Chapter 1

Sampling Theorem and Average Sampling

Some examples are presented to show the generality of the sampling theorem. Index Terms - Fourier transform, generating function, sampling, shift-invariant subspace, Zak transform . Regular and irregular average sampling theorems for spline subspaces are obtained .

Section(1.1) Shift-Invariant Subspace and Sampling Theorem

A fundamental question in signal processing is how to represent a signal in terms of a discrete sequence. Shannon's popular sampling theorem states that finite energy band-limited signals are completely characterized by their samples values. Realizing that the Shannon interpolating function $sinc(t) = sin(t)/t$ is in fact a scaling function of an MRA, Walter [268] found a sampling theorem for a class of wavelet subspaces. Suppose $\phi(t)$ is a continuous orthonormal scaling function of an MRA $\{V_m\}_{m\in\mathbb{Z}}$ such that $|\phi(t)| \le O((1 + |t|)^{-1-\varepsilon})$ for some $\varepsilon > 0$.

Let $\hat{\phi}^*(\omega) = \sum_n \phi(n)e^{-in\omega}$. Walter showed that there is an $S(t) \in V_0$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n) \qquad (1)$$

holds for any $f(t) \in V_0$ if $\hat{\phi}^* \neq 0$. Following Walter's [268] work, Janssen [233] studied the shift sampling case by using the Zak-transform. Xia and Zhang [277] discussed the so-called sampling property ($S(t) = \phi(t)$). Walter [294], Xia [295], and Chen–Itoh [288], [289] studied the more general case "oversampling." Chen *et al.* [285], [287], Chen and Itoh [286], Liu [255], and Liu and Walter [234] even studied irregular sampling in wavelet subspaces. Furthermore, Aldroubi and Unser [282],[283],[284], [291] studied the sampling procedure in shift invariant subspaces. They established a more comprehensive sampling theory for shift-invariant subspaces. One of their important result states that when $\phi(t)$ ($\in L^2(\mathbb{R})$) is a generating function, the orthogonal projection $g_p(t)$ of a function $g(t) \in L^2(\mathbb{R})$ on the shift-invariant subspace $V_0(\phi)$ is given by

$$g_p(t) = \sum_{n\in\mathbb{Z}} \langle g(.), \tilde{\phi}(.-n)\rangle \phi(t-n) \qquad (2)$$

where $\{\tilde{\phi}(.-n)\}_n$ is the biorthogonal basis of $\{\phi(.-n)\}_n$ in $V_0(\phi)$, and $\langle .,.\rangle$ is the $L^2(\mathbb{R})$-inner product. They then found that the $\phi(t)$can be replaced by an interpolating generating function $S(t)$ if

$\phi(t) \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, $\quad \sum_k \hat{\phi}(\omega + 2k\pi) \neq 0$, and the Fourier transform $\hat{\phi}(\omega)$ of $\phi(t)$ satisfies $|\hat{\phi}(t)| \le O((1 + |\omega|)^{-1-\varepsilon}$ for some $\varepsilon > 0$ (see [284]). In fact, these constraints are related to those of Walter sampling theorem due to the fact $\sum_k \hat{\phi}(\omega + 2k\pi) = \hat{\phi}^*(-\omega)$ in some sense .

Our purpose in this correspondence is to find a weaker constraint on the generating function $\phi(t)$ such that a formula similar to (1) [or(2) with interpolating generating function $S(t)$ instead of $\phi(t)$] holdsfor any function $f(t)$ in the shift-invariant subspaces $V_0(\phi)$. We find a condition for (1) that is sufficient and necessary. In thisway, we are able to remove the continuity and regularity constraintsimposed on the generating function $\phi(t)$ by Walter [268] or imposedon its Fourier transform $\hat{\phi}(\omega)$ by Aldroubi and Unser [284]. We also make a case to show the generality of our result in Section .We now introduce some notations used in this correspondence. Fora measurable subset $E \subset \mathbb{R}$, $|E|$ denotes the measure of E. For a measurable function $f(t)$, we write

$$\|f(t)\|_0 = \sup_{|E|=0} \inf_{\mathbb{R}\ominus E} |f(t)| \tag{3}$$

$$\|f(t)\|_0 = \inf_{|E|=0} \sup_{\mathbb{R}\ominus E} |f(t)| \tag{4}$$

$$\chi_E(t) = \begin{cases} 1 & t \in E \\ 0 & otherwise. \end{cases} \tag{5}$$

Suppose an $L^2(\mathbb{R})$ function $\phi(t)$ is such that the sampling $\{\phi(n)\}_n$ makes sense and $\{\phi(n)\}_n \in l^2$. Then, the series $\sum_n \phi(n)e^{-in\omega}$ converges to an $L^2[0,2\pi]$ function $\hat{\phi}^*(\omega)$ in $L^2[0,2\pi]$ sense. Let us now consider the shift-invariant subspace sequence $\{V_j(\phi)\}_j$ generated by $\phi(t)$

$$V_j(\phi) = \left\{\sum_k c_k\phi(2^j t - k) : \{c_k\}_k \in l^2\right\} \subset L^2(\mathbb{R}). \tag{6}$$

For $f(t) = \sum_k c_k\phi(t-k) \in V_0(\phi)$, we let $f(n) = \sum_k c_k\phi(n-k)$.

Then, $\{f(n)\}_n$ $(\in l^\infty)$ is well defined since $\{c_k\}_k$ and $\{\phi(k)\}_k$ are both l^2 sequences. In fact, $f(n)$ as the Fourier coefficients of the $L^1[0,2\pi]$ function $\hat{\phi}^*(\omega)\sum_k c_k e^{-ik\omega}$ tends to 0 at infinity by the Riemann–Lebesgue Lemma.

Generally, $\{\phi(t-k)\}_k$ may not be a Riesz basis of $V_0(\phi)$. It is shown that $\{\phi(t-k)\}_k$ is a Riesz basis of $V_0(\phi)$ if and only if

$$0 < \|G_\phi(\omega)\|_0 \le \|G_\phi(\omega)\|_\infty < \infty \tag{7}$$

holds, where $G_\phi(\omega) = \left(\sum_k |\hat{\phi}(\omega + 2k\pi)|^2\right)^{\frac{1}{2}}$, and $\hat{\phi}(\omega)$ is the Fourier transform of $\phi(t)$defined by $\hat{\phi}(\omega) = \int_{\mathbb{R}} \phi(t)e^{-i\omega t}\,dt$. If $\phi(t)$ satisfies (7), it is called a generating function (see [284]).

Theorem(1.1.1)[272] : Suppose $\phi(t) \in L^2(\mathbb{R})$ is a generating function such that the sampling $\{\phi(n)\}_n$ makes sense, and $\{\phi(n)\}_n \in l^2$. Then, there is an

$S(t) \in V_0(\phi)$ such that

$$f(t) = \sum_n f(n)S(t-n) \quad , \text{ for } \quad f(t) \in V_0(\phi) \tag{8}$$

holds in the $L^2(\mathbb{R})$ sense if and only if

$$\frac{1}{\hat{\phi}^*(\omega)} \in L^2[0, 2\pi] \tag{9}$$

holds. In this case , $\hat{S}(\omega) = \hat{\phi}(\omega)/\hat{\phi}^*(\omega)$ holds for a.e. $\omega \in \mathbb{R}$.

Proof : **Step 1**—Sufficiency : Assume $\frac{1}{\hat{\phi}^*(\omega)} \in L^2[0, 2\pi]$. Then , $\hat{\phi}^*(\omega) \neq 0$ holds for a.e. $\omega \in \mathbb{R}$, and there is a $\{c_k\}_k \in l^2$ such that

$$\frac{1}{\hat{\phi}^*(\omega)} = \sum_k c_k e^{ik\omega} \tag{10}$$

holds in the $L^2[0, 2\pi]$ sense. Let $F(\omega) = \hat{\phi}(\omega)/\hat{\phi}^*(\omega)$. Then

$$\begin{aligned} \int_{\mathbb{R}} |F(\omega)|^2 \, d\omega &= \int_{\mathbb{R}} \left|\frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)}\right|^2 d\omega \\ &= \int_0^{2\pi} \frac{\sum_k \left|\hat{\phi}(\omega + 2k\pi)\right|^2}{\left|\hat{\phi}^*(\omega)\right|^2} d\omega \\ &\leq \left\|G_\phi(\omega)\right\|_\infty^2 \int_0^{2\pi} \frac{1}{\left|\hat{\phi}^*(\omega)\right|^2} \, d\omega \, . \end{aligned}$$

It is easy to see $F(\omega) \in L^2(\mathbb{R})$ due to (7). Hence, we can take the Fourier inverse of $F(\omega)$ in $L^2(\mathbb{R})$ denoted by $S(t)$, $i.e.$, we derive

$$\hat{S}(\omega) = \frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} \tag{11}$$

or

$$\hat{\phi}(\omega) = \hat{S}(\omega)\hat{\phi}^*(\omega) \, . \tag{12}$$

Take inverse Fourier transform on both sides of (11) and refer to (10)

$$S(t) = \sum_k c_k \phi(t-k) \ . \tag{13}$$

Formula (13) implies $S(t) \in V_0(\phi)$ [due to the fact that $\{\phi(t-k)\}_k$ is a Riesz basis of $V_0(\phi)$]. For any $f(t) \in V_0(\phi)$, there is a $\{a_k\}_k \in l^2$ such that $f(t) = \sum_k a_k \phi(t-k)$. Then

$$\hat{f}(\omega) = \hat{\phi}(\omega) \sum_k a_k e^{-ik\omega} \tag{14}$$

$$= \left(\hat{\phi}^*(\omega) \sum_k a_k e^{-ik\omega}\right) \hat{S}(\omega) \, . \tag{15}$$

Therefore , $f(t) = \sum_k f(k)S(t-k)$.

Step 2—Necessity*:* On the contrary, if there is an $S(t) \in V_0(\phi)$ such that (8) holds in the $L^2(\mathbb{R})$ sense, then

$$\phi(t) = \sum_n \phi(n)S(t-n) \tag{16}$$

holds in the $L^2(\mathbb{R})$ sense. By taking the Fourier transform on both sides of (16), we obtain $\hat{\phi}(\omega) = \hat{\phi}^*(\omega)\hat{S}(\omega)$. (17)

Equation (17) implies that $\operatorname{supp}\hat{\phi}(\omega) \subset \operatorname{supp}\hat{\phi}^*(\omega)$ holds for a.e. $\omega \in \mathbb{R}$, i.e., $\operatorname{supp}\hat{\phi}(\omega+2k\pi) \subset \operatorname{supp}\hat{\phi}^*(\omega)$ holds for all $k \in \mathbb{Z}$ and for a.e. $\omega \in \mathbb{R}$ because $\hat{\phi}^*(\omega)$ is 2π periodic. Meanwhile

$$\bigcup_k \operatorname{supp}\hat{\phi}(\omega+2k\pi) = \mathbb{R} \tag{18}$$

holds except for a zero measure subset of $\mathbb{R}$. Otherwise, there is a measurable subset δ with measure $|\delta| \neq 0$ such that

$$\delta \subset \mathbb{R} \ominus \bigcup_k \operatorname{supp}\hat{\phi}(\omega+2k\pi) \; . \tag{19}$$

Then , $\hat{\phi}(\omega+2k\pi) = 0$ holds for any $\omega \in \delta$ and for all $k \in \mathbb{Z}$. Hence

$$G_\phi(\omega) = \left(\sum_k \left|\hat{\phi}(\omega + 2k\pi)\right|^2\right)^{\frac{1}{2}} = 0 \tag{20}$$

holds for any $\omega \in \delta$. However , $G_\phi(\omega) \neq 0$ holds for a.e. $\omega \in \mathbb{R}$. It forces (18) to hold for a.e. $\omega \in \mathbb{R}$. Therefore

$$\operatorname{supp}\hat{\phi}^*(\omega) \supset \bigcup_k \operatorname{supp}\hat{\phi}(\omega+2k\pi) \tag{21}$$

holds for a.e. $\omega \in \mathbb{R}$, i.e. , $\hat{\phi}^*(\omega) \neq 0$ for a.e. $\omega \in \mathbb{R}$. Formula (17) is now rewritten to be

$$\frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} = \hat{S}(\omega). \tag{22}$$

Since $\hat{S}(\omega) \in L^2(\mathbb{R})$ [due to $S(\omega) \in L^2(\mathbb{R})$], we derive

$$\begin{aligned} \infty > \int_{\mathbb{R}} \left|\frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)}\right|^2 d\omega &= \int_0^{2\pi} \frac{\sum_k \left|\hat{\phi}(\omega + 2k\pi)\right|^2}{\left|\hat{\phi}^*(\omega)\right|^2} d\omega \\ &\geq \left\|G_\phi^2(\omega)\right\|_0 \int_0^{2\pi} \left|\frac{1}{\hat{\phi}^*(\omega)}\right|^2 d\omega \, . \end{aligned} \tag{23}$$

From (7) and (23), we conclude that $\frac{1}{\hat{\phi}^*(\omega)} \in L^2[0,2\pi]$ holds. This completes the proof.

If $\phi(t)$ satisfies the conditions of the Walter sampling theorem or the proposition of

Aldroubi and Unser, there must be a constant $C \geq 1$ such that $C^{-1} \leq |\hat{\phi}^*(\omega)| \leq C$. Obviously , $\frac{1}{\hat{\phi}^*(\omega)} \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$. Therefore, the Walter sampling theorem and the Aldroubi and Unser proposition can be obtained as a corollary of our theorem (refer to Examples (1.1.3), (1.1.3) ,(1.1.5)). A related problem is the study of truncation error and aliasing error. We do not estimate them here and refer to Walter, Unser and Daubechies [292] and Chen and Itoh [288], [289].As done by Janssen [233] for Walter's sampling theorem, Chen *et al.* [287] for the irregular sampling theorem, and Chen and Itoh [288] for the over sampling theorem, the shift-sampling theorem for shift-invariant subspace can be obtained by using the Zak transform. Suppose $\phi(t) \in L^2(\mathbb{R})$is such that the sampling $\{\phi(\sigma+n)\}_n$ makes sense, and $\{\phi(\sigma+n)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then , the Zak transform $Z_\phi(\sigma,\omega)$ of $\phi(t)$ is defined by

$$Z_\phi(\sigma,\omega) = \sum_n \phi(n+\sigma)e^{-in\omega} . \tag{24}$$

A generating function $\phi(t)$may not satisfy $\frac{1}{\hat{\phi}^*(\omega)} \in L^2[0,2\pi]$ but may satisfy $\frac{1}{Z_\phi(\sigma,\omega)} \in L^2[0,2\pi]$for some $\sigma \in [0,1)$. Then , it can be dealt with by the shift-sampling theorem (see Example (1.1.6)).We now present the shift-sampling theorem without proof (since it is very close to the previous).

Theorem(1.1.2)[272] : Suppose $\phi(t) \in L^2(\mathbb{R})$ is a generating function such that the sampling $\{\phi(\sigma+n)\}_n$ makes sense , and $\{\phi(\sigma+n)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then, there is an $S_\sigma(t) \in V_0(\phi)$ such that

$$f(t) = \sum_n f(n+\sigma)S_\sigma(t-n) , \quad \text{for} \quad f(t) \in V_0(\phi) \tag{25}$$

holds in the $L^2(\mathbb{R})$ sense if and only if

$$\frac{1}{Z_\phi(\sigma,\omega)} \in L^2[0,2\pi] \tag{26}$$

holds. In this case ,$\hat{S}(\omega) = \hat{\phi}(\omega)/Z_\phi(\sigma,\omega)$ holds for a.e. $\omega \in \mathbb{R}$.

Since the Haar function is not continuous and Shannon's *sinc* function is not regular enough, they can not be covered by Walter's sampling theorem. Since the Fourier transform of the Haar function is not regular enough and the Fourier transform of the Shannon function is not continuous, they are covered by [284], although we should note that there is no such restriction for the more general sampling theorems presented in [291]. Both functions are covered by our sampling theorem (see Examples (1.1.3) and (1.1.4)).

Example(1.1.3)[272] :Haar function $\phi(t) = \chi_{[0,1)}$. The piecewise continuity of $\phi(t)$ implies that the sampling $\{\phi(n)\}_n$ makes sense. $\frac{1}{\hat{\phi}^*(\omega)} = 1 \in L^2[0,2\pi]$ implies that our sampling theorem can be applied and $S(t) = \chi_{[0,1)}$.

Example$(1.1.4)$[272] : Shannon function $\phi(t) = sin\pi t/\pi t$. The continuity of $\phi(t)$ implies that the sampling $\{\phi(n)\}_n$ makes sense. $\frac{1}{\hat{\phi}^*(\omega)} = 1 \in L^2[0, 2\pi]$ implies that our sampling theorem can be applied , and $S(t) = sin\pi t/\pi t$ .The following Example (1.1.5) shows that there exists a generating function $\phi(t)$ such that $\hat{\phi}^*(\omega) \longrightarrow 0$ as ω tends to a point $\omega_0(a.e.)$, but $\frac{1}{\hat{\phi}^*(\omega)} = L^2[0, 2\pi]$ holds. It implies that our sampling theorem is substantially more general than Walter sampling theorem.

Example$(1.1.5)$[272]: For a positive number $s < 1/2$, take $\phi(t)$ as the Fourier inverse of $\hat{\phi}(\omega)$ defined by

$$\hat{\phi}(\omega) = \begin{cases} -1 , & \omega \in [-4\pi, -2\pi) \\ 1 , & \omega \in [-2\pi, 0) \\ \omega^s , & \omega \in [0,2\pi) \\ 0 , & otherwise . \end{cases} \tag{27}$$

Then , $G_\phi(\omega)\chi_{[0,2\pi)} = (2 + \omega^{2s})^{1/2}$.

Obviously, $\sqrt{2} \le G_\phi(\omega) \le (2 + \omega^{2s})^{\frac{1}{2}}$. Therefore , $\phi(t)$ is a generating function. The fact $\hat{\phi}(\omega) \in L^1(\mathbb{R})$implies that $\phi(t)$ is continuous. Then , the sampling $\{\phi(n)\}_n$makes sense. Since $\hat{\phi}^*(\omega) = \sum_k \hat{\phi}(\omega + 2k\pi)$ in $L^2[0, 2\pi]$,

we derive $\hat{\phi}^*(\omega)\chi_{[0,2\pi)} = \omega^s$. However , $\hat{\phi}^*(\omega) \longrightarrow 0$ as $\omega \longrightarrow 0^+ (a.e.)$. Hence, neither Walter's sampling theorem nor Aldroubi and Unser's Proposition can be applied to deal with the $\phi(t)$ [since both of them require the condition $C^{-1} \le \hat{\phi}^*(\omega) \le C$ for some $C \ge 1$]. However $\omega^{-s} \in L^2[0, 2\pi]$ implies that our sampling theorem is available. The $\hat{S}(\omega)$ is given by

$$\hat{S}(\omega) = \begin{cases} -(\omega + 4\pi)^{-s} , & \omega \in [-4\pi, -2\pi) \\ (\omega + 2\pi)^{-s} , & \omega \in [-2\pi, 0) \\ 1 , & \omega \in [0, 2\pi) \\ 0 , & otherwise. \end{cases} \tag{28}$$

The following Example (1.1.6) (by [233]) shows the usefulness of shift-sampling theorem. It is also very interesting to find some works on centered spline interpolating in [271] , [290].

Example$(1.1.6)$[272] : B-spline of order 2 scaling function

$$N_2(t) = \frac{t^2}{2}\chi_{[0,1)}(t) + \frac{6t - 2t^2 - 3}{2}\chi_{[1,2)}(t) + \frac{(3-t)^2}{2}\chi_{[2,3)}(t). \tag{29}$$

$N_2(t)$ is a generating function (see [249]).

$\widehat{N}_2^*(\omega) = e^{i\omega}\left(e^{i\omega} + 1\right)^{1/2}$ implies that $1/N_2^*(\omega) = 2/e^{i\omega}\left(e^{i\omega} + 1\right)$ is not an $L^2[0, 2\pi]$ function.

However , $1/Z_{N_2}(1/2, \omega) = 8/\left(1 + 6e^{i\omega} + e^{2i\omega}\right) \in L^2[0, 2\pi]$ implies that the shift-sampling theorem is available. The $\hat{S}_{1/2}(\omega)$ is given by

$$\hat{S}_{\frac{1}{2}}(\omega) = 8\left(\frac{1- e^{-i\omega}}{i\omega}\right)^2 /\left(1 + 6e^{i\omega} + e^{2i\omega}\right). \quad (30)$$

Section(1.2) Spline Subspace and Average Sampling

Sampling theory is one of the most powerful results in signal analysis. The objective of sampling is to reconstruct a signal from its samples. For example, if f is band-limited to $[-\Omega,\Omega]$, then f is uniquely determined and can be reconstructed by its samples at $x_k = k\pi/\Omega$, which is the classical Shannon sampling theorem. Although the assumption that a signal is band-limited is eminently useful, it is not always realistic since a band-limited signal is of infinite duration. Thus, it is natural to investigate other signal classes for which a sampling theorem holds. A simple model is to consider shift-invariant subspaces, e.g., wavelet subspaces, which generalize the space of band-limited signals. In fact, there have been many results concerning the sampling in shift invariant subspaces for both regular and irregular sampling (see[240,243,254,257,268,270,271,272,273,274,275,276,277]). In particular, for the spline subspace $V_N = \{\sum_{k\in\mathbb{Z}} c_k\varphi_N(. - c_k):\{c_k\} \in \ell^2)$ generated by $\varphi_N = \chi_{[0,1]} * \dots * \chi_{[0,1]}$ (N convolutions) , $N \geq 1$, it was shown that for any

$$f \in V_N , f(x) = \sum_{k\in\mathbb{Z}} f\left(k + \frac{N+1}{2}\right) S(x-k),$$

where

$$\hat{S}(\omega) = \frac{\hat{\varphi}_N(\omega)}{\sum_{k\in\mathbb{Z}} \varphi_N\left(k + \frac{N+1}{2}\right) e^{-ik\omega}} .$$

In [243,254], Aldroubi , Grochenig and Liu studied irregular sampling in spline subspaces . In practice , measured sampled values may not be values of a signal f precisely at times x_k , but only local averages of f near x_k . Specifically, measured sampled values are

$$\langle f, u_k\rangle = \int f(x)u_k(x)dx ,$$

for some collection of averaging functions $u_k(x)$, $k \in \mathbb{Z}$, which satisfy the following properties:

$$\text{supp}u_k \subset \left[x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}\right], u_k(x) \geq 0, \text{ and } \int u_k(x)dx = 1.$$

It is clear that from local averages one should obtain at least a good approximation of the original signal if b is small enough. Wiley, Butzer and Lei studied the approximation error when local averages are used as sampled values [278,279]. Furthermore, Feichtinger and Grijchenig [280,281] proved that a signal is uniquely determined by its local averages under certain conditions. We study the reconstruction of functions in spline subspaces from local averages.

We show that every $f \in V_N$, is uniquely determined by its local averages on the intervals $\left[x_k - \frac{1}{2}\ ,\ x_k + \frac{1}{2}\right]$ for certain sampling points $\{x_k\}$.

Definition $(1.2.1)[269]$:The Fourier transform and the Zak transform of $f \in L^2(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}dx \quad \text{and} \quad Zf(x,\omega) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-ik\omega},$$

respectively. Recall that a family of functions $\{f_k : k \in \mathbb{Z}\}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \ ,$$

for every $f \in \mathcal{H}$. The numbers A and B are called the lower and upper frame bounds, respectively. A frame that ceases to be a frame when any one of its elements is removed is called an exact frame. It is well known that exact frames and Riesz bases are identical. First, we study average sampling with regular sampling points.

Lemma $(1.2.2)[269]$: Let V_0 be a closed subspace of $L^2(\mathbb{R})$ and $\{\varphi(.-k) : k \in \mathbb{Z}\}$ be a frame for V_0 with bounds A and B. Suppose that φ is continuous and $\sum_{k\in\mathbb{Z}}|\varphi(x-k)|^2 \leq L < +\infty$. Then for any frame $\{S_k : k \in \mathbb{Z}\}$ of

$$V_0 \ , \quad \sum_{k\in\mathbb{Z}} |S_k(x)|^2 \ \text{ is bounded on } \mathbb{R}.$$

Proof : Let $\{\tilde{\varphi}(.-k) : k \in \mathbb{Z}\}$ be the dual frame of $\{\varphi(.-k) : k \in \mathbb{Z}\}$. Then for any $f \in V_0$, $f(x) = \sum_{k\in\mathbb{Z}}\langle f, \tilde{\varphi}(.-k)\rangle\varphi(x-k)$. Therefore,

$$\|f\|_\infty^2 \leq \sup_x \sum_{k\in\mathbb{Z}} |\langle f, \tilde{\varphi}(.-k)\rangle|^2 \sum_{k\in\mathbb{Z}} |\varphi(x-k)|^2 \leq \frac{L}{A}\|f\|_2^2 \quad .$$

Suppose that $\{\boldsymbol{S_k(x)} : \boldsymbol{k} \in \mathbb{Z}\}$ is of upper frame bound M . Then for any $x \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} |S_k(\boldsymbol{x})|^2 = \sup_{\|c\|_2=1} \left|\sum_{k\in\mathbb{Z}} c_k S_k(\boldsymbol{x})\right|^2$$

$$\leq \sup_{\|c\|_2=1} \frac{L}{A} \left\|\sum_{k\in\mathbb{Z}} c_k S_k\right\|_2^2$$

$$\leq \frac{LM}{A} .$$

This completes the proof.

Theorem $(1.2.3)[269]$: Let $\hat{S}(\omega) = (\hat{\varphi}_N(\omega))/(Z_{\varphi_{N+1}}(N/2,\omega))$. Then $\{S(.-k)\}$ is a Riesz basis for V_N and for any $f \in V_N$,

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f, u(.-k)\rangle S(x-k) \ , \tag{31}$$

where $u(x) = \chi_{[N/2-1,N/2]}(x)$ and the convergence ***is*** both in $L^2(\mathbb{R})$ and uniform *on* $\mathbb{R}$.

Proof : Let $\overset{2}{\varphi}_N(\omega) = (\hat{\varphi}_N(\omega))/\sum_{k\in\mathbb{Z}}|\hat{\varphi}_N(\omega + 2k\pi)|^2$ and $\hat{h}(\omega) = \overline{Z_{\varphi_{N+1}}\left(\frac{N}{2},\omega\right)}\overset{2}{\varphi}_N(\omega)$. Then $\{\varphi_N(.-k)\}$ and $\bar{\varphi}_N(.-k)\}$ are dual Riesz bases for V_N. Since $Z_{\varphi_{N+1}}(N/2,\omega)$ has no zero on $[-\pi,\pi]$ (see [271,249]), it is easy to check that $\{h(.-k)\}$ and $\{S(.-k)\}$ are also dual Riesz bases for V_N. On the other hand, for any $f \in V_N$, suppose that $\hat{f}(\omega) = C(\omega) + \hat{\varphi}_N(\omega)$ for some $C(\omega) = \sum_{k\in\mathbb{Z}} c_k\, e^{-ik\omega} \in L^2[-\pi,\pi]$. Then

$$\langle f, h(.-k)\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\overline{\hat{h}(\omega)}e^{\,ik\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} C(\omega)\hat{\varphi}_N(\omega)Z_{\varphi_{N+1}}\left(\frac{N}{2},\omega\right)\overline{\hat{\overset{2}{\varphi}}_N(\omega)}e^{\,ik\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} C(\omega)Z_{\varphi_{N+1}}\left(\frac{N}{2},\omega\right)e^{\,ik\omega}\, d\omega$$

$$= \sum_{n\in\mathbb{Z}} c_n\varphi_{N+1}\left(\frac{N}{2}+k-n\right)$$

$$= \sum_{n\in\mathbb{Z}} c_n\int_0^1 \varphi_N\left(\frac{N}{2}+k-n-x\right)dx$$

$$= \int_0^1 f\left(\frac{N}{2}+k-x\right)dx$$

$$= \langle f, u\,(.-k)\rangle\,.$$

Hence, for any $f \in V_N$,

$$f(x) = \sum_{k\in\mathbb{Z}}\langle f, h(.-k)\rangle S(x-k) = \sum_{k\in\mathbb{Z}}\langle f, u\,(.-k)\rangle S(x-k)\,.$$

By Lemma (1.2.2) , the above series is convergent uniformly on $\mathbb{R}$.

Theorem (1.2.4)[269] : Suppose that $\{x_k\}$ is a real sequence such that $0 < \alpha \le x_{k+1} - x_k + 1/ \le \beta < 1$, for some two constants α and β. Then there is a frame $\{S_k\}$ *for* V_N *such* that *for any* $f \in V_N$,

$$f(x) = \sum_{k\in\mathbb{Z}}\langle f, u\,(.-x_k)\rangle S_k(x)\,,$$

Where $u(x) = \chi_{[-1/2,1/2]}(x)$ and the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$.

Proof : By [249] , $\{\varphi_N(.-k) : k \in \mathbb{Z}\}$ is a Riesz basis for V_N with bounds $A_N = \sum_k |\hat{\varphi}_N(\pi + 2k\pi)|^2$ and $B_N = 1$. Suppose that $\{\tilde{\boldsymbol{\varphi}}_{\boldsymbol{N}}(.-\boldsymbol{k}) : \boldsymbol{k} \in \mathbb{Z}\}$ is the dual Riesz basis of $\{\varphi_N(.-k) : k \in \mathbb{Z}\}$. Define

$$q(x,y) = \sum_{n\in\mathbb{Z}} \varphi_N(x-n)\tilde{\varphi}_N(y-n). \tag{32}$$

Then $q(x,y)$ is well defined for each $q(x,y) \in \mathbb{R}^2$, $q(x,.) \in V_N$, and $f(x) = \langle f, q(x,.)\rangle$ for any $f \in V_N$ (see [275]). Put $h_k(x) = \int_{x_k-1/2}^{x_k+1/2} q(x,y)\,dx$. By (32), $h_k \in V_N$. For any $f \in V_N$ we have

$$\begin{aligned}\langle f, h_k\rangle &= \int_{-\infty}^{+\infty} f(y)\,dy \int_{x_k-1/2}^{x_k+1/2} \overline{q(x,y)}\,dx \\ &= \int_{x_k-1/2}^{x_k+1/2}\int_{-\infty}^{+\infty} f(y)\overline{q(x,y)}\,dydx \qquad (33)\\ &= \int_{x_k-1/2}^{x_k+1/2} f(x)\,dx\end{aligned}$$

where Fubini's theorem is used. Suppose that $f(x) = \sum_{n\in\mathbb{Z}} c_n\varphi_N(x-n)$ and $g(x) = \sum_{n\in\mathbb{Z}} c_n\varphi_{N+1}(x-n)$ for some $\{c_n\} \in \ell^2$. Then

$$\begin{aligned}\int_{x_k-1/2}^{x_k+1/2} f(x)\,dx &= \sum_{n\in\mathbb{Z}} c_n \int_{x_k-1/2}^{x_k+1/2} \varphi_N(x-n)\,dx \\ &= \sum_{n\in\mathbb{Z}} c_n \int_0^1 \varphi_N\left(x_k+\frac{1}{2}-n-t\right)dt \qquad (34)\\ &= \sum_{n\in\mathbb{Z}} c_n\varphi_{N+1}\left(x_k+\frac{1}{2}-n\right) \\ &= g\left(x_k+\frac{1}{2}\right).\end{aligned}$$

By [243], there exist two constants $C_1, C_2 > 0$ depending only on N and $\{x_k\}$ such that

$$C_1\|g\|_2^2 \le \sum_{k\in\mathbb{Z}} \left|g\left(x_k+\frac{1}{2}\right)\right|^2 \le C_2\|g\|_2^2 \tag{35}$$

Since $\{\varphi_N(.-k) : k \in \mathbb{Z}\}$ and $\{\varphi_{N+1}(.-k) : k \in \mathbb{Z}\}$ are Riesz bases for V_N and V_{N+1} , respectively, we have

$$A_N\|c\|_2^2 \le \|f\|_2^2 \le B_N\|c\|_2^2 \tag{36}$$

and

$$A_{N+1}\|c\|_2^2 \le \|g\|_2^2 \le B_{N+1}\|c\|_2^2 \tag{37}$$

Putting (33),(34),(35),(36),(37) together, we have

$$\frac{C_1 A_{N+1}}{B_N} \|f\|_2^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, h_k\rangle|^2 \leq \frac{C_2 B_{N+1}}{A_N} \|f\|_2^2$$

Hence, $\{ h_k)$ is a frame for V_N . Let $\{S_k\}$ be the dual frame.

Then for any $F \in V_N$,

$$f(X) = \sum_{k\in\mathbb{Z}} \langle f, h_k\rangle S_k(x) = \sum_{k\in\mathbb{Z}} \langle f, u\,(.- x_k)\rangle S_k(x)$$

By Lemma (1.2.2) , the above series is convergent uniformly on $\mathbb{R}$.

Chapter 2

Channeled Sampling and Sampling Theorems

First, we give a single channel sample formula in V_0 , which extends results by G. G. Walter and W. Chen and S. Itoh. We then find necessary and sufficient conditions for two-channel sampling formula to hold in V_1. We show that some subspaces may not have a regular point. We also present a reconstruction algorithm which is slightly different from the known one but is more efficient. We study the aliasing error and prove that every smooth square integrable function can be approximated by its sampling series .

Section(2.1) Translation Invariant Subspaces and Channeled Sampling

The classical Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem [264] states that if a signal $f(t)$ with finite energy is band-limited with the bandwidth π , then it can be completely reconstructed from its discrete values by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{sin\pi(t-n)}{\pi(t-n)}$$

which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$, which has been extended in many directions (see [244], [265] and [267]). In 1992, G. G. Walter [268] developed a sampling theorem in wavelet subspaces, noticing that the sampling function $sin\pi t/\pi t$ in the WSK theorem is a scaling function of a multi resolution analysis. He assumed that the scaling function $\phi(t)$ is a continuous real valued function with $\phi(t) = O(|t|^{-1-\epsilon})(\epsilon > 0)$ for $|t|$ large, which does not hold for $sin\pi t/\pi t$. Following G. G. Walter's work, A. J. E. M. Janssen [233] used the Zak transform to generalize Walter's work. Later, W. Chen and S. Itoh [263] extended Walter's result by requiring only the condition $\{\phi(n)\} \in l^2$ on the scaling function without any decaying property. However, there were some gaps in the proof of the main result in [263].

We first re-examine the results in [263] and then extend it to single and double channel sampling formulas in the translation invariant subspaces of a multi resolution analysis.

Definition $(2.1.1)[262]$: A function $\phi(t) \in L^2(\mathbb{R})$ is called a scaling function of a multi resolution analysis (MRA in short) $\{V_j\}$ if the closed subspaces V_j of $L^2(\mathbb{R})$,

$$V_j = \overline{span}\{\phi(2^j t - k): k \in \mathbb{Z}\}, \qquad j \in \mathbb{Z}$$

satisfy the following properties ,

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$;

(ii) $\overline{\cup V_j} = L^2(\mathbb{R})$;

(iv) $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$;

(v) $\{\phi(t-k): k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Then $\{\phi(2^j t - k): k \in \mathbb{Z}\}$ becomes (iii) $\overline{\cap V_j} = \{0\}$;

(a Riesz basis of V_j for each j). The wavelet subspace W_j is defined to be the orthogonal complement of V_j in V_{j+1} so that $V_{j+1} = V_j \oplus W_j$. Then there exists a wavelet $\psi(t) \in L^2(\mathbb{R})$ that induces a Riesz basis $\{\psi(2^j t - k): k \in \mathbb{Z}\}$ of W_j . Moreover, $\{\phi(2^j t - k), \psi(2^j t - k): k \in \mathbb{Z}\}$ forms a Riesz basis of V_{j+1} . For any $\phi(t) \in L^2(\mathbb{R})$, we let

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \phi(t) e^{-it\xi}\, dt \quad \text{and}$$

$$\mathcal{F}^{-1}(\hat{\phi})(t) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{\,it\xi}\, d\xi$$

be the Fourier and inverse Fourier transforms of $\phi(t)$ and $\hat{\phi}(\xi)$ respectively. For a measurable function $f(t)$ on a set $X \subset \mathbb{R}$, we let

$$\|f(t)\|_0 = \sup_{|E|=0} \inf_{X/E} |f(t)| \ \text{and} \ \|f(t)\|_\infty = \inf_{|E|=0} \sup_{X/E} |f(t)|$$

be the essential infimum of $|f(t)|$ on X and the essential supremum of $|f(t)|$on X respectively.

Proposition (2.1.2)[262] : (see [231]) Let $\phi(t) \in L^2(\mathbb{R})$. Then

(i) $\{\phi(t - k): k \in \mathbb{Z}\}$ is a Bessel sequence if and only if there is a constant $B > 0$ such that

$$\sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 \le B, \qquad a.e.\ in\ [0,2\pi];$$

(ii) $\{\phi(t - k): k \in \mathbb{Z}\}$ is a Riesz sequence if and only if there are constants $B \ge A > 0$ such that

$$A \le \sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 \le B, \qquad a.e.\ in\ [0,2\pi].$$

We call A and B lower and upper Riesz bounds for a Riesz sequence $\{\phi(t - k): k \in \mathbb{Z}\}$ respectively. For later use we give a corollary of Proposition (2.1.2) .

Corollary (2.1.3)[262] : Let $\phi(t) \in L^2(\mathbb{R}), M(\xi) \in L^\infty(\mathbb{R})$, and

$$C(\phi)(t) := \mathcal{F}^{-1}(\hat{\phi}M)(t) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \hat{\phi}(\xi) M(\xi) e^{\,it\xi}\, d\xi \ \text{. Then}$$

(i) $\{C(\phi)(t - k): k \in \mathbb{Z}\}$ is a Bessel sequence if $\{\phi(t - k): k \in \mathbb{Z}\}$ is a Bessel sequence.

(ii) $\{C(\phi)(t - k): k \in \mathbb{Z}\}$ is a Riesz sequence if $\{\phi(t - k): k \in \mathbb{Z}\}$*is a* Riesz sequence and $\|M(\xi)\|_0 > 0$.

Proof : (i) Let $\{\phi(t - k): k \in \mathbb{Z}\}$ be a Bessel sequence with

$$\sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 \le B, \qquad a.e.\ in\ [0,2\pi].$$

Then

$$\sum_k \left|\widehat{C(\phi)}(\xi + 2k\pi)\right|^2 = \sum_k \left|\hat{\phi}(\xi + 2k\pi)M(\xi + 2k\pi)\right|^2$$

$$\leq \sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 \|M(\xi)\|_\infty^2 \leq B\|M(\xi)\|_\infty^2, a.e.\ in\ [0,2\pi]$$

so that $\{C(\phi)(t-k): k \in \mathbb{Z}\}$ is a Bessel sequence by Proposition (2.1.2).

(ii) Let $\{\phi(t-k): k \in \mathbb{Z}\}$ be a Riesz sequence with bounds A and B. Then, as in (i) we have

$$A\|M(\xi)\|_0^2 \leq \sum_k \left|\widehat{C(\phi)}(\xi + 2k\pi)\right|^2$$

$$= \sum_k \left|\hat{\phi}(\xi + 2k\pi)M(\xi + 2k\pi)\right|^2 \leq B\|M(\xi)\|_\infty^2$$

so that $\{C(\phi)(t-k): k \in \mathbb{Z}\}$ is a Riesz sequence by Proposition (2.1.2).

In this section we give a single channel sampling in V_0, which extends results in G. G. Walter [268] and W. Chen and S. Itoh [263].

Lemma (2.1.4)[262]: [231] *Let* $\phi(t) \in L^2(\mathbb{R})$ be such that $\{\phi(t-k): k \in \mathbb{Z}\}$ is a Bessel sequence. Then, for any $\{c_k\} \in l^2$, $\sum_k c_k\phi(t-k)$ converges in $L^2(\mathbb{R})$ and

$$\mathcal{F}\left(\sum_k c_k\phi(t-k)\right) = \sum_k c_k e^{-ik\xi}\hat{\phi}(\xi) = \left(\sum_k c_k e^{-ik\xi}\right)\hat{\phi}(\xi)\ .$$

Let $\mathcal{F}^*$ be the discrete Fourier transform on $l^p(p = 1,2)$ defined by

$\mathcal{F}^*(\{c_k\})(\xi) = \sum_k c_k e^{-ik\xi}$ Then, $\mathcal{F}^*(\{c_k\})(\xi)$ belongs to $C[0,2\pi]$ or $L^2[0,2\pi]$ if $\{c_k\} \in l^1$ or l^2 respectively. We denote $\mathcal{F}^*(\{\phi(k)\})(\xi)$ by $\hat{\phi}^*(\xi)$ for $\phi(t) \in L^2(\mathbb{R})$ when $\phi(k)(k \in \mathbb{Z})$ are well defined.

Lemma (2.1.5)[262] : If $\{a_k\},\{b_k\} \in l^2$, and $\mathcal{F}^*(\{a_k\})(\xi) \in L^\infty[0,2\pi]$, then

$$\left\{\sum_j a_j b_{k-j}\right\} \in l^2 \quad \text{and}$$

$$\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi) = \mathcal{F}^*\left(\left\{\sum_j a_j b_{k-j}\right\}\right)(\xi)\ .$$

Proof : Since $\mathcal{F}^*(\{a_k\})(\xi) \in L^\infty[0,2\pi]$ and $\mathcal{F}^*(\{b_k\})(\xi) \in L^2[0,2\pi]$, $\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi) \in L^2[0,2\pi]$. Hence we can expand $\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi)$ into its Fourier series $\sum_n c_n e^{-in\xi}$ in $L^2[0,2\pi]$, where

$$c_n = \frac{1}{2\pi}\langle \mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{b_k\})(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]}$$

$$= \frac{1}{2\pi}\langle \sum_k a_k e^{-ik\xi}, \left(\sum_k \overline{b_k} e^{ik\xi}\right) e^{-in\xi}\rangle_{L^2[0,2\pi]}$$

$$= \frac{1}{2\pi} \langle \sum_k a_k e^{-ik\xi}, \sum_k \overline{b_{n-k}} e^{-ik\xi} \rangle_{L^2[0,2\pi]} = \sum_k a_n b_{n-k}$$

by Parseval's identity. Hence the conclusion follows.

Theorem(2.1.6)[262] : Suppose that $\phi(t)$is a scaling function for an MRA $\{V_j\}$ such that $\phi(n)$'s are well defined and $\{\phi(n)\} \in l^2$. Then, there exists $S(t) \in V_0$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of V_0 and

$$f(t) = \sum_n f(n) S(t-n) \ in \ L^2(\mathbb{R}) \ , \qquad f(t) \in V_0 \tag{1}$$

if and only if $0 < \|\hat{\phi}^*(\xi)\|_0 \leq \|\hat{\phi}^*(\xi)\|_\infty < \infty$. In this case, we have $\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}$.

Proof : Assume $0 < \alpha = \|\hat{\phi}^*(\xi)\|_0 \leq \beta = \|\hat{\phi}^*(\xi)\|_\infty < \infty$. Then $\frac{1}{|\hat{\phi}^*(\xi)|} \leq \frac{1}{\alpha} \ a.e. \ in \ [0, 2\pi]$ so that $\frac{1}{\hat{\phi}^*(\xi)} \in [0, 2\pi]$. Let $\frac{1}{\hat{\phi}^*(\xi)} = \sum_k c_k e^{-ik\xi}$ be its Fourier series, where $\{c_k\} \in l^2$ and set $F(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}$. Then $F(\xi) \in L^2(\mathbb{R})$ and

$$F(\xi) = \left(\sum_k c_k e^{-ik\xi} \right) \hat{\phi}(\xi) = \sum_k \left(c_k e^{-ik\xi} \hat{\phi}(\xi) \right)$$

by Lemma (2.1.4) . Hence $S(t) = \mathcal{F}^{-1}(F)(t) = \sum_k c_k \phi(t-k) \in V_0$. Now, we show that $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence. Since $\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}$, we have

$$\frac{A_\phi}{\beta^2} \leq \sum_k \left| \hat{S}(\xi + 2k\pi) \right|^2 = \frac{\sum_k \left| \hat{\phi}(\xi + 2k\pi) \right|^2}{\left| \hat{\phi}^*(\xi) \right|^2} \leq \frac{B_\phi}{\alpha^2} \ a.e. \ in \ [0, 2\pi]$$

where A_ϕ and B_ϕ are Riesz bounds for $\{\phi(t-k) : k \in \mathbb{Z}\}$. Hence $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence by Proposition (2.1.2) (ii). For any $f(t) = \sum_k c_k \phi(t-k) \in V_0$ where $\{a_k\} \in l^2$, we have by Lemma (2.1.4),

$$\hat{f}(\xi) = \left(\sum_k a_k e^{-ik\xi} \right) \hat{\phi}(\xi) = \left(\sum_k a_k e^{-ik\xi} \right) \hat{\phi}^*(\xi) \hat{S}(\xi)$$

Since $\|\hat{\phi}^*(\xi)\|_\infty < \infty$,

$$\left(\sum_k a_k e^{-ik\xi} \right) \hat{\phi}^*(\xi) = \sum_n f(n) e^{-in\xi} \tag{2}$$

Where $\{f(t) = \sum_k a_k \phi(t-k)\} \in l^2$ by Lemma (2.1.5) . Hence

$$\hat{f}(\xi) = \left(\sum_n f(n) e^{-in\xi} \right) \hat{S}(\xi) = \sum_n \left(f(n) e^{-in\xi} \hat{S}(\xi) \right) \tag{3}$$

by Lemma (2.1.4) since $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz sequence. Thus we have (1) by taking inverse Fourier transform on (3). Then $\overline{span}\{S(t-k) : k \in \mathbb{Z}\} = V_0$ so that $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0. Conversely assume that there exists $S(t) \in V_0$ such that $\{S(t-k):$ $: k\in\mathbb{Z}$ is a Riesz basis of $V0$ and (1) holds. In particular $\phi t=k\phi nSt-n$ so that

$$\hat{\phi}(\xi)=\sum_n\left(\phi(n)e^{-in\xi}\hat{S}(\xi)\right)=\left(\sum_n f(n)e^{-in\xi}\right)\hat{S}(\xi)=\hat{\phi}^*(\xi)\hat{S}(\xi)\ . \qquad (4)$$

Hence

$$\sum_k\left|\hat{\phi}(\xi+2k\pi)\right|^2=\left|\hat{\phi}^*(\xi)\right|^2\sum_k\left|\hat{S}(\xi+2k\pi)\right|^2$$

so that

$$\frac{A_\phi}{BS}\le\left|\hat{\phi}^*(\xi)\right|^2\le\frac{B_\phi}{AS}\ \ a.e.\ \ in\ \ [0,2\pi]$$

where (A_ϕ, B_ϕ) and (AS, BS) are Riesz bounds for $\{\phi(t-k): k\in\mathbb{Z}\}$ and $\{S(t-k): k\in\mathbb{Z}\}$ respectively. Thus $0<\left\|\hat{\phi}^*(\xi)\right\|_0\le\left\|\hat{\phi}^*(\xi)\right\|_\infty<\infty$.
If $\{\phi(n)\}\in l^1$, then $\hat{\phi}^*(\xi)=\hat{\phi}^*(\xi+2\pi)\in C[0,2\pi]$ so that

$$\left\|\hat{\phi}^*(\xi)\right\|_0=\min_{[0,2\pi]}\left|\hat{\phi}^*(\xi)\right| \quad and \quad \left\|\hat{\phi}^*(\xi)\right\|_\infty=\max_{[0,2\pi]}\left|\hat{\phi}^*(\xi)\right|.$$

Hence we have:

Corollary $(2.1.7)[262]$: Suppose that $\phi(t)$ is a scaling function for an MRA $\{V_j\}$ such that $\phi(n)$'s are well defined and $\{\phi(n)\}\in l^1$. Then there exists $S(t)\in V_0$ such that $\{S(t-k): k\in\mathbb{Z}\}$ is a Riesz basis of $V_0$and (1) holds if and only if $\hat{\phi}^*(\xi)\neq 0$ in $[0,2\pi]$. In [268], G. Walter requires that $\phi(t)$ is a continuous on $\mathbb{R}$ and $\phi(t)=O(|t|^{-1-\epsilon})(\epsilon>0)$ for $|t|$ large. Then $\{\phi(n)\}\in l^1$ so that the results in [268] is a special case of Corollary (2.1.7) . On the other hand , W.Chen and S. Itoh [263] claimed: under the same hypothesis as in Theorem (2.1.6) , there exists $S(t)\in V_0$ with which (1) holds if and only if $\hat{\phi}^*(\xi)^{-1}\in L^2[0,2\pi]$. However,there are some gaps in the argumentsin [263]. In the proof of sufficiency for Theorem 1 in [263] , $\left(\sum_k a_k e^{-ik\xi}\right)\hat{\phi}^*(\xi)$ belongs to $L^1[0,2\pi]$ but not necessarily in $L^2[0,2\pi]$ (unless $\left\|\hat{\phi}^*(\xi)\right\|_\infty<\infty$) so that $\{f(n)\}=\{\sum_k a_k\phi(n-k)\}\in l^\infty$ and the (2) becomes only a formal Fourier series expansion of a function in $L^1[0,2\pi]$ (see [263]). Even if $\hat{\phi}^*(\xi)^{-1}\in L^2[0,2\pi]$ and $\left\|\hat{\phi}^*(\xi)\right\|_\infty<\infty$, (2) holds but (3) may not hold since $\{S(t-k): k\in\mathbb{Z}\}$ is not a Bessel sequence unless $\left\|\hat{\phi}^*(\xi)\right\|_0>0$. Also, in the proof of necessity, we may not have (4) unless $\{S(t-k): k\in\mathbb{Z}\}$ is a Riesz sequence.We may extend Theorem (2.1.6) by the same reasoning to a single channel sampling as:

Theorem $(2.1.8)[262]$: Let $M(\xi)$ be a measurable function on $\mathbb{R}$ such that $0<\|M(\xi)\|_0\le\|M(\xi)\|_\infty<\infty$. Suppose that $\phi(t)$ is a scaling function for an MRA $\{V_j\}$ such that $C(\phi)(n)$*'s* are well defined and $\{C(\phi)(n)\}\in l^2$ where

$C(\phi)(t) := \mathcal{F}^{-1}(\hat{\phi}M)(t)$: Then, there exists $S(t) \in V_0$ such that $\{S(t-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0 and

$$f(t) = \sum_n C(f)(n)S(t-n) \quad in \quad L^2(\mathbb{R})\,, \qquad f(t) \in V_0 \tag{5}$$

if and only if $0 < \|\widehat{C(\phi)}^*\|_0(\xi) \le \|\widehat{C(\phi)}^*\|_\infty < \infty$. In this case, we have $\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\widehat{C(\phi)}^*}$.

Example $(2.1.9)[262]$: Shannon function $\phi(t) = \frac{sin\pi t}{\pi t}$ is continuous on $\mathbb{R}$ and $\{\phi(n)\} = \{\delta_{n0}\} \in l^1$. Since $\hat{\phi}^*(\xi) = 1$ on $[0,2\pi]$ but $|\phi(t)| = O(|t|^{-1})$ for $|t|^{-1}$ large, the WSK sampling theorem is not covered by [263] or [268] but follows Corollary (2.1.7).

Example $(2.1.10)[262]$: Let $\phi(t)$ be the continuous scaling function considered by Chen and Itoh (Example 3 in [263]) such that

$$\hat{\phi}(\xi) = \begin{cases} -1\,, & -4\pi \le \xi < -2\pi\,, \\ 1\,, & -2\pi \le \xi < 0\,, \\ \xi^s\,, & 0 \le \xi < 2\pi\,, \\ 0\,, & otherwise \end{cases}$$

with $0 < s < \frac{1}{2}$. Then we can easily see that $\phi(n) = O\left(\frac{1}{n}\right)$ for $|n|$ large so that $\{\phi(n)\} \in l^2 \backslash l^1$. Even though $\hat{\phi}^*(\xi) = \xi^s$ on $[0,2\pi]$ so that $\hat{\phi}^*(\xi) \in L^\infty[0,2\pi]$ and $\hat{\phi}^*(\xi)^{-1} \in L^2[0,2\pi]$, $\|\hat{\phi}^*(\xi)\|_0 = 0$ so that we cannot expect a sampling formula from $\phi(t)$ suggested either by Theorem (2.1.6).

Example $(2.1.11)[262]$: Let $M(\xi) = e^{-ia\xi}$ with $0 < a < 1$ so that $1 = \|e^{-ia\xi}\|_0 = \|e^{-ia\xi}\|_\infty$, and $\phi(t)$ a scaling function Then $C(\phi)(t) = \phi(t-a)$ and $\{\phi(n-a)\} \in l^2$ so that $Z_\phi(a,\xi) = \sum_n \phi(n-a)e^{-in\xi} \in L^2[0,2\pi]$. Hence if $0 < \|Z_\phi(a,\xi)\|_0 \le \|Z_\phi(a,\xi)\|_\infty < \infty$, then we obtain the shift-sampling $f(t) = \sum_n f(n-a)S(t-n)$.

We let $\phi(t)$ be a scaling function for an MRA $\{V_j\}$ and $\psi(t)$ the associated wavelet. Let $M_1(\xi)$ and $M_2(\xi)$ be in $L^\infty(\mathbb{R})$ and $C_i(f)(t) = \mathcal{F}^{-1}(\hat{f}M_i)(t)$ for $i = 1,2$ and $f(t) \in L^2(\mathbb{R})$.

Assume that $C_i(\phi)(n)'s$ and $C_i(\psi)(n)'s$ are well defined and $\{C_i(\phi)(n)\}$ and $\{C_i(\psi)(n)\}$ are in l^2. Let

$$A_{11}(\xi) = \sum_n C_1(\phi)(n)e^{-in\xi}\,, \qquad A_{12}(\xi) = \sum_n C_2(\phi)(n)e^{-in\xi}\,,$$

$$A_{21}(\xi) = \sum_n C_1(\psi)(n)e^{-in\xi}\,, \qquad A_{22}(\xi) = \sum_n C_2(\psi)(n)e^{-in\xi}\,,$$

and $A(\xi) := [A_{ij}(\xi)]_{i,j=1}^2$. Then $A_{ij}(\xi) \in L^2[0,2\pi]$and $A_{ij}(\xi) = A_{ij}(\xi + 2\pi)$.

We always assume that $\|A_{ij}(\xi)\|_\infty < \infty$ for $i,j = 1,2$ and $detA(\xi) = 0\ a.e.$

in $[0,2\pi]$. Set $A^{-1}(\xi) = B(\xi) = [B_{ij}(\xi)]_{i,j=1}^{2}$. Then $B(\xi) = B(\xi + 2\pi)$ is well defined a.e. in $\mathbb{R}$.

Lemma $(2.1.12)[262]$: Let $\lambda_{1,B}(\xi)$ and $\lambda_{2,B}(\xi)$ be eigenvalues of $B(\xi)B(\xi)^{*} with\ \lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi)$. If $\|detA(\xi)\|_0 > 0$, then

$$0 < \|\lambda_{1,B}(\xi)\|_0 \le \|\lambda_{2,B}(\xi)\|_\infty < \infty .$$

Proof : Since $B(\xi)B(\xi)^{*}$ is nonsingular Hermitian a.e. in $[0,2\pi]$,

$$0 < \lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi) \ a.e.\ in\ [0,2\pi].$$

Since $A_{ij}(\xi) \in L^\infty[0,2\pi]$ and $\|detA(\xi)\|_0 > 0$, all entries of $B(\xi)$ and so $B(\xi)B(\xi)^{*}$ are also in $L^\infty[0,2\pi]$ so that the characteristic equation of $B(\xi)B(\xi)^{*}$ is of the form $\lambda(\xi)^2 + f(\xi)\lambda(\xi) + g(\xi) = 0$ where $f(\xi)$ and $g(\xi)$ are real-valued functions in $L^\infty[0,2\pi]$.

Hence $0 < \|\lambda_{2,B}(\xi)\|_\infty < \infty$. Since

$$\lambda_{1,B}(\xi)\lambda_{2,B}(\xi) = det[B(\xi)B(\xi)^{*}] = |detA(\xi)|^{-2},$$

$$\|detA(\xi)\|_\infty^{-2} \le \lambda_{1,B}(\xi)\lambda_{2,B}(\xi) \le \|detA(\xi)\|_0^{-2} \ a.e.\ in\ [0,2\pi]$$

so that

$$\|detA(\xi)\|_\infty^{-2}\|\lambda_{2,B}(\xi)\|_\infty^{-1} \le \lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi) \le \|\lambda_{2,B}(\xi)\|_\infty \ a.e.\ in\ [0,2\pi].$$

For any $\phi(t) \in L^2(\mathbb{R})$,

$$\|\phi\|_{L^2(\mathbb{R})} = \|\hat{\phi}\|_{L^2(\mathbb{R})} = \int_0^{2\pi} \sum_k |\hat{\phi}(\xi + 2k\pi)|^2 d\xi$$

so that $\{\hat{\phi}(\xi + 2k\pi)\}_{k\in\mathbb{Z}} \in l^2$ for a.e. in $[0,2\pi]$.

Definition$(2.1.13)[262]$: For any $\phi(t)$ and $\psi(t)$ in $L^2(\mathbb{R})$, we call

$$G(\xi) = \begin{bmatrix} \sum_k |\hat{\phi}(\xi + 2k\pi)|^2 & \sum_k \hat{\phi}(\xi + 2k\pi)\overline{\hat{\psi}(\xi + 2k\pi)} \\ \sum_k \overline{\hat{\phi}(\xi + 2k\pi)}\hat{\psi}(\xi + 2k\pi) & \sum_k |\hat{\psi}(\xi + 2k\pi)|^2 \end{bmatrix}$$

the Gramian of $\{\phi, \psi\}$, which is well defined a.e. in $[0,2\pi]$.

Then as a Hermitian matrix, $G(\xi)$ has real eigenvalues.

Theorem$(2.1.14)[262]$: In [266] let $\lambda_1, G(\xi)$ and $\lambda_2, G(\xi)$ be eigenvalues of the Gramian $G(\xi) of \{\phi, \psi\}$ such that $\lambda_1, G(\xi) \le \lambda_2, G(\xi)$.

Then $\{\phi(t - k), \psi(t - k) : k \in \mathbb{Z}\}$ is a Riesz sequence if and only if there are constants $B \ge A > 0$ such that

$$A \le \lambda_1, G(\xi) \le \lambda_2, G(\xi) \le B \ a.e.\ in\ [0,2\pi]. \tag{6}$$

Lemma $(2.1.15)[262]$: Set $\begin{bmatrix} F_1(\xi) \\ F_2(\xi) \end{bmatrix} = B(\xi) \begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix}$ on $\mathbb{R}$. If $\|detA(\xi)\|_0 > 0$, then $F_i(\xi) \in L^2(\mathbb{R}), S_i(t) = \mathcal{F}^{-1}(F_i)(t) \in V_1$ for $i = 1,2$, and $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}$ is a Riesz sequence.

Proof : Since $B_{ij}(\xi) \in L^\infty(\mathbb{R}), F_i(\xi) = B_{i1}(\xi)\hat{\phi}(\xi) + B_{i2}(\xi)\hat{\psi}(\xi) \in L^2(\mathbb{R})$ for $i = 1,2$. Since $B_{ij}(\xi) = B_{ij}(\xi + 2\pi) \in L^2[0,2\pi]$, we may expand $B_{ij}(\xi)$ into its Fourier series $B_{ij}(\xi) = \sum_k b_{ij,k}e^{-ik\xi}$ Where $\{b_{ij,k}\} \in l^2$. Then by Lemma (2.1.4),

$$F_i(\xi) = \left(\sum_k b_{i1,k}e^{-ik\xi}\right)\hat{\phi}(\xi) + \left(\sum_k b_{i2,k}e^{-ik\xi}\right)\hat{\psi}(\xi)$$
$$= \sum_k \left(b_{i1,k}e^{-ik\xi}\hat{\phi}(\xi) + b_{i2,k}e^{-ik\xi}\hat{\psi}(\xi)\right)$$

so that

$$S_i(t) = \mathcal{F}^{-1}(F_i)(t) = \sum_k \left(b_{i1,k}\phi(t-k) + b_{i2,k}\psi(t-k)\right) \in V_1 \ .$$

Let

$$S(\xi) = \begin{bmatrix} \sum_k \left|\hat{S}_1(\xi + 2k\pi)\right|^2 & \sum_k \hat{S}_1(\xi + 2k\pi)\overline{\hat{S}_2(\xi + 2k\pi)} \\ \sum_k \overline{\hat{S}_1(\xi + 2k\pi)}\hat{S}_2(\xi + 2k\pi) & \sum_k \left|\hat{S}_2(\xi + 2k\pi)\right|^2 \end{bmatrix}$$

be the Gramian of $\{S_1, S_2\}$ and $\lambda_{1,S}(\xi) \le \lambda_{2,S}(\xi)$ the eigenvalues of $S(\xi)$. Then we have by periodicity of $B(\xi)$, $S(\xi) = B(\xi)G(\xi)B(\xi)^*$. Let $U_S(\xi)$ and $U_G(\xi)$ be unitary matrices, which diagonalize $S(\xi)$ and $G(\xi)$respectively , $i.e.$,

$$S(\xi) = U_S(\xi)\begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi) \end{bmatrix} U_S(\xi)^*$$

and

$$G(\xi) = U_G(\xi)\begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi) \end{bmatrix} U_G(\xi)^* .$$

Then

$$\begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi) \end{bmatrix} = R(\xi)\begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi) \end{bmatrix} R(\xi)^*$$

where

$$R(\xi) = U_S(\xi)^* B(\xi) U_G(\xi) = \begin{bmatrix} R_{11}(\xi) & R_{12}(\xi) \\ R_{21}(\xi) & R_{22}(\xi) \end{bmatrix}$$

so that

$$\lambda_{1,S}(\xi) = \lambda_{1,G}(\xi)|R_{11}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{12}(\xi)|^2 \ , \tag{7}$$
$$\lambda_{2,S}(\xi) = \lambda_{1,G}(\xi)|R_{21}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{22}(\xi)|^2 \ . \tag{8}$$

On the other hand,

$$R(\xi)\,R(\xi)^* = U_S(\xi)^* B(\xi) B(\xi)^* U_S(\xi)$$
$$= U_S(\xi)^* U_B(\xi)\begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi) \end{bmatrix} U_B(\xi)^* U_S(\xi) , \tag{9}$$

where $U_B(\xi)$ is the unitary matrix such that

$$B(\xi)B(\xi)^* = U_B(\xi)\begin{bmatrix}\lambda_{1,B}(\xi) & 0\\ 0 & \lambda_{2,B}(\xi)\end{bmatrix}U_B(\xi)^*$$

with $\lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi)$. Set $U_S(\xi)^*U_B(\xi) = \left[D_{ij}(\xi)\right]_{i,j=1}^2$ which is also a unitary matrix. Then we have from diagonal entries of both sides of (9),

$$|R_{11}(\xi)|^2 + |R_{12}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{11}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{12}(\xi)|^2\,, \qquad (10)$$

$$|R_{21}(\xi)|^2 + |R_{22}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{21}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{22}(\xi)|^2\,. \qquad (11)$$

Then we have from (6), (7), (8), (11) and (11)

$$\lambda_{1,S}(\xi) \ge \lambda_{1,G}(\xi)(|R_{11}(\xi)|^2 + |R_{12}(\xi)|^2) \ge \lambda_{1,G}(\xi)\lambda_{1,B}(\xi)\ a.e.\ in\ [0,2\pi],$$

$$\lambda_{2,S}(\xi) \ge \lambda_{2,G}(\xi)(|R_{21}(\xi)|^2 + |R_{22}(\xi)|^2) \ge \lambda_{2,G}(\xi)\lambda_{2,B}(\xi)\ a.e.\ in\ [0,2\pi],$$

since $|D_{11}(\xi)|^2 + |D_{12}(\xi)|^2 = |D_{21}(\xi)|^2 + |D_{22}(\xi)|^2 = 1\ a.e.\ in\ [0,2\pi]$. Hence

$$0 < \left\|\lambda_{1,G}(\xi)\right\|_0\left\|\lambda_{1,B}(\xi)\right\|_0 \le \lambda_{1,S}(\xi) \le \lambda_{2,S}(\xi)$$
$$\le \left\|\lambda_{2,G}(\xi)\right\|_\infty\left\|\lambda_{2,B}(\xi)\right\|_\infty < 1\ a.e.\ in\ [0,2\pi]$$

by Lemma (2.1.12) so that $\{S_i(t-n) : i = 1,2\ and\ n \in \mathbb{Z}\}$ is a Riesz sequence by Theorem (2.1.14) .

Theorem $(2.1.16)[262]$:Under the above setting, there exist $S_i(t) \in V_1 (i = 1,2)$ such that $\{S_i(t-n) : i = 1,2$ and $n \in \mathbb{Z}\}$ is a Riesz basis of V_1 for which two-channel sampling formula

$$f(t) = \sum_n C_1(f)(n)S_1(t-n) + \sum_n C_2(f)(n)S_2(t-n)\,, \in V_1 \qquad (12)$$

holds if and only if $\|detA(\xi)\|_0 > 0\ on\ [0,2\pi]$. In this case

$$S_i(t) = \mathcal{F}^{-1}\left(B_{i1}(\xi)\hat{\phi}(\xi) + B_{i2}(\xi)\hat{\psi}(\xi)\right)(t) \quad \text{for} \quad i = 1,2\ . \qquad (13)$$

Proof : Assume $\|detA(\xi)\|_0 > 0$ on $[0,2\pi]$ and define $S_i(t)$ by (13). Then $S_i(t) \in V_1 (i = 1,2)$ and $\{S_i(t-n): i = 1,2\ and\ n \in \mathbb{Z}\}$ is a Riesz sequence by Lemma (2.1.15) . For any $f(t) \in V_1$

$$f(t) = \sum_k c_{1,k}\phi(t-k) + \sum_k c_{2,k}\psi(t-k) \qquad (14)$$

where $\left\{c_{i,k}\right\}_k \in l^2$ for $i = 1,2$ since $\{\phi(t-k), \psi(t-k) : k\,n \in \mathbb{Z}\}$ is a Riesz basis for V_1. Applying the bounded linear operator $C_i(.)$ to (14) gives

$$C_i(f)(t) = \sum_k c_{1,k}C_i(\phi)(t-k) + \sum_k c_{2,k}C_i(\psi)(t-k). \qquad (15)$$

On the other hand, we have by Lemma (2.1.4)

$$\hat{f}(\xi) = \left(\sum_k c_{1,k}e^{-ik\xi}\right)\hat{\phi}(\xi) + \left(\sum_k c_{2,k}e^{-ik\xi}\right)\hat{\psi}(\xi)$$

Since

$$\begin{bmatrix}\hat{\phi}(\xi)\\ \hat{\psi}(\xi)\end{bmatrix} = A(\xi)\begin{bmatrix}\hat{S}_1(\xi)\\ \hat{S}_2(\xi)\end{bmatrix},$$

$$\begin{aligned}\hat{f}(\xi) &= \left[\left(\sum_k c_{1,k}e^{-ik\xi}\right)A_{11}(\xi) + \left(\sum_k c_{2,k}e^{-ik\xi}\right)A_{21}(\xi)\right]\hat{S}_1(\xi)\\ &\quad + \left[\left(\sum_k c_{1,k}e^{-ik\xi}\right)A_{12}(\xi) + \left(\sum_k c_{2,k}e^{-ik\xi}\right)A_{22}(\xi)\right]\hat{S}_2\\ &= \sum_n\left(\sum_k c_{1,k}C_1(\phi)(n-k) + \sum_k c_{2,k}C_1(\psi)(n-k)\right)e^{-in\xi}\hat{S}_1(\xi)\\ &\quad + \sum_n\left(\sum_k c_{1,k}C_2(\phi)(n-k) + \sum_k c_{2,k}C_2(\psi)(n-k)\right)e^{-in\xi}\hat{S}_2(\xi)\\ &= \sum_n C_1(f)(n)e^{-in\xi}\hat{S}_1(\xi) + \sum_n C_2(f)(n)e^{-in\xi}\hat{S}_2(\xi) \qquad (16)\end{aligned}$$

by (15), where $\{C_i(f)(n)\} \in l^2$ $(i = 1,2)$ by Lemma (2.1.5) . Taking inverse Fourier transform on (16) gives (12), which implies $V_1 = \overline{span}\{S_i(t-n): i = 1,2 \ and\ n \in \mathbb{Z}\}$ so that $\{S_i(t-n): i = 1,2$ and $n \in \mathbb{Z}\}$ is a Riesz basis of V_1. Conversely assume that there exist $S_i(t) \in V_1$ $(i = 1,2)$ such that $\{S_i(t-n): i = 1,2 \ and\ n \in \mathbb{Z}\}$ is a Riesz basis of V_1 and (12) holds. In particular,

$$\phi(t) = \sum_n C_1(\phi)(n)S_1(t-n) + \sum_n C_2(\phi)(n)S_2(t-n),$$

$$\phi(t) = \sum_n C_1(\psi)(n)S_1(t-n) + \sum_n C_2(\psi)(n)S_2(t-n).$$

By taking Fourier transform and using Lemma (2.1.4) , we have

$$\begin{bmatrix}\hat{\phi}(\xi)\\ \hat{\psi}(\xi)\end{bmatrix} = A(\xi)\begin{bmatrix}\hat{S}_1(\xi)\\ \hat{S}_2(\xi)\end{bmatrix}.$$

We then have as in the proof of Lemma (2.1.15)

$$G(\xi) = A(\xi)S(\xi)A(\xi)^*,$$

where $G(\xi)$ and $S(\xi)$ are Gramians of $\{\phi,\psi\}$and $\{S_1,S_2\}$ respectively. Hence $detG(\xi) = det\,S(\xi)|detA(\xi)|^2$ so that

$$|detA(\xi)|^2 = \frac{detG(\xi)}{det\,S(\xi)} = \frac{\lambda_{1,G}(\xi)\lambda_{2,G}(\xi)}{\lambda_{1,S}(\xi)\lambda_{2,S}(\xi)} \geq \frac{\lambda_{1,G}(\xi)^2}{\lambda_{2,S}(\xi)^2} \quad a.e.\ \text{in} \quad [0,2\pi],$$

where $\lambda_{1,G}(\xi) \leq \lambda_{2,G}(\xi)$ and $\lambda_{1,S}(\xi) \leq \lambda_{2,S}(\xi)$are eigenvalues of $G(\xi)$ and $S(\xi)$ respectively. Therefore,

$$|detA(\xi)| \geq \frac{\lambda_{1,G}(\xi)}{\lambda_{2,S}(\xi)} \geq \frac{\|\lambda_{1,G}(\xi)\|_0}{\|\lambda_{2,S}(\xi)\|_\infty} \quad a.e.\ \text{in} \quad [0,2\pi]$$

so that $\|detA(\xi)\|_0 > 0$ since both $\phi(t-n), \psi(t-n) : n \in \mathbb{Z}$ and

$\{S_i(t-n): i = 1,2 \ and \ n \in \mathbb{Z}\}$ are Riesz sequences.

Example $(\mathbf{2.1.17})[\mathbf{262}]$: For Haar orthogonal system $\phi(t) = \chi_{[0,1)}(t)$ and $\psi(t) = \chi_{\left[0,\frac{1}{2}\right)}(t) - \chi_{\left[\frac{1}{2},1\right)}(t)$. Let $M_1(\xi) = 1$ and $M_2(\xi) = e^{-ia\xi}$ with $0 < a \le \frac{1}{2}$. Then $\boldsymbol{C_1(\phi)(t)} = \boldsymbol{\chi_{[0,1)}(t)}$, $\boldsymbol{C_2(\phi)(t)} = \boldsymbol{\chi_{[0,1)}(t-a)}$, $\boldsymbol{C_1(\psi)(t)} = \boldsymbol{\chi_{\left[0,\frac{1}{2}\right)}(t) - \chi_{\left[\frac{1}{2},1\right)}(t)}$ and $C_2(\psi)(t) = \chi_{\left[0,\frac{1}{2}\right)}(t-a) - \chi_{\left[\frac{1}{2},1\right)}(t-a)$. Then $A(\xi) = \begin{bmatrix} 1 & e^{-i\xi} \\ 1 & -e^{-i\xi} \end{bmatrix}$ so that $|detA(\xi)| = 2$, which satisfies the condition of Theorem (2.1.16) . Hence we have a sampling formula

$$f(t) = \sum_n f(n)S_1(t-n) + \sum_n f(n-a)S_2(t-n)$$

Corollary $(\mathbf{2.1.18})[\mathbf{296}]$: If $\{a_k\}, \{a_k+\epsilon_k\} \in l^2$, and $\mathcal{F}^*(\{a_k\})(\xi) \in L^\infty[0,2\pi]$, then

$$\left\{\sum_j a_j\left(a_{k-j}+\epsilon_{k-j}\right)\right\} \in l^2 \quad \text{and}$$

$$\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{a_k+\epsilon_k\})(\xi) = \mathcal{F}^*\left(\left\{\sum_j a_j\left(a_{k-j}+\epsilon_{k-j}\right)\right\}\right)(\xi) .$$

Proof : Since $\mathcal{F}^*(\{a_k\})(\xi) \in L^\infty[0,2\pi]$ and $\mathcal{F}^*(\{a_k+\epsilon_k\})(\xi) \in L^2[0,2\pi]$, $\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{a_k+\epsilon_k\})(\xi) \in L^2[0,2\pi]$. Hence we can expand $\mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{a_k+\epsilon_k\})(\xi)$ into its Fourier series $\sum_n c_n e^{-in\xi}$ in $L^2[0,2\pi]$, where

$$c_n = \frac{1}{2\pi}\langle \mathcal{F}^*(\{a_k\})(\xi)\mathcal{F}^*(\{a_k+\epsilon_k\})(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]}$$

$$= \frac{1}{2\pi}\langle \sum_k a_k e^{-ik\xi} , \left(\sum_k \overline{(a_k+\epsilon_k)e^{\,ik\xi}}\right) e^{-in\xi}\rangle_{L^2[0,2\pi]}$$

$$= \frac{1}{2\pi}\langle \sum_k (a_k+\epsilon_k)e^{-ik\xi} , \sum_k \overline{(a_{n-k}+\epsilon_{n-k})e^{-ik\xi}}\rangle_{L^2[0,2\pi]} = \sum_k a_n(a_{n-k}+\epsilon_{n-k})$$

by Parseval's identity. Hence the conclusion follows.

Corollary $(\mathbf{2.1.19})[\mathbf{296}]$: Suppose $\lambda_{n,B}(\xi)$ and $\lambda_{n+1,B}(\xi)$ are eigenvalues of $B(\xi)^2$ with $\lambda_{n,B}(\xi) \le \lambda_{n+1,B}(\xi)$. If $\|detA(\xi)\|_0 > 0$, then

$$0 < \left\|\lambda_{n,B}(\xi)\right\|_0 \le \left\|\lambda_{n+1,B}(\xi)\right\|_\infty < \infty .$$

Proof **:** Since $B(\xi)^2$ is nonsingular Hermitian a.e. in $[0,2\pi]$,

$$0 < \lambda_{n,B}(\xi) \le \lambda_{n+1,B}(\xi) \ a.e. \ in \ [0,2\pi].$$

Since $A_{ij}(\xi) \in L^\infty[0,2\pi]$ and $\|detA(\xi)\|_0 > 0$, all entries of $B(\xi)$ and so $B(\xi)^2$ are also in $L^\infty[0,2\pi]$ so that the characteristic equation of $B(\xi)^2$ is of the form $\lambda(\xi)^2 + f(\xi)\lambda(\xi) + g(\xi) = 0$ where $f(\xi)$ and $g(\xi)$ are real-valued functions in $L^\infty[0,2\pi]$. Hence $0 < \left\|\lambda_{n+1,B}(\xi)\right\|_\infty < \infty$. Since

$$\lambda_{n,B}(\xi)\lambda_{n+1,B}(\xi) = det[B(\xi)B(\xi)^*] = |detA(\xi)|^{-2} ,$$

$$\|detA(\xi)\|_{\infty}^{-2} \leq \lambda_{n,B}(\xi)\lambda_{n+1,B}(\xi) \leq \|detA(\xi)\|_{0}^{-2} \ a.e. \ in \ [0,2\pi]$$

so that

$$\|detA(\xi)\|_{\infty}^{-2}\left\|\lambda_{n+1,B}(\xi)\right\|_{\infty}^{-1} \leq \lambda_{n,B}(\xi) \leq \lambda_{n+1,B}(\xi) \leq \left\|\lambda_{n+1,B}(\xi)\right\|_{\infty} \ a.e. \ \text{in} \ [0,2\pi].$$

For any $\phi(t) \in L^2(\mathbb{R})$,

$$\|\phi\|_{L^2(\mathbb{R})} = \left\|\hat{\phi}\right\|_{L^2(\mathbb{R})} = \int_0^{2\pi} \sum_k \left|\hat{\phi}(\xi + 2k\pi)\right|^2 d\xi$$

so that $\left\{\hat{\phi}(\xi + 2k\pi)\right\}_{k\in\mathbb{Z}} \in l^2$ for a.e. in $[0,2\pi]$.

Section(2.2) Multivariate Shift Invariant and Sampling Theorems

We consider shift invariant subspaces of $L^2(\mathbb{R}^d)$ of the form

$$V_0 = \overline{span}\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}, \quad (17)$$

where $\phi \in L^2(\mathbb{R}^d)$ and A is a $d \times d$ matrix , det$A \neq 0$. Fix some $x_0 \in \mathbb{R}^d$. If there is a frame $\{S(\cdot - A_n) : n \in \mathbb{Z}^d\}$ for V_0 such that for any $f \in V_0$,

$$f(x) = \sum_{n \in \mathbb{Z}^d} f(x_0 + A_n)S(x - A_n) , \quad (18)$$

where the convergence is both in $L^2(\mathbb{R}^d)$ and pointwisely on $\mathbb{R}^d$, then we say that x_0 is a regular point for V_0 and the (regular) sampling theorem holds on V_0. Note that a function in $L^2(\mathbb{R}^d)$ is only defined almost everywhere. For the sampled values $f(x_0 + A_n)$ to make sense, we require that f be continuous near $x_0 + A_n$. Similarly, we can consider irregular sampling, i.e., $x_0 + A_n$is replaced by $x_0 + A_n + a_n$ for some $a_n \in \mathbb{R}^d$ satisfying $\|a_n\|_\infty \leq \delta$,$\forall n$. Now some problems arise:

(i) Does every V_0 of the form (17) have a regular point?

(ii) Characterize all regular points for given V_0.

(iii) Find conditions for the irregular sampling theorem to hold.

There are many results concerning the last two problems. For example, see [242, 243, 244, 250, 251, 252, 233, 255, 234, 240]. Most of them are focused on univariate functions.

We study the sampling theorem for multivariate functions. We first prove that every subspace V_0 of the form (17) must have a frame like $\{(\cdot - A_n) : n \in \mathbb{Z}^d\}$ for some $\psi \in V_0$. Then we give an equivalent condition for f to be continuous near sampling points $x_0 + A_n$. We give a characterization of regular points for V_0 and give a representation of $S(x)$. Also, we illustrate that there is some shift invariant subspace which has no regular point. We study irregular sampling for V_0 and show that if the generating function ϕ satisfies some conditions, then we can find some $\delta > 0$ such that every $f \in V_0$ can bereconstructed from irregular sampled values $f(x_0 + A_n + a_n)$ provided $\|a_n\|_\infty \leq \delta$. Our result covers many of known ones, such as Kadec's 1/4-theorem and some results in [233, 257, 240]. We also present a reconstruction algorithm which is slightly different from the known one but more efficient.

Definition (2.2.1)[241] : A stands for some fixed $d \times d$ matrix with det $A \neq 0$. The Fourier transform and the Zak transform of $f \in L^2(\mathbb{R}^d)$ are defined by $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-i2\pi\langle x,\omega\rangle}\, dx$ and

$$(Z_A f)(x,\omega) = \sum_{n \in \mathbb{Z}^d} f(x + A_n) e^{-i2\pi\langle A_n,\omega\rangle}, \qquad \text{respectively.}$$

We call a function f is $A\mathbb{Z}^d$-periodic if $f = f(\cdot + A_n)$, a.e., $n \in \mathbb{Z}^d$.

$$[\hat{f},\hat{g}](\omega) = \sum_{n \in \mathbb{Z}^d} \hat{f}(\omega + A^{-t}n)\bar{\hat{g}}(\omega + A^{-t}n), \text{where } A^{-t} \text{ is the}$$

inverse of A^t , the transpose of A. $E_\phi = \{\omega : [\hat{\phi},\hat{\phi}](\omega) > 0\}$.

Example (2.2.2)[241]: Let $\phi(x) = \frac{\sin\pi x}{\pi x}$. then $V_0 = \{f : \operatorname{supp}\hat{f} \subset [-1/2,1/2]\}$. Since $(Z\phi)(x,\omega) = e^{i2\pi x\omega}(Z\phi)(\omega,-x)$ thanks to Lemma (2.2.11), it is easy to check that $|(Z\phi)(x,\omega)| = 1, a.e.$

Example (2.2.3)[241] : Subspaces generated by centered B-splines defined by

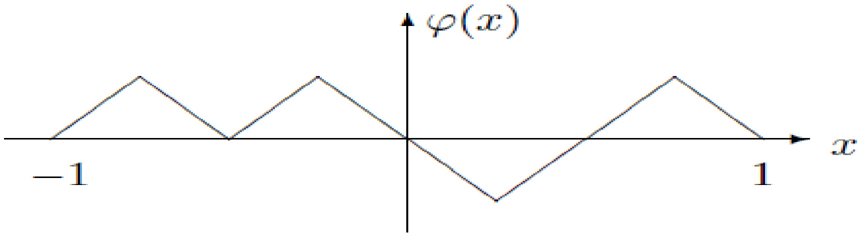
$$\widehat{\phi_m}(\omega) = \left(\frac{\sin\pi\omega}{\pi\omega}\right)^{m+1}, \qquad m \geq 1,$$

or equivalently , $\phi_m = \chi_{[-1/2,1/2]} * \cdots * \chi_{[-1/2,1/2]}$ (m convolutions).

Let $V_m = \overline{span}\{\phi_m(\cdot - n) : n \in \mathbb{Z}\}$. It can be shown (see [249, 240]) that $\{\phi_m(\cdot - n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_m and $(Z\phi_m)(0,\omega)$ has no zero on $\mathbb{R}$. Therefore, 0 is a regular point. On the other hand, since ϕ_m is symmetric with respect to $x = 0$, it is easy to see that $(Z\phi_m)(\frac{1}{2},\pi) = 0$. Hence $x = \frac{1}{2}$ is not a regular point (see [233]).

Example (2.2.4)[241] :

$$\text{Let } \phi(x) = \begin{cases} x + 1, & -1 < x \leq -3/4, \\ -x - 1/2, & -3/4 < x \leq 1/2, \\ x + 1/2, & -1/2 < x \leq 1/4, \\ -x, & -1/4 < x \leq 1/4, \\ x - 1/2, & 1/4 < x \leq 3/4, \\ 1 - x, & 3/4 < x \leq 1, \\ 0, & otherwise. \end{cases}$$



Then $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is an orthogonal basis for the space V_0 it spans . On the other hand, it is easy to check that

$$(Z\phi)(x,\omega) = \begin{cases} -x(1 - e^{i2\pi\omega}), & 0 < x \le 1/4, \\ (x - 1/2)(1 - e^{i2\pi\omega}), & 1/4 < x \le 1/2, \\ (x - 1/2)(1 + e^{i2\pi\omega}), & 1/2 < x \le 3/4, \\ (1 - x)(1 + e^{i2\pi\omega}), & 3/4 < x \le 1. \end{cases}$$

Hence $(Z\phi)(x,0) = 0$ for $0 < x \le 1/2$ and $(Z\phi)(x,\pi) = 0$ for $1/2 \le x \le 1$. Therefore , V_0 has no regular point.

Lemma(2.2.5)[241]: Suppose that $\phi \in L^2(\mathbb{R}^d)$ and $V_0 = \overline{span}\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$.

(i) $\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$ is a frame for V_0 with bounds C_1 and C_2 if and only if $C_1\chi_{E_\phi}(\omega) \le |detA|^{-1}[\hat{\phi},\hat{\phi}](\omega) \le C_2\chi_{E_\phi}(\omega), a.e.$

Moreover , $\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$ is a Riesz basis for V_0 if and only if the above inequalities are satisfied with $E_\phi = \mathbb{R}^d$.

(ii) $\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$ is a Bessel sequence with upper bound C_2 if and only if $|detA|^{-1}[\hat{\phi},\hat{\phi}](\omega) \le C_2\chi_{E_\phi}(\omega), a.e.$

Lemma (2.2.6)[241] : Suppose that $\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$ is a frame for V_0 . Let

$$\hat{\tilde{\phi}}(\omega) = \begin{cases} \hat{\phi}(\omega)/[\hat{\phi},\hat{\phi}](\omega), & \omega \in E_\phi, \\ 0, & otherwise. \end{cases}$$

Then $\{|detA|\tilde{\phi}(\cdot - A_n): n \in \mathbb{Z}^d\}$ is the dual frame of $\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$.

Lemma (2.2.7)[241] : Let $\phi \in L^2(\mathbb{R}^d)$ and V_0 be defined by (17). Then we have

$$V_0 = \{f \in L^2(\mathbb{R}^d) : \hat{f}(\omega) = C(\omega)\,\hat{\phi}(\omega), C(\omega) \text{ is } A^{-t}\mathbb{Z}^d - \text{periodic}\}. \qquad (19)$$

Moreover, there is some $\psi \in V_0$ such that $\{\psi(\cdot - A_n): n \in \mathbb{Z}^d\}$ is a frame for V_0.

Lemma (2.2.8)[241] : Suppose that $\{\phi(\cdot - A_n): n \in \mathbb{Z}^d\}$ is a frame for some V_0. Fix some $x_0 \in \mathbb{R}^d$ and $\delta > 0$. Then the following two assertions are equivalent:

(i) For any $\{c_n : n \in \mathbb{Z}^d\} \in \ell^2, \sum_{n\in\mathbb{Z}^d} c_n\phi(x - A_n)$ converges pointwisely to a function continuous on

$E = \cup_{n\in\mathbb{Z}^d}\{x : \| x - x_0 - A_n\|_\infty \le \delta \}$.

(ii) ϕ is continuous on E and $\sup_{\| x-x_0\|_\infty\le\delta} \sum_{n\in\mathbb{Z}^d} |\phi(x - A_n)|^2 < \infty$.

Proof : (i)⇒(ii). Obviously, ϕ is continuous on E. Since $\sum_{n\in\mathbb{Z}^d} c_n\phi(x - A_n)$ is convergent for any $\{c_n : n \in \mathbb{Z}^d\} \in \ell^2$, we have

$$\sum_{n\in\mathbb{Z}^d} |\phi(x - A_n)|^2 < +\infty, \forall x \in E.$$

Define

$$\Lambda_x c = \sum_{n\in\mathbb{Z}^d} c_n\phi(x - A_n)\,, \forall c = \{c_n : n \in \mathbb{Z}^d\} \in \ell^2.$$

Then Λ_x is a bounded linear functional on ℓ^2 with the norm

$$\left(\sum_{n\in\mathbb{Z}^d}|\phi(x-A_n)|^2\right)^{/2}.$$

Note that for fixed $c, \Lambda_x c$ is continuous on $x_0 + [-\delta,\delta]^d$, we have

$$\sup_{\|x-x_0\|_\infty\le\delta}|\Lambda_x c| < +\infty.$$

By the Banach-Steinhaus theorem , $\sup_{\|x-x_0\|_\infty\le\delta}\|\Lambda_x c\| < +\infty$. Consequently ,

$$\sup_{\|x-x_0\|_\infty\le\delta}\sum_{n\in\mathbb{Z}^d}|\phi(x-A_n)|^2 < \infty.$$

(ii)⇒(i). By the Cauchy inequality , $\sum_{n\in\mathbb{Z}^d} c_n\phi(x-A_n)$ is convergent uniformly on E for any $c \in \ell^2$. Now the conclusion follows.

Lemma $(2.2.9)[241]$: Suppose that ϕ is continuous and $\sum_{n\in\mathbb{Z}^d}|\phi(x-A_n)|^2$ is bounded on $\mathbb{R}^d$. Then we have for any $x \in \mathbb{R}^d$ and almost every $\omega \in \mathbb{R}^d\backslash E_\phi$, $(Z_A\phi)(x,\omega) = 0$.

Proof : Let $C(\omega) = 1 - \chi_{E_\phi}(\omega) = \sum_{n\in\mathbb{Z}^d} c_n e^{-i2\pi\langle A_n,\omega\rangle}$. Then $C(\omega)\hat{\phi}(\omega) = 0$.Therefore ,

$$\sum_{n\in\mathbb{Z}^d} c_n\phi(x-A_n) = 0, \qquad \forall x \in \mathbb{R}^d .$$

It follows that

$$\int_{A^{-t}[-1/2,1/2]^d\backslash E_\phi}|(Z_A\phi)(x,\omega)|^2\,d\omega = \int_{A^{-t}[-1/2,1/2]^d}|C(\omega)|^2|(Z_A\phi)(x,\omega)|^2\,d\omega$$

$$= |\det A|^{-1}\sum_{m\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d}c_n\phi(x+A_m-A_n)\right|^2 = 0.$$

Hence $(Z_A\phi)(x,\omega) = 0$ for almost every $\omega \in A^{-t}[-1/2,1/2]^d\backslash E_\phi$. Now the conclusion follows since $(Z_A\phi)(x,\omega)$ is $A^{-t}\mathbb{Z}^d$-periodic with respect to ω.

Lemma $(2.2.10)[241]$: Let $\{\phi(\cdot - A_n) : n\in\mathbb{Z}^d\}$ and $\{S_n : n\in\mathbb{Z}^d\}$ be two frames for some V_0. Suppose that ϕ is continuous and $\sum_{n\in\mathbb{Z}^d}|\phi(x-A_n)|^2 \le L < +\infty$. Then $S_n(x)$ is continuous and $\sum_{n\in\mathbb{Z}^d}|S_n(x)|^2$ is bounded on $\mathbb{R}^d$.

Proof : Let $\{|\det A|\tilde{\phi}(\cdot - A_n) : n \in \mathbb{Z}^d\}$ be the dual frame of $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$. For any $f \in V_0$, we have

$f = \sum_{n\in\mathbb{Z}^d}|\det A|\langle f,\tilde{\phi}(\cdot - A_n)\rangle\phi(\cdot - A_n)$. By Lemma (2.2.8), f is continuous. Moreover ,

$$|f(x)|^2 \le \sum_{n\in\mathbb{Z}^d}\left|\langle f, |\det A|\tilde{\phi}(\cdot - A_n)\rangle\right|^2\sum_{n\in\mathbb{Z}^d}|\phi(x-An)|^2$$

$$\le \frac{L}{C_1}\|f\|_2^2 , \qquad \forall x\ 2\ \mathbb{R}^d,$$

where C_1 is the lower frame bound of $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$. Let C_2'

be the upper frame bound of $\{S_n : n \in \mathbb{Z}^d\}$. Then

$$\sum_{n \in \mathbb{Z}^d} |S_n(x)|^2 = \sup_{\|c\|_2=1} \left| \sum_{n \in \mathbb{Z}^d} c_n S_n(x) \right|^2 \leq \sup_{\|c\|_2=1} \frac{L}{C_1} \left\| \sum_{n \in \mathbb{Z}^d} c_n S_n \right\|_2^2$$

$$\leq \frac{LC_2'}{C_1}, \qquad \forall x.$$

Lemma $(2.2.11)[241]$: For any $f \in L^2(\mathbb{R}^d)$, we have

$$(Z_A f)(x,\omega) = e^{i2\pi\langle x,\omega\rangle} |\, detA|^{-1} \big(Z_{A^{-t}} \hat{f}\big)(\omega, -x), \qquad a.e. \tag{20}$$

Proof : By the Poisson summation formula (see [253]), we have

$$\sum_{n \in \mathbb{Z}^d} f(A_n) = |\, detA|^{-1} \sum_{n \in \mathbb{Z}^d} \hat{f}(A^{-t}n), \forall f \in S(\mathbb{R}^d),$$

where $S(\mathbb{R}^d)$ is the Schwartz class which consists of infinitely continuously differentiable functions with rapidly decaying at the infinity. Substituting $f(\cdot + x)e^{-i2\pi\langle \cdot + x,\omega\rangle}$ for f, we get

$$\sum_{n \in \mathbb{Z}^d} f(A_n + x)\, e^{-i2\pi\langle A_n + x,\omega\rangle} = |\, detA|^{-1} \sum_{n \in \mathbb{Z}^d} \hat{f}(A^{-t}n + \omega)\, e^{i2\pi\langle A^{-t}n, x\rangle}.$$

Now (20) follows. Since $S(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, (20) holds for any $f \in L^2(\mathbb{R}^d)$.

Lemma $(2.2.12)[241]$: Suppose that $\phi \in L^2(\mathbb{R}^d)$ and $V_0 = \overline{span}\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$. Then $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$ is a frame for V_0 with bounds C_1 and C_2 if and only if

$$C_1 \chi_{E_\phi}(\omega) \leq \int_{A[0,1]^d} |(Z_A\phi)(x,\omega)|^2\, dx \leq C_2 \chi_{E_\phi}(\omega), \qquad a.e.$$

Also , $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$ is Bessel sequence if and only if the right hand inequality holds.

Proof : By Lemma (2.2.11), we have

$$\int_{A[0,1]^d} |(Z_A\phi)(x,\omega)|^2\, dx = \int_{A[0,1]^d} |\, detA|^{-2} |(Z_{A-t}\hat{\phi})(\omega,-x)|^2\, dx$$

$$= |\, detA|^{-1} [\, \hat{\phi}, \hat{\phi}\,](\omega).$$

Now the conclusion follows from Lemma (2.2.5) .

Theorem $(2.2.13)[241]$: Let $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$ be a frame for some V_0. Suppose that ϕ is continuous and $\sum_{n \in \mathbb{Z}^d} |\phi(x - A_n)|$ is bounded on $\mathbb{R}^d$. Then x_0 is a regular point for V_0 if and only if there are two positive constants C_1' and C_2' such that

$$C_1' \chi_{E_\phi}(\omega) \leq |Z_A\phi(x_0,\omega)| \leq C_2' \chi_{E_\phi}(\omega), a.e. \tag{21}$$

If (2.1) is satisfied , let

$$\hat{S}(\omega) = \begin{cases} \hat{\phi}(\omega)/Z_A\phi(x_0,\omega) , & \omega \in E_\phi, \\ 0, & otherwise. \end{cases} \tag{22}$$

Then(18) holds and the convergence is both in $L^2(\mathbb{R}^d)$ and uniform on $\mathbb{R}^d$.

Proof : Suppose that x_0 is a regular point. Then there is a frame $\{S(\cdot - A_n) : n \in \mathbb{Z}^d\}$ for V_0 such that

$$f(x) = \sum_{n \in \mathbb{Z}^d} f(x_0 + A_n)S(x - A_n), \forall f \in V_0 .$$

In particular ,

$$\phi(x) = \sum_{n \in \mathbb{Z}^d} \phi(x_0 + A_n)S(x - A_n).$$

Hence

$$\hat{\phi}(\omega) = (Z_A\phi)(x_0, \omega)\hat{S}(\omega).$$

Therefore, $[\hat{\phi}, \hat{\phi}](\omega) = |(Z_A\phi)(x_0, \omega)|^2[\hat{S}, \hat{S}](\omega)$.Now (21) follows by Lemmas (2.2.5) and (2.2.9). Next we prove the sufficiency. Let $\hat{S}(\omega)$ be defined by (22) and

$$\hat{\tilde{S}}(\omega) = \begin{cases} \hat{S}(\omega)\overline{Z_A\phi(x_0, \omega)}/[\hat{\phi}, \hat{\phi}](\omega), & \omega \in E_\phi, \\ 0, & otherwise. \end{cases} \tag{23}$$

Then

$$[\hat{S}, \hat{S}](\omega) = \frac{[\hat{\phi}, \hat{\phi}](\omega)}{|Z_A\phi(x_0, \omega)|^2} = \frac{1}{\left[\hat{\tilde{S}}, \hat{\tilde{S}}\right](\omega)}, \quad \omega \in E_\phi .$$

By Lemma (2.2.5) , (2.2.6) and (2.2.7) , $\{S(\cdot - A_n) : n \in \mathbb{Z}^d\}$ and $\{|\det A|\tilde{S}(\cdot - A_n) : n \in \mathbb{Z}^d\}$ are a pair of dual frames for V_0 . For any $f = \sum_{n \in \mathbb{Z}^d} c_n\phi(\cdot - A_n) \in V_0$, we have

$$\langle f, |\det A|\tilde{S}(\cdot - A_n)\rangle = \int_{n \in \mathbb{Z}^d} C(\omega)\hat{\phi}(\omega)|\det A|\overline{\hat{\tilde{S}}(\omega)}e^{i2\omega\langle A_n, \omega\rangle}\, d\omega$$

$$= |\det A| \int_{A^{-t}[-1/2,1/2]^d} C(\omega)[\hat{\phi}, \hat{\tilde{S}}](\omega)e^{i2\omega\langle A_n, \omega\rangle}\, d\omega$$

$$= |\det A| \int_{A^{-t}[-1/2,1/2]^d} (Z_A\phi)(x_0, \omega)e^{i2\omega\langle A_n, \omega\rangle}\, d\omega$$

$$= \sum_{m \in \mathbb{Z}^d} c_m\phi(x + A_n - A_m)$$

$$= f(x + A_n).$$

Hence

$$f(x) = \sum_{n \in \mathbb{Z}^d} \langle f, |\det A|\tilde{S}(\cdot - A_n)\rangle S(x - A_n) = \sum_{n \in \mathbb{Z}^d} f(x_0 + A_n)S(x - A_n).$$

The uniform convergence follows by Lemma (2.2.10).

