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Structural stability on bifurcation theory

الاستقرار البنيوي عن نظرية التفرعات

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Dedication

To my beloved grandmother ,for her great love,

To my dear parents, for their everlasting support,

To my lovely

ALSOMUAL MOHAMEDAHMED,

for his support and encouragement

To my friends, for their great support

I dedicate this work with great love ,to all of them.

thank you…

Acknowledgements

I thank allah for giving me the health and strength which enabled me to complete this study. I sincerely thank my supervisor Dr. Emad Aldeen Abd Allah Abd Alrahim for his professional guidance ,support and advice throughout the process of this study. last but not least , I thank my family and friends for their support and advice.

The Contents

Abstract

In this research we classify the continuily and discrete time linear systems .we Also discuss the equilibrium point for some standard systems by using the Flow-box theorem and Hartman-Grobman theorems with some applications in term of center manifolds.

We also discuss some new methods for studying the global phase portrait in general , and then introduce the index theory of the vectors fields in two dimensions with some examples . We illustrate the Bifurcation theory and we study some standard Bifurcations. We also discuss the structural stability of one–dimension system with some applications and examples.

الخلاصة

في هذا البحث قمنا بتصنيف الاستمرارية ومنظومة خطية الزمن المتقطع .ايضا ناقشنا نقطة الاتزان لبعض المنظومات الاساسية بواسطة استخدام مبرهنة انسياب –الصندوق ومبرهنات هارتمان- جروبمان مع بعض التطبيقات بدلالة متعددات طيات المركز .ايضا ناقشنا بعض الطرق الجديدة لأجل دراسة عامة لصورة فترة شاملة وإذا ادخلنا نظرية الدليل لحقول المتجهات في بعدين مع بعض الامثلة.اوضحنا نظرية التفرعات ودرسنا بعض التفرعات الاساسية وأيضا ناقشنا الاستقرار البنيوي لنظام ذو بعد واحد مع بعض التطبيقات والأمثلة.

Introduction

The theory of differential equations is the field of mathematics ,that is, more than 300 years old , motivated greatly by challenges arising from different applications ,and leading to the birth of other fields of mathematics .In this research , we don’t aim to show a panoramic view of this enormous field , we only intend to reveal its relation to the theory of dynamical systems. Therefore, the organized of this research is as follows :

In chapter one we discuss briefly some definitions and properties of the two dynamical systems . we also classify the continuous and discrete time linear systems with some examples and applications. In chapter two we investigate the behavior of trajectories in a neighborhood of steady states .

*- We introduce the concept of the equilibrium point of a system

*- We discuss the Flow-box theorem and Hartman-Grobman Theorem for global version Theorem , follows from the Hartman-Grobman theorem for the center manifolds, we introduce the approximation of center manifolds Theorem with some examples and applications .In chapter three we show some methods for studying the global phase portrait by using the local phase portraits determined at different points with some examples in several dimensions .

We also discuss the index Theory of the vector fields which express an important tool for investigating the global phase portrait in two dimensions with some examples and applications .The behavior of trajectory at infinity is also discussed .

In chapter four we investigate and study the Bifurcation theory which enable us to understand that how the phase portrait changes as the values of the parameters of the solution of differential equation are varied with some standard illustrated examples. Also the structural stability of one dimensional systems is discussed with some applications and the global bifurcation are showed where global structures appear as the parameter is varied with some applications.

Chapter 1

Topological Classification of Dynamical Systems

The evolution of the theory of differential equations started by developing methods for solving differential equations, using these methods the solution of different special differential equations can be given analytically, i.e. formulas can be derived.

However, it turned out that the solution of systems of ordinary differential equations can be obtained analytically only in very special cases and even in the case when formulas can be derived for the solution , it is difficult to determine the properties of the solutions based on the formulas .

For example, two dimensional linear systems can be characterized by plotting trajectories are not plotted analytically, instead the main characteristics of the trajectories are shown , similarly to the case of plotting graphs of functions in calculus , when only the monotonicity , local maxima, minima and the convexity are taken into account , the exact value of the function does not play important role .

Section(1.1): Equivalences of dynamical systems

Two dynamical systems will be called equivalent if their orbits can be mapped onto each other by a suitable mapping. The equivalence relations for discrete and continuous time dynamical systems will be defined , there we use the nation $\mathbb{T}$ for $\mathbb{R}$ or for $\mathbb{Z}$.

Definition(1.1.1):-

Let $M, N \subset \mathbb{R}^n$ be open sets , a function $h: m \to n$ is called a homorphism (sometimes a C^0- diffeomorphism) if it is continuous , bijective and its inverse is also continuous. The functions is called a C^k- diffeomorphism if it is k times continuously differentiable ,bijective and its inverse is also k time continuously differentiable.

Definition(1 .1.2):-

Let $M, N \subset \mathbb{R}^n$ be open connected sets ,the dynamical systems $\varphi = \mathbb{T}XM \to M$ and $\psi = \mathbb{T}XN \to N$ are called C^k equivalent , (for K = 0 topologically equivalent) if there exists a C^k–diffeomorphism $h: M \to N$ (for k=0 a homorphism), that maps orbits on to each other by preserving the direction of time , this means that there exists a continuous function $a: \mathbb{T}XM \to \mathbb{T}$, for which $t \to a(\ t, p\ )$ is a strictly increasing bijection and for all $t \epsilon\, \mathbb{T}$ and $p \in M$ we have

$$h(\varphi(t,p)) = \psi(a(t,p), h(p)).$$

Definition (1 .1.3) :-

The dynamical systems φ and ψ are called C^k flow equivalent if in the general definition above , the function does not depend on p that there exists a strictly increasing bijection b: $\mathbb{T} \to \mathbb{T}$, for which $a\ (t, p)\ =\ b$ (t) for all $p \in M$. thus the reparametrization of time is the same on each orbits .

Definition (1 .1.4):-

The dynamical systems $\varphi$and ψ are called C^k conjugate if in the general above $a(t, p) = t$ for all t $\in \mathbb{T}$ and $p \in$ M, thus there is no re parameterization of time along the orbits. in this case the condition of equivalence takes the form

$h\big((\varphi(t,p)\big) = \psi(t, h(p))$.

Definition(1.1.5):-

The dynamical systems φ and ψ are called c^k orbitally equivalent if in the general definition above $M = N$ and $h = id$, that is the orbits are the same in the two systems and time is reparametrized.

Proposition (1.1.6):-

1-If the dynamical systems φ and ψ are C^k conjugate then they are C^k flow equivalent.

2-If the dynamical systems φ and ψ are C^k flow equivalent then they are C^k equivalent.

3-If the dynamical systems φ and ψ are orbitally equivalent , then they are C^k – equivalent .

let$\varphi: z \times M \to M$ and $\psi: z \times M \to N$ be discrete time dynamical systems let us define function f and g by $f(p) = \varphi(i,p)$and $g(p) = \psi(i,p)$then the definition of a dynamical system simply implies that $\varphi(n,p) = f(p)$and $\psi(n,p) = f(p)$where f^n and g^n denote the composition of the functions with themselves n times $f^n = fofo \dots of$ and similarly g .

Proposition(1.1.7):-

The statements below are equivalent:

1- The dynamical systems ϕ and ψ are c^k conjugate .

2- The dynamical systems ϕ and ψ are c^k flow equivalent.

3- The dynamical systems ϕ and ψ are c^k equivalent .

4- There exists a C^k diffeomorphism $h: M \to N$, for which $hof = goh$.

Proof:

According to the previous proposition the first three statements follow from each other from top to the bottom . First we prove that the last statement implies the first .Then it will be shown that the third statement implies the last using that $hof = goh$ one obtains

$$h \circ f^2 = h \circ f \circ f \underbrace{=}_{hof=goh} g \circ h \circ f \underbrace{=}_{h\circ f=g\circ h} g \circ g \circ h = g^2 \circ h.$$

Similarly ,the condition $h\, o\, f^{n-1} = g^{n-1} o\, h$ implies $h(f^n(p)) = g^n(h(p))$, that is $h\,(\phi(n,p) = \psi(n,h(p))$ holds for all n and p that is exactly the C^k conjugacy of the dynamical systems ϕ and ψ .

Let us assume that the dynamical systems $\phi$and ψ are C^k equivalent . let us observe first if $r: \mathbb{Z} \to \mathbb{Z}$ is a strictly increasing bijection then there exists $k\epsilon\ \mathbb{Z}$,such that $r(n) = n + k$ for all $n\epsilon\, \mathbb{Z}$ namely , strict monotonicity implies $r(n+1) > r(n)$, while since r is a bijection there is no integer between

$r(n+1)$ and $r(n)$ thus $r(n+1) = r(n)+1$. Then introducing $k = r(0)$ we get by induction that $r(n) = n + k$ thus for the function a in the definition of C^k equivalence holds that for all $p \epsilon m$ there exists an integer $k_p \boldsymbol{\epsilon} \mathbb{Z}$ for which a $(n, p) = n + k_p$. Thus the C^k equivalence of ϕ and ψ means that for all $n \epsilon \mathbb{Z}$ and for all $p \epsilon M$

$$h(\phi(n,p) = \psi(n + k_p, h(p))$$

holds, that is $h(f^n(p))=g^{n+k_p}(h(p))$.

Applying this relation for $n = 0$ we get $h(p) = g^{k_p}(h(p))$. Then applying with n = 1 equation .

$$h\big(f(p)\big) = g^{1+k_p}\big(h(p)\big) = g\big(g^{k_p}\big(h(p)\big) = g(h(p)).$$

Proposition(1.1.8):-

The dynamical systems ϕ and ψ are orbitally equivalent if and only if they are equal .Also let $\mathbb{T} = \mathbb{R}$ and let $\phi: \mathbb{R} \times M \to M$ and $\psi: \mathbb{R} \times N \to N$ continuous time dynamical systems .Then there are continuously differentiable functions $f: M \to \mathbb{R}^n$ *and* $g: N \to \mathbb{R}^n$, *such that the solution of* $\dot{x} = f(x)$ *are given by* ϕ and those of $\dot{y} = g(y)$ are given by ψ .

proposition (1.1.9):-

1.let k > 1 then the dynamical systems ϕ and ψ are C^k conjugate if and only if there exists a C^k diffeomorphism $h: M \to N$ C^k, for which

$$h'.f = g \circ h \text{ holds}$$

2. Assume that the function $t \to a(t,p)$ is differentiable then the dynamical systems ϕ and $\psi$are orbitally equivalent if and only if there exists continuous function : $M \to \mathbb{R}^+$, for which $g = f.v$.

3. The converse of the implications are not true.

Section(1.2): C^k – Classification of linear systems

In this section we classify continuous time linear systems of the form $\dot{x} = Ax$ and discrete time linear systems of the form $x_{n+1} = Ax_n$ according to equivalence relations .

Let us introduce the spaces

$$L(\mathbb{R}^n) = \{A: \mathbb{R}^n \to \mathbb{R}^n \; l\text{inear mapping}\}$$

and

$$GL(\mathbb{R}^n) = \{A \in l(\mathbb{R}^n): \det A \neq 0\}$$

for the continuous and for the discrete time cases . If $A \epsilon L(\mathbb{R}^n)$, then the matrix A is considered to be the right hand side of the linear differential equation $\dot{x} = Ax$, while $A \epsilon L(\mathbb{R}^n)$ is considered to be a linear map determining the discrete time system $x_{n+1} = Ax_n$. Thus the space $L(\mathbb{R}^n)$ represents continuous time and $GL(\mathbb{R}^n)$ represent discrete time linear systems .In the linear case the dynamical system can be explicitly given in terms of the matrix . If $A \epsilon L(\mathbb{R}^n)$, then the dynamical system determined by A is $\varphi(t,p) = e^{At}p$.

If $A \epsilon GL(\mathbb{R}^n)$ then the dynamical system generated by A is $\psi(n,p) = A^n p$.

Definition(1.2.1):-

The matrices A and B are called linearly equivalent , if there exists $\alpha > 0$ and an inveretible matrix P, for which $A = \propto P\,B\,P^{-1}$ holds .

Proposition(1.2.2):-

let $\mathbb{T} = \mathbb{R}$ and $K \geq 1$.

1. The matrices $A, B \epsilon L(\mathbb{R}^n)$ are C^k conjugate if and only if they are similar.
2. The matrices $A, B \epsilon L(\mathbb{R}^n)$ are C^k equivalent if and only if they are linearly equivalent.

Proof :

1. Assume that the matrices A and B are C^k conjugate, that is there exists a C^k-diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, such that $h\big(\phi(t,p)\big) = \psi\big(t, h(p)\big)$, i.e $h(e^{At}p) = e^{Bt}h(p)$. Differentiating this equation with respect to p we get $h'(e^{At}p).e^{At} = e^{Bt}h'(p)$, then substituting $p = 0$yields $h'(0)e^{At} = e^{Bt}.h'(0)$. Differentiating now with respect to t leads to $h'(0)e^{At}.A = e^{Bt}.B.h'(0)$, from which by substituting t = 0 we get $h'(0)A = B.h'(0)$. The matrix $h'(0)$is

invertible, because h is a diffeomorphism hence multiplying the equation by this inverse we arrive to $A = h'(0)^{-1}Bh'(0)$, i.e the matrices A and B are similar.

Assume now A and B are similar , that is there exists an invertible matrix P for which $A = P^{-1}BP$. Then the linear function $h(p) = P_p$ is a C^k −diffeomorphism taking the orbits onto other by preserving time , namely $Pe^{At}p = Pe^{P-1BPt}\, p = \, e^{Bt}P_p$.

3. Assume that the matrices A and B are C^k equivalent that is there exists C^k- diffeomorphism $h = \mathbb{R}^n \to \mathbb{R}^n$ and differentiable function $a = \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $h\big(\phi(t,p)\big) = \psi(a(t,p), h(p))$, that is $h(e^{At}P) = e^{Ba(t,p)}\, h(p)$.

Differentiating this equation with respect to p and substituting $p = 0$ yields $h'(0)e^{At} = \, e^{Ba(t,0)}.h'(0)$. Differentiating now with respect to t we get $h'(0)e^{At}.A = e^{Ba(t,0)}.B\dot{a}(t,0).h'(0)$, from which by substituting t = 0 we obtain $h'(0)A = B\dot{a}(0,0).h'(0)$.The matrix $h'(0)$is invertible, because h is a diffeomorphism, hence multiplying the equation by this inverse and introducing $\propto = \dot{a}(0,0)$we arrive to $A = \propto h'(0)^{-1}Bh'(0)$, i.e the matrices A and B are linearly equivalent .

Assume that the matrices A and B are linearly equivalent that there exist an invertible matrix P and $\propto > 0$, for which $A = \propto P^{-1}BP$. Then the linear function $h(p) = P_p$ is a C^k −diffeomorphism taking the orbits on to each other with time reparametrisation $a(t,p) = \propto t$ namely

$$Pe^{At} = Pe^{\propto p-1}B^{pt}\, p = \, e^{B\propto t}p_p.$$

Remark(1.2.3): -According to above proposition the classification given by C^k conjugacy and equivalence is too fine when k≥ 1 . Namely the matrices $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$are neither C^kconjugate nor C^k equivalent

Proposition(1.2.4):-Let $\mathbb{T} = \mathbb{Z}$ and $k \geq 1$. The matrices $A, B \in GL(\mathbb{R}^n)$are C^kconjugate, if and only if they are similar.

Section(1.3): C^0classification of linear systems

Definition (1.3.1):-

Continuous time case in n = 1 dimension the differential equation $\dot{x} = ax$. If $a < 0$ then the origin is asymptotically stable i.e all solutions tend to the origin as $t \to \infty$. If $a > 0$ then the origin is un stable i.e all solutions tent to infinity as t $\to \infty$. If a = 0 then every point is a steady state.

Definition(1.3.2):-

To define the discrete time case in $n = 1$ dimension let us consider the discrete time dynamical system given by the recursim $x_{n+1} = ax_n$ for different value of $a \in \mathbb{R} \setminus \{0\}$. We note that the set $GL(\mathbb{R})$ can be identified with the set $\mathbb{R} \setminus \{0\}$ C^0equivalence divides GL ($\mathbb{R}$) into the following six classes:

1. If $a > 1$, then for a positive initial condition x_0 the sequence is strictly increasing hence 0 is an unstable fixed point.
2. If $a = 1$, then every point is fined point .
3. If $0 < a < 1$, then 0 is a stable fixed point every converges to zero monotonically.
4. If $-1 < a < 0$, then 0 is a stable fixed point every solution converges to zero an alternating sequence.
5. If $a = 1$, then the solution is an alternating sequence.
6. If $a < 1$, then 0 is an unstable fixed point .

Definition(1.3.3):- Let $\{u_1, \dots, u_n\} \subset \mathbb{R}^n$be the basis determining the real Jordan canonical form of the matrix A. Let $\lambda_k$be the eigenvalue corresponding to U_k . The subspaces

$$E_s(A) = \langle \{u_k : Re\lambda_k < 0\} \rangle, E_u(A) = \langle \{u_k : Re\lambda_k > 0\} \rangle ,$$
$$E_c(A) = \langle \{u_k : Re\lambda_k = 0\} \rangle$$

are called the stable, unstable and center subspaces of the linear system $\dot{x} = Ax$. ($\langle . \rangle$denotes the subspace spanned by the vectors given between the brackets.)

Theorem (1.3.4):-

The subspaces $\mathrm{E_s}$ (A), $\mathrm{E_u}$ (A), $\mathrm{E_c}$ (A) have the following properties.

1. $\mathrm{E_s(A)} \oplus \mathrm{E_u(A)} \oplus \mathrm{E_c(A)} = \mathbb{R}^n$
2. They are invariant under A (that is $\mathrm{A(E_i\ (A))} \subset \mathrm{E_i(A)}, \mathrm{i} = \mathrm{s,u,c}$), and under $\mathrm{e^{At}}$.
3. For all $\mathrm{p} \in \mathrm{E_s(A)}$ we have $\mathrm{e^{At}p} \to 0.$ if $\mathrm{t} \to +\infty$ moreover, there exists $\mathrm{K}, \propto > 0,$ for which $\left|\mathrm{e^{At}p}\right| \leq \mathrm{Ke^{-\propto t}|p|}, \text{if } \mathrm{t} \geq 0$
4. For all $\mathrm{p} \in \mathrm{E_u(A)}$ we have $\mathrm{e^{At}p} \to 0.$ if $\mathrm{t} \to -\infty$ moreover, there exists $\mathrm{L}, \beta > 0,$ for which $\left|\mathrm{e^{At}p}\right| \leq \mathrm{L\ e^{\beta t}|p|}, \text{if } \mathrm{t} \leq 0.$

Definition(1.3.5):-

Let $s(A) = \dim\big(E_s(A)\big), u(A) = \dim\big(E_u(A)\big) and\ c(A) = \dim\big(E_c(A)\big)$denote the dimensions of the stable, unstable and center subspaces of a matrix A, respectively. The spectrum , i.e. the set of eigenvalues of the matrix A will be denoted by $\sigma(A)$ .The following set of matrices is important from the classification point of view the elements of

$EL(\mathbb{R}^n) = \{A \in L(\mathbb{R}^n)\colon Re\lambda \neq 0, \forall \lambda \in \sigma(A)\}$, are called hyperbolic matrices in the continuous time case. First, these hyperbolic systems will be classified according to C^0-conjugacy. In order to carry out that, we will need the Lemma below.

Lemma(1.3.6) :-

1- If $s(A) = n$, then the matrices A and - I are C^0- conjugate.

2- If $u(A) = n$, then the matrices A and I are C^0-conjugate.

Theorem(1.3.7) (Kuiper):-

Let A , $B \in L(\mathbb{R}^n)$ be matrices with $c(A) = c(B) = n$ .These are C^0 equivalent , if and only if they are linearly equivalent.

Theorem(1.3.8) :- The matrices A , $B \in L(\mathbb{R}^n)$ are C^0 equivalent , if and only if $s(A) = s(B), u(A) = u(B)$ and their restriction to their center subspaces are linearly equivalent ($i.e. A|_{E_c}$ and $B|_{E_c}$ are linearly equivalent).

Definition(1.3.9):-

Let $\{u_1, \dots, u_n\} \subset \mathbb{R}^n$ be the above basis and let λ_k be the eigenvalue corresponding to u_k (note that u_k may not be an eigenvector). The subspaces

$$E_s(A) = \langle \{u_k : |\lambda_k| < 1\} \rangle$$

$$E_u(A) = \langle \{u_k : |\lambda_k| > 1\} \rangle$$

$$E_c(A) = \langle \{u_k : |\lambda_k| = 1\} \rangle$$

are called the stable , unstable and center subspaces belonging to the matrix $A \in GL(\mathbb{R}^n)$.

Theorem(1.3.10) :-

The subspaces $E_s(A), E_u(A), E_c(A)$ have the following properties :-

1. $E_s(A) \oplus E_u(A) \oplus E_c(A) = \mathbb{R}^n$
2. They are invariant under A .
3. For any $p \in E_s(A)$ we have $A^n p \to 0$, if $n \to +\infty$.
4. For any $p \in E_s(A)$ we have $A^{-n} p \to 0$, if $n \to +\infty$.

Definition(1.3.11) :-

Let $s(A) = \dim(E_s(A)), u(A) = \dim(E_u(A))$ and $c(A) = \dim(E_c(A))$

denote the dimensions of the stable, unstable and center subspaces of a matrix A, respectively. The following set of matrices is important from the classification point of view. The elements of

$$HL(\mathbb{R}) = \{A \in GL(\mathbb{R}^n) : |\lambda| \neq 1 \ \forall \lambda \in \sigma(A)\},$$

are called hyperbolic matrices in the continuous time case.

Lemma(1.3.12):-

Let the hyperbolic matrices $A, B \in GL(\mathbb{R}^n)$ be C^0conjugate, that is there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, for which $h(Ax) = Bh(x)$ for all

$x \in \mathbb{R}^n$. Then the following statements hold :

1. $h(0) = 0$,
2. $h\,(E_s(A)) = E_s(B)$, that is h takes the stable subspace to stable subspace; $h\,(E_u(A)) = E_u(B)$, that is h takes the unstable subspace to unstable subspace
3. $s(A) = s(B)$, $u(A) = u(B)$.

Example (1.3.13):-

Consider the one-dimensional linear equations given by the numbers (one-by-one matrices) $A = \frac{1}{2}$ and $B = \frac{1}{2}$. Both have one dimensional stable subspace, that is $s(A) = s(B) = 1$, since the orbits of both systems are formed by geometric sequences converging to zero.

Lemma (1.3.14):-

Assume that s$(A) = s(B) = n\,(or\, u(A) = u(B) = n)$. Then A and B are C^0conjugate, if and only if sgn det $A = sgn$ det B.

Theorem(1.3.15):- The hyperbolic linear maps $A, B \in HL(\mathbb{R}^n)$ are C^0 conjugate, if and only if

- $s(A) = s(B)$
- sgn det $A|_{E_s(A)} = sgn \det B|_{E_s(B)}$
- sgn det $A|_{E_u(A)} = sgn \det B|_{E_u(B)}$

Example(1.3.15) :- According to this theorem the $A = \frac{1}{2} I$ and $B = \frac{1}{2} I$,where I is the $n \times n$ unit matrix , are C^o conjugate if and only if n is even, namely in this case the determinant of the matrix $-$I is also positive while for odd values of n it is negative. thus as we have already shown, the matrices,

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

are C^o conjugate.

Chapter 2

Local classification, normal forms and the Hartman-Grobman theorem

Consider the n-dimensional system of autonomous differential equations

$$\dot{x}(t) = f\big(x(t)\big). \tag{2.1}$$

There is no general method for solving this system, hence the most important way of getting information about the solutions is to determine the phase portrait. The constant solutions $x(t) \equiv p$ can be obtained by solving the system of algebraic equations $f(p) = 0$. The solution p of this system is called an equilibrium or steady state of the dynamical system . The behaviour of trajectories in a neighbourhood of a steady state can be investigated by linearisation that can be explained simply as follows.

