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Variational Formulations for Linear Boundary-Value Problems

صيغ التغاير لمسائل القيم الحدية الخطية

A thesis Submitted in Partial Fulfillment for the Degree of M.Sc in Mathematic

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الآيـة

قال تعالى: ﴿وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ ۖ
وَسَتُرَدُّونَ إِلَىٰ عَالِمِ الْغَيْبِ وَالشَّهَادَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ﴾

صدق الله العظيم

سورة التوبة الآية (105)

Dedication

To my beloved family, my Sir.

Dr. Mohamed Hassan Mohamed Khabir and dear friends.

Acknowledgement

My thanks , all thanks to my God , to

My Sir . Dr. Mohamed Hassan Mohamed Khabir for advising , encouraging and helping me overcoming all difficulties faced, he spare no effort in finishing this work, my dear friends and colleagues,

To all those who stood by me

May God bless and please all those who shed light upon my way.

Abstract

In this thesis , we construct a variational formulation to a two- dimensional Laplace- Dirichlet problem, by transforming the continuous problem (CP) into an integral formulation known as a variational problem (VP) in Sobolev Spaces . We state some theorems and lemmas for the existence and uniqueness of the solution of the variational problem (VP) . We prove the existence and uniqueness of the solution of the variational problem (VP) . We also use the hypothesis of Lax-Milgram theorem and the Ce'a's lemma to estimate the approximation error between the exact and the approximate solution to the (VP) . We state some description of an ordinary finite elements most commonly used in applications of engineering Sciences . Finally as a case , we construct a variational formulation for a one- dimensional Dirichlet and Neumann problems using Lagrange finite elements P1 .

الخلاصة

فى هذا البحث أنشانا صيغة التغاير لمسألة لابلاس درشلت فى بعدين , ويتم إنشاء هذه الصيغة عن طريق تحويل المسألة المستمرة لصيغة تكامل تعرف بصيغة التغاير فى فضاءات سبولوف . وأيضا قمنا بإدراج بعض المبرهنات والقضايا لوجود ووحدانية الحل لمسألة التغاير . كما أثبتنا وجود ووحدانية الحل لهذه المسألة . واستعملنا فرضية مبرهنة لاكس مليجرام وقضية سيس لتقدير الخطأ التقريبى بين الحل الفعلى والحل التقريبى لمسألة التغاير . وقمنا بوصف العناصر المنتهيه العاديه الأكثر إستعمالاً فى تطبيقات العلوم الهندسية . وأخيراً أنشانا صيغة التغاير لمسألة درشلت ونيومان فى بعد واحد مستخدمين عناصر لاجرانج المنتهية P1.

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1.1 Introduction:

In this chapter some essential tools are presented to facilitate understanding and manipulation of this particularly famous and efficient technique for the numerical analysis of equations having partial derivatives.

This chapter presents the formulas for vector analysis as well as the main Hilbert and functional analysis theorems that may be applied to problems subsequently worked out in this thesis.

1.2 Adapted Functional Spaces and Their Properties:

This section recalls the definition of certain functional spaces that would be used to state certain fundamental results.

Definition (1):

Let Ω be an open domain of $\mathbb{R}^n$ and the Sobolev space $H^1(\Omega)$ is defined as:

$$H^1(\Omega) = \left\{ v: \Omega \to \boldsymbol{R}, v \in L^2(\Omega), \frac{\partial v}{\partial x_i} \in L^2(\Omega), \qquad (i = 1, n) \right\}. \tag{1.1}$$

Definition (1) is then generalized by introducing the Sobolev space $H^m(\Omega)$ as follows:

Definition (2):

$\forall m \in N$ and the result is:

$$H^m(\Omega) = \left\{ v: \Omega \subset \boldsymbol{R}^n \to \boldsymbol{R}, v \in L^2(\Omega), \frac{\partial^k v}{\partial x_{i_1} \dots \partial x_{i_k}} \in L^2(\Omega), \forall k = 0, m \right\}. \tag{1.2}$$

Theorem (1):

For any integer m, the Sobolev space $H^m(\Omega)$ is a Hilbert space.

Space $\mathscr{D}(\Omega)$, for which the notion of support (noted $Supp\ v$) is introduced as the smallest closed subset containing all the points where a given function v is nonzero, is another functional space essential for the functional analysis of equations having partial derivatives:

$$supp\ v \equiv \overline{\{x \in \mathbb{R}^n / v(x) \neq 0\}}^{\mathbb{R}^n}. \qquad (1.3)$$

To illustrate the notion of support, consider the example of a function of a real variable defined as:

$$H(x) = \begin{cases} 1 & \text{given } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad (1.4)$$

In this case, the function H is non-zero at the open domain $]0,1[$ but having closed interval $[0,1]$ as support:

$$supp\ H \equiv \overline{\{x \in \mathbf{R} / H(x) \neq 0\}}^{\mathbb{R}^n} = [0,1].$$

Space $\mathscr{D}(\Omega)$ is therefore defined as:

Definition (3):

$$\mathscr{D}(\Omega) = \{v: \Omega \subset \boldsymbol{R}^n \to \boldsymbol{R}, v \in C^{\infty}(\Omega), supp\ v \subset (\Omega)\}. \qquad (1.5)$$

Terminology: The $\mathscr{D}(\Omega)$ space is the space of functions C^{∞} over Ω with a compact support strictly included in Ω.

The following fundamental density theorem is thus obtained:

Theorem (2):

Space $\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$.

Finally, the closure of space $H^1(\Omega)$ in $\mathscr{D}(\Omega)$ is associated to the former and is noted as $H_0^1(\Omega)$. The following definition and property are thus obtained:

Definition (4):

$$H_0^1(\Omega) = \overline{H^1(\Omega)}^{\mathscr{D}(\Omega)}. \tag{1.6}$$

The following result is then shown:

Theorem (3):

$$H_0^1(\Omega) = \{v\colon \Omega \subset \boldsymbol{R^n} \to \boldsymbol{R}, v \in H^1(\Omega), \qquad v = 0 \ \text{ on } \ \partial\Omega\}. \tag{1.7}$$

The Green formula:

Theorem (4):

Let Ω be an open-bounded domain of $\mathbf{R}^n$ with continuous boundary $\partial\Omega = \Gamma$, only admitting discontinuities of the first kind for the tangent vector (i. e. typical angular points). Given that u and v are two functions of the defined variables $(x_1, \ldots, x_n)$ on Ω having real values and belonging to $C^1(\Omega) \cap C^0(\overline{\Omega})$.

The result is:

$$\int_\Omega \frac{\partial u}{\partial x_i} \cdot v \, \mathrm{d}\Omega = -\int_\Omega u \cdot \frac{\partial v}{\partial x_i} \, \mathrm{d}\Omega + \int_{\partial\Omega} u \cdot v \, n_i \, \mathrm{d}\Gamma, \tag{1.8}$$

where n_i denotes the component according to the ith coordinate x_i of normal external vector **n** to open domain Ω.

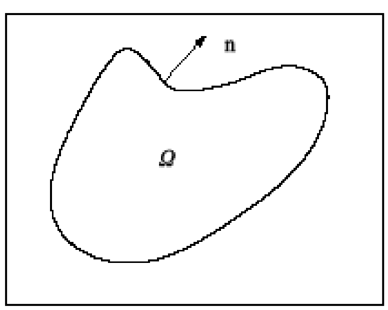

Fig. 1.1 Integration domain Ω and normal external **n.**

It is to be noted that the Green formula (1.8) is nothing but a generalization of the formula of integration by parts in dimension 1.

A variation of the Green formula:

Theorem (5):

Let Ω be an open-bounded domain of $\mathbf{R}^n$ with continuous boundary $\partial\Omega = \Gamma$ only admitting discontinuity of the first kind for the tangent vector (i. e.: typical angular points). Given that u and v are two functions of the defined variables $(x_1, \dots, x_n)$ on Ω having real values such that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $v \in C^1(\Omega)C^0(\overline{\Omega})$, the result obtained is:

$$\int_{\Omega} \Delta u \cdot v \, \mathrm{d}\Omega = -\int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}\Gamma, \tag{1.9}$$

where n denotes the normal vector external to open domain Ω and $\frac{\partial u}{\partial n}$ the projection of the gradient vector in the direction of normal n.

It is to be noted again that the use of formula (1.9) is valid for functions having a weaker regularity, namely for $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$.

1.3 A Set of Fundamental Inequalities:

We recall some fundamental inequalities that emerge from the analysis and that are used intensely within the framework of functional analysis of equations having partial derivatives.

Cauchy-Schwartz Inequality:

Theorem (6):

Let u and v be two functions belonging to $L^2(\Omega)$. The result obtained is:

$$\int_{\Omega} u \cdot v \, \mathrm{d}\Omega \leq \left[\int_{\Omega} u^2 \, \mathrm{d}\Omega\right]^{1/2} \cdot \left[\int_{\Omega} v^2 \, \mathrm{d}\Omega\right]^{1/2}. \tag{1.10}$$

Hölder's Inequality:

Theorem (7):

Let p and q be two conjugated real numbers that satisfy: $\frac{1}{p}+\frac{1}{q}=1$. Let u be a function belonging to $L^p(\Omega)$ and ν a function belonging to $L^q(\Omega)$. The result obtained is:

$$\int_\Omega u\cdot\nu\,\mathrm{d}\Omega\le\left[\int_\Omega u^p\,\mathrm{d}\Omega\right]^{1/p}\cdot\left[\int_\Omega \nu^q\,\mathrm{d}\Omega\right]^{1/q}. \tag{1.11}$$

Poincaré's Inequality:

Theorem (8):

Let Ω be an open-bounded domain of $\mathbf{R}^n$ and u a function belonging to Sobolev space $H_0^1(\Omega)$. The constant $C(\Omega)$ is such that:

$$\int_\Omega |u|^2\,\mathrm{d}\Omega\le C\int_\Omega |\nabla u|^2\,\mathrm{d}\Omega. \tag{1.12}$$

Observation: Poincaré's inequality is also valid if function u is zero only for part of boundary $\sum\subset\partial\Omega$. In this case, $H_\sum^1(\Omega)$ which consists of functions belonging to $H^1(\Omega)$ and that are zero on boundary $\sum$ replaces the $H_0^1(\Omega)$ space.

Korn's Inequality:

Theorem (9):

Let Ω be a "sufficiently regular" open-bounded domain of $\mathbf{R}^3$. Let $\boldsymbol{\nu}$ be a field of vectors defined over Ω with components $\nu_i,(i\in 1,2,3)$ such that ν_i belongs to $H^1(\Omega)$. It is thus stated that field $\boldsymbol{\nu}$ belongs to space $[H^1(\Omega)]^3$

then, the constant $C > 0$ is such that:

$$\int_{\Omega} [\varepsilon(\mathrm{v}) \cdot \varepsilon(\mathrm{v}) + \mathrm{v} \cdot \mathrm{v}]\, d\Omega \geq C \int_{\Omega} [\nabla \mathrm{v} \cdot \nabla \mathrm{v} + \mathrm{v} \cdot \mathrm{v}]\ \mathrm{d}\Omega, \tag{1.13}$$

where:

$$\varepsilon(\mathbf{v}) \cdot \varepsilon(\mathbf{v}) = \frac{1}{2} \sum_{ij} \left[\left(\frac{\partial v_i}{\partial x_i} \right)^2 + \left(\frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right) \right]. \tag{1.14}$$

It is to be noted that ∇v denotes the second order tensor of components $\frac{\partial v_i}{\partial x_j}, 1 \leq i, j \leq 3$.

Elementary Concepts on Distributions:

The Sobolev spaces $H^m(\Omega)$ were introduced in definition (2). The elements of these spaces require an essential observation related to their intrinsic nature.

Indeed when considering the elements of space $H^1(\Omega)$, it is observed that it should relate to functions whose square as well as the square of each partial derivative can be integrated.

To propose a formalism which is broad enough, yet as simple as possible, the functions framework is studied followed by the distributions defined on $\mathbf{R}^2$.

Therefore, given that Ω indicates an open domain of $\mathbf{R}^2$, for any function f belonging to $L^2(\Omega)$, the linear form T_f is considered and is defined by:

$$T_f\colon \mathscr{D}(\Omega) \to \mathbf{R}$$

$$\varphi \rightsquigarrow T_f(\varphi) \equiv \int_{\Omega} f\varphi, \tag{1.15}$$

where $\mathscr{D}(\Omega)$ refers to the set of functions C^{∞} with compact support strictly within Ω.

It can then be noted that definition (1.27) of the linear form T_f is licit since:

$$\left|T_f(\varphi)\right| \leq \int_\Omega |f\varphi| \leq \left(\int_\Omega f^2\right)^{1/2} \left(\int_{supp\ \varphi} \varphi^2\right)^{1/2}, \tag{1.16}$$

where the Cauchy-Schwartz (1.10) inequality would have been used.

In so far as φ belongs to $\mathscr{D}(\Omega)$, the integral of its square is convergent over its support and the definition of T_f is obtained there from.

Observation and Intuitive Definition: The action of the linear form T_f over any function φ belonging to $\mathscr{D}(\Omega)$ may be interpreted as the inner product in $L^2(\Omega)$, expressed as $(.,.)_{L^2(\Omega)}$, from f by φ:

$$\int_\Omega f\varphi \equiv (f,\varphi)_{L^2(\Omega)}. \tag{1.17}$$

This explains why the following notation is adopted:

$$T_f(\varphi) = (f,\varphi)_{L^2(\Omega)} \equiv \langle T_f, \varphi\rangle. \tag{1.18}$$

To complete this analogy, it is necessary to verify the extent to which the total characterisation of any function f belonging to $L^2(\Omega)$ can be carried out by knowing the linear forms T_f.

It is therefore necessary to introduce the application J and is defined by:

$$J: L^2(\Omega) \to \mathscr{D}'(\Omega)$$

$$f \rightsquigarrow T_f, \tag{1.19}$$

where $\mathscr{D}(\Omega)$ represents, at this stage of the construction, the set of the linear forms which are defined on $\mathscr{D}(\Omega)$.

It is then easily shown, that application J defined by (1.19) is itself linear.

The injectivity of J can now be studied.

Let f then be a function belonging to $L^2(\Omega)$ and an element of the kernel of J. By definition, the following is obtained:

$$\big(J(f) \equiv T_f = 0\big) \Leftrightarrow \big(\textstyle\int_\Omega f\varphi = 0\,, \forall\varphi \in \mathscr{D}(\Omega)\big) \qquad (1.20)$$

In the present case, the difficulty to draw a conclusion lies in the fact that function f which is being sought for, belongs to space $L^2(\Omega)$ which strictly holds $\mathscr{D}(\Omega)$

In other words, it is not, a priori, certain that function f which needs to be worked out, would be found among all functions φ that establish the integral equation (1.20).

Consequently, the particular case $\varphi = f$ which helps in concluding that f is equally zero cannot be directly chosen from formulation (1.20).

