Chapter 1

Completely bounded mappings of C^*- algebras

We establish the necessity of the known sufficient condition for isometry of the map, namely that all Glimm ideals of A are primal. However, when the map is restricted to tensors with length bounded by a fixed quantity, a weaker necessary and sufficient condition is established.

Section (1-1) : Basic construction and solution of isomety problem

We consider in some detail an example of a unital C^*-algebra A in which all Glimm ideals are primitive except for one particular Glimm ideal G_∞ which is 3-primal but not 4-primal. This example is an elaboration of an example in Example (1.1.4) which has a Glimm ideal that is 2-primal but not 3-primal, and it is also a prototype for variants which seem to be able to exhibit many of the phenomena that can occur in general.

The basic idea is to build a 4-point compactification of a locally compact Hausdorff space, where in each way of approaching the points at infinity one actually has three limiting values (but not the fourth). This requires four 'directions' of approach to infinity. A way to visualise such a space is to consider a disjoint union T= $\bigcup_{j=1}^{4} R_j$ of four semi-infinite closed rays in the plane with four points adjoined as follows. For example T ={(x, y) : xy=0 , $x^2 + y^2 \geq 1\}\subset R^2$with (say) R_1={(x,0) : x ≥1}. Label the four extra points ω_i ($1\leq i \leq 4$). A basis of neighbourhoods of each ω_i is given by the sets

$$\{\omega_i\} \cup_{j\neq 1}\{ t \in R_j\{t \in R_j : |t| > r\},$$

Where r >1. So, for example, the sequence (n,0) in the ray R_1 of the space

T∪ $\{\omega_1\omega_2, \omega_3 , \omega_4 \}$ would have each of ω_2, $\omega_3$and ω_4 as limits as n→∞(but not ω_1).

A 'discrete' version of this space would start with $T\cap\mathbb{Z}^2$ in place of T. Clearly one can map $T\cap\mathbb{Z}^2$ to N by mapping the four directions to equivalence classes in N modulo 4.

We now construct a C^*-algebra ((pronounced "C-star") are an important area of research in functional analysis, a branch of mathematics. A C*-algebra is a complex algebra A of continuous linear operators on a complex Hilbert space with two additional properties: A is a topologically closed set in the norm topology of operators. And A is closed under the operation of taking adjoints of operators) [5]. As such that Prim(A) is (homeomorphic to) T $\cup\{\omega_1, \omega_2, \omega_3, \omega_4\}$. We consider the C^*-algebra B of bounded continuous functions $x: T \to M_3(\mathbb{C})$ and we defineAto be the C^*-sub algebra of B consisting of all those elements $x \in B$ for which there exist scalars λ_1(x), λ_2 (x), λ_3 (x), λ_4 (x) such that:

$$\lim_{R_j \ni t \to \infty} x(t) = \text{diag } \lambda_{j+1}(x), \lambda_{j+2}(x), \lambda_{j+3}(x)) \ (1 \leq j \leq 4),$$

where we understand the subscripts λ_{j+1} ($1 \leq i \leq 3$) to be reduced modulo 4 to lie in the range 1,2,3,4. Next, we introduce notation for what we call 'constant' elements of A. Given four scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ we write c($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) for the element x∈ A where

$$x(t)= \text{diag}(\lambda_{J+1} , \lambda_{J+2} , \lambda_{J+3}) \ t \in R_j, \ j \in \{1,2,3,4\})$$

(where we again understand the subscripts modulo 4). The set A_c of all constant elements of A forms an a belian C^*- subalgebra isomorphic to $\mathbb{C}^4$

and A=C_0(T,$M_3(\mathbb{C})$)+ A_c.

We call this A a '4-spoke' example. The centre Z(A)of A consists of elements x where each x(t) is a multiple of the identity (and hence λ_i(x) does not depend on i) and so Z(A)is canonically isomorphic to the algebra of scalar-valued continuous functions on the one-point compactification of T. The space Glimm(A) can then be identified with this one point compactification , or

$\{G_t$: t ∈T}∪$\{G_\infty\}$ where G_t ={x ∈A: x(t)=0} and G_∞=C_0(T, $M_3(\mathbb{C})$).

As A/G∞ is a belian, the irreducible representations of A whose kernels contain G_∞ are X → λ_i (x). The remaining irreducible representations of Arestrict to irreducible representations of C_0(T, $M_3(\mathbb{C})$)and hence have the form

π_t:A→M_3(C) where M_3(x) =x(t) for x ∈ A and t ∈T. Thus, as Ker π_t = G_t,

$$\text{Prim(A)}=\{G_t : t \in T\}\cup\{\ker \lambda_i : 1\leq i \leq 4\}.$$

As a topological space Prim(A) is homeomorphic to T∪{ ω_i : 1≤ i ≤4 }. For example, to see that as t ∈R_j tends to infinity we have G_t =ker π_t→ker λ_i for each i ≠j, let us fix i ≠j and consider an open neighbourhood U of ker λ_i in Prim(A). Then there is a closed two-sided idea J of A with U={I ∈Prim(A): J$\not\subseteq I$}. Since kerλ_i ∈U, there is some x ∈ J with λ_i(x) ≠0. Thus there exists r>1 so that if t ∈R_j and |t|>r then x(t) ≠0. It follows that ker π_t∈U whenever t ∈R_j and |t| >r.

The four ideals J_i ={x∈ A: x(t)=0 for all t ∈R_j} have product {0} but (for Example) J_1 is not contained in G_∞ because c(1,0,0,0) ∈ J_1. Hence G∞ is not

4-primal. To show that G_∞ is 3-primal, note that there are only four primitive ideals of A which contain G_∞, namely ker λ_i(i = 1,2,3,4). So it suffices to show that, for each i,

$$\ker \lambda_{i+1}\cap \ker \lambda_{i+2} \cap \ker \lambda_{i+3}$$

is primal. But we have just shown that ker π_t converges in Prim(A) to each of ker λ_{i+1}, ker λ_{i+2} and ker λ_{i+3}, as t ∈ R_i tends to infinity.

Proposition(1.1.1) [1]

For A as above, denote by $\tilde{a}_j, \tilde{b}_j$ ∈ A for 1≤j≤ 4 the following elements:

$$\tilde{a}_1= c(0,1,1,1),\ \tilde{b}_1= c(1,0,0,0),$$

$$\tilde{a}_2= c(1,0,1,1),\ \tilde{b}_2 = c(0,1,0,0),\quad (1)$$

$$\tilde{a}_3= c(1,1,0,1),\ \tilde{b}_3= c(0,0,1,0),$$

$$\tilde{a}_4= c(1,1,1,0),\ \tilde{b}_4= c(0,0,0,1).$$

Then, for u =$\sum_{j=1}^{4} \tilde{a}_j \otimes \tilde{b}_j$ ∈ A⊗A and T =θ(u) ∈$\mathcal{E}\ell$(A),

$$||u||_{z,h} > ||u||_{cb}\ .$$

Our verification of the proposition will require an analysis of norms of elementary operators similar to T but acting on M_3 and M_4. We will use e_{ij} for the n×n matrix with 1 in the (i,j) position and zeros elsewhere. We also use δ_{ij} for the Kronecker delta symbol.

Proof :-

We know that $||T||_{c,p} = sup_{\pi \epsilon \hat{A}} ||T^{n}||_{c,p}$ where the supremum is over all irreducible representations π of A and T^{π}:B(H_{π}) → B(H_{π}) is given by

$$T^{\pi}\ (y) = \sum_{j=1}^{4} \pi(\tilde{a}_j) y\, \pi(\tilde{b}_j).$$

When $\pi = \pi_t$ (for any t ∈T) we have $T^{\pi} = T_3$ and so $||T^{\pi}||_{cb}$= 4/3. For $\pi = \lambda_j$ we have T^{π} = 0. Hence

$||T||_{cb}$ = 4/3.

We now claim that

$||u||_{z,h} = ||T_4\ ||_{cb}$ =3/2 for u $= \sum_{j=1}^{4} \tilde{a}_j \otimes \tilde{b}_j$.

$||u||_{z,h} = Sup_G |u^G||_h$ where the sup is over all Glimm ideals of A and $u^G \in$ (A/G)⊗h (A/G) is u^G =4 j=1($\tilde{a}_j$ +G)⊗($\tilde{b}_j$ +G). The case G = G∞ yields A/G as a four-dimensional commutative algebra. We can identify it as the diagonal in $M_4(\mathbb{C})$. Then the elements $\tilde{b}_j$+ G can be taken to correspond to $e_{jj} \in M_4$ (ℂ) and $\tilde{a}_j$+G to I4 $-e_{jj}$. By injectivity of the Haagerup norm we can compute $||u^G||_h$ in $M_4 \otimes M_4$ where, by Haagerup's it gives $||T_4||_{cb}$ which equals 3/2.

Thus $|u||_{z,h} \geq 3/2$ (and in fact we could easily show equality as all the other Glimm ideals are primitive, being the kernels of the representations t for t ∈T). Thus we have

$|u||_{z,h} \geq 3/2 > 4/3 = ||T||_{cb} = ||\theta_z(u)\ ||_{cb}$.

Example(1.1. 2) [1]

Consider the (elementary) operator T_n: $M_n \to M_n$ given by

$$T_{nx} = \sum_{j=1}^{n} (I_n - e_{jj}) x e_{jj} = x - \sum_{j=1}^{n} e_{jj} x e_{jj}. \text{ Then}$$

$$||T_n|| = ||T_n||_{c,b} = 2(n-1)/n.$$

Proof:-

Note that $T_n e_{ij}$ = (1−δ_{ij}) e_{ij} and $T_n I_n$ = 0.

We can rewrite T_{nx} = ((n − 1)/n)x −$\sum_{j=1}^{n}(e_{ij} - I_n/n)x(e_{jj} - I_n/n)$ = ((n −1)/n)x − S_nx where S_n is a completely positive operator. Hence

$$||S_n||_{c,p} = ||S_n|| = ||S_n\ (I_n)|| = (n-1)/n$$

and thus

$|||T_n|| \leq ||T_n||_{c,p}$ 2(n−1)/n. To show that we have equality in both of these inequalities, we introduce the unit vector ξ = (1,1,...,1)/$\sqrt{n} \in \mathbb{C}^n$ and the rank one projection operator $\xi^* \otimes \xi$ of $\mathbb{C}^n$ onto the span of ξ. As a matrix, $\xi^*\otimes\xi$has 1/n in each entry. So we can see that $< T_n\ (\xi^*\otimes\xi)\ \xi, \xi >$=(n−1)/n. Since 2($\xi^*\otimes\xi$)− I_n is a norm one operator and $< T_n(2(\xi^*\otimes\xi)-\text{In})\ \xi,\xi >=2< T_n$ $(\xi^*\otimes\xi)\ \xi, \xi >$=2(n−1)/n we have $||T_n||$≥2(n−1)/n.

Remark(1.1 3)[1]

The proof can be generalised to produce similar examples where all Glimm ideals are n-primal but not all are (n+1)-primal. The elementary operator would have length n + 1 and the algebra A would be replaced by an '(n + 1)-spoke' algebra constructed from a T having n + 1 rays to infinity R_i (1≤ i ≤n + 1) and matrices $M_n(\mathbb{C})$. There would be n+1 multiplicative linear functionals x→λ_i(x) at 'infinity' with x(t) tending to a diagonal using n of the n+1 values λ_i(x) as t →∞in any R_i. One would obtain u ∈ A⊗A such that $|u||_{z,h}$= 2n/(n+1), = 2(n−1)/n, $||\theta_z(u)\ ||_{cb}$ and hence $|u||_{z,h}$/= $||\theta_z(u)\ ||_{cb}$ = $n^2/(n^2 - 1)$. In Proposition (1.1.1), the (minimal) length of the tensor u is 4, the elementary operator T = Z(u) is self-adjoint T*(x) = T(x*)* = T (x)) and$|u||_{z,h}/||T||_{cb}$= 9/8. For the same algebra A, we now exhibit a tensor u with length 2 on which Z fails to be isometric. In this case, the corresponding elementary operator T is not self-adjoint but$|u||_{z,h}/||T||_{cb}$ = $\frac{4}{1+\sqrt{5}}$ > 9/8.

Example(1.1.4) [1]

For A the '4-spoke' $\mathbb{C}^*$-algebra introduced above, take T :A → A to be the generalised derivation given by Tx= α x−xb where

α = c(0,ι,0,− ι) , b = c(−1,0,1,0) ($\iota = \sqrt{-1}$). Then

$||T||_{cb}$ = 1/2+ $\sqrt{5/4}$< 2 =||α⊗1−1⊗b||$_{z,h}$.

Proof:-

The norm of a generalised derivation S : x→ αx−xb on $\mathcal{B}$(H) (any Hilbert space H, any α ,b∈ $\mathcal{B}$ (H)) is

$$||S|| = inf_{\lambda \epsilon \mathbb{C}}(||\alpha - \lambda|| + ||b - \lambda||).$$

For k = 2,3,..., the operator $S^{(k)}$ on M_k ($\mathcal{B}$(H)) given by $S^{(k)}((x_{ij})_{i,j=1}^{k}) = (Sx_{ij})_{i,j=1}^{k}$ may be regarded as the generalised derivation on $\mathcal{B}(H^k)$ defined by the amplifications of a and b. So, by Stampfli's formula again,|| $S^{(k)}$||= ||S|| . Hence$||S||_{cb}$ =||S||. As before we compute ||T|| via the representations π_t(t ∈T). When t ∈ R_4 we end up with

$$\pi_t(a) = \text{diag}(0, \iota, 0) \qquad \pi_t(b) = \text{diag}(-1,0,1).$$

One can see geometrically that

$$|| \pi_t(a) - (\iota/2) I_3 || + || \pi_t(b) - (\iota/2) I_3 || = \frac{1}{2} + |1 + \frac{1}{2}| + \sqrt{\frac{5}{4}}$$

achieves the minimum in the Stampfli formula, but in any case$||T^{\pi}||$ is bounded above by this number for $\pi = \pi_t$ and t ∈ R_4. A similar analysis applies for all R_i(1≤ i ≤4). In the one-dimensional irreducible representations $\pi = \lambda_j$ we have $||T^{\pi}||$ =1 and so we end up with

$$||T||_{cb} = Sup_{\pi \epsilon \hat{A}} ||T^{\pi}||_{cb} \leq 1/2 + \sqrt{\frac{5}{4}}.$$

Finally, to show that $||a\otimes 1-1\otimes b||_{z,h}$ =||a||+||b|| =2 we concentrate on the quotient A/ G_∞. We then must consider (following the pattern of proof in Example (1.1.2) the norm (= cb-norm) of the generalised derivation on M_4 given by

x→ diag(0, ι,0,− ι)x −xdiag(−1,0,1,0). One may verify that the norm is 2 using Stampfli's formula quoted above.

We show that if a unital C^*-algebra A has a non-primal Glimm ideal then the mapping θ_z is not an isometry. In order to utilise the computations of Example (1.1.2) in a more general setting, we shall need the following lemma:

Lemma(1.1. 5) [1]

Let b_j(1≤j≤n) be orthogonal, positive elements of norm one in a C^*- algebra A (that is, $b_j \geq 0$,
$||b_j||$=1 and $b_j b_k$ = 0 for j ≠ k,1≤j,k≤n) and let X denote their linear span. Let d_j(1≤j≤n) be orthogonal positive elements of a C^*-algebra B and let Y denote their linear span. Assume $||d_j||$≤1 for 1≤j≤n. We can define a linear map∅:X → Y by∅ (b_j) = d_j and it has the following properties:

(i)|| ∅||≤1.

(ii) The map ∅⊗∅:X$\otimes_h$ X → Y $\otimes_h$Y (with Haagerup tensor norms in each case) has norm at most one.

(iii) If $||d_j||$ =1 for each j, then∅⊗∅ is an isometry between X$\otimes_h$X and Y $\otimes_h$Y.

Proof:-

Consider the commutative C^*-algebra generated by theb_j(1≤j≤n). It is isomorphic to an algebra of continuous functions C_0 (K_X) on some locally compact Hausdorff space K_X where the$b_j$must be positive functions that are non-zero on disjoint open sets. It is clear then that the norm of a linear combination $\alpha_j b_j$ is max_j $|\alpha_j|| b_j||$ =max$_j$ $|\alpha_j|$. (In particular the b_j are linearly independent and is well-defined.) For similar reasons, we may view Y ⊆ C_0 (K_X) for a locally compact Hausdorff space K_Y, and

$$||\sum_{j=1}^{n} \alpha_j d_j \;||= \max_j|\alpha_j|||d_j||\leq max_j|\alpha_j|.$$

This shows|| ∅||≤1.

For the second part, note that when we compute the Haagerup tensor norm of u $=\sum_{i=1}^{N} a_i \otimes c_i \in$ X⨂X, we consider an infimum of expressions

$$\frac{1}{2}(\sum_{i=1}^{N} a_i a_i^* + ||\sum_{i=1}^{N} c_i^* c_i \,||)$$

over all representations of u and we can find a representation where this infimum is attained (without going outside representations in X⨂X). We can compute that applying ⨂ to this same representation produces a representation of (∅⨂∅)(u) ∈ Y⨂Y where the corresponding expression is reduced. For example if we write $a_j = \sum_{j=1}^{n} a_{ij} b_j \in$ X then, for $k \epsilon K_Y$,

$$(\sum_{i=1}^{N} \emptyset(a_i)\emptyset(a_j)^*)(k) = \sum_{i=1}^{N}(\sum_{j=1}^{n} \bar{a}_{ij} d_j^*(k))$$

$$=\sum_{j=1}^{n}\sum_{i=1}^{N}|a_{ij}\;|^2|d_j(k)|^2$$

because $d_j(k)$ is non-zero for at most one j. The supremum of this latter sum over k $\in K_Y$ is at most

$$\max_j \sum_{i=1}^{N}|a_{ij}|^2 .$$

Thus

||∅⨂∅|| ≤1.

For the third part, we can apply the second part to the inverse map of ∅ if $||d_j||$=1 for all j.

Lemma (1.1.6) [1]

Let b_j(1≤j≤n) be positive elements of a C^*-algebra with $b_j b_k$= 0 for j≠ k (1≤j,k≤n). Let

u $=(\sum_{j=1}^{n} b_j)\otimes(\sum_{j=1}^{n} b_j) - (\sum_{j=1}^{n} b_j \otimes b_j)$.

(i) If $||b_j|| \leq 1$ for 1≤j≤n, then $||$u$||_h \leq$2(n−1)/n.

(ii) if $||b_j||$ =1 for 1≤j≤n, then $||u||_h$ = 2(n−1)/n.

Proof:-

We can deduce this from Lemma (1.1.5) and Example (1.1.2) We identify

$\mathbb{C}^n$with the diagonals in M_n and consider $\emptyset$: $\mathbb{C}^n \to$ Y = span{b_j : 1≤j≤n} given by $\emptyset$ (e_{jj}) = b_j. Using injectivity of the Haagerup norm, Haagerup's theorem and Example (1.1. 2), we have

$||I_n \otimes I_n|| - (\sum_{j=1}^{n} e_{ij} \otimes e_{ij})\,||_h = ||T_n||_{cb}$ = 2(n−1)/n.

But the tensor in the left-hand side maps to u under the mapping $\emptyset \otimes \emptyset$ of Lemma (1.1.5).

Theorem (1.1.7) [1]

Let A be a unital C^*-algebra containing a Glimm ideal G that is not n-primal for some n≥2. Then there exists u=($\sum_{j=1}^{n} a_j \otimes b_j$) ∈ A⊗A with $||u||_{z,h} > ||\,\theta_Z\,(u)||_{cb}$.

Proof:-

By reducing n if necessary, we may assume that G is (n−1)-primal but not n-primal (where we adopt the convention that all closed two-sided ideals in A are 1-primal).

There must exist n primitive ideals P_j of A (1 ≤ j ≤ n) with G ⊆ P_j for all j but J $\bigcap_{j=1}^{n} P_j$ not primal. However

$$R_j = \bigcap_{1 \le k \le n, k \ne j} P_k$$

is primal for each 1≤j≤n. Note that since R_1, R_2,..., R_n are primal but J is not, it follows that $P_j \not\supseteq P_k$ for j ≠k.

There must exist open neighbor hoods U_j of P_j in Prim(A) (1≤j≤n) so that

$$\bigcap_{j=1}^{n} U_j = \emptyset.$$

For, if no such neighbor hoods existed there would be a net $(Q_\alpha)_\alpha$ in Prim(A) converging to each of the P_j (1≤j≤n) and hence to every primitive ideal containing J, contradicting the non-primality of J . Now there are closed two-sided ideals J_j in A so that U_j = Prim(J_j) (hence U_j ={Q ∈ Prim(A) : $J_j \not\subseteq$Q}).

Let I_j = $J_j R_j$ for 1≤j≤n. The ideal I_j cannot be contained in J because then we would have $J_j R_j \subseteq P_j$ and since the primitive ideal P_j is necessarily prime, it would follow that $J_j \subseteq P_j$ or $R_j \subseteq P_j$. Since $P_j \in U_j$, we have$J_j \not\subseteq P_j$. By primeness of P_j, if $R_j \subseteq P_j$, then $P_K \subseteq P_j$ for some k ≠ j (again not so).

Let Ψ :A → A/J denote the quotient map. Let K_j =Ψ(I_j), a non-zero closed ideal of A/J. Note that $K_j K_K$ = 0 for j≠ k (as R_j $R_K \subseteq$ J).

For 1≤j≤n, choose a positive element $d_j \in K_j$ of norm one and $g_j \in I_j$ positive of norm one with Ψ(g_j) = $d_j^{1/3}$. Since $d_j^{1/3}$ $d_k^{1/3}$ = 0 for j ≠ k, we can find c_j ∈ A (1≤j≤n) withΨ(c_j) = Ψ(g_j) = $d_j^{1/3}$ and c_j c_k =0 for j ≠ k. Let b_j= $c_j d_j c_j$ ∈ I_j^+. Then $b_j' b_k'$ = 0 for j ≠k and Ψ (b_j') = d_j(1≤j,k≤n).

Let f:[0,∞) →[0,∞) be f(t)= min(t,1), a uniform limit on any compact subset of [0,∞) of polynomials without constant term. Define b_j = $f(b_j')$ by functional calculus. Then we have $b_j \in I_j^+$, Ψ(b_j) = d_j,$||b_j||$=1 and $b_j b_k$= 0 for j ≠k. Consider now

U =($\sum_{j=1}^n b_j) \otimes (\sum_{j=1}^n b_j) - \sum_{j=1}^n b_j \otimes b_j$ = $\sum_{j=1}^n((\sum_{j=1}^n b_k) - b_j) \otimes b_j$
as in Lemma(1.1.6)[1] Note that the canonical quotient map from A/G to A/J induces a contraction from A/G $\otimes_h$A/G to A/J ⊗h A/J. and , we have $||u||_{z,h} \geq ||u^G||_h$. Applying Lemma(1.1. 6) to A/J, we deduce $||u||_{z,h} \geq ||u^G||_h \geq ||u^j||_h$ = 2(n−1)/n.

On the other hand,

$$||\theta_Z(u)||_{cb} = \sup\{||u^p||_h : P \in \text{Prim}(A)\}.$$

Let P ∈ Prim(A). There exists j ∈{ 1,...,n} such that P∉ U_j and hence $b_j \in I_j \subseteq J_j$ ⊆ P. Applying Lemma(1.1.6) again (this time to A/P with at most n −1 non-zero b_k +P), we have

$$||u^P|| \leq 2(n-2)/(n-1).$$

Thus

$$||\theta_Z(u)||_{cb} \leq 2(n-2)/(n-1) < 2(n-1)/n = ||u||_{z,h}.$$

Combining Theorem(1.1.7) with Theorem (1.1.4), we obtain the following result:

Theorem(1.1. 8) [1]

Let A be a unital C^*-algebra. The mapping θ_Z : A$\otimes_{z,h}$ A → CB(A) is an isometry if and only if every Glimm ideal of A is primal.

If A is a non-unital C^*-algebra then it is customary to consider the multiplier algebra M(A). If Z now denotes the centre of M(A), then we have the natural contraction θ_Z:M(A) $\otimes_{z,h}$M(A) → CB(M(A)) for the unital C^*-algebra M(A). But, since A is an ideal in M(A), we obtain a contraction Θ_Z:M(A) $\otimes_{z,h}$ M(A) → CB(A) by defining Θ_Z(u) =θ_Z(u)$|_A$.

Corollary(1.1.9) [1]

Let A be a non-unital C^*-algebra. The map Θ_Z:M(A) $\otimes_{z,h}$M(A) → CB(A) is an isometry if and only if every Glimm ideal of M(A) is primal.

Proof:-

Let u ∈ M(A)⊗M(A). By taking a faithful non-degenerate representation of A on a Hilbert space H, we may assume the inclusions A ⊆ M(A) ⊆ A'' ⊆ B(H). By tensoring with M_n (ℂ) and using Kaplansky's density theorem, one obtains that
$||\theta_Z(u)||_{cb}$= || $\Theta_Z(u)||_{cb}$. The result now follows from Theorem(1.1.8)

We can state a necessary condition for Θ_Z to be an isometry in terms of Glimm ideals of A, something that involves an extension of the notion of

Glimm ideal to the non-unital case. In a (not necessarily unital) C^*-algebra A , aGlimm ideal is the kernel of an equivalence class in Prim(A), where primitive ideals P and Q are defined to be equivalent if f(P)= f(Q) for all

$f \in C^b$ (Prim(A)). By the Dauns–Hofmann theorem, this definition is consistent with the one already given in the unital case.

Lemma(1.1.10) [1]

Let A be a (non-unital) C^*-algebra containing a Glimm ideal G that is not n-primal (some n≥2). Then M(A) also contains a Glimm ideal that is not n-primal.

Proof:-

In this proof, we elaborate an argument and use different notation. By the Dauns–Hofmann theorem, there is an isomorphism Φof the algebra C^b(Prim(A)) onto the centre Z(M(A)) of M(A) such that for $f \in C^b$(Prim(A)), a ∈ A and P ∈ Prim(A),

$$(\Phi(f)a)+P = f(P)(a+P)$$

in A/P. Temporarily fix P ∈ Prim(A) with P ⊇ G and define a multiplicative linear functional ϕon C^b(Prim(A)) byϕ(f) = f(P). Clearly ϕ is independent of the choice of P ⊇ G. Let J = ker ($\phi \circ \phi^{-1}$), a maximal ideal of Z(M(A)), and let H = M(A)J, a Glimm ideal of M(A).

We have H ∩A = M(A)JA = AJ. Let a ∈ A, z ∈ J and let Q be any primitive ideal of A containing G. InA/Q we have

$$za+Q = (\Phi^{-1}(z))(Q)(a+Q) = (\Phi^{-1}(z))(a+Q) = 0.$$

Hence **AJ** ⊆ G. (In fact **AJ** = G, but we will not need that.)

Suppose that H is n-primal. For any closed idealsI_1, I_2,..., I_n ⊆ A, with product $I_1 I_2 \cdots I_n$ ={ 0} we must have I_i ⊆ H (for some 1≤i≤n) and so I_i ⊆ H ∩A = AJ ⊆ G. Thus G is n-primal, a contradiction showing that H cannot be n-primal.

From Lemma(1.1.10) and Corollary (1.1.9), we can make the following assertion:

Corollary (1.1.11) [1]

Let A be a (non-unital) C^*-algebra. If the map θ_Z:M(A) $\otimes_{z,h}$M(A) → CB(A) is an isometry then every Glimm ideal of A is primal.

Proof :-

For an odd integer n≥3, let W_n be the simply connected, 2-step nilpotent, Lie group considered and let A = $C^*(W_n)$. Then A has a Glimm ideal which is not (n+1)-primal and so, by Corollary (1.1.11), Z is not an isometry in this case. The next example, together with Corollary (1.1.9), shows that the necessary condition in Corollary (1.1.11) is not sufficient for Θ_Z to be an isometry.

Example (1.1.12) [1]

There is a C^*-algebra A with compact, Hausdorff, primitive ideal space (and hence with every Glimm ideal primal) such that M(A) has a Glimm ideal which is not 2-primal.

Proof:-

Let X be a non-compact, locally compact Hausdorff space such that the Stone–$\check{C}$ech remainder βX\X has at least two distinct points y and z (e.g. we could take X = ℕ or X =ℝ). Let B be the C^*-algebra C(βX, M_2 (ℂ)), and let B_1 be the C^*-sub algebra consisting of those functions f ∈ B for which there exist complex numbers λ_1 (f), λ_2 (f), λ_3 (f) such that f(y)= diag(λ_1 (f), $\lambda_2(f)$) and f(z)= diag($\lambda_2(f)$, λ_3 (f)). Let π_x: $B_1 \to M_2$ denote the representation $\pi_x(f)$ = f (x). Then Prim(B_1) ={kerπ_x : x ∈X\{y ,z}}∪{ ker λ_1,ker λ_2,ker λ_3} and, for G = ker λ_1 ∩ker λ_2 ∩ker λ_3,

$$\text{Glimm}(B_1) = \{\ker \pi_x : x \in X\backslash\{y, z\}\} \cup \{G\}.$$

The Glimm ideal G is not 2-primal. To see this, let U and V be disjoint neighbor hoods of y and z, respectively, in βX. Let K_U be the closed ideal of B_1 consisting of those functions vanishing off U, and similarly let K_V consist of those f ∈ B_1 vanishing off V, also a closed ideal of B_1. Then K_U, K_V ⊈G, but $K_U K_V$ ={ 0}.

Let A ={ $f \in B_1$: $\lambda_1(f)$ =$\lambda_3(f)$ = 0}, a closed ideal in B_1.

We have Prim(A) ={ker $\pi_x|_A$: x ∈ βX\{y,z}}∪{ ker ($\lambda_2|_A$)}.

Furthermore, Prim(A) is homeomorphic to the compact Hausdorff space obtained from βX by identifying the points y and z. In particular, therefore, every Glimm ideal of A is primitive and hence primal.

Now let J = C_0(X, $M_2(\mathbb{C})$). Then M(J) = C^b(X,$M_2(\mathbb{C})$) by Akemann et al.. Note that the restriction map f⟼ $f|_x$ is a $*$-isomorphism between B = C(βX, $M_2(\mathbb{C})$) and C^b(X, $M_2(\mathbb{C})$). Since J is an essential ideal in A, it is also an essential ideal in M(A) and so we now have J ⊆ A ⊆ M(A) ⊆ M(J) = B. Elementary computations show that M(A) = B_1.

Section (1-2):- Specific Result

If every Glimm ideal of a unital C^*-algebra A is 2-primal (so that $\theta_Z$is injective) but not every Glimm ideal is primal, then one may look for a relationship between the degree of primality of the Glimm ideals of A and the length of the shortest tensors u ∈ A⊗A on which θ_Z fails to be isometric. We begin by considering the question of whether n-primality of all the Glimm ideals of A is sufficient for θ_Z to be isometric on tensors u =$\sum_{j=1}^{\ell} a_j \otimes b_j$ ∈ A⊗A, where n and ℓ are related in some way.

We will use results in the sequel in order to be able to calculate Haagerup norms. By injectivity of the Haagerup norm, we can always make our computation in $\mathcal{B}$(H) for some H and in this setting we have equality of the Haagerup norm of a tensor u =$\sum_{j=1}^{\ell} a_j \otimes b_j$and the cb-norm of the elementary operator T = θ(u) on $\mathcal{B}$(H) . The difficulty addressed is to be able to recognise when a tensor u is represented in an optimal way, meaning a way that gives equality in the infimum

$$||u||_h \inf \frac{1}{2}(||a||^2 + ||b||^2),$$

where we now adopt the shorthand b =[b_1, b_2,..., b_ℓ]^t for the (column) ℓ-tuple of the bj's and a =[a_1, a_2,..., a_ℓ] for the (row) ℓ-tuple of the aj's. Recall that.

$$||a||^2 = \left|\left|\sum_{j=1}^{\ell} a_j\, a_j^*\right|\right| \text{ while } ||b||^2 = \left|\left|\sum_{j=1}^{\ell} b_j^* \otimes b_j\right|\right|.$$

The infimum for $||u||_h$ can also be written using the geometric mean version

$||u||_h$ = inf||a||b||

but there is no loss in restricting to representations u $=\sum_{j=1}^{\ell} a_j \otimes b_j$ where ||a|| = ||b|| and so the geometric and arithmetic means of $||a||^2$ and $||b||^2$ agree

The results use numerical range ideas to characterise the situation where we have equality in

$$|| \theta(u)|| \leq ||\theta(u)||_{cb} \leq ||a||^2 + ||b||^2 \quad (2)$$

and then an extension of this characterisation to amplifications $\theta\,(u)^k$ of $\theta(u)$ in order to deal with the equality in the second inequality only. we use the notation $W_m(b)$ for the matrix numerical range

$$W_m(b) = \{(<b_j^* b_i \xi, \xi>)_{i,j=1}^{\ell} = \{(< b_i \xi, b_j \xi >)_{i,j=1}^{\ell} : \xi \in H, ||\xi|| = 1\}$$

associated with a column b. This subset of M_ℓ^+ (the positive semidefinite $\ell\times\ell$ matrices) is in fact the joint spatial numerical range of the ℓ^2 operators $b_j^* b_i$ but it is convenient to consider it as a set of matrices. It is easy to see that each matrix in $W_m(b)$ has trace at most $||b||^2$ and that this is the supremum of the traces. The 'extremal matrix numerical range' $W_{me}(b)$ is defined as the subset of the closure of $W_m(b)$ consisting of those matrices with trace equal to . (In case H is finite dimensional, $W_m(b)$ is already closed and the extremal matrix numerical range corresponds to restricting $\xi \in$ H to be in the eigenspace for the maximum eigen value of $\sum_j b_j^* b_j$.) The criterion equality in (2) is

$$W_m(a^*) \cap W_{me}(b) \neq \emptyset$$

(where $a^* = [\, a_1^*, a_2^*, ..., a_\ell^*]^t$ is a column).

Let co(S) denote the convex hull of a set S. Equality in the second inequality of (2) occurs if and only if

$$co(W_{me}(a^*)) \cap co(W_{me}(b)) \neq \emptyset \quad (3)$$

We have u $\in B(H) \otimes B(H)$ of length , it can be written as u $=\sum_{j=1}^{\ell} a_j \otimes b_j$ so as to get

$||u||_h = ||a||^2 = ||b||^2 = \frac{||a||^2+||b||^2}{2}$ (4)

with the same ℓ. Via Haagerup's theorem

$||u||_h = ||\theta(u)||_{cb}$,

we see that (3) and (4) are equivalent for u $\in \mathcal{B}$ (H)$\otimes$ $\mathcal{B}$ (H). We will use this equivalence several times to detect when representations of such u satisfy (4).

Lemma (1.2.1) [1]

Consider a Hilbert space H which is a (Hilbert space) direct sum of Hilbert spaces H_i (i ∈ **I** = some index set). Let $a_{i,j}$, $b_{i,j}$ ∈ B(H_i) for each i ∈ **I**

with $\sup_i ||a_{i,j}|| < \infty$ and $\sup_i ||b_{i,j}||$ n$< \infty$ for $1 \le j \le \ell$. Consider $a_j = (a_{i,j})_{i \in \mathbf{I}}$ as a 'block diagonal' element in B(H), $b_j = (b_{i,j})_{i \in \mathbf{I}}$ similarly and u $= \sum_{j=1}^{\ell} a_j \otimes b_j \in$ B(H)$\otimes$B(H). For a subset F ⊆ **I**, let H_F be the direct sum of those H_i for i ∈ F and let a_j,F $= (a_{i,j})_{i \in \mathbf{I}} \in \mathcal{B}(H_F)$, b_j,F similarly defined and $u_F = \sum_{j=1}^{\ell} a_j \otimes b_j$,F ∈ B($H_F$)$\otimes$B($H_F$). Then

$||u||_h = \sup\{u_F||_h$: F ⊆I,F has at most ℓ^2 +1 elements}.

Proof:-

As remarked above, we know that

$||u||_h = ||\theta(u)||_{cb}$ for $\theta(u) \in \varepsilon\ell$(B(H)) and similarly for $||u_F||_h$.

Let (P) be an increasing net of projections converging in the strong operator topology to the identity operator on H. Since, for the strong operator topology, multiplication is jointly continuous on norm-bounded sets, we have

$$||\theta(u)|| = \lim_\mu ||\theta(u_\mu)||_{cb}$$

where

$$u_\mu = \sum_{j=1}^{n} (P_\mu a_j P_\mu \otimes (P_\mu b_j P_\mu).$$

Furthermore, for each k≥2, the k-fold amplification of P_μ converges strongly to the identity on Hk and so

$||\theta(u)||_{cb} = \lim_\mu ||\theta(u_\mu)||_{cb}$.

We may therefore assume that I is finite.