Introducing the function $y(t) = x(t) - p$ the differential equation takes the form

$$\dot{y}(t) = \dot{x}(t) = f\big(x(t)\big) = f(p) + f'(p)y(t) + r\big(y(t)\big) = f'(p)y(t) + r\big(y(t)\big),$$

where r denotes the remainder term . For small y, i.e when x is close to p, the remainder can be neglected with respect to the linear term (assuming that it is not too small). Hence it can be expected that the local phase portrait in a neighbourhood of the equilibrium p is determined by the linear equation

$$\dot{y}(t) = f'(p)y(t) \tag{2.2}$$

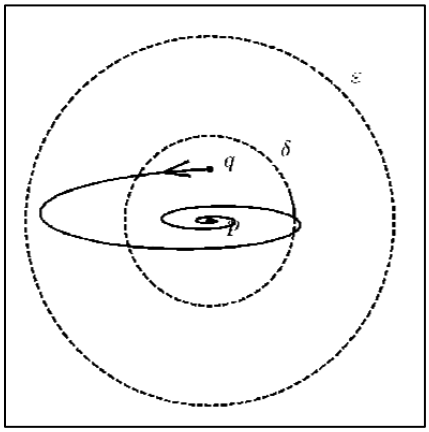
that is called the linearised equation at the point p, since it is given by the Jacobian of f. In order to make this argumentation rigorous we have to define two things. First, what does it mean that the linear term is not too small, second, what does the local phase portrait mean . Concerning these questions, the stability of steady states is investigated in an introductory differential equation course. First, we briefly summarise these notions and the corresponding results.

Let us denote by $t \rightarrow \varphi(t,p)$ the solution of (2.1) that satisfies the initial condition $x(0) = p$, and denote by $I(p)$ the interval in which this solution is defined.

Definition (2.1):- An equilibrium point $p \in \mathbb{R}^n$ of a system (2.1) is called stable, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$q \in \mathbb{R}^n, |q - p| < \delta, t \geq 0 \; imply \; |\varphi(t,q) - p| < \varepsilon.$$

The equilibrium is called asymptotically stable , if it is stable and $|\varphi(t,q) - p| \rightarrow 0$ as $t \rightarrow +\infty$ for the above q , see Figure (2.1).The equilibrium is called unstable ,if it is not stable.



Figure(2.1): An asymptotically stable equilibrium.

The linearisation determines stability in the following cases.

Theorem(2.2) :-

1. If the real parts of the eigenvalues of the matrix $f'(p)$ are negative, then p is an asymptotically stable equilibrium of system (2.1)
2. If the matrix $f'(p)$ has at least one eigenvalue with positive real part, then p is an unstable equilibrium of the system (2.1) .

Definition(2.3):-

Let $(M,\varphi),(N,\psi)$ be dynamical systems and $p \in M$, $q \in N$ be given points . The dynamical system φ at the point p and the dynamical system ψ at the point q are C^k equivalent (conjugate), if there is a neighbourhood U of the point p, a neighbourhood V of the point q and a C^k- diffeomorphism h : U → V taking the orbits into each other (by preserving time) and satisfying h(p) = q.

The main idea of local investigation is that the terms of the power series expansion of f at the point , i.e. $f(x) = f(p) + f'(p) \cdot (x - p) + \ldots$ determines the local phase portrait at the point p. These can be summarized briefly as follows :

- Flow-box Theorem: The non-zero zeroth order term determines the local phase portrait.
- Hartman–Grobman Theorem: A hyperbolic linear term determines the local phase portrait.
- Theory of normal forms: The resonant higher order terms determine the local phase portrait.

Theorem(2.4): (Flow-box Theorem)

If $f(p) \neq 0$, then the system $\dot{x} = f(x)$ at the point p and the system $\dot{x} = f(x)$ at the origin are locally C^k conjugate (assuming that $f \in C^k$). That is, ***when*** *$f(p) \neq 0$, i.e. p* is not an equilibrium point, then the zeroth order term of the expansion determines the local phase portrait.

Theorem(2.5): (Hartman–Grobman)

Let $D \subset \mathbb{R}^n$ be a connected open set, $f : D \to \mathbb{R}^n$ be a C^1function , $p^* \in D$ be an equilibrium point ,i.e $f(p^*) = 0$ and assume that the matrix A is hyperbolic, i.e. the real parts of its eigenvalues are non-zero. Then system (2.1) at the point $p*$ and system (2.2) *at the* origin are locally topologically conjugate. That is there exist a neighbourhood $U \subset \mathbb{R}^n$ of p^*, a neighbourhood $V \subset \mathbb{R}^n$of the origin and a homeomorphism $h : U \to V$, for which

$$h\big(\varphi(t,p)\big) = e^{At}h(p) \qquad (2.3)$$

for all $p \in U$ and for all $t \in \mathbb{R}$, for which $\varphi(t,p) \in U$ holds. That is in a short form $h \circ \varphi_t = e^{At} \circ h$.

Proof:

1. It is shown that $p^* = 0$ can be assumed without loss of generality.

2. The function f is extended to the whole space $\mathbb{R}^n$ in such a way that it is equal to its own linear part outside a suitably chosen ball. It is shown that the extended system and the linear part are (globally) topologically conjugate. This statement is referred to as the global version of the Hartman–Grobman Theorem .
3. The global version is reduced to the discrete time version of the Hartman–Grobman Theorem.
4. Proof of the discrete time version of the Hartman–Grobman theorem, that is also referred to as Hartman–Grobman Theorem for maps.

The following notations will be used:

$$B_r = \{p \in \mathbb{R}^n : |p| < r\}$$

$$C^0(\mathbb{R}^n, \mathbb{R}^n) = \{g : \mathbb{R}^n \to \mathbb{R}^n : g \text{ is continuous}\}$$

$$C^1(\mathbb{R}^n, \mathbb{R}^n) = \{g : \mathbb{R}^n \to \mathbb{R}^n : g \text{ is continuously differentiable}\}$$

$$C_b^0(\mathbb{R}^n, \mathbb{R}^n) = \{g : \mathbb{R}^n \to \mathbb{R}^n : g \text{is bounded and continuous}\}$$

for $a \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ and for $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\| a \|_0 = \sup_{\mathbb{R}^n} |a| \qquad \| b \|_1 = \| b \|_0 + \| b' \|_0$$

Before proving the theorem we formulate the above mentioned versions of the Hartman– Grobman Theorem.

Theorem(2.6) :(Hartman–Grobman global version)

Let $A \in L(\mathbb{R}^n)$be a hyperbolic linear map, $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $a(0) = 0$, $a'(0) = 0$ and let $f(p) = A_p + a(p)$. The solution of system (2.1) is denoted by $\varphi(. , p)$. The equilibrium point is the origin. Then there exists a number $\nu > 0$, such that for a function a with compact support and satisfying $\| a \|_1 < \nu$, there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying*

$$h\big(\varphi(t,p)\big) = e^{At} h(p) \qquad (2.4)$$

for all $p \in \mathbb{R}^n$ and for all $t \in \mathbb{R}^n$.

Theorem(2.7) :(Hartman–Grobman for maps)

Let $L \in GL(\mathbb{R}^n)$ be a hyperbolic linear map, that is the absolute value of its eigenvalues are not equal to 1 and to 0. Then there exists a number $\mu > 0$, such that for a function $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\| F \|_1 < \mu$, there exists $g \in C_b^0 (\mathbb{R}^n, \mathbb{R}^n)$, for which $H = id + g$ is a homeomorphism and

$$H \circ (L + F) = L \circ H. \qquad (2.5)$$

Now we turn to the proof of the Hartman–Grobman theorem (Theorem 2.5).

STEP 1 of the proof of Theorem(2.5):

Assume that the Hartman–Grobman theorem is proved when $p^* = 0$. Then the theorem holds for any p^*.

Proof : Let l :$\mathbb{R}^n \to \mathbb{R}^n$ be the translation $l(p) = p - p^*$. Let y : $\mathbb{R} \to \mathbb{R}^n$, $y(t) = x(t) - p^*$. Then $y(t) = x(t) = f(x(t)) = f(y(t) + p^*)$, that is y is a solution of the differential equation

$$\dot{y}(t) = g\big(y(t)\big), \qquad (2.6)$$

$g = f \circ l^{-1}$. The equilibrium of this equation is the origin .

Let $\psi(., q)$ denote the solution of this differential equation satisfying the initial condition $y(0) = q$. It is easy to see that $\psi(t, q) = \varphi(t, q + p^*) - p^*$, that is

$$l \circ \varphi_t = \psi_t \circ l . \qquad (2.7)$$

Since by the hypothesis the Hartman–Grobman theorem is true for equation (2.6), there is a homeomorphism h_1, for which

$$h_1 \circ \psi_1 = e^{At} \circ h_1 . \qquad (2.8)$$

where $A = g'(0) = f'(p^*)$. Composing equation (2.10) with l from the right $h_1 \circ \psi_t \circ l = e^{At} \circ h_1 \circ l$. Applying (2.9) we get $h^1 \circ \psi_t = e^{At} \circ h_1 \circ l$. Introducing $h = h_1 \circ l$ we get the desired statement, because, being the composition of two homeomorphisms, h itself is a homeomorphism.

STEP 2 of the proof of Theorem (2.5): Using the extension lemma below we prove that the global version of the Hartman– Grobman theorem (Theorem(2.6)) implies the local version(Theorem (2.5)).

Lemma(2.8):- Let $f \in C^1(B_R, \mathbb{R}^n)$ a*nd let* $A = f'(0)$. *For any number* $\nu > 0$ *there exist* $r > 0$*and* $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, *for which*

1. $|p| < r$ *implies* $a(p) = f(p) - Ap$,

2. $|p| > 2r$ *implies* $a(p) = 0$,

3. $\| a \|_1 < \nu$.

STEP 3 of the proof of Theorem(2.5):

In this part we prove that the global version of the Hartman–Grobman theorem (Theorem(2.6))follows from the Hartman–Grobman theorem for maps.

Proof of Theorem (2.6) :Applying the variation of constants formula to the differential equation $\dot{x}(t) = Ax(t) + a\big(x(t)\big)$ with initial condition $x(0) = p$ one obtains

$$\varphi(t,p) = e^{At}p + \int_0^t e^{A(t-s)}a(\varphi(s,p))ds\,.$$

Substituting $t = 1$

$$\varphi(1,p) = e^{A}p + \int_0^1 e^{A(1-s)}a\big(\varphi(s,p)\big)ds\,. \qquad (2.9)$$

Let

$$L = e^{A} \text{ and } \mathrm{F}(p) = \int_0^1 e^{A(1-s)}a\big(\varphi(s,p)\big)ds\,.$$

Let us choose the number $\mu > 0$ to the matrix L according to Theorem (2.7).Then it is easy to show that there exists a number $\nu > 0$, for which $\| a \|_1 < \nu$ implies $\| F \|_1 < \mu$.

Since the eigenvalues of the matrix A have non-zero real part, the absolute value of the eigenvalues of L are not equal to 1. Thus the assumptions of Theorem (2.7) are fulfilled. Therefore there exists a unique function $g \in C_b^0\ (\mathbb{R}^n, \mathbb{R}^n)$, for which $(\mathrm{id} + g) \circ (L + F) = L \circ (\mathrm{id} + g)$. (2.10)

Now we show that by choosing $h = \mathrm{id} + g$ the statement to be proved, i.e. $h \circ \varphi t = e^{At} \circ h$ holds. In order to prove that it is enough to show that the function

$$\alpha(p) = e^{-At} h(\varphi(t,p))$$

Coincides with the function h. This will be verified if we prove that the function $\alpha - \mathrm{id}$ satisfies (2.10) and $\alpha - \mathrm{id} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ holds as well, because the function g is unique in the given function space . These two statement will be proved below. According to equation (2.10)

$$h(\varphi(1,p)) = e^A h(p)$$

holds for all $p \in \mathbb{R}^n$. Hence

$$\begin{aligned}[\alpha \circ (L + F)](p) &= \alpha(\varphi(1,p)) = e^{-At} h(\varphi(t, \varphi(1,p))) = e^{-At} h(\varphi(t+1,p)) \\ &= e^{-At} h(\varphi(1, \varphi(t,p))) = e^{-At} e^A h(\varphi(t,p)) = e^A e^{-At} h(\varphi(t,p)) \\ &= (L \circ \alpha)(p),\end{aligned}$$

which proves that the function $\alpha - \mathrm{id}$ satisfies (2.12). On the other hand ,$(\alpha - id)(p) = e^{-At} h(\varphi(t,p)) - p = e^{-At}(h(\varphi(t,p)) - \varphi(t,p)) + e^{-At}\varphi(t,p) - p$ The first term of the right hand side is bounded (in the variable p), since $h - id$ is bounded in $\mathbb{R}^n$. The second term is also bounded because for large values of $|p|$ we have $a(p) = 0$, hence the equation is linear there, implying $\varphi(t,p) = e^{At}p$. This proves that $\alpha - id \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ that verifies the statement.

STEP 4 of the proof of Theorem (2.5):

Proof of Theorem(2.5) :The proof is divided into five steps:

1. Let E_s and E_u be the stable and unstable subspaces belonging to the linear mapping L. It is known that these are invariant, i.e. $L(E_s) \subset E_s$ and $L(E_u) \subset E_u$, and they span the whole space, that is $E_s \oplus E_u = \mathbb{R}^n$. Let

$$L_s = L|_{E_s} \quad , \quad L_u = L|_{E_u} .$$

It can be shown that by the suitable choice of the norm (or, in other words, by the suitable choice of the basis in the subspaces) one can achieve $\| L_s \| < 1$ and $\| L_u^{-1} \| < 1$. Let

$$r = \max\{\| L_s \|, \| L_u^{-1} \| < 1\} < 1 .$$

2. In this step we prove that there exists a positive number $\mu > 0$, for which in the case $\| F \|_1 < \mu$ the function $L + F$ is invertible. In order to prove that we apply the following global inverse function theorem.

Global inverse function theorem: *Let* $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be a function, for which ϕ' exists for all $p \in \mathbb{R}^n$ and there exists $K > 0$, such that $\| \phi'(p)^{-1} \| \leq K$. Then ϕ is a homeomorphism.

Let $\phi = L + F$, then $\phi' = L + F'$. Hence there exists $\mu > 0$, such that the conditions of the global inverse function theorem hold for ϕ, if $\| F \|_1 < \mu$. (We do not check these conditions in detail.) Thus the function L + F is a homeomorphism .

3. Now we transform equation (2.5) to a form, for which the Contraction Mapping Principle can be applied to determine the function g. It is easy to see that (2.7) is equivalent to equation

$$F + g \circ (L + F) = L \circ g . \qquad (2.11)$$

Composing this equation by the function $(L + F)^{-1}$ from right we get

$$g = -F \circ (L + F)^{-1} + L \circ g \circ (L + F)^{-1} . \qquad (2.12)$$

and then composing by the function L^{-1} from left we arrive to

$$L^{-1} \circ F + L^{-1} \circ g \circ (L + F) = g . \qquad (2.13)$$

Since $E_s \oplus E_u = \mathbb{R}^n$. for both F and g it can be introduced the functions $F_s , g_s : \mathbb{R}^n \to E_s$ and $F_u , g_u : \mathbb{R}^n \to E_u$, in such a way that

$$g = g_s + g_u \text{and } F = F_s + F_u$$

Hold . It is obvious that $g \in C_b^0 (\mathbb{R}^n, \mathbb{R}^n)$ implies $g_s , g_u \in C_b^0 (\mathbb{R}^n, \mathbb{R}^n)$as well. Define the operator T for a function $g \in C_b^0 (\mathbb{R}^n, \mathbb{R}^n)$ as follows.

$$T(g) = L\, o g_s o (L + F)^{-1} - F_s o (L + F)^{-1} + L^{-1} o g_u o (L + F) + L^{-1}\, o\, F_u \qquad (2.14)$$

We show that if g is a fixed point of T, then $H = id + g$is a solution of equation (2.5). Namely, for an arbitrary $p \in \mathbb{R}^n$

$$(L\,og_s o(L\,+\,F)^{-1}-\,F_s o(L\,+\,F)^{-1}(p)\in E_s \text{ and}$$

$$(L^{-1}og_u o(L\,+\,F)+L^{-1}o\,F_u\,)(p)\in E_u$$

hold . Hence according to $g\,=\,g_s\,+g_u$ the equality $T(g)\,=\,g$ can hold only if the following two equations hold

$$L\,og_s o(L\,+\,F)^{-1}-\,F_s o(L\,+\,F)^{-1}=g_s \text{ and}$$

$$L^{-1}og_u o(L\,+\,F)+L^{-1}o\,F_u=g_u$$

These equations yield

$$L\,og_s\,=g_s(L\,+\,F)+\,F_s \text{and } L\,og_u\,=g_u\,o(L\,+\,F)+\,F_u$$

Adding these equations and using the linearity of L one obtains (2.11), which is equivalent to equation (2.5).

4. In this step we prove that the operator T maps the space $C_b^0\ (\mathbb{R}^n,\mathbb{R}^n)$ into itself, and choosing a suitable norm on the space the mapping T is a contraction Hence the existence and uniqueness of the function g follows from the Contraction Mapping Principle. The operator T is obviously defined on the whole space $C_b^0\ (\mathbb{R}^n,\mathbb{R}^n)$, and for every $g\ \in C_b^0\ (\mathbb{R}^n,\mathbb{R}^n)$ the function $T(g)$ is a continuous function on $\mathbb{R}^n$. We show that $T(g)$ is a bounded function verifying that T maps the space $C_b^0\ (\mathbb{R}^n,\mathbb{R}^n)$ into itself . The boundedness follows from the fact that each term in the right hand side of (2.14) is a bounded function. For example, in the case of the last term

$$|(L^{-1}oF_u)(p)|\ =\ |L^{-1}F_u(p)|\ \leq \|\ L^{-1}\ \|\|\ F\ \|_0$$

holds for all $p\in\mathbb{R}^n$ .The proof is similar for the other terms.

Now we prove that T is a contraction. The norm is defined as follows: For $g\ \in C_b^0(\mathbb{R}^n,\mathbb{R}^n)$let

$\|g\|_*=\|g_s\|_0+\|g_u\|_0$.

It can be easily proved that this defines a norm and the space $C_b^0(\mathbb{R}^n,\mathbb{R}^n)$endowed with this norm is complete. In order to prove that T is contractive we use the relations

$$(T(g))_s=L\,og_s o(L\,+\,F)^{-1}-\,F_s o(L\,+\,F)^{-1}$$

$$(T(g))_u=L^{-1}og_u o(L\,+\,F)+L^{-1}o\,F_u.$$

and the fact that for $g, \bar{g} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ we have that $(g + \bar{g})_s = g_s + \bar{g}_s$, and $(g + \bar{g})_u = g_u + \bar{g}_u$. For arbitrary $g, \bar{g} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$

$\|T(g) - T(\bar{g})\|_* = \|T(g) - T(\bar{g}))_s\|_0 + \|T(g) - T(\bar{g}))_u\|_0 =$

$$\| L\, o g_s o (L + F)^{-1} - L\, o \bar{g}_s o (L + F)^{-1} \|_0 + \| L^{-1}\, o g_u o (L + F) - L^{-1} o\, \bar{g}_u o (L + F) \|_0 =$$

$$\sup_{p \in \mathbb{R}^n} \left| L_s g_s((L + F)^{-1}(p)) - L_s \bar{g}_s((L + F)\,(p)) \right| +$$

$$\sup_{p \in \mathbb{R}^n} \left| {L_u}^{-1} g_u((L + F)(p)) - {L_u}^{-1} \bar{g}_u((L + F)\,(p)) \right| \leq$$

$$\|L_s\| \|g_s - \bar{g}_s\|_0 + \left\|{L_u}^{-1}\right\| \|g_u - \bar{g}_u\|_0 \leq r(\|(g - \bar{g})_s\|_0 + \|(g - \bar{g})_u\|_0)$$

$$\leq r\|g - \bar{g}\|_* \ .$$

Thus $r < 1$ ensures that T is a contraction, hence it has a unique fixed point $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$

5. It remained to show that the function $H = id + g$ is a homeomorphism. This will be proved by constructing the inverse of H as a solution of an equation similar to (2.5), and by proving that this equation has a unique solution. Repeating the steps 3 and 4 one can show that there exists a unique function $g^* \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$, such that

$$(L + F)\ o\ (id + g^*) = (id + g^*)\ o\ L\ . \tag{2.15}$$

On the other hand, substituting F ≡ 0 into equation (2.5) we get

$$Lo(id\ +\ g) =\ (id\ +\ g) o\ L \tag{2.16}$$

The only solution of this equation is $g \equiv 0$.

We show that if $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$,is the unique solution of equation (2.5), then the inverse of $H = id + g$ is the function $H^* = id + g^*$. Equations (2.7) and (2.15) yield

$$L\ o\ H\ o\ H^* = Ho\ (L\ +\ F)\ oH^*\ =\ H\ oH^* o\ L\ .$$

Let $\bar{g} = H\ oH^* -$ id . Then g is a solution of (2.16). If we prove that $\bar{g} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$then by the uniqueness $\bar{g} \equiv 0$, that is $id = H\ oH^*$. This can be shown as follows:

$$\bar{g} = H\, oH^* - \text{id} = (\text{id} + g)\, o(\text{id} + g^*) - id = g^* + g\, o\, (\text{id} + g^*)\,.$$

The function in the right hand side is obviously continuous and bounded, which proves the statement. It can be proved similarly that $\text{id} = H^* o\, H$, hence H is a continuous bijection implying that it is a homeomorphism .

The examples below illustrate the necessity of the assumption on hyperbolicity. Namely, if the linear part is not hyperbolic, then the linearised system and the non- linear system may have different local phase portraits.

Example (2.8):- Consider the two dimensional system

$$\dot{x} = -y - xy^2 - x^3, \qquad (2.17)$$

$$\dot{y} = x - y^3 - x^3 y, \qquad (2.18)$$

The origin is an equilibrium of this system. The linearized system at the origin is given by the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which determines a center. Hence the linear part is not hyperbolic . The phase portrait of the non-linear system can be obtained by polar coordinate transformation. Introducing the functions r and φ by the transformation formulas $x(t) = r(t)cos\big(\phi(t)\big)$ and

$y(t) = r(t)\sin(\phi(t))$, one obtains after differentiation

$$\dot{x} = \dot{r}\cos(\phi) - r\dot{\phi}\sin(\phi), \qquad \dot{y} = \dot{r}\sin(\dot{\phi}) + r\,\dot{\phi}\cos(\phi).$$

Multiplying the first equation by $\cos(\phi)$ and the second equation by $\sin(\phi)$, then adding the equations and using the differential equations for x and for y we arrive to $\dot{r} = -r^3$. Carrying out a similar calculation, but now the first equation is multiplied by $\sin(\phi)$ and the second one by $\cos(\phi)$ we get $\phi = 1$. Hence the function r is strictly decreasing and tends to zero, while ϕ is a strictly increasing function tending to infinity.

Now we discuss the Normal forms :Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a k times continuously differentiable function that will be denoted shortly by $f \in C^k$. Sometimes it is assumed that f is infinitely many times differentiable , this will be denoted by $f \in C^\infty$. If in addition, $f$is analytic, that is it is equal to the sum of its Taylor series, then the notation $f \in C^\omega$ is used. The main idea of deriving

normal forms can be briefly summarised as follows. In order to simplify the differential equation $\dot{x}(t) = f(x(t))$ let us introduce the function $y(t)$ by the transformation $x = h(y)$, where h is a diffeomorphism. The differential equation can be easily derived for the function y in the following way. Differentiating the relation $x(t) = h(y(t))$ we get $\dot{x} = h'(y) \,.\, \dot{y}$, on the other hand $, \dot{x} = f(x) = f(h(y))$ implies h′(y).$\dot{y} = f(h(y))$, hence for y the differential equation $\dot{y}(t) = g(y(t))$ holds, where f and g are related by

$$h' \cdot g = f \; o \; h. \tag{2.19}$$

This means that the dynamical systems corresponding to the differential equations $\dot{x}(t) = f(x(t))$ and $\dot{y}(t) = g(y(t))$ are C^k conjugate, once $h \in C^k$. The aim is to choose the diffeomorphism h in such a way that the Taylor expansion of g contains as few terms as possible (in this case the phase portrait of $\dot{y}(t) = g(y(t))$ is easier to determine than that of system $\dot{x}(t) = f\big(x(t)\big)$). In other words, by the transformation h we try to find those terms in the expansion of f that play role in determining the phase portrait.

The local phase portrait will be investigated in a neighborhood of an equilibrium point p. As a preliminary step the equilibrium is shifted to the origin and the linear part is transformed to Jordan canonical form. That is the function $y = x - p$ is introduced first. For this function the differential equation takes the form $\dot{y} = g(y)$, where $g(y) = f(y + p)$. In order to get the Jordan canonical form let us introduce the invertible matrix $P \in R^{n \times n}$, and let y be defined by the linear transformation $x = Py$. Then $P\,\dot{y} = fP(y)$, that is $\dot{y} = g(y)$, where $g(y) = P^{-1} \cdot f(Py)$.

