) > 0$ or $\bar{r}_i(f, A, x) = \bar{s}_i(f, A, x) = 0$ for all $i \in \{l+1, \dots, n\}$. Then $r_i > 0$ or $r_i = s_i = 0$ for all $i \in \{l+1, \dots, n\}$ and $(r, s) \in \Delta(f, A, x)$. So applying Corollary (5.3.26) to each term of the sum $f \circ F = \sum_{(r,s)} f_{r,s}$ shows that $f \circ F(x, \cdot)$ is bounded on B_x , and hence $\infty \in LC(f|_A, |\mu|^q|_A, x)$.

Conversely, suppose that $\infty \in \text{LC}(f|_A, |\mu|^q|_A, x)$. Then $f \circ F(x, \cdot)$ is bounded on B_x . So for each $i \in \{l+1, \dots, n\}$ we may choose $y_{\leq i} \in \Omega(f, A, x)$ and y' in the set (45), and thereby conclude that $\bar{r}_i(f, A, x) > 0$ or $\bar{r}_i(f, A, x) = \bar{s}_i(f, A, x) = 0$ by Lemmas (5.3.25) and (5.3.41). **Theorem (5.3.44)[190]:** Let $q > 0$ and $f, \mu \in \mathcal{C}(D)$ for some subanalytic set $D \subset \mathbb{R}^{m+n}$, and put $E = \Pi_m(D)$ and $\mathcal{I} = \{\text{LC}(f, |\mu|^q, x) : x \in E\}$. Then $\mathcal{I}$ is a finite set of open subintervals of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, and for each $I \in \mathcal{I}$ there exists $g_I \in \mathcal{C}(E)$ such that

$$\{x \in E : I \subset \text{LC}(f, |\mu|^q, x)\} = \{x \in E : g_I(x) = 0\}. \quad (49)$$

Moreover, if f and μ are subanalytic, then each of the functions g_I can be taken to be subanalytic.

Proof. in the subanalytic case. Suppose that $q > 0$ and that f and μ are real-valued subanalytic functions on $D \subset \mathbb{R}^{m+n}$. Put $E = \Pi_m(D)$ and $\mathcal{I} = \{\text{LC}(f, |\mu|^q, x) : x \in E\}$. Apply Proposition (5.3.31) to $\mathcal{F} = \{f, \mu\}$. This constructs an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$, there exist a subanalytic analytic isomorphism $F : B \rightarrow A$ over $\mathbb{R}^m$ and a rectilinear rational monomial map ψ on B over $\mathbb{R}^m$ such that $f \circ F$, $\mu \circ F$ and $\det \frac{\partial F}{\partial y}$ are ψ -prepared.

Focus on one $A \in \mathcal{A}$, along with its associated maps $F : B \rightarrow A$ and ψ on B , where ψ is l -rectilinear over $\mathbb{R}^m$. Define $\nu : B \rightarrow \mathbb{R}$ by

$$\nu(x, y) = |\mu \circ F(x, y)|^q \left| \det \frac{\partial F}{\partial y}(x, y) \right|.$$

On B write

$$\begin{aligned} f \circ F(x, y) &= a(x) y^\alpha u(x, y), \\ \nu(x, y) &= b(x) y^\beta v(x, y), \end{aligned}$$

for some analytic subanalytic functions a and b , tuples $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in (\text{span}_{\mathbb{Q}}\{1, q\})^n$, and ψ -units u and v . We may assume that a and b have constant sign. If $a = 0$ or $b = 0$, let $I_A = (0, \infty]$. Otherwise, let I_A be the set consisting of all $p \in (0, \infty)$ such that $\alpha_i p + \beta_i > -1$ for all $i \in \{l+1, \dots, n\}$, and also consisting of ∞ if $\alpha_i \geq 0$ for all $i \in \{l+1, \dots, n\}$. Note that I_A is a subinterval of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$. Also note that by Corollary (5.3.26),

$$\text{LC}(f|_A, |\mu|^q|_A, \Pi_m(A)) = \text{LC}(f \circ F, \nu, \Pi_m(A)) = \Pi_m(A) \times I_A.$$

Now, for each $x \in E$, the set $\text{LC}(f, \mu, x)$ is a subinterval of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$ because it equals the intersection of the sets I_A for all $A \in \mathcal{A}$ with $x \in \Pi_m(A)$. This, and the fact that $\mathcal{A}$ is finite, also implies that $\mathcal{I}$ is finite. To finish, let $I \in \mathcal{I}$, and note that $\{x \in E : I \subset \text{LC}(f, \mu, x)\}$ equals

$$\{x \in E : I \subset I_A \text{ for all } A \in \mathcal{A} \text{ with } x \in \Pi_m(A)\},$$

which is a subanalytic set, and hence is the zero locus of a subanalytic function.

In the constructible case. Let $f, \mu \in \mathcal{C}(D)$ for a subanalytic set $D \subset \mathbb{R}^{m+n}$, fix $q > 0$, and write $E = \Pi_m(D)$. Apply Proposition (5.3.38) to $\mathcal{F} = \{f, \mu\}$, and use Notation (5.3.39). We claim that for each $A \in \mathcal{A}$, the set

$$\mathcal{I}_A := \{\text{LC}(f|_A, |\mu|^q|_A, x) : x \in \Pi_m(A)\}$$

is a finite set of open subintervals of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, and that for each $I \in \mathcal{I}_A$ there exists $g_{A,I} \in \mathcal{C}(\Pi_m(A))$ such that

$$\{x \in \Pi_m(A) : I \subset \text{LC}(f|_A, |\mu|^q|_A, x)\} = \{x \in \Pi_m(A) : g_{A,I}(x) = 0\}.$$

The claim implies the theorem because for each $x \in E$,

$$\text{LC}(f, |\mu|^q, x) = \bigcap_{\substack{A \in \mathcal{A} \text{ s.t.} \\ x \in \Pi_m(A)}} \text{LC}(f|_A, |\mu|^q|_A, x),$$

so the claim shows that $\mathcal{I}$ is a finite set of open subintervals of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, and that for each $I \in \mathcal{I}$,

$$\begin{aligned} \{x \in E : I \subset \text{LC}(f, |\mu|^q, x)\} &= \{x \in E : I \subset \text{LC}(f|_A, |\mu|^q|_A, x) \text{ for all } A \in \mathcal{A} \text{ with } x \in \Pi_m(A)\} \\ &= \left\{ x \in E : \sum_{\substack{A \in \mathcal{A} \\ I \subset J}} \sum_{\substack{J \in \mathcal{I}_A \text{ s.t.} \\ I \subset J}} (g'_{A,J}(x))^2 \right\}, \end{aligned}$$

where each $g'_{A,J} : E \rightarrow \mathbb{R}$ is defined by extending $g_{A,J}$ by 0 on $E \setminus \Pi_m(A)$.

To show the claim, focus on one $A \in \mathcal{A}$. Lemma (5.3.43) shows that each member of $\mathcal{I}_A$ is an open subinterval of $(0, \infty]$ with endpoints in $(\text{span}_{\mathbb{Q}}\{1, q\} \cap [0, \infty)) \cup \{\infty\}$, and that $\mathcal{I}_A$ is finite because

$$\text{LC}(f|_A, |\mu|^q|_A, x) = \text{LC}(f|_A, |\mu|^q|_A, x')$$

for all $x, x' \in \Pi_m(A)$ such that $\Delta^{\text{cr}}(f, A, x) = \Delta^{\text{cr}}(f, A, x')$ and $\Delta^{\text{cr}}(\mu, A, x) = \Delta^{\text{cr}}(\mu, A, x')$. Fix $I \in \mathcal{I}_A$. We may define $g_{A,I} = 0$ if I is empty, so assume that I is nonempty. Let $a = \inf I$ and $b = \sup I$. Lemma (5.3.43) implies that for any $x \in \Pi_m(A)$, when the infimum of $\text{LC}(f|_A, |\mu|^q|_A, x)$ is finite, this infimum is determined by the inequalities (48) for all $i \in \{l+1, \dots, n\}$ for which $\bar{r}_i(f, A, x)$ is positive; and similarly, when the supremum of $\text{LC}(f|_A, |\mu|^q|_A, x)$ is finite, this supremum is determined by the inequalities (48) for all $i \in \{l+1, \dots, n\}$ for which $\bar{r}_i(f, A, x)$ is negative. Therefore $I \subset \text{LC}(f|_A, |\mu|^q|_A, x)$ if and only if each of the following two conditions hold.

1. If $I \cap (0, \infty)$ is nonempty, then

$$f_{r,s}(x, y_{\leq l}) = 0 \text{ and } \mu_{r',s'}(x, y_{\leq l}) = 0 \text{ for all } y_{\leq l} \in \Pi_l(B_x),$$

for every $(r, s) \in \Delta^{\text{cr}}(f, A)$ and $(r', s') \in \Delta^{\text{cr}}(\mu, A)$ such that for all $i \in \{l+1, \dots, n\}$,

$$\begin{cases} r_i a + r'_i q + \gamma_i < -1, & \text{if } r_i > 0, \\ r'_i q + \gamma_i \leq -1, & \text{if } r_i = 0, \\ r_i b + r'_i q + \gamma_i < -1, & \text{if } r_i < 0, \end{cases}$$

with the understanding that we are allowing computations in the extended real number system since a or b could be ∞ .

2. If $\infty \in I$, then at least one of the following two conditions hold.

(a) We have

$$\mu_{r',s'}(x, y_{\leq l}) = 0 \text{ for all } y_{\leq l} \in \Pi_l(B_x),$$

for every $(r', s') \in \Delta^{\text{cr}}(\mu, A)$.

(b) We have

$$f_{r,s}(x, y_{\leq l}) = 0 \text{ for all } y_{\leq l} \in \Pi_l(B_x),$$

for every $(r, s) \in \Delta^{\text{cr}}(f, A)$ such that for all $i \in \{l+1, \dots, n\}$, either $r_i < 0$, or else $r_i = 0$ and $s_i > 0$.

Therefore $g_{A,l}$ can be constructed using Theorem (5.3.4).

We now turn our attention to stating and showing the preparation theorem.

Notation (5.3.45)[190]: When considering the situation described in Notation (5.3.39), we shall now also write $G = (G_1, \dots, G_{m+n}): A \rightarrow B$ for the inverse of F , and for each $j \in \{l+1, \dots, n\}$ write

$$G_{m+j}(x, y) = H_j(x) |\tilde{y}|^{\beta_j} V_j(x, y)$$

on A , where H_j is an analytic subanalytic function, $\beta_j \in \mathbb{Q}^n$, and V_j is a φ -unit.

Lemma (5.3.46)[190]: Consider the situation described in Notations (5.3.39) and (5.3.45). Let $f \in \mathcal{F}$ and $(r, s) \in \Delta(f, A)$, where $r = (r_{l+1}, \dots, r_n)$ and $s = (s_{l+1}, \dots, s_n)$. We may express $f_{r,s} \circ G$ in the form

$$f_{r,s} \circ G(x, y) = \sum_{k \in K_{r,s}(f, A)} T_k(x, y) \quad (50)$$

on A , where $K_{r,s}(f, A)$ is a finite index set and for each $k \in K_{r,s}(f, A)$,

$$T_k(x, y) = g_k(x) G_{>m}(x, y)^{R_k} \left(\prod_{j=1}^n (\log |\tilde{y}|^{\beta_j})^{S_{k,j}} \right) u_k(x, y) \quad (51)$$

for some $g_k \in \mathcal{C}(\Pi_m(A))$, tuples $R_k = (R_{k,1}, \dots, R_{k,n}) \in \mathbb{Q}^n$ and $S_k = (S_{k,1}, \dots, S_{k,n}) \in \mathbb{N}^n$ satisfying $R_{k,j} = r_j$ and $S_{k,j} \leq s_j$ for all $j \in \{l+1, \dots, n\}$, and φ -units u_k .

Proof. By (37) we may write $f_{r,s}(x, y)$ as a finite sum of terms of the form

$$g(x) y^R (\log y)^S h(x, y) \quad (52)$$

on B , where $g \in \mathcal{C}(\Pi_m(A))$, the tuples $R = (R_1, \dots, R_n) \in \mathbb{Q}^n$ and $S = (S_1, \dots, S_n) \in \mathbb{N}^n$ satisfy $R_j = r_j$ and $S_j = s_j$ for all $j \in \{l+1, \dots, n\}$, and h is a ψ -function. Pulling back (52) by G gives

$$g(x) G_{>m}(x, y)^R (\log G_{>m}(x, y))^S h \circ G(x, y)$$

on A . In the above equation, by writing

$$\log G_{m+j}(x, y) = \log H_j(x) + \log |\tilde{y}|^{\beta_j} + \log V_j(x, y)$$

for each $j \in \{1, \dots, n\}$, and then distributing, we obtain the desired form given in (50) and (51), except that each u_k is only a φ -function, not necessarily a φ -unit. But then by writing $u_k = (u_k - c) + c$ for some sufficiently large constant c so that $u_k - c$ and c are both units, and then separating each term in (50) into two terms, we may further assume that each u_k in (50) is a φ -unit.

Lemma (5.3.47)[190]: Consider a single term T_k given in (51). We may express $T_k \circ F$ as a finite sum

$$T_k \circ F(x, y) = \sum g_\zeta(x) y^{R_k} (\log y)^{S_\zeta} h_\zeta(x, y) \quad (53)$$

on B for some $g_\zeta \in \mathcal{C}(\Pi_m(A))$, tuples $S_\zeta = (S_{\zeta,1}, \dots, S_{\zeta,n}) \in \mathbb{N}^n$ satisfying $S_{\zeta,j} = S_{k,j}$ for each $j \in \{1, \dots, n\}$, and bounded functions h_ζ .

Proof. Since

$$|\tilde{y}|^{\beta_j} = \frac{G_{m+j}(x, y)}{H_j(x) V_j(x, y)}$$

for each $j \in \{1, \dots, n\}$, it follows from (51) that

$$T_k \circ F(x, y) = g_k(x) y^{R_k} \left(\prod_{j=1}^n \left(\log \frac{y_j}{H_j(x) V_j \circ F(x, y)} \right)^{S_{k,j}} \right) u_k \circ F(x, y)$$

on B . In the above equation, write

$$\log \frac{y_j}{H_j(x) V_j \circ F(x, y)} = \log y_j - \log H_j(x) - \log V_j \circ F(x, y)$$

for each $j \in \{1, \dots, n\}$, and then distribute.

Theorem (5.3.48)[190]: Let Φ be a finite subset of $\mathcal{C}(D) \times \mathcal{C}(D) \times (0, \infty)$ for some subanalytic set $D \subset \mathbb{R}^{m+n}$. For each $(f, \mu, q) \in \Phi$ let

$$I(f, \mu, q) = \{LC(f, |\mu|^q, x) : x \in \Pi_m(D)\},$$

and let $\mathcal{F} = \{f, \mu : (f, \mu, q) \in \Phi\}$. Then there exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ into subanalytic cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exist a rational monomial map φ on A over $\mathbb{R}^m$ and rational numbers $\beta_{i,j}$, where $i, j \in \{1, \dots, n\}$, for which we may express each $f \in \mathcal{F}$ in the form

$$f(x, y) = \sum_{k \in K(f, A)} T_k(x, y) \quad (54)$$

on A , where $K(f, A)$ is a finite index set and for each $k \in K(f, A)$,

$$T_k(x, y) = g_k(x) \left(\prod_{i=1}^n |\tilde{y}_i|^{r_{k,i}} \left(\log \prod_{j=1}^n |\tilde{y}_j|^{\beta_{i,j}} \right)^{S_{k,i}} \right) u_k(x, y) \quad (55)$$

for some $g_k \in \mathcal{C}(\Pi_m(A))$, rational numbers $r_{k,i}$, natural numbers $s_{k,i}$, and φ -units u_k , where we are writing $(x, \tilde{y})$ for the coordinates on A with center θ , with θ being the center for φ . Moreover, for each $f \in \mathcal{F}$ and $A \in \mathcal{A}$ there exists a partition $\mathcal{P}(f, A)$ of $K(f, A)$ described as follows.

For each $A \in \mathcal{A}$, $(f, \mu, q) \in \Phi$, $K \in \mathcal{P}(f, A)$, $\Lambda \in \mathcal{P}(\mu, A)$, and $I \in \mathcal{I}(f, \mu, q)$, at least one of the following two statements holds:

1. for all $(k, \lambda) \in K \times \Lambda$, we have $\Pi_m(A) \times I \subset LC(T_k, |T_\lambda|^q, \Pi_m(A))$;
2. for all $x \in \Pi_m(A)$ such that $I \subset LC(f, |\mu|^q, x)$, either $\sum_{k \in K} T_k(x, y) = 0$ for all $y \in A_x$ or $\sum_{\lambda \in \Lambda} T_\lambda(x, y) = 0$ for all $y \in A_x$;

and if Statement 2 does not hold, then

$$\Pi_m(A) \times (I \setminus \{\infty\}) \subset LC(T'_k, |T'_\lambda|^q, \Pi_m(A)) \quad (56)$$

for all $(k, \lambda) \in K \times \Lambda$ and all functions T'_k and T'_λ of the form

$$T'_k(x, y) = \prod_{i=1}^n |\tilde{y}_i|^{r_{k,i}} \left(\log \prod_{j=1}^n |\tilde{y}_j|^{\beta'_{k,i,j}} \right)^{s'_{k,i}},$$

$$T'_\lambda(x, y) = \prod_{i=1}^n |\tilde{y}_i|^{r_{\lambda,i}} \left(\log \prod_{j=1}^n |\tilde{y}_j|^{\beta'_{\lambda,i,j}} \right)^{s'_{\lambda,i}}$$

where the $\beta'_{k,i,j}, \beta'_{\lambda,i,j} \in \mathbb{Q}$ and $s'_{k,i}, s'_{\lambda,i} \in \mathbb{N}$ are arbitrary and the $r_{k,i}, r_{\lambda,i}$ are as in (55).

Proof. Apply Proposition (5.3.38) to $\mathcal{F}$. Fix $A \in \mathcal{A}$, and use the notation found in Notations (5.3.39) and (5.3.45) and in Lemmas (5.3.46) and (5.3.47). Lemma (5.3.46) shows that each $f \in \mathcal{F}$ may be written in the form given in (54) and (55), where each T_k is defined as in (51) and

$$K(f, A) = \bigcup_{(r,s) \in \Delta(f, A)} K_{r,s}(f, A).$$

For each $f \in \mathcal{F}$, define

$$\mathcal{P}(f, A) = \{K_{r,s}(f, A)\}_{(r,s) \in \Delta(f, A)}.$$

Now also fix $(f, \mu, q) \in \Phi$, $K \in \mathcal{P}(f, A)$, $\Lambda \in \mathcal{P}(\mu, A)$ and $I \in \mathcal{I}(f, \mu, q)$. Write $K = K_{r,s}(f, A)$ and $\Lambda = K_{r',s'}(\mu, A)$ for some $(r,s) \in \Delta(f, A)$ and $(r',s') \in \Delta(\mu, A)$. We are done if Statement 2 in the last sentence of the theorem holds, so assume otherwise. Therefore we may fix $x_0 \in \Pi_m(A)$ such that $I \subset \text{LC}(f, |\mu|^q, x_0)$, $(r,s) \in \Delta(f, A, x_0)$ and $(r',s') \in \Delta(\mu, A, x_0)$. Lemma (5.3.43) gives the following.

$$\left\{ \begin{array}{l} \text{For all } p \in I \cap (0, \infty) \text{ and all } i \in \{l+1, \dots, n\}, \\ \bar{r}_i(f, A, x_0)p + \bar{r}_i(\mu, A, x_0)q + \gamma_i > -1. \end{array} \right. \quad (57)$$

$$\left\{ \begin{array}{l} \text{If } \infty \in I, \text{ then for all } i \in \{l+1, \dots, n\}, \\ \bar{r}_i(f, A, x_0) > 0 \text{ or } \bar{r}_i(f, A, x_0) = \bar{s}_i(f, A, x_0) = 0. \end{array} \right. \quad (58)$$

Let $\kappa \in K$ and $\lambda \in \Lambda$. Write T_κ and T_λ as in (7.7) with $k = \kappa$ and $k = \lambda$, respectively, and write

$$T_\kappa \circ F(x, y) = \sum_{\zeta} g_{\zeta}(x) y^{R_{\kappa}} (\log y)^{S_{\zeta}} h_{\zeta}(x, y), \quad (59)$$

$$T_{\lambda} \circ F(x, y) = \sum_{\eta} g_{\eta}(x) y^{R_{\lambda}} (\log y)^{S_{\eta}} h_{\eta}(x, y), \quad (60)$$

as in (53). Note that for each $i \in \{l+1, \dots, n\}$,

$$R_{\kappa,i} = r_i \geq \bar{r}_i(f, A, x_0) \text{ and } R_{\lambda,i} = r'_i \geq \bar{r}_i(\mu, A, x_0). \quad (61)$$

So (57) holds with $R_{\kappa,i}$ and $R_{\lambda,i}$ in place of $\bar{r}_i(f, A, x_0)$ and $\bar{r}_i(\mu, A, x_0)$, respectively. Therefore by Corollary (5.3.26), Lemma (5.3.42), (59) and (60), it follows that

$$\Pi_m(A) \times (I \setminus \{\infty\}) \subset \text{LC}(T_{\kappa}, |T_{\lambda}|^q, \Pi_m(A)).$$

Note that the proof of this fact depends only on the values of r and r' , being independent of the values $\beta_1, \dots, \beta_n$, s and s' , so (56) follows.

Now suppose that $\infty \in I$. Note that for each ζ and $i \in \{l+1, \dots, n\}$, we have $S_{\zeta,i} \leq S_{\kappa,i} \leq s_i$. Combining this with (61) shows that for each $i \in \{l+1, \dots, n\}$, either $R_{\kappa,i} > 0$ or else $R_{\kappa,i} = S_{\zeta,i} = 0$ for all ζ . Therefore Corollary (5.3.26) and (59) show that $T_{\kappa} \circ F(x, \cdot)$ is bounded on B_x for each $x \in \Pi_m(A)$. So $\Pi_m(A) \times \{\infty\} \subset \text{LC}(T_{\kappa}, |T_{\lambda}|^q, \Pi_m(A))$.

This completes the proof of the theorem, except for the fact that A need not be a cell over $\mathbb{R}^m$. To remedy this, simply construct an open partition of A over $\mathbb{R}^m$ consisting of cells over $\mathbb{R}^m$ (for instance, using Proposition (5.3.9)), and then restrict to each of these cells.