We assume next that u is written so as to get equality in the Haagerup norm infimum $||u||_h = (||a||^2||b||^2)/2$, hence (3) holds. Since we are in the case where I is finite,

$$||a||^2 = \max_{i\in I}||a_{\{i\}}||^2 \max_{i\in I}||\sum_{j=1}^{\ell} a_{j,i}(a_{j,i})^*||, \qquad (5)$$

where now $a_{\{i\}} = [a_{1,i}, a_{2,i}, ..., a_{\ell,i}]$ relates to the summand i. A unit vector $\xi \in H = \bigoplus_i H_i$ gives an element of W_μ (a^*) which is a convex combination of elements of u_μ ($a^*_{\{i\}}$) (i ∈I). Hence, since closed bounded subsets ofM are compact and I is finite,

$$\mathrm{co}(\overline{W_m(a^*)}) = \mathrm{co}(\overline{\bigcup_{\iota\in I} W_m(a^*_{\{\iota\}})}) = \mathrm{co}\bigcup_{i\in I}\overline{W_m(a^*_{\{\iota\}})}$$

To get elements of the extremal matrix numerical range $W_{m,e}$ (a^*), we must only use those i ∈I where the maximum in (5) is attained and matrices from $\mathrm{co}(W_{m,e}(a_{\{i\}}))$ in the convex combination. Thus, if Ia denotes the subset of i ∈I where the maximum in (5) is attained, we have

$$\mathrm{co}(W_{m,e}(a^*)) = \mathrm{co}(\bigcup_{i\in I_a} W_{m,e}(a^*_{\{i\}})). \qquad (6)$$

Applying the same argument to b as applied above to a^*, we obtain a (possibly different) $I_b \subseteq I$ so that

$$\mathrm{co}(W_{m,e}(b)) = \mathrm{co}(\bigcup_{i\in I_a} W_{m,e}(b_{\{i\}})). \qquad (7)$$

We claim that there are non-empty subsets $F_a \subseteq I_a$ and $F_b \subseteq I_b$ such that $|F_a| + |F_b| \leq (\ell^2-1)+2 = \ell^2+1$ and

$$(\mathrm{co}(\bigcup_{i\in F_a} W_{m,e}(a^*_{\{i\}}))\cap(co(\bigcup_{i\in I_a} W_{m,e}(b^*_{\{i\}}))\neq\emptyset. \qquad (8)$$

To see this, note that all the matrices we are considering (in the extremal matrix numerical ranges) are hermitian $\ell\times\ell$ matrices with the same trace $||a||^2 = ||b||^2$ and hence they lie in an affine space of real dimension ℓ^2-1 (or affine dimension ℓ^2). By Carathéodory's theorem, any element in the convex hull of a subset S of $\mathbb{R}_n$ can be represented as a convex combination of n+1 or fewer elements of S. A slightly less well-known fact is that if the convex hulls of two non-empty sets S_1, $S_2 \subset \mathbb{R}_n$ (or an affine space equivalent to it) intersect, then we can find a convex combination of n_1 elements in S_1 to equal a convex combination of n_2 elements of S_2, where n_1, $n_2 \geq 1$ and n_1+$n_2 \leq$ n+2. This follows by applying Carathéodory's theorem to the origin, which belongs to the convex hull of

$$\{(x,1) : x \in S_1\} \cup \{ (-y,-1) : y \in S_2\} \subset \mathbb{R}^{n+1}.$$

We can apply this fact because we have (3) valid, and therefore the subsets F_a and F_b exist as claimed

. Let α be in intersection (8) and let F = $F_a \cup F_b$.

Let a_F =[$a_{1,F}$, $a_{2,F}$,..., $a_{\ell,F}$] and b_F =[$b_{1,F}$, $b_{2,F}$,..., $b_{\ell,F}]^t$.

Applying (6) and (7) to a_F^* and b_F, respectively, and noting that F $\cap I_a \supseteq F_a$ and F $\cap I_b \supseteq F_b$, we obtain
$||a_F||$ =$||a||$, $||b_F||$ = $||b||$ and that

$$\alpha \in \mathrm{co}(W_{m,e}\ (a_F^*)) \cap \mathrm{co}(W_{m,e}(b_F)).$$

Hence, by criterion (3) we have
$||u_F||_h$ = $||a_F||^2$ =$||b_F||^2||a||^2$=$||u||_h$.

Since F has at most ℓ^2 +1 elements, the result now follows.

Proposition (1.2.2)[1]

Let A be a unital C^*-algebra and ℓ a positive integer. Suppose that every Glimm ideal in A is (ℓ^2 +1)-primal. Let u =$\sum_{j=1}^{\ell} a_j \otimes b_j \in A \otimes A$. Then $||\theta_Z(u)||_{cb}$ = $||u||_{z,h}$

Proof :-

we know that

$||u||_{z,h}$= sup{$u^G||_h$: G ∈ Glimm(A)} while

$||\theta_Z(\mathrm{u})||_{cb}$ = sup{$||u^J||_{z,h}$: J minimal primal in A}.

Let G ∈ Glimm(A) and consider uG ∈ (A/G) $\otimes_h$ (A/G). In order to compute $||u^G||_h$ we embed A/G faithfully as an algebra of operators, and use injectivity of the Haagerup norm **.** We take as our faithful representation the reduced atomic representation

σ_r : A/G 2→$\prod_\pi \mathcal{B}(H_\pi)$⊂$\mathcal{B}(\bigoplus_\pi H_\pi)$
(one irreducible representation π from each equivalence class in $\widehat{A/G}$).

Let ε > 0. By Lemma (1.2.1), there exist inequivalent irreducible representationsπ_1,..., $\pi_n$of A/G such that n≤ ℓ^2+1 and
$||\mathrm{u}||_h - \varepsilon < ||((\sigma_r \otimes \sigma_r)(\, u^G))_F||_h$,

where H_F = $H_{\pi_1} \oplus \cdots \oplus H_{\pi_n}$ and
=$\pi_1 \oplus \cdots \oplus \pi_n$. Let P_i = ker $\pi_1$for 1≤i≤n and let I =n i=$\bigcap_{i=1}^n P_i$. By hypothesis, I is a primal ideal of A.

Since σ induces a faithful representation of A/I (given by a+I →σ (a) for a ∈ A), we have $||(\,\sigma \otimes \sigma\,)(\,u^G)||_h = ||\,u^I\,||_h$ by injectivity of the Haagerup norm. Now let J be a minimal primal ideal of A contained in I. We have

$|| \, u^G \, ||_h - \varepsilon < ||u^I||_h \leq || \, u^J \, ||_h \leq ||(\theta_Z(\mathrm{u})||_{cb}$.

Since ε and G were arbitrary, $||\mathrm{u}||_{z,h} \leq ||(\theta_Z(\mathrm{u})||_{cb}$. AsZ is a contraction, the result follows.

Our aim now is to show that the converse of Proposition (1.2.2) holds, and for that we need some preparation.

Lemma (1.2.3) [1]

Given a positive definite n×n matrix α of trace 1, there exist n^2 affinely independent rank one (self-adjoint) projections p_i∈M_n (1≤i≤ n^2) so that

$$\alpha = \sum_{i=1}^{n^2} t_i \, \rho_i$$

is a convex combination of the $\rho_i$with t_i> 0 for each i (and n^2 i=1 t_i = 1).
Proof:-

Note that positive semidefinite trace 1 matrices $\rho \in M_n$ correspond to states of M_n via $x \to$ trace(x_ρ)and the rank one projections correspond to the pure states. We argue by induction on n. Of course the n = 1 case is obvious and so we consider n>1.

Recall that we can write any rank one projection ρ in M_n as $\rho = \xi^* \otimes \xi$ for a unit vector ξ in the range of ρ. We can assume the given matrix x is diagonal with (positive) diagonal entries $\alpha_{11} \geq \alpha_{22} \cdots \geq \alpha_{nn} > 0$ in descending order (by replacing the original α by $u^* \alpha u$ for some suitable unitary $u \in M_n$and applying u(·)u^* to the rank one projections we find). Since n>1, α_{11}< 1. Choose $\delta > 0$ so that $\delta < \alpha_{nn} / \alpha_{11} \leq 1$ and $\alpha_{11}(1+(n-1)\ \delta^2)<1$. Let ξ be a primitive mth root of unity with m = 2n−1. Let

$\xi_i = (1, \delta\xi^i, \delta\xi^{2i}, ..., \delta\xi^{(n-)i})/\sqrt{1+(n-1)\delta^2}$ ($1 \leq i \leq m$)and observe that

$$\alpha = \sum_{i=1}^{m} \frac{\alpha_{11\sqrt{1+(n-1)\delta^2}}}{m} (\xi_1^* \xi_i) + (1 - \alpha_{11}\sqrt{1+(n-1)\delta^2})\, \alpha'$$

, where α is essentially a positive definite diagonal matrix of trace 1 in M_{n-1}. Strictly speaking, α' is in M_n and has 0 in the (1,1) entry, but we are able to apply the inductive hypothesis to it. We end up with αas a convex combination of a total of m+(n−1)^2= n^2 rank one projections.

Working with the first row and column (and using a Vandermonde determinant argument), we can check that the projections $\xi_i^* \otimes \xi_i$are affinely independent among themselves and also when we add in the (n−1)^{2}projections we get from the inductive step.

There is a simpler argument which does not quite prove the preceding lemma. The affine dimension of the state space is n^2 and so it is possible to find n^2 affinely independent rank one projections. One can argue that the average of such a collection of projections has to be positive definite. If not, there is a unit vector $\xi \in \mathbb{C}_n$ with $(\beta\xi,\xi)$=0 and then each of the projections p would necessarily satisfy $(p\xi,\xi)$=0. That is the projections would be restricted to lie in an affine space of dimension strictly less than n^2 (in fact in a face of the state space). So β has to be non-singular. For us, it is more convenient to be able to express any pre-assigned, positive definite matrix α with trace(α) = 1 as a convex combination of n^2rank one projections (though we could actually manage with a non-specific β). A variant of the inductive argument above is needed in the next lemma.

Lemma (1.2.4) [1]

For $\ell \geq 2$ and $(\ell-1)^2+2 \leq N \leq \ell^2+1$ there exists u =$\sum_{j=1}^{\ell} a_j \otimes b_j \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that $||u||_h = 1$(where $\mathbb{C}^n$ is considered as the commutative C^*algebra of functions on a discrete space with N points) and such that for any non-empty subset F $\subset$\{ 1,2,...,N\} of N −1 points or fewer,

$$||u_F||_h < 1$$

where $u_F=\sum_{j=1}^{\ell} a_{j,F} \otimes b_{j,F}$ and $a_{j,F}$, $b_{j,F}$ are the restrictions of a_j, b_j to F.

Proof:-

We will adopt a similar notation to that in Lemma(1.2.1) and take I =\{ 1, 2,...,N\}, H_i = C (each i ∈ I) and H = $\oplus_{i\in I} H_i$. Our a_j will be diagonal elements of $\mathcal{B}$(H) with diagonal entries $(a_{j,i})_{i\in I}$ and similarly $b_j = (b_{j,i})_{i\in I}$ (for scalars $a_{j,i}\ b_{j,i} \in \mathbb{C}$). We abbreviate a =[a_1, a_2,..., a_ℓ] and b =[b_1, b_2,..., $b_\ell]^t$.

Let m = $2(\ell-1)$ and n = N −(($\ell-1)^2$ +2). Our $a_{j,i}$ will be zero for m<i≤N and $b_{j,i}$ will be zero for 1≤i≤n. As 0≤n≤m<N, we shall be able to arrange that for each i ∈\{ 1,...,N\} there will be a j with $a_{j,i}$= 0 or $b_{j,i}$= 0 (or both). We will arrange that

$$||a||^2 = ||\sum_{j=1}^{\ell} a_j a_j^*|| = \max_{1\leq i\leq m} \sum_{j=1}^{\ell} |a_{j,i}|^2 = 1$$

and that the maximum is achieved in each position 1≤ i≤ m (so that$\sum_{j=1}^{\ell} |a_{j,i}|^2$ = 1 for 1≤ i≤ m). We will also arrange that $||b||^2 = ||\sum_{j=1}^{\ell} b_j^* \, b_j|| = \max_{n \le i \le N} \sum_{j=1}^{\ell} |b_{j,i}|^2 = 1$

and each $\sum_{j=1}^{\ell} |b_{j,i}|^2 = 1$ for n < i≤ N. We will use (3) to ensure $||u||_h = (||a||^2 + ||b||^2)/2 = 1$ by ensuring that $\alpha \in$ co($W_{m,e}$ (a^*))∩co($W_{m,e}$ (b)) with α the diagonal $\ell \times \ell$ matrix with diagonal entries all equal to $1/\ell$. In fact, will be the only matrix in the intersection. But we achieve this in such a way that all N summands in H are required and therefore for any choice of F giving N−1 or fewer summands we do not satisfy criterion (3) (and hence

$||u_F||_h$ is strictly less than 1 by Timoney . For the a_j (1≤j≤ ℓ), it is helpful to think of ℓ rows a_1,..., a_ℓ which we will specify column by column (where each column has length ℓ). We take a primitive mth root of unity and, for i ∈{ 1,..., m }, we define

$$(a_{1,i}, a_{2,i},..., a_{\ell,i}) = (1, \xi^{i}, \xi^{2i}, \ldots, \xi^{(\ell-1)i})/\sqrt{\ell}.$$

Recall that $a_{j,i}$ is to be zero for i>m and 1≤j≤ ℓ . Any unit vector $\xi \in$ H supported in the summands H_i(1≤i≤m) gives a matrix in $W_{m,e}$(a^*), specifically the matrix

$$\sum_{j=1}^{\ell} |\xi_i|^2 (\eta_i^* \otimes \eta_i)$$

(a convex combination of the $\eta_i^* \otimes \eta_i$, from which we see that $W_{m,e}(a^*)$ is convex)

Where

$$\eta_i = (1, \xi^{-i}, \xi^{-2i}, \ldots, \xi^{-(\ell-1)i})/\sqrt{\ell}.$$

Taking each $\xi_i = 1/\sqrt{m}$we get the matrix α. For future reference, notice that $\xi^{\ell-1}$= −1 and so, for $1 \le i \le m$, the matrix $\eta_i^* \otimes \eta_i$ has the real number $(-1)^i/\ell$ in the (1, ℓ) position.

As with the a_j, it is helpful to think of the b_j as ℓ rows which we will specify column by column. The first two non-zero columns (column n+1 and column n+2) are as follows:

$$\theta_1=(\sqrt{2},0,...,0,\iota)/\sqrt{3} \text{ and } \theta_2=(\sqrt{2},0,...,0,\iota)/\sqrt{3},$$

where $\iota=\sqrt{-1}$. We choose the remaining $(\ell-1)^2$ columns by using Lemma (1.2.3). According to that lemma, we can find $(\ell-1)^2$ affinely independent rank one projections $\rho_k(1\leq k\leq(\ell-1)^2)$ in$M_{\ell}-1$ so that the diagonal $(\ell-1)\times(\ell-1)$ matrix

$$\beta=\begin{pmatrix} \frac{2}{2\ell-3} & 0 & \cdots & 0 \\ 0 & \frac{2}{2\ell-3} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & \frac{1}{2\ell-3} \end{pmatrix}=\sum_{k=1}^{(\ell-1)^2} t_k\rho_k$$

is a convex combination of all of the ρ_k (that is, $t_k>0$ for all k and $\sum_k t_k=1$). (Note that only the final diagonal entry of β is reduced to the value $1/(2\ell-3)$.) Take unit vectors $\mu_k(1\leq k\leq(\ell-1)^2)$ in to be in the ranges ofρ_k, and extend them to vectors $(0,\mu_k)=\tilde{\mu}_k\in\mathbb{C}^{\ell}$. Let

$(b_{1,i}, b_{2,i},..., b_{\ell,i})=\tilde{\mu}_k$ (i = n+2+k,$1\leq k\leq \ell\,(\ell-1)^2$).

We can check that

$$\alpha=\frac{3}{4\ell}(\theta_1^*\otimes\theta_1)+\frac{3}{4\ell}(\theta_2^*\otimes\theta_2)+\sum_{k=1}^{(\ell-1)^2}\frac{2\ell-3}{2\ell}t_k(\mu_k^*\otimes\tilde{\mu}_k),$$

a convex combination. Thus $\alpha\in W_{m,e}$(b). Since $\alpha\in W_{m,e}(a^*)$ also, the criterion (3) guarantees that$||u||_2=1$.

We show next that α is the unique element of co($W_{m,e}(a^*)$)∩ co($W_{m,e}$(b)) = $W_{m,e}(a^*)\cap W_{m,e}(b)$.Suppose that

$$\sum_{i=1}^{m} c_i\eta_1^*\otimes\tilde{\mu}_k,$$

where c_i,r,s, t_k ≥0,$\sum_i$ c_i = 1 and r+s+k t_k = 1, and let the common value be the $\ell \times \ell$ matrix γ . By considering the (1,1)-entry of γ , we see that 1/ ℓ= 2(r+s)/3. On the other hand, recalling that the (1, ℓ)-entry of γ must be real, we see that −r +s = 0. Thus r = s = 3/(4 ℓ). By considering the first row of γ and also the entries $\gamma_{\ell-1,1}$,$\gamma_{\ell-2,1}$,..., $\gamma_{2,1}$, we obtain that V_c = e_1 where V is the m×m matrix whose (i,j)-entry is$\xi^{j(i-1)}$, c = $(c_1,..., c_m)^t$ and e_1 = $(1,0,...,0)^t$. By inspection, one solution is c_i = c_i =⋯=c_i = 1/m (giving γ =α), and this solution is unique because the determinant of V is a non-zero alternant of Vandermonde.

What remains, in order to show that $||u_F||_h$ < 1 for any non-empty proper subset F of {1,2,...,N}, is to show that we cannot find a common element of the convex hulls of the corresponding extremal matrix numerical ranges when we remove any summand H_i(or more than one H_i). However, by the uniqueness established above, the matrix α is the only possible candidate for being such a common element. Removing the summand H_i implies removing one of the η_i if 1≤i≤m, and one of θ_1,θ_2 or some $\tilde{\mu}_k$ if n<i≤N. (If N<ℓ^2+1, then there will be some i falling into both groups.) But to get α on the a_F^* side, we need all of the $\eta_1^* \otimes \eta_i$ (1≤ i≤ m) because they form an affinely independent set (since the equationV_d= 0 has unique solution d = 0). Thus F must contain all i in the range 1≤ i≤ m. On the other hand, it is easily checked that the set $\{\theta_1^* \otimes \theta_1, \theta_2^* \otimes \theta_2\} \cup \{\tilde{\mu}_k^* \otimes \tilde{\mu}_k : 1 \le k(\ell - 1)^2\}$ is affinely independent. Hence, to get α on the $b_F$side, F must contain all i in the range n<i≤N. So if F is a proper subset of {1,2,...,N}, then we cannot satisfy the criterion (3) and so$||u_F||_h$ < 1.

Theorem (1.2.5) [1]

Let A be a unital C^*-algebra. Fix ℓ ≥1. Then

$$||\theta_z(u)||_{cp}=||u||_{z,h}$$

holds for each u =$\sum_{j=1}^{\ell} a_j \otimes b_j$∈ A⊗A if and only if every Glimm ideal in A is (ℓ^2 +1)-primal.

Proof:-

One direction is already done in Proposition (1.2.2) above. For the converse, suppose that A has a Glimm ideal G which is not (ℓ^2 +1)-primal. If G is not 2-primal then there exists u = a ⊗b ∈ A⊗A such that $||u||_{z,h} \neq 0$ and θ_z(u) = 0 .If G is 2-primal (so ℓ >1) then there exists ℓ' ∈{ 2,..., ℓ } and N ∈{(ℓ'−1)^2+2,..., ℓ'^2+1} such that G is (N −1)-primal but not N-primal. Since a tensor with ℓ'summands may be regarded as a tensor with ℓ summands , by the addition of zeros, we may as well assume (for notational convenience) that $\ell' = \ell$. As in the proof of Theorem (1.2.5), there exist primitive ideals P_1,..., P_N of A such that G ⊆ P_i for 1≤i≤N and J := $P_1 \cap \cdots \cap P_N$ is not primal. Furthermore, there exist mutually orthogonal positive elements b_1,..., b_N of A such that $||b_i|| = ||b_i + j||$=1 for 1≤i≤ N and such that for each P ∈ Prim(A) there exists i ∈{ 1,...,N} for which b_i ∈ P. We now re-label these N elements as d_1,..., d_N (to avoid confusion with the elements b_1,..., b_ℓ which we are about to import from Lemma (1.2.4) Let $v = \sum_{j=1}^{\ell} a_j \otimes b_j \in \mathbb{C}^N \otimes \mathbb{C}^N$ have the properties of Lemma (1.2.4), let

β= max{$||v_F||_h$: F a proper non-empty subset of {1,...,N}} < 1

and let

u := $\sum_{j=1}^{\ell}(a_{j,i}\, d_i) \otimes (\sum_{j=1}^{N} b_{j,i}\, d_j)$∈ A⊗A.

On the one hand,

$||u||_{z,h} \geq ||u^G||_h \geq ||u^J||_h = ||v||_h = 1$,

where the penultimate equality follows by applying Lemma (1.1.5) to the linear map $\emptyset: \mathbb{C}^N \to$ span{d_1+J,..., d_N+J} given by $\emptyset(e_{ii})$ = d_i+J (where e_{ii} is the ith standard basis vector). On the other hand, if P ∈ Prim(A) then there exists i'∈{ 1,...,N} such that$d_{i'}$∈ P. Let F ={ 1,...,N}\{ i' }. Then

$||u^P||_h = ||\sum_{j=1}^{\ell}(\sum_{i\neq i'} a_{j,i}\,(d_i + P) \otimes \sum_{i\neq i'} b_{j,i}\, d_i + P)||_h \leq ||v_F||_h \leq \beta$,

where the penultimate inequality follows by applying Lemma (1.1.5) to the linear map $\emptyset$: span{e_{ii}: $i \neq i'$}→span{$d_i + P$: $i \neq i'$} given by $\emptyset$:(e_{ii}) =$d_i + P$.

Hence

$\geq||\theta_z(u)\,||_{cb} = \sup_{\mathrm{P}\in\mathrm{Prim(A)}}||u^P\;||_h < 1.$

Finally, we note that we can extend Theorem (1.2.5) to the non-unital case in the same way as Corollary (1.1.9) extends Theorem (1.1.8).

Chapter 2

Operator Coefficients

Although some definitions are recalled, we will be assumed to be familiar with the basics of free probability theory and more precisely with its combinatorial aspect (non-crossing partitions, free cumulants ,R-diagonal operators ...). For the vocabulary of non-commutative L^p spaces nothing more than the definitions of the p-norm, the Cauchy–Schwarz inequality $|\tau(ab)| \leq a\|a\|_2\|b\|_2$ and the fact that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ will be used.

We proved that the same inequality holds for the norms in the associated non-commutative L_p spaces when p is an even integer, p ≥ d and when the generators of the free group are more generally replaced by ∗-free R-diagonal operators. In particular it applies to the case of free circular operators. We also get inequalities for the non-holomorphic case, with a rate of growth of order d + 1 as for the classical Haagerup inequality. The proof is of combinatorial nature and is based on the definition and study of a symmetrization process for partitions.

Let F_r be the free group on r generators and $|\cdot|$ the length function associated to this set of generators and their inverses. The left regular representation of F_r on $\ell^d(F_r)$ is denoted by λ, and the C^* -algebra generated by $\lambda(F_r)$ is denoted by $C^*_\lambda(F_r)$. ,Haagerup proved the following result, now known as the Haagerup inequality: for any function f: $F_r \to \mathbb{C}$ supported by the words of length d,

$$\Big\|\sum_{g\in F_r} f(g)\lambda(g)\Big\|_{C^*_\lambda(F_r)} \leqslant (d+1)\|f\|_2. \quad (1)$$

This inequality has many applications and generalizations. It indeed gives a useful criterion for constructing bounded operators in $C^*_\lambda(F_r)$ since it implies in particular that for f: $F_r \to \mathbb{C}$,

$$\Big\|\sum_{g\in F_r} f(g)\lambda(g)\Big\|_{C^*_\lambda(F_r)} \leqslant 2\sqrt{\sum_{g\in F_r} (|g|+1)^4 |f(g)|^2},$$

and the so-called Sobolev norm $\sqrt[2]{\sum_{g\in F_r}(|g|+1)^4|f(g)|^2}$ is much easier to compute that the operator norm of λ(f)=∑ f (g)λ(g). The groups for which the same kind of inequality holds for some length function (replacing the term(d+1)in (1) by some power of(d+1)) are called groups with property and have been extensively studied; they play for example a role in the proof of the Baum–Connes conjecture for discrete cocompact lattices of $SL_3(\mathbb{R})$.

Another direction in which the Haagerup inequality was studied and extended is the theory of operator spaces. It concerns the same inequality when the function f is allowed to take operator values. This question was first studied by Haagerup and Pisier , and the most complete inequality was then proved by Buchholz . One of its interests is that it gives an explanation of the (d+1) term in the classical inequality. Indeed, in the operator valued case, the term (d+1) $\|f\|_2$ is replaced by a sum of d+1 different norms off (which are all dominated by $\|f\|_2$ when f is scalar valued). More precisely if S denotes the canonical set of generators of F_r and their inverses, a function f: $F_r \to M_n(\mathbb{C})$ supported by the words of length d can be viewed as a family $(: a_{(h_1,...,h_d)})_{(h_1,...,h_d)\in S^d}$ of matrices indexed by S d in the following way: $a_{(h_1,...,h_d)})_{(h_1,...,h_d)}$ if$|h_1 ... h_d|$=d and a(h_1,..., h_d)=0 otherwise.

The family of matrices a = $(a_h)_{h\in S^d}$ can be seen in various natural ways as a bigger matrix, for any decomposition of $S^d \simeq S^l \otimes S^{d-l}$.If the a_h's are viewed as operators on a Hilbert space H(H=$\mathbb{C}^n$), then let us denote by M_l the operator from $H\otimes \ell^2(S)^{\otimes d-l}$ to $H\otimes \ell^2(S)^{\otimes l}$

Having the following block-matrix decomposition:

$$M_l = (a_{(s,t)})_{s\in S^l,\, t\in S^{d-l}}.$$

Then the generalization is

Theorem (0.1)[2]

Let f:$F_r \to M_n(\mathbb{C})$supported by the words of length d and define $(a_h)_{h\in S^d}$ and M_l for 0≤l≤d as above. Then

$$\left\|\sum_{g\in F_r)} f(g)\lambda(g)\right\|_{c_\lambda^*} \leq (d+1)\|f\|_2$$

The same result has also been extended in [16] to theLp-norms up to constants that are not bounded as d→∞.

Theorem (0.2) [2]

Let f: F_r →ℂ be a function supported on W_d^+ . Then

$$\left\|\sum_{g\in W_d^+} f(g)\lambda(g)\right\|_{C_\lambda^*(F_r)} \leqslant \sqrt{e}\sqrt{d+1}\|f\|_2.$$

A similar result has been obtained when the operators λ(g_1),...,λ(g_r) are replaced by free R-diagonals elements. These results are proved using combinatorial methods: to get bounds on operator norms the authors first get bounds for the norms in the

non-commutativeL_p-spaces forpeven integers, and makeptend to infinity. For an even integer, the L_p-norms are expressed in terms of moments and these moments are studied using the free cumulants.

In this paper we generalize and improve these results to the operator-valued case. As for the generalization of the usual Haagerup inequality, the operator valued inequality we get gives an explanation of the term $\sqrt{d+1}$: for operator coefficients this term has to be replaced by the ℓ^2 combination of the norms$\|M_l\|$introduced above. A precise statement is the following. We state the result for the free group F_∞ on countably many generators$(g_i)_{i\in\mathbb{N}}$, but it of course applies for a free group with finitely many generators.

Theorem (0.3) [2]

Fo r d ∈ℕ, denote by $W_d^+ \subset F_\infty$the set of words of length din the gi's (but not their inverses).Fork=(k_1,..., k_d)∈$\mathbb{N}^d$ let $g_k = g_{k_1}...g_{k_d} \in W_d^+$. Let a =$(a_k)_{k\in\mathbb{N}^d}$ be a finitely supported family of matrices, and for 0 ≤l ≤ d denote by M_l = $(a_{(k_1,...,k_l),(k_{l+1},...,k_d)})$the corresponding $\mathbb{N}^l\times\mathbb{N}^{d-1}$ block-matrix. Then

$$\left\|\sum_{k\in\mathbb{N}^d} a_k \otimes \lambda(g_k)\right\| \le 4^5\sqrt{e}\left(\sum_{l=0}^{d}\|M_l\|^2\right)^{1/2} . \qquad (2)$$

Note that even when$a_k \in \mathbb{C}$, this really is (up to the constant 4^5 an improvement of Theorem (2.1.2). Indeed it is always true that for any l, $\|M_l\|^2 \le T_r(M_l^* M_l)\sum_k |a_k|^2$. There is equality when l=0 or d but the inequality is in general strict when 0<l<d. Thus if the a_k's are scalars such that$\|(a_k)\|_2$=1 and $\|M_l\| \le 1/\sqrt{d}$ for 0<l<d , the inequality in Theorem (0.3) becomes k$\|\sum_{k\in\mathbb{N}^d} a_k\lambda(g_k)\| \le 4^5\sqrt{3e}\|(a_k)\|_2$. Since the reverse inequality $\|\sum_{k\in\mathbb{N}^d} a_k\lambda(g_k)\| \ge \|(a_k)\|_2$ always holds, we thus get that$\|\sum_{k\in\mathbb{N}^d} a_k\lambda(g_k)\| \simeq \|(a_k)\|_2$with constants independent of d. An example of such a family is given by the following construction: if p is a prime number and $a_{k_1,\dots,k_d}$ = exp(2iπ$k_1 \dots k_d$/p)/$p^{d/2}$ for any k_i ∈{1,...,p} and a_k =0 otherwise then obviously$\sum_k |a_k|^2$ =1, where as a computation see (Lemma 2.2.9) shows that$\|M_l\|^2 \le \frac{d}{p}\, if$ 0<l<d. It is thus enough to take p≥ d^2.

The same arguments apply for the more general setting of∗-free R-diagonal elements (∗-free means that the C^*-algebras generated are free). Moreover we get significant results

already for theL_p-norms for peven integers. Recall that on aC^*-algebra A equipped with a trace τ ,the p-norm of an element x∈A is defined by $\|x\|_p = \tau(|x|^p)^{1/p}$for 1≤p<∞, and

that for p=∞ the L^∞norm is just the operator norm. In the following the algebra $M_n \otimes$A will be equipped with the trace$T_r \otimes$τ. The most general statement we get is thus:

Theorem (0.4) [2]

Let c be an R-diagonal operator and$(c_k)_{k\in\mathrm{N}}$ a family of∗-free copies of c on a tracial C^*-probability space(A,τ). Let (a_k)

K ∈ Nd be as above a finitely supported family of matrices and M_l =($a_{(k_1,\dots,k_l),(k_{l+1},\dots,k_d)}$) zzfor 0≤l≤ d the corresponding$\mathbb{N}^l \times \mathbb{N}^{d-1}$block-matrix.

For k=(k_1,..., k_d)∈$\mathbb{N}^d$ denote c_k=c_{k_1}... c_{k_d}. Then for any integer m,

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k \right\|_{2m} \leqslant 4^5 \|c\|_2^{d-2} \|c\|_{2m}^2 e\sqrt{1+\frac{d}{m}} \left(\sum_{l=0}^{d} \|M_l\|_{2m}^2 \right)^{1/2}. \quad (3)$$

For the operator norm,

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k \right\| \leqslant 4^5 \|c\|_2^{d-2} \|c\|^2 \sqrt{e} \left(\sum_{l=0}^{d} \|M_l\|^2 \right)^{1/2}. \quad (4)$$

When theC_{pK}'s are circular these inequalities are valid without the factor $4^5\|c\|_2^{d-2}\|c\|^2$.

For The outline of the proof of Theorem (0.4), we first prove the statement for the L_p-norms when p=2m is an even integer (letting p→∞ leads to the statement for the operator norm). This is done with the use of free cumulants that express moments in terms of non-crossing partitions . More precisely to any integer n, any non-crossing partition π of the set {1,...,n} and any family b_1,..., b_n∈Athe free cumulant $k_\pi[b_1,...,b_n]\in\mathbb{C}$ is defined . When π=1nis the partition into only one block, k_π is denoted by k_n. The free cumulants have the following properties:

•Multiplicativity :If π={V_1,..., V_s}, k_π [b_1,..., b_n]=$\prod_{i} k_{|v_i|}[(b_k)_{k\in v_i}]$.

•Moment-cumulant formula :τ(b_1,..., b_n)= $\sum_{\pi\in NC(n)} k_\pi$ [$b_1,\ldots,b_n$].

•Characterization of freeness: A family $(A_i)_i$of subalgebras is free iff all mixed cumulants vanish, i.e. for any n, any $b_k\in A_{ik}$and any π ∈NC(n)then $k_\pi[b_1,..., b_k]$=0 unless $i_k=i_l$

for any k and l in a same block of π.

The first two properties characterize the free cumulants (and hence could be taken as a definition), whereas the third one motivates their use in free probability theory. Since the∗-distribution of an Operator c∈(A,τ)is characterized by the trace of the polynomials in c and c^*, the cumulants involving only c and c^*(that is the cumulants k_π $[(b_i)]$withb_i ∈{c, c^*}for any i) depend only on the ∗-distribution of c.

In order to motivate the combinatorial study of certain non-crossing partitions in the first section, let us shortly sketch the proof of the main result.With the notation of Theorem (2.1.4) let ,A=$\sum a_k \otimes c_k$.For k=(k(1),...,k(d))∈$\mathbb{N}^d$ set $\tilde{a}_k$=$a_{(k(d),...,k(1))}$ and $\tilde{c}_k$=$c_{k(d)}$ ... $c_{k(1)}$ so that $(\tilde{c}_k)^*$=$c^*_{k(1)}\cdots c^*_{k(d)}$. Then

A^*=$\sum_k \tilde{a}^*_k \otimes \tilde{c}^*_k$, and for p=2m the pth power of the p-norm of A is just the trace T_r⊗τ of $(AA^*)^m$, which can be expressed by linearity as the sum of the terms of the form T_r $(a_{k_1}\tilde{a}^*_{k_2}\cdots a_{k_{2m-1}}\tilde{a}^*_{k_{2m}}) \otimes$ τ$(c_{k_1}\tilde{c}^*_{k_2}\ldots c_{k_{2m-1}}\tilde{c}^*_{k_{2m}})$.The expression $c_{k_1}\tilde{c}^*_{k_2}\ldots c_{k_{2m-1}}\tilde{c}^*_{k_{2m}}$is the product of 2dmterms of the form c_i or c^*_i(for i∈ℕ).Apply the moment-cumulant formula to its trace. Using the characterization of freeness with cumulants and then the multiplicativity of cumulants and the fact that cumulants only depend on the ∗-distribution we thus get

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi\in NC(2dm)} \kappa_\pi[c_{d,m}] \underbrace{\sum_{(k_1,\ldots,k_{2m})\prec\pi} Tr\big(a_{k_1}\tilde{a}^*_{k_2}\cdots\tilde{a}^*_{k_{2m}}\big)}_{\stackrel{\text{def}}{=} S(a,\pi,d,m)},$$

where for k ∈$\mathbb{N}^{2dm}$ and π ∈NC(2dm)we write k ≺π if k_i =k_j when ever i and j belong to the same block of π and where

$$c_{d,m} = \overbrace{\underbrace{c,\ldots,c}_{d}, \underbrace{c^*,\ldots,c^*}_{d}, \ldots, \underbrace{c,\ldots,c}_{d}, \underbrace{c^*,\ldots,c^*}_{d}}^{2m \text{ groups}}.$$

Up to this point we did not use the assumption that c is R-diagonal, since the R-diagonal operators are those operators for which the list of non-zero cumulants is very short , we get that the previous sum can be restricted to a sum over the partitions in the subset NC^*(d, m)⊂NC(2dm), which is defined and extensively studied in part 1.2:

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi\in NC^*(d,m)} \kappa_\pi[c_{d,m}] S(a,\pi,d,m). \qquad (5)$$

The term k_π $[c_d$,m]is easy to dominate (Lemma 2.2.5). When the a_k's are scalars the second term S(a,π,d,m)can be dominated by$\|(a_k)\|_2^{d-2}$ (by the usual Cauchy–Schwarz inequality). But here the fact that we are dealing with operators and not scalars forces to derive a more sophisticated Cauchy–Schwarz type inequality that may control explicitly the expressions S(a,π,d,m)in terms of norms of the operators M_l. This is one of the main technical results in this section, Corollary (2.2.4). This Corollary states that

$$\left|S(a,\pi,d,m)\right| \leqslant \prod_{l=0}^{d} \|M_l\|_{2m}^{2m\mu_l} \qquad (6)$$

for some non-negativefor some non-negative μ_l with $\sum_l \mu_l$ =1. Moreover the μ_l are explicitely described by some combinatorial properties of π. This inequality is proved through a process of "symmetrization" of partitions. The basic observation is that if one applies a simple Cauchy–Schwarz inequality to S(a,π,d,m)(Lemma 2.2.1) , this corresponds on the level of partitions to a certain combinatorial operation of symmetrization. This observation was already used implicitely, in some special case: Buchholz indeed notices that for d=1 and if π is a pairing (i.e. has blocks of size 2), this Cauchy–Schwarz inequality corresponds to some transformation of pairings (for which he does not give a combinatorial description), and that iterating this inequality eventually leads to an domination of the form (6) (for d=1) but in which he does not compute the exponents μ_0 and μ_l.