Then the Jacobian of system $\dot{y} = g(y)$ at the origin $g'(0) = P^{-1} \cdot f'(0) \cdot P$ can be considered as the Jordan canonical form of the Jacobian $f'(0)$ with a suitably chosen matrix P. Hence we can assume without loss of generality that the equilibrium of system $\dot{x}(t) = f(x(t))$ is the origin (i.e. $f(0) = 0$) and the Jacobian $f'(0)$ is in Jordan canonical form .Now, the aim is to transform the

system to the simplest possible form. We illustrate first the method in the one dimensional case.Let $f(x) = Ax + a_r \cdot x^r + o(x^r)$, where $r \geq 2$ and $o(x^r)$ denotes a function containing only the terms that are higher order than r, that is a function, for which $o(x^r)/x^r$converges to zero as $x \to 0$. Let us look for the function h in the form $h(x) = x + h_r \cdot x^r$. Let us check , based on equation (2.19), whether the function g can be chosen in the form $g(x) = Ax + o(x^r)$. Since $h'(x) = 1 + r \cdot h_r \cdot x^{r-1}$, therefore assuming g(x) = Ax + o(x^r) the left hand side of (2.19) takes the form

$$h'(x)g(x) = Ax + Arh_r x^r + o(x^r),$$

on the other hand, the right hand side of (2.21) is

$$fh(x) = Ax + Ah_r x^r + a_r x^r + o(x^r).$$

If the coefficients of the terms of order r are equal, then $Arh_r = Ah_r + a_r$, from which the unknown coefficient h_r of function h can be determined as

$$h_r = \frac{a_r}{A(r-1)}$$

if $A \neq 0$ holds. Thus in the one dimensional case, if the linear part is non-zero, then all the higher order terms can be transformed out, which means that the system is locally C^∞ conjugate to its linear part. In fact, using the above procedure the nonlinear terms can be transformed out step by step after each other as follows. First, the second degree polynomial $H_2(x) = x + h_2 \cdot x_2$ is used, with $h_2 = a_2/A$. Then the function G_2takes the form

$G_2(x) = Ax + o(x^2)$ and satisfies (2.19), that is $H_2' \cdot G_2 = f \, oH_2$.

Then for the function $G_2(x) = Ax + b_3x^3 + o(x^3)$ we determine a function $H_3(x) = x + h_3 \cdot x^3$. In this function $h_3 = \frac{b_3}{2A}$, hence

$G_3(x) = Ax + o(x^3)$ and satisfies (2.19), that is $H_3' \cdot G_3 = G_2 oH_3$. Now let $h = H_2 oH_3$, then $h' = (H_2' oH_3) \cdot H_3'$, therefore

$$f \, o \, h = f \, o \, H_2 oH_3 = (H_2' \cdot G_2) \, o \, H_3 = (H_2' o \, H_3) \cdot (G_2 o \, H_3))$$
$$= (H_2' o \, H_3) \cdot H_3' \cdot G_3 = h' \cdot G_3.$$

Thus we have $f\ o\ h = \ h' \cdot \ G_3$, which means that the transformation $h \ = \ H_3 o\ H_2$ leads to an equation that does not contain second and third degree terms. Following this procedure, assume that the function G_{k-1} containing only terms of degree $k-1$ and higher, is already determined. Then by a suitable coefficient h_k of the function H_k we can achieve that the function G_k determined by the equation

$$H_k' \cdot \ G_k = \ G_{k-1} o\ H_k$$

contains no terms of degree smaller than k. Hence the mapping h given by the infinite composition $h = \ H_2\ o\ H_3 o$ . ... transforms the equation to its linear part, i.e. equation (2.19) holds with the linear function $g(x) = Ax$. Here a difficult question arises, namely the convergence of the infinite composition. At the end of this chapter a theorem will be formulated about the convergence. Now, we extend the procedure to the case of n-dimensional systems. Thus let us consider the system $\dot{x}(t) = f(x(t))$, the equilibrium of which is the origin, and assume that the linearised system, i.e. the Jacobian $A = f'(0)$ is a diagonal matrix. That is

$$f(x) \ = \ Ax \ + \ a(x) \ + \ o(x^r),$$

where

$$A \ = \begin{pmatrix} \lambda_1 & . & 0 \\ . & \ddots & . \\ 0 & . & \lambda_n \end{pmatrix}$$

is diagonal and the nonlinear part $a(x)$ contains only terms of degree r and higher, that is $a(x)$ can be expressed as the linear combination of terms in the form

$$e_i \ \cdot \ x_1^{m_1} \ \cdot \ x_2^{m_2} \ \cdot \cdots \cdot \ x_n^{m_n} \ : \ m_1 \ + \ m_2 \ + \cdots + m_n \ = \ r,$$

where $e_i \in R^n$ is the ith unit vector (its ith coordinate is 1, its other coordinates are 0). For example, in the case $n \ = \ 2\ , r \ = \ 2$ the function $a(x)$ is in the space

$$V_2 = span\left\{\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}\right\}$$

Thus system $\dot{x}(t) = f(x(t))$ can be written in the form

$$\dot{x}_1 = \lambda_1 x_1 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + o(x^3),$$
$$\dot{x}_2 = \lambda_2 x_2 + b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2 + o(x^3).$$

For an arbitrary n and r the function $a(x)$ is an element of the similarly defined space

$$V_r = span\{e_i.x_1^{m_1} \cdot x_2^{m_2} \cdot \cdots \cdot x_n^{m_n} : m_1 + m_2 + \cdots + m_n = r, i = 1,2,..,n\}$$

The homeomorphism h is searched in the form $h(x) = x + H(x)$, where $H(x) \in V_r$ is also a linear combination of terms of order r. The aim is to choose a function H, for which equation (2.19) holds with a function g that can be written in the form $g(x) = Ax + o(xr)$.The left hand side of equation (2.19) is

$$(f\ o\ h)(x) = Ax + AH(x) + a(x + H(x)) + o(x^r) = Ax + AH(x) + a(x) + o(x^r)$$

because $a(x + H(x) + = a(x) + o(x^r)$, by using the power series expansion of a. On the other hand

$$h'(x).g(x) = (I + H'(x)) \cdot (Ax + o(x^r)) = Ax + H'(x) \cdot Ax + o(x^r).$$

Since the terms of degree r are equal on the left and right hand side, we get the following equation for H

$$H'(x)Ax - AH(x) = a(x).$$

This equation is referred to as the homological equation. Let us introduce the linear mapping $L_A : V_r \to V_r$ that associates to a function H the left hand side of the homological equation, i.e.

$$(L_AH)(x) = H'(x)Ax - AH(x).$$

The homological equation has a unique solution to each function $a \in V_r$, if and only if L_A is bijective, that is the following proposition holds.

Proposition(2.9):-

If 0 is not an eigenvalue of the mapping L_A, then the terms of degree r can be transformed out.

Let us investigate the effect of the mapping L_A on the basis elements of V_r. Introducing the notation $x^m = x_1^{m_1} \cdot x_2^{m_2} \cdot \cdots \cdot x_n^{m_n}$ one can prove the following statement .

Lemma(2.10):-

$$L_A(e_i x^m) = e_i x^m \left(\sum_{k=1}^{n} m_k \lambda_k - \lambda_i \right)$$

where the numbers m_k are the coordinates of m, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues in the diagonal of the matrix A.

Proof:

Since the function H is given by

$$H(x) = e_i x^m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1^{m_1} x_1^{m_2} \ldots x_1^{m_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we have

$$AH(x) = e_i x^m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_i e_i x^m.$$

On the other hand

$$H'(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_1 \dfrac{x^m}{x_1} \quad m_2 \dfrac{x^m}{x_2} \ldots m_n \dfrac{x^m}{x_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

and

$$Ax = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{pmatrix},$$

implying

$$H'(x)Ax = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^m \sum_{k=1}^{n} m_k \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_i x^m \sum_{k=1}^{n} m_k \lambda_k.$$

The equations above yield

$H'(x)Ax - AH(x) = e_i x^m (\sum_{k=1}^{n} m_k \lambda_k - \lambda_i)$

that was to be proved.

According to this lemma the eigenvalues of the mapping L_A can be written in the form $\sum_{k=1}^{n} m_k \lambda_k - \lambda_i$, the corresponding eigenvectors are the functions of the form $e_i x^m$. Since these are exactly the functions that span the vector space V_r, we found so many eigenvalues as the dimension of the vector space V_r.

Therefore all eigenvalues of L_A can be written in the form $\sum_{k=1}^{n} m_k \lambda_k - \lambda_i$. Hence if these are nonzero numbers, then the mapping L_A is bijective. Based on this formula let us introduce the following notion.

Definition(2.11):-

The set of eigenvalues of the matrix A are called resonant , if there exists $i \in \{1,2,\ldots,n\}$, and there are nonnegative integers $m_1, m_2, \ldots, m_n$, for which $m_1 + m_2 + \ldots + m_n \geq 2$ and $\lambda_i = \sum_{k=1}^{n} m_k \lambda_k$. If this holds, then the term $e_i x^m$ is called a resonant term.Thus if the eigenvalues of the matrix *A* are non-resonant, then the mapping L_A is bijective. This implies that there is a sequence of diffeomorphisms, the composition of which is a diffeomorphism that shows that

the system and its linear part are C^{∞} conjugate. (Consequently, they are C^{k} conjugate for any k.) The convergence of the infinite composition can be verified under suitable conditions. This is formulated in the next theorem.

Theorem(2.12) :(Poincare)

1. If the eigenvalues of the matrix A are non-resonant, then all non-linear terms can be formally transformed out from the system .
2. If the eigenvalues of the matrix A are non-resonant and the convex hull of the eigenvalues in the complex plane does not contain the origin ,then the system $\dot{x} = f(x)$ and its linear part $\dot{y} = Ay$ are locally C^{∞} conjugate at the origin.

Example(2.13):-

If the system is n = 2 dimensional, then the resonant terms of degree 2 can be occur as follows.

$$m = (0,2), \quad 0 \cdot \lambda_1 + 2 \cdot \lambda_2 = \lambda_1 \, or \, \lambda_2$$
$$m = (1,1), \quad 1 \cdot \lambda_1 + 1 \cdot \lambda_2 = \lambda_1 \, or \, \lambda_2$$
$$m = (2,0), \quad 2 \cdot \lambda_1 + 0 \cdot \lambda_2 = \lambda_1 \, or \, \lambda_2$$

For example, in the case $\lambda_1 = 0 \, or \, \lambda_2 = 0$, we get resonance with m = (1,1), hence the term $x_1 x_2$ cannot be transformed out.

Example(2.14) :-

Let us consider the linear system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2$$

That has a saddle point. In this system the stable subspace E_s is the horizontal axis, while the unstable subspace $E_u$is the vertical axis .Note that there is no other invariant one-dimensional subspace (i.e. there is no other line through the origin that is not left by the trajectories).

The example below shows how the invariant subspaces are changed by a non-linear perturbation of the above linear system.

Example(2.15):-

Let us consider the non-linear system

$$\dot{x}_1 = -x_1,$$

$$\dot{x}_2 = -x_{2+}x_1^2$$

The first equation is independent, its solution can be given $as\ x_1(t) = e^{-t}C_1$. Substituting this solution into the second equation we arrive to an in homogeneous linear differential equation that can be solved as

$x_2(t) = e^t \cdot C_2 + \frac{C_1^2}{3}\ (e^t - e^{-2t})$. These solutions satisfy the initial conditions $x_1(0) = C_1$, $x_2(0) = C_2$. Note that if $C_2 + \frac{C_1^2}{3} = 0$ holds for the initial conditions, then for any time t the same relation $x_2(t) + \frac{x_1^2(t)}{3} = 0$ holds. That is the set

$$W_s = \{(C_1, C_2) \in \mathbb{R}^2 : C_2 + \frac{C_1^2}{3} = 0\}$$

is invariant , i.e. the trajectories do not leave it . The solutions starting from this set tend to the origin as time goes to infinity. This is why it is called stable manifold, this is the set that took over the role of the stable subspace. The invariant manifold is easier to determine, since the vertical axis is invariant and along this line trajectories tend to infinity. Thus the invariant manifold is the $W_s = \{(0, C_2) \in \mathbb{R}^2 : c_2 \in \mathbb{R}\}$ subspace.

Theorem (2.16) :(Stable and unstable manifold)

There is a neighbourhood U of the origin and there exist continuously differentiable functions $\Psi : E_s \cap U \to E_u$and $: E_u \cap U \to E_s$, for which the k-dimensional local stable and $n - k$ −dimensional local unstable manifolds $W_s^{loc} = \{(q, \Psi(q)) \in \mathbb{R}^2 : q \in E_s \cap U\}\ and\ W_u^{loc} = \{(r, \Phi(r)) \in \mathbb{R}^2 : r \in E_u \cap U\}$ have the following properties. (We note that the vectors q and $\Phi(r)$ above have k coordinates, while the vectors r and $\Psi(q)$ have $n - k$ coordinates.)

1. W_s^{loc} is positively invariant, W_u^{loc} is negatively invariant.

2. The manifold W_s^{loc} is tangential to the subspace E_s at the origin, and the manifold W_u^{loc} is tangential to the subspace E_u at the origin.
3. If $p \in W_s^{loc}$, then $\lim_{t\to\infty} \varphi(t,p) = 0$.
4. If $\in W_u^{loc}$, then $\lim_{t\to-\infty} \varphi(t,p) = 0$.

Definition(2.17): (Global stable and unstable manifolds)

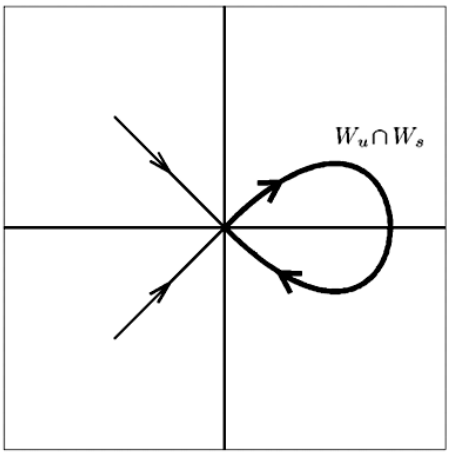
The global stable and unstable manifolds can be defined by using the local manifolds W_s^{loc} and W_u^{loc} . The global manifolds typically cannot be given as graphs of suitable functions. The stable manifold is defined as the set of points from which the solution tends to the origin. Since these trajectories lie in the local stable manifold when they are close to the origin, it is reasonable to define the global stable manifold as follows:

$$W_s := \bigcup_{t\le 0} \varphi(t, W_s^{loc}).$$

The unstable manifold is the set of points from which the solution tends to the origin as $t \to -\infty$.These trajectories lie in the local unstable manifold when they are close to the origin, hence the global unstable manifold is defined as follows .

$$W_u := \bigcup_{t\le 0} \varphi(t, W_u^{loc}).$$

The global stable and unstable manifolds may have common points , this situation is shown in Figure below The system, the phase portrait of which is shown in the Figure, has a homoclinic orbit that is the intersection of the stable and unstable manifolds.



Figure(2.2):Homoclinic orbit as the intersection of the stable and unstable manifolds.

Example(2.18):- Consider the system

$$\dot{x} = xy + x^3,$$
$$\dot{y} = -y - 2x^2$$

and investigate its local phase portrait in the neighbourhood of the origin. The Jacobian obtained by linearisation is $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, its eigenvalues are 0 and -1. Thus the originis not a hyperbolic equilibrium, therefore the linearisation does not determine the local phase portrait. Since there is negative eigenvalue, the system has a one dimensional stable manifold, along which the trajectories tend to the origin. At the end of this chapter it will be shown that there is an invariant center manifold that is tangential to the center subspace belonging to the eigenvalue 0, and the behaviour of the trajectories in this manifold can easily be determined by investigating the phase portrait of a one-dimensional system.

Let us consider again a general autonomous system $\dot{x} = f(x)$, and assume that the origin is an equilibrium, that is $f(0) = 0$, and the Jacobian is written in the form $f'(0) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where the eigenvalues of B have zero real part, the eigenvalues of C have non-zero real part. Thus using the notation $x = \begin{pmatrix} y \\ z \end{pmatrix}$ the system takes the form

$$\dot{y} = By + g(y, z) \qquad (2.20)$$

$\dot{z} = Cz + h(y,z),$ (2.21)

where it is assumed that the derivatives of g and h are zero at the origin.

Theorem(2.19):(Center manifold) There is a neighbourhood U of the origin and there exists a differentiable map $\psi : E_c \cap U \to E_s \oplus E_u$ that satisfy the following conditions.

1. $\psi(0) = 0, \psi'(0) = 0.$
2. The local center manifold $W_c^{loc} = \{(p_c, \psi(p_c)) : p_c \in E_c \cap U\}$ is locally invariant, i.e. if $p \in W_c^{loc}$ and $\varphi(t,p) \in U$, then $\varphi(t,p) \in W_c^{loc}$

Proof :

The technical details make the proof of the theorem considerably long, hence we present here only the main ideas of the proof based on the book by Chow and Hale[8]. Consider a solution $\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$ starting from the point $(p_c , \psi(p_c))$.

The invariance of the manifold requires

$\dot{y} = By + g(y, \psi(y))$ (2.22)

$\dot{z} = Cz + h(y, \psi(y))$. (2.23)

Applying the variation of constants formula to the second equation with a starting point $t = \tau$

$z(t) = e^{C(t-\tau)} \cdot z(\tau) + \int_\tau^t e^{C(t-\tau)} h(y(s), \varphi(y(s)))$ds.

Let $\tau < 0$, then substituting t = 0 into this equation

$\psi(p_c) = z(0) = e^{-C\tau} \cdot z(\tau) + \int_\tau^0 e^{-Cs} h\left(y(s), \varphi\left(y(s)\right)\right)$ds . (2.24)

For simplicity, let us consider only the case when the eigenvalues of C have negative real part . (The general case can be dealt with in a similar way.) For the solutions in the center manifold $z(t)$ cannot tend to infinity as $t \to -\infty$, therefore

$$\lim_{\tau \to -\infty} e^{-C\tau} z(\tau) = 0,$$

hence equation (2.24) implies

$$\psi(p_c) = \int_{-\infty}^{0} e^{-Cs}\ h\,(\,y(s), \varphi(y(s)))\mathrm{ds}.$$

Thus the procedure is as follows. For a given function ψ solve differential equation (2.24) subject to the initial condition $y(0) = p_c$, and define the operator

$$(T(\psi))(p_c) = \int_{-\infty}^{0} e^{-Cs}\ h\,(\,y(s), \psi(y(s)))\mathrm{ds}.$$

Choosing an appropriate Banach space this operator is a contraction, hence by Banach's fixed point theorem it has a fixed point . This fixed point yields the function ψ determining the local center manifold.

Theorem (2.20):(Center manifold reduction) Consider system (2.20)-(2.21). Let ψ be the function determining the local center manifold. By using the so called center manifold reduction introduce the system

$$\dot{u} = Bu + g\big(u, \psi(u)\big)^{\cdot} \qquad (2.25)$$

$$\dot{v} = Cv, \qquad (2.26)$$

in which the linearisation is used in the hyperbolic part.Then system (2.27)-(2.26) and system (2.25)-(2.26) are locally topologically equivalent in the origin.

Remark (2.21):-

This theorem can be considered as the generalisation of the Hartman– Grobman theorem to the case of non-hyperbolic linear part . The theorem enables us to reduce the dimension, because the phase portrait of the linear system $\dot{v} = Cv$ can easily be determined, hence in order to characterise the full phase portrait it is enough to determine the phase portrait of the lower dimensional non-linear system $\dot{u} = Bu + g(u, \psi(u))$.

Now we introduce approximation of the center manifold in order to apply center manifold reduction the function ψ determining the center manifold has to be determined. In most of the cases it is not possible to calculate this function

explicitly.Assume that the solution$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$lies in the center manifold. The invariance of the manifold implies $z(t) = \psi\big(y(t)\big)$.Differentiating this equation we get $\dot{z} = \psi'(y) \cdot \dot{y}$, therefore

$\dot{z} = Cy + h(y,\psi)(y)) = \psi'(y)) \cdot (By + g(y,\psi(y)))$, (2.27)

which can be considered as an equation determining the function ψ. This equation does not enable us to use Banach's fixed point theorem to prove the existence of the center manifold, however, it is useful in calculating the coefficients of the power series expansion of the function ψ. According to the following theorem, if this equation holds for the terms in the power series of a function $\tilde{\psi}$ up to degree r, then this function approximates ψ in order r.

Theorem(2.22):(Approximation of the center manifold)

Let $\tilde{\psi} : E_c \cap U \rightarrow E_s \oplus E_u$ be a function, for which:

1. $\tilde{\psi}(0) = 0$, $\tilde{\psi}'(0) = 0$.
2. (2.27)holds in order r.

Then $|\psi(y) - \tilde{\psi}(y)| = O(|y|^r)$, that is the power series expansions of ψ and $\tilde{\psi}$ coincide up to degree r.

Example(2.23):-

$$\dot{x} = xy + x3 \qquad (2.28)$$

$$\dot{y} = -y - 2x^2. \qquad (2.29)$$

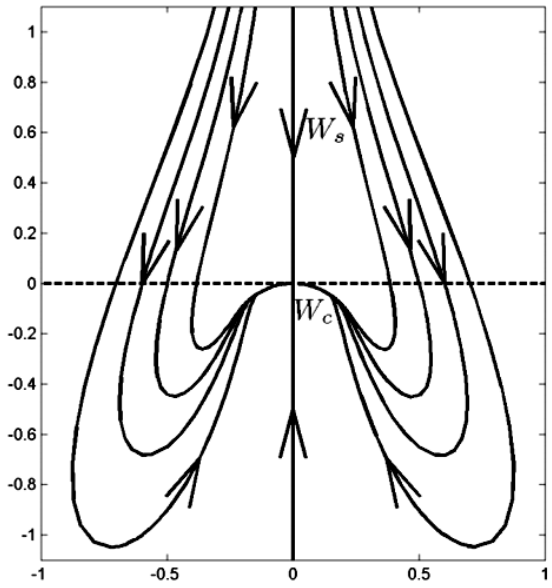
The Jacobian is $f'(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, the stable subspace E_s is determined by (0,1), the center subspace E_c is given by (1,0).

The approximation of the function ψ determining the center manifold can be given in the form $\psi(x) = a_2x^2 + a_3x^3 +$ ... because the center manifold is tangential to the center subspace at the origin, that is ψ can be given as a function of x and $\psi(0) = 0 = \psi'(0)$. The invariance of the manifold implies$y(t) = \psi(x(t))$, the derivative of which yields $\dot{y}(t) = \psi'(x(t)) \cdot \dot{x}(t)$. Substituting the derivatives of x and y from the differential equations to

this equation one obtains $-\psi(x) - 2x^2 = \psi'(x)(x \cdot \psi(x) + x^3)$. Using the power series expansion of ψ the coefficients of the corresponding terms are equal in the left and right hand sides. Using the coefficients of the quadratic term x_2 on both sides we get $-a_2 - 2 = 0$ yielding $a_2 = -2$. Hence the function defining the center manifold can be given as $\tilde{\psi}(x) = -2x^2 + O(x^3)$. Thus the approximation of the center manifold up to second degree can be given as $\widetilde{\psi}(x) = -2x^2$. Substituting $\psi(x) = -2x^2 + O(x^3)$. Into the first equation of the reduced system

$$\dot{x} = x(-= 2x^2 + a_3x^3 + \dots) + x^3 = -x^3 + O(x^4)$$

The local phase portrait at the origin does not depend on the terms of $O(x^4)$, because they do not have influence on the direction field. In this one-dimensional system the trajectories tend to the origin, hence along the center manifold the trajectories tend to the origin. Since the other eigenvalue of the system is negative, all solutions in a neighborhood of the origin tend to the origin, thus according to the center manifold reduction the origin is asymptotically stable.The phase portrait is shown in Figure(2.3) .



The phase portrait of system (2.28)-(2.29).

Chapter 3

Global phase portrait, periodic orbits and index of a vector field

Section(3.1):Investigating the global phase portrait by using the local ones

In the previous chapter we studied how can the local phase portrait of a differential equation $\dot{x} = f(x)$ be determined in the neighbourhood of a given point. The results presented can be summarized briefly as follows:

- If the given point p is not an equilibrium, then the flow-box theorem can be applied, hence the local phase portrait at the point p is given by straight lines parallel to the vector $f(p)$.
- If the given point p is an equilibrium, then the local phase portrait can be characterized by using the linearization $f'(p)$:
  - If the linear part is hyperbolic, that is the eigenvalues of $f'(p)$ have non-zero real part, then according to the Hartman-Grobman Theorem the phase portrait is locally conjugate to that of the linearised system.
  - If the linear part is not hyperbolic, then higher order terms play role in determining the phase portrait . The effect of these terms can be studied by using the following tools :

    ∗Finding the normal form of the system.