Theorem (5.3.48) was formulated in such a way so as to be as strong and general as possible, but at the cost of having a technical formulation that may obscure the fact that it implies the simpler Theorem (5.3.2). The corollary of Theorem (5.3.48) given below directly implies Theorem (5.3.2) and its analog for $p = \infty$ described in (19), and it generalizes the interpolation theorem [172, Theorem 2.4].

The proof of the corollary makes use of the following observation: for the set $\mathcal{F}$ from Theorem (5.3.48), if $f \in \mathcal{F}$ is subanalytic, then the restriction of f to A is φ -prepared (as opposed to being in the more general form allowed by (54) and (55)). This observation follows from the way the proof of Theorem (5.3.48) uses Proposition (5.3.38), and from the way the proof of Proposition (5.3.38) uses Proposition (5.3.31).

Corollary (5.3.49)[190]: Suppose that $P \subset (0, \infty]$, that $D \subset \mathbb{R}^{m+n}$ is subanalytic, and that Φ is a finite set of triples (f, μ, q) for which $f : D \rightarrow \mathbb{R}$ is constructible, $\mu : D \rightarrow \mathbb{R}$ is

subanalytic, and $q > 0$. Define $E \in \Pi_m(D)$ and $\mathcal{F} = \{f : (f, \mu, q) \in \Phi\}$. Then to each $f \in \mathcal{F}$ we may associate a function $f^* \in \mathcal{C}(D)$ in such a way so that the following statements hold.

1. There exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ there exists a rational monomial map φ on A over $\mathbb{R}^m$ such that for every $(f, \mu, q) \in \Phi$, the function μ is φ -prepared and we may express f^* as a finite sum

$$f^*(x, y) = \sum_k T_k(x, y) \quad (62)$$

on A , where each function T_k is of the form (55).

2. The following hold for all $(f, \mu, q) \in \Phi$.

(a) We have $f = f^*$ on $\{(x, y) \in D : P \subset \text{LC}(f, |\mu|^q, x)\}$.

(b) For all $A \in \mathcal{A}$ and all terms T_k in the sum (62), we have $\Pi_m(A) \times P \subset \text{LC}(T'_k, |\mu|^q, \Pi_m(A))$. (Hence $E \times P \subset \text{LC}(f^*, |\mu|^q, x)$).

3. If $\infty \notin P$, then we may take each function T_k to be of the simpler form

$$T_k(x, y) = g_k(x) \left(\prod_{i=1}^n |\tilde{y}_i|^{r_{k,i}} (\log |\tilde{y}_i|)^{s_{k,i}} \right) u_k(x, y), \quad (63)$$

and the fact that $\Pi_m(A) \times P \subset \text{LC}(T_k, |\mu|^q, \Pi_m(A))$ only depends on the values of the $r_{k,i}$, and not the values of the $s_{k,i}$, in the following sense: we have $\Pi_m(A) \times P \subset \text{LC}(T'_k, |\mu|^q, \Pi_m(A))$ for any function T'_k on A of the form

$$T'_k(x, y) = \prod_{i=1}^n |\tilde{y}_i|^{r_{k,i}} (\log |\tilde{y}_i|)^{s'_{k,i}}$$

where the $r_{k,i}$ are as in (63) and the $s'_{k,i}$ are arbitrary natural numbers.

Proof. Let $\mathcal{A}$ be the open partition of D obtained by applying Theorem (5.3.48) to Φ ; we use the notation of the theorem. Because μ is subanalytic for every $(f, \mu, q) \in \Phi$, it follows that we may partition the members of $\mathcal{A}$ further in the x -variables to assume that for each $A \in \mathcal{A}$ and each $(f, \mu, q) \in \Phi$, either $\mu(x, y) = 0$ for all $(x, y) \in A$, or else for each $x \in \Pi_m(A)$ there exists $y \in A_x$ such that $\mu(x, y) \neq 0$. Therefore for all $(f, \mu, q) \in \Phi$, $I \in \mathcal{I}(f, \mu, q)$, $A \in \mathcal{A}$ and $K \in \mathcal{P}(f, A)$, at least one of the following two statements holds.

1. For every $k \in K$ we have $\Pi_m(A) \times I \subset \text{LC}(T_k, |\mu|^q, \Pi_m(A))$.
2. We have $\sum_{k \in K} T_k(x, y) = 0$ on $\{(x, y) \in A : I \subset \text{LC}(f, |\mu|^q, x)\}$.

For each $(f, \mu, q) \in \Phi$ and $A \in \mathcal{A}$, define $K^*(f, A)$ to be the union of all $K \in \mathcal{P}(f, A)$ for which there exists $I \in \mathcal{I}(f, \mu, q)$ such that $P \subset I$ and the above Statement 1 holds. For each $(f, \mu, q) \in \Phi$, define f^* by

$$f^*(x, y) = \begin{cases} \sum_{k \in K^*(f, A)} T_k(x, y), & \text{if } (x, y) \in A \text{ with } A \in \mathcal{A}, \\ f(x, y), & \text{if } (x, y) \in D \setminus \bigcup \mathcal{A}. \end{cases}$$

Observe that Statements 1 and 2 of the corollary hold.

To show Statement 3, suppose that $\infty \notin P$. By writing

$$\log \prod_{j=1}^n |\tilde{y}_j|^{\beta_{i,j}} = \sum_{j=1}^n \beta_{i,j} \log |\tilde{y}_j|$$

in (55) and then distributing, we may write each term T_k as a finite sum of terms of the form (63) with the same values of the $r_{k,i}$ but possibly different values of the $s_{k,i}$. But only the values of the $r_{k,i}$ are relevant by (56) since $\infty \notin P$.

The analog of Theorem (5.3.2) for $p = \infty$ mentioned in the Introduction can be stated as follows: if $D \subset \mathbb{R}^{m+n}$ is subanalytic and $f \in \mathcal{C}(D)$ is such that $\text{Int}^\infty(f, \Pi_m(D)) = \Pi_m(D)$, then there exists an open partition $\mathcal{A}$ of D over $\mathbb{R}^m$ into cells over $\mathbb{R}^m$ such that for every $A \in \mathcal{A}$ we may express f as a finite sum $f(x, y) = \sum_k T_k(x, y)$ on A for terms T_k with $\text{LC}^\infty(T_k, \Pi_m(A)) = \Pi_m(A)$ that are of the form

$$T_k(x, y) = g_k(x) \left(\prod_{i=1}^n |\tilde{y}_i|^{r_i} \left(\log \prod_{j=1}^n |\tilde{y}_j|^{\beta_{i,j}} \right)^{s_{k,i}} \right) u_k(x, y), \quad (64)$$

as denoted in the previous section. This statement was shown in Corollary (5.3.49). A more literal analog of Theorem (5.3.2) for $p = \infty$ would require the terms T_k to be of the simpler form

$$T_k(x, y) = g_k(x) \left(\prod_{i=1}^n |\tilde{y}_i|^{r_i} (\log |\tilde{y}_i|)^{s_{k,i}} \right) u_k(x, y); \quad (65)$$

however, this more literal analog is false, and the purpose of this section is to show this by giving a counterexample. It follows that in Statement 3 of Corollary (5.3.49), one may not drop the assumption that $\infty \notin P$; and in Theorem (5.3.48), one may not replace (56) with the statement $\Pi_m(A) \times I \subset \text{LC}(T'_k, |T'_\mu|^q, \Pi_m(A))$.

For the rest of the section, write $(x, y) = (x, y_1, y_2)$ for coordinates on $\mathbb{R}^3$, and define $f : D \rightarrow \mathbb{R}$ by

$$f(x, y) = \log \left(\frac{y_1}{y_2} \right), \quad (66)$$

where

$$D = \{(x, y) \in \mathbb{R}^3 : 0 < x < 1, 0 < y_1 < 1, xy_1 < y_2 < y_1\}. \quad (67)$$

Note that the function $f(x, \cdot)$ is bounded on D_x for every $x \in (0, 1)$, and that the function f is already a single term of the form given in (64) on D . The obvious way to express f as a sum of terms of the form (65) is to write

$$f(x, y) = \log y_1 - \log y_2$$

on D ; however, the terms $\log y_1$ and $\log y_2$ now become unbounded on each fiber D_x . It should therefore seem feasible that f is a counterexample for the more literal analog of Theorem (5.3.2) for $p = \infty$. To show that this is in fact the case, we show the following assertion.

Lemma (5.3.50)[190]: Let

$$A = \{(x, z) \in \mathbb{R}^2 : 0 < x < 1, x < z < 1\},$$

and define an analytic isomorphism $\eta : (0, 1)^2 \rightarrow A$ by

$$\eta(x, t) = (x, x^t).$$

Suppose that $g : A \rightarrow \mathbb{R}$ is a function of the form

$$g(x, z) = \sum_{i \in I} (\log x)^i x^{\alpha_i} z^{\beta_i} g_i(x, z) \quad (68)$$

where $I \subset \mathbb{N}$ is finite and nonempty, the α_i and β_i are integers, and each g_i is a function on A that is not identically zero and is of the form

$$g_i(x, z) = G_i \left(x, z, \frac{x}{z} \right)$$

for an analytic function G_i on $[0, 1]^3$ represented by a single convergent power series, say

$$G_i(X) = \sum_{\gamma \in \mathbb{N}^3} G_{i,\gamma} X^\gamma, \quad \text{for } X \in [0, 1]^3.$$

Then there exist $\epsilon \in (0,1]$, a nonzero real number a , a natural number r , and integers p and q such that for all $t \in (0,\epsilon)$,

$$\lim_{x \rightarrow 0} \frac{g \circ \eta(x,t)}{x^{p+qt} (\log x)^r} = a. \quad (69)$$

Proof. By factoring out the lowest powers of x and z in (68), we may assume that the α_i and β_i are all natural numbers. But then each monomial $x^{\alpha_i} z^{\beta_i}$ can be incorporated into the function g_i , so we may in fact assume that the numbers α_i and β_i are all zero. For each $i \in I$,

$$g_i \circ \eta(x,t) = G_i(x, x^t, x^{1-t}) = \sum_{\gamma \in \mathbb{N}^3} G_{i,\gamma} x^{\gamma_1 + t\gamma_2 + (1-t)\gamma_3} = \sum_{k=0}^{\infty} \sum_{l=-k}^{\infty} G_i^{[k,l]} x^{k+lt},$$

where

$$G_i^{[k,l]} = \sum_{\substack{\gamma \in \mathbb{N}^3 \text{ s.t.} \\ \gamma_1 + \gamma_3 = k, \gamma_2 - \gamma_3 = l}} G_{i,\gamma}.$$

So

$$g_i \circ \eta(x,t) = \sum_{i \in I} (\log x)^i g_i \circ \eta(x,t) = \sum_{i \in I} \sum_{k=0}^{\infty} \sum_{l=-k}^{\infty} G_i^{[k,l]} x^{k+lt} (\log x)^i. \quad (70)$$

Note that for each $i \in I$, the function g_i is not identically zero and η is a bijection, so $g_i \circ \eta$ is not identically zero, which implies that $G_i^{[k,l]} \neq 0$ for some k and l .

Let (p,q) be the lexicographically minimum member of the set

$$\bigcup_{i \in I} \{(k,l) \in \mathbb{N} \times \mathbb{Z} : k+l \geq 0 \text{ and } G_i^{[k,l]} \neq 0\}, \quad (71)$$

and define $r = \max\{i \in I : G_i^{[p,q]} \neq 0\}$, $a = G_r^{[p,q]}$, and $\epsilon = \frac{1}{p+q+1}$. We claim that for all $(k,l) \neq (p,q)$ in the set (71) and all $t \in (0,\epsilon)$,

$$k+lt > p+qt. \quad (72)$$

The claim and (70) together imply (69). To show the claim, consider $(k,l) \neq (p,q)$ in (71). If $k = p$, then $l > q$, in which case (72) holds for all $t > 0$. So suppose that $k \geq p+1$. Simplifying the inequality $(p+1)(1-t) > p+qt$ shows that it is equivalent to the inequality $t < \epsilon$. So for all $t \in (0,\epsilon)$,

$$k+lt = k(1-t) + (k+l)t \geq (p+1)(1-t) + 0t > p+qt,$$

which shows the claim.

In the following proof, we shall say that two functions $g, h : A \rightarrow \mathbb{R} \setminus \{0\}$ are equivalent on A if the range of g/h is contained in a compact subset of $(0,\infty)$.

Assertion (5.3.51)[190]: For the function $f : D \rightarrow \mathbb{R}$ defined in (66) and (67), there does not exist an open cover $\mathcal{A}$ of D over $\mathbb{R}$ such that for each $A \in \mathcal{A}$, f may be written as a finite sum of terms T_k of the form (65) with each $T_k(x, \cdot)$ bounded on A_x for all $x \in \Pi_m(A)$.

Proof. Suppose for a contradiction that there exists an open cover $\mathcal{A}'$ of D over $\mathbb{R}$ such that for each $A' \in \mathcal{A}'$, f may be written as a finite sum $f(x,y) = \sum_k T_k(x,y)$ on A' for terms T_k of the form (65) with each $T_k(x, \cdot)$ bounded on A'_x for all $x \in \Pi_m(A')$; note that we associate to A' a certain rational monomial map φ' on A' over $\mathbb{R}$ that is used to defined the terms T_k . By Proposition (5.3.9) there exists an open cover $\mathcal{A}$ of D over $\mathbb{R}^0$ such that for each $A \in \mathcal{A}$ there exist a unique $A' \in \mathcal{A}'$ containing A and a prepared

rational monomial map φ on A over $\mathbb{R}^0$ such that for each function g_k occurring in (65), say of the form

$$g_k(x) = \sum_i g_{k,i}(x) \prod_j \log g_{k,i,j}(x) \quad (73)$$

for subanalytic functions $g_{k,i}$ and $g_{k,i,j}$, the functions $g_{k,i}$ and $g_{k,i,j}$ are all $\varphi_{\leq 1}$ -prepared on $\Pi_1(A)$.

The functions xy_1 and y_1 are not equivalent for x near 0, so we may fix $A \in \mathcal{A}$ of the form

$$A = \{(x, y) : 0 < x < b_0, 0 < y_1 < b_1(x), a_2(x, y_1) < y_2 < b_2(x, y_1)\}$$

with a_2 and b_2 not equivalent on $\Pi_2(A)$. Let φ be the rational monomial map on A over $\mathbb{R}^0$ associated with A . Note that x is not equivalent on $\Pi_1(A)$ to a constant, that y_1 is not equivalent on $\Pi_2(A)$ to a function of x , and that y_2 is not equivalent on A to a function of (x, y_1) , so φ must have center 0. For the same reason, if A' is the unique member of $\mathcal{A}'$ containing A , and if φ' is the rational monomial map over $\mathbb{R}$ associated with A' , then φ' must also have center 0. We are only interested in the restriction of f to A , so we may therefore simply assume that $A' = A$ and $\varphi = \varphi'$. So we may write

$$\log\left(\frac{y_1}{y_2}\right) = \sum_k g_k(x) y_1^{r_{k,1}} y_2^{r_{k,2}} (\log y_1)^{s_{k,1}} (\log y_2)^{s_{k,2}} u_k(x, y_1, y_2) \quad (74)$$

on A for the constructible functions g_k given in (73), rational numbers $r_{k,1}$ and $r_{k,2}$, natural numbers $s_{k,1}$ and $s_{k,2}$, and φ -units u_k ; and we may write

$$a_2(x, y_1) = x^\alpha y_1 u(x, y_1) \quad \text{and} \quad b_2(x, y_1) = x^\beta y_1 v(x, y_1)$$

on $\Pi_2(A)$ for some rational numbers α and β satisfying $0 \leq \beta < \alpha \leq 1$ and some $\varphi_{\leq 2}$ -units u and v .

Fix positive constants c and d satisfying $c > u(x, y_1)$ and $d < u(x, y_1)$ on $\Pi_2(A)$. Since $\alpha > \beta$, by shrinking b_0 we may assume that

$$A = \{(x, y) : 0 < x < b_0, 0 < y_1 < b_1(x), cx^\alpha y_1 < y_2 < dx^\beta y_1\}.$$

Pulling back Eq. (74) by the map $(x, y_1, y_2) \mapsto (x, y_1, y_1 y_2)$ gives

$$\log\left(\frac{1}{y_2}\right) = \sum_k g_k(x) y_1^{r_{k,1}+r_{k,2}} y_2^{r_{k,2}} (\log y_1)^{s_{k,1}+s_{k,2}} \left(1 + \frac{\log y_2}{\log y_1}\right)^{s_{k,2}} u_k(x, y_1, y_1 y_2) \quad (75)$$

on the set

$$\{(x, y_1, y_2) : 0 < x < b_0, 0 < y_1 < b_1(x), cx^\alpha < y_2 < dx^\beta\}.$$

By assumption, each term of (75) is bounded for each fixed value of x , so letting y_1 tend to 0 for each fixed value of (x, y_2) shows that for each k , either $r_{k,1} + r_{k,2} > 0$ or $r_{k,1} + r_{k,2} = s_{k,1} + s_{k,2} = 0$ (and $s_{k,1} + s_{k,2} = 0$ means that $s_{k,1} = s_{k,2} = 0$). So letting y_1 tend to 0 in (75) gives

$$\log\left(\frac{1}{y_2}\right) = \sum_k g_k(x) y_2^{r_{k,2}} v_k(x, y_2) \quad (76)$$

on

$$\{(x, y_2) : 0 < x < b_0, cx^\alpha < y_2 < dx^\beta\},$$

where each v_k is a ψ -unit with ψ defined by $\psi(x, y_2) = \lim_{y_1 \rightarrow 0} \varphi(x, y_1, y_1 y_2)$.

By pulling back (76) by the map $(x, y_2) \mapsto (x, cx^\beta y_2^{\alpha-\beta})$ and expanding logarithms

using (73), we may write

$$\log y_2 = \sum_i (\log x)^i x^{\alpha_i} y_2^{\beta_i} f_i(x, y_2) \quad (77)$$

on

$$\{(x, y_2) : 0 < x < b_0, x < y_2 < C\} \quad (78)$$

for some $C > 0$, rational numbers α_i and β_i , and ψ -functions f_i (for an appropriately modified ψ), where i ranges over some finite set of natural numbers. By pulling back by $(x, y_2) \mapsto (x^r, y_2^r)$ for a suitable positive integer r , we may further assume that all the α_i and β_i are integers, and that the components of $\psi(x, y_2)$ are also all monomial in (x, y_2) with integer powers. Thus each component of ψ is either of the form x^p for some positive integer p , is of the form y_2^q for some positive integer q , or is of the form $x^p/y_2^q = x^{p-q}(x/y_2)^q$ for some positive integers p and q with $p \geq q$. So we may assume that $\psi(x, y_2) = (x, y_2, x/y_2)$, and therefore write $f_i(x, y_2) = F_i(x, y_2, x/y_2)$ for some analytic function F_i defined on the closure of $\{(x, y_2, x/y_2) : (x, y_2) \in A\}$. Fix $\delta > 0$ sufficiently small so that

$$\{(x, y_2) : 0 < x < \delta^2, 0 < y_2 < \delta, x/y_2 < \delta\} \quad (79)$$

is contained in (78) and that F_i is represented by a single convergent power series on $[-\delta^2, \delta^2] \times [-\delta, \delta] \times [-\delta, \delta]$. Thus restricting to (79) and then pulling back by $(x, y_2) \mapsto (\delta^2 x, \delta y_2)$ gives an equation of the form

$$\log y_2 = \sum_i (\log x)^i x^{\alpha_i} y_2^{\beta_i} F_i\left(x, y_2, \frac{x}{y_2}\right) \quad (80)$$

on

$$\{(x, y_2) : 0 < x < 1, x < y_2 < 1\},$$

with each F_i represented by a single convergent power series on $[-1, 1]^3$ centered at the origin.

Applying Lemma (5.3.50) to the right side of (80) shows that there exist $\epsilon \in (0, 1]$, a nonzero real number a , a natural number r , and integers p and q such that for all $t \in (0, \epsilon)$,

$$\lim_{x \rightarrow 0} \frac{t \log x}{x^{p+qt} (\log x)^r} = a.$$

Considering this limit for any fixed value of $t \in (0, \epsilon)$ shows that $r = 1$ and that $p + qt$, so in fact $p = q = 0$ since $t \in (0, \epsilon)$ is arbitrary. But then $t = a$ for all $t \in (0, \epsilon)$, which is a contradiction that completes the proof.

Chapter 6

Existence of Primitives Lipschitz Maps and Integration

We show that if X is a quasi-Banach space with trivial dual then every continuous function $f : [0,1] \rightarrow X$ has a primitive, answering a question of M.M. Popov. We construct the first known examples of functions in $e^{(1)}([a,b], X)$ that fail to be Lipschitz. On the positive side, we obtain a criterion for Riemann integrability of quasi-Banach valued maps based on an approximation method by polynomial functions. Finally, with an eye to finding a class of functions whose integral interacts well with differentiation, we give sufficient conditions that guarantee the fulfillment of the fundamental theorem of calculus, and show the Lebesgue differentiation theorem for the integral in the sense of Vogt.

Section (6.1): Continuous Functions in a Quasi-Banach Space

Let X be a quasi-Banach space and let $f : [0,1] \rightarrow X$ be a continuous function. We say that f has a primitive if there is a differentiable function $F : [0,1] \rightarrow X$ so that $F'(t) = f(t)$ for $0 \leq t \leq 1$. M.M. Popov has asked where every continuous Function $f : [0,1] \rightarrow L_p$ where $0 < p < 1$ has a primitive; more generally, he asks the same question for any space with trivial dual [202]. We show here that the answer to this question is positive. We remark that by an old result of Mazur and Orlicz [201],[134], every continuous f is Riemann-integrable if and only if X is a Banach space.

Let us suppose for convenience that X is p -normed where $0 < p < 1$, and let $I = [0,1]$. Let $C(I; X)$ be the usual quasi-Banach space of continuous functions $f : I \rightarrow X$ with the quasi-norm $\|f\|_\infty = \max_{0 \leq t \leq 1} \|f(t)\|$. We also introduce the space $C^1(I; X)$ of all functions $f \in C(I; X)$ which are differentiable at each t and such that the function $g : I^2 \rightarrow X$ is continuous where $g(t, t) = f'(t)$ for $0 \leq t \leq 1$ and

$$g(s, t) = \frac{f(s) - f(t)}{s - t}$$

when $s \neq t$. It is easily verified that $C^1(I; X)$ is a quasi-Banach space under the quasi-norm

$$\|f\|_{C^1} = \|f(0)\| + \sup_{0 \leq s < t \leq 1} \frac{\|f(t) - f(s)\|}{s - t}.$$

Let $C_0^1(I; X)$ be the closed subspace of $C^1(I; X)$ of all f such that $f(0) = 0$. We consider the map $D : C_0^1(I; X) \rightarrow C(I; X)$ given by $Df = f'$. The following result is proved in [130].

