Chapter 1

Banach Spaces and Isometric Extensions Problems with Sharp Corner Points

In this chapter for any Banach space y. we define collection of "sharp corner points" of the unit ball $B_1(Y^*)$. Which is empty if Y is strictly convex and dim Y≥ 2. Then we prove that any surjective isometry between two unit spheres of banach spaces X and Y has linear isometric extension on the whole space if Y is a Gateanux differentiability space (in particular. Separable spaces or reflexive spaces) and the intersection of "sharp corner points" and weak^* - exposed points of $B(Y^*)$ is weak - dense in the latter .

Section (1.1): Some Lemmas:

The famous Mazur-Ulam theorem stated that any surjective isometry V between two real normed spaces with $V(\theta) = \theta$ (zero element) must be linear. P. Mankiewicz proved that any surjective isometry between the convex bodies (i.e. open connected subsets) of two normed spaces can be extended to a surjective affine isometry on the whole space.

In 1987, D. Tingley proposed the following problem .

Problem (1.1.1) [1]:

Let X and Y be real normed spaces with unit spheres $S_1(X)$ and $S_1(Y)$, respectively. Suppose that $V: S_1(X) \to S_1(Y)$ is a surjective isometry. Is V_0 necessarily the restriction of a linear or affine isometry on X?

We only consider the isometric extension problem in real normed spaces, since it is clearly negative in the complex case. This problem is interesting and easy to understand. Moreover, it is very important. If this problem has a positive answer, then the local geometric property of a mapping on the unit sphere will determine the property of the mapping on the whole space.

However, it is very difficult to solve. As Professor E. Odell said ''this is a very difficult problem that remains unsolved after 25 years''. D. Tingley only proved that any isometry V_0 between the unit spheres $S_1(X_{(n)})$ and $S_1(Y_{(m)})$ necessarily maps the antipodal points to antipodal points, that is $V_0(-x) = -V_0(x)$ for any $x \in S_1(X_{(n)})$ (both $X_{(n)}$ and $Y_{(m)}$ are real finite-dimensional normed spaces).

For quite a while (about 15 years), there has been no progress at all on this problem, until it was solved in Hilbert space and $\ell^p(\Gamma)$ space $(1 \leq p \leq \infty)$ .In the past decade, the isometric extension problem was considered in various classical Banach spaces and many good results were obtained, through studying the specific form of norm and a lot of special skills .

By now, the isometric extension problem has been solved affirmatively if X is any classical Banach space and Y is a general Banach space . However, little progress has been obtained if X and Y are both general Banach spaces, even in the two-dimensional case. Recently, the isometric extension problem was considered in somewhere-flat Banach spaces and polyhedral Banach spaces and some impressive results were obtained . Moreover, this problem was also considered in the F-spaces .

We attempt to study the isometric extension problem in general Banach spaces through some geometric properties of the Banach spaces including weak*-exposed points, Gâteaux differentiability, and so on.

Theorem (1.1.2) [1]:

Let X be a Banach space and Y be a Gâteaux differentiability space. If $\mathcal{P}(Y^*)$ is the set of weak*-exposed points in $B_1(Y^*)$ and $\mathcal{P}(Y^*) \cap$ S $\mathcal{C}(Y^*)$ is weak*-dense in $\mathcal{P}(Y^*)$, then any surjective isometry between two unit spheres $S_1(X)$ and $S_1(Y)$ can be extended to a linear isometry on the whole space.

From this theorem, we deduce a result concerning the isometric extension of isometry between unit spheres $S_1(X)$ and $S_1(Y)$, where X is a general Banach space and Y is an Asplund generated space.

Theorem (1.1.3) [1]:

Let X be a Banach space and Y be an Asplund generated space. Suppose that V_0 is an isometric mapping from the unit sphere $S_1(X)$ into $S_1(Y)$, which satisfies the following condition:

$(*)$ For any $x_1, x_2 \in S_1(X)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$,

$\| \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \| = 1 \Rightarrow \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \in V_0[S(X)]$.

Let $Z = \overline{\text{span}}\{V_0x : x \in S_1(X)\}$. Suppose that $\mathcal{P}(Z^*) \cap$ S $\mathcal{C}(Z^*)$ is weak*-dense in $\mathcal{P}(Z^*)$. Then V_0 can be extended to a linear isometry on the whole space.

Consequently, we obtain that if $Y = (\ell^1), c_0(\Gamma), c(\Gamma), \ell^\infty(\Gamma)$ or some $C(\Omega)$ (for example, the set of ‘‘G_δ-points’’ is dense in Ω), then the answer for the isometric extension problem is also affirmative.

In this section, all normed spaces are over $\mathbb{R}$ and Y^* denote the dual space of a normed space Y. $S_1(Y)(B_1(Y))$ denotes the unit sphere (unit ball) of a normed space Y.

Let Y be a normed space and $y_0^* \in S_1(Y^*)$:

$$A(y_0^*) \coloneqq \{y \in S_1(Y): y_0^* = 1\};$$

$$\mathcal{A}(Y^*) \coloneqq \{y^* \in S_1(Y^*): A(Y^*) \neq \phi\};$$

$$P(y_0^*) \coloneqq \{y \in S_1(Y): y_0^*(y) = 1, y^*(y) < 1 \text{ for any } y^* \in S_1(Y^*) \text{with } y^* \neq y_0^*\};$$

$$\mathcal{P}(y^*) \coloneqq \{y^* \in S_1(Y^*): P \neq \phi\}.$$

Remark (1.1.4) [1]:

Let Y be a normed space and $y_0^* \in S_1(Y^*)$. $A(y_0^*)$ is the set of ‘‘norm-attaining points’’ of y_0^*. $A(Y^*)$ is the subset of $S_1(Y^*)$ in which any y^* norm-attains at some point in $S_1(Y)$. $P(y_0^*)$ is the set of ‘‘peak-functions’’ $J(y) \in Y^{**}$, which have (only) a peak at y_0^* (where J is the canonical mapping from Y to Y^{**}). $y_0^* \in \mathcal{P}(Y^*)$ is called the weak*-exposed point of unit ball $B_1(Y^*)$. It is evident that any $y_0 \in P(y_0^*)$ is a smooth point of $S_1(Y)$. Conversely, if y_0 is a smooth point of $S_1(Y)$, there exists a unique $y_0^* \in \mathcal{P}(Y^*)$ with $y_0^*(y_0) = 1$.