Moreover, and to overcome this difficulty, the result of the density theorem (2) can be applied, namely: The space $\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$.

This theorem is then applied in the following way: For any given function ψ belonging to $L^2(\Omega)$, there exists a sequence of functions ψ_n belonging to $\mathscr{D}(\Omega)$ such that:

$$\lim_{n\to\infty}\left[\int_\Omega |\psi_n - \psi^2|^2\right] = 0. \qquad (1.21)$$

The interest of the “proximity” between the sequence of functions ψ_n and the function ψ dwells in the “contamination” of the properties of the sequence ψ_n which are passed on to the function ψ.

Indeed, it is adequate to write that the integral equation (1.20) is satisfied for the sequence of functions ψ_n belonging to $\mathscr{D}(\Omega)$:

$$\int_\Omega f\psi_n = 0, \quad \forall n \in \mathbf{N}. \qquad (1.22)$$

Consequently, for the arbitrary function ψ belonging to $L^2(\Omega)$, the following is obtained:

$$\left|\int_\Omega f\psi\right| = \left|\int_\Omega (\psi - \psi_n)\right| \leq \|f\|_{L^2(\Omega)} \|\psi_n - \psi\|_{L^2(\Omega)}, \qquad (1.23)$$

where the Cauchy-Schwartz (1.10) inequality has been again used.

It is sufficient to see to it that n then tends towards $+\infty$ in the inequality (1.23) so as to conclude that:

$$\int_\Omega f\psi = 0, \quad \forall\psi \in L^2(\Omega). \qquad (1.24)$$

Therefore, since f and ψ are now found in the same space $L^2(\Omega)$ within the integral equation (1.24), the particular case $\psi = f$ can assuredly be chosen, allowing as such to reach the conclusion that f = 0.

Consequently, the linear application J contains a kernel reduced to the null element and becomes injective:

$$\left(T_{f_1} = T_{f_2}\right) \Rightarrow (f_1 = f_2). \qquad (1.25)$$

The characterisation of the functions f belonging to $L^2(\Omega)$ is then completed via the injection J (1.19). It is now possible to define a first type of distributions which are defined on Ω.

Definition (5):

The linear form T_f defined by (1.15) is called regular distribution associated to any function f which belongs to $L^2(\Omega)$.

Observation: To reach the definition of a distribution which is sufficiently general, it is adequate to presently keep in mind that the aim of this study is to constitute a mathematical tool that is likely to “derive” functions presenting points of discontinuity similar to the function defined by H (1.4).

In this prospect, the usual derivation properties must be preserved. Therefore, the new concept of "derivation" will have to imply the "continuity" of the distributions.

This explains why it is the proper time, at this stage, to introduce the definition of the continuous distributions.

The regular distribution support whose definition is given by (1.15) is maintained as the medium of presentation.

The continuity at the point 0 of the linear form T_f is defined:

Definition (6):

Given φ_n a sequence of functions belonging to $\mathscr{D}(\Omega)$, it follows that the linear form T_f is continuous at the point 0 if:

$$(\varphi_n \to 0 \ \text{ in} \mathscr{D}\,(\Omega)) \Rightarrow \left(T_f(\varphi_n) \to 0 \ \text{ in } \ \mathbf{R}\right). \tag{1.26}$$

Considering the linearity of the form T_f, the continuity at 0 is equivalent to any continuous point .0 of $\mathscr{D}(\Omega)$.

Indeed, to prove this fact, it is sufficient to express the difference $T_f(\varphi) - T_f(\varphi_0)$ in the form $T_f(\varphi - \varphi_0)$, which emphasises the reference element $\psi \equiv \varphi - \varphi_0$.

In other words, the continuity of the form T_f at the point $\varphi = \varphi_0$ is equivalent to the continuity of T_f at the point $\psi \equiv \varphi - \varphi_0 = 0$.

To ensure that the definition of the continuity 6 is complete, the convergence of a sequence of functions .n belonging to $\mathscr{D}(\Omega)$ needs to be specified.

Definition (7):

A sequence of functions φ_n converge towards 0 in $\mathscr{D}(\Omega)$ given that:

1. $\exists$ a fixed compact (independent of n) K_0 such that: $Supp\varphi_n \subset K_0$,
2. $D^k\varphi_n$ uniformly converges to 0, $\forall k \in N$,

where $D^k\varphi_n$ represents the k-differential of the sequence of functions φ_n.

The continuity which corresponds to the regular distribution T_f, as defined by (1.15) is then verified.

Hence let φ_n be a sequence of functions which belongs to $\mathscr{D}(\Omega)$ and which converges towards 0, as per the definition (7) (1, 2).

It is obvious that only the property of the uniform convergence of the sequence of functions φ_n is required to establish the convergence of the sequence $T_f(\varphi_n)$ in **R**, according to the definition of (1.26):

$$\left|T_f(\varphi_n)\right| \leq \underset{x\in\Omega}{\mathrm{Sup}}|\varphi_n(x)| \int_{K_0} |f|. \quad (1.27)$$

Therefore, in the case of the regular distribution T_f, the definition of the convergence in $\mathscr{D}(\Omega)$ is compatible so as to secure the continuity property which corresponds to linear forms.

At this point, it is legitimate to know why the uniform convergence of the sequence φ_n needs to be extended to the successive differentials $D^k\varphi_n$ according to the definition 7 (2).

In this view, it is important to have a more global vision of the distributions, defined presently as:

Definition (8):

A distribution T is a linear form defined on $\mathscr{D}(\Omega)$, continuous as understood in the definition (6).

Consequently the effect of the distribution T upon any function φ belonging to $\mathscr{D}(\Omega)$ can be observed, according to the following convention:

$$T(\varphi) \equiv \langle T, \varphi \rangle, \quad \forall\varphi \in \mathscr{D}(\Omega). \quad (1.28)$$

Furthermore, all the distributions defined on Ω are referred to by $\mathscr{D}'(\Omega)$.

It can be observed that the angle bracket $\langle T, \varphi \rangle$ can no more be interpreted in the general case of a distribution T belonging to $\mathscr{D}(\Omega)$, as the inner product in $L^2(\Omega)$ of the distribution T by the function φ.

It exclusively concerns a notation whish has been retained by analogy with the regular distributions T_f associated to the functions f belonging to $L^2(\Omega)$.

The definition (8) of a distribution T includes new mathematical tools which cannot be associated anymore to the functions f via the regular distributions T_f.

The most popular example, in this case, is the Dirac distribution δ, defined by:

$$\delta: \mathscr{D}(\mathbf{R}) \rightarrow \mathbf{R}, \quad \delta(\varphi) \equiv \langle \delta, \varphi \rangle \varphi(0). \tag{1.29}$$

The definition of δ makes it possible to establish, by simple inspection, that it is a distribution belonging to $\mathscr{D}'(\mathbf{R})$, that is, a linear form defined on $\mathscr{D}(\mathbf{R})$ and continuous, as understood in the definition (1.26).

It can then be shown that no functions f belonging to $L^2_{\text{loc}}(\mathbf{R})$ exist such that:

$$\delta(\varphi) \equiv \varphi(0) = \int_R f\varphi, \quad \forall \varphi \in \mathscr{D}(\mathbf{R}). \tag{1.30}$$

Indeed, proceeding by reductio ad absurdum, it is assumed that there exists a function f belonging to $L^2_{\text{loc}}(\mathbf{R})$ so that (1.30) is satisfied.

The particular case of the functions φ belonging to $\mathscr{D}(\mathbf{R})$ is chosen so that $\varphi(0) = 0$.

For each of these functions φ, there exists a function ϕ in $\mathscr{D}(\mathbf{R})$ so that:

$$\phi(x) = \frac{\varphi(x)}{x}. \tag{1.31}$$

Indeed, the only difficulty for the function ϕ dwells in its regularity in the neighbourhood of $x = 0$.

Yet, in so far as φ belongs to $\mathscr{D}(\mathbf{R})$, while having a non-zero value when $x = 0$, the expression of ϕ can be written again in the form:

$$\phi(x) = \frac{\varphi(0) + \int_0^x \varphi'(t)\,\mathrm{d}t}{x} = \frac{\int_0^x \varphi'(t)\mathrm{d}t}{x}. \quad (1.32)$$

Therefore, when x tends to 0, the following is obtained:

$$\phi(x) = \frac{\int_0^x \varphi'(t)\mathrm{d}t}{x}. \quad (1.33)$$

by applying Hôpital's rule.

However, $\varphi'(0)$ is bounded since φ is an element of $\mathscr{D}(\mathbf{R})$. Moreover, the function ϕ defined by (1.33) is bounded in the neighbourhood of $x = 0$.

In the case when $x = 0$, the function ϕ is C^∞ over **R**, this is sufficient to ensure that it belongs to $\mathscr{D}(\mathbf{R})$.

The equation (1.30) is then expressed again by using the function ϕ defined by (1.31):

$$\varphi(0) = 0 \cdot \phi(0) = 0 = \int_R xf\phi, \quad \forall \phi \in \mathscr{D}(\mathbf{R}). \quad (1.34)$$

It is then inferred from density arguments that xf is equal to the null function, therefore implying that the function f is itself zero.

Consequently, the degenerate equality (1.30) is written as:

$$\varphi(0) = 0, \quad \forall \varphi \in \mathscr{D}(\mathbf{R}). \quad (1.35)$$

This is obviously absurd since non-zero functions when $x = 0$ exist in $\mathscr{D}(\mathbf{R})$ and thus δ does not constitute a regular distribution.

It is at present possible to define the derivation, understood as in the sense of distributions.

Definition (9):

Let T be a distribution belonging to $\mathscr{D}'(\Omega)$. The distribution $\frac{\partial T}{\partial x_i}$ is defined, which is a partial derivative, understood in the sense of distributions in the direction x_i, $(i = 1,2)$, of the distribution T, as follows:

$$\langle\frac{\partial T}{\partial x_i},\varphi\rangle \equiv -\langle T,\frac{\partial \varphi}{\partial x_i}\rangle, \quad \forall\varphi \in \mathscr{D}(\Omega). \qquad (1.36)$$

It is immediately observed that according to the definition (1.38), $\frac{\partial T}{\partial x_i}$ indeed constitutes a distribution belonging to $\mathscr{D}'(\Omega)$.

In fact, the properties of the distribution T confer the linear and continuity properties defined by (1.26) to $\frac{\partial T}{\partial x_i}$, maintaining as such its distribution status.

Observation: The definition (9) constitutes a generalisation of the form of derivation which is usual in the sense of functions.

To prove this, a function f belonging to $L^2(\Omega)$ and its associated regular distribution T_f can be considered and it can be assumed, moreover, that f is C^1 according to the classical sense that takes the differentiation of functions on Ω.

The partial derivative in the sense of the distributions $\frac{\partial T_f}{\partial x_i}$ can hence be worked out.

$\forall\varphi \in \mathscr{D}(\Omega)$, the following is obtained:

$$\langle\frac{\partial T_f}{\partial x_i},\varphi\rangle \equiv -\langle T_f,\frac{\partial \varphi}{\partial x_i}\rangle \equiv -\int_\Omega f\frac{\partial \varphi}{\partial x_i}\mathrm{d}s. \qquad (1.37)$$

In so far as the function f has been equally accepted as C^1 on Ω in the classical sense, its regular distribution $T_{\frac{\partial f}{\partial x_i}}$ can be associated to each of its classical partial derivative $\frac{\partial f}{\partial x_i}$, since $\frac{\partial f}{\partial x_i}$ belongs to $L^2_{\mathrm{loc}}(\Omega)$.

Therefore, by using integration by parts (see Green's Formula (1.8)), the following is obtained:

$$\langle T_{\frac{\partial f}{\partial x_i}},\varphi\rangle \equiv \int_\Omega f\frac{\partial f}{\partial x_i}\varphi\mathrm{d}s = -\int_\Omega f\frac{\partial \varphi}{\partial x_i}\mathrm{d}s, \qquad (1.38)$$

where the functions φ with a strictly included compact support in Ω have been applied.

In other words, such functions are equal to zero on the boundary $\partial\Omega$ of Ω. This explains the integral absence of a boundary in the integration by parts (1.38).

In the end, when bringing (1.37) and (1.38) closer, the following is obtained:

$$\frac{\partial T_f}{\partial x_i} = T_{\frac{\partial f}{\partial x_i}}, \quad \text{in} \quad \mathscr{D}'(\Omega). \tag{1.39}$$

The interpretation of the equation (1.39) is worked out, as shown below:

The derivative $\frac{\partial T_f}{\partial x_i}$ is usually referred to as "the derivative of f", understood as in the sense of distributions since it is obvious that the derivation of the function f in this sense would prove meaningless; the exclusive working out of the derivation of the distribution of T_f as a distribution makes sense.

Furthermore, the distribution $T_{\frac{\partial f}{\partial x_i}}$ is characteristic of the usual partial derivative $\frac{\partial f}{\partial x_i}$, when the injection J defined by (1.19) enables the association between this partial derivative and its regular distribution.

This becomes completely licit as soon as it is assumed that the function f is C^1, as understood in the sense of functions. In other words, its first partial derivatives are continuous over Ω and as a result, belong to $L^2_{\text{loc}}(\mathbf{R})$.

Therefore, the equality (1.41) shows that the derivation in the sense of the distributions of a function, (i. e., its associated regular distribution T_f), coincides with the usual derivation of its functions when the distribution "is a function" which can be continuously differentiated.

The distribution of the derivative $T_{\frac{\partial f}{\partial x_i}}$ and the derivative of the distribution $\frac{\partial T_f}{\partial x_i}$ are equal.

This proves that the new derivation as well as its generalisation regarding to the classical derivative in the sense of its functions are consistent.

The example of the function H defined by (1.4) and basically belonging to $L^2(\mathbf{R})$ can again be considered to work out its derivative, as understood in the sense of distribution.

In other words, as H basically belongs to $L^2_{\text{loc}}(\mathbf{R})$, it is significant to consider its regular distribution T_H defined by:

$$\forall \varphi \in \mathscr{D}(\Omega): \langle T_H, \varphi \rangle \equiv \int_{\mathbf{R}} H(x)\varphi(x)dx = \int_0^1 \varphi(x)dx. \qquad (1.40)$$

The derivative T'_H of the regular distribution T_H is then worked out, as shown below:

$$\forall \varphi \in \mathscr{D}(\mathbf{R}): \langle T'_H, \varphi \rangle \equiv -\langle T_H, \varphi' \rangle = -\int_0^1 \varphi'(x)dx = \varphi(0) - \varphi(1). \qquad (1.41)$$

Hence:

$$\forall \varphi \in \mathscr{D}(\mathbf{R}): \langle T'_H, \varphi \rangle = \langle \delta_0 - \delta_1, \varphi \rangle, \qquad (1.42)$$

where a notation (1.31) analogous to the distribution notation δ has been adopted, by specifying that the Dirac distribution is δ_0 and δ_1 characteristic of the points $x = 0$ and $x = 1$.