In our more general setting it also appears that repeating this operation in an appropriate way turns every non-crossing partition π ∈NC^*(d, m)into one very simple and fully symmetric partition for which the expression S(a,π,d,m)is exactly the (2m-power of the 2m-) norm of one of the M_l's. This is stated and proved in Corollary (2.1.6) and Lemma (2.1.2) One important feature of our study of the symmetrization operation on NC^*(d, m)is the fact that we are able to determine some combinatorial invariants of this operation (see part 1.3). This allows to keep track of the exponents of the$\|M_l\|_{2m}$ that progressively appear during the symmetrization process, and to compute the coefficients μ_l in (6).

The second technical result that we prove and use is a finer study of NC^*(d, m).The main conclusion is Theorem (2.1.7) which expresses that partitions in NC^*(d, m)have mainly blocks of size 2 and that NC^*(d, m)is not very far from the set $NC(m)^{(d)}$ of non-decreasing chains of non crossing partitions on m (in the sense that there is a natural surjection $NC^*(d, m) \to NC(m)^{(d)}$ such that the fiber of any point has a cardinality dominated by a term not depending on d). This combinatorial result is then generalized in Theorem 1.13 and Lemma (2.1.16), and then used to transform the sum in (5) into a sum over NC(m)(d) for which the combinatorics .

We prove also the following results, which are extensions to the non-holomorphic case of the previous results and their proofs. Let c be an R-diagonal operator and $(c_k)_{k\in\mathbb{N}}$ Na family Of ∗-free copies of con a tracial C^*-probability space (A ,τ).For $\varepsilon=(\varepsilon_1,\dots,\varepsilon_d)\in\{1,*\}^d$ and $k=(k_1,\dots,k_d)\in\mathbb{N}^d$ denote $c_{k,\varepsilon} = c_{k_1}^{\varepsilon_1}\dots c_{k_d}^{\varepsilon_d}$. The result is an extension of Haagerup's inequality for the space generated by the $c_{k,\varepsilon}$for the k, ε satisfying $k_i = k_{i+1} \Rightarrow \varepsilon_i = \varepsilon_{i+1}$, i.e. for which $\lambda(g)_{k,\varepsilon}$ has length d. Denote by I_d the set of such(k, ε).

Theorem(0.5)[2]

Let$(a_{(k,\varepsilon)})_{(k,\in)\in\mathbb{N}\times\{1,*\}^d}$ be a finitely supported family such that $a_{(k,\varepsilon)}=0$ for (k, ε)$\notin I_d$.For $0\le l\le d$, let M_l be the matrix formed as above from $(a_{(k,\varepsilon)})$for the decomposition$(\mathbb{N}\{1,*\})^d=(\mathbb{N}\times\{1,*\})^l\times(N\times\{1,*\})^{d-l}$. Then for any $p\in 2\mathbb{N}\cup\{\infty\}$

$$\left\|\sum_{(k,\varepsilon)\in(\mathbb{N}\times\{1,\varepsilon\})^d} a_{k,\varepsilon}\otimes c_{k,\varepsilon}\right\|_p \leqslant 4^5\|c\|_p^2\|c\|_2^{d-2}(d+1)\max_{0\leqslant l\leqslant d}\|M_l\|_p.$$

Similarly for self-adjoint operators we have:

Theorem (0.6) [2]

.Let μ be a symmetric compactly supported probability measure on ℝ, and ca self-adjoint element of a tracial C^*-algebra distributed as μ.

Let$(c_k)_{k\in\mathbb{N}}$be self-a djoint free copies of c and $(a_{k_1,\dots,k_d})_{k_1,\dots,k_d\in\mathbb{N}}$be a finitely supported family of matrices such that $a_{k_1,\dots,k_d}$=0 if k_i =$k_{i+1}$for some 1 ≤i<d. Then for any p∈ 2ℕ∪{∞},

$$\left\| \sum_{(k_1,\dots,k_d)\in\mathbb{N}^d} a_{k_1,\dots,k_d} \otimes c_{k_1} \dots c_{k_d} \right\|_p \leqslant 4^5 \|c\|_p^2 \|c\|_2^{d-2} (d+1) \max_{0\leqslant l\leqslant d} \|M_l\|_p. \qquad (7)$$

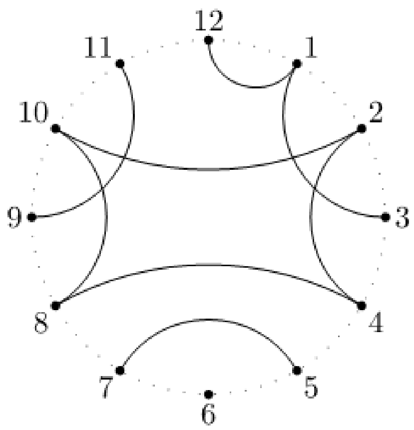


Fig. 1. A graphical representation of the partition{{1,3,12},{2,4,8,10},{5,7},{6},{9,11}}.

For the case of the semicircular law and scalar coefficientak, this result is not new. It is due to, and was reproved using combinatorial methods by Biane and Speicher, . Our proof is a generalization of their proof and uses it. Note also that the condition that $a_{k_1,\dots,k_d}$=0 if k_ii =$k_{i+1}$for some i is crucial to get (7): indeed if $a_{k_1,\dots,k_d}$=0 except for $a_{1,\dots,1}$=1 then we have the equality $|| \sum_{k\in\mathbb{N}^d} a_k \otimes c_{k_1}, \dots c_{k_d} ||_p = \left\| C_1^d \right\|_p$, whereas $\max_l \|M_l\|_p$=1 and if μ is not a Bernoulli measure $||C||_p^2||C||_2^{d-2}$(d+1)=o($||C||_{dp}^d$) When d→∞. The inequality (7) thus does not hold for this choice of(a_k), even up to a constant. These results are of some interest since they prove a new version of Haagerup's inequality in a broader setting, but they are still unsatisfactory since one would expect to be able to replace the term(d+1)$\max_{0\leq l\leq d}\|M_l\|_p$ by $\sum_{l=0}^{d}\|M_l\|$.

The paper is organized as follows: the first part only deals with combinatorics of non-crossing partitions. In the second part we use the results of the first part to get inequalities for the expressions S(a,π, d,m) . In the third and last part we finally prove the main results stated above.

Section (2-1): symmetrization of Non crossing partition

the norm of an element of the form $\sum_{i=(i_1,\dots,i_d)} a_i \otimes \lambda\,(g_{i_1},\dots,g_{i_d}$ *is* less than 4^5 $\sqrt{e}(\|M_0\|^2+\cdots+\|M_d\|^2)^{1/2}$, where $M_0,\dots,M_d$ are d + 1 different block-matrices naturally constructed from the family $(a_i)_{i\in I^d}$ for each decomposition of $I^d \simeq I^l\times I^{d-l}$ with l = 0, . . . , d. It is also proved that the same inequality holds for the norms in the associated non-commutative L_p spaces when p is an even integer, p ≥ d and when the generators of the free group are more generally replaced by ∗-free R-diagonal operators. In particular it applies to the case of free circular operators. We also get inequalities for the non-holomorphic case, with a rate of growth of order d + 1 as for the classical Haagerup inequality. The proof is of combinatorial nature and is based on the definition and study of a symmetrization process for partitions. For any integer n, we denote by [n] the interval {1, 2, ...,n}, which we identify with ℤ/nℤ and which is endowed with the natural cyclic order: for $k_1,\dots,k_p\in$[n] we say that $k_1<k_2<\cdots<k_p$ for the cyclic order if there are integers $\ell_1,\dots,\ell_1$ such that $\ell_1<\ell_2<\cdots<\ell_p, k_i=l_i$ mod n and $l_p-l_1\leq$ n. In other words, if the elements of [n]are represented on the vertices of a regular polygon with n vertices labelled by elements of[n]as in **Fig. 1**, then we say that $k_1<k_2<\cdots<k_p$if the sequence $k_1,\dots,k_p$can be read on the vertices of the regular polygon by following the circle clockwise for at most one full circle.

If π is a partition of [n], and i ∈[n], the element of the partition π to which i belongs is denoted by π(i). We also write $i\sim\pi$ j if i and j belong to the same block of the partition π.

If the elements of [n] are represented on the vertices of a regular polygon with n vertices, a partition π of[n]is then represented on the regular polygon by drawing a path between i and j if i~π j. See Fig. 1 for an example.

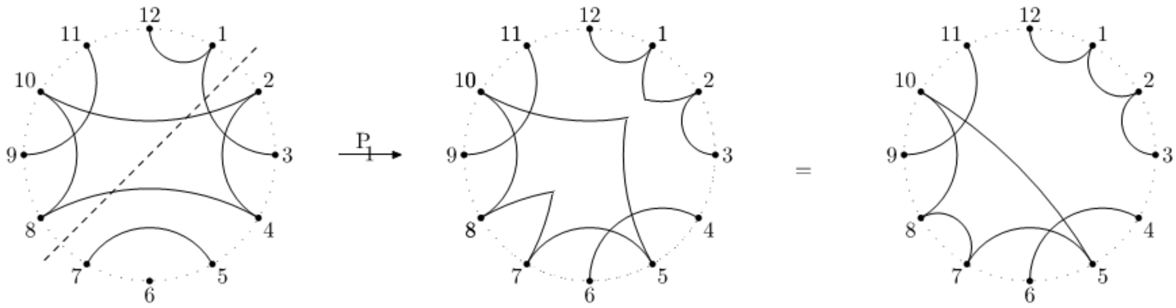


Fig. 2. The operationP1on the partition{{1,3,12},{2,4,8,10},{5,7},{6},{9,11}}.

We introduce the operationsP_k on the set of partitions of an even number n=2N.This definition is motivated by Lemma (2.2.1).

Definition (2.1.1) [2]

Let k∈[2N], and l_k the subinterval of [2N]of length N and ending with k, l_k =

$\{k-N+1, k-N+2, \dots, k\}$and $S_k^{(N)}$(or simply $S_k$when no confusion is possible) the symmetry $S_k(i)$ =2k+1−i (note that S_k is an involution of [2N]that exchanges l_k and[2N]\ l_kIk). For a partition π of[2N], s_k(π)is the symmetric of π: A∈s_k(π)if s_k^{-1}(A)= s_k(A)∈π. In other words i~s_k (π)j if and only If s_k(i)~πs_k(j).

For any partition π of [2N], we denote by P_k(π) the partition of [2N] that we view as a symmetrization(in mathematics, symmetrization is a process that converts any function in *n* variables to a symmetric function in *n* variables. Conversely, anti-symmetrization converts any function in *n* variables into an antisymmetric function)[6] of π around d_k , and which is formally defined by the following: if one denotes π′=P_k(π), then

For I , j ∈ l_k,i~π′j if and only if i~πj, (8)

For i, j ∈[2N]\ l_k,i~π′j if and only if s_k (i)~πs_k(j), (9)

For i∈l_k,j / ∈l_k,i~π′j if and only if i~πs_k(j)and ∃l/∉Il_k ,i~πl. (10)

It is straightforward to check that this indeed defines a partition of[2N], and that it is symmetric with respect tok, that is sk(π′)= π′.

The operation $P_k$is perhaps more easily described graphically: represent π on a regular polygon as above, and draw the symmetry line going through the middle of the segment[k, k+1].A graphical representation of P_k(π)is then obtained by erasing all the half-polygon not containing kand replacing it by the mirror-image of the half-polygon containingk. See Fig. 2 for an example.

The following lemma expresses the fact that applying sufficiently many times appropriate operators P_k, one can make a partition symmetric with respect to all thesk's. See Fig. 3 to see an example of this symmetrization process.

Lemma(2.1.2) [2]

Let m be a positive integer. Let $k \in \mathbb{N}$ such that 2k ≥m. Then for any partition π of [2m], the partition $\pi_k = P_{2^k} P_{2^{k-1}} \ldots P_2 P_1 P_m(\pi)$is one of the four following partitions:

$$\pi_k = 0_{2m} = \{\{j\}, j \in [2m]\},$$

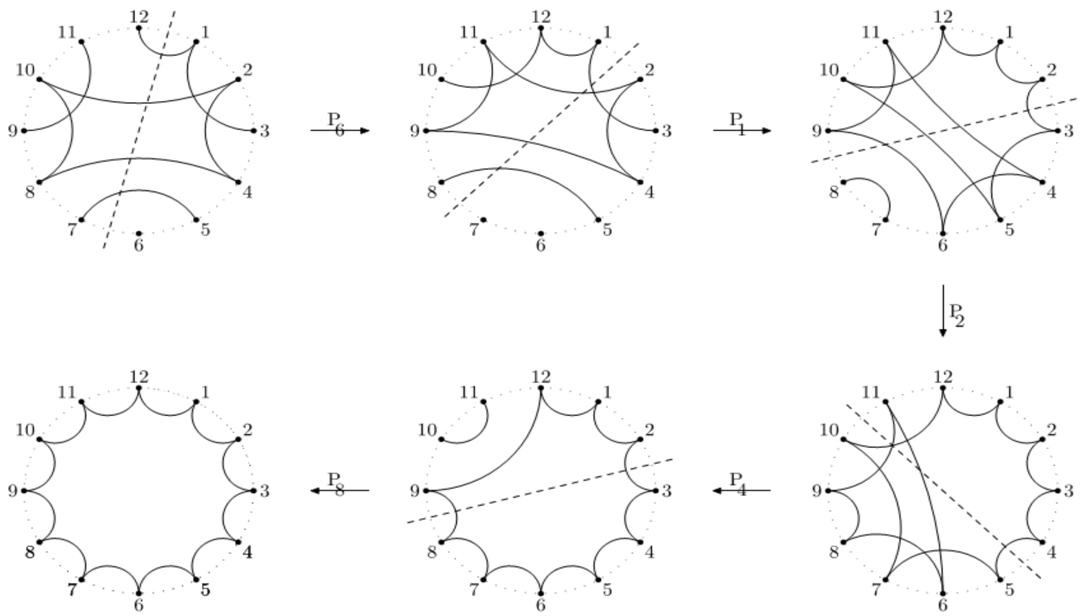


Fig. 3. The symmetrization process starting from the partition{{1,3,12},{2,4,8,10},{5,7},{6},{9,11}}.

π_k=c_m={2j;2j+1},j∈[m]},

π_k=r_m={2j−1;2j},j∈[m]},

π_k=1_{2m}={[2m]}.

Proof:-

LetA=I_m∩π(1)\{1} and B=([2m]\Im)∩π(1). The four cases correspond respectively to the four following cases:

(i) A=B=∅.

(ii) A=∅andB≠∅ .

(iii) A≠ ∅ andB=∅.

(iv) A≠∅ andB≠ ∅ .

In the first case, it is straightforward to prove by induction on k that π_k includes the blocks{i}for anyi∈{1,...,2^{k+1} }.

If A=∅ and B≠∅ , then P_m(π)includes the block {0,1}and this implies that $P_1 P_m$(π) includes the blocks {0,1} and {2,3}, which in turn implies that $P_1 P_2 P_m$(π) includes the blocks{0,1}, {2,3} and{4,5} . . . More generally π_k includes the blocks {0,1},{2,3},..., {2^{k+1}, 2^{k+1}+1} (this can be proved by induction). For 2^{k+1} ≥2mthis is exactly π_k =c_m.

In the same way, in the third case it is easy to prove by induction on k that π_k includes the blocks {$2l-1$,$2l$}for$l$∈{1,..., 2^k }.

The fourth case follows from a similar proof by induction that π_k(1)contains{0,1,2,..., 2^{k+1} 2^{k+1} +1}. The details are not provided.

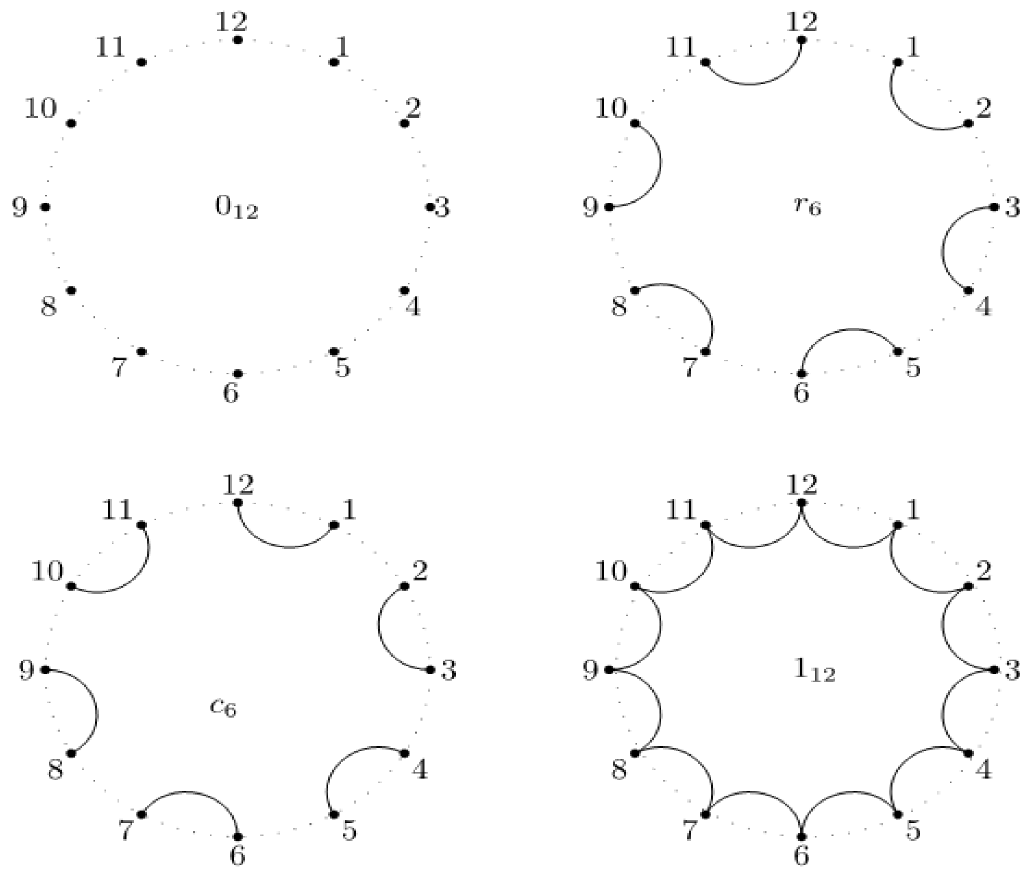


Fig. 4. The partitions $0_{12}, r_6, c_6$and 1_{12}.

Although $P_k(\pi)$ is defined for any partition π, we will be mainly interested in the case when π is a non-crossing partition, and more precisely when $\pi \in NC^*(d,m)$.

We first recall the definition of a non-crossing partition. A partition π of[N]is called non crossing if for any distinct $i < j < k < l \in [N], i \sim \pi\, k$ and $j \sim \pi l$ implies $i \sim \pi j$ (in this definition either take for<the usual order on{1,...,N}or the cyclic order since it gives to the same notion). More intuitively π is non-crossing if and only if there is a graphical representation of π (on a regular polygon with n vertices as explained in the beginning) such that the paths lie inside the polygon and only intersect (possibly) at the vertices of the regular polygon. For example the partitions of Figs. 1, 2 are crossing, whereas the partitions in Figs. 4, 5, 6 are all non-crossing. The set of non-crossing partitions of $[N]$ is denoted by *NC(N).* The cardinality of $NC(N)$is known to be

equal to the Catalan number $(2N)!/(N!\,(N+1)!)$, but we will only use that it is less that 4^{N-1}. we introduce the subset $NC^*(d,m)$ of $NC(2dm)$.

In the following, for a real number x one denotes by x the biggest integer smaller than or equal to x.

Divide the set[2dm]into 2mintervals $J_1 ... J_2$mof sized: the first one is

$$J_1 = \{1,2,\ldots,d\}, \text{ and the } kth \text{ is } J_k = \{(k-1)d+1,\ldots,k_d\}.$$

To each element of [2dm]we assign a label in{1,...,d}in the following way: in any interval J_k of size d as above, the elements are labelled from 1 to d if k is odd and from d to 1 if k is even. We shall denote by $A_k$the set of elements labelled by k.

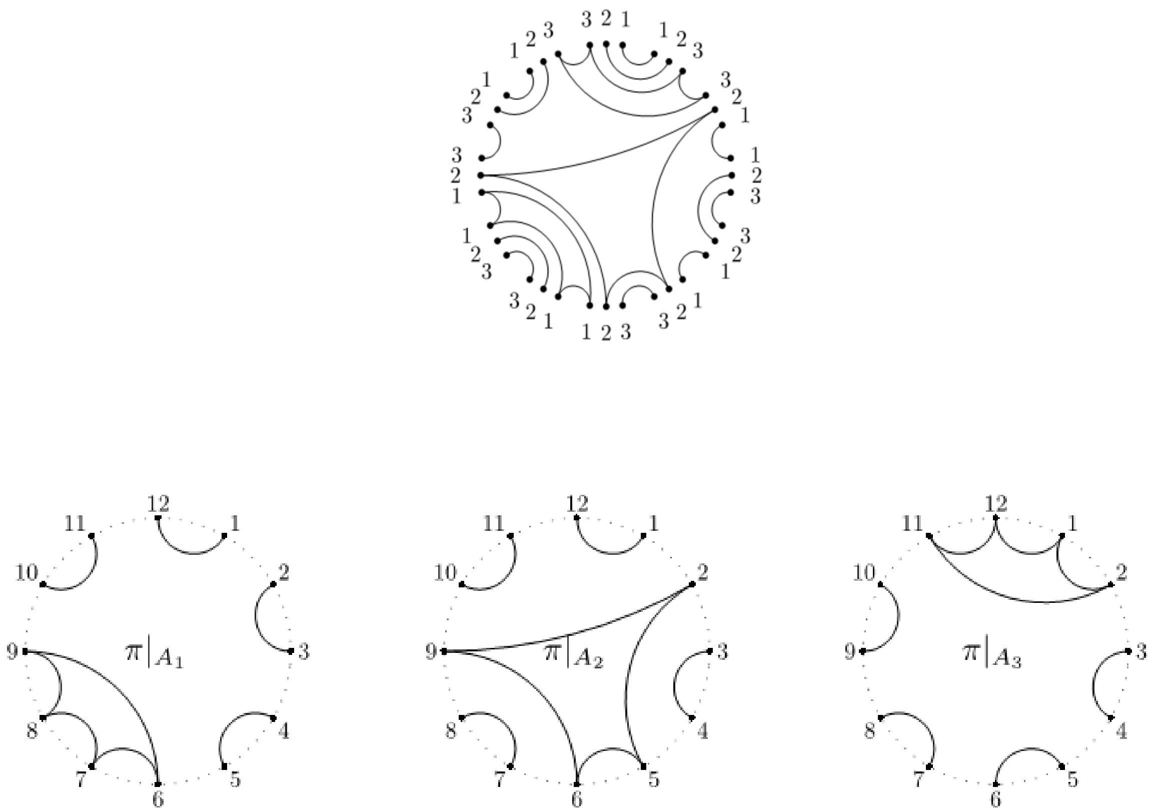


Fig. 5. A graphical representation of a partition π in N C*(3,6)and the corresponding restrictions $\pi|_{A_1}$,$\pi|_{A_2}$and$\pi|_{A_3}$

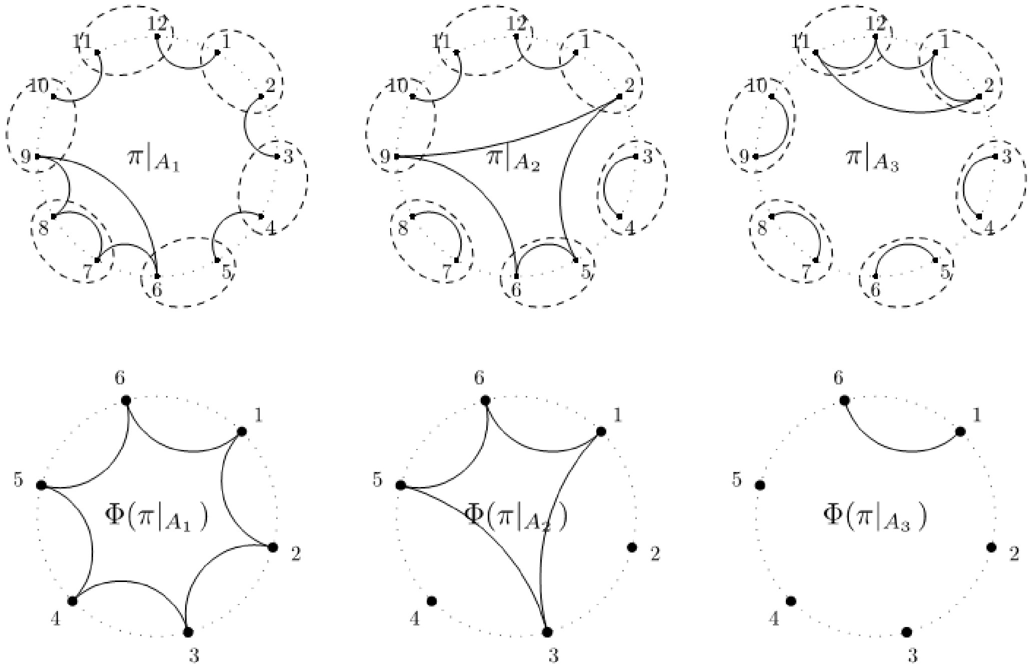


Fig. 6. The map P for the partition π∈C*(3,6)of Fig. 5.

Definition (2.1.3) [2]

A non-crossing partition π of[$2dm$] belongs to NC*(d, m) if each block of the partition has an even cardinality, and if within each block, two consecutive elements I and j belong to intervals of sized of different parity. Formally, the last condition means that [(i−1)/d]≠[(j−1)/d]mod 2 or equivalently k(i)≠k(j)mod 2 when i ∈J_k(i) and Here are some first elementary properties ofN C*(d, m):

Lemma(2.1.4) [2]

If d=1, a non-crossing partition π ∈NC(2m)belongs toNC*(1,m)if and only

if it has blocks of even cardinality. A non-crossing partition of[2dm]is in NC*(d, m)if and only if it has blocks of even cardinality and it connects only elements with the same labels(i.e. it is finer than the partition {A_1,..., A_d}).

Proof:-

The first statement is a particular case of the second statement, which we now prove. For any i∈[2dm] denote by $k(i)$the integer such that i ∈$J_{k(i)}$: $k(i) = 1 + [(i-1)/d]$. Let $\pi \in NC^*(d,m)$. Then by the definition of NC*(d, m)every block of π contains as many elements i such that $k(i)$ is odd than elements i such that $k(i)$ is even. We have to prove that if s and t are two consecutive elements of a block of π, then s and t have the same labellings. Assume for example that s belongs to an odd interval, i.e. $k(s)$is odd, and denote by $l(s)$the label of s Then s= $(k(s)-1)d + l(s)$.In the same way ,k(t)is then even and if l(t)is the label of t ,we have that t =k(t)d+1−l(t). Hence the number of elements $i \in \{s+1, \ldots, t-1\}$ such that $k(i)(= 1 + [(i-1)/d])$is odd is equal to $d - l(s) + d \cdot (k(t) - k(s) - 1)/2$, and the number of elements i such that $k(i)$is even is equal to d$-l(t) + d \cdot (k(t) - k(s) - 1)/2$. But since π is non-crossing, the interval $\{s+1, \ldots, t-1\}$is a union of blocks of π and therefore contains as many elements i such that $k(i)$is odd than elements i such that $k(i)$is even. This implies $l(s) = l(t)$. The proof is the same if $k(s)$ is even.

Now assume that π∈ $NC(dm)$has blocks of even cardinality and that π is finer than the partition {A_1,..., A_d}.Let s and t be two consecutive elements of a block of π. Then there is i such that s, t∈ A_i. Since π is non-crossing and π is finer than {A_1,..., A_d},the set {s+1,...,t−1}∩A_i is a union of blocks of π, and in particular it has an even cardinality. But {s+1,...,t−1}∩A_i is the set of elements labelled by i in the union of the intervals J_k for $k(s) < k < k(t)$(for the cyclic order). Hence its cardinality is $k(t) - k(s) - 1$. Hence k(t)−k(s) is odd. Since s and t are arbitrary, this proves that $\pi \in NC^*(d,m)$.

Thus to any $\pi \in NC^*(d,m)$we can assign d partitions π$|_{A_1}$,..., $|_{A_d}$, which are the restrictions of π to A_1,..., A_d, respectively. It is immediate that for any i ∈{1,...,d}, πA_i∈NC*(1,m). See **Fig. 5** for an example. To study NC*(d, m), we thus begin with the study of NC*(1,m).

The first lemma shows that if k is a multiple of d, then P_k maps NC*(d, m)into itself:

Lemma (2.1.5) [2]

If k ∈N and π∈ℕ**C**(2N)then P_k(π)∈NC(2N). If k ∈ ℕ then for any π ∈NC*(d, m), P_{kd}(π)∈NC*(d, m). Moreover if π ∈NC*(d, m), then for any i∈{1,...,d}:

P_{kd} d(π) $|_{A_i}$=$P_k|_{A_i}$).

We have the following corollary of Lemma (2.1.2)

Corollary (2.1.6) [2]

Let π ∈NC*(d, m). Then for 2^k ≥m, the partition π_k =$P_{2^k d}P_{2^{k-1}d}$...

P_{2d} $P_d P_{md}$ (π) is one of the 2d+1partitions $\sigma_i^{(d,m)}$ for l =0,1,...,d and $\widetilde{\sigma}_i^{(d,m)}$for l =1,2,...,d defined by

$$\sigma_i^{(d,m)}|_{A_i}= \begin{cases} c_m & if\ 1 \le i \le l, \\ r_m & if\ \ l \le i \le d, \end{cases}$$

$$\widetilde{\sigma}_i^{(d,m)}kd|_{A_i}= \begin{cases} c_m & if\ 1 \le i \le l, \\ 1_{2m} & if\ i=j, \\ r_m & if\ l \le i \le d. \end{cases}$$

Moreover for any integer i, P_{id} (π)=π when π is one of the partitions $\sigma_i^{(d,m)}$ for l =0,1,...,d and $\widetilde{\sigma}_i^{(d,m)}$l for l=1,2,...,d.

Proof:-

Let k and π as above. By Lemma (2.1.5), $\pi_k|_{A_i}$ =$P_{2^k}P_{2^{k-1}}$... P_2 $P_1 P_m$ ($\pi|_{A_i}$), which is by Lemma (2.1.2) one of 0_{2m}, r_m, c_m and 1_{2m}. But since 0_{2m}does not have blocks of even sizes, only the three r_m, $c_m$and $1_{2m}$are possible.

Let 1≤i<j≤d.If $\pi_k|_{A_i}$= r_m or 1_{2m}then in particular i∼π_k2d+1−i. Since π_k is noncrossing ,j ≁π_k 1−j, which implies that $\pi_k|_{A_j}$= c_m, 1_{2m}. Thus $\pi_k|_{A_j}$=r_m .In the same way if $\pi_k|_{A_j}$ =$c_m$or 1_{2m}then $\pi_k|_{A_j}$=c_m. This concludes the proof.

Similarly, the second claims follows from the fact (easy to verify) that P_i (π)=π for any i∈[2m]when π=1_{2m}, $r_m$orc_m.

An important subset of $NC^*(d,m)$is the subset $NC_2^*(d,m)$of partitions in $NC^*(d,m)$ with blocks of size 2. $NC_2^*(d,m)$is naturally in bijection with the non-decreasing chains (for the natural lattice structure on NC(m)) of length d of non-crossing partitions of[m]. Let us denote by $NC(m)^{(d)}$this set of non-decreasing chains in $NC(m)$,for the order of refinement, given by $\pi \leq \pi'$ if π' is finer that π. The bijective map $NC_2^*(d, m) \to NC(m)^{(d)}$ extends naturally to a (of course non-bijective) map $NC^*(d,m) \to NC(m)^{(d)}$which is of interest. We now describe the construction of this map.

Let π ∈$NC^*(1,m)$, that is a non-crossing partition of [2m]with blocks of even size. Then Φ(π)is the partition of[m]defined by the fact that $\sim \Phi_{(\pi)}$ is the transitive closure of the relation that relates $k and\ l$ if $2k \sim_\pi 2l$ or

$2k - 1 \sim_\pi 2l$ or $2k \sim_\pi 2l - 1 or 2k - 1 \sim_\pi 2l - 1$. That is Φ(π)is the partition obtained by identifying the 2k−1 and 2k in[2m]to get k∈[m].

If π ∈NC^*(d, m), we define the map PbyP(π)=(Φ(π|A1),...,Φ(π$|_{A_d}$)). See Fig. 6. The map P is a good tool to make a finer study of NC^*(d, m).

The result is that partitions in NC^*(d, m)are not far from belonging to NC_2^*(d, m):

Theorem (2.1.7) [2]

For any σ ∈NC_2^*(d, m)there are less than 4^{2m}partitions π ∈NC^*(d, m)such That $p(\pi) = p(\sigma\ )$.

Moreover for such a π, the partition σ is finer than π and the number of blocks of π of size2 is greater thandm−2m, and every block has size at most 2m.

Proof :-

Let σ ∈NC_2^*(d, m).If π ∈NC^*(d, m) satisfies p(π)=σ then Lemma (2.1.8) applied to $\sigma|_{A_i}$and $\pi\ |_{A_i}$for i =1,..., d_p roves that σ is finer than π, and 1.9 implies that π has at least $dm - 2m$ blocks of size 2. The fact that every block of π has size at

Most m just follows from the definition of NC^*(d, m):π is indeed finer than{ A_1,..., A_d}with$|A_j|$=2m.

We now prove the first statement of Theorem (2.1.7) Let A be the subset of$[2dm]$ given by Lemma (2.1.11) Then there is an injection:

π ∈N C^*(d, m), p(π)= p(σ) →$NC[2dm]$\A, π→ π $|_{[2dm]\backslash A}$.

In particular since there are less than 4^Nnon-crossing partitions on[N], the first statement of the theorem follows with 4^{2m} replaced by 4^{4m}because [2dm]\A has cardinality less than 4m.To get the 4^{2m} just replace [2dm]\A by a set B that contains exactly one element of σ(k)for any k∈[2dm]\A. Then B has cardinality less than 2m because [2dm]\A is a union of blocks (=pairs) of σ, and the previous map is still an injection since π ∈N C^*(d,)and P(π)=P(σ) implies that σ is finer that π.

Motivated by Lemma(2.2.1) we are interested in invariants of the operations p_{kd} on NC^*(d, m).For π ∈NC^*(1,m)let B(π)be the number of blocks in Φ(π). This is the fundamental observation:

This theorem will follow from a series of lemmas. Here is the first one, which treats the case d=1:

Lemma (2.1.8) [2]

Let σ ∈NC_2^*(1,m)and π∈ NC^*(1,m)such that Φ(π)=Φ(σ). Then σ is finer than π.

More precisely if π ∈NC^*(1,m) and if {k_1<k_2<⋯<k_p}is a block of Φ(π), then for

Any i, $2k_i$ ~π$2k_{i+1}$−1 (with the convention k_{p+1}=k_1).

Proof:-

The first statement follows easily from the second one. We thus focus on the second statement. At least as far as partitions in NC_2^*(1,m)are concerned, this is explained in the discussion .The proof is the same for a general π ∈NC^*(1,m), but for completeness we still provide a proof.

It is clear that Φ(π)(k)={k} implies that 2k$\sim_\pi$2k−1. Thus to prove the statement we have to prove that if k and l are consecutive and distinct elements of a block of Φ(π)then $2k \sim_\pi 2l - 1$.

The first element in $\pi(2k)$ after $2k$ is odd, that is of the form 2p−1, because 2k is even and the parity alternates in blocks of π. We claim that $p = l$. Note that we necessarily have $k < l \leq p$(again for the cyclic order) because $k \sim_{\Phi(\pi)} p$. Suppose that k<l<p. We get to a contradiction: indeed since $l \sim_{\Phi(\pi)} k$ and$\{2l-1,2l\} \subset \{2k+1,2k+2,\ldots,2p-2\}$there is at least one $j \in \{2k+1,2k+2,\ldots,2p-2\}$ and $i \in \{2p-1,2p\ldots 2k\}$ such that i $\sim\pi_j$. But by definition of p, j $\nsim\pi$ 2k and j $\nsim\pi$ 2p−1. This contradicts the fact that π is non crossing.

We can now check that p is well defined:

Lemma (2.1.9) [2]

The map P from N C^*(d, m)takes values in NC(m)$^{(d)}$.