Let us introduce the multi-index [253]: $I_d = \{(i_1, \cdots, i_d): i_k = 0 \text{ or } 1\}$,

$$\alpha = (\alpha_1, \cdots, \alpha_d), \alpha_i \geq 0, |\alpha| = \alpha_1 + \cdots + \alpha_d, x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d},$$

$(X^\beta f)(x) = x^\beta f(x) = x_1^{\beta_1} \cdots x_d^{\beta_d} f(x)$, and

$D_x^\alpha f(x) = \frac{\partial|\alpha|}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}} f(x)$. We write $D^\alpha f$ simply when there is no confusion . When we write $D^\alpha f$, we mean that $D^\alpha f$ exists almost every where in the classical sense and for any $\beta = (\beta_1,\cdots,\beta_d)$ with $\beta_i = \alpha_i$ for $i \neq k$ and $\beta_k = \alpha_k - 1 \geq 0$, $D^\beta f$ is locally absolutely continuous on almost all straight lines parallel to the k-th coordinate axis.

In the following, we give two criteria on irregular sampling and an algorithm to reconstruct a function from irregular sampled values, which is slightly different from known results but more efficient in some cases.

Example $(2.2.14)[241]$: Subspace of bandlimited functions given in Example (2.2.2) .Since $(Z\phi)(x,\omega) = e^{i2\pi x\omega}(Z\hat{\phi})(\omega,-x)$, it is easy to

check that $(Z\phi)(x,\omega) = 1, (Z\phi')(x,\omega) = 2\pi\omega, a.e., |\omega| \leq 1/2$.

So the condition $\varepsilon > 0$ in (26) is equivalent to $\delta < \frac{1}{4}$, which is just the kadec's 1/4-theorem.

Lemma$(2.2.15)[241]$: Let $f \in L^2(\mathbb{R}^d)$ be such that $D^\alpha f \in L^2(\mathbb{R}^d)$. Then

$$f(x + y) - f(x) = \sum_{\alpha\in I_d\{0\}} \int_{E_{y,\alpha}} (D^\alpha f)(x + t)\, dt,$$

where $x, y \in \mathbb{R}^d$, and $E_{y,\alpha} = \{\in \mathbb{R}^d : 0 \leq t_k \leq \alpha_k y_k \text{ or } \alpha_k y_k \leq t_k \leq 0\}$ is an $|\alpha|$-dimensional rectangle.

Lemma $(2.2.16)[241]$: If f is differentiable on $[a,b]$, $f, f_0 \in L^2[a,b]$ and there is some $c \in [a,b]$ such that $f(c) = 0$, then

$$\int_a^b |f(x)|^2\, dx \leq \frac{4\,\Delta^2}{\pi^2}\int_a^b |f'(x)|^2\, dx, \text{where } \Delta = max\{c - a, b - c\}.$$

Lemma $(2.2.17)[241]$: Suppose that E is a rectangle in $\mathbb{R}^d$ with side lengths $a = (a_1,\cdots,a_d)$ and $D^\alpha f \in L^2(E)$ for any $\alpha \in I_d$. Then for any

$$y = (y_1,\cdots,y_d) \in E, \|f - f(y)\|_{L^2(E)} \leq \sum_{\alpha\in I_d\setminus\{0\}} \frac{2^{|\alpha|}a^\alpha}{\pi^{|\alpha|}} \|D^\alpha f\|_{L^2(E)}.$$

Proof : Let $I_{d,k} = \{(i_1,\cdots,i_d) \in I_d : i_p = 0, p > k\}$ and $A_k f(x) = f(y_1,\cdots,y_k,x_{k+1},\cdots,x_d), 1 \leq k \leq d$. It suffices to prove the following inequality.

$$\|f - A_k f(y)\|_{L^2(E)} \leq \sum_{\alpha\in I_{d,k}\setminus\{0\}} \frac{2^{|\alpha|}a^\alpha}{\pi^{|\alpha|}} \|D^\alpha f\|_{L^2(E)}, \forall 1 \leq k \leq d. \quad (24)$$

For $k = 1$, we have

$$\|f - A_1 f(y)\|_{L^2(E)}^2 = \int_E |f(x_1,x_2,\cdots,x_d) - f(y_1,x_2,\cdots,x_d)|^2\, dx$$

$$= \int_F dx_2 \ \cdots \ dx_d \int_{b_1}^{a_1+b_1} |f(x_1, x_2, \cdots, x_d) - f(y_1, x_2, \cdots, x_d)|^2 \, dx_1$$

(F is some measurable subset of $\mathbb{R}^{d-1}$, and $b_1 \le y_1 \le a_1 + b_1$.)

$$\le \frac{4a_1^2}{\pi^2} \int_F dx_2 \ \cdots \ dx_d \int_{b_1}^{a_1+b_1} \left| \frac{\partial}{\partial x_1} f(x) \right|^2 dx_1$$

$$= \frac{4a_1^2}{\pi^2} \int_E \left| \frac{\partial}{\partial x_1} f(x) \right|^2 dx,$$

where Lemma (2.2.16) is used. Hence (24) holds for $k = 1$. Suppose that (24) holds for some $1 \le k < d$. Then

$$\|A_k f\|_{L^2(E)} \le \|f\|_{L^2(E)} + \|f - A_k f\|_{L^2(E)} \le \sum_{\alpha \in I_{d,k} \setminus \{0\}} \frac{2^{|\alpha|} a^\alpha}{\pi^{|\alpha|}} \|D^\alpha f\|_{L^2(E)} .$$

A similar argument shows that

$$\|A_K f - A_{K+1} f\|_{L^2(E)}^2 \le \frac{4a_{K+1}^2}{\pi^2} \int_E \left| \frac{\partial}{\partial x_{k+1}} f(y_1, \ldots, y_K, x_{k+1}, \ldots, x_d) \right|^2 dx$$

$$= \frac{4a_{K+1}^2}{\pi^2} \left\| A_K \frac{\partial}{\partial x_{k+1}} f \right\|_{L^2(E)}^2 .$$

Hence

$$\|f - A_{k+1} f\|_{L^2(E)} \le \|f - A_k f\|_{L^2(E)} + \|A_K f - A_{K+1} f\|_{L^2(E)}$$

$$\le \sum_{\alpha \in I_{d,k} \setminus \{0\}} \frac{2^{|\alpha|} a^\alpha}{\pi^{|\alpha|}} \|D^\alpha f\|_{L^2(E)} + \frac{2a_{k+1}}{\pi} \sum_{\alpha \in I_{d,k} \setminus \{0\}} \frac{2^{|\alpha|} a^\alpha}{\pi^{|\alpha|}} \left\| D^\alpha \frac{\partial}{\partial x_{k+1}} f \right\|_{L^2(E)}$$

$$= \sum_{\alpha \in I_{d,k+1} \{0\}} \frac{2^{|\alpha|} a^\alpha}{\pi^{|\alpha|}} \|D^\alpha f\|_{L^2(E)} .$$

By induction, (24) holds for any $1 \le k \le d$.

Theorem $(2.2.18)[241]$: Let $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$ be a frame for some V_0. Suppose that $D^\alpha \phi \in L^2(\mathbb{R}^d), \forall \alpha \in I_d$ and $|(Z_A D^\alpha \phi)(x, \omega)| \le L_\alpha < +\infty$, $a.e.$ Let x_0 be a regular point and (21) be satisfied for some $C_1', C_2' > 0$. If $\delta > 0$ is such that one of the following conditions is satisfied,

$$\Delta = \sum_{\alpha \in I_d \setminus \{0\}} 2^{|\alpha|/2} \delta^{|\alpha|} L_\alpha < C_1' , \tag{25}$$

or

$$\varepsilon = \inf_{\omega \in E_\phi} \left(\frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} |(Z_A \phi)(x + x_0, \omega)|^2 \, dx\omega \right)^{1/2} - \sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{4\delta}{\pi} \right)^{|\alpha|} L_\alpha > 0 , \tag{26}$$

then for any sequence $\{a_n : n \in \mathbb{Z}^d\} \subset \mathbb{R}^d$ satisfying $\|a_n\|_\infty \le \delta, \forall n,$ there is a frame $\{S_n : n \in \mathbb{Z}^d\}$ for V_0 such that

$$f(x) = \sum_{n \in \mathbb{Z}^d} f(x_0 + A_n + a_n) S_n(x)\,, \tag{27}$$

where the convergence is both in $L^2(\mathbb{R}^d)$ and uniform on $\mathbb{R}^d$.

Proof: Let $\tilde{\phi}$ be defined as in Lemma (2.2.6). Set

$$q(x, y) = \sum_{n \in \mathbb{Z}^d} |\det A| \phi(x - A_n) \tilde{\phi}(y - A_n)\,.$$

Then q(x, y) is well-defined on $\mathbb{R}^{2d}$, due to Lemma (2.2.10). Note that

$$f(x) = \langle f, q(x,\cdot)\rangle.$$

We need only to show that $\{q(x_0 + A_n + a_n,\cdot) : n \in \mathbb{Z}^d\}$ is a frame for V_0. In fact , if it is the case, then (27) holds with $\{S_n : n \in \mathbb{Z}^d\}$ being the dual frame and the uniform convergence follows by Lemma (2.2.10) . For any $f \in V_0$, let $c_n = |\det A| \langle f, \tilde{\phi}(\cdot - A_n)\rangle$. Then

$$C(\omega) = \sum_{n \in \mathbb{Z}^d} c_n e^{-i2\pi\langle A_n,\omega\rangle} = \left[\hat{f}, \hat{\tilde{\phi}}\right](\omega) = 0 \ a.e. \text{ on } \ \mathbb{R}^d \ \ E_\phi \ \text{ and}$$

$$\frac{1}{C_2}\|f\|_2^2 \le \|c\|_2^2 = \sum_{n \in \mathbb{Z}^d} |c_n|^2 \le \frac{1}{C_1}\|f\|_2^2\ ,$$

where C_1 and C_2 are the frame bounds for $\{\phi(\cdot - A_n) : n \in \mathbb{Z}^d\}$.First, we assume (25) is satisfied. By Lemma (2.2.15), we have

$$\left(\sum_{n \in \mathbb{Z}^d} |f(x_0 + A_n + a_n) - f(x_0 + A_n)|^2\right)^{1/2}$$

$$= \left(\sum_{n \in \mathbb{Z}^d} \left|\sum_{\alpha\in I_d\setminus\{0\}} \int_{E_{a_n,\alpha}} (D^\alpha f)(x_0 + A_n + x)\, dx\right|^2\right)^{1/2}$$

$$\le \sum_{\alpha\in I_d\setminus\{0\}} \left(\sum_{n \in \mathbb{Z}^d} \left|\int_{E_{a_n,\alpha}} (D^\alpha f)(x_0 + A_n + x)\, dx\right|^2\right)^{1/2}$$

$$\le \sum_{\alpha\in I_d\setminus\{0\}} \left(\sum_{n \in \mathbb{Z}^d} \delta^{|\alpha|} \int_{E_{\delta,\alpha}} |(D^\alpha f)(x_0 + A_n + x)|^2\, dx\right)^{1/2}$$

$$\left(E_{\delta,\alpha} = \{x : -\alpha_k\delta \le x_k \le \alpha_k\delta\}\right)$$

$$= \sum_{\alpha\in I_d\setminus\{0\}} \left(\delta^{|\alpha|} \int_{E_{\delta,\alpha}} \int_{A^{-t}[-1/2,1/2]^d} |\det A| \cdot |(Z_A D^\alpha f)(x_0 + x, \omega)|^2\, d\omega\, dx\right)^{1/2}$$

$$= \sum_{\alpha\in I_d\setminus\{0\}} \left(\boldsymbol{\delta^{|\alpha|}} \int_{E_{\delta,\alpha}} \int_{A^{-t}[-1/2,1/2]^d} |\boldsymbol{detA}| \cdot |C(\omega)(\boldsymbol{Z_A D^\alpha \phi})(\boldsymbol{x_0}+\boldsymbol{x},\boldsymbol{\omega})|^2 \, \boldsymbol{d\omega\, dx} \right)^{\boldsymbol{1/2}}$$

$$\leq \sum_{\alpha\in I_d\setminus\{0\}} 2^{|\alpha|/2} \delta^{|\alpha|} L_\alpha \|c\|_2$$

$$= \Delta \, \|c\|_2 \quad . \tag{28}$$

Since

$$\sum_{n\,\in\,\mathbb{Z}^d} |f(x_0 + A_n)|^2 = |\, detA| \int_{A^{-t}[-1/2,1/2]^d} |C(\omega)(Z_A D^\alpha f)(x_0,\omega)|^2 \, d\omega \geq {C_1'}^2 \|c\|_2^2 \,, \tag{29}$$

we have

$$\sum_{n\,\in\,\mathbb{Z}^d} |f(x_0 + A_n + a_n)|^2 \geq (C_1' - \Delta)^2 \|c\|_2^2 \geq \frac{(C_1' - \Delta)^2}{C_2} \|f\|_2^2 \,. \tag{30}$$

Similarly we can prove that

$$\sum_{n\,\in\,\mathbb{Z}^d} |f(x_0 + A_n + a_n)|^2 \leq \frac{(C_1' + \Delta)^2}{C_2} \|f\|_2^2 \ . \tag{31}$$

Hence $\{q(x_0 + A_n + a_n, \cdot) : n \in \mathbb{Z}^d\}$ is a frame for V_0 .

Next we assume that (26) is satisfied. Put $E_n = x_0 + A_n + [-\delta,\delta]^d$. By Lemma (2.2.17), we have

$$\left(\sum_{n\,\in\,\mathbb{Z}^d} \frac{1}{(2\delta)^d} \int_{E_n} |f(x) - f(x_0 + A_n + a_n)|^2 \, dx \right)^{1/2}$$

$$\leq \left(\sum_{n\,\in\,\mathbb{Z}^d} \frac{1}{(2\delta)^d} \left(\sum_{\alpha\in I_d\setminus\{0\}} \left(\frac{4\delta}{\pi}\right)^{|\alpha|} \|D^\alpha f\|_{L^2(E_n)} \right)^2 \right)^{1/2}$$

$$\leq \frac{1}{(2\delta)^{d/2}} \sum_{\alpha\in I_d\setminus\{0\}} \left(\frac{4\delta}{\pi}\right)^{|\alpha|} \left(\sum_{n\,\in\,\mathbb{Z}^d} \|D^\alpha f\|^2_{L^2(E_n)} \right)^{1/2}$$

$$= \frac{1}{(2\delta)^{d/2}} \sum_{\alpha\in I_d\setminus\{0\}} \left(\frac{4\delta}{\pi}\right)^{|\alpha|} \left(\int_{[-\delta,\delta]^d} \sum_{n\,\in\,\mathbb{Z}^d} |(D^\alpha f)(x_0 + x + A_n)|^2 \, dx \right)^{1/2}$$

$$= \frac{\boldsymbol{|detA|^{1/2}}}{(2\delta)^{d/2}} \sum_{\alpha\in I_d\setminus\{0\}} \left(\frac{4\delta}{\pi}\right)^{|\alpha|} \left(\int_{[-\delta,\delta]^d} \int_{E^0} |(Z_A D^\alpha f)(x + x_0,\omega)|^2 \, dx d\omega \right)^{1/2}$$

$$= \frac{|\boldsymbol{detA}|^{1/2}}{(\boldsymbol{2\delta})^{d/2}} \sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{\boldsymbol{4\delta}}{\boldsymbol{\pi}}\right)^{|\alpha|} \left(\int_{[-\delta,\delta]^d} \int_{E^0} |\boldsymbol{C(\omega)(Z_A D^\alpha \phi)(x + x_0, \omega)}|^2 \, \boldsymbol{dxd\omega} \right)^{1/2}$$

$$\leq \sum_{\alpha \in I_d \setminus \{0\}} \left(\frac{4\delta}{\pi}\right)^{|\alpha|} L_\alpha \|c\|_2 \qquad (E^0 = A^{-t}[-1/2,1/2]^d).$$

$$= \Delta_1 \|c\|_2$$

On the other hand, we see from (26) that

$$\sum_{n \in \mathbb{Z}^d} \frac{1}{(2\delta)^d} \int_{E_n} |f(x)|^2 \, dx = \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} \sum_{n \in \mathbb{Z}^d} |f(x_0 + x + A_n)|^2 \, dx$$

$$= \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} \int_{E^0} |detA| \cdot |(Z_A f)(x + x_0, \omega)|^2 \, d\omega \, dx$$

$$= \frac{1}{(2\delta)^d} \int_{[-\delta,\delta]^d} \int_{E^0} |detA| \cdot |C(\omega)(Z_A \phi)(x + x_0, \omega)|^2 \, d\omega \, dx$$

$$\geq (\varepsilon + \Delta_1)^2 \|c\|_2^2 \ .$$

Hence

$$\left(\sum_{n \in \mathbb{Z}^d} |f(x_0 + A_n + a_n)|^2 \right)^{1/2}$$

$$= \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{(2\delta)^d} \int_{E_n} |f(x_0 + A_n + a_n)|^2 \, dx \right)^{1/2}$$

$$= \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{(2\delta)^d} \int_{E_n} |f(x) - (f(x) - f(x_0 + A_n + a_n))|^2 \, dx \right)^{1/2}$$

$$\geq \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{(2\delta)^d} \left| \left(\int_{E_n} |f(x)|^2 \, dx \right)^{\frac{1}{2}} - \left(\int_{E_n} |f(x) - f(x_0 + A_n + a_n)|^2 \, dx \right)^{1/2} \right|^2 \right)^{\frac{1}{2}}$$

$$\geq \left| \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{(2\delta)^d} \int_{E_n} |f(x)|^2 \, dx \right)^{\frac{1}{2}} - \left(\sum_{n \in \mathbb{Z}^d} \frac{1}{(2\delta)^d} \int_{E_n} |f(x) - f(x_0 + A_n + a_n)|^2 \, dx \right)^{\frac{1}{2}} \right|^2$$

$$\geq \varepsilon \|c\|_2 \geq \frac{\varepsilon}{C_2^{1/2}} \|F\|_2 \ .$$

Proposition $(2.2.19)[241]$: [see 251] Suppose that $\{\phi_n : n \in \mathbb{Z}\}$ is a frame for some Hilbert space H with bounds A and B. Let ρ be a constant such that $0 < \rho < \frac{2}{B}$. For any $f \in \mathcal{H}$, define

$$Sf = \sum_{n \in \mathbb{Z}^d} \langle f, \phi_n \rangle \phi_n \ , \qquad f_0 = Sf \ ,$$

$$f_k = f_{k-1} + S(f - f_{k-1}) \ , \quad k \geq 1,$$

Then $\lim_{k\to\infty} f_k = f$ and $\|f - f_k\| \leq \gamma^{k+1} kfk$, where $\gamma = \max\{|1-\rho A|, |1-\rho B|\} < 1$. The relaxation parameter ρ plays an important role in the above algorithm. If we know the exact value of the frame bounds, then $\rho = 2/(A+B)$ is the best choice since is minimized in this case. For Theorem (2.2.18), the operator S can be defined by

$$(Sf)(x) = \rho \sum_{n \in \mathbb{Z}^d} f(x_0 + A_n + a_n) q(x_0 + A_n + a_n, x) .$$

If (25) is satisfied, the frame bounds for $\{q(x_0 + A_n + a_n, \cdot), n \in \mathbb{Z}^d\}$ are $\frac{(C_1' - \Delta)^2}{C_2}$ and $\frac{(C_2' - \Delta)^2}{C_1}$. So we can choose $\rho = 2/\left(\frac{(C_1' - \Delta)^2}{C_2} + \frac{(C_2' - \Delta)^2}{C_1}\right)$, which leads to

$$\gamma = \frac{C_2(C_2' + \Delta)^2 - C_1(C_1' - \Delta)^2}{C_2(C_2' + \Delta)^2 + C_1(C_1' - \Delta)^2} \ . \tag{32}$$

In the above algorithm, the functions $q(x_0 + A_n + a_n, \cdot)$ are still irregular. So we have to compute them one by one. On the other hand, the decaying factor is close to 1 if C_2/C_1 *or* C_2'/C_1' is very large, which corresponds to a very slow convergence rate.

Section(2.3) An Aspect of the Sampling Theorem

The sampling theorem shows that a function satisfying certain conditions can be reconstructed from a sequence of sampled values. For example, the classical Shannon sampling theorem says that for each

$$f \in B_{1/2} = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset \left[-\frac{1}{2}, \frac{1}{2}\right] \right\},$$

$$f(x) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)} \ ,$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$, and the Fourier transform is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} \, dx \, .$$

Let $\psi(x) = \frac{\sin \pi x}{\pi x}$. Then $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for $B_{1/2}$, a shift invariant subspace of $L^2(\mathbb{R})$. A natural extension of the classical Shannon sampling theorem is to study the sampling theorem in shift invariant subspaces of

$L^2(\mathbb{R})$. Recall that $\{\phi_k : k \in \mathbb{Z}\} \subset L^2(\mathbb{R})$ is a frame for the closed subspace V it spans, if there are two positive numbers A and B such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, \phi_k\rangle|^2 \leq B\|f\|^2 \ ,\forall f \in V .$$

A and B are called the lower and upper frame bounds, respectively. In this case, $f \in V$ if and only if there is some $\{c_k\} \in \ell^2$ such that $f = \sum_{k\in\mathbb{Z}} c_k \phi_k$, where the convergence is in $L^2(\mathbb{R})$. We refer to [239] for an overview on frames and Riesz bases. In particular, if $\phi_k = \phi(\cdot - k)$, then V is called a shift invariant subspace generated by ϕ. Now a natural question arises: characterize the shift invariant subspace V generated by $\phi \in L^2(\mathbb{R})$ such that the sampling theorem holds, i.e. there is a frame $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ for V such that

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\psi(x - k) \ , \qquad \forall f \in V,$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$. Since functions in $L^2(\mathbb{R})$ are only defined almost everywhere, for the sampled values to make sense, we require that every function in V be continuous.

Many authors have contributed to this topic. For example, see [228, 229, 232,233,234,235,236,237,238]. In particular, we proved the following result.

Proposition (2.3.1)[226] : (see [240]) Suppose that $\phi \in L^2(\mathbb{R})$ and $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a frame for the space V it spans. Then the following two assertions are equivalent:

(i) $\sum_{k\in\mathbb{Z}} c_k \phi(x - k)$ converges pointwisely to a continuous function for any $\{c_k : k \in \mathbb{Z}\} \in \ell^2$ and there is a frame $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ for V such that

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\psi(x - k) \ , \qquad \forall f \in V,$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$.

(ii) ϕ is continuous , $\sum_{k\in\mathbb{Z}} |\phi(x - k)|^2$ is bounded on $\mathbb{R}$ and

$$A\chi_{E_\phi}(\omega) \leq \left|(Z_\phi)(0, \omega)\right| \leq B\chi_{E_\phi}(\omega) \ , \qquad a.e.$$

for some constants $A, B > 0$, where

$$(Z_\phi)(x, \omega) = \sum_{k\in\mathbb{Z}} \phi(x + k)e^{-2\pi i k\omega}$$

is the Zak transform of ϕ and

$$E_\phi = \left\{\omega \in \mathbb{R} : \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + n)|^2 > 0\right\} .$$

Definition (2.3.2)[226] : A closed subspace V in $L^2(\mathbb{R})$ is called a sampling space if there is a frame $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ for V such that $\sum_{k\in\mathbb{Z}} c_k \psi(x - k)$ converges pointwisely to a continuous function for any $\{c_k : k \in \mathbb{Z}\} \in \ell^2$ and

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\psi(x - k) \;, \qquad \forall f \in V \;,$$

where the convergence is both in $L^2(\mathbb{R})$and uniform on $\mathbb{R}$.

By Proposition (2.3.1), many sampling spaces can be given. First, we point out that a sampling space may not have a Riesz basis of the form $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$. To see this, let $\hat{\psi}(\omega) = \chi_{[-1/4,1/4]}$ and $V = B_{1/4}$. For any $f \in V \subset B_{1/2}$, we have

$$\sum_{n\in\mathbb{Z}} |f(x + n)|^2 = \| f\|_2^2 < +\infty$$

and

$$f(n) = \int_{-1/4}^{1/4} \hat{f}(\omega)e^{i2\pi n\omega}\, d\omega = \int_{-1/2}^{1/2} \hat{f}(\omega)e^{i2\pi n\omega}\, d\omega \;.$$

Hence

$$\hat{f}(\omega) = \sum_{n\in\mathbb{Z}} f(n)e^{-i2\pi n\omega} = \sum_{n\in\mathbb{Z}} f(n)\hat{\psi}(\omega)e^{-i2\pi n\omega}\,.$$

Therefore,

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\psi(x - n)\,.$$

Since $\sum_{n\in\mathbb{Z}}|\psi(x+n)|^2 = \|\psi\|_2^2$ and $\sum_{n\in\mathbb{Z}}|f(n)|^2 = \|f\|_2^2$, the above series converges uniformly on $\mathbb{R}$. Hence V is a sampling space. However, for any $\phi \in V$, $\sum_{n\in\mathbb{Z}} | \hat{\phi}(\omega+n)|^2 = 0 \quad a.e.$ on $[1/4, 1/2]$. Consequently, $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ cannot be a Riesz basis for .

Proposition $(2.3.3)[226]$: (see [240]) Suppose $\psi \in L^2(\mathbb{R})$. Then the following two assertions are equivalent.

(i) For any $\{c_k : k \in \mathbb{Z}\} \in \ell^2$, $\sum_{k\in\mathbb{Z}} c_k\psi(x - k)$ converges pointwisely to a continuous function.

(ii) ψ is continuous and $\sum_{k\in\mathbb{Z}} |\psi(x-k)| \le L < \infty$ for some constant L.

Proposition $(2.3.4)[226]$: Suppose that $\{x_k : k \in \mathbb{Z}\}, \{y_k : k \in \mathbb{Z}\} \in \ell^2$ and

$$X(\omega) = \sum_{k\in\mathbb{Z}} x_k e^{-2\pi ik\omega} \;, Y(\omega) = \sum_{k\in\mathbb{Z}} y_k e^{-2\pi ik\omega}$$

are their Fourier transforms, respectively. Then

$$\sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} x_k y_{n-k} \right|^2 = \int_{-1/2}^{1/2} |X(\omega)Y(\omega)|^2\, d\omega\,.$$

When one side of the above equation is finite, the Fourier transform of

$$(x * y)(n) = \sum_{k\in\mathbb{Z}} x_k y_{n-k} \qquad \text{is} \quad X(\omega)Y(\omega)\,.$$

Proposition $(2.3.5)[226]$: (see [230], [231]) Suppose that $\psi \in L^2(\mathbb{R})$ and $V = \overline{\operatorname{span}}\{\psi(\cdot - k) : k \in \mathbb{Z}\}$. Then $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is a frame for V with bounds A and B if and only if

$$A\chi_{E_\psi}(\omega) \le \sum_{n\in\mathbb{Z}} |\psi(\omega + n)|^2 \le B\chi_{E_\psi}(\omega), \qquad a.e.$$

Moreover, $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis for V if $\mathbb{R}\backslash E_\psi$ has measure 0 .

Theorem $(2.3.6)[226]$: Let V be a sampling space. Then there is a sampling space U such that $V \subset U$ and U has a Riesz basis of the form $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$.

Proof : Let $\{\psi_1(\cdot - n) : n \in \mathbb{Z}\}$ be a frame for V and A and B be the lower and upper frame bounds, respectively. By Proposition (2.3.1) ,

$$L = \sup_{x\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\psi_1(x - n)|^2 < +\infty \quad . \tag{33}$$

Put $\hat{\psi}_2(\omega) = 1 - \chi_{E_{\psi_1}}(\omega)$ for $|\omega| \le 1/2$ and 0 for others. Let $\phi = \psi_1 + \psi_2$. Then we have

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + n)|^2 = \sum_{n\in\mathbb{Z}} |\hat{\psi}_1(\omega + n)|^2 + \sum_{n\in\mathbb{Z}} |\hat{\psi}_2(\omega + n)|^2, \qquad a.e.$$

But

$$A\chi_{E_{\psi_1}}(\omega) \le \sum_{n\in\mathbb{Z}} \left|\hat{\psi}_1(\omega + n)\right|^2 \le B\chi_{E_{\psi_1}}(\omega) , \qquad a.e.$$

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}_2(\omega + n)|^2 = 1 - \chi_{E_{\psi_1}}(\omega), \qquad a.e.$$

Hence

$$\min\{A, 1\} \le \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + n)|^2 \le \max\{B, 1\} , \qquad a.e.$$

It follows from Proposition (2.3.5) that $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is a Riesz basis for the space U it spans.

Next we show that U is a sampling space. Since $\psi_2 \in B_{1/2}$, we have

$$\sum_{n\in\mathbb{Z}} |\psi_2(x - n)|^2 = \|\psi_2\|_2^2 < +\infty \quad . \tag{34}$$

By (33) and (34), we have $\sum_{n\in\mathbb{Z}} |\phi(x + n)|^2 \le L' < \infty$. Hence, $\sum_{n\in\mathbb{Z}} c_n\phi(x + n)$ converges uniformly on $\mathbb{R}$ for any $\{c_n : n \in \mathbb{Z}\} \in \ell^2$. On the other hand, for any $f \in U$, there is some $\{c_n : n \in \mathbb{Z}\} \in \ell^2$ such that

$$f = \sum_{n\in\mathbb{Z}} c_n\phi(\cdot + n). \text{ Let } f_1 = \sum_{n\in\mathbb{Z}} c_n\psi_1(\cdot + n), f_2 = \sum_{n\in\mathbb{Z}} c_n\psi_2(\cdot + n)$$

and $C(\omega) = \sum_{n\in\mathbb{Z}} c_n e^{-i2\pi n\omega}$. Then we have

$$f_2(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}_2(\omega) e^{i2\pi n\omega}\, d\omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} C(\omega)\hat{\psi}_2(\omega) e^{i2\pi n\omega}\, d\omega\,. \tag{35}$$

Hence

$$\hat{f}_2(n) = C(\omega)\hat{\psi}_2(\omega) = \sum_{n\in\mathbb{Z}} f_2(n) e^{-i2\pi n\omega}\ \hat{\psi}_2(\omega)\,, \quad a.e. \tag{36}$$

Therefore,

$$f_2(x) = \sum_{n\in\mathbb{Z}} f_2(n)\psi_2(x - n)\,. \tag{37}$$

Since both sides are continuous, the above equation holds for any $x \in \mathbb{R}$. On the other hand, we see from (36) that $\sum_{n\in\mathbb{Z}} f_2(n) e^{-i2\pi n\omega} = 0\ a.e.$ on E_{ψ_1} . Hence

$$\sum_{n\in\mathbb{Z}} f_2(n)\psi_1(x - n) = 0\,, \qquad a.e\,. \tag{38}$$

Again, the above equation holds for any $x \in \mathbb{R}$ since both sides are continuous. Now we see from (37) and (38) that

$$f_2(x) = \sum_{n\in\mathbb{Z}} f_2\big((n)\psi_2(x-n) + \psi_1(x-n)\big) = \sum_{n\in\mathbb{Z}} f_2(n)\phi(x-n). \tag{39}$$

$$By\ (37)\,,\ \sum_{n\in\mathbb{Z}} |f_2(n)|^2 = \|\hat{f}_2\|_2^2 < +\infty\,.$$

Hence the series in (39) is convergent both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$. Similarly we can prove that

$$f_1(x) = \sum_{n\in\mathbb{Z}} f_1(n)\phi(x - n)\quad , \tag{40}$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$. Hence

$$f(x) = f_1(x) + f_2(x) = \sum_{n\in\mathbb{Z}} f(n)\phi(x - n)$$

with the same convergence. Consequently, U is a sampling space.At last, let us prove that $V \subset U$. For any $f \in V$, there is some $C_1(\omega) \in L^2[-1/2, 1/2]$ such that $\hat{f}\,(\omega) = C_1(\omega)\hat{\psi}_1(\omega)$. Let $C(\omega) = C_1(\omega)\cdot\chi_{E_{\psi_1}}(\omega)$. Then we have $\hat{f}(\omega) = C(\omega)\hat{\psi}_1(\omega) = C(\omega)(\hat{\psi}_1(\omega) + \hat{\psi}_2(\omega)) = C(\omega)\hat{\phi}(\omega)$. Hence $f \in U$. This completes the proof.

Theorem $(2.3.7)[226]$: Suppose that $f \in L^2(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. If there are two positive numbers A and B such that

$$A\left|\sum_{n\in\mathbb{Z}} \hat{f}(\omega + n)\right|^2 \le \sum_{n\in\mathbb{Z}} |\hat{f}(\omega + n)|^2\ , a.e. \tag{41}$$

and

$$\left(\sum_{n\in\mathbb{Z}} |\hat{f}(\omega + n)|\right)^2 \le B\left|\sum_{n\in\mathbb{Z}} \hat{f}(\omega + n)\right|^2, a.e. \tag{42}$$

then f belongs to some sampling space.

Proof : Since $f \in L^1(\mathbb{R})$, we have

$$\int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} |\hat{f}(\omega+n)|\ d\omega = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} |\hat{f}(\omega)|\ d\omega = \int_{-\infty}^{\infty} |\hat{f}(\omega)|\ d\omega < \infty.$$

Thus $\sum_{n\in\mathbb{Z}} |\hat{f}(\omega + n)|$ is convergent pointwisely almost everywhere. And so are the series $\sum_{n\in\mathbb{Z}} |\hat{f}(\omega + n)|$ and $\sum_{n\in\mathbb{Z}} |\hat{f}(\omega + n)|^2$.

Since $f \in L^1(\mathbb{R}), f(x)$ is continuous. For any $k \in$ Z, we have

$$f(k) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{2\pi ikw}\, d\omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} \hat{f}(\omega + n)\cdot e^{2\pi ik\omega}\ d\omega. \tag{43}$$

Note that $\sum_{n\in\mathbb{Z}} f(\omega + n) \in L^2[-1/2, 1/2]$, thanks to (41). We have

$$(Zf)(0,\omega) = \sum_{n\in\mathbb{Z}} f(\omega + n) \ , \qquad a.e. \tag{44}$$

On the other hand, since

$$\int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} |\hat{f}(\omega + n)|^2\, d\omega = \int_{-\infty}^{\infty} |f(\omega)|^2\ d\omega < \infty,$$

we have

$$F(\omega) = \sum_{n\in\mathbb{Z}} \left|\hat{f}(\omega + n)\right|^2 \in L^1[-1/2, 1/2].$$

Now we can rewrite inequalities (41) and (42) in the following form.

$$A|(Zf)(0,\omega)|^2 \le F(\omega) \le \left(\sum_{n\in\mathbb{Z}} \left|\hat{f}(\omega+n)\right|\right)^2$$
$$\le B|(Zf)(0,\omega)|^2, a.e. \tag{45}$$

Let

$$E_f = \{\omega \in \mathbb{R} : (Zf)(0,\omega) \neq 0\} \tag{46}$$

$$\psi(\omega) = \begin{cases} \hat{f}(\omega)/(Zf)(0,\omega)\ , & \omega \in E_f\ , \\ 1\ , & \omega \in \left[-\frac{1}{2}, \frac{1}{2}\right)\backslash E_f\ , \\ 0 & otherwise\ . \end{cases} \tag{47}$$

Since $(Zf)(0,\omega)$ has period 1, the set E_f is shift invariant, i.e. $E_f = E_{f+k} = \{\omega + k : \omega \in E\}$ for each $k \in Z$.

By (47) , we have

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}(\omega + n)| = \begin{cases} \dfrac{\sum_{n\in\mathbb{Z}} |\hat{f}(\omega+n)|}{|(Zf)(0,\omega)|} & ,\omega \in E_f \cap \left[-\frac{1}{2},\frac{1}{2}\right), \\ 1 \quad , & \omega \in \left[-\frac{1}{2},\frac{1}{2}\right)\backslash E_f \ . \end{cases} \tag{48}$$

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}(\omega + n)|^2 = \begin{cases} F(\omega)/|(Zf)(0,\omega)|^2, \omega \in E_f \cap \left[-\frac{1}{2},\frac{1}{2}\right), \\ 1 \quad , \qquad \omega \in \left[-\frac{1}{2},\frac{1}{2}\right)\backslash E_f \ . \end{cases} \tag{49}$$

It follows from (45) that

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}(\omega + n)| \le \max\{1, \sqrt{B}\} \ , \qquad a.e. \tag{50}$$

and

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}(\omega + n)|^2 \le \max\{1, B\} \ , \qquad a.e. \tag{51}$$

Hence $\hat{\psi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). By$ (45) and (49), we have

$$\min\{1, A\} \le \sum_{n\in\mathbb{Z}} |\hat{\psi}(\omega + n)|^2 \le \max\{1, B\} \ , \qquad a.e.$$

Therefore , $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis for the subspace V it spans .
On the other hand, since $\hat{\psi} \in L^1(\mathbb{R})$, ψ is continuous . For each $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned} \psi(x + k) &= \int_{-\infty}^{\infty} \hat{\psi}(\omega) e^{2\pi i x\omega} \cdot e^{2\pi i k\omega}\, d\omega \\ &= \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} \hat{\psi}(\omega + n) e^{2\pi i x(\omega+n)} \cdot e^{2\pi i k\omega} d\omega. \end{aligned} \tag{52}$$

Since

$$\left| \sum_{n\in\mathbb{Z}} \hat{\psi}(\omega + n) e^{2\pi i x(\omega+n)} \right| \le \sum_{n\in\mathbb{Z}} |\hat{\psi}(\omega + n)| \ ,$$

we see from (50) that as a function of ω ,
$\sum_{n\in\mathbb{Z}} \hat{\psi}(\omega + n) e^{2\pi i x(\omega+n)} \in L^2[-1/2, 1/2]$. It follows from (52) that

$$\sum_{k\in\mathbb{Z}} |\psi(x + k)|^2 = \int_{-1/2}^{1/2} \left| \sum_{n\in\mathbb{Z}} \hat{\psi}(\omega + n) e^{2\pi i x(\omega+n)} \right|^2 d\omega \le \max\{1, B\} < \infty.$$

Thus every function in V is continuous. Moreover, for any $g \in V$, there is some $\{c_k\} \in l^2$ such that

$$g(x) = \sum_{k\in\mathbb{Z}} c_k \psi(x - k), \tag{53}$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$. By setting $x = 0$ in (52) , we get

$$(Z\psi)(0,\omega) = \sum_{k\in\mathbb{Z}} \psi(k)e^{-2\pi ik\omega} = \sum_{n\in\mathbb{Z}} \hat{\psi}(\omega + n).$$

But $\sum_{n\in\mathbb{Z}} \hat{\psi}(\omega+n) = 1 \ a.e.$, thanks to (44). Therefore ,$(Z\psi)(0,\omega) = 1$, a.e. Thus for $g \in V$ given by (53) , $g(n) = \sum_k c_k \psi(n - k)$,$\forall\, n \in \mathbb{Z}$. By Proposition(2.3.4),

$$(Zg)(0,\omega) = C(\omega)(Z\psi)(0,\omega) = C(\omega)\,, \qquad (54)$$

$where\ C(\omega) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi ik\omega}$. Hence $c_k = g(k)$for any $k \in \mathbb{Z}$ and $g(x) = \sum_{k\in\mathbb{Z}} g(k)\psi(x - k)$.Therefore , V is a sampling space. Moreover, since $\hat{f}(\omega) = (Zf)(0,\omega)\hat{\psi}(\omega)$, we have $f \in V$. This completes the proof.

Example$(2.3.8)[226]$: Suppose that supp $\hat{f} \subset [-1/2, 1/2]$. Then for $\omega \in [-1/2, 1/2)$ the series in (41) and (42) contain only one term $for\, n = 0$, respectively. Thus (41) and (42) are satisfied for $A = B = 1$. Moreover, $\hat{\psi}$ defined by (47) satisfies

$$\hat{\psi}(\omega) = \begin{cases} 1\,, & \omega \in \left[-\frac{1}{2}, \frac{1}{2}\right), \\ 0\,, & otherwise. \end{cases} \qquad (55)$$

Hence

$$\psi(x) = \int_{-1/2}^{1/2} e^{2\pi ix\omega}\, d\omega = \frac{sin\, \pi x}{\pi x}$$

and

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) \frac{sin\, \pi(x - k)}{\pi(x - k)} .$$

This is the classical Shannon sampling theorem.

Example $(2.3.9)[226]$: If $k \geq 2$, $\hat{f}(\omega) = \left(\frac{sin\, \pi\omega}{\pi\omega}\right)^k$, then all series in (41) and (42) in Theorem (2.3.7) are continuous functions with no zeros. Thus in this case (41) and (42) are satisfied for some $A, B > 0$.

Example$(2.3.10)[226]$: Let $\{c_k\}_k$ be a sequence of positive numbers, $\sum_{k\in\mathbb{Z}} c_k < \infty$, and $h(\omega)$ be a continuous function with period 1 and $h(0) = 0$. Let

$$\hat{f}(\omega) = c_k h(\omega)\,, \qquad \omega \in \left[k - \frac{1}{2}, k + \frac{1}{2}\right] , \qquad k \in \mathbb{Z}\,.$$

For any $\omega \in [-1/2, 1/2)$, we have

$$\sum_{n\in\mathbb{Z}} \hat{f}(\omega + n) = h(\omega) \sum_{n\in\mathbb{Z}} c_n \,, \left|\sum_{n\in\mathbb{Z}} \hat{f}(\omega + n)\right|^2 = |h(\omega)|^2 \left(\sum_{n\in\mathbb{Z}} c_n\right)^2,$$

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(\omega + n)\right|^2 = |h(\omega)|^2 \sum_{n\in\mathbb{Z}} c_n^2 \quad ,$$

$$\left(\sum_{n\in\mathbb{Z}} \left|\hat{f}(\omega + n)\right|^2\right)^2 = |h(\omega)|^2 \left(\sum_{n\in\mathbb{Z}} c_n\right)^2 .$$

Let

$$A = \frac{\sum_{n\in\mathbb{Z}} c_n^2}{(\sum_{n\in\mathbb{Z}} c_n)^2} \quad , \qquad B = 1 .$$

Then (41) and (42) in Theorem (2.3.7) are satisfied. In other words , f belongs to some sampling space. However , since

$$F(\omega) = \sum_{n\in\mathbb{Z}} |f(\omega + n)|^2$$

is continuous and has zeros , $\{f(\cdot - k) : k \in Z\}$ itself is not a frame for the closed space it spans.

Chapter 3

Reconstruction in Multiply Generated Shift-Invariant Spaces with Symmetric Averaging Functions

The problem of reconstructing a function f from a set of nonuniforml distributed, weighted-average sampled values $\{\int_{\boldsymbol{R}^d} f(x)\psi_{x_j}(x)\,dx : j \in J\}$ is studied in the context of shift-invariant subspaces of $L^p(\boldsymbol{R}^d)$ generated by p-frames. The special but important case where the weighted-average sampled values are of the form$\{\int_{\boldsymbol{R}^d} f(x)\psi(\cdot - x_j)\,dx : j \in J\}$ is also studied. We show that every square integrable function can be approximated by its average sampling series. As special cases we also obtain new error bounds for regular sampling. Examples are given. In fact, any shift-invariant space V_φ with a stable generator φ is the range space of a bounded one-to-one linear operator T between $L^2(0,1)$ and $L^2(R)$. Thus, regular and irregular sampling formulas in V_φ are obtained by transforming, via T , expansions in $L^2(0,1)$ with respect to some appropriate Riesz bases .

Section(3.1)Nonuniform Average Sampling

The reconstruction of a function f on $\mathbb{R}^{\boldsymbol{d}}$ from its samples $\{f(x_j) : j \in J\}$, where J is a countable index set, is a common task in many applications in signal or image processing. The sampling set $X = \{x_j : j \in J\}$ is often nonuniform and prevents the use of standard methods from Fourier analysis. For example, the loss of data packets during transmission through the Internet or from satellites can be viewed as a nonuniform Sampling / reconstruction problem. In geophysical exploration, the Earth's magnetic field is measured by a combination of airborne, fast-moving acquisition devices, as well as scattered stationary devices resulting in highly nonuniform sampling patterns, and a huge data set. The goal is to reconstruct the magnetic field and use it to reveal geological features. In the sampling and reconstruction problem, the function f is usually assumed to belong to a shift-invariant space of the form

$$V_p(\Phi) = \left\{ \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^d} c_{ik}\phi_i(\cdot - k) : c_i = (c_{ik}) \in \ell^p(\mathbb{Z}^d), i = 1, \dots, r \right\}, \qquad (1)$$

Where $\Phi = (\phi_1, \dots, \phi_r)$ is called the generator of V. If $r = 1, d = 1,\ p = 2,$ and $\phi(x) = sin(\pi x)/\pi x$, then $V^2(\phi)$ is the classical space of band-limited functions often used as a model in sampling theory (see [201], [208], [214], [221], [224]). However, since band-limited functions are analytic, they have infinite support, thus local errors may propagate, and the reconstruction algorithms can be computationally inefficient. Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason, the sampling and reconstruction problems have been investigated in spline subspaces [200], [211], [219], wavelet subspaces [151], [154], [200], [204], [206],[207], [212], [214], [215], [224], and general shift-invariant spaces [135], [197], [198], [201], [222]. The assumption that the sample values $\{f(x_j) : j \in J\}$ can be measured exactly is not always valid. To take into account the characteristics of the acquisition devices , a weighted-average value in the neighborhood of x_j is assumed. This means that the sampled data is of the form

$$g_{x_j} = \int_{\mathbb{R}^d} f(x)\overline{\psi_{x_j}(x)}\ dx \quad , \tag{2}$$

Where $\int_{\mathbb{R}^d} \psi_{x_j} = 1$. Each function ψ_{x_j} reflects the characteristic of the sampling device used to measure the average sampling value of f in the neighborhood of x_j .

One of the goals of a sampling theory is to find conditions on the sampling set $X = \{x_j : j \in J\}$ such that a small change in the function f produces a small change in the sample values $\left\{g_{x_j} : j \in J\right\}$, and such that f can be reconstructed from $\{g_{x_j} : j \in J\}$ exactly and in a stable way. Equivalently, we must find conditions on X such that

$$c_p\|f\|_{L^p} \le \left(\sum_{x_j \in X} \left|g_{x_j}(f)\right|^p\right)^{\frac{1}{p}} \le C_p\|f\|_{L^p} \ , \tag{3}$$

where g_{x_j} are defined by (2) and where c_p and C_p are positive constants independent of f . Another important goal in sampling theory is to find fast algorithms for reconstructing the function f from its sample values.

When the sampling set is uniform, the weighted-average sampling and reconstruction problem has been studied in [220] for the particular case where the functionals in (2) are of the form $\psi_{x_j} = \psi(.-x_j)$ (i.e., a single device ψ is used to obtain all the measurements) , the sampling is critical (i.e., no oversampling), and in (1) $p = 2, r = 1$, and $d = 1$.

The case of uniform sampling with multiple devices has been studied by Sun and Zhou [217], under the assumption that

$$\operatorname{supp}\psi_{x_j} \subset \left[x_j - \frac{\delta}{2}, x_j + \frac{\delta}{2}\right], \qquad \psi_{x_j} \ge 0 . \tag{4}$$

Define the Fourier transform $\hat{f}$ of an integrable function f by

$\hat{f}(\xi) = \int f(\tau)e^{i2\pi\tau\xi}\, d\tau$. For nonuniform sampling, Grochenig [211] proved that if $|x_{j+1} - x_j| \le \delta < \sqrt{2}/2$, then any band-limited function f with $\operatorname{supp}(\hat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$ is uniquely determined from its averages $\langle f, \psi_{x_j}\rangle$, provided that (4) holds. He also showed that f can be reconstructed by iterative algorithms. Sun and Zhou [218] also studied average sampling under assumption (4) and $\psi_{x_j}(\cdot + x_j)$ even and nondecreasing on $[0, \delta/2]$. They gave density conditions on

X under which f satisfies (3) and derived frame algorithms for the reconstruction. They also gave bounds on the error of reconstruction when a nonband-limited function is reconstructed by the frame algorithms. In [219], Sun and Zhou showed that if the maximal gap between consecutive sampling points is smaller than a characteristic length, then a function in a spline subspace is uniquely determined from local averages obtained from averaging functions satisfying (4). For $p = 2$ and $r = 1$ in (1), Aldroubi gave conditions on the density of X and the diameter of the support of the sampling functionals ψ_{x_j}, under which a function f can be reconstructed by iterative approximation-projection algorithms (A-P algorithms for short) [196]. In [196], estimates were also derived for the convergence rates of the A-P algorithms in terms of the generating function ϕ and the diameter of the support of the functionals ψ_{x_j} . It should be noted that A-P algorithms are not frame algorithms and do not require knowledge of the frames associated with $\{\psi_{x_j} : x_j \in X\}$. A-P algorithms are robust, their convergence is geometric, and they perform optimally even if the samples are corrupted by noise [135] ,[196],[197].

We will consider the sampling problem in $V_p(\Phi)$, where $\{\phi_i(\cdot - j) : j \in \mathbb{Z}^{\boldsymbol{d}}, i = 1,\dots,r\}$ is a p-frame for $V_p(\Phi)$, i.e., there exists a positive constant A (depending on Φ and p) such that

$$A^{-1}\|f\|_{L^p} \leq \sum_{i=1}^{r} \left\| \left(\int_{\mathbb{R}^d} f(x)\phi_i(x-j)\,dx \right)_{j\in\mathbb{Z}^{\boldsymbol{d}}} \right\|_{\ell^p} \leq A\|f\|_{L^p},\ f \in V_p(\Phi). \quad (5)$$

We also assume that

$$\Phi = (\phi_1,\dots,\phi_r) \in W(L^1)^{(r)},\ i.e., \qquad \phi_i \in W(L^1), i = 1,\dots,r. \quad (6)$$

Under these conditions, the space $V_p(\Phi)$ in (1) is well defined and it is a closed linear subspace of $L^p(\mathbb{R}^d)$ (see [197]). For this case, the well-posedness sampling condition (3) can then be written as

$$c\|f\|_{L^p} \leq \left(\sum_{x_j\in X} |\langle f,\psi_{x_j}\rangle|^p \right)^{1/p} \leq C\|f\|_{L^p},\ f \in V_p(\Phi), \quad (7)$$

which is similar to a frame condition.However, the set $\{\psi_{x_j} : x_j \in X\}$ does not necessarily form a frame for $V_p(\Phi)$ since the functions $\psi_{x_i}, x_i \in X$, are not necessarily in $V_p(\Phi)$.The sampling theory in such spaces is new, since all previous results consider spaces in which $r = 1$ (single generator) , and assume $\{\phi(\cdot - j) : j \in \mathbb{Z}^{\boldsymbol{d}}\}$ to be a Riesz basis, instead of a (possibly redundant) frame. Moreover, for average sampling in shift-invariant spaces, only the case $p = 2$ has been considered so far [196].

We show that a function $f \in V_p(\Phi)$ can be reconstructed from its average samples by an iterative A-P algorithm, provided that the sampling set X satisfies a density condition that depends on Φ and the set $\{\psi_{x_j} : x_j \in X\}$. Our results treat the case of averaging functions in which the only requirement is that supp ψ_{x_j} is compact for each $x_j \in X$ (Theorem (3.1.8)). But we also treat the important case where $\psi_{x_j} = \psi(\cdot - x_j)$ for each $x_j \in X$ (Theorem (3.1.7)).However , for this case, we do not assume that ψ has compact support. We prove that the A-P algorithms converge even if the samples are corrupted by noise and that the reconstruction result is optimal in some sense (Theorem (3.1.9)). and , we present estimates for the rate of convergence of the A-P algorithms of Theorems (3.1.7) and (3.1.8) in terms of the generator Φ and the sampling functions $\{\psi_{x_j} : x_j \in X\}$.

For the sampling problem we need to impose regularity requirements on the space $V_p(\Phi)$.Wiener amalgam spaces are useful in this context and they are defined as follows: A measurable function f belongs to
$W(L^p)$, $1 \le p < \infty$, if it satisfies

$$\|f\|_{W(L^p)} = \left(\sum_{k \in \mathbb{Z}^d} \text{ess sup}\{|f(x+k)|^p : x \in [0,1]^d\}\right)^{1/p} < \infty. \quad (8)$$

If $p = \infty$, a measurable function f belongs to $W(L^\infty)$ if it satisfies

$$\|f\|_{W(L^\infty)} = \lim_{k \in \mathbb{Z}^d}\{\text{ess sup}\{|f(x+k)| : x \in [0,1]^d\}\} < \infty . \quad (9)$$

In this case , $W(L^\infty)$ coincides with $L^\infty(\mathbb{R}^d)$. Endowed with this norm , $W(L^p)$ becomes a Banach space [208], [209]. The subspace of continuous functions $W_0(L^p) = W(C, L^p) \subset W(L^p)$ is a closed subspace of $W(L^p)$ and thus also a Banach space [208], [209]. We have the following inclusions between the various spaces:

$$W_0(L^p) \subset W_0(L^q) \subset W(L^q) \subset L^q(\mathbb{R}^d), 1 \le p \le q \le \infty. \quad (10)$$

The following convolution relations hold for $1 \le p \le \infty$ [196]:

(i) If $f \in L^p(\mathbb{R}^d)$ and $g \in W(L^1)$, then $f * g \in W(L^p)$ and

$$\|f * g\|_{W(L^p)} \le C\|f\|_{L^p}\|g\|_{W(L^1)} . \quad (11)$$

(ii) If $c = (c_k) \in \ell^p(\mathbb{Z}^d)$ and $\phi \in W(L^1)$, then $\sum_{k\in\mathbb{Z}^d} c_k\phi(\cdot - k) \in W(L^p)$ and

$$\left\|\sum_{k\in\mathbb{Z}^d} c_k\phi(\cdot - k)\right\|_{W(L^p)} \le \|c\|_{\ell^p}\|\phi\|_{W(L^1)} . \quad (12)$$

(iii) If $f \in L^p(\mathbb{R}^d)$ and $g \in W(L^1)$, then the sequence $d = (d_k)$ defined by $d_k = \int_{\mathbb{R}^d} f(x)\overline{g(x-k)}\, dx, k \in \mathbb{Z}^d$, belongs to $\ell^p(\mathbb{Z}^d)$ and

$$\|d\|_{\ell^p} \le \|f\|_{L^p}\|g\|_{W(L^1)} \quad . \tag{13}$$

In addition to the requirement that the generator Φ of $V_p(\Phi)$ satisfies (12) and (13) , we also require ϕ_i , $i = 1,\ldots,r$, to be continuous. Thus, together, the requirements are that Φ satisfies (12) and belongs to $W_0(L^1)^{(r)}$ (here $W_0(L^1)^{(r)}$ denotes the Cartesian product $W_0(L^1) \times \cdot \cdot \cdot \times W_0(L^1)$ of r copies of $W_0(L^1)$). With these requirements, it is well known that the space $V_p(\Phi)$ is a space of continuous L^p-functions and we have the following properties [135], [199]:

(i) The space $V_p(\Phi)$ is a closed linear subspace of $L^p(\mathbb{R}^d)$ and there exists a positive constant B (depending on Φ and p) such that

$$B^{-1}\|f\|_{L^p} \le \inf_{f=\sum_{i=1}^{r} \phi_i *' c_i} \sum_{i=1}^{r} \|c_i\|_{\ell^p} \le B\|f\|_{L^p} \; \forall f \in V_p(\Phi), \tag{14}$$

where $\phi_i *' c_i = \sum_{k\in\mathbb{Z}^d} c_{ik}\phi_i(\cdot - k)$ and $c_i = (c_{ik}) \in \ell^p(\mathbb{Z}^d)$.

(ii) The space $V_p(\Phi)$ is a closed linear subspace of $W_0(L^p)$ and we have the norm equivalence $\|f\|_{L^p} \approx \|f\|_{W(L^p)}$.

(iii) There exists $\tilde{\phi}_1,\ldots,\tilde{\phi}_r \in W_0(L^1) \cap V_p(\Phi)$ such that, for every $f \in V_p(\Phi)$,

$$f = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \tilde{\phi}_i(\cdot - j)\rangle \phi_i(\cdot - j) = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \phi_i(\cdot - j)\rangle \tilde{\phi}_i(\cdot - j). \tag{15}$$

Hence the operator P, defined by

$$Pf = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \tilde{\phi}_i(\cdot - j)\rangle \phi_i(\cdot - j) \,, f \in L^p(\mathbb{R}^d) , \tag{16}$$

is a bounded projection from $L^p(\mathbb{R}^d)$ onto $V_p(\Phi)$.

(iv) If $X = \{x_j : j \in J\}$ is separated , i.e., $\inf_{j\neq l} |x_j - x_l| > 0$, then

$$\left(\sum_{x_j\in X} |f(x_j)|^p\right)^{1/p} \le C\|f\|_{L^p} \text{ for all } f \in V_p(\Phi) . \tag{17}$$

We will assume throughout that the sampling set X is separated and that the sampling functionals ψ_{x_j} satisfy the following properties:

(i) $\sup_j \left\|\psi_{x_j}\right\|_{W(L^1)} < \infty$, and

(ii) $\int_{\mathbb{R}^d} \psi_{x_j} = 1$.

Fast approximation-projection (A-P) iterative algorithms for the reconstruction of functions from their samples have been introduced by Feichtinger and Gröchenig for the case of band-limited functions [210]. These schemes have been extended by Aldroubi and Feichtinger to general shift-invariant spaces [197]. We will develop the theory of fast A-P iterative reconstruction schemes for the case of average sampling. First, we need to introduce the notion of γ -density useful in this regard.

Definition$(3.1.1)$[195]: A set $X = \{x_j : j \in J\}$ is γ_0-dense in $\mathbb{R}^d$ if

$$\mathbb{R}^d = \bigcup_j B_\gamma(x_j) \text{ for every } \gamma > \gamma_0 , \tag{18}$$

where $B_\gamma(x_j)$ are balls centered at x_j and with radius γ .

This definition implies that the distance of any sampling point to its next neighbor isat most $2\gamma_0$. Thus, strictly speaking, γ_0 is the inverse of a density, i.e., if γ_0 increases ,the number of points per unit cube decreases. A special but important case for average sampling is when the sampling functions ψ_{x_j} are obtained by translation of a single function ψ. Thus , $\psi_{x_j} = \psi(\cdot - x_j)$ and the weighted samples are of the form $g_{x_j} = \langle f, \psi(\cdot - x_j)\rangle$. For this case, the iterative algorithm that we develop uses a quasi-reconstruction operator $A_{X,a}$ in the iteration scheme. To define this operator, we start from a partition of unity $\{\beta_j\}_{j\in J}$ defined as follows:

Definition$(3.1.2)$[195]: A bounded uniform partition of unity (BUPU) associated with $\{B_\gamma(x_j)\}_{j\in J}$ is a set of functions $\{\beta_j\}_{j\in J}$ that satisfy:

(i) $0 \le \beta_j \le 1$, $\forall j \in J$,

(ii) $\operatorname{supp}\beta_j \subset B_\gamma(x_j)$, and

(iii) $\sum_{j\in J}\beta_j = 1$.

The operator $A_{X,a}$ is then defined by

$$A_{X,a}f = \sum_{j\in J}\langle f, \psi_a(\cdot - x_j)\rangle\beta_j = \sum_{j\in J}(f * \psi_a^*)(x_j)\beta_j , \tag{19}$$

where $\psi_a(\cdot) = (1/a^d)\psi(\cdot/a)$ and where $\psi_a^*(x) = \overline{\psi_a(-x)}$. Obviously the quasi-reconstruction operator $A_{X,a}f$ does not belong to the space $V_p(\Phi)$. However, we can use this operator in an A-P iterative scheme to reconstruct the exact function $f \in V_p(\Phi)$ as follows:

Lemma $(3.1.3)$[195]: Let $\phi \in W_0(L^1)$ and let $f = \sum_k c_k\phi(.-k)$ where $c = (c_k) \in \ell^p(\mathbb{Z}^d)$. Then

(i) the oscillation (or modulus of continuity)

$\operatorname{osc}_\gamma(f)(x) = \sup_{1|y|\le\gamma}|f(x+y) - f(x)|$ belongs to $W(L^p)$,

(ii) the oscillation $osc_\gamma(\phi)$ satisfies

$$\left\|osc_\gamma(\phi)\right\|_{W(L^1)} \le C'(\gamma,d)\|\phi\|_{W(L^1)} , \tag{20}$$

where $C'(\gamma,d) \le \lceil 2\gamma + 4\rceil^d$ and

$$\left\|osc_\gamma(\phi)\right\|_{W(L^1)} \to 0 \qquad as\, \gamma \to 0 ,$$

(iii) the oscillation $osc_\gamma(f)$ satisfies

$$\left\|osc_\gamma(\phi)\right\|_{W(L^p)} \le \|C\|_{\ell^p}\left\|osc_\gamma\phi\right\|_{W(L^1)} \text{ for all } c\, \ell^p. \tag{21}$$

In particular , $\left\|osc_{\gamma}(\phi)\right\|_{W(L^{p})} \rightarrow 0 \;\; as \;\; \gamma \rightarrow 0.$

Lemma(3.1.4)[195]: Let X be any sampling set with γ-density $\gamma\,(X)$, let $\{\beta_j : j \in J\}$ be a BUPU associated with X (see Definition (3.1.2)), and let $\phi \in W_0(L^1)$. Then there exists a constant $C = C(\gamma, d)$ such that for any $f = \sum_k c_k\phi(\,\cdot - k)$, we have

$$\|Q_X f\|_{L^p} \le \|Q_X f\|_{W(L^p)} \le C(\gamma, d)\|C\|_{\ell^p}\|\phi\|_{W(L^1)} \forall\, c = (c_k) \in \ell^p(\mathbb{Z}^{\boldsymbol{d}}),$$

where the constant $C(\gamma, d) \le (\lceil 2\gamma + 4\rceil^d + 1)$ does not depend explicitly on the sampling set X or on the partition of unity in definition (19). Here $\lceil t\rceil$ denotes the smallest integer greater than or equal to t.

Proof: Let $f = \sum_k c_k\phi(\,\cdot - k)$ where $c = (c_k) \in \ell^p(\mathbb{Z}^{\boldsymbol{d}})$ and (see [135] ,[197]). From (12), we have $f \in W(L^p)$ and

$$\begin{aligned}|f(x) - (Q_X f)(x)| &= \left|f(x) - \sum_{j\in J} f(x_j)\beta_j(x)\right| \\ &= \left|f(x)\sum_{j\in J}\beta_j(x) - \sum_{j\in J} f(x_j)\beta_j(x)\right| \\ &\le \sum_{j\in J}|f(x) - f(x_j)|\beta_j(x) \\ &\le \sum_{j\in J} osc_{\gamma}(f)(x)\beta_j(x) \\ &\le osc_{\gamma}(f)(x)\sum_{j\in J}\beta_j(x) = osc_{\gamma}(f)(x).\end{aligned}$$

From this pointwise estimate and Lemma (3.1.3) , we get that

$$\|f - Q_X f\|_{W(L^p)} \le \left\|osc_{\gamma}(f)\right\|_{W(L^p)} \le \|C\|_{\ell^p}\left\|osc_{\gamma}\phi\right\|_{W(L^1)}. \tag{22}$$

Thus using (12), (20), and (22), we obtain

$$\begin{aligned}\|Q_X f\|_{W(L^p)} &\le \|f - Q_X f\|_{W(L^p)} + \|f\|_{W(L^p)} \\ &\le (\lceil 2\gamma + 4\rceil^d + 1)\|C\|_{\ell^p}\|\phi\|_{W(L^1)}.\end{aligned} \tag{23}$$

Lemma (3.1.5)[195]: Let $\psi \in L^1(\mathbb{R}^{\boldsymbol{d}})$ such that $\int_{\mathbb{R}^{\boldsymbol{d}}}\psi(x)\,dx = 1$, and define $\psi_a(\,\cdot\,) = a^{-d}\psi(\,\cdot/a)$ where $a > 0$ is any positive real number. Then, for every $\phi \in W(L^1)$, $\|\phi - \phi * \psi_a^*\|_{W(L^1)} \to 0 \; as \; a \to 0^+$.

Proof : We will estimate the $W(L^1)$-norm of $\phi^a = \phi - \phi * \psi_a^*$. Since $\int_{\mathbb{R}^{\boldsymbol{d}}}\psi(x)\,dx = 1$ and $\psi_a(x) = a^{-d}\psi(x/a)$, we have

$$\phi^a(x) = \phi(x) - \phi * \psi_a^*(x) = \int_{\mathbb{R}^{\boldsymbol{d}}}(\phi(x) - \phi(x + t))\overline{\psi_a(t)}\,dt.$$

Therefore

$$|\phi^a(x)| \leq \int_{\mathbb{R}^d} |\phi(x) - \phi(x+t)||\psi_a(t)|\,dt$$

$$= \left(\int_{|t|\leq 1} + \int_{|t|\geq 1} \right) |\phi(x) - \phi(x+t)||\psi_a(t)|dt$$

$$= I_1(x) + I_2(x). \tag{24}$$

By direct computations, we have

$$\|I_2\|_{W(L^1)} = \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} \int_{|t|\geq 1} |\phi(x) - \phi(x+t)||\psi_a(t)|dt$$

$$\leq \sum_{k\in\mathbb{Z}^d} \int_{|t|\geq 1} \left(\sup_{x\in k+[0,1)^d} |\phi(x)| + \sup_{x\in k+[0,1)^d} |\phi(x+t)| \right) |\psi_a(t)|\,dt$$

$$\leq \int_{|t|\geq 1} \left(\|\phi\|_{W(L^1)} + \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} |\phi(x+t)| \right) |\psi_a(t)|dt$$

$$\leq (1+2^d)\|\phi\|_{W(L^1)} \int_{|t|\geq 1} |\psi_a(t)|\,dt$$

$$\leq (1+2^d)\|\phi\|_{W(L^1)} \int_{|t|\geq a^{-1}} |\psi\,(t)|\,dt\,, \tag{25}$$

and

$$\|I_1\|_{W(L^1)} = \int_{|t|\leq 1} \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} |\phi(x) - \phi(x+t)||\psi_a(t)|dt$$

$$\leq \int_{|t|\leq 1} \sum_{k\in\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} osc_{|t|}\phi(x)|\psi_a(t)|dt$$

$$= \int_{|t|\leq 1} \left\|osc_{|t|}\phi(x)\right\|_{W(L^1)} |\psi_a(t)|dt = I\,. \tag{26}$$

By Lemma (3.1.3), for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\left\|osc_{\,\delta}\phi(x)\right\|_{W(L^1)} < \varepsilon \ \ \forall\, \delta < \delta_0\,.$$

Write

$$I = \left(\int_{|t|\leq\delta} + \int_{\delta_0 \leq |t| \leq 1} \right) \left\|osc_{|t|}\phi(x)\right\|_{W(L^1)} |\psi_a(t)|dt = I_3 + I_4\,. \tag{27}$$

Then

$$I_3 \leq \varepsilon \int_{|t|\leq\delta_0} |\psi_a(t)|\,dt \leq \varepsilon\|\psi\|_1\,,$$

and

$$I_4 \le \int_{1\ge|t|\ge\delta_0} \|osc_1\phi(x)\|_{W(L^1)}|\psi_a(t)|\,dt$$

$$\le \|osc_1\phi(x)\|_{W(L^1)} \int_{|s|\ge\delta_0/a} |\psi(s)|\,ds \to 0 \;\; as \;\; a \to 0^+.$$

By (27) , $I \to 0$ as $a \to 0^+$. Combining (24), (25), and (26), we have

$$\|\phi^a\|_{W(L^1)} \le \|I_1\|_{W(L^1)} + \|I_2\|_{W(L^1)} \to 0 \quad as \;\; a \to 0^+.$$

Lemma $(3.1.6)[195]$: Let P be a bounded projection from $L^p\ (\mathbb{R}^{\boldsymbol{d}})$ onto $V_p(\Phi)$. Then there exist $\gamma_0 > 0$ and $a_0 > 0$ such that for every separated γ -dense set X with $\gamma \le \gamma_0$ and for every positive $a \le a_0$, the operator $I - PA_{X,a}$ is a contraction on $V_p(\Phi)$.

Proof :

Let $f = \sum_{i=1}^{r}\sum_{k} c_{ik}\phi_i(\cdot - k) \in V_p(\Phi)$. We have

$$\left\| f - PA_{X,a}f\right\|_{L^p} = \left\| f - PQ_Xf + PQ_Xf - PA_{X,a}f\right\|_{L^p}$$
$$\le \|Pf - PQ_Xf\|_{L^p} + \left\|PQ_Xf - PA_{X,a}f\right\|_{L^p}$$
$$\le \|P\|_{op}(\|f - Q_Xf\|_{L^p} + \|Q_Xf - Q_X(f * \psi_a^*)\|_{L^p}). \qquad (28)$$

Using (22) and the upper bound inequality of (14), the first term of the last inequality in (28) can be estimated as follows:

$$\|f - Q_Xf\|_{L^p} \le \|f - Q_Xf\|_{W(L^p)} \le B \max_{1\le i\le r}\left\|osc_\gamma(\phi_i)\right\|_{W(L^1)} \|f\|_{L^p} . \qquad (29)$$

The second term $\|Q_Xf - Q_X(f*\psi_a^*)\|_{L^p}$ can be estimated as follows. Write $\phi_i^a = \phi_i - \phi_i * \psi_a^*$ for $i = 1,\dots,r$. Since each $\phi_i \in W_0(L^1)$ and $\psi \in L^1$, (11) implies that $\phi_i^a \in W_0(L^1)$. Noting that

$$Q_Xf - Q_X(f*\psi_a^*) = Q_X\left(\sum_{i=1}^{r}\sum_{k} c_{ik}\phi_i^a(\cdot - k)\right),$$

and using Lemma (3.1.4) , we obtain

$$\|Q_Xf - Q_X(f*\psi_a^*)\|_{L^p} \le \left\|Q_X\left(\sum_{i=1}^{r}\sum_{k} c_{ik}\phi_i^a(\cdot - k)\right)\right\|_{W(L^p)}$$
$$\le C(\gamma,d)_{ri} = 1\|c_i\|_{\ell^p}\|\phi_i^a\|_{W(L^1)} .$$

Hence, by (14),

$$\|Q_Xf - Q_X(f*\psi_a^*)\|_{L^p} \le C(\gamma,d)B\|f\|_{L^p}\max_{1\le i\le r}\|\phi_i^a\|_{W(L^1)}. \qquad (30)$$

By combining (28), (29), and (30), we get

$$\left\|f - PA_{X,a}f\right\|_{L^p} \le \|P\|_{op}\left(\max_{1\le i\le r}\left\|osc_\gamma(\phi_i)\right\|_{W(L^1)}\right.$$
$$\left. + (\lceil 2\gamma + 4\rceil^d + 1)\max_{1\le i\le r}\|\phi_i^a\|_{W(L^1)}\right)B\|f\|_{L^p} . \qquad (31)$$

Let $\varepsilon > 0$ be any positive real number. Using Lemma (3.1.3)(ii), we may choose γ_0 so small so that $\max_{1\le i\le r}\|osc_\gamma(\phi_i)\|_{W(L^1)} \le \varepsilon/2$ for all $\gamma \le \gamma_0$. Then, by Lemma (3.1.5) , we may choose a_0 so small that
$(\lceil 2\gamma_0 + 4\rceil^d + 1) \max_{1\le i\le r}\|\phi_i^a\|_{W(L^1)} \le \varepsilon/2$ for all $a \le a_0$. Therefore, we can choose γ_0 and a_0 so that, for any $\gamma \le \gamma_0$ and $a \le a_0$, we have

$$\|f - PA_{X,a} f\| \le B\varepsilon\|P\|_{op}\|f\|_{L^p} \text{ for all } f \in V_p(\Phi). \quad (32)$$

To get a contraction, we choose $B\varepsilon\|P\|_{op} < 1$.

Theorem (3.1.7)[195]: Let Φ be in $W_0(L^1)^{(r)}$, let ψ be a function in $W(L^1)$ such that $\int_{\mathbb{R}^d}\psi = 1$, and let P be a bounded projection from L^p onto $V_p(\Phi)$. Then there exists a density $\gamma = \gamma(\Phi,\psi) > 0$ and $a_0 > 0$ such that any $f \in V_p(\Phi)$ can be recovered from its weighted average samples $\{\langle f, \psi_a(\cdot - x_j)\rangle : j \in J\}$ on any γ -dense set $X = \{x_j : j \in J\}$ and for any $0 < a < a_0$, by the following A-P iterative algorithm:

$$\begin{cases} f_1 = PA_{X,a}f \ , \\ f_{n+1} = PA_{X,a}(f - f_n) + f_n \ . \end{cases} \quad (33)$$

In this case, the iterate fn converges to f uniformly and also in the $W(L^p)$ - and $L^p -$ norms. Moreover, the convergence is geometri c, that is ,

$$\|f - f_n\|_{L^p} \le \|f - f_n\|_{W(L^p)} \le C_1\alpha^n\|f - f_1\|_{W(L^p)}$$

for some $\alpha = \alpha(\gamma, a, \Phi, \psi) < 1$ and $C_1 < \infty$.

Proof: Let $e_n = f - f_n$ be the error after n iterations of algorithm (33). Then the sequence e_n satisfies the recursion

$$e_{n+1} = f - f_{n+1} = f - f_n - PA_{X,a}(f - f_n) = (I - PA_{X,a})e_n \ . \quad (34)$$

Using Lemma (3.1.6) , we may choose γ_0 and a_0 so small that
$\| I - PA_{X,a}\|_{op} = \alpha < 1$. Therefore, by (34), we obtain

$$\|e_{n+1}\|_{L^p} \le \alpha\|e_n\|_{L^p} \quad (35)$$

and

$$\|e_n\|_{L^p} \le \alpha^{n-1}\|e_1\|_{L^p} \ .$$

Thus $\|e_n\|_{L^p} \to 0 \ as \ n \to \infty$. Since, for $V_p(\Phi)$, the $W(L^p)$- and L^p-norms are equivalent, the inequality above also holds in the $W(L^p)$- norm and the proof is completed.

Theorem (3.1.7) treats the case of a single averaging function ψ_a shifted to the points $\{x_j\}$ for obtaining the measurements $\langle f, \psi_a(\cdot - x_j)\rangle$.
In practice, this is the situation when a single measuring device is used to obtain the discrete data. For this case , ψ_a is what is called the *impulse* response of the measuring device. More generally, we can allow the
averaging function ψ_{x_j} to depend on the point x_j. Thus, the averaging functions can be described by the infinite vector $= (\psi_{x_j})_{j\in J}$. For this case, and under some

uniformity on the size of the averaging functions ψ_{x_j} , we can recover the function f exactly by using the quasi-reconstruction operator

$$A_X f = \sum_{j\in J} \langle f, \psi_{x_j}\rangle \beta_j \quad (36)$$

in the following A-P iterative algorithm:

Theorem $(3.1.8)$[195]: Let Φ be in $W_0(L^1)^{(r)}$, let P be a bounded projection from L^p onto $V_p(\Phi)$,and let the averaging sampling functional s $\psi_{x_j} \in W(L^1)$ satisfy $\int_{\mathbb{R}^d} \psi_{x_j} = 1$ and $\int_{\mathbb{R}^d} \left|\psi_{x_j}\right| \le M$, where $M > 0$ is independent of x_j . Then there exists a density $\gamma = \gamma(\Phi, M) > 0$ and $a_0 = a_0(\Phi, M) > 0$ such that if $X = \{x_j : j \in J\}$ is separated and γ -dense in $\mathbb{R}^d$, and if the average sampling functionals ψ_{x_j} satisfy $\operatorname{supp} x_j \subset x_j + [-a, a]^d$ for some $0 < a < a_0$, then any $f \in V_p(\Phi)$ can be recovered from its weighted-average samples $\{\langle f, \psi_{x_j}\rangle : j \in J\}$ by the following iterative algorithm:

$$\begin{cases} f_1 = PA_X f \ , \\ f_{n+1} = PA_X(f - f_n) + f_n \ . \end{cases} \quad (37)$$

In this case, the iterate fn converges to f uniformly and also in the $W(L^p)$- and L^p-norms. Moreover, the convergence is geometric, that is,
$\|f - f_n\|_{L^p} \le \|f - f_n\|_{W(L^p)} \le C_1\alpha^n\|f - f_1\|_{W(L^p)}$ for some $\alpha = \alpha(\gamma, a, \Phi, M) < 1$ and $C_1 < \infty$.

Proof : Let $f = \sum_{i=1}^{r}\sum_k c_{ik}\phi_i(\cdot - k) \in V_p(\Phi)$. We have

$$\begin{aligned} \|f - PA_X f\|_{L^p} &= \|f - PQ_X f + PQ_X f - PA_X f\|_{L^p} \\ &\le \|Pf - PQ_X f\|_{L^p} + \|PQ_X f - PA_X f\|_{L^p} \\ &\le \|P\|_{op}(\|f - Q_X f\|_{L^p} + \|Q_X f - A_X f\|_{L^p}) . \end{aligned} \quad (38)$$

The second term $\|Q_X f - A_X f\|_{L^p}$ of the last inequality can be estimated as follows: Write $f_i = \sum_k c_{ik}\phi_i(\cdot - k)$ for $i = 1, \dots, r$. Clearly, $f_i \in V_p(\Phi)$ for $i = 1, \dots, r$ and $f = \sum_{i=1}^{r} f_i$. For each f_i, we have the following pointwise estimate:

$$\begin{aligned} |(Q_X f_i - A_X f_i)(x)| &= \left|\sum_j \left(f_i(x_j) - \langle f_i, \psi_{x_j}\rangle\right)\beta_j(x)\right| \\ &= \left|\sum_j \left(\int_{\mathbb{R}^d} (f_i(x_j) - f_i(\xi))\overline{\psi_{x_j}(\xi)}\, d\xi\right)\beta_j(x)\right| \\ &\le \sum_j \int_{\mathbb{R}^d} \left|f_i(x_j) - f_i(\xi)\right|\left|\psi_{x_j}(\xi)\right| d\xi \beta_j(x) \end{aligned}$$

$$\leq \sum_j osc_a(f_i)(x_j) \int_{\mathbb{R}^d} |\psi_{x_j}(\xi)| \, d\xi \beta_j(x)$$
$$\leq M \sum_j osc_a(f_i)(x_j)\, \beta_j(x)$$
$$\leq M \sum_j \left(\sum_k |c_{ik}| osc_a(\phi_i)(x_j - k) \right) \beta_j(x). \qquad (39)$$

From this pointwise estimate and Lemma (3.1.4) it follows that $\|Q_X f_i - A_X f_i\|_{L^p} \leq MC(\gamma, d)\|c_i\|_{\ell^p} \|osc_a(\phi_i)\|_{W(L^1)}$.Thus we conclude that

$$\|Q_X f - A_X f\|_{L^p} \leq MC(\gamma, d) \sum_{i=1}^{r} \|c_i\|_{\ell^p} \|osc_a(\phi_i)\|_{W(L^1)} \text{. Hence, by (14) ,}$$

$$\|Q_X f - A_X f\|_{L^p} \leq MC(\gamma, d) B \|f\|_{L^p} \max_{1\leq i \leq r} \|osc_a(\phi_i)\|_{W(L^1)}. \qquad (40)$$

By combining (29), (38), and (40) we get

$$\|f - PA_X f\|_{L^p} \leq \|P\|_{op} \Big(\max_{1\leq i\leq r} \left\| osc_\gamma(\phi_i) \right\|_{W(L^1)} + M(\lceil 2\gamma + 4\rceil^d + 1)$$
$$\times \max_{1\leq i\leq r} \|osc_a(\phi_i)\|_{W(L^1)} \Big) B \|f\|_{L^p} .$$

The rest of the proof is similar to the last part of the proof of Lemma (3.1.6). Let $\varepsilon > 0$ be any positive real number. Using Lemma (3.1.3)(ii), we may choose γ_0 so small so that $\max_{1\leq i\leq r} \left\| osc_\gamma(\phi_i) \right\|_{W(L^1)} \leq \varepsilon/2$ for all $\gamma \leq \gamma_0$. Then we may choose a_0 so small that $M(\lceil 2\gamma_0 + 4\rceil^d + 1) \max_{1\leq i\leq r} \|osc_a(\phi_i)\|_{W(L^1)} \leq \varepsilon/2$ for all $\gamma \leq \gamma_0$. Therefore, we can choose γ_0 and a_0 so that for any $\gamma \leq \gamma_0$ and $\gamma \leq \gamma_0$, we have

$$\|f - PA_X f\|_{L^p} \leq B\varepsilon \|P\|_{op} \|f\|_{L^p} \quad \text{for all } f \in V_p(\Phi) . \qquad (41)$$

To get a contraction, we choose $B\varepsilon \|P\|_{op} < 1$.

In practice, the sampled data is often corrupted by noise. Moreover, the assumption that the function f belongs to some specific space $V_p(\Phi)$ is often an idealization. Thus, it is important to know whether the A-P algorithms (33) and (37) still converge under nonideal circumstances. To investigate these situations, we only assume that the data $f' = \{ f'_j : j \in J \}$ belong to ℓ^2, but we do not assume that $f' = \{ f'_j : j \in J \}$ are local averages of a function $f \in V_p(\Phi)$. For this case we use the initialization

$$f_1 = PQ_X\{f'_j\} = P\left(\sum_{j\in J} f'_j \beta_j \right) \in V_p(\Phi) , \qquad (42)$$

where $\{\beta_j : j \in J \}$ is the BUPU in definition (19). Algorithm (33) becomes

$$f_{n+1} = f_1 + (\mathrm{I} - \mathrm{PA}_{\mathrm{X,a}}) f_n , \qquad (43)$$

and algorithm (37) becomes

$$f_{n+1} = f_1 + (I - P\mathrm{A_X}) f_n . \qquad (44)$$

Theorem(3.1.9)[195]: Under the same assumptions as in Theorem (3.1.7), algorithm (43), with the initialization (42), converges to a function $f_\infty \in V_p(\Phi)$ which satisfies $P(A_{X,a}f_\infty - Q_X\{f_j'\} = 0$.Correspondingly, under the assumptions of Theorem (3.1.8), algorithm (44) converges to a function $f_\infty \in V_p(\Phi)$ which satisfies $P\left(\mathrm{A_X}\, f_\infty - Q_X\{f_j'\}\right) = 0$.

Proof: By Lemma (3.1.6), the operator $I - PA_{X,a}$ *is a* contraction on $V_p(\Phi)$. It follows that the sequence of functions f_n in (43) is convergent to a function f_∞ in $V_p(\Phi)$. By taking the limits of both sides of (43), and using (42), we get

$$P(A_{X,a}f_\infty - Q_X\{f_j'\}) = 0.$$

The proof of the second part of Theorem(3.1.9) is almost identical, except using the contractive property of the operator $I - PA_X$ on $V_p(\Phi)$.

Theorem(3.1.10)[195]: Assume that Φ and ψ satisfy the conditions of Theorem (3.1.7) , and that $|\nabla\phi_i| \in W(L^1)$ for every $i = 1,\ldots,r$ and $\|\psi\|_{1,\eta} = \int_{\mathbb{R}^d} |\psi(t)||t|^\eta \, dt < \infty$ for some $0 < \eta \le 1$. Then the convergence rate α in Theorem (3.1.7) satisfies

$$\alpha \le B\|P\|_{op}\left(3^d\gamma \max_{1\le i\le r}\||\nabla\phi_i|\|_{W(L^1)} + (6^d + 1)a^\eta\|\psi\|_{1,\eta} \times\left((1 + 2^d)\max_{1\le i\le r}\|\phi_i\|_{W(L^1)} + 3^d\max_{1\le i\le r}\||\nabla\phi_i|\|_{W(L^1)}\right)\right),$$

where B is the upper bound constant in (14). We have a corresponding result for the situation in Theorem (3.1.8).

Proof : Consider ϕ and ϕ^a as in Lemma (3.1.5) . Assume further that $|\nabla_\phi| \in W(L^1)$. Let us first estimate $osc_\delta(\phi)W(L^1)$ for $0 < \delta \le 1$. Note that

$$\phi(x + y) - \phi(x) = \int_0^1 y \cdot \nabla_\phi(x + sy)\, ds\,.$$

Therefore

$$|\phi(x + y) - \phi(x)| \le \int_0^1 |y|\left|\nabla_\phi(x + sy)\right| \le |y| \sup_{|t|\le|y|}\left|\nabla_\phi(x + sy)\right|,$$

which leads to the following estimate to $osc_\delta(\phi)$:

$$osc_\delta(\phi)(x) = \sup_{|y|\le\delta}|\phi(x + y) - \phi(x)| \le \sup_{|y|\le\delta}|y| \sup_{|t|\le|y|}|\nabla_\phi(x + t)| \le \delta \sup_{|y|\le\delta}|\nabla_\phi(x + t)|.$$

Thus, for every $k \in \mathbb{Z}^d$,

$$\sup_{x\in k+[0,1)^d} osc_\delta(\phi)(x) \le \sup_{x\in k+[0,1)^d} \delta \sup_{|y|\le\delta}\left|\nabla_\phi(x + t)\right| \le \delta \sup_{y\in k+[-1,2)^d}\left|\nabla_\phi(y)\right|.$$

Hence,

$$\|osc_\delta(\phi)\|_{W(L^1)} = \sum_{k\,\in\,\mathbb{Z}^d} \sup_{x\in k+[0,1)^d} osc_\delta(x) \;\le \delta \sum_{k\,\in\,\mathbb{Z}^d} \sup_{y\in k+[-1,2)^d} |\nabla_\phi(y)|$$

$$\le\; 3^d \sum_{k\,\in\,\mathbb{Z}^d} \sup_{y\in k+[0,1)^d} |\nabla_\phi(y)| =\; 3^d\delta \big\||\nabla_\phi|\big\|_{W(L^1)}. \qquad (45)$$

Next we estimate the $W(L^1)$-norm of $\phi^a =\; \phi - \phi * \psi_a^*$. By (26) and (45),

$$\|I_1\|_{W(L^1)} \le \int_{|t|\le 1} 3^d|t|\, \big\||\nabla_\phi|\big\|_{W(L^1)} |\psi_a(t)|\, dt$$

$$=\; 3^d\big\||\nabla_\phi|\big\|_{W(L^1)} \int_{|t|\le 1} |t||\psi_a(t)|\; dt$$

$$=\; 3^d\big\||\nabla_\phi|\big\|_{W(L^1)} a \int_{|t|\le a^{-1}} |t||\psi\,(t)|\; dt. \qquad (46)$$

Combining (24), (25), and (46), we obtain the following estimate for the $W(L^1)$-norm of ϕ^a:

$$\|\phi^a\|_{W(L^1)} \le\; (1\;+\;2^d)\|\phi\|_{W(L^1)} \int_{|t|\le a^{-1}} |\psi(t)|\; dt$$

$$+\, 3^d\big\||\nabla_\phi|\big\|_{W(L^1)} a \int_{|t|\le a^{-1}} |t||\psi(t)|\; dt. \qquad (47)$$

If $\|\psi\|_{1,\eta} =\; \int_{\mathbb{R}^d} |\psi(t)||t|^\eta\; dt \;<\; \infty$ for some $0 \;<\; \eta \;\le\; 1$, then, by (47),

$$\|\phi^a\|_{W(L^1)} \le\; (1\;+\;2^d)\|\phi\|_{W(L^1)} a^\eta \|\psi\|_{1,\eta} \;+ 3^d\big\||\nabla_\phi|\big\|_{W(L^1)} a^\eta \|\psi\|_{1,\eta}$$

$$=\; a^\eta((1\;+\;2^d)\|\phi\|_{W(L^1)}) + 3^d\big\||\nabla_\phi|\big\|_{W(L^1)})\|\psi\|_{1,\eta}. \qquad (48)$$

The desired result in Theorem(3.1.10) then follows from(31),(45),and (48).

Theorem $(3.1.11)$[195]: Assume that Φ and $\psi_X = (\psi_{x_j})_{j\,\in\,J}$ satisfy the conditions of Theorem (3.1.8) and that $|\nabla\phi_i| \;\in\; W(L^1)$ for every $i \;=\; 1,\ldots,r$. Then the convergence rate α in Theorem (3.1.8) satisfies

$$\alpha \;\le\; 3^d B\|P\|_{op}(\gamma \;+\; M(6^d \;+\; 1)a) \max_{1\le i\le r} \||\nabla\phi_i|\|_{W(L^1)}, \qquad (49)$$

where B is the upper bound constant in (14) and M is the upper bound in Theorem (3.1.8).

Section(3.2)Average Sampling in Shift Invariant Subspaces

The sampling theory says that if a function $f(x)$ satisfies certain conditions, then it is uniquely determined and can be reconstructed from its sampled values at a sequence of sampling points $\{x_k\colon k \in \mathbb{Z}\}$, i.e., there exist some functions $S_k(x)$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} f(x_k)S_k(x).$$

For example, every band-limited function
$f \in B_\Omega = \{ f : supp\hat{f} \subset [-\Omega, \Omega] \}$ can be reconstructed by the formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(k\pi/\Omega)(sin(\Omega x - k\pi))/(\Omega x - k\pi).$$

This is the classical Shannon sampling theorem. Although the assumption that a function is band-limited is eminently useful, it is not always realistic since a band-limited function is of infinite duration. Thus, it is natural to investigate other function classes for which a sampling theorem holds. A simple model is to consider shift-invariant subspaces, which generalize the space of band-limited functions and have the form

$$V_0 = \overline{span}\{\phi(\cdot - k) : k \in \mathbb{Z}\}$$

for some generating function $\phi(x)$. In fact, there have been many results concerning the sampling in shift-invariant subspaces for both regular and irregular sampling, see [135,140, 150,154,157,159,263,162,169,176,178,179,182,183,191,192,194].

For physical reasons, e.g., the inertia of the measurement apparatus, measured sampled values obtained in practice may not be values of a function f precisely at times x_k , but only local averages of f near x_k .Specifically, measured sampled values are

$$\langle f, u_k \rangle = \int f(x)u_k(x)\, dx$$

for some collection of averaging functions $u_k(x), k \in \mathbb{Z}$, which satisfy the following properties:

$$\text{supp}u_k \subset \left[x_k - \frac{\delta}{2}, x_k + \frac{\delta}{2}\right], u_k(x) \geq 0 \text{, and } \int u_k(x)\, dx = 1.$$

Observe that the averaging procedure is allowed to vary form point to point.

It is clear that from local averages one should obtain at least a good approximation of the original function if δ is small enough. Wiley [193], Butzer and Lei [166,167] studied the approximation error when local averages are used as sampled values. Furthermore, Gröchenig [174] proved that if sampling points x_k satisfy

$$0 < x_{k+1} - x_k \leq \delta < \frac{1}{\sqrt{2}\Omega} ,$$

then every $f \in B_\Omega$ is uniquely determined and can be reconstructed by local averages $\langle f, u_k \rangle$ around x_k . Specifically, there are some functions $S_k(x) \in B_\Omega$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f, u_k \rangle S_k(x). \tag{50}$$

In [166], Feichtinger and Gröchenig proved that if

$$\delta = \sup_{k\in\mathbb{Z}}(x_{k+1} - x_k) < \frac{\pi}{\Omega},$$

then every $f \in B_\Omega$ is uniquely determined by

$$(1/(y_k - y_{k-1})) \int_{y_{k-1}}^{y_k} f(x)\,dx \quad \text{with} \quad y_k = (y_k + y_{k+1})/2$$

and can be reconstructed with a formula similar to (50). If $u_k(x)$ are taken to be translations of a generating function, i.e., $u_k(x) = u(x - x_k)$ for some averaging function $u(x)$, then the average sampling procedure can be viewed as prefiltering, which is widely studied in literature. In [161,162,191], Aldroubi and Unser studied the reconstruction of signals by means of prefiltering and sampling in more general sense.

In [184,186,189], we studied average sampling in shift invariant subspaces with arbitrary averaging functions and gave the optimal upper bound for the support length of averaging functions for some special cases. In [156,158,160], Aldroubi, Feichtinger, Sun and Tang studied density conditions on sampling points and fast iterative reconstruction algorithms, for which the performance were analyzed when the data were corrupted by noise. We study the reconstruction of functions in shift invariant subspaces from local averages with equally spaced sampling points and symmetric averaging functions. Specifically, the averaging function $u_k(x)$ is symmetric with respect to $x = x_k$ and nonincreasing on $[x_k, x_k + \delta/2]$. A simple example is

$$u_k(x) = \frac{1}{\delta_k}\chi_{[x_k-\delta_k/2, x_k+\delta_k/2]}(x), 0 < \delta_k < \delta.$$

We present an average sampling theorem and give explicit error bounds for the aliasing error and the truncation error. Since the classical point sampling can be viewed as a special case of average sampling, i.e., $u_k(x)$ are δ −functions concentrated at x_k , our results also give new error bounds for regular sampling. At the end of section, we give some examples.

The Fourier transform and the Zak transform of $f \in L^2(\mathbb{R})$ are defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}\,dx \quad \text{and} \quad Zf(x,\omega) = \sum_{k\in\mathbb{Z}} f(x + k)e^{-ik\omega},$$

respectively;

$$[\hat{f}, \hat{g}](\omega) = \sum_{k\in\mathbb{Z}} f(\omega + 2k\pi)\bar{\hat{g}}(\omega + 2k\pi).$$

We call $u(x)$ an averaging function if $u(x) \geq 0, u(x) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} u(x)\,dx = 1$.

Recall that a family of functions $\{\phi_k : k \in \mathbb{Z}\}$ belonging to a Hilbert space $\mathcal{H}$ is said to be a frame if there exist positive constants A and B such that $A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, \phi_k\rangle|^2 \leq B\|f\|^2$ for every $f \in \mathcal{H}$. The numbers A and B are called the lower and upper frame bounds, respectively. $\{\phi_k : k \in \mathbb{Z}\}$ is said to be a Riesz basis for $\mathcal{H}$ if it is complete in $\mathcal{H}$ and there are positive constants A and B such that for any

$$c = \{c_k : k \in \mathbb{Z}\} \in \ell^2, A\|c\|_2^2 \leq \left\| \sum_{k \in \mathbb{Z}} c_k \phi_k \right\|_2^2 \leq B\|c\|_2^2.$$

A frame that ceases to be a frame when any one of its elements is removed is said to be an exact frame. It is well known that exact frames and Riesz bases are identical. Let $\{\phi_k : k \in \mathbb{Z}\}$ be a frame for some Hilbert space $\mathcal{H}$. The frame operator S is defined by

$$Sf = \sum_{k \in \mathbb{Z}} \langle f, \phi_k \rangle \phi_k \ , \qquad \forall f \in \mathcal{H} \quad .$$

It can be proved that S is a bounded, invertible, and self-adjoint operator on H. Let $\tilde{\phi}_k = S^{-1}\phi_k$. Then $\{\tilde{\phi}_k : k \in \mathbb{Z}\}$ is also a frame for $\mathcal{H}$, called the dual frame of $\{\phi_k : k \in \mathbb{Z}\}$. For any $f \in H$, we have

$$f = \sum_{k \in \mathbb{Z}} \langle f, \phi_k \rangle \tilde{\phi}_k = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_k \rangle \phi_k \ .$$

We refer to [142,152] for details on the frame theory.

Proposition$(3.2.1)$[155]: (see [175]). If f(x) is differentiable on $[a,b]$, $f, f' \in L^2[a,b]$, and $f(a)f(b) = 0$, then

$$\int_a^b |f(x)|^2 \, dx \leq \frac{4}{\pi^2}(b - a)^2 \int_a^b |f'(x)|^2 \, dx.$$

Lemma$(3.2.2)$[155]: If $f(x)$ *is* differentiable on $[a,b], f, f' \in L^2[a,b]$ and there is some $c \in [a,b]$ such that $f(c) = 0$, then

$$\int_a^b |f(x)|^2 \, dx \leq \frac{4\delta^2}{\pi^2} \int_a^b |f'(x)|^2 \, dx \, ,$$

where $\delta = \max\{c - a, b - c\}$.

Proposition $(3.2.3)$[155]: Let f be an integrable function on $[a,b]$ and let $F(x) = \int_a^x f(t)\,dt, |F(x)| \leq M(x - a)$ for $a < x \leq b$ (M a positive constant); furthermore, let g be a nonnegative, nonincreasing and integrable function. Then

$$\left| \int_a^b f(x)g(x)\,dx \right| \leq M \int_a^b g(x)\,dx.$$

Lemma$(3.2.4)$[155]: Let $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ be a Riesz basis for some $V_0 \subset L^2(\mathbb{R})$. Suppose that ϕ is locally absolutely continuous and $\phi' \in L^2(\mathbb{R})$. Then for any $\{c_n : n \in \mathbb{Z}\} \in \ell^2$, $f = \sum_{n \in \mathbb{Z}} c_n \phi(\cdot - n)$ is locally absolutely continuous and $f'(x) = \sum_{n \in \mathbb{Z}} c_n \phi'(x - n)$ *a.e.*

Proof : For any $\{c_n : n \in \mathbb{Z}\} \in \ell^2$ and $a < b$, we have

$$\int_a^b \sum_{n\in\mathbb{Z}} |c_n\phi'(x-n)|^2\,dx \le \int_a^b \left(\sum_{n\in\mathbb{Z}} |c_n|^2\right)^{1/2} \left(\sum_{n\in\mathbb{Z}} |\phi'(x-n)|^2\right)^{1/2} dx$$

$$\le \left(\sum_{n\in\mathbb{Z}} |c_n|^2\right)^{1/2} (b-a)^{1/2} \left(\int_a^b \sum_{n\in\mathbb{Z}} |\phi'(x-n)|^2\,dx\right)^{1/2} < +\infty,$$

thanks to $\phi \in L^2(\mathbb{R})$. Hence $\sum_{n\in\mathbb{Z}} c_n\phi'(x-n)$ is both absolutely convergent almost everywhere and convergent in $L^1[a,b]$ for any $a<b$.