Theorem (6.1.1)[199]: If X has trivial dual then for every $x \in X$ there exists $f \in C_0^1(I; X)$ such that $Df = 0$ and $f(1) = x$.

From this we deduce the answer to the question of Popov.

Theorem (6.1.2)[199]: If X has trivial dual then the map $D : C_0^1(I; X) \rightarrow C(I; X)$ is surjective. In particular every continuous $f : I \rightarrow X$ has a primitive.

Proof. From Theorem (6.1.1) and the Open Mapping Theorem we deduce the existence of a constant $M \geq 1$ so that if $x \in X$ there exists $f \in C_0^1(I; X)$ so that $Df = 0$, $f(1) = x$ and $\|f\|_{C^1} \leq M \|x\|$.

Now suppose $g \in C(I; X)$ with $\|g\|_\infty < 1$. For any $\epsilon > 0$ we show the existence of $f \in C_0^1(I; X)$ with $\|Df - g\|_\infty < \epsilon$ and $\|f\|_{C^1} < 4^{1/p} M$. Once this is achieved the Theorem follows again from a well-known variant of the Open Mapping Theorem.

Since g is uniformly continuous, there is a piecewise linear function h so that

$\|g - h\|_\infty < \epsilon$ and $\|h\|_\infty < 1$. Since h has finite-dimensional range there exists $H \in C_0^1(I; X)$ with $DH = h$. Now let n be a natural number, and let $x_{kn} = H(k/n) - H((k-1)/n)$. For $k = 1, 2, \dots, n$ define $f_{k,n} \in C_0^1(I; X)$ so that $Df = 0$, $\|f_{k,n}\|_{C_0^1} \leq M \|x_{kn}\|$ and $f_{k,n}(1) = x_{kn}$. Then we define $f_n \in C_0^1(I; X)$ by

$$F_n(t) = H(t) - H\left(\frac{k-1}{n}\right) - f_{kn}(nt - k + 1)$$

For $(k-1)/n \leq t \leq k/n$. Clearly $DF_n = DH = h$. It remains to estimate $\|F_n\|_{C_0^1}$.

Let

$$\eta(\epsilon) = \sup_{|t-s| \leq \epsilon} \frac{\|H(t) - H(s)\|}{|t-s|}$$

It is easy to see that $\lim_{\epsilon \rightarrow 0} \eta(\epsilon) = \|h\|_\infty < 1$. Now suppose $\frac{k-1}{n} \leq s < t \leq \frac{k}{n}$ for some $1 \leq k \leq n$. Then

$$\begin{aligned} \|F_n(t) - F_n(s)\| &\leq \left(\eta\left(\frac{1}{n}\right)^p + n^p \|f_{kn}\|_{C_0^1}^p\right)^{1/p} (t-s) \\ &\leq \left(\eta\left(\frac{1}{n}\right)^p + M^p n^p \|x_{kn}\|^p\right)^{1/p} (t-s) \\ &\leq (M^p + 1)^{1/p} \eta\left(\frac{1}{n}\right) (t-s). \end{aligned}$$

Since $F_n(\frac{k}{n}) = 0$ for $0 \leq k \leq n$ we obtain that for any $0 \leq s < t \leq 1$,

$$\|F_n(t) - F_n(s)\| \leq 2^{1/p} (M^p + 1)^{1/p} \eta\left(\frac{1}{n}\right) \min(t-s, \frac{1}{n}).$$

By taking n large enough we have $\|F_n\|_{C_0^1} < 4^{1/p} M$. Thus the theorem follows.

We close with a few remarks on the general problem of classifying those quasi-Banach spaces X so that the map $D : C_0^1(I; X) \rightarrow C(I; X)$ is surjective; let us say that such a space is a D -space. The following facts are clear:

Proposition (6.1.3)[199]: (i) Any quotient of a D -space is a D -space.

(ii) If X and Y are D -spaces then $X \oplus Y$ is a D -space.

Proof. (i) Let E be a closed subspace of X and let $\pi : X \rightarrow X/E$ be the quotient map. Let $\tilde{\pi} : C(I; X) \rightarrow C(I; X/E)$ be the induced map $\tilde{\pi}f = f \circ \pi$. We start with the observation that $\tilde{\pi}$ is surjective. If $g \in C(I; X/E)$ with $\|g\|_\infty < 1$ then we can find $f \in C(I; X)$ with $\|f\|_\infty < 2^{1/p-1}$ and $\|\tilde{\pi}f - g\|_\infty < 1$. To do this suppose N is an integer and let f_N be a function which is linear on each interval $[(k-1)/N, k/N]$ for $1 \leq k \leq N$ and such that $\pi f_N(k/N) = g(k/N)$ with $\|f_N(k/N)\| < 1$ for $0 \leq k \leq N$. For large enough N we have $\|g - \tilde{\pi}f_N\|_\infty < 1$ and our claim is substantiated.

Now if X is a D -space and $g \in C(I; X/E)$ then there exists $f \in C(I; X)$ with $\tilde{\pi}f = g$. Let $F \in C_0^1(I; X)$ with $DF = f$. Then if $G = \tilde{\pi}F$ we have $DG = g$.

(ii) is trivial.

In [130] the notion of the core is defined: if X is a quasi-Banach space then $\text{core} X$ is the maximal subspace with trivial dual.

Theorem (6.1.4)[199]: If $\text{core} X = \{0\}$ then X is a D -space if and only if X is a Banach space (i.e. is locally convex).

Proof. Suppose $\text{core} X = \{0\}$ and X is a D -space. Suppose $DF = 0$ where $F \in C_0^1(I; X)$. Let Y be the closed subspace generated by $\{F(s) : 0 \leq s \leq 1\}$. We show $Y = \{0\}$; if not there exists a nontrivial continuous linear functional y^* on Y . Then $D(y^* \circ F) = 0$ so that

$y^*(F(s))=0$ for $0 \leq s \leq 1$. But then $y^*=0$ on Y . We conclude that $Y = \{0\}$ and so $F=0$. Hence D is one-one and surjective and by the Closed Graph Theorem D is an isomorphism.

Let M be a constant so that $\|DF\|_\infty \leq 1$ implies $\|F\|_{C^1} \leq M$ for $F \in C_0^1(I; X)$. Let ϕ be any C^∞ -real function on $\mathbf{R}$ with $\phi(t)=0$ for $t \leq 0$ and $\phi(t)=1$ for $t \geq 1$. Let $K = \max_{0 \leq t \leq 1} |\phi'(t)|$. For any N and any $x_1, \dots, x_N \in X$ with $\max \|x_k\| \leq 1$, we define $F(t) = \sum_{k=1}^N \phi(Nt - k + 1)x_k$. Then $F \in C_0^1(I; X)$ and $\|DF\|_\infty \leq NK$. Hence $\|F(1)\| \leq NMK$, i.e.

$$\left\| \frac{1}{N}(x_1 + \dots + x_N) \right\| \leq MK.$$

This implies X is locally convex.

Combining Proposition (6.1.3) and Theorem (6.1.4) gives that if X is a D -space then $X/\text{core } X$ is a Banach space. It is, however, possible to construct an example to show that the converse to this statement is false, and there does not seem, therefore to be any nice classification of D -spaces in general.

To construct the example we observe the following theorem. First for any quasi-Banach space X let $a_N(X) = \sup\{\|x_1 + \dots + x_N\| : \|x_i\| \leq 1\}$ (so that $a_N(X) \geq N$).

Theorem (6.1.5)[199]: Suppose X is a D -space; then for some constant C we have $a_N(X) \leq C a_N(\text{core } X)$.

Proof. Let $b_N = a_N(\text{core } X)$. Suppose $x_1, \dots, x_N \in X$ with $\|x_i\| \leq 1$ and define as in Theorem (6.1.4), $F(t) = \sum_{k=1}^N \phi(Nt - k + 1)x_k$. Then $\|DF\|_\infty \leq NK$ and so by the Open Mapping Theorem, for some constant $M = M(X)$, there exists $G \in C_0^1(I; X)$ with $DG = DF$ and $\|G\|_{C^1} \leq MNK$. Then $\|G(k/N) - G((k-1)/N)\| \leq MK$ for $1 \leq k \leq N$.

Let $H(t) = F(t) - G(t)$. Since $DH = 0$ and $X/\text{core } X$ is a Banach space H has range in $\text{core } X$. Now for $1 \leq k \leq N$, $H(k/N) - H((k-1)/N) = x_k - G(k/N) + G((k-1)/N)$ so that $\|H(k/N) - H((k-1)/N)\| \leq (M^p K^p + 1)^{1/p}$. Hence if $C^p = M^p K^p + 1$, we have $\|H(1)\| \leq C b_N$ or $\|x_1 + \dots + x_N\| \leq C b_N$.

To construct our example we start with the Ribe space Z ([200],[203]) which is a space with a one-dimensional subspace L so that Z/L is isomorphic to ℓ_1 . A routine calculation shows $a_N(Z) \geq cN \log N$ for some $c > 0$. Then let Y be any quasi-Banach space with trivial dual so that $a_N(Y) = o(N \log N)$ (for example a Lorentz space $L(1, p)$ where $1 < p < \infty$). Let $j: L \rightarrow Y$ be an isometry and let X be the quotient of $Y \times Z$ by the subspace of all (jz, z) for $z \in L$. Then Z embeds into X so that $a_n(X) \geq cN \log N$ but $\text{core } X \sim Y$ so that X cannot be a D -space. However $X/\text{core } X$ is isomorphic to Z/L which is a Banach space.

Corollary (6.1.6)[221]: If X has trivial dual then the map $D_j: C_0^1(I_j; X) \rightarrow C(I_j; X)$ are surjective. In particular every series of continuous $\sum f_j: \sum I_j \rightarrow X$ has a primitive.

Proof. From Theorem (6.1.1) and the Open Mapping Theorem we deduce the existence of a constant $M \geq 1$ so that if $x \in X$ there exists $f_j \in C_0^1(I_j; X)$ so that $\sum D_j f_j = 0$, $\sum f_j(1) = x$ and $\|\sum f_j\|_{C^1} \leq M \|x\|$.

Now suppose $g_j \in C(I_j; X)$ with $\|\sum g_j\|_\infty < 1$. For any $\epsilon > 0$ we show the existence of

$f_j \in C_0^1(I_j; X)$ with $\|\sum D_j f_j - \sum g_j\|_\infty < \epsilon$ and $\|\sum f_j\|_{C^1} < 4^{1/p} M$. Once this is achieved the Theorem follows again from a well-known variant of the Open Mapping Theorem.

Since $\sum g_j$ is uniformly series of continuous, there is a piecewise linear function $\sum h_j$ so that $\|\sum g_j - \sum h_j\|_\infty < \epsilon$ and $\sum \|h_j\|_\infty < 1$. Since $\sum h_j$ has finite-dimensional range there exists $H_j \in C_0^1(I_j; X)$ with $\sum D_j H_j = \sum h_j$. Now let n be a natural number, and let $x_{kn} = \sum H_j(k/n) - \sum H_j((k-1)/n)$. For $k=1, 2, \dots, n$ define $(f_j)_{k,n} \in C_0^1(I_j; X)$ so that $\sum D_j f_j = 0$, $\|(\sum f_j)_{k,n}\|_{C_0^1} \leq M \|x_{kn}\|$ and $(\sum f_j)_{k,n}(1) = x_{kn}$. Then we define $(f_j)_n \in C_0^1(I; X)$ by

$$(\sum F_j)_n(t) = \sum H_j(t) - \sum H_j\left(\frac{k-1}{n}\right) - (\sum f_j)_{kn}(nt - k + 1)$$

For $(k-1)/n \leq t \leq k/n$. Clearly $\sum D_j (\sum F_j)_n = \sum D_j H_j = \sum h_j$. It remains to estimate $\|(\sum F_j)_n\|_{C_0^1}$.

Let

$$\eta(\epsilon) = \sup_{|t-s| \leq \epsilon} \frac{\|\sum H_j(t) - \sum H_j(s)\|}{|t-s|}$$

It is easy to see that $\lim_{\epsilon \rightarrow 0} \eta(\epsilon) = \|\sum h_j\|_\infty < 1$. Now suppose $\frac{k-1}{n} \leq s < t \leq \frac{k}{n}$ for some $1 \leq k \leq n$. Then

$$\begin{aligned} \|(\sum F_j)_n(t) - (\sum F_j)_n(s)\| &\leq \left(\eta\left(\frac{1}{n}\right)^p + n^p \|(\sum f_j)_{kn}\|_{C^1}^p\right)^{\frac{1}{p}} (t-s) \\ &\leq \left(\eta\left(\frac{1}{n}\right)^p + M^p n^p \|x_{kn}\|^p\right)^{\frac{1}{p}} (t-s) \\ &\leq (M^p + 1)^{\frac{1}{p}} \eta\left(\frac{1}{n}\right) (t-s). \end{aligned}$$

Since $(\sum F_j)_n(\frac{k}{n}) = 0$ for $0 \leq k \leq n$ we obtain that for any $0 \leq s < t \leq 1$,

$$\|(\sum F_j)_n(t) - (\sum F_j)_n(s)\| \leq 2^{1/p} (M^p + 1)^{1/p} \eta\left(\frac{1}{n}\right) \min(t-s, \frac{1}{n}).$$

By taking n large enough we have $\|(\sum F_j)_n\|_{C_0^1} < 4^{1/p} M$. Thus the theorem follows.

Corollary (6.1.7)[221]: (i) quotient of a D -space are D -space.

(ii) If X_j and Y_j are D -spaces then $X_j \oplus Y_j$ are D -space.

Proof. (i) Let E be a closed subspace of X_j and let $\pi: X_j \rightarrow X_j/E$ be the quotient map. Let $\tilde{\pi}: C(I; X_j) \rightarrow C(I; X_j/E)$ be the induced map $\tilde{\pi}f = f \circ \pi$. We start with the observation that $\tilde{\pi}$ is surjective. If $g \in C(I; X_j/E)$ with $\|g\|_\infty < 1$ then we can find $f \in C(I; X_j)$ with $\|f\|_\infty < 2^{1/p-1}$ and $\|\tilde{\pi}f - g\|_\infty < 1$. To do this suppose N is an integer and let f_N be a function which is linear on each interval $[(k-1)/N, k/N]$ for $1 \leq k \leq N$ and such that $\pi f_N(k/N) = g(k/N)$ with $\|f_N(k/N)\| < 1$ for $0 \leq k \leq N$. For large enough N we have $\|g - \tilde{\pi}f_N\|_\infty < 1$ and our claim is substantiated.

Now if X_j is a D -space and $g \in C(I; X_j/E)$ then there exists $f \in C(I; X_j)$ with $\tilde{\pi}f = g$. Let $F \in C_0^1(I; X_j)$ with $DF = f$. Then if $G = \tilde{\pi}F$ we have $DG = g$.

(ii) is trivial.

Corollary (6.1.8)[221]: Suppose X is a D -space; then for some constant $\tilde{C}$ we have

$$\sum_{j=1}^m a_{N_j}(X) \leq \tilde{C} \sum_{j=1}^m a_{N_j}.$$

Proof. Let $b_{N_j} = a_{N_j}(\text{core } X)$. Suppose $\{x_{N_j}\} \in X$ with $\|x_i\| \leq 1$ and define as in Theorem (6.1.4), $\sum F_j(t) = \sum_{j=1}^m \sum_{k=1}^{N_j} \phi(N_j t - k + 1)x_k$. Then $\sum \|DF_j\|_\infty \leq \sum N_j K$ and so by the Open Mapping Theorem, for some constant $M = M(x)$, there exists $G \in C_0^1(I; X)$ with $DG_j = DF_j$ and $\|G_j\|_{C^1} \leq MN_j K$. Then $\|G_j(k/N_j) - G_j((k-1)/N_j)\| \leq MK$ for $1 \leq k \leq N_j$. Let $H(t) = F_j(t) - G_j(t)$. Since $DH = 0$ and $X/\text{core } X$ is a Banach space H has range in $\text{core } X$. Now for $1 \leq k \leq N_j$, $H(k/N_j) - H((k-1)/N_j) = x_k - G_j(k/N_j) + G_j((k-1)/N_j)$ so that $\|H(k/N_j) - H((k-1)/N_j)\| \leq (M^p K^p + 1)^{1/p}$. Hence if $C^p = M^p K^p + 1$, we have $\|H(1)\| \leq \tilde{C} b_{N_j}$ or $\|x_1 + \dots + x_n\| \leq \tilde{C} b_n$.

Section (6.2): Primitives for Continuous Functions in Quasi-Banach Spaces

If X is a Banach space, every continuous map $f: [a, b] \rightarrow X$ is Riemann-integrable and the corresponding integral function, $F(t) = \int_a^t f(u) du$ is differentiable at every $t \in [a, b]$ with derivative $F'(t) = f(t)$, that is, F is a primitive of f . However, when X is a non-locally convex F-space, a classical theorem of Mazur and Orlicz [201] informs us about the existence of continuous X -valued functions on $[a, b]$ failing to be integrable. Popov investigated in [202] the properties of the Riemann integral for functions $f: [a, b] \rightarrow X$ where X is an F-space and showed that while some usual properties of this integral remain true in the non-locally convex setting, other properties and techniques, like the usual way of getting primitives for integrable functions, may be false. His work naturally led to the question whether every continuous function $f: [a, b] \rightarrow X$ has a primitive. Kalton provided an affirmative answer for the quasi-Banach spaces X which, like the L_p spaces for $p < 1$, have trivial dual [199], but the main question remained unsolved. In the first part of this section we solve Popov's problem by showing that if the space l_p with $0 < p < 1$ embeds isomorphically in a quasi-Banach space X with separating dual, then there exists an integrable continuous function $f: [0, 1] \rightarrow X$ failing to have a primitive. This will follow as a consequence of our main theorem.

Proposition (6.2.1)[204]: Let X be a quasi-Banach space. For a given pair (τ, x) we have the following.

(i) The function $f = f(\tau, x): [0, 1] \rightarrow X$ is continuous at 1, hence continuous on $[0, 1]$, if and only if $x_k \rightarrow 0$.

(ii) Suppose that X is p -convex for some $0 < p \leq 1$. If (x_k) is bounded and the sequence (λ_k) verifies $\sum_{k=1}^\infty \lambda_k^p < \infty$, then f is Riemann-integrable on $[0, 1]$.

(iii) $F = F(\tau, x)$ can be extended continuously to $[0, 1]$ by putting $F(1) = \sum_{k=1}^\infty \lambda_k x_k$ if and only if the series $\sum_{k=1}^\infty \lambda_k x_k$ converges in X .

(iv) Suppose (x_k) is bounded. Then $F: [0, 1] \rightarrow X$ is Lipschitz if and only if there is $K > 0$ so that for all integers m, n with $m < n$,

$$\frac{\left\| \sum_{m+1 \leq k \leq n} \lambda_k x_k \right\|}{\sum_{m+1 \leq k \leq n} \lambda_k} = \frac{\left\| \sum_{m+1 \leq k \leq n} \lambda_k x_k \right\|}{t_n - t_m} \leq k. \quad (1)$$

In this case F extends to a Lipschitz function on the whole interval $[0,1]$.

(v) Suppose $x_k \rightarrow 0$. Then $F : [0,1] \rightarrow X$ is differentiable with zero left-derivative at $t=1$ if and only if

$$\lim_{n \rightarrow \infty} \frac{\left\| \sum_{k \geq n+1} \lambda_k x_k \right\|}{\sum_{k \geq n+1} \lambda_k} = \lim_{n \rightarrow \infty} \frac{\left\| \sum_{k \geq n+1} \lambda_k x_k \right\|}{1 - t_n} = 0. \quad (2)$$

Proof. The proof of statement (i) is straightforward and so we skip it.

(ii) By the Aoki–Rolewicz theorem we can assume that the quasi-norm on X is p -subadditive for some $0 < p \leq 1$. We will make use of this throughout the remainder of the proof. Put $B = \sup_k \|x_k\|$ so that $\|f(t)\| \leq 2B$ for all $t \in [0,1]$. Since

$$\left\| \sum_{k=m+1}^n \lambda_k x_k \right\|^p \leq \sum_{k=m+1}^n \lambda_k^p \|x_k\|^p \leq B^p \sum_{k=m+1}^n \lambda_k^p \xrightarrow{m,n \rightarrow \infty} 0$$

the series $\sum_k \lambda_k x_k$ is Cauchy, so it converges. We will show that $\sum_k \lambda_k x_k$ is the Riemann integral of f in the interval $[0,1]$.

Fix $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \lambda_k^p \leq \epsilon^p / (3(1+2^p 3^{1-p})B^p)$.

Now, since f is Riemann-integrable in $[0, t_N]$, there exists $\delta > 0$ such that for all Riemann sums, $\sigma(f, \pi)$, of f associated with a partition π of $[0, t_N]$ with diameter at most δ , $\left\| \sigma(f, \pi) - \sum_{k=1}^N \lambda_k x_k \right\|^p \leq \epsilon^p / 3$.

Associated with a partition of $[0,1]$,

$$\pi_1 = \{0 = a_0 < \dots < a_{l-1} < a_l < \dots < a_L = 1\},$$

of diameter at most $\delta_1 = \min\{\delta, \epsilon / (2 \cdot 3^{1/p} B)\}$, we consider a Riemann sum $\sigma_1(f, \pi_1) = \sum_{l=1}^L f(b_l) \mu_l$, where $\mu_l = a_l - a_{l-1} \leq \delta_1$ and $b_l \in [a_{l-1}, a_l]$. Using the p -subadditivity of the quasi-norm we estimate $\left\| \sum_{k=1}^{\infty} \lambda_k x_k - \sigma_1(f, \pi_1) \right\|^p$ by splitting it into four chunks:

$$\left\| \sum_{k=N+1}^{\infty} \lambda_k x_k \right\|^p + \left\| \sum_{k=1}^N \lambda_k x_k - \sum_{l=1}^{\ell-1} f(b_l) \mu_l \right\|^p + \|f(b_\ell) \mu_\ell\|^p + \left\| \sum_{l=\ell+1}^L f(b_l) \mu_l \right\|^p,$$

where $\ell \leq L$ is such that $a_{\ell-1} \leq t_N < a_\ell$.