Proof:-

Let π∈ N C^*(d, m);we have to prove that if 1 ≤ i<j ≤ d then Φ(π $|_{A_j}$) is finer than

Φ(π $|_{A_i}$).

Let{ k_1< k_2<⋯< k_p}be a block of Φ(π $|_{A_i}$). Suppose that Φ(π $|_{A_i}$)(k_1) ⊈{ k_1, k_2... k_p}. Then there exist $1 \leq s \leq p$ and $l \notin$\{ k_1, k_2... k_p\}such that $k_s$and l are consecutive elements of Φ(π $|_{A_j}$)(k_1)(for the cyclic order). If 1 ≤t ≤ p is such that k_t <l< k_{t+1} (with again the convention k_{p+1}= k_1), we have by Lemma (2.1.8) that 2 $k_t \sim_{\pi|_{A_i}} 2k_{t+1}$−1 and $2k_s$ $)^m \sim_{\pi|_{A_j}} 2l-1$, which contradicts the fact that π is non-crossing. This shows that Φ(π $|_{A_j}$)(k_1)⊆{ k_1, k_2... k_p}=Φ(π $|_{A_j}$)(k_1). Since $k_1$was arbitrary, the proof is complete.

Here is a last elementary lemma concerning general non-crossing partitions:

Lemma(2.1.10) [2]

Let N ∈ℕ and π ∈NC(N)with α blocks. Then the number of k∈[N]such that

k$\sim_\pi$k+1 is greater or equal toN−2(α−1).

Proof:-

For π ∈NC(N), let us denote by c(π)the number of k∈[N]such that $k\sim_{\pi}k+1$. We prove by induction on α that if π ∈NC(N)has α blocks, then c(π)N−2(α−1).If α=1, this is clear since c(π)=N.

Assume that the statement of the lemma is true for all N and all π ∈NC(N)with α blocks. Take π ∈NC(N)withα+1 blocks. Since π is non-crossing there is a block of π, say A, which is an interval of size S .If$\pi|_{[N]\backslash A}$ is regarded as an element of NC(N−S)then c(π)≥ S−1+c($\pi|_{[N]\backslash A}$)−1. By the induction hypothesis c($\pi|_{[N]\backslash A}$) ≥N−S−2(α−1), which implies c(π) ≥N−2αand thus concludes the proof.

We have The next lemma, and Theorem (2.1.7) will easily follow from it:

Lemma (2.1.11) [2]

Let σ ∈NC_2^*(d, m). Then there is a subset A of [2dm]of size greater than $2dm-4m$, which is a union of blocks of σ, and such that for any π∈NC^*(d, m)with p (π)= p(σ) and any k ∈A , π(k)=σ(k).

Proof:-

For any 1≤j≤d, denote by σj =Φ($\sigma|_{A_j}$). Denote by d σ_{d+1} +1=0_m.Fix now 1 ≤ i ≤ d and{k_1<k_2<⋯<k_p}a block of σ_i . As usual we take the convention that k_{p+1}=k_1.We claim that if $k_s\sim\sigma_{i+1}\ k_{s+1}$then for any π ∈$NC^*$(d, m) with p(π)= p(σ), π(2dk_s $-i+1$)= {2dk_s−i+1,2dk_s+$1-2d+i$}(=σ(2d$k_s$$-i+1$)by Lemma (2.1.8).

Let us first check that this claim implies the lemma. By Lemma (2.1.9) , σ_{i+1}finer than σ_i and in particular its restriction to {k_1, k_2... k_p}makes sense. By Lemma (2.1.10), the number of s's in {1,...,p}such that $k_s\sim\sigma_{i+1}k_s$+1is greater than p−2($|\sigma_{i+1}\ |_{\{k_1,k_2\ldots k_p\}}|$−1)where |σ| is the number of blocks of σ. Thus summing over all blocks of σ_i we get at least 2m−4($|\sigma_{i+1}|-|\sigma_i|$)Elements k in$A_i$such that π(k)=σ(k)for any π ∈NC^*(d,m)with p(π)= p(σ). This allows to conclude the proof since

$\sum_{i=1}^{d}(2m-4(|\sigma_{i+1}|-|\sigma_i|)) = 2md-4(|\sigma_{d+1}|-|\sigma_1| > 2md-4m.$

Note that A is constructed as a union of blocks of σ.

We now only have to prove the claim. Assume that $k_s \sim\sigma_{i+1}k_{s+1}$ and take π ∈NC^*(d, m) such that P(π)=P(σ). By Lemma (2.1.8) applied to Φ(σ$|_{A_i}$)= σ_i,2dk_s −i+1∼π2dk_s+1− 2d+i. Thus we only have to prove that if ks ∼σi+1

ks+1there is no k∈{{k_1, k_2... k_p}}\{ks+1}such that 2dk_s−$i+1\sim\pi 2dk-2d$+i.

But if $k_s\sim\sigma_{i+1}k_{s+1}$ then i ≠d(because σd+1=0_m) and by Lemma(2.1.9)$k_s$andk_s+1 are consecutive elements inσ$_{i+1}(k_s)$. Thus by Lemma (2.1.8) , 2dk_s −i∼π2dks+1−2d+i+1. The condition that π is non-crossing implies the claim since fork∈{{k_1, k_2... k_p}}\{k_{s+1}},

2dk_s−i<2dk_s−i+1<2dk_s+1−2$d+i+1 < 2dk-2d+i$,

that is(2dk_s−$i+1$,2$dk-2d+i$)and(2dk_s−i,2dk_s+1−2$d+i+1$)are crossing.

We can now prove the theorem.

Lemma (2.1.12):-

For any π ∈NC^*(1,m)

B(π)=$\frac{1}{2}$(B(p_k(π)+B (p_{k+m} (π))).

This lemma is a consequence of the following description, which proves that for any k ,the set of blocks of Φ(π)but one is in bijection with the set of blocks of π that do not contain k and that begin with an odd element (after k for the cyclic order):

Proof :-

We use Lemma (2.1.13) with k+1 instead of k. For any π ∈NC^*(1,m)we denote by $F(\pi,k)$the set of odd l∈[2m]\{$k+1$}such that $l \sim_\pi l \Rightarrow l \le l'$ <$k+1$. We know that |F(π ,k)|=B(π)−1. Moreover let us decompose F(π ,k)as the disjoint union of F_1 (π, k) andF_2(π, k)defined by : l∈F_1(π, k)if and only l∈ F(π ,k)and π(l)⊂ l_{k+m} ; and F_2(π, k)is the set of l∈F(π ,k)such that π(l)∩ $I_l \neq \emptyset$.

If l∈l_{k+m} then l∈F($p_{k+m}(\pi), k$) if and only if l ∈F (π ,k)because if k+1≤ l'<l, then$l$~p_{k+m}(π) l if and only if $l \sim_{\pi} l$.

Take now $l \notin I_{k+m}$. By definition of F(· , k), l is inF(p_{k+m}(π), k)if and only if$l$is odd and l is the first element (after k+1 for the cyclic order) of a block of p_{k+m} (π)contained in I_k, which is equivalent to the fact that s_k(l)=2k+1−l is even and is the last element of a block of π contained in I_{k+m}. Such a block then has first element odd, and thus belongs to F_1(π, k)except if it is equal tok+1. To summarize, we have thus proved that

|F(p_{k+m}(π),k| = |F(π ,k)∩ I_{k+m}| + |F_1(π ,k)| +1

If k+1 is odd and π(k+1)⊂ I_{k+m}={k+1,k+2,...,k+m}, and

|F(p_{k+m}(π),k| = |F(π ,k)∩ I_{k+m}| + |F_1(π ,k)| (12)

Otherwise.

We now compute |F(p_k(π), k)|.If l ∈I_k then as above l ∈F(p_k(π), k)if and only if l ∈ F(π ,k).If l/∈I_k then l ∈F(p_k(π), k)if and only if l is odd and l is the first element strictly After k+1 (in the cyclic order) of a block of p_k(π)not containing k+1. By construction of p_k(π) this is equivalent to the fact that s_k(l)=2k+1−l is even, belongs to I_k, is different from K and is the last element before in a block of π. The first element (strictly after kin the cyclic order) of such a block is then in F_2(π, k) except if it is equal to k+1. Reciprocally, if l is the last element of a block containing an element of F_2(π, k) then l =$s_k(l')$∈ F(p_k (π), k) except if l =k. The same is true if π(k+1)⊄ I_{k+m} ,k+1 is odd and if l denotes the last element in π(k+1). Thus

|F(p_k(π),k| = |F(π ,k)∩ I_k| + |F_2(π ,k)| −I_k is even +I_k k is even and

−1kis even and π(k+1) ⊄ $I_{k+m.}$

|F(p_k(π),k| = |F(π ,k)∩ I_k| + |F_2(π ,k)| −I_k is even and π(K+1)⊂ Summing this last equality with (11) or (12) yields

|Fp_k(π), k)|+|Fp_{k+m} (π), k|

= F(π, k)∩ I_k F_2(π, k) F(π ,k)∩ $I_{k+m}F_1$(π, k) =2F(π,k).

This concludes the proof since by Lemma (2.1.13) for any σ ∈NC^*(1,m),

|F(σ ,k)|= B(σ)−1.

Another relevant subset of NC(2dm)is the set NC(d ,m)of partitions π with blocks of even cardinality and that connect only elements of different intervals I_k. In other words for all i, j ∈[2dm], I ≁π j if i, j ∈J_k.

The following observation is very simple but, it is the motivation for the introduction of NC(d ,m):

Lemma (2.1.13) [2]

Let $k \in [2m]$ and $\pi \in N(1,m)$. Then $B(\pi) - 1$is equal to the number of

$l \in [2m]\setminus\{k\}$such that l is odd and such that for any $l' \sim$πl, $l \leq l'$<k (for the cyclic order).

Proof :-

Indeed the set of odd l's different from k such that $l' \sim_{\pi} l \Rightarrow l \leq l'$ <k(for the cyclic order) is in bijection with the blocks of Φ(π)that do not contain(k+1)/2.

The direct map consists in mapping to any such l the block Φ(π)(⌊(l+1)/2⌋) and the reverse map gives to any block A of Φ(π)no containing ⌊(k+1)/2⌋the smallest l greater thank(again for the cyclic order) such that⌊(l+1)/2⌋∈A. The reader can check using Lemma (2.1.8) that these maps are indeed inverses of each other.

Lemma (2.1.14) [2]

Let π ∈NC(2dm)with blocks of even cardinality. Then π ∈NC(d ,m)if and

only ifπ does not connect two consecutive elements of a same subinterval J_i

. In other words, i∼π_i +1only if i is a multiple of d.

Proof:-

The only if part of the proof is obvious. The converse follows from the fact that a non-crossing partition always contains an interval (if π is non-crossing with blocks of even size, and s<t∈J_i with s∼π_t and t ≠s+1, apply this fact to$\pi|_{\{s,s+1,...,t-1\}}$).

We show the following.

Theorem (2.1.15) [2]

The cardinality of NC(d ,m)is less than $(4d+4)^{2m}$.

Moreover for any π ∈NC(d ,m)the number of blocks of π of size 2 is greater than(d−2)m.

The idea of the proof is similar to the proof of we try to reduce to the subset of NC(d ,m)consisting of partitions into pairs. For this we introduce the map Q=Q(N) from the set of non-crossing partitions of [2N] into blocks of even sizes to the set of non-crossing partitions of[2N]into pairs. The map Q has the property that if π ∈NC(2N)has blocks of even sizes, then Q(π)is finer than π and any block$\{k_1,..., k_{2p}\}$of π with 1≤ k_1<⋯<k_{2p} ≤2N Becomes p blocks of Q(π), namely$\{k_1, k_2\}$,...,$\{k_{2p-1}, k_{2p}\}$. It is straight forward to check that this indeed defines a non-crossing partition of [2N] into pairs. Note that unlike here the element 1∈[2N]plays a specific role in the definition of Q and we abandon the cyclic symmetry of [2N]. But this has the advantage to allow to define an order relation on the set of pairs of elements of[2N]: we will say that a pair (i, j) covers a pair(k, l) if$1 \leq i < k < l < j \leq 2N$.

A noteworthy property of Q is that if σ=Q(π)then two blocks (=pairs) of σ cannot be contained in the same block of π if one covers the other. In other words if 1≤i<k<l<j≤2N with i$\sim_\sigma$ j and k$\sim_\sigma$ l then i$\sim_\pi$k.

Following the notation of, the image Q(NC(d, m))is denoted by (d, m); it is the set of partitions of π into pairs that do not connect elements of a same subinterval J_k for k=1,...,2m. We are not aware of any nice combinatorial description of I(d, m)as for NC_2^*(d, m), but a precise bound for its cardinality is known: ,the cardinality of I(d, m)is equal to τ(T_d (s)^{2m})where T_d is the dth Tchebycheff polynomial and s is a semicircular element of variance 1 in a tracial C^*-algebra (A, τ). In particular since T_d(s)=d+1 we have that |I(d, m)| ≤ (d+1)^{2m}. Theorem (2.1.15) will thus follow from the following more general statement:

Lemma(2.1.16) [2]

Suppose that [2N] is divided into k non-empty intervalsS_1,..., S_k and let σ be anon-crossing partition of[2N]into pairs that do not connect elements of a same subinterval S_i. Then there are at most 4^{k-2}non-crossing partitions π of [2N]that do not connect elements of a same subinterval S_i and such that Q(π)= σ . Moreover for such a π there are at most2k−4 element S_1 ∈[2N]for which π(i)is not a pair.

Proof:-

We prove this statement by induction on N. For simplicity of notation we will assume that the intervalsS_1,..., $S_k$are ordered, i.e .that if i∈S_s and j ∈S_t with s<t then i<j.

If N=1 and σ is as above then σ=12, k=2, and there is only one π ∈NC(2) with

Q(π)=σ. This proves the assertion for N=1.

Assume that the above statement holds for 1,2,...,N−1 and take σ as above. Consider the set{{s_i , t_i},i=1...p}of outermost blocks (=pairs) of σ, i.e. the set of pairs of σ that are not being covered by another block of σ. If we order the s_i's and t_i's so that s_i<t_i and s_i<s_{i+1}then we have that s_1=1, s_{i+1}=$t_i + 1$ and t_p=2N.

By the property of Q mentioned above, a partition π ∈NC(2N)that does not connect elements of the same interval S_j (for j=1,...,k) satisfies Q(π)=σ if and only if the following properties are satisfied:

•For any 1≤i≤p, {s_i +1,..., t_i −1}is a union of blocks of π, the non-crossing partition π$|_{\{s_i+1,\dots,t_i-1\}}$ does not connect elements of the same subinterval Sj ∩{s_i+1,..., t_i −1} For j=1,...,k, and Q(π$|_{\{s_i+1,\dots,t_i-1\}}$)=σ$|_{\{s_i+1,\dots,t_i-1\}}$.

•Any block of$\pi|_{\{s_i+1,...,t_i-1\}}$is a union of pairs $\{s_i, t_i\}$and does not contain 2 elements of a same interval S_j.

Define K_+ (i) and K_- (i) for 1≤i≤p by s_i ∈$S_{k-(i)}$ and t_i ∈$S_{k+(i)}$. Then for any 1 ≤i ≤p, K_- (i) < k+(i)and for i<p, K_+ (i) ≤ K_- (i+1).

Since $\{s_i$+1,..., $s_i-1\}$ intersects at most K_+ (i) − K_- (i)+1 different intervals S_j, we have by the induction hypothesis that the number of non-crossing partitions of$\{s_i$+1,..., t_i −1}that satisfy the first point above is at most $4^{K_+ \text{(i)} - K_- \text{(i)}+1}$,and for such a partition at most 2(K_+ (i)− K_- (I)−1)elements of$\{s_i$+1,..., t_i −1}do not belong to a pair.

Moreover the set of non-crossing partitions of$\{s_1, t_1, s_2, t_2,..., s_p t_p\}$that satisfy the second point is in bijection with the set of non-crossing partitions of$\{s_i$,i=1...p}such that $s_i \nsim s_{i+1}$if K_+ (i) =K_- (i+1) .Its cardinality is in particular less than (or equals) the number of non-crossing partitions of[p], which is less than 4^{p-1}. Therefore the total number of non-crossing partitions

π of[2N]that do not connect elements of a same subinterval S_j and such that Q(π)=σ is less than

$$4^{p-1} \prod_{i=1}^{p} 4^{k+(i)-K_- \text{(i)}-1} \leq 4^{k-2}.$$

We used the inequality $\sum_{i=1}^{p} K_+ \text{(i)} - K_- \text{(i)} \leq k-1-p$. To prove that for such a π at most 2k−4 elements of [2N]do not belong to a pair of π, note

that for an element j∈[2N]the block π(j)is not a pair either if j∈$\{s_1, t_1,... s_p, t_p\}$or if j belongs to a block of $\pi|_{\{s_i+1,...,t_i-1\}}$which is not a pair for some 1≤i≤p. If K_+ (i) < K_- (i+1) for some i then we are done since 2p+$\sum_{i=1}^{p} 2K_+ \text{(i)} - 2K_- \text{(i)} - 2 \leq 2k-4$. To conclude the proof we thus have to check that ifk+(i)=k−(i+1)for any 1≤i<p then there are at least

2 elements of$\{s_1, t_1,..., s_p, t_p\}$that belong to a pair of π. But this amounts to showing that a non-crossing partition of[p]such that i $\nsim$ i+1 for any 1≤i<p contains at least one singleton, which is clear.

The following lemma is also an easy extension of Lemma (2.1.2). Remember that the partitions $\sigma_l^{(d,m)}$and $\tilde{\sigma}_l^{(d,m)}$are defined in Corollary (2.1.6).

Lemma (2.1.17) [2]

Fix integers d and m. For any k∈[2m] and π ∈NC(d ,m)the partition P_{kd}(π)

also belongs to NC(d ,m) Let k ∈ℕ such that $2^k \geq$ m. Then for any partition π ∈ NC(d ,m), the partition π_k = $P_{2^k}P_{2^{k-1}}...P_1P_2P_m$ (π) is one of the 2d+1partitions $\sigma_l^{(d,m)}$for 0≤l ≤d or $\tilde{\sigma}_l^{(d,m)}$for 1≤l≤d.

Proof:-

The first point is straightforward.

The proof of the second point is the same as Lemma(2.1.2): depending on the fact that {1,2,...,dm}∩π(i)\{i} and{dm+1,...,2dm}∩π(i)are empty or not for i =1,...,d, we prove by induction on k that π_k has the right properties.

Section(2-2):-

Inequalities and A main Result

For any partition π of [2N], and any k=(K_1,..., K_{2N})∈$\mathbb{N}^{2N}$, we write k≺ π if for any i, j ∈[2N]such that i∼π j, k_i = k_j.

Let a=$(a_k)_{k\in\mathbb{N}^N}$ be a finitely supported family of matrices. For any k=(K_1,..., K_N)∈ $\mathbb{N}^N$let $\tilde{a}_k$ =a(K_N, K_{N-1},..., K_1).

For such a and for a partition π of[2N], we denote by S(a,π,N,1)the following quantity:

S(a,π,N,1)=$\sum_{k,l\in\mathbb{N}^N,(k,l)\prec\pi} Tr a_k \tilde{a}_l^*$. (13)

More generally for integers m, d, for a finitely supported family of matrices a=$(a_k)_{k\in\mathbb{N}^d}$ and a partition π of [2dm], we define

S(a,π,d,m)= $\sum_{K_1,\dots,K_{2m}\in\mathbb{N}^d (K_1,\dots,K_{2m})\prec\pi} Tr(a_{k_1}\tilde{a}_{k_2}^* a_{k_3} \dots a_{k_{2m-1}}\tilde{a}_{k_{2m}}^*)$. (14)

In this equation and in the rest of the section an element k=(K_1,..., K_{2m}) ∈ $(\mathbb{N}^d)^{2m}$ is identified with an element of $\mathbb{N}^{2md}$. Therefore the expression k≺ π has a meaning for π ∈NC(2dm).

The following application of the Cauchy–Schwarz inequality is what motivates the introduction of the operations P_k on the partitions of[2N]. The same use of the Cauchy–Schwarz inequality has been made in the second part of [4].

Lemma (2.1.1)[2]

For a partition π of[2N]and a finitely supported family of matrices a=$(a_k)_{k\in\mathbb{N}^N}$,

|S(a,π,N,1)| ≤S(a, P_0 (π), N,1)$^{1/2}$(S(a, P_N(π), N,1)$^{1/2}$.

More generally for a partition π of[2dm], for a finitely supported family of matrices a= $(a_k)_{k\in\mathbb{N}^N}$ and any integer i,

|S(a,π,d,m)| ≤(S(a, P_{di} (π), d, m))$^{1/2}$ (S(a, $P_{(m+i)d}$(π), d, m)$^{1/2}$. (15)

Proof:-

The second statement for i=0 follows from the first one by replacing N by dm. Indeed for any and k=(k_1,..., k_m)∈($\mathbb{N}^d)^m \simeq \mathbb{N}^{dm}$, denote β$_k$=$a_{k_1}\tilde{a}_{k_2}^* a_{k_3}\dots a_{k_m}$ if m is odd and β$_k$= $a_{k_1}\tilde{a}_{k_2}^* a_{k_3}\dots\tilde{}\ \tilde{a}_{k_m}^*$ if m is even. We claim that S(a,π,d,m)=S(β,π,dm,1).We give a proof

When m is odd, the case when m is even is similar. It is enough to prove that if k=(k_1,..., k_m)∈ $\mathbb{N}^d)^m$ then$\tilde{\beta}_k^*$=$\tilde{a}_{k_1}^* a_{k_2}\dots\tilde{a}_{k_m}^*$.But if r: $\mathbb{N}^d\to\mathbb{N}^d$ denotes the map r(s_1,..., s_d)=(s_d,..., s_1) We have that

$\tilde{\beta}_k^* = \tilde{\beta}^*_{r(k_m)}, \ldots, r(k_1)$

$a_r(k_m) = \tilde{a}^*_{r(k_2)} a_{r(k_1)})^*$

$a_{r(k_1)} \tilde{a}^*_{r(k_2)} \ldots \tilde{a}^*_{r(k_m)}$.

$\tilde{a}^*_{r(k_1)} a_{k_2} \ldots \tilde{a}^*_{k_m}$.

For a general i the following argument based on the trace property allow to reduce to the case i =0: for a partition π of [2dm] and any n∈[2dm] denote

$\tau_n(\pi)$ the partition such that $s \sim \tau_n(\pi) t$ if and only if $s+n \sim_\pi t+n$, so that

$P_{n+k}(\pi) = (\tau_n^{-1} \circ (P_k) \circ \tau_n)(\pi)$ for any integer k. Moreover by the trace property S(a,π,d,m)=S(a, τ_{di} (π), d, m) if n is even and S(a,π,d,m)=S($\tilde{a}^*$, τ_{di} (π), d, m) if i is even (here $\tilde{a}^*$ denotes the family ($\tilde{a}_k^*)_{k\in\mathbb{N}^d}$). Therefore if one assumes that the inequality (15) is satisfied for any π and any a but only for i=0, then we can deduce it for a general i in the following way. Denote b=$(a_k)_{k\in\mathbb{N}^d}$ if i is even and b=($\tilde{a}_k^*)_{k\in\mathbb{N}^d}$ if i is odd and:

$|S(a,\pi,d,m)|^2 = |S(b,\tau_{di}(\pi),d,m|^2$

$\leq S(b,P_0\tau_{di}(\pi)),d,m)S(b,P_{dm}\tau_{di}(\pi),d,m)$

$= S(b,\tau_{di}(P_{di}(\pi),d,m)S(b,\tau_{di}(P_{dm+id}(\pi),d,m)$.

$= S(a,P_{di}(\pi),d,m)S(a,P_{(m+id)}(\pi),d,m)$.

We now prove the first statement. We take the same notation as in Definition (2.1.1).Let us clarify the notation for the rest of the proof. In the whole proof, for a set X we see $a_k \in \mathbb{N}^X$ as a function from X to ℕ, and for an integer N we will identify $\mathbb{N}^N$ With $\mathbb{N}^{[N]}$.In particular, if X and Yare disjoint subsets of a set Z, and if k $\in\mathbb{N}^X$ and l$\in\mathbb{N}^Y$, [k, l]will denote the element of $N\ X \cup Y$ corresponding to the function on X∪Y that has k as restriction to X and las restriction to Y.

Let us denote by A the union of the blocks of π that are contained in I_N={1,...,N},by B the union of the blocks of π that are contained in $[2N]\backslash I_N = \{N+1,\ldots,2N\} = I_{2N}$and By C the rest of [2N]. In the following equations, swill vary in $\mathbb{N}^A$, t in $\mathbb{N}^{I_N \setminus A}$, u in $\mathbb{N}^B$ And v in $\mathbb{N}^{I_{2N}}$ \B . For such s, t, u and v and with the previous notation, $[s, t, u, v] \prec \pi$ if and only if $s \prec \pi|_A$, [t ,v]$\prec\pi|_C$ and u $\prec\pi$ $|_B$.F or k $\in\mathbb{N}^{I_{2N}}$ (i.e .k is a function k: $I_{2N} \to \mathbb{N}$), we will also abusively denote $\tilde{a}_k \stackrel{\text{def}}{=} \tilde{a}_{k(N+1),\dots,k(2N))}$.With this notation the definition in (13) becomes

$$S(a, \pi, N, 1) = \sum_{\substack{s\in\mathbb{N}^A, t\in\mathbb{N}^{I_N\setminus A}, u\in\mathbb{N}^B, v\in\mathbb{N}^{I_{2N}\setminus B} \\ [s,t,u,v]\prec\pi}} Tr(a_{[s,t]}\tilde{a}^*_{[u,v]} \ s \ \in N,$$

$$= \sum_{\substack{t,v \\ [t,v]\prec\pi|_C}} Tr((\sum_{s\prec\pi|_A} a_{[s,t]})(\sum_{u\prec\pi|_B} \tilde{a}_{[u,v]})^*.$$

Thus

$$|S(a, \pi, N, 1)| \leq \sum_{\substack{t,v \\ [t,v]\prec\pi|_C}} Tr((\sum_{s\prec\pi|_A} a_{[s,t]})(\sum_{u\prec\pi|_B} \tilde{a}_{[u,v]})^*.$$

Applying the Cauchy–Schwarz inequality for the trace, we get

$$S(a, \pi, N, 1) \ = \sum_{\substack{t,v \\ [t,v]\prec\pi|_C}} || \sum_{s\prec\pi|_A} a_{[s,t]} ||_2 || \sum_{u\prec\pi|_B} \tilde{a}_{[u,v]} ||_2.$$

The classical Cauchy–Schwarz inequality yields

$S(a,\pi,N,1) \leq (1)^{1/2}(2)^{1/2}$

Where

$$(1) = \sum_{[t,v]\prec\pi|_C} \left\| \sum_{s\prec\pi|_A} a_{[s,t]} v \right\|_2^2$$

$$(2) = \sum_{[t,v]\prec\pi|_C} \left\| \sum_{u\prec\pi|_B} \tilde{a}_{[u,v]} \right\|_2^2.$$

We claim that(1)=S(a, P_N(π), N,1) and(2)=S(a, P_0(π), N,1).We only prove the first equality, the second is proved similarly(or follows from the first).But

$$\begin{aligned}
(1) &= \sum_{[t,v]\prec\pi|_C} \left\| \sum_{s\prec\pi|_A} a_{[s,t]} \right\|_2^2 \\
&= \sum_{[t,v]\prec\pi|_C} Tr\left(\left(\sum_{s\prec\pi|_A} a_{[s,t]}\right)\cdot\left(\sum_{s\prec\pi|_A} a_{[s,t]}\right)^*\right) \\
&= Tr\left(\sum_{[t,v]\prec\pi|_C}\sum_{s\prec\pi|_A}\sum_{s'\prec\pi|_A} a_{[s,t]}a^*_{[s',t]}\right) \\
&= Tr\left(\sum_{[t,v]\prec\pi|_C}\sum_{s\prec\pi|_A}\sum_{s'\prec\pi|_A} a_{[s,t]}\tilde{a}^*_{r([s',t])}\right),
\end{aligned}$$

where on the last line for any k=(k_1,..., k_N)∈ $\mathbb{N}^{IN}$, r(k)∈ $\mathbb{N}^{IN}$ is defined by r(k)= (k_N, k_{N-1},..., k_1).

By definition of B, for any j∈I_{2N}\B there is i ∈I_N\A such that i $\sim_\pi j$. Thus for any t ∈$\mathbb{N}^{I_N\setminus A}$ there is exactly one or zero v∈ N I[2N]\B such that[t, v]≺πC, depending whether t ≺π IN\A or not.

The claim that (1)=S(a, P_N(π), N,1)thus follows from the observation that for k, l ∈$\mathbb{N}^N$, (k, l)≺ P_N(π) if and only there are s , s' ∈ $\mathbb{N}^A$and t ∈$\mathbb{N}^{I_N\setminus A}$such that k=[s, t], l =r([s',t]) and s ≺π$|_A$, s' ≺ π $|_A$ and t ≺ π $|_{I_N\setminus A}$.

We now have to observe that the quantities S(a ,$\sigma_l^{(d,m)}$,d, m) for l =0,...,d and $\tilde{\sigma}_l^{*(d,m)}$,d ,m) for l=0,...,d have simple expressions.

A (finitely supported) family of matrices a= $(a_k)_{k\in\mathbb{N}^d}$ can be made in various natural ways in to a bigger matrix, for any decomposition of $\mathbb{N}^d \simeq \mathbb{N}^l \times \mathbb{N}^{d-1}$.If the a_k's are viewed as operators on a Hilbert space H(H=$\mathbb{C}^\alpha$ if the a_k's are in M_α(C)), then let us denote by M_l the operator from H⊗$\ell^2(\mathbb{N})^{\otimes d-1}$ To H⊗ℓ^2 $(\mathbb{N})^{\otimes l}$having the following block-matrix decomposition:

$$(a_{[s,t]})\ s\ \in N\{1,\ldots,l\}\,,t\ \in N\{l+1,\ldots,d\}.$$

Note that since (a_k) has finite support, the above matrix has only finitely many non-zero entries, and hence corresponds to a finite rank operator. In particular, it belongs tos_p(H ⊗$\ell^2(\mathbb{N})^{\otimes l}$) for any p∈(0,∞].

Lemma(2.2.2)[2]

.Let d, m, a=$(a_k)_{k\in\mathbb{N}^d}$ and M_l as above, and $\sigma_l$and$\tilde{\sigma}_l$ defined in Corollary(2.1.6) Then for l∈{0,1,...,d}:

S (a, $\sigma_l^{(d,m)}$,d,m)=$||M||^{2m}_{S_{2m(H\otimes\ell^2(\mathbb{N})^{\otimes d-1};H\otimes\ell^2(\mathbb{N})^{\otimes l})}}$.

Moreover for $l \in \{1,\ldots,d\}$

S(a,$\tilde{\sigma}_l^{(d,m)}$,d,m)$||M_{l-1}||^{2m}_{s_{2m}(H\otimes\ell^2(\mathbb{N})^{\otimes d-1};H\otimes\ell^2}(\mathbb{N})^{\otimes l}$,

Remark.

It is also true that

S(a, $\tilde{\sigma}_l^{(d,m)}\tilde{\sigma}_l^{(d,m)}$,d,m) ≤ $||M_{l-1}||^{2m}_{s_{2m}(H\otimes\ell^2(\mathbb{N})^{\otimes d-1};H\otimes\ell^2}(\mathbb{N})^{\otimes l}$, but we will only use the inequality stated in the lemma. This inequality follows from the one stated by conjugating by the rotation k∈[2dm]→k +d.

Proof :-

We fix l ∈{0,...,d}. For any s=(s_1,..., s_l)∈ $\mathbb{N}^l$

we denote by As =$(a_{s,t})_{t\in\mathbb{N}^{d-l}}$ viewed as a row matrix. As an operator ,As thus acts from H⊗ℓ^2 (N)$^{\otimes\mathbb{N}^{d-l}}$ to H .For s, s'∈$\mathbb{N}^l$, if r(k_1,..., k_d)=(k_d,..., k_1)

$$A_s\, A_{s'}^* = \sum_{t\in\mathbb{N}^{d-l}} a_{s,t}\tilde{a}_{s',t}^* = \sum_{t\in\mathbb{N}^{d-l}} a_{s,t}\tilde{a}_{r(s',t)}^*.$$

Hence for $s^{(1)}, s^{(2)}, \dots, s^{(m)}$∈$\mathbb{N}^l$,if $s^{(m+1)}= s^{(1)}$,

$$\prod_{i=1}^{m} A_{s^{(i)}}\, A_{s^{(i+1)}}^* = \sum_{t^{(1)},\dots,t^{(m)}\in\mathbb{N}^{d-l}} a_{s^{(1)}}\, t^{(1)}\tilde{a}_{r(s^{(2)},t^{(1)})}^* a_{s^{(2)}}t^{(2)}\tilde{a}_{r(s^{(3)},t^{(2)})}^* \dots \tilde{a}_{r(s^{(1)},t^{(m)})}^*.$$

But for k ∈$\mathbb{N}^{[2dm]}$, k ≺$\sigma_l^{(d,m)}$ if and only if there exist $s^{(1)}, s^{(2)},\dots, s^{(m)}$∈$\mathbb{N}^l$

And $t^{(1)}, t^{(2)}, \dots, t^{(m)}$ ∈$\mathbb{N}^{d-l}$such that for all i, (k_{2di+1}, k_{2di+2},..., k_{2di+d}) =($s^{(i)}$, $t^{(i)}$) and

(k_{2di+2d}, $k_{2di+2d-1}$,..., $k_{2di+d+1}$)=($s^{(i+1)}$, $t^{(i)}$). Thus summing over s

$s^{(1)}, s^{(2)},\dots, s^{(m)}$∈$\mathbb{N}^l$in the preceding equation leads to

$$\sum_{s^{(1)},s^{(2)},\dots,s^{(m)}\in\mathbb{N}^l} \prod_{i=1}^{m} A_{s^{(i)}}\, A_{s^{(i+1)}}^* = \sum_{(k_1,\dots,k_{2m})\prec\sigma_l^{(d,m)}} a_{k_1}\tilde{a}_{k_2}^* a_{k_3} \dots a_{k_{2m-1}}\tilde{a}_{k_{2m}}^*.$$

Taking the trace and using the trace property we get

$$S\left(a, \sigma_l^{(d,m)}, d, m\right) = \sum_{s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in \mathbb{N}^l} Tr\left(\prod_{i=1}^{m} A_{s^{(i)}}^* A_{s^{(i)}}\right)$$

$$= Tr\left[\left(\sum_{s \in \mathbb{N}^l} A_s^* A_s\right)^m\right]$$

$$= Tr\left[\left(M_l^* M_l\right)^m\right]$$

where the last identity follows from the fact that M_l=$\sum A_s \otimes a_s 1$. This concludes the proof for $\sigma_l^{(d,m)}$.For $\tilde{\sigma}_l^{(d,m)}$with 1≤l≤d, the same kind of computations yield to

$$S\left(a, \tilde{\sigma}_l^{(d,m)}, d, m\right) = \sum_{s_l \in \mathbb{N}} Tr\left[\left(\sum_{s \in \mathbb{N}^{l-1}} A_{(s,s_l)}^* A_{(s,s_l)}\right)^m\right].$$

To conclude we only have to use Lemma (2.2.3) below.

Lemma (2.2.3) [2]

Let X_1, X_2... X_N be matrices. Then for any integer m≥1,

$$\sum_{i=1}^{N} Tr\left(\left(X_i^* X_i\right)^m\right) \leqslant Tr\left(\left(\sum_{i=1}^{N} X_i^* X_i\right)^m\right).$$

Proof:-

This is a general inequality for the non-commutative L_P-norms. Indeed, for any α, N∈ ℕ, and p∈[2,∞],the map

$$T: \quad M_{N,1}\left(M_\alpha(\mathbb{C})\right) \rightarrow M_N\left(M_\alpha(\mathbb{C})\right),$$

$$\begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \mapsto \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_N \end{pmatrix}$$

is a contraction for all p-norms. For p=2, this is easy because T is an isometry. For p=∞this is also obvious. For a general p∈(2,∞)the claim follows by interpolation. Applied for p=2m, this concludes the proof since for an integer m,

$$\left\| \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \right\|_{2m}^{2m} = Tr\left(\left(\sum_{i=1}^{N} X_i^* X_i\right)^m\right)$$

And

$$\left\| \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_N \end{pmatrix} \right\|_{2m}^{2m} = \sum_{i=1}^{N} Tr((X_i^* X_i)^m).$$

We are now able to state and prove the ollowing result. Recall that for a partition π of N C^*(1,m), B(π)was defined as the number of blocks of the partition Φ(π)(the map Φ was defined after Corollary(2.1.6).