There consequently results:

$$\frac{\mathrm{d}T_H}{\mathrm{d}x} \equiv T'_H = \delta_0 - \delta_1, \quad \text{in} \quad D'(\mathbf{R}). \qquad (1.43)$$

Generalisation to the k-order derivation: The first partial derivative in the sense of the distributions (1.38) can be extended to the order k by introducing the k order partial derivative distribution, denoted $\frac{\partial^k T}{\partial x_1^{k_1} \partial x_2^{k_2}}$, according to the following definition:

$$\langle \frac{\partial^k T}{\partial x_1^{k_1} \partial x_2^{k_2}}, \varphi \rangle \equiv (-1)^{|k|} \langle T, \frac{\partial^k \varphi}{\partial x_1^{k_1} \partial x_2^{k_2}} \rangle, \quad \forall \varphi \in D(\Omega), \tag{1.44}$$

where it has been observed that:

$$|k| = k_1 + k_2.$$

The definition (1.44) underlines the fact that all the weight of the derivation in the sense of distributions is assumed by the functions φ belonging to $\mathscr{D}(\Omega)$. This facilitates the working out of the derivation in the sense of distributions, even if they are particularly irregular!

This principally accounts for the existence of the functional framework $\mathscr{D}(\Omega)$, requiring the regularity C^∞ of the functions φ, which define the effect of any distribution T belonging to $\mathscr{D}'(\Omega)$.

Remarks: The definition of the Sobolev spaces (see definitions (1) and (2)) consequently needs to be re-examined by considering the partial derivatives which occur in the definition of these spaces, like derivatives in the sense of distributions.

For example, when a "function" f belongs to space $H^1(\Omega)$, it is now clear that the first partial derivatives, in the sense of the distributions $\frac{\partial T_f}{\partial x_i}$ of the regular distribution T_f , are associated with the functions of $L^2(\Omega)$ as usual through the canonical injection J defined by (1.19).

Finally, to simplify writings and oral expressions, it is specified that function f and its regular distribution T_f are similar. It amounts to say that for Sobolev space $H^1(\Omega)$, its elements constitute the distributions f, which belong to $L^2(\Omega)$ and whose partial derivatives $\frac{\partial f}{\partial x_i}$ are also elements of $L^2(\Omega)$, through injection J defined in (1.19).

Chapter Two

Nature of the Finite Elements Method

2.1 Introduction :

Developments in the field of numerical analysis during the 20th century gave rise to various methods that provided approximate solutions to equations having partial derivative.

Be it the finite differences method, the spectral methods, the finite volumes method or even the singularities method, it cannot be denied that the finite elements method is the most efficient one.

Undoubtedly, the other methods find their use in specific fields of application, but the finite elements have literally shattered the capacity of modifying the usually complex nature of problems with partial derivatives.

It is most certainly its tremendous adaptation to solve equations-whose inherent complexity, partly due to the domains of integration, when it comes to solving real problems emanating from industry-that caused the finite elements method to undergo such significant developments in the second half of the 20th century.

At this stage, it is necessary to consider the issues involved in the numerical approximation of equations having partial derivatives, irrespective of the method used.

2.2 **Structure and Functional Framework of Equations Having Partial Derivatives:**

To give concrete expression to the demonstration, the two-dimensional problem of Laplace-Dirichlet is considered.

More specifically, let Ω be a bounded open domain of $\mathbf{R}^2$ and it is required to find function u defined from Ω to **R** and solution of:

$$(\mathbf{CP})\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where f is a given function.

At this stage, it is important to note that such a formulation is incomplete because neither the nature of the regularity of boundary $\partial\Omega$ of the integration domain Ω nor that of the second member f is specified though the regularity of solution u of (**CP**) depends much upon it, as does the regularity of research perimeter V in which solution u can be considered.

In this way, for reasons that will be explained later, the integration domain Ω will be assumed to possess a boundary $\partial\Omega$ whose regularity is of the order of C^2. In other words, the curvature is a continuous function of the curvilinear abscissa that describes boundary $\partial\Omega$.

Moreover, assuming that the second member f belongs to $C^0(\Omega)$, it is then legitimate to consider the search for solutions of (**CP**) as elements of $C^2(\Omega)$, thus ensuring that the Laplacian is itself continuous, (it then implies classical solutions).

In this case, the Poisson equation can be considered again, not in the form of a functional equation, but at each point M of Ω, in the form:

Find u belonging to $C^2(\Omega)$ which is the solution to:

$$(\mathbf{CP})\begin{cases}-\Delta u(M) = f(M) & \forall M \in \Omega,\\ \quad u(M) = 0 & \forall M \in \partial\Omega,\end{cases} \tag{2.2}$$

It is obvious that the second member f does not always exhibit regularity C^0. For instance, consider the case where f belongs to $L^2(\Omega)$. In this case, the Laplacian of solution u (which is equal to-f) must also be an element of $L^2(\Omega)$.

This is why it is necessary to find solution u to the (**CP**) in Sobolev space $H^2(\Omega)$, because if this is the case, the Laplacian of u is indeed an element of $L^2(\Omega)$.

The Poisson equation can no more be considered, a priori, point-by-point as in the case of regularity C^0 for f but in the form of a functional equation. In the present case, the Poisson equation needs to be considered as an equality in $L^2(\Omega)$, that is, as a root mean square equality, or as an “energy” balance:

$$\left(\Delta u + f = 0 \;\text{ in }\; L^2(\Omega)\right) \Leftrightarrow \left(\int_\Omega [\Delta u + f]^2 \mathrm{d}\Omega = 0\right). \qquad (2.3)$$

The Poisson equation would nevertheless still be studied as a global equation (2.3) written in $L^2(\Omega)$ rather than as a local equation (2.2).

Construction of a Variational Formulation:

The essential principles constituting the finite elements method will now be studied. The basic idea prevailing in this method is to consider the unknown u no more as a scalar field which, at each point M of Ω associates a real number $u(M)$ that needs to be determined, but as an element belonging to a space of functions V in which different research trajectories would be contemplated so as to lead to the identification of the solution.

Concerning the approximation, it is no more required to determine a numerical sequence $(\tilde{u}_1, \dots, \tilde{u}_N)$ which provides an approximation of the finite differences type for values $(u_1, \dots, u_N)$ of solution u to $(\mathbf{CP})$ along points $M_j, (j = 1, N)$ that have been chosen on an adequate mesh and covering integration domain Ω.

However, it is more meaningful to elaborate a method that would lead to an approximation function $\tilde{u}$. It is obvious, in fine, that knowing solution u, or rather its approximation $\tilde{u}$, would facilitate the evaluation of $\tilde{u}$ at any point M of domain Ω and this evaluation would not be limited to a set of points lying on an already defined mesh, as is the case for finite differences.

A second major characteristic of the finite elements method is the transformation of $(\mathbf{CP})$ into an integral formulation known as variational $(\mathbf{VP})$.

To proceed, a function v, called test function, defined from Ω to **R** and describing the functional space V that will be elaborated later and that is not defined a priori.

Equation (2.1) is then multiplied by the test function v and the two members of the equation are integrated on Ω:

$$-\int_\Omega \Delta u \cdot v \,\mathrm{d}\Omega = \int_\Omega f \cdot v \,\mathrm{d}\Omega, \quad \forall v \in V. \qquad (2.4)$$

Such a transformation is guided by the historic heritage from the finite elements method introduced as a generalisation of the Principle of Virtual Power in Continuum Mechanics (see Cours de Mecanique des Solides, [G. Duvaut]).

Moreover, the transformation of the local writing of problem (**CP**) into a global or integral formulation (VP) is motivated by the need to reach a formalism that properly fits the concept of research trajectories in a functional space V.

This is precisely the case within an integral formulation, in so far as the functions do not directly reveal their numerical values at points M of Ω and only the concept of the "average value" of the functions is apparent.

The Green formula (1.9) of theorem (5) is then applied, thus enabling equation (2.4) to be written as:

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial n} \, v \, d\Gamma = \int_{\Omega} f \cdot v \, d\Omega, \quad \forall v \in V. \tag{2.5}$$

The present stage consists in the definition of the characteristics of space V.

A first point concerns the complete preservation of information between the writing of the formulation of the continuous problem (**CP**) and that of the variational formulation (**VP**).

As such, it is observed that the Dirichlet condition $u = 0$ along boundary $\partial\Omega$ of Ω cannot be analysed directly within the integral writing (2.5).

Considering that the future solution u of the variational problem (**VP**) must be one of the functions v of V, it is compulsory that all functions v of V satisfy the Dirichlet condition:

$$v = 0 \quad \text{on} \quad \partial\Omega. \tag{2.6}$$

This yields equation (2.5) written as:

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} f \cdot v \, d\Omega, \quad \forall v \in V. \tag{2.7}$$

The second point concerns the existence of the integrals of formulation (2.7). is essential to impose sufficient conditions of convergence to the integrals of equation (2.7).

The convergence of the second member of equation (2.7) is easily obtained by verification, via the Cauchy-Schwartz inequality:

$$\left|\iint_{\Omega} f \cdot \nu \,\mathrm{d}\Omega\right| \leq \int_{\Omega} |f \cdot \nu| \,\mathrm{d}\Omega \leq \left[\iint_{\partial\Omega} |f|^2 \,\mathrm{d}\Omega\right]^{1/2} \cdot \left[\iint_{\Omega} |\nu|^2 \,\mathrm{d}\Omega\right]^{1/2}. \qquad (2.8)$$

Therefore, since f is a given function belonging to $L^2(\Omega)$, it is sufficient to consider that ν is also an element of $L^2(\Omega)$, so as to ensure the convergence of the second member of equation (2.7).

In the case of the convergence of the first integral of the left member of equation (2.7), the absolute convergence of the integral is always taken into consideration and the Cauchy-Schwartz inequality is once again used:

$$\left|\iint_{\Omega} \nabla u \cdot \nabla \nu \,\mathrm{d}\Omega\right| \leq \int_{\Omega} |\nabla u \cdot \nabla \nu| \,\mathrm{d}\Omega \leq \left[\iint_{\partial\Omega} |\nabla u|^2 \,\mathrm{d}\Omega\right]^{1/2} \cdot \left[\iint_{\Omega} |\nabla \nu|^2 \,\mathrm{d}\Omega\right]^{\frac{1}{2}} \qquad (2.9)$$

Convergence of the first member of (2.7) is hence assured if the gradients of test function v belonging to V are compulsorily elements that belong to $L^2(\Omega)$.

In conclusion, it has been proved that the sufficient conditions for the convergence of the integrals of equation (2.7) are:

$$\nu \in L^2(\Omega) \qquad \text{and} \qquad \nabla \nu \in [L^2(\Omega)]^2.$$

These reasons consequently explain the choice of the variational space V as the Sobolev space $H^1(\Omega)$ and to which the homogenous Dirichlet condition (2.6) must necessarily be added.

In other words, the following is stated:

$$V \equiv H_0^1(\Omega) \equiv \{\nu : \Omega \to R, \nu \in L^2(\Omega), \nabla \nu \in [L^2(\Omega)]^2, \nu = 0 \ \text{ on } \ \partial\Omega\} \qquad (2.10)$$

All the results when grouped enable the expression of the variational formulation (**VP**) that will be considered in the sequel:

$$
(\mathbf{VP})\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ solution of} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\Omega = \int_\Omega f \cdot v \, d\Omega, \quad \forall v \in H_0^1(\Omega) \end{cases} \tag{2.11}
$$

2.3 Existence, Uniqueness of a Weak Solution:

Concerning variational formulations, there is a sufficient general formalism for which, under some conditions, the existence and uniqueness of the solution may be guaranteed.

It is the object of the Lax-Milgram theorem that is pointed out in the following form:

Theorem (10): (Lax – Milgram)

Let V be a Hilbert space in relation to a given norm $\|.\|$, $a(.,.)$ a bilinear form defined on $V \times V$ and L a linear form defined on V verifying the following properties:

1. $a(.,.)$ is continuous: $\exists C_1 > 0$ such that: $|a(u,v)| \leq C_1 \|u\| \cdot \|v\|, \forall (u,v) \in V \times V$.
2. $a(.,.)$ is V-elliptical: $\exists C_2 > 0$ such that: $a(v,v) \geq C_2 \|v\|^2, \forall v \in V$.
3. L is continuous: $\exists C_3 > 0$ such that: $L(v) \leq C_3 \|v\|, \forall v \in V$.

Then, there is one and only one solution u belonging to V, solution to the variational problem:

$$
Find\ u \in V\ solution\ of: a(u,v) = L(v), \qquad \forall v \in V\ . \tag{2.12}
$$

Observations:

i) The 3 constants $C_i, (i = 1,2,3)$ intervening in each of the three clauses of the Lax-Milgram theorem must absolutely be independent from the generic element v covering the space V.

ii) It is essential to note that during the application of the Lax-Milgram theorem, all the properties required necessitate the use of a unique norm of the Hilbert space V, (noted as $\|.\|$), mainly in order to establish its Hilbertian property.

Yet, it is possible that for the sake of convenience, a change of norms is required to prove one of the properties of the Lax-Milgram theorem.

In this case, it is appropriate to ensure that all the applied norms are equivalent, namely:

Given $\|\cdot\|_1$ and $\|\cdot\|_2$ two appropriate norms for space V, it should be established that there are two constants α and β that are strictly positive and independent from ν such that:

$$\forall \nu \in V: \alpha\|\nu\|_2 \leq \|\nu\|_1 \leq \beta\|\nu\|_2.$$

The following two lemmas are fundamental for the a priori analysis of the regularity of weak solutions to a variational formulation having a one space dimension. Their demonstrations may be consulted .

Lemma (2):

Let I be an open interval of **R** and f a function belonging to $L^1_{\text{loc}}(\mathbf{R})$ verifying:

$$\int_I f(x)\varphi(x)\,\mathrm{d}x = 0, \quad \forall \varphi \in C_0^1(I). \tag{2.13}$$

then: $f = C^{te}$ almost everywhere.

$C_0^1(I)$ refers to the defined functions and C^1 to those defined over the interval I, having a compact support and strictly included in I.

Lemma (3):

Consider $\mathrm{g} \in L^2_{\text{loc}}(\boldsymbol{I})$; for y_0 fixed in I, the following is expressed:

$$\nu(x) = \int_{y_0}^{x} \mathrm{g}(t), \quad \forall x \in I. \tag{2.14}$$

Then $\nu \in C(I)$ (given that I is a bounded interval then ν belongs to $H^1(I)$) and

$$\int_I \nu\varphi' = -\int_I \mathrm{g}\varphi, \quad \forall \varphi \in C_0^1(I). \tag{2.15}$$

Finally, a trace theoremthat is very useful for the application of the Lax-Milgram theorem is recalled, mainly in the framework of the Laplacian-Neumann-Dirichlet problem.