Clearly,

$$\left\| \sum_{k \geq N+1} \lambda_k x_k \right\|^p \leq B^p \sum_{k \geq N+1} \lambda_k^p.$$

To find a bound for the second summand we observe that, since $f(t_N) = 0$, $\sum_{l=1}^{\ell-1} f(b_l) \mu_l$ is a Riemann sum of f also in $[0, t_N]$, with diameter at most $\delta_1 \leq \delta$. Hence,

$$\left\| \sum_{k=1}^N \lambda_k x_k - \sum_{l=1}^{\ell-1} f(b_l) \mu_l \right\|^p \leq \frac{\epsilon^p}{3}$$

For the third term, simply note that

$$\|f(b_\ell) \mu_\ell\|^p \leq (2B)^p \delta_1^p \leq \frac{\epsilon^p}{3}.$$

The fourth term requires some more work. The underlying idea behind the technicalities is to transform an expression involving the lengths of the intervals of the partition π_1 into another expression involving the lengths of the intervals I_k . To that end, let

$$\mathcal{F} = \{l > \ell : \exists n \text{ such that } a_{l-1} \leq t_n < a_l\}.$$

Let M be the first element of $\mathcal{F}$. Notice that $L \in \mathcal{F}$ so that $L = \max \mathcal{F}$. If $i \in \mathcal{F} \setminus \{L\}$, we denote by $m(i)$ and $n(i)$, respectively, the smallest and the largest of the indices n with the property that $a_{i-1} \leq t_n < a_i$. We have

$$\|f(b_i)\mu_i\|^p \leq (2B)^p \left((t_{m(i)} - a_{i-1})^p + (a_i - t_{n(i)})^p + \sum_{k=m(i)+1}^{n(i)} \lambda_k^p \right),$$

where the last term is null if $m(i) = n(i)$.

Analogously, if $m(L)$ denotes be the first index n with $a_{i-1} \leq t_n < a_i$,

$$\|f(b_L)\mu_L\|^p \leq (2B)^p \left((t_{m(L)} - a_{L-1})^p + \sum_{k=m(L)+1}^{\infty} \lambda_k^p \right).$$

Let i and j be consecutive terms in $\mathcal{F}$ and denote $i = i(j)$. From the definition of $\mathcal{F}$ we infer that there is no n such that $a_i \leq t_n < a_{j-1}$. Hence $n(i) = m(j) - 1$ and $[a_i, a_{j-1}) \subseteq I_{m(j)}$. Then,

$$\left\| \sum_{l=i+1}^{j-1} f(b_l)\mu_l \right\|^p = \|x_{m(j)}\|^p \left(\sum_{l=i+1}^{j-1} f_{I_{m(j)}}(b_l)\mu_l \right)^p \leq (2B)^p \left(\sum_{l=i+1}^{j-1} \mu_l \right)^p = (2B)^p (a_{j-1} - a_i)^p.$$

In the same fashion, for the indices to the left of M , we have $m(M) - 1 = N$, $[a_\ell, a_{M-1}) \subseteq I_{m(M)}$ and

$$\left\| \sum_{l=\ell+1}^{M-1} f(b_l)\mu_l \right\|^p \leq (2B)^p (a_{M-1} - a_\ell)^p.$$

Notice that $n(i(j)) = m(j) - 1$ and $x^p + y^p + z^p \leq 3^{1-p}(x + y + z)^p$. Adding the four inequalities above,

$$\begin{aligned} \left\| \sum_{l=\ell+1}^L f(b_l)\mu_l \right\|^p &\leq (2B)^p \left(\sum_{\substack{k \geq N+1 \\ k \neq m(j)}} \lambda_k^p + ((a_{M-1} - a_\ell)^p + (t_{m(M)} - a_{M-1})^p) \right) \\ &\quad + (2B)^p \sum_{\substack{j \in \mathcal{F} \\ j \neq M}} \left((a_{i(j)} - t_{n(i(j))})^p + (a_{j-1} - a_{i(j)})^p + (t_{m(j)} - a_{j-1})^p \right) \\ &\leq (2B)^p \left(\sum_{\substack{k \geq N+1 \\ k \neq m(j)}} \lambda_k^p + ((a_\ell - a_N)^p + (t_{M-1} - a_\ell)^p + (t_{m(M)} - a_{M-1})^p) \right) \\ &\quad + (2B)^p \sum_{\substack{j \in \mathcal{F} \\ j \neq M}} \left((a_{i(j)} - t_{m(j)-1})^p + (a_{j-1} - a_{i(j)})^p + (t_{m(j)} - a_{j-1})^p \right) \\ &\leq (2B)^p \left(\sum_{\substack{k \geq N+1 \\ k \neq m(j)}} \lambda_k^p + 3^{1-p} \sum_{j \in \mathcal{F}} \lambda_{m(j)}^p \right) \leq 3^{1-p} (2B)^p \sum_{k \geq N+1} \lambda_k^p. \end{aligned}$$

Gathering all the inequalities,

$$\left\| \sum_{k=1}^{\infty} \lambda_k x_k - \sigma_1(f, \pi_1) \right\|^p \leq \epsilon^p / 3 + \epsilon^p / 3 + B^p (1 + 3^{1-p} 2^p) \sum_{k=N+1}^{\infty} \lambda_k^p \leq \epsilon^p.$$

(iii) If $F(1)$ can be defined continuously, then the sequence $F(t_n) = \sum_{k=1}^n \lambda_k x_k$ must

converge to $F(1)$, i.e., $\sum_{k=1}^{\infty} \lambda_k x_k = F(1)$. To show the converse, suppose that $\sum_{k=1}^{\infty} \lambda_k x_k$ converges and put $F(1) = \sum_{k=1}^{\infty} \lambda_k x_k$. We will show that $\lim_{t \rightarrow 1} F(t) = F(1)$. Since $\lambda_k x_k \rightarrow 0$ due to the convergence of $\sum_{k=1}^{\infty} \lambda_k x_k$, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\lambda_n x_n\| \leq 2^{-1/p} \epsilon$ and $\left\| \sum_{k \geq n+1} \lambda_k x_k \right\| \leq 2^{-1/p} \epsilon$ for all $n \geq N$.

Then if $t \geq t_N$ and $n \geq N$ is such that $t \in I_n$, from (7) we get

$$\begin{aligned} \|F(1) - F(t)\|^p &\leq \|x_n\|^p \left(\int_t^{t_n} f_{I_n}(u) du \right)^p + \left\| \sum_{k=n+1}^{\infty} \lambda_k x_k \right\|^p \\ &\leq \|x_n\|^p \left(\int_{I_n} f_{I_n}(u) du \right)^p + \frac{\epsilon^p}{2} \leq \|x_n\|^p \lambda_n^p + \frac{\epsilon^p}{2} \leq \frac{\epsilon^p}{2} + \frac{\epsilon^p}{2} = \epsilon^p. \end{aligned}$$

(iv) If $F : [0,1] \rightarrow X$ is Lipschitz, then whenever $m < n$,

$$\left\| \sum_{k=m+1}^n \lambda_k x_k \right\| = \|F(t_n) - F(t_m)\| \leq \|F\|_{Lip} (t_n - t_m).$$

For the reverse implication, given any $0 \leq s < t < 1$ find integers $m \leq n$ such that $s \in I_m$ and $t \in I_n$. Then, knowing that for $t \in I_n$ we can estimate

$$\int_{t_{n-1}}^t f_{I_n}(u) du \leq t - t_{n-1}, \quad (3)$$

and

$$\int_t^{t_n} f_{I_n}(u) du \leq t_n - t, \quad (4)$$

with the help of inequality (1) we obtain the following Lipschitz condition,

$$\begin{aligned} \|F(t) - F(s)\|^p &\leq \|x_m\|^p \left(\int_s^{t_m} f_{I_m}(u) du \right)^p + \left\| \sum_{k=m+1}^n \lambda_k x_k \right\|^p + \|x_n\|^p \left(\int_{t_n}^t f_{I_n}(u) du \right)^p \\ &\leq B^p (t_m - s)^p + K^p (t_n - t_m)^p + B^p (t - t_n)^p \\ &\leq \max\{K^p, B^p\} \left((t_m - s)^p + (t_n - t_m)^p + (t - t_n)^p \right) \\ &\leq 3^{1-p} \max\{K^p, B^p\} (t - s)^p. \end{aligned}$$

Finally, we note that (1) implies that $\sum_k \lambda_k x_k$ is a Cauchy series, so it converges. Using (iii), F can be extended continuously to $[0,1]$.

A clarification might be in order here. If the space X is locally convex and the sequence $x = (x_k)$ is bounded, then the series $\sum_k \lambda_k x_k$ converges and condition (1) is fulfilled. But in this case there is a simpler way to look at the function F which allows us to write $F(t) = \int_0^t f(u) du$ even for $t = 1$ without using the series as a bypass. The reason is that in Banach spaces we have the tool of the Bochner integral, and the function f is Bochner-integrable. If X is not locally convex, the tool of the Bochner integral is no longer available and we should be more careful when writing the identity $F(1) = \sum_k \lambda_k x_k = \int_0^1 f(u) du$. However, with an additional condition on (λ_k) we can interpret $F(1)$ as a Riemann-integral, which as we know can be defined in quasi-Banach spaces.

(v) Suppose $F'(-1) = 0$. We have

$$\frac{\left\| \sum_{k=n+1}^{\infty} \lambda_k x_k \right\|}{1-t_n} = \frac{\|F(1) - F(t_n)\|}{1-t_n} \rightarrow \|F'(-1)\| = 0$$

Conversely, suppose that (2) holds. Implicitly we are assuming that the series $\sum_k \lambda_k x_k$ converges, so we define $F(1) = \sum_k \lambda_k x_k$. We will show that $\lim_{t \rightarrow -1} (F(1) - F(t))/(1-t) = 0$.

For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{\left\| \sum_{k \geq n+1} \lambda_k x_k \right\|}{1-t_n} \leq \frac{\epsilon}{2^{1/p-1}} \text{ and } \|x_n\| \leq \frac{\epsilon}{2^{1/p-1}(2-\sqrt{2})}$$

Given $t \geq t_{N-1}$ we have that $t \in I_n$ for some $n \geq N$. Then,

$$\frac{\|F(1) - F(t)\|^p}{(1-t)^p} \leq \frac{\|x_n\|^p \left(\int_t^{t_n} f_{I_n}(u) du\right)^p + \left\| \sum_{k=n+1}^{\infty} \lambda_k x_k \right\|^p}{(1-t)^p} \leq \frac{2^{p-1} \epsilon^p (t_n - t)^p + 2^{p-1} \epsilon^p (1-t_n)^p}{(1-t)^p} \leq \epsilon^p$$

Corollary (6.2.2)[204]: Let X be a quasi-Banach space with separating dual. Suppose the pair $(\tau, \mathbf{x})$ is such that:

- (i) $x_k \rightarrow 0$,
- (ii) condition (1) holds, and
- (iii) condition (2) does not hold.

Then we have the following.

(a) The function $F(\tau, \mathbf{x}): [0,1] \rightarrow X$ is Lipschitz and differentiable at every $t \in [0,1)$ but fails to be differentiable at $t = 1$.

(b) The composition of F with the natural inclusion of X into its Banach envelope $\hat{X}$ is (Lipschitz and) differentiable at every $t \in [0,1)$.

Theorem (6.2.3)[204]: Suppose $0 < p < 1$. Then there exists a continuous Riemann-integrable function $f: [0,1] \rightarrow l_p$ whose integral function $F: [0,1] \rightarrow l_p$, $t \mapsto \int_0^t f(s) ds$ verifies:

- (a) F is Lipschitz, i.e., there is $C > 0$ so that $\|F(s) - F(t)\|_p \leq C|s - t|$ for all $s, t \in [0,1]$;
- (b) F is differentiable at every $t \in [0,1)$ with derivative $F'(t) = f(t)$;
- (c) F Fails to have left derivative at $t = 1$.

we exploit the construction used below in the proof of Theorem (6.2.3) to show that, unlike for Banach spaces, every non-locally convex quasi Banach space X with separating dual admits a continuously differentiable function $f: [a,b] \rightarrow X$ which is not Lipschitz. Finally, gather remarks on the general problem of classifying those quasi-Banach spaces X for which every continuous function $f: [a,b] \rightarrow X$ has a primitive. We refer the reader to [202, 199] for background and to [200, 134] for the needed terminology and notation on quasi-Banach spaces.

Proof. The proof of Theorem (6.2.3) relies on the following construction inspired by [202]. Let $\tau = (t_k)_{k=1}^{\infty}$ be an increasing sequence of scalars contained in $(0,1)$ tending to 1. With $t_0 = 0$, let us denote the interval $[t_{k-1}, t_k]$ by I_k and its length by λ_k , i.e., $\lambda_k = |I_k| = t_k - t_{k-1}$. This way we can write $[0,1) = \bigcup_{k=1}^{\infty} I_k$ (disjoint union). For each $k \in \mathbb{N}$ let $f_{I_k}: [0,1] \rightarrow R$ be the nonnegative piecewise linear function supported on the interval I_k having a node at the midpoint of the interval $c_k = (t_k + t_{k-1})/2$ with $f_{I_k}(c_k) = 2$ and $f_{I_k}(t_{k-1}) = f_{I_k}(t_k) = 0$, i.e.,

$$f_{I_k}(t) = \begin{cases} \frac{4}{t_k - t_{k-1}}(t - t_{k-1}) & \text{if } t \in [t_{k-1}, t_k) \\ \frac{4}{t_k - t_{k-1}}(t - t_k) & \text{if } t \in [t_k, t_{k+1}) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{x} = (x_n)_{n=1}^\infty$ be a sequence of vectors in a quasi-Banach space X . We define the function $f = f(\tau, \mathbf{x}) : [0, 1] \rightarrow X$ as

$$f(t) = \begin{cases} f_{I_k}(t)x_k & \text{if } t \in I_k, \\ 0 & \text{if } t = 1. \end{cases} \quad (5)$$

Note that f is continuous and Riemann-integrable on $[0, 1)$ since for each $s < 1$ the set $f([0, s])$ is a finite-dimensional subspace of X . Let $F = F(\tau, \mathbf{x})$ be the corresponding integral function on $[0, 1)$,

$$F(t) = \int_0^t f(u) du. \quad (6)$$

The additivity of the Riemann-integral with respect to the interval gives that for $t \in I_n$

$$F(t) = \sum_{k=1}^{n-1} \lambda_k x_k + \int_{t_{n-1}}^t f(u) du = \sum_{k=1}^n \lambda_k x_k - \int_t^{t_n} f(u) du. \quad (7)$$

Again, since $F([0, s])$ maps into a finite-dimensional subspace of X for each $s < 1$, F is differentiable with derivative $F'(t) = f(t)$ at every $t \in [0, 1)$. The next proposition deals mainly with the behavior of the functions f and F at the point $t = 1$ depending on the choice of $(\tau, \mathbf{x})$.

For $0 < p < 1$ fixed, pick $b = \frac{1}{p} - 1$ and any $a > \frac{1}{p} - 1$. Consider the pair $(\tau, \mathbf{x})$, where $\tau = (t_k)_{k=1}^\infty$ is the sequence

$$t_k = 1 - \frac{1}{(k+1)^a}, \quad k = 1, 2, \dots$$

and $\mathbf{x} = (x_k)_{k=1}^\infty$ is the sequence in ℓ_p obtained by scaling down its unitary basis vectors $(e_k)_{k=1}^\infty$ according to the formula

$$x_k = \frac{1}{(k+1)^b} e_k, \quad k = 1, 2, \dots$$

Next define $f = f(\tau, \mathbf{x}) : [0, 1] \rightarrow \ell_p$ and $F = F(\tau, \mathbf{x}) : [0, 1) \rightarrow \ell_p$ as in Eqs. (5) and (6) respectively.

Proposition (6.2.1)(i) gives that F is continuous on $[0, 1]$.

Note that the series $\sum_{k=1}^\infty \lambda_k x_k$ converges in X if and only if $(\frac{\lambda_k}{(k+1)^b})_{k=1}^\infty = \sum_{k=1}^\infty \frac{\lambda_k}{(k+1)^b} e_k \in \ell_p$.

. Now,

$$\left\| \sum_{k=1}^\infty \frac{\lambda_k}{(k+1)^b} e_k \right\|_p^p = \left\| \sum_{k=1}^\infty \frac{t_k - t_{k-1}}{(k+1)^b} e_k \right\|_p^p = \sum_{k=1}^\infty \left| \left(\frac{1}{k^a} - \frac{1}{(k+1)^a} \right) \frac{1}{(k+1)^b} \right|^p.$$

Since the sequences $(\frac{1}{k^a} - \frac{1}{(k+1)^a})_{k=1}^\infty$ and $(\frac{1}{k^{a+1}})_{k=1}^\infty$ are equivalent infinitesimals as $k \rightarrow \infty$, the last series has the same character as $\sum_{k=1}^\infty (k+1)^{-(a+b+1)p}$, which converges because $(a+b+1)p > 1$. Using Proposition (6.2.1)(iii), we extend $F : [0, 1) \rightarrow \ell_p$ to a continuous

function on $[0,1]$ by putting $F(1) = \sum_{k=1}^{\infty} \lambda_k x_k$. In fact, Proposition (6.2.1)(ii) yields that f is Riemann-integrable on $[0,1]$ and so $F(1) = \int_0^1 f(u) du$.

To see that F is Lipschitz on $[0,1]$ we use the simple inequality $(t^p - s^p)^{1/p} \leq t - s$ for all $0 \leq s < t$. Thus,

$$\begin{aligned} \left\| \sum_{k=m+1}^n \lambda_k x_k \right\|_p &\approx \left(\sum_{k=m+1}^n \frac{1}{k^{(a+b+1)p-1}} - \frac{1}{(n+1)^{(a+b+1)p-1}} \right)^{1/p} \\ &= \left(\frac{1}{(m+1)^{(a+b+1)p-1}} - \frac{1}{(n+1)^{(a+b+1)p-1}} \right)^{1/p} \\ &\leq \frac{1}{(m+1)^{a+b+1-1/p}} - \frac{1}{(n+1)^{a+b+1-1/p}} = \frac{1}{(m+1)^a} - \frac{1}{(n+1)^a} = t_n - t_m, \end{aligned}$$

whence condition (1) is fulfilled. As we argued above, F is differentiable at every $t \in [0,1)$. Since ℓ_p has separating dual, if F has a left derivative at $t=1$ it must be $F'(-1) = 0$. However, this fact fails by appealing to Proposition 6.2.2(v) since

$$\begin{aligned} \left\| \sum_{k \geq n+1} \lambda_k x_k \right\|_p^p &\approx \sum_{k \geq n+1} \frac{1}{(k+1)^{(a+b+1)p}} \\ &\approx \sum_{k \geq n+1} \left(\frac{1}{k^{(a+b+1)p-1}} - \frac{1}{(k+1)^{(a+b+1)p-1}} \right) = \frac{1}{(n+1)^{(a+b+1)p-1}}, \\ \frac{\left\| \sum_{k \geq n+1} \lambda_k x_k \right\|_p}{1 - t_n} &\approx \frac{1}{n^{b+1-1/p}} = 1, \end{aligned}$$

so that

hence $\lim_{n \rightarrow \infty} \frac{\left\| \sum_{k \geq n+1} \lambda_k x_k \right\|_p}{1 - t_n}$ cannot be 0.

We are now ready to show that for a wide class of quasi-Banach spaces, including those with separating dual that contain a copy of ℓ_p for some $0 < p < 1$, there exists a function $f : [0,1] \rightarrow X$ as in the title of the section. This will follow combining Theorem (6.2.3) with our next lemma. In [130], Kalton introduced the notion of core of a quasi-Banach space X as the biggest subspace of X with trivial dual. Note that if X^* separates the points of X then $\text{core}(X) = \{0\}$, so the lemma applies to quasi-Banach spaces with separating dual.

Lemma (6.2.4)[204]: Let X be a quasi-Banach space with $\text{core}(X) = \{0\}$. Let J be an interval of the real line. Suppose that F is differentiable with $F'(t) = 0$ for all $t \in J$. Then there is $C \in \mathbb{R}$ so that $F(t) = C$ for all $t \in J$.

Proof. Assume $F(a) = 0$ for some $a \in J$ and that $F(u) \neq 0$ for some $u \in J$. Let Y be the closed linear subspace of X generated by $\{F(t) : t \in J\}$. Since $Y \neq 0$, by hypothesis there exists a nontrivial bounded linear functional $y^* : Y \rightarrow \mathbb{R}$. The composition $y^* \circ F : J \rightarrow \mathbb{R}$ is nonzero and differentiable at every $s \in J$ with derivative

$$\begin{aligned} (y^* \circ F)'(s) &= \lim_{h \rightarrow 0} \frac{(y^* \circ F)(s+h) - (y^* \circ F)(s)}{h} \\ &= y^* \left(\lim_{h \rightarrow 0} \frac{F(s+h) - F(s)}{h} \right) = y^*(F'(s)) = 0 \end{aligned}$$

By the fundamental theorem of calculus,

$$(y^* \circ F)(t) = (y^* \circ F)(a) + \int_a^t (y^* \circ F)'(s) ds = 0$$

for all $t \in J$, a contradiction.

Theorem (6.2.5)[204]: Suppose the space ℓ_p with $0 < p < 1$ embeds in a quasi-Banach space X with trivial core. Then there exists a continuous Riemann integrable function $f : [0,1] \rightarrow X$ failing to have a primitive.

Proof. Let $0 < p < 1$ and assume, without loss of generality, that ℓ_p is a subspace of X . Let f and F be the functions in Theorem 6.2.1. Suppose there exists a differentiable function $G : [0,1] \rightarrow X$ so that $G'(t) = f(t)$ for all $t \in [0,1]$. Then $(F - G)'(t) = 0$ for all $t \in [0,1]$. By Lemma 6.2.5 it must be $F(t) = G(t) + C$ for all $t \in [0,1]$, where C is some real number. Using continuity, we extend this identity to $[0,1]$. But then F would be differentiable at every $t \in I$, which contradicts our previous construction.

Let $(X, \|\cdot\|)$ be an infinite-dimensional real quasi-Banach space. Let I be the unit interval $[0,1]$ and $e(I, X)$ be the usual quasi-Banach space of continuous functions $f : I \rightarrow X$ with the quasi-norm $\|f\| : \max_{0 \leq s \leq 1} \|f(s)\|$. We will denote by $e^{(1)}(I, X)$ the space of all X -valued functions f having a derivative at every point of I , and such that $f' \in e(I, X)$. The closed subspace of $e^{(1)}(I, X)$ consisting of the functions that vanish at zero will be denoted by $e_0^{(1)}(I, X)$.

When X is a Banach space, a function $f \in e^{(1)}(I, X)$ is Lipschitz in I thanks to the mean value theorem. This result breaks down for non-locally convex spaces [205], allowing thus the possibility of having functions in the class $e^{(1)}(I, X)$ that are not Lipschitz!