Corollary (2.2.4) [2]

Letπ∈NC^*(d, m). Then if a andM_l are as in Lemma(2.2.2**),**

|S(a,π,d,m)|≤ $\prod_{i=0}^{d} ||M||_{S_{2m}(H\otimes\ell^2\mathbb{N}^{\otimes d-l}H\otimes\ell^2\mathbb{N}^{\otimes l})}^{2m\mu l}$

whereμ_l =(B(π$|_{A_{l+1}}$)−B(π$|_{A_l}$))/(m−1)where we take the convention that B(π$|_{A_0}$)=1 and B(π$|_{A_{d+l}}$)=m.

Proof:-

The idea is, as in , to iterate the inequality of Lemma(2. 2.1), except that here the combinatorial invariants of the map π ↦ (P_{kd}(π), P_{kd+md}(π))(Lemma 2.1.12) allow us to precisely determine the exponents of each$||M_l||_{2m}$. In the rest of the proof since no confusion is possible, we will simply denote $\sigma_l = \sigma_l^{(d,m)}$ and $\tilde{\sigma}_l = \tilde{\sigma}_l^{(d,m)}$, and S will denote the set $\{\sigma_l, 0 \le l \le d\} \cup \{\tilde{\sigma}_l, 0 \le l \le d\}$. Fix $\pi \in NC^*(d, m)$.

Maybe the clearest way to write out a proof is using the basic vocabulary of probability theory**.** Let us consider the (homogeneous) Markov chain $(\pi_n)_{n\ge 0}$ on (the finite state space) $NC^*(d, m)$ given by $\pi_0=\pi$ and $\pi_{n+1}=P_{id}(\pi_n)$ where i is uniformly distributed in [2m] and independent from $(\pi_k)_{0\le k\le n}$ (note that $\pi_{n+1} \in NC^*(d, m)$ if $\pi_n \in NC^*(d, m)$ by Lemma (2.1.5). Corollary (2.1.6) implies that the sequence $(\pi_n)_n$ is almost surely eventually equal to one of the σ_l or $\tilde{\sigma}_l$. Its second statement indeed expresses that if $\pi_n \in S$ then $\pi_N = \pi_n$ for all $N \ge n$; it suffices therefore to prove that $P_n \overset{\text{def}}{=} P(\pi_n \notin S) \to 0$ as $n\to\infty$. But if k is fixed with $2^{k-2} \ge m$, its first statement implies that $P_k \le 1-(1/2m)^k = c < 1$ for any starting state π_0. From the equality $P_{n+k} = P_n \mathbb{p}(\pi_{n+k} \notin S | \pi_n / \notin S)$ and the Markov property we get that $P_{n+k} \le c p_n$ for any integer $n\in\mathbb{N}$, from which we deduce that $p_n \le c^{\llcorner n/k \lrcorner} \to 0$ as $n\to\infty$.

Let us denote $\lambda_1(\pi)=P(\lim_n \pi_n = \sigma_l)$ and $\tilde{\lambda}_l(\pi)=P(\lim_n \pi_n = \tilde{\sigma}_l)$ for $0 \le l \le d$ (take $\tilde{\lambda}_0(\pi)=0$); note that $\lambda_l(\pi)+\tilde{\lambda}_l(\pi)=1$.

Lemma (2.1.12) and the last statement of Lemma (2.1.5) show that for any $i \in \{1,\dots,d\}$ the sequence

$B(\pi_n|A_i)$ is a martingale. In particular since $\pi_0=\pi$, $B(\pi_n|A_i)=E[B(\pi_n|A_i)]$ for any $n \ge 0$. Letting $n\to\infty$ we get

$$B(\pi|_{A_i}) = \sum_{l=0}^{d} \lambda_l(\pi) B(\sigma_l|_{A_i}) + \sum_{l=1}^{d} \tilde{\lambda}_l(\pi) B(\tilde{\sigma}_l|_{A_i})$$

We used the fact that $B(\sigma_l|A_i)=B(\tilde{\sigma}_l|A_i)=1+(m-1)\,1_l< i$. This follows from the observations that since $\Phi(c_m)=\Phi(1_{2m})=1_m$, $B(c_m)=|1_m|=1$ and that since $\Phi(r_m)=0_m$, $B(r_m)=m$. Subtracting the equalities above for i and i+1 gives

$$\lambda_i(\pi)+\tilde{\lambda}_i(\pi)(m-1)=B(\pi\,\pi|_{A_{i+1}})-B(\pi|_{A_i}) \qquad (16)$$

with the convention that $B(\pi|_{A_0})=1$ and $B(\pi|A_i)=m$.

On the other hand Lemma (2.2.1) implies that the sequence $M_n=\log|S(a, \pi_n, d, m)|$ is a sub martingale. As above letting $n\to\infty$ in the inequality $M_0 \leq \mathbb{E}[M_n]$ yields

$$\operatorname{Log}|S(a,\pi,d,m)| \leq \sum_{l=0}^{d} \lambda_1(\pi) \log|S(a, \sigma_l, d, m)| + \sum_{l=1}^{d} \tilde{\lambda}_1 \log|S(a, \tilde{\sigma}_l, d, m)$$

If we denote simply by $||M_l||_{2m}$ the quantity $||M_l||_{S_{2m}(H\otimes l^2(\mathbb{N})^{\otimes l}}$, then by Lemma (2.2.2) this inequality becomes

$$|S(a,\pi,d,m)| \leq \prod_{l=0}^{d} ||M_l||_{2m}^{2m(\lambda_1(\pi)+\tilde{\lambda}_l(\pi)}$$

This inequality, combined with (16), concludes the proof.

We first treat the "holomorphic" setting for which the results we get are completely satisfactory.

It is a generalization to operator coefficients of the main result . When the coefficients a_k are taken to be scalars, the techniques of our Theorem(0.4) give a new proof and an improvement of , Kemp and Speicher introduce free Poisson variables to get an upper bound, whereas the proof is more combinatorial and lies is the study of N(d, m) that is done for definitions and facts on free Cumulants and $\mathfrak{R}$-diagonal operators. We just recall that the $*$-distribution of a variable c in a C^*-probability space is characterized by its free cumulants, which are the family of complex Numbers $k_n[c^{\varepsilon_1},..., c^{\varepsilon_n}]$,for $n \in N$ and $\varepsilon_i \in\{1,*\}$. Moreover the R-diagonal operators are exactly the operators c for which the cumulants $k_n[c^{\varepsilon_1},..., c^{\varepsilon_n}]$vanish except ifnis even and if 1's and$*$'s alternate in the sequence$\varepsilon_1,..., \varepsilon_n$. Since the family $\lambda(g_1), . . . , \lambda(g_r)$(where$g_1,..., g_r$ are the generators of the free group F_r) form an example of$*$-free $\mathfrak{R}$-diagonal operators , Theorem (0.3) is a particular case of Theorem (0.4), that is why do not include a proof.

The start of the proof is the same as in the proof of Theorem (2.2.5) , and was sketched in the Introduction. Fix $p=2m\in2\mathbb{N}$.

As in (14), if $k=(k_1,\ldots,k_d)\in\mathbb{N}^d$ denote by $\tilde{a}_k=a(k_1,\ldots,k_d)$ and $\tilde{c}_k=c(k_d,\ldots,k_1)=c_{k_d}\ldots c_{k_1}$. First develop the norms:

$$\left\|\sum_{k\in\mathbb{N}^d} a_k\otimes c_k\right\|_{2m}^{2m}=\sum_{k_1,\ldots,k_{2m}\in\mathbb{N}^d} Tr\left(a_{k_1}a_{k_2}^*\cdots a_{k_{2m}}^*\right)\tau\left(c_{k_1}c_{k_2}^*\cdots c_{k_{2m}}^*\right)$$

$$=\sum_{k_1,\ldots,k_{2m}\in\mathbb{N}^d} Tr\left(a_{k_1}\tilde{a}_{k_2}^*\cdots\tilde{a}_{k_{2m}}^*\right)\tau\left(c_{k_1}\tilde{c}_{k_2}^*\cdots\tilde{c}_{k_{2m}}^*\right).$$

Take $k_1,\ldots,k_{2m}\in\mathbb{N}^d$; if $k_l=(k_l(1),k_l(2),\ldots,k_l(d))$ then

$c_{k_1}\tilde{c}^*_{k_2\ldots}\tilde{c}^*_{k_{2m}}=c_{k_1(1)}\,c_{k_1(2)}\ldots c_{k_1(d)}\,\tilde{c}^*_{k_{2(1)}},\ldots\tilde{c}^*_{k_2(d)}\ldots\tilde{c}^*_{k_{2m}(d)}$

and by the fundamental property of cumulants:

$\tau(c_{k_1}\tilde{c}^*_{k_2\ldots}\tilde{c}^*_{k_{2m}})=\sum_{\pi\in NC(2dm)}k_\pi[\,c_{k_1(1)},\ldots,c_{k_1(d)}\,\tilde{c}^*_{k_{2(1)}},\ldots\tilde{c}^*_{k_2(d)}\ldots\tilde{c}^*_{k_{2m}(d)}].$

Denote $k=(k_1,\ldots,k_{2m})\in(\mathbb{N}^d)^{2m}\simeq\mathbb{N}^{2dm}$. Since freeness is characterized by the vanishing of mixed cumulants , $k_\pi[c_{k_1},\ldots,\tilde{c}^*_{k_{2m}(d)}]$is non-zero only if $k\prec\pi$, and in this case we claim that it is equal to $k_\pi\,[c_d,m]$where

$$c_d,m=\overbrace{\underbrace{c,\ldots c,}_{d}\underbrace{c^*,\ldots,}_{d}c^*\,\underbrace{c,\ldots,c}_{d}\,\underbrace{c^*,\ldots.,c^*}_{d}}^{2m\ goups}\qquad(17)$$

Relabel indeed the sequence $k_1(1),\ldots,k_{2m}(d)$ by $k_1,\ldots,k_{2dm}$, and denote also by $\varepsilon_1,\ldots,\varepsilon_{2dm}$ the corresponding sequence of 1's and $*$'s, in such a way that $k_\pi[c_{k_1(1)},\ldots,c^*_{k_{2m}(d)}]=k_\pi[(c^{\varepsilon_i}_{k_i})_{1\le i\le 2dm}]$ and $k_\pi[c_d,m]=k_\pi[(c^{\varepsilon_i})1\le i\le i2dm]$. By the definition of k_π,we have

$k_\pi[(c^{\varepsilon_i}_{k_i})_{1\le i\le 2dm}]=\prod_{V\in\pi}k_{|V|}\,[c^{\varepsilon_i})_{i\in V}]$

where the products runs over by the blocks of π. Similarly

$k_\pi\,[c_d,m]=\prod_{V\in\pi}k_{|V|}\,[c^{\varepsilon_i})_{i\in V}]$

Our claim thus follows from the observation that if k $\prec\pi$ then for any block V of π there is an

Index s such that k_i=s for all I ∈V, and the equality $k_{|V|}[(c_s^{\varepsilon_i})_{i\in V}] = k_{|V|}[(c^{\varepsilon_i})_{i\in V}]$ expresses just the fact that c and c_s have the same∗-distribution and therefore the same cumulants. The next claim is that since c is $\mathfrak{R}$-diagonal , $k_\pi[c_d,m]$ is non-zero only if $\pi \in Nc^*(d, m)$. Since with the previous notation $k_\pi[c_d,m] = \prod_{V\in\pi} k_{|V|}\ [c^{\varepsilon_i})_{i\in V}]$, this amounts to showing that if there is a block V of π which is not of even cardinality or for which 1's and ∗'s do not alternate

in the sequence$(\varepsilon_i)_{i\in V}$, then $k_{|V|}\ [(c^{\varepsilon_i})_{i\in V}]=0$. But this is exactly the definition of R-diagonal operators. Thus we get

$$||\sum_{k\in\mathbb{N}^d} a_k \otimes c_k||_{2m}^{2m} = \sum_{\pi\in nc^*(d,m)} k_\pi(c_d, m) \sum_{(k_1,\ldots,k_{2m})\prec\pi} Tr(a_{k_1} a_{k_2}^* \ldots \tilde{a}_{k_{2m}}^*$$

or with the notation introduced in (14)

$$||\sum_{k\in\mathbb{N}^d} a_k \otimes c_k||_{2m}^{2m} = ||\sum_{\pi\in NC^*(d,m)} a_k \otimes c_k||_{2m}^{2m}. \tag{18}$$

We can now use the study ofNC^*(d, m)that we did Recall in particular that there is a mapP: $NC^*(d, m)\to NC(m)^{(d)}$ the properties of which are summarized in Theorem (2.1.7) Take$(\sigma_1,\ldots, \sigma_d)\in NC(m)^{(d)}$ and denote $\mu_l=(|\sigma_{l+1}|-|\sigma_l|)/(m-1)$where |σ| denotes the number of blocks of σ with the convention$|\sigma_0|=1$ and$|\sigma_{d+1}|=m$. If $\pi\in NC^*(d, m)$and

$P(\pi)=(\sigma_1,\ldots, \sigma_d)$then by Corollary (2.2.4),$|S(a,\pi,d,m)| \leq \prod_{l=0}^{d} ||\mu_l||_{2m}^{2m}$ .Thus by the first part of Theorem (2.1.7), we have that

$$\left|\sum_{\pi\in NC^*(d,m),\ \mathcal{P}(\pi)=(\sigma_1,\ldots,\sigma_d)} \kappa_\pi[c_{d,m}] S(a,\pi,d,m)\right|$$
$$\leqslant 4^{2m} \prod_{l=0}^{d} \|M_l\|_{2m}^{2m\mu_l} \max_{\mathcal{P}(\pi)=(\sigma_1,\ldots,\sigma_d)} \left|\kappa_\pi[c_{d,m}]\right|.$$

But by the second statement of Theorem (2.1.7)and Lemma (2.2.5) below (recall that for $\tau(c)=\kappa_1[c]=0$ sincecis $\Re$-diagonal)

$$|\, k_\pi\,[cd,m]| \leq ||c||_{2m}^{2m}(\frac{16||c||_{2m}}{|c||_2})^{4m}$$

which implies

$$|\sum_{\pi\in nc^*(d,m),P(\pi)=(\sigma_1,\ldots,\sigma_d)} k_\pi[c_d,m],S(a,\pi,d,m)|$$

$$\leq 4^{10m}\prod_{l=0}^{d}||\mu_l||_{2m}^{2m\mu_l} \leq ||c||_2^{2m}(\frac{16||c||_{2m}}{|c||_2}).^{4m} \qquad (19)$$

But for any non-negative integers $s_0,\ldots,s_d)\in$such that $\sum_i s_i=m-1$, the number of $(\sigma_1,\ldots,\sigma_d)\in NC(m)^{(d)}$such that$|\sigma_{l+1}|-|\sigma_l|=s_l$ for any $0\leq l\leq d$(with the conventions $|\sigma_0|=1$ and$|\sigma_{d+1}|=m$) is equal to $(1/m)\binom{m}{s_0}\binom{m}{s_1}\ldots\binom{m}{s_d}$. Thus from (18) we Deduce

$$\sum_{k\in\mathbb{N}^d} a_k\otimes c_k\,||_{2m}^{2m}$$

$$\leq 4^{10m}\prod_{l=0}^{d}||\mu_l||_{2m}^{2m\mu_l}\sum_{s_0+\cdots+s_d=m-1}(1/m)\binom{m}{s_0}\binom{m}{s_1}\prod_{l=0}^{d}||\mu_l||_{2m}^{2ms_l/(m-1)}.$$

Denote for simplicity $\gamma_l=||\mu_l||_{2m}^{2m/(m-1)}$. Since the number of $s_0,\ldots,s_d)\in\mathbb{N}$ such that $s_0+\cdots+s_d=m-1$ is equal to$\binom{m+d-1}{d}$, this inequality becomes

$$||\sum_{k\in\mathbb{N}^d} a_k\otimes c_k\,||_{2m}^{2m}$$

$$\leq 4^{10m}||c||_2^{2dm}\frac{||c||_{2m}}{|c||_2})^{4m}\binom{m+d-1}{d}\sup_{s_0+\cdots+s_d=m-1}\left(\frac{1}{m}\right)\binom{m}{s_0}\binom{m}{s_1}\ldots\binom{m}{s_d}\prod_{l=0}^{d}\gamma_l^{sl}.$$

Now use the fact that for any integers Nand n, $\binom{N}{n} \leq (N/n)^n (N/(N-n))^{N-n}$ with the Convention $(N/0)^0=1$. For a fixed N, this can be proved by induction on $n \leq N/2$ using the fact that $x \in \mathbb{R}^+ \longmapsto x\log(1+1/x)$ is increasing. Thus

$$\binom{m+d-1}{d} \leq \binom{m+d}{d} \leq \left(1+\frac{m}{d}\right)^d \left(1+\frac{d}{m}\right)^m.$$

But sice log is concave, if $s_0 + \cdots + s_d = m-1$

$\prod_{l=0}^{d} \left(\frac{m}{m-s_l}\right)^{m-s_l} = \exp\left((md+1) \sum_0^d \frac{s_1}{m-1} \log(m\gamma_l/s_l)\right).$

$$\leq \exp\left((md+1)\log\left(\sum_0^d m/(md+1)\right)\right)$$

$=\exp) \ (md+1)\log 1+(m-1)/\ (md+1))) \ \leq \exp(m)$

And

$\prod_{l=0}^{d} \left(\frac{m\gamma_l}{s_l}\right)^{s_l} = \exp\left((m-1) \sum_0^d \frac{s_1}{m-1} \log(m\gamma_l/s_l)\right).$

$$\leq \exp\left((m-1)\log\left(m/(m-1) \sum_0^d \gamma_l\right)\right)$$

$= (\gamma_0 + \ \ldots + \gamma_l\ )^{m-1} \left(\frac{m}{m-1}\right)^{m-1}.$

But $m/(m-1)^{m-1} \leq m \ for \ any \ m \geq 1$ leads to

$$\left|\left| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right|\right|_{2m}^{2m}$$

$$\leq 4^{10m} ||c||_2^{2dm} \frac{||c||_{2m}}{|c||_2})^{4m} \left(1+\frac{m}{d}\right)^d \left(1+\frac{d}{m}\right)^m \exp\ (m)(\gamma_0 + \ldots + \gamma_l)^{m-1}. \quad (20)$$

Noting that since $2m/(m-1) \geq 2$,

$(\gamma_0 + \ldots + \gamma_l)^{m-1} = ||(||M_l||_{2m})_l||^{2m}_{\ell^{2m/(m-1)}(\{0,\ldots,d\})}. \leq = ||(||M_l||_{2m})_l||_{\ell^2\{0,\ldots,d\})}$

And taking the 2mth root in 20 one finally gets

$$||\sum_{k\in\mathbb{N}^d} a_k \otimes c_k\,||_{2m}$$

$$\leq 4^5\sqrt{e(1+d/m)}\Big(1+\frac{m}{d}\Big)^{d/2m}\,||c||_2^{d}\,\frac{||c||_{2m}}{|c||_2})^2\,||(||\mathrm{M_l}||_{2\mathrm{m}})_\mathrm{l}||_{\ell^2\{0,\dots,d\})}.$$

To conclude for the case m<∞, just note that $\left(1+\frac{m}{d}\right)^{d/m} \leq e$.

Letting m→∞and noting that$\left(1+\frac{m}{d}\right)^{d/m}$→1 concludes the proof for the operator norm.

When the c_k's are circular, since $k_\pi[c_d, m]1$ if π ∈NC_2^*(d, m) and $k_\pi[c_d, m]$ =0 otherwise, we can replace (19) by

$|\sum_{\pi\in nc^*(d,m),P(\pi)=(\sigma_1,\dots,\sigma_d)} k_\pi[c_d, m]$,S(a,π,d,m)| $\leq \prod_{l=0}^{d} ||\mu_l\,||_{2m}^{2m\mu_l}$.

Following the rest of the arguments we get the claimed results.

We still have to prove this lemma that was used in the above proof.

Lemma (2.2.5)[2]

Let π ∈NC(n)a non-crossing partition that has at least K blocks of size2and in

which all blocks have a size at most N.

Let$c_1,\dots,c_n$be elements of a tracial C^*-probability space (A ,τ) that are centered: $\tau(c_k)=0$ for all k. Let$m_p=\max_k||c||_p$ for p=2,N. Then

$|k_\pi[c_1,\dots, c_n]| \leq m_2^{2K}(16m_N)^{n-2k}$.

Proof:-

Since both π⟼ k_π and the right-hand side of (21) are multiplicative, we only have to prove (21) when π=$1_n$with n≤N. Then as usual $k_\pi$is denoted by k_n.If n=1 it is obvious since $k_1(c_1)=\varphi(c_1)=0$.

If n=2, then K=1 and $k_2(c_k, c_l)=\tau(c_k\, c_l)-\tau(c_k)\tau\,(c_l)=\tau(c_kc_l)$. By the Cauchy–

Schwarz inequality we get $|k_2(c_k, c_l)| \leq m_2^2$.

We now focus on the case n >2, and then K=0. This is essentially done in the but we have to replace the inequality $|\tau(c_{k_1} \dots c_{k_l})| \leq m_l^{\infty}$ by Hölder's inequality$|\tau(c_{k_1} \dots c_{k_l})| \leq m_N^l$ for any $l \leq n \leq N$. Following the proof of Lemma 4.3 in [12], we thus get that

$$k_n[c_1,\dots, c_n] \leq 4^{n-1} \sum_{\sigma \in NC(n)} m_n^n \leq 4^{2n} m m_N^n.$$

Here we consider Theorems (0.5) and (0.6). We only sketch their proofs. The idea is the same as in the holomorphic setting, except that here the relevant subset of non-crossing partitions is the set NC(d ,m)introduced and studied . proof of Theorem (0.6).

We will use that if c has a symmetric distribution , then c has vanishing odd cumulants. This means that $k_\pi[c,\dots,c]=0$ unless π has only blocks of even cardinality. To check this, by the multiplicativity of free cumulants, we have to prove that $k_n[c,\dots,c]= k_{1_\pi}[c,\dots,c] =0$ if n is odd. But this is clear: since−c and c have the same distribution, $k_n[c,\dots,c]= k_n[-c,\dots,-c]$. On the other hand since κ n is n-linear , $k_n[-c,\dots,-c]=$

$(-1)^n k_n[c,\dots,c]$.

Take $(c_k)_{k\in\mathbb{N}}$ N and $(a_k)_{k\in\mathbb{N}^d}$ as in Theorem (0.6) and define $\tilde{a}_k$ and $c_{k_1}, \dots, k_d$ as in the proof of Theorem (0.4). Assume for simplicity that c_k is normalized by $||c_k||_2=1$. Denote by I the set of $k=(k_1,\dots, k_d)\in \mathbb{N}^d$ such that for any $1\leq i<d k_i \neq k_{i+1}$. Then for p=2m we have that

$$\Big|\Big|\sum_{k\in\mathbb{N}^d} a_k \otimes c_k\Big|\Big|_{2m}^{2m} = \sum_{k_1 \dots k_{2m} \in I} Tr\big(a_{k_1} \tilde{a}_{k_2}^* \dots \tilde{a}_{k_{2m}}^*\big) \tau\big(c_{k_1} c_{k_2} \dots c_{k_{2m}}\big).$$

Expanding the moment

$$\tau\big(c_{k_1} c_{k_2} \dots c_{k_{2m}}\big) || \sum_{\pi \in NC(2dm)} k_\pi(c_{k_1}(1), \dots, c_{k_1}(d), c_{k_2}(d), \dots, c_{k_{2m}}(d)].$$

By freeness of the family$((a_k)_{k\in\mathbb{N}}$, by the assumption on the vanishing of odd moments and by Lemma (2.1.14) such a cumulant is equal to 0 except If $\pi\in NC(d,m)$ and $(k_1,\dots, k_{2m})\prec\pi$, in which case it is equal to $k_\pi[c ,c\dots,c]$. We get

$|| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k ||_{2m}^{2m} = \sum_{\pi\in NC(2dm)} k_\pi(c,\dots,c)s(a,\pi,d,m).$

But by Lemma (2.1.17), Lemma 2.2 and an iteration of Lemma (2.2.1) we get that for any π∈ NC(d,m)

$$S(a,\pi,d,m) \le \max_{0\le l\le d} ||M_l||_{2m}^{2m}.$$

On the other hand (remembering that$||c||_2$), Theorem (2.1.15) and Lemma (2.2.5) imply that for π ∈NC(d ,m),

$|k_\pi[c,\dots,c]| \le (16||c||_{2m})^{4m}.$

This yields

$||\sum_{k\in\mathbb{N}^d} a_k \otimes c_k||_{2m}^{2m} \le \sum_{\pi\in NC(d,m)} (16||c||_{2m})^{4m}.$

But by Theorem (2.1.15) NC(d,m)has cardinality less than $4^{2m}(d+1)^{4m}$. Taking the 2mth root in the preceding equation we thus get

$|| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k ||_{2m} \le 4^5(d+1)\, ||c||_{2m}^2 = \max_{0\le l\le d} ||M_l||_{2m}.$

This proves Theorem(0.6) for the case when p∈2N.For p=∞ just make p→∞

For Theorem (0.5) the proof is the same except that we have to be slightly more careful in the beginning. Recall that Id is the set of $(k_1,\varepsilon_1,\dots,k_d,\varepsilon_d)\in(\mathbb{N}\times\{1,*\})^d$ such that $\lambda(gk_1)^{\varepsilon_1}\dots\lambda(gk_d)^{\varepsilon_d}$ corresponds to an element of length d in the free group F_∞For a family of matrices $(a_k,\varepsilon)_{(k,\varepsilon)\in I_d}$ denote by

$$\tilde{a}_{k,\varepsilon} = a(k_d,\dots,k_1)(\bar{\varepsilon}_d,\dots,\bar{\varepsilon}_1)$$

Where $\bar{*}=1$ and$\bar{1}=*$. The motivation for this notation is the following: for (k,ε)∈ $I_d$denote by $c_{k,\varepsilon}=c_{k_1}^{\varepsilon_1}\dots c_{k_d}^{\varepsilon_d}$ so that if $\check{C}_{k,\varepsilon}$, is defined as $\check{a}_{k,\varepsilon}$,wehavethat$\check{C}_{k,\varepsilon}^*=C_{k,\varepsilon}$

For $k=(\varepsilon_1,\dots,k_{2m})\in(\mathbb{N}^d)^{2m}$, $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{2m})\in(\{1,*\})^{2m}$and π ∈NC(2dm)with blocks of even cardinality we will also write(k, ε)≺π if $k_i=k_j$ for all $i\sim_\pi j$ and if in addition for each block$\{i_1<\dots<i_{2p}\}$of π, 1's and ∗'s alternate in the sequence $\varepsilon_{i_1},\varepsilon_{i_2},\dots,\varepsilon_{i_{2p}}$.Last we denote, for π ∈NC(d, m)

$\tilde{S}$ (a ,π ,d ,m)= $\sum_{(k,\varepsilon)\prec\pi} Tr(a_{k_1,\varepsilon_1} \check{a}^*_{k_2,\varepsilon_2} a_{k_3\varepsilon_3} \dots \check{a}^*_{k_{2m},\varepsilon_{2m}})$.

The proofs of Lemma (2.2.1)and Lemma (2.2.2) still apply with this notation:

Lemma (2.2.6)[2]

Let π ∈ NC(d ,m), and take a finitely supported family of matrices a=

$(ak,\varepsilon)_{(k,\varepsilon)\in I_d}$ as above. For any integer i

$|\tilde{S}$ (a,π,d,m)$| \leq (\tilde{S}(a, P_{id}(\pi), d, m))^{1/2}(\tilde{S}(a, P_{(i+d)}(\pi), d, m))^{1/2}$

Lemma (2.2.7)[2]

Let d , m , $a = (a_{K,\varepsilon})_{(K,\varepsilon)\in I_d}$, and M_l be as in Theorem0.5, and σ_l and $\tilde{\sigma}_l$ as defined in Corollary1.4. Then for $l \in \{0,1, \dots, d\}$:

$$\tilde{S}\left(a, \sigma_l^{(d,m)}, d, m\right) = \|M_l\|^{2m}_{S_{2m}(H\otimes\ell^2(\mathbb{N})^{\otimes d-l};H\otimes\ell^2(\mathbb{N})^{\otimes l})}.$$

Moreover forl∈{1,...,d}

$\tilde{S}\left(a, \tilde{\sigma}_l^{(d,m)}, d, m\right) = \|M_l\|^{2m}_{S_{2m}(H\otimes\ell^2(\mathbb{N})^{\otimes d-l};H\otimes\ell^2(\mathbb{N})^{\otimes l})}$.

We leave the proofs to the reader.

Sketch of the proof of Theorem (0.5).Use the same notation as above. Take m∈ℕ. Then as for the self-adjoint case we expand the 2m-norm as follows:

$|| \sum_{(k,\varepsilon)\in I_d} a_{k,\varepsilon} \otimes c_{k,\varepsilon} ||^{2m}_{2m}$ =
$\sum_{(k_1,\varepsilon_1)\in I_d} Tr(a_{k_1,\varepsilon_1} \check{a}^*_{k_2,\varepsilon_2} \dots \check{a}^*_{k_{2m},\varepsilon_{2m}}). \tau(c_{k_1,\varepsilon_1} c_{k_2,\varepsilon_2}, \dots, c_{k_{2m},\varepsilon_{2m}})$.

By the freeness, the definition of I_d ,Lemma (2.1.14) and the fact that the c_k's are R-diagonal, the expression of the moment $\tau(c_{k_1,\varepsilon_1} \dots c_{k_{2m}\varepsilon_{2m}})$becomes simply

$\tau(c_{k_1,\varepsilon_1} \dots c_{k_2\varepsilon_{2m}})$=π ∈NC(d ,m) = $\sum_{\pi\in NC(d,m)} 1_{(k,\varepsilon)\prec k_\pi}[c^{\varepsilon_1(1)}_{k_1(1)}, \dots, c^{\varepsilon_{2m}(d)}_{k_{2m}(d)}]$.

Where if (k, ε)≺π and α_n(c)= κ_{2n} [c, c^*,c, c^*,..., c^*, c] we have that

$$k_\pi[c_{k_1(1)}^{\varepsilon_1(1)},\dots,c_{k_{2m}(d)}^{\varepsilon_{2m}(d)}] = \prod_{v\ block\ of\ \pi} \alpha_{\frac{|v|}{2}}(c).$$

In particular this quantity (which we will abusively denote by $k_\pi((c))$ does not depend on(k, ε). We therefore get

$$||\sum_{k\in\mathbb{N}^d} a_k \otimes c_k ||_{2m}^{2m} \sum_{\pi\in NC(d,m)} k_\pi[C]\ \tilde{S}\ (a\ ,\pi\ ,d\ ,m).$$

From this point the proof of Theorem (0.6) applies except that we use Lemma (2.2.7) and an iteration of Lemma (2.2.6) instead of Lemma (2.2.2) and an iteration of Lemma (2.2.1).

Here we get some lower bounds on the norms we investigated before. For example the following minoration is classical:

Lemma (2.2.8)[2]

Let$(c_k)_{k\in\mathbb{N}}$ be circular∗-free elements with$||C||_1$=1. Then for any finitely supported family of matrices$(a_{k_1,\dots,k_d})_{k_1,\dots,k_d\in\mathbb{N}}$ the following inequality holds:

$||\sum_{k_1,\dots,k_d\in\mathbb{N}} a_{k_1,\dots,k_d} \otimes c_{k_1},\dots,c_{k_d}|| \geq \max_{0\leq l\leq d} \|M_1\|.$

Proof:-

We use the following (classical) realization of free circular elements on a Fock space. Let $H=H_1 \oplus_2 H_2$ be a Hilbert space with an orthonormal basis given by $(e_k)_{k\in\mathbb{N}}\cup(f_k)_{k\in\mathbb{N}}$ ($(e_k)$is a basis of H_1 and(f_k)ofH_2). Let $F(H) = \mathbb{C}_{\Omega\oplus} \oplus_{n\geq 1} H^{\oplus n}$ be the full Fock space constructed on Hand for k∈ ℕ s(k)(resp $\tilde{}_s$(k)) the operator of creation bye_k(resp. f_k). Define finally $c_k=s_k+\tilde{s}_k^*$. It is well that $(c_k)_{k\in\mathbb{N}}$form of∗-free family of circular variables for the state ⟨Ω, Ω⟩ which is tracial on theC^* -algebra generated by the c_k's.

Let K be the Hilbert space on which the a_k's act ($K=\mathbb{C}^\alpha$if $a_k\in M_\alpha(\mathbb{C})$). Then if $P_k$denotes the orthogonal projection from $F(H)\to H_2^{\otimes k}$,for0≤l ≤d the operator $(id\otimes P_l)\circ \sum_{k_1,\dots,k_d\in\mathbb{N}} a_{k_1,\dots,k_d} \otimes c_{k_1},\dots,c_{k_d}|_{k\otimes H_1^{\otimes d-1}}$ corresponds to M_l if it is

viewed as an operator from K$\otimes H_1^{\otimes \mathrm{d-l}} \simeq k \otimes \ell^{2(\mathbb{N})^{\otimes d-1}}$ to
k $\otimes$ $\mathrm{H}_2^{\otimes \ell^2}$ $\simeq$k$\otimes\ell^2(\mathbb{N})^{\otimes l}$ for the identification $H_1 \simeq \ell^2$

And $H_2 \simeq \ell^2$with the orthonormal bases(e_k)and(f_k).This proves the lemma.

We also prove the following lemma which was stated in the introduction.

Lemma (2.2.9)[2]

Let p be a prime number and define a_{k_1},..., a_{k_d}=exp(2iπk_1... k_d/p)for any $k_i \in$ {1,...,p}.Then $||(a_k)||_2$ =$p^{d/2}$and for any l≤ $l \leq d$-1 the matrix $M_l$defined by M_l=(a(k_1,..., k_l),(k_{l+1},..., k_d))∈ $M_{p^l p^{d-1}}(\mathbb{C})$,satisfies $||M_l|| \leq p^{d/2}\sqrt{(d-1)/p}$.

Proof:-

Since $||M_l||^2$= $||M_l M_l^*||$ we compute the matrix$M_l M_l^* \in p^l p^l(\mathbb{C})$.

For any s=(s_1,..., s_l)and t =(t_1,..., t_l)∈$\{1,...,\mathrm{p}\}^l$thes, tth entry of$M_l M_l^*$is equal to

$$\sum_{(k_{l+1},\ldots\ldots k_d)\in[l,\ldots,p]^{d-1}} exp(2\mathrm{i}\pi(s_1\ldots s_l - t_1\ldots t_l)/k_{l+1} \ldots k_d/\mathrm{p}).$$

If s_1...s_l =t_1... t_l mod p then this quantity is equal to p^{d-l}

Where as otherwise, ω=exp(2iπ(s_1... s_l −t_1... t_l)/p)is a primitive pth root of 1, and it is straightforward to check that for such an ω,

$$\sum_{(k_{l+1},\ldots\ldots k_d)\in[l,\ldots,p]^{d-1}} \omega^{k_{l+1}\ldots k_d} = \sum_{(k_{l+1},\ldots\ldots k_d)\in[l,\ldots,p]^{d-1}} (\omega^{k_{l+1}\ldots k_d})^{k_d}.$$

$$\sum_{(k_{l+1},\ldots\ldots k_d)} Pl_{k_{l+1}\ldots k_{d-1}=0 \bmod p}$$

$$=\mathrm{p}(p^{d-l-1} - (p-1)^{d-l-1}$$

We therefore have that

$$M_l M_l^* (p^{d-1} - p(p-1)^{d-l-1})(1)_{s,t\in\backslash p]^l} + p(p-1)^{d-l-1}(1_{s_1\ldots s_l = t_1\ldots t_l})_{s,t\in\backslash p]^l}$$

The norm of an N×N matrix with entries all equal to 1 is N.

Moreover if$[p]^l$={(s_1,..., s_l)}is decomposed depending on the value ofs_1... s_l modulo p, the matrix $(1_{s_1\ldots s_l = t_1\ldots t_l})_{s,t\in\backslash p]^l}$ is a block-diagonal matrix with blocks having all entries equal to 1. Its norm is therefore equal to

$\max_{i\in[p]} |\{s_1, \ldots, s_l) \in [p]^l, s_1 \ldots s_l = i \bmod p\}| = |\{s_1, \ldots, s_l) \in [p]^l, s_1 \ldots s_l = 0\}|$

=$p^l - (p-1)^l$.

$$p(p^{d-l-1} - (p-1)^{d-l-1}) + p(p-1)^{d-l-1}(p^l - (p-1)^l)$$

$$p^d - p(p-1)^{d-1} \leq (d-1)p^{d-1}.$$

Chapter 3

C^*-algebras and Haageup property

Let Γ be a discrete group and Λ a normal subgroup of Γ, we show that the inclusion $A \rtimes_{\alpha,r} \Lambda \subseteq A \rtimes_{\alpha,r} \Gamma$ has the relative Haagerup property if and only if the quotient group Γ/Λ has the Haagerup property. In particular, the inclusion $C_r^*(\Lambda) \subseteq C_r^*(\Gamma)$ has the relative Haagerup property if and only if Γ/Λ has the Haagerup property; $C_r^*(\Gamma)$ has the Haagerup property if and only if Γ has the Haagerup property. We also characterize the Haagerup property for Γ in terms of its Fourier algebra $A(\Gamma)$.