Theorem (11):

Assume that Ω is an open bounded domain of $\mathbf{R}^2$, having boundary $\Gamma = \partial\Omega$ which is "sufficiently regular", (at least C^1-per piece).

Application γ defined by:

$$\begin{gathered} \gamma : H^1(\Omega) \to L^2(\Gamma) \\ \nu \rightsquigarrow \nu|\Gamma, \end{gathered} \tag{2.16}$$

is linear continuous.

In other words, there is a constant $C > 0$ independent from ν, such that:

$$\forall \nu \in H^1(\Omega) : \|\nu\|_{L^2(\Gamma)} \leq \|\nu\|_{H^1(\Omega)}. \tag{2.17}$$

Application to the Laplacian-Dirichlet Problem:

A first application of the Lax-Milgram theorem is proposed in order to establish the existence and uniqueness of the solution to the variational formulation (**VP**) defined by (2.11), associated with the continuous problem (**CP**) defined by (2.1) when the given data f belongs to $L^2(\Omega)$.

Application of the Lax-Milgram theorem requires the identification of space V, the bilinear form $a(.,.)$ and that of the linear form $L(.)$.

Variational formulation (**VP**) defined by (2.11) suggests the introduction of the following quantities:

Let V be the search space of solution u to the variational problem defined by: $V = H_0^1(\Omega)$.

Space $H_0^1(\Omega)$ is provided with the natural norm $\|.\|_{H^1(\Omega)}$ of functions belonging to $H^1(\Omega)$

Thus, $\forall \nu \in H^1(\Omega)$, the following is written:

$$\|v\|^2_{H^1(\Omega)} \equiv \int_\Omega v^2 \, \mathrm{d}\Omega + \int_\Omega \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}\Omega + \int_\Omega \left(\frac{\partial v}{\partial y}\right)^2 \mathrm{d}\Omega. \qquad (2.18)$$

This norm is Hilbertian for space $H^1(\Omega)$,as well as for $H_0^1(\Omega)$ as a closed vectorial subspace in $H^1(\Omega)$.

Let a be the bilinear form defined by:

$$\begin{aligned} a &: V \times V \to R \\ (u, v) &\rightsquigarrow a(u,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\Omega\,. \end{aligned} \qquad (2.19)$$

Likewise, let L be the linear form defined by:

$$\begin{aligned} L &: V \to R \\ v &\rightsquigarrow L(v) \equiv \int_\Omega f v \, \mathrm{d}\Omega\,. \end{aligned} \qquad (2.20)$$

Thus, variational formulation (**VP**) defined by (2.11) is written in the form:

$$\text{Find } u \in V \text{ solution } of : a(u,v) = L(v), \qquad \forall v \in H_0^1(\Omega)\,. \qquad (2.21)$$

Then, a verification of the clauses of the Lax-Milgram theorem 10 is carried out.

1. $a(.,.)$ is a continuous bilinear form:

The bilinearity of form $a(.,.)$ is obvious.

As for its continuity, consider any two elements u and v belonging to $H_0^1(\Omega)$.

The following is obtained:

$$|a(u,v)| \leq \int_\Omega |\nabla u \cdot \nabla v| \leq \left(\int_\Omega |\nabla u|^2\right)^{1/2} \cdot \left(\int_\Omega |\nabla v|^2\right)^{1/2}, \qquad (2.22)$$

where the Cauchy-Schwartz inequality would have been used.

However,

$$\int_\Omega |\nabla u|^2 = \int_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)}, \qquad (2.23)$$

where $\|.\|_{L^2(\Omega)}$ refers to the natural norm in $L^2(\Omega)$, namely:

$$\forall u \in L^2(\Omega): \|u\|_{L^2(\Omega)} \equiv \left(\int_\Omega |u|^2\right)^{1/2}. \tag{2.24}$$

The following is then inferred:

$$\int_\Omega |\nabla u|^2 \leq \|u\|^2_{H^1(\Omega)}. \tag{2.25}$$

Inequality (2.22) then leads to:

$$|a(u,v)| \leq |u\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)}, \tag{2.26}$$

and the continuity constant C_1 of theorem 10 is basically equal to one.

2. $a(.,.)$ is a V-elliptical form:

In order to establish the V-ellipticity of the bilinear $a(.,.)$ form, the quantity $a(v,v)$ defined from (1.77) needs to be minorated.

Also, any function $H_0^1(\Omega)$ yields:

$$a(v,v) = \int_\Omega |\nabla v|^2 = \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\Omega)}. \tag{2.27}$$

In order to obtain a lower bound of $a(v,v)$ in relation to the $H^1(\Omega)$ norm, it is pointed out that for all functions v belonging to $H_0^1(\Omega)$, the Poincare inequality (1.12) is available.

In other words, a constant $C(\Omega) > 0$ exists such that:

$$\int_\Omega |v|^2 \mathrm{d}\Omega \leq C(\Omega) \int_\Omega |\nabla v|^2 \mathrm{d}\Omega. \tag{2.28}$$

To each side of inequality (2.28), the square of norm $L^2(\Omega)$ of the module of ∇v is added so as to yield the square of norm $H^1(\Omega)$ of function v:

$$\|v\|^2_{H^1(\Omega)} \equiv \|v\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\Omega)}. \tag{2.29}$$

$$\leq \left(1 + C(\Omega)\right)\left[\left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\Omega)}\right]. \tag{2.30}$$

$$\leq \left(1 + C(\Omega)\right)a(v, v). \tag{2.31}$$

It then becomes:

$$a(v, v) \geq C_2 \|v\|^2_{H^1(\Omega)}, \tag{2.32}$$

where the V-ellipticity constant C_2 is defined by: $C_2 = \frac{1}{1+C(\Omega)}$.

3. $L(.)$ is a continuous linear form:

Once again, the linearity of form L is obvious.

Control of linear form L is quite simple being given that f is a function belonging to $L^2(\Omega)$:

$$|L(v)| \leq \int_\Omega |fv| \, \mathrm{d}\Omega \leq \|f\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \cdot \|v\|_{H(\Omega)}. \tag{2.33}$$

Continuity constant C_3 of linear form L is thus equal to $\|f\|_{L^2(\Omega)}$.

Result: According to the Lax-Milgram theorem, only one function belongs to $H_0^1(\Omega)$ solution of the variational formulation (**VP**) defined by (2.21). In case the continuous problem (**CP**) defined by (2.1) is replaced by the Laplace-Neumann-Dirichlet problem then the boundary Γ of Ω is constituted of two complementary parts Γ_1 and Γ_2, respectively dedicated to the definition of the Dirichlet and the Neumann conditions.

In such a case, the continuous problem (**CP**) takes the following form:

$$(\mathbf{CP}) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial n} = \mathrm{g} & \text{on } \Gamma_2, \end{cases} \tag{2.34}$$

where it is assumed that f and g are two given functions respectively belonging to $L^2(\Omega)$ and to $L^2(\Gamma_2)$.

As a consequence, it is easily established that the new associated variational formulation is written as:

$$(\mathbf{VP})\begin{cases} \text{Find } u \in H^1_{\Gamma_1}(\Omega) \text{ solution to} \\ \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\Omega = \int_\Omega f \cdot \mathrm{d}\Omega + \int_{\Gamma_2} \mathrm{g} \cdot v \mathrm{d}\Omega, \qquad \forall v \in H^1_{\Gamma_1}(\Omega), \end{cases} \tag{2.35}$$

where Sobolev $H^1_{\Gamma_1}(\Omega)$ space is defined by:

$$H^1_{\Gamma_1}(\Omega) \equiv \{\mathrm{v}: \Omega \to \boldsymbol{R}, \mathrm{v} \in L^2(\Omega), \nabla v \in [L^2(\Omega)]^2, v = 0 \ \text{ on } \ \Gamma_1\}. \tag{2.36}$$

In fact, in this case, the action of form L on any function v belonging to $H^1_{\Gamma_1}(\Omega)$ is expressed as:

$$L(v) \equiv \int_\Omega f \cdot v \, \mathrm{d}\Omega + \int_{\Gamma_2} \mathrm{g} \cdot v \, \mathrm{d}\Gamma, \quad \forall v \in H^1_{\Gamma_1}(\Omega). \tag{2.37}$$

The control of $L(v)$ is then carried out using:

$$L(v) \leq \int_\Omega |f \cdot v| \, \mathrm{d}\Omega + \int_{\Gamma_2} |g \cdot v| \, \mathrm{d}\Gamma, \tag{2.38}$$

$$\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|\mathrm{g}\|_{L^2(\Gamma_2)} \|v\|_{L^2(\Gamma_2)}. \tag{2.39}$$

Thus, a new difficulty results from the application of the g Neumann condition defined on the Γ_2 boundary.

Since, control of $L(v)$ should be performed only in relation to norm $H^1(\Omega)$ of function v. This is why the term resulting from the Neumann condition and providing a measure of v for norm $L^2(\Gamma_2)$ should consequently be modified.

The trace theorem (11) mentioned above is the one that would enable a control over $L(v)$ in relation to the only measure of function v for norm $H^1(\Omega)$. It is to be noted that C_4 is the continuity constant of the trace application γ defined by (2.16).

Then, inequality (2.39) may be modified as follows:

$$|L(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C_4 \|\mathrm{g}\|_{L^2(\Gamma_2)} \|v\|_{H^1(\Omega)}. \tag{2.40}$$

$$\leq C_5 \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1_{\Gamma_1}(\Omega), \tag{2.41}$$

where it would have been set that: $C_5\|f\|_{L^2(\Omega)} + C_4\|g\|_{L^2(\Gamma_2)}$.

These are the essential points that needed to be specified for extending the Laplace-Dirichlet problem to that of Laplace-Neumann-Dirichlet.

Other minor modifications, that do not represent any major difficulties, concern the adaptation of the results while shifting the functional framework of $H^1_0(\Omega)$ to that of $H^1_{\Gamma_1}(\Omega)$.

This is why, once the point about the control of linear form $L(.)$ defined by (2.37) is made, the application of Lax-Milgram theorem guarantees the existence and uniqueness of solution $u \in H^1_{\Gamma_1}(\Omega)$ to the variational problem (**VP**) defined by (2.35).

Equivalence Between Strong and Weak Formulations:

An additional point concerning the equivalence between different formulations needs to be mentioned within the whole transformation process that has been presented above.

More precisely, it is not obvious to declare that any solution to variational problem (**VP**) (2.11) is a solution to continuous problem (**CP**) (2.1).

The subtleties of the concept of equivalence between the two formulations may be tested by continuing to assume that the second member f is a function belonging to $L^2(\Omega)$ and it is then only necessary to believe that if the solution to the continuous problem is searched for in the Sobolev space $H^2(\Omega)$, then that of the variational problem (**VP**) is searched for in $H^1(\Omega)$ and $H^2(\Omega) \subset H^1(\Omega)$.

In other words, any solution to the continuous problem may be a solution to a variational problem with regards to its regularity, whereas, a priori, there is no justification for a solution to a variational problem (**VP**) to be the solution to a continuous problem (**CP**).

In fact, the concept of equivalence between the two formulations is completely dependent on the functional frameworks governing the respective areas of research for solutions to a continuous problem ($\mathbf{CP}$) on one hand and to a variational problem ($\mathbf{VP}$) on the other hand.

Chapter Three

Some Fundamental Classes of Finite Elements

3.1 Variational Formulation and Approximations:

This chapter is dedicated to the approximation of variational formulations and to different choices generated by the finite elements method.

The whole process leads to the estimation of an approximate solution $\tilde{u}$ for a variational formulation **(VP)** as well as for a continuous problem **(CP)**, both of which produce that form.

The shift from variational problem **(VP)** to the approximate variational problem $(\widetilde{\boldsymbol{VP}})$ is performed by substituting the pair of functions (u, v) belonging to $V \times V$ by their approximations $(\tilde{u}, \tilde{v})$ belonging to $\tilde{V} \times \tilde{V}$.

Thus $(\widetilde{\boldsymbol{VP}})$ is written as:

$$(\widetilde{\mathbf{VP}})\begin{cases} \text{Find } \tilde{u} \in \tilde{V} \text{ solution of:} \\ \displaystyle\int_{\Omega} \nabla\tilde{u} \cdot \nabla\tilde{v} d\Omega = \int_{\Omega} f \cdot \tilde{v} d\Omega, \quad \forall \tilde{v} \in \tilde{V} \end{cases} \tag{3.1}$$

Care should be taken to avoid the misleading simplicity of the approximation process since the approximate variational formulation $(\widetilde{\boldsymbol{VP}})$ is not simply a writing composition in relation to formulation **(VP)**.

On the contrary, it is a real progress in the capacity to resolve variational problem **(VP)** by approximation but it also relates to a loss of information that should be estimated subsequently.

In order to really appreciate the critical progress that this represents in terms of resolution, we introduce a basis $(\varphi_i)_{i=1,K_h}$ of the approximation space $\tilde{V}$ we consider which a finite dimensional vector space.

In this case, unknown $\tilde{u}$ may be broken down on the basis of functions φ_i as below:

$$\tilde{u} = \sum_{j=1}^{K_h} \tilde{u}_j \varphi_j. \tag{3.2}$$

In other words, since equation (3.1) is true, $\forall \tilde{v} \in \tilde{V}$, each basis functions $\varphi_i, (i = 1, K_h)$, may be chosen from among the approximate test functions $\tilde{v}$ and this leads to state: $\tilde{v} = \varphi_i$.

The approximate variational equation (3.1) is then written as:

$$(\widetilde{\mathbf{VP}}) \begin{cases} \text{Find } \tilde{u}_j, (j = 1, K_h) \text{ solution of:} \\ \sum_{j=1}^{K_h} \left(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j d\Omega \right) \tilde{u}_j = \int_\Omega f \cdot \varphi_i d\Omega, \quad \forall i = 1, K_h. \end{cases} \tag{3.3}$$

Then the following is stated:

$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j d\Omega \quad \text{and} \quad B_i = \int_\Omega f \cdot \varphi_i d\Omega. \tag{3.4}$$

Then, approximate variational problem $(\widetilde{\mathbf{VP}})$ is stated as:

$$(\widetilde{\mathbf{VP}}) \begin{cases} \text{Find } \tilde{u}_j, (j = 1, K_h), \text{ solution of:} \\ \sum_{j=1}^{K_h} A_{ij}\, \tilde{u}_j = B_i, \quad \forall i = 1, K_h. \end{cases} \tag{3.5}$$

This last form clearly shows the reduction that occurs as a result of the approximation process, by starting with variational problem **(VP)**, to give a problem having a finite dimension and whose resolution consists of a linear system of K_h equations with K_h unknowns.

As seen previously (see 3.3), the Galerkin method is used to associate the variational formulation with the Laplace-Dirichlet problem and to give an approximate formulation $(\widetilde{\mathbf{VP}})$ that is only a linear system needing to be reversed.