Theorem (6.2.6)[204]: Let X be a non-locally convex quasi-Banach space. Then there exists $F : I \rightarrow X$ such that

- (a) F is differentiable on I ;
- (b) F' is continuous and Riemann-integrable on I and $F(t) = \int_0^t F'(u) du$ for all $t \in I$;
- (c) F is not Lipschitz on I .

Proof. As above, by the Aoki–Rolewicz theorem we can assume that X is a p -Banach space for some $0 < p < 1$. Hence for any $(\mu_j)_{j=1}^k$ in $(0, \infty)$ and $(y_j)_{j=1}^k$ in X such that $\sum_{j=1}^k y_j = 1$ and $\|y_j\| \leq 1$, we have

$$\left\| \sum_{j=1}^k \mu_j y_j \right\| \leq \left(\sum_{j=1}^k \mu_j^p \right)^{1/p} \leq k^{1/p-1}.$$

For every $k \in \mathbb{N}$ we set

$$C_k = \sup \left\{ \left\| \sum_{j=1}^k \mu_j y_j \right\| : \mu_j > 0, \sum_{j=1}^k \mu_j = 1, y_j \in X, \|y_j\| \leq 1 \right\}.$$

Clearly $(C_k)_{k=1}^\infty$ is an increasing sequence and, since X is not locally convex, $C_k \rightarrow +\infty$. Moreover $C_k \leq k^{1/p-1}$.

Pick out a sequence $(D_k)_{k=1}^\infty$ such that $0 < D_k < C_k$ and $C_k - D_k \rightarrow 0$. This yields $C_k \sim D_k$, i.e., $\lim_{k \rightarrow \infty} C_k / D_k = 1$. From our choice of C_k , for each k there exist positive scalars $(\mu_{k,j})_{j=1}^k$ and vectors $(y_{k,j})_{j=1}^k$ in X such that $\sum_{j=1}^k \mu_{k,j} = 1$, $\|y_{k,j}\| \leq 1$ and $\left\| \sum_{j=1}^k \mu_{k,j} y_{k,j} \right\| \geq D_k$. Every natural number n can be written in a unique way in the form

$$n = \frac{2k^2+1}{2} \pm \frac{2j-1}{2}, \quad (8)$$

for some $k \in \mathbb{N}$ and $1 \leq j \leq k$. In fact, for a fixed k we have

- the set $\left\{\frac{2k^2+1}{2} - \frac{2j-1}{2} : 1 \leq j \leq k\right\}$ covers all the integers between $k(k-1)+1$ and k^2 ;
- the set $\left\{\frac{2k^2+1}{2} + \frac{2j-1}{2} : 1 \leq j \leq k\right\}$ covers all the integers between k^2+1 and $k(k+1)$,

so that the numbers $\left\{\frac{2k^2+1}{2} \pm \frac{2j-1}{2} : 1 \leq j \leq k\right\}$ run over all the integers between $k(k-1)+1$ and $k(k+1)$.

For each $n \in \mathbb{N}$, let $k = k(n)$, $j = j(n)$, and $\varepsilon = \varepsilon(n) \in \{-1, 1\}$ uniquely determined by the representation (8).

First, for $b > 2(1-p)/p$ fixed, we define

$$\lambda_n = \frac{1}{2} \mu_{k,j} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right).$$

Note that

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n &= \sum_{k=1}^{\infty} \sum_{j=1}^k \sum_{\varepsilon=\pm 1} \frac{1}{2} \mu_{k,j} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) = \sum_{k=1}^{\infty} \sum_{j=1}^k \mu_{k,j} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) = 1 - \lim_{k \rightarrow \infty} \frac{1}{(1+k)^b} = 1. \end{aligned}$$

Let $t_n = \sum_{m=1}^n \lambda_m$ and $\tau = (t_n)_{n=1}^{\infty}$ so that $t_n - t_{n-1} = \lambda_n$.

Pick any $0 < a < \min\{1, p/(1-p)\}$. Let $\mathbf{x} = (x_n)_{n=1}^{\infty}$ in X be given by $x_n = \varepsilon A_k y_{k,j}$, where $A_k = C_k^{a-1}$.

With this pair $(\tau, \mathbf{x})$ we construct maps $f = f(\tau, \mathbf{x})$ and $F = F(\tau, \mathbf{x})$ from $[0, 1]$ into X .

Since $A_k \rightarrow 0$ and $\|y_{k,j}\| \leq 1$, applying Proposition (6.2.1)(i) we obtain that f is continuous on $[0, 1]$. The function f is also Riemann integrable on $[0, 1]$ from Proposition (6.2.1)(ii) since

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n^p &= \sum_{k=1}^{\infty} \sum_{j=1}^k \sum_{\varepsilon=\pm 1} \frac{1}{2^p} \mu_{k,j}^p \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)^p = 2^{1-p} \sum_{k=1}^{\infty} \sum_{j=1}^k \mu_{k,j}^p \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)^p \\ &\leq 2^{1-p} \sum_{k=1}^{\infty} k^{1-p} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)^p \approx \sum_{k=1}^{\infty} k^{1-p} \frac{1}{k^{(b+1)p}} = \sum_{k=1}^{\infty} \frac{1}{k^{bp+2p-1}}, \end{aligned}$$

and this last series converges because $bp+2p-1 > 2(1-p)+2p-1=1$. In particular, F is well defined and continuous on the closed interval $[0, 1]$.

To compute $F(1)$, observe that for fixed k ,

$$\sum_{n=k(k-1)+1}^{k(k+1)} \lambda_n x_n = \sum_{j=1}^k \sum_{\varepsilon=\pm 1} \varepsilon \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k \mu_{k,j} y_{k,j} = \sum_{j=1}^k 0 = 0.$$

Hence $\sum_{n=1}^{k(k+1)} \lambda_n x_n = 0$, which yields $F(1) = \sum_{n=1}^{\infty} \lambda_n x_n = 0$.

That F is not Lipschitz in $[0, 1]$ follows from Proposition (6.2.1)(iv). Indeed,

$$\begin{aligned} \frac{\left\| \sum_{n=k^2+1}^{k(k+1)} \lambda_n x_n \right\|}{\sum_{n=k^2+1}^{k(k+1)} \lambda_n} &= \frac{\left\| \sum_{j=1}^k \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k \mu_{k,j} y_{k,j} \right\|}{\sum_{j=1}^k \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) \mu_{k,j}} = \frac{A_k \left\| \sum_{j=1}^k \mu_{k,j} y_{k,j} \right\|}{\sum_{j=1}^k \mu_{k,j}} = A_k \left\| \sum_{j=1}^k \mu_{k,j} y_{k,j} \right\| \\ &\geq A_k D_k \sim A_k C_k = C_k^a \rightarrow \infty. \end{aligned}$$

As we argued above, F is differentiable in $[0,1)$ with derivative $F'(t)=f(t)$ for all $t \in [0,1)$. To conclude we show that $F'(1)=0$ aided by Proposition (6.2.1)(v). Let $n \in \mathbb{N}$. Suppose that $n+1=(2k^2+1)/2+(2j-1)/2$ with $k \in \mathbb{N}$ and $1 \leq j \leq k$. Taking into account that the functions of the form $t \mapsto \frac{t}{a+bt}$ ($a, b > 0$) are increasing in $t \in (0, +\infty)$ and that $\sum_{i=j}^k \mu_{k,i} \leq 1$,

$$\begin{aligned} \frac{\left\| \sum_{m \geq n+1} \lambda_m x_m \right\|}{\sum_{m \geq n+1} \lambda_m} &= \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k \left\| \sum_{i=j}^k \mu_{k,i} y_{k,i} \right\|}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) \sum_{i=j}^k \mu_{k,i}} \leq \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_{k-j+1} \sum_{i=j}^k \mu_{k,i}}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) \sum_{i=j}^k \mu_{k,i}} \\ &\leq \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_{k-j+1}}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)} \leq \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_k}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)}. \end{aligned}$$

If $n+1=(2k^2+1)/2-2j-1/2$, with $k \in \mathbb{N}$ and $1 \leq j \leq \mathbb{N}$,

$$\begin{aligned} \frac{\left\| \sum_{m \geq n+1} \lambda_m x_m \right\|}{\sum_{m \geq n+1} \lambda_m} &= \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k \left\| \sum_{i=j+1}^k \mu_{k,i} y_{k,i} \right\|}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) \left(1 + \sum_{i=1}^j \mu_{k,i} \right)} \\ &\leq \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_{k-j} \sum_{i=j+1}^k \mu_{k,i}}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) \left(1 + \sum_{i=1}^j \mu_{k,i} \right)} \\ &\leq \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_{k-j}}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)} \leq \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_k}{\frac{1}{(1+k)^b} + \frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right)}. \end{aligned}$$

In both cases,

$$\frac{\left\| \sum_{m \geq n+1} \lambda_m x_m \right\|}{\sum_{m \geq n+1} \lambda_m} = \frac{\frac{1}{2} \left(\frac{1}{k^b} - \frac{1}{(1+k)^b} \right) A_k C_k}{\frac{1}{2k^b} + \frac{1}{2(1+k)^b}} \sim \frac{b}{2} A_k C_k k^{-1}.$$

But

$$A_k C_k k^{-1} = C_k^a k^{-1} \leq k^{a(1/p-1)-1},$$

and $k^{a(1/p-1)-1} \rightarrow 0$ since $a(1/p-1) < 1$.

Corollary (6.2.7)[204]: Let X be a quasi-Banach space. We have that $e^{(1)}(I, X) \subseteq Lip(I, X)$ if and only if X is locally convex.

Based on the results in the previous section it makes sense to define the space $e_{Lip}^{(1)}(I, X)$ of all $f \in e_0^{(1)}(I, X)$ which are Lipschitz, equipped with the quasi-norm

$$\|f\|_{Lip} = \sup_{0 \leq s < t \leq 1} \frac{\|f(t) - f(s)\|}{t - s}. \quad (9)$$

We will also consider the space $e_{Kal}^{(1)}(I, X)$ of all $f \in e_{Lip}^{(1)}(I, X)$ with $f(0)=0$ such that the

function $g : I^2 \rightarrow X$ given by

$$g(s, t) = \begin{cases} \frac{f(s) - f(t)}{s - t} & s \neq t \\ f'(t) & s = t \end{cases}$$

is continuous. Of course, when X is a Banach space, $e_{Kal}^{(1)}(I, X) = e_{Lip}^{(1)}(I, X) = e_0^{(1)}(I, X)$.

Kalton introduced the space $e_{Kal}^{(1)}(I, X)$ in [199] (with a different notation) to provide the only affirmative answer known as of today to the question of Popov. He showed that if X has trivial dual then the map

$$\mathcal{D} : e_{Kal}^{(1)}(I, X) \rightarrow e(I, X), \quad f \mapsto \mathcal{D}(f) = f'$$

is surjective and so every continuous function $f : I \rightarrow X$ has a primitive (that belongs to $e_{Kal}^{(1)}(I, X)$). His proof relied heavily on a pathology that we find in quasi-Banach spaces X with trivial dual, namely, they admit nonconstant functions $f \in e_{Kal}^{(1)}(I, X)$ with zero derivative at every point [130]. Kalton's result opened the problem of classifying those quasi-Banach spaces X , which he named D-spaces, for which the operator $\mathcal{D} : e_{Kal}^{(1)}(I, X) \rightarrow e(I, X)$ is surjective.

Our first goal in this section is to show that if a non-locally convex quasi-Banach space X has separating dual then X is not a D-space. In fact, with the help of the next preparatory result we will obtain something stronger, that the operator $\mathcal{D}$ cannot be onto even when it is defined on the bigger space $e_{Lip}^{(1)}(I, X)$.

Lemma (6.2.8)[204]: Let X be a quasi-Banach space.

- (i) The linear map $\mathcal{D} : e_{Lip}^{(1)}(I, X) \rightarrow e(I, X)$ given by $\mathcal{D}(f) = f'$ is bounded.
- (ii) The space $e_{Lip}^{(1)}(I, X)$ is closed in $Lip(I, X)$.
- (iii) The space $e_{Kal}^{(1)}(I, X)$ is closed in $e_{Lip}^{(1)}(I, X)$.
- (iv) The space $e_{Lip}^{(1)}(I, X)$ is complete with the quasi-norm $\|\cdot\|_{Lip}$.

Theorem (6.2.9)[204]: Suppose X is a non-locally convex quasi-Banach space with $core(X) = \{0\}$. Then the map $\mathcal{D} : e_{Lip}^{(1)}(I, X) \rightarrow e(I, X)$ is not surjective. In particular, there exists a continuous function $f : I \rightarrow X$ that fails to have a primitive in $e_{Lip}^{(1)}(I, X)$.

Proof. The operator $\mathcal{D}$ is bounded, and one-to-one thanks to Lemma (6.2.4). If $\mathcal{D}$ were surjective, from the open mapping theorem we deduce the existence of a constant $K \geq 1$ so that

$$\|f\|_{Lip} \leq K \|f'\|, \quad \forall f \in e_{Lip}^{(1)}(I, X).$$

In particular, for every Lipschitz function $f \in e_{Lip}^{(1)}(I, X)$ we would have

$$\|f(t) - f(s)\| \leq K \|f'\| |t - s|, \quad \forall s, t \in I.$$

By the mean value formula for quasi-Banach spaces [205] the space X should be locally convex, a contradiction.

We close with some remarks and open problems. To simplify our discussion let us make a definition.

Definition (6.2.10)[204]: A quasi-Banach space X will be said to have property (P) (or that X is a P-space) if every continuous function $f : I \rightarrow X$ has a primitive.

Trivially, Banach spaces are P-spaces. Quasi-Banach spaces with trivial dual are D-spaces [199] hence they are P-spaces too. On the other hand, Theorem (6.2.3) tells us that no space with separating dual containing a copy of ℓ_p for $p < 1$ is a P-space.

Problem. Does there exist a non-locally convex quasi-Banach space with separating dual

having property (P)?

The answer to this question will determine the way in which some of the topics in this section are related. We can entertain some digression.

When X is a Banach space, the vector space $e_0^{(1)}(I, X)$ is complete both with the norm

$$\|f\|_{e_0^{(1)}} = \|f'\|$$

and the norm (9). In fact, it is well-known that these two norms are equivalent in $e_0^{(1)}(I, X)$. However, when X is not locally convex and has separating dual the space $e_0^{(1)}(I, X)$ is complete under the quasi-norm $\|\cdot\|_{Lip}$ but could fail to be complete under the natural norm of the space, $\|\cdot\|_{e_0^{(1)}}$.

Theorem (6.2.11)[204]: A quasi-Banach space X with separating dual is a P-space if and only if $e_0^{(1)}(I, X)$ is complete with the quasi-norm $\|\cdot\|_{e_0^{(1)}}$.

Proof. The operator

$$\mathcal{D}: (e_{Lip}^{(1)}(I, X), \|\cdot\|_{e_0^{(1)}}) \rightarrow (e(I, X), \|\cdot\|_\infty), \quad f \mapsto \mathcal{D}(f) = f',$$

is a linear isometry of dense rank in $e(I, X)$. If $(e_0^{(1)}(I, X), \|\cdot\|_{e_0^{(1)}})$ were complete, the image of $\mathcal{D}$ would be closed and so $\mathcal{D}$ would be onto, i.e., X would be a P-space. Conversely, if X is a P-space then $\mathcal{D}$ is onto and we deduce that $(e_0^{(1)}(I, X), \|\cdot\|_{e_0^{(1)}})$ is complete.

Corollary (6.2.12)[221]: Suppose X is a non-locally convex quasi-Banach space with $\text{core}(X) = \{0\}$. Then the map $\mathcal{D}_j: e_{Lip}^{(1)}(I_j, X) \rightarrow e(I_j, X)$ are not surjective. In particular, there exists series of continuous function $\sum_{j=1}^m f_j: I_j \rightarrow X$ that fails to have a primitives in $e_{Lip}^{(1)}(I_j, X)$.

Proof. The operator $\mathcal{D}_j$ is bounded, and one-to-one as in (6.2.4). If $\mathcal{D}_j$ were surjective, from the open mapping theorem we deduce the existence of a constant $K \geq 1$ so that

$$\left\| \sum_{j=1}^m f_j \right\|_{Lip} \leq K \left\| \sum_{j=1}^m f_j' \right\|, \quad \forall \sum_{j=1}^m f_j \in e_{Lip}^{(1)}(I_j, X).$$

In particular, for every Lipschitz function $\sum_{j=1}^m f_j \in e^{(1)}(I_j, X)$ we would have

$$\left\| \sum_{j=1}^m f_j(t) - \sum_{j=1}^m f_j(s) \right\| \leq K \left\| \sum_{j=1}^m f_j' \right\| |t - s|, \quad \forall s, t \in I_j.$$

By the mean value formula for quasi-Banach spaces [205] the space X should be locally convex, a contradiction.

Section (6.3): Quasi-Banach Spaces and the Fundamental Theorem of Calculus

It is a part of the mathematical folklore that continuous functions from a compact interval of the real line into a Banach space are Riemann integrable and that the fundamental theorem of calculus holds.

Theorem (6.3.1)[206]: Suppose X is a Banach space and that $f: [a, b] \rightarrow X$ is a continuous function. Then:

(i) The integral function $F(t) = \int_a^t f$ is differentiable at every $t \in [a, b]$ and $F'(t) = f(t)$.

(ii) (Barrow's rule) the element $\int_a^b f$ of X can be computed as $F(b) - F(a)$, where F is any primitive of f .

The definition of the Riemann integral extends verbatim for functions $f: [a, b] \rightarrow X$

where X is a quasi-Banach space, i.e., a locally bounded topological vector space that is complete for the metric induced by its quasi-norm. We recall the construction to render our exposition self-contained. For a partition $\mathcal{P} = \{t_k\}_{k=1}^n$ of the interval $[a, b]$ with $a = t_0 < t_1 < \dots < t_n = b$, and a collection of points $\Lambda = \{\lambda_k\}_{k=1}^n$ with $\lambda_k \in [t_{k-1}, t_k]$, the Riemann sum of f associated to $\mathcal{P}$ and Λ is the vector

$$\sigma_f(\mathcal{P}, \Lambda) = \sum_{k=1}^n f(\lambda_k)(t_k - t_{k-1}).$$

Then, f is said to be Riemann integrable on $[a, b]$ if there exists an element $\int_a^b f$ in X such that

$$\lim_{\|\mathcal{P}\| \rightarrow 0} \sigma_f(\mathcal{P}, \Lambda) = \int_a^b f.$$

That is, for any $\epsilon > 0$ there exists $\delta > 0$ such that for each partition $\mathcal{P}$ of $[a, b]$ with $\|\mathcal{P}\| = \max_k (t_k - t_{k-1}) < \delta$ and each Λ , we have $\left\| \sigma_f(\mathcal{P}, \Lambda) - \int_a^b f \right\| < \epsilon$. The linear space of Riemann integrable functions on the interval $I = [a, b]$ will be denoted by $\mathcal{R}(I, X)$.

The well-intentioned attempt to generalize the fundamental theorem of calculus to non-locally convex spaces faces major obstructions from a very early stage since by a result of Mazur and Orlicz such spaces admit continuous functions failing to be Riemann integrable [201]. This initial drawback may be overcome by choosing to study for the sake of it the differentiability properties of the functions $F(t) = \int_a^t f$ whenever f is Riemann integrable on an interval $[a, b]$ and $t \in [a, b]$.

The first mover in this direction was Popov. He investigated in [202] the properties of the Riemann integral for functions $f : [a, b] \rightarrow X$ where X is an F -space and showed that while some usual properties of this integral remain true in the non-locally convex setting, other properties and techniques, like the usual way of getting primitives for integrable functions, may be false. His work also contains an example of a continuous Riemann integrable function $g : [0, 1] \rightarrow \ell_p$ for $0 < p < 1$ whose integral function $G(t) = \int_0^t g$ does not possess a right derivative at $t = 0$.

Which means that part (i) of Theorem (6.3.1) breaks down for $X = \ell_p$ when $0 < p < 1$! To the best of our knowledge this connection, however trivial, had not been made explicit before.

Bayoumi [209] claimed to have extended the fundamental theorem of calculus to locally bounded topological vector spaces via the notion of quasi-differentiability (or Bayoumi-differentiability, according to himself). These appeared shortly afterwards in [210], a book rightfully devoted to the study of the theory of functions in the lack of local convexity. Unfortunately, Bayoumi's quasi-differential is nothing other than the Fréchet derivative in disguise [207] and so, in view of Popov's example, his extension of Theorem (6.3.1)(i) to quasi-Banach spaces contained in [209] and [210] cannot hold. A close look at the proofs reveals two important errors. The first one is an approach to the Riemann integral for functions with values in a quasi-Banach space that mimics the construction for normed spaces based on the boundedness of the integral operator on the step functions. The other glitch has been recently noticed in [211] and consists of taking for granted that the Riemann integral of a continuous function $f : [a, b] \rightarrow X$ fulfills the familiar estimate,

$$\left\| \int_a^b f \right\| \leq \int_a^b \|f(s)\| ds, \quad (10)$$

which is expected of any integral worth defining. This assumption permits one to write for each fixed $t \in [a, b]$,

$$\left\| \frac{F(t+h) - F(t)}{h} - f(t) \right\| \leq \frac{1}{|h|} \int_t^{t+h} \|f(s) - f(t)\| ds = \|f(u) - f(t)\|,$$

with u between t and $t+h$. From here, an appeal to the continuity of f at t yields the differentiability of F at this point. Of course, this is true when X is a Banach space. But unfortunately we do not have an inequality like (10) in the lack of local convexity because a quasi-norm does not satisfy the triangle law in the usual sense. The very same reason hinders the construction of the Bochner integral in quasi-Banach spaces.

Aside from fixing the above misconception, this section is motivated by the work on the subject of Maurey [214], Kalton [199], and Popov [202], and, continuing in the spirit of [204, 208], aims at making headway in the theory of integration for quasi-Banach spaces and its applications. To that end, we get started with the analysis of the shortcomings that frustrate the efforts to define a satisfactory integral in quasi-Banach spaces. We show that local convexity is not only a sufficient condition for the integral operator to be bounded but it is also necessary. For that we introduce a new class of spaces, namely Orlicz spaces of functions taking values in a quasi-Banach space, modeled on a standard Orlicz function.