Similarly, $\sum_{n\in\mathbb{Z}} c_n\phi(x-n)$ is convergent almost everywhere. Suppose that $\sum_{n\in\mathbb{Z}} c_n\phi \times (x_0-n)$ is convergent for some x_0. Then for any $x \in \mathbb{R}$, we have

$$\sum_{n\in\mathbb{Z}} c_n(\phi(x-n)-\phi(x_0-n)) = \int_{x_0}^{x} \sum_{n\in\mathbb{Z}} c_n\phi'(t-n)\,dt.$$

Hence $f(x) = \sum_{n\in\mathbb{Z}} c_n\phi(x-n)$ is well defined on $\mathbb{R}$. Moreover, the above equation also implies that f is locally absolutely continuous and

$$f'(x) = \sum_{n\in\mathbb{Z}} c_n\phi'(x-n)\ a.e.$$

This completes the proof .

Theorem$(3.2.5)[155]$: Let $\{\phi(\cdot-n): n \in \mathbb{Z}\}$ be a Riesz basis for V_0 with bounds A and B. Suppose that ϕ is locally absolutely continuous, $\phi' \in L^2(\mathbb{R}), |Z_{\phi'}(x,\omega)| \le L, a.e.$, and there are two positive constants C_1 and C_2 such that

$$C_1 \le |Z_\phi(0,\omega)| \le C_2\ a.e. \tag{51}$$

Let $\{u_K(x): k \in \mathbb{Z}\}$ be a sequence of averaging functions such that $\operatorname{supp} u_K \subset [k-\frac{\delta}{2},\ k+\frac{\delta}{2}], u_K(x+k)$ is even and nonincreasing on $[0,\frac{\delta}{2}]$.

If $0 < \delta < \frac{\pi C_1}{L}$, there is a frame $\{S_K(x): k \in \mathbb{Z}\}$ for V_0 such that for any $f \in V_0$,

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f, u_K\rangle S_K(x), \tag{52}$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$.

Proof: For any $f \in V_0$, there is some $c = \{c_k: k \in \mathbb{Z}\} \in \ell^2$ such that $f(x) = \sum_{n\in\mathbb{Z}} c_n\phi(x-n)$. By the definition of Riesz basis, we have

$$\frac{1}{B}\|f\|_2^2 \le \|c\|_2^2 = \sum_{n\in\mathbb{Z}} |c_n|^2 \le \frac{1}{A}\|f\|_2^2.$$

By Lemma (3.2.4), f is locally absolutely continuous and $f'(x) = \sum_{n\in\mathbb{Z}} c_n\phi'(x-n)\ a.e.$ By Proposition (3. 2.1), for any $0 < x < \frac{\delta}{2}$,

$$\int_0^x |f(t+k) - f(k)|^2\,dt \le \frac{4x^2}{\pi^2}\int_0^x |f'(t+k)|^2\,dt \le \frac{2\delta}{\pi^2}\int_0^{\delta/2} |f'(t+k)|^2\,dt \cdot x\,.$$

It follows from Proposition (3. 2.3) that

$$\int_0^{\frac{\delta}{2}} |f(x+k) - f(k)|^2\, u_k(x+k)dx$$

$$\le \frac{2\delta}{\pi^2}\int_0^{\delta/2} |f'(x+k)|^2\,dx \int_0^{\delta/2} u_k(x+k)\,dx = \frac{\delta}{\pi^2}\int_0^{\delta/2} |f'(x+k)|^2\,dx.$$

A similar argument shows that

$$\int_{-\frac{\delta}{2}}^{0} |f(x+k) - f(k)|^2\, u_k(x+k)dx = \frac{\delta}{\pi^2}\int_{-\frac{\delta}{2}}^{0} |f'(x+k)|^2\,dx.$$

Hence

$$\sum_{k\in\mathbb{Z}} |\langle f, u_k\rangle - f(k)|^2 = \sum_{k\in\mathbb{Z}}\left|\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \big(f(x+k) - f(k)\big)\, u_k(x+k)dx\right|^2$$

$$\le \sum_{k\in\mathbb{Z}}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |f(x+k) - f(k)|^2\, u_k(x+k)dx \le \sum_{k\in\mathbb{Z}}\frac{\delta}{\pi^2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |f'(x+k)|^2\,dx$$

$$= \frac{\delta}{\pi^2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}} c_n\phi'(x+k-n)\right|^2 dx$$

$$= \frac{\delta}{\pi^2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|C(\omega)Z_{\phi'}(x,\omega)\right|^2 d\omega dx$$

$$\le \frac{\delta}{\pi^2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} L^2\|c\|_2^2\,dx = \frac{\delta^2L^2}{\pi^2}\|c\|_2^2\,,$$

where $C(\omega) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\omega}$. But

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 = \sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}} c_n\phi\,(k-n)\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|C(\omega)Z_\phi(0,\omega)\right|^2 d\omega \ge C_1^2\|c\|_2^2\,.$$

Therefore,

$$\sum_{k\in\mathbb{Z}} |\langle f, u_k\rangle|^2 \ge \left(C_1 - \frac{\delta L}{\pi}\right)^2 \|c\|_2^2 \ge \frac{1}{B}\left(C_1 - \frac{\delta L}{\pi}\right)^2 \|f\|_2^2 .$$

Similar arguments show that

$$\sum_{k\in\mathbb{Z}} |\langle f, u_k\rangle|^2 \ge \frac{1}{A}\left(C_2 + \frac{\delta L}{\pi}\right)^2 \|f\|_2^2 \quad .$$

Let $\tilde{u}_k(x)$ be the orthogonal projection of $u_k(x)$ onto V_0 . Then $\langle f, \tilde{u}_k\rangle = \langle f, u_k\rangle$ for any $f \in V_0$. It follows that

$$\frac{1}{B}\left(C_1 - \frac{\delta L}{\pi}\right)^2 \|f\|_2^2 \le \sum_{k\in\mathbb{Z}} |\langle f, \tilde{u}_k\rangle|^2 \le \frac{1}{A}\left(C_2 + \frac{\delta L}{\pi}\right)^2 \|f\|_2^2 \quad .$$

Consequently, $\{\tilde{u}_k(x): k \in \mathbb{Z}\}$ is a frame for V_0. Let $\{S_k(x): k \in \mathbb{Z}\}$ be the dual frame. Then for any $f \in V_0$,

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f, \tilde{u}_k\rangle S_k(x) = \sum_{k\in\mathbb{Z}} \langle f, u_k\rangle S_k(x) .$$

To prove the uniform convergence,we need only to show that $\sum_{k\in\mathbb{Z}} |S_k(x)|^2$ is bounded on $\mathbb{R}$. Since

$$\left|Z_\phi(x,\omega) - Z_\phi(0,\omega)\right| \le \left|\int_0^x \left|Z_{\phi'}(t,\omega)\right| dt\right| \le L|x| \text{, we have}$$

$|Z_\phi(x,\omega)| \le L|x| + C_2$. It follows that for $|x| \le 1/2$ (and thus for any $x \in \mathbb{R}$),

$$\sum_{k\in\mathbb{Z}} |\phi(x-k)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Z_\phi(x,\omega)\right|^2 d\omega \le \left(\frac{L}{2} + C_2\right)^2 . \tag{53}$$

Hence, for any

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \phi(x-k) \in V_0 \quad ,$$

$$\|f\|_\infty^2 \le \sup_x \sum_{k\in\mathbb{Z}} |\phi(x-k)|^2 \|c\|_2^2 \le \left(\frac{L}{2} + C_2\right)^2 \frac{1}{A}\|f\|_2^2 .$$

Therefore,

$$\left(\sum_{k\in\mathbb{Z}} |S_k(x)|^2\right)^{1/2} = \sup_{\|c\|_2=1} \left|\sum_{k\in\mathbb{Z}} c_k S_k(x)\right| \le \sup_{\|c\|_2=1} \frac{L/2 + C_2}{\sqrt{A}} \left\|\sum_{k\in\mathbb{Z}} c_k S_k\right\|_2 .$$

Since $\{S_k(x): k \in \mathbb{Z}\}$ is a frame for V_0 with upper bound $B/(C_1 - \delta L/\pi)^2$, we have

$$\left(\sum_{k\in\mathbb{Z}} |\boldsymbol{S}_k(\boldsymbol{x})|^2\right)^{\mathbf{1/2}} \le \frac{\left(\frac{\boldsymbol{L}}{\mathbf{2}} + \boldsymbol{C}_2\right)\sqrt{B}}{\left(C_1 - \frac{\delta L}{\pi}\right)\sqrt{A}} , \forall x \in \mathbb{R} . \tag{54}$$

This completes the proof.

Proposition $(3.2.6)[155]$: Suppose that $\phi \in L^2(\mathbb{R})$ and $V_0 = \overline{\text{span}}\{\phi(\cdot - n): n \in \mathbb{Z}\}$. Then $\{\phi(\cdot - n): n \in \mathbb{Z}\}$ *is* a Riesz basis for V_0 with bounds A and B if and only if

$$A \leq [\hat{\phi}, \hat{\phi}](\omega) \leq B \ a.e. \tag{55}$$

Moreover, if the above inequalities hold and let

$$\hat{\tilde{\phi}}(\omega) = \frac{\hat{\phi}(\omega)}{[\hat{\phi}, \hat{\phi}](\omega)} , \tag{56}$$

then $\{\tilde{\phi}(\cdot - k): k \in \mathbb{Z}\}$ is the dual Riesz basis of $\{\phi(\cdot - k): k \in \mathbb{Z}\}$ with bounds $\frac{1}{B}$ and $\frac{1}{A}$. Consequently,

$$f(x) = \sum_{k \in \mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \tilde{\phi}(x - k), \forall f \in V_0.$$

Lemma $(3.2.7)[155]$: Let the hypotheses be as in Theorem(3. 2.5) . Moreover, suppose that ϕ satisfies the first order Strang–Fix condition [181] , $i.e.$, $\hat{\phi}(2k\pi) = \delta_{k,0}$, $k \in \mathbb{Z}$, and $x\phi(x) \in L^2(\mathbb{R})$. Let $V_h = \{f : f(h\,\cdot) \in V_0\}$ and P_h be the orthogonal projection operator from $L^2(\mathbb{R})$ onto V_h . *If* $f, f' \in V_h$, then

$$\|f - P_h f\|_2 \leq h\left(\frac{2}{\pi} + \frac{2\pi}{A}\|x\phi(x)\|_2^2\right)\|f'\|_2 ,$$

$$\|(f - P_h f)'\|_2 \leq \left(1 + \frac{L}{\pi\sqrt{A}}\right)\|f'\|_2 .$$

Proof : Define $\tilde{\phi}$ as in (56). By Proposition (3.1.6), it is easy to check that $\{h^{-1/2}\phi(\cdot/h - k): k \in \mathbb{Z}\}$ and $\{h^{-1/2}\tilde{\phi}(\cdot/h - k): k \in \mathbb{Z}\}$ are dual Riesz bases for V_h . Hence

$$(P_h f)(x) = \sum_{k \in \mathbb{Z}} \langle f, \frac{1}{h}\phi\left(\frac{1}{h}\cdot - k\right)\rangle \tilde{\phi}\left(\frac{x}{h} - k\right)$$

and

$$(P_h \hat{f})(\omega) = \sum_{k \in \mathbb{Z}} \hat{f}\left(\omega + \frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega + 2k\pi)\hat{\tilde{\phi}}(h\omega). \tag{57}$$

For any $|\omega| \leq \pi$, we have

$$\begin{aligned}\left|1 - \bar{\hat{\phi}}(\omega)\hat{\tilde{\phi}}(\omega)\right| &= \sum_{k \neq 0} \left|\hat{\phi}(\omega + 2k\pi)\right|^2 / [\hat{\phi}, \hat{\phi}](\omega)\\ &= \sum_{k \neq 0} \left|\int_0^{\omega} \hat{\phi}'(\xi + 2k\pi)\, d\xi\right|^2 / [\hat{\phi}, \hat{\phi}](\omega)\\ &\leq \frac{1}{A}|\omega| \int_{-\pi}^{\pi} \sum_{K \neq 0} \left|\hat{\phi}'(\xi + 2k\pi)\right|^2 d\xi \ \left(by\ (55)\right)\\ &\leq \frac{1}{A}|\omega| \cdot \left\|\hat{\phi}'\right\|_2^2 = \frac{2\pi}{A}|\omega| \cdot \|x\phi(x)\|_2^2 .\end{aligned} \tag{58}$$

Noting that

$$0 \ \le \ \bar{\hat{\phi}}(\omega)\hat{\tilde{\phi}}(\omega) = \frac{\left|\hat{\phi}(\omega)\right|^2}{[\,\hat{\phi},\hat{\phi}\,](\omega)} \le 1 \quad , \tag{59}$$

we see from (58) that

$$\left\|\hat{f}(\omega)\left(1 \ - \ \bar{\hat{\phi}}(h\omega)\ \hat{\tilde{\phi}}(h\omega)\right)\right\|_2^2$$

$$\le \int\limits_{|\omega|\le\pi/h} \left|\hat{f}(\omega)\right|^2 \left|1 \ - \ \bar{\hat{\phi}}(h\omega)\ \hat{\tilde{\phi}}(h\omega)\right|^2 d\omega \ + \int\limits_{|\omega|>\pi/h} \left|\hat{f}(\omega)\right|^2 d\omega$$

$$\le \frac{4\pi^2}{A^2}\|x\phi(x)\|_2^4 h^2|\omega| \int\limits_{|\omega|\le\pi/h} \left|\omega\hat{f}(\omega)\right|^2 d\omega + \frac{h^2}{\pi^2}\int\limits_{|\omega|>\pi/h} \left|\omega\hat{f}(\omega)\right|^2 d\omega$$

$$\le h^2\left(\frac{8\pi^3}{A^2}\|x\phi(x)\|_2^4 + \frac{2}{\pi}\right)\|f'\|_2^2 \ . \tag{60}$$

On the other hand, by setting $\hat{f}_1(\omega) \ = \ \hat{f}\,(\omega) - \hat{f}(\omega)\chi_{[-\pi/h,\pi/h]}(\omega)$, we have

$$\int\limits_{-\infty}^{+\infty} \left|\sum_{k\neq 0} \hat{f}\left(\omega \ + \ \frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega \ + \ 2k\pi)\hat{\tilde{\phi}}(h\omega)\right|^2 d\omega$$

$$= \int\limits_{-\infty}^{+\infty} \left|\sum_{k\in\mathbb{Z}} \hat{f}_1\left(\omega \ + \ \frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega \ + \ 2k\pi)\hat{\tilde{\phi}}(h\omega)\right|^2 d\omega$$

$$= \int\limits_{-\pi/h}^{\pi/h} \left|\sum_{k\in\mathbb{Z}} \hat{f}_1\left(\omega \ + \ \frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega \ + \ 2k\pi)\right|^2 \left[\,\hat{\tilde{\phi}},\hat{\tilde{\phi}}\,\right](h\omega)d\omega$$

$$\le \int\limits_{-\pi/h}^{\pi/h} \sum_{k\in\mathbb{Z}} \left|\hat{f}_1\left(\omega \ + \ \frac{2k\pi}{h}\right)\right|^2 [\,\hat{\phi},\hat{\phi}\,](h\omega)\left[\,\hat{\tilde{\phi}},\hat{\tilde{\phi}}\,\right](h\omega)d\omega$$

$$= \int\limits_{|\omega|>\pi/h} \left|\hat{f}(\omega)\right|^2 d\omega \ \le \int\limits_{|\omega|>\pi/h} \left|\omega\hat{f}(\omega)\right|^2 . \frac{h^2}{\pi^2} d\omega \le \frac{2h^2}{\pi}\|f'\|_2^2 \ . \tag{61}$$

Putting (57), (60), and (61) together, we have

$$\|f \ - \ P_h f\|_2 \ = \ \frac{1}{\sqrt{2\pi}}\|(f \ - \ P_h f)\hat{}\|_2$$

$$= \frac{1}{\sqrt{2\pi}}\left\|\hat{f}(\omega)\left(1 - \ \bar{\hat{\phi}}(h\omega)\hat{\tilde{\phi}}(h\omega)\right) - \sum_{k\neq 0} \hat{f}\left(\omega + \frac{2k\pi}{h}\right)\ \bar{\hat{\phi}}(h\omega + 2k\pi)\hat{\tilde{\phi}}(h\omega)\right\|_2$$

$$\le h\left(\frac{2}{\pi}\pi + \frac{2\pi}{A}\|x\phi(x)\|_2^2\right)\|f'\|_2 \ .$$

Next we prove the second inequality. By [163],

$$\left|Z_{\phi'}(x,\omega)\right| = \left|\sum_{k\in\mathbb{Z}} \hat{\phi}'(\omega \ + 2k\pi)e^{i2k\pi x}\right| \ \ a.e.$$

Hence

$$[\hat{\phi}',\hat{\phi}'](\omega)=\sum_{k\in\mathbb{Z}}\left|\hat{\phi}'(\omega+2k\pi)\right|^2=\int_0^1\left|Z_{\phi'}(x,\omega)\right|^2dx\le L^2 .$$

By (56), it is easy to see that $[\hat{\tilde{\phi}}',\hat{\tilde{\phi}}'](\omega)=[\hat{\phi}',\hat{\phi}'](\omega)/|[\hat{\phi},\hat{\phi}](\omega)|^2$. Hence,

$$\sum_{k\in\mathbb{Z}}\left|(\omega+2k\pi)\hat{\tilde{\phi}}(\omega+2k\pi)\right|^2=\left[\hat{\tilde{\phi}}',\hat{\tilde{\phi}}'\right](\omega)$$

$$=\frac{[\hat{\phi}',\hat{\phi}'](\omega)}{\left|[\hat{\phi},\hat{\phi}](\omega)\right|^2}\le\frac{L^2}{\left|[\hat{\phi},\hat{\phi}](\omega)\right|^2}.$$

It follows that

$$\int_{-\infty}^{+\infty}\left|\omega\sum_{k\neq0}\hat{f}\left(\omega+\frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega+2k\pi)\hat{\tilde{\phi}}(h\omega)\right|^2d\omega$$

$$=\int_{-\pi/h}^{\pi/h}\frac{1}{h^2}\left|\sum_{k\in\mathbb{Z}}\hat{f}_1\left(\omega+\frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega+2k\pi)\right|^2\left[\hat{\tilde{\phi}}',\hat{\tilde{\phi}}'\right](h\omega)d\omega$$

$$\le\frac{1}{h^2}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\sum_{k\in\mathbb{Z}}\left|\hat{f}_1\left(\omega+\frac{2k\pi}{h}\right)\right|^2[\hat{\phi},\hat{\phi}](\omega)\frac{L^2}{\left|[\hat{\phi},\hat{\phi}](\omega)\right|^2}d\omega$$

$$\le\frac{L^2}{Ah^2}\int_{|\omega|>\pi/h}\left|\hat{f}_1(\omega)\right|^2d\omega \qquad \text{(by (55))}$$

$$\le\frac{L^2}{A\pi^2}\int_{|\omega|>\pi/h}\left|\omega\hat{f}(\omega)\right|^2d\omega\le\frac{2L^2}{A\pi}\|f'\|_2^2 .$$

On the other hand, we see from (59) that

$$\left\|\omega\hat{f}(\omega)\left(1-\bar{\hat{\phi}}(h\omega)\hat{\tilde{\phi}}(h\omega)\right)\right\|_2\le\|\omega\hat{f}(\omega)\|_2=\sqrt{2\pi}\|f'\|_2.$$

Hence

$$\|(f-P_hf)'\|_2=\frac{1}{\sqrt{2\pi}}\|\omega(f-P_hf)^\wedge(\omega)\|_2$$

$$=\frac{1}{\sqrt{2\pi}}\left\|\omega\hat{f}(\omega)\left(1-\bar{\hat{\phi}}(h\omega)\hat{\tilde{\phi}}(h\omega)\right)-\omega\sum_{k\neq0}\hat{f}\left(\omega+\frac{2k\pi}{h}\right)\bar{\hat{\phi}}(h\omega+2k\pi)\hat{\tilde{\phi}}(h\omega)\right\|_2$$

$$\le\left(1+\frac{L}{\pi\sqrt{A}}\right)\|f'\|_2 .$$

Theorem $(3.2.8)$[257] : Let the hypotheses be as in Theorem (3.2.5). Moreover, suppose that $\hat{\phi}(2k\pi)=\delta_{k,0}$ and $x\phi(x)\in L^2(\mathbb{R})$. Let

$$(R_hf)(x)=\sum_{k\in\mathbb{Z}}\langle f,\frac{1}{h}u_k\left(\frac{1}{h}\cdot\right)\rangle S_k\left(\frac{x}{h}\right) . \qquad (62)$$

(i) If f is locally absolutely continuous and $f \in L^2(\mathbb{R})$, then

$$\|f - R_h f\|_2 \le Ch\|f'\|_2 , \tag{63}$$

where

$$C = \left(1 + \frac{\sqrt{B}}{C_1 - \delta L/\pi}\right)\left(\frac{2}{\pi} + \frac{2\pi}{A}\|x\phi(x)\|_2^2\right) + \frac{(1 + \delta)\sqrt{B}}{C_1 - \delta L/\pi\sqrt{2}}\left(1 + \frac{L}{\pi\sqrt{A}}\right).$$

(ii) If $u_k \in L^2(\mathbb{R})$ and $\|u_k\|_2^2 \le M < +\infty$, $k \in \mathbb{Z}$, then

$$\lim_{h \to 0^+} \|f - R_h f\|_2 = 0 , \qquad \forall f \in L^2(\mathbb{R}) .$$

Proof: Let $a_k = \int_{-1/2}^{1/2} f(x + k)\, dx$. Then

$$\sum_{k \in \mathbb{Z}} |\langle f, u_k\rangle - a_k|^2 = \sum_{k \in \mathbb{Z}} \left| \int_{-\delta/2}^{\delta/2} \int_{-1/2}^{1/2} (f(x + k) - f(y + k)) u_k(x + k) dy\, dx \right|^2$$

$$= \sum_{k \in \mathbb{Z}} \left| \int_{-\delta/2}^{\delta/2} \int_{-1/2}^{1/2} \int_{0}^{x-y} f'(y + k + t)\, dt u_k(x + k)\, dy\, dx \right|^2$$

$$\le \sum_{k \in \mathbb{Z}} \int_{-\delta/2}^{\delta/2} \int_{-1/2}^{1/2} \int_{-|x-y|}^{|x-y|} |f'(y + k + t)|^2\, dt |x - y| u_k(x + k)\, dy\, dx$$

$$\le \sum_{k \in \mathbb{Z}} \frac{1 + \delta}{2} \int_{-\delta/2}^{\delta/2} \int_{-1/2}^{1/2} \int_{-(1+\delta)/2}^{(1+\delta)/2} |f'(y + k + t)|^2\, dt u_k(x + k)\, dy\, dx$$

$$= \frac{(1 + \delta)^2}{2} \|f'\|_2^2 .$$

Noting that

$$\sum_{k \in \mathbb{Z}} |ak|^2 \le \sum_{k \in \mathbb{Z}} \int_{-1/2}^{1/2} |f(x + k)|^2\, dx = \|f\|_2^2 ,$$

we have

$$\sum_{k \in \mathbb{Z}} |\langle f, u_k\rangle|^2 \le \left(\|f\|_2 + \frac{1 + \delta}{\sqrt{2}}\|f'\|_2\right)^2 . \tag{64}$$

Since $\{S_k(x): k \in \mathbb{Z}\}$ is a frame for V_0 with upper bound $\frac{B}{\left(C_1 - \frac{\delta L}{\pi}\right)^2}$,

we have

$$\|R_h f\|_2^2 = \left\| \sum_{k\in\mathbb{Z}} \langle f, \frac{1}{h} u_k\left(\frac{1}{h}\cdot\right)\rangle S_k\left(\frac{x}{h}\right) \right\|_2^2 = h \left\| \sum_{k\in\mathbb{Z}} \langle f(h\,\cdot), u_k\rangle S_k(x) \right\|_2^2$$

$$\leq \frac{B}{(C_1 - \delta L/\pi)^2}\; h \sum_{k\in\mathbb{Z}} |\langle f(h\,\cdot), u_k\rangle|^2$$

$$\leq \frac{B}{(C_1 - \delta L/\pi)^2}\left(h\|f(h\,\cdot)\|_2 + \frac{1+\delta}{\sqrt{2}}\|hf'(h\,\cdot)\|_2\right)^2$$

$$= \frac{B}{(C_1 - \delta L/\pi)^2}\|f\|_2 + \frac{1+\delta}{\sqrt{2}} h\|f'\|_2.$$

Let $V_h = \{f : f(h\,\cdot) \in V_0\}$ and P_h be the orthogonal projection operator form $L^2(\mathbb{R})$ onto V_h. Then $R_h P_h = P_h$ and so

$$\|f - R_h f\|_2 \leq \|f - P_h f\|_2 + \|R_h(f - P_h f)\|_2$$

$$\leq \left(1 + \frac{\sqrt{B}}{C_1 - \delta L/\pi}\right)\|f - P_h f\|_2 + \frac{(1+\delta)\sqrt{B}}{(C_1 - \delta L/\pi)\sqrt{2}} \cdot \|(f - P_h)' f\|_2.$$

Now (63) follows from Lemma (3.2.7) . Next we prove (ii). For any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, u_k\rangle|^2 \leq \sum_{k\in\mathbb{Z}} \int_{-\delta/2}^{\delta/2} |f(x+k)|^2\,dx \int_{-\delta/2}^{\delta/2} |u_k(x+k)|^2\,dx$$

$$\leq M \int_{-\delta/2}^{\delta/2} \sum_{k\in\mathbb{Z}} |f(x+k)|^2\,dx \leq M\lceil\delta\rceil \|f\|_2^2 \;,$$

where $\lceil\delta\rceil = \min\{n : n \geq \delta, n \in \mathbb{Z}\}$. Similarly to (i) we can prove that $\|f - P_h f\|_2 \leq C'\|f - P_h f\|_2$, where $C'$is a constant independent of h. By the wavelet theory (see [177]), we know that $\|f - P_h f\|_2 \to 0$ as $h \to 0$. Therefore, $\lim_{h\to 0}\|f - P_h f\|_2 = 0$. This completes the proof.

In practice we can handle only finite sums. The error made by cutting off infinite sums is the truncation error. Specifically, it is defined by

$$(T_N f)(x) = f(x) - \sum_{|k|\leq N} \langle f, u_k\rangle S_k(x)\,.$$

For the truncation error, we have

Theorem (3.2.9)[257]: Let the hypotheses be as in Theorem (3.2.5). Then for any $f \in V_0$, we have $f \in L^2(\mathbb{R})$ and

$$\|T_N f\|_2 \leq \frac{\sqrt{B}}{C_1 - \delta L/\pi}\left(\left\|\tau_{N,\delta} f\right\|_2 + \frac{1+\delta}{\sqrt{2}}\left\|\tau_{N,\delta} f'\right\|_2\right), \tag{65}$$

$$\|T_N f\|_\infty \leq \frac{(L/2 + C_2)\sqrt{B}}{(C_1 - \delta L/\pi)\sqrt{A}}\left(\left\|\tau_{N,\delta} f\right\|_2 + \frac{1+\delta}{\sqrt{2}}\left\|\tau_{N,\delta} f'\right\|_2\right), \tag{66}$$

where $\tau_{N,\delta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$(\tau_{N,\delta}f)(x) = \begin{cases} f(x), & |x| \geq N - \dfrac{\delta}{2}, \\ 0, & otherwise. \end{cases}$$

Proof : For any $f \in V_0$, there is some $c = \{c_k : k \in \mathbb{Z}\} \in \ell^2$ such that $f(x) = \sum_{k\in\mathbb{Z}} c_k \phi \times (x - k)$. By Lemma (3.2.4), f is locally absolutely continuous and $f'(x) = \sum_{k\in\mathbb{Z}} c_k \phi'(x - k)$ a.e. Thus

$$\sum_{n\in\mathbb{Z}} |f'(x + n)|^2 = \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} c_k \phi'(x + n - k) \right|^2$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| C(\omega) Z_{\phi'}(x, \omega) \right|^2 d\omega \leq L^2 \|c\|_2^2 a.e.$$

Therefore, $f' \in L^2(\mathbb{R})$. Similarly to (64) we can prove that

$$\sum_{|k|>N} |\langle f, u_k\rangle|^2 \leq \left(\left\|\tau_{N,\delta} f\right\|_2 + \frac{1+\delta}{\sqrt{2}} \left\|\tau_{N,\delta} f'\right\|_2 \right)^2 .$$

Now the conclusion follows by (54) and the fact that $\{S_k(x) : k \in \mathbb{Z}\}$ is a frame for V_0 with upper bound $B/\left(C_1 - \delta L/\pi\right)^2$.

Theorem (3.2.10)[155]: Let the hypotheses be as in Theorem (3.2.5). Put $\hat{S}(\omega) = \hat{\phi}(\omega)/Z_\phi(0, \omega)$ and

$$(T_N f)(x) = f(x) - \sum_{|k|\leq N} f(k) S(x - k).$$

Then

$$\|T_N f\|_2 \leq \frac{\sqrt{B}}{C_1} \left(\left\|\tau_{N,1} f\right\|_2 + \frac{1}{\pi} \left\|\tau_{N,1} f'\right\|_2 \right),$$

$$\|T_N f\|_\infty \leq C_0 \left(\left\|\tau_{N,1} f\right\|_2 + \frac{1}{\pi} \left\|\tau_{N,1} f'\right\|_2 \right),$$

where $\tau_{N,1}$ is defined as in Theorem (3.1.9) and C_0 is a constant determined in the proof.

Proof : By [256], for any $f \in V_0$,

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) S(x - k).$$

Let $a_k = \int_{-1/2}^{1/2} f(x + k)\, dx$. Then $\sum_{|k|>N} |a_k|^2 \leq \left\|\tau_{N,1} f\right\|_2^2$. By Lemma (3.2.2), it is easy to see that

$$\sum_{|k|>N} |f(k) - a_k|^2 \leq \frac{1}{\pi^2} \left\|\tau_{N,1} f'\right\|_2^2 .$$

Hence,

$$\sum_{|k|>N} |f(k)|^2 \le \left(\left\|\tau_{N,1}f\right\|_2 + \frac{1}{\pi}\left\|\tau_{N,1}f'\right\|_2\right)^2.$$

Since $[\hat{S},\hat{S}](\omega) = [\hat{\phi},\hat{\phi}](\omega)/|Z_\phi(0,\omega)|^2, \{S(\cdot - k): k \in \mathbb{Z}\}$ has an upper frame bound B/C_1^2 , thanks to Proposition (3.1.6). Hence

$$\|T_N f\|_2 = \left\|\sum_{|k|>N} f(k)S(\cdot - k)\right\| \le \frac{\sqrt{B}}{C_1}\left(\left\|\tau_{N,1}f\right\|_2 + \frac{1}{\pi}\left\|\tau_{N,1}f'\right\|_2\right).$$

On the other hand, by (53) , $\sum_{k\in\mathbb{Z}} |\phi(x - k)|^2$ is bounded on $\mathbb{R}$ and

$$C_0^2 = \sup_x |S(x-k)|^2 = \sup_x \frac{1}{2\pi}\int_{-\pi}^{\pi} |ZS(x,\omega)|^2\, d\omega$$

$$= \sup_x \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|Z_\phi(x,\omega)|^2}{|Z_\phi(0,\omega)|^2}\, d\omega < \infty\,.$$

Hence

$$\|T_N f\|_\infty \le \left(\sum_{|k|>N} |f(k)|^2\right)^{1/2} \sup_x \left(\sum_{k\in\mathbb{Z}} |S(x - k)|^2\right)^{1/2}$$

$$\le C_0\left(\left\|\tau_{N,1}f\right\|_2 + \frac{1}{\pi}\left\|\tau_{N,1}f'\right\|_2\right).$$

The proof is over.

Example $(3.2.11)[155]$: Subspaces generated by the centered B-splines ϕ_m defined by $\hat{\phi}_m(\omega) = \left(\frac{\sin\omega/2}{\omega/2}\right)^{m+1}, \quad m \ge 1\,.$

It was shown that $\{\phi_m(\cdot - k): k \in \mathbb{Z}\}$ is a Riesz basis for the subspace $V_0^{(m)}$ it spans and $Z_\phi(0,\omega)$ has no zero on $[-\pi,\pi]$ for any $m \ge 1$ (see[170]). Therefore ϕ_m meets the requirements of Theorem (3. 2.5). Since

$$|Z_{\phi_1'}(x,\omega)| = |1 - e^{-i\omega}| \le 2 \text{ and } \phi_m'(x) = \int_{-1/2}^{1/2} \phi_{m-1}'(x - t)\, dt,$$

we have

$$L^{(m)} = \sup_{x,\omega}\left|Z_{\phi_m'}(x,\omega)\right| = \sup_{x,\omega}\left|\int_{-1/2}^{1/2} Z_{\phi_{m-1}'}(x-t,\omega)\, dt\right| \le 2\,, m \ge 1.$$

Let $C_1^{(m)} = \min_{|\omega|\le\pi} |Z_{\phi_m}(0,\omega)|$. By Theorem (3.2.5), every $f \in V_0^{(m)}$ is uniquely determined by its local averages $\langle f, u_k\rangle$for any sequence of symmetric averaging functions $\{u_k(x): k \in \mathbb{Z}\}$ satisfying

$\mathrm{supp}\, u_k \subset [k - \pi C_1^{(m)}/4\ ,\ k + \pi C_1^{(m)}/4]$. In Table 1, we give the values of $\pi C_1^{(m)}/2$ for $m \leq 8$.

Example $(3.2.12)[155]$: Let $\phi(x) = sin\pi x/\pi x$. Then $V_0 = B_\pi = \{f : \text{supp}\, \hat{f} \subset [-\pi,\pi]\}$. It is easy to check that

$C_1 = C_2 = 1$ and $L = \pi$. Hence (52) holds for

$$u_k(x) = \left(\frac{1}{\delta_k}\right)\chi_{\left[k-\frac{\delta_k}{2}\ ,\ k+\frac{\delta_k}{2}\right]}(x)\quad,\quad 0 < \delta_k < \delta < 1\,,$$

which is just [187]. We refer to [187,188] for more results on average sampling for band-limited functions.

Corollary $(3.2.13)[296]$: Let the hypotheses be as in Theorem (3.2.5). Then for any $f_j \in V_0$, we have $f_j \in L^2(\mathbb{R})$ and

$$\sum_{j\in\mathbb{Z}}\|T_N f_j\|_2 \leq \frac{\sqrt{B}}{C_1 - \delta L/\pi}\left(\sum_{j\in\mathbb{Z}}\|\tau_{N,\delta}f_j\|_2 + \frac{1+\delta}{\sqrt{2}}\sum_{j\in\mathbb{Z}}\|\tau_{N,\delta}f_j{}'\|_2\right),$$

$$\sum_{j\in\mathbb{Z}}\|T_N f_j\|_\infty \leq \frac{(L/2 + C_2)\sqrt{B}}{(C_1 - \delta L/\pi)\sqrt{A}}\left(\sum_{j\in\mathbb{Z}}\|\tau_{N,\delta}f_j\|_2 + \frac{1+\delta}{\sqrt{2}}\sum_{j\in\mathbb{Z}}\|\tau_{N,\delta}f_j{}'\|_2\right),$$

where $\tau_{N,\delta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$\sum_{j\in\mathbb{Z}}(\tau_{N,\delta}f_j)(x) = \begin{cases}\sum_{j\in\mathbb{Z}} f_j\,(x)\quad, & |x| \geq N - \dfrac{\delta}{2}\quad,\\ 0\quad, & otherwise\,.\end{cases}$$

Proof : For any $f_j \in V_0$, there is some $c = \{c_k : k \in \mathbb{Z}\} \in \ell^2$ such that $\sum_{j\in\mathbb{Z}} f_j\,(x) = \sum_{k\in\mathbb{Z}} c_k\phi \times (x - k)$. By Lemma (3.2.4), f_j is locally absolutely continuous and $\sum_{j\in\mathbb{Z}} f_j{}'(x) = \sum_{k\in\mathbb{Z}} c_k\phi'(x - k)$ a.e. Thus

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left|f_j{}'(x+n)\right|^2 = \sum_{n\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}} c_k\phi'(x+n-k)\right|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|C(\omega)Z_{\phi'}(x,\omega)\right|^2 d\omega \leq L^2\|c\|_2^2 a.e.$$

Therefore, $f_j{}' \in L^2(\mathbb{R})$. Similarly to (64) we can prove that

$$\sum_{|k|>N}\sum_{j\in\mathbb{Z}}\left|\langle f_j, u_k\rangle\right|^2 \leq \left(\sum_{j\in\mathbb{Z}}\|\tau_{N,\delta}f_j\|_2 + \frac{1+\delta}{\sqrt{2}}\sum_{j\in\mathbb{Z}}\|\tau_{N,\delta}f_j{}'\|_2\right)^2 .$$

Now the conclusion follows by (54) and the fact that $\{S_k(x) : k \in \mathbb{Z}\}$ is a frame for V_0 with upper bound $B/\left(C_1 - \delta L/\pi\right)^2$.

Section(3.3) Riesz Bases in $L^2(0,1)$ Related to Sampling

The Whittaker–Shannon–Kotel'nikov sampling theorem states that any function f in the classical Paley–Wiener space PW_π ,

$$PW\pi = f \in L^2(\mathbb{R}) \cap C(\mathbb{R}): \operatorname{supp} \hat{f} \subseteq [-\pi,\pi],$$

i.e., bandlimited to $[-\pi,\pi]$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)sinc(t-n) \quad , \tag{67}$$

where sinc denotes the cardinal sine function , $sinc(t) = sin\,\pi t/\pi t$.

This theorem and its numerous offspring have been proved in many different ways, e.g., using Fourier expansions, the Poisson summation formula, contour integrals, etc. (see [145,153). But the most elegant proof is probably the one due to Hardy [144], using that the Fourier transform F is an isometry between PW_π and $L^2[-\pi,\pi]$. For any $f \in PW_\pi$ one has

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\mathcal{W})e^{i\mathcal{W}t}\, d\mathcal{W} = \langle \hat{f}, \frac{e^{-i\mathcal{W}t}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]} \ , \ t \in \mathbb{R} \ ,$$

so any value $f(t_n)$ of f is the inner product in $L^2[-\pi,\pi]$ of $\hat{f}$ and the complex exponential $e^{-it_n\mathcal{W}}/\sqrt{2\pi}$. The key point in Hardy's proof is that an expansion converging in $L^2[-\pi,\pi]$ is transformed by $\mathcal{F}^{-1}$ into another expansion which converges in the topology of PW_π . This implies, in particular, that it converges absolutely and uniformly on $\mathbb{R}$. Recall that the Paley–Wiener space PW_π is a reproducing kernel Hilbert space (RKHS) whose reproducing kernel is $k(t,s) = sinc(t - s)$. This technique has been coined in [145] as the Fourier duality in Paley–Wiener spaces. Thus, expanding $\hat{f}$ with respect to the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ and transforming by $\mathcal{F}^{-1}$ we obtain the Shannon sampling formula (67). An irregular sampling formula in PW_π at a sequence $\{t_n\}_{n\in\mathbb{Z}}$ of real points may be obtained by perturbating the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in such a way that the sequence of complex exponentials $\{e^{-it_nw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for PW_π . This is the case if, for instance, the sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ verifies the Kadec condition: $\sup_{n\in\mathbb{Z}} |t_n - n| < 1/4$. Moreover, the Paley–Wiener–Levinson sampling theorem states that any function $f \in PW_\pi$ can be recovered from its samples $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange-type interpolation series

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t - t_n)} \ ,$$

where G stands for the infinite product

$$G(t) = (t - t_0)\prod_{n=1}^{\infty}\left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right) \quad [143] .$$

On the other hand, the Paley–Wiener space PW_π is a particular case of a shift-invariant space, i.e., a closed subspace in $L^2(\mathbb{R})$ generated by the integer shifts of a single function $\phi \in L^2(\mathbb{R})$. Whenever the sequence $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ forms, at least, a frame sequence in $L^2(\mathbb{R})$ (i.e., it is a frame for its closed linear span), the corresponding shift-invariant space can be described as

$$V_\phi = \left\{\sum_{n\in\mathbb{Z}} a_n\phi(\cdot - n) : \{a_n\} \in \ell^2(\mathbb{Z})\right\} .$$

The generator ϕ is stable if the sequence $\{\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ. For PW_π , a stable generator is $\phi = sinc$.Wavelet subspaces are important examples of shift-invariant spaces generated by the scaling function of the corresponding multiresolution analysis. See [137,138,149] for the general theory of shift-invariant spaces and their applications. In addition, sampling theory in shift-invariant spaces and, in particular, in wavelet subspaces has been largely studied in the recent years. Let us cite, for instance, the works of Aldroubi and Gröchenig [135], Aldroubi and Unser [136], Chen, Itoh and Shiki [140,141], Janssen [147], Sun and Zhou [150,154], or Walter [148,151] among others. The main aim in this section is to show that the Fourier duality for Paley–Wiener spaces can be generalized to the case of a shift-invariant space V_ϕ with a stable generator ϕ. To this end, we define a bounded one-to-one linear operator T between $L^2(0,1)$ and $L^2(\mathbb{R})$ as

$$T : L^2(0,1) \longrightarrow L^2(\mathbb{R})$$
$$F \longrightarrow f \text{ such that } f(t) = \langle F, K_t\rangle_{L^2(0,1)} ,$$

where the kernel transform $t \in \mathbb{R} \longmapsto K_t \in L^2(0,1)$ is given by the Zak transform of $\bar{\phi}$ namely, $K_t(x) = Z_{\bar{\phi}}(t,x)$, $a.e. x \in (0,1)$. Recall that the Zak transform of $f \in L^2(\mathbb{R})$ is formally defined as

$$(Zf)(t,w) = \sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi i n \mathcal{W}}, t, \mathcal{W} \in \mathbb{R} .$$

The shift-invariant space V_ϕ coincides with the range space of . Thus, sampling expansions in V_ϕ can be seen as transformed expansions via T of expansions in $L^2(0,1)$ with respect to appropriate Riesz bases. Taking into account the definition of T , these bases should have the particular form $\{K_{t_n}\}_{n\in\mathbb{Z}}$. Taking the sampling points $\{t_n = a + n\}_{n\in\mathbb{Z}}$, we obtain the regular sampling in V_ϕ, whereas perturbing this sequence as $\{t_n = a + n + \delta_n\}_{n\in\mathbb{Z}}$, we obtain the irregular sampling. Let $\phi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\phi = \left\{\sum_{n\in\mathbb{Z}} a_n\phi(\cdot - n) : \{a_n\} \in \ell^2(\mathbb{Z})\right\} \subset L^2(\mathbb{R}) ,$$

i.e., the sequence $\{\phi(\cdot\ -n)\}_{n\in\mathbb{Z}}$is a Riesz basis for V_ϕ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\phi(\cdot\ -n)\}_{n\in\mathbb{Z}}$is a Riesz sequence, i.e., a Riesz basis for V_ϕ if and only if

$$0\ <\ \|\Phi\|_0 \le\ \|\Phi\|_\infty\ <\infty\ \ ,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w)=\sum_{k\in\mathbb{Z}}|\ \hat{\phi}(w\ +k)|^2$ in $[0,1]$, and $\|\Phi\|_\infty$ it's essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [142]. We assume along the section that, for each $t\in\mathbb{R}$, the series $\sum_{n\in\mathbb{Z}}|\phi(t\ -n)|^2$ converges. Thus, by using the Riesz' subsequence theorem [142] we can choose the point wise lim $f(t)\ \ =\sum_{n\in\mathbb{Z}}a_n\phi(t\ -n)$ for each $t\ \in\ \mathbb{R}$, as the representative element of the class $\sum_{n\in\mathbb{Z}}a_n\phi(\cdot\ -n)$ in $L^2(\mathbb{R})$. Moreover, if ϕ is a continuous function and the series $\sum_{n\in\mathbb{Z}}|\phi(t\ -n)|^2$converges uniformly in compact subsets of $\mathbb{R}$, we can take any $f\ \in\ V_\phi$ as a continuous function in $\mathbb{R}$. Besides, V_ϕ is a RKHS since the evaluation functionals are bounded in V_ϕ. Indeed, for each fixed $t\ \in\ \mathbb{R}$ we have

$$|f(t)|^2\le\frac{1}{\|\Phi\|_0}\sum_{n\in\mathbb{Z}}|\phi(t-n)|^2\ \ \|f\|^2\ \ ,\qquad f\in\ V_\phi\,, \tag{68}$$

where we have used Cauchy–Schwartz's inequality in $f(t)=\sum_{n\in\mathbb{Z}}a_n\phi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0\sum_{n\in\mathbb{Z}}|a_n|^2\ \le\|f\|^2\le\ \|\Phi\|_\infty\sum_{n\in\mathbb{Z}}|a_n|^2\ ,\qquad f\in\ V_\phi\ .$$

Inequality (68) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence in $\mathbb{R}$. The convergence is uniform in subsets of the real line where $\|K_t\|^2_{L^2(0,1)}\ =\ \sum_{n\in\mathbb{Z}}|\phi(t\ -n)|^2$ is bounded. The reproducing kernel of V_ϕ is given by $k(t,s)=\sum_{n\in\mathbb{Z}}\phi(t-n)\overline{\phi^*(s-n)}$ where the sequence $\{\phi^*(\cdot\ -n)\}_{n\in\mathbb{Z}}$ denotes the dual Riesz basis of $\{\phi(\cdot\ -n)\}_{n\in\mathbb{Z}}$. Recall that the function ϕ^* has Fourier transform $\widehat{\phi^*}=\ \hat{\phi}/\Phi$ [136].

For each $t\ \in\ \mathbb{R}$, consider the function $K_t\in\ L^2(0,1)$ defined by the Fourier series

$$K_t\ \ =\ \sum_{n\in\mathbb{Z}}\phi(t\ +n)e^{-2\pi inx}\ \ .$$

Notice that $K_t(x)\ =\ Z_{\bar{\phi}}(t,x)\ \ a.e.\ x\ \in\ (0,1)$, where Z denotes the Zak transform of $\bar{\phi}$. See [143,146] for properties and uses of the Zak transform. Thus, for each $F\ \in\ L^2(0,1)$ we can define the function

$$f:\mathbb{R}\ \longrightarrow\ \mathbb{C}\,,$$
$$t\ \longrightarrow\ f\ (t)=\ \langle F,K_t\rangle_{L^2(0,1)}.$$

If we denote by T the linear transform which maps $F\ \in\ L^2(0,1)$ into f, i.e.,$T\ (F)\ =\ f$, then we can identify the range space of T as the shift-invariant V_ϕ, i.e.,$T\big(L^2(0,1)\big)=V_\phi$. Indeed, for $F\in L^2(0,1)$ we have that

$$[T(F)(t)] = \langle F, K_t\rangle_{L^2(0,1)} = \sum_{n\in\mathbb{Z}} \langle F, e^{-2\pi inx}\rangle_{L^2(0,1)}\,\phi(t+n)\,, t\in\mathbb{R}\,,$$

which belongs to V_ϕ. Furthermore, for each $f\in V_\phi$ there exists a sequence $\{a_n\}\in \ell^2(\mathbb{Z})$ such that $f=\sum_{n\in\mathbb{Z}} a_n\phi(\cdot+n)$ in $L^2(\mathbb{R})$. Since $\{e^{-2\pi inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0,1)$, there exists a function $F\in L^2(0,1)$ such that $\langle F, e^{-2\pi inx}\rangle_{L^2(0,1)} = a_n$ for every $n\in\mathbb{Z}$. Hence, $T(F)=f$. Moreover, the following result holds:

Theorem $(3.3.1)$[134]: The mapping T is an invertible bounded operator between $L^2(0,1)$ and V_ϕ .

Proof: The operator T is bijective since it maps the orthonormal basis $\{e^{-2\pi inx}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$into the Riesz basis $\{\phi(t+n)\}_{n\in\mathbb{Z}}$ in V_ϕ . Concerning the continuity, for $F\in L^2(0,1)$, we have

$$\|T(F)\|^2_{L^2(\mathbb{R})} = \left\|\sum_{n\in\mathbb{Z}}\langle F, e^{-2\pi inx}\rangle_{L^2(0,1)}\,\phi(t+n)\right\|^2_{L^2(\mathbb{R})}$$
$$\le \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} \left|\langle F, e^{-2\pi inx}\rangle\right|^2 = \|\Phi\|_\infty \|F\|^2_{L^2(0,1)}\ ,$$

where we have used the upper Riesz basis condition for $\{\phi(.+n)\}_{n\in\mathbb{Z}}$. Having in mind the periodicity relations of the Zak transform, the function K_t satisfies $K_{t+m}(x) = e^{2\pi imx}K_t(x)$ in $L^2(0,1)$, where $t\in\mathbb{R}$ and $m\in\mathbb{Z}$. Now, for $f\in V_\phi$ consider $F = T^{-1}(f)\in L^2(0,1)$. For each $n\in\mathbb{Z}$we have

$$T\left[F(x)e^{2\pi inx}\right](t) = \langle F(\cdot)e^{2\pi in\cdot}, K_t(\cdot)\rangle_{L^2(0,1)} = \langle F, K_{t-n}\rangle_{L^2(0,1)} = f(t-n)\,.$$

Since T is a bounded invertible operator, the sequence $\{f(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ if and only if $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. The following theorem which can be found in [139] gives a characterization of Bessel sequences, Riesz bases and frames in $L^2(0,1)$having the form $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$. From now on, $\|F\|_\infty$ (respectively $\|F\|_0$) will denote the essential supremum (respectively infimum) of $|F|$ in $(0,1)$.

Theorem$(3.3.2)$[134]: Given a function $F\in L^2(0,1)$, the following results hold:

(i) The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(0,1)$ if and only if the function F satisfies $\|F\|_\infty<\infty$.

(ii) The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$is a Riesz basis for $L^2(0,1)$ if and only if the function F satisfies $0<\|F\|_0\le\|F\|_\infty<\infty$. In this case, the optimal Riesz bounds of $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ are $\|F\|_0^2$ and $\|F\|^2_\infty$.

(iii) The sequence $\{F(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a frame in $L^2(0,1)$ if and only if is a Riesz basis for $L^2(0,1)$.Thus we have the following corollary in V_ϕ.

Corollary$(3.3.3)$[134]: Given a function $g \in V_\phi$, consider $G = T^{-1}(g) \in L^2(0,1)$. Then, the sequence $\{g(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ if and only if $0 < \|G\|_0 \le \|G\|_\infty < \infty$.

Regular sampling in V_ϕ arises by considering appropriate Riesz bases in $L^2(0,1)$. Namely, for a fixed $a \in [0,1)$, the regular samples at $\{a+n\}_{n\in\mathbb{Z}}$ of $f \in V_\phi$ are given by $f(a+n) = \langle F, K_{a+n}\rangle_{L^2(0,1)} = \langle F, K_a e^{2\pi inx}\rangle_{L^2(0,1)}, n \in \mathbb{Z}$, where $F = T^{-1}(f)$. The sequence $\{K_a(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$ has the biorthonormal sequence $\{e^{2\pi inx}/\overline{K}_a(x)\}_{n\in\mathbb{Z}}$ provided $1/K_a \in L^2(0,1)$. Hence, stable regular sampling in V_ϕ reduces to studying whenever the sequence $\{K_a(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$, and this depends on the function K_a as stated in Theorem (3.3.2).Expanding $F = T^{-1}(f)$ with respect to the Riesz basis$\{e^{2\pi inx}/\overline{K}_a(x)\}_{n\in\mathbb{Z}}$, via the invertible bounded operator T,we obtain a regular sampling formula for f.

Lemma $(3.3.4)$[134]: Given $a \in [0,1)$, there exists a function $S_a \in V_\phi$ satisfying the interpolation condition $S_a(a+n) = \delta_{n,0}$, where $n \in \mathbb{Z}$, if and only if the function $1/K_a$ belongs to $L^2(0,1)$. In this case $S_a = T(1/\overline{K}_a)$.

Proof: Assume that there exists a function $S_a \in V_\phi$ satisfying the interpolation condition $S_a(a+n) = \delta_{n,0}$, where $n \in \mathbb{Z}$. For $F_a = T^{-1}(S_a)$ we have

$$S_a(a+n) = \langle F_a, K_{a+n}\rangle_{L^2(0,1)} = \langle F_a, e^{2\pi inx}K_a\rangle_{L^2(0,1)}$$
$$= \int_0^1 F_a(x)\overline{K_a(x)}e^{-2\pi inx}\,dx = \delta_{n,0},$$

which implies that $F_a(x)\overline{K_a(x)} = 1$ $a.e.$ in $(0,1)$, and consequently the function $1/K_a$ belongs to $L^2(0,1)$. Conversely, if $1/K_a$ is in $L^2(0,1)$, we define $S_a = T(1/\overline{K}_a)$. For $n \in \mathbb{Z}$ it satisfies

$$S_a(a+n) = \langle \frac{1}{\overline{K}_a}, K_{a+n}\rangle_{L^2(0,1)} = \langle 1, e^{2\pi inx}\rangle_{L^2(0,1)} = \delta_{n,0}.$$

Thus we can characterize stable regular sampling in V_ϕ.

Theorem $(3.3.5)$[134]: Consider $a \in [0,1)$ such that the function $1/K_a \in L^2(0,1)$. The following conditions are equivalent:

(i) $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$.

(ii) There exists a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_ϕ such that, for each $f \in V_\phi$, we have then pointwise expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n)S_n(t), t \in \mathbb{R}.$$

Furthermore, in this case the sampling functions are $S_n(t) = S_a(t-n)$, where $S_a = T(1/\overline{K}_a)$. The sampling series converges in the $L^2(\mathbb{R})$-norm sense, absolutely and uniformly in subsets of $\mathbb{R}$ where K_t is bounded.

Proof: First we prove that (i) implies (ii). Consider $S_a = T(1/\overline{K}_a)$. Condition (i) implies that $0 < \|1/\overline{K}_a\|_0 \leq \|1/\overline{K}_a\|_\infty < \infty$ and , as a consequence, Corollary (3.3.3) gives that $\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ. For each $f \in V_\phi$, there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that $f(t) = \sum_{n\in\mathbb{Z}} a_n S_a(t-n)$ where the convergence is also point wise for each $t \in \mathbb{R}$ since V_ϕ is a RKHS. Taking $t = a + m$, and using the interpolatory condition $S_a(a+n) = \delta_{n,0}$, we obtain that $a_m = f(a+m)$ for any $m \in \mathbb{Z}$.

Conversely, assume that the condition (ii) holds.Taking $f(t) = S_a(t-m), m \in \mathbb{Z}$, we obtain that $S_m(t) = S_a(t-m)$ and , as a consequence, $\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ. Since $S_a = T(1/\overline{K}_a)$, Corollary (3.3.3) gives condition (i).

Absolute convergence comes from the inconditional character of a Riesz basis. The uniform convergence is a standard result in the setting of the RKHS theory.

A straightforward calculation gives the Fourier transform of Sa. Indeed ,

$$\hat{S}_a(\mathcal{W}) = \widehat{T(1/\overline{K}_a)}(\mathcal{W}) = \frac{\hat{\phi}(\mathcal{W})}{Z_\phi(a, w/2\pi)} \; a.e. \text{ in } \mathbb{R} \quad .$$

Usually, one may consider irregular sampling as a perturbation of the regular sampling. In the present setting, we can try to recover any function $f \in V_\phi$ from its perturbed samples $\{f(a+n+\delta_n)\}_{n\in\mathbb{Z}}$, where $a \in [0,1)$ and $\{\delta_n\}_{n\in\mathbb{Z}}$ is a sequence in $(-1,1)$. Since $f(a+n+\delta_n) = \langle F, K_{a+n+\delta_n}\rangle_{L^2(0,1)}, n \in \mathbb{Z}$, where $F = T^{-1}(f) \in L^2(0,1)$, a challenge problem is to prove that $\{K_{a+n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

One possibility is to use a perturbation technique on the Riesz basis $\{K_{a+n}\}_{n\in\mathbb{Z}} = \{K_a(x)e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ which gives the sequence of regular samples $\{f(a+n)\}_{n\in\mathbb{Z}}$. As a consequence, we need a perturbation result for those Riesz bases in $L^2(0,1)$ appearing in Theorem (3.3.2).

For an infinite matrix $M = \{m_{n,k}\}_{n,k\in\mathbb{Z}}$ defining a bounded operator in $\ell^2(\mathbb{Z})$ we denote its operator norm as $\|M\|_2 = \sup_{\|c\|_{\ell^2(\mathbb{Z})}=1} \|Mc\|_{\ell^2(\mathbb{Z})}$.

Theorem $(3.3.6)[134]$: Let $F = \sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k x}$ be in $L^2(0,1)$ such that $0 < \|F\|_0 \leq \|F\|_\infty < \infty$. Let $\{F_n\}_{n\in\mathbb{Z}}$ be a sequence of functions in $L^2(0,1)$ with Fourier expansions $F_n = \sum_{k\in\mathbb{Z}} a_k(n) e^{-2\pi i k x}, n \in \mathbb{Z}$. Suppose that the infinite matrix $D = \{d_{n,k}\}_{n,k\in\mathbb{Z}}$ with entries $d_{n,k} = a_{n-k}(n) - a_{n-k}, n, k \in \mathbb{Z}$, satisfies the condition $\|D\|_2 < \|F\|_0$. Then, the sequence $\{F_n(x)e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

Proof : To this end we use the following result on perturbation of Riesz bases in a Hilbert space $\mathcal{H}$ which can be found in [142]: let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for $\mathcal{H}$ with Riesz bounds A, B, and let $\{g_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that

$$\sum_{k=1}^{\infty} |\langle f_k - g_k, f\rangle|^2 \le R\, \|f\|^2 , \qquad \text{for each } f \in \mathcal{H},$$

then $\{g_k\}_{k=1}^{\infty}$is a Riesz basis for $\mathcal{H}$. For any $f = \sum_{j\in\mathbb{Z}} \overline{c_j}\, e^{2\pi ijx}$ in $L^2(0,1)$ we have

$$\sum_{n\in\mathbb{Z}} \left|\langle F_n(x)e^{2\pi inx} - F(x)e^{2\pi inx}, f\rangle\right|^2$$

$$= \sum_{n\in\mathbb{Z}} \left|\langle \sum_{k\in\mathbb{Z}} (a_k(n) - a_k) e^{2\pi i(n-k)x}, \sum_{j\in\mathbb{Z}} \overline{c_j}\, e^{2\pi ijx}\rangle\right|^2$$

$$= \sum_{n\in\mathbb{Z}} \left|\sum_{k\in\mathbb{Z}} (a_{n-k}(n) - a_{n-k}) c_k\right|^2 = \sum_{n\in\mathbb{Z}} \left|\sum_{k\in\mathbb{Z}} d_{n,k} c_k\right|^2$$

$$= \|D\, c\|^2_{\ell^2(\mathbb{Z})} \le \|D\|_2^2 \|f\|^2 .$$

Taking into account that in our case $A = \|F\|_0^2$, we obtain that $\{F_n(x)e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$.

As a consequence of the above perturbation theorem in $L^2(0,1)$, we obtain an irregular sampling result in V_ϕ.

Theorem (3.3.7)[134]: Given $a \in [0,1)$ such that $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$. *Let* $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ be a sequence in $(-1,1)$ such that the infinite matrix $D_\Delta = \{d_{n,k}\}_{n,k\in\mathbb{Z}}$ whose entries are given by

$$d_{n,k} = \overline{\phi(a + n - k + \delta_n)} - \overline{\phi(a + n - k)}, \qquad n,k \in \mathbb{Z},$$

satisfies $\|D_\Delta\|_2 < \|K_a\|_0$. Then, there exists a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ *for* V_ϕ such that any function $f \in V_\phi$ can be expanded as

$$f(t) = \sum_{n\in\mathbb{Z}} f(a + n + \delta_n) S_n(t), \qquad t \in \mathbb{R}.$$

The convergence of the series is absolute and uniform in subsets of $\mathbb{R}$ where $\|K_t\|$ is bounded. Also, it converges in the $L^2(\mathbb{R})$ −norm sense.

Proof : Applying Theorem (3.3.6) to

$$K_a(x) = \sum_{k\in\mathbb{Z}} \overline{\phi(a + k)} e^{-2\pi ikx} \quad \text{and}$$

$$K_{a+\delta_n}(x) = \sum_{k\in\mathbb{Z}} \overline{\phi(a + k + \delta_n)} e^{-2\pi ikx} , \qquad n \in \mathbb{Z},$$

we obtain that $\{K_{a+\delta_n} e^{2\pi inx}\}_{n\in\mathbb{Z}} = \{K_{a+n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$. Denote by $\{G_n\}_{n\in\mathbb{Z}}$ its dual Riesz basis. Now, given $f \in V_\phi$, we expand the function $F = T^{-1}(f) \in L^2(0,1)$ with respect to $\{G_n\}_{n\in\mathbb{Z}}$. Thus ,

$$F = \sum_{n\in\mathbb{Z}} \langle F, K_{a+n+\delta_n}\rangle_{L^2(0,1)} G_n = \sum_{n\in\mathbb{Z}} f(a + n + \delta_n)\, G_n \ \ in\ \ L^2(0,1).$$

Applying the operator T , we get

$$f = \sum_{n\in\mathbb{Z}} f(a + n + \delta_n)\, T(G_n) \quad in \quad L^2(\mathbb{R}).$$

Furthermore, since T is an invertible bounded operator, the sequence $\{S_n = T(G_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ. The pointwise convergence properties of the series come out as in Theorem (3.3.5).

The next result yields a uniform bound of the norm $\|D_\Delta\|_2$ regardless the sequence $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ in $[\alpha,\beta] \subset [-1,1]$.

Theorem (3.3.8)[134]: For any sequence $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ in $[\alpha,\beta]$ the following inequality holds:

$$\|D_\Delta\|_2 \le \sup_{\{d_n\}\subset[\alpha,\beta]} \sum_{n\in\mathbb{Z}} |\phi(a + n + d_n) - \phi(a + n)|. \tag{69}$$

Proof : Assume that the second term in the above inequality is finite. Otherwise, the inequality trivially holds. For any $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell(\mathbb{Z})$ we have

$$\|D_\Delta c\|^2_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \left|\sum_{k\in\mathbb{Z}} d_{n,k}\, c_k\right|^2 \le \sum_{n\in\mathbb{Z}} \sum_{l,j\in\mathbb{Z}} |d_{n,l}|\, |c_l|\, |\overline{d_{n,j}}|\, |\overline{c_j}\,|$$

$$= \sum_{l,j\in\mathbb{Z}} |c_l|\, |c_j\,| \sum_{n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}\,| \le \sum_{l,j\in\mathbb{Z}} \frac{|c_l\,|^2 + |c_j\,|^2}{2} \sum_{n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}|$$

$$= \sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{j,n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}\,| \le \sup_{l\in\mathbb{Z}} \left(\sum_{j,n\in\mathbb{Z}} |d_{n,l}|\, |d_{n,j}\,| \right) \|c\|^2_{\ell^2(\mathbb{Z})}$$

$$\le \sup_{l\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} |d_{n,l}| \right) \sum_{j\in\mathbb{Z}} |d_{n,j}|\, \|c\|^2_{\ell^2(\mathbb{Z})} .$$

Having in mind that

$$\sum_{j\in\mathbb{Z}} |d_{n,j}| = \sum_{j\in\mathbb{Z}} \left|\phi(a + j - k + \delta_j) - \phi(a + j - k)\right|$$

$$= \sum_{n\in\mathbb{Z}} |\phi(a + n + \delta_{n+k}) - \phi(a + n)| \quad ,$$

we obtain the desired result.

A comment about the second term in (69) is in order. Namely, looking for an estimation of the ratio between $\sum_{n\in\mathbb{Z}} |\phi(a + n + d_n) - \phi(a + n)|$ and $(\sup_n |d_n|)^\lambda$ for a fixed $\lambda > 0$, led Chen et al. to introduce in [6] the classes of functions $L_a^\lambda[\alpha,\beta]$. Next we give a particular example when Theorem (3.3.8) works. Namely, suppose that the stable generator $\phi \in C^1(\mathbb{R})$ and for some $\varepsilon > 0$ it satisfies $\phi'(t) = O\left(|t\,|^{-(1+\varepsilon)}\right)$ as $|t| \to \infty$. Then, it is easy to prove that, for $\delta \in (0,1]$,

$$M_{\phi'}(\delta) = \sum_k \max_{I_k(\delta)} |\phi'(t)| \le M_{\phi'}(1) < \infty \ ,$$

where $I_k(\delta)$ denotes the interval $[a + k - \delta, a + k + \delta]$.

Corollary $(3.3.9)[134]$: Let $\phi \in C^1(\mathbb{R})$ be a stable generator such that $M_{\phi'}(\delta) < \infty$, where $\delta = \sup_{n\in\mathbb{Z}} |\delta_n|$. Then, the condition $\delta \, M_{\phi'}(\delta) < \|K_a\|_0$ implies the existence of a Riesz basis $\{S_n\}_{n\in\mathbb{Z}}$ for V_ϕ such that any function in this space can be expanded as

$$f(t) = \sum_{n\in\mathbb{Z}} f(a + n + \delta_n) S_n(t) \;, \qquad t \in \mathbb{R} \;.$$

The convergence in the series is absolute and uniform in subsets of $\mathbb{R}$ where K_t is bounded. It converges also in the $L^2(\mathbb{R})$-norm sense.

Proof : The mean value theorem gives

$$\sup_{\{d_n\}\subset[-\delta,\delta]} \sum_{n\in\mathbb{Z}} |\phi(a + n + d_n) - \phi(a + n)| \le \delta \, M_{\phi'}(\delta) \;.$$

Theorem (3.3.7) concludes the proof.

Corollary $(3.3.10)[296]$: Given $0 \le \epsilon < 1$, there exists a function $S_{1-\epsilon} \in V_\phi$ satisfying the interpolation condition $S_{1-\epsilon}(n-\epsilon) = \delta_{n,0}$, where $n \in \mathbb{Z}$, if and only if the function $1/K_{1-\epsilon}$ belongs to $L^2(0,1)$. In this case $S_{1-\epsilon} = T\,(1/\overline{K}_{1-\epsilon})$.

Proof: Assume that there exists a function $S_{1-\epsilon} \in V_\phi$ satisfying the interpolation condition $S_{1-\epsilon}(n-\epsilon) = \delta_{n,0}$, where $n \in \mathbb{Z}$. For $F_{1-\epsilon} = T^{-1}(S_{1-\epsilon})$ we have

$$S_{1-\epsilon}(n-\epsilon) = \langle F_{1-\epsilon}, K_{n-\epsilon}\rangle_{L^2(0,1)} = \langle F_{1-\epsilon}, e^{2\pi inx} K_{1-\epsilon}\rangle_{L^2(0,1)}$$

$$= \int_0^1 F_{1-\epsilon}(x)\overline{K_{1-\epsilon}(x)} e^{-2\pi inx}\, dx = \delta_{n,0} \,,$$

which implies that $F_{1-\epsilon}(x)\overline{K_{1-\epsilon}(x)} = 1 \; a.e.$ in $(0,1)$, and consequently the function $1/K_{1-\epsilon}$ belongs to $L^2(0,1)$. Conversely, if $1/K_{1-\epsilon}$ is in $L^2(0,1)$, we define $S_{1-\epsilon} = T\,(1/\overline{K}_{1-\epsilon})$. For $n \in \mathbb{Z}$ it satisfies

$$S_{1-\epsilon}(n-\epsilon) = \langle \frac{1}{\overline{K}_{1-\epsilon}}, K_{n-\epsilon}\rangle_{L^2(0,1)} = \langle 1, e^{2\pi inx}\rangle_{L^2(0,1)} = \delta_{n,0} \;.$$

Thus we can characterize stable regular sampling in V_ϕ .

Chapter 4

Dual Frames in $L^2(0,1)$ and Multivariate Generalized Sampling

Involved samples are expressed as the frame coefficients of an appropriate function in $L^2(0,1)$ with respect to some particular frame in $L^2(0,1)$. Since any shift-invariant space with stable generator is the image of $L^2(0,1)$ by means of a bounded invertible operator, our generalized sampling is derived from some dual frame expansions in $L^2(0,1)$. An $L^2(\mathbb{R}^d)$ theory involving the frame theory is exhibited.Sampling formulas which are frame expansions for the shift-invariant space are obtained. In the case of overcomplete frame formulas, the search of reconstruction functions with prescribed good properties is allowed.Finally,approximation schemes using these generalized sampling formulas are included.

Section(4.1) Generalized Sampling in Shift-Invariant Spaces

Suppose that s linear-time invariant systems (filters) $\mathcal{L}_j$, $j = 1,2,\ldots,s,$ are defined on a shift-invariant space V_ϕ of $L^2(\mathbb{R})$

$$V\phi = \left\{f(t) = \sum_{n\in\mathbb{Z}} a_n\phi(t-n) : \{a_n\} \in \ell^2(\mathbb{Z})\right\},$$

where the function $\phi \in L^2(\mathbb{R})$ is a stable generator for V_ϕ. The main aim in this work is to recover any function $f \in V_\phi$ by means of a stable sampling formula. More precisely, by using a frame expansion which involves the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$, where the sampling period $r \in \mathbb{N}$ necessarily satisfies $r \le s$. Whenever $s > r$ we are in the oversampling case. The advantages of the oversampling technique in practical applications are well-known (see [119], [125] or [130]).

This problem goes back to [126] where a sampling formula is given, which allows to recover a bandlimited function f by using the sequence of samples $\{(\mathcal{L}_j f)(sn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$, which involves s filtered versions of . Note that, according to the Whittaker–Shannon–Kotel'nikov sampling theorem, the space of functions bandlimited to an interval $[-\sigma,\sigma]$, i.e., the classical Paley–Wiener space $PW_\sigma = \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \sup \hat{f} \subseteq [-\sigma,\sigma]\}$,where $\hat{f}$ stands for the Fourier transform $f(w) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i t w}\, dt$, is an example of a shift-invariant space where the generator is a scaled version of the cardinal sine function $sinct = \sin \pi t/\pi t$. Wavelet subspaces are also important examples of shift-invariant spaces.

Papoulis' result has been extended to a general shift-invariant space by using the filter banks technique. Concretely [85] extended Papoulis' result for some important particular [107, 131] extended it in the general case.

The case where the number of channels s is larger than the sampling period r, i.e., the oversampling case, has been also considered in [85] by means of an example. In [132] studied this general setting for classical bandlimited functions.

We propose a new approach involving the theory of frames in a separable Hilbert space $\mathcal{H}$. Recall that a sequence $\{f_k\}$ is a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 \, B\|f\|^2 \quad \text{for all} \quad f \in \mathcal{H}.$$

Given a frame $\{f_k\}$ for $\mathcal{H}$ the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_k c_k f_k$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for over complete frames) is sacrificed. Suitable frame coefficients c_k which depend continuously and linearly on f are obtained by using the dual frames $\{g_k\}$ of $\{f_k\}$, i.e., $\{g_k\}$ is another frame for $\mathcal{H}$ such that $f = \sum_k \langle f, g_k\rangle f_k = \sum_k \langle f, f_k\rangle g_k$ for each $f \in \mathcal{H}$. For more details on the frame theory see the superb monograph [82] and references therein.

The shift-invariant space V_ϕ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T} : L^2(0,1) \to V_\phi$, which maps the orthonormal basis $\{e^{-2\pi inw}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ for V_ϕ.

Our starting point is to write the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ as the frame coefficients with respect to a particular frame in $L^2(0,1)$ of the function $F = \mathcal{T}^{-1} f \in L^2(0,1)$. Searching for its dual frames we obtain those expansions for F in $L^2(0,1)$ having the samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ as frame coefficients. Thus, applying the isomorphism $\mathcal{T}$ to the above frame expansions of F we will obtain sampling expansions for $f = \mathcal{T} F$ in V_ϕ involving its samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$.

The use of several different dual frames allow us to obtain a variety of reconstruction functions. Thus we can try to find some reconstruction functions with "good properties." For instance, following an idea in [85], those with compact support. All these steps will be carried out throughout the remaining sections. Let $\phi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space

$$V_\phi = \left\{ \sum_{n\in\mathbb{Z}} a_n \phi(\cdot - n) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\} \subset L^2(\mathbb{R}),$$

i.e., the sequence $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_ϕ. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_ϕ if and only if $0 < \|\Phi\|_0 \leq \|\Phi\|_\infty < \infty$, Where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(\mathcal{W}) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(w+k)|^2$ in $(0,1)$, and $\|\Phi\|_\infty$ it's essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$are the optimal Riesz bounds [82].

We assume throughout the section that the functions in the shift-invariant space V_ϕ are continuous on $\mathbb{R}$. Equivalently, that the generator ϕ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\phi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [113]). Thus, any $f \in V_\phi$ is defined on $\mathbb{R}$ as the pointwise sum$f(t) = \sum_{n\in\mathbb{Z}} a_n \phi(t-n)$.

Besides, V_ϕ is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_ϕ.Indeed, for each fixed $t \in \mathbb{R}$ we have

$$|f(t)|^2 \leq \frac{\|f\|^2}{\|\Phi\|_0} \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 , f \in V_\phi , \tag{1}$$

where we have used Cauchy–Schwartz's inequality in $f(t) = \sum_{n\in\mathbb{Z}} a_n \phi(t-n)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \|f\|^2 \leq \|\Phi\|_\infty \sum_{n\in\mathbb{Z}} |a_n|^2 , f \in V_\phi .$$

Inequality (1) shows that convergence in the $L^2(\mathbb{R})$-norm implies pointwise convergence which is uniform on $\mathbb{R}$.The reproducing kernel of V_ϕ is given by $k(t,s) = \sum_{n\in\mathbb{Z}} \phi(t-n)\overline{\phi^*(s-n)}$ where the sequence $\{\phi^*(\cdot - n)\}_{n\in\mathbb{Z}}$ denotes the dual Riesz basis of $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$. Recall that the function ϕ^* has Fourier transform $\widehat{\phi^*} = \hat{\phi}/\Phi$ [115].On the other hand, the space V_ϕ is the image of $L^2(0,1)$ by means of the isomorphism $\mathcal{T} : L^2(0,1) \to V_\phi$ which maps the orthonormal basis $\{e^{-2\pi i n \mathcal{W}}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ for V_ϕ (see [120]), i.e. ,

$$(\mathcal{T} F)(t) = \sum_{n\in\mathbb{Z}} \langle F(\cdot), e^{-2\pi i n \cdot}\rangle_{L^2(0,1)} \phi(t-n) , F \in L^2(0,1).$$

Notice that for each $F \in L^2(0,1)$ the function $f = \mathcal{T} F$ is given by $f(t) = \langle F, K_t\rangle_{L^2(0,1)} , t \in \mathbb{R}$. The kernel transform $t \in \mathbb{R} \to K_t \in L^2(0,1)$ is defined as $K_t(x) = \overline{Z_\phi}(t,x)$, where Z_ϕ denotes the Zak transform of ϕ. Recall that the Zak transform of $f \in L^2(\mathbb{R})$ is formally defined in $\mathbb{R}^2$ as $(Zf)(t,\mathcal{W}) = \sum_{n\in\mathbb{Z}} f(t+n) e^{-2\pi i n \mathcal{W}}$. See [92] for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}$ will be used later: For $F \in L^2(0,1), r \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have

$$\mathcal{T}\left[F(\cdot) e^{-2\pi i r n \cdot}\right](t) = \mathcal{T}[F](t-rn), t \in \mathbb{R}. \tag{2}$$

We close this section citing [81,116,127] for the general theory of shift-invariant spaces and their applications. Whenever the generator ϕ is a B-spline, the corresponding shift-invariant space has been proved to be very fruitful in signal processing applications [80]. Besides, sampling in shift-invariant spaces has been a topic largely studied in recent years, see, for instance, the papers by Aldroubi and Gröchenig [78], Aldroubi and Unser [115], Chen et al. [117], Janssen [124], Sun and

Zhou [128], or Walter [120]. Average sampling in shift-invariant spaces is also an important topic related to generalized sampling (see, for instance, [114] or [129] and references therein) . Throughout the section we distinguish two types of linear-time invariant system $\mathcal{L}$:

(a) The impulse response l of L belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, for any $f \in V_\phi$ we have

$$(\mathcal{L}f)(t) = [f * l](t) = \int_{-\infty}^{\infty} f(x) l(t-x)\, dx = \langle f(\cdot), \varphi(\cdot - t)\rangle_{L^2(\mathbb{R})}, t \in \mathbb{R} ,$$

where $\varphi(t) = \overline{l(-t)}$. Notice that $\mathcal{L}f$ is a continuous and bounded function in $L^2(\mathbb{R})$.

(b) The impulse response l has the form $l = \sum_{k=0}^{N} c_k \delta^{(k)}(t + d_k)$, where $\delta^{(k)}$ denotes the kth derivative of the Dirac delta and c_k , d_k are constants for $k = 0, 1, \dots, N$. For each $f \in V_\phi$ we have

$$(\mathcal{L}f)(t) = \sum_{k=0}^{N} c_k f^{(k)}(t + d_k) \ , \ t \in \mathbb{R} .$$

In this case we also assume that $\phi^{(N)}$ exists on $\mathbb{R}$, and $\sum_{n\in\mathbb{Z}} |\phi^{(k)}(t - n)|^2$ is uniformly bounded on $\mathbb{R}$ for each $k = 0, 1, \dots, N$. Given a linear-time invariant system $\mathcal{L}$ of the type (a) or (b), next lemma assures that, for each fixed $t \in \mathbb{R}$, the Zak transform $(Z\mathcal{L}_\phi)(t, \mathcal{W}) = \sum_{n\in\mathbb{Z}} \mathcal{L}_\phi(t + n) e^{-2\pi i n \mathcal{W}}$ defines a function in $L^2(0, 1)$.

Lemma$(4.1.1)$[89]: Let $\mathcal{L}$ be a linear-time invariant system of the type (a) or (b) above. For any $t \in \mathbb{R}$ the sequence $\{(\mathcal{L}_\phi)(t + n)\} n \in \mathbb{Z}$ belongs to $\ell^2(\mathbb{Z})$.

Proof : Whenever $\mathcal{L}$ is of the type (b), the result trivially holds. Assume that $\mathcal{L}$ is a system of the type (a) with impulse response l. Then, for any $t \in \mathbb{R}$ we have

$$\left|(\mathcal{L}_\phi)(t + n)\right|^2 = \sum_{n\in\mathbb{Z}} \left| \int_{-\infty}^{\infty} l(x)\phi(t + n - x)\, dx \right|^2$$

$$\leq \left(\int_{-\infty}^{\infty} \left(\sum_{n\in\mathbb{Z}} |l(x)\phi(t + n - x)|^2 \right)^{\frac{1}{2}} dx \right)^2$$

$$\leq \left(\int_{-\infty}^{\infty} |l(x)| \left(\sum_{n\in\mathbb{Z}} |\phi(t + n - x)|^2 \right)^{\frac{1}{2}} dx \right)^2 \leq M \|l\|_1^2,$$

where $M = \sup_{x\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\phi(x - n)|^2$, and we have used a version of the Minkowski inequality for integrals [122]. Now, consider s linear-time invariant systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$, of the type (a), (b), or both. For notational ease we choose $t = 0$ without loss of generality. The apparently more general set of samples

$\{\mathcal{L}_j f\ (rn+e_j\ )\}_{n\in\mathbb{Z}\ ,j=1,2,\ldots,s}$, where $e_j \in \mathbb{R}$ for $j = 1,2,\ldots,s$, is reduced to the case considered here by taking the appropriate shifted systems.For $j = 1,2,\ldots,s$, the function g_j in $L^2(0,1)$ defined by

$$g_j(\mathcal{W}) = \sum_{n\in\mathbb{Z}} \mathcal{L}_j\ \phi(n)e^{-2\pi in\mathcal{W}} = \big(Z\mathcal{L}_j\phi\big)(0,\mathcal{W})\ , \qquad (3)$$

plays an important role throughout this section. Indeed, next lemma gives an expression for the samples $\{\boldsymbol{\mathcal{L}_j f\ (rn)}\}_{\boldsymbol{n\in\mathbb{Z}\ ,j=1,2,\ldots,s}}$, which involves the functions $g_j\ ,j = 1,2,\ldots,s$ and the function $F = \mathcal{T}^{-1}f$ in $L^2(0,1)$:

Lemma(4.1.2)[89] : Let f be a function in V_ϕ such that $f = \mathcal{T}\,F$ where $F \in L^2(0,1)$. For every $j = 1,2,\ldots,s$, we have

$$\big(\mathcal{L}_j f\ \big)(rn) = \langle F(\cdot), \bar{g}_j\ (\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}\ , n \in \mathbb{Z}\ . \qquad (4)$$

Proof : Assume that $\mathcal{L}_j$ is a filter of the type (a). For each $n \in \mathbb{Z}$ we have

$$\begin{aligned}\big(\mathcal{L}_j f\ \big)(rn) &= \langle f\ (\cdot), \varphi_j\ (\cdot\ - rn)\rangle_{L^2(\mathbb{R})}\\ &= \langle \sum_{k\in\mathbb{Z}} \langle F(\cdot), e^{-2\pi ik\cdot}\rangle_{L^2(0,1)}\ \phi(\cdot\ - k), \varphi_j\ (\cdot - rn)\rangle_{L^2(\mathbb{R})}\\ &= \sum_{k\in\mathbb{Z}} \langle F(\cdot), e^{-2\pi ik\cdot}\rangle_{L^2(0,1)}\ \mathcal{L}_j\phi(rn\ - k).\end{aligned}$$

Parseval's equality and a change in the summation index gives

$$\begin{aligned}\big(\mathcal{L}_j f\ \big)(rn) &= \langle F(\cdot), \sum_{k\in\mathbb{Z}} \bar{\mathcal{L}}_j\phi(rn - k)e^{-2\pi ik\cdot}\rangle_{L^2(0,1)}\\ &= \langle F(\cdot), \bar{g}_j\ (\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}.\end{aligned}$$

Assume now that $\mathcal{L}_j$ is a filter of the type (b). Under our hypotheses on $\mathcal{L}_j$ we have that $f^{(k)}(t) = \sum_{l\in\mathbb{Z}} \langle F(\cdot), e^{-2\pi il\cdot}\rangle_{L^2(0,1)}\ \phi^{(k)}(t\ - l)$. Hence, for each $n \in \mathbb{Z}$, one gets

$$\begin{aligned}\big(\mathcal{L}_j f\ \big)(rn) &= \sum_{k=0}^{N} c_k f^{(k)}\ (rn\ +\ d_k)\\ &= \sum_{k=0}^{N} c_k \sum_{l\in\mathbb{Z}} \langle F(\cdot), e^{-2\pi il\cdot}\rangle_{L^2(0,1)} \phi^{(k)}(rn\ +\ d_k - l)\\ &= \langle F(\cdot), \sum_{k=0}^{N} \bar{c_k} \sum_{l\in\mathbb{Z}} \bar{\phi}^{(k)}(rn\ + d_k - l)e^{-2\pi il\cdot}\rangle_{L^2(0,1)}\\ &= \langle F(\cdot), \bar{g}_j\ (\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}.\end{aligned}$$

Observe that, under appropriate hypotheses, the Poisson summation formula gives a different expression for the functions g_j . For instance, assuming that $\sum_{n\,\in\,\mathbb{Z}} |\widehat{\mathcal{L}_j\phi}(\mathcal{W}\ + n)| \in L^2(0,1)$, one has

$$g_j(\mathcal{W}) = \sum_{n \in \mathbb{Z}} \widehat{\mathcal{L}_j \phi}(\mathcal{W} + n) = \sum_{n \in \mathbb{Z}} \hat{l}_j(\mathcal{W} + n)\hat{\phi}(\mathcal{W} + n) \ in\, L^2(0,1)\,, \qquad (5)$$

where we have used that $l_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ when $\mathcal{L}_j$ is a system of the type (a). Lemma (4.1.2) leads us to study when the sequence $\{a_j\,(\cdot)\, e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z}\,,j=1,2,\dots,s}$ (or equivalently the sequence $\{a_j\,(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$), where $a_j \in L^2(0,1)$ for each $j = 1,2,\dots,s$, is a Bessel sequence or a frame for $L^2(0,1)$.To this end, associated with the functions $a_j\,, j = 1,2,\dots,s$, we introduce the $s \times r$ matrix function defined for $\mathcal{W} \in (0,1)$ as

$$\boldsymbol{A}(\mathcal{W}) = \begin{pmatrix} a_1(\mathcal{W}) & a_1\left(\mathcal{W} + \frac{1}{r}\right) & \cdots & a_1\left(\mathcal{W} + \frac{r-1}{r}\right) \\ a_2(\mathcal{W}) & a_2\left(\mathcal{W} + \frac{1}{r}\right) & \cdots & a_2\left(\mathcal{W} + \frac{r-1}{r}\right) \\ \vdots & \vdots & & \vdots \\ a_s(\mathcal{W}) & a_s\left(\mathcal{W} + \frac{1}{r}\right) & \cdots & \left(a_s\mathcal{W} + \frac{r-1}{r}\right) \end{pmatrix}$$

$$= \left[a_j\left(\mathcal{W} + \frac{k-1}{r}\right)\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r}} \quad ,$$

and its related constants

$$\alpha_A = \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{min}\,[A^*(\mathcal{W})A(\mathcal{W})]\,, \qquad \beta_A = \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{max}\,[A^*(\mathcal{W})A(\mathcal{W})]\,,$$

where $A^*(\mathcal{W})$ denotes the transpose conjugate of the matrix $A(\mathcal{W})$, and λ_{min} (respectively, λmax) the smallest (respectively, the largest) eigenvalue of the positive semi definite matrix $A^*(\mathcal{W})A(\mathcal{W})$. Observe that $0 \le \alpha_A \le \beta_A \le \infty$. Notice that in the definition of the matrix $A(\mathcal{W})$ we are considering the 1-periodic extensions of the involved functions $a_j\,, j = 1,2,\dots,s$.

Lemma (4.1.3)[89]: Let a_j be in $L^2(0,1)$ for $j = 1,2,\dots,s$ and let $A(\mathcal{W})$be its associated matrix. Then:

(i) The sequence $\{\bar{a}_j\,(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ if and only if $a_j \in L^\infty(0,1)$ for $j = 1,\dots,s$. In this case, the optimal Bessel bound is β_A/r.

(ii) The sequence $\{\bar{a}_j\,(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_A \le \beta_A < \infty$.. In this case, the optimal frame bounds are α_A/r and β_A/r.

Proof : Notice that the equivalence between the spectral and the Frobenius norms (see [123]) gives $a_j \in L^\infty(0,1)$ for $j = 1,2,\dots,s$ if and only if $\beta_A < \infty$. For $p = r$ or $s, L_p^2(0,1/r)$ denotes the space of the functions $H = [h_1,\dots,h_p]^\top$ such that

$$\|H\|^2_{L_p^2(0,1/r)} = \int_0^{1/r} |H(\mathcal{W})|^2\, d\mathcal{W} = \sum_{j=1}^{p} \left\|h_j\right\|^2_{L^2(0,1/r)} < \infty\,, \text{ where } |H(\mathcal{W})|^2 \text{ is}$$

the Euclidean norm of $H(\mathcal{W})$ in $\mathbb{C}^p$. For any $F \in L^2(0,1)$ we have

$$\langle F(\cdot), \bar{a}_j\,(\cdot) e^{2\pi i r n \cdot}\rangle_{L^2(0,1)} = \int_0^{1/r} \sum_{k=1}^{r} a_j\left(\mathcal{W} + \frac{k-1}{r}\right) F\left(\mathcal{W} + \frac{k-1}{r}\right) e^{-2\pi i r n \mathcal{W}}\, d\mathcal{W}.$$

Denote $F(\mathcal{W}) = \left[F(\mathcal{W}), F\left(w + \frac{1}{r}\right), \dots, F\left(\mathcal{W} + \frac{r-1}{r}\right)\right]^{\intercal}$, whenever $A(\mathcal{W})F(\mathcal{W}) \in L_s^2(0, 1/r)$, we obtain

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left|\langle F(\cdot), \bar{a}_j\,(\cdot) e^{2\pi i r n \cdot}\rangle_{L^2(0,1)}\right|^2 = \frac{1}{r}\sum_{j=1}^{s} \left\|\sum_{k=1}^{r} a_j\left(\cdot + \frac{k-1}{r}\right) F\left(\cdot + \frac{k-1}{r}\right)\right\|_{L^2(0,1/r)}^2$$

$$= \frac{1}{r}\|A(\cdot)F(\cdot)\|_{L_s^2(0,1/r)}^2 = \frac{1}{r}\int_0^{1/r} F^*(\mathcal{W})A^*(\mathcal{W})A(\mathcal{W})F(\mathcal{W})\, d\mathcal{W}. \qquad (6)$$

If $\beta_A < \infty$ then, for each $F \in L^2(0,1)$, we have

$$\int_0^{1/r} F^*(\mathcal{W})A^*(\mathcal{W})A(\mathcal{W})F(\mathcal{W})\, d\mathcal{W} \le \beta_A \|F\|_{L_r^2(0,1/r)}^2 = \beta_A\|F\|_{L^2(0,1)}^2,$$

from which $\{\bar{a}_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,\dots,s}$ is a Bessel sequence of $L^2(0,1)$ and the optimal Bessel bound is less than or equal to β_A/r. Let $K < \beta_A$. Then, there exists a set $\Omega_K \subset (0, 1/r)$ of positive measure such that $\lambda_{max}[A^*(\mathcal{W})A(\mathcal{W})] \ge K$ for $\mathcal{W} \in \Omega_K$. Let $F \in L^2(0,1)$ such that its associated vector function $F(\mathcal{W})$ is 0 if $\mathcal{W} \in (0, 1/r)\setminus\Omega_K$ and $F(\mathcal{W})$ is an eigenvector of norm 1 associated with the largest eigenvalue of $A^*(\mathcal{W})A(\mathcal{W})$ if $\mathcal{W} \in \Omega_K$. We have that $A(\mathcal{W})F(\mathcal{W}) \in L_s^2(0, 1/r)$ and, using (6), we obtain

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left|\langle F(\cdot), \bar{a}_j(\cdot) e^{2\pi i r n \cdot}\rangle_{L^2(0,1)}\right|^2 \ge \frac{1}{r}\int_0^{1/r} K|F(\mathcal{W})|^2\, d\mathcal{W} = \frac{K}{r}\|F\|_{L^2(0,1)}^2.$$

Therefore if $\beta_A = \infty$ then $\{\bar{a}_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,\dots,s}$ is not a Bessel sequence in $L^2(0,1)$, and if $\beta_A < \infty$ then the optimal Bessel bound is β_A/r. This completes the proof of (i).To prove part (ii) of the Lemma, assume first that $0 < \alpha_A \le \beta_A < \infty$. By using part (i), the sequence $\{\bar{a}_j(\cdot)e^{2\pi i r n \cdot}\}_{n\in\mathbb{Z}, j=1,\dots,s}$ is a Bessel sequence in $L^2(0,1)$. Moreover, using (64) and the Rayleigh–Ritz theorem (see [123]), for each $F \in L^2(0,1)$ we obtain

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \left|\langle F(\cdot), \bar{a}_j(\cdot) e^{2\pi i r n \cdot}\rangle_{L^2(0,1)}\right|^2 \ge \frac{\alpha_A}{r}\|F\|_{L_r^2(0,1/r)}^2 = \frac{\alpha_A}{r}\|F\|_{L^2(0,1)}^2.$$

Hence, the sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,\ldots,s}$ is a frame for $L^2(0,1)$ with optimal lower frame bound bigger or equal that α_A/r. Conversely, if $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,\ldots,s}$ is a frame for $L^2(0,1)$ we know by part (i) that $\beta_A < \infty$. In order to prove that $\alpha_A > 0$, consider any constant $K > \alpha_A$. Then, there exists a set $\Omega_K \subset (0,1/r)$ with positive measure such that $\lambda_{min}[A^*(\mathcal{W})A(\mathcal{W})] \leq K$ for $\mathcal{W} \in \Omega_K$. Let $F \in L^2(0,1)$ such that its associated vector function $F(\mathcal{W})$ is 0 if $w \in (0,1/r)\setminus\Omega_K$ and $F(\mathcal{W})$ is an eigenvector of norm 1 associated with the smallest eigenvalue of $A^*(\mathcal{W})A(\mathcal{W})$ if $w \in \Omega_K$. Since F is bounded, we have that $A(\mathcal{W})F(\mathcal{W}) \in L^2_S(0,1/r)$. From (6) we get

$$\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\left|\langle F(\cdot),\bar{a}_j(\cdot)e^{2\pi irn\cdot}\rangle_{L^2(0,1)}\right|^2 \leq \frac{K}{r}\int_0^{1/r}|F(\mathcal{W})|^2\,d\mathcal{W} = \frac{K}{r}\|F\|^2_{L^2(0,1)}.$$

Denoting by A the optimal lower frame bound of $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z}}, j = 1,\ldots,s$, we have obtained that $K/r \geq A$ for each $K > \alpha_A$. Thus $\alpha_A/r \geq A$ and consequently, $\alpha_A > 0$. Moreover, under the hypotheses of part (ii) we deduce that α_A/r and β_A/r are the optimal frame bounds.

In order to complete the statement of Lemma (4.1.3),it is worth mentioning that one can also prove that the sequence $\{\bar{a}_j(\cdot)e^{2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,\ldots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if it is a frame for $\boldsymbol{L^2(0,1)}$ and $r = s$. Consider the functions $g_j, j = 1,2,\ldots,s$, given in (3), and its related matrix **G**. It is worth to point out that, Lemmas (4.1.2), (4.1.3), and the isomorphism $\mathcal{T}$gives the following result: There exist two constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\left|\mathcal{L}_j f(rn)\right|^2 \leq B\|f\|^2 \quad \text{for all} \quad f \in V_\phi \tag{7}$$

if and only if $0 < \alpha_G \leq \beta_G < \infty$. Equation (7) coincides with the definition of *stable uniform averaging sampler* given by Aldroubi and co-workers in [105]. In [105] a necessary and sufficient condition for (7) is given for a shift-invariant space with several generators. That condition is equivalent to this given above as one can easily check.

The main aim in this section is to recover any function f in the shift-invariant space V_ϕ from its samples $\{(\mathcal{L}_j f)(rn)\}_{n\in\mathbb{Z},j=1,2,\ldots,s}$ by means of a stable sampling formula, i.e., the sampling formula will be an expansion with respect to an appropriate frame for V_ϕ. Having in mind Lemma (4.1.2), for each $j = 1,2,\ldots,s$ we have

$$(\mathcal{L}_j f)(rn) = \langle\sum_{k=0}^{r-1}F\left(\cdot + \frac{k}{r}\right)g_j\left(\cdot + \frac{k}{r}\right), e^{-2\pi inr\cdot}\rangle_{L^2(0,1/r)}, n \in \mathbb{Z},$$

where $f = \mathcal{T}F$. Assuming that $g_j \in L^\infty(0,1)$, for each $j = 1,2,\ldots,s$, we obtain that

$$r\sum_{n\in\mathbb{Z}}(\mathcal{L}_j f)(rn)e^{-2\pi inr\mathcal{W}} = \sum_{k=0}^{r-1} F\left(\mathcal{W}+\frac{k}{r}\right) g_j\left(\mathcal{W}+\frac{k}{r}\right) \; in \; L^2(0,1/r).$$

The above expansions also hold in $L^2(0,1)$ by considering the 1-periodic extensions of F and g_j , $j = 1,2,\ldots,s$. Thus we have the matrix expression

$$G(\mathcal{W})F(\mathcal{W}) = r\left[\sum_{n\in\mathbb{Z}}(\mathcal{L}_1 f)(rn)e^{-2\pi inr\mathcal{W}},\ldots,\sum_{n\in\mathbb{Z}}(\mathcal{L}_s f)(rn)e^{-2\pi inr\mathcal{W}}\right]^{\top} \text{ in } L^2(0,1), \qquad (8)$$

where $G(\mathcal{W}) = [g_j(\mathcal{W} + \frac{k-1}{r})]_{j=1,2,\ldots,s,k=1,2,\ldots,r}$ and

$F(\mathcal{W}) = \left[F(\mathcal{W}), F(\mathcal{W} + 1r), \ldots, F(\mathcal{W} + \frac{r-1}{r})\right]^{\top}$. In order to recover F, assume that there exists a vector $[a_1(\mathcal{W}),\ldots,a_s(\mathcal{W})]$ with entries in $L^\infty(0,1)$ such that $[a_1(\mathcal{W}),\ldots,a_s(\mathcal{W})]G(\mathcal{W}) = [1,0,\ldots,0]$ a.e. in $(0,1)$.