The resolution of this linear system offers approximation $\tilde{u}$ of the solution to variational problem **(VP)** and consequently an approximation of the solution to continuous problem **(CP)**.

In fact, many mathematical models for engineering sciences lead to a formalism similar to the one demonstrated for the Laplace-Dirichlet problem.

A generic family of variational problems **(VP)** describing this formalism can be abstractly expressed in the form of:

$$\textbf{(VP)}\ \text{Find } u \text{ belonging to } V \text{ solution of: } a(u,v) = L(v), \quad \forall v \in V, \qquad (3.6)$$

where:

- V is a vector space of functions,
- $a(.,.)$ is a bilinear form on $V \times V$,
- $L(.)$ is a linear form on V.

The approximation of the variational formulation (3.6) is essential in the wake of these conditions that "lock" the continuous and variational problems.

To this end, Galerkin suggests a method that consists in considering a subspace $\tilde{V}, (\tilde{V} \subset V)$, of finite dimension K_h that enables overcoming this incapacity of resolution of formulations having the same structure as formulation (3.6).

In this case, the abstract variational formulation (3.6) is transformed into the following approximation $(\widetilde{\mathbf{VP}})$:

Find $\tilde{u}$ belonging to $\tilde{V}$ solution of:

$$a(\tilde{u},\tilde{v}) = L(\tilde{v}), \quad \forall \tilde{v} \in \tilde{V}. \qquad (3.7)$$

Since approximation space $\tilde{V}$ of finite dimension K_h has been introduced, it is now natural, to consider a base of functions $\varphi_i, (i = 1, K_h)$, and to look for approximation $\tilde{u}$ that replaces solution u belonging to V in the form:

$$\tilde{u} = \sum_{j=1,K_h} \tilde{u}_j \varphi_j. \qquad (3.8)$$

In the approximate variational formulation $(\widetilde{\mathbf{VP}})$ defined by (3.7), the specific choice of functions $\tilde{v}$ equal to basis functions φ_i ($i = 1$ to K_h), now allows rewriting formulation (3.7) as follows:

$$(\widetilde{\mathbf{VP}}) \left[\begin{array}{l} \text{Find } \tilde{u} = {}^t[\tilde{u}_1, \dots, \tilde{u}_{K_h}] \text{ belonging to } \tilde{V} \text{ solution of:} \\ a\left(\sum_{j=1,K_h} \tilde{u}_j \varphi_j, \varphi_i \right) = L(\varphi_i), \quad \forall i = 1 \ \text{ to } \ K_h. \end{array} \right. \tag{3.9}$$

The bilinear properties of form $a(.,.)$ and of the linear properties of form $L(.)$ are thus applied.

Therefore, the variational formulation $(\widetilde{\mathbf{VP}})$ is expressed in the form of:

$$(\widetilde{\mathbf{VP}}) \left[\begin{array}{l} \text{Find } \tilde{u} = {}^t[\tilde{u}_1, \dots, \tilde{u}_{K_h}] \text{ belonging to } \tilde{V} \text{ solution of:} \\ \sum_{j=1,K_h} a(\varphi_j, \varphi_i)\, \tilde{u}_j = L(\varphi_i), \quad \forall i = 1 \ \text{ to } \ K_h. \end{array} \right. \tag{3.10}$$

The A_{ij} and b_i quantities are finally introduced and defined by:

$$A_{ij} = a(\varphi_j, \varphi_i), \quad b_i = L(\varphi_i). \tag{3.11}$$

From there, approximate variational formulation $(\widetilde{\mathbf{VP}})$ takes its following final form:

$$(\widetilde{\mathbf{VP}}) \left[\begin{array}{l} \text{Find } \tilde{u} = {}^t[\tilde{u}_1, \dots, \tilde{u}_{K_h}] \text{ belonging to } \tilde{V} \text{ solution of:} \\ \qquad \sum_{j=1,K_h} A_{ij} \tilde{u}_j = b_i, \quad \forall i = 1 \ \text{ to } \ K_h. \end{array} \right. \tag{3.12}$$

As from then, it is noted that formulation (3.12) is none other than a linear system composed of matrix A of generic elements A_{ij} and of a second member b having component b_i.

Thus, it is established that any variational formulation **(VP)** expressed in form (3.6) and having form $a(.,.)$ and $L(.)$ are respectively bilinear and linear and can be solved by an approximation whose solution is equivalent to a linear system (3.12).

Of what nature is the data that can be defined within parameters and that is required to determine an effective solution to linear system (3.12) and consequently an approximation to variational formulation (3.6)?

Calculation of coefficients A_{ij} and that of second member b_i requires knowing basis functions φ_i, ($i = 1$ to K_h), of approximation space $\tilde{V}$.

Of course, this knowledge closely depends on the definition of space $\tilde{V}$ whose dimension K_h is finite.

From then on, the introduction of an elementary geometry G_m, ($m = 1$ to M), is relevant since it generates a mesh of integration domain Ω, de facto providing the nodes of the geometrical discretisation (See Fig. 3.1).

These concepts gave rise to the Lagrange finite elements defined by triplet $\left(G, \sum, P(G)\right)$ where:

- G defines the geometry of the elementary mesh (segment, triangle),
- $\sum = (M_1, \dots, M_{K'}), (K' < K)$ denotes the group of nodes delimiting the elementary mesh G,
- $P(G)$ is the approximation space consisting of polynomials defined over G.

Finally, $(G, \sum, P)$ triplet has to satisfy the unisolvence property defined as follows:

$$\begin{gathered} \forall(\xi_1, \dots, \xi_{K'}) \in R^{K'}, \exists!\, p \in P(G) \\ \text{such that:} P(\xi_k) = \xi_k, \;\; \forall k = 1 \;\text{ to }\; K'. \end{gathered} \tag{3.13}$$

In other words, there exists only one function p belonging to $P(G)$ going through the given K' values $(\xi_1, \dots, \xi_{K'})$ at K' nodes delimiting elementary mesh G.

When the definition of functions belonging to $P(G)$ defined by a generating mesh G is known, the process of construction of approximation spaces $\tilde{V}$ within the framework of the Lagrange finite elements consists in stating:

$$\tilde{V} \equiv \{\tilde{v}: \Omega \to \boldsymbol{R}, \tilde{v} \in C^0(\Omega), \tilde{v}|_G \in P(G)\}, \tag{3.14}$$

where the boundary conditions to which functions $\tilde{v}$ of $\tilde{V}$ may be subjected, depending on the problem considered, are disregarded.

Consequently, when disregarding boundary conditions that vary from one problem to another, the dimension of space $\tilde{V}$ defined by (3.14) is inferred from dimension K' of $P(G)$, from the number of mesh and from the number of nodes intervening in the geometrical discretisation of the integration domain Ω.

The generalisation of Lagrange finite elements is then done as follows:

Triplet $(G\sum,,P(G))$ defines a finite element with:

An elementary mesh of geometrical discretisation G of $\boldsymbol{R}^n, (n = 1{,}2\ or\ 3)$.

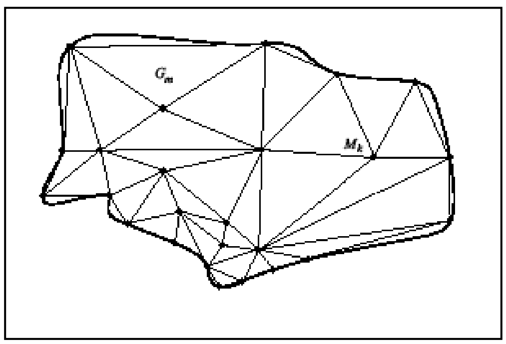

Fig. 3.1 Example of Mesh by Triangular Finite Elements

- A set $\sum$ of degrees of freedom σ_k, ($k = 1$ to K'), consisting of linear forms on the space of defined functions on G.
- A vector space $P(G)$ of finite dimension equivalent to K'.
- The unisolvance property: For any K'-tuple of R^K having real numbers, a unique element p exists and belongs to $P(G)$ such that:

$$\sigma_k(p) = \xi_k, \quad \forall k = 1 \text{ to } K'.$$

Convergence of the Finite Elements Method:

As shown in paragraph 1.2.6 of Chap. 1, the significance of the different approximation levels resulting from the process of the modelling cascade of a numerical approximation, should encourage the numerical analyst to make a lucid and humble use of measuring tools which are necessary for the estimation of an approximation error in relation to the relevant numerical method.

In this case, the finite elements method offers a body of theoretical results which facilitates the estimation of the approximation error between solution u of variational problem **(VP)** and its approximation u. which is the solution to the approximate variational problem $\left(\widetilde{\mathbf{VP}}\right)$.

Considering the nature of the mathematical objects involved (functions u and $\tilde{u}$), a family of results are proposed in this paragraph to enable an estimation of the distance between solution u and its approximation ($\|u - \tilde{u}\|$), according to an ad hoc norm.

The family of variational problems **(VP)** having the abstract form below is used as an aid to the demonstration:

$$\text{Find } u \in V \text{ solution of: } a(u, v) = L(v),\ \forall v \in V\ . \tag{3.15}$$

As in the previous paragraph, $\tilde{V}$ denotes the approximation space internal to V with finite dimension ($\tilde{V} \subset V$) having the generic element noted as $\tilde{v}$ and the approximate solution $\tilde{u}$ of solution u being a particular case approximation functions $\tilde{v}$ belonging to $\tilde{V}$.

In other words, approximate formulation $\left(\widetilde{\mathbf{VP}}\right)$ of variational problem **(VP)** is expressed as:

$$\text{Find } \tilde{u} \in V \text{ solution of: } a(\tilde{u}, \tilde{v}) = L(v),\ \forall \tilde{v} \in \tilde{V}\ . \tag{3.16}$$

Given the assumptions of the Lax-Milgram theorem 10 (See Chap. 1) associated with a Hilbertian norm noted as $\|\cdot\|$, the following lemma is obtained:

Lemma (4):

Variational Problem $\left(\widetilde{\mathbf{VP}}\right)$ (3.16) only admits one and only one solution $\tilde{u}$. Moreover, this solution satisfies the orthogonality relationship:

$$a(u - \tilde{u}, \tilde{v}) = 0, \quad \forall \tilde{v} \in \tilde{V}. \tag{3.17}$$

Proof:

The existence and uniqueness of solution $\tilde{u}$ belonging to $\tilde{V}$ is immediate since the approximation is internal, $(\tilde{V} \subset V)$.

In fact, it is first noticed that the approximation space of finite dimension $\tilde{V}$ included in V is consequently a closed vector sub-space of V and thus presents a Hilbertian structure. The inclusion of $\tilde{V}$ in V allows the use of properties required for the application of a Lax-Milgram theorem to $\tilde{V}$.

As for the orthogonality relationship (2.12), the variational equation **(VP)** has to be expressed by substituting v by $\tilde{v}$:

$$a(u, \tilde{v}) = L(\tilde{v}), \qquad \forall \tilde{v} \in \tilde{V} \tag{3.18}$$

The difference between equs. (3.18) and (3.16) immediately leads to orthogonality relationship (3.17).

The first result of the estimation of approximation error $\|u - \tilde{u}\|$ is known as the Céa's lemma.

Lemma (5):

According to the hypothesis of Lax-Milgram theorem 10, if it is additionally assumed that approximation $\tilde{u}$ of exact solution u is internal, $(\tilde{V} \subset V)$, then the estimation of the error is obtained as:

$$\|u - \tilde{u}\| \leq C \inf_{\tilde{v} \in \tilde{V}} \|u - \tilde{v}\|. \tag{3.19}$$

Proof :

The demonstration of Céa's lemma is based on the double control of quantity $a(u - \tilde{u}, u - \tilde{u})$ by using the property of V-ellipticity and the continuity of bilinear form $a(.,.)$.

For a start, according to orthogonality relationship (3.17), the following result is obtained by choosing $\tilde{v} = \tilde{u}$:

$$a(u - \tilde{u}, \tilde{u}) = 0. \tag{3.20}$$

From then on, quantity $a(u-\tilde{u},u-\tilde{u})$ may be expressed as:

$$\forall\tilde{v}\in\tilde{V}: a(u-\tilde{u},u-\tilde{u}) = a(u-\tilde{u},u) - a(u-\tilde{u},\tilde{u}) = a(u-\tilde{u},u) \quad (3.21)$$

$$= a(u-\tilde{u},u) - a(u-\tilde{u},\tilde{v}) = a(u-\tilde{u},u-\tilde{v}). \quad (3.22)$$

Thus, by keeping the notations of the Lax-Milgram theorem 10, the following is obtained:

$$C_2\|u-\tilde{u}\|^2 \leq a(u-\tilde{u},u-\tilde{u}) = a(u-\tilde{u},u-\tilde{v}) \leq C_1\|u-\tilde{u}\|\cdot\|u-\tilde{v}\| \quad (3.23)$$

After simplification, the following is obtained:

$$\|u-\tilde{u}\| \leq \frac{C_1}{C_2}\|u-\tilde{v}\|, \quad \forall\tilde{v}\in\tilde{V}. \quad (3.24)$$

and expected constant C is none other than the ratio between C_1 and C_2.

The use of inequality control (3.24) is even more convincing considering that the upper bound of norm $\|u-\tilde{v}\|$ is minimised.

This explains how the result of Céa's lemma brings out the lower bound of quantities $\|u-\tilde{v}\|$ for any function $\tilde{V}$ belonging to $\tilde{V}$.

The next step consists in the characterisation of approximation space .V to determine the estimation of the error produced by Céa's lemma.

As mentioned in paragraph 2.1, Lagrange finite elements offer a simple solution for the systematic production of approximation space $\tilde{V}$ of finite dimension.

This process is based on the unique determination of an approximation function by considering its values taken at a finite number of points $M_k,(k=1,K)$ situated on a given mesh of integration domain Ω.

This brings about the general introduction of the interpolation operator π_h defined as:

$$\pi_h: C^0(\overline{\Omega}) \to \tilde{V} \quad (3.25)$$

$$v \rightsquigarrow \pi_h v \equiv \sum_{k=1,K} v(M_k)\varphi_k,$$

where φ_k denotes the basis function of approximation space $\tilde{V}$ characteristic of node M_k and satisfying the following property:

$$\varphi_k(M_l) = \varphi_l \, , (\varphi_{kl} \text{ being the Krönecker symbol}). \qquad (3.26)$$

It is thus simple to verify that function $\pi_k v$, interpolated from v to K nodes M_k of the mesh of integration domain Ω is the unique function of $\tilde{V}$ proving:

$$\pi_h v(M_k) = v(M_k), \qquad \forall k = 1, K \, . \qquad (3.27)$$

It is thus licit to express the control inequality of Céa's lemma when specifically choosing $\tilde{v} = \pi_h u$:

$$\|u - \tilde{u}\| \leq C\|u - \tilde{v}\| = C\|u - \pi_h u\|. \qquad (3.28)$$

Therefore, according to inequality control (3.28), the approximation error and the interpolation error are of the same order of magnitude.