One of the earliest applications of integration as a tool in geometric functional analysis has been the fundamental role it played in determining which Banach spaces X have the property that Lipschitz maps f from the unit interval $[0,1]$ into X are differentiable almost everywhere. This problem, known in full generality as Tamarkin's question and which led to the forging of the Radon–Nikodym property, remains unexplored for quasi-Banach spaces due to the absence of one of the most important tools for the analyst, the Hahn–Banach theorem. Thus, with the intention to find a class of Lipschitz functions from $[0,1]$ into X with good differentiability properties, we investigate which additional conditions guarantee the Riemann integrability and the fulfillment of the fundamental theorem of calculus for a continuous function with values on a quasi-Banach space. We provide a criterion in terms of approximation by polynomials, which leads to the introduction of the new class of functions called analytic of order r .

We discuss the validity of the second part of the fundamental theorem of calculus for the Riemann integral. The conclusion is that, while Barrow's rule breaks down in spaces with trivial dual like the spaces $L_p[0,1]$ for $p < 1$, a slightly weaker version of Theorem (6.3.1)(ii) still works as long as X^* has enough linear functionals to separate the points of X , like in ℓ_p for $p < 1$.

Finally, we revisit the notion of integral specifically designed for p -normed spaces with $p < 1$ by Vogt in 1967 [220] and use it to show the first “Lebesgue differentiation theorem” for functions mapping in a non-locally convex space.

The unfamiliar reader with quasi-Banach spaces and F -spaces will find the few required prerequisites in the books [200, 134].

Given a quasi-Banach space X and (Ω, Σ, μ) a measure space, we will denote by $L_0(\mu, X)$ the topological linear space of all Σ -measurable functions $f: \Omega \rightarrow X$ of

separable range mapping into the quasi-Banach space X , equipped with its standard topology that gives the convergence in measure, with the usual convention about identifying functions equal almost everywhere. We will also consider the linear subspace $S(\mu, X)$ of the simple functions in $L_0(\mu, X)$, i.e., the Σ -measurable functions $s: \Omega \rightarrow X$ of the form

$$s = \sum_{i=1}^n x_i \chi_{A_i},$$

where $\{x_i\}_{i=1}^n \subset X$, $\{A_i\}_{i=1}^n \subset \Sigma$ with $\mu(A_i) < \infty$, and n is an arbitrary integer. The following is the main theorem of this section.

Theorem (6.3.2)[206]: Suppose X is a quasi-Banach space and let (Ω, Σ, μ) be a non-purely atomic measure space. Suppose that for some F -space E which embeds continuously in $L_0(\mu, X)$, with $S(\mu, X) \subseteq E$ we have:

(a) There exists a continuous linear operator $\mathcal{I}: E \rightarrow X$ so that for every $\sum_{i=1}^n x_i \chi_{A_i} \in S(\mu, X)$,

$$\mathcal{I}\left(\sum_{i=1}^n x_i \chi_{A_i}\right) = \sum_{i=1}^n x_i \mu(A_i).$$

(b) Whenever a function $\phi \in L_0(\mu, X)$ satisfies $\|\phi(\omega)\| \leq \|\psi(\omega)\|$ almost everywhere for some $\psi \in E$, it implies that $\phi \in E$.

Then X is locally convex (and so isomorphic to a Banach space).

Proof. Let x_0 be any norm-one vector in X , and define the sets

$$F = \{f \in L_0(\mu, \mathbb{R}) : x_0 f \in E\},$$

and

$$E_0 = \{\phi \in E : \phi(\omega) \in \mathbb{R} x_0 \mu \text{ a.e. } \omega \in \Omega\},$$

which are in bijective correspondence through the natural mapping

$$F \rightarrow E_0, \quad f \rightarrow x_0 f.$$

Note that E_0 is a closed subspace of E , so that F equipped with the topology it inherits via the above bijection is an F -space that embeds continuously in $L_0(\mu, \mathbb{R})$. Of course, neither of them is trivial since F contains the real-valued simple functions $S(\mu, \mathbb{R})$.

Suppose $g \in L_\infty(\mu, X)$, i.e., $g \in L_0(\mu, X)$ with

$$\|g\|_\infty = \inf_{\mu(A)=0} \sup_{\omega \in \Omega/A} \|g(\omega)\| < \infty.$$

Then, for any $f \in F$ we have

$$\|g(\omega)f(\omega)\| \leq \|g\|_\infty |f(\omega)| = \|\|g\|_\infty x_0 f(\omega)\|, \quad \text{a.e. } \omega \in \Omega.$$

Since the function $\|g\|_\infty x_0 f(\omega)$ belongs to E_0 , the hypothesis (b) yields that $gf \in E$. Combining the closed graph theorem with the uniform boundedness principle gives that the bilinear operator

$$T: L_\infty(\mu, X) \times F \rightarrow E, \quad (g, f) \rightarrow gf,$$

is continuous.

Pick an atomless set $A \in \Sigma$ with $0 < \mu(A) < \infty$. Using the continuity of T we deduce that the set

$$T(B_{L_\infty(\mu, X)} \times \{\chi_A\}) = \{g\chi_A : \|g\|_\infty \leq 1\}$$

is bounded in E . Therefore its image under the operator $\mathcal{I}$ will be bounded in X . In other words, there exists a positive constant C so that

$$\|\mathcal{I}(g\chi_A)\| \leq C, \quad \forall g \in L_\infty(\mu, X) \text{ with } \|g\|_\infty \leq 1. \quad (11)$$

For $n \in \mathbb{N}$ arbitrary, let $\{x_i\}_{i=1}^n$ be any norm-one vectors in X , and let $\{\lambda_i\}_{i=1}^n$ be nonnegative scalars with $\sum_{i=1}^n \lambda_i = 1$. Using Sierpiński's theorem on the range of a real nonatomic measure (see [216]), we pick recursively a partition $\{A_1, \dots, A_n\}$ of A such that $\mu(A_i) = \lambda_i / \mu(A)$ for $i = 1, \dots, n$. Thus, the simple function $g = \sum_{i=1}^n x_i \chi_{A_i}$ verifies $\|g\|_\infty \leq 1$, and so, by (2.1), $\|\mathcal{I}(g)\| = \|\mathcal{I}(g\chi_A)\| \leq C$. That is, $\left\| \sum_{i=1}^n x_i \mu(A_i) \right\| \leq C$, which implies $\left\| \sum_{i=1}^n \lambda_i x_i \right\| \leq C \mu(A)$. We have showed that the origin of X has a convex neighborhood, i.e., X is locally convex as claimed.

Let φ be an Orlicz function, that is, a right-continuous, increasing function on $[0, +\infty)$ such that $\varphi(0) = 0$. Define

$$L_\varphi(\mu, X) = \left\{ f \in L_0(\mu, X) : \int_\Omega \varphi\left(\frac{\|f(x)\|}{\rho}\right) d\mu(x) < \infty \text{ for some } \rho > 0 \right\}.$$

The properties of φ yield that if for $f \in L_\varphi(\mu, X)$ we put

$$\|f\|_\varphi = \inf \left\{ \rho > 0 : \int_\Omega \varphi\left(\frac{\|f(x)\|}{\rho}\right) d\mu(x) < \rho \right\},$$

Then $\|\cdot\|_\varphi$ satisfies the axioms of a Δ -norm in $L_\varphi(\mu, X)$ (see [11]):

- $\|f\|_\varphi > 0$ if $f \neq 0$.
- $\|\alpha f\|_\varphi \leq \|f\|_\varphi$ whenever $|\alpha| \leq 1$ and $f \in L_\varphi(\mu, X)$;
- $\lim_{\alpha \rightarrow 0} \|\alpha f\|_\varphi = 0$ for any $f \in L_\varphi(\mu, X)$;
- $\|f + g\|_\varphi \leq \kappa(\|f\|_\varphi + \|g\|_\varphi)$ for all f and g in $L_\varphi(\mu, X)$, where $\kappa \geq 1$ is the modulus of concavity of the quasi-norm in X .

Moreover, $\|\cdot\|_\varphi$ is equivalent to an F-norm under which $L_\varphi(\mu, X)$ is complete, whence $(L_\varphi(\mu, X), \|\cdot\|_\varphi)$ is an F-space. We will put $M_\varphi(\mu, X)$ for the closure of $S(\mu, X)$ in $L_\varphi(\mu, X)$, and note that if the Orlicz function φ satisfies the Δ_2 -condition,

$$\varphi(2t) \leq C\varphi(t), \quad \forall t \geq 0, \quad (12)$$

for some constant C , then $M_\varphi(\mu, X) = L_\varphi(\mu, X)$. The rigorous proof of these facts is similar in spirit to the case of Orlicz spaces of scalar valued functions. The details are left to the reader, who can look up the classical work [215] on the subject.

The following theorem gains in interest if we realize that it evinces that the Bochner integral cannot be defined in non-locally convex spaces like the spaces $L_p(\mu, X)$ for $0 < p < \infty$ when X is quasi-Banach.

Theorem (6.3.3)[206]: Let X be a quasi-Banach space. Suppose there exist a non-purely atomic measure space (Ω, Σ, μ) and an Orlicz function φ so that the integral operator $\mathcal{I} : M_\varphi(\mu, X) \rightarrow X$ given by

$$\mathcal{I}\left(\sum_{i=1}^n x_i \chi_{A_i}\right) = \sum_{i=1}^n x_i \mu(A_i), \quad s = \sum_{i=1}^n x_i \chi_{A_i} \in S(\mu, X),$$

is continuous. Then X is locally convex.

Proof. It suffices to apply Theorem (6.3.2) to $E = M_\varphi(\mu, X)$.

Now we will delve deeper into the matter and show that the Riemann integral

operator also cannot be extended when mapping into a quasi-Banach space X . To that end, let $I = [a, b]$ be a fixed compact interval of the real line and $C(I, X)$ be the usual quasi-Banach space of continuous functions $f : I \rightarrow X$ with the quasi-norm $\|f\|_\infty = \max_{t \in I} \|f(t)\|$. We will denote by $S(I, X)$ the linear space of all step functions $s : I \rightarrow X$ and by $\bar{S}(I, X)$ its closure in $L_\infty(I, X)$. Recall that an X -valued function s defined on $[a, b]$ is called a step function if there is a partition $a = t_0 < t_1 < \dots < t_n = b$ such that for each k the function s assumes only one value on the interval $[t_{k-1}, t_k)$.

If $f \in C(I, X)$ then f is uniformly continuous on I hence $C(I, X) \subset \bar{S}(I, X) \subset L_\infty(I, X)$. Each s in $S(I, X)$ can be written in the form

$$s = \sum_{k=1}^{n-1} x_k \chi_{[t_{k-1}, t_k)} + x_n \chi_{[t_{n-1}, t_n]}, \quad (13)$$

where $\mathcal{P} = \{t_k\}_{k=0}^n$ is a partition of I with $a = t_0 < t_1 < \dots < t_n = b$ and $x_k \in X$ for $k = 1, \dots, n$. For such an s put

$$\int_a^b s = \mathcal{I}_{\mathcal{R}}(s) = \sum_{k=1}^n x_k (t_k - t_{k-1}) \in X. \quad (14)$$

When X is a Banach space,

$$\|\mathcal{I}_{\mathcal{R}}(s)\| = \left\| \sum_{k=1}^n x_k (t_k - t_{k-1}) \right\| \leq \sum_{k=1}^n \|x_k\| (t_k - t_{k-1}) \leq (b-a) \|s\|_\infty, \quad (15)$$

and so $\mathcal{I}_{\mathcal{R}}$ defines a bounded linear map from $S(I, X)$ into X . Thus $\mathcal{I}_{\mathcal{R}}$ extends uniquely to a continuous linear operator $\bar{\mathcal{I}}_{\mathcal{R}} : \bar{S}(I, X) \rightarrow X$ satisfying

$$\|\bar{\mathcal{I}}_{\mathcal{R}}(f)\| \leq (b-a) \|f\|_\infty, \quad \forall f \in \bar{S}(I, X). \quad (16)$$

A tedious but straightforward argument shows that $\bar{\mathcal{I}}_{\mathcal{R}}(f)$ may be computed as

$$\bar{\mathcal{I}}_{\mathcal{R}}(f) = \lim_{\|\mathcal{P}\| \rightarrow 0} \sum_{k=1}^n f(c_k)(t_k - t_{k-1}),$$

where, for each partition $\mathcal{P} = \{t_k\}_{k=0}^n$ of I , the point c_k may be chosen arbitrarily inside $[t_{k-1}, t_k]$ for $1 \leq k \leq n$. It follows that a function $f \in \bar{S}(I, X)$ if and only if $f \in \mathcal{R}(I, X)$, and $\bar{\mathcal{I}}_{\mathcal{R}}(f) = \int_a^b f$.

However, if as it is done in [209, 210], we try to reproduce this operator approach to the Riemann integral when X is a quasi-Banach space, we get in trouble. Indeed, assuming that the quasi-norm on X is p -subadditive for some $p < 1$, the inequality path that we must follow to bound the quasi-norm $\|\mathcal{I}_{\mathcal{R}}(s)\|$ of a step function $s : I \rightarrow X$ becomes

$$\|\mathcal{I}_{\mathcal{R}}(s)\|^p = \left\| \sum_{k=1}^n x_k \right\|^p (t_k - t_{k-1})^p \leq \max_{1 \leq k \leq n} \|x_k\|^p \sum_{k=1}^n (t_k - t_{k-1})^p, \quad (17)$$

But now the amount $\sum_{k=1}^n (t_k - t_{k-1})^p$ depends on the partition of the interval and, unlike for Banach spaces, tends to infinity as n increases. Consequently from (17) we cannot infer an estimate of the form

$$\left\| \int_a^b s \right\| \leq C \|s\|_\infty, \quad s \in S(I, X), \quad (18)$$

for some constant $C > 0$ as stated in [209] and incorrectly proved in [210].

Actually, the following theorem prevents such an inequality from being true at all,

unless the space is already locally convex.

Theorem (6.3.4)[206]: Let X be a quasi-Banach space. Suppose the Riemann integral operator $\mathcal{I}_{\mathcal{R}} : S(I, X) \rightarrow X$ defined by (14) satisfies (18). Then X is locally convex.

Proof. The argument runs as the last part of the proof of Theorem (6.3.2), but we include it nevertheless for completeness. For n arbitrary, let $\{x_i\}_{i=1}^n$ be any vectors contained in the closed unit ball B_x of X , and let $\{\lambda_i\}_{i=1}^n$ be nonnegative scalars with $\sum_{i=1}^n \lambda_i = 1$. Pick a partition $\mathcal{P} = \{t_k\}_{k=0}^n$ of I with $a = t_0 < t_1 < \dots < t_n = b$ and $\lambda_i = t_i - t_{i-1}$ for $1 \leq i \leq n$. Then, the hypothesis yields

$$\left\| \sum_{i=1}^n \lambda_i x_i \right\| = \left\| \sum_{i=1}^n (t_{i-1} - t_i) x_i \right\| = \left\| \mathcal{I} \left(\sum_{i=1}^n x_i \chi_{[t_{i-1}, t_i)} \right) \right\| \leq C \left\| \sum_{i=1}^n x_i \chi_{[t_{i-1}, t_i)} \right\|_{\infty} \leq C,$$

which implies that the origin has a convex neighborhood.

Alternative proof. Since $\|f\|_1 \leq (b-a)\|f\|_{\infty}$ for all $f \in \mathcal{C}(I, X)$ and $S(I, X)$ is dense in $L_1(I, X)$ we deduce from (18) that there exists a linear bounded operator: $\mathcal{I} : L_1(I, X) \rightarrow X$ such that $\mathcal{I}(x \chi_{[c,d]}) = (d-c)x$ whenever $a \leq c < d \leq b$ and $x \in X$. We infer that $\mathcal{I}(x \chi_E) = |E|x$ for every measurable set $E \subseteq [a, b]$ and $x \in X$. Using Theorem (6.3.3) we obtain that X is locally convex.

Another alternative proof. Since $\mathcal{C}(I, X) \subseteq \overline{S(I, X)}$ we can extend $\mathcal{I}_{\mathcal{R}}$ to a bounded linear operator: $\mathcal{I} : \mathcal{C}(I, X) \rightarrow X$. It is straightforward to check that $\mathcal{I}(f)$ is the Riemann integral of the continuous function f . By Mazur–Orlicz theorem, X is locally convex.

Throughout the section, X will denote a quasi-Banach space, unless otherwise specified. Recall that a p -Banach space, $0 < p \leq 1$, is a quasi-Banach space $(X, \|\cdot\|)$ whose quasi-norm is p -subadditive, that is,

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad \forall x, y \in X.$$

In [213] Gramsch proved that the X -valued analytic functions are Riemann-integrable on a compact interval $I = [a, b]$ of the real line. His proof is based on the following sufficient condition for Riemann integrability.

Theorem (6.3.5)[206]: Let X be a p -Banach space ($0 < p \leq 1$) and $f : I \rightarrow X$. Suppose that

$$f(t) = \sum_{n=1}^{\infty} x_n f_n(t), \quad \forall t \in I, \quad (19)$$

Where $(x_n)_{n=1}^{\infty} \subset X$, $(f_n)_{n=1}^{\infty} \subset \mathcal{R}(I, \mathbb{R})$ and $\sum_{n=1}^{\infty} \|x_n\|^p \|f_n\|_{\infty}^p < \infty$. Then, $f \in \mathcal{R}(I, X)$ with integral

$$\int_a^b f = \sum_{n=1}^{\infty} x_n \int_a^b f_n.$$

Since the class of analytic functions is very restrictive, it makes sense to study weaker conditions that guarantee Riemann integrability. We will attain such a criterion through a concept that originated in [130].

Definition (6.3.6)[206]. Let $0 < r < \infty$. A function $f : I \rightarrow X$ will be called analytic of order r on I if for every integer k with $0 \leq k \leq r$, there exist functions $f^{(k)} : I \rightarrow X$ and $\rho_k : I \times I \rightarrow X$, such that the following Taylor expansions hold

$$f^{(k)}(t) = \sum_{j=0}^{\lfloor r \rfloor - k} \frac{f^{(k+j)}(s)}{j!} (t-s)^j + (t-s)^{r-k} \rho_k(t, s).$$

Moreover, the functions $f^{(k)}$ (the derivatives of $f = f^{(0)}$) and the Taylor remainders ρ_k

must be continuous, and $\rho_k(t, t) = 0$ for all $t \in I$. The class of all analytic functions of order r will be denoted by $\mathcal{C}^{(r)}(I, X)$.

In $\mathcal{C}^{(r)}(I, X)$ we consider the topology of uniform convergence of the functions, their derivatives and their Taylor remainders in the above expansion. We will denote by $\mathcal{C}^{(\infty)}(I, X)$ the intersection of all spaces $\mathcal{C}^{(r)}(I, X)$ for $r > 0$.

Theorem (6.3.7)[206]: Let $0 < r < \infty$, $U \subseteq \mathbb{R}^d$ open, and $V \subset U$ relatively compact. There exist a continuous linear operator $T = (T_n)_{n=1}^{\infty} : \mathcal{C}^{(r)}(U, X) \rightarrow c_0(X)$ and a sequence of functions $(\psi_n)_{n=1}^{\infty}$ in $\mathcal{C}^{(\infty)}(V, \mathbb{R})$ with $\|\psi_n\|_{\infty} \leq Cn^{-r}$ for some constant C independent of $n \in \mathbb{N}$ so that every $g \in \mathcal{C}^{(r)}(U, X)$ can be expanded in the form

$$g(t) = \sum_{n=1}^{\infty} T_n(g) \psi_n(t), \quad \forall t \in V. \quad (20)$$

The convergence of this series is understood in the sense of $\mathcal{C}^{(r)}(V, X)$.

Turpin and Waelbroeck made good use of this approximation to prove in [219] that a function in $\mathcal{C}^{(r)}(U, X)$ is integrable in the sense of Vogt with respect to a finite measure with compact support (cf. Section 5). Let us explain how their ideas can be adapted to imply Riemann integrability. The key ingredient is the following lemma.

Lemma (6.3.8)[206]: Suppose $g_1 \in \mathcal{C}^{(r)}([a, b], X)$ and $g_2 \in \mathcal{C}^{(r)}([b, c], X)$ are such that $g_1^{(k)}(b^-) = g_2^{(k)}(b^+)$ for all $k = 0, 1, \dots, \lfloor r \rfloor$. We define functions $h_k : [a, c] \rightarrow X$ for $0 \leq k \leq \lfloor r \rfloor$ by

$$h_k(t) = \begin{cases} g_1^{(k)}(t) & \text{if } t \in [a, b], \\ g_2^{(k)}(t) & \text{if } t \in [b, c]. \end{cases}$$

Then, $h_0 \in \mathcal{C}^{(r)}([a, c], X)$ and $h_0^{(k)} = h_k$ for all $k = 0, 1, \dots, \lfloor r \rfloor$.

Proof. We must show that the functions h_k have a suitable Taylor expansion. We will do this for h_0 , and the same argument will be valid for h_k , with $1 \leq k \leq \lfloor r \rfloor$.

To that end, define for $s, t \in [a, c]$,

$$\rho(s, t) = \begin{cases} (t-s)^{-r} (h_0(t) - \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(s)}{j!} (t-s)^j) & \text{if } s \neq t, \\ 0 & \text{if } s = t. \end{cases}$$

Our goal is to show that ρ is continuous. Since f and g are $\mathcal{C}^{(r)}$ -functions, we need only see that $\lim_{s \rightarrow b^-, t \rightarrow b^+} \rho(s, t) = 0$. For $a \leq s < b < t \leq c$ put

$$\rho(s, t) = \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(b)}{j!} (t-s)^j,$$

and

$$\tau(s, t) = \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(s)}{j!} (t-s)^j,$$

so that

$$\rho(s, t) = (t-s)^{-r} ([h_0(t) - \sigma(s, t)] - [\tau(s, t) - \sigma(s, t)]).$$

Now, on the one hand,

$$h_0(t) - \sigma(s, t) = o((t-b)^r) + \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(b)}{j!} [(t-b)^j - (t-s)^j],$$

and, on the other hand,

$$\begin{aligned}
\tau(s,t) - \sigma(s,t) &= \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(s) - h_j(b)}{j!} (t-s)^j \\
&= o((b-s)^r) + \sum_{j=0}^{\lfloor r \rfloor} \sum_{k=j+1}^{\lfloor r \rfloor} \frac{h_k(b)}{(k-j)!} (s-b)^{k-j} \frac{1}{j!} (t-s)^j \\
&= o((b-s)^r) + \sum_{k=0}^{\lfloor r \rfloor} \sum_{j=0}^{k-1} \frac{h_k(b)}{k!} \binom{k}{j} (s-b)^{k-j} (t-s)^j \\
&= o((b-s)^r) + \sum_{k=0}^{\lfloor r \rfloor} \frac{h_k(b)}{k!} [(s-b)^k - (t-s)^k].
\end{aligned}$$

Subtracting the two equations,

$$\rho(s,t) = (t-s)^{-r} \left[o((t-b)^r) + o((b-s)^r) \right].$$

We conclude by noting that $(t-s)^{-r} \leq (t-b)^{-r}$ and $(t-s)^{-r} \leq (b-s)^{-r}$.