    ∗ Using center manifold reduction.

The aim of this section is to show methods for studying the global phase portrait by using the local phase portraits determined at different points. In the case of one dimensional systems we can achieve full classification by using this approach. This will be shown in the following illustration.

Defintion(3.1.1):(Global phase portraits of one dimensional systems)

The phase portrait of a one dimensional system can be simply obtained based on the sign of the right hand side. This is illustrated in the case of a simple example.

Example(3.1.2):-Consider the differential equation $\dot{x} = x - x^3$. The zeros of the function $f(x) = x - x^3 \; are \; -1, 0, and \; 1$. The sign of the function can simply be determined in the segments determined by these points. In the intervals $(-\infty, -1)$ and (0,1) the function is positive, while in the intervals (−1,0) and (1,+∞) it is negative. The graph of the function is shown in Figure (3.1).Thus the equation has three equilibria −1, 0, and 1. There are two orbits directed positively, these are (−∞,−1) and (0,1), and there are two negatively directed orbits (−1,0) and (1,+∞). Hence by determining the sign of the function one can get the global phase portrait shown in Figure(3.1).

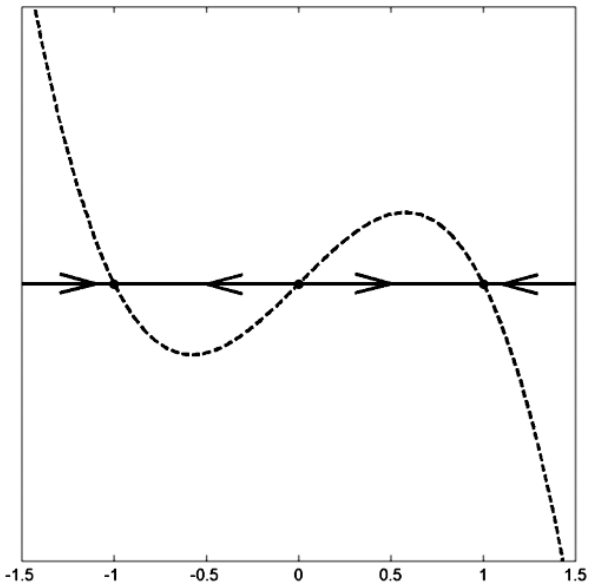
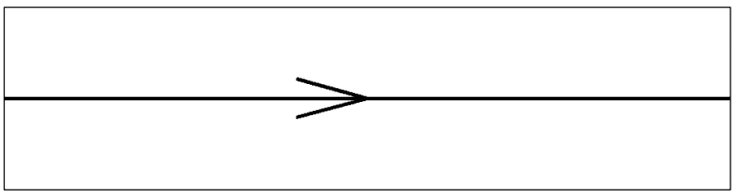


Figure (3.1): The phase portrait of the differential equation $\dot{x} = x - x^3$.

Let us turn now to the classification of the phase portraits of the one dimensional dynamical systems of the form $\dot{x} = f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function . The classification is based on the number of zeros of the function f as it is suggested by the above example.

- If the function f has no zeros, then there is no steady state. Then the phase portrait is equivalent to that of the equation $\dot{x} = 1$ that is shown in Figure (3.2). (We note that the phase portrait is equivalent to this one also in the case when f is negative.)
- If the f function has one zero, then there is a unique equilibrium. Then the dynamical system is equivalent to one of the three equations below, their phase portrait are shown in Figure (3.3).
  - $\dot{x} = -x$
  - $\dot{x} = x$
  - $\dot{x} = x^2$
- If the function f has two zeros, then there are two equilibria. Each of them can be stable, unstable or neutral (stable from one side and unstable from the other side). Then there are four combinations of the two steady states: 1. a stable and an unstable point, (2) a stable and a neutral point, (3) an unstable and a neutral point, (4) two neutral points. The phase portraits corresponding to these cases are shown in Figure (3.4).

Equations having n equilibria can be classified similarly. We note that there can be infinitely many steady states, for example in the case of $f(x) = \sin x$, moreover, the equilibria may accumulate as for the function $f(x) = x^2 \cdot sin\frac{1}{x}$. These cases are not dealt with here.



Figure(3.2): The phase portrait of $\dot{x} = f(x)$ when the function f has no zeros.

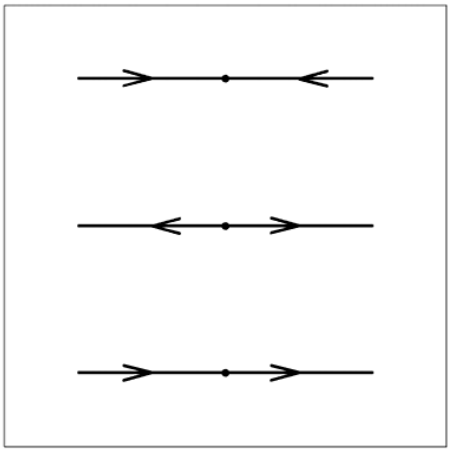


Figure (3.3): The three possible phase portraits of $\dot{x} = f(x)$ when the function f has one zero.

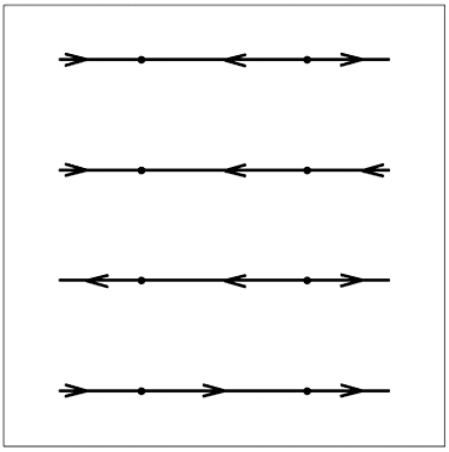


Figure (3.4): The four possible phase portraits of $\dot{x} = f(x)$ when the function f has two zeros.

Definition(3.1.3): (Global phase portraits of two dimensional systems)

Consider the system

$$\dot{x} = P(x, y),$$
$$\dot{y} = Q(x, y)$$

Of two differential equations, where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable functions .The goal here is to characterize the global phase portrait. The following methods will be presented.

- Determining the direction field and the nullclines.
- Transforming the system to polar coordinates or to a complex variable.

- Finding a first integral or a Lyapunov function.
- Exploiting the symmetry of the vector field (P, Q).

Beyond these methods it is always useful to solve the differential equations numerically from suitably chosen initial conditions.

Now we study direction field and nullclines method:

The direction field of the differential equation is the function $(P, Q) : \mathbb{R}^2 \to \mathbb{R}$ that associates a two dimensional vector to each point of the phase plane. This vector is the tangent of the trajectory at the given point. In order to characterize the phase portrait it is often enough to know if the vectors of the direction field point up or down, and to the left or to the right. In order to see this the nullclines

$$N_1 := \{p \in \mathbb{R}^2 : P(p) = 0\} \; N_2 := \{p \in \mathbb{R}^2 : Q(p) = 0\}$$

can help. The null cline N_1 divides the phase plane into two parts (these are not necessarily connected sets). In one of these, in which P > 0, trajectories move to the right (since $\dot{x} > 0$ there), in the other part, in which P < 0, trajectories move to the left (since $\dot{x} < 0$ there). Similarly, the null cline N_2 divides the phase plane into two parts (these are not necessarily connected sets). In one of these, in which Q > 0, trajectories move up (since $\dot{y} > 0$ there), in the other part, in which Q < 0, trajectories move down (since $\dot{y} < 0$ there). Thus the nullclines N_1 and N_2 divide the phase plane into four parts, in each of them it can be decided if the trajectories move up or down, and to the left or to the right. (We use the terminology that ”the trajectory moves”, in fact the point $\varphi(t, p)$ moves along the trajectory as t is varied.)

The intersection points of the nullclines are the equilibria where both P and Q are equal to zero. In the neighbourhood of the steady states the phase portraits can be determined by linearisation. In order to get the global picture it is useful to determine the behaviour of the separatrices of the saddle points, i.e.

the trajectories converging to saddle points as $t \to +\infty \; or \; as \; t \to -\infty$.We illustrate this approach by the following examples.

Example(3.1.4):- Consider the system

$$\dot{x} = x - xy \quad , \quad \dot{y} = x^2 - y.$$

The equation of the null cline N_1 is $x(1-y)= 0$. That is this null cline consists of two lines, the line $\{(x,y) \in \mathbb{R}^2 : x = 0\}$ and the line $\{(x,y) \in \mathbb{R}^2 : y = 1\}$. These lines divide the phase plane into four domains. In the upper right and lower left domain the trajectories move to the left, while in the other two domains they move to the right. Since the line $x = 0$ is contained in the null cline, it is an invariant line (this can also be easily seen from the fact that $x = 0$ implies $\dot{x} = 0$). The equation of the null cline N^2 is $y = x^2$, hence this is a parabola. Above the parabola $\dot{y} < 0$, hence trajectories are moving down, while below the parabola $\dot{y} > 0$, hence trajectories are moving up. The nullclines divide the phase plane into 8 parts. In Figure (3.2)an arrow represents the direction field in each region. The equilibria and their types can simply be determined. The point (0,0) is a saddle, the steady states (1,1) and (−1,1) are stable foci. The stable manifold of the saddle is the invariant line $x = 0$. Using the arrows of the direction field it can be shown that the two trajectories in the unstable manifold of the saddle point tend to the focus points. In a similar way, it can be shown that the trajectories starting in the right half plane tend to the stable focus (1,1) as $t \to +\infty$, and those starting in the left half plane tend to the stable focus (−1,1) as $t \to +\infty$. Hence we get the phase portrait shown in Figure (3.2). The phase portrait of the system in Example (3.2)is determined by using the direction field.

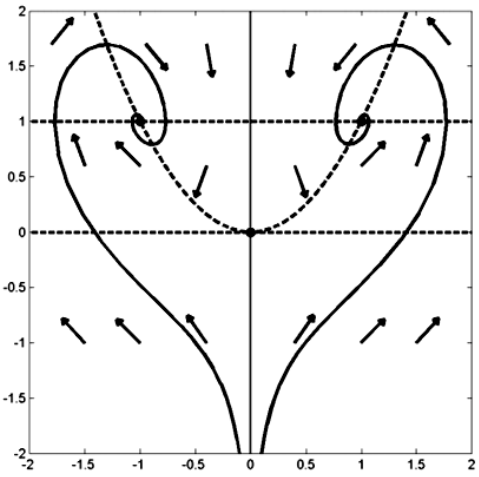
Example(3.1.5):- Consider the system

$$\dot{x} = 2x + y^2 - 1, \quad \dot{y} = 6x - y^2 + 1.$$

First, let us determine the equilibria. Adding the two equations $8x = 0$, then from the first one $y = \pm 1$. Therefore the equilibria are (0,1) and (0,−1). The Jacobian of the system is

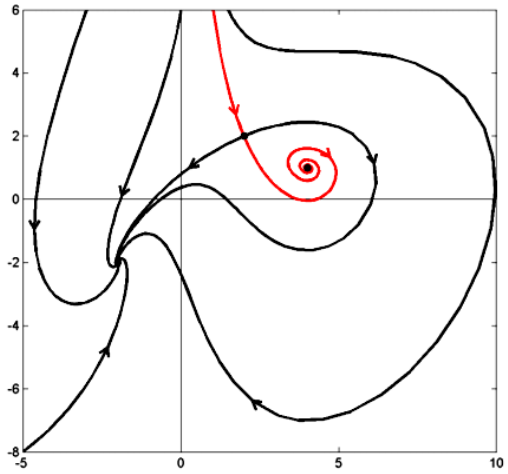
$$J = \begin{pmatrix} 2 & 2y \\ 6 & 2y \end{pmatrix}$$

At the steady state (0,1) the Jacobian is $J(0,1) = J = \begin{pmatrix} 2 & 2 \\ 6 & -2 \end{pmatrix}$, for which $Tr\big(J(0,1)\big) = 0, Det(J(0,1)) = -16$, hence (0,1) is a saddle point. At the steady state (0,−1) the



Jacobian is $\mathrm{J}(0,-1) = \begin{pmatrix} 2 & -2 \\ 6 & 2 \end{pmatrix}$, for which $\mathrm{Tr}(\mathrm{J}(0,-1)) = 4$, $Det(J(0,-1)) = 16, Tr^2(J(0,-1)) - 4Det(J(1,2)) < 0$, hence (0,−1) is an unstable focus. The equation of the null cline N_1 is $x = (1 - y^2)/2$, hence this determines a parabola dividing the plane into two regions. In the domain on the right hand side the trajectories move to the right, and in the domain on the left hand side the trajectories move to the left. The equation of the null cline $N_2$is $x = (y^2 - 1)/6$, which is also a parabola dividing the plane into two regions. In the domain on the right hand side the trajectories move up, and in the domain on the left hand side the trajectories move down. The two nullclines divide the plane into five parts. The trajectories in the unstable manifold of the saddle point tend to infinity. The trajectories in the stable manifold of the saddle tend to the unstable focus and to infinity as $t \to -\infty$. (These are shown in red in

Figure (3.6). Computing some trajectories numerically we get the phase portrait shown in Figure(3.6).



Now we study Transforming the system to polar coordinates or to a complex variable method :

Introduce the functions r and ϕ to system $\dot{x} = P(x,y), \dot{y} = Q(x,y)$ by the transformation formulas $x(t) = r(t)cos(\phi(t)), y(t) = r(t)\sin(\phi(t))$. These yield

$$\dot{x} = \dot{r}\, cos(\phi) - r\dot{\phi}\sin(\phi)\ , \qquad \dot{y} = \dot{r}\sin(\phi) + r\,\dot{\phi}cos(\phi).$$

Multiplying the first equation by $\cos(\phi)$ and the second one by $\sin(\phi)$, then adding the two equations and using the differential equations one obtains

$$\dot{r} = P(\, rcos(\phi)\cos(\phi), r\sin(\phi)) + Q(\, r\cos(\phi), r\sin(\,\phi))\sin(\phi).$$

In a similar way, multiplying the first equation by $\sin(\phi)$ and the second one by $\cos(\phi)$, then subtracting the two equations and using the differential equations one obtains

$$\dot{\phi} = Q(r\, cos(\phi), r\dot{\phi}\sin(\phi))\cos(\phi) - P(\, r\cos(\phi), r\sin(\,\phi))\sin(\phi).$$

In certain cases the differential equations obtained for r and ϕ are simple enough to determine the phase portrait belonging to them. The simplest example is the center when the equations are $\dot{x} = -y$ and $\dot{y} = x$. Then the equations for the polar coordinates take the form $\dot{r} = 0$ and $\dot{\phi} = 1$, showing that the orbits are circles centered at the origin. Another useful transformation can be

$z(t) = x(t) + iy(t)$.For this new function the differential equation takes the form

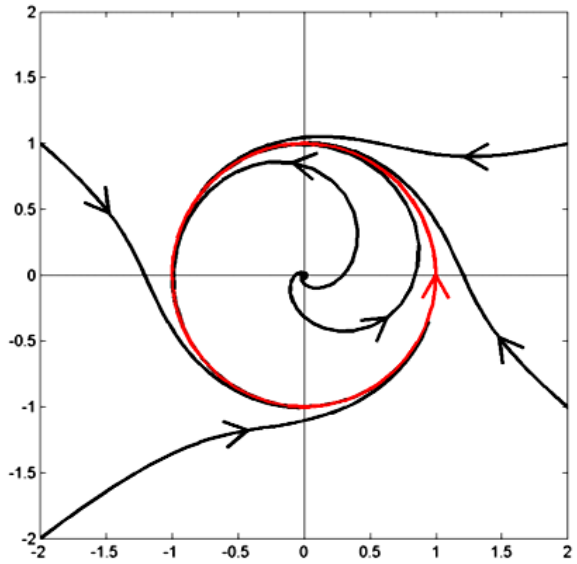
$$\dot{z} = \dot{x} + i\dot{y} = P(Re(z), Im(z)) + iQ(Re(z), Im(z)).$$

The transformation is effective if the right hand side can be expressed in terms of z without using the real and imaginary parts. The simplest example is the center when the equations are $\dot{x} = -y$ and $y = x$ yielding to $\dot{z} = iz$. The solution of this equation is $z(t) = \exp(it)$, that gives explicitly the time dependence of x and , moreover, it implies simply that $|z(t)|$ is constant in time, hence the orbits are circles centered at the origin.

Example(3.1.6):- Consider the system

$$\dot{x} = x(1 - x^2 - y^2) - y, \dot{y} = y(1 - x^2 - y^2) + x.$$

Following the above approach introduce r and ϕ by using the transformation formulas $x(t) = r(t)\cos(\phi(t)), y(t) = r(t)\sin(\phi(t))$. Then for the polar coordinates we get the differential equations $\dot{r} = r(1 - r^2)$and $\phi = 1$. Since $r = 1$implies $\dot{r} = 0$, the circle $r = 1$ is invariant and $\phi = 1$ yields that the trajectory along the circle rotate counterclockwise. If $r < 1$, then $\dot{r} > 0$, thus the radius is increasing inside the circle that is the trajectories tend to the circle. If $r > 1$, then $r < 0$, thus the radius is decreasing outside the circle that is these trajectories also tend to the circle. Therefore the phase portrait looks like as it is shown in Figure (3.7).



In the following we discuss the method of findding the First integral and Lyapunov function : The function $V: \mathbb{R}^2 \to \mathbb{R}$ is a first integral to system $\dot{x} = P(x,y), \dot{y} = Q(x,y)$, if $P\partial_1 V + Q\partial_2 V = 0$. Using a first integral the phase portrait can easily be obtained , because trajectories lie on the level curves of the first integral. As an example let us consider the Lotka–Volterra system

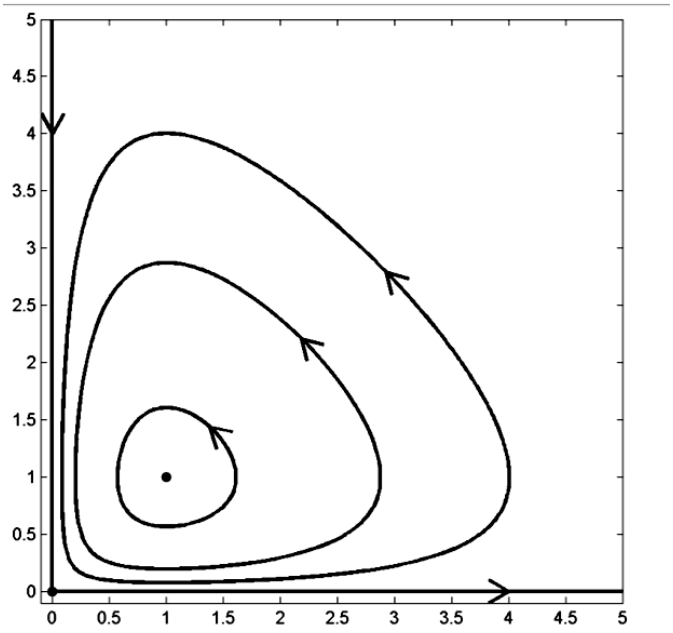
$$\dot{x} = x - xy, \qquad \dot{y} = xy - y.$$

Dividing the two equations by each other and assuming that y can be expressed in terms of x (at least in a suitable domain) we get the following differential equation for the function $x \to y(x)$: $\frac{dy}{dx} = \frac{y(x-1)}{x(1-y)}$ .This can be solved by using the method of separating the variables. Taking the terms containing y to the left hand side $\frac{1-y}{y} dy = \frac{x-1}{x} dx$.

Integrating the equation we get $\ln(|y|) - y = x - ln(|x|) + K$, where K is an arbitrary constant . Hence the function $V(x,y) = ln(|x|) - x + ln(|y|) - y$ is a first integral , that can be checked easily by differentiation (if the above derivation does not seem to be reliable).

Namely, $L_f V(x,y) = (1/x - 1)x(1-y) + (1/y - 1)y(x-1) = 0$, where $L_f V$ is the Lie-derivative of V along the vector field given by f . We note that searching for a first integral in the form $V(x,y) = F(x) + G(y)$, we arrive to

the same function V . It can be shown that the level curves of this function V in the positive quadrant are closed curves, hence the orbits with positive coordinates are periodic as it is shown in Figure(3.8).



Figure(3.8): The phase portrait of the Lotka–Volterra system.

There is no general method for finding a first integral, however, for an important class of dynamical systems, namely for Hamiltonian systems the first integral can be explicitly given.

The two dimensional system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is called Hamiltonian if there exists a differentiable function H: $\mathbb{R}^2 \to \mathbb{R}$.

For which $P = \partial_2 H$ and $Q = -\partial_1 H$. Applying the theorem about the necessary condition of the existence of primitive functions we get that if the system is Hamiltonian, then $\partial_1 P = -\partial_2 Q$, that is $\partial_1 P + \partial_2 Q = 0$, yielding that the divergence of the right hand side is zero. The Jacobian of a Hamiltonian system takes the form

$$J = \begin{pmatrix} \partial_{12} H & \partial_{22} H \\ \partial_{11} H & \partial_{21} H \end{pmatrix}$$

The trace of this matrix is $Tr(J) = 0$, hence linearisation shows that an equilibria is either a saddle or a center. On the other hand ,

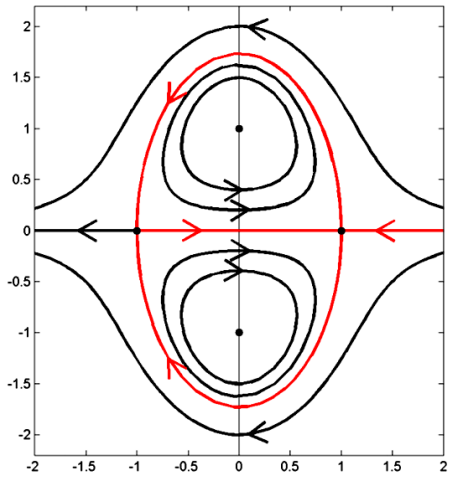
$Det(J) = Det\big(H''(x,y)\big)$hence the equilibrium is a saddle if $Det(H''(x,y)) < 0$, i.e. when $H''(x,y)$ is indefinite. If $Det(H''(x,y)) > 0$, that is (x,y) is an extremum of H, then the equilibrium is a center, because in a neighbourhood of an extremum the level curves of H are closed curves.

Example(3.1.7) :- Consider the system

$$\dot{x} = 1 - x^2 - y^2, \dot{y} = 2xy.$$

Since $\partial_1 P(x,y) + \partial_2 Q(x,y) = -2x + 2x = 0$, the system is Hamiltonian. Integrating the function P with respect to y the Hamiltonian function can be given in the form $H(x,y) = y - x^2y - y^3/3 + C(x)$, where C is an arbitrary differentiable function. Differentiating this function with respect to x and using the relation $Q = -\partial_1 H$ we get $C'(x) = 0$, hence the function C is a constant function, which can be assumed to be the zero function without loss of generality. Therefore the Hamiltonian is $H(x,y) = y - x^2y - y^3/3$. Before determining the level curves of H it is useful to find the steady states and draw the direction field. For the equilibria the second equation yields $x = 0\ or\ y = 0$. Substituting these values into the first equation one obtains $y = \pm 1\ or\ x = \pm 1$. Therefore the equilibria are (0,1), (0,−1), (1,0) and (−1,0). The type of the steady states is determined by the sign of $Det(H''(x,y))$. In this case $Det(H''(x,y)) = -4x^2 + 4y^2$. Hence (1,0) and (−1,0) are saddle points, (0,1) and (0,−1) are centers. The level curve $H = 0$ consists of two parts. On one hand it contains the x coordinate axis and on the other hand , the ellipse $1 = x^2 + y^2/3$ connecting the two saddle points in the upper and lower half plane. The level curves corresponding to negative values of H are closed curves around the centers and lying inside the ellipse. The level curves corresponding to positive values of H tend to $+\infty$ and to $-\infty$ along the x- axis and pass round the ellipse in the upper and lower half plane. The direction of the trajectories can easily be obtained from the second equation according to the sign of $\dot{y}$. An important special case of Hamiltonian systems is $\ddot{x} + U'(x) = 0$ describing a

mechanical system with one degree of freedom. The corresponding first order system takes the form $\dot{x} = y,\ \dot{y} = -U'(x)$. The first integral of this system can be given as $V\ (x, y) = y^2/2 + U(x)$, namely, its Lie derivative along the vector field is $L_f V\ (x, y) = -yU'(x) + yU'(x) = 0$.