As it will be proved later (see Theorem (4.2.2) below), a necessary and sufficient condition for the existence of such a vector (not necessarily unique) is that $\alpha_{\boldsymbol{G}} > 0$. If we left multiply (8) by $[a_1(\mathcal{W}),\ldots,a_s(\mathcal{W})]$ we get

$$F(\mathcal{W}) = r[a_1(\mathcal{W}),\ldots,a_s(\mathcal{W})]\left[\sum_{n\in\mathbb{Z}}(\mathcal{L}_1 f)(rn)e^{-2\pi inr\mathcal{W}},\ldots,\sum_{n\in\mathbb{Z}}(\mathcal{L}_s f)(rn)e^{-2\pi inr\mathcal{W}}\right]^{\top}$$

$$= r\sum_{j=1}^{s} a_j(\mathcal{W})\sum_{n\in\mathbb{Z}}(\mathcal{L}_j f)(rn)e^{-2\pi inr\mathcal{W}} = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)a_j(\mathcal{W})e^{-2\pi inr\mathcal{W}},$$

in the $L^2(0,1)$-sense. Finally, the isomorphism $\mathcal{T}$ gives

$$f(t) = r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)(\mathcal{T}a_j)(t-rn) \quad \text{in} \quad V_\phi,$$

where we have used (2). In addition, much more can be said about the above sampling expansion. In fact, the following result holds:

Theorem (4.1.4)[89]: Assume that $g_j \in L^\infty(0,1)$ for $j = 1,2,\ldots,s$. If there exists a vector $[a_1(\mathcal{W}),\ldots,a_s(\mathcal{W})]$ with entries in $L^\infty(0,1)$such that

$$[a_1(\mathcal{W}),\ldots,a_s(\mathcal{W})]G(\mathcal{W}) = [1,0,\ldots,0] \; a.e. \; in \; (0,1) \qquad (9)$$

then, for each $f \in V_\phi$, we have

$$f(t) = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)S_j(t-rn), t \in \mathbb{R}, \qquad (10)$$

where $S_j = r\mathcal{T}a_j$, $j = 1,\ldots,s$. Moreover, the sequence $\{S_j(t - rn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$ is a frame for V_ϕ with frame bounds $r\alpha_A\|\Phi\|_0$ and $r\beta_A\|\Phi\|_\infty$. The convergence of the series in (10) is in the $L^2(\mathbb{R})$ sense, absolute and uniform on $\mathbb{P}$.

Proof : Given $f \in$, consider $F = \mathcal{T}^{-1}f$ in $L^2(0,1)$. Above we have proved that

$$F(\mathcal{W}) = r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}\langle F(\cdot), \bar{g}_j(\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)} a_j(\mathcal{W})e^{-2\pi inr\mathcal{W}} \; in \; L^2(0,1). \qquad (11)$$

Thus, the sequences $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ and $\{g_j\,(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ are Bessel sequences for $L^2(0,1)$ satisfying the representation property (11). In [82] we obtain that they are dual frames for $L^2(0,1)$.Next, applying the isomorphism $\mathcal{T}$ to (11) one gets the sampling expansion (10) in V_ϕ, where $S_j = r\mathcal{T}a_j\ , j = 1,2,\dots,s$. Moreover, the sequence $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for V_ϕ. From Lemma (4.1.3) the optimal frame bounds for $\{raj\ (\cdot)e - 2\pi irn\ \cdot\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ are $r\alpha_A$ and $r\beta_A$.

Hence, $r\alpha_A\|\mathcal{T}^{-1}\|^{-2} = r\alpha_A\|\Phi\|_0$ and $r\beta_A\|T\|^2 = r\beta_A\|\Phi\|_\infty$ are frame bounds for $\{S_j(\cdot - nr)\}_{n\in\mathbb{Z}}\ , j = 1,2,\dots,s = \{\mathcal{T}[ra_j\ (\cdot)e^{-2\pi irn\cdot}]\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ (see [82]).

Point wise convergence in the sampling series is absolute due to the unconditional convergence of a frame expansion. The uniform convergence on $\mathbb{R}$ is a consequence of (11). Notice that the frame bounds in Theorem (4.1.4) are optimal whenever $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_ϕ because, in this case, $\mathcal{T}$ is an unitary operator. In the general case, the optimal frame bounds could be computed orthonormalizing the Riesz basis $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ as $\{\tilde{\phi}(\cdot - n)\}_{n\in\mathbb{Z}}$, where the orthonormal generator $\tilde{\phi}$ has Fourier transform $\hat{\tilde{\phi}} := \hat{\phi}/\sqrt{\Phi}$ (see [82]), and using (5).The functions S_j for $j = 1,2,\dots,s$ are determined from the Fourier coefficients of $\boldsymbol{a_j}$ with respect to the orthonormal basis $\{\boldsymbol{e^{2\pi in\cdot}}\}_{\boldsymbol{n}\in\mathbb{Z}}$. Indeed**,**

$$S_j(t) = r(\mathcal{T}a_j)(t) = r\sum_{n\in\mathbb{Z}}\langle a_j(\cdot), e^{-2\pi in\cdot}\rangle_{L^2(0,1)}\,\phi(t-n)\,. \qquad (12)$$

The Fourier transform in (12) gives $\hat{S}_j\,(\mathcal{W}) = ra_j(\mathcal{W})\hat{\phi}(\mathcal{W}), j = 1,2,\dots,s,$ where we have used [82].Observe that condition (9) is equivalent to $A^\top(\mathcal{W})G(\mathcal{W}) = I_r$ a.e. in $(0,1)$. In particular, this matrix equality implies that $\operatorname{rank}[G(\mathcal{W})] = r$ a.e. in $(0,1)$ and, as a consequence, necessarily $s \ge r$. In the next result we give a characterization of the existence of a sampling formula like (10). It is also proved that Theorem (4.1.4) provides all these formulas.

Theorem $(\mathbf{4.1.5})[\mathbf{89}]$: Assume that $g_j \in L^\infty(0,1)$ *for* $j = 1,2,\dots,s$. Then the following statements are equivalent:

(i) There exists a frame for $V_\phi$having the form $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ such that for each $f \in V_\phi$,

$$f = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(L_jf)(rn)S_j(\cdot - rn)\ \ in\ L^2(\mathbb{R})\,, \qquad (13)$$

(ii) $\alpha_{\boldsymbol{G}} > 0$. If these equivalent conditions hold, the reconstruction functions are given by $S_j = r\mathcal{T}a_j$, where the functions $a_j \in L^\infty(0,1), j = 1,2,\dots,s,$ satisfy $A^\top(\mathcal{W})G(\mathcal{W}) = I_r\ a.e.\,in\ (0,1)$.

Proof: First, assume that $\{S_j\ (t - rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for V_ϕ for which formula (13) holds. Applying the isomorphism $\mathcal{T}^{-1}$ to (13) we get

$$F(\mathcal{W}) = r\sum_{n\in\mathbb{Z}}\sum_{j=1}^{s}(\mathcal{L}_j f)(rn)a_j(\mathcal{W})e^{-2\pi irn\mathcal{W}} \quad in \quad L^2(0,1),$$

where $ra_j = \mathcal{T}^{-1}S_j, j = 1,2,\dots,s$. The sequence $\{ra_j(\mathcal{W})e^{-2\pi irn\mathcal{W}}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ and therefore, the functions $a_j \in L^\infty(0,1)$. Since

$$F(\mathcal{W}) = r\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\langle F(\cdot),\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}\ a_j(\mathcal{W})e^{-2\pi inr\mathcal{W}},$$

and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Bessel sequence for $L^2(0,1)$, we obtain that the sequences $\{ra_j(\cdot)e - 2\pi irn\ \cdot\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ are dual frames for $L^2(0,1)$ (see [82]). In particular, according to Lemma (4.1.3), we deduce that $\alpha_{\boldsymbol{G}} > 0$. This proves (ii). Besides, for each $F_1, F_2 \in L^2(0,1)$ we have [82]:

$$\langle F_1,F_2\rangle_{L^2(0,1)} = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\langle F_1(\cdot),a_j(\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)}\langle\bar{g}_j(\cdot)e^{-2\pi irn\cdot},F_2(\cdot)\rangle_{L^2(0,1)}. \quad (14)$$

Having in mind that

$$\langle F_1(\cdot),a_j(\cdot)e^{-2\pi irn\cdot}\rangle_{L^2(0,1)} = \langle\sum_{k=0}^{r-1}F_1\left(\cdot + \frac{k}{r}\right)a_j\left(\cdot + \frac{k}{r}\right),e^{-2\pi inr\cdot}\rangle_{L^2(0,1/r)},$$

$$\langle\bar{g}_j(\cdot)e^{-2\pi irn\cdot},F_2(\cdot)\rangle_{L^2(0,1)} = \langle e^{-2\pi inr\cdot},\sum_{k=0}^{r-1}F_2\left(\cdot+\frac{k}{r}\right)g_j\left(\cdot+\frac{k}{r}\right)\rangle_{L^2(0,1/r)},$$

Parseval's equality allows us to write the right-hand side in (14) as

$$\sum_{j=1}^{s}\langle\sum_{k=0}^{r-1}F_1\left(\cdot + \frac{k}{r}\right)\bar{a}_j\left(\cdot + \frac{k}{r}\right),\sum_{k=0}^{r-1}F_2\left(\cdot + \frac{k}{r}\right)g_j\left(\cdot + \frac{k}{r}\right)\rangle_{L^2(0,1/r)}$$

$$= \int_0^{1/r} F_1^{\intercal}(\mathcal{W})\bar{a}_j(\mathcal{W})g_j^*(w)\bar{F}_2(\mathcal{W})\,d\mathcal{W} = \int_0^{1/r} F_1^{\intercal}(\mathcal{W})A^*(\mathcal{W})\bar{G}(\mathcal{W})\bar{F}_2(\mathcal{W})\,d\mathcal{W}.$$

Since the left-hand side in (14) equals $\int_0^{1/r}F_1^{\intercal}(\mathcal{W})\bar{F}_2(\mathcal{W})\,d\mathcal{W}$, we obtain that $A^{\intercal}(\mathcal{W})G(\mathcal{W}) = I_r\ a.e.\ in\ (0,1)$.Conversely, assume that $\alpha_{\boldsymbol{G}} > 0$. Hence, the inverse matrix $[G*(\mathcal{W})\,G(\mathcal{W})]^{-1}$ exists a.e. in $(0,1)$. Consider the firstrow $[a_1(\mathcal{W}),\dots,a_s(\mathcal{W})]$ of the pseudo-inverse matrix $G^{\dagger}(\mathcal{W}) = [G^*(\mathcal{W})\,G(\mathcal{W})]^{-1}G^*(\mathcal{W})$ of $G(\mathcal{W})$. Its entries a_j are essentially bounded in $(0,1)$ since the functions g_j and $det^{-1}[G^*(\mathcal{W})G(\mathcal{W})]$ are essentially bounded in $(0,1)$. From $G^{\dagger}(\mathcal{W})G(\mathcal{W}) = I_r$ we obtain that $[a_1(\mathcal{W}),\dots,a_s(\mathcal{W})]\,G(\mathcal{W}) = [1,0,\dots,0]$ a.e. in $(0,1)$. Thus, (i) comes out by using Theorem (4.1.4) .

When ever the functions g_j are continuous on $\mathbb{R}$, the condition $\alpha_{\boldsymbol{G}} > 0$ is equivalent to $det[G^*(\mathcal{W})(w)G(w)] \neq 0\ on\ \mathbb{R}$. It can be proved that the first row

$[a_1(\mathcal{W}),\dots,a_s(\mathcal{W})]$ of the pseudo-inverse matrix $G^{\dagger}(\mathcal{W})$gives precisely the canonical dual frame $\{ra_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ of the frame $\{g_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$. Other suitable solutions for (9) are given by the first row of the matrix $G^{\dagger}(\mathcal{W})+U(\mathcal{W})[I_s-G(\mathcal{W})G^{\dagger}(\mathcal{W})]$, where $U(\mathcal{W})$ is any $r\times s$ matrix function with entries in $L^{\infty}(0,1)$. When ever $r=s$ we are in the Riesz bases setting, and the following result holds:

Corollary (4.1.6)[89]: Assume that $r=s$ and $\alpha_{\boldsymbol{G}}>0$. Then, there exists a unique frame $\{S_j(t-sn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ for V_ϕ for which the sampling formula (13) holds. In this case, this frame is a Riesz basis for $V_\phi$with Riesz bounds $s\|\Phi\|_0/\beta_G$ and $s\|\Phi\|_\infty/\alpha_{\boldsymbol{G}}$. Moreover, the functions $a_j, j=1,2,\dots,s$, form the first row of the matrix $\boldsymbol{G}^{-1}$. The functions $S_j, j=1,2,\dots,s$, satisfy the interpolation property $(\mathcal{L}_l S_j)(sn)=\delta_{j,l}\delta_{n,0}$, where $j,l=1,2,\dots,s$ and $n\in\mathbb{Z}$.

Proof : In this case, the unique solution of $[\boldsymbol{a_1}(\boldsymbol{\mathcal{W}}),\dots,\boldsymbol{a_s}(\boldsymbol{\mathcal{W}})]\boldsymbol{G}(\boldsymbol{\mathcal{W}})=[\mathbf{1},\mathbf{0},\dots,\mathbf{0}]$ is given by the first row of $G^{\dagger}=G^{-1}$. By using that $G(\mathcal{W})G^{-1}(\mathcal{W})=I_s$ we obtain

$$\begin{aligned}\langle sa_j(\cdot)e^{-2\pi isn\cdot},\bar{g}_l(\cdot)e^{-2\pi ims\cdot}\rangle_{L^2(0,1)}\\ &= s\int_0^1 a_j(\mathcal{W})g_l(\mathcal{W})e^{2\pi i(m-n)s\mathcal{W}}\,d\mathcal{W}\\ &= s\int_0^{\frac{1}{s}}\sum_{k=0}^{s-1} a_j\left(\mathcal{W}+\frac{k}{s}\right)g_l\left(\mathcal{W}+\frac{k}{s}\right)e^{2\pi i(m-n)s\mathcal{W}}\,d\mathcal{W}\\ &= \delta_{l,j}\,\delta_{n,m}.\end{aligned}$$

Therefore, the dual frames $\{sa_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ and $\{\bar{g}_j(\cdot)e^{-2\pi irn\cdot}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$are biorthogonal. Hence [82], they form a pair of biorthogonal Riesz bases. The Riesz bounds for $\{S_j(t-rn)\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ follow from Theorem (4.1.4) having in mind that, in this case, $\alpha_A\beta_G=\alpha_G\beta_A=1$.

Finally, sampling formula (13) for S_j gives

$S_j(t)=\sum_{n\in\mathbb{Z}}\sum_{l=1}^{s}(\mathcal{L}_l S_j)(sn)S_l(t-sn)$. The uniqueness of the coefficients of an expansion with respect to a Riesz basis implies $(\mathcal{L}_l S_j)(sn)=\delta_{j,l}\delta_{n,0}$.

First, recall that $\{\phi(\cdot-n)\}_{n\in\mathbb{Z}}$ is a frame sequence with bounds $0<A\le B<\infty$, i.e.,a frame for its closed linear span , if and only if $A\le\Phi(\mathcal{W})\le B$ $a.e.$ in $(0,1)\setminus N$, where $N=\{w\in(0,1):\Phi(\mathcal{W})=0\}$ [82]. In this case, $\mathcal{T}$ is a bounded surjective operator from $L^2(0,1)$ onto V_ϕ. For any $f=\mathcal{T}F\in V_\phi$ we have $\hat{f}=F\hat{\phi}$. Since $\hat{\phi}(w+n)=0$ a.e. in N and $n\in\mathbb{Z}$, we deduce that $\mathcal{T}F_1=\mathcal{T}F_2$, where $F_1,F_2\in L^2(0,1)$, if and only if $F_1=F_2$ in $L^2((0,1)\setminus N)$. Under the new hypothesis $[a_1(\mathcal{W}),\dots,a_s(\mathcal{W})]G(\mathcal{W})=[1,0,\dots,0]$ $a.e.in$ $(0,1)\setminus N$, the sampling result (10) in Theorem (4.1.4) also holds. One can check that the proof in Theorem (4.1.4)

applies, having in mind that the operator $\tilde{\mathcal{T}} : L^2((0,1)\setminus N) \to V_\phi$ defined for $F \in L^2((0,1)\setminus N)$ as $\tilde{\mathcal{T}}F = \mathcal{T}\tilde{F}$, where $\tilde{F}(\mathcal{W}) = \begin{cases} F(\mathcal{W}) \ if \ \ \mathcal{W} \in (0,1)\setminus N, \\ 0 \ if \ \ \mathcal{W} \in N, \end{cases}$
is an isomorphism satisfying the shifting property (2). In this case, the sequence $\{S_j(t - rn)\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$, where $S_j = r\,\tilde{\mathcal{T}} a_j$, $j = 1,2,\dots,s$, is also a frame for V_ϕ. Notice that the function g_j defined in (3) is nothing but the Zak transform $(\boldsymbol{Z\mathcal{L}_j\phi})(\mathbf{0},\cdot)$. As in Lemma (4.1.2), one can prove for any $f \in V_\phi$ that $(\mathcal{L}_j f)(rn + \varepsilon n) = \langle F(\cdot), \overline{(Z\mathcal{L}_j\phi)}(\varepsilon n,\cdot)e^{-2\pi i rn\cdot}\rangle_{L^2(0,1)}$, where $F = \mathcal{T}^{-1}f$, and $\{\varepsilon n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$. As a consequence, stable generalized irregular sampling in V_ϕ depends on whether the sequence $\{\overline{(Z\mathcal{L}_j\phi)}(\varepsilon n,\mathcal{W})e^{-2\pi i rn\mathcal{W}}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ is a frame for $L^2 0,1)$. This sequence can be seen as a perturbation of the frame $\{\overline{(Z\mathcal{L}_j\phi)}(0,\mathcal{W})e^{-2\pi i rn\mathcal{W}}\}_{n\in\mathbb{Z}, j=1,2,\dots,s}$ appearing in Theorem (4.1.4). Hence, by using similar techniques as those in [120], the theory on perturbation of frames (see [82]) yields generalized irregular sampling in the shift-invariant space V_ϕ, for suitable error sequences $\{\varepsilon n\}_{n\in\mathbb{Z}}$. This is work in progress and will appear elsewhere [121].

In the over sampling setting, i.e., $s > r$, Theorem (4.1.4) allows us different choices for the vector $a(\mathcal{W}) := [a_1(\mathcal{W}),\dots,a_s(\mathcal{W})]$ and consequently, different reconstruction functions S_j . One may use this flexibility in order to obtain appropriate sampling functions S_j. For instance, if the generator ϕ and the impulse responses of the linear-time invariant systems $\mathcal{L}_j$ have compact support, the functions g_j are trigonometric polynomials and we can choose $a(\mathcal{W})$ in order to obtain sampling functions S_j with compact support (which involves low computational complexities and avoids truncation errors). We illustrate this assertion in the case of cubic splines:

The cubic B-spline is defined as $N_4 = N_1 * N_1 * N_1 * N_1$, where N_1denotes the characteristic function of the interval $(0,1)$. It is known that N_4 is a stable generator for the cubic splines in $L^2(\mathbb{R})$ with nodes at the integers (see[118]). Consider the $s = 3$ linear-time invariant systems defined as

$$\mathcal{L}_1 f(x) = \int_x^{x+\frac{1}{3}} f(t)dt \ , \mathcal{L}_2 f(x) = \int_{x+\frac{1}{3}}^{x+\frac{2}{3}} f(t)dt \, , \mathcal{L}_3 f(x) = \int_{x+\frac{2}{3}}^{x+1} f(t)\, dt \, ,$$

and the sampling period $r = 2$. Denoting by

$$g_j(z) = \sum_{n\in\mathbb{Z}} \mathcal{L}_j\phi(n)z^n \ , G(z) = \left[g_j\left(ze^{-2\pi i(k-1)/r}\right)\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r}}$$

if there exists a vector $b(z) := [b_1(z), b_2(z), b_3(z)]$ whose entries are polynomials, and such that $b(z)G(z) = [z^l, 0]$ for some non-negative integer l, then the vector $a(w) = e^{2\pi i k\mathcal{W}} b(e^{-2\pi i\mathcal{W}})$, whose entries are trigonometric polynomials, satisfies

$a(\mathcal{W})G(\mathcal{W}) = [1,0]$. Thus we have obtained reconstruction functions S_j with compact support (see (12)). In particular, solving a linear system of 12 equations with 12 unknowns we find a vector $b(z)$ whose entries are polynomials of degree 3, satisfying $b(z)G(z) = [z,0]$. The corresponding sampling functions S_j , $j = 1,2,3$, are

$$S_1(t) = 418^{-1}[-5395N_4(t+1) + 22687N_4(t) + 188N_4(t-1) - 705N_4(t-2)],$$

$$S_2(t) = 418^{-1}[7943N_4(t+1) - 41438N_4(t) - 892N_4(t-1) + 3345N_4(t-2)],$$

$$S_3(t) = 418^{-1}[-1750N_4(t+1) + 21715N_4(t) + 1160N_4(t-1) - 4350N_4(t-2)].$$

The associated sampling formula for $f \in V_{N_4}$ reads:

$$f(t) = \sum_{n\in\mathbb{Z}}[\mathcal{L}_1 f(2n)S_1(t-2n) + \mathcal{L}_2 f(2n)S_2(t-2n) + \mathcal{L}_3 f(2n)S_3(t-2n)],$$

$t \in \mathbb{R}$, uniformly on $\mathbb{R}$.

Section(4.2) Shift-Invariant Spaces and its Approximation Properties

The classical Whittaker–Shannon–Kotel'nikov sampling theorem states that any function f band-limited to $[-1/2,1/2]$,i.e.,

$f(t) = \int_{-1/2}^{1/2}\hat{f}(\mathcal{W})e^{2\pi i t W}\, d\mathcal{W}, t \in \mathbb{R}$, may be reconstructed from its sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, sinc(t-n)\, , t \in \mathbb{R},$$

where sinc denotes the cardinal sin function, $sinc(t) = sin\pi t/\pi t$. Thus, the Paley–Wiener space of functions band-limited to $[-1/2,1/2]$ is generated by the integer shifts of the sinc function. The WSK sampling formula has its counterpart in d dimensions. It reads:

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha)sinc(t_1-\alpha_1)\ldots sinc(t_d-\alpha_d)\, , t = (t_1,\ldots,t_d) \in \mathbb{R}^d,$$

where now the function f is band-limited to the d-dimensional cube

$$[-1/2,1/2]^d, i.e., \qquad f(t) = \int_{[-1/2,1/2]^d} \hat{f}(x)e^{2\pi i x^{\top} t}\, dx, t \in \mathbb{R}^d.$$

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [80,106]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval.

Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. See, for instance, [78,106,113] and the references therein. In many practical applications, signals are assumed to belong to some shift-invariant space of the form $V_\phi^2 = \overline{span}_{L^2}\{\phi(t - \alpha): \alpha \in \mathbb{Z}^d\}$ where the function ϕ in $L^2(\mathbb{R}^d)$ is called the generator of V_ϕ^2 . Assuming that $\phi \in L^2(\mathbb{R}^d)$ is a stable generator, i.e., the sequence $\{\phi(t - \alpha)\}_{\alpha \in \mathbb{Z}^d}$ is a Riesz basis for V_ϕ^2, the shift-invariant space V_ϕ^2 can be described as

$$V_\phi^2 = \left\{ \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \phi(t - \alpha): \{a_\alpha\} \in \ell^2(\mathbb{Z}^d) \right\} \subset L^2(\mathbb{R}^d) . \qquad (15)$$

On the other hand, in many common situations the available data are samples of some filtered versions of the signal itself. This leads to generalized sampling (or average sampling following some recent authors [105]) in V_ϕ^2: Suppose that s linear time-invariant systems (filters) $\mathcal{L}_j , j = 1, 2, \ldots, s,$ are defined on the shift-invariant subspace V_ϕ^2 of $L^2(\mathbb{R}^d)$. In mathematical terms we are dealing with (continuous) operators which commute with shifts. The goal is to recover any function f in V_ϕ^2 from an appropriate subsequence of the set of samples $\{(\mathcal{L}_j f)(\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$, by means of a sampling formula which is a frame expansion in V_ϕ^2. Recall that a sequence $\{f_n\}$ is a frame for a separable Hilbert space H if there exist two constants $A, B > 0$ (frame bounds) such that $A\| f\|^2 \leq \sum_n|\langle f , f_n\rangle|^2 \leq B\| f\|^2$ for all $f \in \mathcal{H}$.Given a frame $\{f_n\}$ for H the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_n c_n f_n$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for over complete frames) is sacrificed. Suitable frame coefficients c_n which depend continuously and linearly on f are obtained by using the dual frames $\{g_n\}$ of $\{f_n\}$, i.e., $\{g_n\}$ is another frame for H such that $f = \sum_n\langle f , g_n\rangle f_n = \sum_n\langle f , f_n\rangle g_n$ for each $f \in \mathbb{H}$. For more details on the frame theory see the superb monograph [82] and the references therein.

Under appropriate hypotheses, any function in a shift-invariant space in $L^2(\mathbb{R}^d)$ can be recovered from its samples in the lattice $\mathbb{Z}^d$ of $\mathbb{R}^d$ (see [103]). If we sample the function on the sub-lattice $M\mathbb{Z}^d$, where M denotes a matrix of integer entries with positive determinant, we are using the sampling rate $1/(detM)$ and, roughly speaking, we will need the generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ from $s \geq detM$ linear systems $\mathcal{L}_j$ for the recovery of f . The one-dimensional case has been treated in [85,89,107]: Under suitable hypotheses, we can recover any function in V_ϕ^2 from the sequence of generalized samples $\{(\mathcal{L}_j f (rn)\}_{n\in\mathbb{Z}, j=1,2,\ldots,s}$, where the number of channels is $s \geq r \in N$. In this work we obtain, in the light of the $L^2(\mathbb{R}^d)$-theory, sampling formulas for V_ϕ^2 of the type

$$f(t) = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_j(t - M\alpha), t \in \mathbb{R}^d, \quad (16)$$

where the sequence of reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ forms a frame for the shift-invariant space V_ϕ^2. To this end, first observe that the shift-invariant space V_ϕ^2 is the image of $L^2[0,1)^d$ under the isomorphism $T_\phi : L^2[0,1)^d \rightarrow V_\phi^2$, which maps the orthonormal basis $\{e^{-2\pi i\alpha^{\top}x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2[0,1)^d$ onto the Riesz basis $\{\phi(t - \alpha)\}_{\alpha\in\mathbb{Z}^d}$ for V_ϕ^2.

Next we express the generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ as the inner products of the function $F = \mathcal{T}_\phi^{-1} f \in L^2[0,1)^d$ with respect to a particular frame in $L^2[0,1)^d$. Searching for its dual frames we obtain those expansions for F in $L^2[0,1)^d$having the samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ as the frame coefficients. These frame expansions have precisely the form

$$F = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)d_j(x)e^{-2\pi i\alpha^{\top}M^{\top}x} \ in \ L^2[0,1)^d, \quad (17)$$

where the functions $d_j \in L^2[0,1)^d, j = 1,2,\ldots,s$, are obtained by solving a matrix equation

$$[d_1(x),\ldots,d_s(x)]G(x) = [1,0,\ldots,0] \ a.e. in \ [0,1)^d, \quad (18)$$

where $G(x)$ is an $s \times (detM)$ matrix of functions defined in $[0,1)^d$ (the so-called modulation matrix in the filter-bank jargon) which only depends on the generator ϕ and on the systems $\mathcal{L}_j, j = 1,2,\ldots,s$ (see (26) infra). Finally, applying the isomorphism T_ϕ to the frame expansion (17) for F we will obtain the aforesaid sampling expansions for $f = T_\phi F$ in V_ϕ^2 , where $S_j = T_\phi d_j, j = 1,2,\ldots,s$. Besides, the perturbation theory for frames gives generalized irregular sampling for appropriate sequences of perturbed generalized samples
$\{(\mathcal{L}_j f)(M\alpha + \varepsilon_j,\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$.

Moreover, in the oversampling case, i.e., whenever $s > detM$, we are dealing with overcomplete frames and several different dual frames allow us to obtain a variety of reconstruction functions. Thus we can try to find some reconstruction functions $S_j, j = 1,2,\ldots,s$, with “good properties", such as compact support, exponential decay, etc. As one can see in the present section, this relies on the search of solutions of (18) with prescribed properties. From a mathematical point of view, this is equivalent to solving (18) whenever the entries of the matrix function $G(x)$ belong to a prescribed algebra of functions .

This section shows that a generalized sampling formula like (16) allows to construct an $L^2(\mathbb{R}^d)$-approximation scheme as follows: For a suitable smooth function f (in a Sobolev space), consider the operator Γ, formally defined as

$$(\Gamma f)(t) = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_j(t - M\alpha), t \in \mathbb{R}^d.$$

The aim is to obtain a good approximation of f by means of the scaled operator Γ^h given by $\Gamma^h = \sigma_h\Gamma\sigma_{1/h}$, where $\sigma_h f = f(\cdot/h), h > 0$.

For f in an appropriate Sobolev space we obtain an estimation for the L^2-approximation error of the type $\|\Gamma^h f - f\|_2 = \mathcal{O}(h^r)$ as $h \to 0^+$, where $r \in \mathbb{N}$ denotes the approximation order which coincides with the order of the Strang–Fix conditions satisfied by the generator ϕ.

Looking for an estimation like the one above with respect to the L^∞-norm leads to extend the sampling formula (16) to the larger space $\boldsymbol{V_\phi^\infty} = \{\overline{\boldsymbol{span}}_{L^\infty}\boldsymbol{\phi}(\boldsymbol{t} - \boldsymbol{\alpha}): \boldsymbol{\alpha} \in \mathbb{Z}^{\boldsymbol{d}}\}$. Thus, for any function f in an appropriate Sobolev space, we obtain an analogous estimation for the L^∞-approximation error: Namely, $\|\boldsymbol{\Gamma^h f} - \boldsymbol{f}\|_\infty = \boldsymbol{\mathcal{O}}(\boldsymbol{h^r})$ as $h \to 0^+$ where now the approximation order r depends both on the order of the Strang–Fix conditions satisfied by the generator ϕ, and on the greatest order of the partial derivatives appearing in the systems $\mathcal{L}_j$, if any.

We introduce the needed preliminaries on the shift-invariant space V_ϕ^2, on the linear time-invariant systems $\mathcal{L}_j$, and on the lattices in $\mathbb{Z}^d$ in order to derive a generalized sampling theory in V_ϕ^2. Moreover, we study some sequences in $L^2[0,1)^d$ which play a crucial role in what follows. Let $\phi \in L^2(\mathbb{R}^d)$be a stable generator for the shift-invariant space

$$V_\phi^2 = \left\{\sum_{\alpha\in\mathbb{Z}^d} a_\alpha\phi(\cdot - \alpha): \{a_\alpha\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)\right\} \subset L^2(\mathbb{R}^d),$$

i.e., the sequence $\{\phi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz basis for V_ϕ^2. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\phi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz sequence in $L^2(\mathbb{R}^d)$, i.e., a Riesz basis for V_ϕ^2 if and only if $\mathbf{0} < \|\boldsymbol{\Phi}\|_{\mathbf{0}} \leq \|\boldsymbol{\Phi}\|_\infty < \infty$, where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(\mathcal{W}) = \sum_{\beta\in\mathbb{Z}^d}|\hat{\phi}(w + \beta)|^2 \; in \; [0,1)^d$, and $\|\Phi\|_\infty$ it's essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [82].

Besides, V_ϕ^2 is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_ϕ^2. Indeed, for each fixed $t \in \mathbb{R}^d$ we have

$$|f(t)|^2 \leq \frac{\|f\|^2}{\|\Phi\|_0}\sum_{\alpha\in\mathbb{Z}^d}|\phi(t - \alpha)|^2, f \in V_\phi^2, \qquad (19)$$

where we have used Cauchy–Schwartz's inequality on $f(t) = \sum_{\alpha\in\mathbb{Z}^d} a_\alpha \phi(t - \alpha)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{\alpha\in\mathbb{Z}^d} |a_\alpha|^2 \le \|f\|^2 \le \|\Phi\|_\infty \sum_{\alpha\in\mathbb{Z}^d} |a_\alpha|^2 , f \in V_\phi^2 .$$

Inequality (19) shows that convergence in the $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is uniform on $\mathbb{R}^d$.The reproducing kernel of V_ϕ^2 is given by $k(t,s) = \sum_{\alpha\in\mathbb{Z}^d} \phi(t-\alpha)\overline{\phi^*(s-\alpha)}$ where the sequence $\{\phi^*(\cdot -\alpha)\}_{\alpha\in\mathbb{Z}^d}$ denotes the dual Riesz basis of $\{\phi(\cdot -\alpha)\}_{\alpha\in\mathbb{Z}^d}$. Recall that the Fourier transform of the function ϕ^* is $\widehat{\phi^*} = \frac{\widehat{\phi}}{\Phi}$. On the other hand, the space V_ϕ^2 is the image of $L^2[0,1)^d$ by means of the isomorphism $\mathcal{T}_\phi : L^2[0,1)^d \to V_\phi^2$ which maps the orthonormal basis $\{e^{-2\pi i\alpha^\intercal x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2[0,1)^d$ onto the Riesz basis $\{\phi(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ for V_ϕ^2 . For $F \in L^2[0,1)^d$ we have $(\mathcal{T}_\phi F)(t) = \sum_{\alpha\in\mathbb{Z}^d} \hat{F}(\alpha)\phi(t-\alpha), t \in \mathbb{R}^d$, where $\hat{F}(\alpha)$, $\alpha \in \mathbb{Z}^d$, are the Fourier coefficients of F , i.e., for each $\alpha \in \mathbb{Z}^d$, $\hat{F}(\alpha) = \int_{[0,1)^d} F(x) e^{2\pi i\alpha^\intercal x}\, dx.$ Notice that any function $f = \mathcal{T}_\phi F$ in V_ϕ^2 , where $F \in L^2[0,1)^d$, can be expressed as $f(t) = \langle F, \overline{Z_\phi}(t,\cdot)\rangle_{L^2[0,1)^d}, t \in \mathbb{R}^d$, where Z_ϕ denotes the Zak transform of ϕ. Recall that the Zak transform of $f \in L^2(\mathbb{R}^d)$ is formally defined in $\mathbb{R}^{2d}$ as $(Zf)(t,x) = \sum_{\beta\in\mathbb{Z}^d} f(t+\beta)e^{-2\pi i\beta x}$. See [92] for properties and uses of the Zak transform. The following shifting property of $\mathcal{T}_\phi$ will be used later: For $F \in L^2[0,1)^d$ and $\alpha \in \mathbb{Z}^d$ we have

$$\mathcal{T}_\phi\left[F(\cdot)e^{-2\pi i\alpha^\intercal \cdot}\right](t) = \mathcal{T}_\phi[F](t-\alpha), t \in \mathbb{R}^d . \qquad (20)$$

We consider s linear time-invariant systems $\mathcal{L}_j$ in $L^2(\mathbb{R}^d)$ such that $\mathcal{L}_j f = h_j * f$, $j = 1,2,\dots,s$, of the following types:

(a) The impulse response h_j of $\mathcal{L}_j$ belongs to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Thus, for any $f \in V_\phi^2$ we have

$$(\mathcal{L}_j f)(t) = [f * h_j](t) = \int_{\mathbb{R}^d} f(x)h_j(t-x)\,dx, t \in \mathbb{R}^d.$$

(b) The impulse response h_j is a linear combination of partial derivatives of shifted delta functionals, i.e.,

$$(\mathcal{L}_j f)(t) = \sum_{|\beta|\le N_j} c_{j,\beta} D^\beta f\left(t + d_{j,\beta}\right) , \qquad t \in \mathbb{R}^d.$$

If there is a system of this type, we also assume that $\sum_{\alpha\in\mathbb{Z}^d} |D^\beta\phi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ for $|\beta| \le N_j$ .Whenever the linear system $\mathcal{L}_j$ is of type (a), the Minkowski inequality for integrals shows that the sequence $\{(\mathcal{L}_j\phi)(t+\beta)\}_{\beta\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ for any fixed $t \in \mathbb{R}^d$ (see [89]). Trivially, the same applies for $\mathcal{L}_j$ of type (b). Therefore, for any fixed $t \in \mathbb{R}^d$, the function

$(Z\mathcal{L}_j\phi)(t,x)=\sum_{\beta\in\mathbb{Z}^d}(\mathcal{L}_j\phi)(t+\beta)e^{-2\pi i\beta^\top x}$ belongs to $L^2[0,1)^d$ and the following expression for $\mathcal{L}_j$ holds: For any $f=\mathcal{T}_\phi F\in V_\phi^2$ we have

$$\left(\mathcal{L}_j f\right)(t)=\langle F,\overline{Z\mathcal{L}_j}\phi(t,\cdot)\rangle_{L^2[0,1)^d},t\in\mathbb{R}^d. \quad (21)$$

The proof is analogous to the one of [89]. In particular, for any $\alpha\in\mathbb{Z}^d$ we have

$$(\mathcal{L}_j f)(\alpha)=\langle F,\overline{Z\mathcal{L}_j}\phi(0,\cdot)e^{-2\pi i\alpha^\top\cdot}\rangle_{L^2[0,1)^d}=\int_{[0,1)^d}F(x)g_j(x)e^{2\pi i\alpha^\top x}\,dx, \quad (22)$$

where the functions $g_j, j=1,2,\dots,s$, given by

$$g_j(x)=Z\mathcal{L}_j\phi(0,x)=\sum_{\beta\in\mathbb{Z}^d}\left(\mathcal{L}_j\phi\right)(\beta)e^{-2\pi i\beta^\top x}\in L^2[0,1)^d, \quad (23)$$

will play a central role throughout this section.

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e., $Lat(M)=\{M\alpha:\alpha\in\mathbb{Z}^d\}\subset\mathbb{Z}^d$.

Without loss of generality we can assume that $detM>0$; otherwise we can consider $M'=ME$ where E is some $d\times d$ integer matrix satisfying $\det E=-1$. Trivially, $LatM=LatM'$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1} respectively. The following useful generalized orthogonal relationship holds (see [86]):

$$\sum_{k\in\mathcal{N}(M^\top)}e^{-2\pi i\alpha^\top M^{-\top}k}=\begin{cases}detM, & \alpha\in Lat(M),\\ 0, & \alpha\in\mathbb{Z}^d\setminus Lat(M),\end{cases} \quad (24)$$

Where $\mathcal{N}(M^\top)=\mathbb{Z}^d\cap\{M^\top x: x\in[0,1)^d\}$. The set $\mathcal{N}(M^\top)$ has $detM$ elements (see [108] or [109]). One of these elements is zero, say $i_1=0$; we denote the rest of elements by $i_2,\dots,i_{detM}$ ordered in any form.Note that the sets, defined as $Q_k=M^{-\top}i_k+M^{-\top}[0,1)^d, k=1,2,\dots,detM$, satisfy (see [109]):

$$Q_k\cap Q_{k'}=\emptyset \quad if \quad k\neq k' \quad and \quad Vol\left(\bigcup_{k=1}^{detM}Q_k\right)=1.$$

Thus, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic we have

$$\int_{[0,1)^d}F(x)\,dx=\sum_{k=1}^{detM}\int_{Q_k}F(x)\,dx.$$

Given s linear time-invariant systems $\mathcal{L}_j, j=1,2,\dots,s$, the aim is to recover any function $f\in V_\phi^2$ from its generalized samples at a lattice $Lat(M)$ of $\mathbb{Z}^d$, i.e., from the sequence of samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$. (80) gives

$$\left(\mathcal{L}_j f\right)(M\alpha)=\langle F,\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2[0,1)^d},\alpha\in\mathbb{Z}^d \ and \ j=1,2,\dots,s. \quad (25)$$

As a consequence, the recovery of the function $F=\mathcal{T}_\phi^{-1}f\in L^2[0,1)^d$, and hence of $f\in V_\phi^2$, from the sequence of generalized samples leads us to study the properties (completeness, Bessel, frame, or Riesz basis properties) of the sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ in $L^2[0,1)^d$.

Next, we carry out the study of the completeness, Bessel, frame, or Riesz basis properties of the sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ in $L^2[0,1)^d$. To this end, we introduce the $s \times (detM)$ matrix of functions

$$\boldsymbol{G}(x) = \begin{bmatrix} g_1(x) & g_1(x+M^{-\intercal}i_2) & \cdots & g_1(x+M^{-\intercal}i_{detM}) \\ g_2(x) & g_2(x+M^{-\intercal}i_2) & \cdots & g_2(x+M^{-\intercal}i_{detM}) \\ \vdots & \vdots & & \vdots \\ g_s(x) & g_s(x+M^{-\intercal}i_2) & \cdots & g_s(x+M^{-\intercal}i_{detM}) \end{bmatrix}$$

$$= \left[g_j(x+M^{-\intercal}i_k)\right]_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,detM}}, \tag{26}$$

and its related constants

$$A_G = \underset{x\in[0,1)^d}{ess\,inf}\,\lambda_{min}\,[G^*(x)G(x)]\ , B_G = \underset{x\in[0,1)^d}{ess\,sup}\lambda_{max}[G^*(x)G(x)]\,,$$

where $G^*(x)$ denotes the transpose conjugate of the matrix $G(x)$, and λ_{min} (respectively λ_{max}) the smallest (respectively the largest)eigen value of the positive semi definite matrix$G^*(x)G(x)$. Observe that $0 \le A_G \le B_G \le \infty$. Note that in the definition of the matrix $G(x)$ we are considering the $\mathbb{Z}^{\boldsymbol{d}}$-periodic extension of the involved functions $g_j\,, j = 1,2,\ldots,s$. The following result remains true for arbitrary functions g_j in $L^2[0,1)^d$, j = 1, 2, . . . , s, not necessarily given by (23).

Lemma (4.2.1)[77] : Let g_j be in $L^2[0,1)^d$ for $j = 1,2,\ldots,s$ and let $G(x)$ be its associated matrix as in (26). Then:

(a) The sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ is a complete system for $\boldsymbol{L^2[0,1)^d}$ if and only if the rank of the matrix $G(x)$ is detM a.e. in $[0,1)^d$.

(b) The sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ is a Bessel sequence for $L^2[0,1)^d$ if and only if $g_j \in L^\infty[0,1)^d$ (or equivalently $B_G < \infty$).

In this case, the optimal Bessel bound is $B_G/(detM)$.

(c) The sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ is a frame for $L^2[0,1)^d$ if and only if $0 < A_G \le B_G < \infty$.

In this case, the optimal frame bounds are $A_G/(detM)$ and $B_G/(detM)$.

(d) The sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ is a Riesz basis for $L^2[0,1)^d$ if and only if it is a frame and $s = det\,M$.

Proof : Properties (a), (b) and (c) depend on the behavior of the ℓ^2-norm of the sequence of inner products $\{\langle F\,, \overline{g_j(x)}e^{-2\pi i\alpha^\intercal M^\intercal \cdot}\rangle_{L^2[0,1)^d}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\ldots,s}$ for any function $F \in L^2[0,1)^d$. First, we obtain a representation for this ℓ^2-normby using that the sequence $\{e^{2\pi i\alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d}$ is an orthogonal basis for $L^2(M^{-\intercal}[0,1)^d$. For any $F \in L^2[0,1)^d$ we have

$$\langle F(x), \overline{g_j(x)} e^{-2\pi i \alpha^\intercal M^\intercal x} \rangle_{L^2[0,1)^d} = \int_{[0,1)^d} F(x) g_j(x) e^{2\pi i \alpha^\intercal M^\intercal x}\, dx$$
$$= \sum_{k=1}^{detM} \int_{Q_k} F(x) g_j(x) e^{2\pi i \alpha^\intercal M^\intercal x}\, dx$$
$$= \int_{M^{-\intercal}[0,1)^d} \sum_{k=1}^{detM} F(x + M^{-\intercal} i_k) g_j(x + M^{-\intercal} i_k) e^{2\pi i \alpha^\intercal M^\intercal x}\, dx, \quad (27)$$

where we have considered the $\mathbb{Z}^d$-periodic extension of F . Then,

$$\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \langle F(x), \overline{g_j(x)} e^{-2\pi i \alpha^\intercal M^\intercal x} \rangle_{L^2[0,1)^d} \right|^2$$
$$= \frac{1}{detM} \sum_{j=1}^{s} \left\| \sum_{k=1}^{detM} F(x + M^{-\intercal} i_k) g_j(x + M^{-\intercal} i_k) \right\|^2_{L^2(M^{-\intercal}[0,1)^d)} .$$

Denoting $F(x) = [F(x), F(x + M^{-\intercal} i_2), \ldots, F(x + M^{-\intercal} i_{detM})]^\intercal$ the equality above reads

$$\sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left| \langle F(x), \overline{g_j(x)} e^{-2\pi i \alpha^\intercal M^\intercal x} \rangle_{L^2[0,1)^d} \right|^2 = \frac{1}{detM} \| G(x) F(x) \|^2_{L^2_s(M^{-\intercal}[0,1)^d)}, \quad (28)$$

Where we have denoted $L^2_s(M^{-\intercal}[0,1)^d) = L^2(M^{-\intercal}[0,1)^d) \times \cdots \times L^2(M^{-\intercal}[0,1)^d)$ (s times) with the usual norm. On the other hand, using that the function g_j is Z^d -periodic, we obtain that the set $\{g_j(x + M^{-\intercal} i_k + M^{-\intercal} i_1), g_j(x + i_k + M^{-\intercal} i_2), \ldots, g_j(x + M^{-\intercal} i_k + M^{-\intercal} i_{detM})\}$ has the same elements as $\{g_j(x + M^{-\intercal} i_1), g_j(x + M^{-\intercal} i_2), \ldots, g_j(x + M^{-\intercal} i_{detM})\}$. Thus the matrix $G(x + M^{-\intercal} i_k)$ has the same columns of $G(x)$, possibly in a different order. Hence, $rank G(x) = detM$ a.e. in $[0,1)^d$ if and only if rank$G(x) = detM$ a.e. in $M^{-\intercal}[0,1)^d$. Moreover,

$$A_G = \operatorname*{ess\,inf}_{x \in M^{-\intercal}[0,1)^d} \lambda_{min} [G^*(x) G(x)] \,, B_G = \operatorname*{ess\,sup}_{x \in M^{-\intercal}[0,1)^d} \lambda_{max} [G^*(x) G(x)] . \quad (29)$$

To prove (a), assume that there exists a set $\Omega \subseteq M^{-\intercal}[0,1)^d$ with positive measure such that $rank\, G(x) < detM, x \in \Omega$. Then, there exists a measurable function $v(x), x \in \Omega$, such that $\boldsymbol{G(x) v(x) = 0}$ and $|v(x)| = 1$ in Ω. This function can be constructed as in [97]. Define $F \in L^2[0,1)^d$ such that $F(x) = v(x)$ if $x \in \Omega$, and $F(x) = 0$ if $x \in M^{-\intercal}[0,1)^d \setminus \Omega$. Hence, from (28) we obtain that the system is not complete.

Conversely, if the system is not complete, by using (28) we obtain a $F(x)$ different from 0 in a set with positive measure such that $G(x)F(x) = 0$. Thus $rank G(x) < detM$ on a set with positive measure. Parts (b) and (c) in Lemma (4.2.1) have been proved in [89] for the univariate case. By using (28) and (29), the proofs for the general case are completely analogous. To prove (d) we assume that $detM = s$ and that the sequence is a frame. We see that it is also a Riesz basis by proving that the analysis operator

$$\Lambda : L^2[0,1)^d \to \ell_S^2 , \Lambda(F) = \{\langle F(x) , \overline{g_j(x)} e^{-2\pi i \alpha^\intercal M^\intercal x} \rangle_{L^2[0,1)^d}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$$

is surjective (see [82]). To this end, notice that when $detM = s$ the matrix $G(x)$ is a square matrix and hence, the condition $A_G > 0$ implies that the inverse matrix $G^{-1}(x)$ exists and its entries are essentially bounded. Let $\{c_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ be an element of ℓ_S^2 . For $j = 1,2,\dots,s$ we define the function

$$\xi_j(x) = (detM) \sum_{\alpha\in\mathbb{Z}^d} c_{j,\alpha} e^{-2\pi i \alpha^\intercal M^\intercal x},$$

and let F be the function such that $F(x) = G^{-1}(x)[\xi_1(x),\dots,\xi_s(x)]^\intercal, x \in M^{-\intercal}[0,1)^d$. This function belongs to $L^2[0,1)^d$ because the entries of $G^{-1}(x)$ are essentially bounded. We have that $\boldsymbol{G}(x)F(x) = [\xi_1(x),\dots,\xi_s(x)]^\intercal$, and using (27) we obtain that

$$\langle F(x) , g_j(x) e^{-2\pi i \alpha^\intercal M^\intercal x} \rangle = \int_{M^{-\intercal}[0,1)^d} \sum_{k=1}^{detM} F(x + M^{-\intercal} i_k) g_j(x + M^{-\intercal} i_k) e^{2\pi i \alpha^\intercal M^\intercal x}\, dx$$

$$= \int_{M^{-\intercal}[0,1)^d} \xi_j(x) e^{2\pi i \alpha^\intercal M^\intercal x}\, dx = c_{j,\alpha} \quad ,$$

and consequently, $\Lambda(F) = \{c_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$. Conversely, assume that the sequence $\{\overline{\boldsymbol{g}_j(\boldsymbol{x})} \boldsymbol{e}^{-2\pi i \alpha^\intercal M^\intercal x}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a Riesz basis. Let $\{f_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ be its dual Riesz basis. Then, by using (27) we obtain

$$\int_{M^{-\intercal}[0,1)^d} \sum_{k=1}^{detM} f_{j',0}(x + M^{-\intercal} i_k) g_j(x + M^{-\intercal} i_k) e^{2\pi i \alpha^\intercal M^\intercal x}\, dx = \delta_{\alpha,0}\delta_{j,j'} .$$

Therefore, for $j, j' = 1,2,\dots,s$, we have

$$\sum_{k=1}^{detM} f_{j',0}(x + M^{-\intercal} i_k) g_j(x + M^{-\intercal} i_k) = (detM)\delta_{j,j'} \ a.e. in \ [0,1)^d.$$

Thus the matrix $G(x)$ has a right inverse; in particular, $s \le detM$. As a consequence of (a) we have $s \ge detM$ and, finally, $s = detM$. Next we discuss the meaning of Lemma (4.2.1), whenever the functions $g_j, j = 1,2,\dots,s$, are given by (23), in terms of the average sampling terminology introduced by Aldroubi et al. in [105]. Thus, following [105] , we say that:

(i) The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-determining filtering sampler for V_ϕ^2 if the only function $f \in V_\phi^2$ satisfying $\mathcal{L}_j f(M\alpha) = 0$ for all $j = 1,2,\dots,s$ and $\alpha \in \mathbb{Z}^d$ is the zero function.

(ii) The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_ϕ^2 if there exist two positive constants C_1 and C_1 such that

$$C_1\|f\|^2 \leq \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \left|\mathcal{L}_j f(M\alpha)\right|^2 \leq C_2\|f\|^2 \quad for \quad all \quad f \in V_\phi^2.$$

If $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_ϕ^2, then any function $f \in V_\phi^2$ can be recovered, in a stable way, from the sequence of generalized samples. Roughly speaking, the operator which maps $f \in V_\phi^2$ into the sequence of samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ has a bounded inverse. An M-determining filtering sampler for V_ϕ^2 can distinguish between two distinct functions in V_ϕ^2, but the recovery is not necessarily stable. Notice that from (25), parts (a) and (c) of Lemma (4.2.1) read as follows:

(i) The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-determining filtering sampler for V_ϕ^2 if and only if $rank G(x) = det M$ a.e. in $[0,1)^d$ (and hence, necessarily, $s \geq det M$).

(ii) The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_ϕ^2 if and only if $0 < A_G \leq B_G < \infty$. These properties can be expressed in terms of the function $det[G^*(x)G(x)]$. Indeed, as $rank G(x) = det M$ if and only if $det[G^*(x)G(x)] \neq 0$, we have that the set of systems $\{\boldsymbol{\mathcal{L}_1}, \boldsymbol{\mathcal{L}_2}, \dots, \boldsymbol{\mathcal{L}_s}\}$ is an M-determining filtering sampler for $\boldsymbol{V_\phi^2}$ if and only if $det[G^*(x)G(x)] \neq 0$ *a.e. in* $[0,1)^d$ Provided that the function $g_J \in L^\infty[0,1)^d$ for each $j = 1,2,\dots,s$ (or equivalently $B_G < \infty$), since $det[G^*(x)G(x)]$ is the product of the eigenvalues, we have that

$$(\lambda_{min}[G^*(x)G(x)])^{det M} \leq det G^*(x)\boldsymbol{G}(x) \leq$$
$$(\lambda_{max}[G^*(x)G(x)])^{(det M)-1}\lambda_{min}[G^*(x)G(x)] \leq B_G^{(det M)-1}\,\lambda_{min}[G^*(x)G(x)],$$

and therefore, the set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_ϕ^2 if and only if

$$\operatorname*{ess\,inf}_{x \in [0,1)^d} det[G^*(x)G(x)] > 0.$$

If the functions $g_j, j = 1,2,\dots,s$, are continuous on $\mathbb{R}^d$, the above condition reads: $det[G^*(x)G(x)] \neq 0$ for all $x \in [0,1)^d$.Hence, the set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_ϕ^2 if and only if

$$rank G(x) = det M \quad for\ all \quad x \in [0,1)^d \ . \tag{30}$$

In the above section we have proved that, provided that the functions $g_j \in L^\infty[0,1)^d$ for each $j = 1,2,\dots,s$, the set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M −stable filtering sampler for V_ϕ^2 if and only if $A_G > 0$. In this section we obtain the corresponding stable sampling formulas leading to the recovery of any function $f \in V_\phi^2$ from the sequence of its generalized samples $\{\mathcal{L}_j f\ (M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$. The sampling formula will be unique in the case $s = detM$. These, explicitly given, sampling formulas consist of the major difference with the analogous results included in [105].

Now we prove that the expression (22) allows us to obtain F from the generalized samples $\{\mathcal{L}_j f\ (M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$. Applying the isomorphism $\mathcal{T}_\phi$ we get generalized regular sampling formulas in V_ϕ^2 . Assume that $g_j \in L^\infty[0,1)^d$ for $j = 1,2,\dots,s$; then, $F(x)g_j(x) \in L^2[0,1)^d$. Hence using (24) and (22), for $j = 1,2,\dots,s$ we obtain that

$$(detM)\sum_{\alpha\in\mathbb{Z}^d}\left(\mathcal{L}_j f\right)(M\alpha)e^{-2\pi i\alpha^\top M^\top x} = \sum_{\alpha\in\mathbb{Z}^d}\left(\mathcal{L}_j f\right)(\alpha)e^{-2\pi i\alpha^\top x}\sum_{k\in\mathcal{N}(M^\top)} e^{-2\pi i\alpha^\top M^{-\top} k}$$
$$= \sum_{k\in\mathcal{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\left(\mathcal{L}_j f\right)(\alpha)e^{-2\pi i\alpha^\top(x+M^{-\top}k)}$$
$$= \sum_{k\in\mathcal{N}(M^\top)} F(x+M^{-\top}k)g_j(x+M^{-\top}k).$$

This can be written in matrix form as

$$G(x)\mathbb{F}(x) = (detM)\left[\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_1 f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x},\dots,\sum_{\alpha\in\mathbb{Z}^d}\mathcal{L}_s f(M\alpha)e^{-2\pi i\alpha^\top M^\top x}\right]^\top$$

in $L^2[0,1)^d$, where the matrix function $G(x)$ is given in (26) and $\mathbb{F}(\boldsymbol{x})$ denotes the vector $\mathbb{F}(\boldsymbol{x}) = [F(x), F(x+M^{-\top}i_2),..,F(x+M^{-\top}i_{detM})]^\top$. In order to recover the function F, let $[d_1(x),\dots,d_s(x)]$ be a vector with entries in $L^\infty[0,1)^d$ such that $[d_1(x),\dots,d_s(x)]G(x) = [1,0,\dots,0]\ a.e.\ in\ [0,1)^d$. Later, we will show that a necessary and sufficient condition for the existence of such a vector is that $A_G > 0$. As a consequence, we get

$$F(x) = (detM)\sum_{J=1}^{S}\sum_{\alpha\in\mathbb{Z}^d}\left(\mathcal{L}_j f\right)(M\alpha)e^{-2\pi i\alpha^\top M^\top x}\ in\ [0,1)^d. \qquad (31)$$

Finally, the isomorphism $\mathcal{T}_\phi$ gives

$$f(t) = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\boldsymbol{\mathcal{L}_j f})(M\alpha)(\mathcal{T}_\phi d_j)(t-M\alpha), t\in\mathbb{R}^d,$$

where we have used the shifting property (20) and that the shift-invariant space V_ϕ^2 is a RKHS. In addition, much more can be said about the above sampling expansion. In fact, the following result holds:

Theorem $(4.2.2)[77]$: Assume that the functions g_j given in (23) belong to $L^\infty[0,1)^d$ for each $j = 1,2,\dots,s$. Let $\boldsymbol{G}(x)$ be the associated matrix defined in $[0,1)^d$ as in (26). The following statements are equivalents:

(a) $A_G > 0$;

(b) The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M −stable filtering sampler for V_ϕ^2;

(c) There exists a vector $[d_1(x), \dots, d_s(x)]$with entries $d_j \in L^\infty[0,1)^d$ satisfying

$$[d_1(x), \dots, d_s(x)]G(x) = [1, 0, \dots, 0] a.e. in [0,1)^d , \tag{32}$$

(d) There exists a frame for V_ϕ^2 having the form $\{S_j(\cdot - M\alpha)\}_{\alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ such that for any $f \in V_\phi^2$

$$f = (detM) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(\cdot - M\alpha) \quad in \quad L^2(\mathbb{R}^d) . \tag{33}$$

In case the equivalent conditions are satisfiedwe have that the reconstruction functions $S_j, j = 1, 2, \dots, s$, in the sampling formula (33) are necessarily given through a vector $[d_1(x), \dots, d_s(x)]$ satisfying (32), by

$$S_j(t) = \sum_{\alpha \in \mathbb{Z}^d} \hat{d}_j(\alpha) \phi(t - \alpha) \quad , t \in \mathbb{R}^d , \tag{34}$$

Where $\hat{d}_j(\alpha), \alpha \in \mathbb{Z}^d$, are the Fourier coefficients of d_j , i.e., $d_j(x) = \sum_{\alpha \in \mathbb{Z}^d} \hat{d}_j(\alpha) e^{-2\pi i \alpha^\intercal x}$. The sampling series in (33) also converges absolutely and uniformly on $\mathbb{R}^d$. If $s = detM$ then the sequence $\{S_j(\cdot - M\alpha)\}_{\alpha \in \alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ is a Riesz basis for V_ϕ^2 and the sampling functions $S_j , j = 1, 2, \dots, s$, satisfy the interpolation property $(\mathcal{L}_{j'} S_j)(M\alpha) = \delta_{j,j'} \delta_{\alpha,0}$, where $j, j' = 1, 2, \dots, s$ and $\alpha \in \mathbb{Z}^d$.

Proof : Part (c) in Lemma (4.2.1) proves that conditions (a) and (b) are equivalent. If $A_G > 0$ then $\text{ess inf}_{x \in [0,1)^d} det[G^*(x)G(x)] > 0$ and , consequently, there exists the pseudo-inverse matrix $G^\dagger(x) = [G^*(x)G(x)]^{-1}G^*(x)$; its entries are essentially bounded and its first row satisfies (32); therefore (a) implies (c). If $[d_1(x), \dots, d_s(x)]$ satisfies (32) with $d_j \in L^\infty[0,1)^d$, we have proved earlier that formula (33) holds in $L^2(\mathbb{R}^d)$ where S_j is equal to $\mathcal{T}_\phi d_j$ or, equivalently, is given by (34). Since we have assumed that $d_j \in L^\infty[0,1)^d$ for each $j = 1, 2, \dots, s$, Lemma (4.2.1) (b) proves that $\{\overline{g_j(x)} e^{-2\pi i \alpha^\intercal M^\intercal x}\}_{\alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ is a Bessel sequence in $L^2[0,1)^d$. The same argument proves that $\{(detM) d_j(x) e^{-2\pi i \alpha^\intercal M^\intercal x}\}_{\alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ is also a Bessel sequence in $L^2[0,1)^d$. These two Bessel sequences satisfy (see (25) and (31)):

$$F(x) = (detM) \sum_{J=1}^{S} \sum_{\alpha \in \mathbb{Z}^d} \langle F, \overline{g_J(x)} e^{-2\pi i \alpha^\intercal M^\intercal x} \rangle_{L^2[0,1)^d} d_j(x) e^{-2\pi i \alpha^\intercal M^\intercal x}, F \in L^2[0,1)^d.$$

Hence, they form a pair of dual frames for $L^2[0,1)^d$ (see [82]). Since $S_j(t - M\alpha) = \mathcal{T}_\phi[d_j(\cdot) e^{-2\pi i \alpha^\intercal M^\intercal \cdot}](t)$ and $\mathcal{T}_\phi$ is an isomorphism, the sequence $\{S_j(t - M\alpha)\}_{\alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ is a frame for V_ϕ^2 ; hence (c) implies (d). Notice that since we have assumed that $\{\{\overline{g_j(x)} e^{-2\pi i \alpha^\intercal M^\intercal x}\}_{\alpha \in \mathbb{Z}^d, j=1,2,\dots,s}$ is a Bessel sequence with bound

$B_G/(detM)$ and $(\mathcal{L}_j f\,)(M\alpha) = \langle F, \overline{g_j(x)} e^{-2\pi i\alpha^\intercal M^\intercal x}\rangle_{L^2[0,1)^d}$, we have

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\mathcal{L}_j f(M\alpha)\right|^2 \le \frac{B_G}{detM}\|f\|^2 \le \frac{B_G\left\|\mathcal{T}_\phi^{-1}\right\|^2}{detM}\|f\|^2 \ , f \in V_\phi^2.$$

If $\{S_j(t - M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a frame for V_ϕ^2 , then formula (33) gives a stable way to recover $f \in V_\phi^2$ from its generalized samples. Indeed,

$$\|f\|^2 = (detM)^2\left\|\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left(\mathcal{L}_j f\right)(M\alpha)S_j(\cdot - M\alpha)\right\|^2$$

$$\le (detM)^2 C\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\mathcal{L}_j f(M\alpha)\right|^2 \ ,$$

where C is a Bessel bound for $\{S_j(t - M\alpha)\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$. Hence the set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M −stable filtering sampler for V_ϕ^2 . Therefore (d) implies (b). The pointwise convergence in the sampling series is absolute due to the unconditional convergence of a frame expansion; it is uniform on $\mathbb{R}^d$ as a consequence of (19).

If $s = detM$ then, according to Lemma (4.2.1) (d), the frame $\left\{S_j\left(t - M\alpha\right)\right\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s} = \left\{\mathcal{T}_\phi d_j e^{-2\pi i\alpha^\intercal M^\intercal \cdot}\right\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a Riesz basis for V_ϕ^2 . Applying formula (33) for $f = S_{j'}$ and having in mind the uniqueness of the coefficients in a Riesz basis, we get the interpolatory property $\left(\mathcal{L}_{j'} S_j\right)(M\alpha) = \delta_{j,j'}\delta_{\alpha,0}$. The equivalence between conditions (a), (b) and (d) in Theorem (4.2.2) was established in [105] for average sampling, at the lattice $\mathbb{Z}^d$, in finitely-generated shift-invariant spaces by using another techniques. Notice that our generalized sampling on the more general sampling lattice $M\mathbb{Z}^d$ can be seen as a problem of generalized sampling in a finitely-generated shift-invariant space on the sampling lattice $\mathbb{Z}^d$. Indeed, the generalized sampling of the functions $f = \sum_{\alpha\in\mathbb{Z}^d} a_\alpha \phi(\cdot - \alpha)$ at $M\mathbb{Z}^d$ can be thought as a bounded map from $\ell(\mathbb{Z}^d)$ to $(\ell^2(M\mathbb{Z}^d))^s$:

$$\{a_\alpha\}_{\alpha\in\mathbb{Z}^d} \longmapsto \left\{\sum_{\alpha\in\mathbb{Z}^d} a_\alpha \mathcal{L}_j \phi(M\beta - \alpha)\right\}_{1\le j\le s, \beta\in\mathbb{Z}^d} \ ,$$

or also as

$$\{a_\alpha^t\}_{1\le t\le detM, \alpha\in\mathbb{Z}^d} \longmapsto \left\{\sum_{t=1}^{detM}\sum_{\alpha\in\mathbb{Z}^d} a_\alpha^t \mathcal{J}_j \phi_t(\beta - \alpha)\right\}_{1\le j\le s, \beta\in\mathbb{Z}^d} \ ,$$

where $\mathcal{J}_j f(u) = [\mathcal{L}_j\{f(u + M^{-1}\cdot)\}](0), \phi_t(\cdot) = \phi(M\cdot - \mathcal{J}_j)$, and $\{\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_{detM}\} = \mathbb{Z}^d \cap (M[0,1)^d)$, which can be seen as generalized sampling at $\mathbb{Z}^d$ of the functions f having the form: $f = \sum_{t=1}^{detM}\sum_{\alpha\in\mathbb{Z}^d} a_\alpha^t \phi_t(\cdot - \alpha)$. The sampling formulas (33), explicitly given by using (c), are the novelty of the result proved here. The solutions of (32) with entries in $L^\infty[0,1)^d$ are exactly the first row of the $(detM)\times\ s$ matrices of the form

$$D(x)=\ G^\dagger(x)+U(x)[I_s-G(x)G^\dagger(x)]\quad , \tag{35}$$

where $G^\dagger(x)$ is the pseudo-inverse matrix of $G(x), G^\dagger(x)\ =\ [G^*(x)G(x)]^{-1}G^*(x)$, and $U(x)$ is an arbitrary $(detM)\ \times\ s$ matrix with entries in $L^\infty[0,1)^d$. Indeed, if the vector $[d_1(x),\dots,d_s(x)]$ satisfies (32), it can be easily checked that the $(detM)\times s$ matrix $D(x)=[d_j(x+M^\top i_k)]_{\substack{k=1,2,\dots,detM\\ j=1,2,\dots,s}}$

is a left inverse of the matrix $G(x)$, and it can be expressed in the form (35) by taking $U(x)\ =\ D(x)$. Conversely, any matrix of the form (35) is a left inverse of **G***(x)* and its first row satisfies (32). Finally, notice that if the functions $g_j\ , j\ =\ 1,2,\dots,s$, are continuous on $\mathbb{R}^d$, the condition (a) in Theorem (4.2.2) reads: $rank\boldsymbol{G}(x)\ =\ detM$ for all $x\ \in\ \mathbb{R}^d$ (see (30)).

Given an error sequence $\varepsilon\ =\ \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ in $\mathbb{R}^d$, the aim in this section is to study when it is possible to recover any function $f\ \in\ V_\phi^2$ from the sequence of perturbed samples $\{(\mathcal{L}_jf\ )(M\alpha\ +\ \varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$. Having in mind expression (21) for the systems $\mathcal{L}_j\ , j\ =\ 1,2,\dots,s$, for $f\ =\ \mathcal{T}_\phi F\ \in\ V_\phi^2$ we have

$$(\mathcal{L}_jf)(M\alpha+\varepsilon_{j,\alpha})=\langle F\ ,\left(\overline{Z\mathcal{L}_j}\phi\right)\left(\varepsilon_{j,\alpha},\cdot\right)e^{-2\pi i\alpha^\top M^\top\cdot}\rangle_{L^2[0,1)^d}\ ,\alpha\in\mathbb{Z}^d, \tag{36}$$

where we have used that $\left(Z\mathcal{L}_j\phi\right)\left(M\beta+\ \varepsilon_{j,\beta}\ ,x\right)=\ \left(Z\mathcal{L}_j\phi\right)\left(\varepsilon_{j,\beta}\ ,x\right)e^{2\pi i\beta^\top M^\top x}$ for any $\beta\ \in\ \mathbb{Z}^d$. (36) leads us to study the frame property of the perturbed sequence $\{(\overline{Z\mathcal{L}_j}\phi)(\varepsilon_{j,\alpha}\ ,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}\quad in\quad L^2[0,1)^d$. On the other hand, we know that, whenever $0\ <\ A_G\ \ \le\ \ B_G\ \ <\infty$, the sequence $\{(\overline{Z\mathcal{L}_j}\phi)(0,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a frame for $L^2[0,1)^d$ with optimal frame bounds $A_G/(detM)$ and $B_G/(detM)$. In the case of $s\ =\ detM$, the above sequence is a Riesz basis for $L^2[0,1)^d$.

One possibility is to use frame perturbation theory in order to find the suitable error sequences for which the sequence $\{(\overline{Z\mathcal{L}_j}\phi)(\varepsilon_{j,\alpha}\ ,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a frame for $L^2[0,1)^d$. The following result on frame perturbation, which proof can be found in [82] will be used later:

Lemma $(4.2.3)[77]$: Let $\{\ f_n\}_{n=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{\ g_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R\ <\ A$ such that $\sum_{n=1}^\infty|\langle\ f_n-g_n, f\rangle|^2\ \le R\|f\|^2$ for each $f\ \in\mathcal{H}$, then $\{\ g_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ with bounds $A\left(1-\sqrt{R/A}\right)^2$ *and* $B\left(1-\sqrt{R/B}\right)^2$.I f $\{\ f_n\}_{n=1}^\infty$ is a Riesz basis, then $\{\ g_n\}_{n=1}^\infty$ is a Riesz basis. Given an error sequence $\varepsilon\ =\ \{\varepsilon_{j,\alpha}\ \}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}\subset\ \mathbb{R}^d$ we define on $\ell^2(\mathbb{Z}^d)$ the operator $D_\varepsilon\ =\ \left[D_{\varepsilon,1}\ ,\dots,D_{\varepsilon,s}\ \right]$, where

$$D_{\varepsilon,j}c = \left\{ \sum_{\beta\in\mathbb{Z}^d} \left[\mathcal{L}_j\phi(M\alpha - \beta + \varepsilon_{j,\alpha}) - \mathcal{L}_j\phi(M\alpha - \beta) \right] c_\beta \right\}_{\alpha\in\mathbb{Z}^d}$$

for each $c = \{c_\beta\}_{\beta\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. The operator norm is defined as usual

$$\|D_\varepsilon\| = \sup_{c\in\ell^2(\mathbb{Z}^d)\setminus\{0\}} \frac{\|D_\varepsilon c\|_{\ell_s^2(\mathbb{Z}^d)}}{\|c\|_{\ell^2(\mathbb{Z}^d)}} \text{ , where } \|D_\varepsilon c\|^2_{\ell_s^2(\mathbb{Z}^d)} = \sum_{j=1}^{s} \left\|D_{\varepsilon,j}c\right\|^2_{\ell^2(\mathbb{Z}^d)}$$

for each $c \in \ell^2(\mathbb{Z}^d)$.

Theorem $(4.2.4)[77]$: Assume that $g_j \in L^\infty[0,1)^d$ for $j = 1,2,\dots,s$ with $A_G > 0$. If the error sequence $\varepsilon = \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ satisfies the inequality $\|D_\varepsilon\|^2 < A_G/(detM)$, then there exists a frame $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ for V_ϕ^2 such that, for any $f \in V_\phi^2$

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) S^\varepsilon_{j,\alpha}(t), t \in \mathbb{R}^d, \tag{37}$$

where the convergence of the series is in the $L^2(\mathbb{R}^d)$ −sense, absolute and uniform on $\mathbb{R}^d$. Moreover, when $s = detM$ the sequence $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ a Riesz basis for V_ϕ^2 , and the interpolation property $(\mathcal{L}_s S^\varepsilon_{j,\alpha})(M\beta + \varepsilon_{j,\beta}) = \delta_{j,l}\delta_{\alpha,\beta}$ holds.

Proof : The sequence $\{(\overline{Z\mathcal{L}_j}\phi)(0,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a frame (a Riesz basis if $s = detM$) for $L^2[0,1)^d$ with frame (Riesz) bounds $A_G/(detM)$ and $B_G/(detM)$. For any $F(x) = \sum_{\gamma\in\mathbb{Z}^d} c_\gamma e^{-2\pi i\gamma^\top x}$ in $L^2[0,1)^d$ we have

$$\sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \langle (\overline{Z\mathcal{L}_j}\phi)(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot} - (\overline{Z\mathcal{L}_j}\phi)(0,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}, F(\cdot)\rangle_{L^2[0,1)^d} \right|^2$$

$$= \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \langle \sum_{\beta\in\mathbb{Z}^d} \left[\overline{\mathcal{L}_j\phi}(\beta + \varepsilon_{j,\alpha}) - \overline{\mathcal{L}_j\phi}(\beta) \right] e^{-2\pi i(M\alpha-\beta)^\top\cdot}, \sum_{\gamma\in\mathbb{Z}^d} c_\gamma e^{-2\pi i\gamma^\top\cdot} \rangle_{L^2[0,1)^d} \right|^2$$

$$= \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \sum_{\beta\in\mathbb{Z}^d} \left[\overline{\mathcal{L}_j\phi}(\beta + \varepsilon_{j,\alpha}) - \overline{\mathcal{L}_j\phi}(\beta) \right] \bar{c}_{M\alpha-\beta} \right|^2$$

$$= \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \sum_{\beta\in\mathbb{Z}^d} \left[\mathcal{L}_j\phi(M\alpha - \beta + \varepsilon_{j,\alpha}) - \mathcal{L}_j\phi(M\alpha - \beta) \right] c_\beta \right|^2$$

$$= \sum_{j=1}^{s} \left\| D_{\varepsilon,j}\{c_\gamma\}_{\gamma\in\mathbb{Z}^d} \right\|^2_{\ell^2(\mathbb{Z}^d)} \le \|D_\varepsilon\|^2 \left\| \{c_\gamma\}_{\gamma\in\mathbb{Z}^d} \right\|^2_{\ell^2(\mathbb{Z}^d)} = \|D_\varepsilon\|^2 \|F\|^2_{L^2[0,1)^d} .$$

By using Lemma (4.2.3) , the sequence $\{(\overline{Z\mathcal{L}_j}\phi)(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ is a frame for $L^2[0,1)^d$ (a Riesz basis if $s = detM$). Let $\{h^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d, j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F \in L^2[0,1)^d$ we have

$$F=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^{d}}\langle F(\cdot),(\overline{Z\mathcal{L}_j}\phi)(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^{\top}M^{\top}\cdot}\rangle_{L^2[0,1)^d}h_{j,\alpha}^{\varepsilon}$$
$$=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^{d}}(\mathcal{L}_j f)(M\alpha+\varepsilon_{j,\alpha})h_{j,\alpha}^{\varepsilon}\ \ in\ \ L^2[0,1)^d.$$

Applying the isomorphism $\mathcal{T}_\phi$, one gets (37) in $L^2(\mathbb{R}^d)$ where $S_{j,\alpha}^{\varepsilon}=\mathcal{T}_\phi h_{j,\alpha}^{\varepsilon}$. Since $\mathcal{T}_\phi$ is an isomorphism between $L^2[0,1)^d$ and V_ϕ^2, the sequence $\{S_{j,\alpha}^{\varepsilon}\}_{\alpha\in\mathbb{Z}^d,j=1,2,\dots,s}$ is a frame for V_ϕ^2 (a Riesz basis if $s=detM$). Point wise convergence in the sampling series is absolute due to the unconditional character of a frame. The uniform convergence on $\mathbb{R}^d$ is a consequence of the reproducing property (19) in V_ϕ^2 . The interpolatory property in the case $s=detM$ follows from the uniqueness of the coefficients with respect to a Riesz basis. Formula (37) in Theorem (4.2.4) is useless from a practical point of view, since the frame $\{S_{j,\alpha}^{\varepsilon}\}_{\alpha\in\mathbb{Z}^d,j=1,2,\dots,s}$, which depends on the error sequence $\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,j=1,2,\dots,s}$, is impossible to determine. As a consequence, in order to recover any function $f\in V_\phi^2$ from the samples $\{\mathcal{L}_j f(M\alpha+\varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d,j=1,2,\dots,s}$, we should use the frame algorithm (see [87]).

In order to approximate the sequence $\{a_\alpha\}_{\alpha\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$ defining $f\in V_\phi^2$, the frame algorithm can be implemented in the $\ell^2(\mathbb{Z}^d)$ setting as in [90].

Following the techniques in [90] (see also [86,103]), whenever the generator ϕ and the impulse responses of the systems $\mathcal{L}_j$, $j=1,2,\dots,s$, are compactly supported one could obtain a bound for $\|D_\varepsilon\|$ in terms of $\delta=\lim_{j,\alpha}\|\varepsilon_{j,\alpha}\|_\infty$.

Finally, it is worth to mention the recent related [83,99,110].

We denote by $W_r^2(\mathbb{R}^d)=\{f:\|D^\gamma f\|_2<\infty,|\gamma|\le r\}$ the usual Sobolev space, and by $|f|_{j,2}=\sum_{|\beta|=j}\|D^\beta f\|_2$, $0\le j\le r$, the corresponding semi norm of a function $f\in W_r^2(\mathbb{R}^d)$. We assume here that all the systems $\mathcal{L}_j$, $j=1,2,\dots,s$, are of type (a), i.e., $\mathcal{L}_j f=h_j*f$, belonging the impulse response h_j to the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$. Recall that a Lebesgue measurable function $h:\mathbb{R}^d\to C$ belongs to the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$ if

$$|h|_2=\left(\int_{[0,1)^d}\left(\sum_{\alpha\in\mathbb{Z}^d}|h(t-\alpha)|\right)^2dt\right)^{1/2}<\infty.$$

Notice that $\mathcal{L}^2(\mathbb{R}^d)\subset L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Moreover, $\|\{h*f(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_2\le|h|_2\|f\|_2$ (see [96]); thus the sequence of generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,j=1,2,\dots,s}$ belongs to $\ell^2(\mathbb{Z}^d)$ for any $f\in L^2(\mathbb{R}^d)$. Besides, we assume that the generator ϕ satisfies the Strang–Fix conditions of order r, i.e.,

$$\hat{\phi}(0)\neq 0\ ,\qquad D^\beta\hat{\phi}(\alpha)=0\ ,\qquad |\beta|<r\ ,\qquad \alpha\in\mathbb{Z}^d\setminus\{0\}.$$

Given a vector $d = [d_1, \ldots, d_s]$ with entries in $L^\infty[0,1)^d$ and satisfying (32), first we note that the operator $\Gamma_d : (L^2(\mathbb{R}^d)), \|\cdot\|_2) \to (V_\phi^2, \|\cdot\|_2)$ given by

$$(\Gamma_d f)(t) = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,d}(t - M\alpha), t \in \mathbb{R}^d,$$

is a well-defined bounded operator onto V_ϕ^2 . Besides, $\Gamma_d f = f$ for all $f \in V_\phi^2$. Under appropriate hypotheses we prove that the scaled operator $\Gamma_d^h = \sigma_h \Gamma_d \sigma_{1/h}$, where $\sigma_h f = f(\cdot/h)$ for $h > 0$, approximates, in the L^2 −norm sense, any function f in the Sobolev space $W_r^2(\mathbb{R}^d) as\ h \to 0^+$. Concretely we have:

Theorem $(4.2.5)[77]$: Assume that $\boldsymbol{ess\ sup}_{t\in\mathbb{R}^d}\sum_{\alpha\in\mathbb{Z}^d}|\boldsymbol{\phi}(\boldsymbol{t}+\boldsymbol{\alpha})|(\mathbf{1}+|\boldsymbol{t}+\boldsymbol{\alpha}|)^{\boldsymbol{r}} < \infty$ for some $r \in \mathbb{N}$. Let d be a vector with entries in $L^\infty[0,1)^d$ and satisfying (32). If the generator ϕ satisfies the Strang–Fix conditions of order r, then, for each $f \in W_r^2(\mathbb{R}^d)$ and $h > 0$, the L^2 −approximation error satisfies $\|f - \Gamma_d^h f\|_2 \le K|f|_{r,2}h^2$, where the constant K is independent of f and h.

Proof : Using that $\Gamma_d^h\xi = \xi$ for each $\xi \in \sigma_h V_\phi^2$ then, for each $f \in L^2(\mathbb{R}^d)$ and $\in \sigma_h V_\phi^2$, Lebesgue's Lemma [84] gives

$$\|f - \Gamma_d^h f\|_2 \le (1 + \|\Gamma_d\|)\min_{\xi \in \sigma_h V_\phi^2}\|f - \xi\|_2 ,$$

where we have used that $\|\Gamma_d^h\| = \|\Gamma_d\|$. Now, for each $f \in W_r^2(\mathbb{R}^d)$ and $h > 0$ there exists a function $\xi_h \in \sigma_h V_\phi^2$ such that $\|\xi_h - f\|_2 \le \widetilde{K}|f|_{r,2}h^2$, where the constant $\widetilde{K}$ is independent of f and h (see [98]), from which we obtain the desired result. Notice that the approximation property given in Theorem (4.2.5)is similar to those given by integral operators in [98].

For the efficiency and stability of the reconstruction process given in Theorem (4.2.2), it is very desirable for the reconstruction functions $S_j, j = 1, 2, \ldots, s$ to be well localized; see [88,93,101] and the references therein. In this section we study two particular cases, reconstruction functions with exponential decay and reconstruction functions with compact support, by using directly formulas (34).Thus we prove that whenever the generator ϕ and the functions $\boldsymbol{\mathcal{L}_j\phi}, \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{s}$, decay exponentially fast, there are many sampling formulas like (33) involving reconstruction functions S_j with exponential decay, i.e., there exist constants $C > 0$ and $q \in (0, 1)$ such that $|S_j(t)| \le Cq^{|t|}, t \in \mathbb{R}^d$.First we introduce some complex notation, more convenient for this study. We denote $\boldsymbol{z}^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_d^{\alpha_d}$ for $z = [z_1, \ldots, z_d] \in \mathbb{C}^d, \alpha = [\alpha_1, \ldots, \alpha_d] \in \mathbb{Z}^{\boldsymbol{d}}$, and the d −torus by $T^d = \{z \in \mathbb{C}^d : |z_1| = |z_2| = \cdots = |z_d| = 1\}$. We define

$$g_j(z) = \mu \in Zd\mathcal{L}_j\phi(\mu)z^{-\mu} , G(z) = \left[g_j\left(z_1 e^{2\pi i m_1^\intercal ik}, \ldots, z_d e^{2\pi i m_d^\intercal ik}\right)\right]_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,detM}},$$

where $m_1,\dots,m_d$ denote the columns of the matrix M^{-1}. Note that for the vector $z = [e^{2\pi i x_1},\dots,e^{2\pi i x_d}]$ we have $G(x) = G(z)$. Provided that the functions g_j are continuous on $\mathbb{R}^d$, we have the following result: There exists a vector $d = [d_1,\dots,d_s]$ with entries essentially bounded in T^d and satisfying

$$d(z)G(z) = [1,0,\dots,0] \quad \text{for all} \quad z \in T^d \tag{38}$$

if and only if

$$\text{rank}G(z) = detM \qquad \text{for all} \qquad z \in T^d. \tag{39}$$

For $j = 1,2,\dots,s$, the corresponding reconstruction function $S_{j,d}$ in the sampling formula (33) is

$$S_{j,d}(t) = \sum_{\alpha\in\mathbb{Z}^d} \hat{d}_j(\alpha)\phi(t-\alpha)\ , \tag{40}$$

where $\hat{d}_j(\alpha), \alpha \in \mathbb{Z}^d$, are the Laurent coefficients of the functions d_j , $i.e.$, $d_j(z) = \sum_{\alpha\in\mathbb{Z}^d} \hat{d}_j(\alpha)z^{-\alpha}$. Let $\mathcal{H}$ denote the algebra of all holomorphic functions in a neighborhood of the d –torus T^d. Note that the elements in $\mathcal{H}$ are characterized as admitting a Laurent series where the sequence of coefficients decays exponentially fast [96].The following theorem shows that, whenever the generator ϕ and the functions $\mathcal{L}_j\phi, j = 1,2,\dots,s$ have exponential decay, if the vector d has entries in $\mathcal{H}$ then the reconstruction function S_j ,d has also exponential decay. It also proves that condition (39) is also sufficient for the existence of a vector d with entries in H and satisfying (38). Its proof uses the standard technique for proving extensions of Wiener $1/f$ Lemma in group algebras.

Theorem $(4.2.6)$[77] : Assume that the generator ϕ and the functions $\mathcal{L}_j\phi, j = 1,2,\dots,s$, have exponential decay. Then, there exists a vector $d = [d_1,\dots,d_s]$ with entries in $\mathcal{H}$ and satisfying $d(z)G(z) = [1,0,\dots,0]$ for all $z \in T^d$ if and only if $rankG(z) = detM$ for all $z \in T^d$. In this case, all of such vectors d are given as the first row of a $(detM)\times s$ matrix $D(z)$ of the form

$$D(z) = G^{\dagger}(z) + U(z)[I_s - G(z)G^{\dagger}(z)]\ , \tag{41}$$

where $U(z)$ is any $(detM)\times s$ matrixwith entries in Hand $G^{\dagger}(z) = [G^*(z)G(z)]^{-1}G^*(z)$. The corresponding reconstruction functions $S_{j,d}$, $j = 1,2,\dots,s$, given by (98) have exponential decay.

Proof : The hypotheses say that $\boldsymbol{g_j \in \mathcal{H}, j = 1,2,\dots,s}$; thus $det[G^*(z)G(z)] \in \mathcal{H}$. Assuming that $rankG(z) = detM$ for all $z \in T^d$ we have that $det[G^*(z)G(z)] \neq 0$ for all $z \in T^d$ and then, the matrix $[G^*(z)G(z)]^{-1}$ has entries in $\mathcal{H}$. As a consequence, the entries of $G^{\dagger}(z) = [G^*(z)G(z)]^{-1}G^*(z)$ belong to $\mathcal{H}$. Now it is easy to check, as we did in this Section, that all the vectors d with entries in $\mathcal{H}$ and satisfying (38) are given as the first row of matrices $D(z)$ satisfying (41), where the entries of $U(z)$ belong to $\mathcal{H}$. Since $d_j \in \mathcal{H}, j = 1,2,\dots,s$, their Laurent coefficients $\hat{d}_j(\alpha)$ have exponential decay, i.e., there exist

$C > 0$ and $q \in (0,1)$ such that $|\hat{d}_j(\alpha)| \le Cq^{|\alpha|}, \alpha \in \mathbb{Z}^d, j = 1,2,\dots,s$. Without loss of generality, we can also assume that $|\phi(t)| \le Cq^{|\alpha|}$, for all $t \in \mathbb{R}^d$; then the reconstruction functions $S_{j,d}(t) = \sum_{\alpha\in\mathbb{Z}^d} \hat{d}_j(\alpha)\phi(t-\alpha), j = 1,2,\dots,s$, satisfy

$$|S_{j,d}(t)| \le C \sum_{\alpha\in\mathbb{Z}^d} q^{|\alpha|}\phi(t-\alpha) \le C^2 \left(\sum_{\alpha\in\mathbb{Z}^d} q^{2|\alpha|}\right) q^{|\alpha|}, \qquad t \in \mathbb{R}^d.$$

Notice that, in particular, the solution obtained from the pseudo-inverse matrix$G^{\dagger}(z)$, which is unique in the case $s = detM$, gives reconstruction functions $S_{j,d}$with exponential decay.

Theorem $(4.2.7)$[77]: Let $G(z)$ be an $s \times m$ matrix whose entries are Laurent polynomials. Then, there exists an $m \times s$ matrix $D(z)$ whose entries are also Laurent polynomials satisfying $D(z)G(z) = I_m$ if and only if $rankG(z) = m$ for all $z \in (\mathbb{C}\backslash\{0\})^d$. From this theorem, we derive the following corollary:

Corollary $(4.2.8)$[77]: Assume that the generator ϕ and the functions $\mathcal{L}_j\phi, j = 1,2,\dots,s$, have compact support. Then, there exists a vector $d = [d_1,\dots,d_s]$ whose entries are Laurent polynomials and satisfying $d(z)G(z) = [1,0,\dots,0]$ if and only if $rankG(z) = detM$ for all $z \in (\mathbb{C}\backslash\{0\})^d$. The reconstruction functions $S_{j,d}, j = 1,2,\dots,s$, obtained from such vectors d by (40) have compact support. A vector $d(z)$ satisfying $d(z)G(z) = [1,0,\dots,0]$ whose entries are Laurent polynomials can be obtained by solving a linear system whose unknowns are precisely the coefficients of $d_j(z), j = 1,2,\dots,s$. From one of these vectors, say $\tilde{d} = [\tilde{d}_1,\dots,\tilde{d}_s]$, we can get all of them. Indeed, it is easy to check that they are given by the first row of the $(detM) \times s$ matrices of the form

$$D(z) = \widetilde{D}(z) + U(z)[I_s - G(z)\widetilde{D}(z)], \tag{42}$$

where $\widetilde{D}(z) = [\tilde{d}_j([z_1 e^{2\pi i m_1^{\intercal} ik},\dots,z_d e^{2\pi i m_d^{\intercal} ik}])]_{\substack{k=1,2,\dots,detM \\ j=1,2,\dots,s}}$ and $U(z)$ is any $(detM) \times s$ matrix with Laurent polynomial entries. The interested reader can find in [100,111,112] methods to check if the condition in the theorem holds, and also another method to find a particular solution $\widetilde{D}(z)$of (42). Both involve the use of Grobner bases.

Finally, notice that having reconstruction functions with compact support implies low computational complexity and truncation errors are avoided. A related topic is the local reconstruction in shift-invariant spaces which invokes only finite samples to reconstruct a function on a bounded interval: See [102,104]. The aim in section is to validate the sampling formulas

$$f(t) = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,d}(t - M\alpha), t \in \mathbb{R}^d,$$

obtained in this Section for the shift-invariant space V_ϕ^2 , in a larger space. To this end, assume that the generator $\boldsymbol{\phi} \in \boldsymbol{\mathcal{L}}^\infty(\mathbb{R}^{\boldsymbol{d}}) \cap \boldsymbol{C}(\mathbb{R}^{\boldsymbol{d}})$. Recall that a Lebesgue measurable function $\boldsymbol{\varphi}: \mathbb{R}^{\boldsymbol{d}} \to \boldsymbol{C}$ belongs to the Banach space $\mathcal{L}^\infty(\mathbb{R}^d)$ if $|\varphi|_\infty = \operatorname{ess\,sup}_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |\varphi(t-\alpha)| < \infty$. For $1 \le p \le \infty$ we have that $\mathcal{L}^\infty(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$, in particular, $\mathcal{L}^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Observe that if there are constants $C > 0$ and $\delta > 0$ such that $|\varphi(t)| \le \frac{C}{1+|t|^{d+\delta}}$ $, t \in \mathbb{R}^d$ $, then\ \varphi \in \mathcal{L}^\infty(\mathbb{R}^d)$ .Let V_ϕ^2 be the L^∞-closure of the linear span of the integer translates of $\phi, i.e.$,
$V_\phi^\infty = \overline{span}_{L^\infty}\{\phi(t-\alpha): \alpha \in \mathbb{Z}^{\boldsymbol{d}}\}$. As the integer translates of ϕ are ℓ^2 −stable (they form a Riesz sequence in $L^2(\mathbb{R}^d)$), then this space can be expressed as $V_\phi^\infty = \{\phi *' a : a \in c_0(\mathbb{Z}^{\boldsymbol{d}})\}$, where $\phi *' a$ denotes the semi-discrete convolution $\sum_{\alpha\in\mathbb{Z}^d} a(\alpha)\varphi(\cdot - \alpha)$ and $c_0(\mathbb{Z}^{\boldsymbol{d}})$ denotes the space of sequences on $\mathbb{Z}^{\boldsymbol{d}}$ vanishing at ∞ (see [98]). As a consequence, $V_\phi^\infty$is a set of continuous functions on $\mathbb{R}^d$ and the set inclusion $V_\phi^2 \subset V_\phi^\infty$ holds . Let A be the set of functions of the form $f(x) = \sum_{\alpha\in\mathbb{Z}^d} a(\alpha)e^{-2\pi i\alpha^\top x}$ with $a \in \ell^1(\mathbb{Z}^{\boldsymbol{d}})$. The space A, normed by $\|f\|_A = \|a\|_1$ and with point wise multiplication is a commutative Banach algebra. If $f \in A$ and $f(x) \neq 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in A by Wiener's Lemma. Consider s linear time-invariant systems $\mathcal{L}_j$ $, j = 1,2,\ldots,s$. In addition, assume that $D^\beta\phi \in \mathcal{L}^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d), |\beta| \le m,$ where m is the largest order among the partial derivatives appearing in the systems of type (b) ($m = 0$ if no partial derivatives appear).Thus we have that $\{\mathcal{L}_j\phi(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^{\boldsymbol{d}})$ for the systems of the type (b).
This is also true for the systems of type (a) since $\|\{\phi * h(\alpha)\}\|_1 \le \|h\|_1|\phi|_\infty$ (see[96]).
As aconsequence, the Fourier transforms of these sequences, which are precisely the functions g_j $, j = 1,2,\ldots,s,$ defined in (23), belong to the algebra A. The next result describes when (90)has a solution d with entries in the algebra A:

Lemma $(\mathbf{4.2.9})[\mathbf{77}]$ **:**There exists a vector $d = [d_1,\ldots,d_s]$ with entries d_j in the algebra A , $j = 1,2,\ldots,s,$ and satisfying

$$d(x)G(x) = [1,0,\ldots,0], x \in [0,1)^d \tag{43}$$

if and only if $rankG(x) = detM$ for all $x \in \mathbb{R}^d$.

Proof : The proof is the same as the one in [166, Lemma 1] although for a slightly different matrix G. For any vector d satisfying the above lemma, Theorem (4.2.2) gives the corresponding sampling formula in V_ϕ^2 :

$$f(t) = (detM)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,d}(t - M\alpha), t \in \mathbb{R}^d, \tag{44}$$

where $S_{j,d} = \mathcal{T}_\phi d_j$ $, j = 1,2,\ldots,s.$ In particular, formula (44) holds for the space $span\{\phi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d}$. The reconstruction functions $S_{j,d}, j = 1,2,\ldots,s,$ are determined from the Fourier coefficients of d_j , $i.e.$,

$\hat{d}_j(\alpha) = \int_{[0,1)^d} d_j(x) e^{2\pi i \alpha^{\top} x}\, dx$, $\alpha \in \mathbb{Z}^{\boldsymbol{d}}$. More specifically,

$$S_{j,d}(t) = \sum_{\alpha \in \mathbb{Z}^d} \hat{d}_j(\alpha)\phi(t - \alpha), t \in \mathbb{R}^d . \tag{45}$$

Since $\hat{d}_j \in \ell^1(\mathbb{Z}^{\boldsymbol{d}}), j = 1, 2, \ldots, s,$ and $\phi \in \mathcal{L}^\infty(\mathbb{R}^d)$, we obtain that the reconstruction functions $S_{j,d} \in \mathcal{L}^\infty(\mathbb{R}^d)$, $j = 1, 2, \ldots, s$ (notice that $|\varphi *' a|_\infty \leq |\varphi|_\infty \|a\|_1$, see [96]). By using a density argument, in the next theorem we extend the sampling formula (44) to the whole space V_ϕ^∞ in a point wise sense.

Theorem $(4.2.10)$[77]: Let $\boldsymbol{d} = [d_1, \ldots, d_s]$ be a vector with entries d_j in the algebra A , j $= 1, 2, \ldots,$ s, and satisfying (43). Then, for any $f \in V_\phi^\infty$ the following sampling formula holds point wise:

$$f(t) = (detM) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,d}(t - M\alpha), t \in \mathbb{R}^d, \tag{46}$$

where the reconstruction function $S_{j,d}$, given by (45), belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$ for $j = 1, 2, \ldots, s$. Moreover, assuming that $\phi, D^\beta \phi \in C_0(\mathbb{R}^d)$ for $|\beta| \leq m$, (i.e., ϕ and its derivatives are continuous on $\mathbb{R}^d$ vanishing at infinity), then the convergence of the sampling series in (46) is also absolute and uniform on $\mathbb{R}^d$.

Proof : Consider the Banach space $C_b^m(\mathbb{R}^d)$ of all functions f which, together with all their partial derivatives $D^\beta f$ of order $|\beta| \leq m$,are continuous and bounded on $\mathbb{R}^d$ with the norm $\|f\|_{C_b^m} = \max_{|\beta| \leq m} \sup_{t \in \mathbb{R}^d} |D^\beta f(t)|$. For any vector d with entries in A and satisfying (43) there exists a constant $K > 0$ such that, for each

$$f \in C_b^m (\mathbb{R}^d), \qquad |(\Gamma_d f)(t)| \leq K \|f\|_{C_b^m} \text{ for all } t \in \mathbb{R}^d , \tag{47}$$

where $(\Gamma_d f)(t) = (detM) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,d}(t - M\alpha)$. For the proof, see [91]. Let $f \in V_\phi^\infty$ and $a \in c_0(\mathbb{Z}^{\boldsymbol{d}})$ such that $f(t) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\phi(t - \alpha)$. For $n \in \mathbb{N}$ we define $f_n(t) = \sum_{|\alpha| \leq n} a(\alpha)\phi(t - \alpha)$.From the assumptions on ϕ we have that $f_n \in C_b^m(\mathbb{R}^d)$. Moreover, for $|\beta| \leq m$ and $n > l > 0$, we have

$$|D^\beta (f_n - f_l)(t)| \leq \sum_{l < |\alpha| \leq n} |a(\alpha)| \, |D^\beta \phi(t - \alpha)| \leq \sup_{l < |\alpha| \leq n} |a(\alpha)| \, |D^\beta \phi|_\infty, t \in \mathbb{R}^d.$$

Since the sequence $a \in c_0(\mathbb{Z}^{\boldsymbol{d}})$, $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in the Banach space $C_b^m(\mathbb{R}^d)$, we deduce that f_n converges in the C_b^m –norm to f as $n \to \infty$. In particular $f \in C_b^m(\mathbb{R}^d)$.