This is why it suffices to estimate the interpolation error as a tool to measure the approximation error according to the nature and property of each Lagrange finite element.

The Bramble-Hilbert lemma is then introduced since it relies on these considerations to render the application of Céa's lemma fully operational.

In this present work, the demonstration is limited to the terms of the lemma having straight and unflattened finite elements and variational space V is considered to correspond to Sobolev space $H^1(\Omega)$.

Indeed, numerous problems arising from engineering sciences correspond to this functional framework (probably more regularly), knowing that, in any event, certain applications that may not fit in this framework would require mathematical techniques coming from a functional analysis that is way more than what this book can handle.

Lemma (6):

Let h be the size of the elementary mesh of a Lagrange finite element. If approximation space $\tilde{V}$ contains the space of polynomials P_h having a degree less than or equal to k, in relation the pair of variables (x, y), then, for a finer discretisation and for any "sufficiently regular" solution u (at least in $H^1(\Omega)$) to variational problem **(VP)** of form (3.15), the following is obtained:

$$\|u - \pi_h u\|_{H^1(\Omega)} = O(h^k) \quad \text{and} \quad \|u - \tilde{u}\|_{H^1(\Omega)} = O(h^k). \tag{3.29}$$

Evidently, all the technicalities of the result of this lemma rest upon the estimation of the norm measuring the gap between solution u and its interpolation $\pi_h u$.

The preamble exposed using Céa's lemma was actually meant to underline the necessity of estimating this last norm in order to conclude on the approximation error of the finite elements method, at least in the context previously described.

3.2 Description of Ordinary Finite Elements:

This section is dedicated to the introduction of the principal finite elements most commonly used in applications of engineering sciences.

Each finite element is described in a systematic manner according to the following model:

1. The definition of elementary geometric mesh G,
2. The definition and dimension of approximation space $P(G)$
3. The definition of all linear forms σ_i on the space of functions defined on G.
4. The determination of the functions of the canonical basis of space $P(G)$, i.e. functions $\left(p_1, \dots, p_{\dim P(G)}\right)$ satisfying: $G_i\left(p_j\right) = \delta_{ij}$, where δ_{ij} denotes the Kronecker symbol.

3.2.1 **Finite Elements with a Space Variable:**

The finite elements presented are described by an elementary mesh consisting of the interval $G \equiv [0,1]$.

Finite P_0 Element:

1) Space $P(G) \equiv P_0$ is constituted by polynomials p being defined as a constant on interval [0,1].

The dimension of P_0 is equal to 1.

2) Linear form σ is considered and is defined by:

$$\sigma: p \rightarrow \int_0^1 p(x)dx. \tag{3.30}$$

3) The unique function of the canonical basis for this element is the constant function equal to 1 on interval [0,1].

To be sure, the definition of the function of the canonical basis is expressed according to the following agreed definition:

$$(\sigma(p) = 1) \Leftrightarrow \left(\int_0^1 p(x)dx = 1\right), \quad \text{where } p(x) = C^{te}, \ \ \forall x \in [0,1]. \tag{3.31}$$

It is immediately deduced that

$$p(x) = 1, \qquad \forall x \in [0,1].$$

Functions $\tilde{v}$ belonging to $\tilde{V}$ are constant functions for each elementary mesh for this first finite element.

It would be noted that the constant on each mesh element corresponds to the mean value of function $\tilde{v}$ on the corresponding mesh.

Finite P_1 Element:

1) Approximation space $P(G) \equiv P_1$ consists of affine functions defined on elementary mesh [0,1].

The dimension of space P_1 is equal to 2.

2) Both linear forms being considered are defined by:

$$\sigma_1: p \rightarrow p(0), \quad \sigma_2: p \rightarrow p(1). \tag{3.32}$$

3) In order to determine the functions of the canonical basis of space P_1 the property of both basis functions (p_1, p_2) are expressed as:

$$\begin{aligned}\sigma_1(p_1) = 1 \;\Leftrightarrow\; p_1(0) = 1, \quad \sigma_1(p_2) = 0 \;\Leftrightarrow\; p_2(0) = 0,\\ \sigma_2(p_1) = 0 \;\Leftrightarrow\; p_1(1) = 0, \quad \sigma_2(p_2) = 1 \;\Leftrightarrow\; p_2(1) = 1.\end{aligned} \tag{3.33}$$

It is then easily inferred that basis functions (p_1, p_2), solutions to (3.33) belonging to space P_1, consisting of defined affine functions on interval [0,1], correspond to:

$$p_1(x) = 1 - x, p_2(x) = x. \tag{3.34}$$

Finite P_2 Elements:

1) Approximation space $P(G) \equiv P_2$ consists of polynomials having degrees less than or equal to two and defined on elementary mesh [0,1].

The dimension of P_2 is equal to 3.

2) The three linear forms defined below are considered:

$$\sigma_1 : p \to p(0), \quad \sigma_2 : p \to p\left(\frac{1}{2}\right), \quad \sigma_3 : p \to p(1). \tag{3.35}$$

3) Now the properties of functions (p_1, p_2, p_3) of the canonical basis belonging to p_2 are expressed:

$$\begin{cases}\sigma_1(p_1) = 1 \Leftrightarrow p_1(0) = 1, & \sigma_1(p_2) = 0 \Leftrightarrow p_2(0) = 0,\\ \sigma_1(p_3) = 0 \Leftrightarrow p_3(0) = 0, & \sigma_2(p_1) = 0 \Leftrightarrow p_1\left(\frac{1}{2}\right) = 0,\\ \sigma_2(p_2) = 1 \Leftrightarrow p_2\left(\frac{1}{2}\right) = 1, & \sigma_2(p_3) = 0 \Leftrightarrow p_3\left(\frac{1}{2}\right) = 0,\\ \sigma_3(p_1) = 0 \Leftrightarrow p_1(1) = 0, & \sigma_3(p_2) = 0 \Leftrightarrow p_2(1) = 0,\\ \sigma_3(p_3) = 1 \Leftrightarrow p_3(1) = 1. & \end{cases} \tag{3.36}$$

Then, use is made of the fact that each polynomial p_i of degree less than or equal to two is in the form of: $ax^2 + bx + c$.

The 9 coefficients of the 3 polynomials (p_1, p_2, p_3) are obtained from the 9 relationships (3.36).

The following is then obtained:

$$p_1(x) = (2x-1)(x-1), \;\; p_2(x) = 4x(1-x), \;\; p_3(x) = x(2x-1). \quad (3.37)$$

Hermite's Finite Element:

1) Approximation space $P(G) \equiv P_3$ consists of polynomials having degrees less than or equal to three and defined on elementary mesh [0,1].

The dimension of P_3 is equal to 4.

2) Let the four linear forms be defined by:

$$\sigma_1: p \to p(0), \quad \sigma_2: p \to \frac{\mathrm{d}p}{dx}(0), \quad \sigma_3: p \to p(1), \quad \sigma_4: p \to \frac{\mathrm{d}p}{dx}(1). \qquad (3.38)$$

3) The four functions (p_1, p_2, p_3, p_4) of canonical basis P_3 are determined. This result is achieved by expressing the sixteen relationships of the form $\sigma_i(p_j) = \delta_{ij}$:

$$\begin{cases} \sigma_1(p_1) = 1 \Leftrightarrow p_1(0) = 1, & \sigma_1(p_2) = 0 \Leftrightarrow p_2(0) = 0, \\ \sigma_1(p_3) = 0 \Leftrightarrow p_3(0) = 0, & \sigma_1(p_4) = 0 \Leftrightarrow p_4(0) = 0, \\ \sigma_2(p_1) = 0 \Leftrightarrow p_1'(0) = 0, & \sigma_2(p_2) = 1 \Leftrightarrow p_2'(0) = 1, \\ \sigma_2(p_3) = 0 \Leftrightarrow p_3'(0) = 0, & \sigma_2(p_4) = 0 \Leftrightarrow p_4'(0) = 0, \\ \sigma_3(p_1) = 0 \Leftrightarrow p_1(1) = 0, & \sigma_3(p_2) = 0 \Leftrightarrow p_2(1) = 0, \\ \sigma_3(p_3) = 1 \Leftrightarrow p_3(1) = 1, & \sigma_3(p_4) = 0 \Leftrightarrow p_4(1) = 0, \\ \sigma_4(p_1) = 0 \Leftrightarrow p_1'(1) = 0, & \sigma_4(p_2) = 0 \Leftrightarrow p_2'(1) = 0, \\ \sigma_4(p_3) = 0 \Leftrightarrow p_3'(1) = 0, & \sigma_4(p_4) = 1 \Leftrightarrow p_4'(1) = 1. \end{cases} \qquad (3.39)$$

The sixteen relationships (3.39) again produce the sixteen coefficients of the four polynomials (p_1, p_2, p_3, p_4) of the canonical basis of P_3.

After a few calculations, the following is obtained:

$$\begin{array}{ll} p_1(x) = (x-1)^2(2x+1), & p_2(x) = x(x-1)^2, \\ p_3(x) = x^2(3-2x), & p_4(x) = (x-1)x^2. \end{array} \qquad (3.40)$$

3.2.2 Finite Elements with Two Space Variables:

Triangular Finite Elements:

This sub-section is dedicated to the introduction of finite elements whose elementary mesh G is a triangle with vertices M_1, M_2 and M_3 on plan $(O; x, y)$ (See Fig. 3.2).

Finite P_0 Element:

1) Approximation space $P(G) \equiv P_0$ consists of constant functions on triangle G.

The dimension of P_0 is equal to 1.

2) Linear form σ defined below is considered:

$$p \rightarrow \frac{1}{\text{Area}(G)} \iint_G P(x, y) \mathrm{d}x \mathrm{d}y = 1. \qquad (3.41)$$

3) Basis function p of P_0 satisfying property $\sigma(p) = 1$ is determined:

$$(\sigma(p) = 1) \Leftrightarrow \left(\frac{1}{\text{Area}(G)} \iint_G P(x, y) \mathrm{d}x \mathrm{d}y = 1 \right),$$

where:

$$p(x, y) = C^{te}, \quad \forall (x, y) \in G. \qquad (3.42)$$

Then, the function of canonical basis p is the constant function equal to 1 on the whole of triangle G.

Finite P_1 Element:

1) Approximation space P_1 consists of polynomial functions having degrees less than or equal to one for the pair of variables (x, y).

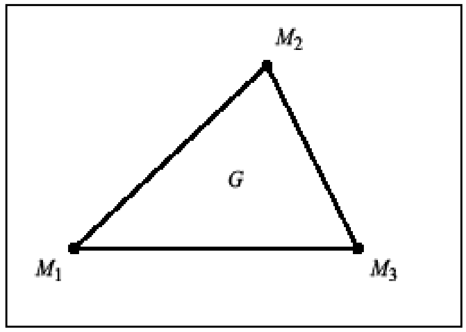

Fig. 3.2 Triangular Elementary Mesh

In other words, any function p of P_1 is expressed in the form:

$$p(x, y) = ax + by + c, \tag{3.43}$$

where (a, b, c) is a triplet of $\boldsymbol{R}^3$.

The previous definition leads to the conclusion that dimension P_1 is equal to 3.

2) The three linear forms defined below are considered:

$$\sigma_1: p \to p(M_1), \quad \sigma_2: p \to p(M_2), \quad \sigma_3: p \to p(M_3) \tag{3.44}$$

3) The identification of the three functions of canonical basis (p_1, p_2, p_3) corresponds to the three barycentric functions $(\lambda_1, \lambda_2, \lambda_3)$ whose existence is established in the work of Daniel Euvrard [5].

Though, it is pointed out that, by definition, the polynomial functions of degree less than or equal to one for the pair (x, y) prove the canonical property:

Finite P_2 Element:

1) Approximation space $P(G) \equiv P_2$ consists of polynomial functions having degrees less or equal to 2 for the pair of variables (x, y).

In other words, any function p of P_2 is written in the form:

$$p(x, y) = ax^2 + by^2 + cxy + dx + ey + f\ , \tag{3.45}$$

where $(a, b, c, d, e, f\ )$ is of any value and belongs to $\boldsymbol{R}^6$.

The previous definition (3.46) leads to the conclusion that the dimension of P_2 is equal to 6.

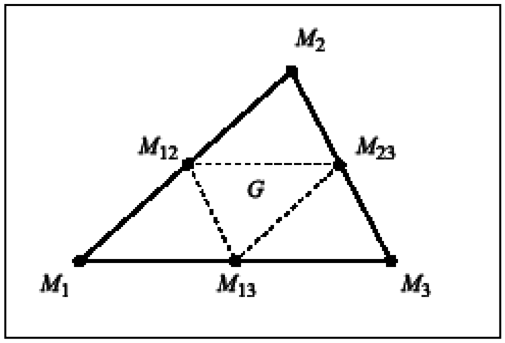

Fig. 3.3 Triangular Mesh for a Finite P_2 Element

2) To define the six linear forms $\sigma_i, (i = 1 \text{ to } 6)$, three new nodes (M_1, M_2, M_3, M_4) are introduced and placed in the middle of each side of triangle G (See Fig. 3.3).

It is then possible to introduce the six linear forms defined by:

$$\sigma_1 : p \to p(M_1), \quad \sigma_2 : p \to p(M_2), \tag{3.47}$$
$$\sigma_3 : p \to p(M_3), \quad \sigma_4 : p \to p(M_{12}), \tag{3.48}$$
$$\sigma_5 : p \to p(M_{13}), \quad \sigma_6 : p \to p(M_{23}). \tag{3.49}$$

3) The construction of the functions of canonical basis $(p_1, p_2, p_3, p_4, p_5, p_6)$ is worked out in the following way:

Function p_1 may be taken as an example. This second degree polynomial in relation to the pair (x, y) must be zero at the points below: M_2, M_3, M_{12}, M_{13} and M_{23}.

Therefore, polynomial p_1, whose trace is a second degree trinomial of the oblique variable defining the parameter of segment M_2M_3, is identically zero over segment M_2M_3, being zero at the three points M_2, M_3 and M_{23}.

Moreover, as segment M_2M_3 is characterised by equation $\lambda_1 = 0$, it means that λ_1 can be factorised in the polynomial expression p_1.

In the same way, polynomial p_1 is zero at nodesM_{13} and M_{12}. Since the barycentric functions λ_i are affine in x and in y, in these two nodes, λ_1 is exactly equal to $1/2$ on segment $M_{13}M_{12}$.

In other words, by factorising p_1 by the quantity $\lambda_1 - \frac{1}{2}$, it is ensured that p_1 is really zero at nodes M_{13} and M_{12}.

Therefore, the polynomial structure of function p_1 is written as:

$$p_1(M) = \alpha\lambda_1(M)\left(\lambda_i(M) - \frac{1}{2}\right), \tag{3.50}$$

where α is a constant which must be determined so that polynomial p_1 may equal 1 at its characteristic node, namely at node M_1.