Thus the trajectories lie on the level curves of the function . Thus in order to determine the phase portrait one has to find the level curves of V and then determine the direction of the trajectories based on the sign of $\dot{x}$ and $\dot{y}$. The level curves of V can be found by using the graph of the function U in the plane (x, U). Then taking the y-axis orthogonal to this plane we plot the parabola $y^2/2$ in the plane (y, U). Finally, moving this parabola along the graph of the function U we get a surface, which is the graph of the function V . Then the level curves of V can be simply obtained by cutting this surface by horizontal planes. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & 1 \\ -U''(x) & 0 \end{pmatrix}$$

Its trace is $Tr(J) = 0$, and determinant is $Det(J) = U''(x)$. Therefore if $U''(x) < 0$, then the steady state $(x, 0)$ is a saddle, while in the case $U''(x) > 0$ the point $(x, 0)$ is a minimum point of V , hence it is a center.

Now we study the exploting phase portrait of a vector field with symmetry method **:** The information obtained from the direction field can be supplemented by the symmetry of the vector field. One of the simplest examples is the non-linear center, the existence of which cannot be deduced from the direction field or from linearisation. In this case it may help to know that the orbits are symmetric to an axis through an equilibrium. This fact can be verified from the differential equations without knowing the solutions. Namely, let us consider a general (possibly n-dimensional) system $\dot{x}(t) = f\big(x(t)\big)$. Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a transformation taking the orbits into each other. If the mapping T maps the positive part $(t > 0)$ of an orbit starting from the point $p \in \mathbb{R}^n$ onto the negative part of the orbit starting from $T(p)$, then

$$\varphi(-t, T(p)) = T(\varphi(t, p))$$

Holds for all t. Differentiating this relation with respect to t we get $-\dot{\varphi}(-t, T(p)) = T'(\varphi(t, p))\, \dot{\varphi}(t, p)$. Then substituting $t = 0$ yields

$$-f(T(p)) = T(p) f(p).$$

In order to check this relation the solutions are not needed. Let us consider two important 2-dimensional consequences of this relation. Namely, let us formulate the condition ensuring that the trajectories are symmetric to one of the coordinate axes. The reflection to the vertical axis can be given as $T(x, y) = (-x, y)$. Its derivative is the matrix $T' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence using the above formula, the trajectories of the system $\dot{x} = P(x, y), y = Q(x, y)$ are symmetric to the vertical axis ,if

$-\big(P(-x, y), Q(-x, y)\big) = T'\big(P(x, y), Q(x, y)\big)$, that is

$$P(-x, y) = P(x, y), \qquad -Q(-x, y) = Q(x, y).$$

The reflection to the horizontal axis can be given as $T(x, y) = (x, -y)$. Its derivative is the matrix $T' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence using the above formula, the

trajectories of the system $\dot{x} = P(x,y), \dot{y} = Q(x,y)$ are symmetric to the horizontal axis, if $-(P(x,-y), Q(x,-y)) = T(P(x,y), Q(x,y))$, that is

$$-P(x,-y) = P(x,y), Q(x,-y) = Q(x,y).$$

Example(3.1.8) :- Consider the system

$$\dot{x} = 1 - x^2 - y^2, \qquad \dot{y} = 2x.$$

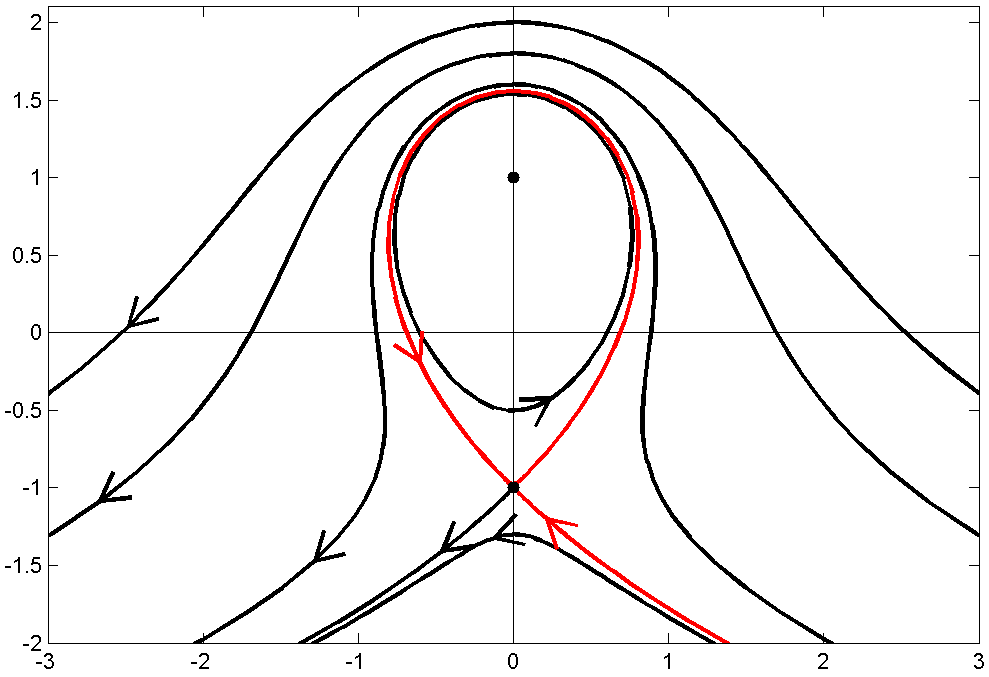
Let us find first the equilibria. From the second equation $x = 0$, then the first one implies $y = \pm 1$. Therefore the equilibria are (0,1) and (0,−1). The Jacobian of the system is

$$J = \begin{pmatrix} -2x & -2y \\ 2 & 0 \end{pmatrix}$$

At the point (0,1) the Jacobian is $J(0,1) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, for which $Tr(J(0,1)) = 0, Det(J(0,1)) = 4, Tr^2(J(0,1)) - 4Det(J(0,1)) < 0$, hence the type of the steady state(0,1) cannot be decided by using linearisation (the eigenvalues are pure imaginary). At the point (0,−1) the Jacobian takes the form $J(0,-1) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, for which $Tr(J(0,-1)) = 0, Det(J(0,-1)) = -4$, hence (0,−1) is a saddle point.

The equation of the nullcline N_1 is $x^2 + y^2 = 1$, hence this null cline is a circle centered at the origin, dividing the plane into two parts. In the outer part the trajectories move to the left, while inside the circle they move to the right. The nullcline N_2 is the vertical axis, dividing the plane into two parts. In the left half plane $\dot{y} < 0$, hence trajectories are moving down there, in the right half plane $\dot{y} < 0$, hence trajectories are moving up there. The nullclines divide the phase plane into four parts. In Figure (3.11) an arrow shows the direction of the trajectories in each part. One of the trajectories in the unstable manifold of the saddle point passes round the point (0,1), the other one tends to −∞. One of the trajectories in the stable manifold of the saddle point passes round the point (0,1), the other one comes from −∞. For the full characterization of the

behaviour of the trajectories observe that the phase portrait is symmetric to the vertical axis.This is shown by the fact that $P(-x, y) = P(x, y)$ and $-Q(-x, y) = Q(x, y)$ hold. Hence the equilibrium (0,1) is surrounded by periodic orbits, i.e. it is a center and a trajectory in the stable and in the unstable manifold of the saddle point forms a homoclinic orbit. Computing some trajectories numerically we get the phase portrait shown in Figure(3.9).



The phase portrait of the system in Example (3.8).

The above examples show that for two dimensional systems the global phase portrait cannot be typically obtained from the local phase portraits. Namely, there global structures may occur. In two dimensional systems these can be :

- periodic orbits
- homoclinic orbits
- heteroclinic orbits.

Section (3.2): Periodic orbits

Consider the system of differential equations

$$\dot{x}(t) = f(x(t)),$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. The solution starting from the point p is denoted by $\varphi(t, p)$.

Definition(3.2.1):- The point p is called a periodic point if there exists $T > 0$, for which $\varphi(T, p) = p$ and p is not an equilibrium. The orbit of p is called a periodic orbit, the smallest value of T is called the period of the orbit.

Remark (3.2.2): (Existence of periodic orbits)

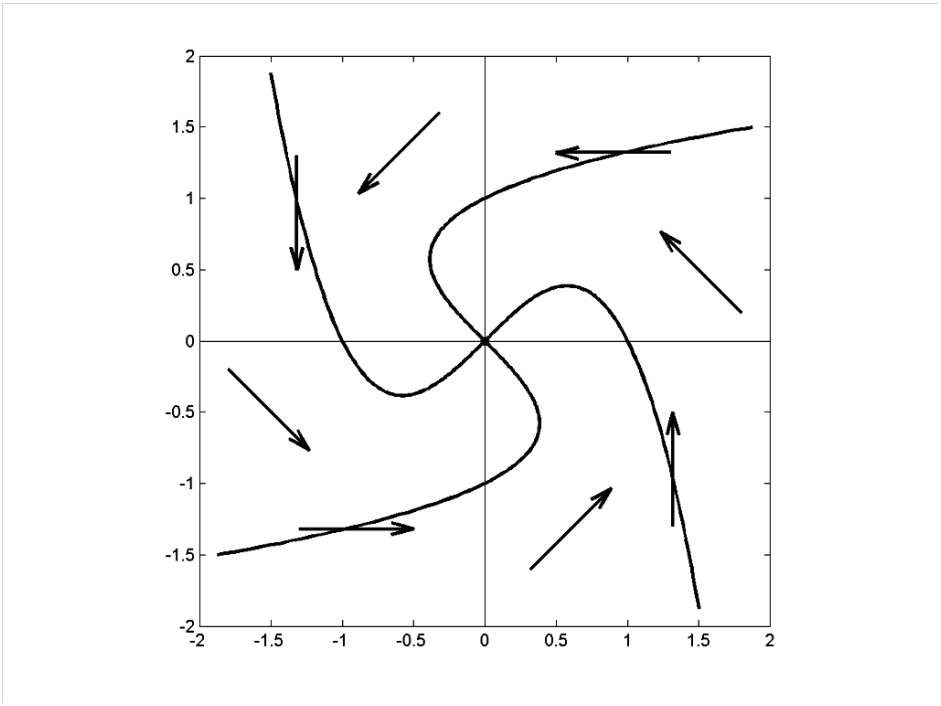
The existence of periodic orbits in two dimensional systems can be verified by using the Poincar´e-Bendixson theorem .Before stating the theorem a motivating example is shown to illustrate the key idea of the proof.

Example(3.2.3) :-

Consider the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3.$$

Plotting the nullclines $y = x - x^3$ and $x = y^3 - y$ one can see that the trajectories rotate around the origin, however, it cannot be directly seen if they are tending to the origin or to infinity ,see Figure (3.10).



Figure(3.10):The nullclines and the direction field of the system in Example (3.2.3). In order to investigate this question introduce the Lyapunov function

$$V\,(x,y):=\ x^2+y^2.$$

For an arbitrary solution $(x(t),y(t))\ let\ V^*\,(t)\ =\ x^2(t)\ +\ y^2(t)$. Then using the differential equations $V^*(t)\ =\ 2\big(x^2(t)+y^2(t)-x^4(t)-y^4(t)\big)$.The sign of this expression can help in understanding the behavior of the trajectories.

The following two statements can be proved by elementary calculations $if x^2+y^2<1$, then $x^2+y^2-x^4-y^4>0$, and $x^2+y^2>2$ implies $x^2+y^2-x^4-y^4<0$. This means that the trajectories are spiraling away from the origin inside the unit disc, while they are spiraling inward outside the circle of radius $\sqrt{2}$. Hence the annular domain bordered by the circles of radius 1 and $\sqrt{2}$ is positively invariant, i.e. trajectories cannot leave this region. Consider the segment of the horizontal axis with end points 1 and $\sqrt{2}$ an define the Poincar´e map $P\colon[1,\sqrt{2}]\ \rightarrow\ [1,\sqrt{2}]$ in this set as follows. For a given $q\ \in\ [1,\sqrt{2}]\ let\ P(q)\in[1,\sqrt{2}]$ be the point to which the trajectory starting from q returns after one rotation, see the schematic Figure (3.11).

It can be proved that the Poincare map is continuous, hence it maps the interval $[1,\sqrt{2}]$ into itself continuously, thus it has a fixed point $p\ \in\ [1,\sqrt{2}]$, for which $P(p)=p\ holds$.

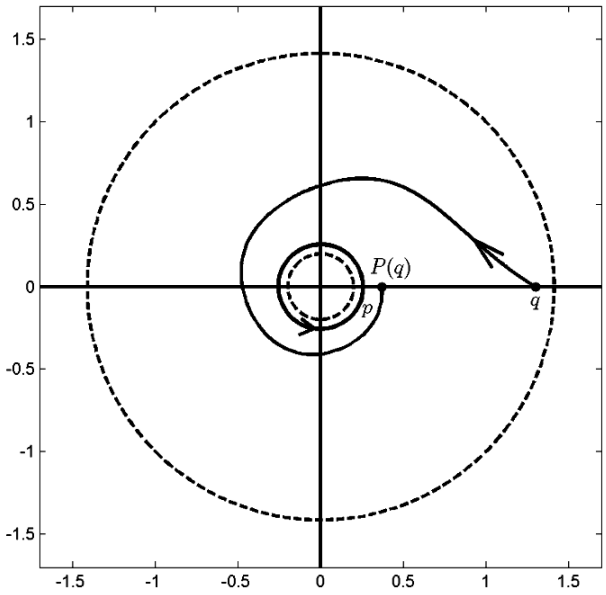


Figure (3.11): The Poincare map and a periodic orbit starting from p.

This point is a periodic point, because the trajectory starting from this point returns to the same point .Generalising the idea used in this example, the following theorem can be proved .

Theorem :(3.2.4) (Poincare –Bendixson (weak form)). Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function and consider the two dimensional system $\dot{x}(t) = f(x(t)).If\ K \subset \mathbb{R}^n$ is a positively invariant, compact set that contains no equilibria, then there is a periodic orbit in it .

Lemma(3.2.5):- Let $D \subset \mathbb{R}^2$ be a simply connected open set and $G : D \to \mathbb{R}^2$ be a differentiable function . Then

$$\int_D (\partial_1 G_1 + \partial_2 G_2) = \int_{\partial D} (-G_2 , G_1) .$$

Using this lemma we prove first Bendixson's criterion.

Theorem(3.2.6) : (Benison's criterion)

Assume that $H \subset \mathbb{R}^2$ is a simply connected open domain, in which the divergence $divf = \partial_1 f_1 + \partial_2 f_2$ has a given sign and it is zero at most in the

points of a curve. Then system $x(t) = f(x(t))$ has no periodic orbit lying completely in H.

Proof:

Assume that there is a periodic orbit Γ in the set H. Denote by D the interior of Γ. Applying Stokes's theorem for the function f in the domain D we get a contradiction. Namely, the left hand side has a given sign (according to the sign of the divergence), while the right hand side is zero that can be proved as follows. The vector $(-f_2, f_1)$ is orthogonal to the curve ∂D, because this curve is the periodic orbit itself. Hence denoting the periodic orbit by $\gamma(t)$ the right hand side in Stokes's theorem takes the form

$$\int_{\Gamma}(-f_2, f_1) = \int_0^T \left(-f_2\big(\gamma(t)\big) \cdot \dot{\gamma}_1(t) + f_1\big(\gamma(t)\big) \cdot \dot{\gamma}_2(t)\right) dt$$

$$= \int_0^T (-\dot{\gamma}_2\dot{\gamma}_1 + \dot{\gamma}_1\,\dot{\gamma}_2)\, dt = 0.$$

This theorem can be generalised to derive the Bendixson–Dulac criterion.

Theorem(3.2.7): (Bendixson–Dulac criterion)

Let $H \subset \mathbb{R}^2$ be a simply connected open domain and let $B: H \to \mathbb{R}$ be a differentiable function, for which $\operatorname{div}(Bf)$ has a given sign and it is zero at most in the points of a curve. Then system $\dot{x}(t) = f(x(t))$ has no periodic orbit lying completely in H.

Example(3.2.8) :-

Consider the differential equation

$$\dot{x} = x - xy^2 + y^3$$

$$\dot{y} = 3y - x^2y + x^3.$$

Then $\partial_1 f_1(x, y) + \partial_2 f_2(x, y) = 1 - y^2 + 3 - x^2 = 4 - (x^2 + y^2)$, which is positive inside the circle of radius 2 and centered at the origin. Hence in this disk the system has no periodic orbit.

Definition(3.2.9):-

Let P and Q be polynomials of degree . How many limit cycles can be at most in the following system :

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

In the case $n = 1$, i.e. when the system is linear, there is no limit cycle (periodic orbits may exist only in the case of a center, however these periodic orbits are not isolated).

Definition (3.2.10):-

Let $L \subset \mathbb{R}^n$ be an $(n-1)$ -dimensional hyperplane with normal ν . Aconnected subset $\Sigma \subset L$ is called a transversal section , if $\langle \nu, \partial_t \varphi(0, q) \rangle \neq 0$ for all $q \in \Sigma$.

Let Σ be a transversal section containing p in its interior. Using the implicit function theorem the following statement can be shown.

Proposition(3.2.11): (Existence of the Poincare map)

The point p has a neighbourhood $U \subset \Sigma$ and there exists a continuously differentiable function $\Theta: U \to \mathbb{R}$, for which $\Theta(p) = T$ and $\varphi(\Theta(q), q) \in \Sigma$, for all $q \in U$.

Definition(3.2.12):-

The function $P: U \to \Sigma, P(q) = \varphi(\Theta(q), q)$ is called the Poincar´e map (belonging to the transversal section Σ).

The Poincare map depends on the choice of the point p and that of the transversal section Σ. The next proposition shows that different Poincare maps can be transformed to each other by a suitable coordinate transformation.

Proposition(3.2.13) :- Let $p_1, p_2 \in \Gamma$, and let Σ_1, Σ_2 be transversal sections containing the corresponding points . Let $U_i \subset \Sigma_i$ $(i = 1,2)$ be open sets , in which the Poincare maps $P_i: U_i \to \Sigma_i (i = 1,2)$ are defined. Then there exists a neighbourhood $V \subset U_1$ of the point P_1 and there exists a differentiable function $S: V \to U_2$, for which $P_2(S(p)) = S((P_1(p))$ holds for all $p \in V$.

Corollary(3.2.14):- The eigenvalues of the matrices $P_1'(p_1)$ and $P_2'(p_2)$ coincide .

If $P(U) \subset U$, then the Poincar´e map P defines a discrete time dynamical system (more precisely a semi-dynamical system) $\psi: N \times U \to U$ in the following way. Let $\psi(n,q) = P^n(q)$, where $P^n = P \circ P \circ \ldots \circ P$ denotes the composition with n terms. (The notion semi-dynamical system is used, because it is not defined for negative values of n.) since $P(p) = p$, the point p is a steady state of the dynamical system ψ. It will be shown that its stability determines that of the periodic orbit Γ. Hence we will recall the definition of stability for a fixed point of a discrete time dynamical and the results about linearisation at a fixed point. Let $G \subset \mathbb{R}^k$ be an open set and let $g: G \to G$ be a diffeomorphism.

Then g defines a discrete time dynamical system $\psi: \mathbb{Z} \times G \to G$, by the definition $\psi(n,q) = g^n(q)$, where $g^{\mathrm{n}} = g \circ g \circ \ldots \circ$ is a g composition of n terms. If $n < 0$, then let $g^n = g^{-1} \circ g^{-1} \circ \ldots \circ g^{-1}$ be a composition of $-n$ terms.

Definition (3.2.15):-

The point $p \in G$ is an equilibrium or fixed point of the dynamical system ψ, if $g(p) = p$ (that is $\psi(n,p) = p$, for all $n \in \mathbb{Z}$).

Definition (3.2.16):-

The fixed point p is called stable, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that $|q - p| < \delta$ and $n \in N$ imply $|g^n(q) - p| < \varepsilon$. The fixed point p is called unstable, if it is not stable. The fixed point p is called asymptotically stable, if it is stable and $|q - p| < \delta$ implies $\lim_{n\to\infty} g^n(q) = p$.

Theorem (3.2.17): (Stability of the fixed point)

If $|\lambda| < 1$ for all eigenvalues of the matrix $g'(p)$, then the fixed point p is asymptotically stable.

Let us return now to the investigation of the Poincare map $P: U \to \Sigma$. It will be studied how the stability of a fixed point of the Poincare map determines the stability of the periodic orbit Γ containing the given fixed point . A first, and important observation is that the periodic orbit cannot be asymptotically stable, because the distance between two solutions starting from two different points of the periodic orbit does not tend to zero. Despite of this fact, trajectories may converge to the periodic orbit, hence it can be stable as a set, in a suitable sense. For example, it can be an ω limit set or an attractor. Therefore in order to characterize its stability the notion of orbital stability will be introduced. The following usual definition for the distance of a set and a point is used.

$$d(q, \Gamma) = \inf\{|q - \gamma(t)| : t \in [0, T]\}.$$

Definition(3.2.17) :-

The periodic solution γ is called orbitally stable, if for all $\varepsilon > 0$ there exists $\delta > 0$, for which $d(q, \Gamma) < \delta$ and $t \geq 0$ imply $d(\varphi(t, q), \Gamma) < \varepsilon$.

The periodic solution γ is called orbitally asymptotically stable, if it is orbitally stable and

$$\lim_{t \to \infty} d(\varphi(t, q), \Gamma) = 0 .$$

The periodic orbit Γ is a limit cycle, if there is a point $q \notin \Gamma$, for which $\Gamma \subset \omega(q)$ or $\Gamma \subset \alpha(q)$. The periodic orbit Γ is a stable limit cycle, if γ orbitally asymptotically stable.

Let Σ be a transversal section containing the point p, let $U \subset \Sigma$ be a neighbourhood of p and let $P: U \to \Sigma$ be the Poincar´e map.

Theorem (3.2.18):-

If p is a (asymptotically) stable fixed point of the Poincare map p, then the periodic solution γ is orbitally (asymptotically) stable. Thus the stability of the periodic orbit is determined by the eigenvalues of the derivative of the Poincar´e map at its fixed point.

Definition(3.2.19):-The eigenvalues of the $((n-1)\times(n-1))$ matrix $P'(p)$ are called the characteristic multipliers of the periodic orbit Γ.

Theorem(3.2.20) :(Andronov–Witt)

If the absolute values of all characteristic multipliers of Γ are less than 1, then Γ is a stable limit cycle.

For two dimensional systems there is a single characteristic multiplier P′(p). It can be shown, that this can be given in the form

$$P'(p) = exp\left(\int_0^T divf\ (\gamma(t))\right),$$

where $\operatorname{div} f = \partial_1 f_1 + \partial_2 f_2$ denotes the divergence of f. Thus, if the divergence is negative along the periodic orbit, then it is a stable limit cycle, while if it is positive along the periodic orbit, then it is an unstable limit cycle.

In the two dimensional case the periodic orbit has only one characteristic multiplier and an invariant manifold that is either stable or unstable. In higher dimension it may happen that the periodic orbit is attracting in one direction and repelling in another direction, hence the local phase portrait has a saddle-like structure. In that case the local behaviour of trajectories in a neighbourhood of the periodic orbit can be characterised by the dimension of the stable, unstable and center manifolds. The introduction of invariant manifolds of periodic orbits is illustrated first with the following example.

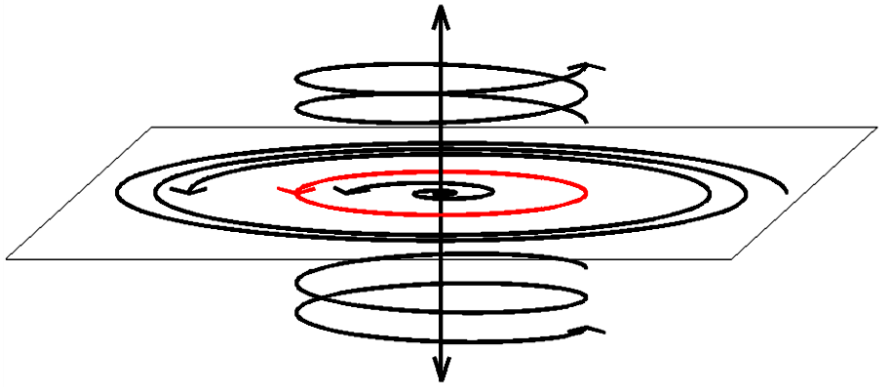
Example(3.2.21) :- Consider the following three variable system

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$
$$\dot{z} = z$$

It can be easily seen that the horizontal (x, y) plane and the vertical z axis are invariant, that is trajectories do not leave them. The unit circle centered at the origin and lying in the horizontal plane is a periodic orbit. The trajectories starting in the horizontal plane tend to this limit cycle. The vertical cylinder containing this circle is also invariant, however, the trajectories moving in this

cylinder spiral towards infinity, because in the upper half space z is increasing and in the lower half space it is decreasing. Therefore the stable manifold of the periodic orbit is the horizontal plane and its unstable manifold is the vertical cylinder as it is shown in Figure (3.11).