Using that the sampling formula holds for $f_n \in span\{\phi(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d}$ and inequality (47) we obtain that, for all $t \in \mathbb{R}^d$,

$0 \leq |f_n(t) - \Gamma_d f(t)| = |\Gamma_d[(f_n - f)](t)| \leq K\|f_n - f\|_{C_b^m} \to 0 \; as \; n \to \infty$, and then $\Gamma_d f(t) = f(t)$ for all $t \in \mathbb{R}^d$. This proves that the sampling formula (46) holds point wise. It remains to prove the absolute and uniform convergence of the series in (46). Let $|\beta| \leq m$. Assuming that $D^{\beta}\phi \in C_0(\mathbb{R}^d)$ we have that $D^{\beta} f_k \in C_0(\mathbb{R}^d)$. Since $D^{\beta} f_m$ converges uniformly to $D^{\beta} f$ on $\mathbb{R}^d$, and $C_0(\mathbb{R}^d)$ is a closed subspace in $L^{\infty}(\mathbb{R}^d)$, we obtain that $D^{\beta} f \in C_0(\mathbb{R}^d)$. From this fact and using the Lebesgue dominated convergence theorem (whenever $\mathcal{L}_j$ is a system of type (a)), we obtain that $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d} \in C_0(\mathbb{R}^d)$ for each $j = 1, 2, \ldots, s$. Hence, by using that $S_{j,d} \in \mathcal{L}^{\infty}(\mathbb{R}^d)$ and the inequality

$$\sum_{|\alpha|>n} \left|(\mathcal{L}_j f)(M\alpha) S_{j,d}(t - M\alpha)\right| \leq \sup_{|\alpha|>n} \left|(\mathcal{L}_j f)(M\alpha)\right| |S_{j,d}|_{\infty}, t \in \mathbb{R}^d, n \in \mathbb{N},$$

we obtain that the series in (46) converges absolutely and uniformly on $\mathbb{R}^d$. Observe that, under the assumed hypotheses, in the proof of the theorem we have obtained that

$$V_{\phi}^{\infty} \subset C_b^m(\mathbb{R}^d) \quad . \tag{48}$$

In the case that the continuous functions ϕ and $D^{\beta}\phi, |\beta| \leq m$, belong to the Wiener space $W(L^{\infty}, \ell^1) := \{ f : \sum_{n\in\mathbb{Z}} \|f\chi_{[n,n+1)}\|_{\infty} < \infty\}$, then the generator ϕ and its derivatives $D^{\beta}\phi, |\beta| \leq m$, belong to $\mathcal{L}^{\infty}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. Finally, notice that our space V_{ϕ}^{∞} differs from the shift-invariant space V_{ϕ}^{∞} introduced in [79]. Following this reference,under appropriate hypotheses, a similar sampling result can be proved for functions in V_{ϕ}^{∞} having locally uniform convergence.

We denote by $W_{\infty}^r(\mathbb{R}^d) = \{ f : \|D^{\gamma} f\|_{\infty} < \infty, |\gamma| \leq r\}$ the usual Sobolev space, and by $|f|_{j,\infty} = \sum_{|\beta|=j} \|D^{\beta} f\|_{\infty}$,$0 \leq j \leq r$, the corresponding seminorm of a function $f \in W_{\infty}^r(\mathbb{R}^d)$.

Theorem (4.2.11)[77]: Let $\varphi \in W_{\infty}^j(\mathbb{R}^d)$, $0 \leq j < r$ and $\tilde{\varphi} \in L^1(\mathbb{R}^d)$ be compactly supported functions and let Q be the quasi-projection operator given by

$$(Qf)(t) = \sum_{\alpha\in\mathbb{Z}^d} \langle f, \tilde{\varphi}(\cdot - \alpha)\rangle_{L^2(\mathbb{R}^d)} \varphi(t - \alpha). \tag{49}$$

If $Q\pi = \pi$ for every polynomial π of degree at most $r - 1$, then $|f - Q_h f|_{j,\infty} \leq K h^{r-j} | f|_{r,\infty}, f \in W_{\infty}^r(\mathbb{R}^d)$, where $Q_h = \sigma_h Q \sigma_{1/h}$ and K is a constant independent of $h > 0$ and f .From this result, and assuming that the generator ϕ satisfies the Strang–Fix conditions of order r, we deduce by using Theorem (4.2.10) that, for any function $f \in C_0(\mathbb{R}^d) \cap W_{\infty}^r(\mathbb{R}^d)$

$$\|\Gamma_d^h f - f\|_{\infty} = \mathcal{O} h^{r-m} \quad as \quad h \to 0^+ .$$

Theorem (4.2.12)[77]: Assume that ϕ is a compactly supported generator in $W_{\infty}^r(\mathbb{R}^d)$ where $r > m$, with m being the largest order of the partial derivatives appearing in the systems $\mathcal{L}_j$. Let $d = [d_1, \ldots, d_s]$ be a vector with entries in the algebra A and satisfying (43). If the generator ϕ satisfies the Strang–Fix conditions of order r then, for each $f \in C_0(\mathbb{R}^d) \cap W_{\infty}^r(\mathbb{R}^d)$ and $0 < h \leq 1$, the following inequality holds: $\left\|f-\Gamma_d^h f\right\|_\infty \leq K|f|_{r,\infty} h^{r-m}$,Where $\Gamma_d^h = \sigma_h \Gamma_d \sigma_{1/h}$ and the constant K is independent of f and h.

Proof : From Theorem (4.2.10) we have that $\Gamma_d^h \xi = \xi$, for any $\xi \in \sigma_h V_\phi^\infty$ and $h > 0$. From (48) , $\xi \in C_b^m(\mathbb{R}^d)$, and from (47), there exists a constant $M > 0$ such that $\left\|\Gamma_d^l\right\|_\infty \leq M\|l\|_{C_b^m}$ for all $l \in C_b^m(\mathbb{R}^d)$. Hence, for any $f \in C_b^m(\mathbb{R}^d)$ and $0 < h \leq 1$, we obtain that

$$\left\|f-\Gamma_d^h f\right\|_\infty \leq \|f-\xi\|_\infty + \left\|f-\Gamma_d^h f\right\| = \|f-\xi\|_\infty + \left\|\Gamma_d^h(\xi-f)\right\|_\infty$$
$$= \|f-\xi\|_\infty + \left\|\Gamma_d \sigma_{1/h}(\xi-f)\right\|_\infty \leq \|f-\xi\|_\infty + M\left\|\sigma_{1/h}(\xi-f)\right\|_{C_b^m}$$
$$\leq \|f-\xi\|_\infty + M\|\xi-f\|_{C_b^m} \leq (1+M)\|\xi-f\|_{C_b^m}, \xi \in \sigma_h V_\phi^\infty , \tag{50}$$

where we have used that $\sigma_{1/h}(\xi - f) \in C_b^m(\mathbb{R}^d)$.Given $\varphi = \phi$ satisfying the Strang–Fix conditions of order r, there exists a compactly supported function $\tilde{\varphi} \in L^1(\mathbb{R}^d)$ satisfying the conditions of Theorem (4.2.11). An example of such a function $\tilde{\varphi}$ can be found in [98]. Let Q be the quasi-projection operator defined in (48). Note that for $f \in C_0(\mathbb{R}^d)$ we have that $\left\{\langle f, \tilde{\varphi}(\cdot - \alpha)\rangle_{L^1(\mathbb{R}^d)}\right\}_{\alpha\in\mathbb{Z}^d} \in c_0(\mathbb{Z}^d)$ and hence $Qf \in V_\phi^\infty$. Moreover, from Theorem (4.2.11), for $j = 0,1,\ldots,m$, we have that $\left| f - Q_h f \right|_{j,\infty} \leq K_j h^{r-j} \left| f \right|_{r,\infty}$, $f \in W_\infty^r(\mathbb{R}^d)$,where the constants K_j , $0 \leq j \leq m$, are independent of f and h. By using (50), for any $f \in C_0(\mathbb{R}^d) \cap W_\infty^r(\mathbb{R}^d)$, we obtain

$$\left\|f-\Gamma_d^h f\right\|_\infty \leq C \inf_{\xi\in\sigma_h V_\phi^\infty} \|\boldsymbol{\xi}-\boldsymbol{f}\|_{C_b^m} \leq C\|Q_h f - f\|_{C_b^m}$$
$$= C \max_{|\beta|\leq m}\left\|D^\beta Q_h f - D^\beta f\right\|_\infty \leq C\sum_{j=0}^{m}|Q_h f - f|_{j,\infty}$$
$$\leq C|f|_{j,\infty}\sum_{j=1}^{m} K_j h^{r-j} \leq C\left(\sum_{j=1}^{m} K_j\right)|f|_{j,\infty} h^{r-m} ,$$

where the constant C is independent of f and h .

Corollary $(4.2.13)$[296] : Suppose that ϕ is a compactly supported generator in $W_\infty^{m+\epsilon}(\mathbb{R}^d)$ where $\epsilon > 0$, with largest order of the partial derivatives appearing in the systems $\mathcal{L}_j$. Let $d = [d_1,\ldots,d_s]$ be a vector with entries in the algebra A and satisfying (43). If the generator ϕ satisfies the Strang–Fix conditions of order $m+\epsilon$ then, for each $f \in C_0(\mathbb{R}^d) \cap W_\infty^{m+\epsilon}(\mathbb{R}^d)$ and the following inequality holds: $\left\|f-\Gamma_d^{(1-\epsilon)} f\right\|_\infty \leq K|f|_{m+\epsilon,\infty}(1-\epsilon)^\epsilon$,Where $\Gamma_d^{(1-\epsilon)} = \sigma_{(1-\epsilon)}\Gamma_d\sigma_{1/(1-\epsilon)}$ and the constant K is independent of f and $(1-\epsilon)$.

Proof : From Theorem (4.2.10) we have that $\Gamma_d^{(1-\epsilon)}\xi = \xi$, for any $\xi \in \sigma_{(1-\epsilon)} V_\phi^\infty$ and $\epsilon > 0$. From (48) , $\xi \in C_b^m(\mathbb{R}^d)$, and from (47), there exists a constant $M > 0$

such that $\left\|\Gamma_d^l\right\|_\infty \leq M\|l\|_{C_b^m}$ for all $l \in C_b^m(\mathbb{R}^d)$. Hence, for any $f \in C_b^m(\mathbb{R}^d)$ and we obtain that

$$\left\|f-\Gamma_d^{(1-\epsilon)} f\right\|_\infty \leq \|f-\xi\|_\infty+\left\|f-\Gamma_d^{(1-\epsilon)} f\right\|=\|f-\xi\|_\infty+\left\|\Gamma_d^{(1-\epsilon)}(\xi-f)\right\|_\infty$$
$$=\|f-\xi\|_\infty+\left\|\Gamma_d \sigma_{1/(1-\epsilon)}(\xi-f)\right\|_\infty \leq\|f-\xi\|_\infty+M\left\|\sigma_{1/(1-\epsilon)}(\xi-f)\right\|_{C_b^m}$$
$$\leq \|f-\xi\|_\infty+M\|\xi-f\|_{C_b^m} \leq(1+M)\|\xi-f\|_{C_b^m}, \xi \in \sigma_{(1-\epsilon)} V_\phi^\infty,$$

where we have used that $\sigma_{1/(1-\epsilon)}(\xi-f) \in C_b^m(\mathbb{R}^d)$.Given $\varphi=\phi$ satisfying the Strang–Fix conditions of order $m+\epsilon$, there exists a compactly supported function $\tilde{\varphi} \in L^1(\mathbb{R}^d)$ satisfying the conditions of Theorem (4.2.11). An example of such a function $\tilde{\varphi}$ can be found in [98]. Let Q be the quasi-projection operator defined in (48). Note that for $f \in C_0(\mathbb{R}^d)$ we have that $\left\{\langle f, \tilde{\varphi}(\cdot-\alpha)\rangle_{L^1(\mathbb{R}^d)}\right\}_{\alpha \in \mathbb{Z}^d} \in c_0(\mathbb{Z}^d)$ and hence $Qf \in V_\phi^\infty$. Moreover, from Theorem (4.2.11), for $j=0,1,\ldots,m$, we have that $\left|f-Q_{(1-\epsilon)} f\right|_{j,\infty} \leq K_j(1-\epsilon)^{m+\epsilon}|f|_{m+\epsilon,\infty}$, $f \in W_\infty^{m+\epsilon}(\mathbb{R}^d)$,where the constants K_j , $0 \leq j \leq m$, are independent of f and $1-\epsilon$. By using (50), for any $f \in C_0(\mathbb{R}^d) \cap W_\infty^{m+\epsilon}(\mathbb{R}^d)$, we obtain

$$\left\|f-\Gamma_d^{(1-\epsilon)} f\right\|_\infty \leq C \inf_{\xi \in \sigma_{(1-\epsilon)} V_\phi^\infty}\|\boldsymbol{\xi}-\boldsymbol{f}\|_{C_b^m} \leq C\left\|Q_{(1-\epsilon)} f-f\right\|_{C_b^m}$$
$$=C \max_{|\beta| \leq m}\left\|D^\beta Q_{(1-\epsilon)} f-D^\beta f\right\|_\infty \leq C \sum_{j=0}^m\left|Q_{(1-\epsilon)} f-f\right|_{j,\infty}$$
$$\leq C|f|_{j,\infty} \sum_{j=1}^m K_j(1-\epsilon)^{m+\epsilon} \leq C\left(\sum_{j=1}^m K_j\right)|f|_{j,\infty}(1-\epsilon)^\epsilon,$$

where the constant C is independent of f and $1-\epsilon$.

Chapter 5

Approximation with Sampling and Recovery of Bandlimited Functions

We explain the algorithm underlying ΣΔ quantization in its simplest version, we review the mathematical results that are known, and we generalize the simple first-order ΣΔ scheme to higher orders, leading to better bounds. A generalization of Kadec's 1/4 theorem to higher dimensions is considered. Finally, the developed techniques are used to approximate biorthogonal functions of particular exponential Riesz bases for $L_2[-\pi,\pi]$, and a wellknown theorem of Levinson is recovered as a corollary.

Section(5.1) A family of Stable Sigma-Delta Modulators of Arbitrary Order

Digital signal processing has revolutionized the storage and transmission of audio and video signals as well as still images, in consumer electronics and in more scientific settings (such as medical imaging). The main advantage of digital signal processing is its robustness: although all the operations have to be implemented with, of necessity, not quite ideal hardware, the a priori knowledge that all correct outcomes must lie in a very restricted set of well-separated numbers makes it possible to recover them by rounding off appropriately. Bursty errors can compromise this scenario (as is the case in many communication channels, as well as in memory storage devices), making the "perfect" data unrecoverable by rounding off. In this case, knowledge of the type of expected contamination can be used to protect the data, prior to transmission or storage, by encoding them with error correcting codes; this is done entirely in the digital domain. These advantages have contributed to the present widespread use of digital signal processing.

Many signals, however, are not digital but analog in nature; audio signals,for instance, correspond to functions $f(t)$, modeling rapid pressure oscillations, which depend on the "continuous" time t (i.e. t ranges over $\mathbb{R}$ or an interval in $\mathbb{R}$, and not over a discrete set), and the range of f typically also fills an interval in $\mathbb{R}$. For this reason, the first step in any digital processing of such signals must consist in a conversion of the analog signal to the digital world, usually abbreviated as A/D conversion. For different types of signals, different A/D schemes are used; in this paper, we restrict our attention to a particular class of A/D conversion schemes adapted to audio signals. Note that at the end of the chain, after the signal has been processed, stored, retrieved, transmitted,..., all in digital form, it needs to be reconverted to an analog signal that can be understood by a human hearing system; we thus need a D/A conversion there.

The digitization of an audio signal rests on two pillars: sampling and Quantization , both of which we now briefly discuss. We start with sampling. It is standard to model audio signals by bandlimited functions, i.e. functions $f \in L^2(\mathbb{R})$ for which the Fourier transform

$$\hat{f}(\xi) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t}\, dt$$

vanishes outside an interval $|\xi| \leq \Omega$. Note that our Fourier transform is normalized so that it is equal to its inverse, up to a sign change,

$$f(t) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi t}\, dt$$

The bandlimited model is justified by the observation that for the audio signals of interest to us, observed over realistic intervals $[-T, T]$, $\left\| \chi_{|\xi|>\Omega} \left(\chi_{|\xi|\leq T} f \right)^{\wedge} \right\|_2$ is negligible compared with $\left\| \chi_{|\xi|\leq\Omega} \left(\chi_{|\xi|\leq T} f \right)^{\wedge} \right\|_2$ for $\Omega \simeq 2\pi \cdot 20{,}000\, Hz$. Here and later in this section , $\|\cdot\|_2$ denotes the $L^2(\mathbb{R})$ norm. For bandlimited functions one can use a well-known sampling theorem, the derivation of which is so simple that we include it here for completeness: since $\hat{f}$ is supported on $[-\Omega, \Omega]$, it can be represented by a Fourier series converging in $L^2(-\Omega, \Omega)$; $i.e.,$

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi\pi/\Omega} \qquad \text{for} \quad |\xi| \leq \Omega \quad ,$$

where

$$c_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{in\xi\pi/\Omega} \quad = \quad \frac{1}{\Omega}\sqrt{\frac{\pi}{2}}\, f\left(\frac{n\pi}{\Omega}\right) \; .$$

We thus have

$$\hat{f}(\xi) \quad = \quad \frac{1}{\Omega}\sqrt{\frac{\pi}{2}} \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\Omega}\right) e^{-in\xi\pi/\Omega} \; \chi_{|\xi|\leq\Omega} \quad ,$$

which by the inverse Fourier transform leads to

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\Omega}\right) sinc(\Omega t - n\pi) \; . \qquad (1)$$

This formula reflects the well-known fact that an Ω-bandlimited function is completely characterized by sampling it at the corresponding Nyquist frequency $\frac{\Omega}{\pi}$. However, (18) is not useful in practice, because $sinc(x) = x^{-1} \sin x$ decays too slowly. If, as is to be expected, the samples $f\left(\frac{n\pi}{\Omega}\right)$ are not known perfectly, and have to be replaced, in the reconstruction formula (1) for $f(t)$, by $\tilde{f}_n = f\left(\frac{n\pi}{\Omega}\right) + \varepsilon_n$, with all $|\varepsilon_n| \leq \varepsilon$, then the corresponding approximation $\tilde{f}_n$ may differ appreciably from $f(t)$. Indeed, the infinite sum $\sum_n \varepsilon_n sinc(\Omega t - n\pi)$ need not converge. Even if we assume that we sum only over the finitely many n Satisfying $\left| n\frac{\Omega}{\pi} \right| \leq T$ (using the tacit assumption that the $f\left(\frac{n\pi}{\Omega}\right)$ decay rapidly for n outside this interval) , we will still not be able to ensure a better bound than $|f(t) - \tilde{f}(t)| \leq C\varepsilon \log T$, since

T may well be large, this is not satisfactory. To circumvent this, it is useful to introduce oversampling. This amounts to viewing $\hat{f}$ as an element of $L^2(-\lambda\Omega, \lambda\Omega)$, with $\lambda > 1$; for $|\xi| \leq \lambda\Omega$ we can then represent $\hat{f}$ by a Fourier series in which the coefficients are proportional to

$$f\left(\frac{n\pi}{\lambda\Omega}\right) \ , \qquad \hat{f}(\xi) = \frac{1}{\lambda\Omega}\sqrt{\frac{\pi}{2}}\sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-in\xi\pi/\lambda\Omega} \text{ for } |\xi| \leq \lambda\pi \, .$$

Introducing a function g such that $\hat{g}$ is C^∞ , and $\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}$ for $|\xi| \leq \pi$, $\hat{g}(\xi) = 0$ for $|\xi| > \lambda\pi$, we can write

$$\hat{f}(\xi) = \frac{\pi}{\lambda\Omega}\sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\lambda\Omega}\right) e^{-in\xi\pi/\lambda\Omega}\, \hat{g}\left(\frac{\pi\xi}{\Omega}\right) ,$$

resulting in

$$f(t) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\lambda\Omega}\right) g\left(\frac{\Omega}{\pi}t - \frac{n}{\lambda}\right) . \qquad (2)$$

Because g is smooth with fast decay, this series now converges absolutely and uniformly; moreover if the $f\left(\frac{n\pi}{\lambda\Omega}\right)$ are replaced by

$\tilde{f}_n = f\left(\frac{n\pi}{\lambda\Omega}\right) + \varepsilon_n$ in (2) ,with $|\varepsilon_n| < \varepsilon$, then the difference between the approximation $\tilde{f}_n(x)$ and $f(x)$can be bounded uniformly:

$$\left|f(t) - \tilde{f}(t)\right| \leq \varepsilon\frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \left|g\left(\frac{\Omega}{\pi}t - \frac{n}{\lambda}\right)\right| \leq \varepsilon C_g \qquad (3)$$

where $C_g = \lambda^{-1}\|g'\|_{L^1} + \|g\|_{L^1}$ does not depend on T. Oversampling thus buys the freedom of using reconstruction formulas, like (2), that weigh the different samples in a much more localized way than (1) (only the $f\left(\frac{n\pi}{\lambda\Omega}\right)$ with $\left|t - \frac{n\pi}{\lambda\Omega}\right|$ "small" contribute significantly). In practice, it is customary to sample audio signals at a rate that is about 10 or 20% higher than the Nyquist rate; for high quality audio, a traditional sampling rate is 44,000 Hz. The above discussion shows that moving from "analog time" to "discrete time" can be done without any problems or serious loss of information: for all practical purposes , f is completely represented by the sequence $\left(f\left(\frac{n\pi}{\lambda\Omega}\right)\right)_{n\in\mathbb{Z}}$. At this stage, each of these samples is still a real number. The transition to a discrete representation for each sample is called *quantization*. The simplest way to "quantize" the samples $f\left(\frac{n\pi}{\lambda\Omega}\right)$ would be to replace each by a truncated binary expansion. If we know a priori that $|f(t)| \leq A < \infty$ for all t (a very realistic assumption), then we can write

$$f\left(\frac{n\pi}{\lambda\Omega}\right) = -A + A\sum_{k=0}^{\infty} b_k^n 2^{-k} \quad ,$$

with $b_k^n \in \{0, 1\}$ for all k, n. Ifwe can "spend" κ bits per sample, then a natural solution is to just select the $(b_k^n)_{0\le k\le \kappa-1}$; constructing $\tilde{f}(x)$ from

the approximations $\tilde{f}_n = -A + A\sum_{k=0}^{k-1} b_k^n 2^{-n}$ then leads to

$|f(t) - \tilde{f}(t)| \le C2^{-\kappa+1}A$, where C is independent of κ or f. Quantized representations of this type are used for the digital representations of audio signals, but they are not the solution of choice for the A/D conversion step. (Instead, they are used after the A/D conversion, once one is firmly in the digital world.) The main reason for this is that it is very hard (and therefore very costly) to build analog devices that can divide the amplitude range $[-A, A]$ into $2^{-\kappa+1}$ precisely equal bins.

It turns out that it is much easier (= cheaper) to increase the oversampling rate, and to spend fewer bits on each approximate representation $\tilde{f}_n$ of $f\left(\frac{n\pi}{\lambda\Omega}\right)$. By appropriate choices of $\tilde{f}_n$ one can then hope that the error will decrease as the oversampling rate increases. Sigma-Delta (abbreviated by ΣΔ) quantization schemes are a very popular way to do exactly this. In the most extreme case, every sample $f\left(\frac{n\pi}{\lambda\Omega}\right)$ in (1) is replaced by just one bit, i.e. by a q_n with $q_n \in \{-1, 1\}$; we shall restrict our attention to such 1-bitΣΔ quantization schemes. Although multi-bit ΣΔ schemes are becoming more popular in applications, there are many instances where 1-bit ΣΔ quantization is used .

The following is an outline of the content of the section. We explain the algorithm underlying ΣΔ quantization in its simplest version, we review the mathematical results that are known, and we formulate several questions. We generalize the simple first-order ΣΔ scheme to higher orders, leading to better bounds. In particular, we show, for any $k \in \mathbb{N}$, an explicit mathematical algorithm that defines, for every function f that is bandlimited (i.e. the inverse Fourier transform of a finite measure supported in $[-\Omega, \Omega]$) with absolute value bounded by $a < 1$, and for all $n \in \mathbb{Z}$, "bits"$q_n^{(k)} \in \{-1, 1\}$ such that, uniformly in t,

$$f(t) - \frac{1}{\lambda}\sum_n q_n^{(k)} g\left(\frac{\Omega}{\pi}t - \frac{n}{\lambda}\right) \le C_g^{(k)} g\lambda^{-k} . \qquad (4)$$

Moreover, we prove that our algorithm is *robust* in the following sense.

Since we have to make a transition from real-valued inputs $f\left(\frac{n\pi}{\lambda\Omega}\right)$ to the discrete valued $q_n \in \{-1, 1\}$, we have to use a discontinuous function as part of our algorithm. In our case, this will be the sign function, $\text{sign}(A) = 1$ if $A \ge 0$, $\text{sign}(A) = -1$ if $A < 0$. In practice, one cannot build, except at very high cost, an implementation of *sign* that "toggles" at exactly 0; we shall therefore allow every occurrence of $\text{sign}(A)$ to be replaced by $Q(A)$, where Q can vary from one time step to the next, or from one component of the algorithm to another, with only the restrictions that $Q(A) = \text{sign}(A)$ for $|A| \geq \tau$ and $|Q(A)| \leq 1$ for $|A| \leq \tau$, where $\tau > 0$ is known. (Note that this allows for both continuous and discontinuous Q, if we impose *a priori* that $Q(t)$ can take the values 1 and -1 only, then the restrictions reduce to the first condition.) Moreover, whenever our algorithm uses multiplication by some real-valued parameter P, we also allow for the replacement of P by $\boldsymbol{P}(\mathbf{1}+\boldsymbol{\epsilon})$, where can again vary, subject only to $|\epsilon| \leq \mu < 1$, where the tolerance μ is again known a prioiri. We can now formulate what we mean by robustness: despite all this wriggle room, we prove that (4) holds independently of the (possibly time-varying) values of all the ϵ and Q, within the constraints.

For the sake of convenience, we shall set (by choosing appropriate units if necessary) $\Omega = \pi$ and $A = 1$. We are thus concerned with coarse quantization of functions $f \in C2 = \{h \in L^2, \|h\|_{L^\infty} \leq 1,$
support $\hat{h} \subset [-\pi, \pi]\}$; for most of our results we also can consider the larger class $C_1 = \{h : \hat{h}$ is a finite measure supported in $[-\pi, \pi], \|h\|_{L^\infty} \leq 1\}$.With these normalizations (3) simplifies to

$$f(t) = \frac{1}{\lambda} \sum_n f\left(\frac{n}{\lambda}\right) g\left(t - \frac{n}{\lambda}\right), \quad (5)$$

with g as described before ; $i.e.$,

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \text{ for } |\xi| \leq \pi, \hat{g}(\xi) = 0 \text{ for } |\xi| > \lambda\pi \text{ and } \hat{g} \in C^\infty. \quad (6)$$

It is not immediately clear how to construct sequences $q^\lambda = \left(q_n^\lambda\right)_{n\in\mathbb{Z}}$, with $q_n^\lambda \in \{-1, 1\}$ for each $n \in \mathbb{Z}$, such that

$$\tilde{f}_{\boldsymbol{q}^\lambda}(t) = \frac{1}{\lambda} \sum q_n^\lambda g\left(t - \frac{n}{\lambda}\right) \quad (7)$$

provides a good approximation to f. Taking simply $q_n^\lambda = sign\left(f\left(\frac{n}{\lambda}\right)\right)$ does not work because there exist infinitely many independent bandlimited functions ϕ that are everywhere positive (such as the lowest order prolate spheroidal wave functions [67], [68] for arbitrary time intervals and symmetric frequency intervals contained in $[-\pi, \pi]$); picking the signs of samples as candidate q_n^λ would make it impossible to distinguish between any two functions in this class. First order ΣΔ-quantization circumvents this by providing a simple iterative algorithm in which the q_n^λ are constructed by taking into account not only $f\left(\frac{n}{\lambda}\right)$ but also past $f\left(\frac{m}{\lambda}\right)$; we shall see below how this leads to good approximate $\tilde{f}_{\boldsymbol{q}^\lambda}$. Concretely, one introduces an auxiliary sequence $(u_n)_{n\in\mathbb{Z}}$ (sometimes described as giving the "internal state" of the ΣΔ quantizer) iteratively defined by

$$\begin{cases} u_n = u_{n-1} + f\left(\frac{n}{\lambda}\right) - q_n^\lambda \\ q_n^\lambda = sign\left(u_{n-1} + f\left(\frac{n}{\lambda}\right)\right), \end{cases} \tag{8}$$

and with an "initial condition" u_0 arbitrarily chosen in $(-1,1)$. In circuit implementation, the range of n in (8) is $n \geq 1$. However, for theoretical reasons, we view (8) as defining the u_n and q_n for all n. At first glance, this means the u_n are defined implicitly for $n < 0$. However, as we shall see below , it is possible to write u_n and q_n directly in terms of u_{n+1} and f_{n+1} when $n < 0$. We shall now show by a simple inductive argument that the u_n of (8) are all bounded by 1. We prove this in two steps:

Lemma$(5.1.1)$[49]: For any $f \in C_1$ and $|u_0| < 1$, the sequence $(u_n)_{n\in\mathbb{Z}}$ defined by the recursion (8) is uniformly bounded, $|u_n| < 1$ for all $n \geq 0$.

Proof : Suppose $|u_{n-1}| < 1$. Because $f \in C_1$, we have $\left|f\left(\frac{n}{\lambda}\right)\right| \leq 1$, so that $\left|f\left(\frac{n}{\lambda}\right) + u_{n-1}\right| < 2$. It then follows that

$$\left|f\left(\frac{n}{\lambda}\right) + u_{n-1} - sign\left(f\left(\frac{n}{\lambda}\right) + u_{n-1}\right)\right| < 1.$$

For negative n, we first have to transform the system (8) into a recursion in the other direction. To do this, observe that for $n \geq 1$,

$$u_{n-1} + f\left(\frac{n}{\lambda}\right) > 0 \Rightarrow u_n - f\left(\frac{n}{\lambda}\right) = u_{n-1} - 1 < 0$$

$$u_{n-1} + f\left(\frac{n}{\lambda}\right) < 0 \Rightarrow u_n - f\left(\frac{n}{\lambda}\right) = u_{n-1} + 1 > 0.$$

In all cases we have, thus, $sign\left(u_n - f\left(\frac{n}{\lambda}\right)\right) = -sign\left(u_{n-1} + f\left(\frac{n}{\lambda}\right)\right)$. The recursion (8) therefore implies, for $n \geq 1$,

$$u_{n-1} = u_n - f\left(\frac{n}{\lambda}\right) - sign\left(u_n - f\left(\frac{n}{\lambda}\right)\right), \tag{9}$$

which we can now extend to all n, making it possible to compute u_n for $n < 0$ corresponding to the "initial" value $u_0 \in (-1,1)$. The same inductive argument then proves that these u_n are also bounded by 1. We have thus:

Proposition $(5.1.2)$[49]: The recursion (8), with $|u_0| < 1$ and $f \in C_1$, defines a sequence $(u_n)_{n\in\mathbb{Z}}$ for which $|u_n| < 1$ for all $n \in \mathbb{Z}$.

From this we can immediately derive a bound for the approximation error

$$|f(t) - \tilde{f}_{q^\lambda}(t)|.$$

Proposition $(5.1.3)$[49]: For $f \in C_1, \lambda > 1$, define the sequence q^λ through the recurrence (8), with u_0 chosen arbitrarily in $(-1,1)$. Let g be a function satisfying (6). Then

$$\left| f(t) - \frac{1}{\lambda}\sum_n q_n^\lambda g\left(t - \frac{n}{\lambda}\right)\right| \le \frac{1}{\lambda}\|g'\|_{L^1} . \tag{10}$$

Proof : Using (5), summation by parts, and the bound $|u_n| < 1$, we derive

$$\begin{aligned}\left| f(t) - \frac{1}{\lambda}\sum_n q_n^\lambda g\left(t - \frac{n}{\lambda}\right)\right| &= \frac{1}{\lambda}\left|\sum_n \left(f\left(\frac{n}{\lambda}\right) - q_n^\lambda\right) g\left(t - \frac{n}{\lambda}\right)\right| \\ &= \frac{1}{\lambda}\left|\sum_n u_n\left(g\left(t - \frac{n}{\lambda}\right) - g\left(t - \frac{n+1}{\lambda}\right)\right)\right| \\ &\le \frac{1}{\lambda}\sum_n \left| g\left(t - \frac{n}{\lambda}\right) - g\left(t - \frac{n+1}{\lambda}\right)\right| \\ &\le \frac{1}{\lambda}\sum_n \int_{t-\frac{n+1}{\lambda}}^{t-\frac{n}{\lambda}} |g'(y)|\, dy = \frac{1}{\lambda}\|g'\|_{L^1} .\end{aligned}$$

This extremely simple bound is rather remarkable in its generality. What makes it work is, of course, the special construction of the q_n^λ via (8); the q_n^λ are chosen so that, for any N, the sum $\sum_{n=1}^N q_n^\lambda$ closely tracks $\sum_{n=1}^N f\left(\frac{n}{\lambda}\right)$, since

$$\left|\sum_{n=1}^N f\left(\frac{n}{\lambda}\right) - \sum_{n=1}^N q_n^\lambda\right| = |u_N - u_0| < 2 .$$

If we choose $u_0 = 0$ (as is customary), then we even have

$$\left|\sum_{n=1}^N f\left(\frac{n}{\lambda}\right) - \sum_{n=1}^N q_n^\lambda\right| = |u_N| < 1 , \tag{11}$$

this requirement (which can be extended to negative N) clearly fixes the q_n^λ unambiguously. The “Σ” in the name ΣΔ-modulation or ΣΔ-quantization stems from this feature of tracking “sums” in defining the q_n^λ ; ΣΔ-modulation can be viewed as a refinement of earlier Δ-modulation schemes, to which the sum-tracking was added. There exists a vast literature on ΣΔ-modulation in the electrical engineering community; see e.g. the review books [58] and [71]. This literature is mostly concerned with the design of, and the study of good design criteria for, more complicated ΣΔ-schemes. The one given by (8) is the oldest and simplest [58], but is not, as far as we know, used in practice. We shall see below how better bounds than (10), i.e. bounds that decay faster as $\lambda \to \infty$, can be obtained by replacing (8) by other recursions, in which higher order differences play a role. Before doing so, we spend the remainder of section on further comments on the first-order scheme and its properties.

In practice, one cannot use filter functions g that satisfy the condition in (6) because they require the full sequence $\left(q_n^\lambda\right)_{n\in\mathbb{Z}}$ to approximate even one value $f(t)$. It would be closer to the common practice to use G that are compactly supported (and for which the support of $\hat{G}$ is therefore all of $\mathbb{R}$, in contrast with (6)). In this case, the reconstruction formula (5) no longer holds, and the approximation error has additional contributions. Suppose G is supported in $[-R, R]$, so that, for a given t, only the q_n^λ With $|t - \frac{n}{\lambda}| < R$ can contribute to the sum $\sum_n q_n^\lambda G(t - q_n^\lambda)$. Then we have

$$\left| f(t) - \frac{1}{\lambda}\sum_n q_n^\lambda G\left(t - \frac{n}{\lambda}\right)\right| \le \left| f(t) - \frac{1}{\lambda}\sum_n f\left(\frac{n}{\lambda}\right) G\left(t - \frac{n}{\lambda}\right)\right| \tag{12}$$

$$+\frac{1}{\lambda}\sum_n \left(f\left(\frac{n}{\lambda}\right) - q_n^\lambda\right) G\left(t - \frac{n}{\lambda}\right).$$

The second term can be bounded as before. We can bound the first term by introducing again an "ideal" reconstruction function g, satisfying supp $\hat{g} \subset [-\lambda\pi, \lambda\pi]$ and $\hat{g}|_{[-\pi,\pi]} \equiv (2\pi)^{-1/2}$. Then

$$\left| f(t) - \frac{1}{\lambda}\sum_n f\left(\frac{n}{\lambda}\right) G\left(t - \frac{n}{\lambda}\right)\right|$$

$$= \frac{1}{\lambda}\left|\sum_n f\left(\frac{n}{\lambda}\right)\left[g\left(t - \frac{n}{\lambda}\right) - G\left(t - \frac{n}{\lambda}\right)\right]\right|$$

$$\le \frac{1}{\lambda}\sum_n \left| g\left(t - \frac{n}{\lambda}\right) - G\left(t - \frac{n}{\lambda}\right)\right| \le \|G - g\|_{L^1} + \lambda^{-1}\|G' - g'\|_{L^1}.$$

By imposing on G that the L^1 distance of G and G'/λ to g and g'/λ, respectively, be less than C/λ for at least one suitable g, we see that this term becomes comparable to the estimate for the first term. (This means that G depends on λ; the support of G typically increases with λ.)

In practical applications, one is generally interested only in approximating $f(t)$ for t after some starting time $t_0, t > t_0$. If finite filters are used this means that one needs the q_n^λ only for n exceeding some corresponding n_0. There is then no need to consider the "backwards" recursion (9), introduced to extend Lemma (5.1.1) (bound on the $|u_n|$ uniform in $\boldsymbol{n} \ge \mathbf{0}$) to Proposition(5.1.2) (bound on the $|\boldsymbol{u_n}|$ uniform in n).

Note that in practice, and except at the final D/A step mentioned in the introduction, bandlimited models for audio signals are always represented in sampled form. This means that once a digital sequence $(q_n^\lambda)_{n\in\mathbb{Z}}$ is determined , all the filtering and manipulations will be digital, and an estimate closer to the electrical engineering practice would seek to bound errors of the type

$$\left| f\left(\frac{m}{\lambda}\right) - \sum_n q_n^\lambda G_{m-n}^\lambda \right|, \tag{13}$$

using discrete convolution with finite filters G^{λ}, rather than expressions of the type (10) or (11). If we were interested in optimizing constants relevant for practice, we should concentrate on (13) directly. For our present level of modeling however, in which we want to study the dominant behavior as a function of λ, working with (10) or (11), or their equivalent forms for higher order schemes, below, will suffice, since (13) will have the same asymptotic behavior as (11), for appropriately chosen G_m^{λ}. Unless specified otherwise, we shall assume, for the sake of convenience, that we work with reconstruction functions g satisfying (6). Since such g are supported on all of ℝ, we will always need to define q_n for all $n \in \mathbb{Z}$ (rather than ℕ). For first-order ΣΔ, we could easily "invert" the recursion so as to reach $n < 0$. For the higher order ΣΔ considered from this Section onwards, such an inversion is not straightforward; instead we will simply give, for every algorithm that defines q_n for $n \geq 0$, a parallel prescription that defines q_n for $n < 0$.

In practice, one observes better behavior for $|f(t) - \tilde{f}q^{\lambda}(t)|$ than that proved in Proposition (5.1.3). In particular, it is believed that, for arbitrary

$$f \in C_1,\ \lim_{T\to\infty} \frac{1}{2T} \int_{|t|\leq T} \left| f(t) - \frac{1}{\lambda} \sum_n q_n^{\lambda} g\left(t - \frac{n}{\lambda}\right) \right|^2 dt \leq \frac{C}{\lambda^3}, \qquad (14)$$

with C independent of $f \in C_1$ or of the initial condition u_0 for the recursion (8). Whether the conjecture (14) holds, either for each $f \in C_1$, or in the mean (taking an average over a large class of functions in C_1 or C_2) is still an open problem.

It is not surprising that a better bound than (10) would hold, since we used very little in its derivation. In particular, we never used explicitly that the $f\left(\frac{n}{\lambda}\right)$ were samples of the entire (because bandlimited) function f.

For some special cases, i.e. for very restricted classes of functions f, (14) has been proved. In particular, it was proved by ℝ. Gray [72] that if one restricts oneself to $f = f_a$, where $a \in [-1, 1]$ and $f_a(t) \equiv a$, then

$$\int_{-1}^{1} \left[\lim_{T\to\infty} \frac{1}{2T} \int_{|t|\leq T} \left| f_a(t) - \frac{1}{\lambda} \sum_n q_n^{\lambda} g\left(t - \frac{n}{\lambda}\right) \right|^2 dt \right] da \leq \frac{C}{\lambda^3}, \qquad (15)$$

in Gray's analysis the integral over t is a sum over samples, and g is replaced by a discrete filter G^{λ} (see above), but his analysis applies equally well to our case. A different proof can be found in [63]. Gray's result was later extended by Gray, Chou and Wong [72] to the case where the input function $f(t)$ is a sinusoid, $f(t) = a \sin bt$, with $|b| < \pi$.

For general bandlimited functions, there were no results, to our knowledge, until the work of [61], [62], [63], who proved, by a combination of tools from number theory and harmonic analysis, that, for all $f \in C_1$ and all t for which

$$f'(t) \neq 0, \qquad \left| f(t) - \sum_n q_n^\lambda g^\lambda \left(t - \frac{n}{\lambda}\right) \right| \leq C\lambda^{-\frac{4}{3}+\varepsilon}. \tag{16}$$

In Güntürk's analysis the value of C depends on $|f'(t)|$ as well as ϵ; his g^λ (into which the $1/\lambda$ factor from (10) has been absorbed) is compactly supported, and has to satisfy various technical conditions. Although there is no mathematical proof for the moment, numerical simulations of intermediate results in $Güntürk's$ work suggest that (16) may still hold, for general $f \in C_1$, if the upper bound $C\lambda^{-\frac{4}{3}+\varepsilon}$ is replaced by $C\lambda^{-\frac{4}{3}+\varepsilon}$. For more details concerning the whole analysis and this discussion in particular, we refer the reader to [62], [63].

Remarkably, an iterative procedure very similar to (8) can be used to compute the binary expansion of a number in $(0, 1)$. Consider the recursion

$$\begin{cases} \tilde{u}_n = 2\tilde{u}_{n-1} + x_n - \tilde{b}_n \\ \tilde{b}_n = sign(2\tilde{u}_{n-1} + x_n) \end{cases} \tag{17}$$

with initial condition $\tilde{u}_{-1} = \alpha/2$, $b_0 = sign(\alpha)$, and with the sequence $(x_n)_n$ defined by $x_0 = \alpha, x_n = 0$ for $n > 0$; here α is any number in $(-1, 1)$. By induction one derives again that $|\tilde{u}_n| < 1$ for all n, so that

$$\begin{aligned} \left| 2\alpha - \sum_{n=0}^{N} 2^{-n}\tilde{b}_n \right| &= \left| \alpha - \sum_{n=0}^{N} 2^{-n}(x_n - \tilde{b}_n) \right| \\ &= \left| 2\tilde{u}_{-1} + \sum_{n=0}^{N} 2^{-n}(\tilde{u}_n - 2\tilde{u}_{n-1}) \right| \\ &= |2^{-N}\tilde{u}_N| < 2^{-N} \to 0 \ as\ N \to \infty, \end{aligned}$$

which converges exponentially like a binary expansion. (Since the $\tilde{b}_n \in \{-1, 1\}$, $\sum_{n=0}^{\infty} 2^{-n}\tilde{b}_n$ is not quite a binary expansion; however, for $n \geq 1$, the $b_n = (1 + \tilde{b}_n)/2 \in \{0, 1\}$ are the digits for the binary expansion of $\frac{1+\alpha}{2}$.)

The only difference between the two recursions is the presence of the multiplications by 2 in (17). When the recursive equations are converted into block diagrams for circuits that would implement these recursions in practice , the diagram for (17) would require only one item more (a multiplier by 2) than the diagram for (8). The similarity of the two algorithms or circuits seems to contradict the claim in the introduction, that ΣΔ quantization is much cheaper to implement than binary quantization of less frequent samples. However, the two algorithms behave very differently when imperfections, in particular imperfect quantizers, are introduced. Quantizers are never perfect. Although we desire to use $q(x) = sign(x)$ for our 1-bit quantizer, in practice we may have, $e.g.$, $q(x) = sign(x + \delta)$, where δ is unknown except for the specification $|\delta| < \tau$; the value of δ may vary from one circuit to another, and it may even, due to thermal fluctuations, vary from one time step n to the next. More generally, we may have $Q(x) = sign(x)$ for $|x| \geq \tau$, whereas for $|x| \leq \tau$, we have only the bound $|Q(x)| \leq 1$. (Note that if Q is restricted to take only the values 1 and -1, the second condition is automatically satisfied, implyingthat for $|t| < \tau$, the behavior of $Q(t)$ can be completely arbitrary.) A good algorithm or circuit is one that will perform well even without very stringent requirements on τ; if extremely tight specifications on τ are necessary to make everything work well, then this will translate into an expensive circuit.

Let us replace the sign function in (8) by such a nonideal quantizer; the new recursion is then

$$\begin{cases} u_n = u_{n-1} + f\left(\frac{n}{\lambda}\right) - q_n \\ q_n = Q_n\left(u_{n-1} + f\left(\frac{n}{\lambda}\right)\right), \end{cases} \tag{18}$$

and let us assume that , for all $n \in \mathbb{Z}$,

$$\begin{cases} Q_n(x) = sign(x) \quad \text{for} \quad |t| \geq \tau \\ |Q_n(x)| \leq 1 \quad \text{for} \quad |t| \leq \tau . \end{cases} \tag{19}$$

It turns out that the u_n are then still bounded, uniformly, independently of the detailed behavior of Q_n, as long as (19) is satisfied:

Lemma (5.1.4)[49] : Let f be $\in C_1$, let u_n, q_n be as defined in (18), and let Q_n satisfy (19) for all n. If $|u_0| \leq 1 + \tau$, then $|u_n| \leq 1 + \tau$ for all $n \geq 0$.

Proof : We use induction again. Suppose $|u_{n-1}| \leq \tau + 1$. Because $f \in C_1, \left|f\left(\frac{n}{\lambda}\right)\right| \leq 1$. We now distinguish three cases. If $u_{n-1} + f\left(\frac{n}{\lambda}\right) > \tau$, then $u_n = u_{n-1} + f\left(\frac{n}{\lambda}\right) - 1 \in (\tau - 1, \tau + 1)$. Likewise, if $u_{n-1} + f\left(\frac{n}{\lambda}\right) < -\tau$, then $u_n = u_{n-1} + f\left(\frac{n}{\lambda}\right) + 1 \in (-\tau - 1, -\tau + 1)$. Finally , if $-\tau \leq u_{n-1} + f\left(\frac{n}{\lambda}\right) \leq \tau$, then $\left|Q_n\left(u_{n-1} + f\left(\frac{n}{\lambda}\right)\right)\right| \leq 1$, so that

$$u_n = u_{n-1} + f\left(\frac{n}{\lambda}\right) - Q_n\left(u_{n-1} + f\left(\frac{n}{\lambda}\right)\right) \in (-\tau - 1, \quad \tau + 1).$$

Note that Lemma (5.1.4) holds regardless of how large τ is; even $\tau \gg 1$ is allowed. To discuss the case $n \leq 0$, we need to reconsider the recursion , because for generic Q_n , we can no longer “invert” the relationship between u_n and u_{n-1}. Therefore, we simply posit the following recursion for $n < 0$, inspired by (9),

$$\begin{cases} u_n = u_{n+1} - f\left(\frac{n+1}{\lambda}\right) + q_n \\ q_n = -Q_n\left(u_{n+1} - f\left(\frac{n+1}{\lambda}\right)\right). \end{cases} \tag{20}$$

An immediate generalization of Lemma (5.1.4) is then

Lemma $(\mathbf{5.1.5})[\mathbf{49}]$ **:** Let f be in C_1, let u_n , q_n be as defined in (18) or (20), and let Q_n satisfy (18) for all $|n| > 1$. Assume also that $|u_0| \le 1 + \tau$. Then $|u_n| \le \tau + 1$ for all $n \in \mathbb{Z}$.

By the same argument as in the proof of Proposition (5.1.3), Lemma (5.1.5) has as an immediate consequence the following :

Corollary $(\mathbf{5.1.6})[\mathbf{49}]$**:** Let f be in C_1 , let λ be > 1, and suppose g satisfies (6).Suppose, also, the sequence $(q_n^\lambda)_{n\in\mathbb{Z}}$ is generated by (18), with imperfect quantizers $Q_n(t)$ that satisfy (19). Then , for all $t \in \mathbb{R}$,

$$f(t) - \frac{1}{\lambda}\sum_n q_n^\lambda g\left(t - \frac{n}{\lambda}\right) \le \frac{1 + \tau}{\lambda}\|g'\|_{L^1} . \tag{21}$$

If one replaces the "perfect" reconstruction function g by a suitable compactly supported G^λ , as in this subsection , then one can also derive estimates similar to (21), exploiting the compactness of the support of G^λ. Although we must pay some penalty for the imperfection of the quantizer in all these cases (the constants increase), the precision that can be attained is nevertheless not limited by the imperfection: by choosing λ sufficiently large, the approximation error can be made arbitrarily small.

The same is not true for the binary expansion-type schemes (17). Suppose we use (17) to generate bits $\tilde{b}_n \in \{-1, 1\}$, and consider the approximation $\alpha_N = \sum_{n=0}^N 2^{-n}\tilde{b}_n$ to the input α, as before; however, the quantizer has been changed to, say , $Q_n(t) = sign(t - \delta_n)$, with $|\delta_n| < \tau$. Suppose now $\alpha = \frac{\delta_0}{2}$,for the sake of definiteness, assume $\delta_0 > 0$. Then (34), with this imperfect quantizer , will give $\tilde{b}_0 = -1$, so that $\alpha_N = \tilde{b}_0 + \sum_{n=1}^N 2^{-n}\tilde{b}_n \le -2^{-N}$ for all N, implying $|\alpha - \alpha_N| > \frac{\delta_0}{2}$ for all N. The mistake made by the imperfect quantizer cannot be recovered by computing more bits, in contrast to the self-correcting property of the ΣΔ-scheme. In order to obtain good precision overall with the binary quantizer , one must therefore impose very strict requirements on τ , which would make such quantizers very expensive in practice (or even impossible if τ is too small). On the other hand [73], ΣΔ-quantizers are robust under such imperfections of the quantizer, allowing for good precision even if cheap quantizers are used (corresponding to less stringent restrictions on). It is our understanding that it is this feature that makes ΣΔ-schemes so successful in practice.

It would be better, however, to see the approximation error decay faster with λ, faster even than the $\lambda^{-\frac{3}{2}}$ estimate conjectured to hold for first order ΣΔ-quantization of bandlimited functions. For this faster decay we must turn to higher order schemes.

Proposition$(\mathbf{5.1.7})[\mathbf{49}]$**:**Take $f \in C_1$, take $\lambda > 1$, and suppose g satisfies (6). Suppose that the $q_n^\lambda \in \{-1, 1\}$ are such that there exists a bounded sequence $(u_n)_{n\in\mathbb{Z}}$ for which

$$f\left(\frac{n}{\lambda}\right) - q_n^{\lambda} = \Delta_n^k(u) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} u_{n-l} \,. \tag{22}$$

Then , for all $x \in \mathbb{R}$,

$$\left| f(t) - \frac{1}{\lambda} \sum_{n\in\mathbb{Z}} q_n^{\lambda} g\left(t - \frac{n}{\lambda}\right) \right| \le \frac{1}{\lambda^k} \|u\|_{l^\infty} \left\| \frac{d^k g}{dt^k} \right\|_{L^1} . \tag{23}$$

Proof : It follows from (22) that

$$\left| f(t) - \frac{1}{\lambda} \sum_{n} q_n^{\lambda} g\left(t - \frac{n}{\lambda}\right) \right| = \frac{1}{\lambda} \left| \sum_{n} \Delta_n^k(u) g\left(t - \frac{n}{\lambda}\right) \right|$$

$$= \frac{1}{\lambda} \left| \sum_{n} u_n \Delta_n^{-k} \left(g\left(t - \frac{\cdot}{\lambda}\right) \right) \right| , \tag{24}$$

Where Δ^{-k} is the k-th order forward difference. Thus (see [59, p. 137]),

$$\Delta_n^{-k} \left(g\left(t - \frac{\cdot}{\lambda}\right) \right) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} g\left(t - \frac{n+l}{\lambda}\right)$$

$$= (-1)^k \frac{1}{\lambda^{k-1}} \int_0^{k/\lambda} g^{(k)}\left(t - \frac{n+k}{\lambda} + s\right) \phi_k(\lambda s)\, ds , \tag{25}$$

where ϕ_k is the k-th order B-spline, $\phi_k = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ (k convolution factors). Note that ϕ_k is positive, and supported on $[0, k]$ (so that we can just as well replace the integration limits by $-\infty$ and ∞). Moreover ,$\sum_{m\in\mathbb{Z}} \phi_k(y + m) = 1$ for all $y \in \mathbb{R}$. It follows that we can estimate

$$\left| f(t) - \frac{1}{\lambda} \sum_{n\in\mathbb{Z}} q_n^{\lambda} g\left(t - \frac{n}{\lambda}\right) \right|$$

$$\le \frac{1}{\lambda^k} \|u\|_{l^\infty} \sum_{n} \int_{-\infty}^{\infty} \left| g^{(k)}\left(t - \frac{n+k}{\lambda} + s\right) \right| \phi_k(\lambda s)\, ds$$

$$= \frac{1}{\lambda^k} \|u\|_{l^\infty} \int_{-\infty}^{\infty} \left| g^{(k)}(y) \right| \phi_k(\lambda y - \lambda t + n + k)\, dy$$

$$= \frac{1}{\lambda^k} \|u\|_{l^\infty} \left\| g^{(k)} \right\|_{L^1} .$$

The key to better decay in λ for the approximation rate is thus to construct algorithms of type (22) with $k > 1$ and uniformly bounded u_n . A ΣΔ algorithm which has such uniform bounds on the “internal state variables” is called “stable” in the electrical engineering literature; see e.g. [66]. We are thus concerned here with establishing the existence of stable ΣΔ schemes of arbitrary order. We first discuss the cases $k = 2$ and 3, before proceeding to general k. We shall consider the recursion

$$\begin{cases} v_n = v_{n-1} + x_n - q_n \\ u_n = u_{n-1} + v_n \\ q_n = sign[F(u_{n-1}, v_{n-1}, x_n)], \end{cases} \tag{26}$$

where the function F still needs to be specified. We are interested in applying this to the case where the x_n are samples of a function $f \in C_1$; however, our discussion of the boundedness of u_n, v_n is valid for arbitrary input sequences $(x_n)_{n\in\mathbb{Z}}$, provided $|x_n| \leq a < 1$.

Several choices for F have been considered in the literature; see e.g. [58]. One family of choices described in [58] is

$$F(u,v,x) = \gamma u + v + x, \tag{27}$$

where γ is a fixed parameter. A detailed discussion of the mathematical properties of this family is given in [70]. Another very interesting choice, proposed by N. Thao [69], is

$$F(u,v,x) = \frac{6x - 7sign(x)}{3} + \left(v + \frac{x + 3sign(x)}{2}\right)^2 + 2(1 - |x|)u\,. \tag{28}$$

In both cases, one can prove that there exists a bounded set $A_a \subset \mathbb{R}^2$ so that if $|x_n| \leq a$ for all n, and $(u_0, v_0) \in A_a$, then $(u_n, v_n) \in Aa$ for all $n \in \mathbb{N}$; see [70]. It follows that we have uniform boundedness for the u_n if $x_n = f\left(\frac{n}{\lambda}\right)$ for bandlimited f with $\|f\|_{L^\infty} \leq a$, implying $a\,\lambda^{-2}$ bound according to (23). As in the first order case, it turns out that for (28) this λ^{-2} bound can be improved by a more detailed analysis; for constant input one achieves, in a root-mean squared sense, $a\,\lambda^{-9/4+\epsilon}$ bound. Numerical observations suggest that this result can be improved to $a\,\lambda^{-5/2}$ decay rate for appropriately "balanced" F; they also suggest that this result can be extended to general band-limited functions (instead of constants). We refer to[70], [74], [75] for a detailed analysis and discussion of these schemes.

Robustness is an issue for second-order (and higher-order) schemes, just as it was for the first-order case. In fact, the problem becomes trickier because the quantization scheme should be able to deal not only with imperfect quantizers, but also with imprecisions in the multiplicative factors defining F in (28) (below). The analysis in [70] shows that we do indeed have such robustness, for a wide family of second-order sigma-delta schemes.

Proving more refined bounds than (23) for higher order $\Sigma\Delta$ schemes, even for constant input, turns out to be much harder than for first order (where already the analysis leading to (16) is highly nontrivial – see [62], [63]). This is mainly because even for $x_n \equiv x$ constant, the dynamical system (26) is much more complex than (8). In particular, the map

$$R_{1,x} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$u \longmapsto u + x - sign(u + x)$$

has $[-1,1]$ as an invariant set, regardless of the value of $x \in [-1,1]$. In contrast , the maps

$$R_{2,x} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\binom{u}{v} \mapsto \begin{pmatrix} u + x - sign\left(u + \frac{v}{2} + x\right) \\ v + u + x - sign\left(u + \frac{v}{2} + x\right) \end{pmatrix} \tag{29}$$

have invariant sets Γ_x that depend on the value of $x \in (-1,1)$. The sets Γ_x have fascinating properties which are still poorly understood; for instance, for each fixed x, Γ_x seems to be a tile for $\mathbb{R}^2$ under translations by $2\mathbb{Z}^2$. (This tiling property is observed for many F, and we conjecture that it holds for a large family of F, even though we can prove only a few special cases – see below.) For $x \neq 0$, the Γ_x for (27) can have interesting fractal boundaries; for "large" x, these Γ_x are disconnected.

On the other hand, the sets Γ_x for (28) are connected neighborhoods of (0, 0) bounded by four parabolic arcs ; because of the explicit characterization of these sets, a proof that the $2\mathbb{Z}^2$-translates of Γ_x tile $\mathbb{R}^2$ is straightforward in this case. The smoothness of the boundaries also makes it possible to refine (23) for this choice constant input (see [74]).

Neither of the two schemes (27) or (28) is easy to generalize to higher order. We shall therefore concentrate our attention here on yet another choice for F,

$$F(u,v,x) = v + x + M \operatorname{sign}(u), \tag{30}$$

with $M > 1$ to be fixed below. In addition, we shall also allow the *sign*functions in (26) and (30) to be imperfect quantizers, and the multiplication by M to be imperfect as well. Our recursion thus reads, for $n > 0$,

$$\begin{cases} v_n = v_{n-1} + x_n - q_n \\ u_n = u_{n-1} + v_n \\ q_n = Q_n^1[v_{n-1} + x_n + M(1 + \epsilon_n)Q_n^2(u_{n-1})] , \end{cases} \tag{31}$$

where we assume that Q_n^1 , Q_n^2satisfy (2), and $|\epsilon_n| \leq \mu < 1$.

The approach in [19] can be used to show that this second-order recursion does produce uniformly bounded u_n , v_n . We shall provide a different argument here, that, unlike the analysis in [19], generalizes to arbitrary order. Prescribing initial values u_0 , v_0 (or equivalently u_0, u_{-1}) the recursion (31) determines q_n, u_n, $v_n, n \geq 1$. In addition, we also need to give a prescription for $n \leq 0$. Observe that the equations for u_n, v_n can be rewritten as

$u_n = 2u_{n-1} - u_{n-2} + x_n - q_n$; this suggests a symmetry between u_n and u_{n-2}. We use this to define the following recursion for u_n, q_n with $n < 0$,

$$\begin{cases} u_n = 2u_{n+1} - u_{n+2} + x_{n+2} - q_{n+2} \\ q_{n+2} = Q_n^1[u_{n+1} - u_{n+2} + x_{n+2} + M(1 + \varepsilon_n)Q_n^2(u_{n+1})], \end{cases}$$

to be used for $n \leq -2$. If we introduce also $v_n = u_n - u_{n+1}$ for $n < 0$, this becomes

$$\begin{cases} v_n = v_{n+1} + x_{n+2} - q_{n+2} \\ u_n = u_{n+1} + v_n \\ q_{n+2} = Q_n^1[v_{n+1} + x_{n+2} + M(1+\varepsilon_n)Q_n^2.(u_{n+1})], \end{cases} \quad (32)$$

We define $v_{-1} = -v_0$ and use this together with the already prescribed values u_0, u_{-1} in (32). This recursion will then serve to determine the values of q_j, u_j, v_j for $j \le 0$. The sequences $(u_n), (q_n)$ will then satisfy, for all n,

$$\Delta^2 u_n = x_n - q_n.$$

We introduce an algorithm to generate q_n for $n < 0$ because our approximation formula (5), using g supported on all of $\mathbb{R}$, requires them; in practice one uses only compactly supported G, and q_n with $n \le 0$ are not needed. Since the negatively-indexed q_n are kept for only theoretical reasons, we would be justified in keeping the sign function"clean" in their recursion, i.e. without the $Q_n^1, Q_n^2, \varepsilon_n$ "imperfections"; we left them in for the sake of generality. It is clear, by comparing (32) with (31), that if we can prove that (31) implies uniform bounds on $|u_n|, |v_n|$ for $n > 0$, starting from some initial condition $|u_0| \le U_0, |v_0| \le V_0$ (with U_0, V_0 to be determined), then the same uniform bounds on $|u_n|, |v_n|$ for $n < 0$ will follow, provided $|u_{-1}| \le U_0, |v_{-1}| \le V_0$. Since $v_{-1} = -v_0$, we need to impose only the additional constraint $|u_0 + v_{-1}| = |u_0 - v_0| \le U_0$ for this to hold. This will allow us to restrict our arguments to the $n > 0$ case. We then have:

Lemma (5.1.8)[49]: If $|v_0| \le M(1+\mu) + 1 + \tau$, then $|v_n| \le M(1+\mu) + 1 + \tau$ for all $n \in \mathbb{Z}$.

Proof: By induction. Suppose $|v_{n-1}| \le M(1+\mu) + 1 + \tau$. If $|v_{n-1} + x_n| > M(1+\epsilon_n) + \tau$, then

$$|v_n| = |v_{n-1} + x_n - Q_n^1(v_{n-1} + x_n)| = |v_{n-1} + x_n| - 1$$
$$\le |v_{n-1}| + a - 1 < M(1+\mu) + 1 + \tau,$$

where we have used that $|v_{n-1} + x_n| > \tau$. If $|v_{n-1} + x_n| \le M(1+\epsilon_n) + \tau$, then $|v_n| \le |v_{n-1} + x_n| + 1 \le M(1+\epsilon_n) + \tau + 1 \le M(1+\mu) + 1 + \tau$.

Lemma(5.1.9)[49]: Suppose $u_k \le \tau$, and $u_{k+1}, u_{k+2}, \ldots, u_{k+L} > \tau$. Define κ to be the smallest integer strictly larger than $\frac{2M}{1-a} + 1$. If $L \ge \kappa$, then there exists at least one $l \in \{1, \ldots, \kappa\}$ such that $v_{k+l} + x_{k+l+1} < -M(1-\mu) + 1 + a + \tau$.

Proof: Suppose $v_{k+1} + x_{k+2}, \ldots, v_{k+\kappa-1} + x_{k+\kappa}$ are all $\ge -M(1-\mu) + 1 + a + \tau$. Because $u_{k+1}, \ldots, u_{k+\kappa-1}$ are all $> \tau$, we have $q_{k+2} = \ldots = q_{k+\kappa} = 1$, which implies

$$\begin{aligned} v_{k+\kappa} + x_{k+\kappa+1} &= v_{k+1} + \sum_{l=2}^{\kappa}(x_{k+l} - q_{k+l}) + x_{k+\kappa+1} \\ &\le M(1+\mu) + 1 + \tau + (\kappa-1)(a-1) + a \\ &< M(1+\mu) + 1 + \tau + a - (1-a)\frac{2M}{1-a} \\ &= -M(1-\mu) + 1 + a + \tau. \end{aligned}$$

Lemma $(\mathbf{5.1.10})[\mathbf{49}]$**:** Let $u_k, u_{k+1}, \dots, u_{k+L}$ be as in Lemma (5.1.9). If

$$v_{k+l} + x_{k+l+1} < -M(1-\mu) + 1 + a + \tau$$

for some $l \in \{1, \dots, L\}$, then for all l' satisfying $l \le l' \le L$,

$$v_{k+l'} + x_{k+l'+1} < -M(1-\mu) + 1 + a + \tau\,.$$

Proof: By induction. Suppose $v_{k+n} + x_{k+n+1} < -M(1-\mu) + 1 + a + \tau$ with $n \in \{1, \dots, L-1\}$; we prove that this implies $v_{k+n+1} + x_{k+n+2} < -M(1-\mu) + 1 + a + \tau$. If $v_{k+n} + x_{k+n+1} \ge -M(1+\varepsilon_{n+k+1}) + \tau$, then $q_{k+n+1} = 1$ (since $u_{k+n} > \tau$), hence

$$\begin{aligned} v_{k+n+1} + x_{k+n+2} &< -M(1-\mu) + 1 + a + \tau - 1 + x_{k+n+2} \\ &< -M(1-\mu) + 1 + a + \tau. \end{aligned}$$

On the other hand, if

$$v_{k+n} + x_{k+n+1} < -M(1+\varepsilon_{n+k+1}) + \tau,$$

then

$$\begin{aligned} v_{k+n+1} + x_{k+n+2} &< -M(1+\varepsilon_{n+k+1}) + \tau + 1 + x_{k+n+2} \\ &\le -M(1-\mu) + 1 + a + \tau. \end{aligned}$$

Lemma $(\mathbf{5.1.11})[\mathbf{49}]$**:** Let $u_k, u_{k+1}, \dots, u_{k+L}$ be as above. Then the v_{k+l} decrease monotonically in l, *with* $v_{k+l-1} - v_{k+l} \ge 1 - a$, until $v_{k+l} + x_{k+l+1}$ drops below $-M(1-\mu) + 1 + a + \tau$. All subsequent $v_{k+l'}$ with $l' \le L$ remain negative.

Proof: As long as $v_{k+n} + x_{k+n+1} \ge -M(1-\mu) + 1 + a + \tau$ with $n \le L$, we have $q_{k+n+1} = 1$, so $v_{k+n} - v_{k+n+1} = -x_{k+n+1} + 1 \ge 1 - a$. If $v_{k+l} + x_{k+l+1} < -M(1-\mu) + 1 + a + \tau$, then $v_{k+l'} + x_{k+l'+1} < -M(1-\mu) + 1 + a + \tau$ by Lemma (5.1.10) if $l \le l' \le L$, so that $v_{k+l'} < -M(1-\mu) + 1 + 2a + \tau \le 0$.

Proposition$(\mathbf{5.1.12})[\mathbf{49}]$**:** Suppose $|x_n| \le a < 1$ for all $n \in \mathbb{Z}$. Let u_n, v_n, and q_n be defined as in (31) and (32), *with* $M \ge \frac{2a+\tau+1}{1-\mu}$. Then, if $|v_0| \le M(1+\mu) + 1 + \tau$, there exists $|v_n| \le M(1+\mu) + 1 + \tau$ for all $n \in \mathbb{Z}$. Moreover, if $|u_0|, |v_0| \le \tau/2$, then $|u_n| \le \tau + \frac{[M(1+\mu)+\tau+3/2-a/2]^2}{2(1-a)}$ for all $n \in \mathbb{Z}$.

Proof: We first discuss the case $n > 0$. The bound on v_n is proved in Lemma (5.1.8), we now turn to u_n. Suppose $u_{k+1}, \dots, u_{k+L}$ is a stretch of $u_n > \tau$, preceded by $u_k \le \tau$. We have then, for all $m \in \{1, \dots, L\}$,

$$u_{k+m} = u_k + \sum_{l=1}^{m} v_{k+l} \le \tau + \sum_{l=1}^{m} v_{k+l}\,.$$

By Lemma (5.1.11), these v_{k+l} decrease monotonically by at least $(1-a)$ at every step until they drop below a certain negative value, after which they stay negative. Consequently, $u_{k+l} \le u_{k+1} - (1-a)(l-1) \le M(1+\mu) + 1 + \tau - (1-a)(l-1)$, at least until this last expression drops below zero. It follows that

$$u_{k+m} \le \tau + \max_{n\ge 1} \sum_{l=1}^{n} [M(1 + \mu) + 1 + \tau - (1 - a)(l - 1)] \quad (33)$$

$$\le \tau + \frac{[M(1 + \mu) + 3/2 - 1/2 + \tau]^2}{2(1 - a)}$$

The initial condition $|u_0| \le \tau/2$ ensures that the upper bound (33) holds for all $u_n, n \ge 0$. The lower bound $, u_n \ge -\tau - \frac{[M(1+\mu)+3/2-a/2+\tau]^2}{2(1-a)}$ for $n \ge 0$, is proved entirely analogously.

To treat $n < 0$, note that the "initial conditions" for the recursion (32) satisfy $|v_{-1}| = |v_0| \le \tau/2$, and $|u_{-1}| = |u_0 - v_0| \le \tau$. It follows that we can repeat the same arguments to derive an identical bound on $|u_n|$ for $n \le -1$.

A third-order $\Sigma\Delta$ scheme. Let us consider the construction we discussed for second order, but take it one step further. For $n > 0$ define the recursion

$$\begin{cases} u_n^{(1)} = u_{n-1}^{(1)} + x_n - q_n \\ u_n^{(2)} = u_{n-1}^{(2)} + u_n^{(1)} \\ u_n^{(3)} = u_{n-1}^{(3)} + u_n^{(2)} \\ q_n = Q_n^1 \left[u_{n-1}^{(1)} + x_n + M_1(1 + \varepsilon_n^1) Q_n^2 \left(u_{n-1}^{(2)} + M_2(1 + \varepsilon_n^2) Q_n^3 \left(u_{n-1}^{(3)}\right)\right)\right] \end{cases} \quad (34)$$

where Q_n^1, Q_n^2, Q_n^2 satisfy (19) , $|\varepsilon_n^1|, |\varepsilon_n^2| \le \mu$, and where M_1, M_2 will be fixed below in such a way as to ensure uniform boundedness of the $(|u_n^{(3)}|)_{n\in\mathbb{N}}$, provided we start from appropriate initial conditions $u_0^{(1)}, u_0^{(2)}, u_0^{(3)}$. We assume again that $|x_n| \le a < 1$ for all $n \ge 0$.

We shall keep this discussion to a sketch only; a formal proof of this third order case will be implied by the formal proof for arbitrary order in the next.

This preliminary discussion will help us understand the more general construction, however.

First of all, exactly the same argument as in the proof of Lemma (5.1.8) establishes that $| u_n^{(1)} | \le M_1(1 + \mu) + 1 + \tau = M'_1$.

Next, imagine a long stretch of $u_{n+1}^{(2)}, u_{n+2}^{(2)}, \ldots,$ all $> M_1(1 + \mu) + 1 + \tau$. Then the corresponding q_{n+l+1} are all automatically equal to 1, unless $u_{n+l}^{(1)} + x_{n+l} < -M1(1 + \epsilon_{n+l}^1) + \tau$. Arguments similar to those in the proofs of Lemmas (5.1.9),(5.1.10),(5.1.11) then show that if $u_{n+1}^{(1)} > -M_1(1 - \mu) + 1 + a + \tau \ge 0$, the $u_{n+l}^{(1)}$ will decrease monotonically, by at least $(1 - a)$ at each step, until $u_{n+l}^{(1)} + x_{n+l+1}$ drops below $-M_1(1 - \mu) + 1 + a + \tau$ (in at most $\kappa_1 = \left\lfloor \frac{2M_1}{1-a} \right\rfloor + 2$ steps), after which all the subsequent $u_{n+l'}^{(1)}$ in the stretch are negative, provided we chose $M_1 \ge \frac{1+2a+\tau}{1-\mu}$. As before, this argument leads to

$$|u_n^{(2)}| \le M'_2 = M_2(1+\mu) + \tau + \frac{M'_1 + (1-a)/2}{2(1-a)} .$$

One could then imagine repeating the same argument again to prove the desired bound on the $|u_n^{(3)}|$: prove that if one has a long stretch of $u_{l+1}^{(3)},\dots,u_{l+L}^{(3)}$ that are all positive, then necessarily the corresponding $u_{l+m}^{(2)}$ must dip to negative values and remain negative, in such a way that the total possible growth of the $u_{l+m}^{(3)}$ must remain bounded. We will have to make up for a missing argument, however: when we followed this reasoning at the previous level, we were helped by the a priori knowledge that consecutive $u_n^{(1)}$ just differ by some minimal amount, $|u_{n+1}^{(1)} - u_n^{(1)}| \ge 1-a$. We used this to ensure a minimum speed for the dropping $u_{l+m}^{(1)}$, and thus to bound the $u_{l+m}^{(2)}$. In our present case, we have no such a priori bound on $|u_{n+1}^{(2)} - u_n^{(2)}|$, so that we need to find another argument to ensure sufficiently fast decrease of the $u_{l+m}^{(2)}$. What follows sketches how this can be done.

Suppose $u_l^{(3)} \le \tau, u_{l+1}^{(3)},\dots,u_{l+L}^{(3)} > \tau$. Then we must have, within the first κ_2 indices of this stretch (with κ_2, independent of L, to be determined below) that some $u_{l+m}^{(2)} \le -M_2(1-\mu)+\tau$. Indeed, if $u_{l+1}^{(2)},\dots,u_{l+\kappa_2-1}^{(3)} > -M_2(1-\mu)+\tau$, then the corresponding q_{l+m} are 1, unless $u_{l+m}^{(1)} - 1 < -M_1(1-\mu)+a+\tau$. As before, this forces the $u_{l+m}^{(1)}$ down, until they hit below $-M_1(1-\mu)+a+\tau$ in at most κ_1 steps, after which they remain below this negative value. This forces the $u_{l+m}^{(2)}$ to decrease, and one can determine κ_2 so that if $u_{l+1}^{(2)},\dots,u_{l+\kappa_2-1}^{(2)} > -M_2(1-\mu)+\tau$, then $u_{l+\kappa_2}^{(2)} \le -M_2(1-\mu)+\tau$ must follow. Once $u_{l+l'}^{(2)}$ has dropped below $-M_2(1-\mu)+\tau$, the picture changes. We can get $q_{l+l'+k} = -1$, and the argument that kept the $u_{l+m}^{(1)}$ down can then no longer be applied. In fact, some of the $u_{l+m}^{(1)}$ with $m > l'$ may exceed τ again, causing the $u_{l+m}^{(2)}$ toincrease. However, as soon as we have κ_1 consecutive $u_n^{(2)} > -M_2(1-\mu)+\tau$,we must have, for at least one of the corresponding indices, that $u_n^{(1)} < -M_1(1-\mu)+1+a+\tau$, which forces the subsequent $u_n^{(1)}$ below this value too, and we are back in our cycle forcing the $u_n^{(2)}$ down, until they hit below $-M_2(1-\mu)+\tau$. So if $-M_2(1-\mu)+\tau+\kappa_1 M'_1 \le 0$, then the $u_n^{(2)}$ do not get a chance to grow to positive values within the first κ_1 indices after $u_{l+l'}^{(2)} < -M_2(1-\mu)+\tau$ .This forces all the $u_{l+m}^{(2)}$ to be negative for $m = l'+1,\dots,L$; since $l' \le \kappa_2$,this then leads, by the same argument as on the previous level, to a bound on $u_{l+m}^{(3)}$.

We present this argument formally, for schemes of arbitrary order; the proof consists essentially of careful repeats of the last paragraph at every level. This then also leads to estimates for the bounds M_j' ,and corresponding conditions on the M_j. We assume again that $|x_n| \leq a < 1$ for all $n \in \mathbb{N}$. To define the $\Sigma\Delta$ scheme of order J for which we shall prove uniform boundedness of all internal variables, we need to introduce a number of constants. As before, the $\Sigma\Delta$ -scheme will use nonideal quantizers with an inherent imprecision limited by τ , and all the multipliers in the algorithm will be known only up to a factor $(1 + \epsilon)$, where $|\epsilon| \leq \mu < 1$. We pick α so that $2\alpha < 1 - \mu$, and we define

$$M_1 = 2\frac{1 + a + \tau}{1 - \mu} \qquad \kappa_1 = \left\lfloor\frac{2M_1 + 1 + a}{1 - a}\right\rfloor + 2$$

$$B = \frac{4}{1 - \mu - \alpha} \qquad M_j = M_1 B^{j-1} \nu^{(j-1)^2} \tag{35}$$

$$\nu = \left\lfloor \max\left(\frac{4B}{\kappa_1(1 - \mu)} + \frac{\kappa_1^2}{B}, 1 + \frac{B(3 - \alpha - \mu)}{\alpha\kappa_1}, \kappa_1\right)\right\rfloor + 1$$

where j ranges from 1 to J. For $n \geq 0$, the scheme itself is then defined as follows

$$\begin{cases} u_n^{(1)} = u_{n-1}^{(1)} + x_n - q_n \\ u_n^{(j)} = u_{n-1}^{(j)} + u_n^{(j-1)} ,j = 2,\dots,J \\ q_n = Q_n^1\{u_{n-1}^{(1)} + M_1(1 + \epsilon_n^1)Q_n^2[u_{n-1}^{(2)} + M_2(1 + \epsilon_n^2)Q_n^3(u_{n-1}^{(3)} + \cdots \\ \qquad +M_{J-2}\left(1 + \epsilon_n^{J-2}\right)Q_n^{J-1}(u_{n-1}^{(J-1)} + \cdots \\ \qquad +M_{J-1}(1 + \epsilon_n^{J-1})Q_n^J(u_{n-1}^{(J)})) \cdots)]\} , \end{cases} \tag{36}$$

where $|\varepsilon_n^1|, |\varepsilon_n^1|, \dots, |\varepsilon_n^{J-1}| \leq \varepsilon$ and $Q_n^1, \dots Q_n^J$ satisfy (19) for all n. We start with initial conditions $u_0^{(1)}, \dots, u_0^{(J)}$, and we apply (36) recursively to determine $q_j, u_j^{(1)}, \dots, u_j^{(J)}$, for $j = 1, 2, \dots$. Prescribing these initial conditions is equivalent to prescribing $u_0^{(J)}, \dots, u_{-J+1}^{(J)}$.

For $n < 0$, we mirror this system, obtaining

$$\begin{cases} u_n^{(1)} = u_{n+1}^{(1)} + (-1)^J \left(x_{n+J} - q_{n+J}\right) \\ u_n^{(j)} = u_{n+1}^{(j)} + u_n^{(j-1)} ,j = 2,\dots,J \\ q_{n+J} = (-1)^J Q_n^1\{u_{n+1}^{(1)} + M_1(1 + \epsilon_n^1)Q_n^2[u_{n+1}^{(2)} + M_2(1 + \epsilon_n^2)Q_n^3(u_{n+1}^{(3)} + \cdots \\ \qquad +M_{J-2}\left(1 + \epsilon_n^{J-2}\right)Q_n^{J-1}(u_{n+1}^{(J-1)} + M_{J-1}(1 + \epsilon_n^{J-1})Q_n^J(u_{n+1}^{(J)})) \cdots)]\}. \end{cases} \tag{37}$$

To set the recursion running for $n < 0$, we prescribe the mirrored initial conditions $u_{-J+1}^{(j)} = \sum_{l=1}^{j}(-1)^{j-l}u_0^{(l)}\binom{j - 1}{l - 1}$. These conditions are chosen to guarantee that $u_0^{(J)}, \dots, u_{-J+1}^{(J)}$ are given the same values as in the prescription for the forward recurrence. We now use (37) recursively to generate the $q_n, n \leq 0$. If we take, for simplicity, $u_0^{(j)} = 0$ for $j = 1,\dots J$, then the "initial conditions" for the $n < 0$ recursion have likewise $u_{-J+1}^{(j)} = 0$ for $j = 1,\dots J$. If we relax our constraints on the initial conditions somewhat, imposing $u_0^{(j)} \leq A_j$ for appropriate A_j , then we also impose that $\left|\sum_{l=1}^{j}(-1)^{j-l}u_0^{(l)}\binom{j-1}{l-1}\right| \leq A_j$ .In both cases, one readily sees, as before, that the proof of a uniform bound for the $|u_n^{(J)}|$ in the $n > 0$ recursion simultaneously provides the same uniform bound for the $|u_n^{(J)}|$ in the $n < 0$ recursion.

Proposition(5.1.13)[49]: Suppose $|x_n| \leq a < 1$ for all $n \in \mathbb{Z}$. *Let* M_j for $j = 1,\dots,J,$ be defined as in (35), let the imperfect quantizers $Q_n^1,\dots Q_n^J$ Satisfy (36) for all $n \in \mathbb{Z}$, and let the sequences $(q_n)_{n\in\mathbb{Z}}$ and $\left(u_n^{(j)}\right)_{n\in\mathbb{Z}}$ $,j = 1,\dots,J,$ be as defined by (34) or (37), with initial conditions $u_0^{(j)} = 0$ for $j = 1,\dots,J$. Then $|u_n^{(J)}| \leq (2-\alpha)M_1B^{J-1}\nu^{(J-1)^2}$ for all $n \in \mathbb{Z}$.

The proof of Proposition (5.1.13) is essentially along the lines sketched for the third-order case, albeit more technical in order to deal with general J. The whole argument is one big induction on j. We start by stating two lemmas for the lowest value of j, to start off the induction argument.

Lemma(5.1.14)[49]: $|u_n^{(1)}| \leq M_1(1+\mu)+1+a+\tau$ for all $n \in \mathbb{Z}$.

Proof: The argument is very similar to that used in the proof of Lemma (5.1.8), except that x_n does not appear in the definition of q_n.We work by induction. Suppose $|u_{n-1}^{(1)}| \leq M_1(1+\mu)+1+a+\tau$. If $|u_{n-1}^{(1)}| > M_1(1+\epsilon_n^1)+\tau$, then q_n and $u_{n-1}^{(1)}$ have the same sign, so that $|u_n^{(1)}| \leq |u_{n-1}^{(1)}|-1+|x_n| \leq |u_{n-1}^{(1)}|-1+a \leq |u_{n-1}^{(1)}| \leq M_1(1+\mu)+1+a+\tau$. If $|u_{n-1}^{(1)}| \leq M_1(1+\epsilon_n^1)+\tau$,then $|u_n^{(1)}| \leq |u_{n-1}^{(1)}|+1+a \leq M_1(1+\mu)+1+a+\tau$.

Lemma(5.1.15)[49]: If $u_{n+1}^{(2)},\dots,u_{n+N}^{(2)} > M_2(1+\mu)+\tau$, with $N \geq \kappa_1$, then there must exist $l \in \{1,\dots,\kappa_1\}$ such that $u_{n+l}^{(1)} < -M_1(1-\mu)+\tau$. Moreover, for all $l' \in \{l,\dots,N\}, u_{n+l'}^{(1)} < -M_1(1-\mu)+\tau+1+a$. A similar statement holds if $u_{n+1}^{(2)},\dots,u_{n+N}^{(2)} < -M_2(1+\mu)-\tau$, and other signs are reversed accordingly.

Proof: The argument is again similar to the proofs of Lemmas (5.1.9),(5.1.10). Suppose $u_{n+1}^{(1)},\dots,u_{n+k_1-1}^{(1)}$ are all $\geq -M_1(1-\mu)+\tau$. Then we have $q_{n+2} =\cdots= q_{n+k_1} = 1$. Hence

$$u_{n+k_1}^{(1)} = u_{n+1}^{(1)} + \sum_{l=2}^{k_1}(x_{n+l}-q_{n+l})$$
$$\leq M_1(1+\mu)+1+a+\tau-(k_1-1)(1-a) < -M_1(1-\mu)+\tau .$$

This establishes that $u_{n+l}^{(1)} < -M_1(1-\mu)+\tau$ for some $l \in \{1,\dots,k_1\}$.

Next, suppose that $u_{n+r}^{(1)} < -M_1(1-\mu)+\tau+1+a$, for some r with

$l \le r \le N-1$. If $u^{(1)}_{n+r} \ge -M_1(1-\mu)+\tau$, then $q_{n+r+1} = 1$, hence

$$u^{(1)}_{n+r+1} = u^{(1)}_{n+r} + x_{n+r+1} - 1 < u^{(1)}_{n+r} < -M_1(1-\mu) + \tau + 1 + a\,,$$

if $u^{(1)}_{n+r} < -M_1(1-\mu)+\tau$, then

$$u^{(1)}_{n+r+1} < -M_1(1-\mu) + \tau + 1 + |x_{n+r+1}| \le -M_1(1-\mu) + \tau + 1 + a.$$

In both cases, $u^{(1)}_{n+r+1} < -M_1(1-\mu)+\tau+1+a$, and we continue by induction.

Next we introduce auxiliary constants, for $j = 1,\ldots,J$:

$$\kappa_j = \nu^{2(j-1)}\kappa_1$$

$$M_1' = (1+\mu)M_1 + \tau + 1 + a\,,\ M_j' = (1+\mu)\,M_j + \tau + \kappa_{j-1}M_{j-1}' \text{ for } j \ge 2$$

$$M_1'' = (1-\mu)M_1 - \tau - 1 - a\,,\ M_j'' = (1-\mu)\,M_j - \tau - \kappa_{j-1}M_{j-1}'' \text{ for } j \ge 2 \qquad (38)$$

$$(\widetilde{M}_j = M_j(1+\mu) + \tau$$

$$\widetilde{m}_j = M_j(1-\mu) - \tau\ .$$

These have been tailored so that

Lemma $(5.1.16)$[49]: The constants defined above by (37) satisfy, for $j = 2,\ldots,J$,

$$(1-\mu)\,M_j > \tau + \kappa_{j-1}(2-\alpha)\,M_{j-1}\ , \qquad (39)$$

$$M_j' \le (2-\alpha)\,M_j \qquad (40)$$

$$\kappa_j - \kappa_{j-1} \ge \frac{\widetilde{m}_j + M_j'}{M_{j-1}''}\ . \qquad (41)$$

Proof: The first equation is proved by straight substitution:

$$\begin{aligned}
&(1-\mu)\,M_j - \tau - \kappa_{j-1}(2-\alpha)M_{j-1} \\
&= B^{j-1}\nu^{(j-1)^2}M_1\left[1-\mu-\frac{\tau}{\nu^{(j-1)^2}B^{j-1}M_1}-\frac{(2-\alpha)\kappa_1}{B\nu}\right] \\
&\ge B^{j-1}\nu^{(j-1)^2}M_1\left[1-\mu-\frac{\tau/M_1+(2-\alpha)\kappa_1}{B\nu}\right] \qquad (42)\\
&\ge B^{j-1}\nu^{(j-1)^2}M_1\left[1-\mu-\frac{2(2-\alpha)(1-\alpha-\mu)}{4}\right] \ge \alpha\,M_j\ .
\end{aligned}$$

The second equation is proved by induction. First we consider the case $j = 2$:

$$M_2' - (2-\alpha)M_2 = (\mu+\alpha-1)M_2 - \tau - \kappa_1 M_1' < -\alpha M_2 - \tau - \kappa_1 M_1' < 0.$$

Now suppose that $M_j' \le (2-\alpha)\,M_j$ holds for some $j \ge 2$. Then (42) immediately implies that

$$M_{j+1}'' > (1-\mu)\,M_{j+1} - \tau - \kappa_j(2-\alpha)\,M_j \ge \alpha\,M_{j+1}\ ,$$

leading to

$$M_{j+1}' > 2\,M_{j+1} - M_{j+1}'' \le (2-\alpha)\,M_{j+1}\ .$$

It remains to prove the third inequality. Because the definition of M_{j+1}'' is slightly different for $j = 2$ than for $j > 2$, we handle the case $j = 2$ separately.Now

$$\begin{aligned}
M_1''(\kappa_2-\kappa_1) - \widetilde{m}_2 - M_2' &= M_1''\kappa_2 - 2\,M_1\kappa_1 - 2\,M_2 \\
&= (a+1+\tau)\nu^2\kappa_1 - 2\,M_1\kappa_1 - 2\nu B\,M_1
\end{aligned}$$

$$= (\alpha + 1 + \tau)\left[\nu\left(\nu\kappa_1 - \frac{4B}{1-\mu}\right) - \frac{4\kappa_1}{1-\mu}\right] > 0\,,$$

where we have used $\nu > \frac{4B}{\kappa_1(1-\mu)} + \kappa_1^2 B$.

For $j > 2$ we use $M_j' \le (2-\alpha)\,M_j$ and $M_{j-1}'' \ge \alpha\, M_{j-1}$ to upper bound the right-hand side of (40), and we replace the various κ_s and M_s by their definitions; then we see that the equation holds if $\nu\kappa_1(1-\nu^{-2}) \ge B(3-\alpha-\mu)\alpha^{-1}$, or, equivalently, if $\nu^2 \ge B(3-\alpha-\mu)\nu(\alpha\kappa_1)^{-1} + 1$. From the definition of ν one easily checks that this is indeed the case, completing the proof.

Corollary $(\mathbf{5.1.17})[\mathbf{296}]$: Show that

$$\lambda > \frac{\ln 1}{\ln q_n}.$$

Proof : $\left|f\left(\frac{n}{\lambda}\right)\right| + |u_{n-1}| + \left|sign\left(f\left(\frac{n}{\lambda}\right) + u_{n-1}\right)\right| < 1\,,$

$$1 + 1 + \left|sign\left(f\left(\frac{n}{\lambda}\right) + u_{n-1}\right)\right| < 1\,,$$

$$2 + \left|sign\left(f\left(\frac{n}{\lambda}\right) + u_{n-1}\right)\right| < 1\,,$$

$$\left|sign\left(f\left(\frac{n}{\lambda}\right) + u_{n-1}\right)\right| < -1$$

From (8) we get $q_n^\lambda > 1$ taking ln we get $\ln q_n^\lambda > \ln 1$ then $\lambda \ln q_n > \ln 1$ and $\lambda > \frac{\ln 1}{\ln q_n}$.

Section(5.2) Multidimensional Bandlimited Functions

The subject of recovery of bandlimited signals from discrete data has its origins in the Whittaker– Kotel'nikov –Shannon (WKS) sampling theorem (stated below), historically the first and simplest such recovery formula. Without loss of generality, the formula recovers a function with a frequency band of $[-\pi,\pi]$ given the function's values at the integers. The WKS theorem has drawbacks. Foremost, the recovery formula does not converge given certain types of error in the sampled data, as Daubechies and DeVore mention in [49]. They use oversampling to derive an alternative recovery formula which does not have this defect. Additionally for the WKS theorem, the data nodes have to be equally spaced, and nonuniform sampling nodes are not allowed. As discussed in [48], nonuniform sampling of bandlimited functions has its roots in the work of Paley, Wiener, and Levinson. Their sampling formulae recover a function from nodes $(t_n)_n$, where $\left(e^{it_nx}\right)_n$ forms a Riesz basis for $L_2[-\pi,\pi]$. More generally, frames have been applied to nonuniform sampling, particularly in the work of Benedetto and Heller in [44,45], [48].

We derive a multidimensional oversampling formula (see (46)), for nonuniform nodes and bandlimited functions with a fairly general frequency domain; investigates the stability of (46) under perturbation of the sampled data. This Section presents a computationally feasible version of (46) in the case where the nodes are asymptotically uniformly distributed. Kadec's theorem gives a criterion for the nodes $(t_n)_n$ so that $\left(e^{it_nx}\right)_n$ forms a Riesz basis for $L_2[-\pi,\pi]$.Generalizations of Kadec's 1/4 theorem to higher dimensions are considered in this Section, and an asymptotic equivalence of two generalizations is given. Investigates approximation of the biorthogonal functionals of Riesz bases. Additionally, we give a simple proof of a theorem of Levinson.

We use the d-dimensional L_2 Fourier transform

$$\mathcal{F}(f)(\cdot) = \int_{\mathbb{R}^d} f(\xi)e^{-i\langle\cdot,\xi\rangle}\, d\xi, f \in L_2(\mathbb{R})^d,$$

where the inverse transform is given by

$$\mathcal{F}^{-1}(f)(\cdot) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(\xi)e^{i\langle\cdot,\xi\rangle}\, d\xi, f \in L_2(\mathbb{R})^d,$$

This is an abuse of notation. The integral is actually a principal value where the limit is in the L_2 sense. This map is an onto isomorphism from $L_2(\mathbb{R}^d)$ to itself.

Definition$(5.2.1)[42]$: Given a bounded measurable set E with positive measure, we define

$$PW_E = \{f \in L_2(\mathbb{R}^d) \,|supp(\mathcal{F}^{-1}(f)) \subset E\}.$$

Functions in PW_E are said to be bandlimited.

Definition $(5.2.2)[42]$: The function sinc : $\mathbb{R} \to \mathbb{R}$ is defined by $sinc(x) = \frac{sin(x)}{x}$. We also define the multidimensional sinc function SINC : $\mathbb{R}^d \to \mathbb{R}^d$ by $\mathrm{SINC}(x) = \mathrm{sinc}(x_1)\cdots\mathrm{sinc}(x_d), x = (x_1,\ldots,x_d)$. We recall some basic facts about PW_E :

(i) PW_E is a Hilbert space consisting of entire functions, though in this section we only regard the functions as having real arguments.

(ii) In PW_E , L_2 convergence implies uniform convergence. This is an easy consequence of the Cauchy–Schwarz inequality.

(iii) The function $sinc(\pi(x - y))$ is a reproducing kernel for $PW_{[-\pi,\pi]}$ That is, if $\in PW_{[-\pi,\pi]}$, then we have

$$f(t) = \int_{-\infty}^{\infty} f(\tau)sinc\pi(t-\tau)d\tau \;, t \in \mathbb{R}. \tag{43}$$

(iv) The WKS sampling theorem (see [69]). If $f \in PW_{[-\pi,\pi]}$, then

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\, sinc\pi(t-n)\, , t \in R,$$

where the sum converges in $PW_{[-\pi,\pi]}$, and hence uniformly.

If $(f_n)_{n\in\mathbb{N}}$ is a Schauder basis for a Hilbert space H, then there exists a unique set of functions $(f^*{}_n)_{n\in\mathbb{N}}$ (the biorthogonalsof $((f_n)_{n\in\mathbb{N}})$such that $\langle f_n, f^*{}_m\rangle = \delta_{nm}$.

The biorthogonals also form a Schauder basis for H. Note that biorthogonality is preserved under a unitary transformation.

Definition (5.2.3)[42]: A sequence $(f_n)_n \subset H$ such that the map $Le_n = f_n$ is an onto isomorphism is called a Riesz basis for H. The following definitions and facts concerning frames are found in [57].

Definition (5.2.4)[42]: A frame for a separable Hilbert space H is a sequence $(f_n)_n \subset H$ such that for some $0 < A < B$,

$$A\|f\|^2 \leq \sum_n |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \forall f \in H. \tag{44}$$

The numbers A and B in (44) are called the lower and upper frame bounds. Let H be a Hilbert space with orthonormal basis $(e_n)_n$. The following conditions are equivalent to $(f_n)_n \subset H$ being a frame for H.

(i) The map $L : H \to H$ defined by $Le_n = f_n$ is bounded linear and onto. This map is called the synthesis operator.

(ii) The map $L^* : H \to H$ (the analysis operator) given by $f \mapsto \sum_n \langle f, f_n \rangle e_n$ is an isomorphic embedding.

Given a frame $(f_n)_n$ with synthesis operator L, the map $S = LL^*$ given by $Sf = \sum_n \langle f, f_n \rangle f_n$ is an onto isomorphism. S is called the frame operator associated to the frame. It follows that S is positive and self-adjoint. The basic connection between frames and sampling theory of bandlimited functions (more generally in a reproducing kernel Hilbert space) is straightforward. If $(e^{itn(\cdot)})_n$ is a frame for $f \in PW_{[-\pi,\pi]}$ with frame operator S, and $f \in PW_{[-\pi,\pi]}$, then

$$S(\mathcal{F}^{-1}(f)) = \sum_n \langle \mathcal{F}^{-1}(f), f_n \rangle f_n = \sum_n \mathcal{F}(\mathcal{F}^{-1}(f))(t_n) f_n$$
$$= \sum_n f(t_n) f_n,$$

implying that $\mathcal{F}^{-1}(f) = \sum_n f(t_n) S^{-1} f_n$, so that $f = \sum_n f(t_n) \mathcal{F}^{-1}(S^{-1} f_n)$. Note that in the case when $t_n = n$, we recover the WKS theorem.

Definition (5.2.5)[42]: A sequence $(f_n)_n$ satisfying the second inequality in (44) is called a Bessel sequence.

Definition (5.2.6)[42]: An exact frame is a frame which ceases to be one if any of its elements is removed. It can be shown that the notions of Riesz bases, exact frames, and unconditional Schauder bases coincide.

Definition (5.2.7)[42]: A subset S of $\mathbb{R}^d$ is said to be uniformly separated if

$$\inf_{x,y \in S, x \neq y} \|x - y\|_2 > 0.$$

Definition (5.2.8)[42]: If $S = (x_k)_k$ is a sequence of real numbers and f is a function with S in its domain, then f_S denotes the sequence $(f(x_k))_k$. In [49], Daubechies and DeVore derive the following formula

$$f(t) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right) g\left(t - \frac{n}{\lambda}\right), t \in \mathbb{R}, \quad (45)$$

where g is infinitely smooth and decays rapidly. Thus oversampling allows the representation of bandlimited functions as combinations of integer translates of g rather than the sinc function. In (45) is a generalization of the WKS theorem. The rapid decay of g yields a certain stability in the recovery formula, given bounded perturbations in the sampled data [49].We derive a multidimensional version of (45).Daubechies and DeVore regard $\mathcal{F}^{-1}(f)$ as an element of $L_2[-\lambda\pi, \lambda\pi]$ for some $\lambda > 1$. In their proof the obvious fact that $[-\pi, \pi] \subset [-\lambda\pi, \lambda\pi]$ allows for the construction of the bump function $\mathcal{F}^{-1}(g) \in C^\infty(\mathbb{R})$ which is 1 on $[-\pi, \pi]$ and 0 off $[-\lambda\pi, \lambda\pi]$. If their result is to be generalized to a sampling theorem for PW_E in higher dimensions, a suitable condition for E allowing the existence of a bump function is necessary. If $E \subset \mathbb{R}^d$is chosen to be compact such that for all $\lambda > 1, E \subset int(\lambda E)$, then in [51], a C^∞-version of the Urysohn lemma, implies the existence of a smooth bump function which is 1 on E and 0 off λE . It is to such regions that we generalize (45):

Theorem $(5.2.9)[42]$: Let $0 \in E \subset \mathbb{R}^d$ be compact such that for all $\lambda > 1, E \subset int(\lambda E)$. Choose $S = (t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $(f_n)_{n\in\mathbb{N}}$, defined by $f_n(\cdot) = e^{i\langle\cdot, t_n\rangle}$, is a frame for $L_2(E)$ with frame operator S. Let $\lambda_0 > 1$ with $\mathcal{F}^{-1}(g) : \mathbb{R}^d \to \mathbb{R}, \mathcal{F}^{-1}(g) \in C^\infty$ where $\mathcal{F}^{-1}(g)|_E = 1$ and $\mathcal{F}^{-1}(g)|_{(\lambda_0 E)^c} = 0$. If $\lambda \geq \lambda_0$ and $\in PW_E$, then

$$f(t) = \frac{1}{\lambda^d}\sum_{k\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} B_{kn} f\left(\frac{t_n}{\lambda}\right)\right) g\left(t - \frac{t_k}{\lambda}\right), t \in \mathbb{R}^d, \quad (46)$$

where $B_{kn} = \langle S^{-1} f_n, S^{-1} f_k\rangle_E$. Convergence of the sum is in $L_2(\mathbb{R}^d)$, hence also uniform. Further, the map $B : \ell_2(N) \to \ell_2(N)$ defined by $(y_k)_{k\in\mathbb{N}} \mapsto (\sum_{n\in\mathbb{N}} B_{kn}\, y_n)_{k\in\mathbb{N}}$ is bounded linear, and is an onto isomorphism iff $(f_n)_{n\in\mathbb{N}}$ is a Riesz basis for $L_2(E)$.