Section (3.1): Haagerup Property for C^*-algebras

In the following we first recall some facts about crossed products. Let Γ be a discrete group and $\mathcal{A}$ be a unital C^*-algebra. An action of Γ on $\mathcal{A}$ is a group homomorphism α from Γ into the group of $*$-automorphisms(is an isomorphism from a mathematical object to itself. It is, in some sense, a symmetry of the object, and a way of mapping the object to itself while preserving all of its structure. The set of all automorphisms of an object forms a group, called the automorphism group. It is, loosely speaking, the symmetry group of the object.)[4] on $\mathcal{A}$. A C^*-algebra equipped with a Γ-action is called a Γ-C^*-algebra. For a Γ-C^*-algebra $\mathcal{A}$, we denote by $C_c(\Gamma, \mathcal{A})$ the linear space of finitely supported function on Γ with values in $\mathcal{A}$. A typical element S in $C_c(\Gamma, \mathcal{A})$ is written as a finite sum

$$S = \sum_{s \in \Gamma} a_s S,$$

where $a_s \in \mathcal{A}$. We equip $C_c(\Gamma, \mathcal{A})$ with an α-twisted convolution product and $*$-operation as follows: for $S = \sum_{s \in \Gamma} a_s S$, $\mathrm{T} = \sum_{t \in \Gamma} b_t t \in C_c(\Gamma, \mathcal{A})$, we define

$$S *_\alpha \mathrm{T} = \sum_{s,t \in \Gamma} a_s \alpha_s(b_t) st \quad \text{and} \quad S^* = \sum_{s \in \Gamma} \alpha_{s^{-1}}(a_s^*) S^{-1}.$$

Note that when $\mathcal{A} = \mathbb{C}$ and the action α is trivial, we simply recover the group ring $\mathbb{C}[\Gamma]$. Now the question is: how shall we complete $C_c(\Gamma, \mathcal{A})$? Just as for group C^*-algebras, there are two natural choices, a universal and reduced completion.

A covariant representation $(u, \pi, \mathcal{H})$ of the Γ-C^*-algebra $\mathcal{A}$ consists of a unitary representation $(u, \mathcal{H})$ of Γ and a $*$-representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ such that $u_s\pi(a)u_s^* = \pi(\alpha_s(a))$ for every $s \in \Gamma$ and $a \in \mathcal{A}$. For a covariant representation of $(u, \pi, \mathcal{H})$, we denote by $u \times \pi$ the associated $*$-representation of $C_c(\Gamma, \mathcal{A})$.

Definition (3.1.1)[3]

The full crossed product of a C^*-dynamical system $(\mathcal{A}, \alpha, \Gamma)$, denoted by $\mathcal{A} \rtimes_\alpha \Gamma$, is the completion of $C_c(\Gamma, \mathcal{A})$ with respect to the norm

$$\|\chi\|_{\mathrm{u}} = \sup\|\pi(\chi)\|,$$

where the supremum is over all cyclic $*$-homomorphisms $\pi : C_c(\Gamma, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$.

To define the reduced crossed product, we begin with a faithful representation $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Define a new representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H} \otimes \ell_2(\Gamma))$ by

$$\pi(a)(v \otimes \delta_{\mathrm{g}}) = \alpha_{\mathrm{g}^{-1}}(a)(v) \otimes \delta_{\mathrm{g}},$$

where $\{\delta_{\mathrm{g}}\}_{\mathrm{g}\in\Gamma}$ the canonical orthonormal basis of $l^2(\Gamma)$. Under the identification $\mathcal{H} \otimes l^2(\Gamma) \cong \bigoplus_{\mathrm{g}\in\Gamma} \mathcal{H}$ we have simply taken the direct sum representation

$$\pi(a) = \bigoplus_{\mathrm{g}\in\Gamma} \alpha_{\mathrm{g}^{-1}}(a) \in \mathcal{B}\left(\bigoplus_{\mathrm{g}\in\Gamma} \mathcal{H}\right).$$

For all elementary tensors we can check that

$$(I \otimes \lambda_\Gamma(s))\pi(a)(I \otimes \lambda_\Gamma(s)^*)(v \otimes \delta_{\mathrm{g}}) = \pi(\alpha_s(a))(v \otimes \delta_{\mathrm{g}}).$$

Hence we get an induced covariant representation $(I \otimes \lambda_\Gamma) \times \pi$, called a regular representation.

Definition (3.1.2) [3]

The reduced crossed product of a C^*-dynamical system $(\mathcal{A}, \Gamma, \alpha)$, denoted by $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ is defined to be the norm closure of the image of a regular representation $C_c(\Gamma, \mathcal{A}) \to \mathcal{B}(\mathcal{H} \otimes l^2(\Gamma))$. We often denote a typical element $x \in C_c(\Gamma, \mathcal{A}) \subseteq \mathcal{A} \rtimes_{\alpha,r} \Gamma$ as a finite sum $x = \sum_{s\in\Gamma} a_s\lambda_\Gamma(s)$.

Let τ be a faithful tracial state on a unital C^*-algebra $\mathcal{A}$. By the GNS-construction, τ defines an $\mathcal{A}$-Hilbert bimodule, denoted by $L_2(\mathcal{A}, \tau)$, which has a unit central vector $\xi_\tau \in L_2(\mathcal{A}, \tau)$ such that $\tau(a) = \langle a\xi_\tau, \xi_\tau \rangle$ for all $a \in \mathcal{A}$. More precisely, let

$$\mathcal{N} = \{a \in \mathcal{A} : \tau(a^*a) = 0\}.$$

$\mathcal{N}$ is a two-sided $*$-ideal in $\mathcal{A}$. For each $a \in \mathcal{A}$, we often denote the associated vector $a + \mathcal{N}$ by a_τ . Define an inner product on the quotient $\mathcal{A}/\mathcal{N}$ by

$$\langle a_\tau, b_\tau \rangle = \tau(b^*a).$$

Let $L_2(\mathcal{A}, \tau)$ be the Hilbert space completion of $\mathcal{A}/\mathcal{N}$. Let $\xi_\tau = I + \mathcal{N}$, we have

$$\begin{aligned} \tau(a) &= \langle a + \mathcal{N}, I + \mathcal{N} \rangle \\ &= \langle a\xi_\tau, \xi_\tau \rangle, \ \ \forall a \in \mathcal{A}, \end{aligned}$$

We also denote by $\|\cdot\|_{2,r}$ the associated Hilbert norm on $\mathcal{A}$ (we simply write $\|\cdot\|_2$ when τ is fixed and when there is no danger of confusion). Thus for each $a \in \mathcal{A}$,

$$\|a\|_2 = \|a_\tau\| = \tau(a^*a)^{1/2} \leq \|a\|,$$

If $\Phi: \mathcal{A} \to \mathcal{A}$ is a unital completely positive map such that $\tau \circ \Phi \leq \tau$, then Φ can be extended to a contraction $T_\Phi : L_2(\mathcal{A}, \tau) \to L_2(\mathcal{A}, \tau)$ via the equality

$$T_\Phi(a_\tau) = \Phi(a)_\tau, \ \ \forall a \in \mathcal{A}.$$

Indeed,

$$\begin{aligned} \|T_\Phi(a_\tau)\|^2 &= \|\Phi(a)_\tau\|^2 \\ &= \tau\big(\Phi(a)^*\Phi(a)\big) \\ &\leq \tau\big(\Phi(a^*a)\big) \quad \big(\text{since } \Phi(a)^*\Phi(a) \leq \Phi(a^*a)\big) \\ &\leq \tau(a^*a) \quad (\text{since } \tau \circ \Phi \leq \tau) \\ &= \|a_\tau\|^2. \end{aligned}$$

We say Φ is L_2-compact if T_Φ is a compact operator on $L_2(\mathcal{A}, \tau)$.

Definition (3.1.3) [3]

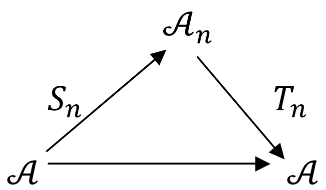
Let $\mathcal{A}$ be a unital C^*-algebra admitting a faithful tracial state τ. We say that $\mathcal{A}$ has the Haagerup approximation property with respect to τ (shortly $\mathcal{A}$ has the Haagerup property) if there exists a sequence $\{\Phi_n\}_{n\geq 1}$ of unital completely positive maps from $\mathcal{A}$ into itself such that

(i) $\tau \circ \Phi_n \leq \tau$ and Φ_n is L_2-compact for every n;
(ii) For every $a \in \mathcal{A}, \|\Phi_n(a) - a\|_2 \to 0$ as $n \to +\infty$.

The following lemma will be needed in the proof of Theorem (3.1.5).

Lemma (3.1.4) [3]

(i) Assume that $\mathcal{A}$ is a unital separable C^*-algebra with a faithful tracial state τ and there exist a sequence of unital C^*-algebras $\mathcal{A}_n$, each of which has the Haagerup property with respect to faithful tracial states τ_n, and the following approximately commutative diagram of unital completely positive maps



which $\tau_n \circ S_n \leq \tau, \tau \circ T_n \leq \tau_n$ and

$$\|T_n \circ S_n(a)\|_2 \to 0, \ \forall a \in \mathcal{A}.$$

Then $\mathcal{A}$ has the Haagerup property with respect to τ.

(ii) Suppose that $\mathcal{A}$ has the Haagerup property with respect to a faithful tracial state τ. Then for each $k \in \mathbb{N}, M_k(\mathcal{A})$ has the Haagerup property with respect to the induced trace τ_k.

Proof:-

(i) Suppose that $\mathcal{A}$ is separable with a countable dense subset $\{a_n : n \geq 1\}$. Then for each n, the argument in (ii) gives a unital completely positive map $\Phi_n : \mathcal{A} \to \mathcal{A}$ such that Φ_n is L_2-compact and satisfies

$$\left\|\Phi_{\mathrm{n}}(a_j) - a\right\|_2 < \frac{1}{\mathrm{n}} \quad \text{for} \quad j = 1,2,\dots,\mathrm{n}.$$

Since $\|\cdot\|_2 \leq \|\cdot\|$ on $\mathcal{A}$ and $\|\Phi_{\mathrm{n}}\| = 1$, then for $x \in \mathcal{A}$ and $\epsilon > 0$, and $\frac{\epsilon}{3}$-argument gives $\|\Phi_{\mathrm{n}}(x) - \chi\|_2 < \epsilon$ for all large enough n. So $\mathcal{A}$ has the Haagerup property with respect to τ.

(ii) Since $\mathcal{A}$ has the Haagerup property with respect to τ, there exists a sequence of unital completely positive maps $\{\Phi_{\mathrm{n}}\}$ with the conditions in Definition (3.1.3). So each

$$id_{M_k} \otimes \Phi_{\mathrm{n}}: M_k(\mathcal{A}) \to M_k(\mathcal{A})$$

is a unital completely positive map with

$$\tau_k \circ \left(id_{M_k} \otimes \Phi_{\mathrm{n}}\right) \leq \tau_k.$$

Indeed, for $a = \left(a_{ij}\right) \in M_k(\mathcal{A})$ we have

$$\begin{aligned}\tau_k \circ \left(id_{M_k} \otimes \Phi_{\mathrm{n}}\right)(a^* a) &= \frac{1}{k}\sum_{i,j} \tau \circ \Phi_{\mathrm{n}}\left(a_{ij}^* a_{ij}\right) \\ &\leq \frac{1}{k}\sum_{i,j} \tau\left(a_{ij}^* a_{ij}\right) \\ &= \tau_k(a^* a).\end{aligned}$$

For $a = \left(a_{ij}\right) \in M_k(\mathcal{A})$, we check that

$$\begin{aligned}\left\|id_{M_k} \otimes \Phi_{\mathrm{n}}(a) - a\right\|_{2,\tau_k}^2 &= \left\|\Phi_{\mathrm{n}}\left(a_{ij}\right) - a_{ij}\right\|_{2,\tau_k}^2 \\ &= \frac{1}{k}\sum_{i,j} \left\|\Phi_{\mathrm{n}}\left(a_{ij}\right) - a_{ij}\right\|_{2,\tau}^2 \\ &\to 0, \quad \text{as} \quad \mathrm{n} \to +\infty.\end{aligned}$$

In the following we will show that each id_{M_k} is L_2-compact. We omit the suffix ' n' of Φ_{n} for simplicity. Define

$$\prod : \; L_2(M_k(\mathcal{A}),\tau_k) \to L_2(M_k, tr_k) \otimes L_2(\mathcal{A},\tau)$$

by

$$\prod\left(a_{\tau_k}\right) = \sum_{\mathrm{i,j}} e_{i,j} \otimes \left(a_{i,j}\right)_\tau, \quad \text{for any } a = \left(a_{i,j}\right) \in M_k(\mathcal{A}).$$

We can check that for $a = \left(a_{i,j}\right) \in M_k(\mathcal{A})$,

$$\begin{aligned}
\left\|a_{\tau_k}\right\|^2 &= \tau_k(a^* a) \\
&= \frac{1}{k}\sum_{i,j} \tau\left(a_{i,j}^* a_{i,j}\right) \\
&= \sum_{i,j} tr_k\left(e_{i,j}^* e_{i,j}\right)\tau\left(a_{i,j}^* a_{i,j}\right) \\
&= \left\|\sum_{i,j} e_{i,j} \otimes \left(a_{i,j}\right)_\tau\right\|^2 .
\end{aligned}$$

This implies that $\prod$ is an isometric isomorphism between $L_2(M_k(\mathcal{A}),\tau_k)$ and $L_2(M_k, tr_k) \otimes L_2(\mathcal{A},\tau)$. Since Φ is L_2-compact, then for any $\epsilon > 0$ there exists a finite dimensional linear map Q on $L_2(\mathcal{A},\tau)$ such that

$$\|\mathrm{T}_\Phi(\chi_\tau) - Q(\chi_\tau)\| \le \epsilon \cdot \|\chi_\tau\|, \;\; \forall x \in \mathcal{A}$$

Now for any $a = \left(a_{ij}\right) \in M_k(\mathcal{A})$, we can identify a_{τ_k} with $\sum_{i,j} e_{i,j} \otimes \left(a_{i,j}\right)_\tau$ from the above isomorphism. So we have

$$\begin{aligned}
\left\|\mathrm{T}_{id_{M_k}\otimes\Phi(a\tau_k)} - i\mathrm{d}_{M_k} \otimes Q(a\tau_k)\right\|^2 &= \left\|\sum_{i,j} e_{i,j} \otimes (T_\Phi - Q)\left(\left(a_{i,j}\right)_\tau\right)\right\|^2 \\
&= \frac{1}{k}\left\|\sum_{i,j} (\mathrm{T}_\Phi - Q)\left(\left(a_{i,j}\right)_\tau\right)\right\|^2
\end{aligned}$$

$$\leq \frac{1}{k} \sum_{i,j} \epsilon^2 \left\| \left((a_{i,j})_\tau \right) \right\|^2$$

$$= \epsilon^2 \cdot \left\| a_{\tau_k} \right\|^2.$$

Therefore $M_k(\mathcal{A})$ has the Haagerup property.

In the following result, we suppose that $\mathcal{A}$ is a unital separable C^*-algebra with a faithful tracial state τ and a τ-preserving action α of a countable discrete group Γ. Through the result is a C^*-version, the proof is quite different.

Theorem (3.1.5) [3]

If Γ is amenable and $\mathcal{A}$ has the Haagerup property, then $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ has also the Haagerup property.

Proof :-

If $\mathrm{F_n} \subseteq \Gamma$ is a finite set for every $n \in \mathbb{N}$, we define $\varphi_n : \mathcal{A} \rtimes_{\alpha,r} \Gamma \to \mathcal{A} \otimes_{\min} M_{\mathrm{F_n}}(\mathbb{C})$ such that

$$\varphi_{\mathrm{n}}\left(a\lambda_\Gamma(s)\right) = \sum_{\mathrm{p} \in F_{\mathrm{n}} \cap \mathrm{s} F_n} \alpha_{\mathrm{p}}^{-1}(a) \otimes e_{\mathrm{p,s^{-1}p}}$$

and $\psi_{\mathrm{n}} : \mathcal{A} \otimes_{\min} M_{\mathrm{F_n}}(\mathbb{C}) \to \mathcal{A} \rtimes_{\alpha,r} \Gamma$ such that

$$\psi_{\mathrm{n}}\left(a \otimes e_{\mathrm{p,q}}\right) = \frac{1}{|F_{\mathrm{n}}|} \alpha_{\mathrm{p}}(a) \lambda_\Gamma(\mathrm{pq^{-1}}).$$

We know that $\varphi_{\mathrm{n}}, \psi_{\mathrm{n}}$ are unital completely positive maps. First we want to show that $\tau_{\mathrm{n}} \circ \varphi_{\mathrm{n}} \leq \tau', \tau' \circ \psi_{\mathrm{n}} \leq \tau_{\mathrm{n}}$ where τ_{n} and τ' are the induced traces of τ on $\mathcal{A} \otimes_{\min} M_{\mathrm{F_n}}(\mathbb{C})$ and $\rtimes_{\alpha,r} \Gamma$ respectively. For that, let $x = \sum \chi_{\mathrm{g}} \lambda_\Gamma(\mathrm{g}) \in \mathcal{A} \rtimes_{\alpha,r} \Gamma$ with $\chi_{\mathrm{g}} = 0$ except for finitely many g's. We have

$$\chi^* \chi = \sum_{\mathrm{s}} \left(\sum_{h} \alpha_{\mathrm{h^{-1}}} \left(\chi_h^* \chi \mathrm{hs}\right) \right) \lambda_\Gamma(s),$$

and

$$\tau_n \circ \varphi_n(\chi^*\chi) = \tau_n\left(\sum_s \sum_{p\in F_n\cap sF_n} \alpha_{p^{-1}}\left(\sum_h \alpha_{h^{-1}}(\chi_h^*\chi_{hs})\right)\otimes e_{p,s^{-1}p}\right)$$

$$= \frac{1}{|F_n|}\sum_{p\in F_n} \tau\left(\alpha_{p^{-1}}\sum_h \alpha_{h^{-1}}(\chi_h^*\chi_h)\right)$$

$$= \frac{1}{|F_n|}\sum_{p\in F_n}\sum_h \tau(\chi_h^*\chi_h)$$

$$= \frac{1}{|F_n|}\sum_{p\in F_n\cap} \tau'(\chi^*\chi) = \tau'(\chi^*\chi),$$

where the third equality follows from that α is τ-preserving. To prove $\tau' \circ \psi_n \leq \tau_n$, that we only need to check that for any set $\{a_p\}_{p\in F_n} \subseteq \mathcal{A}$,

$$\tau' \circ \psi_n\left(\sum_{p,q\in F_n} a_p^* a_q \otimes e_{p,q}\right) \leq \tau_n\left(\sum_{p,q\in F_n} a_p^* a_q \otimes e_{p,q}\right).$$

But

$$\tau' \circ \psi_n\left(\sum_{p,q\in F_n} a_p^* a_q \otimes e_{p,q}\right) = \frac{1}{|F_n|}\tau'\left(\sum_{p,q\in F_n} \alpha_p\left(a_p^* a_q\right)\lambda_\Gamma(pq^{-1})\right)$$

$$= \frac{1}{|F_n|}\sum_{p\in F_n} \tau\left(a_p^* a_p\right)$$

$$= \tau_n\left(\sum_{p\in F_n} a_p^* a_q \otimes e_{p,q}\right).$$

Secondly, since Γ is amenable, we can choose F_n as a Folner sequence and φ_n, ψ_n, are the corresponding maps constructed above. Let $x = \sum \chi_g \lambda_\Gamma(g) \in C_c(\Gamma, \mathcal{A})$, we want to prove that

$$\|\psi_n \circ \varphi_n(\chi) - \chi\|_{2,\tau'} \to 0.$$

Indeed,

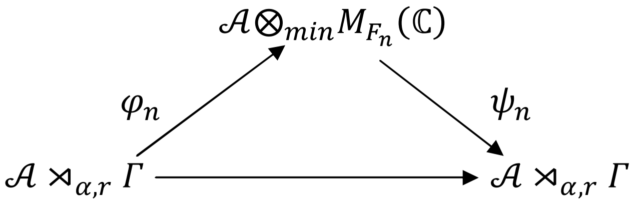
$$\|\psi_n \circ \varphi_n(\chi) - \chi\|_{2,\tau'} \leq \|\psi_n \circ \varphi_n(\chi) - \chi\|_{\mathcal{A} \rtimes_{\alpha,r} \Gamma}$$

$$= \left\| \sum_g \left(\frac{|F_n \cap F_n|}{|F_n|} \right) \chi_g \lambda_\Gamma(g) \right\|_{\mathcal{A} \rtimes_{\alpha,r} \Gamma}$$

$$\to 0,$$

since $\{F_n\}$ is a Folner sequence and only finite many χ_g's are non-zero. Since $\|\cdot\|_2 \leq \|\cdot\|$ on the cross product and each $\psi_n \circ \varphi_n$ is unital completely positive, so a routine $\epsilon/3$-argument shows the convergence for general χ in $\mathcal{A} \rtimes_{\alpha,r} \Gamma$.

Since $\mathcal{A}$ has the Haagerup property, implies that $\mathcal{A} \otimes_{\min} M_{F_n}(\mathbb{C})$ has also the Haagerup property. From above, we have the following approximately commutative diagram of unital completely positive maps



which $\tau_n \circ \varphi_n \leq \tau', \tau' \circ \psi_n \leq \tau_n$ and

$$\|\psi_n \circ \varphi_n(\chi) - \chi\|_{2,\tau'} \to 0, \quad \forall \chi \in \mathcal{A} \rtimes_{\alpha,r} \Gamma.$$

in Lemma (3.1.4) shows that $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the Haagerup property.

We will concern at the Haagerup property for reduced group C^*-algebras.

Theorem (3.1.6) [3]

Let Γ be a countable discrete group. Then the following properties are equivalent.

(i) Γ has the Haagerup property;

(ii) The reduced group C^*-algebra $C_{\mathrm{r}}^*(\Gamma)$ has the Haagerup property;

(iii) The group von Neumann algebra $L(\Gamma)$ has the Haagerup property.

Proof:

The equivalence of (i) and (ii) was proved first by Choda. So we only need to show the equivalence of (i) and (ii).

(i)⇒(ii) Suppose that Γ has the Haagerup property and take a sequence $\{\varphi_i\}$ of positive definite functions such that $\varphi_i(e) = 1, \varphi_i \in c_0(\Gamma)$ and $\varphi_i \to 1$ pointwise. We define corresponding multipliers $m_{\varphi_i}: \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ by

$$m_{\varphi_i}\left(\sum_{t\in\Gamma} a_t\lambda_\Gamma(t)\right) = \sum_{t\in\Gamma} \varphi_{\mathrm{i}}(t)a_t\lambda_\Gamma(t).$$

The multipliers $\{m_{\varphi_i}\}$ can be extended to unital completely positive maps from $C_{\mathrm{r}}^*(\Gamma)$ to $C_{\mathrm{r}}^*(\Gamma)$. In the following, we will prove that the multipliers $\{m_{\varphi_i}\}$ satisfy the conditions of the Haagerup property for $C_{\mathrm{r}}^*(\Gamma)$. Since each φ_i vanishes at the infinity of , then for each $k \in \mathbb{N}$, we have a finite subset $F_{\mathrm{i}}^{(k)} \subseteq \Gamma$such that

$$|\varphi_{\mathrm{i}}(t)| < \frac{1}{k}, \quad \forall t \in \Gamma\backslash F_{\mathrm{i}}^{(k)}.$$

Set, for each $k \in \mathbb{N}$

$$m_{\varphi_{\mathrm{i}}^{(k)}}\left(\sum_{t\in\Gamma} a_t\lambda_\Gamma(t)\right) = \sum_{t\in F_{\mathrm{i}}^{(k)}} \varphi_{\mathrm{i}}(t)a_t\lambda_\Gamma(t).$$

Then $\left\{m_{\varphi_{\mathrm{i}}^{(k)}}\right\}_{k=1}^{+\infty}$ is a sequence of finite rank linear maps on $C_{\mathrm{r}}^*(\Gamma)$. For each $\chi = \sum_{t\in\Gamma} a_t\lambda_\Gamma(t) \in \mathbb{C}[\Gamma] \subseteq C_{\mathrm{r}}^*(\Gamma)$,

$$\left\|m_{\varphi_{\mathrm{i}}}(\chi) - m_{\varphi_{\mathrm{i}}^{(k)}}(\chi)\right\|_2^2 = \left\|\sum_{t\in\Gamma\backslash F_{\mathrm{i}}^{(k)}} \varphi_{\mathrm{i}}(t)a_t\,\lambda_\Gamma(t)\right\|_2^2$$

$$= \sum_{t\in\Gamma\setminus F_{\mathrm{i}}^{(k)}} |\varphi_{\mathrm{i}}(t)|^2 |a_t|^2$$

$$\leq \left(\sum_{t\in\Gamma\setminus F_{\mathrm{i}}^{(k)}} |a_t|^2 \right) \Big/ k^2$$

$$\leq \|\chi\|_2^2 / k^2,$$

where τ is the canonical trace on $C_{\mathrm{r}}^*(\Gamma)$. Note that since multipliers $m_{\varphi_{\mathrm{i}}}, m_{\varphi_{\mathrm{i}}^k}$ are contractive completely positive, so their associated operators on $L_2(C_{\mathrm{r}}^*(\Gamma), \tau)$ are contractive. Hence the norm $\left\| \mathrm{T}_{m_{\varphi_{\mathrm{i}}}} - \mathrm{T}_{m_{\varphi_{\mathrm{i}}^k}} \right\|$ is determined on the dense subspace $\mathbb{C}[\Gamma]$ of $L_2(C_{\mathrm{r}}^*(\Gamma), \tau)$ and hence

$$\left\| \mathrm{T}_{m_{\varphi_{\mathrm{i}}}} - \mathrm{T}_{m_{\varphi_{\mathrm{i}}^k}} \right\| \leq \frac{1}{k} \to 0, \quad \text{as } k \to +\infty.$$

Hence $m_{\varphi_{\mathrm{i}}}$ is L_2-compact. Since for any $t \in \Gamma, |\varphi_{\mathrm{i}}(t)| \leq \varphi_{\mathrm{i}}(e) = 1$, so it is obvious that $\tau \circ m_{\varphi_{\mathrm{i}}} \leq \tau$. Let $c \in C_{\mathrm{r}}^*(\Gamma)$ and $\epsilon > 0$, then there is a finite subset F of Γ and elements $a_t \in \mathcal{A}(t \in F)$ such that

$$\|\chi - \chi_{\mathrm{F}}\| \leq \frac{\epsilon}{3},$$

where $\chi_F = \sum_{t\in F} a_t \lambda_\Gamma(t)$. Therefore, for each $\chi \in C_{\mathrm{r}}^*(\Gamma)$, we have

$$\left\| m_{\varphi_{\mathrm{i}}}(\chi) - \chi \right\|_2 \leq \left\| m_{\varphi_{\mathrm{i}}}(\chi) - m_{\varphi_i}(\chi_F) \right\|_2 + \left\| \mathrm{T}_{m_{\varphi_{\mathrm{i}}}}(\chi_F) - \chi_F \right\|_2 + \|\chi_F - \chi\|_2$$

$$\leq \frac{\epsilon}{3} + \left(\sum_{t\in F} |\varphi_{\mathrm{i}}(t) - 1|^2 \, |a_t|^2 \right)^{\frac{1}{2}} + \frac{\epsilon}{3}$$

$$< \epsilon$$

for sufficiently large i, by the assumption for the sequence $\{\varphi_{\mathrm{i}}\}$. Hence $C_{\mathrm{r}}^*(\Gamma)$ has the Haagerup property.

(ii)⇒(i) Assume that $C_r^*(\Gamma)$ has the Haagerup property. Then there is a sequence $\{\varphi_i\}_{i\geq 1}$ of unital completely positive maps from $C_r^*(\Gamma)$ to $C_r^*(\Gamma)$ which satisfy the conditions in Definition (3.1.3). For each i ∈ ℕ, set

$$\Phi_i(t) = \tau\left(\lambda_\Gamma(t)^*\Phi_i(\lambda_\Gamma(t))\right),$$

$\varphi_i(e) = 1$. We know that m_{φ_i} is a unital completely positive map from $C_r^*(\Gamma)$to $C_r^*(\Gamma)$. So .i is positive definite. For each $t \in \Gamma$, we have

$$\begin{aligned} |\Phi_i(t) - 1| &= \left|\tau\left(\lambda_\Gamma(t)^*\Phi_i(\lambda_\Gamma(t)) - 1\right)\right| \\ &= \left|\tau\left(\lambda_\Gamma(t)^*\Phi_i(\lambda_\Gamma(t)) - \lambda_\Gamma(t)\right)\right| \\ &\leq \left\|\Phi_i(\lambda_\Gamma(t)) - \lambda_\Gamma(t)\right\|_2 \to 0. \end{aligned}$$

In the following we will show that each φ_i (i fixed) vanishes at the infinity of Γ. By the Cauchy.Schwarz inequality,

$$|\Phi_i(t)| \leq \left\|\Phi_i(\lambda_\Gamma(t))\right\|_2 = \left\|\mathrm{T}_{\Phi_i}(\lambda_\Gamma(t)_\tau)\right\|,$$

where $\{\lambda_\Gamma(t)_\tau\}$ is the standard basis for $L_2(C_r^*(\Gamma), \tau)$. Now, it is a well-known and elementary fact that if T is a compact operator on a Hilbert space with orthonormal basis $\{e_\mu\}_{\mu\in\Delta}$, then, given $\epsilon > 0$, there is a finite subset F of Δ such that $\|\mathrm{T}(e_\mu)\| < \epsilon$ for all $\mu \in \Delta\backslash F$. Thus it follows from the compactness of $T\Phi_i$ and the above argument that each φ_i vanishes at the infinity of Γ and Γ has the Haagerup property.

Section (3.2): Relative Haagerup property for C^*-algebras

We now extend the definition of the Haagerup property from the above single C^*-algebra case to the relative case of inclusions of C^*-algebras, by using a similar strategy to the way notions of amenability and Property (T) were extended from single algebras to inclusions of algebras.

Let $1 \in \mathcal{B} \subseteq \mathcal{A}$ be C^*-algebras, τ be a fixed faithful trace on $\mathcal{A}$, which acts by left multiplication on $L_2(\mathcal{A}, \tau)$ in the GNS representation of τ. Let $\chi_\tau \in L_2(\mathcal{A}, \tau)$

be the appropriate vector for each $\chi \in \mathcal{A}$. Suppose that there exists a τ-preserving conditional expectation $\mathrm{E}_{\mathcal{B}}$ from $\mathcal{A}$ onto $\mathcal{B}$. Let $\Phi: \mathcal{A} \in \mathcal{A}$ be an $\mathrm{E}_{\mathcal{B}}$-preserving $\mathcal{B}$-bimodule unital completely positive map. Then the Cauchy-Schwartz inequality $\Phi(\chi)^*\Phi(\chi) \leq \Phi(\chi^*\chi), \chi \in \mathcal{A}$, yields the construction $\mathrm{T}_{\Phi} \in \mathcal{B}(L_2(\mathcal{A},\tau))$,

$$\mathrm{T}_{\Phi}(\chi_\tau) = (\Phi(\chi))_\tau, \quad \chi \in \mathcal{A}.$$

The $\mathcal{B}$-linearity of Φ yields $\mathrm{T}_{\Phi}(\chi_\tau) = (\Phi(\chi))_\tau = \chi_\tau, \ \forall \chi \in \mathcal{B}$; hence $\mathrm{T}_{\Phi}|_{L_2(\mathcal{B},\tau)}$. For $a \in \mathcal{A}, \ \chi \in \mathcal{B}$, we have

$$\begin{aligned}\langle T_{\Phi}(a_\tau), \chi_\tau\rangle &= \tau(\chi^*\Phi(a)) \\ &= \tau\left(E_{\mathcal{B}}(\chi^*\Phi(a))\right) \\ &= \tau\left(\chi^* E_{\mathcal{B}}(\Phi(a))\right) \\ &= \tau(\chi^* E_{\mathcal{B}}(a)) \\ &= \tau(E_{\mathcal{B}}(\chi^* a)) \\ &= \tau(\chi^* a) \\ &= \langle a_\tau, \chi_\tau\rangle.\end{aligned}$$

This implies that $\mathrm{T}_{\Phi}^*(\chi_\tau) = \chi_\tau, \ \forall \chi \in \mathcal{B}$. Therefore

$$\mathrm{T}_{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{T}|_{L_2(\mathcal{B},\tau)^\perp} \end{pmatrix}$$

subject to the orthogonal decomposition $L_2(\mathcal{A},\tau) = L_2(\mathcal{B},\tau) \oplus L_2(\mathcal{B},\tau)^\perp$. We define $e_{\mathcal{B}} = \mathrm{T}_{\mathrm{E}_{\mathcal{B}}}$, so for any $\chi \in \mathcal{A}, e_{\mathcal{B}}(\chi_\tau) = \mathrm{T}_{\mathrm{E}_{\mathcal{B}}}(\chi_\tau) = (\mathrm{E}_{\mathcal{B}}(\chi))_\tau$. Note that $e_{\mathcal{B}}$ is just the associated projection from $L_2(\mathcal{A},\tau)$ onto $L_2(\mathcal{B},\tau)$. An operator $ae_{\mathcal{B}}b, a, b \in \mathcal{A}$, acts on $L_2(\mathcal{A},\tau)$ by

$$ae_{\mathcal{B}}bx_\tau = (aE_{\mathcal{B}}(b\chi))_\tau, x\mathcal{A}.$$

Set

$$\mathcal{F}_{\mathcal{B}}(\mathcal{A}) = \left\{ \mathrm{T} \in \mathcal{B}' \cap \mathcal{B}\big(L_2(\mathcal{A},\tau)\big) : \mathrm{T} = \sum_{\mathrm{i}\in F} a_{\mathrm{i}} e_{\mathcal{B}} b_{\mathrm{i}}\,, F \text{ finite set and } a_{\mathrm{i}}, b_{\mathrm{i}} \in \mathcal{A} \right\}$$

and let $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ be the norm closure of $\mathcal{F}_{\mathcal{B}}(\mathcal{A})$ in $\mathcal{B}\big(L_2(\mathcal{A},\tau)\big)$.

Definition (3.2.1) [3]

We say that the inclusion $\mathcal{B} \subseteq \mathcal{A}$ has the relative Haagerup property if there exists a sequence $\{\Phi_{\mathrm{n}}\}_{\mathrm{n}\geq 1}$ of $\mathrm{E}_{\mathcal{B}}$-preserving, $\mathcal{B}$-bimodule, unital completely positive maps from $\mathcal{A}$ to itself such that

(i) $\lim_{\mathrm{n}\to+\infty} \|\Phi_{\mathrm{n}}(\chi) - \chi\|_2 = 0$ for every $\chi \in \mathcal{A}$;
(ii) $\mathrm{T}_{\Phi_{\mathrm{n}}} \in \mathcal{K}_{\mathcal{B}}(\mathcal{A})$ for all n.

Lemma (3.2.2) [3]

Suppose that $\mathcal{A}$ is a unital C^*-algebra with a faithful tracial trace τ and α is a τ-preserving action of a countable discrete group Γ on $\mathcal{A}$. If Λ is a normal subgroup of Γ, then the corresponding $\sum_{t\in\Lambda} a_t \lambda_\Lambda(t) \to \sum_{t\in\Lambda} a_t \lambda_\Gamma(t)$ extends to an isometric C^*-algebraic embedding $J : \mathcal{A} \rtimes_{\alpha,r} \Lambda \to \mathcal{A} \rtimes_{\alpha,r} \Gamma$. Moreover, there is a τ'-preserving conditional expectation E from $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ onto the range of this embedding and $\mathrm{E}\big(a\lambda_\Gamma(t)\big) = a\lambda_\Gamma(t)$ if $t \in \Lambda$ and zero otherwise.