Let us turn now to the definition of the stable, unstable and center manifolds of a periodic orbit. Let $M \subset \mathbb{R}^n$ be a connected open set and let $\varphi: \mathbb{R} \times M \to M$ be a dynamical system. Let $p \in M$ be a periodic point with period T, that is $\varphi(T,p) = p$. The notations $\gamma(t) := \varphi(t,p)$ and $\Gamma := \{\gamma(t): t \in [0,T]\}$ are used again. The characteristic multipliers of the periodic orbit Γ are denoted by $\mu_1, \mu_2, \ldots, \mu_{n-1}$.



Theorem(3.2.22) :(Stable and unstable manifold theorem)

Assume that there are k characteristic multipliers of Γ within the unit disk (that is having absolute value less than one) and outside the unit disk there are $n-1-k$ (having absolute value greater than one). Then for some $\delta > 0$ *let* $W_s^{loc}(\Gamma)$ be the set of those points $q \in \mathbb{R}^n$, for which

1. $d(\varphi(t,q),\Gamma) \to 0 \; as \; t \to \infty$,
2. $d(\varphi(t,q),\Gamma) < \delta, if \; t \geq 0$.

Moreover, let $W_u^{loc}(\Gamma)$ be the set of those points q $\in \mathbb{R}^n$, for which

1. $d(\varphi(t,q),\Gamma) \to 0$ as $t \to -\infty$,
2. $d(\varphi(t,q),\Gamma) < \delta$, if $t \leq 0$.

Then there exist $\delta > 0$, such that $W_s^{loc}(\Gamma)$ is a $k+1$ dimensional positively invariant differentiable manifold and $W_u^{loc}(\Gamma)$ is an $(n-k)$-dimensional negatively invariant differentiable manifold, that are called the stable and unstable manifolds of the periodic orbit Γ. These manifolds intersect transversally at the points of Γ.

The global stable and unstable manifolds can be defined similarly to the case of equilibria, using the local stable and unstable manifolds $W_s^{loc}(\Gamma)$ and $W_u^{loc}(\Gamma)$. The stable manifold is defined in such a way that the trajectories starting in that manifold tend to the periodic orbit Γ. Since these trajectories are in the local stable manifold in a neighbourhood of the periodic orbit, the global stable manifold can be defined by

$$W_s(\Gamma) := \bigcup_{t \le 0} \varphi(t, W_s^{loc}(\Gamma)(\Gamma)) \ .$$

The unstable manifold is defined in such a way that the trajectories starting from it tend to Γ $as\ t \to -\infty$. Since these trajectories are in the local unstable manifold in a neighbourhood of the periodic orbit, the global unstable manifold can be defined by

$$W_u(\Gamma) := \bigcup_{t \ge 0} \varphi(t, W_u^{loc}(\Gamma)) \ .$$

If there are characteristic multipliers with absolute value 1, then besides these invariant manifolds the periodic orbit has a center manifold denoted by $W_c(\Gamma)$. Its dimension is $m+1$ if there are m characteristic multipliers along the unit circle.

Example(3.2.23):- Consider the following three variable system

$$\dot{x} = -y$$
$$\dot{y} = x$$
$$\dot{z} = -z$$

It can be easily seen that the horizontal plane (x, y) and the vertical axis z are invariant, that is trajectories do not leave them . In the horizontal plane the

orbits are circles centered at the origin. Consider the unit circle as a periodic orbit. Its center manifold is the horizontal plane, and the stable manifold is the vertical cylinder containing the circle. The later statement is proved by the fact that the cylinder is invariant and for positive values of z its time derivative is negative, while for negative values the time derivative is positive. These invariant manifolds are shown in Figure (3.12).

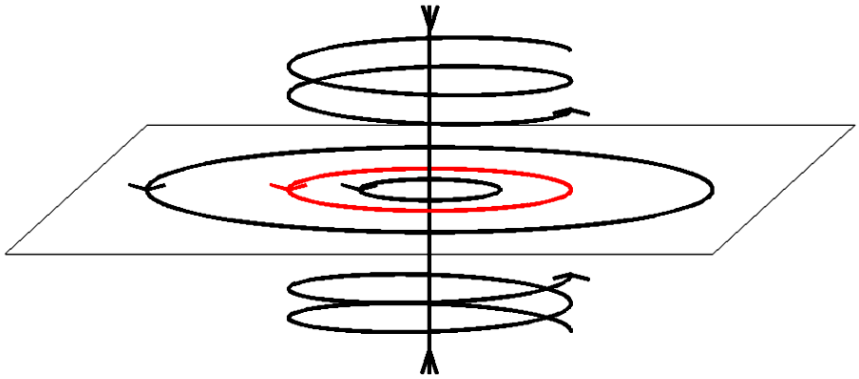


Figure (3.12):Invariant manifolds for the differential equation in Example (3.2.23).

Section(3.3): Index theory for two dimensional systems

An important tool for investigating the global phase portrait in two dimension is the index of the vector field. The index is a topological invariant that can be defined also in higher dimension, however it is dealt with here only for two dimensional systems. Consider the system

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

and take a continuous simple closed curve $\gamma: [a, b] \to \mathbb{R}^2$ (it is not assumed to be an orbit of the differential equation). The index of γ, denoted by ind (γ) is an integer that gives the number of rotations of the vector field $(P(\gamma(s)), Q(\gamma(s)))$ while s moves in the interval $[a, b]$. It can be seen easily that the index of a curve making a round around a (stable or unstable) node is 1, while that around a saddle point the index is -1. These statements can be proved also formally once the index is defined by a formula. This is what will be shown now.

Denote by $\Theta(x,y)$ the angle of the vector $\big(P(x,y),Q(x,y)\big)$ with the x coordinate axis at a given point (x,y). Given a curve $\gamma\colon [a,b]\to\mathbb{R}^2$

let $\Theta^*(s) = \Theta(\gamma(s)) for\ s\in[a,b]$. Then $\tan\Theta^*(s) = \frac{Q(\gamma(s))}{P(\gamma(s))}$.

Differentiating with respect to s one obtains

$$\frac{1}{\cos^2\Theta^*(s)}\cdot\dot{\Theta}^*(s) = \frac{(\partial_1 Q(\gamma(s))\cdot\dot{\gamma}_1(s)+\partial_2 Q(\gamma(s))\cdot\dot{\gamma}_2(s))\,P(\gamma(s))}{P^2(\gamma(s))}$$

$$-\frac{(\partial_1 P(\gamma(s))\cdot\dot{\gamma}_1(s)+\partial_2 P(\gamma(s))\cdot\dot{\gamma}_2(s))\,Q(\gamma(s))}{P^2(\gamma(s))}$$

Using that

$$\frac{1}{\cos^2\Theta^*} = \mathrm{tg}^2\,\Theta^* + 1 = \frac{Q_2}{P_2} + 1 = \frac{Q^2+P^2}{P^2}$$

the derivative $\dot{\Theta}^*$ can be expressed as

$$\dot{\Theta}^*(s) = \frac{P(\partial_1 Q\dot{\gamma}_1+\partial_2 Q\dot{\gamma}_2) - Q(\partial_1 P\dot{\gamma}_1+\partial_2 P\dot{\gamma}_2)}{P^2+Q^2}$$

The rotation of the vector (P(γ(s)),Q(γ(s))) can be given by the integral

$$\frac{1}{2\pi}\int_a^b \dot{\Theta}^*(s)ds$$

hence the index of the curve can be defined as follows.

Definition (3.3.1):-

Let $\gamma\colon[a,b]\to\mathbb{R}^2$ be a continuous simple closed curve, that do not pass through any equilibrium .Then its index with respect to the system $\dot{x} = P(x,y),\dot{y} = Q(x,y)\ is$

$$ind(\gamma) = \frac{1}{2\pi}\int_a^b \frac{P(\partial_1 Q\dot{\gamma}_1+\partial_2 Q\dot{\gamma}_2) - Q(\partial_1 P\dot{\gamma}_1+\partial_2 P\dot{\gamma}_2)}{P^2+Q^2}$$

Example(3.3.2) :-Compute the index of a curve encircling an unstable node. The origin is an unstable node of the system

$$\dot{x} = x$$

$$\dot{y} = y.$$

Let $\gamma(t)$=(cost,sint), $t \in [0,2\pi]$ be the parametrisation of the unit circle centered at the origin , therefore its rotation is 2π, hence the index of the curve is 1. This can be computed also by the formal definition as follows. In this case

$$P(x,y) = x\,, \partial_1 P = 1, \partial_2 P = 0$$

$$Q(x,y) = y\,, \partial_1 Q = 0, \partial_2 Q = 1$$

and $\dot{\gamma}_1$ = −sint, $\dot{\gamma}_2$= cost. Hence the definition yields

$$\text{ind}(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\text{cost}(\text{cost}) - \text{sint}(-\text{sint})}{\cos^2 t + \sin^2 t}\,dt = 1\,.$$

Example(3.3.3) :-

Compute the index of a curve encircling a stable node. The origin is a stable node of the system

$$\dot{x} = -x$$

$$\dot{y} = -y.$$

Let $\gamma(t)$ = (cost,sint), $t \in [0,2\pi]$ be the parametrisation of the unit circle centered at the origin. Therefore its rotation is 2π , hence the index of the curve is 1. This can be computed also by the formal definition as follows. In this case

$$P(x,y) = -x, \partial_1 P = -1, \partial_2 P = 0$$

$$Q(x,y) = -y, \partial_1 Q = 0, \partial_2 Q = -1$$

and $\dot{\gamma}_1$ = −sint, $\dot{\gamma}_2$= cost. Hence the definition yields

$$ind(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{-\text{cost}(-\text{cost}) + \text{sint}(\text{sint})}{\cos^2 t + \sin^2 t}\,dt = 1$$

Proposition(3.3.4):- For an arbitrary curve γ and for an arbitrary vector field (P, Q) the following statements hold:

1. The index $ind(\gamma)$ depends continuously on the curve γ, if it does not pass through an equilibria.
2. The index $ind(\gamma)$ depends continuously on the functions P and Q, if the curve does not pass through an equilibria.

Since the index is an integer value, the continuous dependence implies the following .

Corollary (3.3.5):-

For an arbitrary curve γ and vector field (P, Q) the index $ind(\gamma)$ is constant as the curve or the vector field is changed, if the curve does not pass through an equilibria of the vector field.

This corollary enables us to prove global results about the phase portrait.

Proposition(3.3.6):-

If there is no equilibrium inside γ, then $ind(\gamma) = 0$.

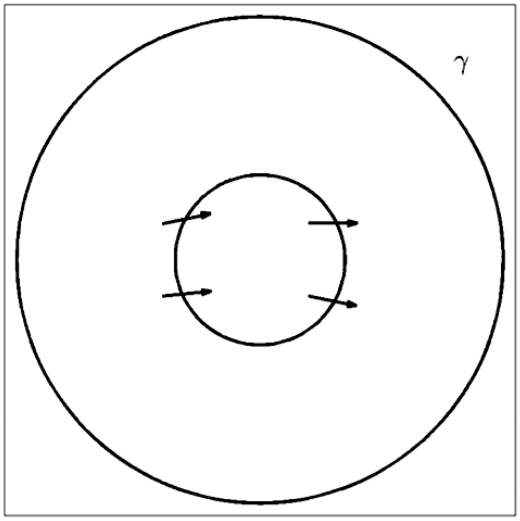


Figure (3.13): The index of a curve is zero, if it does not contain any equilibrium in its interior.

Proof :

If there is no equilibrium inside γ, then it can be shrunk to a single point without crossing an equilibrium. In Figure (3.13) one can see that if γ is shrunk to a certain small size then the rotation of the vector field along that curve is 0, hence the index of the curve is 0. Since the index is not changed as the curve is shrunk, the index of γ is also 0.

The corollary above also enables us to define the index of an equilibrium.

Definition(3.3.7):-

Let (x_0, y_0) be an isolated equilibrium of system $\dot{x} = P(x, y)\ \dot{y} = Q(x, y)$. Then the index of this steady state is defined as the index of a curve encircling

(x_0, y_0) but not containing any other equilibrium in its interior. (According to the corollary this is well-defined.)

Proposition (3.3.8):-

The index of a saddle point is −1, while the index of a node or a focus is 1.

The following proposition can also be proved by varying the curve continuously.

Proposition (3.3.9):-

Let (x_i, y_i), $i = 1,2,\dots k$ be the equilibria in the interior of the curve γ. Then the index of the curve is equal to the sum of the indices of the equilibria, that is

$$ind(\gamma) = \sum_{i=1}^{k} ind\ (x_i, y_i).$$

It can be seen in Figure (3.14) that in the case when γ is a periodic orbit, the rotation of the vector field along gamma is 2π, yielding the following.

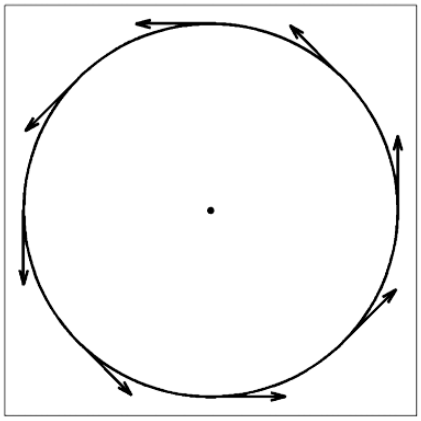


Figure (3.14): Computing the index of a periodic orbit.

Proposition(3.3.10):-

If γ is a periodic orbit, then $ind(\gamma) = 1$.

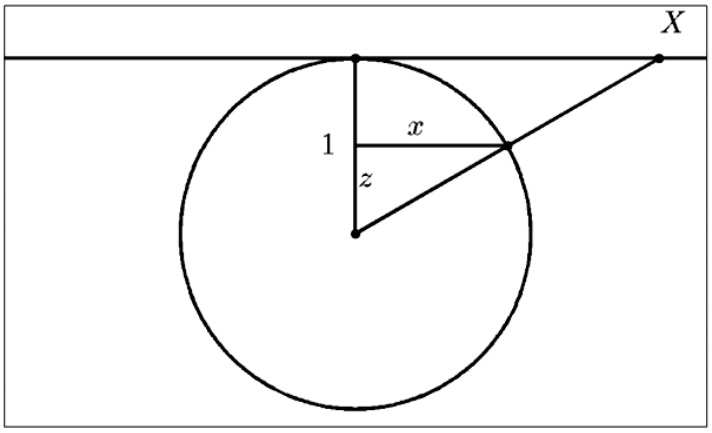
Corollary(3.3.11):-

If γ is a periodic orbit, then it contains at least one equilibrium in its interior. Moreover, the sum of the indices of these equilibria is 1.

We note that the index can be defined also for differential equations on other two dimensional manifolds, for example, on the sphere, or torus. The Poincar´e's index theorem states that the sum of the indices on a compact manifold is equal to the Euler characteristic of the manifold, which is 2 for the sphere and 0 for the torus. Thus the sum of the indices of equilibria on the sphere is 2.

Proposition(3.3.12): (Behaviour of trajectories at infinity)

One of the difficulties of classifying two dimensional phase portraits is caused by non- compactness of the phase plane, that makes it difficult to catch the behaviour of trajectories at infinity. This problem can be handled by projecting the phase plane to a compact manifold. Here the Poincare center projection is presented that maps the plane to the upper hemisphere and takes the points at infinity to the equator of the sphere. By using that the structure of the phase portrait at infinity can be studied in detail. Let us consider the unit sphere centered at the origin and a plane tangential to the sphere at the point (0,0,1) (i.e. at the north pole). Let the origin of the plane be at the tangent point, the coordinates in the plane are denoted by (x, y). The space coordinates are denoted by X, Y, Z. The points of the plane are projected to the upper hemisphere as follows. Take a point (x, y) in the plane, connect to the origin of the space and associate to (x, y) the intersection point of the ray (from the origin to (x, y)) and the upper hemisphere. In Figure (3.15) there are similar triangles leading to the coordinate transformation $x = X/Z$.



Figure(3.15): The projection of the plane to the upper hemisphere.

Thus let us introduce the space coordinates X, Y, Z in terms of the (x, y) coordinates of the plane as follows

$$X = xZ$$

$$Y = yZ$$

$$Z = \frac{1}{\sqrt{x^2 + y^2 + 1}}$$

We will now investigate how this coordinate transformation effects the phase portrait of the differential equations

$$\dot{x} = P(x, y) \qquad (3.15)$$

$$\dot{y} = Q(x, y) \qquad (3.16)$$

Differentiating the identities defining the projection leads to the system of differential equations

$$\dot{X} = \dot{x}Z + x\dot{Z}$$

$$\dot{Y} = \dot{y}Z + y\dot{Z}$$

$$\dot{Z} = -Z^3(x\,\dot{x} + y\,\dot{y}).$$

Substituting the differential equations (3.15)-(3.16) into these equations and using the equations of the transformation one obtains

$$\dot{X} = Z \cdot P\left(\frac{X}{Z}, \frac{Y}{Z}\right) - Z \cdot X \cdot \left(XP\left(\frac{X}{Z}, \frac{Y}{Z}\right) + YQ\left(\frac{X}{Z}, \frac{Y}{Z}\right)\right)$$

The differential equations for Y and Z can be obtained similarly. For the sake of simplicity we write P instead of $\left(\frac{X}{Z}, \frac{Y}{Z}\right)$, and similarly for Q. Then the differential equations for the new variables take the form

$$\dot{X} = ZP - ZX(XP + YQ) \qquad (3.17)$$

$$\dot{Y} = ZQ - ZY(XP + YQ) \qquad (3.18)$$

$$\dot{Z} = -Z^3(XP + YQ) \qquad (3.19)$$

It can easily be checked that the unit sphere is invariant for this system. The phase portrait on the sphere is equivalent to that on the plane (x,y). However, we can see the trajectories at infinity on the sphere, these are the trajectories moving along the equator, i.e. in the plane $Z = 0$.It will be shown for some linear systems how the known phase portrait can be extended to infinity.

Example (3.3.13) :-

Consider the linear system

$$\dot{x} = x$$

$$\dot{y} = -y$$

with a saddle point. Then $P(x, y) = x, Q(x, y) = -y$, hence $ZP = X, ZQ = -Y$ and $Z(XP + YQ) = X^2 - Y^2$. Substituting these into system (3.17)-(3.19), one obtains the system

$$\dot{X} = X(1 - X^2 + Y^2)$$

$$\dot{Y} = Y(-1 - X^2 + Y^2)$$

$$\dot{Z} = -Z(X^2 - X^2)$$

For the new variables. Let us investigate the phase portrait on the sphere. The finite equilibria are (0,0,1) and (0,0,−1). These are saddle points, because the original point in the plane is also a saddle. Let us turn now to the equilibria at infinity. The third coordinate of these is $Z = 0$, hence these are given by equation $X^2 + Y^2 = 1$. This equation yields $X = 0$ and $Y = \pm 1$ or $X = \pm 1$ and $Y = 0$. Hence there are four equilibria at infinity, namely (1,0,0), (−1,0,0),

(0,1,0), (0,−1,0). Their type can be obtained as follows. In the point (1,0,0) project the equation to the plane $X = 1$, then we get the two variable system

$$\dot{Y} = Y(-2 + Y^2)$$

$$\dot{Z} = Z(1 - Y^2)$$

Its steady state is (0,0) (corresponding to (1,0,0)). The Jacobian of the system at (0,0) is

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

hence (0,0) is a stable node, implying that (1,0,0) is a stable node. The remaining three steady states can be studied in a similar way. The point (−1,0,0) is also a stable node, while the points (0,1,0) and (0,−1,0) are unstable nodes.(The phase portrait is projected from the upper hemisphere to the plane of the equator vertically).One can observe that there are 6 equilibria in the whole sphere all together. Two of them are saddles (with index -1) and the remaining four are nodes (with index 1) .

Hence the sum of the indices is $2(-1) + 4 = 2 = \chi(S^2)$ yielding the Euler characteristic of the sphere, as it is stated by Poincare's index theorem.

Chapter 4

Introduction to bifurcation theory and structural stability

In the previous chapters several methods were shown for the investigation of the phase portrait of systems of differential equations. Systems occurring in applications typically contain parameters, hence it is a natural question how the phase portrait changes as the values of the parameters are varied. Changing a parameter the solution of the differential equation changes, however, qualitative change in the behaviour of the solutions occurs only at a few (isolated) values of the parameter. As an illustration, consider the differential equation $\dot{x} = ax$, in which $a \in \mathbb{R}$ is a parameter. The solution can be easily given as $x(t) = e^{at}x(0)$ that obviously depends on a , but as the value of a is changed from $a = 1$ to $a = 1.01$, then there is no qualitative change in the solution. However, changing the value of a around zero we can observe qualitative change in the behaviour. For $a = 0$ all solutions are constant functions, for negative values of a the solutions tend to zero, while for positive values of a they tend to infinity. The bifurcation is the qualitative change in the phase portrait. The bifurcation occurs at those parameter values for which the phase portrait is not topologically equivalent to those belonging to nearby parameter values. This is formulated in the following definition . Consider the equation $\dot{x}(t) = f(x(t), \lambda)$, where $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a continuously differentiable function and $\lambda \in \mathbb{R}^k$ is a parameter.

Definition (4.1):-

The parameter value $\lambda_0 \in \mathbb{R}^k$ is called regular, if there exists $\delta > 0$, for which $|\lambda - \lambda_0| < \delta$ implies that the system $f(\cdot, \lambda)$ is topologically equivalent to the system $f(\cdot, \lambda_0)$. At the parameter value $\lambda_0 \in \mathbb{R}^k$ there is a bifurcation if it is not regular.

Example (4.2):- Consider the differential equation $\dot{x} = \lambda - x$, in which $\lambda \in \mathbb{R}$is a parameter. For a given value of λ the equilibrium is the point $x = \lambda$. This point is globally asymptotically stable for all values of λ , that is trajectories are tending to this point. The phase portrait for different values of λ can be shown in a coordinate system, where the horizontal axis is for λ and for a given value of λ the corresponding phase portrait is given on the vertical line at λ, as it is shown in Figure (4.1).This Figure shows that the phase portrait is the same for all values of λ, that is all values of λ are regular, i.e. there is no bifurcation. The topological equivalence of the phase portraits corresponding to different values of λ can be formally verified by determining the homeomorphism the orbits to each other. For example, the orbits for $\lambda = 0$ can be taken to those belonging to $\lambda = 1$ by the homeomorphism $h(p) = p - 1$.

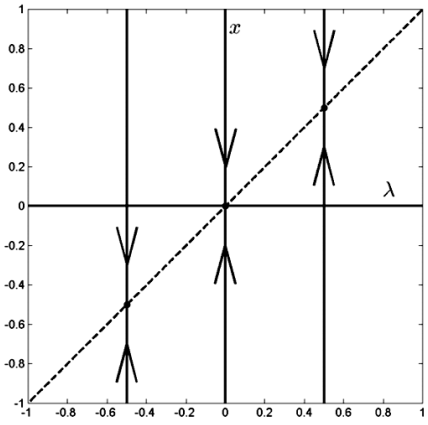


Figure (4.1) :The phase portrait of the differential equation $\dot{x} = \lambda - x$ for different values of the parameter λ .

The animation shows how the phase portrait of the differential equation in Example (4.2) changes as λ is varied between −1 and 1 .

Let us now turn to differential equations where bifurcation may occur. The simplest of them is the fold bifurcation that is also called saddle-node bifurcation.