As a consequence, we obtain the following extension lemma.

Lemma (6.3.9)[206]: There exists a continuous linear operator $E : \mathcal{C}^{(r)}(I, X) \rightarrow \mathcal{C}^{(r)}(\mathbb{R}, X)$ such that $f^{(k)}(t) = [E(f)]^{(k)}(t)$ for all $f \in \mathcal{C}^{(r)}(I, X)$, $t \in I$ and $0 \leq k \leq \lfloor r \rfloor$. Moreover, if we fix a compact neighborhood of I , say J , we can get $\text{supp } E(f) \subseteq J$ for all f .

Proof. Let $I = [a, b]$. We pick $-\infty < a_1 < a < b < b_1 < \infty$ such that $[a_1, b_1] \subseteq J$. Let $f \in \mathcal{C}^{(r)}([a, b], X)$. Using a standard polynomial interpolation technique we construct functions $f_l \in \mathcal{C}^{(r)}([a_1, a], X)$, $f_r \in \mathcal{C}^{(r)}([b, b_1], X)$ such that $f_l^{(k)}(a) = f^{(k)}(a)$, $f_r^{(k)}(b) = f^{(k)}(b)$ and $f_l^{(k)}(a_1) = f_r^{(k)}(b_1) = 0$. Define

$$E(f)(t) = \begin{cases} f(t) & \text{if } t \in [a, b], \\ f_l(t) & \text{if } t \in [a_1, a], \\ f_r(t) & \text{if } t \in [b, b_1], \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma (6.3.8), $E(f) \in \mathcal{C}^{(r)}(\mathbb{R}, X)$. Finally, we observe that the assignment of an interpolating polynomial through the mapping $f \mapsto (f_l, f_r)$ is linear and continuous.

Now we are able to give an analogous of Turpin and Waelbroeck's theorem for analytic functions of order r on I .

Theorem (6.3.10)[206]: Let $0 < r < \infty$. There exist a bounded linear operator $S = (S_n)_{n=1}^{\infty} : \mathcal{C}^{(r)}(I, X) \rightarrow c_0(X)$ and a sequence of functions $(\psi_n)_{n=1}^{\infty}$ in $\mathcal{C}^{(\infty)}(I, \mathbb{R})$ with $\|\psi_n^{(k)}\|_{\infty} \leq C_k n^{k-r}$ for all $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, where the constants C_k are independent of $n \in \mathbb{N}$, so that whenever $g \in \mathcal{C}^{(r)}(I, X)$,

$$g(t) = \sum_{n=1}^{\infty} S_n(g) \psi_n(t), \quad \forall t \in I. \quad (21)$$

The convergence of this series is understood in the sense of $\mathcal{C}^{(r)}(I, X)$.

Proof. Let V be a bounded open set such that $I \subseteq V$. By appealing to Theorem (6.3.7), there are $T = (T_n) : \mathcal{C}^{(r)}(\mathbb{R}, X) \rightarrow c_0(X)$ and $(\psi_n) \in \mathcal{C}^{(\infty)}(V, \mathbb{R})$ such that

$$f(t) = \sum_{n=1}^{\infty} T_n(f) \psi_n(t), \quad \text{in } \mathcal{C}^{(r)}(V, X).$$

But, a careful reading of the proof of [218] evinces that $\|\psi_n^{(k)}\|_{\infty} \leq C_k n^{k-r}$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Appealing to Lemma (6.3.9), we simply need to consider $S = T \circ E$.

Theorem (6.3.11)[206]: Let X be a p -Banach space ($0 < p < 1$). Suppose that $f : I \rightarrow X$ is

analytic of order r on I . If $r > 1/p$, then $f \in \mathcal{R}(I, X)$. Moreover, if we define $F(t) = \int_a^t f(u)du$, the $f \in C^{(1)}(I, X)$ and $F' = f$.

Proof. We use Theorem (6.3.10) to write $f = \sum_{n=1}^{\infty} x_n \psi_n$, where $(x_n)_{n=1}^{\infty} = S(f)$. Then,

$$\sum_{n=1}^{\infty} \|x_n\|^p \|\psi_n\|_{\infty}^p \leq \|S(f)\|_{\infty}^p C_0^p \sum_{n=1}^{\infty} n^{-rp} < \infty.$$

By Theorem (6.3.5), $f \in \mathcal{R}(I, X)$ and $F(t) = \sum_{n=1}^{\infty} x_n \int_a^t \psi_n(u)du$.

For $s \neq t$ define

$$\phi_n(s, t) = \frac{1}{t-s} \int_s^t \psi_n(u)du,$$

and

$$g(s, t) = \frac{F(t) - F(s)}{t-s}.$$

Put $\phi_n(t, t) = \psi(t)$ and $g(t, t) = f(t)$. We have

$$g(s, t) = \sum_{n=1}^{\infty} x_n \phi_n(s, t), \quad \forall s, t \in [a, b]^2.$$

Since $\|\phi_n\|_{\infty} = \|\psi_n\|_{\infty}$ and $\sum_{n=1}^{\infty} \|x_n\|^p \|\psi_n\|_{\infty}^p < \infty$, the above series converges uniformly on I^2 . Hence, since ϕ_n are continuous functions, g is continuous, i.e., $f \in C^{(1)}(I, X)$. Moreover $F'(t) = g(t, t) = f(t)$.

As usual, we can apply this result to pseudo-convex spaces, noting that such spaces are projective limits of locally p -convex spaces.

Corollary (6.3.12)[206]: Suppose that X is a pseudo-convex F -space and that $f \in C^{(\infty)}(I, X)$. Then $f \in \mathcal{R}(I, X)$. Moreover, if we define $F(t) = \int_a^t f(u)du$, then $F' = f$.

When dealing with a quasi-Banach space X we run the risk of having no bounded linear functionals on X besides the zero map. If this is the case, a beautiful theorem of Kalton informs us that for every $x \in X$ there exists a continuously differentiable function F from $I = [a, b]$ into X such that $F(a) = 0$, $F(b) = x$, and $F' = 0$ (see [130]). This prevents the second part of the fundamental theorem of calculus from holding for these particular spaces since, by another result of Kalton [199], when $X^* = \{0\}$ every continuous function $f : I \rightarrow X$ has primitives. The validity of Theorem (6.3.1)(ii) is also biased for quasi-Banach spaces X with separating dual from the moment we know that not every continuous function $f : I \rightarrow X$ has a primitive [208]. However, in this case the following version of Barrow's rule does hold.

Theorem (6.3.13)[206]: Let X be a quasi-Banach space with separating dual. Let F be differentiable on I so that $F' \in \mathcal{R}(I, X)$. Then,

$$\int_a^b F' = F(b) - F(a). \quad (22)$$

Proof. Given any $x^* \in X^*$, the composite function $x^* \circ F : I \rightarrow \mathbb{R}$ is differentiable with derivative $(x^* \circ F)'(t) = x^* \circ F'(t)$ for all $t \in I$. Using the (real) fundamental theorem of calculus and the fact that the Riemann integral commutes with linear functionals we have

$$\begin{aligned} x^*(F(b) - F(a)) &= (x^* \circ F)(b) - (x^* \circ F)(a) \\ &= \int_a^b x^* \circ F'(t) dt = x^* \left(\int_a^b F'(t) dt \right). \end{aligned}$$

Since X^* separates points, we deduce Eq. (22).

Theorem (6.3.14)[206]: Let X be a p -Banach space ($0 < p < 1$). Suppose that f is analytic of order r on I . If $r > (p+1)/p$, then $f' \in \mathcal{R}(I, X)$ and

$$f(b) - f(a) = \int_a^b f'(u) du. \quad (23)$$

Proof. We use Theorem (6.3.10) to write $f = \sum_{n=1}^{\infty} x_n \psi_n$, where $(x_n)_{n=1}^{\infty} = S(f)$. We have $f' = \sum_{n=1}^{\infty} x_n \psi'_n$ uniformly. From $\|\psi'_n\|_{\infty} \leq C_1 n^{1-r}$ and $p(r-1) > 1$, we obtain $\sum_{n=1}^{\infty} \|x_n\|^p \|\psi'_n\|_{\infty}^p < \infty$. Applying Theorem (6.3.5) we get that $f \in \mathcal{R}(I, X)$ and

$$\int_a^b f'(u) du = \sum_{n=1}^{\infty} x_n \int_a^b \psi'_n(u) du = \sum_{n=1}^{\infty} x_n (\psi_n(b) - \psi_n(a)) = f(b) - f(a).$$

In [220] Vogt introduced a concept of integrability quite different from that of Riemann. Let (Ω, Σ, μ) be a measure space and let X be a p -Banach space. A function $f : \Omega \rightarrow X$ is said to be integrable in the sense of Vogt, and we write $f \in L_V^1(\mu, X)$ (also, $f \in L_V^1(I, X)$ when μ is the Lebesgue measure on a subset $I \subseteq \mathbb{R}^d$) if f admits an expression of the following guise

$$f(t) = \sum_{n=1}^{\infty} x_n f_n(t) \quad a.e. t \in I, \quad (24)$$

where $\mathbf{x} = (x_n)_{n=1}^{\infty}$ in X and $\mathbf{f} = (f_n)_{n=1}^{\infty}$ in $L_1(\mu, \mathbb{R})$ verify the condition

$$N(\mathbf{x}, \mathbf{f}) = \sum_{n=1}^{\infty} \|x_n\|^p \|f_n\|_1^p < \infty. \quad (25)$$

The space $L_V^1(\mu, X)$ equipped with the gauge

$$\|f\|_{1V} = \inf \{N(\mathbf{x}, \mathbf{f})^{1/p} : (24) \text{ and } (25) \text{ hold}\}$$

is a p -Banach space. Moreover, for $E \in \Sigma$ the expression

$$\sum_{n=1}^{\infty} x_n \int_E f_n d\mu$$

does not depend on the decomposition (24) chosen for f , and so it is consistent to define the Vogt integral of f on E as

$$\int_E f d\mu = \sum_{n=1}^{\infty} x_n \int_E f_n d\mu$$

The crucial fact in the work of Vogt is the possibility to identify isometrically $L_V^1(\mu, X)$ with the completion of the tensor product $X \otimes L_1(\mu, \mathbb{R})$ endowed with the quasi-norm

$$\|\Phi\| = \inf \left\{ \left(\sum_{n=1}^N \|x_n\|^p \|f_n\|_1^p \right)^{1/p} : \Phi = \sum_{n=1}^N x_n \otimes f_n, \quad N \in \mathbb{N} \right\}.$$

Simple functions are dense in $L_V^1(\mu, X)$. In general $L_V^1(\mu, X) \subseteq L_1(\mu, X)$ and, as a consequence of the next proposition, the two spaces coincide for all measure spaces if and only if X is a Banach space.

Proposition (6.3.15)[206]: Let X be a p -Banach space for some $0 < p \leq 1$. Suppose that there exist a non-purely atomic measure space (Ω, Σ, μ) and an Orlicz function φ such that $M_{\varphi}(\mu, X) \subseteq L_V^1(\mu, X)$. Then $p = 1$ (i.e., X is a Banach space).

Proof. Define $\mathcal{I} : M_{\varphi}(\mu, X) \rightarrow X$ by $\mathcal{I}(f) = \int_{\Omega} f d\mu$ (integral in the sense of Vogt). For

$s = \sum_{i=1}^n x_i \chi_{A_i} \in S(\mu, X)$ we obtain $\mathcal{I}(s) = \sum_{i=1}^n x_i \mu(A_i)$. By Theorem (6.3.2), X is locally convex.

The corresponding statement to the fundamental theorem of calculus for functions in $L_1(\mu, \mathbb{R})$ is the Lebesgue differentiation theorem. It is well-known that Lebesgue differentiation theorem works for a Banach space X and for functions in $L_1(\mathbb{R}^d, X)$. It seems natural to ask if this theorem will remain valid for a p -Banach space X and for functions in $L_V^1(\mathbb{R}^d, X)$.

We begin our discussion with some ideas from harmonic analysis (see e.g. [212]). Let $0 < s < \infty$. The Lorentz function space $L_{s,w}(\mu, \mathbb{R})$ consists of all measurable functions f verifying

$$\mu(\{\omega \in \Omega : |f(\omega)| > t\}) \leq \frac{C^s}{t^s}, \quad (26)$$

for some constant C that does not depend on $t \in \mathbb{R}^+$. Denote by $\|f\|_{s,w}$ the best constant C such that (26) holds. Then, $(L_{s,w}(\mu, \mathbb{R}), \|\cdot\|_{s,w})$ is a quasi-Banach space. If $s > 1$, the space $L_{s,w}(\mu, \mathbb{R})$ is locally convex, i.e., there exists a constant $D(s)$ such that for all $N \in \mathbb{N}$,

$$\left\| \sum_{n=1}^N f_n \right\|_{s,w} \leq D(s) \sum_{n=1}^N \|f_n\|_{s,w}. \quad (27)$$

The space $L_{1,w}(\mathbb{R}^d, \mathbb{R})$ appears in a natural way when studying the Lebesgue differentiation theorem since the Hardy–Littlewood maximal operator $\mathcal{M}_{HL}$ is bounded from $L_1(\mathbb{R}^d, \mathbb{R})$ into $L_{1,w}(\mathbb{R}^d, \mathbb{R})$. Explicitly, there exists a constant L such that for every measurable function $g : \mathbb{R}^d \rightarrow [0, +\infty)$,

$$\|\mathcal{M}_{HL}(g)\|_{1,w} \leq L \|g\|_1,$$

where $\mathcal{M}_{HL}(g)$ is defined for $t \in \mathbb{R}^d$ as

$$\mathcal{M}_{HL}(g)(t) = \sup \left\{ \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} g(u) du : \mathcal{Q} \text{ is a cube, } \mathcal{Q} \ni t \right\}.$$

Now we define a maximal operator for vector-valued functions. If X is a p -Banach space, $f : \mathbb{R}^d \rightarrow X$ is locally Vogt integrable ($f \in L_V^1(K, X)$ for every compact set $K \subset \mathbb{R}^d$, for short $f \in L_{V,loc}^1(\mathbb{R}^d, X)$), and $t \in \mathbb{R}^d$,

$$Mf(t) = \sup_{t \in \mathcal{Q}, \mathcal{Q} \text{ cube}} \left\| \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} f(u) du \right\|.$$

In what follows, $\mathcal{Q} \rightarrow t$ means the directed set of cubes containing t as interior point.

Theorem (6.3.16)[206]: Let X be a p -Banach space ($0 < p < 1$). If $f \in L_{V,loc}^1(\mathbb{R}^d, X)$, then

$$\lim_{\mathcal{Q} \rightarrow t} \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} f(u) du = f(t),$$

almost everywhere t in $\mathbb{R}^d$.

Proof. The result is true for $f \in S(\mu, X)$, and $S(\mu, X)$ is dense in $L_V^1(\mathbb{R}^d, X)$. Moreover, the maximal operator M satisfies the following p -subadditivity condition

$$M(f + g) \leq \left((Mf)^p + (Mg)^p \right)^{1/p}.$$

Thus it suffices to show that M maps $L_V^1(\mathbb{R}^d, X)$ into $L_{1,w}(\mathbb{R}^d, \mathbb{R})$.

Let $f \in L_V^1(\mathbb{R}, X)$, $\mathbf{x} = (x_n)_{n=1}^\infty$ in X and $\mathbf{f} = (f_n)_{n=1}^\infty$ in $L^1(\mu, \mathbb{R})$ such that (24) and (25) hold.

Using p -convexity of the space we get

$$Mf(t) \leq \left(\sum_{n=1}^{\infty} \|x_n\|^p [\mathcal{M}_{HL}(|f_n|)(t)]^p \right)^{1/p}.$$

Hence, denoting $s = 1/p$,

$$\begin{aligned} \|Mf\|_{1,w} &\leq \left\| \sum_{n=1}^{\infty} \|x_n\|^p [\mathcal{M}_{HL}(|f_n|)]^p \right\|_{s,w}^s \\ &\leq [D(s)]^s \left(\sum_{n=1}^{\infty} \|x_n\|^p \|\mathcal{M}_{HL}(|f_n|)\|_{s,w}^p \right)^s \\ &\leq [D(s)]^s \left(\sum_{n=1}^{\infty} \|x_n\|^p \|\mathcal{M}_{HL}(|f_n|)\|_{1,w}^p \right)^s \\ &\leq L[D(s)]^s \left(\sum_{n=1}^{\infty} \|x_n\|^p \|f_n\|_1^p \right)^s = L[D(1/p)]^{1/p} N(\mathbf{x}, \mathbf{f})^{1/p}. \end{aligned}$$

Taking the infimum we obtain $\|Mf\|_{1,w} \leq L[D(1/p)]^{1/p} \|f\|_{1,v}$ as desired.

Corollary (6.3.17)[221]: Suppose X is a quasi-Banach space and let (Ω, Σ, μ) be a non-purely atomic measure space. Suppose that for some F -space E which embeds continuously in $L_0(\mu, X)$, with $S(\mu, X) \subseteq E$ we have:

(a) There exists a continuous linear operator $\mathcal{I}: E \rightarrow X$ so that for every $\sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i]} \in S(\mu, X)$,

$$\mathcal{I} \left(\sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i]} \right) = \sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \mu[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i].$$

(b) Whenever a function $\phi \in L_0(\mu, X)$ satisfies $\|\phi(\omega)\| \leq \|\psi(\omega)\|$ almost everywhere for some $\psi \in E$, it implies that $\phi \in E$.

Then X is locally convex (and so isomorphic to a Banach space).

Proof. Let x_0 be any norm-one vector in X , and define the sets

$$F = \{f \in L_0(\mu, \mathbb{R}) : x_0 f \in E\},$$

and

$$E_0 = \{\phi \in E : \phi(\omega) \in \mathbb{R} x_0 \mu \text{ a.e. } \omega \in \Omega\},$$

which are in bijective correspondence through the natural mapping

$$F \rightarrow E_0, \quad f \rightarrow x_0 f.$$

Note that E_0 is a closed subspace of E , so that F equipped with the topology it inherits via the above bijection is an F -space that embeds continuously in $L_0(\mu, \mathbb{R})$. Of course, neither of them is trivial since F contains the real-valued simple functions $S(\mu, \mathbb{R})$.

Suppose $g \in L_\infty(\mu, X)$, i.e., $g \in L_0(\mu, X)$ with

$$\|g\|_\infty = \inf_{\mu(\epsilon)=0} \sup_{\omega \in \Omega/\epsilon} \|f(\omega)\| < \infty.$$

Then, for any $f \in F$ we have

$$\|g(\omega)f(\omega)\| \leq \|g\|_\infty \|f(\omega)\| = \|\|g\|_\infty x_0 f(\omega)\|, \quad \text{a.e. } \omega \in \Omega.$$

Since the function $\|g\|_\infty x_0 f(\omega)$ belongs to E_0 , the hypothesis (b) yields that $gf \in E$. Combining the closed graph theorem with the uniform boundedness principle gives that the bilinear operator

$$T : L_\infty(\mu, X) \times F \rightarrow E, \quad (g, f) \rightarrow gf,$$

is continuous.

Pick an atomless set $\epsilon \in \Sigma$ with $0 < \mu(\epsilon) < \infty$. Using the continuity of T we deduce that the set

$$T(B_{L_\infty(\mu, X)} \times \{\chi_\epsilon\}) = \{g\chi_\epsilon : \|g\|_\infty \leq 1\}$$

is bounded in E . Therefore its image under the operator $\mathcal{I}$ will be bounded in X . In other words, there exists a positive constant C so that

$$\|\mathcal{I}(g\chi_\epsilon)\| \leq C, \quad \forall g \in L_\infty(\mu, X) \text{ with } \|g\|_\infty \leq 1.$$

For $n \in \mathbb{N}$ arbitrary, let $\{x_{i-1} + \sum_{j=1}^i \delta_{j-1}\}_{i=1}^n$ be any norm-one vectors in X , and let $\{\lambda_i\}_{i=1}^n$ be nonnegative scalars with $\sum_{i=1}^n \lambda_i = 1$. Using Sierpiński's theorem on the range of a real nonatomic measure (see [18]), we pick recursively a partition $\{[\epsilon_0, \epsilon_0 + \epsilon_1], \dots, [\epsilon_{n-1}, \epsilon_{n-1} + \epsilon_n]\}$ of ϵ such that $\mu[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i] = \lambda_i / \mu(\epsilon)$ for $i = 1, \dots, n$. Thus, the simple function $g = \sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i]}$ verifies $\|g\|_\infty \leq 1$, and so, by (11), $\|\mathcal{I}(g)\| = \|\mathcal{I}(g\chi_\epsilon)\| \leq C$. That is, $\left\| \sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \mu[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i] \right\| \leq C$, which implies $\left\| \sum_{i=1}^n \lambda_i (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \right\| \leq C \mu(\epsilon)$. We have showed that the origin of X has a convex neighborhood, i.e., X is locally convex as claimed.

Corollary (6.3.18)[221]: Let X be a quasi-Banach space. Suppose there exist a non-purely atomic measure space (Ω, Σ, μ) and an Orlicz function φ so that the integral operator: $\mathcal{I} : M_\varphi(\mu, X) \rightarrow X$ given by

$$\begin{aligned} \mathcal{I} \left(\sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i]} \right) &= \sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \mu[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i], \\ s &= \sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i]} \in S(\mu, X), \end{aligned}$$

is continuous. Then X is locally convex.

Proof. It suffices to apply Theorem (6.3.2) to $E = M_\varphi(\mu, X)$.

Corollary (6.3.19)[221]: Let X be a quasi-Banach space. Suppose the Riemann integral operator $\mathcal{I}_R : S(I, X) \rightarrow X$ defined by (14) satisfies (18). Then X is locally convex.