Proof:

Let $(u, id_{\mathcal{A}}, \mathcal{H})$ be a covariant representation. By Fell's absorption principle, we may view $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ as the C^*-algebra generated by $\mathcal{A} \otimes I$ and $(u \otimes \lambda)(\Gamma)$. Similarly, $\mathcal{A} \rtimes_{\alpha,r} \Lambda$ is a C^*-algebra generated by $\mathcal{A} \otimes I$ and $(u \otimes \lambda)(\Lambda)$. Let $Q = \Gamma/\Lambda$, thus $\Gamma = \bigcup_{q\in Q} \Lambda \mathrm{t}_{\mathrm{q}}$ is the decomposition of Γ into disjoint right cosets, where $t_{\mathrm{q}} = 1$ for the trivial coset Λ. Then $\delta_{st_{\mathrm{q}}} \to \delta_s \otimes s_{t_{\mathrm{q}}}$ extends to a unitary from $\ell_2(\Gamma)$ onto $\ell_2(\Lambda) \otimes \ell_2(Q)$, so we have an identification

$$L_2(\Gamma) \cong L_2(\Lambda) \otimes_2 L_2(Q)$$

such that

$$\lambda_\Gamma(t) = \lambda_\Lambda(t) \otimes I, \quad t \in \Lambda. \tag{1}$$

Since $\mathcal{A} \rtimes_{\alpha,r} \Lambda$ is a C^*-algebra generated by $\mathcal{A} \otimes I$ and $(u \otimes \lambda)(\Lambda)$, we can define J as the restriction of the $*$-isomorphism $\chi \to \chi \otimes I$ of $\mathcal{B}(\mathcal{H} \otimes \ell_2(\Lambda))$ into $\mathcal{B}(\mathcal{H} \otimes \ell_2(\Lambda) \otimes \ell_2(Q))$. Thus,

$$J\left(\sum_{t\in\Lambda} a_t \lambda_\Lambda(t)\right) = \sum_{t\in\Lambda} a_t \lambda_\Lambda(t) \otimes I = \sum_{t\in\Lambda} a_t \lambda_\Gamma(t)$$

Define an isometry $V : \mathcal{H} \otimes \ell_2(\Lambda) \to \mathcal{H} \otimes \ell_2(Q)$ by

$$V(\eta) = \eta \otimes \delta_1.$$

In particular, $V(\xi \otimes \delta_s) = \xi \otimes \delta_1$. Then we define

$$E(\chi) = J(V^*\chi V), \quad \forall \chi \in \mathcal{A} \rtimes_{\alpha,r} \Gamma.$$

It is routine to check that E is the required conditional expectation and it is τ'-preserving. Also, $E(a\lambda_\Gamma(t)) = a\lambda_\Gamma(t)$ if $t \in \Lambda$ and zero otherwise.

In the following, we always set $\mathcal{B} = \mathcal{A} \rtimes_{\alpha,r} \Lambda$ for simplicity and $E_\mathcal{B}$B the conditional expectation from $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ onto $\mathcal{B} = \mathcal{A} \rtimes_{\alpha,r} \Lambda$ in Lemma (3.2.2).

Lemma (3.2.3) [3]

For each continuous normalized positive definite function $\tilde{\varphi}$ on Γ/Λ, there is an $E_\mathcal{B}$-preserving, unital completely positive map Φ from $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ into itself such that

$$\Phi(a\chi b) = a\Phi(\chi)b, \quad \forall a, b \in \mathcal{B} = \mathcal{A} \rtimes_{\alpha,r} \Lambda, \quad \chi \in \mathcal{A} \rtimes_{\alpha,r} \Gamma.$$

Proof:

For any $g \in \Gamma$, we can define $\varphi(g) = \tilde{\varphi}(g\Lambda)$. Define

$$\Phi : C_c(\Gamma, \mathcal{A}) \mapsto C_c(\Gamma, \mathcal{A})$$

by

$$\Phi\left(\sum_{s\in\Gamma} a_s\lambda_\Gamma(s)\right) = \sum_{s\in\Gamma}\varphi(s)a_s\lambda_\Gamma(s)$$

Suppose that $\chi = \sum_{s\in\Gamma} a_s\lambda_\Gamma(s) \in C_c(\Gamma,\mathcal{A})$ is positive in $\mathcal{A}\rtimes_{\alpha,r}\Gamma$. That for any finite sequence $s_1,\dots,s_{\mathrm{n}} \in \Gamma$, the operator matrix $\left[\alpha_{s_{\mathrm{i}}}^{-1}\left(a_{s_{\mathrm{i}}s_{\mathrm{j}}^{-1}}\right)\right]_{i,j} \in M_{\mathrm{n}}(\mathcal{A})$ is positive. So the Schur product $\left[\varphi\left(s_{\mathrm{i}}s_{\mathrm{j}}^{-1}\right)\alpha_{s_{\mathrm{i}}}^{-1}\left(a_{s_{\mathrm{i}}s_{\mathrm{j}}^{-1}}\right)\right]_{\mathrm{i,j}}$ is positive, as the pointwise product of two positive matrices $\left[\varphi\left(s_{\mathrm{i}}s_{\mathrm{j}}^{-1}\right)\right]_{\mathrm{i,j}}$ and $\left[\alpha_{s_{\mathrm{i}}}^{-1}\left(a_{s_{\mathrm{i}}s_{\mathrm{j}}^{-1}}\right)\right]_{\mathrm{i,j}}$. $\Phi(\chi) = \sum_{s\in\Gamma}\varphi(s)a_s\lambda_\Gamma(s) \in C_c(\Gamma,\mathcal{A})$ is also positive in $\mathcal{A}\rtimes_{\alpha,r}\Gamma$. This following natural commutative diagram implies that Φ is a unital completely positive map,

$$\begin{array}{ccc}
M_n\otimes(\mathcal{A}\rtimes_{\alpha,r}\Gamma) & \xrightarrow{\varphi_n} & M_n\otimes(\mathcal{A}\rtimes_{\alpha,r}\Gamma) \\
\| & & \| \\
M_n(\mathcal{A})\rtimes_{I\otimes\alpha,r}\Gamma & \xrightarrow{\varphi_n} & M_n\otimes(\mathcal{A})\rtimes_{I\otimes\alpha,r}\Gamma
\end{array}$$

where $(\mathrm{I}\otimes\alpha)(\mathrm{g}) = id_{M_{\mathrm{n}}}\otimes\alpha_{\mathrm{g}}$. Since $\varphi(\mathrm{g}) = 1$, for any $\mathrm{g}\in\Lambda$, it follows that $\mathrm{E}_{\mathcal{B}}\Phi = \mathrm{E}_{\mathcal{B}}$. Now we only need to check that Φ has $\mathcal{B}$-bimodule property. By the continuity of multiplication and of Φ, we only need to check on elements in $C_c(\Gamma,\mathcal{A})$. Suppose that $\chi = \sum_{s\in\Gamma} a_s\lambda_\Gamma(s)$ and $a = \sum_{t\in\Lambda} b_t\lambda_\Lambda(t)$ in $C_c(\Gamma,\mathcal{A})$, we have

$$\Phi(a\chi) = \Phi\left(\sum_{s\in\Gamma,t\in\Lambda} b_t\alpha_t(a_s)\lambda_\Gamma(ts)\right) = \sum_{s\in\Gamma,t\in\Lambda}\varphi(ts)b_t\alpha_t(a_s)\lambda_\Gamma(ts)$$

and

$$a\Phi(\chi) = \left(\sum_{t\in\Lambda} b_t\lambda_\Lambda(t)\right)\left(\sum_{s\in\Gamma}\varphi(s)a_s\lambda_\Gamma(s)\right) = \sum_{s\in\Gamma,t\in\Lambda}\varphi(s)b_t\alpha_t(a_s)\lambda_\Gamma(ts).$$

Since $s\Lambda = \Lambda s = \Lambda ts = ts\Lambda$, so $\varphi(s) = \varphi(ts)$ and $a\Phi = \Phi(a\chi)$. Similarly $\Phi(\chi b) = \Phi(\chi)b$.

We have The following theorem.

Theorem (3.2.4) [3]

Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial trace τ and let α be a τ-preserving action of a countable discrete group Γ on $\mathcal{A}$. If Λ is a normal subgroup of Γ, then the inclusion $\mathcal{A} \rtimes_{\alpha,r} \Lambda \subseteq \mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the relative Haagerup property if and only if the quotient group Γ/Λ has the Haagerup property.

Proof:

Necessity. Suppose that the inclusion $\mathcal{A} \rtimes_{\alpha,r} \Lambda \hookrightarrow \mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the relative Haagerup property, and let $\{\Phi_n\}_{n\geq 1}$ be as in Definition (3.2.1). Define $\varphi_n : \Gamma \mapsto \mathbb{C}$ by

$$\varphi_n(g) = \tau\left(\Phi_n(\lambda_\Gamma(g))\lambda_\Gamma(g^{-1})\right).$$

For any fixed n, clearly $\varphi_n(e) = 1$ and the positivity of τ yields

$$\sum_{i,j=1}^{m} a_i\tilde{a}_j\varphi_n(g_j^{-1}g_i) = \sum_{i,j=1}^{m} \tau\left(\tilde{a}_j\lambda_\Gamma(g_j)\right)\Phi_n\left(\lambda_\Gamma(g_j^{-1})\lambda_\Gamma(g_i)\right)a_i\lambda_\Gamma\left((g_i^{-1})\right)$$

$$\geq 0$$

for all $g_1, \ldots, g_m \in \Gamma$ and $a_1, \ldots, a_m \in \mathbb{C}$; hence φ_n is positive definite on Γ. Since each Φ_n is an $\mathcal{A} \rtimes_{\alpha,r} \Lambda$-bimodule map, so for $g \in \Gamma, g_0 \in \Lambda$ we have

$$\begin{aligned}\varphi_n(gg_0) &= \tau\left(\Phi_n(\lambda_\Gamma(gg_0))\lambda_\Gamma(g_0^{-1}g^{-1})\right)\\ &= \tau\left(\Phi_n(\lambda_\Gamma(g))\lambda_\Gamma(g^{-1})\right)\\ &= \Phi_n(g)\end{aligned}$$

and

$$\begin{aligned}\Phi_n(g_0g) &= \tau\left(\Phi_n(\lambda_\Gamma(gg_0))\lambda_\Gamma(g^{-1}g_0^{-1})\right)\\ &= \tau\left(\lambda_\Gamma(g_0)\Phi_n(\lambda_\Gamma(g))\lambda_\Gamma(g^{-1})\lambda_\Gamma(g_0^{-1})\right)\end{aligned}$$

$$= \tau\left(\Phi_{\mathrm{n}}(\lambda_{\Gamma}(\mathrm{g}))\lambda_{\Gamma}(\mathrm{g}^{-1})\right)$$

$$= \Phi_{\mathrm{n}}(\mathrm{g}).$$

Therefore $\varphi_{\mathrm{n}}(\mathrm{g}_0\mathrm{g}) = \varphi_{\mathrm{n}}(\mathrm{g}\mathrm{g}_0)$ and so we can define $\tilde{\varphi}_{\mathrm{n}}: \Gamma/\Lambda \mapsto \mathbb{C}$ by

$$\widetilde{\Phi}_{\mathrm{n}}(\mathrm{g}\Lambda) = \varphi_{\mathrm{n}}(\mathrm{g}), \quad \forall \mathrm{g} \in \Gamma.$$

Indeed, if $\mathrm{g}\Lambda = \mathrm{h}\Lambda$, we have $\mathrm{h}^{-1}\mathrm{g} \in \Lambda$. So by the above equations,

$$\tilde{\varphi}_{\mathrm{n}}(\mathrm{g}\Lambda) = \varphi_{\mathrm{n}}(\mathrm{g}) = \varphi_{\mathrm{n}}\big(\mathrm{h}(\mathrm{h}^{-1}\mathrm{g})\big) = \varphi_{\mathrm{n}}(\mathrm{h}) = \tilde{\varphi}_{\mathrm{n}}(\mathrm{h}\Lambda).$$

So $\tilde{\varphi}_{\mathrm{n}}$ is also a positive definite function on Γ/Λ and $\tilde{\varphi}_{\mathrm{n}}(\Lambda) = 1$. For any $\mathrm{g} \in \Gamma$, we have

$$\begin{aligned}
|\tilde{\varphi}_{\mathrm{n}}(\mathrm{g}\Lambda) - 1| &= |\varphi_{\mathrm{n}}(\mathrm{g}) - 1| \\
&= \left|\tau\left(\Phi_{\mathrm{n}}(\lambda_{\Gamma}(\mathrm{g}))\lambda_{\Gamma}(\mathrm{g}^{-1})\right) - 1\right| \\
&= \left|\tau\left(\Phi_{\mathrm{n}}(\lambda_{\Gamma}(\mathrm{g}))\lambda_{\Gamma}(\mathrm{g}^{-1})\right) - \tau\big(\lambda_{\Gamma}(\mathrm{g})\lambda_{\Gamma}(\mathrm{g}^{-1})\big)\right| \\
&= \left|\tau\left(\left(\Phi_{\mathrm{n}}\big(\lambda_{\Gamma}(\mathrm{g})\big) - \lambda_{\Gamma}(\mathrm{g})\right)\lambda_{\Gamma}(\mathrm{g}^{-1})\right)\right| \\
&= \left|\langle \Phi_{\mathrm{n}}\big(\lambda_{\Gamma}(\mathrm{g})\big) - \lambda_{\Gamma}(\mathrm{g}), \lambda_{\Gamma}(\mathrm{g})\rangle_2\right| \\
&\leq \left\|\Phi_{\mathrm{n}}\big(\lambda_{\Gamma}(\mathrm{g})\big) - \lambda_{\Gamma}(\mathrm{g})\right\|_2 \to 0, \quad \text{as } \mathrm{n} \to +\infty.
\end{aligned}$$

Finally, to show that Γ/Λ has the Haagerup property, it suffices to cheek that $\tilde{\varphi}_{\mathrm{n}}$ vanishes at the infinity of Γ/Λ. We set $\mathcal{B} = \mathcal{A} \rtimes_{\alpha,r} \Lambda$ and fix n and $\epsilon > 0$. Since $\mathrm{T}_{\Phi_{\mathrm{n}}} \in \mathcal{K}_{\mathcal{B}}(\mathcal{A} \rtimes_{\alpha,r} \Gamma)$, there exist $a_1, \dots, a_{\mathrm{m}}$ in $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ and $b_1, \dots, b_{\mathrm{m}} \in C_c(\Gamma, \mathcal{A})$ such that

$$\left\|\mathrm{T}_{\Phi_{\mathrm{n}}} - \sum_{\mathrm{i}=1}^{\mathrm{m}} a_{\mathrm{i}} e_{\mathcal{B}} b_{\mathrm{i}}\right\| \leq \epsilon/2.$$

In particular,

$$\sup_{g\in\Gamma}\left\|\Phi_n\big(\lambda_\Gamma(g)\big)-\sum_{i=1}^{m}a_i E_{\mathcal{B}}\big(b_i\lambda_\Gamma(g)\big)\right\|_2 \leq \epsilon/2. \qquad (2)$$

Let $\mathcal{S}$ be a complete system of representations in Γ for Γ/Λ. Now for any $b = \sum_{t\in\Gamma} b_t\lambda_\Gamma(t) \in C_c(\Gamma,\mathcal{A})$, we get

$$\begin{aligned} b &= \sum_{t\in\Gamma} b_t\lambda_\Gamma(t) \\ &= \sum_{t\in\Gamma}\lambda_\Gamma(t)\alpha_{t^{-1}}(b_t) \\ &= \sum_{g\in\mathcal{S}}\lambda_\Gamma(g)\left(\sum_{t\in g\Lambda}\lambda_\Gamma(g^{-1}t)\alpha_{t^{-1}}(b_t)\right) \\ &= \sum_{g\in\mathcal{S}}\lambda_\Gamma(g)\left(\sum_{t\in g\Lambda}(\lambda_\Lambda(g^{-1}t)\otimes I)\alpha_{t^{-1}}(b_t)\right) \\ &= \sum_{g\in\mathcal{S}}\lambda_\Gamma(g)E_{\mathcal{B}}(\lambda_\Gamma(g^{-1})b). \end{aligned}$$

So

$$\|b\|_2^2 = \left\|\sum_{g\in\mathcal{S}}\lambda_\Gamma(g)E_{\mathcal{B}}(\lambda_\Gamma(g^{-1})b)\right\|_2^2 = \sum_{g\in\mathcal{S}}\|E_{\mathcal{B}}(\lambda_\Gamma(g^{-1})b)\|_2^2 < +\infty.$$

Let $\delta > 0$. Then there exists a finite set $F_{b,\delta} \subseteq \mathcal{S}$ such that

$$\sum_{g\in\mathcal{S}\setminus F_{b,s}}\|E_{\mathcal{B}}(\lambda_\Gamma(g^{-1})b)\|_2^2 \leq \delta^2. \qquad (3)$$

Let $M = \max_{1\leq i\leq m}\|a_i\|_2$, $\delta = \frac{\epsilon}{2(M+1)m}$ and $F_\epsilon = \bigcup_{i=1}^{m} F_{b_i^*,\delta}$. For any $g \in S\setminus F_\epsilon$, it follows from the inequalities (2) and (3)

$$
\begin{aligned}
|\Phi_{\mathrm{n}}(\mathrm{g})| &= \left|\tau\left(\Phi_{\mathrm{n}}(\lambda_\Gamma(\mathrm{g}))\lambda_\Gamma(\mathrm{g}^{-1})\right)\right| \\
&\le \left|\tau\left(\left(\Phi_{\mathrm{n}}(\lambda_\Gamma(\mathrm{g})) - \sum_{\mathrm{i}=1}^{\mathrm{m}} a_{\mathrm{i}}\mathrm{E}_{\mathcal{B}}(b_{\mathrm{i}}\lambda_\Gamma(\mathrm{g}))\right)\lambda_\Gamma(\mathrm{g}^{-1})\right)\right| \\
&\quad + \sum_{\mathrm{i}=1}^{\mathrm{m}} \left|\tau\left(a_{\mathrm{i}}\mathrm{E}_{\mathcal{B}}(b_{\mathrm{i}}\lambda_\Gamma(\mathrm{g}))\lambda_\Gamma(\mathrm{g}^{-1})\right)\right| \\
&\le \frac{\epsilon}{2} + \sum_{\mathrm{i}=1}^{\mathrm{m}} \|a_{\mathrm{i}}\|_2 \left\|\mathrm{E}_{\mathcal{B}}(b_{\mathrm{i}}\lambda_\Gamma(\mathrm{g}))\right\|_2 \\
&= \frac{\epsilon}{2} + \sum_{\mathrm{i}=1}^{\mathrm{m}} \|a_i\|_2 \|\mathrm{E}_{\mathcal{B}}(\lambda_\Gamma(\mathrm{g}^{-1})b_i^*)\|_2 \\
&\le \frac{\epsilon}{2} + \sum_{\mathrm{i}=1}^{\mathrm{m}} M\frac{\epsilon}{2(M+1)m} \\
&< \epsilon.
\end{aligned}
$$

This implies that $\tilde{\varphi}_{\mathrm{n}} \in c_0(\Gamma/\Lambda)$ and so the quotient group Γ/Λ has the Haagerup property.

Sufficiency. Suppose that the quotient group Γ/Λ has the Haagerup property. So there exists a sequence of normalized positive definite functions $\{\tilde{\varphi}_{\mathrm{n}}\}$ on Γ/Λ that vanish at the infinity of Γ/Λ and such that $\lim_n \tilde{\varphi}_{\mathrm{n}}(\mathrm{g}\Lambda) = 1$, for all $\mathrm{g}\Lambda \in \Gamma/\Lambda$. We can define $\varphi_{\mathrm{n}}: \Gamma \to \mathbb{C}$ by

$$\varphi_{\mathrm{n}}(\mathrm{g}) = \tilde{\varphi}_{\mathrm{n}}(\mathrm{g}\Lambda), \quad \forall \mathrm{g} \in \Gamma.$$

It is routine to check that $\varphi_{\mathrm{n}}(e) = 1, \varphi_{\mathrm{n}}$ is a positive definite function on Γ and

$$\varphi_{\mathrm{n}}(\mathrm{g}\mathrm{g}_0) = \varphi_{\mathrm{n}}(\mathrm{g}_0\mathrm{g}), \quad \lim_{\mathrm{n}} \varphi_{\mathrm{n}}(\mathrm{g}) = 1, \quad \forall \mathrm{g} \in \Gamma, \mathrm{g}_0 \in \Lambda.$$

By Lemma (3.2.3), there exists a sequence $\{\Phi_{\mathrm{n}}\}_{\mathrm{n}\ge 1}$ of $\mathrm{E}_{\mathcal{B}}$-preserving, $\mathcal{B}$-bimodule, unital completely positive maps from $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ to itself. In order to check

$\lim_n \|\Phi_n(\chi) - \chi\|_2 = 0,\ \forall \chi \in \mathcal{A} \rtimes_{\alpha,r} \Gamma$, note that, since $\|T_{\Phi_n}\| \leq 1$, it is enough to consider only the case $\chi = a\lambda_\Gamma(s), a \in \mathcal{A}_1$ and $s \in \Gamma$. Now we compute

$$\left\|\Phi_n(a\lambda_\Gamma(s)) - a\lambda_\Gamma(s)\right\|_2^2 = \|(\Phi_n(s) - 1)a\lambda_\Gamma(s)\|_2^2$$

$$= (\varphi_n(s) - 1)^2 \tau\left(\lambda_\Gamma(s^{-1})a^* a\lambda_\Gamma(s)\right)$$

$$= (\varphi_n(s) - 1)^2 \tau(a^* a) \to 0, \quad \text{as} \quad n \to +\infty.$$

We also need to show $T_{\Phi_n} \in \mathcal{K}_\mathcal{B}(\mathcal{A} \rtimes_{\alpha,r} \Gamma)$. For simplicity of notation, we fix n and denote $\varphi = \varphi_n,\ \Phi = \Phi_n$. Let $\mathcal{S}$ be a complete system of representations for Γ/Λ in Γ. Let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}$ be an increasing sequence of finite sets whose union equals $\mathcal{S}$, and set for each $m \geq 1$

$$T_m = \sum_{g \in S_m} \varphi(g)\lambda_\Gamma(g) e_\mathcal{B} \lambda_\Gamma(g^{-1}).$$

First, we want to show that $T_m \in \mathcal{B}' = (\mathcal{A} \rtimes_{\alpha,r} \Lambda)'$. By linearity and continuity, we only need to check that for $\chi = b\lambda_\Gamma(t) \in C_c(\Gamma, \mathcal{A}) \subseteq \mathcal{A} \rtimes_{\alpha,r} \Gamma$,

$$\lambda_\Gamma(g) e_\mathcal{B} \lambda_\Gamma(g^{-1}) a\lambda_\Lambda(g_0)\chi_\tau = a\lambda_\Lambda(g_0)\lambda_\Gamma(g) e_\mathcal{B} \lambda_\Gamma(g^{-1})\chi_\tau$$

with $g, t \in \Gamma, g_0 \in \Lambda$ and $a, b \in \mathcal{A}$. Now we have

$$\begin{aligned}
\lambda_\Gamma(g) e_\mathcal{B} \lambda_\Gamma(g^{-1}) a\lambda_\Lambda(g_0)\chi_\tau &= \lambda_\Gamma(g) E_\mathcal{B}\left(\lambda_\Gamma(g^{-1}) a\lambda_\Lambda(g_0) b\lambda_\Gamma(t)\right) \\
&= \lambda_\Gamma(g) E_\mathcal{B}\left(\lambda_\Gamma(g^{-1}) a\alpha_{g_0}(b)\lambda_\Gamma(g_0 t)\right) \\
&= \lambda_\Gamma(g) E_\mathcal{B}\left(\alpha_{g^{-1}}\left(a\alpha_{g_0}(b)\right)\lambda_\Gamma(g^{-1} g_0 t)\right) \\
&= \begin{cases} a\alpha_{g_0}(b)\lambda_\Gamma(g_0 t) = a\lambda_\Lambda(g_0) b\lambda_\Gamma(t), & \text{if } g^{-1} g_0 t \in \Lambda; \\ 0, & \text{if } g^{-1} g_0 t \notin \Lambda; \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
a\lambda_\Lambda(g_0)\lambda_\Gamma(g) e_\mathcal{B} \lambda_\Gamma(g^{-1})\chi_\tau &= a\lambda_\Lambda(g_0)\lambda_\Gamma(g) E_\mathcal{B}\left(\lambda_\Gamma(g^{-1}) b\lambda_\Gamma(t)\right) \\
&= a\lambda_\Lambda(g_0)\lambda_\Gamma(g) E_\mathcal{B}\left(\alpha_{g^{-1}}(b)\lambda_\Gamma(g^{-1} t)\right)
\end{aligned}$$

$$= \begin{cases} a\lambda_\Lambda(\mathrm{g}_0)\lambda_\Gamma(\mathrm{g})\alpha_{\mathrm{g}^{-1}}(b)\lambda_\Gamma(\mathrm{g}^{-1}t) = a\lambda_\Lambda(\mathrm{g}_0)b\lambda_\Gamma(t), & \mathrm{f\,g}^{-1}t \notin \Lambda; \\ 0, & \mathrm{f\,g}^{-1}t \notin \Lambda; \end{cases}$$

Since Λ is normal in Γ and $\mathrm{g}_0 \in \Lambda$, so

$$\mathrm{g}^{-1}\mathrm{g}_0 t \in \Lambda \quad \Leftrightarrow \quad t \in \Lambda \mathrm{g}_0^{-1}\mathrm{g} = \Lambda\mathrm{g} \quad \Leftrightarrow \quad \mathrm{g}^{-1}t \in \Lambda.$$

Therefore $\mathrm{T_m} \in \mathcal{B}' = \left(\mathcal{A} \rtimes_{\alpha,r} \Lambda\right)'$.

Secondly, we need to check that $\mathrm{T_m} \mapsto \mathrm{T}_\Phi$ in the operator norm. Let $\chi = \sum_{t\in\Gamma} a_t\lambda_\Gamma(t) \in C_c(\Gamma,\mathcal{A}) \subseteq \mathcal{A} \rtimes_{\alpha,r} \Gamma$. From the proof of necessity, we have

$$\chi = \sum_{\mathrm{g}\in S} \lambda_\Gamma(\mathrm{g})E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi)$$

and so

$$\|\chi\|_2^2 = \sum_{\mathrm{g}\in S} \|E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi)\|_2^2 < +\infty.$$

It follows that for any $\epsilon > 0$, there exists $k(\epsilon) \geq 1$ such that

$$\left\|\chi - \sum_{\mathrm{g}\in S_k} \lambda_\Gamma(\mathrm{g})E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi)\right\|_2^2 = \left\|\sum_{\mathrm{g}\in S\setminus S_k} \lambda_\Gamma(\mathrm{g})E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi)\right\|_2^2$$

$$= \sum_{\mathrm{g}\in S\setminus S_k} \|E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi)\|_2^2$$

$$\leq \epsilon^2\|\chi\|_2^2, \quad \forall k \geq k(\epsilon).$$

Since $\tilde{\varphi}$ vanishes at the infinity of Γ/Λ, there exists a subsequence $\{k_\mathrm{m}\}$ such that $\sup_{\mathrm{g}\in S\setminus S_{k_\mathrm{m}}}|\varphi(\mathrm{g})| \leq \frac{1}{\mathrm{m}}$. Pick an $\mathrm{m} \geq 1$, and choose that $k(\epsilon) \geq k_\mathrm{m}$. We have

$$\left\|\mathrm{T}_\Phi\chi_\tau - \mathrm{T}_{k_\mathrm{m}}\chi_\tau\right\| = \left\|\Phi(\chi)_\tau - \mathrm{T}_{k_\mathrm{m}}\chi_\tau\right\|$$

$$= \left\|\sum_{\mathrm{g}\in S_k} \varphi(\mathrm{g})\lambda_\Gamma(\mathrm{g})E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi) - \sum_{\mathrm{g}\in S_m} \varphi(\mathrm{g})\lambda_\Gamma(\mathrm{g})E_\mathcal{B}(\lambda_\Gamma(\mathrm{g}^{-1})\chi)\right\|_2$$

$$\leq \left\| \sum_{g\in S_{k(\epsilon)}} \varphi(g)\lambda_{\Gamma}(g)E_{\mathcal{B}}(\lambda_{\Gamma}(g^{-1})\chi) \right\|_2 + \left\| \sum_{g\in S_{k(\epsilon)}\setminus S_{k_m}} \varphi(g)\lambda_{\Gamma}(g)E_{\mathcal{B}}(\lambda_{\Gamma}(g^{-1})\chi) \right\|_2$$

$$\leq \epsilon\|\chi\|_2 + \left(\sum_{g\in S_{k(\epsilon)}\setminus S_{k_m}} |\varphi(g)|^2 \|E_{\mathcal{B}}(\lambda_{\Gamma}(g^{-1})\chi)\|_2^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\epsilon + \frac{1}{m}\right)\|\chi\|_2.$$

Therefore $\left\|T_{\Phi} - T_{k_m}\right\| \leq \frac{1}{m}$ and $T_{\Phi} \in \mathcal{K}_{\mathcal{B}}\left(\mathcal{A} \rtimes_{\alpha,r} \Gamma\right)$. This implies that the inclusion $\mathcal{A} \rtimes_{\alpha,r} \Lambda \subseteq \mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the relative Haagerup property.

Corollary (3.2.5) [3]

Let Λ be a normal subgroup of Γ, then the inclusion $C_r^*(\Lambda) \subseteq C_r^*(\Gamma)$ has the relative Haagerup property if and only if the quotient group Γ/Λ has the Haagerup property.

Corollary (3.2.6) [3]

The assumption is the same as that in Theorem (3.2.4). Then the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} \Gamma$ has the relative Haagerup property if and only if Γ has the Haagerup property.

The Haagerup property is a strong negation to Property (T), in that each of equivalent characterizations stands opposite to a characterization of Property (T). Connes and Jones introduced the notion of correspondences for von Neumann algebras. A correspondence of a von Neumann algebra $\mathcal{M}$ is a Hilbert space $\mathcal{H}$ endowed with two normal, commuting, unital actions: one of $\mathcal{M}$ (denoted by $(\chi,\xi) \to \chi\xi$) for χ in $\mathcal{M}$ and ξ in $\mathcal{H}$) and one of the opposite algebra $\mathcal{M}^0$ of $\mathcal{M}$ (denoted by $(y^0,\xi) \to \xi$). Correspondences are analogous to group representations in the theory of von Neumann algebras. At the group level, Jolissaint define an ideal $Bin(\Gamma)$ of the Fourier-Stieltjes algebra $B(\Gamma \times \Gamma)$: $Bin(\Gamma)$ is the subset of elements φ in $B(\Gamma \times \Gamma)$ such that $\varphi(g,\cdot)$ and $\varphi(\cdot,h)$ belong to the Fourier algebra $A(\Gamma)$ for all fixed $g, h \in \Gamma$. Moreover, the set $T\,A(\Gamma)$ of element ψ of $A(\Gamma)$ which is constant on conjugacy classes embeds naturally in $Bin(\Gamma)$ by: $\varphi_{\psi}(\mathrm{g},\mathrm{h}) = \psi(\mathrm{gh}^{-1})$. Denote by $Bin_{+}(\Gamma)\big(\text{resp.} \mathrm{T}\, A_{+}(\Gamma)\big)$ the subset of elements of $Bin(\Gamma)$ $\big(\text{resp.} \mathrm{T}\, A_{+}(\Gamma)\big)$ which are positive definite. Jolissaint characterized Property (T) in terms of functions in $Bin_{+}(\Gamma)$ and $\mathrm{T}\, A_{+}(\Gamma)$.We will give a characterization of Haagerup property for a discrete group Γ which is strong opposite to Jolissaint fs result.we need The following

Lemma (3.2.7) [3]

For any φ in $Bin_{+}(\Gamma)$ there exists a unique binormal positive form on its von Neumann algebra $L(\Gamma)(f_{¡})$, still denoted by φ, such that

$$\varphi(\mathrm{g},\mathrm{h}) = \varphi(\lambda_{\Gamma}(\mathrm{g}) \otimes \lambda_{\Gamma}(\mathrm{h}^{-1})^{0}) = \langle \lambda_{\Gamma}(\mathrm{g})\xi_{\varphi}\lambda_{\Gamma}(\mathrm{h}^{-1}), \xi_{\varphi}\rangle$$

for all g,h in Γ, where ξ_{φ}. denotes the cyclic vector of the correspondence $\mathcal{H}_{\varphi}$. associated to φ.

Definition (3.2.8) [3]

Suppose that $\varphi \in Bin_{+}(\Gamma)$, if the function defined by

$$\psi(\mathrm{g}) \triangleq \varphi(\mathrm{g},\mathrm{g}), \quad \mathrm{g} \in \Gamma.$$

belongs to the abelian C^{*}-algebra $c_0(\Gamma)$, we say $\varphi \in Bin_0(\Gamma)$.

Theorem (3.2.9) [3]

Let Γ be a countable discrete group. Then the following are equivalent:

(i) Γ has the Haagerup property;

(ii) There exists a sequence $\{\varphi_{\mathrm{n}}\}_{\mathrm{n}\geq 1}$of functions in $Bin_0(\Gamma)$ converging pointwise to a non-zero element of $\mathrm{T}\, A_{+}(\Gamma)$.

Proof:

(ii)⇒(i) Suppose that there exists a sequence $\{\varphi_{\mathrm{n}}\}_{\mathrm{n}\geq 1}$ of functions in $Bin_0(\Gamma)$ converging pointwise to a non-zero element φ of $\mathrm{T}\, A_{+}(\Gamma)$, we may assume that $\varphi(1,1) = \varphi_{\mathrm{n}}(1,1) = 1$ for all n. Denote by $\mathcal{H}_{\mathrm{n}}$ the correspondence given by

Lemma (3.2.7) and denote by ξ_{n} the corresponding unit cyclic vector in $\mathcal{H}_{\mathrm{n}}$. The associated representation π_{n} of π_{n} on $\mathcal{H}_{\mathrm{n}}$ is defined by

$$\pi_{\mathrm{n}}(\mathrm{g})\xi = \lambda_{\Gamma}(\mathrm{g})\xi\lambda_{\Gamma}(\mathrm{g}^{-1}), \quad \mathrm{g} \in \Gamma, \xi \in \mathcal{H}_{\mathrm{n}}.$$

Thus

$$\begin{aligned}\psi_{\mathrm{n}}(\mathrm{g}) &\triangleq \varphi_{\mathrm{n}}(\mathrm{g}, \mathrm{g}) \\ &= \langle \pi_{\mathrm{n}}(\mathrm{g})\xi, \xi \rangle \\ &\to \varphi(\mathrm{g}, \mathrm{g}) = 1 \quad \big(\text{since } \varphi \in \mathrm{T}\, A_{+}(\Gamma)\big).\end{aligned}$$

Therefore the abelian C^{*}-algebra $c_0(\Gamma)$ possesses an approximate unit of normalized positive definite functions and this implies that Γ has the Haagerup property.

(i)⇒(ii) The ideas of the **[3]**. Suppose that Γ has the Haagerup property, so there exists a c_0 unitary representation $(\pi, \mathcal{H})$ of Γ which weakly contains the trivial representation 1_{Γ}. Thus there exists a sequence of unit vectors $\{\xi_{\mathrm{n}}\}$ in $\mathcal{H}$ such that

$$\langle \pi(\mathrm{g})\xi_{\mathrm{n}}, \xi_{\mathrm{n}} \rangle \to 1, \quad \forall \mathrm{g} \in \Gamma. \qquad (4)$$

We associate a correspondence with π as see**[3]**. Denote by $\widehat{\mathcal{H}}$ the Hilbert space $\ell^2(\Gamma) \otimes \mathcal{H}$ endowed with the actions:

$$\lambda_{\Gamma}(\mathrm{g}) \cdot (f \otimes \eta) = \lambda_{\Gamma}(\mathrm{g})f \otimes \pi(\mathrm{g})\eta$$

and

$$(f \otimes \eta) \cdot \lambda_{\Gamma}(\mathrm{g}) = f\lambda_{\Gamma}(\mathrm{g}) \otimes \eta$$

for $\mathrm{g} \in \Gamma, f \in \ell^2(\Gamma)$ and $\eta \in \mathcal{H}$. $\widehat{\mathcal{H}}$ is a correspondence of $L(\Gamma)$. Let us denote $\hat{\xi}_n = \delta_1 \otimes \xi_n \in \widehat{\mathcal{H}}$ and define $\{\varphi_n\}$ in $Bin_{+}(\Gamma)$ by

$$\begin{aligned}\varphi_{\mathrm{n}}(\mathrm{g}, h) &\triangleq \langle \lambda_{\Gamma}(\mathrm{g})\hat{\xi}_{\mathrm{n}}\lambda_{\Gamma}(\mathrm{h}^{-1}), \hat{\xi}_{\mathrm{n}} \rangle \\ &= \langle \lambda_{\Gamma}(\mathrm{g})\delta_1\lambda_{\Gamma}(\mathrm{h}^{-1}) \otimes \pi(\mathrm{g})\xi_{\mathrm{n}}, \delta_1 \otimes \xi_{\mathrm{n}} \rangle \\ &= \langle \lambda_{\Gamma}(\mathrm{g})\delta_1\lambda_{\Gamma}(\mathrm{h}^{-1}), \delta_1 \rangle \cdot \langle \pi(\mathrm{g})\xi_{\mathrm{n}}, \delta_1, \xi_{\mathrm{n}} \rangle \\ &= \chi_{\Delta}(\mathrm{g}, \mathrm{h})\langle \pi(\mathrm{g})\xi_{\mathrm{n}}, \xi_{\mathrm{n}} \rangle\end{aligned}$$

$$= \chi_\Delta(g,h)\varphi_n(g,h),$$

where χ_Δ is the characteristic function of the diagonal group Δ in $\Gamma \times \Gamma$. Since π is a c_0 unitary representation, it follows from the above equations that $\varphi_n \in Bin_0(\Gamma)$. Therefore the above equations and Equation (4) show that

$$\varphi_n(g,h) \to \chi_\Delta(g,h), \quad g,h \in \Gamma.$$

It is obvious that $\chi_\Delta \in TA_+(\Gamma)$.