Moreover, it can be noted that expression (3.50) indeed confers a second degree polynomial structure in relation to the pair of variables (x, y) to function p_1 because polynomial λ_1 is of the first degree in relation to the pair (x, y).

The following is then written as:

$$p_1(M_1) \equiv \alpha\lambda_1(M_1)\left(\lambda_i(M) - \frac{1}{2}\right) = \alpha \times \frac{1}{2}. \tag{3.51}$$

To ensure the property $p_1(M_1) = 1$, the value of coefficient α can then be inferred therefrom: $\alpha = 2$.

Polynomial p_1 is finally written as:

$$p_1(M_1) = \lambda_1(M)(2\lambda_1(M) - 1). \tag{3.52}$$

The other polynomials of the canonical basis may be inferred by the same method and the following is obtained:

$$p_1(M) = \lambda_1(M)(2\lambda_1(M) - 1), \quad p_2(M) = \lambda_2(M)(2\lambda_2(M) - 1), \tag{3.53}$$
$$p_3(M) = \lambda_3(M)(2\lambda_3(M) - 1), \quad p_{12}(M) = 4\lambda_1\lambda_2(M), \tag{3.54}$$
$$p_{13}(M) = 4\lambda_1\lambda_3(M), \quad p_{23}(M) = 4\lambda_2\lambda_3(M). \tag{3.55}$$

Finite P_3 Elements:

1) The approximation space $P(G)P_3$ consists of polynomial functions of degree less or equal to three for the pair of variables (x, y).

In other words, any function p of P_3 is written in the form:

$p(x, y) = ax^3 + by^3 + cx^2y + dxy^2 + ex^2 + f\ y^2 + gxy + hx + iy + j,$ (3.56)

where $(a, b, c, d, e, f\ , g, h, i, j)$ is of any value and belongs to $\boldsymbol{R}^{10}$.

The previous definition (3.56) leads to the conclusion that the dimension of P_3 is equal to 10.

2) To define the ten linear forms $\sigma_i, (i = 1 \text{ to } 10)$, seven new nodes $(M_{112}, M_{112}, M_{113}, M_{113}, M_{223}, M_{233}, M_{123})$ are introduced and placed in the third part of each side of triangle G, (See Fig. 3.4).

It is then possible to introduce the ten linear forms by using:

$$\sigma_1 : p \to p(M_1), \qquad \sigma_2 : p \to p(M_2), \tag{3.57}$$
$$\sigma_3 : p \to p(M_3), \qquad \sigma_4 : p \to p(M_{122}), \tag{3.58}$$
$$\sigma_5 : p \to p(M_{122}), \quad \sigma_6 : p \to p(M_{233}), \tag{3.59}$$
$$\sigma_7 : p \to p(M_{233}), \quad \sigma_8 : p \to p(M_{113}), \tag{3.60}$$
$$\sigma_9 : p \to p(M_{133}), \quad \sigma_{10} : p \to p(M_{123}). \tag{3.61}$$

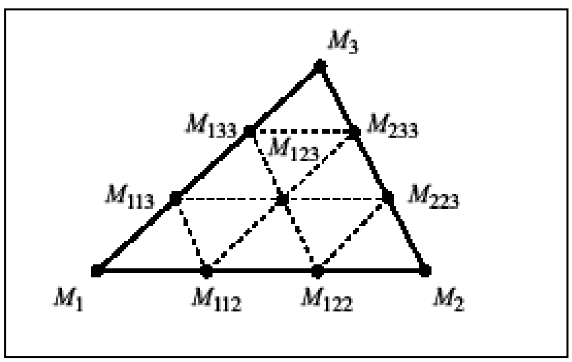

Fig. 3.4 Triangular Mesh for Finite P_3 Element

3) The determination of the ten functions of canonical basis $(p_1, (i = 1\ to\ 10))$ is worked out by using the same principles presented for the triangular finite $\boldsymbol{P_2}$ elements.

The case of polynomial p_1, which is characteristic of the node, M_1 can be studied again, since it satisfies the property: $p_1(M_1) = 1$.

Polynomial p_1 being zero at the other nine nodes, the following factorizations may be inferred:

- λ_1 is factorised in the expression of p_1, since this polynomial must be zero at nodes $(M_2, M_3, M_{223}, M_{233})$.
- $(\lambda_1 - 2/3)$ is factorised in the expression of p_1, since this polynomial must be zero at nodes (M_{112}, M_{113}).
- $(\lambda_1 - 1/3)$ is factorised in the expression of p_1, since this polynomial must be zero at nodes $(M_{122}, M_{133}, M_{123})$.

Therefore, polynomial p_1 takes the following form:

$$p_1(M) = \alpha\lambda_1(M)\left(\lambda_1(M) - \frac{1}{3}\right)\left(\lambda_1(M) - \frac{2}{3}\right), \tag{3.62}$$

where, once again, constant ¥á must be adjusted so that polynomial p_1 is equal to 1 at node M_1.

Moreover, it would be noted that the shape of polynomial p_1 (3.62) is coherent with that of definition (3.56) of the functions belonging to P_3 in accordance with the fact that the barycentric function λ_1 is a first degree polynomial in relation to the pair of variables (x, y).

The following is then easily obtained: $\alpha = \frac{9}{2}$ and the final shape of polynomial p_1 is thus:

$$p_1(M) = \frac{9}{2}\lambda_1(M)\left(\lambda_1(M) - \frac{1}{3}\right)\left(\lambda_1(M) - \frac{2}{3}\right). \tag{3.63}$$

Polynomials p_2 and p_3 are immediately inferred from the expression of polynomial p_1, for reasons of obvious symmetry:

$$p_2(M) = \frac{9}{2}\lambda_2(M)\left(\lambda_2(M) - \frac{1}{3}\right)\left(\lambda_2(M) - \frac{2}{3}\right). \tag{3.64}$$

$$p_3(M) = \frac{9}{2}\lambda_3(M)\left(\lambda_3(M) - \frac{1}{3}\right)\left(\lambda_3(M) - \frac{2}{3}\right). \tag{3.65}$$

Polynomial p_{112} is now studied. This polynomial presents the following factorisations:

- λ_1 is factorised in the expression of p_{112}, since this polynomial must be zero at nodes $(M_2, M_3, M_{223}, M_{233})$.
- λ_2 is factorised in the expression of p_{112}, since this polynomial must be zero at nodes $(M_1, M_3, M_{113}, M_{133})$.
- $(\lambda_1 - 1/3)$ is factorised in the expression of p_1, since this polynomial must be zero at nodes $(M_{122}, M_{133}, M_{123})$.

Hence, the structure of p_{112} is given by:

$$p_{112}(M) = \beta\lambda_1(M)\lambda_2(M)\left(\lambda_1(M) - \frac{1}{3}\right). \tag{3.66}$$

where the constant β must be adjusted so that the polynomial p_{112} is equal to one at node M_{112}.

By therefore writing that $\lambda_1 = 2/3$ and $\lambda_2 = 1/3$ at node M_{112}, the following is hence obtained:

$$\beta = \frac{27}{2}. \tag{3.67}$$

The basis function p_{112} is finally written as:

$$p_{112}(M) = \frac{27}{2}\lambda_1(M)\lambda_2(M)\left(\lambda_1(M) - \frac{1}{3}\right). \tag{3.68}$$

Once again, for reasons of symmetry, the other basis functions p_{ijk}, where (i, j, k) differs from triplet (1,2,3), are written as:

$$p_{112}(M) = \frac{27}{2}\lambda_1(M)\lambda_2(M)\left(\lambda_2(M) - \frac{1}{3}\right). \tag{3.69}$$

$$p_{113}(M) = \frac{27}{2}\lambda_1(M)\lambda_3(M)\left(\lambda_1(M) - \frac{1}{3}\right). \tag{3.70}$$

$$p_{133}(M) = \frac{27}{2}\lambda_1(M)\lambda_3(M)\left(\lambda_3(M) - \frac{1}{3}\right). \tag{3.71}$$

$$p_{233}(M) = \frac{27}{2}\lambda_2(M)\lambda_3(M)\left(\lambda_2(M) - \frac{1}{3}\right). \tag{3.72}$$

$$p_{233}(M) = \frac{27}{2}\lambda_1(M)\lambda_3(M)\left(\lambda_3(M) - \frac{1}{3}\right). \tag{3.73}$$

This study can be concluded by the analysis of the last polynomial function p_{123} of the canonical basis of P_3.

This polynomial presents the following factorisations:

- λ_1 is factorised in the expression of p_{123}, since this polynomial must be zero at nodes $(M_2, M_3, M_{223}, M_{233})$.
- λ_2 is factorised in the expression of p_{123}, since this polynomial must be zero at nodes $(M_1, M_3, M_{113}, M_{133})$.
- λ_3 is factorised in the expression of p_{123}, since this polynomial must be zero at nodes $(M_1, M_2, M_{112}, M_{122})$.

Hence, function p_{123} possesses the following polynomial structure:

$$p_{123}(M) = \gamma\lambda_1(M)\lambda_2(M)\lambda_3(M), \tag{3.74}$$

where constant γ is adjusted so that the polynomial p_{123} may satisfy its characteristic property at node M_{123}, namely: $p_{123}(123) = 1$.

Considering that the barycentric functions λ_1, λ_2 and λ_3 have, all three, the same value of $1/3$ at node M_{123}, the constant γ is then equal to:

$$\gamma = 27. \tag{3.75}$$

The polynomial p_{123} is finally written as:

$$p_{123}(M) = 27\lambda_1(M)\lambda_2(M)\lambda_3(M). \tag{3.76}$$

Chapter Four

Variational Formulations for a One-Dimensional Case

4.1 Dirichlet's Problem:

The aim of this problem is to propose a mathematical and numerical study of the solution to a linear differential equation subjected to Dirichlet boundary conditions.

Find $u \in H^2(0,1)$ being the solution to:

$$(\mathbf{CP})\begin{cases} -u''^{(x)} + u(x) = f(x), 0 \leq x \leq 1, \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

in which f is a given function belonging to $L^2(0,1)$.

Besides, it is pointed out that Sobolev's space $H^2(0,1)$ is defined as:

$$H^2(0,1) = \left\{ v:]0,1[\rightarrow \mathbf{R}, \frac{\mathrm{d}^k v}{\mathrm{d}x^k} \in L^2(0,1), \ \ \forall k = 0,1,2 \right\}. \tag{4.2}$$

Numerical Part-Lagrange Finite $\mathbf{P_1}$ Elements:

The approximation of the variational problem **(VP)** is worked out using the Lagrange finite elements $\mathbf{P_1}$.

This is performed by introducing a regular mesh of $[0,1]$ interval, of constant step h, such as:

$$\begin{cases} x_0 = 0, \ \ x_{N+1} = 1, \\ x_{i+1} = x_i + h, \ \ i = 0 \text{ to } N \end{cases} \tag{4.3}$$

The approximation space $\tilde{V}$ can now be defined as:

$$\tilde{V} = \left\{ \tilde{v} / \tilde{v} \in C^0([0,1]), \ \ \tilde{v}|_{[x_i, x_{i+1}]} \in P_1, \tilde{v}(0) = \tilde{v}(1) = 0 \right\}, \tag{4.4}$$

in which $P_1 \equiv P_1([x_i, x_{i+1}])$ refers to the polynomial space which is defined over $[x_i, x_{i+1}]$, having a degree less or equal to one.

Let $\varphi_i, (i = 1 \text{ to} \dim \tilde{V})$, be the canonical basis of $\tilde{V}$ establishing $\varphi_i(x_j) = \delta_{ij}$, in which δ_{ij} refers to the Kronecker symbol.

Solution:

Let v be a test function, defined on $[0,1]$ having real values and "sufficiently regular". Each time a variational formulation is needed, the regularity of the functions v will be specified a posteriori, so that the formulation is significant enough to be understood.

The differential equation of the continuous problem **(CP)** is then multiplied by v and is integrated along the interval $[0,1]$.

$$-\int_0^1 u''v dx + \int_0^1 uv\, dx = \int_0^1 f\, v dx, \quad \forall v \in V. \tag{4.5}$$

An integration by parts moreover leads to:

$$\int_0^1 u'v' dx + u'(0)v(0) - u'(1)v(1) + \int_0^1 uv\, dx = \int_0^1 f\, v dx, \quad \forall v \in V. \tag{4.6}$$

It is now observed that the homogeneous boundary conditions for $u, (u(0) = u(1) = 0)$ do not appear in the integral formulation (4.6).

In order to retain the whole information of the continuous problem **(CP)** in the future variational formulation **(VP)**, it would therefore be suitable to impose that test functions v fulfil the boundary conditions:

$$v(0) = v(1) = 0\,. \tag{4.7}$$

Such a method indeed ensures that the solution u, as one of the functions v of the searched variational space V, will have all the properties required at the boundary conditions on $[0,1]$.

The following formal variational formulation is thus obtained:

Find u belonging to V being the solution of:

$$\int_0^1 (u'v' + uv)\,dx = \int_0^1 f\,v dx, \quad \forall v \quad \text{such that: } v(0) = v(1) = 0. \quad (4.8)$$

By adding the homogenous Dirichlet boundary conditions (4.7), the variational space V, in which the solution u of the variational formulation **(VP)** will be sought is nothing else but the Sobolev space $H_0^1(0,1)$, which is defined by

$$H_0^1(0,1) = \{]0,1[\rightarrow \mathrm{R}, v \text{ and } \grave{v} \in L^2(0,1), v(0) = v(1) = 0\} \quad (4.9)$$

Finally, the variational formulation **(VP)** is written as:

$$(\mathbf{VP})\begin{cases} \text{Find } u \text{ belonging to } V \text{ being the solution of: } a(u,v) = L(v), \forall v \in V \\ a(u,v) \equiv \int_0^1 [u'(x)v'(x) + u(x)v(x)]dx, \\ L(v) \equiv \int_0^1 f(x)v(x)dx, \\ V \equiv H_0^1(0,1). \end{cases} \quad (4.10)$$

Numerical part-Lagrange Finite Elements $\mathbf{P_1}$:

The dimension of the approximation space $\tilde{V}$ can be determined in various ways. The simplest and smartest way is to state that the functions $\tilde{v}$ of $\tilde{V}$ are basically pecked lines affined by full mesh $[x_i, x_{i+1}]$ and cancelling each one when $x = 0$ and when $x = 1$,

Hence, having $(N + 2)$ points of discretisation for the entire mesh of interval $[0,1]$, two $\tilde{V}$ functions stand out because of their difference in values that may be seen at N interior points $(x_1, \dots, x_N)$.

Any function $\tilde{v}$ of $\tilde{V}$ also needs to satisfy, $.\tilde{v}_0 = \tilde{v}_{N+1} = 0$.

In other words, a function $\tilde{v}$ belonging to $\tilde{V}$ is entirely determined by the N-tuple $(\tilde{v}_1, \dots, \tilde{v}_N)$.