Proof: Define $f_{\lambda,n}(\cdot) = f_n\left(\frac{\cdot}{\lambda}\right)$. Note that $(f_{\lambda,n})_n$ is a frame for $L_2(\lambda E)$ with frame operator S_λ.

Step 1*: We show that*

$$f = \sum_n f\left(\frac{t_n}{\lambda}\right)\left[\mathcal{F}\left(S_\lambda^{-1} f_{\lambda,n}\right)\mathcal{F}^{-1}(g)\right], f \in PW_E . \quad (47)$$

We know $\text{supp}(\mathcal{F}^{-1}(f)) \subset E \subset \lambda E$, so we may work with $\mathcal{F}^{-1}(f)$ via its frame decomposition. We have

$$\mathcal{F}^{-1}(f) = S_\lambda^{-1}S_\lambda(\mathcal{F}^{-1}(f)) = \sum_n \langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E}\ S_\lambda^{-1} f_{\lambda,n}, on\ \lambda E.$$

This yields

$$\mathcal{F}^{-1}(f) = \sum_n \langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E} \left({S_\lambda}^{-1} f_{\lambda,n}\right)\mathcal{F}^{-1}(g), on\ \mathbb{R}^d,$$

since $supp\mathcal{F}(g) \subset \lambda E$. Taking Fourier transforms we obtain

$$f = \sum_n \langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E}\, \mathcal{F}\left[\left({S_\lambda}^{-1}\right)\mathcal{F}^{-1}(g)\right], on\ \mathbb{R}^d. \tag{48}$$

Now

$$\langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E} = \int_{\lambda E} \mathcal{F}^{-1}(f)(\xi) e^{-i\langle \xi, \frac{t_n}{\lambda}\rangle}\, d\xi = f\left(\frac{t_n}{\lambda}\right)$$

which, when substituted into (48), yields (47).

Step 2: We show that

$$f(\cdot) = \sum_n f\left(\frac{t_n}{\lambda}\right)\left[\sum_k \langle {S_\lambda}^{-1} f_{\lambda,n}, {S_\lambda}^{-1} f_{\lambda,k}\rangle_{\lambda E}\, g\left(\cdot - \frac{t_k}{\lambda}\right)\right], \tag{49}$$

where convergence is in L_2.

We compute $\mathcal{F}[({S_\lambda}^{-1} f_{\lambda,n})\mathcal{F}^{-1}(g)]$. For $h \in L_2(\lambda E)$ we have

$$h = S_\lambda\left({S_\lambda}^{-1}h\right) = \sum_k \langle {S_\lambda}^{-1}h, f_{\lambda,k}\rangle_{\lambda E}\, f_{\lambda,k} = \sum_k \langle h, {S_\lambda}^{-1} f_{\lambda,k}\rangle_{\lambda E}\, f_{\lambda,k}.$$

Letting $h = {S_\lambda}^{-1} f_{\lambda,n}$,

$${S_\lambda}^{-1} f_{\lambda,n} = \sum_k \langle {S_\lambda}^{-1} f_{\lambda,n}, {S_\lambda}^{-1} f_{\lambda,k}\rangle_{\lambda E}\, f_{\lambda,k}.$$

This gives

$$\begin{aligned}
\mathcal{F}\left[\left({S_\lambda}^{-1}f_{\lambda,n}\right)\mathcal{F}^{-1}(g)\right](\cdot) &= \sum_k \langle {S_\lambda}^{-1}f_{\lambda,n}, {S_\lambda}^{-1}f_{\lambda,k}\rangle_{\lambda E}\, \mathcal{F}\left[f_{\lambda,k}\mathcal{F}^{-1}(g)\right](\cdot) \\
&= \sum_k \langle {S_\lambda}^{-1}f_{\lambda,n}, {S_\lambda}^{-1}f_{\lambda,k}\rangle_{\lambda E} \int_{\lambda E} e^{i\langle \xi, \frac{t_k}{\lambda}\rangle}\mathcal{F}^{-1}(g)(\xi) e^{-i\langle \xi, \cdot\rangle}\, d\xi \\
&= \sum_k \langle {S_\lambda}^{-1}f_{\lambda,n}, {S_\lambda}^{-1}f_{\lambda,k}\rangle_{\lambda E} \int_{\lambda E} \mathcal{F}^{-1}(g)(\xi) e^{-i\langle \cdot - \frac{t_k}{\lambda}, \xi\rangle}\, d\xi \\
&= \sum_k \langle {S_\lambda}^{-1}f_{\lambda,n}, {S_\lambda}^{-1}f_{\lambda,k}\rangle_{\lambda E}\, g\left(\cdot - \frac{t_k}{\lambda}\right),
\end{aligned}$$

so (49) follows from (47).

Step 3: We show that

$$\langle {S_\lambda}^{-1}f_{\lambda,n}, {S_\lambda}^{-1}f_{\lambda,k}\rangle_{\lambda E} = \frac{1}{\lambda^d}\langle S^{-1}f_n, S^{-1}f_k\rangle_E\ , for\ \ n,k \in \mathbb{N}. \tag{50}$$

First we show $({S_\lambda}^{-1}f_{\lambda,n})(\cdot) = \frac{1}{\lambda^d}\,(S^{-1}f_n)\left(\frac{\cdot}{\lambda}\right)$, or equivalently that

$f_{\lambda,n} = \frac{1}{\lambda^d} S_\lambda\left((S^{-1}f_n)\left(\frac{\cdot}{\lambda}\right)\right)$.We have for any $g \in L_2(\lambda E)$,

$$\langle g, f_{\lambda,k}\rangle_{\lambda E} = \int_{\lambda E} g(\xi)e^{-i\langle\frac{\xi}{\lambda},t_k\rangle}\,d\xi = \lambda^d \int_E g(\lambda x)e^{-i\langle x,t_k\rangle}\,dx$$
$$= \lambda^d\langle g(\lambda(\cdot)), f_k\rangle_E.$$

By definition of the frame operator S_λ, $S_\lambda g = \sum_{k\in\mathbb{N}}\langle g, f_{\lambda,k}\rangle_{\lambda E}\; f_{\lambda,k}$, which then becomes $S_\lambda g = \lambda^d \sum_k \langle g(\lambda(\cdot)), f_k\rangle_E\; f_{\lambda,k}$. Substituting $g = \frac{1}{\lambda^d}(S^{-1}f_n)\left(\frac{\cdot}{\lambda}\right)$ into the equation above we obtain

$$\frac{1}{\lambda^d}S_\lambda\left((S^{-1}f_n)\left(\frac{\cdot}{\lambda}\right)\right) = \sum_k \langle S^{-1}f_n, f_k\rangle_E\; f_{\lambda,k} = \left(S(S^{-1}f_n)\right)\left(\frac{\cdot}{\lambda}\right) = f_{\lambda,n}.$$

We now compute the desired inner product:

$$\langle {S_\lambda}^{-1}f_{\lambda,n}, {S_\lambda}^{-1}f_{\lambda,k}\rangle_{\lambda E} = \frac{1}{\lambda^{2d}}\int_{\lambda E}(S^{-1}f_n)\left(\frac{x}{\lambda}\right)\overline{(S^{-1}f_k)\left(\frac{x}{\lambda}\right)}\,dx$$
$$= \frac{\lambda^d}{\lambda^{2d}}\int_E (S^{-1}f_n)(x)\overline{(S^{-1}f_k)(x)}\,dx \;=\; \frac{1}{\lambda^d}\langle S^{-1}f_n, S^{-1}f_k\rangle_E\,.$$

Note that (49) becomes

$$f(\cdot) = \frac{1}{\lambda^d}\sum_n f\left(\frac{t_n}{\lambda}\right)\left[\sum_k \langle S^{-1}f_n, S^{-1}f_k\rangle g\left(\cdot - \frac{t_k}{\lambda}\right)\right]. \tag{51}$$

Step 4: The map $V : \ell_2(\mathbb{N}) \mapsto \ell_2(\mathbb{N})$ given by $x = (x_k)_{k\in\mathbb{N}} \mapsto (\sum_n B_{kn}x_n)_{k\in\mathbb{N}} = Bx$ is bounded linear and self-adjoint. Let $(d_k)_{k\in\mathbb{N}}$ be the standard basis for $\ell_2(\mathbb{N})$, and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis for $L_2(E)$. Then

$$Vd_j = \left(B_{kj}\right)_{k\in\mathbb{N}} = \sum_k B_{kj}d_k = \sum_k \langle S^{-1}f_j, S^{-1}f_k\rangle_{d_k}$$
$$= \sum_k \langle L^*(S^{-1})^2Le_j\,, e_k\rangle_{d_k}\,,$$

where L is the synthesis f operator, i.e., $S = LL^*$. Define $\varphi : \ell^2(\mathbb{N}) \mapsto L_2(E)$ by $\varphi(d_k) = e_k, k \in \mathbb{N}$. Clearly φ is unitary. It follows that $V = \varphi^{-1}L^*(S^{-1})^2L\varphi$, which concludes Step 4. From here on we identify V with B. Clearly B is an onto isomorphism iff L and L^* are both onto, i.e., iff the map $Le_n = f_n$ is an onto isomorphism.

Step 5: Verification of (46). Recalling Definition (5.2.8), $f_{S/\lambda} = \left(f\left(\frac{t_n}{\lambda}\right)\right)_{n\in\mathbb{N}}$; for each $t \in \mathbb{R}^d$, let $g_\lambda(t) = \left(g\left(t - \frac{t_n}{\lambda}\right)\right)_{n\in\mathbb{N}}$.

Noting that $f\left(\frac{\cdot}{\lambda}\right), g\left(t - \frac{\cdot}{\lambda}\right) \in L_2(\lambda E)$, and recalling that $(f_{\lambda,n})_n$ is a frame for $L_2(\lambda E)$, we have

$$\sum_n \left|f\left(\frac{t_n}{\lambda}\right)\right|^2 = \sum \left|\langle \mathcal{F}^{-1}(\,f\,), f_{\lambda,n}\rangle_{\lambda E}\right|^2 \;\le A_\lambda \|\mathcal{F}^{-1}(\,f\,)\|^2, \tag{52}$$

and

$$\sum_n \left|g\left(t - \frac{t_n}{\lambda}\right)\right|^2 = \sum \left|\langle \mathcal{F}^{-1}\left(g\left(t\; - \frac{\cdot}{\lambda}\;\right)\right), f_{\lambda,n}\rangle_{\lambda E}\right|^2$$
$$\le\; A_\lambda \left\|\mathcal{F}^{-1}\left(g\left(t\; - \frac{\cdot}{\lambda}\;\right)\right)\right\|^2 \quad .$$

Note that (51) becomes

$$f(t) = \frac{1}{\lambda^d}\sum_n f\left(\frac{t_n}{\lambda}\right)\left[\sum_k B_{kn}\, g\left(t - \frac{t_k}{\lambda}\right)\right] = \frac{1}{\lambda^d}\sum_n f\left(\frac{t_n}{\lambda}\right)\overline{\left[\sum_k B_{nk}\, \overline{g\left(t - \frac{t_k}{\lambda}\right)}\right]}$$
$$= \frac{1}{\lambda^d}\sum_n \left(f_{S/\lambda}\right)_n \left(B\overline{g_\lambda(t)}\right)_n = \frac{1}{\lambda^d}\langle f_{S/\lambda}, B\overline{g_\lambda(t)}\rangle = \frac{1}{\lambda^d}\langle Bf_{S/\lambda}, \overline{g_\lambda(t)}\rangle$$
$$= \frac{1}{\lambda^d}\sum_k \left(Bf_{S/\lambda}\right)_k g\left(t - \frac{t_k}{\lambda}\right) = \frac{1}{\lambda^d}\sum_{k\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} B_{kn} f\left(\frac{t_n}{\lambda}\right)\right) g\left(t - \frac{t_k}{\lambda}\right),$$

which proves (46).

Step 6: We verify that convergence in (46) is in $L_2(\mathbb{R})$ (hence uniform). Define

$$f_n(t) \;=\; \frac{1}{\lambda^d}\sum_{1\le k\le n} \left(Bf_{S/\lambda}\right)_k g\left(t - \frac{t_k}{\lambda}\right)$$

and

$$f_{m,n}(t) = \frac{1}{\lambda^d}\sum_{m\le k\le n}\left(Bf_{\frac{S}{\lambda}}\right)_k g\left(t - \frac{t_k}{\lambda}\right).$$

Then

$$\left[\mathcal{F}^{-1}\left(f_{m,n}\right)\right](\xi\,) \;=\; \frac{1}{\lambda^d}\sum_{m\le k\le n}\left(Bf_{\frac{S}{\lambda}}\right)_k \mathcal{F}^{-1}\left[g\left(.-\frac{t_n}{\lambda}\right)\right]$$
$$= \frac{1}{\lambda^d}\sum_{m\le k\le n}\left(Bf_{\frac{S}{\lambda}}\right)_k \mathcal{F}^{-1}(g)(\xi\,)e^{i\langle \xi, \frac{t_k}{\lambda}\rangle},$$

so

$$\left\|\left[\mathcal{F}^{-1}\left(f_{m,n}\right)\right]\right\|_2^2 = \frac{1}{\lambda^d}\int_{\lambda E} |\mathcal{F}^{-1}(g)(\xi\,)|^2 \left|\sum_{m\le k\le n}\left(Bf_{\frac{S}{\lambda}}\right)_k e^{i\langle \xi, \frac{t_k}{\lambda}\rangle}\right|^2 d\xi$$
$$\le \frac{1}{\lambda^d}\left\|\sum_{m\le k\le n}\left(Bf_{\frac{S}{\lambda}}\right)_k f_{\lambda,k}\right\|_2^2.$$

If $(h_n)_n$ is a orthonormal basis for $L_2(\lambda E)$, then the map $Th_K = f_{\lambda,k}$ (the synthesis operator) is bounded linear, so

$$\left\|\left[\mathcal{F}^{-1}\left(f_{m,n}\right)\right]\right\|_2^2 \le \frac{1}{\lambda^d}\left\|T\left(\sum_{m\le k\le n}\left(Bf_{\frac{S}{\lambda}}\right)_k h_K\right)\right\|_2^2 \le \frac{1}{\lambda^d}\|T\|^2 \sum_{m\le k\le n}\left|\left(Bf_{\frac{S}{\lambda}}\right)_k\right|^2.$$

But $Bf_{\frac{S}{\lambda}} \in \ell^2(\mathbb{N})$, so $\|[\mathcal{F}^{-1}(f_{m,n})]\|2 \to 0$ as $m,n \to \infty$. As $\mathcal{F}^{-1}$ is an onto isomorphism, we have $\|f_{m,n}\| \to 0$, implying that $\|f - f_n\| \to 0$ as $n \to \infty$.
Note that (45) is conveniently written as

$$f\ (t) = \frac{1}{\lambda^d}\sum_k \left(Bf_{\frac{S}{\lambda}}\right)_k g\left(t - \frac{t_k}{\lambda}\right), t \in \mathbb{R}^d. \tag{53}$$

Proposition(5.2.10)[42]: If $0 \in E \subset \mathbb{R}^d$ is compact, then the following are equivalent:

(i) $E \subset int(\lambda E)$ for all $\lambda > 1$.

(ii) There exists a continuous map $\varphi : S^{d-1} \to (0,\infty)$ such that $E = \{ty\varphi(y) \mid y \in S^{d-1}, t \in [0,1]\}$. The following is a simplified version of Theorem (5.2.9) , which is proven in a similar fashion:

Theorem (5.2.11)[42]: Choose $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $(f_n)_{n\in\mathbb{N}}$, defined by $f_n(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,t_n\rangle}$, is a frame for $L_2([-\pi,\pi]^d)$. *If* $f \in PW_E$,then

$$f\ (t) = \sum_{k\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} B_{kn} f\ (t_n)\right) SINC\big(\pi(t - t_k)\big), t \in \mathbb{R}^d. \tag{54}$$

The matrix B and the convergence of the sum are as in Theorem (5.2.9).

(46) generalizes (54) in the same way that (45) generalizes the WKS equation. We can write (54) as

$$f\ (t) = \sum_{k\in\mathbb{N}} (Bf_S\ )_k SINC\big(\pi(t - t_k)\big). \tag{55}$$

Frames for$L_2(E)$ satisfying the conditions in Theorems (5.2.9) and (5.2.11) occur in abundance. The following result is due to Beurling in [47].

Theorem(5.2.12)[42]: Let $\Lambda \subset \mathbb{R}^d$ be countable such that

$$r(\Lambda) = \frac{1}{2}\inf_{\lambda,\mu\in\Lambda,\lambda\neq\mu} \|\lambda - \mu\|_2 > 0 \text{ and } R(\Lambda) = \sup_{\xi\in\mathbb{R}^d}\inf_{\lambda\in\Lambda}\|\lambda - \mu\|_2 < \frac{\pi}{2}.$$

If E is a subset of the closed unit ball in $\mathbb{R}^d$ and E has positive measure, then $\{e^{i\langle\cdot,\lambda\rangle} \mid \lambda \in \Lambda\}$ is a frame for $L_2(E)$.

A desirable trait in a recovery formula is stability given error in the sampled data. Suppose we have sample values

$\tilde{f}_n = f\left(\frac{n}{\lambda}\right) + \epsilon_n$ where $sup_n|\epsilon_n| = \epsilon$. If in (45) we replace $f\left(\frac{n}{\lambda}\right)$ *by* $\tilde{f}_n$, and call the resulting expression $\tilde{f}$, then we have

$$\left|f\ (t) - \tilde{f}\ (t)\right| \le \epsilon\frac{1}{\lambda}\sum_{n\in\mathbb{Z}}\left|g\left(t - \frac{n}{\lambda}\right)\right| \le \epsilon\big(\lambda^{-1}\|g'\|_{L_1} + \|g\ \|_{L_1}\big).$$

It follows that (45) is certainly stable under ℓ_∞ perturbations in the data, while the WKS sampling theorem is not. For a more detailed discussion see [49]. Such a stability result is not immediately forthcoming for (46), as the following example illustrates. Restricting to $d = 1$, let $(t_n)_{n\in\mathbb{Z}}$ satisfy $t_2 = D \notin \mathbb{Z}$, and $t_n = n$ for $n \neq 0$. The forthcoming discussion in this Section shows that $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$.

Note that when $(f_n)_n$ is a Riesz basis, the sequence $\left(S^{-1}f_n\right)_n$ is its biorthogonal sequence. The matrix B associated to this basis is computed as follows. The biorthogonal functions $(G_n)_{n\in\mathbb{Z}}$ for $(sinc(\pi(\cdot - n)))_{n\in\mathbb{Z}}$ are $G_n(t) = \frac{(-1)^n n(t-D)sinc(\pi t)}{(n-D)(t-n)}, n \neq 0,$

And $G_0(t) = \frac{sinc(\pi t)}{sinc(\pi D)}$. That these functions are in $PW_{[-\pi,\pi]}$ is verified by applying the Paley–Wiener theorem [56], and the biorthogonality condition is verified by applying (43). Again using (43), we obtain

(i) $B_{m0} = \langle G_0, G_m\rangle = \dfrac{D(-1)^m}{sinc(\pi D)(m-D)}, m \neq 0,$

(ii) $B_{00} = \langle G_0, G_0\rangle = \dfrac{1}{sinc^2(\pi D)},$

(iii) $B_{mn} = \langle G_n, G_m\rangle = \delta_{nm} + \dfrac{D^2(-1)^{n+m}}{(n-D)(m-D)}, else.$

Note that the rows of B are not in ℓ_1, so that as an operator acting on ℓ_∞, B does not act boundedly. Consequently, the equation

$$\tilde{f}(t) = \frac{1}{\lambda}\sum_k \left(B\tilde{f}_{\frac{S}{\lambda}}\right)_k g\left(t - \frac{t_k}{\lambda}\right) \tag{56}$$

is not defined for all perturbed sequences $\tilde{f}_{\frac{S}{\lambda}}$ where

$\left(\tilde{f}_{\frac{S}{\lambda}}\right)_n = \left(f_{\frac{S}{\lambda}}\right)_{n+\epsilon_n}$ where $sup_n|\epsilon_n| = \epsilon$.

Despite the above failure, the following shows that there is some advantage of (46) over (44).

If $\tilde{f}_{\frac{S}{\lambda}}$ is *some* perturbation of $f_{\frac{S}{\lambda}}$ such that $\left\|B\tilde{f}_{\frac{S}{\lambda}} - Bf_{\frac{S}{\lambda}}\right\|_\infty \leq \epsilon$, then

$$\sup_{\xi\in\mathbb{R}^d}\left|f(t) - \tilde{f}(t)\right| = \sup_{\xi\in\mathbb{R}^d}\left|\frac{1}{\lambda}\sum_k \left(B\left(f_{\frac{S}{\lambda}} - \tilde{f}_{\frac{S}{\lambda}}\right)\right)_k g\left(t-\frac{t_k}{\lambda}\right)\right|$$
$$\leq \epsilon \sup_{\xi\in\mathbb{R}^d}\frac{1}{\lambda}\sum_k \left|g\left(t-\frac{t_k}{\lambda}\right)\right| \leq M \tag{57}$$

from here on, we focus on the case where $(t_n)_{n\in\mathbb{N}}$ is an ℓ_∞ perturbation of the lattice $\mathbb{Z}^d$, and $(f_n)_{n\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$. In this case, under the additional constraint that the sample nodes are asymptotically the integer lattice, the following theorem gives a computationally feasible version of (46). The summands in (46) involves an infinite invertible matrix B, though under the constraints mentioned above, we show that B can be replaced by a related finite-rank operator which can be computed concretely. Precisely, one has the following.

Theorem $(5.2.13)[42]$: Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and $S = (t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\lim_{k\to\infty} \|n_k - t_k\|_\infty = 0 .$$

Define $e_k, f_k : \mathbb{R}^d \to \mathbb{C}$ by $e_k(x) = \frac{1}{(2\pi)^{d/2}} e^{i\langle n_k,x\rangle}$ and $\frac{1}{(2\pi)^{d/2}} e^{i\langle t_k,x\rangle}$, and let $(h_k)_k$ be the standard basis for $\ell_2(\mathbb{N})$. *Let* $P_l: \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the orthogonal projection onto span$\{h_1,\dots,h_l\}$. *If* $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, then for all $f \in PW_{[-\pi,\pi]^d}$, we have

$$f(t) = \lim_{l\to\infty} \frac{1}{\lambda^d} \sum_{k=1}^{l} \left[(P_l B^{-1} P_l)^{-1} f_{\frac{S}{\lambda}}\right]_k g\left(t - \frac{t_k}{\lambda}\right), t \in \mathbb{R}^d, \tag{58}$$

where convergence is in L_2 and uniform. Furthermore,

$$(P_l B^{-1} P_l)_{nm} = \begin{cases} sinc\pi(t_{n,1} - t_{m,1}) \cdots sinc\pi(t_{n,d} - t_{m,d}), 1 \le n,m \le l, \\ 0, \qquad otherwise. \end{cases}$$

Convergence of the sum is in L_2 and also uniform.

proof : Step 1*:* B is a compact perturbation of the identity map, namely

$$B = I + \lim_{l\to\infty}(-P_l + (P_l B^{-1} P_l)^{-1}).$$

Since $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, $L^* = (I - T)$ is an onto isomorphism where $T_{e_k} = e_k - f_k$; so B simplifies to $(I - T)^{-1}(I - T^*)^{-1}$. We examine

$$B^{-1} = (I - T^*)(I - T) = I + (T^*T - T - T^*) = I + \Delta ,$$

where Δ is a compact operator. If an operator $\Delta\colon H \to H$ is compact then so is Δ^*, hence $P_l\Delta P_l \to \Delta$ in the operator norm because

$$\|P_l\Delta P_l - \Delta\| \le \|P_l\Delta P_l - P_l\Delta\| + \|P_l\Delta - \Delta\| \le \|\Delta P_l - \Delta\| + \|P_l\Delta - \Delta\|$$
$$= \|P_l\Delta^* - \Delta^*\| + \|P_l\Delta - \Delta\| \to 0 .$$

We have

$$B^{-1} = \lim_{l\to\infty}(I + P_l\Delta P_l) = \lim_{l\to\infty}(I + P_l(B^{-1} - I)P_l)$$
$$= \lim_{l\to\infty}(I - P_l + P_l B^{-1} P_l).$$

Now $(P_l B^{-1} P_l)$ restricted to the first l rows and columns is the Grammian matrix for the set $(f_1,\dots,f_l)$ which can be shown (in a straightforward manner) to be linearly independent. We conclude that

$P_l B^{-1} P_l$ is invertible as an $l \times l$ matrix. By $(P_l B^{-1} P_l)^{-1}$ we mean the inverse as an $l \times l$ matrix and zeroes elsewhere. Observing that the ranges of $P_l B^{-1} P_l$ and $(P_l B^{-1} P_l)^{-1}$are in the kernel of $-P_l$, and that the range of $I - P_l$ is in the kernels of $P_l B^{-1} P_l$ and $(P_l B^{-1} P_l)^{-1}$, we easily compute

$$(I - P_l + (P_l B^{-1} P_l)^{-1})^{-1} = I - P_l + P_l B^{-1} P_l,$$

so that

$$B^{-1} = \lim_{l\to\infty}(I - P_l + (P_l B^{-1} P_l)^{-1})^{-1},$$

implying

$$B = \lim_{l\to\infty}(I - P_l + (P_l B^{-1} P_l)^{-1}) = \lim_{l\to\infty} B_l = \lim_{l\to\infty}(- P_l + (P_l B^{-1} P_l)^{-1}) .$$

Step 2: We verify (58) and its convergence properties. Recalling (53), we have

$$f(t) - \frac{1}{\lambda^d}\sum_{k=1}^{\infty}\left[(I - P_l + (P_l B^{-1} P_l)^{-1}) f_{\frac{S}{\lambda}}\right]_k g\left(t - \frac{t_k}{\lambda}\right)$$
$$= \frac{1}{\lambda^d}\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/\lambda}\right]_k g\left(t - \frac{t_k}{\lambda}\right)$$

implying

$$f(t) - \frac{1}{\lambda^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{\frac{S}{\lambda}}\right]_k g\left(t - \frac{t_k}{\lambda}\right) =$$
$$\frac{1}{\lambda^d}\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/\lambda}\right]_k g\left(t - \frac{t_k}{\lambda}\right) + \frac{1}{\lambda^d}\sum_{k=l+1}^{\infty} f\left(\frac{t_k}{\lambda}\right) g\left(t - \frac{t_k}{\lambda}\right).$$

Therefore,

$$\left\|f(.) - \frac{1}{\lambda^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{\frac{S}{\lambda}}\right]_k g\left(. - \frac{t_k}{\lambda}\right)\right\|_2$$
$$= \left\|\frac{1}{\lambda^d}\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/\lambda}\right]_k g\left(. - \frac{t_k}{\lambda}\right) + \frac{1}{\lambda^d}\sum_{k=l+1}^{\infty} f\left(\frac{t_k}{\lambda}\right) g\left(. - \frac{t_k}{\lambda}\right)\right\|_{[-\lambda\pi,\lambda\pi]^d}$$
$$= \frac{1}{\lambda^d}\left\|\mathcal{F}^{-1}(g)(.)\left(\sum_{k=1}^{\infty}\left[(B - B_l) f_{\frac{S}{\lambda}}\right]_k e^{i\langle ., \frac{t_k}{\lambda}\rangle} + \sum_{k=l+1}^{\infty} f\left(\frac{t_k}{\lambda}\right) e^{i\langle ., \frac{t_k}{\lambda}\rangle}\right)\right\|_{[-\lambda\pi,\lambda\pi]^d}$$

after taking the inverse Fourier transform. Now

$$\left\|f(.) - \frac{1}{\lambda^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{\frac{S}{\lambda}}\right]_k g\left(. - \frac{t_k}{\lambda}\right)\right\|_2$$
$$\le \frac{1}{\lambda^d}\left\|\sum_{k=1}^{\infty}\left[(B - B_l) f_{\frac{S}{\lambda}}\right]_k e^{i\langle ., \frac{t_k}{\lambda}\rangle}\right\|_{[-\lambda\pi,\lambda\pi]^d} + \frac{1}{\lambda^d}\left\|\sum_{k=l+1}^{\infty} f\left(\frac{t_k}{\lambda}\right) e^{i\langle ., \frac{t_k}{\lambda}\rangle}\right\|_{[-\lambda\pi,\lambda\pi]^d}$$
$$\le \frac{M}{\lambda^d}\left\|(B - B_l) f_{\frac{S}{\lambda}}\right\|_{\ell^2(\mathbb{N})} + \frac{M}{\lambda^d}\left(\sum_{k=l+1}^{\infty}\left|f\left(\frac{t_k}{\lambda}\right)\right|^2\right)^{\frac{1}{2}},$$

since $\left(f_k\left(\frac{.}{\lambda}\right)\right)_k$ is a Riesz basis for $L_2[-\lambda\pi, \lambda\pi]^d$. Since $\boldsymbol{B_l} \to B$ as $l \to \infty$ and $\left(\boldsymbol{f}\left(\frac{t_k}{\lambda}\right)\right)_{\boldsymbol{k}} \in \boldsymbol{\ell^2}(\mathbb{N})$, the last two terms in the inequality above tend to zero, which proves the required result. Finally, to compute $(P_l B^{-1} P_l)_{nm}$, recall that

$B^{-1} = (I - T^*)(I - T)$. Proceeding in a manner similar to the proof of (52), we obtain

$$B_{mn}^{-1} = \langle LL^* e_n, e_m\rangle = \langle L^* e_n, L^* e_m\rangle = \langle f_n, f_m\rangle$$
$$= sinc\pi(t_{n,1} - t_{m,1}) \cdot \cdots \cdot sinc\pi(t_{n,d} - t_{m,d}).$$

The entries of $P_l B^{-1} P_l$ agree with those of B^{-1} when $1 \le n, m \le l$.

Theorem $(5.2.14)[42]$: Under the hypotheses of Theorem (5.2.14) ,

$$f(t) = \lim_{l\to\infty} \sum_{k=1}^{l} [(P_l B^{-1} P_l)^{-1} f_S]_k \; SINC(t - t_k), t \in \mathbb{R}^d, \quad (59)$$

where convergence of the sum is both L_2 and uniform. The following lemma forms the basis of the proof of the preceding theorems, as well as the other results in the section.

Lemma $(5.2.15)[42]$: Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$. Define

$e_k, f_k : \mathbb{R}^d \to \mathbb{C}$ by $e_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle n_k, x\rangle}$ and $f_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle t_k, x\rangle}$.

Then for any $r, s \ge 1$, and any finite sequence $(a_k)_{k=r}^{s}$, we have

$$\left\|\sum_{k=r}^{s}\left(\frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle(\cdot), n_k\rangle} - \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle(\cdot), t_k\rangle}\right)\right\|_2$$
$$\le \left(e^{\pi d(sup_{r\le k\le s}\|n_k - t_k\|_\infty)} - 1\right)\left(\sum_{k=r}^{s} |a_k|^2\right)^{1/2}. \quad (60)$$

Proof : Let $\delta_k = t_k - n_k$ where $\delta_k = (\delta_{k_1}, \dots, \delta_{k_d})$. Then

$$\varphi_{r,s}(x) = \sum_{k=r}^{s} \frac{a_k}{(2\pi)^{\frac{d}{2}}}\left[e^{i\langle n_k, x\rangle}, e^{i\langle t_k, x\rangle}\right]$$
$$= \sum_{k=r}^{s} \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle n_k, x\rangle}\left[1 - e^{i\langle \delta_k, x\rangle}\right] . \quad (61)$$

Now for any δ_k,

$$1 - e^{i\langle\delta_k, x\rangle} = 1 - e^{i\delta_{k_1} x_1} \dots e^{i\delta_{k_d} x_d}$$
$$= 1 - \left(\sum_{j_1=0}^{\infty} \frac{(i\delta_{k_1} x_1)^{j_1}}{j_1!}\right) \dots \left(\sum_{j_d=0}^{\infty} \frac{(i\delta_{k_d} x_d)^{j_d}}{j_d!}\right)$$
$$= 1 - \sum_{\substack{(j_1,\dots,j_d)\\ j_i \ge 0}} \frac{(i\delta_{k_1} x_1)^{j_1} \cdot \cdots \cdot (i\delta_{k_d} x_d)^{j_d}}{j_1! \cdot \cdots \cdot j_d!}$$

$$= -\sum_{(j_1,\dots,j_d)\in J} i^{j_1,\dots,j_d}\ \frac{(i\delta_{k_1}x_1)^{j_1}\ \cdot\cdot\cdot\cdot\cdot\ (i\delta_{k_d}x_d)^{j_d}}{j_1!\ \cdot\cdot\cdot\cdot\cdot\ j_d!}\ ,$$

where $J = \{(j_1, \dots, j_d) \in \mathbb{Z}^d \mid j_i \geq 0, (j_1, \dots, j_d) \neq 0\}$. Then (61) becomes

$$\varphi_{r,s}(x) = -\sum_{k=r}^{s} \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle n_k,x\rangle}\left[\sum_{(j_1,\dots,j_d)\in J} i^{j_1,\dots,j_d}\ \frac{(i\delta_{k_1}x_1)^{j_1}\ \cdot\cdot\cdot\cdot\cdot\ (i\delta_{k_d}x_d)^{j_d}}{j_1!\ \cdot\cdot\cdot\cdot\cdot\ j_d!}\right]$$

$$= -\sum_{(j_1,\dots,j_d)\in J} \frac{x_1^{j_1}\ \cdots\ x_d^{j_d}}{j_1!\ \cdots\ j_d!} i^{j_1,\dots,j_d} \sum_{k=r}^{s} \frac{a_k}{(2\pi)^{\frac{d}{2}}} \delta_{k_1}^{j_1}\ \cdot\cdot\cdot\cdot\cdot\ \delta_{k_d}^{j_d}\ e^{i\langle n_k,x\rangle},$$

so

$$|\varphi_{r,s}(x)| \leq \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdot\cdot\cdot\cdot\cdot\ j_d!} \sum_{k=r}^{s} a_k \delta_{k_1}^{j_1}\ \cdot\cdot\cdot\cdot\cdot\ \delta_{k_d}^{j_d}\ \frac{e^{i\langle n_k,x\rangle}}{(2\pi)^{\frac{d}{2}}}\ .$$

For brevity denote the outer summand above by $h_{\boldsymbol{j_1,\dots,j_d}}\ (t)$. Then

$$\left(\int_{[-\pi,\pi]^d} |\varphi_{r,s}(x)|^2\, dt\right)^{\frac{1}{2}} \leq \left(\int_{[-\pi,\pi]^d} \left|\sum_{(j_1,\dots,j_d)\in J} h_{j_1,\dots,j_d}\ (x)\right|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \sum_{(j_1,\dots,j_d)\in J} \left(\int_{[-\pi,\pi]^d} |h_{j_1,\dots,j_d}\ (x)|^2 dx\right)^{\frac{1}{2}},$$

so that

$$\|\varphi_{r,s}\|_2 \leq \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdots\ j_d!}\left(\int_{[-\pi,\pi]^d}\left|\sum_{k=r}^{s} a_k \delta_{k_1}^{j_1}\ \cdot\cdot\cdot\cdot\ \delta_{k_d}^{j_d}\ \frac{e^{i\langle n_k,x\rangle}}{(2\pi)^{\frac{d}{2}}}\right|^2 dx\right)^{\frac{1}{2}}$$

$$= \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdot\cdot\cdot\cdot\cdot\ j_d!}\left(\sum_{k=r}^{s} |a_k|^2 \left|\delta_{k_1}^{j_1}\right|^2\ \cdot\cdot\cdot\cdot\cdot\ \left|\delta_{k_d}^{j_d}\ \right|^2\right)^{\frac{1}{2}}$$

$$\leq \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1!\ \cdots\ j_d!}\left(\sum_{k=r}^{s} |a_k|^2 \left(\sup_{r\leq k\leq s} \|n_k - t_k\|_\infty\right)^{2(j_1,\dots,j_d)}\ \right)^{\frac{1}{2}}$$

$$= \sum_{(j_1,\dots,j_d)\in J} \frac{\pi(sup_{r\leq k\leq s}\|n_k - t_k\|_\infty)^{j_1,\dots,j_d}}{j_1!\ \cdot\cdot\cdot\cdot\cdot\ j_d!}\left(\sum_{k=r}^{s} |a_k|^2\ \right)^{\frac{1}{2}}$$

$$= \left[\prod_{l=1}^{d}\left(\sum_{j_\ell=0}^{\infty} \frac{\pi(sup_{r\leq k\leq s}\|n_k - t_k\|_\infty)^{j_\ell}}{j_\ell!}\right) - 1\ \right]\left(\sum_{k=r}^{s} |a_k|^2\ \right)^{\frac{1}{2}}$$

$$= \left(e^{\pi d(sup_{r\le k\le s}\|n_k - t_k\|_\infty)} - 1\right)\left(\sum_{k=r}^{s}|a_k|^2\right)^{\frac{1}{2}}.$$

Corollary $(5.2.16)[42]$: Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\sup_{k\in\mathbb{N}}\|n_k - t_k\|_\infty = L < \infty$. Define $e_k, f_k : \mathbb{R}^d \to \mathbb{C}$ by $e_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle n_k, x\rangle}$ and $f_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle t_k, x\rangle}$.Then the map $T : L_2[-\pi,\pi]^d \to L_2[-\pi,\pi]^d$, defined by $Te_n = e_n - f_n$, satisfies the following estimate:

$$\|T\| \le e^{\pi L d} - 1 . \tag{62}$$

Proof : Lemma (5.2.15) shows that T is uniformly continuous on a dense subset of the ball in $L_2(E)$, so T is bounded on $L_2[-\pi,\pi]^d$. The inequality (62) follows immediately.

Corollary$(5.2.17)[42]$: Let $(n_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$, and let e_k, f_k and T be defined as in Corollary (5.2.16). For each $l \in \mathbb{N}$, define T_l by $T_{le_k} = e_k - f_k$ for $1 \le k \le l$, and $T_{le_k} = 0$ for $l < k$.

If $\lim_{k\to\infty}\|n_k - t_k\|_\infty = 0$, then $\lim_{l\to\infty} T_l = T$ in the operator norm. In particular, T is a compact operator.

Proof : As

$$(T - T_l)\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} a_k(e_k - f_k) - \sum_{k=1}^{l} a_k(e_k - f_k)$$
$$= \sum_{k=l+1}^{\infty} a_k(e_k - f_k) = T\left(\sum_{k=l+1}^{\infty} a_k e_k\right),$$

the estimate derived in Lemma (5.2.15) yields

$$\left\|(T - T_l)\left(\sum_{k=1}^{\infty} a_k e_k\right)\right\|_2 = \left\|T\left(\sum_{k=l+1}^{\infty} a_k e_k\right)\right\|_2$$
$$\le \left(e^{\pi d sup_{k\ge l+1}\|\delta_k\|_\infty} - 1\right)\left\|\sum_{k=1}^{\infty} a_k e_k\right\|_2 ,$$

so $\|(T - T_l)\|_2 \to 0$ as $l \to \infty$. As T_l has finite rank, we deduce that T is compact.

Theorem $(5.2.18)[42]$: Let $(t_n)_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence of distinct points such that

$$\lim_{|n|\to\infty} sup|n - t_n| = L < \frac{1}{4} .$$

Then the sequence of functions $(f_k)_{k\in\mathbb{Z}}$, defined by $f_k(x) = \frac{1}{\sqrt{2\pi}} e^{it_k x}$, is a Riesz basis for $L_2[-\pi,\pi]$. Theorem (5.2.18) shows that in the univariate case of Theorem (5.2.13), the restriction that $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]$can be dropped. The following example shows that the multivariate case is very different. Let $(e_n)_n$ be an orthonormal basis for a Hilbert space H. Let $f_1 \in H$ with $\|f_1\| = 1$, then $(f_1, e_2, e_3, \dots)$ is a Riesz basis for H iff $\langle f_1, e_1\rangle \neq 0$. Verifying that the map T, given by $e_k \longmapsto e_k$ for $k > 1$ and $e_1 \longmapsto f_1$, is a continuous bijection is routine, so T is an isomorphism via the Open Mapping theorem. In the language of Theorem (5.2.13), $(f_1, e_2, e_3, \dots)$ is a Riesz basis for $L_2[-\pi,\pi]$ iff $0 \neq sinc(\pi t_{1,1}) \cdot \cdots \cdot sinc(\pi t_{1,d})$, that is, iff $t_1 \in (\mathbb{R}\backslash\{\pm 1, \pm 2, \dots\})^d$. Corollary (5.2.16) yields the following generalization of Kadec's theorem in d dimensions.

Corollary(5.2.19)[42]: Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$ and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\sup_{k\in\mathbb{Z}} \|n_k - t_k\|_\infty = L < \frac{ln(2)}{\pi d}. \tag{63}$$

Then the sequence $(f_k)_{k\in\mathbb{N}}$ defined by $f_k(x) = \frac{1}{(2\pi)^{d/2}} e^{i\langle x, t_k\rangle}$ is a Riesz basis for $L_2[-\pi,\pi]^d$.

The proof is immediate. Note that (62) implies that the map T given in Corollary (5.2.16) has norm less than 1. We conclude that the map $(I - T)e_k = f_k$ is invertible by considering its Neumann series.

The proof of Corollary (5.2.16) and Corollary (5.2.19) are straightforward generalizations of the univariate result proved by Duffin and Eachus [50]. Kadec improved the value of the constant in the inequality (63) (for $d = 1$) from $\frac{ln(2)}{\pi}$ to the optimal value of 1/4; this is his celebrated "1/4 theorem" [52].Kadec's method of proof is to expand $e^{i\delta x}$ with respect to the orthogonal basis

$$\left\{1, cos(nx), sin\left(n - \frac{1}{2}\right)x\right\}_{n\in\mathbb{N}}$$

for $L_2[-\pi,\pi]$, and use this expansion to estimate the norm of T.

In the proof of Corollary (5.2.18) and Corollary (5.2.19) we simply used a Taylor series. Unlike the estimates in Kadec's theorem, the estimate in (62) can be used for any sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$such that $sup_{k\in\mathbb{Z}}\|n_k - t_k\|_\infty = L < \infty$, not only those for which the exponentials $(e^{it_n x})_n$ form a Riesz basis. An impressive generalization of Kadec's 1/4 theorem when $d = 1$ is Avdonin's "1/4 in the mean" theorem [43].Sun and Zhou (see [55]) refined Kadec's argument to obtain a partial generalization of his result in higher dimensions:

Theorem (5.2.20)[42]: Let $(a_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$ such that $0 < L < \frac{1}{4}$,

$$D_d(L) = \left(1 - \cos\pi L + \sin\pi L + \frac{\sin\pi L}{\pi L}\right)^d - \left(\frac{\sin\pi L}{\pi L}\right)^d \quad ,$$

and

$\|a_n - n\|_\infty \le L, n \in \mathbb{Z}^d$. *If* $D_d(L) < 1$,

then $\left(\frac{1}{(2\pi)^d} e^{i\langle a_n,(\cdot)\rangle}\right)$ is a Riesz basis for $L_2[-\pi,\pi]^d$ with frame bounds (1 − *DdL)2* and *(1+ DdL)2 .*

In the one-dimensional case, Kadec's theorem is recovered exactly from Theorem (5.2.20). When $d > 1$, the value x_d satisfying $0 < x_d < \frac{1}{4}$ and $D_d(x_d) = 1$ is an upper bound for any value of L satisfying

$0 < L < \frac{1}{4}$ and $D_d(L) < 1$. The value of x_d is not readily apparent, whereas the constant in Corollary (5.2.19) is $\frac{\ln 2}{x_d}$. A relationship between this number and x_d is given in the following theorem (whose proof is omitted).

Theorem (5.2.21)[42]: Let x_d be the unique number satisfying

$0 < x_d < \frac{1}{4}$ and $D_d(x_d) = 1$. Then

$$\lim_{d\to\infty} \frac{x_d - \frac{\ln 2}{\pi d}}{\frac{(\ln 2)^2}{12\pi d^2}} = 1 \ .$$

Thus, for sufficiently large *d*, Theorem (5.2.20) and Corollary (5.2.19) are essentially the same.

We apply the techniques developed previously to approximate the biorthogonal functions to Riesz bases $\left(\frac{1}{\sqrt{2\pi}} e^{it_n(\cdot)}\right)$ for which the synthesis operator is small perturbation of the identity.

Definition (5.2.22)[42] :A Kadec sequence is a sequence $(t_n)_{n\in\mathbb{Z}}$ of real numbers satisfying

$$\sup_{n\in\mathbb{Z}} |t_n - n| = D < \frac{1}{4}.$$

Definition (5.2.23)[42]: Let $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$be a sequence such that

$(f_n)_n = \left(\frac{1}{\sqrt{2\pi}} e^{it_n(\cdot)}\right)_n$ is a Riesz basis for $L_2[-\pi,\pi]$. If $l \ge 0$, the

l-truncated sequence $(t_{l,n})_{n\in\mathbb{Z}}$ is defined by $t_{l,n} = t_n$ if $|n| \le l$ and

$t_{l,n} = n$ otherwise. Define $f_{l,n} = \frac{1}{\sqrt{2\pi}} e^{it_{l,n}(\cdot)}$ for $n \in \mathbb{Z}, l \ge 0$.

Let $P_l : L_2[-\pi,\pi] \to L_2[-\pi,\pi]$ be the orthogonal projection onto span$\{e_{-l},\dots,e_l\}$.

Proposition (5.2.24)[42]: Let $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence such that $(f_n)_n$ (defined above) is a Riesz basis for $L_2[-\pi,\pi]$. If $(e_n)_n$ is the standard exponential orthonormal basis for $L_2[-\pi,\pi]$ and the map L (defined above) satisfies the estimate $\|I - L\| = \delta < 1$, then the following are true:

(i) For $l \geq 0$, the sequence $(f_{l,n})_n$ is a Riesz basis for$L_2[-\pi,\pi]$.

(ii) For $l \geq 0$, the map L_l defined by $L_l e_n = f_{l,n}$ satisfies $\|L_l^{-1}\| \leq \frac{1}{1-\delta}$.

Proof : If $(c_n)_n \in \ell^2(\mathbb{Z})$, then

$$(I-L_l)\left(\sum_n c_n e_n\right) = \sum_n c_n(e_n - L_l e_n) = \sum_{|n|\leq l}(e_n - f_n) = (I-L)P_l\left(\sum_n c_n e_n\right),$$

so that

$$(I - L_l) = (I - L)P_l \quad . \tag{64}$$

From this, $\|I - L_l\| \leq \delta$, which implies (i) and (ii).

Define the biorthogonal functions of $(f_{l,n})_n$ to be $(f_{l,n}^*)_n$. Passing to the Fourier transform, we have $\frac{1}{\sqrt{2\pi}}\mathcal{F}(f_{l,n})(t) = sinc(\pi(t - t_{l,n}))$ and

$$G_{l,n}(t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f_{l,n}^*)(t).$$

Define the biorthogonal functions of $(f_n)_n$ similarly.

Lemma(5.2.25)[42]: If $(t_n)_n \subset \mathbb{R}$ satisfies the hypotheses of Proposition (5.2.24) , then

$$\lim_{l\to\infty} G_{l,n} = G_n \qquad in \quad PW_{[-\pi,\pi]} \ .$$

Proof : Note that

$$\delta_{nm} = \langle f_{l,n}, f_{l,m}^*\rangle = \langle L_l e_n, f_{l,m}^*\rangle = \langle e_n, L_l^* f_{l,m}^*\rangle$$

so that for all m, $f_{l,m}^* = (L_l^*)^{-1}e_m$. Similarly, $f_m^* = (L^*)^{-1}e_m$. We have

$$f_{l,m}^* - f_m^* = ((L_l^*)^{-1}(L^*)^{-1})e_m = (L_l^*)^{-1}(L^* - L_l^*)(L^*)^{-1}e_m.$$

Now (84) implies $L - L_l = (I - P_l)(L - I)$, so that

$$f_{l,m}^* - f_m^* = (L_l^*)^{-1}(L^* - I)(I - P_l)(L^*)^{-1}e_m.$$

Applying Proposition (5.2.24) yields

$$\|f_{l,m}^* - f_m^*\| \leq \frac{1}{1-\delta}\|(L^* - I)(I - P_l)(L^*)^{-1}e_m\| ,$$

which for fixed m goes to 0 as $l \to \infty$. We conclude $\lim_{l\to\infty} f_{l,m}^* = f_m^*$, which, upon passing to the Fourier transform, yields $\lim_{l\to\infty} G_{l,m} = G_m$.

Theorem (5.2.26)[42]: Let $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence (with $t_n \neq 0$ for $n \neq 0$) such that $(f_n)_n = \left(\frac{1}{\sqrt{2\pi}}e^{it_n(\cdot)}\right)_n$ is a Riesz basis for $L_2[-\pi,\pi]$, and let $(e_n)_n$ be the standard exponential orthonormal basis for $L_2[-\pi,\pi]$. If the map L given by $Le_n = f_n$ satisfies the estimate $I - L < 1$, then the biorthogonals G_n of $\frac{1}{\sqrt{2\pi}}\mathcal{F}(f_n)(\cdot) = sinc(\pi(\cdot - t_n))$ $in\ PW_{[-\pi,\pi]}$ are

$$G_n(t) = \frac{H(t)}{(t-t_n)H'(t_n)}, \qquad n \in \mathbb{Z} , \tag{65}$$

where

$$H(t) = (t - t_0)\prod_{n=1}^{\infty}\left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right) . \qquad (66)$$

Proof : We see that $\delta_{nm} = \langle G_{l,m}, S_{l,n}\rangle$, where $S_{l,n}(t) = sinc(\pi(t - t_n))$ when $|n| \leq l$ and $S_{l,n}(t) = sinc(\pi(t - n))$ when $|m| > l$. Without loss of generality, let $|m| < l$. (43) implies that $G_{l,m}(k) = 0$ when $|k| > l$. By the WKS theorem we have

$$G_{l,m}(t) = \sum_{k=-l}^{k=l} G_{l,m}(k) sinc\big(\pi(t - k)\big)$$
$$= \left(\sum_{k=-l}^{k=l} \frac{(-1)^{k-1} t G_{l,m}(k)}{k - t}\right) sinc(\pi t)$$
$$= \frac{w_l(t)}{\prod_{k=1}^{l}(k - t)(-k - t)} sinc(\pi t),$$

where w_l is a polynomial of degree at most $2l$. Noting that

$$sinc(\pi t) = \prod_{k=1}^{\infty}\left(1 - \frac{t^2}{k^2}\right) \text{ and } \prod_{k=1}^{l}(k - t)(-k - t) = (-1)^l (l!)^2 \prod_{k=1}^{l}\left(1 - \frac{t^2}{k^2}\right),$$

we have

$$G_{l,m}(t) = \frac{(-1)^l w_l(t)}{(l!)^2} \prod_{k=l+1}^{\infty}\left(1 - \frac{t^2}{k^2}\right).$$

Again by (43), $\delta_{nm} = G_{l,m}(t_n)$ when $|n| \leq l$ so that

$$\delta_{nm} = \frac{(-1)^l}{(l!)^2} w_l(t_n) \prod_{k=l+1}^{\infty}\left(1 - \frac{t_n^2}{k^2}\right).$$

This determines the zeroes of w_l. We deduce that

$w_l(t) = \frac{c_l \prod_{k=1}^{k=l}(t - t_k)(t - t_{-k})}{t - t_m}$ for some constant c_l . Absorbing constants, we have

$G_{l,m}(t) = \frac{c_l H_l(t)}{t - t_m}$, where

$$H_l(t) = (t - t_0)\prod_{k=1}^{l}\left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right)\prod_{l+1}^{\infty}\left(1 - \frac{t^2}{k^2}\right).$$

Now $0 = H_l(t_m)$, so $G_{l,m}(t) = c_l \frac{H_l(t) - H_l(t_m)}{t - t_m}$. Taking limits,

$c_l = \frac{1}{(H_l)'(t_m)}$. This yields $G_{l,m}(t) = \frac{H_l(t)}{(t - t_m) H_l'(t_m)}$. Define

$$H(t) = (t - t_0)\prod_{k=1}^{\infty}\left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right).$$

Basic complex analysis shows that H is entire, and $H_l \to H$ and $H_l' \to H'$ uniformly on compact subsets of C. Furthermore, $H'(t_k) \neq 0$ for all k, since each t_k is a zero of H of multiplicity one. Together we have

$$\lim_{l\to\infty} G_{l,m}(t) = \frac{H(t)}{(t - t_m)H'(t_m)}, t \in \mathbb{R}.$$

By the foregoing lemma, $G_{l,m} \to G_m$. Observing that convergence in $PW_{[-\pi,\pi]}$ implies pointwise convergence yields the desired result.Levinson proved a version of Theorem (5.2.26) in the case where $(t_n)_{n\in\mathbb{Z}}$ is a Kadec sequence. His original proof is found in [53]. We recall that if $(f_n)_n$ is a Riesz basis arising from a Kadec sequence, then the synthesis operator L satisfies $\|I - L\| < 1$. Levinson's theorem is then recovered from Theorem (5.2.26).

Corollary$(5.2.27)$[296]: Let $0 \in E \subset \mathbb{R}^d$ be compact such that for all $\epsilon_2 > 0, E \subset int((1+\epsilon_2)E)$. Choose $S = (t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}} \subset \mathbb{R}^d$ such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$, defined by $f_{(m+\epsilon_0)}(\cdot) = e^{i\langle\cdot,t_{(m+\epsilon_0)}\rangle}$, is a frame for $L_2(E)$ with frame operator S. Let $\epsilon_5 > 0$ with $\mathcal{F}^{-1}(g) : \mathbb{R}^d \to \mathbb{R}, \mathcal{F}^{-1}(g) \in C^\infty$ where $\mathcal{F}^{-1}(g)|_E = 1$ and $\mathcal{F}^{-1}(g)|_{((1+\epsilon_2)E)^c} = 0$. If $\epsilon_2 > \epsilon_5 > 0$ and $f \in PW_E$, then

$$f(t) = \frac{1}{(1+\epsilon_2)^d}\sum_{k\in\mathbb{N}}\left(\sum_{(m+\epsilon_0)\in\mathbb{N}} B_{k(m+\epsilon_0)}\, f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right) g\left(t - \frac{t_k}{(1+\epsilon_2)}\right), t \in \mathbb{R}^d, \qquad (67)$$

where $B_{k(m+\epsilon_0)} = \langle S^{-1} f_{(m+\epsilon_0)}, S^{-1} f_k\rangle_E$. Convergence of the sum is in $L_2(\mathbb{R}^d)$, hence also uniform. Further, the map $B : \ell_2(N) \to \ell_2(N)$ defined by $(y_k)_{k\in\mathbb{N}} \mapsto \left(\sum_{(m+\epsilon_0)\in\mathbb{N}} B_{k(m+\epsilon_0)}\, y_{(m+\epsilon_0)}\right)_{k\in\mathbb{N}}$ is bounded linear, and is an onto isomorphism iff $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{N}}$ is a Riesz basis for $L_2(E)$.

Proof : Define $f_{(1+\epsilon_2),(m+\epsilon_0)}(\cdot) = f_{(m+\epsilon_0)}\left(\frac{\cdot}{(1+\epsilon_2)}\right)$. Note that$(f_{(1+\epsilon_2),(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a frame for $L_2((1+\epsilon_2)E)$ with frame operator $S_{(1+\epsilon_2)}$.

Step 1: We show that

$$f = \sum_{(m+\epsilon_0)} f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\mathcal{F}\left(S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}\right)\mathcal{F}^{-1}(g)\right], f \in PW_E . \qquad (68)$$

We know $\mathrm{supp}(\mathcal{F}^{-1}(f)) \subset E \subset (1+\epsilon_2)E$, so we may work with $\mathcal{F}^{-1}(f)$ via its frame decomposition. We have

$$\mathcal{F}^{-1}(f) = S^{-1}_{(1+\epsilon_2)} S_{(1+\epsilon_2)}\left(\mathcal{F}^{-1}(f)\right)$$
$$= \sum_{(m+\epsilon_0)} \langle \mathcal{F}^{-1}(f), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\; S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),(m+\epsilon_0)},$$

$on\ (1+\epsilon_2)E$. This yields

$$\mathcal{F}^{-1}(f) = \sum_{(m+\epsilon_0)} \langle \mathcal{F}^{-1}(f), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\left(S^{-1}_{(1+\epsilon_2)}\, f_{(1+\epsilon_2),(m+\epsilon_0)}\right)\mathcal{F}^{-1}(g), on\ \mathbb{R}^d,$$

since $supp\mathcal{F}(g) \subset (1+\epsilon_2)E$. Taking Fourier transforms we obtain

$$f = \sum_{(m+\epsilon_0)} \langle \mathcal{F}^{-1}(f), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\, \mathcal{F}\left[\left(S^{-1}_{(1+\epsilon_2)}\right)\mathcal{F}^{-1}(g)\right], on\ \mathbb{R}^d. \qquad (69)$$

Now

$$\langle \boldsymbol{\mathcal{F}^{-1}}(\boldsymbol{f}), \boldsymbol{f}_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E} = \int_{(1+\epsilon_2)E} \boldsymbol{\mathcal{F}^{-1}}(\boldsymbol{f})(\boldsymbol{\xi})\boldsymbol{e}^{-i\langle\xi,\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\rangle}\, \boldsymbol{d\xi} = \boldsymbol{f}\left(\frac{\boldsymbol{t_{(m+\epsilon_0)}}}{(\boldsymbol{1}+\boldsymbol{\epsilon_2})}\right)$$

which, when substituted into (69), yields (68).

Step 2: We show that

$f(\cdot) =$

$$\sum_{(m+\epsilon_0)} f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\sum_k \langle S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}, S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, g\left(\cdot - \frac{t_k}{(1+\epsilon_2)}\right)\right], (70)$$

where convergence is in L_2.We compute $\mathcal{F}[(S_{(\epsilon_2+1)}^{-1} f_{(\epsilon_2+1),(m+\epsilon_0)})\mathcal{F}^{-1}(g)]$. For $h \in L_2((1+\epsilon_2)E)$ we have

$$h = S_{(1+\epsilon_2)}\big(S_{(1+\epsilon_2)}^{-1} h\big) = \sum_k \langle S_{(1+\epsilon_2)}^{-1} h, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, f_{(1+\epsilon_2),k}$$

$$= \sum_k \langle h, S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, f_{(1+\epsilon_2),k}.$$

Letting $h = S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}$,

$$S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)} = \sum_k \langle S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}, S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, f_{(1+\epsilon_2),k}.$$

This gives

$$\mathcal{F}\big[\big(S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}\big)\mathcal{F}^{-1}(g)\big](\cdot)$$

$$= \sum_k \langle S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}, S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, \mathcal{F}\big[f_{(1+\epsilon_2),k}\mathcal{F}^{-1}(g)\big](\cdot)$$

$$= \sum_{\boldsymbol{k}} \langle \boldsymbol{S}_{(1+\epsilon_2)}^{-1} \boldsymbol{f}_{(1+\epsilon_2),(m+\epsilon_0)}, \boldsymbol{S}_{(1+\epsilon_2)}^{-1} \boldsymbol{f}_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} \int_{(1+\epsilon_2)E} \boldsymbol{e}^{i\langle\xi,\frac{t_k}{(1+\epsilon_2)}\rangle} \boldsymbol{\mathcal{F}^{-1}}(\boldsymbol{g})(\boldsymbol{\xi})\boldsymbol{e}^{-i\langle\xi,.\rangle}\, \boldsymbol{d\xi}$$

$$= \sum_{\boldsymbol{k}} \langle \boldsymbol{S}_{(1+\epsilon_2)}^{-1} \boldsymbol{f}_{(1+\epsilon_2),(m+\epsilon_0)}, \boldsymbol{S}_{(1+\epsilon_2)}^{-1} \boldsymbol{f}_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} \int_{(1+\epsilon_2)E} \boldsymbol{\mathcal{F}^{-1}}(\boldsymbol{g})(\boldsymbol{\xi})\boldsymbol{e}^{-i\langle \cdot - \frac{t_k}{(1+\epsilon_2)},\xi\rangle}\, \boldsymbol{d\xi}$$

$$= \sum_k \langle S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}, S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}\, g\left(\cdot - \frac{t_k}{(1+\epsilon_2)}\right),$$

so (70) follows from (68).

Step 3: We show that

$$\langle S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)}, S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}$$

$$= \frac{1}{(1+\epsilon_2)^d}\langle S^{-1} f_{(m+\epsilon_0)}, S^{-1} f_k\rangle_E \text{, for } \epsilon_0 > 0, k \in \mathbb{N}. \quad (71)$$

First we show $(S_{(1+\epsilon_2)}^{-1} f_{(1+\epsilon_2),(m+\epsilon_0)})(\cdot) = \frac{1}{(1+\epsilon_2)^d}\,(S_{(1+\epsilon_2)}^{-1} f_{(m+\epsilon_0)})\left(\frac{\cdot}{(1+\epsilon_2)}\right)$, or equivalently that

$f_{(1+\epsilon_2),(m+\epsilon_0)} = \frac{1}{(1+\epsilon_2)^d} S_{(1+\epsilon_2)}\left((S^{-1} f_{(m+\epsilon_0)})\left(\frac{\cdot}{(1+\epsilon_2)}\right)\right)$.We have for any $g \in L_2((1+\epsilon_2)E)$,

$$\langle g, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} = \int_{(1+\epsilon_2)E} g(\xi)e^{-i\langle\frac{\xi}{(1+\epsilon_2)},t_k\rangle}\, d\xi$$

$$= (1+\epsilon_2)^d \int_E g\big((1+\epsilon_2)x\big)e^{-i\langle x,t_k\rangle}\, dx = (1+\epsilon_2)^d \langle g\big((1+\epsilon_2)(\cdot)\big), f_k\rangle_E.$$

By definition of the frame operator

$$S_{(1+\epsilon_2)}, \qquad S_{(1+\epsilon_2)}g \ = \sum_{k\in\mathbb{N}} \langle g, f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E} \ f_{(1+\epsilon_2),k},$$

which then becomes $S_{(1+\epsilon_2)}g = \ (1+\epsilon_2)^d \sum_k \langle g\big((1+\epsilon_2)(\cdot)\big), f_k\rangle_E \ f_{(1+\epsilon_2),k}$.
Substituting $g \ = \ \frac{1}{(1+\epsilon_2)^d}(S^{-1}f_{(m+\epsilon_0)})\left(\frac{\cdot}{(1+\epsilon_2)}\right)$ into the equation above we obtain

$$\frac{1}{(1+\epsilon_2)^d} S_{(1+\epsilon_2)}\left(\left(S^{-1}f_{(m+\epsilon_0)}\right)\left(\frac{\cdot}{(1+\epsilon_2)}\right)\right) = \sum_k \langle S^{-1}f_{(m+\epsilon_0)}, f_k\rangle_E \ f_{(1+\epsilon_2),k}$$

$$= \left(S\big(S^{-1}f_{(m+\epsilon_0)}\big)\right)\left(\frac{\cdot}{(1+\epsilon_2)}\right) = \ f_{(1+\epsilon_2),(m+\epsilon_0)}.$$

We now compute the desired inner product:

$$\langle S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),(m+\epsilon_0)}, S^{-1}_{(1+\epsilon_2)} f_{(1+\epsilon_2),k}\rangle_{(1+\epsilon_2)E}$$

$$= \frac{1}{(1+\epsilon_2)^{2d}} \int\limits_{(1+\epsilon_2)E} \left(S^{-1}f_{(m+\epsilon_0)}\right)\left(\frac{x}{(1+\epsilon_2)}\right)\overline{(S^{-1}f_k)\left(\frac{x}{(1+\epsilon_2)}\right)}\, dx$$

$$= \frac{(1+\epsilon_2)^d}{(1+\epsilon_2)^{2d}} \int\limits_E \left(S^{-1}f_{(m+\epsilon_0)}\right)(x)\overline{(S^{-1}f_k)(x)}\, dx \ = \ \frac{1}{(1+\epsilon_2)^d}\langle S^{-1}f_{(m+\epsilon_0)}, S^{-1}f_k\rangle_E \, .$$

Note that (70) becomes

$$f\,(\cdot\,) = \frac{1}{(1+\epsilon_2)^d}\sum_{(m+\epsilon_0)} f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\sum_k \langle S^{-1}f_{(m+\epsilon_0)}, S^{-1}f_k\rangle g\left(\cdot - \frac{t_k}{(1+\epsilon_2)}\right)\right]. \quad (72)$$

Step4: The map $V : \ell_2(\mathbb{N}) \mapsto \ell_2(\mathbb{N})$ given by $x \ = \ (x_k)_{k\in\mathbb{N}} \ \mapsto \ \left(\sum_{(m+\epsilon_0)} B_{k(m+\epsilon_0)}x_{(m+\epsilon_0)}\right)_{k\in\mathbb{N}} \ = \ Bx$ is bounded linear and self-adjoint. Let $(d_k)_{k\in\mathbb{N}}$ be the standard basis for $\ell_2(\mathbb{N})$, and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis for $L_2(E)$. Then

$$Vd_j \ = \ \left(B_{kj}\right)_{k\in\mathbb{N}} = \sum_k B_{kj}d_k \ = \sum_k \langle S^{-1}f_j, S^{-1}f_k\rangle_{d_k} = \sum_k \langle L^*(S^{-1})^2 Le_j \,, e_k\rangle_{d_k},$$

where L is the synthesis f operator, i.e., $S \ = \ LL^*$. Define $\varphi : \ \ell^2\,(\mathbb{N}) \ \mapsto \ L_2(E)$ by $\varphi(d_k) \ = \ e_k, k \ \in \ \mathbb{N}$. Clearly φ is unitary. It follows that $V \ = \ \varphi^{-1}L^*(S^{-1})^2 L\varphi$, which concludes Step 4. From here on we identify V with B. Clearly B is an onto isomorphism iff L and L^* are both onto, i.e., iff the map $Le_{(m+\epsilon_0)} \ = \ f_{(m+\epsilon_0)}$ is an onto isomorphism.

Step 5: Verification of (67). Recalling Definition (5.2.8),

$f_{S/(1+\epsilon_2)} \ = \ \left(f\ \left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right)_{(m+\epsilon_0)\in\mathbb{N}}$, for each $t \ \in \ \mathbb{R}^d$, let

$g_{(1+\epsilon_2)}(t) = \left(g\left(t - \frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right)_{(m+\epsilon_0)\in\mathbb{N}}$. Noting that

$f\ \left(\frac{\cdot}{(1+\epsilon_2)}\right), g\left(t \ - \ \frac{\cdot}{(1+\epsilon_2)}\ \right) \in \ L_2((1+\epsilon_2)E),$ and recalling that $(\ f_{(1+\epsilon_2),(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a frame for $L_2((1+\epsilon_2)E)$, we have

$$\sum_{(m+\epsilon_0)} \left| f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right|^2 = \sum \left|\langle \mathcal{F}^{-1}(\,f\,), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\right|^2 \ \le A_{(1+\epsilon_2)}\|\mathcal{F}^{-1}(\,f\,)\|^2, (73)$$

and

$$\sum_{(m+\epsilon_0)} \left|g\left(t-\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right|^2 = \sum \left|\langle \mathcal{F}^{-1}\left(g\left(t - \frac{\cdot}{(1+\epsilon_2)}\right)\right), f_{(1+\epsilon_2),(m+\epsilon_0)}\rangle_{(1+\epsilon_2)E}\right|^2$$

$$\leq A_{(1+\epsilon_2)} \left\|\mathcal{F}^{-1}\left(g\left(t - \frac{\cdot}{(1+\epsilon_2)}\right)\right)\right\|^2 \quad .$$

Note that (72) becomes

$$f(t) = \frac{1}{(1+\epsilon_2)^d} \sum_{(m+\epsilon_0)} f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\left[\sum_k B_{k(m+\epsilon_0)}\, g\left(t-\frac{t_k}{(1+\epsilon_2)}\right)\right]$$

$$= \frac{1}{(1+\epsilon_2)^d} \sum_{(m+\epsilon_0)} f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\overline{\left[\sum_k B_{(m+\epsilon_0)k}\, \overline{g\left(t-\frac{t_k}{(1+\epsilon_2)}\right)}\right]}$$

$$= \frac{1}{(1+\epsilon_2)^d} \sum_{(m+\epsilon_0)} \left(f_{S/(1+\epsilon_2)}\right)_{(m+\epsilon_0)}\left(B\overline{g_{(1+\epsilon_2)}(t)}\right)_{(m+\epsilon_0)} = \frac{1}{(1+\epsilon_2)^d}\langle f_{\frac{S}{(1+\epsilon_2)}}, B\overline{g_{(1+\epsilon_2)}(t)}\rangle$$

$$= \frac{1}{(1+\epsilon_2)^d}\langle Bf_{S/(1+\epsilon_2)}, \overline{g_{(1+\epsilon_2)}(t)}\rangle$$

$$= \frac{1}{(1+\epsilon_2)^d}\sum_k \left(Bf_{S/(1+\epsilon_2)}\right)_k g\left(t-\frac{t_k}{(1+\epsilon_2)}\right)$$

$$= \frac{1}{(1+\epsilon_2)^d}\sum_{k\in\mathbb{N}}\left(\sum_{(m+\epsilon_0)\in\mathbb{N}} B_{k(m+\epsilon_0)} f\left(\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right) g\left(t-\frac{t_k}{(1+\epsilon_2)}\right),$$

which proves (67).

Step 6: We verify that convergence in (67) is in $L_2(\mathbb{R})$ (hence uniform). Define

$$f_{(m+\epsilon_0)}(t) = \frac{1}{(1+\epsilon_2)^d}\sum_{1\leq k\leq (m+\epsilon_0)} \left(Bf_{S/(1+\epsilon_2)}\right)_k g\left(t-\frac{t_k}{(1+\epsilon_2)}\right)$$

and

$$f_{m,(m+\epsilon_0)}(t) = \frac{1}{(1+\epsilon_2)^d}\sum_{m\leq k\leq (m+\epsilon_0)} \left(Bf_{S/(1+\epsilon_2)}\right)_k g\left(t-\frac{t_k}{(1+\epsilon_2)}\right).$$

Then

$$[\mathcal{F}^{-1}(f_{m,(m+\epsilon_0)})](\xi) = \frac{1}{(1+\epsilon_2)^d}\sum_{m\leq k\leq (m+\epsilon_0)} \left(Bf_{S/(1+\epsilon_2)}\right)_k \mathcal{F}^{-1}\left[g\left(\cdot-\frac{t_{(m+\epsilon_0)}}{(1+\epsilon_2)}\right)\right]$$

$$= \frac{1}{(1+\epsilon_2)^d}\sum_{m\leq k\leq (m+\epsilon_0)} \left(Bf_{S/(1+\epsilon_2)}\right)_k \mathcal{F}^{-1}(g)(\xi)e^{i\langle \xi, \frac{t_k}{(1+\epsilon_2)}\rangle},$$

so

$$\|[\mathcal{F}^{-1}(f_{m,(m+\epsilon_0)})]\|_2^2 = \frac{1}{(1+\epsilon_2)^d}\int_{(1+\epsilon_2)E} |\mathcal{F}^{-1}(g)(\xi)|^2 \left|\sum_{m\leq k\leq (m+\epsilon_0)} \left(Bf_{S/(1+\epsilon_2)}\right)_k e^{i\langle \xi, \frac{t_k}{(1+\epsilon_2)}\rangle}\right|^2 d\xi$$

$$\leq \frac{1}{(1+\epsilon_2)^d}\left\|\sum_{m\leq k\leq (m+\epsilon_0)} \left(Bf_{S/(1+\epsilon_2)}\right)_k f_{(1+\epsilon_2),k}\right\|_2^2.$$

If $\left(h_{(m+\epsilon_0)}\right)_{(m+\epsilon_0)}$ is a orthonormal basis for $L_2((1+\epsilon_2)E)$, then the map

$Th_K = f_{(1+\epsilon_2),k}$ (the synthesis operator) is bounded linear, so

$$\left\|\left[\mathcal{F}^{-1}\left(f_{m,(m+\epsilon_0)}\right)\right]\right\|_2^2 \le \frac{1}{(1+\epsilon_2)^d}\left\|T\left(\sum_{m\le k\le (m+\epsilon_0)}\left(Bf_{S/(1+\epsilon_2)}\right)_k h_K\right)\right\|_2^2$$
$$\le \frac{1}{(1+\epsilon_2)^d}\|T\|^2 \sum_{m\le k\le (m+\epsilon_0)}\left|\left(Bf_{S/(1+\epsilon_2)}\right)_k\right|^2.$$

But $Bf_{S/(1+\epsilon_2)} \in \ell^2(\mathbb{N})$, so $\left\|\left[\mathcal{F}^{-1}\left(f_{m,(m+\epsilon_0)}\right)\right]\right\|2 \to 0$ as $m \to \infty, \epsilon_0 > 0$. As $\mathcal{F}^{-1}$ is an onto isomorphism, we have $\left\|f_{m,(m+\epsilon_0)}\right\| \to 0$, implying that $\left\|f - f_{(m+\epsilon_0)}\right\| \to 0$ as $m \to \infty$. Note that (45) is conveniently written as

$$f(t) = \frac{1}{(1+\epsilon_2)^d}\sum_k \left(Bf_{S/(1+\epsilon_2)}\right)_k g\left(t - \frac{t_k}{(1+\epsilon_2)}\right), t \in \mathbb{R}^d. \quad (74)$$

Corollary$(5.2.28)$[296]: Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and $S = (t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\lim_{k\to\infty}\|(m+\epsilon_0)_k - t_k\|_\infty = 0$. Define $e_k, f_k : \mathbb{R}^d \to \mathbb{C}$ *by* $e_k(x) = \frac{1}{(2\pi)^{d/2}} e^{i\langle (m+\epsilon_0)_k, x\rangle}$ and $\frac{1}{(2\pi)^{d/2}} e^{i\langle t_k, x\rangle}$, and let $(h_k)_k$ be the standard basis for $\ell_2(\mathbb{N})$. *Let* $P_l: \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the orthogonal projection onto span$\{h_1, \dots, h_l\}$. If $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, then for all $f \in PW_{[-\pi,\pi]^d}$, we have

$$f(t) = \lim_{l\to\infty}\frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{S/(1+\epsilon_2)}\right]_k g\left(t - \frac{t_k}{(1+\epsilon_2)}\right), t \in \mathbb{R}^d, (75)$$

where convergence is in L_2 and uniform. Furthermore,

$$(P_l B^{-1} P_l)_{(m+\epsilon_0)m} = \begin{cases} sinc\pi(t_{(m+\epsilon_0),1} - t_{m,1}) \cdots sinc\pi(t_{(m+\epsilon_0),d} - t_{m,d}), & 1 \le (m+\epsilon_0), m \le l, \\ 0, & otherwise. \end{cases}$$

Convergence of the sum is in L_2 and also uniform.

Proof: Step 1*:* B is a compact perturbation of the identity map, namely

$$B = I + \lim_{l\to\infty}(-P_l + (P_l B^{-1} P_l)^{-1}). \quad (76)$$

Since $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, $L^* = (I - T)$ is an onto isomorphism where $T_{e_k} = e_k - f_k$; so B simplifies to $(I-T)^{-1}(I - T^*)^{-1}$. We examine $B^{-1} = (I - T^*)(I - T) = I + (T^*T - T - T^*) = I + \Delta$, where Δ is a compact operator. If an operator $\Delta: H \to H$ is compact then so is Δ^*, hence $P_l\Delta P_l \to \Delta$ in the operator norm because

$$\|P_l\Delta P_l - \Delta\| \le \|P_l\Delta P_l - P_l\Delta\| + \|P_l\Delta - \Delta\| \le \|\Delta P_l - \Delta\| + \|P_l\Delta - \Delta\|$$
$$= \|P_l\Delta^* - \Delta^*\| + \|P_l\Delta - \Delta\| \to 0 .$$

We have $B^{-1} = \lim_{l\to\infty}(I + P_l\Delta P_l) = \lim_{l\to\infty}(I + P_l(B^{-1} - I)P_l)$

$$= \lim_{l\to\infty}(I - P_l + P_l B^{-1} P_l).$$

Now $(P_l B^{-1} P_l)$ restricted to the first l rows and columns is the Grammian matrix for the set $(f_1, \dots, f_l)$ which can be shown (in a straightforward manner) to be linearly independent. We conclude that $P_l B^{-1} P_l$ is invertible as an $l \times l$ matrix. By $(P_l B^{-1} P_l)^{-1}$ we mean the inverse as an $l \times l$ matrix and zeroes elsewhere. Observing that the ranges of $P_l B^{-1} P_l$ and $(P_l B^{-1} P_l)^{-1}$are in the kernel of $1- P_l$, and that the range of $I - P_l$ is in the kernels of $P_l B^{-1} P_l$ and $(P_l B^{-1} P_l)^{-1}$, we easily compute

$$(I - P_l + (P_l B^{-1} P_l)^{-1})^{-1} = I - P_l + P_l B^{-1} P_l,$$

so that $B^{-1} = \lim_{l\to\infty}(I - P_l + (P_l B^{-1} P_l)^{-1})^{-1}$, implying

$$B = \lim_{l\to\infty}(I - P_l + (P_l B^{-1} P_l)^{-1}) = \lim_{l\to\infty} B_l = \lim_{l\to\infty}(- P_l + (P_l B^{-1} P_l)^{-1}) .$$

Step 2: We verify (75) and its convergence properties. Recalling (74), we have

$$f(t) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\left[(I - P_l + (P_l B^{-1} P_l)^{-1}) f_{S/(1+\epsilon_2)}\right]_k g\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$

$$= \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/(1+\epsilon_2)}\right]_k g\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$

implying

$$f(t) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{S/(1+\epsilon_2)}\right]_k g\left(t - \frac{t_k}{(1+\epsilon_2)}\right) =$$

$$\frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/(1+\epsilon_2)}\right]_k g\left(t - \frac{t_k}{(1+\epsilon_2)}\right)$$

$$+ \frac{1}{(1+\epsilon_2)^d}\sum_{k=l+1}^{\infty} f\left(\frac{t_k}{(1+\epsilon_2)}\right) g\left(t - \frac{t_k}{(1+\epsilon_2)}\right).$$

Therefore,

$$\left\| f(.) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{S/(1+\epsilon_2)}\right]_k g\left(. - \frac{t_k}{(1+\epsilon_2)}\right)\right\|_2$$

$$= \left\| \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/(1+\epsilon_2)}\right]_k g\left(. - \frac{t_k}{(1+\epsilon_2)}\right) + \frac{1}{(1+\epsilon_2)^d}\sum_{k=l+1}^{\infty} f\left(\frac{t_k}{(1+\epsilon_2)}\right) g\left(. - \frac{t_k}{(1+\epsilon_2)}\right)\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$

$$= \frac{1}{(1+\epsilon_2)^d}\left\| \mathcal{F}^{-1}(g)(.)\left(\sum_{k=1}^{\infty}\left[(B - B_l) f_{S/(1+\epsilon_2)}\right]_k e^{i\langle ., \frac{t_k}{(1+\epsilon_2)}\rangle} + \sum_{k=l+1}^{\infty} f\left(\frac{t_k}{(1+\epsilon_2)}\right) e^{i\langle ., \frac{t_k}{(1+\epsilon_2)}\rangle}\right)\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$

after taking the inverse Fourier transform. Now

$$\left\| f(.) - \frac{1}{(1+\epsilon_2)^d}\sum_{k=1}^{l}\left[(P_l B^{-1} P_l)^{-1} f_{S/(1+\epsilon_2)}\right]_k g\left(. - \frac{t_k}{(1+\epsilon_2)}\right)\right\|_2$$

$$\le \frac{1}{(1+\epsilon_2)^d}\left\|\sum_{k=1}^{\infty}\left[(B - B_l)f_{S/(1+\epsilon_2)}\right]_k e^{i\langle \cdot\,,\frac{t_k}{(1+\epsilon_2)}\rangle}\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$
$$+\frac{1}{(1+\epsilon_2)^d}\left\|\sum_{k=l+1}^{\infty} f\left(\frac{t_k}{(1+\epsilon_2)}\right)e^{i\langle \cdot\,,\frac{t_k}{(1+\epsilon_2)}\rangle}\right\|_{[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d}$$
$$\le \frac{M}{(1+\epsilon_2)^d}\left\|(B - B_l)f_{S/(1+\epsilon_2)}\right\|_{\ell^2(\mathbb{N})}+\frac{M}{(1+\epsilon_2)^d}\left(\sum_{k=l+1}^{\infty}\left|f\left(\frac{t_k}{(1+\epsilon_2)}\right)\right|^2\right)^{\frac{1}{2}},$$

since $\left(f_k\left(\frac{\cdot}{(1+\epsilon_2)}\right)\right)_k$ is a Riesz basis for $L_2[-(1+\epsilon_2)\pi,(1+\epsilon_2)\pi]^d$. Since $B_l \to B$ as $l \to \infty$ and $\left(f\left(\frac{t_k}{(1+\epsilon_2)}\right)\right)_k \in \ell^2(\mathbb{N})$, the last two terms in the inequality above tend to zero, which proves the required result. Finally, to compute $(P_l B^{-1} P_l)_{nm}$, recall that $B^{-1} = (I - T^*)(I - T)$. Proceeding in a manner similar to the proof of (73), we obtain $B^{-1}_{m(m+\epsilon_0)} = \langle LL^* e_{(m+\epsilon_0)}, e_m\rangle = \langle L^* e_{(m+\epsilon_0)}, L^* e_m\rangle = \langle f_{(m+\epsilon_0)}, f_m\rangle$

$$= \ sinc\pi\big(t_{(m+\epsilon_0),1} - t_{m,1}\big)\cdot \cdots \cdot\ sinc\pi\big(t_{(m+\epsilon_0),d} - t_{m,d}\big).$$

The entries of $P_l B^{-1} P_l$ agree with those of B^{-1} when $1 \le (m+\epsilon_0), m \le l$.

Corollary $(5.2.29)[296]$: Let $((m+\epsilon_0)_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$. Define $e_k, f_k : \mathbb{R}^d \to \mathbb{C}$ by $e_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle (m+\epsilon_0)_k, x\rangle}$ and $f_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{i\langle t_k, x\rangle}$. Then for any $\epsilon_4 > \epsilon_3 \ge 0$, and any finite sequence $(a_k)_{k=(1+\epsilon_3)}^{(1+\epsilon_4)}$, we have

$$\left\|\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)}\left(\frac{a_k}{(2\pi)^{\frac{d}{2}}}\, e^{\,i\langle(\cdot),(m+\epsilon_0)_k\rangle} - \frac{a_k}{(2\pi)^{\frac{d}{2}}}\, e^{\,i\langle(\cdot),t_k\rangle}\right)\right\|_2$$
$$\le \left(e^{\pi d(sup_{(1+\epsilon_3)\le k\le(1+\epsilon_4)}\|(m+\epsilon_0)_k - t_k\|_\infty)} - 1\right)\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)}|a_k|^2\right)^{1/2}. \quad (77)$$

Proof : Let $\delta_k = t_k - (m+\epsilon_0)_k$ where $\delta_k = (\delta_{k_1},\ldots,\delta_{k_d})$. Then

$$\varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x) = \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)}\frac{a_k}{(2\pi)^{\frac{d}{2}}}\left[e^{i\langle(m+\epsilon_0)_k,x\rangle}, e^{i\langle t_k,x\rangle}\right]$$
$$= \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)}\frac{a_k}{(2\pi)^{\frac{d}{2}}}e^{i\langle(m+\epsilon_0)_k,x\rangle}\left[1 - e^{i\langle\delta_k,x\rangle}\right]. \quad (78)$$

Now for any δ_k,

$$1 - e^{i\langle\delta_k,x\rangle} = 1 - e^{i\delta_{k_1}x_1}\ldots e^{i\delta_{k_d}x_d}$$
$$= 1 - \left(\sum_{j_1=0}^{\infty}\frac{(i\delta_{k_1}x_1)^{j_1}}{j_1!}\right)\cdots\left(\sum_{j_d=0}^{\infty}\frac{(i\delta_{k_d}x_d)^{j_d}}{j_d!}\right)$$

$$= 1 - \sum_{\substack{(j_1,\dots,j_d)\\ j_i\geq 0}} \frac{(i\delta_{k_1}x_1)^{j_1} \cdot\cdot\cdot\cdot\cdot (i\delta_{k_d}x_d)^{j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!}$$

$$= - \sum_{(j_1,\dots,j_d)\in J} i^{j_1,\dots,j_d} \frac{(i\delta_{k_1}x_1)^{j_1} \cdot\cdot\cdot\cdot\cdot (i\delta_{k_d}x_d)^{j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!},$$

where $J = \{(j_1,\dots,j_d) \in \mathbb{Z}^d \mid j_i \geq 0, (j_1,\dots,j_d) \neq 0\}$. Then (78) becomes

$$\varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x) = -\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \frac{a_k}{(2\pi)^{\frac{d}{2}}} e^{i\langle (m+\epsilon_0)_k, x\rangle} \left[\sum_{(j_1,\dots,j_d)\in J} i^{j_1,\dots,j_d} \frac{(i\delta_{k_1}x_1)^{j_1} \cdot\cdot\cdot\cdot\cdot (i\delta_{k_d}x_d)^{j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!}\right]$$

$$= -\sum_{(j_1,\dots,j_d)\in J} \frac{x_1^{j_1} \cdot\cdot\cdot x_d^{j_d}}{j_1! \cdot\cdot\cdot j_d!} i^{j_1,\dots,j_d} \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} \frac{a_k}{(2\pi)^{\frac{d}{2}}} \delta_{k_1}^{j_1} \cdot\cdot\cdot\cdot\cdot \delta_{k_d}^{j_d} e^{i\langle (m+\epsilon_0)_k, x\rangle},$$

so

$$\left|\varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x)\right| \leq \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} \sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} a_k \delta_{k_1}^{j_1} \cdot\cdot\cdot\cdot\cdot \delta_{k_d}^{j_d} \frac{e^{i\langle (m+\epsilon_0)_k, x\rangle}}{(2\pi)^{\frac{d}{2}}}.$$

For brevity denote the outer summand above by $h_{j_1,\dots,j_d}(t)$. Then

$$\left(\int_{[-\pi,\pi]^d} \left|\varphi_{(1+\epsilon_3),(1+\epsilon_4)}(x)\right|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_{[-\pi,\pi]^d} \left|\sum_{(j_1,\dots,j_d)\in J} h_{j_1,\dots,j_d}(x)\right|^2 dx\right)^{\frac{1}{2}}$$

$$\leq \sum_{(j_1,\dots,j_d)\in J} \left(\int_{[-\pi,\pi]^d} \left|h_{j_1,\dots,j_d}(x)\right|^2 dx\right)^{\frac{1}{2}},$$

so that

$\left\|\varphi_{(1+\epsilon_3),(1+\epsilon_4)}\right\|_2$

$$\leq \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1! \cdot\cdot\cdot j_d!} \left(\int_{[-\pi,\pi]^d} \left|\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} a_k \delta_{k_1}^{j_1} \cdot\cdot\cdot\cdot \delta_{k_d}^{j_d} \frac{e^{i\langle (m+\epsilon_0)_k, x\rangle}}{(2\pi)^{\frac{d}{2}}}\right|^2 dx\right)^{\frac{1}{2}}$$

$$= \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} \left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2 \left|\delta_{k_1}^{j_1}\right|^2 \cdot\cdot\cdot\cdot\cdot \left|\delta_{k_d}^{j_d}\right|^2\right)^{\frac{1}{2}}$$

$$\leq \sum_{(j_1,\dots,j_d)\in J} \frac{\pi^{j_1,\dots,j_d}}{j_1! \cdot\cdot\cdot j_d!} \left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2 \left(\sup_{(1+\epsilon_3)\leq k\leq(1+\epsilon_4)} \|(m+\epsilon_0)_k - t_k\|_\infty\right)^{2(j_1,\dots,j_d)}\right)^{\frac{1}{2}}$$

$$= \sum_{(j_1,\dots,j_d)\in J} \frac{\pi\left(sup_{(1+\epsilon_3)\leq k\leq(1+\epsilon_4)} \|(m+\epsilon_0)_k - t_k\|_\infty\right)^{j_1,\dots,j_d}}{j_1! \cdot\cdot\cdot\cdot\cdot j_d!} \left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\right)^{\frac{1}{2}}$$

$$= \left[\prod_{l=1}^{d}\left(\sum_{j_\ell=0}^{\infty} \frac{\pi\left(sup_{(1+\epsilon_3)\leq k\leq(1+\epsilon_4)} \|(m+\epsilon_0)_k - t_k\|_\infty\right)^{j_\ell}}{j_\ell!}\right) - 1\right]\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\right)^{\frac{1}{2}}$$

$$= \left(e^{\pi d(sup_{(1+\epsilon_3)\le k\le(1+\epsilon_4)}\|(m+\epsilon_0)_k - t_k\|_\infty)} - 1\right)\left(\sum_{k=(1+\epsilon_3)}^{(1+\epsilon_4)} |a_k|^2\right)^{\frac{1}{2}}.$$

Corollary $(5.2.30)$[296]: Let $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence (with $t_{(m+\epsilon_0)} \neq 0$ for $(m+\epsilon_0) \neq 0$) such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)} = \left(\frac{1}{\sqrt{2\pi}} e^{it_{(m+\epsilon_0)}(\cdot)}\right)_{(m+\epsilon_0)}$ is a Riesz basis for $L_2[-\pi,\pi]$, and let $(e_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ be the standard exponential orthonormal basis for $L_2[-\pi,\pi]$. If the map L given by $Le_{(m+\epsilon_0)} = f_{(m+\epsilon_0)}$ satisfies the estimate $I - L < 1$, then the biorthogonals $G_{(m+\epsilon_0)}$ of $\frac{1}{\sqrt{2\pi}}\mathcal{F}(f_{(m+\epsilon_0)})(\cdot) = sinc(\pi(\cdot - t_{(m+\epsilon_0)}))\ in\ PW_{[-\pi,\pi]}$ are

$$G_{(m+\epsilon_0)}(t) = \frac{H(t)}{(t - t_{(m+\epsilon_0)})H'(t_{(m+\epsilon_0)})}, \qquad (m+\epsilon_0) \in \mathbb{Z}, \quad (79)$$

where

$$H(t) = (t - t_0)\prod_{(m+\epsilon_0)=1}^{\infty}\left(1 - \frac{t}{t_{(m+\epsilon_0)}}\right)\left(1 - \frac{t}{t_{-(m+\epsilon_0)}}\right). \quad (80)$$

Proof : We see that $\delta_{(m+\epsilon_0)m} = \langle G_{l,m}, S_{l,(m+\epsilon_0)}\rangle$, where $S_{l,(m+\epsilon_0)}(t) = sinc(\pi(t - t_{(m+\epsilon_0)}))$ when $|(m+\epsilon_0)| \le l$ and $S_{l,(m+\epsilon_0)}(t) = sinc(\pi(t - (m+\epsilon_0)))$ when $|m| > l$. Without loss of generality, let $|m| < l$. (1) implies that $G_{l,m}(k) = 0$ when $|k| > l$. By the WKS theorem we have

$$G_{l,m}(t) = \sum_{k=-l}^{k=l} G_{l,m}(k) sinc(\pi(t-k))$$
$$= \left(\sum_{k=-l}^{k=l}\frac{(-1)^{k-1} t G_{l,m}(k)}{k - t}\right) sinc(\pi t)$$
$$= \frac{w_l(t)}{\prod_{k=1}^{l}(k-t)(-k-t)} sinc(\pi t),$$

where w_l is a polynomial of degree at most $2l$. Noting that

$$sinc(\pi t) = \prod_{k=1}^{\infty}\left(1 - \frac{t^2}{k^2}\right) \text{ and } \prod_{k=1}^{l}(k-t)(-k-t) = (-1)^l (l!)^2 \prod_{k=1}^{l}\left(1 - \frac{t^2}{k^2}\right),$$

we have

$$G_{l,m}(t) = \frac{(-1)^l w_l(t)}{(l!)^2}\prod_{k=l+1}^{\infty}\left(1 - \frac{t^2}{k^2}\right).$$

Again by (1), $\delta_{(m+\epsilon_0)m} = G_{l,m}(t_{(m+\epsilon_0)})$ when $|(m+\epsilon_0)| \le l$ so that

$$\delta_{(m+\epsilon_0)m} = \frac{(-1)^l}{(l!)^2} w_l(t_{(m+\epsilon_0)})\prod_{k=l+1}^{\infty}\left(1 - \frac{t^2_{(m+\epsilon_0)}}{k^2}\right).$$

This determines the zeroes of w_l. We deduce that

$$w_l(t) = \frac{c_l\prod_{k=1}^{k=l}(t - t_k)(t - t_{-k})}{t - t_m}$$

for some constant c_l . Absorbing constants, we have
$G_{l,m}(t) = \frac{c_l H_l(t)}{t - t_m}$, where

$$H_l(t) = (t - t_0)\prod_{k=1}^{l}\left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right)\prod_{l+1}^{\infty}\left(1 - \frac{t^2}{k^2}\right).$$

Now $0 = H_l(t_m)$, so $G_{l,m}(t) = c_l \frac{H_l(t) - H_l(t_m)}{t - t_m}$. Taking limits,
$c_l = \frac{1}{(H_l)'(t_m)}$. This yields $G_{l,m}(t) = \frac{H_l(t)}{(t - t_m)H_l'(t_m)}$.
Define

$$H(t) = (t - t_0)\prod_{k=1}^{\infty}\left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right).$$

Basic complex analysis shows that H is entire, and $H_l \to H$ and $H_l' \to H'$ uniformly on compact subsets of C. Furthermore, $H'(t_k) \neq 0$ for all k, since each t_k is a zero of H of multiplicity one. Together we have

$$\lim_{l\to\infty} G_{l,m}(t) = \frac{H(t)}{(t - t_m)H'(t_m)}, t \in \mathbb{R}.$$

Corollary$(5.2.31)$[296]: Let $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence such that $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ (defined above) is a Riesz basis for $L_2[-\pi,\pi]$. If $(e_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ is the standard exponential orthonormal basis for $L_2[-\pi,\pi]$ and the map L (defined above) satisfies the estimate $\|I - L\| = \delta < 1$, then the following are true:

(i) For $l \geq 0$, the sequence $(f_{l,(m+\epsilon_0)})_{(m+\epsilon_0)}$ is a Riesz basis for$L_2[-\pi,\pi]$.

(ii) For $l \geq 0$, the map L_l defined by $L_l e_{(m+\epsilon_0)} = f_{l,(m+\epsilon_0)}$ satisfies $\|L_l^{-1}\| \leq \frac{1}{1-\delta}$.

Proof : If $(c_{(m+\epsilon_0)})_{(m+\epsilon_0)} \in \ell^2(\mathbb{Z})$, then

$$\begin{aligned}(I - L_l)\left(\sum_{(m+\epsilon_0)} c_{(m+\epsilon_0)} e_{(m+\epsilon_0)}\right) &= \sum_{(m+\epsilon_0)} c_{(m+\epsilon_0)}\left(e_{(m+\epsilon_0)} - L_l e_{(m+\epsilon_0)}\right) \\ &= \sum_{|(m+\epsilon_0)|\leq l}\left(e_{(m+\epsilon_0)} - f_{(m+\epsilon_0)}\right) \\ &= (I - L)P_l\left(\sum_{(m+\epsilon_0)} c_{(m+\epsilon_0)} e_{(m+\epsilon_0)}\right),\end{aligned}$$

so that

$$(I - L_l) = (I - L)P_l \quad . \tag{81}$$

From this, $\|I - L_l\| \leq \delta$, which implies (i) and (ii).

Define the biorthogonal functions of $(f_{l,(m+\epsilon_0)})_{(m+\epsilon_0)}$ to be $(f^*_{l,(m+\epsilon_0)})_{(m+\epsilon_0)}$. Passing to the Fourier transform, we have $\frac{1}{\sqrt{2\pi}}\mathcal{F}(f_{l,(m+\epsilon_0)})(t) = sinc(\pi(t - t_{l,(m+\epsilon_0)}))$ and $G_{l,(m+\epsilon_0)}(t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f^*_{l,(m+\epsilon_0)})(t)$. Define the biorthogonal functions of $(f_{(m+\epsilon_0)})_{(m+\epsilon_0)}$ similarly.