Chapter 4

Rigidity of C^*- algebras with property (T)

As aconsequence ,the class of all C^*-algebras with the Haagerup property turns out to be quite large. We then apply Popa's results and show the C^*-algebras with property(T) have a certain rigidity property. Unlike the case of von Neumann algebras ,for the reduced group C^*-algebra so f groups with relative property(T),the rigidity property strongly fails in general .Never the less ,for some groups without nontrivial property(T)subgroups ,we show arigidity property in some cases .As examples ,we prove the reduced group C^*-algebras of the (non-amenable) affine groups of the affine planes have arigidity property.

Section (4-1)

Applications of Haagerup property for C*-algebra

The goal of this section is to prove Theorems (4.1.5) and (4.1.9). As the study of nuclearity of C^*-algebras (e.g., a proof of the fact that nuclearity passes to a quotient in order to prove Theorem (4.1.5), we need a deep theorem from the theory of von Neumann algebras. To state and apply the theorem of Connes below, we need the following concepts of von Neumann algebras.

Definition (4.1.1) [4]

Let M be a von Neumann algebra. It is said to be injective if for any unital C^*-algebra A and for any its closed self-adjoint subspace N containing the unit of A, any u.c.p. map from N to M can be extended to a u.c.p. map from A to M.

Definition (4.1.2) [4]

Let M bea von Neumann algebra with the separable predual. It is said to be AFD (abbreviation of "Approximately Finite Dimensional") if there is an increasing sequence of finite dimensional $*$-sub algebras of M whose union is dense in M in the strong operator topology.

The following theorem of Connes states these two properties are equivalent in the separable case.

Theorem (4.1.3) [4]

For a von Neumann algebra M with the separable

Predual , the following are equivalent:

1- The von Neumann algebra M is injective;

2- The von Neumann algebra M is AFD.

Lemma (4.1.4) [4]

Let M be a(not necessarily separable)injective von Neumann algebra with a faithful normal tracial state τ. Then there exists a net $(\Phi_n)_n$of conditional expectations on M which

satisfies the following three conditions:

1- Each image of$\Phi_n$is finite dimensional;

2- Each$\Phi_n$preservesτ;

3- The net $(\Phi_n)_n$converges to the identity map in the pointwise strong operator topology.

Proof:-

Note first that since M has a faithful normal tracial state with each von Neumann sub algebra of M is injective. From this, for each finite Subset F of M, the von Neumann sub algebra W^*(F) of M generated by F is injective and separable. From this, by Connes's theorem, each von Neumann algebra W^*(F)is an AFD von Neumann algebra. Consequently, for each finite sub set F of M, there is a finite dimensional ∗-sub algebra M F of M, such that dist τ(x, MF)<1/|F| for all x∈ F(where |F |is the cardinality of F). Then again , for each finite sub set F of M, there is a τ-preserving conditional expectation EF from M onto MF. Then notice that for any τ-preserving conditional expectation E with the range N, we have

$$\|x - E(x)\|_\tau = dist_\tau(x, N).$$

On the other hand, for each x∈ M, $dist_\tau$(x, MF)converges to zero as F tends to infinity. From this, the net(E F)F satisfies the desired three conditions.

Now we have.

Theorem (4.1.5) [4]

Let (A, τ) be a pair of a unital nuclear C^*-algebra and a faithful tracial state. Then it has the Haagerup property.

Proof:-

Let A be a unital nuclear C^*-algebra ,τ be a faithful tracial state on A. We will show the pair(A, τ)has the Haagerup property. Let π_τ be the GNS representation of τ. Then, since A is nuclear, it is easy to show π_τ(A) is an injective von Neumann algebra. Using Lemma (4.1.4)

we can choose a net $(\Phi_\alpha)_\alpha$of conditional expectations on π_τ(A) satisfying the following three conditions:

(i) Each image of Φ_α has a finite dimension;

(ii) EachΦ_α preserves τ;

(iii) The net $(\Phi_\alpha)_\alpha$ converges to the identity map in the pointwise strong operator topology.

From these conditions, each Φ_α is L^2-compact, and it converges to id_{L^2}(A ,τ)

strongly, as in the definition of the Haagerup property. However, unfortunately, these ranges are not contained in A in general. So we have to modify Φ_α to take its values in A. To do this, we need the following

notations. We identify π_τ(A) with the direct summand A^{**}c(π_τ)of A^{**}

, where c(π_τ)is the central cover of π_τ . Denote the image of Φ_α by E_α , and denote the linear span of $1_{A^{**}}$and E_α by F_α ,which is a finite dimensional C^*-sub algebra of A^{**}

. Denote the canonical inclusion$F_\alpha \hookrightarrow A^{**}$ by ι_α. Since $\iota_\alpha$is a$*$-homomorphism, in particular it is contained in CP(F_α, A^{**}). Then, by using the canonical bijective correspondence between CP(F_α, A^{**})and($A^{**}\otimes F_\alpha$) and the density of $A\otimes F_\alpha$ in $A^{**}\otimes F_\alpha$in the strong operator topology, we can find a bounded net$(\Psi_\beta^{(\alpha)})_\beta$ from CP(F_α,A)that converges to ια in the point wise strong operator topology, as β tends to infinity (by the Kaplan sky density theorem). Then, since each $\Psi_\beta^{(\alpha)}(1)$ is contained in $A\subset A^{**}$ and it converges to $1_{A^{**}} = 1_A \in A$ weakly as β tends to infinity, by retaking a net of c.p. maps from the convex hull of $\{\Psi_\beta^{(\alpha)}\}\beta$, we may assume$(\Psi_\beta^{(\alpha)}(1)_\beta$ converges to 1 in norm. (Recall the Hahn–Banach separation theorem.) We remark that each support of $\tau\circ \Psi_\beta^{(\alpha)}|_{E_\alpha}$ is contained in that of $\tau|_{E\alpha}$, which is equal to E_α, and the former net converges to $\tau|_{E_\alpha}$pointwise as β tends to infinity. From this and the fact E_α has a finite dimension, we can choose a net $c_\beta^{(\alpha)})_\beta$of positive numbers, such that $c_\beta^{(\alpha)})_\beta$ converges to1asβtends to infinity and $\tau\circ\Psi_\beta^{(\alpha)}|_{E_\alpha} \leq c_\beta^{(\alpha)})_\beta \tau|_{E\alpha}$. Put

$$d_\beta^{(\alpha)} := \max\{ c_\beta^{(\alpha)}, ||\Psi_\beta^{(\alpha)}(1)||\}$$

for each α and β. Then by definition of $d_\beta^{(\alpha)}$, each map $(d_\beta^{(\alpha)})^{-1} \Psi_\beta^{(\alpha)}$is a c.c.p. map that decreases τ. Now it is easy to take the desired net from the set $\{c(\pi_\tau)\, \Psi_\beta^{(\alpha)}\circ\Phi_\alpha)_{\alpha,\beta}$, here we identify A with the C^*-subalgebra π_τ(A) of π_τ(A) $=A^{**}c(\pi_\tau)$, not with the canonical C^*-subalgebra of A^{**}).

Indeed, in the above proof, we only use the injectivity of π_τ(A) . From this, we can also apply the proof of Theorem (4.1.5)for some other cases. Here we summarize the cases Theorem (4.1.5)is applicable. Part (1) is pointed out by Professor Narutaka Ozawa.

Corollary (4.1.6) [4]

1-Let A be a unital exact C^*-algebra with a faithful amenable tracial stateτ. Then the pair (A, τ)has the Haagerup property.

2- Let Abe a unital residually finite dimensional C^*-algebra with a faithful tracial state. Then A has a faithful tracial state τ with the Haagerup property.

Proof:-

1-It suffices to show $\pi_\tau(A)''$ is injective. By amenability of τ, the product∗-homomorphism

$$\pi_\tau \times \pi_\tau^{op} : A \odot A^{op} \rightarrow B(L^2(A, \tau))$$

is continuous with respect to the spatial tensor product norm [4] . Then by universality of the double dual, it extends to the normal∗-homomorphism from $(A \otimes A^{op})^{**}$to B(L^2 (A, τ)). Notice that sinceAis exact, it has property C''[4] .So the canonical inclusion

$$A^{**} \odot (A^{op}) \rightarrow A \otimes)^{**}$$

is continuous with respect to the spatial tensor product norm. Consequently, the product∗-homomorphism

$$\pi_\tau(A)'' \odot \pi_\tau^{op}(A^{op}) \rightarrow B\big(L^2(A, \tau)\big)$$

is continuous with respect to the spatial tensor product norm. Note that $(\pi_\tau^{op}(A^{op}))')) = \pi_\tau(A)''$ [4].Now injectivity of $\pi_\tau(A)''$ follows from Lance's trick.

2- By assumption, there exists a faithful tracial state τ onAsuchthat π_τ (A) is a type I von Neumann algebra. Since a type I von Neumann algebra is injective, we obtain the desired result.

Lemma (4.1.7) [4]

.Let A be a C^*-algebra,τ be a faithful tracial state on A, $(A_i)_i$ be an increasing

net of C^*-sub algebras of A whose union is dense in A with respect to the L^2norm determined by τ. Assume for each i, there is a trace-preserving conditional expectation E_i from A on to A_i .Then the pair(A, τ)has the Haagerup property if and only if each pair(A_i,τ)has the Haagerup property.

Now we establish the permanence properties of the Haagerup property. Here we restate Theorem(4.1.9).

Definition(4.1.8)[4]
Let A b aunital C^*-algebra ,τa faithful tracial state on A .The pair(A,τ) is said to have the Haagerup property if there is anet $(\varphi_i)_{i\in I}$ of u.c.p. maps from A to it self satisfying the following conditions:
(1)Each φ_i decreases τ ;i.e. , for any positive element a ∈A ,we have τ(φ_i (a))≤τ(a);
(2)For any a ∈A ,||φ_i(a)−a||$_i$ converges to 0 as it endstoinfinity;
(3)Each φ_i isL^2-compact ;i.e., from the first condition , φ_i extends to abounded operator on its GNS-space L^2(A ,τ),which is compact.

Theorem (4.1.9)[4]

Let $(A_i, \tau_i)_{i\in I}$ be a family of C^*- algebras with the Haagerup property indexed

By as set I. Then the following hold:

1- If I is countable, then the direct product ($\prod_{i\in I} A_i$,τ)has the Haagerup property for any tracial state τ of the form τ= $\sum_{i\in I} c_i\tau_i$, where $(c_i)_{i\in I}$ is a family of positive numbers whose sum is1. More generally, any C^*-subalgebra of ($\prod_{i\in I} A_i$,τ)which contains both $\bigoplus_{i\in I}$and1has the Haagerup property with respect to the restriction tracial state;

2- The spatial tensor product($\bigoplus_{i\in I}$, A_i $\bigoplus_{i\in I}$ τ_i) has the Haagerup property;

3- Thereduced free product (A, τ)= $*_{i\in I}$ (A_i, τ_i)has the Haagerup property.

Proof:-

1- We may assume I=ℕ. For each n∈ ℕ, take an approximation net $(\Phi_n$,j $)_{j\in j_n}$

Of u.c.p. maps of the Haagerup property of(A_n, τ_n). Replace $j_n$by for each n∈ℕ,wemay assume all index sets of the nets are the same one, say J. Then for each n∈ℕ and j∈ J, we define a u.c.p. map $\Psi_{n,j}$on($\prod_{n\in\mathbb{N}} A_n$, τ_n∈ℕ A_n by

$$\Psi_{n,j} := (\bigoplus_{k\leq n} \Phi_{k,j} \oplus \sum_{k>n} c_k\ \tau_k).$$

Then the net $(\Psi_{n,j})_{n,j}$ satisfies the desired condition. Moreover, each range of $\Psi_{n,j}$ is contained in the unitization of n ∈N An. So the second part of the claim also follows.

2- By the previous lemma, it suffices to consider the case I={1,2}.Let(A, τ),(B, ν)be two pairs of C^*-algebras and faithful tracial states both of which have the Haagerup property. By assumption **[4]**,we can choose nets of trace-preserving u.c.p. maps$(\varphi_j)_{j\in J}(\psi_j)_{j\in J}$ which give the Haagerup property of(A, τ), (B, ν) respectively. Then the net $(\phi_j \otimes\psi_j)_{j\in J}$ obviously gives the Haagerup property of (A⊗ B ,τ⊗ ν).

3- Similarly, it suffices to consider the case I={1,2}. With the notations as above, we define

$$\widetilde{\phi}_{j,k}:=c_k\phi_j+(1-c_k)\tau, \qquad \widetilde{\psi}_{j,k}:=c_k\psi_j+(1-c_k)\nu$$

for each j∈J and k ∈N, where ck=1−1/k. (For the notion of reduced free product, We will show the net $(\widetilde{\phi}_{j,k}*\varphi_{j,k})_{(j,k)\in J\times\mathbb{N}}$ gives the Haagerup property of(A, τ)∗(B, ν). Note that by definition of $\widetilde{\varphi}$'s and ˜ $\widetilde{\psi}$'s, it obviously satisfies the conditions listed on Definition(4.1.8)excepting the L^2-compactness. For the L^2-compactness, notice that the GNS-space of the reduced free product 2 ((A, τ)∗(B, ν))is canonically isomorphic to the free product$(L^2(A,\tau),1_A^\tau)*(L^2(B,\nu),1_B^\nu)$ of GNS-spaces .Then since the restriction of $(\varphi_j$to $L^2(A,\tau)^o=L^2(A,\tau)\ominus\mathbb{C}1_A^\tau$ has the norm less than or equal tock and similarly for $\widetilde{\psi}_{j,k}$, $\widetilde{\varphi}_{j,k}*\widetilde{\psi}_{j,k}$ is a c_0-direct sum of compact operators as a bounded operator on$(L^2(A,\tau),1_A^\tau)*(L^2(B,\nu),1_B^\nu)$by definition. Therefore it is a compact operator, as desired.

Next we study the permanence of the Haagerup property under the reduced crossed product construction. As we will see in Theorem (4.2.1), this is no longer true in general. However, in the following AF-setting, we have the permanence property. This is pointed out by the referee.

Theorem (4.1.10) [4]

Let Γ be a group with the Haagerup property acting on a unital C^*-algebraA.

Assume the following hold:

(i) A has a Γ-invariant faithful tracial state τ;

(ii) Γ is he union of an increasing net$(\Gamma_i)_{i\in I}$of subgroups;

(iii) There is an increasing net$(A_i)_{i\in I}$of finite dimensionalC^*-sub algebras of A, whose union is dense in A with respect to the L^2-norm determined by τ, and each A_i is Γ_i-invariant.

Then the reduced crossed product $A\rtimes_r \Gamma$has the Haagerup property.

Proof:-

We will show $A\rtimes_r \Gamma$ has the Haagerup property with respect to the canonical extension$\tilde{\tau}$ of τ. For each i∈I, there exists a unique τ-preserving conditional expectation $E_i$from A On to A_i. Notice that, by uniqueness, it must be Γ_i-equivariant. Hence, E_i extends to a conditional expectation from $A\rtimes_r \Gamma_i$onto$A_i \rtimes_r \Gamma_i$, which preserves $\tilde{\tau}$by definition. At the same time, we also have a$\tilde{\tau}$-preserving conditional expectation From $A\rtimes_r \Gamma$ onto $A\rtimes_r \Gamma_i$. Indeed, first represent $A\rtimes_r \Gamma$ on L^2 (A, τ)⊗ $l^2(\Gamma)$in the canonical way and similarly for$A\rtimes_r \Gamma_i$. Consider the conditional expectation on B(L^2(A, τ)⊗ $l^2(\Gamma)$) induced by the projection p=1⊗$\chi\Gamma_i$. Then the restriction of it to $A\rtimes_r \Gamma$ gives the desired conditional expectation. Composing these two maps, we have a $\tilde{\tau}$-preserving conditional expectation from $A\rtimes_r\Gamma$onto$A_i \rtimes_r \Gamma_i$. [4], each$A_i \rtimes_r \Gamma_i$has the Haagerup property with respect to $\tilde{\tau}$. Then, since the union of $A_i \rtimes_r \Gamma_i$'s is dense in $A\rtimes_r \Gamma$ with respect to the L^2 -topology,Lemma (4.1.11)completes the proof.

Remark (4.1.11) [4]

In this , we give an application of the Haagerup property for C^*-algebras. The results rely heavily on techniques in von Neumann algebras which trace back to the work of Poparom .Popa's theorem says the Haagerup property is a strong negation of relative property (T) in the context of von Neumann algebras. We extend this rigidity theorem to the context of C^*-algebras. The theorem does not depend on the tracial states, therefore it is convenient to introduce the following class of C^*-algebras.

Definition(4.1.12) [4]

.Set H be the class of all C^*-algebras which has a faithful tracial stateτwith the Haagerup property. By the results , the class H is quite large. It contains all nuclear C^*-algebras with a faithful tracial state, many exact C^*-algebras (for example, unital simple exact quasi-diagonalC^*-algebras), residually finite dimensional C^*-algebras with a faithful tracial state, the reduced group C^*-algebras of groups with the Haagerup property, and is closed under taking the direct product, the spatial tensor product and the reduced free product. However, we need to remark that the class His not closed under taking a quotient, even if the quotient has a faithful tracial state. To see this, consider the full group C^*-algebra $C^*(F_\infty)$of the free group $F_\infty$of countably many generators. Then it is residually finite dimensional by Choi's theorem.From this and separability of C^* (F_∞), it is contained in the class H by Corollary (4.1.6). Note that any unital separable C^*-algebra arises as a quotient of C^*(F∞), and as we soon see in Corollary (4.1.17), there is a unital separable C^*-algebra which has a faithful tracial state but is not contained in the class H.

The following theorem, due to Pop .

Theorem (4.1.13) [4]

Let M be a von Neumann algebra with a faithful normal tracial state ,B a von Neumann sub algebra of M .If M has the Haagerup property and the pair(M, B)has relative property(T), then Bis not diffuse.

We now apply Popa's theorem to the context of C^*-algebras. The proofs of the following lemmas are straightforward, so we only give sketches of the proofs.

Lemma (4.1.14) [4]

Let(A, τ) be a pair of a unital C^*-algebra and a faithful tracial state onA. Let

π_τ be the GNS-representation of τ. If the pair (A, τ)has the Haagerup property, then so does $(\pi_\tau(A)'' ,\tau)$.

Proof :-

Note that any trace-preserving u.c.p. map on A extends to a trace preserving u.c.p. map on the GNS-closure, which is L^2 -compact if the original one is. The extensions of approximation maps of the Haagerup property for(A, τ)establish the Haagerup property of$(\pi_\tau(A)'',\tau)$.

Lemma (4.1.15) [4]

Let A be aC^*-algebra, B be aC^*-subalgebra of Aandτ be a tracial state on A. If the pair (A, B) has relative property(T) (in the sense of Leung–Ng), then the pair ($\pi_\tau(A)''$, $\pi_\tau(B)''$)of GNS-closures has relative property(T)in the sense of Popa.

Proof :-

Since the left and right actions of a Hilbert bimodule of a von Neumann

Algebra M are normal, for anyσ-strongly dense subset S of M, any S-central vector of H is indeed M-central. From this, our claim follows easily.

Now, we obtain the rigidity result ,Theorem (4.1.15).

Theorem (4.1.16) [4]

Let A∈H, B be its C^*-subalgebra. If the pair(A, B)has relative property(T),

Then B is residually finite dimensional.

Proof:-

Choose a faithful tracial state τ on A with the Haagerup property. Then by Lemmas (4.1.14) and (4.1.15),($\pi_\tau(A)''$,τ)has the Haagerup property and the pair ($\pi_\tau(A)''$, $\pi_\tau(B)''$) has relative property (T). Hence, by Popa's theorem, $\pi_\tau(B)''$ is a direct sum of matrix algebras.

Corollary (4.1.17) [4]

Let Γ be a property (T) group ,A∈H. Then any unitary representation of Γ on A is weakly equivalent to a direct sum of finite dimensional representations. In particular, if Γ is an infinite property (T) group, then there is no nonzero∗-homomorphism from the reduced group C^*-algebra C_r^*(Γ)into A.

Proof:-

By Leung–Ng's theorem, the full group C^* -algebra C^* (Γ)of Γ has property (T). Since property (T) passes to a quotient , for any representation π of Γ on A ,the C^*-sub algebra of A generated by the image of π, which is isomorphic to a quotient of the full group C^*-algebra of Γ, has property (T). Since it is a C^*-sub algebra of a C^*-algebra in the lass H, it is residually finite dimensional by Theorem (4.1.16) This proves our first claim. For the last statement, recall the reduced group C^* -algebra has a finite dimensional representation if and only if the group is amenable.

Proposition (4.1.18) [4]

If A is a unital C^*-algebra which is residually finite dimensional with property(T)and a faithful tracial state, then A is contained in the class H. Before the proof, we need a comment. Although this is a special case of Corollary (4.1.6) , we prefer to give an independent proof, which is much more elementally, by using a result of Brown about property (T) C^*-algebras .

Proof:-

Let A be as above. Let$\{\pi_i\}_{i\in I}$ be a complete representation system of the set of all equivalent classes of finite dimensional irreducible representations of A. Then $\oplus_{i\in I}\pi_i$ is a faithful representation of A by assumption. Hence we can regard A as a unital C^* -sub algebra of $\prod_{i\in I}\mathbb{M}_{d_i}$, where di is the dimension of π_i. Then by the existence of Kazhdan projections , the unit of the ith direct summand $1_{\mathbb{M}_{d_i}}$is contained in A for all i∈I. Then, by irreducibility ofπ_i , ith direct sum and $\mathbb{M}_{d_i}$ is contained in A for all i∈I. Hence

$\oplus_{i\in I}\mathbb{M}_{d_i}$is contained in A. Then by the existence of a faithful tracial state, I must be countable. Hence A is contained in the class H by Theorem (4.1.9) .

Here we give an infinite dimensional example of C^*-algebra which has both property (T) and the Haagerup property.

Example (4.1.19) [4]

Let n≥3. on the group algebra C[SL_n (ℤ)]of SLn(ℤ), define the C^* -semi norm

‖. ‖ finas follows:

$$\|x\|_{\text{fin}} := \sup\{ \|\pi(x)\| \mid \pi \text{ is a finite representation of } \mathrm{SL}_n(\mathbb{Z})\}.$$

Then define the C^* -algebraC^*_{fin} $(SL_n(\mathbb{Z}))$as the completion ofC[$SL_n(\mathbb{Z})$] with respect to the

Semi norm$\|.\|$fin. Since $SL_n(\mathbb{Z})$is residually finite, the left regular representation is weakly contained in a direct sum of finite dimensional representations. Therefore the seminorm $\|.\|_{\text{fin}}$is (strictly) greater than the reduced norm $\|.\|_{\text{r}}$. Hence this is indeed a norm and consequently $C^*_{\text{fin}}(SL_n(\mathbb{Z}))$ is infinite dimensional. Moreover, since property (T) passes to a quotient C^*_{fin} $(SL_n(\mathbb{Z}))$has property (T). On the other hand, since $C^*_{\text{fin}}(SL_n(\mathbb{Z}))$is residually finite dimensional, it is contained in the class H by Proposition (4.1.18).

Theorem(4.1.20) **[4]**

The Haagerup property for C^* -algebras does depend on the choice of a faithful tracial state.

Proof:-

Let A= $C^*_{\text{fin}}(SL_n(\mathbb{Z}))$, where n≥3. We already know it has a faithful tracial state with the Haagerup property. So to show the claim, it suffices to find a faithful tracial state τ on A without the Haagerup property. Remark that, since the left regular representation λ of $SL_n(\mathbb{Z})$ is weakly contained in a direct sum of finite dimensional representations, δ_e extends to a tracial state of A, say the extension τ_1. Define $\tau=(\tau_1+\tau_2)/2$, whereτ_2 is an arbitrary faithful tracial state on A. We will show the pair (A, τ)does not have the Haagerup property. Assume by contradiction that(A, τ)has the Haagerup property, i.e., there exists a sequence$(\Phi_n)_2$of u.c.p. maps on A satisfying the properties listed on Definition(4.1.8) . Consider $l^2_\tau(SL_n(\mathbb{Z}))$, which is the GNS-space of τ. For any f∈c_c $(SL_n(\mathbb{Z}))$,we have $\|f\|_2^2 \leq \|f\|_\tau^2$, hence the identity map on $c_c(SL_n(\mathbb{Z}))$extends to a bounded operator froml^2_τ $(SL_n(\mathbb{Z}))$into l^2 $(SL_n(\mathbb{Z}))$. Denote the extension by π. Now for each k, we define a complex valued function $\psi_k$on $SL_n(\mathbb{Z})$ by

$$\psi_k(g) := \delta_g, \pi(\Phi_k(g)\ \delta_e^\tau)_2 = \delta g, \lambda\Phi_k(g)\ \delta_e)_2.$$

Then ψ_k converges to 1 pointwise asktends to infinity. On the other hand, since $\lambda\circ\Phi_k$ is u.c.p., ψ_k is positive definite. Hence the convergence of ψ_k is indeed uniform (since $SL_n(\mathbb{Z})$ has property (T)). However, note that the family $(\delta_g)_{g\in SL_n(\mathbb{Z})}$ is an orthonormal basis of $l^2(SL_n(\mathbb{Z}))$, whereas the set $\{\pi(\Phi_k(g)\delta_e^\tau)\}g\in SL_n(\mathbb{Z})$ is relatively compact in $l^2SL_n(\mathbb{Z})$)by the l^2-compactness of$\Phi k$and the boundedness ofπ, which is a contradiction.

In the context of von Neumann algebras, the non-embeddable result of Corollary(4.1.17) still holds for the group von Neumann algebra of a group which has relative property (T) with respect to an infinite subgroup. This is because the group von Neumann algebra of an infinite group is always diffuse. The corresponding result is not true in the context of C^*-algebras, because the reduced group C^*-algebra of an infinite group can be residually finite dimensional. Indeed, many typical relative property (T) groups fail to have the rigidity property.

Lemma (4.1.21) [4]

Let A be a unital C^*-algebra with an action of a group Γ with the Haagerup property. Assume A admits a countable family $(\pi_n)_n$of Γ-equivariant finite dimensional representations which separates the points ofA. Then the reduced crossed product $A\rtimes_r\Gamma$embeds into a C^*-algebra in the class $\mathcal{H}$.

Proof:-

Take a countable separating family$(\pi_n)_n$of Γ-equivariant finite dimensional representations. Then we have aΓ-equivariant embedding

$$\bigoplus_n \pi_n : A \hookrightarrow \prod_n A/\ker \pi_n .$$

By taking the reduced crossed products, we have an embedding

$$A \rtimes_r \Gamma \hookrightarrow \prod_n \big((A/\ker \pi_n) \rtimes_r \Gamma\big).$$

Since each $A/\ker \pi_n$ is finite dimensional and Γ has the Haager up property, the range of the above map is contained in the class$\mathcal{H}$.

Lemma (4.1.22) [4]

can apply to many reduced group C∗-algebras of groups without the Haagerup property. Here we recall the examples of groups which have relative property (T).

Definition (4.1.23) [4]

Let 𝕂 be an algebraic number field (i.e., a finite extension of the rational number field ℚ), R be the ring of integers of 𝕂 (i.e., the ring of all elements of 𝕂 which are roots of a nonzero monic polynomial with the integer coefficients). The three-dimensional Heisenberg group with the coefficients in R, denoted by H_3(R), is the subgroup of SL_3(R)which consists of all upper triangular matrices with the diagonal entries 1. Equivalently, H_3(R)is defined as the Set R_2×R with the group operation

$$(x, \lambda)(y, \mu):= (x+y ,\lambda+\mu)+\omega(x,y),$$

Where ω(x,y):=$x_1y_2-x_2y_1$ is the symplectic form. In the latter picture, SL_2 (R) canonically acts on H_3(R)by acting on the first coordinate. (Since SL_2(R) preserves ω , this indeed defines an action on the groupH_3(R).)

Proposition (4.1.24) [4]

Let 𝕂 be an algebraic number field ,R be the ring of integers of 𝕂.

Then the following hold:

(1) The pair($R^2 \rtimes SL_2$(R), R^2)has relative property(T);

(2) The pair(H_3(R) $\rtimes SL_2$(R), H_3(ℤ))has relative property(T);

(3) The group SL_2(R) has the Haagerup property.

Theorem (4.1.25) [4]

Let 𝕂be an algebraic number field, R be the ring of integers of 𝕂. Then the

reduced group C*-algebras of $R^2 \rtimes SL_2$(R), H_3(R) $\rtimes SL_2$(R)embed into C^*-algebras in the class $\mathcal{H}$.

Proof:-

First note that since both $R^2$and H_3(R) are amenable, the full and reduced group C^*-algebras of these groups are equal. Since R is finitely generated as an additive group, for any natural number n ∈ℕ, R/n R is a finite ring. So $(R/nR)^2$ and H_3(R/nR) are also finite.

Moreover, it is obvious that these quotients are SL_2(R)-equivariant. Consequently, we obtain SL_2(R)-equivariant finite dimensional representations

$$\pi_n : \mathrm{C}_r^*(R^2) \to \mathrm{C}_r^*((R/nR)^2),$$

$$\sigma_n : \mathrm{C}_r^*(H_3(R)) \to \mathrm{C}_r^*(H_3(R/nR)).$$

Now it is easy to check these families separate points. Consequently, we can apply Lemma (4.1.21) to the C*-algebras we have considered.

Section (4.2): Arigidity property of the affine groups of the affine planes

We have seen that, unlike the case of von Neumann algebras, the non-embeddable theorem for the reduced group C^*-algebras of relative property (T) groups fails in general. The difficulty comes from the fact C^*-algebras admit many "mutually singular" faithful tracial states. However, if we overcome this difficulty, then we can prove a rigidity theorem for a group, even if the group has no infinite property (T) subgroups. For instance, we will consider the two classes of groups. The first class consists of groups with Powers's property. For a Powers group without the Haagerup property, the non-embeddable theorem follows from the uniqueness of the tracial state on the reduced group C^*-algebra. By using the free product, it is easy to construct an artificial group in this class without both the Haagerup property and infinite property (T) subgroups (e.g., $(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) * \mathbb{Z})$). However, we does not know an example of group as above which naturally arises. So we also study the other class. These groups do not contain infinite property (T) subgroups and naturally arise in many fields: Namely, we study a rigidity property of the reduced group C^*-algebras of the affine groups $\mathbb{K}^2 \rtimes GL_2(\mathbb{K})$ of the affine planes (or more strongly for the subgroup $\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})$) over the fields $\mathbb{K}$. Note that the group $\mathbb{K}^2 \rtimes \mathrm{GL}_2(\mathbb{K})$ is the automorphism group of the affine plane over $\mathbb{K}$, so this is a very natural object. First we remark that a rigidity property obviously fails when $\mathbb{K}$ is an algebraic extension of a finite field. In this case, $\mathbb{K}$ is an increasing union of finite subfields. From this, the affine group over $\mathbb{K}$ is an increasing union of finite subgroups, in particular it is amenable. We will show that excepting these amenable cases, the affine groups have a rigidity property.

Theorem (4.2.1) [4]

Let $\mathbb{K}$ be a field which is not an algebraic extension of a finite field. Then $C_r^*(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K}))$ cannot be embedded into any C^*-algebra in the class $\mathcal{H}$.

Proof:-

We divide the proof into two cases:

Case 1: $\mathbb{K}$ has characteristic zero.

Assume by contradiction $C_r^*(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K}))$ embeds into a C^*-algebra contained in the class $\mathcal{H}$. Then since the Haagerup property passes to the GNS-closure, and the Haagerup property passes to von Neumann subalgebras, we have a faithful tracial state τ on $C_r^*(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K}))$ such that the GNS-closure $\pi_\tau C_r^*(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K}))''$ has the Haagerup property. We study the tracial state τ. Consider the restriction τ_0 of τ to $C_r^*(\mathbb{K}^2)$. Then τ_0 is an $\mathrm{SL}_2(\mathbb{K})$-invariant tracial state on $C_r^*(\mathbb{K}^2)$. Since the action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathbb{K}^2\{0\}$ is transitive, τ_0 must be of the form

$$\tau_0 = c\chi_{\{0\}} + 1(1-c)\chi_{\mathbb{K}^2}.$$

where $c \in [0,1]$. (Here we identify a state on the reduced group C^*-algebra $C_r^*(\Gamma)$ with the restriction of it to the group Γ, which is a positive definite function of Γ.) By faithfulness of τ_0, we further have $c > 0$. From this, the restriction τ_1 of τ to $C_r^*(\mathbb{Z}^2)$ is of the form

$$\tau_1 = c\chi_{\{0\}} + (1-c)\chi_{\mathbb{Z}^2}$$

with $c > 0$. From this form, the GNS-closure $C_r^*(\mathbb{Z}^2)''$ of $C_r^*(\mathbb{Z}^2)$ has the diffuse direct sum and $L(\mathbb{Z}^2)$. On the other hand, since the pair

$$(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K}), \mathbb{Z}^2)$$

has relative property (T), the pair

$$C_r^*\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big), C_r^*(\mathbb{Z}^2)$$

also has relative property (T). Then by taking GNS-closures, we further have the pair

$$\left(\left(\pi_\tau\left(C_r^*\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big)\right)\right)'', \left(\pi_\tau\big(C_r^*(\mathbb{Z}^2)\big)\right)''\right)$$

has relative property (T). Then notice that $\left(\pi_\tau\big(C_r^*(\mathbb{Z}^2)\big)\right)''$ has a nonzero diffuse direct summand, whereas $\left(\pi_\tau\left(C_r^*\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big)\right)\right)''$ has the Haagerup property. This contradicts Theorem (4.2.2)

Case 2: $\mathbb{K}$ has characteristic p.

This case is also proved by the same method as in Case 1. Take a transcendental element π over the prime field $\mathbb{F}_p$. Then, notice that the ring $\mathbb{F}_p[\pi]$ is isomorphic to the polynomial ring over $\mathbb{F}_p$. Therefore the pair

$$\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K}), \mathbb{F}_p[\pi]^2\big)$$

has relative property (T). Now the same proof as in Case 1 works with $\mathbb{F}_p[\pi]$ plays the same role as $\mathbb{Z}$.

Remark (4.2.2) [4]

Guentner, Higson and Weinberger show the group $\mathrm{SL}_2(\mathbb{K})$ has the Haagerup property for any field $\mathbb{K}$, as a discrete group. From this, the von Neumann algebra $L\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big)$ has the relative Haagerup property with respect to the type I von Neumann subalgebra $L(\mathbb{K}^2)$ in the sense of Popa. Then Popa's theorem shows any von Neumann subalgebra of $L\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big)$ with property (T) is of type I. Consequently, we have any C^*-subalgebra of $C_r^*\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big)$ with property (T) is residually finite dimensional. That is, any property (T) C^*-subalgebra of $C_r^*\big(\mathbb{K}^2 \rtimes \mathrm{SL}_2(\mathbb{K})\big)$ does not say anything in our rigidity theorem Theorem(4.1.15).

However, these C^*-algebras have a rigidity property relative to the class $\mathcal{H}$. This in particular shows the class $\mathcal{H}$ is strictly larger than the complement of the class of C^*-algebras containing a nontrivial property (T) C^*-subalgebra.

Remark (4.2.3) [4]

From Theorem (4.1.1), the class $\mathcal{H}$ is not closed under taking the reduced crossed product by a group with the Haagerup property even if the resulting algebra has a faithful tracial state. (As we have seen in, this is not so obvious.)

Remark (4.2.4) [4]

From Theorems (4.1.2) and (4.2.1), we obtain the reduced group C^*-algebra of $\mathbb{Q}^2 \rtimes L_2(\mathbb{Q})$ cannot embed into that of $\mathbb{Z}^2 \rtimes L_2(\mathbb{Z})$. The corresponding result in the context of von Neumann algebras is not known.

List of symbols

symbol	page
G_∞ : Glimm ideal	1
diag : diagonal	2
Ker : Kernal	3
$\otimes$: Tensor Product	3
Sup : Supremum	4
Inf : infimum	6
max : maximum	7
Prim : prime	10
Co : convex	17
$\oplus$: orthogonal Sum	19
L^p : Lebesgue Space	27
mod : modular	42
Tr : Trace	56
H_2 : Hilbert Space	74
L_2 : Hilbert Space	79
min : minimum	85
AFD : Approximately Finite Dimensional	103
dist : distance	105
op : operator	107
$\odot$: Special tensor product	107
$\ominus$: Direct difference	109
F_∞ : Free group	111
fin :finite	114

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