Example (4.3):(fold or saddle-node bifurcation)

Consider the differential equation $x = \lambda - x^2$, where $\lambda \in \mathbb{R}$ is a parameter. In this case the existence of the equilibrium depends on the parameter λ . If $\lambda < 0$, then there is no equilibrium, for $\lambda = 0$ the origin $x = 0$ is an equilibrium, and for $\lambda > 0$ there are two equilibria $x = \pm\sqrt{\lambda}$. The phase portrait can be shown for different values of λ by using the same method as in the previous example, as it is shown in Figure (4.2).It can be seen in the Figure that the bifurcation is at λ=0 , since the phase portrait is different for positive and negative values of the parameter. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other. For example, the orbits for $\lambda < 0$ can be taken to each other by the homeomorphism $h(p) = p$. For positive values of λ the orbits can be taken to each other by a piece-wise linear homeomorphism. The bifurcation in this example is called fold or saddle-node bifurcation. The latter refers to the bifurcation in the two dimensional system $\dot{x} = \lambda - x^2$, $\dot{y} = -y$ that is illustrated by the animation

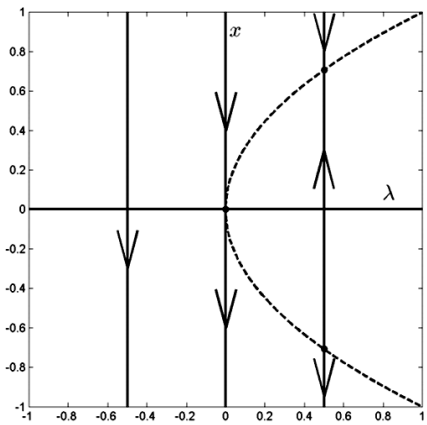


Figure (4.2) Fold or saddle-node bifurcation in the differential equation $\dot{x} = \lambda - x^2$ at $\lambda = 0$.The animation shows how the phase portrait of the

differential equation $\dot{x} = \lambda - x^2$ changes as λ is varied between -1 and 1. Saddle-node bifurcation in the two dimensional system $\dot{x} = \lambda - x^2$, $\dot{y} = -y \; at \; \lambda = 0$.

Example (4.4):(Transcritical bifurcation) Consider the differential equation $\dot{x} = \lambda x - x^2$, where $\lambda \in \mathbb{R}$ is a parameter. The point $x = 0$ is an equilibrium for any value of λ. Besides this point $x = \lambda$ is also an equilibrium, therefore in the case $\lambda \neq 0$ there are two equilibria, while for $\lambda = 0$ there is only one. Hence there is a bifurcation at $\lambda = 0$. The phase portrait can be shown for different values of λ by using the same method as in Example (4.1), as it is shown in Figure (4.3) . It can be seen in the Figure that the bifurcation is at $\lambda = 0$, since the phase portrait for non-zero values of the parameter is different from that of belonging to $\lambda = 0$. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. For negative values of λ the point $x = 0$ is stable and $x = \lambda$ is unstable, while for positive λ values it is the other way around. This bifurcation is called transcritical because of the exchange of stability. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other.

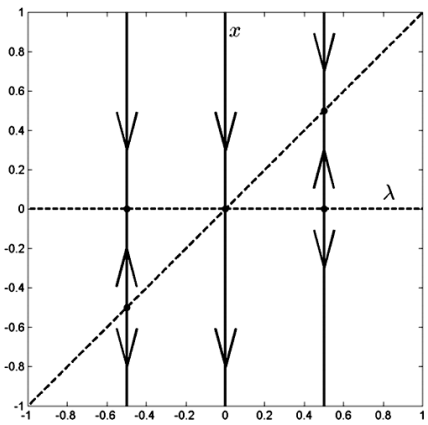
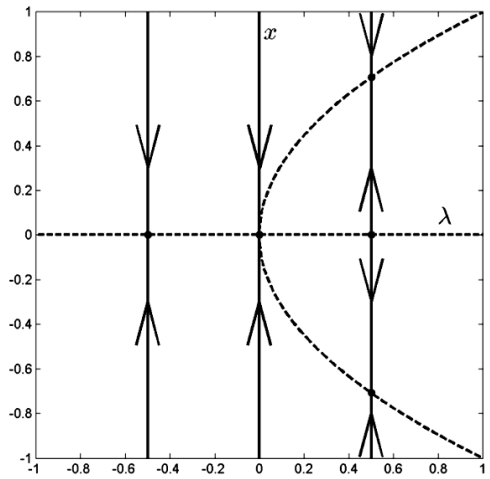


Figure (4.3) :Transcritical-bifurcation in the differential equation

$$\dot{x} = \lambda x - x^2 \; at \; \lambda = 0.$$

The animation shows how the phase portrait of the differential equation $\dot{x} = \lambda x - x^2$ changes as λ is varied between −1 and 1.

Example(4.5): (Pitchfork bifurcation) Consider the differential equation $\dot{x} = \lambda x - x^3$,where $\lambda \in \mathbb{R}$ is a parameter. The point $x = 0$ is an equilibrium for any value of λ . Besides this point $x = \pm \sqrt{\lambda}$ is also an equilibrium if $\lambda > 0$. Thus for $\lambda < 0$ there is a unique equilibrium, while for $\lambda > 0$ there are 3 equilibria. Hence there is a bifurcation at λ=0. The phase portrait can be shown for different values of λ by using the same method as in Example (4.1), as it is shown in Figure(4.4). It can be seen in the Figure that the bifurcation is at $\lambda = 0$, since the phase portrait for non-zero values of the parameter is different from that of belonging to $\lambda = 0$. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. For negative values of λ the equilibrium point $x = 0$ is globally stable. For positive values of λ the points $x = \pm\sqrt{\lambda}$ take over stability. This bifurcation is called pitchfork bifurcation because of the shape of the bifurcation curve. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other.



Figure(4.4):Pitchfork bifurcation in the differential equation $\dot{x} = \lambda x - x^3 \; at \; \lambda = 0$. The animation shows how the phase portrait of the differential equation $\dot{x} = \lambda x - x^3$ changes as λ is varied between −1 and 1.

Example(4.6):-

Consider the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$, where $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ are two parameters. First the value of λ is fixed and it is investigated how the phase portrait changes as ε is varied. The equilibria are located along the curve $\varepsilon = x^3 - \lambda x$, the shape of which depends on the sign of λ . If $\lambda < 0$, then it is strictly monotone, while for $\lambda > 0$, it has a local maximum and minimum. The curve is shown in Figure (4.5) for $\lambda = -1$ and for $\lambda = 1$. It can be seen that in the case $\lambda = -1$ *a*ll values of ε are regular, because the phase portrait does not change, for all values of ε there is a unique globally attracting equilibrium. In the case of $\lambda = 1$ the phase portrait changes at ε_1 and at ε_2, at these parameter values fold bifurcation occurs, because two equilibrium appear and disappear. Thus changing the value of λ the shape of the bifurcation curve in ε changes as λ crosses zero.

Using the three dimensional bifurcation diagram one can determine how the bifurcation curve $\lambda = x^2 - \varepsilon \, / x$ (for the parameter λ) looks like when ε is fixed. First, the value $\varepsilon = 0$ is fixed and take the section of the surface shown in Figure (4.6) with the plane $\varepsilon = 0$. Then we get the curve of the pitchfork

bifurcation shown in Figure (4.4).Then choosing a negative value of ε, say $\varepsilon = -1$ and taking the section of the surface shown in Figure (4.6) with the plane $\varepsilon = -1$ we get the curve $\lambda = x^2 + 1/x$ shown in the left part of Figure (4.7).Choosing now appositive value of ε , say $\varepsilon = 1$ and taking the section of the surface shown in Figure (4.6) with the plane $\varepsilon = 1$ we get the curve $\lambda = x^2 - 1/x$ shown in the right part of Figure (4.7).This Figure also shows that an arbitrarily small perturbation of the equation of the pitchfork bifurcation destroys this bifurcation and fold bifurcation occurs instead. This means that in one-parameter systems the pitchfork bifurcation is not typical because it contains a degeneracy of codimension two, hence it can be observed in systems with two parameters.

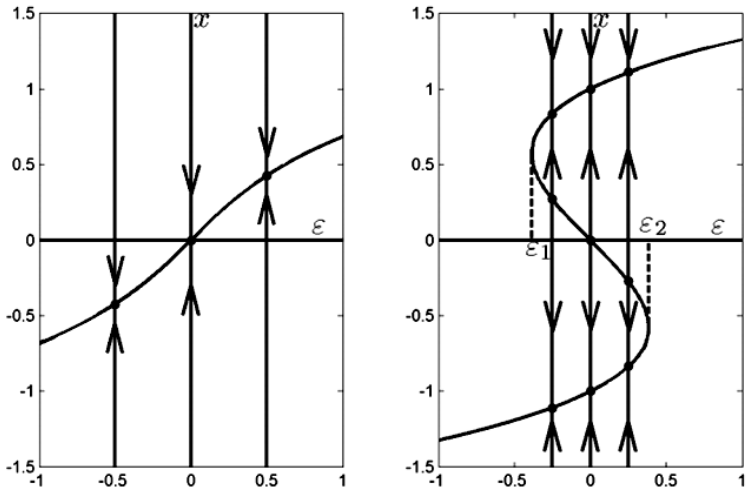


Figure (4.5): The bifurcation diagram with respect to ε in the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ for λ = −1 and for $\lambda = 1$.

The animation shows how the bifurcation diagram with respect to ε of the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ changes as λ is varied between −1 and 1.

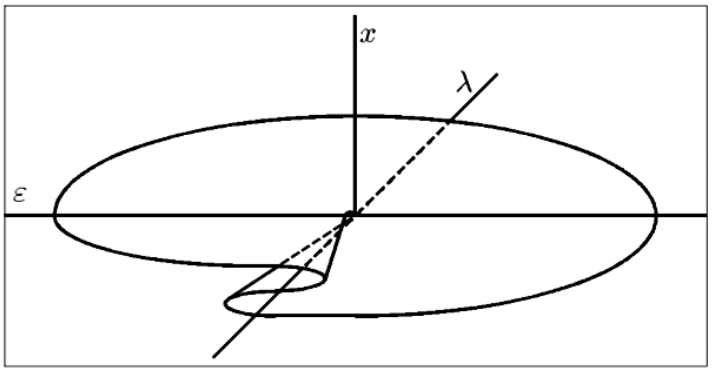


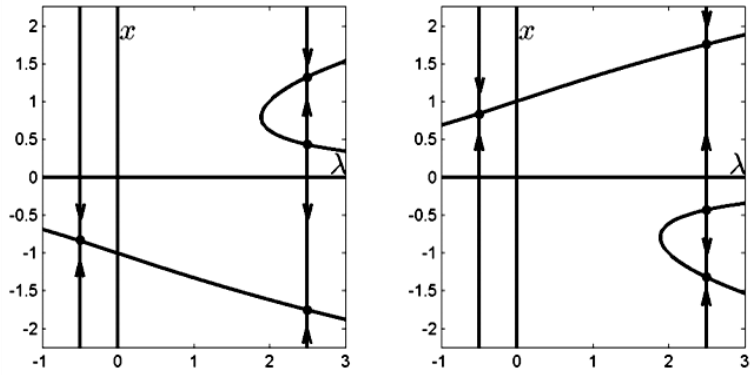
Figure (4.6): The two-parameter bifurcation diagram of the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$.

The animation shows how the bifurcation diagram with respect to λ of the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ changes as ε is varied between −1 and 1.

Now we turn to bifurcations occurring in at least two dimensional systems.

Example (4.7):(Andronov–Hopf bifurcation)

Consider the differential equation $\dot{r} = \lambda r + \sigma r^3$, $\dot{\phi} = 1$ given in polar coordinates, where λ∈ R and $\sigma \in \mathbb{R}$ are parameters. First, fix the value $\sigma = -1$ (any $\sigma < 0$ yields the same phenomenon) and see how the phase portrait changes as the value of λ is varied . The origin is an equilibrium for any value of λ and its stability can be easily determined from the differential equation for r. Namely, in the case $\lambda < 0$, we have $\dot{r} = \lambda r - r^3 < 0$, hence r is strictly decreasing and converges to zero, therefore the solutions tend to the origin. However, for $\lambda > 0$ and $r < \sqrt{\lambda}$ we have $\dot{r} = r(\lambda r - r^2) > 0$, hence r is strictly increasing, therefore the origin is unstable. Moreover, for $r = \sqrt{\lambda}$ we have $\dot{r} = 0$, that is the circle with radius √ λ is a periodic orbit that is orbitally asymptotically stable, because inside the circle $\dot{r} > 0$ and outside $\dot{r} < 0$. This phenomenon is illustrated in Figure (4.8).In the Figure the behaviour of r is shown as λ is varied. The bifurcation is at $\lambda = 0$ and the values $\lambda \neq 0$ are regular.



Figure(4.7): The bifurcation diagram with respect to λ in the differential equation $x = \lambda x - x^3 + \varepsilon$ for ε = −1 and for $\varepsilon = 1$.

The bifurcation in the two dimensional phase space is shown in Figure (4.9) .If λ < 0, then the origin is globally asymptotically stable, while for $\lambda > 0$ the origin is unstable and the stability is taken over by a stable limit cycle, the size of which is increasing as $\sqrt{\lambda}$. This bifurcation is called supercritical Andronov Hopf bifurcation. The animation shows how the phase portrait of the differential equation $\dot{r} = \lambda r - r^3$ changes as λ is varied between −0.1 and 0.1.

Returning to the differential equation $\dot{r} = \lambda r + \sigma r^3$, $\varphi = 1$ consider the case of positive σ values, say let $\sigma = 1$. The origin is an equilibrium again the stability of which is changed in the same way with λ as before, however, the periodic solution now appears for $\lambda < 0$, and it is unstable. The behaviour of r as λ is varied is shown in Figure (4.10).The origin loses its stability for $\lambda > 0$, however, in this case the periodic orbit does not take over the stability, the trajectories tend to infinity. The bifurcation in the two dimensional phase space is shown in Figure (4.11). If λ < 0, then the origin is stable but its domain of attraction is only the interior of the periodic orbit. If λ > 0, then the origin is unstable and the trajectories tend to infinity. This bifurcation is called subcritical Andronov–Hopf bifurcation . The animation shows how the phase portrait of the differential equation $\dot{x} = \lambda r + r^3$ changes as λ is varied between −0.1 and 0.1.

In the previous examples the bifurcation occurred locally in the phase space, in a neighbourhood of an equilibrium . These kind of bifurcations are called local bifurcations.

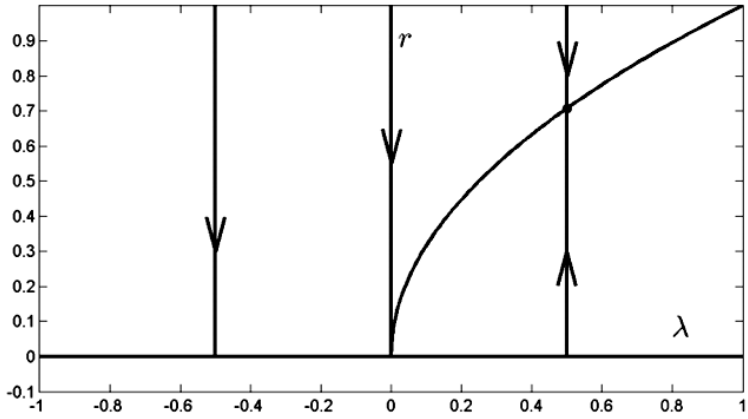


Figure (4.8): Bifurcation of the differential equation $\dot{r} = \lambda r + r^3$ at $\lambda = 0$.

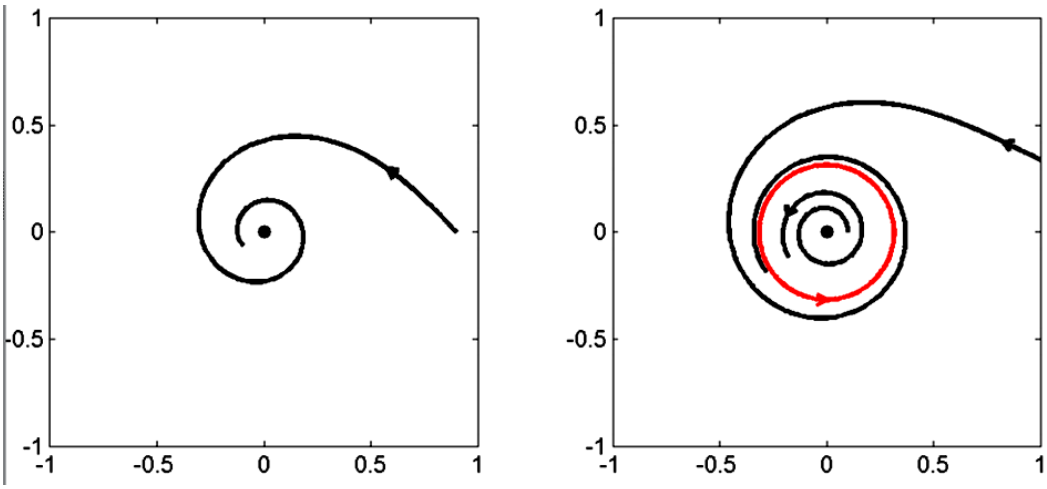


Figure (4.9): Supercritical Andronov–Hopf bifurcation, the origin looses its stability and a stable limit cycle is born.

In the rest of this chapter global (non-local) bifurcations will be shown, where global structures appear as the parameter is varied. These local structures will be periodic,

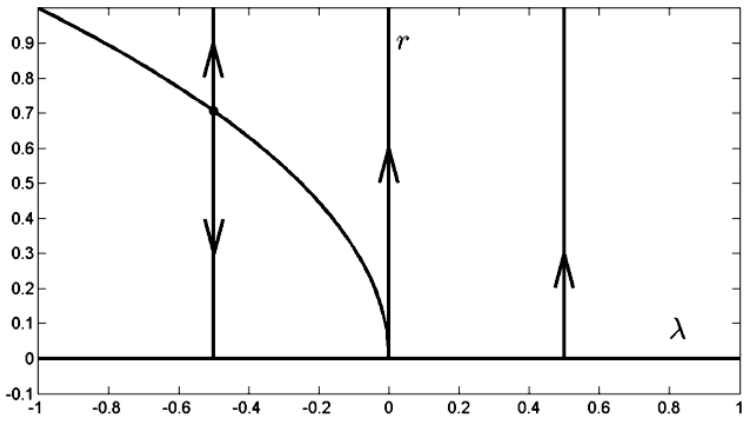


Figure (4.10): Bifurcation of the differential equation $\dot{r} = \lambda r + r^3$ at $\lambda = 0$.

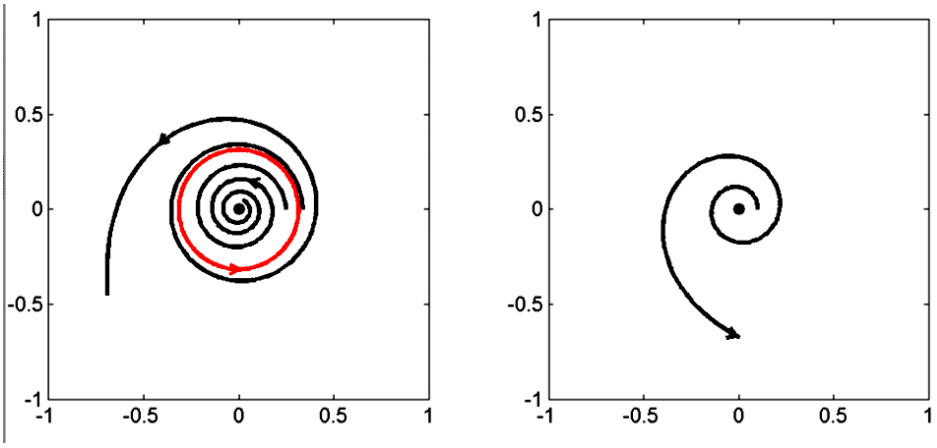


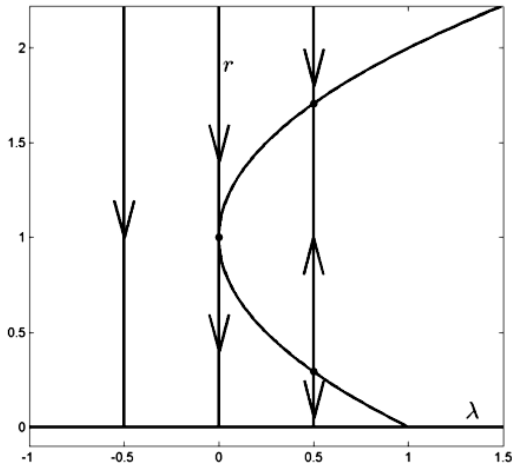
Figure (4.11): Subcritical Andronov–Hopf bifurcation, the origin loses its stability and the unstable limit cycle disappears.

homoclinic and heteroclinic orbits.

Example(4.8) :(Fold bifurcation of periodic orbits)

Consider the differential equation $\dot{r} = r(\lambda-(r-1)^2)$, $\phi = 1$ given in polar coordinates, where $\lambda \in \mathbb{R}$ is a parameter. Similarly to the previous example, investigate first the behaviour of r as λ is varied. The bifurcation diagram is shown in Figure(4.12). If $\lambda < 0$, then $r = r(\lambda-(r-1)^2) < 0$, hence r is strictly decreasing and converges to zero, therefore the solutions tend to the origin. If λ is positive and close to zero ($\lambda < 1$), then for $\lambda = (r-1)^2$ we have $r = 0$, that is the circles with radius $1 \pm \sqrt{\lambda}$ and centered at the origin are periodic orbits. The inner circle is unstable, while the outer one is stable. Thus the bifurcation occurs at $\lambda = 0$. (We note that the inner cycle disappears at

$\lambda = 1$, but this bifurcation is not studied here.) The bifurcation in the two dimensional phase space is shown in Figure (4.13). If $\lambda < 0$, then the origin is globally asymptotically stable. At the bifurcation value $\lambda = 0$ two limit cycles are born with radii close to 1. It is important to note that in the case of Andronov–Hopf bifurcation the radius of the limit cycle born is zero. If $0 < \lambda < 1$, then one of the limit cycles is stable, the other one is unstable. The phenomenon is similar to the fold bifurcation, but in this case instead of equilibria periodic orbits are born. Hence this bifurcation is called fold bifurcation of periodic orbits.



Figure(4.12): Bifurcation of the differential equation $\dot{r} = r(\lambda - (r-1)^2)$ at $\lambda = 0$. The animation shows how the phase portrait of the differential equation $\dot{r} = r(\lambda - (r-1)^2)$ changes as λ is varied between −0.1 and 0.1.

Example(4.9): (Homoclinic bifurcation)

Consider the two dimensional system

$$\dot{x} = y, \qquad \dot{y} = x - x^2 + \lambda y,$$

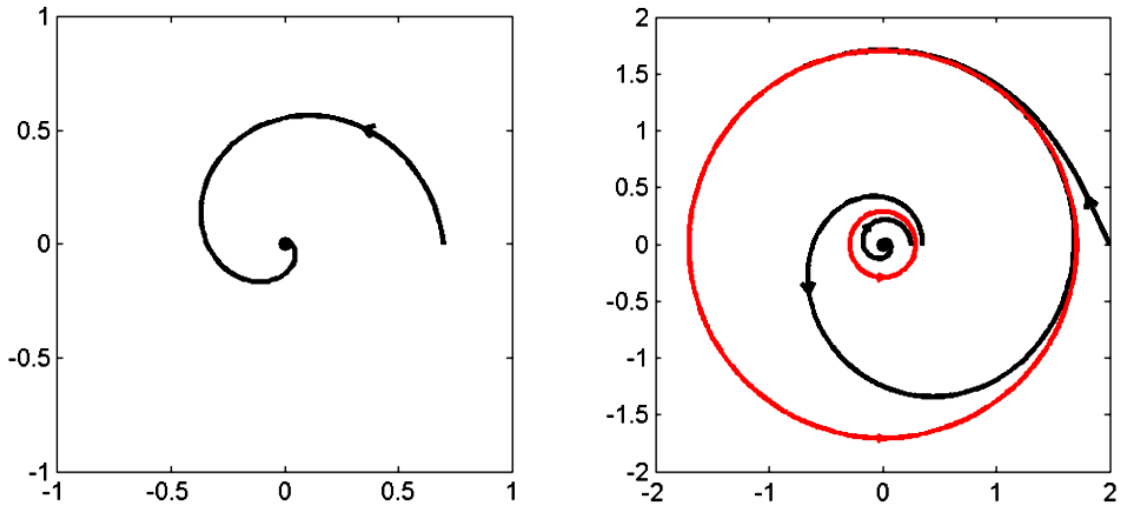


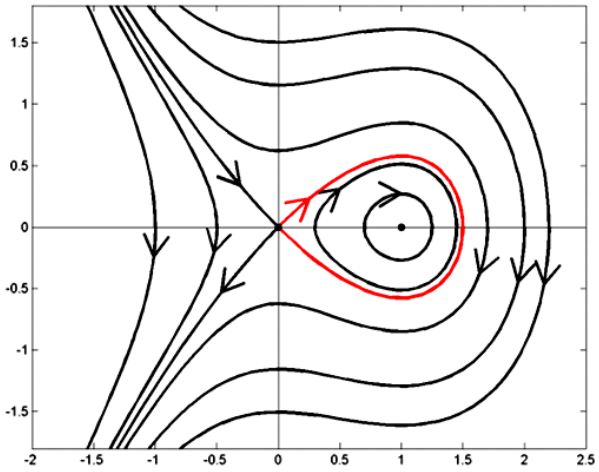
Figure (4.13): Fold-bifurcation for periodic orbits. At $\lambda = 0$ a stable and an unstable periodic orbit are born with radius close to 1. Where $\lambda \in R$ is a parameter.