Corollary $(5.2.32)$[296]: If $(t_{(m+\epsilon_0)})_{(m+\epsilon_0)} \subset \mathbb{R}$ satisfies the hypotheses of Proposition (5.2.25), then

$$\lim_{l\to\infty} G_{l,(m+\epsilon_0)} = G_{(m+\epsilon_0)} \quad \text{in} \quad PW_{[-\pi,\pi]} .$$

Proof : Note that $\delta_{(m+\epsilon_0)m} = \langle f_{l,(m+\epsilon_0)}, f_{l,m}^*\rangle = \langle L_l e_{(m+\epsilon_0)}, f_{l,m}^*\rangle = \langle e_{(m+\epsilon_0)}, L_l^* f_{l,m}^*\rangle$ so that for all $m, f_{l,m}^* = (L_l^*)^{-1} e_m$. Similarly, $f_m^* = (L^*)^{-1} e_m$. We have $f_{l,m}^* - f_m^* = ((L_l^*)^{-1} (L^*)^{-1}) e_m = (L_l^*)^{-1}(L^* - L_l^*)(L^*)^{-1} e_m$. Now (84) implies $L - L_l = (I - P_l)(L - I)$, so that $f_{l,m}^* - f_m^* = (L_l^*)^{-1}(L^* - I)(I - P_l)(L^*)^{-1} e_m$. Applying Proposition (5.2.25) yields $\|f_{l,m}^* - f_m^*\| \le \frac{1}{1-\delta}\|(L^* - I)(I - P_l)(L^*)^{-1} e_m\|$, which for fixed m goes to 0 as $l \to \infty$. We conclude $\lim_{l\to\infty} f_{l,m}^* = f_m^*$, which, upon passing to the Fourier transform, yields $\lim_{l\to\infty} G_{l,m} = G_m$.

Corollary$(5.2.33)[296]$: Let $(2t_{(m+\epsilon_0)})_{(m+\epsilon_0)\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence and $\left(f_{2(m+\epsilon_0)}^2\right)_{(m+\epsilon_0)}$ is a Riesz basis for $L_2[-\pi,\pi]$. If $(e_{2(m+\epsilon_0)})_{(m+\epsilon_0)}$ is the standard exponential orthonormal basis for $L_2[-\pi,\pi]$ and the map L^2 satisfies the estimate $\|I - L^2\| = \delta < 1$, then the following are hold:

(i) For $l \ge 0$, the sequence $\left(f_{2l,2(m+\epsilon_0)}^2\right)_{(m+\epsilon_0)}$ is a Riesz basis for$L_2[-\pi,\pi]$.

(ii) For $l \ge 0$, the map L_{2l}^2 defined by $L_{2l}^2 e_{2(m+\epsilon_0)} = f_{2l,2(m+\epsilon_0)}^2$ satisfies $\|L_{2l}^{-2}\| \le \frac{1}{(1-\delta)^2}$.

Proof : For $(c_{2(m+\epsilon_0)})_{(m+\epsilon_0)} \in \ell^2(\mathbb{Z})$, we have

$$(I - L_{2l}^2)\left(\sum_{(m+\epsilon_0)} c_{2(m+\epsilon_0)} e_{2(m+\epsilon_0)}\right) = \sum_{(m+\epsilon_0)} c_{2(m+\epsilon_0)}\left(e_{2(m+\epsilon_0)} - L_{2l}^2 e_{2(m+\epsilon_0)}\right)$$

$$= \sum_{|(m+\epsilon_0)|\le l} \left(e_{2(m+\epsilon_0)} - f_{2(m+\epsilon_0)}^2\right) \quad = (I - L^2)P_{2l}\left(\sum_{(m+\epsilon_0)} c_{2(m+\epsilon_0)} e_{2(m+\epsilon_0)}\right),$$

so that

$$(I - L_{2l}^2) = (I - L^2)P_{2l} .$$

Hence $\|I - L_{2l}^2\| \le \delta$, which gives (i) and (ii) .

Hence from Definition (5.2.4) we can show that

$$A\|f\|^2 \le \sum_{(m+\epsilon_0)} \left|\left\langle f, \left(f_{2l,2(m+\epsilon_0)}^2 \frac{Le_{(m+\epsilon_0)}}{L_{2l}^2 e_{2(m+\epsilon_0)}}\right)\right\rangle\right|^2 \le (A+\epsilon_1)\|f\|^2,$$

for every $f \in H$, $\epsilon_1 > 0$.

Chapter 6

Channeled Sampling and Sampling Expansion with Symmetric Multi-Channel Sampling in Shift-Invariant Spaces

We find necessary and sufficient conditions under which a regular shifted sampling expansion hold on $V(\phi)$ and also introduce a single channel sampling on $V(\phi)$ together with some illustrating examples. We then find necessary and sufficient conditions under which an irregular or a regular shifted sampling expansion formula holds on $V\ (\phi)$and obtain truncation error estimates of the sampling series. We also find a sufficient condition for a function in $L^2(\mathbb{R})$ that belongs to a sampling subspace of $L^2(\mathbb{R})$. Several illustrating examples are also provided. We use Fourier duality between $V(\phi)$ and $L^2[0, 2\pi]$ to find conditions under which there is a stable asymmetric multi-channel sampling formula on $V(\phi)$.

Section(6.1) Channeled Sampling

The celebrated WSK (Whittaker–Shannon–Kotel'nikov)-sampling theorem says that any signal $f(t)$ of finite energy with band-width π, that is, $f \in PW_\pi$ can be reconstructed via its regularly spaced discrete sample values $\{f(n) : n \in \mathbb{Z}\}$ as $f(t) = \sum_{n\in\mathbb{Z}} f\ (n)\ sinc(t - n)$, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$, where, $sinct = \frac{sin\,\pi t}{\pi t}$ is the cardinal sinc function and PW_π is the Paley–Wiener space:

$$PW_\pi = \{f \in L^2(\mathbb{R}) : supp\,\hat{f}\,(\xi) \subseteq [-\pi, \pi]\}.$$

Here $\mathcal{F}[f](\xi) = \hat{f}\,(\xi)$ is the Fourier transform of $f(t)$, which is normalized as $\hat{f}\,(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{-it\xi} dt$ for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ so that $\mathcal{F}[\cdot]$ is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

As a natural generalization of the WSK-sampling theorem, many authors have developed sampling theory on general shift invariant spaces.

For any $\varphi(t)$ in $L^2(\mathbb{R})$, we let $V(\varphi) = \overline{span}\{\varphi(t - n): n \in \mathbb{Z}\}$ be the closed subspace of $L^2(\mathbb{R})$ generated by integer translates $\{\varphi(t-n): n \in \mathbb{Z}\}$ of $\varphi(t)$ and call $V(\varphi)$ the shift invariant space generated by $\varphi(t)$. Then PW_π is the shift invariant space generated by sinct, of which $\{sinc(t - n): n \in \mathbb{Z}\}$ is anorthonormal basis. For example, Walter11developed a regular sampling theorem on a shift invariant space $V(\varphi)$, where $\varphi(t)$ is a continuous real valued orthonormal generator (in fact, a scaling function of an MRA) with decaying property $\varphi(t) = O(|t|^{-1-\epsilon})\ (\epsilon > 0)$ for $|t|$ large. Following [33], Janssen used Zak transform to generalize Walter's result to regular shifted sampling. Zhou and Sun found a necessary and sufficient condition for a regular sampling expansion to hold on $V(\varphi)$ when $V(\varphi)$ is a space of continuous functions generated by a frame generator $\varphi(t)$. Later noting that sinct does not satisfy the Walter's decaying condition, Chen and Itoh extended Walter's work by removing too much restrictive conditions in [33] like continuity and the decaying property on $\varphi(t)$ when $\varphi(t)$ is a Riesz generator. Zhao, Liu, and Zhao extended further results in [37] by considering frame generators. However, there are some gaps in the arguments of the proofs of results in [37]and [41]. We first find necessary and sufficient conditions under which a regular and a regular shifted sampling expansion to hold on $V(\varphi)$ and then extend them into a single channeled sampling expansion.

In the following, we assume that $\varphi(t)$ is a frame or a Riesz (stable) generator of $V(\varphi)$, that is, $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $V(\varphi)$ so that

$$V(\varphi) = f(t) = \sum_{n\in\mathbb{Z}} c(n)\,\varphi(t-n) : \boldsymbol{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2 \quad ,$$

where f(t) is the $L^2 - limit$ of $\sum_{n\in\mathbb{Z}} c(n)\varphi(t-n)$. We are then concerned on the problem: When is there a function $S(t)$, called an interpolation generating function of $V(\varphi)$ for which the sampling expansion formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n),\, f \in V(\varphi)$$

holds in $L^2(\mathbb{R})$-sense.

For any $\varphi(t)$ in $L^2(\mathbb{R})$ and $c = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2, let$\hat{c}^*(\xi) = \sum_{n\in\mathbb{Z}} c(n)\, e^{-in\xi}$: discrete Fourier transform of c, $(c * \varphi)(t) = \sum_{n\in\mathbb{Z}} c(n)\varphi(t-n)$: discrete-continuous convolution product of c and $\varphi(t)$, $G_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2, \xi \in \mathbb{R}$. Then

$\hat{c}^*(\xi)(\xi) = \hat{c}^*(\xi)(\xi + 2\pi) \in L^2[0, 2\pi]$ and

$$\|\hat{c}^*(\xi)\|^2_{L^2[0,2\pi]} = 2\pi\|c\|^2 = 2\pi\sum_{n\in\mathbb{Z}} |c(n)|^2,$$

$G_\varphi(\xi) = G_\varphi(\xi + 2\pi) \in L^1[0, 2\pi]$ and

$\left\|G_\varphi(\xi)\right\|_{L^1[0,2\pi]} = \|\varphi(t)\|^2_{L^2(\mathbb{R})}$. Moreover, we have (see [4]) that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is

(i) a Bessel sequence with a Bessel bound $B > 0$, i.e.

$$\sum_{n\in\mathbb{Z}} |\langle\psi(t), \varphi(t-n)\rangle|^2 \le B\|\psi\|^2, \psi \in L^2(\mathbb{R}) \left(\|\psi\| = \|\psi\|_{L^2(\mathbb{R})}\right)$$

if and only if

$$2\pi G_\varphi(\xi) \le B \quad a.e. \quad \text{on } \mathbb{R} \,, \tag{1}$$

(ii) a frame of $V(\varphi)$ with frame bounds $B \ge A > 0$, i.e.,

$$A\|\psi\|^2 \le \sum_{n\in\mathbb{Z}} |\langle\psi(t), \varphi(t-n)\rangle|^2 \le B\|\psi\|^2, \psi \in V(\varphi)$$

if and only if

$$A \le 2\pi G_\varphi(\xi) \le B\; a.e. on\; suppG_\varphi \,, \tag{2}$$

(iii) a Riesz basis of $V(\varphi)$ with Riesz bounds $B \ge A > 0$, i.e.,

$$A\|c\|^2 \le \|(c * \varphi)(t)\|^2 \le B\|c\|^2, c \in l^2 \tag{3}$$

if and only if

$$A \leq 2\pi G_{\varphi}(\xi) \leq B \; a.e. on \; \mathbb{R}\,, and \tag{4}$$

(iv) an orthonormal basis of $V(\varphi)$, i.e $\|(\boldsymbol{c} * \varphi)(t)\|^2 = \|\boldsymbol{c}\|^2, \boldsymbol{c} \in l^2$ if and only if $2\pi G_{\varphi}(\xi) = 1$ a.e. on $\mathbb{R}$. Here we use supp f for any $f(\xi)$ in $L^1{}_{loc}(\mathbb{R})$ to denote the support of f viewing f as a distribution on $\mathbb{R}$, that is, $\mathbb{R}\backslash supp\, f = \{\xi \in \mathbb{R} : f(\cdot) = 0$ a.e. on some neighborhood of $\xi\}$.

We begin with two simple lemmas, which play key roles in the following.

Lemma(6.1.1)[35]: For any $c = \{c(n)\}_{n\in\mathbb{Z}}$ and $d = \{d(n)\}_{n\in\mathbb{Z}}$ in l^2, let

$$c * d = \left\{(c * d)(n) = \sum_{k\in\mathbb{Z}} c(k)\, d(n-k)\right\}_{n\in\mathbb{Z}}$$

be the discrete convolution product of **c** and **d**. Then

$$\hat{c}^*(\xi)\, \hat{d}^*(\xi) \sim \sum_{n\in Z} (c * d)(n) e^{-in\xi} \tag{5}$$

which means that $\sum_{n\in\mathbb{Z}} (c * d)(n) e^{-in\xi}$ is the Fourier series expansion of $\hat{c}^*(\xi)\, \hat{d}^*(\xi) \in L^1[0, 2\pi]$. Moreover, $c * d \in c_0$ and

$$\int_0^{2\pi} \left|\hat{c}^*(\xi)\hat{d}^*(\xi)\right|^2 d\xi = 2\pi \sum_{n\in\mathbb{Z}} |(c * d)(n)|^2. \tag{6}$$

Proof : Since $\hat{\boldsymbol{c}}^*(\xi)$ and $\hat{d}^*(\xi) \in L^2[0, 2\pi]$, $\hat{\boldsymbol{c}}^*(\xi)\, \hat{d}^*(\xi) \in L^1[0, 2\pi]$ of which theFourier series is

$$\hat{c}^*(\xi)\hat{d}^*(\xi) \sim \sum_{n\in\mathbb{Z}} \frac{1}{2\pi} \langle \hat{c}^*(\xi)\hat{d}^*(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]}\, e^{-in\xi} \quad \text{from which}$$

(5) follows. Then $c * d \in c_0$ by Riemann–Lebesgue lemma and (6)is an immediate consequence of the Parseval's identity. In particular, (6) implies that $\hat{c}^*(\xi)\, \hat{d}^*(\xi) \in L^2[0, 2\pi]$ if and only if $c * d \in l^2$.

Lemma(6.1.2)[35]: Let $\boldsymbol{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2, \varphi(t) \in L^2(\mathbb{R})$, and assume that $(\boldsymbol{c} * \varphi)(t)$ converges in $L^2(\mathbb{R})$. If either $c \in l^2$ or $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence, then

$$\mathcal{F}[c * \varphi](\xi) = \hat{c}^*(\xi)\, \hat{d}^*. \tag{7}$$

Proof : Since $(\boldsymbol{c} * \varphi)(t) = \sum_{n\in\mathbb{Z}} c\,(n)\varphi(t-n)$ converges in $L^2(\mathbb{R})$, $\mathcal{F}[\boldsymbol{c} * \varphi](\xi) = \sum_{n\in\mathbb{Z}} (c(n)\, e^{-in\xi}\, \hat{\varphi}(\xi))$ converges in $L^2(\mathbb{R})$, that is, $\hat{\boldsymbol{c}}_n(\xi)\, \hat{\varphi}(\xi) = \sum_{|k|\leq n} c\,(k)\, e^{-ik\xi}\, \hat{\varphi}(\xi)$ converges to $\mathcal{F}[\boldsymbol{c} * \varphi](\xi)$ in $L^2(\mathbb{R})$. Hence to show (7), it is enough to show that $\hat{\boldsymbol{c}}_n(\xi)\, \hat{\varphi}(\xi)$ converges to $\hat{\boldsymbol{c}}^*(\xi)\, \hat{\varphi}(\xi)$ in $L^2(\mathbb{R})$. when either $\boldsymbol{c} \in l^1$ or $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence. Now

$$\left\|\hat{c}^*{}_n(\xi)\, \hat{\varphi}(\xi) - \hat{c}^*(\xi)\, \hat{\varphi}(\xi)\right\|^2 = \int_{-\infty}^{\infty} |\hat{c}^*{}_n(\xi) - \hat{c}^*(\xi)|^2 |\hat{\varphi}(\xi)|^2\, d\xi$$

$$= \int_0^{2\pi} | \hat{c}^*{}_n(\xi) - \hat{c}^*(\xi)|^2 G_\varphi(\xi) d\xi$$

$$\leq \left\|\hat{c}^*{}_n(\xi) - \hat{c}^*(\xi)\right\|^2_{L^\infty[0,2\pi]} \int_0^{2\pi} G_\varphi(\xi) d\xi$$

$$\leq \left\|G_\varphi(\xi)\right\|_{L^\infty[0,2\pi]} \int_0^{2\pi} \left|\hat{c}^*{}_n(\xi) - \hat{c}^*(\xi)\right|^2 d\xi$$

so that $lim_{n\to\infty} \|\hat{c}^*{}_n(\xi)\,\hat{\varphi}(\xi) - \hat{c}^*(\xi)\,\hat{\varphi}(\xi)\| = 0$ provided that either $c \in l^1$ so that $\hat{c}^*{}_n$ converges to $\hat{c}^*(\xi)$ uniformly on [0, 2π] or $\{\varphi(t - n) : n \in \mathbb{Z}\}$ is a Bessel sequence so that $G_\varphi(\xi) \in L^\infty[0,2\pi]$ by (1). In the following, we let $\varphi(t)$ be a complex valued square integrable function on ℝ such that $\varphi(t)$ is a frame or a Riesz generator of $V(\varphi)$, that is , $\{\varphi(t - n) : n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $V(\varphi)$. We also assume $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$ and set $\hat{\varphi}^*(\xi) = \sum_{n\in\mathbb{Z}} \varphi(n)\, e^{-in\xi} \in L^2[0,2\pi]$. Then

$$V(\varphi) = (c * \varphi)(t) = \sum_{k\in\mathbb{Z}} c(k)\varphi(t-k) : c \in l^2 \quad ,$$

where each $f(t) = (\boldsymbol{c} * \varphi)(t) = \sum_{k\in\mathbb{Z}} c(k)\varphi(t-k)$ converges in $L^2(\mathbb{R})$. In particular,for each $n \in \mathbb{Z}, \sum_{k\in\mathbb{Z}} c(k)\varphi(n-k)$ converges absolutely, which we may set to be $f(n) = \sum_{k\in\mathbb{Z}} c(k)\varphi(n-k)$.Note that as a shift invariant space, $V(\varphi)$ contains $S(t - n)$ for any n in ℤ if $S(t)$ is in $V(\varphi)$. For a measurable set E in ℝ, we let $|E|$ be the Lebesgue measureof E and $\chi_E(\xi)$ the characteristic function of E. For a measurable function $f(t)$on ℝ , let $\|f\|_0 = \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)|$ and $\|f\|_\infty = \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)|$ be the essential infimum and essential supremum of $f(t)$, respectively.

Theorem(6.1.3)[35]:Assume that $\varphi(t)$ is a frame generator of $V(\varphi)$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$. If there is $S(t)$ in $V(\varphi)$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence(respectively, a frame) of $V(\varphi)$ for which the sampling expansion formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), f \in V(\varphi) \qquad (8)$$

holds in the L^2 sense, then

$$supp\hat{\varphi} = supp\,\hat{S} \subset suppG_\varphi = suppG_S \subset supp\hat{\varphi}^* \qquad (9)$$

and there is a constant $\alpha > 0$
(respectively, there are constants $\beta \geq \alpha > 0$) such that

$$\alpha \leq |\hat{\varphi}^*(\xi)| (\text{respectively}, \alpha \leq |\hat{\varphi}^*(\xi)| \leq \beta) a.e. on\ suppG_\varphi. \qquad (10)$$

Moreover $\hat{S}(\xi) = \dfrac{(\hat{\varphi}(\xi)}{(\hat{\varphi}^*(\xi)}\chi suppG_\varphi(\xi) a.e. on\, \mathbb{R}$. (11)

(b) If $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ and there is $S(t) \in V(\varphi)$ such that (8) holds, then (9), (11) hold and

$$\frac{1}{(\hat{\varphi}^*(\xi)}\,\chi suppG_\varphi(\xi) \in L^2\,[0,2\pi] \ . \tag{12}$$

Proof:(a) Assume that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence of $V(\varphi)$ with a Bessel bound B_S for which the sampling expansion formula(8)

holds .Then $S(t) = \sum_{n\in\mathbb{Z}} a\ (n)\varphi(t-n)$ and

$\varphi(t) = \sum_{n\in\mathbb{Z}} \varphi\ (n)S(t-n)$ for some $a = \{a(n)\}_{n\in\mathbb{Z}}$ in l^1.

Then by Lemma (6.1.2) ,

$$\hat{S}(\xi) = \hat{a}^*(\xi)\hat{\varphi}(\xi)\ and\ \hat{\varphi}(\xi) = \hat{\varphi}^*(\xi)\,\hat{S}(\xi) \tag{13}$$

and so

$$G_S(\xi) = |\hat{a}^*(\xi)|^2 G_\varphi(\xi) \text{ and } G_\varphi(\xi) = |\hat{\varphi}^*(\xi)|^2 G_S(\xi), \tag{14}$$

from which (9) follows immediately. We also have from (13)
$\hat{S}(\xi) = 0$ a.e. on (supp$\hat{\varphi}$)^c and $\hat{S}(\xi) = \frac{(\hat{\varphi}(\xi)}{(\hat{\varphi}^*(\xi)}$ a.e. on supp $(\hat{\varphi}^*(\xi)$so that (11) holds by (9). Now (14) implies

$$|\hat{\varphi}^*(\xi)|^2 = \frac{G_\varphi(\xi)}{G_S(\xi)}\ a.e. on\ suppG_\varphi \tag{15}$$

so that $\frac{A_S}{B_S} \le |(\hat{\varphi}^*(\xi)|^2$ a.e. on $suppG_\varphi$, where (A_S, B_S) are frame bounds of $\{\varphi(t-n) : n \in \mathbb{Z}\}$ [see (2)]. If $\{S(t-n) : n \in \mathbb{Z}\}$ is also a frame of $V\,(\varphi)$ withframe bounds (A_S, B_S), then (15) implies $\frac{A_S}{B_S} \le |\hat{\varphi}^*(\xi)|^2 \le \frac{B_S}{A_S}$ a.e. on suppG_φ. Hence (10) holds.

(b) Assume $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ and (8) holds on $V\,(\varphi)$ for some $S(t) \in V\,(\varphi)$. Then (9) and (11) hold by the same arguments as in the proof of (a).We now have from (11) and

$$\chi suppG_\varphi(\xi) = \chi suppG_\varphi(\xi + 2\pi),$$

$$\infty > \int_{-\infty}^{\infty} \left|\hat{S}(\xi)\right|^2 d\xi = \int_{-\infty}^{\infty} \left|\frac{\hat{\varphi}\ (\xi)}{\hat{\varphi}^*(\xi)}\right|^2 \chi suppG_\varphi(\xi)d\xi$$

$$= \int_0^{2\pi} \frac{\hat{\varphi}\ (\xi)}{|\hat{\varphi}^*(\xi)|^2}\,\chi suppG_\varphi(\xi)d\xi$$

$$\ge \frac{A_\varphi}{2\pi}\int_0^{2\pi} \frac{1}{|\hat{\varphi}^*(\xi)|^2}\,\chi suppG_\varphi(\xi)d\xi$$

so that (12) holds. Theorem (6.1.3) gives some necessary conditions for the sampling expansion formula (8) to hold. Conversely, we have:

Theorem(6.1.4)[35]: Assume that $\varphi(t)$ is a frame generator of $V(\varphi)$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$. If there are constants $\beta \geq \alpha > 0$ such that

$$\alpha \leq |\hat{\varphi}^*(\xi)| \leq \beta \ a.e. \ on \ suppG_\varphi(\xi), \tag{16}$$

then there is a frame generator $S(t)$ of $V(\varphi)$ for which

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n) \tag{17}$$

holds for any $f(t) = (\boldsymbol{c} * \varphi)(t) \in V(\varphi)$ satisfying

$$\hat{\boldsymbol{c}}^*(\xi)\hat{\varphi}^*(\xi)(\xi) \in L^2[0,2\pi]. \tag{18}$$

If moreover $|\hat{\varphi}^*(\xi)| \leq \beta$ a.e. on $\mathbb{R}$, then (8)–(11) hold and $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ for any $f \in V(\varphi)$.

Proof : Inequality (16) implies that $\frac{1}{\hat{\varphi}^*(\xi)}\chi \text{supp} G_\varphi(\xi) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$ So that

$$\frac{1}{\hat{\varphi}^*(\xi)}\chi suppG_\varphi(\xi) = \sum_{n\in\mathbb{Z}} a(n)e^{-in\xi} = \hat{a}^*(\xi) \text{ for some } a = \{a(n)\}_{n\in\mathbb{Z}}$$

in l^2. Define $\hat{S}(\xi)$ by (11) , that is, $\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{\hat{\varphi}^*(\xi)} \chi suppG_\varphi = \hat{a}^*(\xi)\hat{\varphi}(\xi)$. Then

$$\int_{-\infty}^{\infty} |\hat{S}(\xi)|^2 d\xi = \int_0^{2\pi} |\hat{a}^*(\xi)|^2 G_\varphi(\xi)d\xi \leq \|G_\varphi(\xi)\|_\infty |\hat{a}^*(\xi)|^2 d\xi < \infty$$

so that $\hat{S}(\xi) \in L^2(\mathbb{R})$. Since

$$\hat{S}(\xi) = \hat{a}^*(\xi)\hat{\varphi}(\xi) = \sum_{n\in\mathbb{Z}} a(n)e^{-in\xi}\hat{\varphi}(\xi) \tag{19}$$

by Lemma (6.1.2) , we have by Fourier inversion
$S(t) = \sum_{n\in\mathbb{Z}} a(n)\varphi(t-n) \in V(\varphi)$. Now (19) implies

$$supp\hat{S} \subset supp\hat{\varphi} \subset suppG_\varphi \text{ so that } \hat{\varphi}(\xi) = \hat{\varphi}^*(\xi)\hat{S}(\xi) a.e. \text{ on } \mathbb{R} \tag{20}$$

since (20) holds on $suppG_\varphi$ by (11) and $\hat{\varphi}(\xi) = \hat{S}(\xi) = 0$ a.e. on $(suppG_\varphi)^c$. Then as in the proof of Theorem (6.1.3) , (14) holds so that

$$G_S(\xi) = \frac{G_\varphi(\xi)}{|\hat{\varphi}^*(\xi)|^2} \text{ on } supp\,\hat{\varphi}^* \supset suppG_\varphi = suppG_S. \tag{21}$$

Hence, we have by (16) and (21)

$$\frac{A_\varphi}{2\pi\beta_2} \leq GS(\xi) \leq \frac{B_\varphi}{2\pi\alpha_2} \text{ a.e. on } suppG_S \tag{22}$$

so that $\{S(t-n) : n \in \mathbb{Z}\}$ is at least a Bessel sequence of $V(\varphi)$. Now for any $f(t) = (\boldsymbol{c} * \varphi)(t)$ in $V(\varphi)$ with $\boldsymbol{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2,

$$\hat{f}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\,\hat{\varphi}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\,\hat{\varphi}^*(\xi)\,\hat{S}(\xi) \tag{23}$$

by (20). If $\hat{\boldsymbol{c}}^*(\xi)\,\hat{\varphi}^*(\xi) \in L^2[0,2\pi]$, then $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ and

$\hat{\boldsymbol{c}}^*(\xi)\,\hat{\varphi}^*(\xi) = \hat{f}^{\,*}(\xi) = \sum_{n\,\in\mathbb{Z}} f\,(n)e^{-in\xi}$ in $L^2[0,2\pi]$ by Lemma (6.1.1). Hence, we get

$$\hat{f}(\xi) = \hat{f}^{\,*}(\xi)\,\hat{S}(\xi) = \sum_{n\,\in\mathbb{Z}} (f(n)e^{-in\xi}\,\hat{S}(\xi)) \tag{24}$$

by Lemma (6.1.2) since $\{S(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence. Then we have (17) by taking Fourier inversion on (24).On the other hand, we also have from (23) $\hat{f}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\hat{\varphi}^*(\xi)\hat{S}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi)\hat{S}(\xi)$ since $supp\hat{S} \subset suppG_\varphi$. Let

$$\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi) = \hat{d}^*(\xi) = \sum_{n\,\in\mathbb{Z}} (f(n)e^{-in\xi}$$ be the Fourier

series expansion of $\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$. Then

$$\hat{c}^*(\xi)\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi) = \hat{c}^*(\xi)\hat{d}^*(\xi) = \sum_{n\,\in\mathbb{Z}} (c * d)(n)e^{-in\xi}$$

so that $\hat{f}(\xi) = \sum_{n\,\in\mathbb{Z}} (c * d)(n)e^{-in\xi}\,\hat{S}(\xi)$ and so

$$f(t) = \sum_{n\,\in\mathbb{Z}} (\boldsymbol{c} * d)(n)S(t-n), f \in V(\varphi).$$ Hence $V(S) = V(\varphi)$

so that (22) implies $\{S(t-n) : n \in \mathbb{Z}\}$ is a frame of $V(\varphi)$.
Finally, assume

$$\alpha\chi suppG_\varphi(\xi) \leq |\hat{\varphi}^*(\xi)| \leq \beta\ a.e.\,on\,\mathbb{R}\,. \tag{25}$$

Then (18) holds for any $c = \{c(n)\}_{n\,\in\mathbb{Z}}$ in l^2 since $\hat{\varphi}^*(\xi) \in L^\infty[0,2\pi]$. Hence$\{f(n)\}_{n\,\in\mathbb{Z}} \in l^2$ for any $f \in V(\varphi)$ and (17) holds on $V(\varphi)$, that is, (8) holds. (9),(10),(11) then follows from (8) by Theorem (6.1.3).

Corollary (6.1.5)[40]: If $\varphi(t)$ is a frame generator of $V(\varphi)$, $\{\varphi(n)\}_{n\,\in\mathbb{Z}} \in l^1$ and $\hat{\varphi}^*(\xi) \neq 0$ on $suppG_\varphi$, then there is a frame generator $S(t)$ of $V(\varphi)$ for which (8), (9),(10), (11) hold.

Proof : If $\{\varphi(n)\}_{n\,\in\mathbb{Z}} \in l^1$ and $\hat{\varphi}^*\xi) \neq 0$ on $suppG_\varphi$, then $\hat{\varphi}^*(\xi) \in C(\mathbb{R})$ satisfies the condition (25) so that the conclusion follows from Theorem (6.1.4) . In [41], the authors assumed that $\varphi(t)$ is a frame generator of $V(\varphi)$ and $\{\varphi(n)\}_{n\,\in\mathbb{Z}} \in l^2$ and then claimed (see Theorems 1 and 2 in [41]) that there is $S(t)$ in $V(\varphi)$ for which the sampling expansion formula (8) holds if and only if the condition (12) is satisfied. In particular, in [41], the authors assumed nothing on the sequence $\{S(t-n) : n \in \mathbb{Z}\}$. However in [41] have some gaps. Assume first that (8) holds. Then $\varphi(t) = \sum_{n\,\in\mathbb{Z}} \varphi\,(n)S(t-n)$, which needs not imply $\hat{\varphi}(\xi) = \hat{\varphi}^*(\xi)\,\hat{S}(\xi)$ (see [41]) in general unless either $\{S(t-n): n \in \mathbb{Z}\}$ is at least a Bessel sequence or$\{\varphi(n)\}_{n\,\in\mathbb{Z}} \in l^1$ (see Lemma (6.1.2)). Conversely if the condition (12), instead of the condition (16), holds in Theorem (6.1.4) , then we still have

(19),(20),(21) and (23). However $\hat{\boldsymbol{c}}^*(\xi)\,\hat{\varphi}^*(\xi)$ may not be in $L^2[0,2\pi]$ so that (24) may not hold and $\{S(t-n) : n \in \mathbb{Z}\}$ may not be a Bessel sequence in general. Hence ,contrary to the claim in [41], we cannot be sure if (8) holds assuming only the condition (12).

Lemma$(6.1.6)$[35]: Assume that $\varphi(t)$ is a Riesz generator of $V(\varphi)$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$. Assume that there is $S(t) \in V(\varphi)$ for which the sampling expansion formula (8) holds. If either $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ or $\{S(t-n): n \in \mathbb{Z}\}$ is a Bessel sequence, then $\hat{\varphi}^*(\xi)^{-1} \in L^2[0,2\pi]$ and

$$supp\hat{\varphi} = supp\hat{S} \subseteq suppG_\varphi = suppG_S = supp\hat{\varphi}^* = \mathbb{R}, \qquad (26)$$

$$\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{\hat{\varphi}^*(\xi)} \quad \text{a.e. on } \mathbb{R}. \qquad (27)$$

Theorem$(6.1.7)$[35]: Assume that $\varphi(t)$ is a frame generator of $V(\varphi)$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$.Then there is a Riesz generator $S(t)$ of $V(\varphi)$ for which (8) holds if and only if $\varphi(t)$ is also a Riesz generator of $V(\varphi)$ and

$$0 < \|\hat{\varphi}^*(\xi)\|_0 \le \|\hat{\varphi}^*(\xi)\|_\infty < \infty. \qquad (28)$$

Furthermore in this case, we have, in addition to (26) and (27);

$$S(t) \text{ is cardinal, i.e. } S(n) = \delta_{0,n} \text{ for } n \in \mathbb{Z}. \qquad (29)$$

Proof : First assume that (8) holds on $V(\varphi)$ for some Riesz generator $S(t)$ of $V(\varphi)$. Then we have (13), (14) and so (9). Since $suppG_\varphi = suppG_S = \mathbb{R}$, $\{\varphi(t-n) : n \in \mathbb{Z}\}$ must be a Riesz basis of $V(\varphi)$ so that (26) and (27) hold by Lemma (6.1.6).Now (28) comes from (15): $|\hat{\varphi}^*(\xi)|^2 = \frac{G_\varphi(\xi)}{G_S(\xi)}$ a.e. on $\mathbb{R}$ and (29) comes immediately from $S(t) = \sum_{n\in\mathbb{Z}} S(n)S(t-n)$.Conversely, assume that $\varphi(t)$ is a Riesz generator of $V(\varphi)$ and (28) hold. Define $\hat{S}(\xi)$ by (27). Then $\hat{S}(\xi) = \hat{a}^*(\xi)\,\hat{\varphi}(\xi) \in L^2(\mathbb{R})$, where $\hat{a}^*(\xi) = \hat{\varphi}^*(\xi)^{-1} \in L^\infty[0,2\pi]$ so that $S(t) = (a * \varphi)(t) \in V(\varphi)$.The rest of the proof is the same as the one in Theorem (6.1.4) .

Corollary$(6.1.8)$[35]: Assume that $\varphi(t)$ is a frame generator of V (φ) and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$. Then there is a Riesz generator $S(t)$ of $V(\varphi)$ for which (8) holds if and only if $\varphi(t)$ is also a Riesz generator of $V(\varphi)$ and $\hat{\varphi}^*(\xi) \ne 0$ on $[0,2\pi]$.

Proof: If $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$, then $\hat{\varphi}^*(\xi) = \hat{\varphi}^*(\xi+2\pi) \in C[0,2\pi]$ so that $\|\hat{\varphi}^*(\xi)\|_0 = \min_{[0,2\pi]} |\hat{\varphi}^*(\xi)|$ and $\|\hat{\varphi}^*(\xi)\|_\infty = \max_{[0,2\pi]} |\hat{\varphi}^*(\xi)|$.

Hence the condition (27) is equivalent to $\hat{\varphi}^*(\xi) \ne 0$ on $[0,2\pi]$. Therefore, the conclusion follows from [33], Walter assumed that $\varphi(t)$ is a continuous real-valued orthonormal generator with $\varphi(t) = O(|t|^{-1-s})$ $(s > 0)$ for $|t|$ large. Then $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ so that the main Theorem of [33] is a special case in [37], Chen and Itoh claimed in [37] that assuming $\varphi(t)$ is a Riesz generator of $V(\varphi)$ with $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$, (8) holds for some $S(t)$ in $V(\varphi)$if and only if $\hat{\varphi}^*(\xi)^{-1} \in L^2[0,2\pi]$. However, from [41], there are some gaps in [37], which are filled and extended by Theorem (6.1.7). As it was done in [37] and [41].We now assume that $\varphi(t)$ is a complex valued square integrable function on $\mathbb{R}$ such that $\varphi(t)$ is a frame generator and $\{\varphi(\sigma + n)\}_{n\in\mathbb{Z}} \in l^2$ for some σ in $[0,1)$.

Then for any $f(t) = \sum_{n\in\mathbb{Z}} c(n)\varphi(t - n)$ in $V(\varphi)$ with $c = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2, $f(\sigma + n) = \sum_{n\in\mathbb{Z}} c(k)\varphi(\sigma + n - k)$ converges absolutely for each n in $\mathbb{Z}$. Let $Z_\varphi(t,\xi) = \sum_{n\in\mathbb{Z}} \varphi(t + n)e^{-in\xi}$ be the Zak transform of $\varphi(t)$ (see [13]).

Theorem$(6.1.9)$[35]: Assume that $\varphi(t)$ is a frame generator of $V(\varphi)$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$ for some σ in $[0,1)$.

(a) If there is a frame generator $S_\sigma(t)$ of $V(\varphi)$ for which the regular shifted samplingexpansion formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)S_\sigma(t - n), f \in V(\varphi) \tag{30}$$

holds, then there are constants $\beta \geq \alpha > 0$ such that
$\alpha \leq |Z_\varphi(\sigma,\xi)| \leq \beta$ a.e. on $suppG_\varphi$,

$$supp\, \hat{\varphi} = supp\, \hat{S}_\sigma \subset suppG_\varphi = suppG_{S_\sigma} \subset suppZ_\varphi(\sigma,\xi), \text{ and}$$

$$\hat{S}_\sigma(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)}\chi suppG_\varphi(\xi). \tag{31}$$

(b) Conversely, if there are constants $\beta \geq \alpha > 0$ such that
$\alpha\chi suppG_\varphi(\xi) \leq |Z_\varphi(\sigma,\xi)| \leq \beta$ a.e. on $\mathbb{R}$ then there is a frame generator $S_\sigma(t)$ of $V(\varphi)$ for which (30) and (31) hold.

(c) There is a Riesz generator $S_\sigma(t)$ of $V(\varphi)$ for which (30) holds if and only if $\varphi(t)$ is a Riesz generator and $0 < \|Z_\varphi(\sigma,\xi)\|_0 \leq \|Z_\varphi(\sigma,\xi)\|_\infty < \infty$. Furthermore, in this case, we have $S_\sigma(\sigma + n) = \delta_{0,n}$ for n in $\mathbb{Z}$ and $\hat{S}_\sigma(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)}$ a.e. on $\mathbb{R}$.

Corollary $(6.1.10)$[35]: If $\varphi(t)$ is a frame generator of $V(\varphi)$, $\{\varphi(\sigma + n)\}_{n\in\mathbb{Z}} \in l^1$, and $Z_\varphi(\sigma,\xi) \neq 0$ on $suppG_\varphi$, then there is a frame generator $S_\sigma(t)$ of $V(\varphi)$ for which (30) and (31) hold.

Example $(6.1.11)$[35]: The Shannon function $\varphi(t) = sin\pi t/\pi t$ is a continuous real-valued Riesz (in fact orthonormal) generator and $\{\varphi(n)\}_{n\in\mathbb{Z}} = \{\delta_{0,n}\}_{n\in\mathbb{Z}}$. Since $\hat{\varphi}^*((\xi) = 1$ on $[0,2\pi]$ but $|\varphi(t)| = O(|t|^{-1})$ for $|t|$ large so that $\varphi(t)$ does not satisfy the Walter's decaying condition, the WSK sampling theorem is not covered by the sampling theorem in [33] but follows from Corollary (6.1.8). Channeled sampling expansion recovers a signal via discrete sample values taken from one or more channeled (output) signals, which are obtained by passing the original (input) signal through a linear time invariant system of pre-filters. Channeled sampling goes back to the work by Shannon, where sample values are taken from the original signal and its derivative. For general discussion of channeled sampling on Paley–Wiener spaces, we refer to [38], [39]and references therein. Here we consider a single channel sampling on shift invariant spaces. Let $\varphi(t) \in L^2(\mathbb{R})$be a frame generator and $H(\xi) \in L^\infty(supp(\hat{\varphi}))$ a transfer function (or apre-filter). Let $C(f)(t) = \mathcal{F}^{-1}(H(\xi)\hat{f}(\xi)), f \in V(\varphi)$.Then $C(f)(t) \in L^2(\mathbb{R})$ for any $f \in V(\varphi)$. Note that if $\hat{f}(\xi) \in L^1(\mathbb{R})$ or $(\xi) \in L^2(supp\hat{\varphi})$, then $C(f)(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$.

Lemma(6.1.12)[35]: If $\varphi(t) \in L^2(\mathbb{R})$ is such that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence and $H(\xi) \in L^\infty(supp(\hat{\varphi}))$, then $\varphi(t-n) : n \in \mathbb{Z}\}$ is also a Bessel sequence.

Proof : Let $B > 0$ be a Bessel bound of $\varphi(t-n) : n \in \mathbb{Z}\}$. Then

$$2\pi G_{C(\varphi)}(\xi) = 2\pi\sum_{n\in\mathbb{Z}} |H(\xi + 2n\pi)\hat{}\varphi(\xi + 2n\pi)|^2$$
$$\leq \|H(\xi)\chi supp\hat{\varphi}(\xi)\|^2{}_\infty\, 2\pi G\varphi(\xi)$$
$$\leq \|H(\xi)\chi supp\hat{\varphi}(\xi)\|^2{}_\infty B \;\; a.e. on\, \mathbb{R}$$

so that $\{C(\varphi)(t-n) : n \in \mathbb{Z}\}$ is also a Bessel sequence (cf. (1)).

In the following, we assume that $\varphi(t) \in L^2(\mathbb{R})$is a frame generator and $H(\xi) \in L^\infty(supp(\hat{\varphi}))$ is a transfer function such that either $H(\xi) \in L^2(supp(\hat{\varphi}))$, or $\hat{\varphi}(t) \in L^1(\mathbb{R})$.Then $\{C(\varphi)(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence by Lemma (6.1.12) and $C(\varphi)(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ since $H(\xi)\hat{\varphi}(t) \in L^1(\mathbb{R})$. We assume further that$\{C(\varphi)(n)\}_{n\in\mathbb{Z}} \in l^2$. Then for any $f(t) = (\boldsymbol{c} * \varphi)(t) \in V(\varphi)$ with $\boldsymbol{c} = \{\boldsymbol{c_n}\}_{\boldsymbol{n}\in\mathbb{Z}} \in \boldsymbol{l^2}$, $\boldsymbol{C(f)(t) = \mathcal{F}^{-1}(H(\xi)f^\xi) = \mathcal{F}^{-1}(\hat{c}^*(\xi)H(\xi)\hat{\varphi}(\xi)) = (c * C(\varphi))(t)}$ by Lemma (6.1.2) since $\{C(\varphi)(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence. Moreover for any n in $\mathbb{Z}$

$$C(f)(n) = \sum_{n\in\mathbb{Z}} c\,(k)C(\varphi)(n-k)$$

converges absolutely and $lim_{|n|\to\infty}(\boldsymbol{c} * C(\varphi))(n) = 0$ (cf. Lemma (6.1.1)). We then have the following, whose proof is essentially the same as the one in Theorem (6.1.7).

Theorem(6.1.13)[35]: Let $\varphi(t) \in L^2(\mathbb{R})$ be a frame generator and $H(\xi) \in L^\infty(supp(\hat{\varphi}))$ a transfer function such that either $H(\xi) \in L^2(supp(\hat{\varphi}))$ or $\hat{\varphi}(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.Assume $\{C(\varphi)(n)\}_{n\in\mathbb{Z}} \in l^2$. Then there is a Riesz generator $S(t)$ of $V(\varphi)$ for which the channeled sampling expansion formula

$$f(t) = \sum_{n\in\mathbb{Z}} C\,(f)(n)S(t-n), f \in V(\varphi)$$

holds if and only if $\varphi(t)$ is a Riesz generator of $V(\varphi)$ and

$0 < ||\widehat{C(\varphi)}^*(\xi)||_0 \leq ||\widehat{C(\varphi)}^*||_\infty < \infty$. Furthermore in this case, $C(S)(t)$ is interpolatory, i.e $C(S)(n) = \delta_{0,n}$ for $n \in \mathbb{Z}$.And $\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{\widehat{C(\varphi)}^*(\xi)}(\xi)$ a.e. on $\mathbb{R}$.

Example(6.1.14)[35]: Let $\varphi(t) = t\chi_{[0,1)}(t) + (2 - t)\chi_{[1,2)}(t)$ be the cardinal B-spline of degree 1. Then $\varphi(t)$ is a continuous Riesz generator (see [4]) and $\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1- e^{-i\xi}}{i\xi}\right)^2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.Take a transfer function $H(\xi) = e^{i\sigma\xi}$ with $0 \leq \sigma < 1$. Then $C(\varphi)(t) = \varphi(t + \sigma)$ so that $C(\varphi)(\sigma) = \sigma$, $C(\varphi)(\sigma + 1) = 1 - \sigma$, and $C(\varphi)(\sigma + n) = 0$ for $n = 0, 1$. Therefore $\widehat{C(\varphi)}^*\xi) = Z_\varphi(\sigma,\xi) = \sigma + (1 - \sigma)e^{-i\xi}$ so that $\left|\widehat{C(\varphi)}^*\xi\right|_0 = |2\sigma - 1|$ and $|\widehat{C(\varphi)}^*\xi(\xi)|_\infty = 1$. Hence, by Theorem (6.1.13), for any $\sigma \in [0,1)\backslash\left\{\frac{1}{2}\right\}$, there is a Riesz generator $S(t)$ of $V(\varphi)$ for which we have the sampling expansion $f(t) = \sum_n f\ (n+\sigma)S(t-n)\ on\ V(\varphi)$, which converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$ since $sup_{\mathbb{R}}\ \sum_{n\in\mathbb{Z}} |\varphi(t - n)|2 < \infty$.

Example(6.1.15)[35]: Let $\varphi(t) = sinct$ so that $\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(\xi)$. Then $\varphi(t)$ is an orthonormal generator of V $(\varphi) = PW_\pi$. Take a measurable function $H(\xi)$ on $\mathbb{R}$ such that $H(\xi)$ and $H(\xi)^{-1}$ belong to $L^\infty[-\pi,\pi]$. Then $H(\xi) \in L^2[-\pi,\pi]$ and $C(\varphi)(t) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}H(\xi)\chi_{[-\pi,\pi]}(\xi))(t) \in PW_\pi$ so that

$$\sum_{n\in\mathbb{Z}} |C(\varphi)(n)|^2 = ||C(\varphi)(t)||^2 = \frac{1}{2\pi}||H(\xi)||^2_{L^\infty[-\pi,\pi]} < \infty,$$

that is, $\{C(\varphi)(n)\}_{n\in\mathbb{Z}} \in l^2$. On the other hand, by the Poisson summation formula, $\widehat{C(\varphi)}^*(\xi) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \widehat{C(\varphi)}(\xi + 2n\pi) = H(\xi)$ on $[-\pi,\pi]$. Hence by Theorem (6.1.13),there is a Riesz generator $S(t) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi H(\xi)}}\chi_{[-\pi,\pi]}(\xi)\right)$ of PWπ for which we have the sampling expansion $f(t) = \sum_n C\ (f)(n)S(t-n)$ on PW_π, which converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$. It is exactly the single channel sampling introduced in [38].

Corollary (6.1.16)[296]: Suppose that $\varphi_j(t)$ is a frame generators of $V(\varphi_j)$ and $\{\varphi_j(n)\}_{n\in\mathbb{Z}} \in l^2$.Then there is a Riesz generators $S_j(t)$ of $V(\varphi_j)$ for which (8) holds if and only if $\varphi_j(t)$ is also a Riesz generators of $V(\varphi_j)$ and

$$0 < \sum_{j\in\mathbb{Z}}\left\|\widehat{\varphi_J}^*(\xi)\right\|_0 \leq \sum_{j\in\mathbb{Z}}\left\|\widehat{\varphi_J}^*(\xi)\right\|_\infty < \infty.$$

Furthermore in this case, we have, in addition to (26) and (27);

$$S_j(t)\ \text{ is cardinal, i.e. }\ S_j(n) = \delta_{0,n}\ \text{ for }\ n \in \mathbb{Z}\ .$$

Proof : First assume that (8) holds on $V(\varphi_j)$ for some Riesz generator $S_j(t)$ of $V(\varphi_j)$. Then we have (13), (14) and so (9) . Since $suppG_{\varphi_j} = suppG_{S_j} = \mathbb{R}$, $\{\varphi_j(t-n) : j,n \in \mathbb{Z}\}$ must be a Riesz basis of $V(\varphi_j)$ so that (26) and (27) hold by Lemma (6.1.6).Now (28) comes from (15): $\sum_{j\in\mathbb{Z}} |\widehat{\varphi_j}^*(\xi)|^2 = \sum_{j\in\mathbb{Z}} \frac{G_{\varphi_j}(\xi)}{G_{S_j}(\xi)}$ a.e. on $\mathbb{R}$ and (29) comes immediately from $\sum_{j\in\mathbb{Z}} S_j(t) = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} S_j(n)S_j(t-n)$.Conversely, assume that $\varphi_j(t)$ is a Riesz generator of $V(\varphi_j)$ and (28) hold. Define $\widehat{S_j}(\xi)$ by (27). Then $\sum_{j\in\mathbb{Z}} \widehat{S_j}(\xi) = \sum_{j\in\mathbb{Z}} \hat{a}^*(\xi)\,\widehat{\varphi_j}(\xi) \in L^2(\mathbb{R})$, where $\hat{a}^*(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\varphi_j}^*(\xi)^{-1} \in L^\infty[0,2\pi]$ so that $\sum_{j\in\mathbb{Z}} S_j(t) = \sum_{j\in\mathbb{Z}} (a*\varphi_j)(t) \in \sum_{j\in\mathbb{Z}} V(\varphi_j)$. The rest of the proof is the same as the one in Theorem (6.1.4) .

Section (6.2) Sampling Expansion

For any $\varphi(t)$ in $L^2(\mathbb{R})$, let$V(\varphi) = \text{span}\{\varphi(t-n) : n \in \mathbb{Z}\}$ be the closed subspace of $L^2(\mathbb{R})$spanned by integer translates $\{\varphi(t-n) : n \in \mathbb{Z}\}$ of $\varphi(t)$. We call $V(\varphi)$ the shift invariant space generated by $\varphi(t)$ and $\varphi(t)$ a frame or a Riesz or an orthonormal generator if $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is a frame or a Riesz basis or an orthonormal basis of $V(\varphi)$. For more details on the structure of shift invariant spaces with single and multiple generators, we refer to [24] and [31].When $\varphi(t) = sinc\, t = \frac{sin\pi t}{\pi t}$, $\varphi(t)$ is an orthonormal generator of $V(\varphi) = PW_\pi$, the Paley–Wiener space of signals band-limited in $[-\pi,\pi]$ and the celebrated WSK (Whittaker–Shannon–Kotel'nikov) sampling theorem says $f(t) = \sum_{n\in\mathbb{Z}} f(n) sinc(t-n), f \in PW_\pi$, which converges both in$L^2(\mathbb{R})$ and absolutely and uniformly on $\mathbb{R}$. As a natural generalization of WSK-sampling theorem, many authors studied sampling expansion procedure on the general shift invariant space $V(\varphi)$ under various assumptions like continuity and/or decaying condition on the generator $\varphi(t)$. For example, Walter considered a real-valued continuous orthonormal generator satisfying $\varphi(t) = O((1+|t|)^{-s})$ with $s > 1$, Chen, Itoh, and Shiki considered a continuous Riesz generator satisfying $\varphi(t) = O((1+|t|)^{-s}$ with $s > \frac{1}{2}$, and Zhou and Sun considered a continuous frame generator $\varphi(t)$ satisfying $sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 < \infty$. (See [11],[12],[13],[23],[25] and [32]) . Noting that the Paley–Wiener space PWπ is a prototype of the reproducing kernel Hilbert space, we first find conditions under which $V(\varphi)$ can be recognized as a reproducing kernel Hilbert space in section, we find necessary and sufficient conditions under which an irregular sampling expansion and a regular shifted sampling expansion hold on $V(\varphi)$. We can relax most of the restrictions imposed on the generator $\varphi(t)$ before. We also introduce a notion of the sampling space, which was first considered by Zhou and Sun, and find a sufficient condition for a function in $L^2(\mathbb{R})$ to be in some sampling space. Let $\{\varphi n : n \in \mathbb{Z}\}$ be a sequence of elements of a separable Hilbert space H with the inner product (,) and $V = \overline{span}\{\varphi n : n \in \mathbb{Z}\}$ the closed subspace of H spanned by $\{\varphi_n : n \in \mathbb{Z}\}$.

Then $\{\varphi_n : n \in \mathbb{Z}\}$ is called a Bessel sequence (with a Bessel bound B) if there is a constant $B > 0$ such that $\sum_{n\in\mathbb{Z}}|\langle\varphi,\varphi_n\rangle|^2 \leq B||\varphi||^2, \varphi \in H$ (or equivalently $\varphi \in V$),a frame sequence (with frame bounds (A, B)) if there are constants $B, A > 0$ such that $A||\varphi||^2 \leq \sum_{n\in\mathbb{Z}}|\langle\varphi,\varphi_n\rangle|^2 \leq B||\varphi||^2$, $\varphi \in V$, a Riesz sequence (with Riesz bounds (A, B)) if there are constants $B, A > 0$

$$A\|c\|^2 \leq \left\|\sum_{n\in\mathbb{Z}} c(n)\varphi_n\right\|^2 \leq B\|c\|^2, c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$$

where $\|c\|^2 = \sum_{n\in\mathbb{Z}} |c(n)|^2$, an orthonormal sequence if $(\varphi_m, \varphi_n) = \delta_{m,n}$ for all m and n in $\mathbb{Z}$. If $\{\varphi_n : n \in \mathbb{Z}\}$ is a frame sequence or a Riesz sequence or an orthonormal sequence in H, then we say that $\{\varphi_n : n \in \mathbb{Z}\}$ is a frame or a Riesz basis or an orthonormal basis of the Hilbert subspace V in H. On $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we take the Fourier transform to be normalized as

$$\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(t)e^{-it\xi}\,dt, \varphi(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

so that $\mathcal{F}[\cdot]$ becomes a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.
For any $\varphi(t) \in L^2(\mathbb{R})$, let $\Phi(t) = \sum_{n\in\mathbb{Z}}|\varphi(t - n)|^2$,
$G_\varphi(\xi) = \sum_{n\in\mathbb{Z}}|\varphi(\xi + 2n\pi)|^2$. Then $\Phi(t) = \Phi(t + 1) \in L^1[0, 1]$,
$G_\varphi(\xi) = G_\varphi(\xi + 2\pi) \in L^1[0, 2\pi]$ and
$||\varphi(t)||^2_{L^2(\mathbb{R})} = ||\Phi(t)||_{L^1[0,1]} = ||G_\varphi(\xi)||_{L^1[0,1]}$.
Let $suppG_\varphi$ be the support of a locally integrable function $G_\varphi(\xi)$ as a distribution on $\mathbb{R}$ (see [29]), that is, $\mathbb{R}\backslash suppG_\varphi$ is the largest open subset of $\mathbb{R}$ on which $G_\varphi(\cdot) = 0$ a.e. Let $E_\varphi = suppG_\varphi \cap [0, 2\pi]$ and $N_\varphi = [0, 2\pi]\backslash E_\varphi$. For any $c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$, let $\hat{c}^*(\xi) = \sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$ be the discrete Fourier transform of c. Then $\hat{c}^*(\xi) = \hat{c}^*(\xi + 2\pi) \in L^2[0, 2\pi]$ and $\int_0^{2\pi}|\hat{c}^*(\xi)|^2\,d\xi = 2\pi||c||^2$. For any $c = \{c(n)\}_{n\in\mathbb{Z}}$ and $d = \{d(n)\}_{n\in\mathbb{Z}}$ in l^2, the discrete convolution product of c and d is defined by
$c * d = \{(c * d)(n) = \sum_{n\in\mathbb{Z}} c(k)d(n - k)\}$.Then $\hat{c}^*(\xi)\hat{d}^*(\xi)$ belongs to $L^1[0, 2\pi]$ and its Fourier series is $(c * d)(n)e^{-in\xi}$ so that

$$\int_0^{2\pi}\left|\hat{c}^*(\xi)\hat{d}^*(\xi)\right|^2 d\xi = 2\pi||c * d||^2. \tag{32}$$

Proposition(6.2.1)[14]: Let $\varphi(t) \in L^2(\mathbb{R})$and $B, A > 0$. Then
(a) $\{\varphi(t - n) : n \in \mathbb{Z}\}$ is a Bessel sequence with a Bessel bound B if and only if $2\pi G_\varphi(\xi) \leq B$ a.e. on $[0, 2\pi]$,
(b) $\{\varphi(t - n) : n \in \mathbb{Z}\}$ is a frame sequence with frame bounds (A, B) if and only if

$$A \leq 2\pi G_\varphi(\xi) \leq B \; a.e. \; on \; E_\varphi , \tag{33}$$

(c) $\{\varphi(t-n): n \in \mathbb{Z}\}$ is a Riesz sequence with Riesz bounds (A,B) if and only if $A \leq 2\pi G_\varphi(\xi) \cdot B$ a.e. on $[0,2\pi]$,

(d) $\{\varphi(t-n): n \in \mathbb{Z}\}$ is an orthonormal sequence if and only if $2\pi G_\varphi(\xi) = 1$ a.e. on $[0,2\pi]$.

Proof : (See [28]) For any $\varphi(t) \in L^2(\mathbb{R})$and $c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$, let $T(c) = (c * \varphi)(t) = \sum_{k\in\mathbb{Z}} c(k)\varphi(t-k)$ be the semi-discrete convolution product of c and $\varphi(t)$, which may or may not converge in $L^2(\mathbb{R})$. In terms of the operator T, called the pre-frame operator of $\{\varphi(t-n): n \in \mathbb{Z}\}$, (see [28]): $\{\varphi(t-n): n \in \mathbb{Z}\}$is a Bessel sequence with a Bessel bound B if and only if T is a bounded linear operator from l^2 into $V(\varphi)$ and $||T(c)||^2_{L^2(\mathbb{R})} \leq B||c||^2$, $c \in l^2$, $\{\varphi(t-n): n \in \mathbb{Z}\}$ is a frame sequence with frame bounds (A,B) if and only if T is a bounded linear operator from l^2 onto $V(\varphi)$ and

$$A||c||^2 \leq ||T(c)||^2_{L^2(\mathbb{R})} \leq B||c||^2, c \in N(T)^\perp, \quad (34)$$

where $N(T) = Ker\,T = \{c \in l^2 : T(c) = 0\}$ and $N(T)^\perp$ is the orthogonal complement of $N(T)$ in l^2, $\{\varphi(t-n): n \in \mathbb{Z}\}$ is a Riesz sequence with Riesz bounds (A,B) if and only if T is an isomorphism from l^2 onto $V(\varphi)$ and $\boldsymbol{A||c||^2 \leq ||T(c)||^2_{L^2(\mathbb{R})} \leq B||c||^2, c \in l^2}, \{\boldsymbol{\varphi(t-n): n} \in \mathbb{Z}\}$ is an orthonormal sequence if and only if T is a unitary operator from l^2 onto $V(\varphi)$.

Lemma $\mathbf{(6.2.2)}$[14]: Let $\varphi(t) \in L^2(\mathbb{R})$. If $\{\varphi(t-n): n \in \mathbb{Z}\}$ is a Bessel sequence, then for any

$$c = \{c(n)\}_{n\in\mathbb{Z}} \text{ in } l^2, \widehat{c * \varphi}(\xi) = \hat{c}^*(\xi)\hat{\varphi}(\xi) \quad (35)$$

so that

$$||(c*\varphi)(t)||^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\hat{c}^*(\xi)\hat{\varphi}(\xi)|^2 d\xi$$

$$= \int_0^{2\pi} |\hat{c}^*(\xi)|^2 G_\varphi(\xi) d\xi. \quad (36)$$

Proof : See [12] and [28]. Let $\varphi(t)$ be a frame or a Riesz generator. Then T is an isomorphism from $N(T)^\perp$ onto $V(\varphi)$ so that

$$V(\varphi) = \{(c*\varphi)(t): c \in l^2\} = \{(c*\varphi)(t): c \in N(T)^\perp\},$$

where $f(t) = (c*\varphi)(t)$ is the L^2-$limit$ of $\sum_{k\in\mathbb{Z}} c(k)\varphi(t-k)$. By (36), we have $N(T) = \{c \in l^2 : \hat{c}^*(\xi) = 0$ a.e. on $E_\varphi\}$ so that

$$N(T)^\perp = \{c \in l^2 : \hat{c}^*(\xi) = 0 \ a.e. on\, N_\varphi\}. \quad (37)$$

Proposition $\mathbf{(6.2.3)[14]}$ **:** Let $\varphi(t) \in L^2(\mathbb{R})$ be a frame generator and $f(t) = (c*\varphi)(t) \in V(\varphi)$, where $c \in l^2$. Then c $\in N(T)^\perp$ if and only if $c(k) = \langle f(t), \psi(t-k)\rangle_{L^2(\mathbb{R})}, k \in \mathbb{Z}$, where $\{\psi(t-k): k \in \mathbb{Z}\}$ is the canonical dual frame of $\{\varphi(t-k): k \in \mathbb{Z}\}$.

Proof : Note that we have by (35) for any $f(t) = (c \ast \varphi)(t) \in V(\varphi)$,

$$\begin{aligned}\langle f(t), \psi(t-k)\rangle_{L^2(\mathbb{R})} &= \langle \hat{c}^*(\xi)\hat{\varphi}(\xi), e^{-ik\xi}\,\hat{\psi}(\xi)\rangle_{L^2(\mathbb{R})} \\ &= \langle \hat{c}^*(\xi)\,\hat{\varphi}(\xi), \frac{\hat{\varphi}(\xi)}{2\pi G_{\varphi}(\xi)}\chi supp\, G_{\varphi}(\xi) e^{-ik\xi}\rangle_{L^2(\mathbb{R})} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \hat{c}^*(\xi)\,\chi_{E_{\varphi}}(\xi) e^{ik\xi}\,d\xi, k \in \mathbb{Z}\end{aligned}$$

since $\hat{\psi}(\xi) = \frac{\hat{\varphi}(\xi)}{2\pi G_{\varphi}(\xi)}\chi supp\, G_{\varphi}(\xi)$ (see [22]), where $\chi_E(\xi)$ is the characteristic function of a subset E of $\mathbb{R}$. Hence

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \langle f(t), \psi(t-k)\rangle_{L^2(\mathbb{R})} e^{-ik\xi} &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\left(\int_0^{2\pi} \hat{c}^*(\xi)\chi_{E_{\varphi}}(\xi) e^{ik\xi}\,d\xi\right) \\ &= \hat{c}^*(\xi)\,\chi_{E_{\varphi}}(\xi).\end{aligned}$$

Now, $c \in N(T)^{\perp}$ if and only if $\hat{c}^*(\xi) = 0$ a.e. on N_{φ} (see (37)).

That is, $\hat{c}^*(\xi) = \hat{c}^*(\xi)\,\chi_{E_{\varphi}}(\xi)$ a.e. on $[0, 2\pi]$. Hence the conclusion follows. A Hilbert space H consisting of complex-valued functions on a set E is called a reproducing kernel Hilbert space (RKHS in short) if the point evaluation $l_t(f) = f(t)$ is a bounded linear functional on H for each t in E. In an RKHS H, there is a function $k(s,t)$ on $E \times E$, called the reproducing kernel of H satisfying

(i) $k(\cdot, s) \in H$ for each s in E,

(ii) $\langle f(t), k(t,s)\rangle = f(s), f \in H.$

Moreover, any norm converging sequence in an RKHS H converges also uniformly on any subset of E, on which $k(t,t)$ is bounded (see [30]). If a shift invariant space $V(\varphi)$ with a frame generator $\varphi(t)$ is an RKHS, then its reproducing kernel is given by

$$k(t,s) = \sum_{n\in\mathbb{Z}} \varphi(t-n)\overline{\emptyset(s-n)} = \sum_{n\in\mathbb{Z}} \emptyset(t-n)\,\overline{\varphi(s-n)} \tag{38}$$

where $\{\psi(t-n) : n \in \mathbb{Z}\}$ is the canonical dual frame of $\{\varphi(t-n) : n \in \mathbb{Z}\}$. We now find conditions on the generator $\varphi(t)$ under which $V(\varphi)$ can be recognized as an RKHS. Since all functions in an RKHS must be pointwise well defined, we only consider generators $\varphi(t)$ satisfying $\varphi(t)$ is a complex valued square integrable

function well defined every where on $\mathbb{R}$. (39)

If $V(\varphi)$ is recognizable as an RKHS with the reproducing kernel $k(t,s)$ as in (38), where $\varphi(t)$ is a frame generator satisfying (39), then

$$\begin{aligned}\Phi(s) &= \sum_{n\in\mathbb{Z}} |\varphi(s-n)|^2 = \sum_{n\in\mathbb{Z}} \left|\langle (k(t,s), \varphi(t-n)\rangle_{L^2(\mathbb{R})}\right|^2 \\ &\le B\|K(\cdot,s)\|^2_{L^2(\mathbb{R})} = Bk(s,s), s \in \mathbb{R},\end{aligned}$$

where B is an upper frame bound of $\{\varphi(t-n) : n \in \mathbb{Z}\}$. Hence

$$\Phi(t) = \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 < \infty \text{ for any } t \text{ in } \mathbb{R}\ . \qquad (40)$$

Conversely, we have:

Proposition(6.2.4)[14]: Let $\varphi(t)$ be a function satisfying (39) and (40).

(a) If $\varphi(t)$ is a frame generator, then $V_p(\varphi) = \{(c * \varphi)(t) : c \in N(T)^{\perp}\}$ is an RKHS in P which any $f(t) = (c * \varphi)(t)$ is the pointwise limit of

$$\sum_{k\in\mathbb{Z}} c(k)\, \varphi(t-k), \text{which converges also in } L^2(\mathbb{R}).$$

(b) If $\varphi(t)$ is a Riesz generator, then $V(\varphi) = \{(c * \varphi)(t): c \in l^2\}$ is an RKHS in which any $f(t) = (c * \varphi)(t)$ is the pointwise limit of

$$\sum_{k\in\mathbb{Z}} c(k)\, \varphi(t-k), \text{which converges also in } L^2(\mathbb{R}).$$

(c) If $\varphi(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ is a continuous frame generator satisfying $sup_{\mathbb{R}}\Phi(t) < \infty$, then $V(\varphi)$ is an RKHS in which any $f(t) = (c * \varphi)(t)$ is the pointwise limit Of $\sum_{k\in\mathbb{Z}} c(k)\, \varphi(t-k)$, which converges also in $L^2(\mathbb{R})$ and uniformly on R to a continuous function on $\mathbb{R}$ $(so\ V(\varphi) \subset C(R) \cap L^2(\mathbb{R}))$.

Proof : Assume that $\varphi(t)$ is a frame generator satisfying (39) and (40). Then for any c in $l^2, (c * \varphi)(t)$ converges not only in $L^2(\mathbb{R})$ but also absolutely on $\mathbb{R}$ since $\{\varphi(t-n)\}_{n\in\mathbb{Z}} \in l^2$ for any t in $\mathbb{R}$ by (40). Then $V_p(\varphi)$ is a Hilbert space under the $L^2 - inner$ product since for any $f(t) = (c * \varphi)(t)$ with $\in N(T)^{\perp}$, $||f(t)||_{L^2(\mathbb{R})} = 0$ implies $c = 0$ so $f(t) = 0$ on $\mathbb{R}$. Moreover for any $f(\cdot) = \sum_{k\in\mathbb{Z}} c(k)\, \varphi(\cdot - k)$ in $V_p(\varphi)$ and any t in $\mathbb{R}$, $|f(t)| \le ||c||\sqrt{\Phi(t)} \le \frac{1}{\sqrt{A}}\sqrt{\Phi(t)}||f||_{L^2(\mathbb{R})}$ by (34). Hence $V_p(\varphi)$is an RKHS so that (a) is proved. Then (b) follows from (a) since $N(T) = \{0\}$ so that $N(T) = l^2$ and $V(\varphi) = V_p(\varphi)$ when $\varphi(t)$ is a Riesz generator satisfying (39) and (40). Finally let $\varphi(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ be a frame generator and $sup_{\mathbb{R}}\Phi(t) < \infty$. Then for any c in l^2, $c(k)\, \varphi(t-k)$ converges not only in $L^2(\mathbb{R})$ but also absolutely and uniformly on R to a continuous function $f(t)$ on $\mathbb{R}$.

Hence $V(\varphi) \subset C(\mathbb{R}) \cap L^2(\mathbb{R})$. Now for any $f(t) = (c * \varphi)(t)$ in $V(\varphi)$, decompose c into $c = c_1 + c_2$ where $c_1 \in N(T)$ and $c_2 \in N(T)^{\perp}$. Then $||c_1 * \varphi||_{L^2(\mathbb{R})} = 0$ so that $(c_1 * \varphi)(t) = 0$ on $\mathbb{R}$ since $(c_1 * \varphi)(t)$ is continuous on $\mathbb{R}$. Hence $f(t) = (c_2 * \varphi)(t) \in V_p(\varphi)$ so that $V(\varphi) = V_p(\varphi)$is an RKHS as in (a). Hence (c) is proved. Note that when $\varphi(t)$ is a frame generator satisfying (39) and (40) , $\sum_{k\in\mathbb{Z}} c(k)\, \varphi(t-k)$ converges also absolutely on $\mathbb{R}$ for any c in l^2. However V (φ) as a space of the pointwise limits of $\sum_{k\in\mathbb{Z}} c(k)\, \varphi(t-k), c \in l^2$, may not be a Hilbert space under the L^2-$inner$ product since $||f(t)||_{L^2(\mathbb{R})} = 0$ implies $f(t) = 0$ only a.e. on $\mathbb{R}$ unless $\varphi(t)$ is a Riesz generator or a frame generator as in (d).We say that two functions $\varphi_1(t)$ and $\varphi_2(t)$in $L^2(\mathbb{R})$ are equivalent (see [23]) if they generate the same shift invariant space, that is, $V(\varphi_1) = V(\varphi_2)$. It is then easy to see that $\varphi_1(t)$ and $\varphi_2(t)$ equivalent if and only if $\varphi_1(t) \in V(\varphi_2)$ and $\varphi_2(t) \in V(\varphi_1)$. In particular, $\varphi_1(t)$ and $\varphi_2(t)$ are equivalent frame generators if and only if there are c and d in l^2 such that

$$\varphi_2(t) = (c * \varphi_1)(t) \text{ and } \varphi_1(t) = (d * \varphi_2)(t) . \quad (41)$$

Since (41) implies $\widehat{\varphi_2}(\xi) = \hat{c}^*(\xi)\widehat{\varphi_1}(\xi)$, $\widehat{\varphi_1}(\xi) = \hat{c}^*(\xi)\widehat{\varphi_2}(\xi)$ and so $G_{\varphi_2}(\xi) = |\hat{c}^*(\xi)|^2 G_{\varphi_1}(\xi)$, $G_{\varphi_1}(\xi) = |\hat{d}^*(\xi)| G_{\varphi_2}(\xi)$, we must have $supp\widehat{\varphi_1} = supp\widehat{\varphi_2}$, $suppG_{\varphi_1} = suppG_{\varphi_2}$,and $E_{\varphi_1} = E_{\varphi_2}$ if $\varphi_1(t)$ and $\varphi_2(t)$ are equivalent frame generators.

Lemma(6.2.5)[14]: Let $\varphi_1(t)$and $\varphi_2(t)$ be equivalent frame generators. Then $\varphi_1(t)$ is a Riesz generator if and only if $\varphi_2(t)$ is a Riesz generator.

Proof: Since $\varphi_1(t)$and $\varphi_2(t)$ are equivalent frame generators , $E_{\varphi_1} = E_{\varphi_2}$. Now$\varphi_j(t)$ is a Riesz generator if and only if $E_{\varphi_j} = [0, 2\pi]$

(see Propositions (6.2.1)(b) and (6.2.1)(c)) so that $\varphi_1(t)$is a Riesz generator if and only if $\varphi_2(t)$ is a Riesz generator. Concerning the condition in Proposition (6.2.4)(c), Zhou and Sun’s [38] proved.

Lemma(6.2.6)[14]: (see [22]). For any $\varphi(t)$ in $L^2(\mathbb{R})$, the followings are equivalent.

(a) $\varphi(t) \in C(\mathbb{R})$ and $sup_{\mathbb{R}}\Phi(t) < \infty$.

(b) For any c in l^2, $(c * \varphi)(t)$ converges pointwise to a continuous function on $\mathbb{R}$.

(c) For any c in l^2, $(c * \varphi)(t)$ converges uniformly to a continuous function on $\mathbb{R}$.

Moreover for any two equivalent frame generators $\varphi_1(t)$ and $\varphi_2(t)$, $\varphi_1(t) \in C(\mathbb{R})$ and $sup_{\mathbb{R}}\Phi_1(t) < \infty$ if and only if $\varphi_2(t) \in C(\mathbb{R})$ and $sup_{\mathbb{R}}\Phi_2(t) < \infty$.Here, $\Phi_j(t) = \sum_{\mathbb{Z}}|\varphi_j(t - n)|^2$for j = 1, 2. Note that $\varphi(t) \in C(\mathbb{R})$ and $sup_{\mathbb{R}}\Phi(t) < \infty$ if either $\varphi(t) \in C(\mathbb{R})$ and $\varphi(t) = O((1 + |t|)^{-s})$ for $s > \frac{1}{2}$ or $\varphi(t) \in L^2(\mathbb{R})$, and $\hat{\varphi}(\xi)$ has a compact support since when $supp\hat{\varphi}(\xi) \subset [-\sigma, \sigma]$ $(\sigma > 0)$, $\Phi(t) = \sum_{n\in\mathbb{Z}}|\varphi(t - n)|^2 \le \frac{4e^{\sigma}}{\pi}||\varphi||_{L^2(\mathbb{R})}$ by the Plancherel–P´olya inequality (see [30]). We also have:

Proposition (6.2.7)[14]: If $\varphi(t) \in L^2(\mathbb{R})$ is such that

$$H_{\varphi}(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0, 2\pi] , \quad \text{then } \varphi(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$$

and $sup_{\mathbb{R}}\Phi_1(t) < \infty$.

Proof: Since $H_{\varphi}(\xi) \in L^2[0, 2\pi] \subset L^1[0, 2\pi]$ and $||H_{\varphi}(\xi)||_{L^1[0,2\pi]} = ||\hat{\varphi}(\xi)||_{L^1(\mathbb{R})}$, $\hat{\varphi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ so $\varphi(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$. Also $\sum_{n\in\mathbb{Z}}\hat{\varphi}(\xi + 2n\pi)e^{it(\xi + 2n\pi)}$ converges in $L^2[0, 2\pi]$. Hence we have by the Poisson summation formula (see Lemma (6.2.20) below)

$$\sum_{n\in\mathbb{Z}} \hat{\varphi}(\xi + 2n\pi)e^{it(\xi+2n\pi)} = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \varphi(t+n)e^{in\xi} \quad in \quad L^2[0,2\pi]$$

so that

$$\sum_{n\in\mathbb{Z}} |\varphi(t+n)|^2 = \left\|\sum_{n\in\mathbb{Z}} \hat{\varphi}(\xi + 2n\pi)e^{it(\xi+2n\pi)}\right\|^2_{L^2[0,2\pi]}$$

$$\leq \left\|H_\varphi(\xi)\right\|^2_{L^2[0,2\pi]}.$$

Hence $sup_{\mathbb{R}}\Phi(t) \leq \left\|H_\varphi(\xi)\right\|^2_{L^2[0,2\pi]} < \infty$

Theorem $(6.2.8)$[14]: Let $\varphi(t)$ be a frame generator satisfying (39) and (40) so that $V_p(\varphi)$ is an RKHS. Then for any sequence $\{t_n\}_{n\in\mathbb{Z}}$ of points in $\mathbb{R}$ with $t_n < t_{n+1}, n \in \mathbb{Z}$, the followings are equivalent on $V_p(\varphi)$.

(a) There is a frame $\{S_n(t) : n \in \mathbb{Z}\}$ of $V_p(\varphi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(t_n)S_n(t), f \in V_p(\varphi) \tag{42}$$

and $\{f(t_n)\}_{n\in\mathbb{Z}}$ is a moment sequence of a function to $\{S_n(t) : n \in \mathbb{Z}\}$, that is ,

$$f(t_n) = \langle g(t), S_n(t)\rangle_{L^2(\mathbb{R})}, n \in \mathbb{Z} \tag{43}$$

for some $g(t)$ in $V_p(\varphi)$.

(b) There is a Bessel sequence $\{B_n(t) : n \in \mathbb{Z}\}$ in $V_p(\varphi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(t_n)B_n(t), f \in V_p(\varphi) \tag{44}$$

and there is a constant $\beta > 0$ such that

$$\sum_{n\in\mathbb{Z}} |f(t_n)|^2 \leq \beta||f||^2_{L^2(\mathbb{R})}, f \in V_p(\varphi). \tag{45}$$

(c) There are constants $\beta \geq \alpha > 0$ such that

$$\alpha||f||^2_{L^2(\mathbb{R})} \leq \sum_{n\in\mathbb{Z}} |f(t_n)|^2 \leq ||f||^2_{L^2(\mathbb{R})}, f \in V_p(\varphi) \tag{46}$$

(d) $\{k(t,t_n) : n \in \mathbb{Z}\}$ is a frame of $V_p(\varphi)$, where $k(t,s)$ is the reproducing kernel of $V_p(\varphi)$. Furthermore, if any one of the above equivalent statements holds, then the sampling series (42) or (44) converges both in $L^2(\mathbb{R})$ and absolutely and uniformly on any subset E of $\mathbb{R}$ on which $\Phi(t)$ is bounded.

Proof: Since $f(t_n) = \langle f(t), k(t,t_n)\rangle_{L^2(\mathbb{R})}$ for any f in $V_p(\varphi)$ and any n in $\mathbb{Z}$, (c) is equivalent to (d). Assume (d) and let $\{S_n(t) : n \in \mathbb{Z}\}$ be the canonical dual frame of $\{k(t,t_n) : n \in \mathbb{Z}\}$. Then

$$f(t) = \sum_{n\in\mathbb{Z}} \langle f(t), k(t,t_n)\rangle_{L^2(\mathbb{R})} S_n(t) = \sum_{n\in\mathbb{Z}} f(t_n)S_n(t), f \in Vp(\varphi)$$

so that (42) holds. Let U be the frame operator of $\{S_n(t) : n \in \mathbb{Z}\}$. Then $k(t,t_n) = U^{-1}(S_n(t)), n \in \mathbb{Z}$ so that

$$f(t_n) = \langle f(t), k(t,t_n) \rangle_{L^2(\mathbb{R})} = \langle f(t), U^{-1}(S_n(t)) \rangle_{L^2(\mathbb{R})}$$

$$= \langle U^{-1}(f), S_n(t) \rangle_{L^2(\mathbb{R})}, n \in \mathbb{Z}$$

so that $\{f(t_n)\}_{n\in\mathbb{Z}}$ is the moment sequence of $U^{-1}(f)$ to $\{S_n(t) : n \in \mathbb{Z}\}$. Hence (a) holds. Conversely assume (a) and let U be the frame operator of $\{S_n(t) : n \in \mathbb{Z}\}$. Then we have from (42) and (44)

$$f(t) = \sum_{n\in\mathbb{Z}} \langle g(t), S_n(t) \rangle_{L^2(\mathbb{R})} S_n(t) = U(g(t)), f \in Vp(\varphi).$$

so that $g(t) = U^{-1}(f(t))$ and for $f \in V_p(\varphi)$ and $n \in \mathbb{Z}$,

$$\langle f(t), k(t,tn) \rangle_{L^2(\mathbb{R})} = f(tn) = \langle U^{-1}(f), S_n(t) \rangle_{L^2(\mathbb{R})}$$
$$= \langle f, U^{-1}(S_n(t)) \rangle_{L^2(\mathbb{R})}.$$

Hence $k(t,tn) = U^{-1}(S_n(t)), n \in \mathbb{Z}$ so that $\{k(t,t_n) : n \in \mathbb{Z}\}$ must be the canonical dual frame of $\{S_n(t) : n \in \mathbb{Z}\}$. Hence (d) holds. (a) implies (b) with $B_n(t) = S_n(t), n \in \mathbb{Z}$ since (a) also implies (c). Assume (b). Then by (45)

$$\sum_{n\in\mathbb{Z}} \left| \langle f(t), k(t,t_n) \rangle_{L^2(\mathbb{R})} \right|^2 = \sum_{n\in\mathbb{Z}} | f(t_n)|^2 \le \beta ||f||^2_{L^2(\mathbb{R})}, f \in V_p(\varphi)$$

so that both $\{k(t,t_n) : n \in \mathbb{Z}\}$ and $\{B_n(t) : n \in \mathbb{Z}\}$ are Bessel sequences in $V_p(\varphi)$. Then $\{k(t,t_n) : n \in \mathbb{Z}\}$ and $\{B_n(t) : n \in \mathbb{Z}\}$ are dual frames of $V_p(\varphi)$ (see [28]) so that (d) holds. Finally assume, e.g., that (c) holds. Then $\{f(t_n)\}_{n\in\mathbb{Z}} \in l^2$ for any $f \in V_p(\varphi)$. Hence as a frame expansion, the sampling series (42) or (44) converges both in $L^2(\mathbb{R})$ and absolutely on $\mathbb{R}$. Now for the reproducing kernel $k(s,t) = \sum_{n\in\mathbb{Z}} \varphi(t-n)\psi(s-n)$ of $V_p(\varphi)$,

$$k(t,t) = ||k(\cdot,t)||^2_{L^2(\mathbb{R})} = \left\| \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)}\, \psi(\cdot - n) \right\|^2_{L^2(\mathbb{R})} \le B_\psi \Phi(t)$$

where B_ψ is an upper frame bound of $\{\psi(s-n) : n \in \mathbb{Z}\}$. Hence as a series in the RKHS $V_p(\varphi)$, the sampling series (42) or (44) converges also uniformly on any subset E of $\mathbb{R}$ on which $\psi(t)$ is bounded. Inspection of the proof of Theorem (6.2.8) shows that the reconstruction frame $\{S_n(t) : n \in \mathbb{Z}\}$ in (a) is uniquely determined as the canonical dual frame of $\{k(t,t_n) : n \in \mathbb{Z}\}$ but $\{B_n(t) : n \in \mathbb{Z}\}$ in (b) need not be unique but may beany dual frame of $\{k(t,t_n) : n \in \mathbb{Z}\}$. Note also that Theorem (6.2.8) remains true on $V(\varphi) = V_p(\varphi)$ in the cases of Propositions (6.2.4)(b) and (6.2.4)(c).In particular, in the case of Proposition (6.2.4)(c), the sampling series (42) or (44) converges uniformly on $\mathbb{R}$ to a continuous function $f(t)$ on $\mathbb{R}$. Equivalence of (a) and (c) in Theorem (6.2.8) was first proved in [26] assuming that $\varphi(t)$ is a continuous Riesz generator satisfying the growth condition $\varphi(t) = O((1+|t|)^{-s})$ with $> \frac{1}{2}$, which implies $sup_{\mathbb{R}} \Phi(t) < \infty$. In [26], the authors used the Gram matrix A of the frame $\{k(t,t_n) : n \in \mathbb{Z}\}$ in order to realize the reconstruction frame $\{S_n(t) : n \in \mathbb{Z}\}$ claiming that A is invertible (see [26]). But A is not invertible in general unless $\{k(t, t_n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V(\varphi)$. As a special case of Theorem (6.2.8), we now consider regular shifted sampling at sampling points $t_n = \sigma + n, n \in \mathbb{Z}$, where $0 \le \sigma < 1$. In the following, we let (see [13])

$$Z_\varphi(t,\xi) = \sum_{n\in\mathbb{Z}} \varphi\ (t + k)e^{-ik\xi} \text{ be the Zak transform of } \varphi(t)$$

in $L^2(\mathbb{R})$. Then $Z_\varphi\ (t,\xi)$ is well defined a.e. on $\mathbb{R}^2$ and is quasi-periodic in the sense that $Z_\varphi\ (t + 1,\xi) = e^{i\xi}\, Z_\varphi(t,\xi)$ and $Z_\varphi(t, \xi + 2\pi) = Z_\varphi(t,\xi)$. For any measurable function $f(t)$ on $\mathbb{R}$, let $||f||_0 = \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)|$ and $\|f\|_\infty \inf_{|E|=0} \sup_{\mathbb{R}\setminus E} |f(t)|$ be the essential infimum and the essential supremum of $|f(t)|$ respectively where $|E|$ is the Lebesgue measure of E. We first replace the inequality (46) by an equivalent one, which is easier to check.

Lemma(6.2.9)[14]: Let $\varphi\ (t)$ be a frame generator satisfying (39) and (40) and $0 \le \sigma < 1$. Then the followings are equivalent.

(a) There are constants $\beta \ge \alpha > 0$ such that

$$\alpha\, \|f\|^2_{L^2(\mathbb{R})} \le \sum_{n\in\mathbb{Z}} |f(\sigma + n)|^2 \le \beta\, \|f\|^2_{L^2(\mathbb{R})}, f \in V_p(\varphi). \tag{47}$$

(b) There are constants $\beta \ge \alpha > 0$ such that

$$\alpha \le |Z_\varphi(\sigma,\xi)| \le \beta \ a.e. on\ E\varphi. \tag{48}$$

Proof : For any $f(t) = (c * \varphi)(t)$ in $V_p(\varphi)$ with $c \in N(T)^\perp$, we have

$$f(\sigma + n) = \sum_{n\in\mathbb{Z}} c\ (k)\, \varphi\, (\sigma + n - k) = (c * d)(n)$$

where $d = \{\varphi\ (\sigma + n)\}_{n\in\mathbb{Z}}$ so that $\hat{d}^*(\xi) = Z_\varphi(\sigma,\xi)$. Hence by (32) and (37)

$$\sum_{n\in\mathbb{Z}} |f(\sigma + n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{c}^*(\xi)|^2 \left|Z_\varphi\ (\sigma,\xi)\right|^2 d\,\xi$$

$$= \frac{1}{2\pi}\int_{E_\varphi} |\hat{c}^*(\xi)|^2 \left|Z_\varphi\ (\sigma,\xi)\right|^2 d\,\xi.$$

Hence by (36) , (47) is equivalent to

$$2\pi\,\alpha \int_{E\,\varphi} |\hat{c}^*(\xi)|^2 G_\varphi(\xi) d\,\xi \le \frac{1}{2\pi}\int_{E\,\varphi} |\hat{c}^*(\xi)|^2 \left|Z_\varphi\ (\sigma,\xi)\right|^2 d\,\xi$$

$$\le 2\pi\beta \int_{E\,\varphi} |\hat{c}^*(\xi)|^2 G_\varphi(\xi) d\,\xi\,, c \in N(T)^\perp$$

which is also equivalent to

$$2\pi\,\alpha\, G_{\varphi}(\xi) \le \left|Z_{\varphi}(\sigma,\xi)\right|^{2} \le 2\,\pi\,\beta G_{\varphi}(\xi) a.e. \quad on\ E_{\varphi}\,. \qquad (49)$$

Let (A,B) be frame bounds of $\{\varphi(t-n) : n \in \mathbb{Z}\}$. Assume (a). Then we have by (33) and (49), $A\,\alpha \le |Z_{\varphi}(\sigma,\xi)|^2 \le B\beta$ a.e. on E_{φ} so that (b) holds. Conversely assume (b). Then by (33) and (48)

$\frac{2\pi\alpha^2}{B} G_{\varphi}(\xi) \le |Z_{\varphi}(\sigma,\xi)|^2 \le \frac{2\pi\beta^2}{A} G_{\varphi}(\xi)$ a.e. on E_{φ} so that

$\frac{\alpha^2}{B} ||f||^2_{L^2(\mathbb{R})} \le |f(\sigma+n)|^2 \le \frac{\beta^2}{A} ||f||^2_{L^2(\mathbb{R})}, f \in V_p(\varphi)$,i.e. (a) holds.

Combining Theorem (6.2.8) and Lemma (6.2.9) , we have

Theorem (6.2.10)[14]: Let $\varphi(t)$ be a frame generator satisfying (39) and (40) so that $V_p(\varphi)$ is an RKHS and $0 \le \sigma < 1$. Then the followings are equivalent on $V_p(\varphi)$.

(a) There is a frame $\{S_n(t) : n \in \mathbb{Z}\}$ of $V_p(\varphi)$such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n) S_n(t), f \in V_p(\varphi)\ , \qquad (50)$$

Where $\{f(\sigma+n)\}_{n\in\mathbb{Z}}$ is a moment sequence of a function in $V_p(\varphi)$ to $\{S_n(t) : n \in \mathbb{Z}\}$.

(b) There is a frame $\{S(t-n) : n \in \mathbb{Z}\}$ of $V_p(\varphi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n) S(t-n), f \in V_p(\varphi). \qquad (51)$$

(c) There is a Bessel sequence $\{B_n(t) : n \in \mathbb{Z}\}$ in $V_p(\varphi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n) B_n(t)\,, f \in V_p(\varphi) \qquad (52)$$

and there is a constant $\beta > 0$ such that

$$\sum_{n\in\mathbb{Z}} |f(\sigma+n)|^2 \le \beta\, ||f||^2_{L^2(\mathbb{R})}, f \in V_p(\varphi).$$

(d) There are constants $\beta \ge \alpha > 0$ such that

$$\alpha\, ||f||^2_{L^2(\mathbb{R})} \le \sum_{n\in\mathbb{Z}} |f(\sigma+n)|^2 \le \beta\, ||f||^2_{L^2(\mathbb{R})}\,, f \in V_p(\varphi).$$

(e) There are constants $\beta \ge \alpha > 0$ such that $\alpha \le |Z_{\varphi}(\sigma,\xi)| \le \beta$ a.e. on E_{φ} .

(f) $\{k(t,\sigma+n) : n \in \mathbb{Z}\}$ is a frame of $V_p(\varphi)$. Moreover if any one of the above equivalent statements holds , then $S_n(t) = S(t-n), n \in \mathbb{Z}$, where

$$\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_{\varphi}(\sigma,\xi)} \chi_{supp}\, G_{\varphi}(\xi) \quad a.e.\ \ on\ \ \mathbb{R} \qquad (53)$$

and the sampling series (50),(51),(52) converge both in $L^2(\mathbb{R})$ and absolutely and uniformly on any subset E of $\mathbb{R}$ on which $\Phi(t)$ is bounded.

Proof : Equivalences of (a), (c)–(f) come from Theorem (6.2.8) and Lemma (6.2.9). Assume (a) Then as in the proof of Theorem (6.2.8) ,

$U^{-1}(S_n(t)) = k(t,\sigma + n) = k(t - n,\sigma), n \in \mathbb{Z}$, where U is the frame operator of $\{S_n(t): n \in \mathbb{Z}\}, k(t,s)$ is the reproducing kernel of $V_p(\varphi)$and $k(t,\sigma + n) = k(t-n,\sigma)$ for $n \in \mathbb{Z}$ comes from (38). Then as the canonical dual frame of $\{k(t - n,\sigma): n \in \mathbb{Z}\}$, $S_n(t) = S(t - n), n \in \mathbb{Z}$, where $S(t) = S_0(t)$. Hence (b) holds.

Conversely assume (b). Then by (51), $\varphi(t) = \sum_{n\in\mathbb{Z}} \varphi(\sigma+n)\, S(t-n)$,

where $\{\varphi(\sigma + n)\}_{n\in\mathbb{Z}} \in l^2$ by (40). Hence $\hat{\varphi}(\xi)\ Z_\varphi(\sigma,\xi)\, \hat{S}(\xi)$ by (35)and so $G_\varphi(\xi) = |Z_\varphi(\sigma,\xi)|^2 G_S(\xi)$. Hence $|Z_\varphi(\sigma,\xi)|^2\ \frac{G_\varphi(\xi)}{G_S(\xi)}$ on $E_\varphi = E_S$ since $\varphi(t)$ and $S(t)$ are equivalent frame generators, from which we have $\frac{A}{B_S} \le |Z_\varphi(\sigma,\xi)|^2 \le \frac{B}{A_S}$ a.e. on E_φ where (A,B) and (A_S,B_S) are frame bounds of $\{\varphi(t - n) : n \in \mathbb{Z}\}$ and $\{S(t - n) : n \in \mathbb{Z}\}$, respectively. Hence (e) holds. Moreover we have from $\hat{\varphi}(\xi) = Z_\varphi(\sigma,\xi)\hat{S}(\xi)$, $\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)}$ on $supp\, Z_\varphi(\sigma,\xi)$,which implies (53) since $supp\, G_\varphi(\xi) \subset supp\, Z_\varphi(\sigma,\xi)$ by (e)
$supp\hat{\varphi}(\xi) = supp\hat{S}(\xi), supp\hat{\varphi}(\xi) \subset suppG_\varphi(\xi)$. Then $S(t) \in V_p(\varphi)$ since $\frac{1}{Z_\varphi(\sigma,\xi)}\chi_{supp G_\varphi}(\xi) = 0$ a.e. on N_φ (cf. (37)). The mode of convergence of the sampling series (50),(51),(52) was already proved in Theorem (6.2.8). Applying (51) to S(t), we have $S(t) = S(\sigma + n)S(t - n)$ so that $S(\sigma + n) = \delta_{0,n}, n \in \mathbb{Z}$ provided that $\{S(t - n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V_p(\varphi)$. In fact, we have:

Proposition(6.2.11)[14]: Let $\varphi(t)$ be a frame generator satisfying (39) and (40) and $0 \le \sigma < 1$. Assume that any one of the six equivalent conditions in Theorem (6.2.10) holds. Then the followings are equivalent:
(a) $\varphi(t)$is a Riesz generator ;
(b) $\{S_n(t) = S(t - n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V_p(\varphi)$;
(c) $S(\sigma + n) = \delta_{0,n}, n \in \mathbb{Z}$
(d) $\{k(t,\sigma + n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V_p(\varphi)$. Moreover, if any one of the above equivalent conditions holds, $V_p(\varphi) = V(\varphi)$.

Proof: Note that (b) means that $S(t)$ is a Riesz generator of $V_p(\varphi)$. Then equivalence of (a) and (b) follows from Lemma (6.2.5). We saw already that (b) implies (c). Assume (c) and let $S(t) = \sum_{k\in\mathbb{Z}} c(k)\varphi(t-k)$ for some c in $N(T)^\perp$. Then $\hat{c}^*(\xi) = 0\ a.e. on\ N_\varphi(cf. (37))$.Now ,

$$\delta_{0,n} = S(\sigma + n) = \sum_{k\in\mathbb{Z}} c(k)\varphi(\sigma + n - k), n \in \mathbb{Z}, \quad that\ is,$$

$c * d = \delta_0$, where $d = \{\varphi(\sigma + n)\}_{n\in\mathbb{Z}}$ and $\delta_0 = \{\delta_{0,n}\}_{n\in\mathbb{Z}}$. Hence $\hat{c}^*(\xi)\, \hat{d}^*(\xi) = 1$ a.e. on $[0,2\pi]$ so that $\hat{c}^*(\xi) \neq 0$ a.e. on $[0,2\pi]$, i.e.

$N_\varphi = \varphi$. Hence $E_\varphi = [0, 2\pi]$ so that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V_p(\varphi)$, i.e. (a) holds. Equivalence of (b) and (d) follows since $\{Sn(t) : n \in \mathbb{Z}\}$ and $\{k(t, \sigma + n) : n \in \mathbb{Z}\}$ are dual frames each other. Finally when $\varphi(t)$ is a Riesz generator satisfying (39) and (40), $V_p(\varphi) = V(\varphi)$ since $N(T) = \{0\}$ so $N(T)^\perp = l^2$. Combining Theorem (6.2.10) and Proposition (6.2.11) , we have:

Theorem$(6.2.12)$[14]: Let $\varphi(t)$ be a Riesz generator satisfying (39) and (40) so that $V(\varphi)$ is an RKHS and $0 \le \sigma < 1$. Then any one of the equivalent statements on $V(\varphi)$ in Theorem (6.2.10) is also equivalent to each one of the following .$(a)'$ There is a Riesz basis $\{S_n(t) : n \in \mathbb{Z}\}$ of V (φ) with which (50) holds. $(e)'$ $0 < \|Z_\varphi(\sigma, \xi)\|_0 \le \|Z_\varphi(\sigma, \xi)\|_\infty < \infty$.

(f)' $\{k(t, \sigma + n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V(\varphi)$. Moreover if any one of the above equivalent statements holds, then

$$S_n(t) = S(t-n) = B_n(t) , \quad n \in \mathbb{Z} \tag{54}$$

where

$$\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma, \xi)} \quad a.e. \ on \ \mathbb{R} \tag{55}$$

and

$$S(\sigma + n) = \delta_{0,n} , \ n \in \mathbb{Z} . \tag{56}$$

Proof: Theorem (6.2.10) (a) implies (a)' by Proposition (6.2.11) . Conversely assume (a)'. Then the sampling series (50) is a Riesz basis expansion so that

$$f(\sigma + n) = \langle f(t), U^{-1}(S_n) \rangle_{L^2(\mathbb{R})} = \langle U^{-1}(f), S_n(t) \rangle, n \in \mathbb{Z}$$

where U is the frame operator of $\{S_n(t) : n \in \mathbb{Z}\}$. Hence $\{f(\sigma + n)\}_{n \in \mathbb{Z}}$ is a moment sequence of $U^{-1}(f)$ to $\{S_n(t) : n \in \mathbb{Z}\}$ so that Theorem (6.2.10) (a) holds. Equivalence of Theorems(6.2.10) (e) and (6.2.10) (e)' follows since $E_\varphi = [0, 2\pi]$, i.e. $suppG_\varphi(\xi) = \mathbb{R}$ when $\varphi(t)$ is a Riesz generator. Since $k(t, \sigma + n) = k(t - n, \sigma), n \in \mathbb{Z}$, (f)' means that $k(t, \sigma)$ is a Riesz generator of $V(\varphi)$. Then equivalence of Theorems (6.2.10) (f) and (6.2.10) (f)' follows from Lemma (6.2.5) since $\varphi(t)$ and $k(t, \sigma)$ are equivalent frame generators. We have (56) and $S_n(t) = S(t-n), n \in \mathbb{Z}$ by Proposition (6.2.11). Hence (54) holds since (52) and (56) imply

$$S_k(t) = S(t-k) = \sum_{n \in \mathbb{Z}} S(\sigma + n - k) B_n(t)$$

$$= \sum_{n \in \mathbb{Z}} \delta_{k,n} B_n(t) = B_k(t), k \in \mathbb{Z}.$$

Finally (51) implies $\varphi(t) = \sum_{n \in \mathbb{Z}} \varphi(\sigma + n) S(t-n)$ so that $\hat{\varphi}(\xi) = Z_\varphi(\sigma, \xi) \hat{S}(\xi)$ from which (55) follows since $suppZ_\varphi(\sigma, \xi) = \mathbb{R}$ by (e)'. In Theorem (6.2.12), we may express any $f(t)$ in $V(\varphi)$ as

$$f(t) = \sum_{n\in\mathbb{Z}} c(n)\,\varphi(t-n) = \sum_{n\in\mathbb{Z}} f(\sigma+n)S(t-n),$$

which implies by (55) $\hat{f}(\xi) = \hat{c}^*(\xi)\varphi(\xi) = Z_f(\sigma,\xi)Z_\varphi(\sigma,\xi)^{-1}\,\widehat{\varphi}(\xi)$.