This implies that the space is isomorphic to $\boldsymbol{R}^N$. In conclusion, it can be deduced that the dimension of $\tilde{V}$ is equal to N.

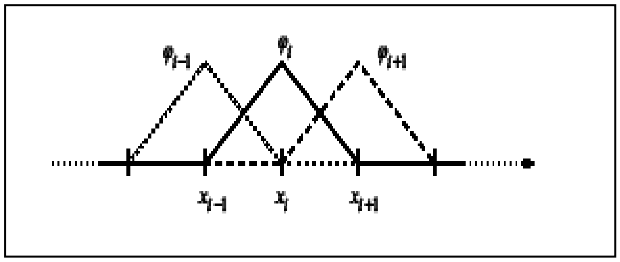

Fig. 4.1 Basis Functions φ_{i-1}, φ_i and φ_{i+1}

The approximated variational formulation is obtained by substituting the approximation functions $(\tilde{u}, \tilde{v})$ to the (u, v) functions in the variational formulation **(VP)**.

Moreover, the approximation expressions given by

$$\tilde{v}(x) = \varphi_i(x), \quad (i = 1 \; to \; \dim \tilde{V}) \text{ and } \tilde{u}(x) \sum_{j=1,dim\tilde{V}} \tilde{u}_j \, \varphi_j \, , \qquad (4.10)$$

are used and the following is obtained:

Find the numerical sequence $(\tilde{u}_j), (\, j = 1$ to $N)$, solution to:

$$\sum_{j=1,N} \left[\int_0^1 (\varphi_i' \varphi_j' + \varphi_i \varphi_j) dx \right] \tilde{u}_j = \int_0^1 f \varphi_i dx, \quad \forall i = 1 \;\text{ to }\; N. \qquad (4.11)$$

The expressions of A_{ij}, and b_j corresponding to the formulas (3.8) are then obtained by identification.

Function $\boldsymbol{\varphi_i}$ Characteristic of a Node Strictly Interior at $[\mathbf{0}, \mathbf{1}]$:

The basis functions φ_i, characteristic of nodes strictly interior to integration interval [0,1], are now considered.

The generic equation of system (4.10) has, a priori, non zero terms, except those corresponding to φ_j functions whose support intercepts those of the φ_i function considered (see Fig. 4.1).

Thus, the basis functions concerned are: φ_{i-1}, φ_i and φ_{i+1}.

This explains why the equation $(\widetilde{\mathbf{VP}}_{\text{int}})$, only has terms $A_{i,i-1}, A_{i,i}$ and $A_{i,i+1}$ and is expressed according to (3.9).

Approximate Calculation of Coefficients $A_{ij}, j = i-1, i, i+1$:

a) Approximation of coefficient A_{ii}.

$$
\begin{aligned}
A_{ii} &= \int_0^1 (\varphi_i'^2 + \varphi_i^2) dx \quad = \int_{supp\ \varphi_i} (\varphi_i'^2 + \varphi_i^2) dx\,, \\
&= \int_{x_{i-1}}^{x_i} (\varphi_i'^2 + \varphi_i^2) dx + \int_{x_i}^{x_{i+1}} (\varphi_i'^2 + \varphi_i^2) dx, \\
&\simeq \left(\frac{1}{h^2} \times h\right) + \frac{h}{2}(0+1) + \left(\frac{1}{h^2} \times h\right) + \frac{h}{2}(1+0), \\
A_{ii} &\simeq \frac{2}{h} + h.
\end{aligned}
\tag{4.11}
$$

This was achieved by considering the fact that the basis functions φ_i of are piecewise affines. Thereafter, the derivatives φ_i' are constant on each mesh having the form $[x_i, x_{i+1}]$.

The integrals bearing on those derivatives can then be calculated either exactly or by using the trapezium quadrature formula being exact for constants functions.

b) Approximation of coefficient $A_{i,i-1}$.

$$
\begin{aligned}
A_{i,i-1} &= \int_0^1 (\varphi_i' \varphi_{i-1}' + \varphi_i \varphi_{i-1}) dx, \\
&= \int_{supp\ \varphi_{i-1} \cap supp\ \varphi_i} (\varphi_i' \varphi_{i-1}' + \varphi_i \varphi_{i-1}) dx\,, \\
&\simeq \left(-\frac{1}{h^2} \times h\right) + \frac{h}{2}[(0+1) \times (1+0)], \\
A_{i,i-1} &\simeq \frac{1}{h}.
\end{aligned}
\tag{4.12}
$$

c) Approximation of coefficient $A_{i,i+1}$.

Calculation of the coefficient $A_{i,i+1}$ is easily obtained as long as the following symmetrical properties are observed:

- MatrixA of coefficient A_{ij} is symmetrical: $A_{i,j} = A_{j,i}$.
- The mesh over interval $[0, L]$ is translation invariant as a consequence of its uniform step of constant discretisation h.

It then becomes:

Symmetry Invariant

$$\begin{array}{c} \downarrow \qquad \downarrow \\ A_{i,i-1} = A_{i-1,i} = A_{i,i+1} \simeq -\frac{1}{h}. \end{array} \tag{4.13}$$

Estimation of the Second Member b_i:

The second member b_i is calculated by considering that every basis function φ_i, characteristic of a strictly interior node has a support consisting of the union of the $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ intervals, (see Fig. 4.1).

It then becomes:

$$\begin{aligned} b_i &= \int_0^1 f\varphi_i dx = \int_{x_{i-1}}^{x_i} f\varphi_i dx + \int_{x_i}^{x_{i+1}} f\ \varphi_i dx\,, \\ &\simeq \frac{h}{2}[0 + f_i] + \frac{h}{2}[f_i + 0]. \\ b_i &\simeq hf_i. \end{aligned} \tag{4.14}$$

The previous results (4.11)-(4.14) are then grouped to obtain the corresponding nodal equation:

$$-\frac{\tilde{u}_{i-1} - 2\tilde{u}_i + \tilde{u}_{i+1}}{h^2} + \tilde{u}_i = f_i, \quad (i = 1 \ \text{ to } \ N). \tag{4.15}$$

4.2 The Neumann Problem:

The aim of this problem is to propose a mathematical and numerical study of the solution to a linear differential problem, subjected to Neumann boundary conditions.

Thus, let u be a function of the real variable defined on $[0,1]$ and has values in **R**. The interest is on the solution to the continuous problem **(CP)** defined by:

Find $u \in H^2(0,1)$ as solution to:

$$(\mathbf{CP})\begin{cases} -u''(x) + u(x) = f(x), & 0 \le x1 \\ u'(0) = u'(1) = 0, \end{cases} \tag{4.16}$$

where f is a given function belonging to $L^2(0,1)$.

Numerical Part-Lagrange Finite Elements $\mathbf{P_1}$:

The approximation of the variational problem **(VP)** is performed by using Lagrange finite elements $\mathbf{P_1}$.

To make that happen, we introduce a regular mesh of the interval [0,1] with a constant step h, so that:

$$\begin{cases} x_0 = 0, x_{N+1} = 1, \\ x_{i+1} = x_i + h, i = 0 \text{ to } N. \end{cases} \tag{4.17}$$

The approximation space $\tilde{V}$ is now defined using:

$$\tilde{V} = \{\tilde{v}: [0,1] \to \mathbf{R}, \tilde{v} \in C^0([0,1]), \tilde{v}|_{[x_i, x_{i+1}]} \in P_1\}, \tag{4.18}$$

where $P_1 \equiv P_1([x_i, x_{i+1}])$ refers to the space of polynomials defined over $[x_i, x_{i+1}]$ having a degree less than or equal to one.

Let $\varphi_i, (i = 1 \text{ to} \dim \tilde{V})$, be the canonical basis of $\tilde{V}$ verifying $\varphi_i(x_j) = \delta_{ij}$, where δ_{ij} refers to the Kronecker symbol.

Solution:

Let v be a test function defined on [0,1] having real values and "sufficiently Regular".

As already mentioned in the presentation of the Dirichlet problem (see section 4.1), the regularity of functions v will be specified a posteriori in order to give sense to the variational formulation, when the latter is established.

The differential equation of the continuous problem **(CP)** is multiplied by v then integrated over the interval [0,1].

$$-\int_0^1 u''v dx + \int_0^1 uv dx = \int_0^1 fv dx, \quad \forall v \in V. \tag{4.19}$$

An integration by parts then gives the following:

$$\int_0^1 u'v' dx + u'(0)v(0) - u'(1)v(1) + \int_0^1 uv dx = \int_0^1 fv dx, \quad \forall v \in V. \tag{4.20}$$

Here, the homogenous Neumann boundary conditions defined in the continuous problem **(CP)**, $(u'(0) = u'(1) = 0)$, appear in the integral formulation (4.20).

As a result and by considering the above two boundary conditions, the following formulation is obtained:

Find u belonging to V being the solution to:

$$\int_0^1 (u'v' + uv) dx = \int_0^1 fv dx, \quad \forall v \in V. \tag{4.21}$$

It is then observed that this variational formulation is strictly analogous to the one obtained within the framework of the Dirichlet problem (4.8)-except the boundary conditions that should no longer be imposed on the test functions v within the framework of the Neumann problem treated here.

That is the reason why, if the functional analysis presented in section 4.1 is used, a sufficient condition guaranteeing the convergence of the integrals in the variational formulation (4.21) consists in defining the variational space V as follows:

$$V \equiv H^1(0,1) \equiv \{v: [0,1] \to \mathbf{R}, v \in L^2(0,1), v' \in L^2(0,1)\}. \tag{4.22}$$

Finally, the variational problem **(VP)** is written as:

$$(\mathbf{VP}) \begin{cases} \text{Find } u \text{ belonging to } V \text{ solution of: } a(u,v) = L(v), \quad \forall v \in V, \text{where:} \\ a(u,v) \equiv \int_0^1 [u'(x)v'(x) + u(x)v(x)] dx, \\ L(v) \equiv \int_0^1 f(x)v(x) dx, \\ V \equiv H^1(0,1). \end{cases} \tag{4.23}$$

Numerical Part . Lagrange Finite Elements P_1:

To calculate the dimension of space $\tilde{V}$, the following remark is necessary:

The definition (4.18) of the approximation space is almost similar to the one considered in the Dirichlet problem

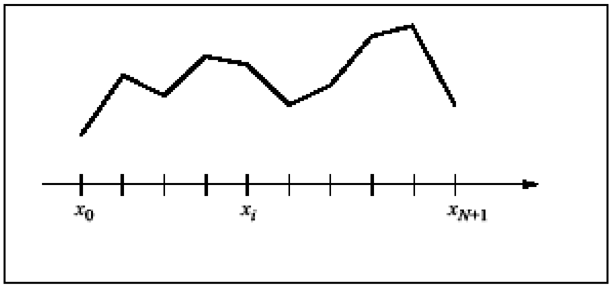

Fig. 4.2 Profile of a Piecewise Affine Function

Thus, using the demonstration performed within the framework of the Dirichlet problem, it is only necessary to note that in space $\tilde{V}$ defined by (4.18), two liberty degrees, due to the two values of any function $\tilde{v}$ of .V in $x = 0$ and in $x = 1$, add two units to the dimension found in the Dirichlet problem.

In other words, finding any function $\tilde{v}$ of $\tilde{V}$ means finding its trace $(\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_N, \tilde{v}_{N+1})$ in $(N + 2)$ discretisation points of the mesh in the interval [0,1], i.e. $(x_0, x_1, \dots, x_N, x_{N+1})$ which on their own fix the definition of . $\tilde{v}$ (see Fig. 4.2).

As a result, space $\tilde{V}$ is isomorphic to $\mathbf{R}^{N+2}$ and the dimension of $\tilde{V}$ is equal to $N + 2$.

As usual, the approximate variational formulation $(\widetilde{\mathbf{VP}})$ is obtained by substituting the approximate functions $(\tilde{u}, \tilde{v})$ for functions (u, v) in the variational formulation **(VP)**.

Moreover, the expressions supplied by the formula (3.50) are used.

Thus, the approximate variational formulation $(\widetilde{\mathbf{VP}})$ is written as:

$$
(\widetilde{\mathbf{VP}}) \begin{cases} \text{Find the numerical sequence } (\tilde{u}_j), (j = 0 \text{ to } N+1), \text{solution to:} \\ \sum_{j=0}^{N+1} \left[\int_0^1 \left(\varphi_i' \varphi_j' + \varphi_i \varphi_j \right) \right] \tilde{u}_j = \int_0^1 f \varphi_i(x), \quad \forall i = 0 \text{ to } N+1. \end{cases} \tag{4.24}
$$

The expressions of A_{ij}, and b_j corresponding to the formulas (3.52) are then obtained by identification.

Function φ_i characteristic of a node strictly interior at $[0,1]$:

When observing variational formulation $(\widetilde{\mathbf{VP}})$ defined by (4.24), it appears that, in the linear system consecutive equations $(N+2)$, the "interior" N equations corresponding to the values of j ranging from 1 to N, are totally identical to those found in the Dirichlet problem,

As mentioned previously in the theoretical part, only the functional frame differs between the two formulation in order to consider the change in boundary conditions.

It is then expected to find the same approximation described by the nodal equations associated to the basis functions .i, characteristic of nodes strictly interior to [0,1] mesh interval.

In other words, the nodal equation (3.53) coefficients have the following value:

$$
A_{ii} \equiv \frac{2}{h} + h, A_{i,i-1} = A_{i,i+1} \equiv -\frac{1}{h}, b_i \equiv h f_i. \tag{4.25}
$$

This is the result of the variational formulation identical formalism between the Dirichlet and the Neumann problem, for basis functions φ_i characteristic of nodes strictly interior at interval [0,1].

The nodal equation of the approximate variational problem $(\widetilde{\mathbf{VP}})$ corresponding to the basis function φ_i, which is characteristic of a strictly interior node x_i and written as:

$$-\frac{\tilde{u}_{i-1}-2\tilde{u}_i+\tilde{u}_{i+1}}{h^2}+\tilde{u}_i=f_i \ (i=1 \text{ to } N). \qquad (4.26)$$

this system is need to be solved using a computer program which is not part of this theisis.

References:

[1] E. B. Becker, G. F. Carey and J. T. Oden (1981), Finite Elements: An Introduction, Volume I, Prentice-Hall, New Jersey.

[2] J. Chaskalovic (2008), Finite Element Methods for Engineering Sciences: Theortical Approach and problem Solving Techniques, Springer- Verlag, Berlin Heidenberg.

[3] M. S. Gockenbach (2006), Understanding and Implementing the Finite Element Method, Siam, Philadelphia.

[4] L. Lapidus and G. F. Pinder (1999), Numerical Solution of Partial Differential Equations in Science and Engineering, John Wiley & Sons, Canada.

[5] O. C. Zienkiewicz, R. L. Taylor and J. Z. Zhu (2005), The Finite Element Method: It's Basis and Fundamentals, Sixth edition, Elsevier Butterworth- Heinemann.