It is known that for $\lambda = 0$ the function $H(x, y) = x^3/3 - x^2/2 + y^2/2$ is a first integral. Namely, let $H^*(t) = H((x(t), y(t))$, then

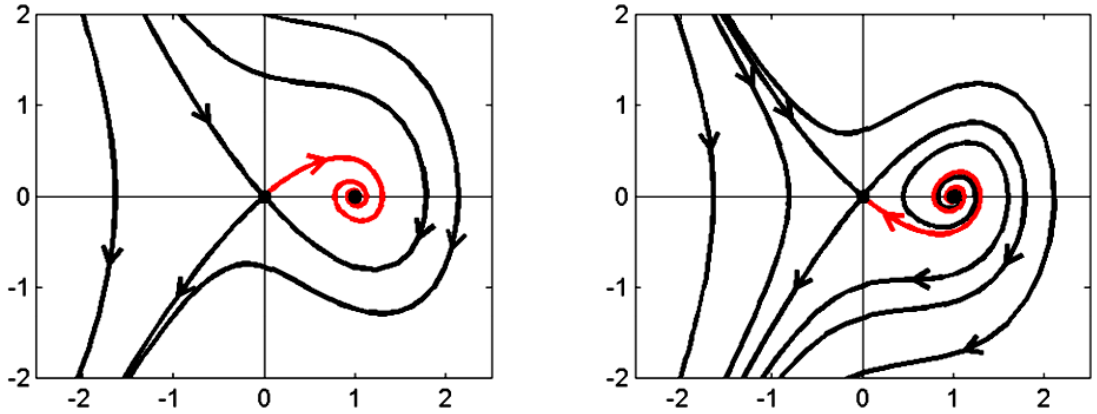
$$\dot{H}^*(t) = x^2(t) \cdot \dot{x}(t) - x(t) \cdot \dot{x}(t) + y(t) \cdot \dot{y}(t) = \lambda y^2(t).$$

Thus for $\lambda = 0$ the value of the function H is constant along trajectories. Moreover, H can also serve as a Lyapunov function. Since for $\lambda > 0$, it is increasing along trajectories and for $\lambda < 0$ it is decreasing along trajectories. The phase portrait is determined first for $\lambda = 0$. Then simply the level curves of H are to be determined, since the orbits lie on the level curves. This way the phase portrait given in Figure (4.14) is obtained. Let us consider now the case $\lambda < 0$. The equilibria remain at (0,0) and at (1,0) as for $\lambda = 0$. The origin remains a saddle point, but the point (1,0) becomes a stable focus . Using that H is decreasing along trajectories, the trajectory along the unstable manifold of the saddle point in the first quadrant tends to the stable focus. Hence also using the direction field one obtains the phase portrait shown in Figure (4.15).The case $\lambda > 0$ can be investigated similarly. The equilibria are again the points (0,0) and (1,0), the origin remains a saddle, but the point (1,0) is an unstable focus. Using that H is now increasing along trajectories, the trajectory along the stable

manifold of the saddle point in the right half plane tends to the unstable focus as time goes to $-\infty$. Hence also using the direction field one obtains the phase portrait shown in Figure (4.15).Thus the bifurcation occurs at $\lambda = 0$. In that case the stable and unstable manifolds of the saddle point form a homoclinic orbit, hence the bifurcation is called homoclinic bifurcation.



Figure(4.14): The phase portrait of system $\dot{x} = y,\ \ \dot{y} = x - x^2 + \lambda y$ for $\lambda = 0$



Figure(4.15): The phase portrait of system $\dot{x} = y,\ \ \dot{y} = x - x^2 + \lambda y$ for $\lambda < 0$ (left) and for $\lambda > 0$ (right).The animation shows how the phase portrait of the differential equation $\dot{x} = y,\ \dot{y} = x - x^2 + \lambda y$ changes as λ is varied between -0.3 and 0.3.

Example (4.10):(Heteroclinic bifurcation)

Consider the two dimensional system

$$\dot{x} = 1 - x^2 - \lambda xy,\ \ \dot{y} = xy + \lambda(1 - x^2),$$

where $\lambda \in \mathbb{R}$ is a parameter. In order to find the steady states multiply the second equation by λ, then add it to the first equation .

This leads to $(1-x^2)(1+\lambda^2)=0$, hence the first coordinate of a steady state can be $x = \pm 1$ and using the second equation $y=0$. Thus the equilibria are $(\pm 1,0)$. The phase portrait is determined first for $\lambda = 0$. Using the direction field the phase portrait can be easily given as it is shown in Figure (4.16). The segment of the x coordinate axis between −1 and 1 is a heteroclinic orbit connecting the two saddle points. It will be investigated what happens to this orbit as the value of λ is varied. It is easy to verify that the steady states $(\pm 1,0)$ remain saddle points as λ is changed . If $\lambda < 0$, then for $y=0$ and $x \in (-1,1)$ we have $x > 0$ and $\dot{y} = \lambda(1-x^2) < 0$, hence the trajectories cross the x axis to the right and down. Hence the trajectory on the right part of the unstable manifold of the saddle point (−1,0) lies in the lower half plane and tends to infinity as it is shown in Figure (4.17).Thus for $\lambda < 0$ the heteroclinic orbit connecting the saddle points does not exist.

The situation is similar in the case $\lambda > 0$. In this case for $y=0$ and $x \in (-1,1)$ we have $\dot{x} > 0$ and $\dot{y} = \lambda(1-x^2) > 0$, hence the trajectories cross the x axis to the right and up. Hence the trajectory on the right part of the unstable manifold of the saddle point (−1,0) lies in the upper half plane and tends to infinity as it is shown in Figure (4.17).Thus varying the value of λ a heteroclinic orbit appears at $\lambda = 0$, hence this bifurcation is called heteroclinic bifurcation The animation shows how the phase portrait of the differential equation $\dot{x} = 1 - x^2 - \lambda xy,\ \dot{y} = xy + \lambda(1-x^2)$ changes as λ is varied between −0.5 and 0.5.

Definition (4.11):-

The pair (x_0, λ_0) is called locally regular, if there exist a neighbourhood $U \subset \mathbb{R}^n$of x_0 and $\delta > 0$, such that for $|\lambda - \lambda_0| < \delta$ the systems $f|_U(\cdot, \lambda_0)$ and $f|_U(\cdot, \lambda)$ are topologically equivalent (that is the phase portraits are

topologically equivalent in U). There is a local bifurcation at (x_0, λ_0), if (x_0, λ_0) is not locally regular.

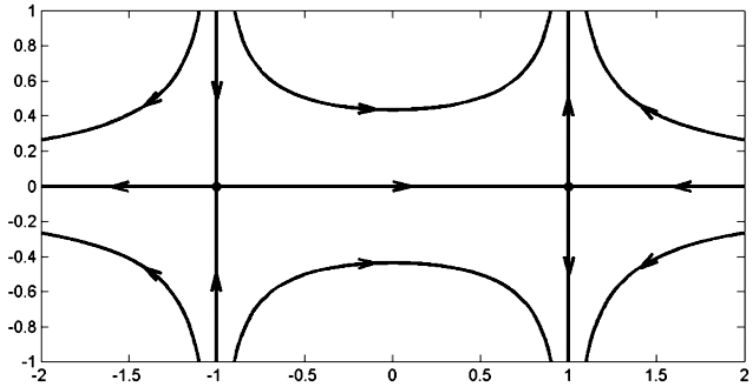


Figure (4.16): The phase portrait of system $\dot{x} = 1 - x^2 - \lambda xy$, $\dot{y} = xy + \lambda(1 - x^2)$ for $\lambda > 0$.

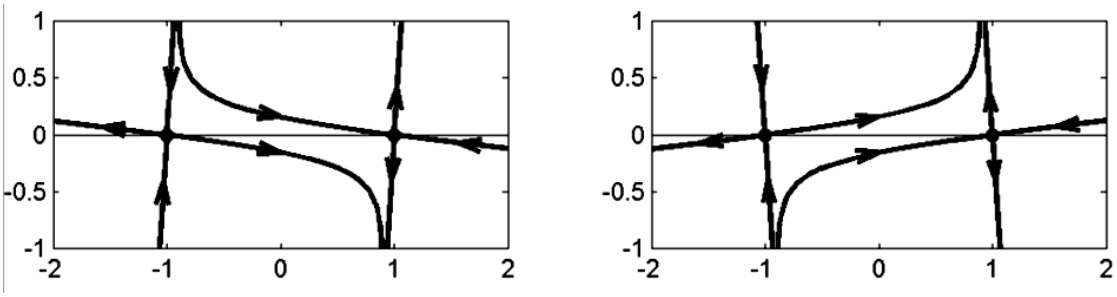


Figure (4.17): The phase portrait of system $\dot{x} = 1 - x^2 - \lambda xy, \dot{y} = xy + \lambda(1 - x^2)$ for $\lambda < 0$ (left) and for $\lambda > 0$ (right).

First, it is shown that local bifurcation may occur only at an equilibrium.

Proposition (4.12):-

If $f(x_0, \lambda_0) \neq 0$, then x_0, λ_0 is locally regular.

Proof:

For simplicity, the proof is shown for the case $n = 1$. Without loss of generality one can assume that $f(x_0, \lambda_0) > 0$. Then the continuity of f implies that there exist a neighbourhood $U \subset \mathbb{R}^n$ of x_0 and $\delta > 0$, such that in the set $\overline{U} \times [\lambda_0 - \delta, \lambda_0 + \delta]$ the value of f is positive. Hence in this set the trajectories are segments directed upward, as it is shown in Figure (4.18).

Hence the phase portraits are obviously topologically equivalent in U for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (The homeomorphism taking the orbits into each other is

the identity.) We note that for $n > 1$ the proof is similar by using the local flow box theorem.

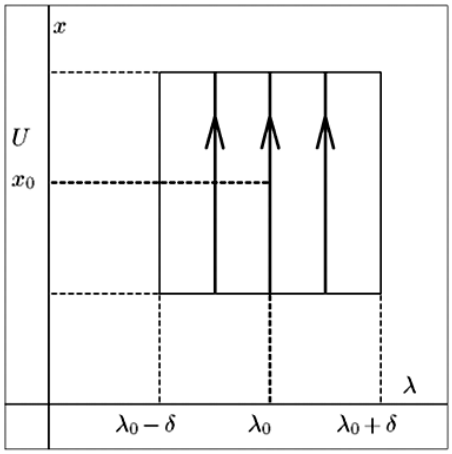


Figure (4.18): The trajectories in a neighbourhood U of a non-equilibrium point are the same for all values of the parameter λ.

It will be shown that at a hyperbolic equilibrium local bifurcation cannot occur.

Proposition (4.13):-

If $f(x_0, \lambda_0) = 0$ and $\partial_x f(x_0, \lambda_0)$ is hyperbolic, then (x_0, λ_0) is locally regular. (Here $\partial_x f(x_0, \lambda_0)$ denotes the Jacobian matrix of f.)

Proof:

For simplicity, the proof is shown again for the case $n = 1$. Without loss of generality one can assume that $\partial_x f(x_0, \lambda_0) < 0$.

Then according to the implicit function theorem there exist $\delta > 0$ and a differentiable function $g: (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathbb{R}$, for which $g(\lambda_0) = x_0$ and $f(g(\lambda), \lambda) \equiv 0$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, moreover, there is neighbourhood U of x_0, such that in other points of the set $U \times (\lambda_0 - \delta, \lambda_0 + \delta)$ the function f is nonzero. Since f is continuously differentiable, the number δ can be chosen so small that $\partial_x f(g(\lambda), \lambda) < 0$ holds for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Hence for these values of λ there is exactly one stable equilibrium in U and the trajectories tend to this point as it is shown in Figure (4.19).Therefore the phase portraitsare obviously to pologically equivalent in U for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (The homeomorphism taking the orbits into each other is a translation taking the

steady states to each other.) We note that for $n > 1$ the proof is similar by using the Hartman–Grobman theorem.

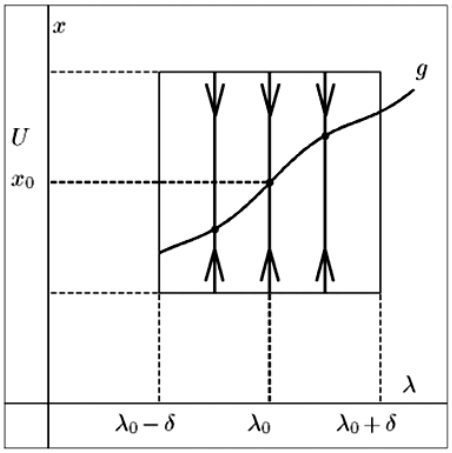


Figure (4.19): The phase portraits in a neighbourhood U of a hyperbolic equilibrium are the same for all values of the parameter λ.

Definition (4.15):-

An element $x \in X$ is called structurally stable, if it has a neighbourhood $U \subset X$, for which $y \in U$ implies x $\sim$ y. An element $x \in X$ is called a bifurcation point, if it is not structurally stable.

In other words, we can say that $x \in X$ is structurally stable, if it is an interior point of an equivalence class, and it is bifurcation point if it is a boundary point of an equivalence class. This interpretation enables us to define the codimension of a bifurcation. The co-dimension of a bifurcation is the codimension of the surface that forms the boundary in a neighbourhood of the given bifurcation point. In Figure (4.20) the point A is structurally stable, point B is a one co-dimensional bifurcation point and point C is a two co-dimensional bifurcation point. This can also be formulated as follows. There is a curve through B that intersects both domains that are separated by the border containing B, while there is no such curve through C. The classes touching C can be reached by a two parameter family, i.e. a surface in the space. This is formulated rigorously in the following definition.

Definition(4.16) :-

A bifurcation point $x \in X$ is called k-co-dimensional, if there is a continuous function g: $\mathbb{R}^k \to X$, for which $g(0) = x$ and the point x has a neighbourhood U and it has an open dense subset V , such that for all $y \in V$ there is an $\alpha \in \mathbb{R}^k$ satisfying $g(\alpha) \sim y$, and k is the smallest dimension with these properties.

Consider the differential equation $\dot{x}(t) = f(x(t))$, where $f: \mathbb{R} \to \mathbb{R}$, is a continuously differentiable function. The space X can be chosen as the space of continuously differentiable functions with a suitable norm or topology. Let us start from C^0 or supremum topology that will be denoted by $\|f\|_0$ and is given by sup| f | on a suitably chosen set. A simple one dimensional example shows that this topology is not a suitable choice. Namely, consider the differential equation $\dot{x} = x$ given by the identity $f(x) = x$. To this function f one can find a function g that is arbitrarily close to f in the C^0 topology and has more than one root . Hence the phase portrait of the differential equation $\dot{y} = g(y)$

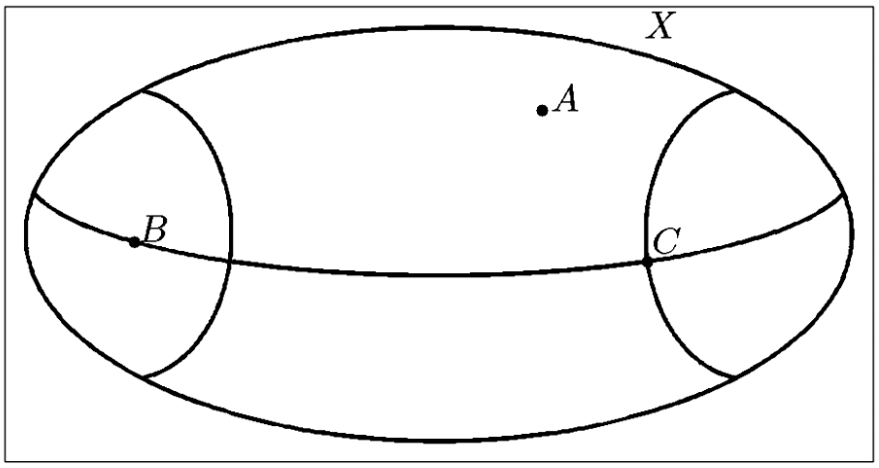


Figure (4.20): Structurally stable (A), one co-dimensional (B) and two co-dimensional (C) bifurcation points in a topological space X.Is not topologically equivalent to the phase portrait of $\dot{x} = x$. This example shows that choosing the C^0 norm on the space X only those systems can be structurally stable that have no equilibrium. This would give a very restrictive definition of structural stability, most of the equivalence classes would not have interior points.

A better norm for our purposes is the C^1 norm that is given by $\|f\|_1 = \|f\|_0 + \|f'\|_0$. Namely, let us consider again the identity function $f(x) = x$ and the corresponding differential equation $\dot{x} = x$. If a function g is close to f in C^1 norm, then it has also a unique zero, hence the corresponding differential equation $\dot{y} = g(y)$ is topologically equivalent to the equation $\dot{x} = x$. Therefore the equation $\dot{x} = x$ having a hyperbolic equilibrium is structurally stable, when the C^1 norm is used . The C^1 norm of a continuously differentiable function is not necessarily finite if the domain of the function is not compact. Therefore investigating structural stability is more convenient for dynamical systems with compact state space. If the state space is one dimensional, then the most straightforward choice for the state space is the circle.

Definition (4.17): (Structural stability of one dimensional systems)

Let us introduce the space $X = C^1(S^1, R)$ that consists of continuously differentiable functions $f: \mathbb{R} \to \mathbb{R}$, which are periodic with period 1, that is $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. This space will be endowed with the norm

$$\|f\|_1 = \max_{[0,1]} |f| + \max_{[0,1]} |f'| .$$

It will be shown that those systems are structurally stable, for which all equilibria are hyperbolic. Introduce the following notation for these systems.

$G = \{f \in X: \ f(x) = 0 \Rightarrow f'(x) \neq 0\}$.

Definition(4.18):-

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. The value y is called a regular value of f, if $f(x) = y$ implies $f'(x) \neq 0$. In the higher dimensional case when $f: \mathbb{R}^n \to \mathbb{R}^n$ the assumption is that $f(x) = y$ implies $\det f'(x) \neq 0$. If y is not a regular value, then it is called a critical value of f.

Lemma (4.19):(Sard): If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, then the set of its critical values has measure zero.

Proposition(4.20) :- The above set G of dynamical systems having only hyperbolic equilib- ria is dense in the space $X = C^1(S^1, \mathbb{R})$.

Proof:

A function is in the set G, if and only if 0 is its regular value. Let $f \in X$ and $\varepsilon > 0$ be arbitrary. It has to be shown that there exists $g \in G$, for which $\| f - g \|_1 < \varepsilon$. If 0 is a regular value of f, then $g = f$ is a suitable choice, since then $f \in G$. If 0 is not a regular value, then chose a positive regular value $c < \varepsilon$. The existence of this c is guaranteed by Sard's lemma .

Then let $g = f - c$, therefore $\| f - g \|_1$ = c < ε and $g(x) = 0$ implies $f(x) = c$, hence the regularity of c yields $f'(x) \neq 0$, which directly gives $g'(x) \neq 0$. Thus 0 is a regular value of g , that is $g \in G$. In the proof of the theorem we use the fact that if a function has a degenerate zero, then a small C^1 perturbation makes it constant zero in a neighbourhood of the zero.

Proposition (4.21):- Let $f \in X$ and assume that for some $x \in (0,1)$ we have $f(x) = 0 = f'(x)$. Then for any $\varepsilon > 0$ and $\alpha > 0$ there exists a function $g \in X$, for which the following statements hold:

1. $f(y) = g(y)$ for all $y \notin (x - \alpha, x + \alpha)$,
2. g is constant 0 in a neighbourhood of x,
3. $\| f - g \|_1 < \varepsilon$.

Proof:

Let $\eta: \mathbb{R} \to [0,1]$ be a C^1 function (in fact it can be chosen as a C^∞ function), that is constant zero outside the interval $[-1,1]$ and constant 1 in the interval $(-1/2,1/2)$. The maximum of $|\eta'|$ is denoted by M. The assumption on f implies that there exists a positive number $\beta < \alpha$, for which

$$|f(y)| < \frac{\varepsilon}{4M} |y - x| . \qquad (4.21)$$

holds for all $y \in (x - \beta, x + \beta)$. Let $\delta < \beta$ be a positive number, for which

$$\max_{[x-\delta,\ x+\delta]} |f| < \frac{\varepsilon}{2} \text{ , and } \max_{[x-\delta, x+\delta]} |f'| < \frac{\varepsilon}{4}. \qquad (4.22)$$

Then let $g \in X$ be given as follows

$$g(y) = g(y) \left(1 - \eta \left(\frac{y - x}{\delta}\right)\right).$$

Now it is checked that g satisfies the conditions . If $|y-x| \geq \alpha$, then $|y-x| > \delta$ yielding $\eta\frac{y-x}{\delta} = 0$, hence the first condition holds, i.e. $g(y) = f(y)$. If $|y-x| < \delta/2$, then $\eta\frac{y-x}{\delta} = 1$, hence the second condition holds, i.e. g is constant zero in a neighbourhood of x . In order to check the last condition we use that

$$f(y) - g(y) = f(y)\eta\left(\frac{y-x}{\delta}\right). \qquad (4.23)$$

Therefore $f-g$ is zero in $[x-\delta, x+\delta]$, hence it is enough to prove that for all $y \in [x-\delta, x+\delta]$ the following holds

$$|f(y)-g(y)| < \frac{\varepsilon}{2} \text{ and } |f'(y)-g'(y)| < \frac{\varepsilon}{32}. \qquad (4.24)$$

For proving both inequalities one can use (4.23).

This yields for $y \in [x-\delta, x+\delta]$ that $|f(y)-g(y)| < |f(y)| < \frac{\varepsilon}{2}$, where the first inequality of (4.22) was used. Differentiating (4.23)

$$f'(y) - g'(y) = f'(y)\eta\left(\frac{y-x}{\delta}\right) + f(y)\eta'\left(\frac{y-x}{\delta}\right)\frac{1}{\delta} .$$

Applying the second equation in (4.22), the inequality (4.21) and that M is the maximum of $|\eta'|$ leads to

$$|f'(y)-g'(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4M}|y-x|\frac{M}{\delta} .$$

Since $y \in [x-\delta, x+\delta]$, we have $|y-x| \leq \delta$, hence the previous estimate can be continued as

$$|f'(y)-g'(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4\mathrm{M}}\delta\frac{\mathrm{M}}{\delta} = \frac{\varepsilon}{2} .$$

Proposition (4.22):-

If all the equilibria of the differential equation $\dot{x} = f(x)$ are hyperbolic, then there are at most finitely many of them in [0,1].

Proof :

Assume that there are infinitely many equilibria in [0,1]. Then one can choose a convergent sequence of equilibria tending to a point $x \in$ [0,1]. Then since f is

continuously differentiable we have $f(x) = 0$ and $f'(x) = 0$, that is x is not a hyperbolic equilibria , which is a contradiction.

Theorem(4.23):-

The dynamical system belonging to the function $f \in X$ is structurally stable, if and only if all equilibria of f are hyperbolic, that is $f \in G$. Moreover, the set G of structurally stable systems is open and dense in the space X.

Proof:

Assume first that f is structurally stable and prove $f \in G$. Since f is equivalent to the systems in a neighbourhood and G is dense, there exists in this neighbourhood a function $g \in G$. Hence all the roots of g are hyperbolic, implying that there are finitely many of them, therefore the equivalence of f and g implies that f has finitely many roots. We show that all of them are hyperbolic. Since the roots are isolated, if one of them were not be hyperbolic, then according to Proposition (4.21) an arbitrarily small C^1 perturbation would make it constant zero. That would mean that arbitrarily close to f there is a function, which is zero in an interval, hence it is not equivalent to f contradicting to the fact that f is structurally stable. This proves the first implication. Assume now that $f \in G$ and prove that f is structurally stable. Proposition (4.22) yields that f has finitely many roots. If it has no zeros at all, then the functions close to f in the C^1 norm cannot have zeros, hence they are equivalent to f. If f has zeros, then it can be easily seen that functions close to f has the same number of zeros and the sign changes at the zeros are the same as those for f. This implies that their phase portraits are equivalent to that belonging to f

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