Hence $\hat{c}^*(\xi) = Z_f(\sigma,\xi)Z_\varphi(\sigma,\xi)^{-1}$, which gives a relation

$c(n) = \sum_{k\in\mathbb{Z}} f(\sigma+k)b(n-k), n \in \mathbb{Z}$ connecting the expansion

Coefficients $\{c(n)\}_{n\in\mathbb{Z}}$ of $f(t)$ and its sample values $\{f(\sigma+n)\}_{n\in\mathbb{Z}}$, where $Z_\varphi(\sigma,\xi)^{-1} = \sum_{n\in\mathbb{Z}} b(n)e^{-in\xi}$.Walter considered V ($\varphi$), where $\varphi(t)$ is a real-valued continuous orthonormal generator such that $\varphi(t) = O((1+|t|)^{-s})$ with $s > 1$ and $Z_\varphi(0,\xi) \neq 0$ on $\mathbb{R}$.Then $Z_\varphi(0,\xi) \in C[0,2\pi]$ since $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ so that the condition (e)' for $\sigma = 0$ in Theorem (6.2.12) holds. Hence we have the sampling series expansion (51) with $\sigma = 0$ on V (φ), which is a Riesz basis expansion and converges both in $L^2(\mathbb{R})$ and absolutely and uniformly on $\mathbb{R}$ since $sup_{\mathbb{R}}\Phi(t) < \infty$. Chen and Itoh, extending Walter's, considered a Riesz generator φ (t) satisfying only $\{\varphi(\sigma+n)\}_{n\in\mathbb{Z}} \in l^2$ for some σ in $[0,1)$ and claimed (see [25]) that there is $S(t) \in V(\varphi)$ such that the sampling expansion formula (51)holds on V (φ) in $L^2(\mathbb{R})$ sense if and only if $Z_\varphi(\sigma,\xi)^{-1} \in L^2$ [0, 2π] and in this case $\hat{S}(\xi) = Z_\varphi(\sigma,\xi)^{-1}\,\widehat{\varphi}(\xi)$. However there are some gaps in the arguments of the proof in [25]. Assuming only $\{\varphi(\sigma+n)\}_{n\in\mathbb{Z}} \in l^2$ for some σ in $[0,1)$, which is weaker than the condition (40), $V(\varphi)$ may not be an RKHS so that the point evaluation of functions in $V(\varphi)$ need not be well defined. Moreover, Chen and Itoh assumed nothing on $\{S(t_n)\}_{n\in\mathbb{Z}}$ but we must assume that $\{S(t-n)\}_{n\in\mathbb{Z}}$ is at least a Bessel sequence to derive $\hat{S}(\xi) = Z_\varphi(\sigma,\xi)^{-1}\,\widehat{\varphi}(\xi)$ from (51) (see Lemma (6.2.2)). Then we also have $G_\varphi(\xi) = \left|Z_\varphi(\sigma,\xi)\right|^2 G_s(\xi)$ a.e. on $\mathbb{R}$ so that $\left\|Z_\varphi(\sigma,\xi)\right\|_0 > 0$, which is already stronger than $Z_\varphi(\sigma,\xi)^{-1}\,L^2[0,2\pi]$. The condition $Z_\varphi(\sigma,\xi)^{-1} \in L^2[0,2\pi]$ guarantees only that $S(t)$ with $\hat{S}(\xi) = Z_\varphi(\sigma,\xi)^{-1}\,\widehat{\varphi}(\xi)$ is in $V(\varphi)$ and satisfies the interpolatory condition $S(\sigma+n) = \delta_{0,n}, n \in \mathbb{Z}$. In [26], the authors proved the equivalences of (a) in Theorem (6.2.10) , (a)' and (e)' in Theorem (6.2.12) together with (55) assuming that $\varphi(t)$ is a continuous Riesz generator satisfying $\varphi(t) = O((1+|t|)^{-s})$ with $s > \frac{1}{2}$. Finally we consider the case (c) in Proposition (6.2.4).

Theorem(6.2.13)[14]: Let $\varphi(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ be a continuous frame generator satisfying $sup_{\mathbb{R}}\Phi(t) < \infty$ and $0 \le \sigma < 1$. Then any one of the equivalent statements on $V_p(\varphi) = V(\varphi)$ in Theorem (6.2.9) is also equivalent to (e)'' there are constants $\beta \ge \alpha > 0$ such that $\alpha\chi suppG_\varphi(\xi) \le \left|Z_\varphi(\sigma,\xi)\right| \le \beta\chi suppG_\varphi(\xi)$ a.e. on $\mathbb{R}$. Moreover, if any one of the above equivalent statements holds, then

$$S_n(t) = S(t-n)\,, \qquad n \in \mathbb{Z} \tag{57}$$

Where

$$\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)}\, \chi suppG_\varphi(\xi) \quad a.e.\ on \quad \mathbb{R} \tag{58}$$

and sampling series (50),(51),(52) converge both in $L^2(\mathbb{R})$ and absolutely and uniformly on $\mathbb{R}$.

Proof: By the same arguments as in Lemma (6.2.9), we can easily see that the condition (d) on $V(\varphi) = V_p(\varphi)$ in Theorem (6.2.10) is now equivalent to

$$2\pi\alpha \int_0^{2\pi} |\hat{c}^*(\xi)|^2 G_\varphi(\xi) d\xi \le \int_{E\varphi} |\hat{c}^*(\xi)|^2 \left|Z_\varphi(\sigma,\xi)\right|^2 d\xi$$

$$\le 2\pi\beta \int_0^{2\pi} |\hat{c}^*(\xi)|^2 G_\varphi(\xi) d\xi, c \in l^2,$$

which is also equivalent to $2\pi\alpha G_\varphi(\xi) \le \left|Z_\varphi(\sigma,\xi)\right|^2 \le 2\pi\beta G_\varphi(\xi)$ a.e. on $[0,2\pi]$. Now by the same arguments as in the proof of Lemma (6.2.9), one can see the equivalence of the condition (d) on $V(\varphi)$ in Theorem (6.2.10) and (e)". (57) and (58) are already proved in Theorem (6.2.10). Finally, the sampling series (52),(51),(52) converge uniformly on $\mathbb{R}$ since $sup_{\mathbb{R}}\Phi(t) < \infty$. Zhou and Sun proved (see Theorem 1 in [38]) the equivalence of the condition (b) in Theorem (6.2.10) and (e)" in the case of $\sigma = 0$. It is interesting to note that the weaker condition (e) in Theorem (6.2.10) implies the stronger condition (e)" for any continuous frame generator $\varphi(t)$ with $sup_{\mathbb{R}}\Phi(t) < \infty$. In [34], Zhou and Sun introduced a notion of "sampling space" as: a closed subspace V of $L^2(\mathbb{R})$ is a sampling space if V has a frame generator $S(t)$ such that $\sum_{k\in\mathbb{Z}} c(k)S(t-k)$ converges pointwise to a continuous function on $\mathbb{R}$ for each $c = \{c(k)\}_{n\in\mathbb{Z}}$ in l^2 and $f(t) = \sum_{k\in\mathbb{Z}} f(k)S(t-k), f \in V$, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$. By Theorem (6.2.13), we can easily see that V is a sampling space if and only if V has a frame generator $\varphi(t)$ in $C(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfying $sup_{\mathbb{R}}\Phi(t) < \infty$ and the condition (e)" in Theorem (6.2.16). Then Theorem (6.2.10)naturally leads to the following relaxed notion of a sampling space.

Definition $(6.2.14)[14]$: A closed subspace V of $L^2(\mathbb{R})$ is called a sampling space if $V = V(\varphi)$ for some frame generator $\varphi(t)$ satisfying (39) and (40) and there are constants $\beta \ge \alpha > 0$ such that $\alpha \le |Z_\varphi(\sigma,\xi)| \le \beta$ a.e. on E_φ for some σ in $[0,1)$. Then Theorem (6.2.10) implies that any sampling space $V = V(\varphi)$ has a frame $\{S(t-n) : n \in \mathbb{Z}\}$ such that

$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n)S(t-n), f \in V_p(\varphi)$ which converges both in $L^2(\mathbb{R})$ and absolutely on $\mathbb{R}$.

Theorem $(6.2.15)$[14]: Let $V\ (\varphi)$ be a sampling space where $\varphi\ (t)$ and σ are the same as in Definition (6.2.14) . If $Z_\varphi(\sigma,\xi) \in L^\infty(\mathbb{R})$, then there is a sampling space $V\ (\psi) \supset V\ (\varphi)$, where ψ (t) is a Riesz generator satisfying (39) and (40), and $\tilde{\alpha} \le |Z_\psi(\sigma,\xi)| \le \tilde{\beta}$ a.e. on $\mathbb{R}$ for some constants $\tilde{\beta} \ge \tilde{\alpha} > 0$.

Proof: If $\varphi\ (t)$ itself is a Riesz generator, then we may take $\psi(t) = \varphi(t)$. Hence assume that $\varphi(t)$ is a frame generator with frame bounds (A_φ, B_φ) but not a Rieszgenerator. Then $E_\varphi = supp\ G_\varphi\ (\xi) \cap [0,2\pi] \subsetneq [0,2\pi]$ and $N_\varphi = [0,2\pi]\backslash E_\varphi$ has a positive measure. Take $\psi(t) = \varphi\ (t) + \zeta\ (t)$, where $\hat{\zeta}\ (\xi) = \frac{B}{\sqrt{2\pi}}\chi_{N_\varphi}(\xi)$ and $||Z_\varphi\ (\sigma,\xi)||_\infty < B$. Then $\zeta\ (t) \in PW_\pi$ and so

$$sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |\zeta(t-n)|^2 < \infty\ .$$ Hence $\psi\ (t)$ satisfies (39) and (40).

On the other hand $G_\psi(\xi) = \sum_{n\in\mathbb{Z}} \left|\ \hat{\psi}\ (\xi + 2n\pi)\right|^2 = G_\psi(\xi) + \hat{\zeta}(\ \xi)^2$ on $[0,2\pi]$ so that $\min\ (A_\varphi, B^2) \le 2\pi G_\psi(\xi) \le \max\ (B_\varphi, B^2)$ a.e. on $[0,2\pi]$. Hence $\psi\ (t)$ is a Riesz generator. Now $Z_\psi(\sigma,\xi) = Z_\varphi\ (\sigma,\xi) + Z_\zeta(\ \sigma,\xi) = Z_\varphi(\sigma,\xi) + Be^{i\sigma\xi}\chi_{N_\varphi}(\xi)$ on $[0,2\pi]$ by Lemma (6.2.20) below. Therefore

$B - ||Z_\varphi\ (\sigma,\xi)||_\infty \le |Z_\psi(\sigma,\xi)| \le B + ||Z_\varphi\ (\sigma,\xi)||_\infty$ $a.e.$ on N_φ so that

$$\min\left(\alpha, B - ||Z_\varphi\ (\sigma,\xi)||_\infty\right) \le |Z_\psi(\sigma,\xi)|$$
$$\le \max\left(\beta, B + ||Z_\varphi\ (\sigma,\xi)||_\infty\right) \text{a.e. on } [0, 2\pi].$$

Finally we have $\hat{\varphi}(\xi) = \hat{\varphi}(\xi)\chi_{supp}G_\varphi\ (\xi) = \hat{\psi}(\xi)\chi_{supp}\ G_\varphi\ (\xi)$ since $supp\hat{\varphi}\ (\xi) \subset suppG_\varphi(\xi)$ and $N_\varphi \cap suppG_\varphi(\xi) = \varphi$. Hence $\varphi\ (t) \in V\ (\psi)$ so $V(\varphi) \subset V\ (\psi)$. Note that the sampling space $V\ (\psi)$ in Theorem (6.2.16) has a Riesz basis $\{S(t-n) : n \in \mathbb{Z}\}$ for which we have a sampling expansion (see Theorem (6.2.12)):

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)S(t - n), f \in V\ (\psi).$$

Finally we give a sufficient condition that a function $f \in L^2(\mathbb{R})$ belongs to some sampling space.

Theorem$(6.2.16)$[14]: (see [34]). Let $f \in L^2(\mathbb{R})$. If $\hat{f}(\xi) \in L^2(\mathbb{R})$ and there exist constants $B \ge A > 0$ and $0 \le \sigma < 1$ such that

$$A|Z_f\ (\sigma,\xi)|^2 \le \sum_{n\in\mathbb{Z}} \left|\hat{f}\ (\xi + 2n\pi)\right|^2 \le \left(\sum_{n\in\mathbb{Z}} |\hat{f}\ (\xi + 2n\pi)|\right)^2$$
$$\le B\left|Z_f\ (\sigma,\xi)\right|^2 , \qquad (59)$$

then f belongs to some sampling space.

Proof : Let $A_f = [0,2\pi] \cap supp\ Z_f\ (\sigma,\xi), B_f = [0,2\pi] \backslash A_f$, and

$$\hat{\varphi}(\xi) = \begin{cases} \frac{\hat{f}(\xi)}{Z_f(\sigma,\xi)} & ,\xi \in supp\, Z_f(\sigma,\xi) \\ \frac{1}{\sqrt{2\pi}} & ,\xi \in B_f \\ 0, \text{otherwise} \end{cases} \quad (60)$$

Then

$$\int_{-\infty}^{\infty} |\hat{\varphi}(\xi)|^2 d\xi = \int_{A_f} \frac{\sum_{n\in\mathbb{Z}} |\hat{f}(\xi + 2n\pi)|^2}{|Z_f(\sigma,\xi)|^2} d\xi + \int_{B_f} \frac{1}{2\pi} d\xi$$

$$\leq \int_{A_f} B\, d\xi + \int_{B_f} \frac{1}{2\pi} d\xi < \infty$$

so that $\hat{\varphi}(\xi) \in L^2(\mathbb{R})$and $\varphi(t) = \mathcal{F}^{-1}(\hat{\varphi})(t) \in L^2(\mathbb{R})$. Then

$$G_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2 = \begin{cases} \frac{\sum_{n\in\mathbb{Z}} |\hat{f}(\xi + 2n\pi)|^2}{|Z_f(\sigma,\xi)|^2} & \xi \in A_f \\ \frac{1}{\sqrt{2\pi}} & ,\xi \in B_f \end{cases}$$

so that $\min\left(\frac{1}{2\pi}, A\right) \leq G\varphi(\xi) \leq \max\left(\frac{1}{2\pi}, B\right)$ $,\xi \in [0,2\pi]$ and $\varphi(t)$ is a Riesz generator. We also have

$$\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| = \begin{cases} \frac{\sum_{n\in\mathbb{Z}} |\hat{f}(\xi + 2n\pi)|}{|Z_f(\sigma,\xi)|} & ,\xi \in A_f \\ \frac{1}{\sqrt{2\pi}} & ,\xi \in B_f \end{cases}$$

so that $\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$. Then $\varphi(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is a continuous Riesz generator and $sup_{\mathbb{R}} \Phi(t) < \infty$ by Proposition (6.2.7). On the other hand , we have by Lemma (6.2.20) below ,

$$Z_\varphi(\sigma,\xi) = \begin{cases} \frac{\sqrt{2\pi}\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)}\hat{f}(\xi+2n\pi)Z_f(\sigma,\xi)}{Z_f(\sigma,\xi)} = 1 & ,\ \xi \in A_f \\ e^{i\sigma\xi}, & \xi \in B_f \end{cases}$$

so that $|Z_\varphi(\sigma,\xi)| = 1$ a.e. on $[0,2\pi]$ and $V(\varphi)$ is a sampling space. Finally we have $\text{supp}\sum_{n\in\mathbb{Z}} |\hat{f}(\xi + 2n\pi)|^2 =$ supp $Z_f(\sigma,\xi)$ by (59) and so $\hat{f}(\xi) = Z_f(\sigma,\xi)\,\hat{\varphi}(\xi)$ by (60), which implies $f \in V(\varphi)$.

We assume that $\varphi(t)$ is a frame generator with frame bounds (A,B) satisfying the conditions (39) and (40) and for some constants $\beta \geq \alpha > 0$ and $\sigma \in [0,1)$ $\alpha \leq |Z_\varphi(\sigma,\xi)| \leq \beta$a.e. on supp$G_\varphi(\xi)$.Then by Theorem (6.2.9), $V_p(\varphi)$ is an RKHS on which we have a sampling expansion (51)

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)S(t-n), f \in V_p(\varphi)$$

where $\{S(t-n) : n \in \mathbb{Z}\}$ is a frame of $V_p(\varphi)$ with frame bounds $\left(\frac{A}{\beta^2}, \frac{B}{\alpha^2}\right)$ and $S(t)$ satisfies (53). Let c = $\{c(n)\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of $\frac{1}{Z_\varphi(\sigma,\xi)}\chi_{supp}G_\varphi(\xi)$ $\in L^\infty[0,2\pi]$ so that $\frac{1}{Z_\varphi(\sigma,\xi)}\chi_{supp}G_\varphi(\xi)=\sum_{n\in\mathbb{Z}} c(n)e^{-in\xi}$. Then $S(t) = (c * \varphi)(t)$ and so we have by (32)

$$\sum_{n\in\mathbb{Z}}|S(t+n)|^2 = \sum_{n\in\mathbb{Z}}|(c*\varphi)(t+n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{c}^*(\xi)Z\varphi(t,\xi)|^2 d\xi.$$

Hence

$$\sum_{n\in\mathbb{Z}}|S(t+n)|^2 \le \frac{1}{\alpha^2}\sum_{n\in\mathbb{Z}}|\varphi(t+n)|^2, t \in \mathbb{R} \quad . \tag{61}$$

Writing (51) as $f(t) = \sum_{|k|\le n} f(\sigma+k)S(t-k) + E_n(f)(t),$

we call

$$E_n(f)(t) = \sum_{|k|\le n} f(\sigma+k)S(t-k) \tag{62}$$

the nth truncation error of $f(t)$. We first consider the L^2-estimate of $E_n(f)(t)$.

Theorem $(6.2.17)[14]$: We have

$$\|E_n(f)(t)\|^2_{L^2(\mathbb{R})} \le \frac{B}{\alpha^2}\sum_{|k|\le n}|f(\sigma+k)|^2$$
$$\le \left(\frac{\beta}{\alpha}\right)^2\frac{A}{B}||f||_{L^2(\mathbb{R})}, f \in V_p(\varphi). \tag{63}$$

Proof :We have by (53) and (62)

$$\|E_n(f)(t)\|^2_{L^2(\mathbb{R})} = \left\|\widehat{E_n(f)}(\xi)\right\|^2_{L^2(\mathbb{R})}$$
$$= \int_0^{2\pi}\left|\sum_{|k|\le n} \mathrm{f}(\sigma+\mathrm{k})e^{-\mathrm{i}k\xi}\right|^2 \frac{G\varphi(\xi)}{|Z\varphi(\sigma,\xi)|^2}\chi_{supp}\, G_\varphi(\xi)d\xi$$

from which (63) follows immediately since $\{k(t,\sigma+n) : n \in \mathbb{Z}\}$ is the canonical dual frame of $\{S(t-n) : n \in \mathbb{Z}\}$. If moreover, $\varphi(t)$ is a Riesz generator, then

$$\frac{A}{\beta^2}\sum_{|k|\le n}|f(\sigma+k)|^2 \le ||E_n(f)(t)||^2_{L^2(\mathbb{R}} \le \frac{B}{\alpha^2}\sum_{|k|\le n}|f(\sigma+k)|^2, f \in V(\varphi).$$

Concerning the point wise estimate of the truncation error (see [36]), we have from (61) and (62)

$$|E_n(f)(t)|^2 \le \frac{1}{\alpha^2}\Phi(t)\sum_{|k|\le n}|f(\sigma+k)|^2, f \in V_p(\varphi)$$

where $\Phi(t) = \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ so that the sampling series (51) converges uniformly on any subset of ℝ on which Φ (t) is bounded . When $E_n(f)(t) \in L^1(\mathbb{R})$, we can improve the L^∞-estimate of the truncation error.

Lemma(6.2.18)[14]: If $H_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0,2\pi]$, then $sup_{\mathbb{R}}\Phi(t) < \infty$ and $\hat{f}(\xi) \in L^1(\mathbb{R})$ for any $f(t) \in V(\varphi)$. In particular , $V(\varphi) = Vp(\varphi)$.

Proof: First $sup_{\mathbb{R}}\Phi(t) < \infty$ by Proposition (6.2.7). For any $f(t) \in V(\varphi)$, $f(t) = (c * \varphi)(t)$ for some c in l^2. Hence $\hat{f}(\xi) = \hat{c}^*(\xi)\,\hat{\varphi}(\xi)$ and

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|\, d\xi = \int_{0}^{2\pi} |\hat{c}^*(\xi)|\, H_\varphi(\xi) d\xi$$

$$\leq ||\hat{c}^*(\xi)||_{L^2[0,2\pi]} ||H_\varphi(\xi)||_{L^2[0,2\pi]} < \infty$$

so that $\hat{f} \in L^1(\mathbb{R})$. Finally $V(\varphi) = V_p(\varphi)$ by Proposition (6.2.4)(d).

Theorem (6.2.19)[14]: If $H_\varphi(\xi) \in L^2[0,2\pi]$, then

$$\|E_n(f)(t)\|_\infty \leq \frac{1}{\alpha}\left\|H_\varphi(\xi)\right\|_{L^2(E_\varphi)}\left(\sum_{|k|>n} |f(\sigma+k)|^2\right)^{\frac{1}{2}}, f \in V(\varphi). \tag{64}$$

Hence the sampling series (51) converges uniformly on ℝ.

Proof : Now $\widehat{E_n(f)}(\xi) \in L^1(\mathbb{R})$ by Lemma (6.2.18) so that

$$E_n(f)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sum_{|k|>n} f(\sigma + k)e^{-ik\xi}\right)\hat{S}(\xi)e^{it\xi}\, d\xi.$$

Hence

$$|En(f)(t)| \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\sum_{|k|>n} f(\sigma + k)e^{-ik\xi}\right|\left|\hat{S}(\xi)\right| d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{E\varphi}\left(\sum_{|k|>n} f(\sigma + k)e^{-ik\xi}\right)\left|Z_\varphi(\sigma,\xi)\right|^{-1} H_\varphi(\xi) d\xi$$

from which (64) follows.

We first introduce a variance of the Poisson summation formula, which is effective in computing $Z_\varphi(\sigma,\xi)$.

Lemma (6.2.20)[14] : Let $F(\xi) \in L^1(\mathbb{R})$ so that $f(t) = \mathcal{F}^{-1}[F](t) \in C(\mathbb{R})$ and $0 \leq \sigma < 1$. Then

$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)}F(\xi + 2n\pi)$ converges absolutely in $L^1[0,2\pi]$ and

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)}F(\xi + 2n\pi) \sim \frac{1}{\sqrt{2\pi}}Z_f(\sigma,\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} f(\sigma + n)e^{-in\xi} \tag{65}$$

which means that $\frac{1}{\sqrt{2\pi}} Z_f(\sigma,\xi)$ is the Fourier series expansion of

$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)$. If moreover $\sum_{n\in Z} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)$ converges in $L^2[0,2\pi]$ or equivalently $\{f(\sigma+n)\}_{n\in\mathbb{Z}} \in l^2$, then

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi) = \frac{1}{\sqrt{2\pi}} Z_f(\sigma,\xi) \text{ in } L^2[0,2\pi]. \tag{66}$$

Proof : Assume that $(\xi) \in L^1(\mathbb{R})$. Then

$$\sum_{n\in\mathbb{Z}} \left\| e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi) \right\|_{L^1[0,2\pi]} = \sum_{n\in\mathbb{Z}} \int_0^{2\pi} |F(\xi+2n\pi)|\, d\xi$$

$$= \sum_{n\in\mathbb{Z}} \int_{2n\pi}^{2(n+1)\pi} |F(\xi)| d\xi = \int_{-\infty}^{+\infty} |F(\xi)| d\xi$$

so that

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi) \text{ converges absolutely in } L^1[0,2\pi] .$$

Hence

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)$$

$$\sim \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \langle \sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi), e^{-ik\xi} \rangle_{L^2[0,2\pi]} \; e^{-ik\xi},$$

where

$$\langle \sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi), e^{-ik\xi} \rangle_{L^2[0,2\pi]}$$

$$= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)\, e^{ik\xi}\, d\xi$$

$$= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)\, e^{ik\xi}\, d\xi$$

$$= \int_{-\infty}^{+\infty} F(\xi)\, e^{i(\sigma+k)\xi}\, d\xi = \sqrt{2\pi}\, f(\sigma+k)$$

by the Lebesgue dominated convergence theorem. Hence (65) holds. Now assume that $F(\xi) \in L^1(\mathbb{R})$ and

$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)$ converges in $L^2[0,2\pi]$. Then (65) becomes an orthonormal basis expansion of $\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi+2n\pi)$ in $L^2[0,2\pi]$

so that (66) holds. Lemma (6.2.20) is a more generalized version of the following result in [25].

Corollary $(6.2.21)$[14]: (see [25]). If $F(\xi)$ is measurable on $\mathbb{R}$ and $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges absolutely in $L^2[0,2\pi]$, then

$$\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi) = \frac{1}{\sqrt{2\pi}} Z_f(0,\xi) \quad \text{where} \quad f(t) = \mathcal{F}^{-1}[F](t).$$

Proof : Assume that $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges absolutely in $L^2[0,2\pi]$. Then $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges absolutely also in $L^1[0,2\pi]$ so that $F(\xi) \in L^1[0,2\pi]$ and $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges in $L^2[0,2\pi]$. Hence the conclusion follows from Lemma (6.2.20) for $\sigma = 0$.

Example $(6.2.22)$[14]: (see [13],[26] and [33]). Let $\varphi_0(t) = \chi_{[0,1)}(t)$ and

$$\varphi_n(t) = \varphi_{n-1}(t) * \varphi_0(t) = \int_0^1 \varphi_{n-1}(t-s)ds, n \geq 1 (\varphi_n(t) = B_{n+1}(t))$$

be the cardinal B-spline of degree n. Then

$$\widehat{\varphi_n}(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\xi}}{i\xi}\right)^{n+1} \text{ and } |\hat{\varphi}n(\xi)| = \frac{1}{\sqrt{2\pi}}\left|sinc\,\frac{\xi}{2\pi}\right|^{n+1}, n \geq 0.$$

It is known in [27] that $\varphi_0(t)$ is an orthonormal generator and $\varphi_n(t)$ for $n \geq 1$ is a continuous Riesz generator. Moreover since φ n(t) has compact support, $sup_{\mathbb{R}}\Phi_n(t) = sup_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\varphi_n(t-k)|^2 < \infty$ so that $V(\varphi_n)$ is an RKHS for $n \geq 0$. Since $\varphi_0(\sigma + n) = \delta_{0,n}$ for $n \in \mathbb{Z}$ and $0 \leq \sigma < 1$, $Z_{\varphi_0}(\sigma,\xi) = 1$ so that by Theorem (6.2.12), we have an orthonormal expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)\varphi_0(t-n), \quad f \in V(\varphi_0)$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$ since

$$\Phi_0(t) = \sum_{n\in\mathbb{Z}} |\varphi_0(t-n)|^2 = 1 \quad \text{on } \mathbb{R}.$$

For $\varphi_1(t) = t\chi_{[0,1)}(t) + (2-t)\chi_{[1,2)}(t)$, and $0 \leq \sigma < 1$, $\varphi_1(t) = \sigma, \varphi_1(\sigma+1) = 1-\sigma, \varphi_1(\sigma+n) = 0$ for $n \neq 0,1$ so that $Z_{\varphi_1}(\sigma,\xi) = \sigma + (1-\sigma)e^{-i\xi}$. Then $\|Z_{\varphi_1}(\sigma,\xi)\|_0 = |2\sigma - 1|$ and $\|Z_{\varphi_1}(\sigma,\xi)\|_\infty = 1$. Hence by Theorem (6.2.12), for any σ with $0 \leq \sigma < 1$ and $\sigma \neq \frac{1}{2}$, we have a Riesz basis expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)S(t-n), \qquad f \in V(\varphi_1)$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$. For

$\varphi_2(t) = \frac{1}{2} t^2 \chi_{[0,1)}(t) + \frac{1}{t^2 2}(6t - 2 - 3)\chi_{[1,2)}(t) + \frac{1}{2}(3-t)^2 \chi_{[2,3)}(t)$, it is known (see [13] and [26]) that

$\left\|Z_{\varphi_2}(0,\xi)\right\|_0 = 0$ but $\frac{1}{2} \leq \left\|Z_{\varphi_2}\left(\frac{1}{2}, \xi\right)\right\|_0 < \left\|Z_{\varphi_2}\left(\frac{1}{2}, \xi\right)\right\|_\infty \leq 1$ so that there is a Riesz basis expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{1}{2} + n\right) S(t-n), f \in V(\varphi_2) \tag{67}$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$. Since the optimal upper Riesz bound of the Riesz sequence $\{\varphi_2(t-k) : k \in \mathbb{Z}\}$ is 1 (see [27]), we have for the sampling series (67)

$$\|E_n(f)(t)\|^2{}_{L^2(\mathbb{R})} \leq 4 \sum_{|k|>n} \left|f\left(\frac{1}{2} + k\right)\right|^2, f \in V(\varphi_2).$$

On the other hand, we have

$$H\varphi_2(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}_2(\xi + 2k\pi)| = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left|sinc\left(\frac{\xi}{2\pi} + k\right)\right|^3$$

$$\leq \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left|sinc\left(\frac{\xi}{2\pi} + k\right)\right|^2 = \frac{1}{\sqrt{2\pi}}$$

Hence, Theorem (6.2.19) gives for the sampling series (67)

$$\|E_n(f)(t)\|_\infty \leq 2\left(\sum_{|k|>n} \left|f\left(\frac{1}{2} + k\right)\right|^2\right)^{\frac{1}{2}}$$

Example $(\mathbf{6.2.23})$[14] : Let $\varphi(t) = e^{\frac{-t^2}{2}}$ be the Gauss kernel . Then $\hat{\varphi}(\xi) = e^{\frac{-\xi^2}{2}}$ and $0 < \|G_\varphi(\xi)\|_0 < \|G_\varphi(\xi)\|_\infty < \infty$ so that $\varphi(t)$ is a continuous Riesz generator satisfying

$$sup_{\mathbb{R}}\Phi(t) = sup_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\varphi(t-k)|^2 < \infty. \text{Since } \hat{\varphi}(\xi) \in L^1(\mathbb{R})$$

and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$, we have by Lemma (6.2.20)

$$Z_\varphi(0,\xi) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} e^{\frac{-1}{2}(\xi + 2n\pi)^2} \text{ so that } 0 < \|Z_\varphi(\xi)\|_0 < \|Z_\varphi(\xi)\|_\infty < \infty.$$

Hence by Theorem (6.2.12), $V(\varphi)$ is an RKHS and there is a Riesz basis expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), f \in V(\varphi)$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$.

Example $(\mathbf{6.2.24})$[14]: (see [22]). Let E be a measurable subset of $[-\pi, \pi]$ and $\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \chi_E(\xi)$. Then $\varphi(t) = \mathcal{F}^{-1}[\hat{\varphi}](\xi) \in PW_\pi$ so that

$$sup_{\mathbb{R}}\Phi(t) = sup_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\varphi(t-k)|^2 < \infty \text{ and}$$

$$G_\varphi(\xi) = \sum_{n\in Z} |\hat{\varphi}(\xi + 2n\pi)|^2 = \frac{1}{2\pi}\chi E(\xi) \text{ on } [-\pi,\pi].$$

Hence $\varphi(t)$ is a (tight) continuous frame generator or an orthonormal generator if $|E| < 2\pi$ or $|E| = 2\pi$, respectively. Since $\hat{\varphi}(\xi) \in L^1(\mathbb{R})$ and $\{\varphi(\sigma + n)\}_{n\in\mathbb{Z}} \in l^2$ for $0 \le \sigma < 1$, we have by Lemma (6.2.20)

$$Z_\varphi(\sigma,\xi) = e^{i\sigma\xi} \sum_{n\in Z} e^{i\sigma 2n\pi} \chi_E(\xi + 2n\pi) = e^{i\sigma\xi} \chi_E(\xi) \text{ on } [0,2\pi]$$

so that $|Z_\varphi(\xi)| = 1$ on $E = E_\varphi$. Hence by Theorem (6.2.13) , V (φ) is an RKHS and there is a frame $\{S(t-n) : n \in \mathbb{Z}\}$ of $V(\varphi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)S(t-n), f \in V(\varphi) \qquad (68)$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$.When $E = [-\pi,\pi]$, $Z_\varphi(\sigma,\xi) = e^{i\sigma\xi}$ on $\mathbb{R}$ so that $\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)} = \frac{1}{\sqrt{2\pi}} \chi_{[-\pi,\pi]}(\xi)e^{-i\sigma\xi}$ Hence, $S(t) = sinc(t-\sigma)$ and so (68) becomes

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)sinc(t - \sigma - n)$$

which is the Wittacker shifted cardinal series on PW_π (see [30]). Moreover, applying Theorems (6.2.17) and (6.2.19), we have for the sampling series (68)

$$\|E_n(f)(t)\|^2_{L^2(\mathrm{R})} \le \sum_{|k|>z} |f(\sigma + k)|^2$$

and

$$\|E_n(f)(t)\|^2_\infty \le \sqrt{\frac{|E|}{2\pi}} \left(\sum_{|k|>Z} |f(\sigma + k)|^2\right)^{\frac{1}{2}}, f \in \mathrm{V}(\varphi).$$

Finally, we give an example of a Riesz generator $\varphi(t)$ with $sup_{\mathbb{R}}\Phi(t) = \infty$.

Example (6.2.25)[14]: For any $\varphi(t) \in L^2(\mathbb{R})\backslash\{0\}$, the Fourier series expansion of $G_\varphi(\xi)(\in L^1[0,2\pi])$ *is* $\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \langle G_\varphi(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]} e^{-in\xi}$

where

$$\langle G_\varphi(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]} = \int_{-\infty}^{\infty} |\hat{\varphi}(\xi)|^2 e^{in\xi} d\xi$$

$$= \int_{-\infty}^{\infty} \varphi(t)\overline{\varphi(t-n)}dt.$$

Hence if supp $\varphi \subset [0,1]$, then $G_\varphi(\xi) = \frac{1}{2\pi}\|\varphi\|$ a.e. on $\mathbb{R}$ so that $\varphi(t)$is a Riesz

For example, we take $\varphi(t) = \sum_{k=1}^{\infty} k\chi_{E_k}(t)$, where $E_k = \left(\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]$. Then

$\|\varphi\| = \sum_{k=1}^{\infty} \frac{k^2}{2^k} < \infty$ and supp $\varphi = [0, 1]$ so that $\varphi(t)$ is a Riesz generator satisfying (39) and (40) but $sup_{\mathbb{R}}\Phi(t) = \infty$ since

$$\Phi(\mathrm{t}) = \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 = \begin{cases} 1 & \text{if } t = 0 \\ k^2 & \text{if } \frac{1}{2^k} < t \le \frac{1}{2^{k-1}} \ (\mathrm{k} \ge 1) \end{cases} \text{ on } [0, 1).$$

Since we have

$Z_{\varphi}(0, \xi) = e^{-i\xi}$ and $Z_{\varphi}(\sigma, \xi) = \varphi(\sigma) \neq 0$ for $0 < \sigma < 1$, Theorem (6.2.12) implies that for any σ with $0 \le \sigma < 1$, there is a Riesz basis expansion

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma + n)S(t-n), f \in V(\varphi),$$

which converges in $L^2(\mathbb{R})$ and absolutely on $\mathbb{R}$.

Corollary (6.2.26)[296] : We have

$$\sum_{j\in\mathbb{Z}} \left\|E_n(f_j)(t)\right\|^2_{L^2(\mathbb{R})} \le \frac{B}{\alpha^2} \sum_{|k|\le n} \sum_{j\in\mathbb{Z}} \left|f_j(\sigma + k)\right|^2$$

$$\le \left(\frac{\beta}{\alpha}\right)^2 \frac{A}{B} \sum_{j\in\mathbb{Z}} ||f_j||_{L^2(\mathbb{R})}, f_j \in V_p(\varphi).$$

Proof :We have by (53) and (62)

$$\sum_{j\in\mathbb{Z}} \left\|E_n(f_j)(t)\right\|^2_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{Z}} \left\|\widehat{E_n(f_j)}(\xi)\right\|^2_{L^2(\mathbb{R})}$$

$$= \int_0^{2\pi} \left|\sum_{|k|\le n} \sum_{j\in\mathbb{Z}} f_j(\sigma + \mathrm{k})\mathrm{e}^{-\mathrm{ik}\xi}\right|^2 \frac{\mathrm{G}\varphi(\xi)}{|\mathrm{Z}\varphi(\sigma, \xi)|^2} \chi_{supp}\, G_{\varphi}(\xi) d\xi$$

from which (63) follows immediately since $\{k(t, \sigma + n) : n \in \mathbb{Z}\}$ is the canonical dual frame of $\{S(t-n) : n \in \mathbb{Z}\}$. If moreover, $\varphi(t)$ is a Riesz generator, then

$$\frac{\boldsymbol{A}}{\boldsymbol{\beta}^2} \sum_{|k|\le n} \sum_{j\in\mathbb{Z}} |\boldsymbol{f_j}(\boldsymbol{\sigma} + \boldsymbol{k})|^2 \le \sum_{j\in\mathbb{Z}} ||\boldsymbol{E_n}(\boldsymbol{f_j})(\boldsymbol{t})||^2_{L^2(\mathbb{R}} \le \frac{\boldsymbol{B}}{\boldsymbol{\alpha}^2} \sum_{|k|\le n} \sum_{j\in\mathbb{Z}} \left|\boldsymbol{f_j}(\boldsymbol{\sigma} + \boldsymbol{k})\right|^2, \boldsymbol{f_j} \in \boldsymbol{V}(\boldsymbol{\varphi}).$$

Concerning the point wise estimate of the truncation error (see [36]), we have from (61) and (62)

$$\sum_{j\in\mathbb{Z}} |E_n(f_j)(t)|^2 \le \frac{1}{\alpha^2} \Phi(t) \sum_{|k|\le n} \sum_{j\in\mathbb{Z}} \left|f_j(\sigma + k)\right|^2, f_j \in V_p(\varphi)$$

where $\Phi(t) = \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ so that the sampling series (51) converges uniformly on any subset of $\mathbb{R}$ on which $\Phi(t)$ is bounded . When $\sum_{j\in\mathbb{Z}} E_n(f_j)(t) \in L^1(\mathbb{R})$, we can improve the L^∞-estimate of the truncation error.

Corollary $(6.2.27)$[296] : Let $F(\xi_j) \in L^1(\mathbb{R})$ so that $f(t) = \mathcal{F}^{-1}[F](t) \in C(\mathbb{R})$ and $\epsilon > 0$. Then

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F\left(\xi_j + 2n\pi\right) \text{ converges absolutely in } L^1[0,2\pi] \text{ and}$$

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi) \sim \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} Z_f(1-\epsilon,\xi_j)$$
$$= \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} f(n-\epsilon)e^{-in\xi_j}$$

which means that $\frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} Z_f(1-\epsilon,\xi_j)$ is the Fourier series expansion of $\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \boldsymbol{e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi)}$. If moreover $\sum_{n\in Z}\sum_{j\in\mathbb{Z}} \boldsymbol{e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi)}$ converges in $L^2[0,2\pi]$ or equivalently $\{f(n-\epsilon)\}_{n\in\mathbb{Z}} \in l^2$, then

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi) = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} Z_f\left(1-\epsilon,\xi_j\right) \text{ in } L^2[0,2\pi].$$

Proof : Assume that $(\xi_j) \in L^1(\mathbb{R})$. Then

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \left\|e^{i(1-\epsilon)(\xi_j+2n\pi)}F\left(\xi_j + 2n\pi\right)\right\|_{L^1[0,2\pi]} = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\int_0^{2\pi} \left|F\left(\xi_j + 2n\pi\right)\right| d\xi_j$$
$$= \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\int_{2n\pi}^{2(n+1)\pi} |F(\xi_j)|d\xi_j = \sum_{j\in\mathbb{Z}}\int_{-\infty}^{+\infty} |F(\xi_j)|d\xi_j$$

so that

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi) \text{ converges absolutely in } L^1[0,2\pi] \ .$$

Hence

$$\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi)$$
$$\sim \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\langle\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F(\xi_j + 2n\pi), \sum_{j\in\mathbb{Z}} e^{-ik\xi_j}\rangle_{L^2[0,2\pi]} \sum_{j\in\mathbb{Z}} e^{-ik\xi_j},$$

where

$$\langle\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F\left(\xi_j + 2n\pi\right), \sum_{j\in\mathbb{Z}} e^{-ik\xi_j}\rangle_{L^2[0,2\pi]}$$
$$= \int_0^{2\pi} \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)}F\left(\xi_j + 2n\pi\right)e^{ik\xi_j}d\xi_j$$

$$= \sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \int_0^{2\pi} e^{i(1-\epsilon)(\xi_j+2n\pi)} F(\xi_j + 2n\pi) e^{ik\xi_j} d\xi_j$$

$$= \sum_{j\in\mathbb{Z}} \int_{-\infty}^{+\infty} F(\xi_j)\, e^{i(k-\epsilon)\xi_j} d\xi_j = \sqrt{2\pi}\, f(k-\epsilon)$$

by the Lebesgue dominated convergence theorem. Hence in lemma (6.2.20) (65) holds. Now assume that $F(\xi_j) \in L^1(\mathbb{R})$ and $\sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)} F(\xi_j + 2n\pi)$ converges in $L^2[0, 2\pi]$. Then (65) becomes an orthonormal basis expansion of $\sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} e^{i(1-\epsilon)(\xi_j+2n\pi)} F(\xi_j + 2n\pi)$ in $L^2[0, 2\pi]$ so that (66) holds.

Section (6.3) Asymmetric Multi-Channel Sampling

Reconstructing a signal from samples which are taken from its several channeled versions is called multi-channel sampling.The multi-channel sampling method goes back to the works of Shannon [19] and Fogel [8], where reconstruction of a band-limited signal from samples of the signal and its derivatives was suggested. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [17]. Since Papoulis' fundamental work, there have been many generalizations and applications of multi-channel sampling. See [2,6,7,15,16,18] and references therein.

Papoulis' result has also been extended to a general shift invariant space by using the filter banks technique (see [5,20,21]). More recently Garcia and Pérez-Villalon [9] derived stable generalized sampling in a shift invariant space. Most previous work related to multi-channel sampling has assumed that the sampling rates of all channels are the same.

We consider an asymmetric multi-channel sampling in a shift invariant space $V(\varphi)$ with a suitable Riesz generator $\varphi(t)$, where each channeled signal is sampled with a uniform but distinct rate. Using Fourier duality between $V(\varphi)$ and $L^2[0, 2\pi]$ [9,10,11], we derive a stable shifted asymmetric multi-channel sampling formula in $V(\varphi)$. The corresponding symmetric multi-channel sampling in $V(\varphi)$ was handled in [10],where $\varphi(t)$ is a continuous Riesz generator satisfying $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 < \infty$. In this case all signals in $V(\varphi)$ are continuous on $\mathbb{R}$ (see[22]). We require only that the Riesz generator $\varphi(t)$ is point wise well defined everywhere on $\mathbb{R}$ and $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 < \infty, t \in \mathbb{R}$. Hence we essentially allow any Riesz generator in $L^2(\mathbb{R})$. On the other hand, we allow more general filters than the ones in [9] by asking only that the impulse responses of filters belong to $L^2(\mathbb{R})$ (or the frequency responses of filters belong to $L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$when $\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in$

$L^2[0,2\pi]$), whereas they belong to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ in [10]. Finally, we give an illustrative example. We take the Fourier transform to be normalized as

$$\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(t) e^{-it\xi}\, dt, \varphi(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

so that $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\cdot]$ extends to a unitary operator from $L^2(\mathbb{R})$onto $L^2(R)$. For any $\varphi(t) \in L^2(\mathbb{R})$, let

$$C_\varphi(t) = \sum_{n\in\mathbb{Z}} |\varphi(t+n)|^2 \text{ and } G_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi+2n\pi)|^2.$$

Then $C_\varphi(t) = C_\varphi(t+1) \in L^1[0,1], G_\varphi(\xi) = G_\varphi(\xi+2\pi) \in L^2[0,2\pi]$ and $\|\varphi(t)\|^2_{L^2(\mathbb{R})} = \|C_\varphi(t)\|_{L^1[0,1]} = \frac{1}{2\pi}\|G_\varphi(\xi)\|_{L^1[0,2\pi]}$

In particular, $C_\varphi(t) < \infty$ for a.e. t in $\mathbb{R}$. We also let

$$Z_\varphi(t,\xi) = \sum_{n\in\mathbb{Z}} \varphi(t+n) e^{-in\xi}$$

be the Zak transform [13] of $\varphi(t)$ in $L^2(\mathbb{R})$). Then $Z_\varphi(t,\xi)$ is well defined a.e. on $\mathbb{R}^2$ and is quasi-periodic in the sense that

$Z_\varphi(t+1,\xi) = e^{i\xi} Z_\varphi(t,\xi)$ and $Z_\varphi(t,\xi+2\pi) = Z_\varphi(t,\xi)$.

A Hilbert space H consisting of complex valued functions on a set E is called a reproducing kernel Hilbert space (RKHS in short) if there is a function $q(s,t)$ on $E \times E$, called the reproducing kernel of H, satisfying

(i) $q(\cdot,t) \in H$ for each t in E,

(ii) $\langle f(s), q(s,t)\rangle = f(t), f \in H$.

In an RKHS H, any norm converging sequence also converges uniformly on any subset of E, on which $\|q(\cdot,t)\|^2_H = q(t,t)$ is bounded. A sequence $\{\varphi_n : n \in \mathbb{Z}\}$ of vectors in a separable Hilbert space H is

(i) a Bessel sequence with a bound B (> 0) if

$$\sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n\rangle|^2 \le B\|\varphi\|^2, \varphi \in H,$$

(ii) a frame of H with bounds $B \ge A$ (> 0) if

$$A\|\varphi\|^2 \le \sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n\rangle|^2 \le B\|\varphi\|^2, \varphi \in H,$$

(iii) a Riesz basis of H with bounds $B \ge A$ (> 0) if it is complete in

$$H \text{ and } A\|\boldsymbol{c}\|^2 \le \left\|\sum_{n\in\mathbb{Z}} c(n)\varphi_n\right\|^2 \le B\|c\|^2\ C = \{c(n)\}_{n\in\mathbb{Z}} \in l^2,$$

$$\text{where} \quad \|\boldsymbol{c}\|^2 = \sum_{n\in\mathbb{Z}} |c(n)|^2 .$$

In the rest of the section, we let $V(\varphi)$ be the shift invariant space, where $\varphi(t)$ is a Riesz generator, that is, $\{\varphi(t-n): n \in \mathbb{Z}\}$ is a Riesz basis of $V(\varphi)$. Then

$$V(\varphi) = \left\{ (\boldsymbol{c} * \varphi)(t) = \sum_{n\in\mathbb{Z}} c(n)\varphi(t-n) : C = \{c(n)\}_{n\in\mathbb{Z}} \in l^2 \right\}.$$

It is well known (see [3]) that $\varphi(t)$ is a Riesz generator if and only if there are constants $B \geq A > 0$ such that $A \leq G_\varphi(\xi) \leq B \; a.e. on \; [0, 2\pi]$. In this case, $\{\varphi(t-n): n \in \mathbb{Z}\}$ is a Riesz basis of $V(\varphi)$ with bounds $B \geq A$. We assume further that

(i) $\varphi(t)$ is everywhere well defined on $\mathbb{R}$;

(ii) $C_\varphi(t) < \infty, t \in \mathbb{R}, i.e., \{\varphi(t+n): n \in \mathbb{Z}\} \in l^2$ for each t in $\mathbb{R}$.

We then allow essentially all Riesz generators since for any $\varphi(t) \in L^2(\mathbb{R})$, $C_\varphi(t) < \infty$ a.e. so that $\varphi(t)$ has an equivalent representative satisfying the above two conditions. Then for each $\boldsymbol{c} \in l^2, (\boldsymbol{c} * \varphi)(t)$ converges both in $L^2(\mathbb{R})$and absolutely for each t in $\mathbb{R}$. Hence $V(\varphi)$ becomes an RKHS with the reproducing kernel (see[14])

$$q(s,t) = \sum_{n\in\mathbb{Z}} \tilde{\varphi}(s-n)\overline{\varphi(t-n)}, \text{where } \{\tilde{\varphi}(t-n): n \in \mathbb{Z}\} \text{ is the dual}$$

Riesz basis of $\{\varphi(t-n): n \in \mathbb{Z}\}$ with bounds $\frac{1}{A} \geq \frac{1}{B}$. As in [10,11] , we introduce an isomorphism $\mathcal{J}$ from $L^2[0, 2\pi]$ onto $V(\varphi)$ defined as:

$$(\mathcal{J}F)(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \langle F(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]} \; \varphi(t-n) = \langle F(\xi), \frac{1}{2\pi} Z_\varphi(t,\xi)\rangle_{L^2[0,2\pi]} .$$

We then have:

(i) $(\widehat{\mathcal{J}F})(\xi) = F(\xi)\,\hat{\varphi}(\xi)$

(ii) $\mathcal{J}(F(\xi)e^{-in\xi}) = (\mathcal{J}F)(t-n), n \in \mathbb{Z}$.

Let $\{L_j[\cdot]: 1 \leq j \leq N\}$ be N LTI (linear time-invariant) systems with impulse responses $\{L_j(t): 1 \leq j \leq N\}$. Develop a stable shifted multi-channel sampling formula for any signal $f(t) \in V(\varphi)$ using discrete sample values from $\{L_j(t): 1 \leq j \leq N\}$, where each channeled signal $L_j[f](t)$ for $1 \leq j \leq N$ is assigned with a distinct sampling rate

$$f(t) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} L_j[f]\big(\sigma_j + r_j n\big) s_{j,n}(t), f(t) \in V(\varphi), \tag{69}$$

where $\{s_{j,n}(t): 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $V(\varphi), \{r_j : 1 \leq j \leq N\}$ are positive integers, and $\{\sigma_j: 1 \leq j \leq N\}$ are real constants. Note that the shifting of sampling instants is unavoidable in some uniform sampling [13] and arises naturally when we allow rational sampling periods in (69).

Here, we assume that each L_j $[\cdot]$ is one of the following three types: the impulse response $l(t)$ of an LTI system is such that

(i) $l(t) = \delta(t + a), a \in \mathbb{R}$ or

(ii) $l(t) \in L^2(\mathbb{R})$ or

(iii) $\hat{l}(\xi) \in L^\infty(\mathbb{R}) \cup L^2(\mathbb{R})$ when $H_\varphi(\xi)=\sum_{n\in\mathbb{Z}} | \hat{\varphi}(\xi + 2n\pi)| \in L^2[0, 2\pi]$. For type (i), $L[f](t) = f(t + a), f \in L^2(\mathbb{R})$ so that $L[\cdot] : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isomorphism. In particular, for any $f(t) = (\boldsymbol{c} * \varphi)(t) \in V(\varphi)$,
$L[f](t) = (\boldsymbol{c} * \psi)(t)$ converges absolutely on $\mathbb{R}$ since

$$C_\psi(t) = \sum_{n\in\mathbb{Z}} |\psi(t + n)|^2 < \infty, t \in \mathbb{R} \text{ , where } \psi(t) = L[\varphi](t) = \varphi(t + a)$$

For types (ii) and (iii), we have:

Lemma $(6.3.1)$[1]: Let $L[\cdot]$ be an LTI system with the impulse response $l(t)$ of the type (ii) *or* (iii) as above and $\psi(t) = L[\varphi](t) = (\varphi * l)(t)$ .Then

(a) $\psi(t) \in C_\infty(\mathbb{R}) := \{u(t) \in C(\mathbb{R}) : \lim_{|t|\to\infty} u(t) = 0\}$,

(b) $sup_{\mathbb{R}}\, C_\psi(t) < \infty$;

(c) for any $f(t) = (\boldsymbol{c} * \varphi)(t) \in V(\varphi), L[f](t) = (\boldsymbol{c} * \psi)(t)$ converges absolutely and uniformly *on* $\mathbb{R}$. *Hence* $L[f](t) \in C(\mathbb{R})$.

Proof: First assume $l(t) \in L^2(\mathbb{R})$. Then $\psi(t) \in C_\infty(\mathbb{R})$ by the Riemann–Lebesgue lemma since $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^1(\mathbb{R})$. Since

$$\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right| \le G_\varphi(\xi)^{\frac{1}{2}} G_l(\xi)^{\frac{1}{2}} \quad ,$$

$$\left\|\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right|\right\|^2_{L^2[0,2\pi]} \le \int_0^{2\pi} G_\varphi(\xi)\, G_l(\xi) d\xi$$
$$\le 2\pi \left\|G_\varphi(\xi)\right\|_{L^\infty(\mathbb{R})} \|l\|^2_{L^2(\mathbb{R})}.$$

Thus for any t in $\mathbb{R}$, we have by the Poisson summation formula (see [14])

$$\sum_{n\in\mathbb{Z}} \hat{\psi}(\xi + 2n\pi)\, e^{it(\xi+2n\pi)} = \sum_{n\in\mathbb{Z}} \psi(t + n)\, e^{-in\xi} \; in \;\; L^2\, [0,2\pi]$$

Therefore for any t in $\mathbb{R}$

$$C_\psi(t) = \sum_{n\in\mathbb{Z}} |\psi(t + n)|^2 = \frac{1}{2\pi} \left\|\sum_{n\in\mathbb{Z}} \psi(t + n)\, e^{-in\xi}\right\|^2_{L^2\,[0,2\pi]}$$
$$= \frac{1}{2\pi} \left\|\sum_{n\in\mathbb{Z}} \hat{\psi}(\xi + 2n\pi)\, e^{it(\xi+2n\pi)}\right\|^2_{L^2\,[0,2\pi]}$$
$$\le \left\|G_\varphi(\xi)\right\|_{L^\infty(\mathbb{R})} \|l\|^2_{L^2(\mathbb{R})}.$$

By Young's inequality on the convolution product,
$\|L[f]\|_{L^\infty(\mathbb{R})} \le \|f\|_{L^2(\mathbb{R})} \|l\|_{L^2(\mathbb{R})}$ so that $L[\cdot] : L^2(\mathbb{R}) \to L^\infty(\mathbb{R})$ is

a bounded linear operator. Hence for any

$$f(t) = (\boldsymbol{c} * \varphi)(t) = \sum_{n\in\mathbb{Z}} c(n)\varphi(t-n) \in V(\varphi),$$

$$L[f](t) = \sum_{n\in\mathbb{Z}} c(n)L[\varphi(t-n)] = \sum_{n\in\mathbb{Z}} c(n)\psi(t-n),$$

which converges absolutely and uniformly on $\mathbb{R}$ by (b). Now assume that $H_\varphi(\xi) \in L^2[0,2\pi]$. The case $\hat{l}(\xi) \in L^2(\mathbb{R})$ is reduced to type (ii). So let $\hat{l}(\xi) \in L^\infty(\mathbb{R})$. Then $\hat{\varphi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ so that $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and so $\psi(\xi) \in C_\infty(R) \cap L^2(\mathbb{R}))$. Since

$\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right| \le \|l\|_{L^\infty(\mathbb{R})} H_\varphi(\xi)$, we have again by thePoisson summation formula

$$C_\psi(t) = \frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}} \hat{\psi}(\xi + 2n\pi)\, e^{it(\xi+2n\pi)}\right\|^2_{L^2\,[0,2\pi]}$$

$$\le \|l\|^2_{L^\infty(\mathbb{R})} \left\|H_\varphi(\xi)\right\|^2_{L^2\,[0,2\pi]}$$

so that $sup_{\mathbb{R}} C_\psi(t) < \infty$. For any $f \in L^2(\mathbb{R})$,

$$\|L[f](t)\|_{L^2(\mathbb{R})} = \|f * l\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\left\|\hat{f}(\xi)\hat{l}(\xi)\right\|_{L^2(\mathbb{R})}$$

$$\le \left\|\hat{l}\right\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

Hence $L[\cdot] : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator so that for any $f(t) = (\boldsymbol{c} * \varphi)(t) \in V(\varphi), L[f](t) = (\boldsymbol{c} * \psi)(t)$ converges in $L^2(\mathbb{R})$. By (b), $(\boldsymbol{c} * \psi)(t)$ also converges absolutely and uniformly on $\mathbb{R}$.

By Lemma (6.3.1)(b) , $\psi(t) \in L^2(\mathbb{R})$. However, $(\boldsymbol{c} * \psi)(t)$ may not converge in $L^2(\mathbb{R})$ unless $\{\psi(t-n): n \in \mathbb{Z}\}$ is a Bessel sequence.

Lemma (6.3.1) (b) improves [10] , in which the proof uses $l(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $sup_{\mathbb{R}} C_\varphi(t) < \infty$, and the integral version of Minkowski inequality. Note that the condition $H_\varphi(\xi) \in L^2[0,2\pi]$ implies $\varphi(t) \in L^2(\mathbb{R}) \cap C_\infty((\mathbb{R})$ and $sup_{\mathbb{R}} C_\varphi(t) < \infty$. (see [14]). Note also that $H_\varphi(\xi) \in L^2[0,2\pi]$ if $\hat{\varphi}(\xi) = O((1+|\xi|)^{-r}), r > 1$, which holds e.g. for $\varphi_n(t) := (\varphi_0 * \varphi_{n-1})(t)$ the cardinal B-spline of degree n (≥ 1), where $\varphi_0 = \chi_{[0,1)}(t)$. We have as a consequence of Lemma (6.3.1): Let $L[\cdot]$ be an LTI system with impulse response $l(t)$ of type (i) or (ii) or (iii) as above and $\psi(t) = L[\varphi](t)$. Then for any $f(t) = (\mathcal{J}F)(t) \in V(\varphi), F(\xi) \in L^2[0,2\pi]$

$$L[f](t) = \langle(\xi), \frac{1}{2\pi}\overline{Z_\psi(t,\xi)}\rangle_{L^2[0,2\pi]} \qquad (70)$$

since $L[\cdot]$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ or $L^\infty(\mathbb{R})$ and $\{\psi(t-n): n \in \mathbb{Z}\} \in l^2$, $t \in \mathbb{R}$. Let $\psi_j(t) = L_j\ [\varphi](t)$ and $g_j(\xi) = \frac{1}{2\pi} Z_{\psi j}(\sigma_j, \xi)$, $1 \le j \le N$ .Then we have by (70)

$$L_j\ [f](\sigma_j + r_j n) = \langle F(\xi), \frac{1}{2\pi} Z_{\psi j}(\sigma_j + r_j n, \xi)\rangle_{L^2\ [0,2\pi]}$$
$$= \langle F(\xi), \overline{g_j(\xi)} e^{-i r_j n \xi}\rangle_{L^2\ [0,2\pi]} \quad (71)$$

for any $f(t) = (\mathcal{J} F)(t) \in V(\varphi)$ and $1 \le j \le N$. Then by (71) and the isomorphism $\mathcal{J}$ from $L^2\ [0,2\pi]$ onto $V(\varphi)$, the sampling expansion (69) is equivalent to

$$F(\xi) = \sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_j(\xi)} e^{-i r_j n \xi}\rangle_{L^2\ [0,2\pi]} S_{j,n}(\xi),$$

$F(\xi) \in L^2\ [0,2\pi]$, where $\{S_{j,n}(\xi): 1 \le j \le N, n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $L^2\ [0,2\pi]$. This observation leads us to consider the problem when is $\{\overline{g_j(\xi)} e^{-i r_j n \xi}: 1 \le j \le N, n \in \mathbb{Z}\}$ a frame or a Riesz basis of $L^2\ [0,2\pi]$? Note that $\left\{ \overline{g_j(\xi)} e^{-i r_j n \xi}: 1 \le j \le N, n \in \mathbb{Z} \right\}$

$$= \left\{ \overline{g_{j,m_j}(\xi)} e^{-irn\xi}: 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$$

where $r = lcm\{r_j : 1 \le j \le N\}$ and $g_{j,mj}(\xi) = g_j(\xi) e^{i r_j (m_j - 1)\xi}$ for $1 \le j \le N$. Let D be the unitary operator from $L^2\ [0,2\pi]$onto $L^2(I)r$, where $I = [0, \frac{2\pi}{r}]$, defined by $DF = \left[F\left(\xi + (k-1)\frac{2\pi}{r}\right) \right]_{k=1}^{r}$, $F(\xi) \in L^2\ [0,2\pi]$. We also let

$$G(\xi) = \left[Dg_{1,1}(\xi), \dots, Dg_{1,\frac{r}{r_1}}(\xi), \dots, Dg_{N,1}(\xi), \dots, Dg_{N,\frac{r}{r_N}}(\xi) \right]^T \quad (72)$$

be a $\left(\sum_{j=1}^{N} \frac{r}{r_j} \right) \times r$matrix on I and $\lambda_m(\xi), \lambda_M(\xi)$be the smallest and the largest eigenvalues of the positive semi-definite $r \times r$ matrix $G(\xi) * G(\xi)$, respectively.

Lemma(6.3.2)[1]: Let $\alpha_G = \|\lambda_m(\xi)\|_0$ and $\beta_G = \|\lambda_M(\xi)\|_\infty$ be the essential infimum of $\lambda_m(\xi)$ and the essential supremum of $\lambda_M(\xi)$ respectively. Then $\left\{ \overline{g_j(\xi)} e^{-i r_j n \xi}: 1 \le j \le N, n \in \mathbb{Z} \right\}$ is

(a) a Bessel sequence in $L^2\ [0,2\pi]$ if and only if $\beta_G < \infty$ or equivalently $\{Z_{\psi_j}(\sigma_j, \xi): 1 \le j \le N\} \in L^\infty\ [0,2\pi]$,

(b) a frame of $L^2\ [0,2\pi]$if and only if $0 < \alpha_G \le \beta_G < \infty$,

(c) a Riesz basis of $L^2\ [0,2\pi]$if and only if $0 < \alpha_G \le \beta_G < \infty$ and

$$\sum_{j=1}^{N} \frac{r}{r_j} = 1.$$

Proof : Since $\left\{ \overline{g_J(\xi)} e^{-ir_j n \xi} : 1 \le j \le N, n \in \mathbb{Z} \right\}$ is a Bessel sequence or a frame or a Riesz basis of $L^2\ [0,2\pi]$ if and only if

$$\left\{ \overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$$

is a Bessel sequence or a frame or a Riesz basis of $L^2\ [0,2\pi]$respectively, all of the conclusions follow from [10]. Note that in [10], the authors use the Fourier transform $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t \xi}\, dt$ so that they use $L^2\ [0,2\pi]$ instead of $L^2\ [0,2\pi]$. Assume that $0 < \alpha_G \le \beta_G < \infty$ so that $\left\{ \overline{g_J(\xi)} e^{-ir_j n \xi} : 1 \le j \le N, n \in \mathbb{Z} \right\}$ or equivalently

$$\left\{ \overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$$

is a frame of $L^2\ [0,2\pi]$. Then we can show easily (see [10]) that

$$\left\{ \overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$$

has a dual frame of the form

$\left\{ S_{j,m_j}(\xi) e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$ with

$S_{j,m_j}(\xi) \in L^\infty\ [0,2\pi]$ for $1 \le j \le N$ and $1 \le m_j \le \frac{r}{r_j}$

satisfying

$$\left[DS_{1,1}(\xi), \ldots, DS_{1,\frac{r}{r_1}}(\xi), \ldots, DS_{N,1}(\xi), \ldots, DS_{N,\frac{r}{r_N}}(\xi) \right] = \frac{r}{2\pi} \left[G(\xi)^\dagger + B(\xi)\left(I - G(\xi)G(\xi)^\dagger\right) \right], \tag{73}$$

where $G(\xi)^\dagger = [G(\xi)^* G(\xi)]^{-1} G(\xi)^*$ is the pseudo-inverse of

$G(\xi), B(\xi)$ is any $r \times \sum_{j=1}^{N} \frac{r}{r_j}$ matrix with entries in $L^\infty(I)$, and I is the $\left(\sum_{j=1}^{N} \frac{r}{r_j} \right) \times \left(\sum_{j=1}^{N} \frac{r}{r_j} \right)$ identity matrix. In particular, when we choose $B(\xi) = 0$ in (73), we have the canonical dual frame of the frame

$$\left\{ \overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}.$$

We are now ready to give the main results of this section. We first discuss the sampling expansion (69), which is a frame expansion in $V(\varphi)$.

Theorem(6.3.3)[1]: Let α_G and β_G be the same as in Lemma (6.3.2). Assume $\beta_G < \infty$. Then the following are all equivalent.

(a) There is a frame $\left\{S_{j,m_j}(t-rn) : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z}\right\}$ of $V(\varphi)$ for which

$$f(t) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r}{r_j}} \sum_{n\in\mathbb{Z}} L_j\,[f]\big(\sigma_j + r_j(m_j - 1) + rn\big) s_{j,mj}(t - rn)$$

$$, f(t) \in V(\varphi). \tag{74}$$

(b) There is a frame $\{s_{j,n}(t) : 1 \le j \le N, n \in \mathbb{Z}\}$ of $V(\varphi)$ for which

$$f(t) = \sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} L_j\,[f]\big(\sigma_j + r_j n\big) s_{j,n}(t), f(t) \in V(\varphi)\ . \tag{75}$$

(c) $0 < \alpha_G$.

Proof: Assume $\beta_G < \infty$. Then by Lemma (6.3.2) $\left\{\overline{g_J(\xi)}e^{-ir_j n\xi} : 1 \le j \le N, n \in \mathbb{Z}\right\}$ is a Bessel sequence in $L^2\,[0,2\pi]$. First (a) implies (b) trivially. Assume (b). Applying the isomorphism $\mathcal{J}^{-1}$ to (7) gives by (71)

$$F(\xi) = \sum_{m_j}^{\frac{r}{r_j}} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_J(\xi)}e^{-ir_j n\xi}\rangle_{L^2\,[0,2\pi]} S_{j,n}(\xi), F(\xi) \in L^2\,[0,2\pi],$$

where $\{s_{j,n}(t) : 1 \le j \le N, n \in \mathbb{Z}\}$is a frame of $L^2\,[0,2\pi]$. Then the Bessel sequence $\left\{\overline{g_J(\xi)}e^{-ir_j n\xi} : 1 \le j \le N, n \in \mathbb{Z}\right\}$ is in fact a dual frame of $\{s_{j,n}(t) : 1 \le j \le N, n \in \mathbb{Z}\}$ (see [3]). Hence (c) must hold by Lemma (6.3.2). Finally assume (c). Then $0 < \alpha_G \le \beta_G < \infty$ that $\left\{\overline{g_{J,m_J}(\xi)}e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z}\right\}$ is a frame of $L^2\,[0,2\pi]$. Then we have a frame expansion on $L^2\,[0,2\pi]$

$$F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r}{r_j}} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_{J,m_J}(\xi)}e^{-irn\xi}\rangle_{L^2\,[0,2\pi]} S_{j,m_j}(\xi)e^{-irn\xi}$$

$$, F(\xi) \in L^2\,[0,2\pi], \tag{76}$$

where $S_{j,mj}(\xi)$'s are given by (73). Then the sampling expansion (74) comes from (76) by applying the isomorphism $\mathcal{J}$ since

$$\begin{aligned}\langle F(\xi), g_{j,m_j}(\xi)e^{-irn\xi}\rangle_{L^2\,[0,2\pi]} &= \langle F(\xi), \frac{1}{2\pi}\overline{Z_{\psi_J}(\sigma_J + r_J(m_J - 1) + rn, \xi)}\rangle_{L^2\,[0,2\pi]} \\ &= L_j\,[f]\big(\sigma_j + r_j(m_j - 1) + rn\big)\end{aligned}$$

for $(\mathcal{J}F)(t) = f(t)$.

Note that when $0 < \alpha_G \le \beta_G < \infty$, the sampling series (74) converges not only in $L^2(\mathbb{R})$ but also uniformly on any subset of $\mathbb{R}$, on which $C_\varphi(t)$ is bounded.

Moreover since $\alpha_G > 0$, the rank of $G(\xi)$ is r a.e. so that $1 \le \sum_{j=1}^{N} \frac{1}{r_j}$,

which means that the total sampling rate $\sum_{j=1}^{N} \frac{1}{r_j}$ of the sampling expansion (74) must be at least 1, the Nyquist sampling rate for signals in $V(\varphi)$. In the extreme case we have:

Theorem (6.3.4)[1]: Let α_G and β_G be the same as in Lemma (6.3.2). Then there is a Riesz basis $\left\{ S_{j,m_{j,n}}(t) : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$ of $V(\varphi)$ for which

$$f(t) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r}{r_j}} \sum_{n \in \mathbb{Z}} L_j[f]\left(\sigma_j + r_j\left(m_j - 1\right) + rn\right) S_{j,m_{j,n}}(t), \quad f(t) \in V(\varphi) \tag{77}$$

if and only if $0 < \alpha_G \le \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{r}{r_j} = 1$. In this case, we also have

(i) $S_{j,m_j}(t - rn) : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}$, and $n \in \mathbb{Z}$

(ii) $L_j[s_{k,m_k}](\sigma_j + r_j(m_j - 1) + rn) = \delta_{j,k}\,\delta_{n,0}$ for $1 \le j, k\ N$ and $n \in \mathbb{Z}$.

Proof: Assume $0 < \alpha_G \le \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{r}{r_j} = 1$. Then by Lemma (6.3.2), $\left\{ \overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$ is a Riesz basis of $L^2[0,2\pi]$. Then we have

$$F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r}{r_j}} \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{g_{J,m_J}(\xi)} e^{-irn\xi} \rangle_{L^2[0,2\pi]} S_{j,m_{j,}}(\xi) e^{-irn\xi}, \quad F(\xi) \in L^2[0,2\pi], \tag{78}$$

where $\left\{ S_{j,m_j}(\xi) e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$ is the dual of $\left\{ \overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z} \right\}$. Applying the isomorphism $\mathcal{J}$ to (78) gives (9), where $S_{j,m_{j,n}}(t) = \mathcal{J}\left(S_{j,m_j}(\xi) e^{-irn\xi}\right) = S_{j,m_j}\left(t - rn\right)$ *and* $\mathcal{J}\left(S_{j,m_j}(\xi)\right) = S_{j,m_j}(t)$.

Conversely assume that the Riesz basis expansion (77) holds on $V(\varphi)$. Applying the isomorphism $\mathcal{J}^{-1}$ to (77) gives

$$F(\xi) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r}{r_j}} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_{J,m_J}(\xi)} e^{-irn\xi}\rangle_{L^2[0,2\pi]} \mathcal{J}^{-1}\left(S_{j,m_j,n}\right)(\xi)$$

$$, F(\xi) \in L^2[0,2\pi]$$

which is a Riesz basis expansion on $L^2\,[0,2\pi]$. Then $\left\{\overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z}\right\}$ must be a Riesz basis of $L^2\,[0,2\pi]$ so that $0 < \alpha_G \le \beta_G < \infty$ and $\sum_{j=1}^{N} \frac{r}{r_j} = 1$ by Lemma (6.3.2). As the dual Riesz basis of $\left\{\overline{g_{J,m_J}(\xi)} e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z}\right\}$, $\left\{\mathcal{J}^{-1}\left(S_{j,m_j,n}\right)(t) : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z}\right\}$ must be of the form $\left\{S_{j,m_j}(\xi) e^{-irn\xi} : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}, n \in \mathbb{Z}\right\}$, where $\left\{S_{j,m_j}(\xi) : 1 \le j \le N, 1 \le m_j \le \frac{r}{r_j}\right\}$ satisfy (73) with $B(\xi) = 0$. Hence $S_{j,m_{j,n}}(t) = \mathcal{J}\left(S_{j,m_j}(\xi) e^{-irn\xi}\right) = S_{j,m_j}\left(t - rn\right),\ 1 \le j \le N,\ and\ n \in \mathbb{Z}$. Finally, we have

$$s_{k,mk}(t) = \sum_{j=1}^{N} \sum_{m_j=1}^{\frac{r}{r_j}} \sum_{n\in\mathbb{Z}} L_j[s_{k,m_k}](\sigma_j + r_j(m_j - 1) + rn) s_{j,mj}(t - rn)$$

so that $L_j\left[s_{k,m_k}\right]\left(\sigma_j + r_j\left(m_j - 1\right) + rn\right) = \delta_{j,k}\,\delta_{n,0}$. When $N = 1$, write $L_1[\cdot], l_1(t), \sigma_1, r_1$, and $\psi_1(t)$ as $L[\cdot], l(t), \sigma, r$, and $\psi(t)$.

Corollary $(6.3.5)[1]$: (see [12].) Let $N = 1$. Then there is a Riesz basis $\{s_n(t) : n \in \mathbb{Z}\}$ *of* $V(\varphi)$ such that

$$f(t) = \sum_{n\in\mathbb{Z}} L[f](\sigma + rn) s_n(t), f(t) \in V(\varphi) \tag{79}$$

if and only if r $= 1$ and

$$0 < \left\|Z_\psi(\sigma,\xi)\right\|_0 \le \left\|Z_\psi(\sigma,\xi)\right\|_\infty . \tag{80}$$

In this case, we also have

(i) $s_n(t) = s(t - n), n \in \mathbb{Z}$,

(ii) $\hat{s}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\psi(\sigma,\xi)}$,

(iii) $L[s](\sigma + n) = \delta_{n,0}, n \in \mathbb{Z}$. (81)

Proof : Note that for $r = 1, G(\xi) = \frac{1}{2\pi} Z_\psi(\sigma,\xi)$ $and\ \lambda_m(\xi) = \lambda_M(\xi) = \left(\frac{1}{2\pi}\right)^2 \left|Z_\psi(\sigma,\xi)\right|^2$ so that $0 < \alpha_G \le \beta_G \infty$ if and only if (80) holds. Therefore, everything except (81) follows from Theorem (6.3.4). Finally applying (79) to $\varphi(t)$ gives

$$\varphi(t) = \sum_{n \in \mathbb{Z}} \psi(\sigma + n) s(t - n)$$

from which we have (81) by taking the Fourier transform. When $l(t) = \delta(t)$ so that $L[\cdot]$ is the identity operator, Corollary (6.3.5) reduces to a regular shifted sampling on $V(\varphi)$ (see [14]).

Corollary$(6.3.6)[1]$: Let $N = 1$ *and* $q\ (\geq 2)$ be an integer. Assume $Z_\psi(\sigma j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq q$, where $\sigma_j = \sigma + \frac{1}{q}(j-1)$. Then the following are all equivalent.

(a) There is a frame $\{s_n(t): n \in \mathbb{Z}\}$ *of* $V(\varphi)$ for which

$$f(t) = \sum_{n \in \mathbb{Z}} L[f]\left(\sigma + \frac{1}{q}n\right) s_n(t), f(t) \in V(\varphi).$$

(b) There is a frame $\{s_j(t-n): 1 \leq j \leq q, n \in \mathbb{Z}\}$ *of* $V(\varphi)$ for which

$$f(t) = \sum_{j=1}^{q} \sum_{n \in \mathbb{Z}} L[f](\sigma_j + n) s_j(t-n), f(t) \in V(\varphi).$$

(c) $\left\|\sum_{j=1}^{q} |Z_\psi(\sigma_j, \xi)|\right\|_0 > 0$.

Proof : Since

$\{L[f]\left(\sigma + \frac{1}{q}n\right): n \in \mathbb{Z}\} = \{L[f](\sigma_j + n): 1 \leq j \leq q, n \in \mathbb{Z}\}$,

we have a shifted symmetric multi-channel sampling

for q LTI systems $\{L_j[\cdot]: 1 \leq j \leq q\}$ with $L_j[\cdot] = L[\cdot], 1 \leq j \leq q$. Then

$g_j(\xi) = \frac{1}{2\pi} Z_\psi(\sigma j, \xi), 1 \leq j \leq q$ and

$G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{j=1}^{q} |Z_\psi(\sigma j, \xi)|^2$. Hence $\alpha_G > 0$ if and only if

$\left\|\sum_{j=1}^{q} |Z_\psi(\sigma_j, \xi)|\right\|_0 > 0$. Therefore, Corollary (6.3.6) is a consequence of Theorem (6.3.3).

Example $(6.3.7)[1]$: Let $\varphi_0 = \chi_{[0,1)}(t)$ be the Haar scaling function and $\varphi_1(t) = (\varphi_0 * \varphi_0)(t) = \chi_{[0,1)}(t) + (2-t)\chi_{[1,2)}(t)$ a B-spline of degree 1. Then $\varphi_1(t)$ is a continuous Riesz generator [4] and $\sup_{\mathbb{R}} C_{\varphi_1}(t) = \sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |\varphi_1(t+n)|^2 < \infty$. First we take $N = 2$, $\sigma_1 = \sigma_2 = 0$, $r_1 = 1, r_2 = 2$, and two LTI systems $L^1[\cdot]$ and $L^2[\cdot]$ with impulse responses $l_1(t) = \chi_{\left[\frac{-1}{2},0\right)}(t)$ and $l_2(t) = \chi_{\left[-1,\frac{-1}{2}\right)}(t)$.

Then it's easy to see that

$$g_1(\xi) = \frac{1}{2\pi} Z_{\psi_1}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_1(n) e^{-in\xi} = \frac{1}{16\pi}\left(1 + 3e^{-i\xi}\right),$$

$$g_2(\xi) = \frac{1}{2\pi} Z_{\psi_2}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_2(n) e^{-in\xi} = \frac{1}{16\pi}\left(1 + 3e^{-i\xi}\right),$$

where $\psi_j(t) = L_j[\varphi](t)$. Hence

$$g_{1,1}(\xi) = g_1(\xi), g_{1,2}(\xi) = g_1(\xi) e^{i\xi}, g_{2,1}(\xi) = g_2(\xi)$$

so that (see (72))

$$G(\xi) = \left[Dg_{1,1}, Dg_{1,2}, Dg_{2,1}\right]^T = \frac{1}{16\pi}\begin{bmatrix} 1 + 3e^{-i\xi} & 1 - 3e^{-i\xi} \\ 3 + e^{i\xi} & 3 - e^{i\xi} \\ 3 + e^{-i\xi} & 3 - e^{-i\xi} \end{bmatrix}$$

and $G(\xi)^* G(\xi) = \frac{1}{(16\pi)^2}\begin{bmatrix} 30 + 18\cos\xi & 8 + 6i\sin\xi \\ 8 - 6i\sin\xi & 30 + 18\cos\xi \end{bmatrix}$.

The eigenvalues of $G(\xi)^* G(\xi)$ are

$\frac{1}{(16\pi)^2}[30 + 18\cos\xi \pm \sqrt{100 - 36\cos^2\xi}\,]$ so that

$\frac{1}{(16\pi)^2} \leq \alpha_G = \|\lambda_m(\xi)\|_0 < \beta_G = \|\lambda_M(\xi)\|_\infty \leq \frac{58}{(16\pi)^2}$.

Hence by Theorem (6.3.3), there is a frame $\{s_j(t - 2n): j = 1,2,3 \text{ and } n \in \mathbb{Z}\}$ of the space of linear splines $V(\varphi_1)$ for which the following asymmetric multi-channel sampling expansion holds:

$$f(t) = \sum_{n \in \mathbb{Z}} \{L_1[f](2n)s_1(t - 2n) + L_1[f](2n + 1)s_2(t - 2n) + L_2[f](2n)s_3(t - 2n)\},\ f \in V(\varphi_1),$$

Which convergesin $L^2(\mathbb{R})$ and absolutely and uniformly on $\mathbb{R}$.

We now take $N = 1$ and $l(t) = \delta(t)$ so that $L[\cdot]$ is the identity operator. Let $q\ (\geq 1)$ be an integer and $0 \leq \sigma < \frac{1}{q}$. Note first that for any fixed t in $\mathbb{R}$,

$$Z_{\varphi_1}(t,\xi) = \sum_{n \in \mathbb{Z}} \varphi_1(t + n)e^{-in\xi} \in C[0,2\pi]$$

since $\varphi_1(t)$ has compact support . Hence $\left\|Z_{\varphi_1}(t,\cdot)\right\|_{L^\infty[0,2\pi]} < \infty$ for each t in $\mathbb{R}$.

Since $Z_{\varphi_1}(\sigma,\xi) = \sigma + (1-\sigma)e^{-i\xi}$ for $0 \leq \sigma < 1$, $\left\|Z_{\varphi_1}(\sigma,\xi)\right\|_0 = 2\left|\sigma - \frac{1}{2}\right|$ and $\left\|Z_{\varphi_1}(\sigma,\xi)\right\|_\infty = 1$. Therefore, by Corollary (6.3.5), for any σ with $0 \leq \sigma < 1$, there is a Riesz basis $\{s(t - n): n \in \mathbb{Z}\}$ of $V(\varphi)$ such that

$$f(t) = \sum_{n \in \mathbb{Z}} f(\sigma + n)s(t - n),\ f(t) \in V(\varphi_1) \text{ if and only if } \sigma \neq \frac{1}{2}.$$

On the other hand, by Corollary (6.3.6), for any $q \geq 2$ and any σ with $0 \leq \sigma < \frac{1}{q}$, there is a frame $\{s_j(t - n): 1 \leq j \leq q, n \in \mathbb{Z}\}$ such that

$$f(t) = \sum_{j=1}^{q}\sum_{\in \mathbb{Z}} f\left(\sigma + \frac{1}{q}(j-1) + n\right)s_j(t - n),\ f(t) \in V(\varphi_1).$$

Corollary(6.3.8)[296]: Let $\alpha_G = \|\lambda_m(\xi)\|_0$ and $\beta_G = \|\lambda_M(\xi)\|_\infty$ *be* the essential infimum of $\lambda_m(\xi)$ and the essential supremum of $\lambda_M(\xi)$ respectively. Then $\left\{g_{(1+\epsilon_1)}(\xi)e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\right\}$ is

(a) a Bessel sequence in $L^2[0,2\pi]$ if and only if $\beta_G < \infty$ or equivalently $\{Z_{\psi_{(1+\epsilon_1)}}(\sigma_{(1+\epsilon_1)},\xi): \epsilon_1 \geq 0\} \in L^\infty[0,2\pi]$,

(b) a frame of $L^2[0,2\pi]$if and only if $0 < \alpha_G \leq \beta_G < \infty$,

(c) a Riesz basis of L^2 $[0,2\pi]$if and only if $0 < \alpha_G \leq \beta_G < \infty$ and

$$\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1.$$

Proof: Since $\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)} n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a Bessel sequence or a frame or a Riesz basis of L^2 $[0,2\pi]$ if and only if

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

is a Bessel sequence or a frame or a Riesz basis of L^2 $[0,2\pi]$respectively, all of the conclusions follow from [10]. Note that in [10], the authors use the Fourier transform $\hat{f}(\xi) = \int_{-\infty}^{\infty} \sum_{d=1}^{m} f(t_d) \prod_{d=1}^{m} e^{-2\pi i t_d \xi}\, dt_d$ so that they use L^2 $[0,2\pi]$ instead of L^2 $[0,2\pi]$.

Assume that $0 < \alpha_G \leq \beta_G < \infty$ so that $\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)} n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ or equivalently

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

is a series of frames of L^2 $[0,2\pi]$. Then we can show easily (see in [10]) that

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

has a series of dual frames of the form

$\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ with $S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) \in L^\infty$ $[0,2\pi]$ for $\epsilon_1 \geq 0$ and $1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$ satisfying

$$\left[DS_{1,1}(\xi),\ldots,DS_{1,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_1}}(\xi),\ldots,DS_{N,1}(\xi),\ldots,DS_{N,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_N}}(\xi)\right] = \frac{(1+\epsilon_2)}{2\pi}\left[G(\xi)^\dagger + B(\xi)\left(I - G(\xi)G(\xi)^\dagger\right)\right], \quad (82)$$

where $G(\xi)^\dagger = [G(\xi)^* G(\xi)]^{-1} G(\xi)^*$ is the pseudo-inverse of $G(\xi), B(\xi)$ is any $(1+\epsilon_2) \times \sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$ matrix with entries in $L^\infty(I)$, and I is the $\left(\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right) \times \left(\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right)$ identity matrix. In particular, when we choose $B(\xi) = 0$ in (82), we have the canonical dual frame of the frame

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}.$$

Corollary (6.3.9)[296]: Let α_G and β_G be the same as in Lemma (6.3.2). Assume $\beta_G < \infty$. Then the following are all equivalent.

(a) There is series of a frames

$$\left\{\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n): \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) =$$

$$\sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} L_{(1+\epsilon_1)}[f]\big(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\big(m_{(1+\epsilon_1)} - 1\big) + (1+\epsilon_2)n\big) s_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n)$$

$$,\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)). \quad (83)$$

(b) There is a series of frames $\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : \epsilon_1 \geq 0\ , n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{N} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} L_{(1+\epsilon_1)}[f]\big(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)} n\big) s_{(1+\epsilon_1),n}(t_d),$$

$$\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)). \quad (84)$$

(c) $0 < \alpha_G$.

Proof : Assume $\beta_G < \infty$. Then by Lemma (6.3.2) $\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)_{(1+\epsilon_1)} n\xi} : \epsilon_1 \geq 0\ , n \in \mathbb{Z}\}$ is a Bessel sequence in $L^2\ [0,2\pi]$. First (a) implies (b) trivially.Assume (b). Applying the isomorphism $\mathcal{J}^{-1}$ to (84) gives by (71)

$$F(\xi) = \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_J(\xi)} e^{-i(1+\epsilon_2)_{(1+\epsilon_1)} n\xi} \rangle_{L^2\ [0,2\pi]} S_{(1+\epsilon_1),n}(\xi),$$

$$F(\xi) \in L^2\ [0,2\pi],$$

where $\{\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$is a frame of $L^2\ [0,2\pi]$. Then the Bessel sequence $\{\overline{g_{(1+\epsilon_1)}(\xi)} e^{-i(1+\epsilon_2)_{(1+\epsilon_1)} n\xi} : 0 \leq \epsilon_1 \leq N\ , n \in \mathbb{Z}\}$ is in fact a dual frame of $\{\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$ (see [8]). Hence (c) must hold by Lemma (6.3.2). Finally assume (c). Then $0 < \alpha_G \leq \beta_G < \infty$ that $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\ , n \in \mathbb{Z}\right\}$ is a frame of $L^2\ [0,2\pi]$. Then we have a series of frame expansions on $L^2\ [0,2\pi]$

$$F(\xi) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \big(\langle F(\xi), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2\ [0,2\pi]} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi}\big)\ , F(\xi) \in L^2\ [0,2\pi], \quad (85)$$

where $S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)$'$s$ are given by (82). Then the sampling expansion (83) comes from (85) by applying the isomorphism $\mathcal{J}$ since

$\langle F(\xi), g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2\ [0,2\pi]} =$

$$\langle F(\xi), \frac{1}{2\pi} \overline{Z_{\psi_J}(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n, \xi)} \rangle_{L^2[0,2\pi]} = L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n)$$

for $\sum_{d=1}^{m}(\mathcal{J}F)(t_d) = \sum_{d=1}^{m} f(t_d)$.

Corollary $(6.3.10)$[296]: Let α_G and β_G be the same as in Lemma (6.3.2). Then there is a Riesz basis $\left\{\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} (L_{(1+\epsilon_1)}[f](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n) S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d)) \quad , \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)) \tag{86}$$

if and only if $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1$. In this case, we also have

(i) $\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$, and $n \in \mathbb{Z}$

(ii) $L_{(1+\epsilon_1)}[s_{k,m_k}](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n) = \delta_{(1+\epsilon_1),k}\,\delta_{n,0}$ for $0 \leq \epsilon_1, k \leq N$ and $n \in \mathbb{Z}$.

Proof : Assume $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1$. Then by Lemma (6.3.2), $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ is a series of Riesz bases of $L^2[0,2\pi]$. Then we have

$$F(\xi) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} \rangle_{L^2[0,2\pi]} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} \quad , F(\xi) \in L^2[0,2\pi], \tag{87}$$

where $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ is the dual of $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$. Applying the isomorphism $\mathcal{J}$ to (87) gives (86), where

$$\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) = \mathcal{J}\left((\xi) e^{-i(1+\epsilon_2)n\xi}\right) = \sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n)$$

and $\mathcal{J}(S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)) = \sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d)$.Conversely assume that the Riesz basis expansion (86) holds on $\sum_{d=1}^{m} V(\varphi(t_d))$. Applying the isomorphism $\mathcal{J}^{-1}$ to (86) gives

$$F(\xi) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} (\langle F(\xi), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi}\rangle_{L^2[0,2\pi]} \mathcal{J}^{-1} \left(S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}\right)(\xi)) , F(\xi) \in L^2[0,2\pi]$$

which is a series of Riesz bases expansions on $L^2\ [0,2\pi]$. Then $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ must be a series of Riesz bases of $L^2\ [0,2\pi]$ so that $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1$ by Lemma (6.3.2). As the series of dual Riesz bases of

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)} e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\},$$

$\left\{\sum_{d=1}^{m} \mathcal{J}^{-1}\left(S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}\right)(t_d) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ must be of the form $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$, where $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right\}$ satisfy (82) with $B(\xi) = 0$. Hence

$$\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) = \mathcal{J}\left(S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) e^{-i(1+\epsilon_2)n\xi}\right) = \sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n), \epsilon_1 \geq 0, n \in \mathbb{Z}.$$

Finally, we have

$$\sum_{d=1}^{m} s_{k,mk}(t_d) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} (L_{(1+\epsilon_1)}[s_{k,m_k}](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)} - 1) + (1+\epsilon_2)n)\, S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n))$$

so that $L_{(1+\epsilon_1)}[s_{k,m_k}]\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\left(m_{(1+\epsilon_1)} - 1\right) + (1+\epsilon_2)n\right) = \delta_{(1+\epsilon_1)_1,k}\, \delta_{n,0}$.

When $N = 1$, write $L_1[\cdot], \sum_{d=1}^{m} l_1(t_d)$, $\sigma_1, (1+\epsilon_2)_1$, and $\sum_{d=1}^{m} \psi_1(t_d)$ as $L[\cdot]$, $\sum_{d=1}^{m} l(t_d)$, σ, $(1+\epsilon_2)$, and $\sum_{d=1}^{m} \psi(t_d)$

Corollary $(6.3.1)[296]$: (see [12].) Let $N = 1$. Then there is a series of Riesz bases $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ such that

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} L[f](\sigma + (1+\epsilon_2)n) s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)) \quad (88)$$

if and only if $\epsilon_2 = 0$ and

$$0 < \left\| Z_\psi(\sigma,\xi) \right\|_0 \leq \left\| Z_\psi(\sigma,\xi) \right\|_\infty . \tag{89}$$

In this case, we also have

(i) $\sum_{d=1}^{m} s_n(t_d) = \sum_{d=1}^{m} s(t_d - n), n \in \mathbb{Z}$,

(ii) $\hat{s}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\psi(\sigma,\xi)}$,

(iii) $L[s](\sigma + n) = \delta_{n,0}, n \in \mathbb{Z}$. (90)

Proof : Note that for $\epsilon_2 = 0$, $G(\xi) = \frac{1}{2\pi} Z_\psi(\sigma,\xi)$ and $\lambda_m(\xi) = \lambda_M(\xi) = \left(\frac{1}{2\pi}\right)^2 \left|Z_\psi(\sigma,\xi)\right|^2$ so that $0 < \alpha_G \leq \beta_G < \infty$ if and only if (89) holds. Therefore, everything except (90) follows from Theorem (6.3.4). Finally applying (88) to $\sum_{d=1}^{m} \varphi(t_d)$ gives

$$\sum_{d=1}^{m} \varphi(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \psi(\sigma + n) s(t_d - n)$$

from which we have (90) by taking the Fourier transform.

Corollary (6.3.12)[296]: Let $N = 1$ *and* $q\ (\geq 2)$ be an integer. Assume $Z_\psi(\sigma_{(1+\epsilon_1)},\xi) \in L^\infty[0,2\pi]$, $0 \leq \epsilon_1 \leq q-1$, where $\sigma_{(1+\epsilon_1)} = \sigma + \frac{1}{q-1}(\epsilon_1)$. Then the following are all equivalent.

(a) There is a series of frame $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} L[f]\left(\sigma + \frac{1}{q-1} n\right) s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(b) There is a series of frame $\left\{\sum_{d=1}^{m} s_{(1+\epsilon_1)}\left(t_d - n\right) : 0 \leq \epsilon_1 \leq q-1, n \in \mathbb{Z}\right\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} \sum_{\epsilon_1=0}^{q-1} \sum_{n \in \mathbb{Z}} L[f](\sigma_{(1+\epsilon_1)} + n) s_{(1+\epsilon_1)}(t_d - n), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(c) $\left\| \sum_{\epsilon_1=0}^{q} |Z_\psi(\sigma_{(1+\epsilon_1)},\xi)| \right\|_0 > 0.$

Proof : Since

$\{L[f]\left(\sigma + \frac{1}{q-1} n\right) : n \in \mathbb{Z}\} = \{L[f](\sigma_{(1+\epsilon_1)} + n) : 0 \leq \epsilon_1 \leq q-1, n \in \mathbb{Z}\}$,

we have a shifted symmetric multi-channel sampling for q LTI systems $\{L_{(1+\epsilon_1)}[\cdot] : 0 \leq \epsilon_1 \leq q-1\}$ with $L_{(1+\epsilon_1)}[\cdot] = L[\cdot], 0 \leq \epsilon_1 \leq q-1$. Then $g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi\left(\sigma_{(1+\epsilon_1)},\xi\right), 0 \leq \epsilon_1 \leq q-1$ and $G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1=0}^{q-1} |Z_\psi(\sigma_{(1+\epsilon_1)},\xi)|^2$. Hence $\alpha_G > 0$ if and only if $\left\| \sum_{\epsilon_1=0}^{q-1} |Z_\psi(\sigma_{(1+\epsilon_1)},\xi)| \right\|_0 > 0$. Therefore, Corollary (6.3.13) is a consequence of Theorem (6.3.3).

Corollary (6.3.13)[296]: Assume $Z_\psi(2-\epsilon,\xi) \in L^\infty[0,2\pi], 0 \leq \epsilon_1 \leq q-1$, then the following are all equivalent.

(a) There is a series of frames $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} L[f](2-\epsilon) s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(b) There is a series of frames $\{\sum_{d=1}^{m} s_{(1+\epsilon_1)}(t_d - n): \epsilon_1 > 0, n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{\epsilon_1\geq 0} \sum_{d=1}^{m} L[f](n-\epsilon) s_{(1+\epsilon_1)}(t_d - n), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(c) $\left\| \sum_{\epsilon_1\geq 0} |Z_\psi(2-\epsilon, \xi)| \right\|_0 > 0.$

Proof : Since $\{L[f](2-\epsilon)\} = \{L[f](n-\epsilon): n \in \mathbb{Z}\}$. Now we have $\{L_{(1+\epsilon_1)}[\cdot]: \epsilon_1 > 0\}$ with $L_{(1+\epsilon_1)}[\cdot] = L[\cdot], \epsilon_1 > 0$. Then $g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi(2-\epsilon, \xi), \epsilon_1 > 0$ and $G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1\geq 0} \left|Z_\psi(2-\epsilon, \xi)\right|^2$. There for $\alpha_G > 0$ if and only if

$$\left\| \sum_{\epsilon_1\geq 0} |Z_\psi(2-\epsilon, \xi)| \right\|_0 > 0.$$

List of Symbols

symbols		page
MRA	Multi Resolution Analysis	1
L^2	Hilbert space	1
L^1	the Lebesgue space on line	1
sup	Supermum	2
inf	Infimum	2
$\ominus$	Direc difference	2
ℓ^2 , ℓ^∞	Hilbert space	2
a.e	Almost every where	3
supp	support	4
L^∞	Essential Lebesgue space	5
V_N	Spline subspace	7
WKS	Whittaker– Kotel'nikov –Shannon	12
$\oplus$	Direct sum	13
ℓ^1	Hilbert space	14
max	maximum	16
min	minimum	16
det	determinant	18
L^p	Lebesgue space	43
V_φ	shift invariant space	43
A-P	approximation-projection	45
ess	essential	46
Osc	oscillation	48
BUPU	bounded uniform partition of unity	49
op	operator	52
PW_π	Paley–Wiener space	71
RKHS	Reproducing kernel Hilbert space	71
Lat	lattice	98
Vol	Volume	98
W_r^2	Sobolev space	108
sign	signature	120
int	interior	142
Loc	Local	170
Ker	Kernal	181
$\|E\|$	Lebesgue measure	187
LTI	linear time-invariant	205

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