Proof. The argument runs as the last part of the proof of Theorem (6.3.2), but we include it nevertheless for completeness. For n arbitrary, let $\{x_{i-1} + \sum_{j=1}^i \delta_{j-1}\}_{i=1}^n$ be any vectors contained in the closed unit ball B_X of X , and let $\{\lambda_i\}_{i=1}^n$ be nonnegative scalars with $\sum_{i=1}^n \lambda_i = 1$. Pick a partition $\mathcal{P} = \{\epsilon_0 + \sum_{i=1}^k \delta_i\}_{k=0}^n$ of I with $\epsilon_0 = \epsilon_0 < \epsilon_0 + \delta_1 < \epsilon_0 + \sum_{i=1}^2 \delta_i < \dots < \epsilon_0 + \sum_{i=1}^n \delta_i = \epsilon_0 + \epsilon$ and $\lambda_i = \delta_i$ for $1 \leq i \leq n$. Then, the hypothesis yields

$$\begin{aligned} \left\| \sum_{i=1}^n \lambda_i (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \right\| &= \left\| \sum_{i=1}^n (-\delta_i) x_i \right\| = \left\| \mathcal{I} \left(\sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_0 + \sum_{i=1}^{i-1} \delta_i, \epsilon_0 + \sum_{i=1}^i \delta_i]} \right) \right\| \\ &\leq C \left\| \sum_{i=1}^n (x_{i-1} + \sum_{j=1}^i \delta_{j-1}) \chi_{[\epsilon_0 + \sum_{i=1}^{i-1} \delta_i, \epsilon_0 + \sum_{i=1}^i \delta_i]} \right\|_\infty \leq C, \end{aligned}$$

which implies that the origin has a convex neighborhood.

Alternative proof. Since $\|f\|_1 \leq (\epsilon) \|f\|_\infty$ for all $f \in \mathcal{C}(I, X)$ and $S(I, X)$ is dense in $L_1(I, X)$ we deduce from (18) that there exists a linear bounded operator: $\mathcal{I} : L_1(I, X) \rightarrow X$ such that $\mathcal{I}(x \chi_{[c, d]}) = (d - c)x$ whenever $\epsilon_0 \leq c < d \leq \epsilon_0 + \epsilon$ and $x \in X$. We infer that $\mathcal{I}(x \chi_E) = |E|x$ for

every measurable set $E \subseteq [\epsilon_0, \epsilon_0 + \epsilon]$ and $x \in X$. Using Theorem (6.3.3) we obtain that X is locally convex.

Another alternative proof. Since $\mathcal{C}(I, X) \subseteq \overline{\mathcal{S}(I, X)}$ we can extend $\mathcal{I}_{\mathcal{R}}$ to a bounded linear operator: $\mathcal{I} : \mathcal{C}(I, X) \rightarrow X$. It is straightforward to check that $\mathcal{I}(f)$ is the Riemann integral of the continuous function f . By Mazur–Orlicz theorem, X is locally convex.

Corollary (6.3.20)[221]: Suppose $g_1 \in \mathcal{C}^{(r)}([\epsilon_0, \epsilon_0 + \epsilon], X)$ and $g_2 \in \mathcal{C}^{(r)}([\epsilon_0 + \epsilon, \epsilon_0 + 2\epsilon], X)$ are such that $g_1^{(k)}((\epsilon_0 + \epsilon)^-) = g_2^{(k)}((\epsilon_0 + \epsilon)^+)$ for all $k = 0, 1, \dots, \lfloor r \rfloor$. We define functions $h_k : [\epsilon_0, \epsilon_0 + 2\epsilon] \rightarrow X$ for $0 \leq k \leq \lfloor r \rfloor$ by

$$h_k(\epsilon_0 + \delta) = \begin{cases} g_1^{(k)}(\epsilon_0 + \delta) & \text{if } (\epsilon_0 + \delta) \in [\epsilon_0, \epsilon_0 + \epsilon], \\ g_2^{(k)}(\epsilon_0 + \delta) & \text{if } (\epsilon_0 + \delta) \in [\epsilon_0 + \epsilon, \epsilon_0 + 2\epsilon]. \end{cases}$$

Then, $h_0 \in \mathcal{C}^{(r)}([\epsilon_0, \epsilon_0 + 2\epsilon], X)$ and $h_0^{(k)} = h_k$ for all $k = 0, 1, \dots, \lfloor r \rfloor$.

Proof. We must prove that the functions h_k have a suitable Taylor expansion. We will do this for h_0 , and the same argument will be valid for h_k , with $1 \leq k \leq \lfloor r \rfloor$.

To that end, define for $(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) \in [\epsilon_0, \epsilon_0 + 2\epsilon]$,

$$\rho(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = \begin{cases} (h_1)^{-r} (h_0(\epsilon_0 + \delta) - \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(\epsilon_0 + \delta - h_1)}{j!} (h_1)^j) & \text{if } (\epsilon_0 + \delta - h_1) \neq (\epsilon_0 + \delta), \\ 0 & \text{if } (\epsilon_0 + \delta - h_1) = (\epsilon_0 + \delta) \end{cases}$$

Our goal is to show that ρ is continuous. Since f and g are $\mathcal{C}^{(r)}$ -functions, we need only see that $\lim_{(\epsilon_0 + \delta - h_1) \rightarrow (\epsilon_0 + \epsilon)^-, (\epsilon_0 + \delta) \rightarrow (\epsilon_0 + \epsilon)^+} \rho(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = 0$. For $0 \leq \delta - h_1 < \epsilon < \delta \leq 2\epsilon$ put

$$\rho(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(\epsilon_0 + \epsilon)}{j!} (h_1)^j,$$

and

$$\tau(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(\epsilon_0 + \delta - h_1)}{j!} (h_1)^j,$$

so that

$$\rho(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = (h_1)^{-r} ([h_0(\epsilon_0 + \delta) - \sigma(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta)] - [\tau(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) - \sigma(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta)]).$$

Now, on the one hand,

$$h_0(\epsilon_0 + \delta) - \sigma(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = o((\delta - \epsilon)^r) + \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(\epsilon_0 + \epsilon)}{j!} [(\delta - \epsilon)^j - (h_1)^j],$$

and, on the other hand,

$$\begin{aligned} \tau(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) - \sigma(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) &= \sum_{j=0}^{\lfloor r \rfloor} \frac{h_j(\epsilon_0 + \delta - h_1) - h_j(\epsilon_0 + \epsilon)}{j!} (h_1)^j \\ &= o((\epsilon - \delta + h_1)^r) + \sum_{j=0}^{\lfloor r \rfloor} \sum_{k=j+1}^{\lfloor r \rfloor} \frac{h_k(\epsilon_0 + \epsilon)}{(k-j)!} (\delta - h_1 - \epsilon)^{k-j} \frac{1}{j!} (h_1)^j \\ &= o((\epsilon - \delta + h_1)^r) + \sum_{k=0}^{\lfloor r \rfloor} \sum_{j=0}^{k-1} \frac{h_k(\epsilon_0 + \epsilon)}{k!} \binom{k}{j} (\delta - h_1 - \epsilon)^{k-j} (h_1)^j \\ &= o((\epsilon - \delta + h_1)^r) + \sum_{k=0}^{\lfloor r \rfloor} \frac{h_k(\epsilon_0 + \epsilon)}{k!} [(\delta - h_1 - \epsilon)^k - (h_1)^k]. \end{aligned}$$

Subtracting the two equations,

$$\rho(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = (h_1)^{-r} [o((\delta - \epsilon)^r) + o((\epsilon - \delta + h_1)^r)].$$

We conclude by noting that $(h_1)^{-r} \leq (\delta - \epsilon)^{-r}$ and $(h_1)^{-r} \leq (\epsilon - \delta + h_1)^{-r}$.

Corollary (6.3.21)[221]: There exists a continuous linear operator $E : \mathcal{C}^{(r)}(I, X) \rightarrow \mathcal{C}^{(r)}(\mathbb{R}, X)$ such that $f^{(k)}(\epsilon_0 + \delta) = [E(f)]^{(k)}(\epsilon_0 + \delta)$ for all $f \in \mathcal{C}^{(r)}(I, X)$, $(\epsilon_0 + \delta) \in I$ and $0 \leq k \leq \lfloor r \rfloor$. Moreover, if we fix a compact neighborhood of I , say J , we can get $\text{supp } E(f) \subseteq J$ for all f .

Proof. Let $I = [\epsilon_0, \epsilon_0 + \epsilon]$. We pick $-\infty < (\epsilon_0)_1 < \epsilon_0 < \epsilon_0 + \epsilon < (\epsilon_0 + \epsilon)_1 < \infty$ such that $[(\epsilon_0)_1, (\epsilon_0 + \epsilon)_1] \subseteq J$. Let $f \in \mathcal{C}^{(r)}([\epsilon_0, \epsilon_0 + \epsilon], X)$. Using a standard polynomial interpolation technique we construct functions $f_l \in \mathcal{C}^{(r)}([\epsilon_0, \epsilon_0 + \epsilon], X)$, $f_r \in \mathcal{C}^{(r)}([\epsilon_0 + \epsilon, (\epsilon_0 + \epsilon)_1], X)$ such that $f_l^{(k)}(\epsilon_0) = f^{(k)}(\epsilon_0)$, $f_r^{(k)}(\epsilon_0 + \epsilon) = f^{(k)}(\epsilon_0 + \epsilon)$ and $f_l^{(k)}((\epsilon_0)_1) = f_r^{(k)}((\epsilon_0 + \epsilon)_1) = 0$. Define

$$E(f)(\epsilon_0 + \delta) = \begin{cases} f(\epsilon_0 + \delta) & \text{if } (\epsilon_0 + \delta) \in [\epsilon_0, \epsilon_0 + \epsilon], \\ f_l(\epsilon_0 + \delta) & \text{if } (\epsilon_0 + \delta) \in [(\epsilon_0)_1, \epsilon_0], \\ f_r(\epsilon_0 + \delta) & \text{if } (\epsilon_0 + \delta) \in [\epsilon_0 + \epsilon, (\epsilon_0 + \epsilon)_1], \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma (6.3.8), $E(f) \in \mathcal{C}^{(r)}(\mathbb{R}, X)$. Finally, we observe that the assignment of an interpolating polynomial through the mapping $f \mapsto (f_l, f_r)$ is linear and continuous.

Corollary (6.3.22)[221]: Let $0 < r < \infty$. There exist a bounded linear operator $S = (S_n)_{n=1}^\infty : \mathcal{C}^{(r)}(I, X) \rightarrow c_0(X)$ and a sequence of functions $(\psi_n)_{n=1}^\infty$ in $\mathcal{C}^{(\infty)}(I, \mathbb{R})$ with $\|\psi_n^{(k)}\|_\infty \leq C_k n^{k-r}$ for all $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, where the constants C_k are independent of $n \in \mathbb{N}$, so that whenever $g \in \mathcal{C}^{(r)}(I, X)$,

$$g(\epsilon_0 + \delta) = \sum_{n=1}^\infty S_n(g) \psi_n(\epsilon_0 + \delta), \quad \forall (\epsilon_0 + \delta) \in I.$$

The convergence of this series is understood in the sense of $\mathcal{C}^{(r)}(I, X)$.

Proof. Let V be a bounded open set such that $I \subseteq V$. By appealing to Theorem (6.3.7), there are $T = (T_n) : \mathcal{C}^{(r)}(\mathbb{R}, X) \rightarrow c_0(X)$ and $(\psi_n) \in \mathcal{C}^{(\infty)}(V, \mathbb{R})$ such that

$$f(\epsilon_0 + \delta) = \sum_{n=1}^\infty T_n(f) \psi_n(\epsilon_0 + \delta), \quad \text{in } \mathcal{C}^{(r)}(V, X).$$

But, a careful reading of the proof of [20] evinces that $\|\psi_n^{(k)}\|_\infty \leq C_k n^{k-r}$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Appealing to Lemma (6.3.9), we simply need to consider $S = T \circ E$.

Corollary (6.3.23)[221]: Let X be a $(1 - \epsilon)$ -Banach space $\epsilon > 0$. Suppose that $f : I \rightarrow X$ is analytic of order r on I . If $r > \frac{1}{1-\epsilon}$, then $f \in \mathcal{R}(I, X)$. Moreover, if we define $F(\epsilon_0 + \delta) = \int_{\epsilon_0}^{\epsilon_0 + \delta} f(u) du$, then $f \in C^{(1)}(I, X)$ and $F' = f$.

Proof. We use Theorem (6.3.10) to write $f = \sum_{n=1}^\infty (x_{n-1} + \sum_{j=1}^n \delta_{j-1}) \psi_n$, where $(x_{n-1} + \sum_{j=1}^n \delta_{j-1})_{n=1}^\infty = S(f)$. Then,

$$\sum_{n=1}^\infty \left\| (x_{n-1} + \sum_{j=1}^n \delta_{j-1}) \right\|^{1-\epsilon} \|\psi_n\|_\infty^{1-\epsilon} \leq \|S(f)\|_\infty^{1-\epsilon} C_0^{1-\epsilon} \sum_{n=1}^\infty n^{-r(1-\epsilon)} < \infty.$$

By Theorem (6.3.5), $f \in \mathcal{R}(I, X)$ and $F(\epsilon_0 + \delta) = \sum_{n=1}^\infty (x_{n-1} + \sum_{j=1}^n \delta_{j-1}) \int_{\epsilon_0}^{\epsilon_0 + \delta} \psi_n(u) du$.

For $\epsilon_0 + \delta - h_1 \neq \epsilon_0 + \delta$ define

$$\phi_n(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = \frac{1}{h_1} \int_{\epsilon_0 + \delta - h_1}^{\epsilon_0 + \delta} \psi_n(u) du,$$

and

$$g(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = \frac{F(\epsilon_0 + \delta) - F(\epsilon_0 + \delta - h_1)}{h_1}.$$

Put $\phi_n(\epsilon_0 + \delta, \epsilon_0 + \delta) = \psi(\epsilon_0 + \delta)$ and $g(\epsilon_0 + \delta, \epsilon_0 + \delta) = f(\epsilon_0 + \delta)$. We have

$$g(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta) = \sum_{n=1}^{\infty} \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \phi_n(\epsilon_0 + \delta - h_1, \epsilon_0 + \delta), \quad \forall (\epsilon_0 + \delta - h_1, (\epsilon_0 + \delta) \in [\epsilon_0, \epsilon_0 + \epsilon]^2.$$

Since $\|\phi_n\|_{\infty} = \|\psi_n\|_{\infty}$ and $\sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} \|\psi_n\|_{\infty}^{1-\epsilon} < \infty$, the above series converges uniformly on I^2 . Hence, since ϕ_n are continuous functions, g is continuous, i.e., $f \in C^{(1)}(I, X)$. Moreover $F'(\epsilon_0 + \delta) = g(\epsilon_0 + \delta, \epsilon_0 + \delta) = f(\epsilon_0 + \delta)$.

Corollary (6.3.24)[221]: Let X be a quasi-Banach space with separating dual. Let F be differentiable on I so that $F' \in \mathcal{R}(I, X)$. Then,

$$\int_{\epsilon_0}^{\epsilon_0 + \epsilon} F' = F(\epsilon_0 + \epsilon) - F(\epsilon_0). \quad (28)$$

Proof. Given any $x^* \in X^*$, the composite function $x^* \circ F : I \rightarrow \mathbb{R}$ is differentiable with derivative $(x^* \circ F)'(\epsilon_0 + \delta) = x^* \circ F'(\epsilon_0 + \delta)$ for all $(\epsilon_0 + \delta) \in I$. Using the (real) fundamental theorem of calculus and the fact that the Riemann integral commutes with linear functionals we have

$$\begin{aligned} x^*(F(\epsilon_0 + \epsilon) - F(\epsilon_0)) &= (x^* \circ F)(\epsilon_0 + \epsilon) - (x^* \circ F)(\epsilon_0) \\ &= \int_{\epsilon_0}^{\epsilon_0 + \epsilon} x^* \circ F'(\epsilon_0 + \delta) d(\epsilon_0 + \delta) = x^* \left(\int_{\epsilon_0}^{\epsilon_0 + \epsilon} F'(\epsilon_0 + \delta) d(\epsilon_0 + \delta) \right). \end{aligned}$$

Since X^* separates points, we deduce Eq. (28).

Corollary (6.3.25)[221]: Let X be a $(1 - \epsilon)$ -Banach space $\epsilon > 0$. Suppose that f is analytic of order r on I . If $r > \frac{2-\epsilon}{1-\epsilon}$, then $f' \in \mathcal{R}(I, X)$ and

$$f(\epsilon_0 + \epsilon) - f(\epsilon_0) = \int_{\epsilon_0}^{\epsilon_0 + \epsilon} f'(u) du.$$

Proof. We use Theorem (6.3.10) to write $f = \sum_{n=1}^{\infty} \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \psi_n$, where $\left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right)_{n=1}^{\infty} = S(f)$. We have $f' = \sum_{n=1}^{\infty} \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \psi'_n$ uniformly. From $\|\psi'_n\|_{\infty} \leq C_1 n^{1-r}$ and $(1-\epsilon)(r-1) > 1$, we obtain $\sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} \|\psi'_n\|_{\infty}^{1-\epsilon} < \infty$. Applying Theorem (6.3.5) we get that $f \in \mathcal{R}(I, X)$ and

$$\begin{aligned} \int_{\epsilon_0}^{\epsilon_0 + \epsilon} f'(u) du &= \sum_{n=1}^{\infty} \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \int_{\epsilon_0}^{\epsilon_0 + \epsilon} \psi'_n(u) du = \\ &= \sum_{n=1}^{\infty} \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) (\psi_n(\epsilon_0 + \epsilon) - \psi_n(\epsilon_0)) = f(\epsilon_0 + \epsilon) - f(\epsilon_0). \end{aligned}$$

Corollary (6.3.26)[221]: Let X be a $(1 - \epsilon)$ -Banach space for some $\epsilon > 0$. Suppose that there exist a non-purely atomic measure space (Ω, Σ, μ) and an Orlicz function φ such that $M_{\varphi}(\mu, X) \subseteq L_V^1(\mu, X)$. Then $\epsilon = 0$ (i.e., X is a Banach space).

Proof. Define $\mathcal{I} : M_{\varphi}(\mu, X) \rightarrow X$ by $\mathcal{I}(f) = \int_{\Omega} f d\mu$ (integral in the sense of Vogt). For $s = \sum_{i=1}^n \left(x_{i-1} + \sum_{j=1}^i \delta_{j-1} \right) \chi_{[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i]}$ we obtain

$$\mathcal{I}(s) = \sum_{i=1}^n \left(x_{i-1} + \sum_{j=1}^i \delta_{j-1} \right) \mu[\epsilon_{i-1}, \epsilon_{i-1} + \epsilon_i].$$

By Theorem (6.3.2), X is locally convex.

Corollary (6.3.27)[221]: Let X be a $(1 - \epsilon)$ -Banach space $\epsilon > 0$. If $f \in L_{V,loc}^1(\mathbb{R}^d, X)$, then

$$\lim_{\varrho \rightarrow (\epsilon_0 + \delta)} \frac{1}{|\varrho|} \int_{\varrho} f(u) du = f(\epsilon_0 + \delta),$$

almost everywhere $(\epsilon_0 + \delta)$ in $\mathbb{R}^d$.

Proof. The result is true for $f \in \mathcal{S}(\mu, X)$, and $\mathcal{S}(\mu, X)$ is dense in $L_V^1(\mathbb{R}^d, X)$. Moreover, the maximal operator M satisfies the following $(1 - \epsilon)$ -subadditivity condition

$$M(f + g) \leq \left((Mf)^{1-\epsilon} + (Mg)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}.$$

Thus it suffices to show that M maps $L_V^1(\mathbb{R}^d, X)$ into $L_{1,w}(\mathbb{R}^d, \mathbb{R})$.

Let $f \in L_V^1(\mathbb{R}, X)$, $\mathbf{x} = \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right)_{n=1}^{\infty}$ in X and $\mathbf{f} = (f_n)_{n=1}^{\infty}$ in $L^1(\mu, \mathbb{R})$ such that (24) and (25) hold. Using $(1 - \epsilon)$ -convexity of the space we get

$$Mf(\epsilon_0 + \delta) \leq \left(\sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} [\mathcal{M}_{HL}(|f_n|)(\epsilon_0 + \delta)]^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}.$$

Hence, denoting $s = \frac{1}{1-\epsilon}$,

$$\begin{aligned} \|Mf\|_{1,w} &\leq \left\| \sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} [\mathcal{M}_{HL}(|f_n|)]^{1-\epsilon} \right\|_{1+\epsilon,w}^{1+\epsilon} \\ &\leq [D(s)]^{1+\epsilon} \left(\sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} \left\| [\mathcal{M}_{HL}(|f_n|)]^{1-\epsilon} \right\|_{1+\epsilon,w} \right)^{1+\epsilon} \\ &\leq [D(s)]^{1+\epsilon} \left(\sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} \left\| \mathcal{M}_{HL}(|f_n|) \right\|_{1,w}^{1-\epsilon} \right)^{1+\epsilon} \\ &\leq L[D(s)]^{1+\epsilon} \left(\sum_{n=1}^{\infty} \left\| \left(x_{n-1} + \sum_{j=1}^n \delta_{j-1} \right) \right\|^{1-\epsilon} \|f_n\|_1^{1-\epsilon} \right)^{1+\epsilon} \\ &= L[D(\frac{1}{1-\epsilon})]^{\frac{1}{1-\epsilon}} N(\mathbf{x}, \mathbf{f})^{\frac{1}{1-\epsilon}}. \end{aligned}$$

Taking the infimum we obtain $\|Mf\|_{1,w} \leq L[D(\frac{1}{1-\epsilon})]^{\frac{1}{1-\epsilon}} \|f\|_{1,V}$ as desired.

List of Symbols

Symbol		page
H^p	Hardy Space	1
H^∞	Hardy Space	1
BMOA	Analytic Bounded Mean Oscillation	1
ℓ_p	Lebesgue Space	1
ess Sup	Essential Supremum	1
max	maximum	1
inf	infimum	3
arg	argument	5
min	minimum	6
L^2	Hilbert Space	15
H^2	Hardy Space	15
L^∞	Lebesgue Space	15
L^1	Lebesgue Space on the Real Line	15
const	constant	15
W_r^1	Sobolev Space	16
Lip	Lipschitz	16
B_{pq}^s	Besov Space	18
dist	distance	22
Hol	Holomorphic	22
card	Cardinal	23
l^p	Hilbert Space	24
$\ominus$	Direct difference	25
Im	Imaginary	25
RKHS	Reproducing Kernel Hilbert Space	29
mod	modular	33
pres	Presburger	33
Dom	Domain	35
dim	dimension	35
Mac	Macintyre	39
HBEP	Hahn-Banach Extension Property	65
L_p	Orlicz Space	69
co	convex	70
a.e	almost every where	77
$\otimes$	Tensor product	121
def	definable	122
sem	semi	123
int	integrability	125
supp	Support	135
loc	Local	139
det	determinant	146
Kal	Kalton	177
HL	Hardy Littlewood	190

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