Lemma (1.1.5) [1]:

Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. Then we have

$$\|x_1 + x_2\| = 2 \Leftrightarrow \|V_0x_1 + V_0x_2\| = 2, \ \ \forall x_1, x_2 \in S_1(X).$$

Proof:

We only need to prove the ''$\Rightarrow$'' part, because V_0^{-1} is also a surjective isometry from $S_1(Y)$ onto $S_1(X)$. Suppose that $\| x_1 + x_2 \| = 2$. By the Hahn-Banach theorem, there exists $x_0^* \in S(X)$ such that $x_0^*(x_1 + x_2) = \| x_1 + x_2 \| = 2$. Hence

$$2 = \|x_1 + x_2\| = |x_0^*(x_1 + x_2)| \leq |x_0^*(x_1)| + |x_0^*(x_2)| \leq 2,$$

and we have

$$x_0^*(x_1) = x_0^*(x_2) = 1. \qquad \textbf{(1)}$$

Let $\bar{x}_n \left(1 - \frac{1}{n}\right) x_1 + \frac{1}{n} x_2$ ($\forall n \in \mathbb{N}$). By Equation. (1) , we get a sequence $\{\bar{x}_n\} \subseteq S_1(X)$. For each $n \in \mathbb{N}$ and $x \in S_1(X)$, suppose that

$$\|\bar{x}_n + x\| = 2. \qquad (2)$$

By the Hahn-Banach theorem and the similar method, there exists $x^*_{(n,x)} \in S_1(X^*)$ such that $x^*_{(n,x)}(\bar{x}_n + x) = 2$, which implies that

$$x^*_{(n,x)}(x_1) = x^*_{(n,x)}(x_2) = x^*_{(n,x)}(x) = 1.$$

Therefore, we obtain

$$\|x_1 + x_2\| = 2. \qquad (3)$$

since

$$2 = x^*_{(n,x)}(x_1 + x) \leq \|x_1 + x\| \leq 2.$$

Note that

$$\|\bar{x}_n - V_0^{-1}(-V_0\bar{x}_n)\| = \|V_0\bar{x}_n + V_0\bar{x}_n\| = \|2V_0\bar{x}_n\| = 2, \forall n \in \mathbb{N}. \qquad (4)$$

By the similar methods we used to deduce (3) from (2), we have that

$$\|x_2 - V_0^{-1}(-V_0\bar{x}_n)\| = 2. \ \ \forall n \in \mathbb{N} \qquad (5)$$

by (4) . Note that V_0 is isometric and (5). We can obtain

$$\|V_0x_2 + V_0\bar{x}_n\| = 2, \ \ \forall n \in \mathbb{N}.$$

Let $n \to \infty$. We get $\| V_0x_1 + V_0x_2 \| = 2$ and complete the proof.

We need to prove the following lemma.

Lemma (1.1.6) [1]:

Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $y_0^* \in \mathcal{P}(Y^*)$, then $V_0^{-1}[A(y_0^*)] \subseteq S_1(X)$ is convex.

Proof:

Since $y_0^* \in \mathcal{P}(Y^*)$, there exists $y_0 \in P(y_0^*)$ ($\subseteq A(y_0^*)$). Therefore, for any $x_1, x_2 \in V_0^{-1}[A(y_0^*)]$ and $\lambda \in [0,1]$, we have

$$2 = y_0^*(y_0 + V_0 x_1) \le \|y_0 + V_0 x_1\| \le 2,$$

that is $\| y_0 + V_0 x_1 \| = 2$. By Lemma (1.1.5), we have that $\| V_0^{-1} y_0 + x_1 \| = 2$, and there exists $x_1^* \in S_1(X^*)$ such that

$$x_1^*(V_0^{-1} y_0 + x_1) = 2,$$

by the Hahn-Banach theorem. Note that $|x_1^*(V_0^{-1} y_0)| \le 1$ and $|x_1^*(x_1)| \le 1$. We get that

$$x_1^*(V_0^{-1} y_0) = x_1^*(x_1) = 1,$$

and thus

$$2 = x_1^*\left(V_0^{-1} y_0 + \frac{V_0^{-1} y_0 + x_1}{2}\right) \le \left\|V_0^{-1} y_0 + \frac{V_0^{-1} y_0 + x_1}{2}\right\| \le 2,$$

that is

$$\left\|V_0^{-1} y_0 + \frac{V_0^{-1} y_0 + x_1}{2}\right\| = 2.$$

By Lemma (1.1.5), we obtain

$$\left\|y_0 + V_0\left(\frac{V_0^{-1} y_0 + x_1}{2}\right)\right\| = 2.$$

Therefore, there exists $y_1^* \in S_1(Y^*)$ such that

$$y_1^*(y_0) + y_1^*\left[V_0\left(\frac{V_0^{-1} y_0 + x_1}{2}\right)\right] = 2,$$

by the Hahn-Banach theorem. From the similar arguments as above, we get that

$$y_1^*(y_0) = y_1^*\left[V_0\left(\frac{V_0^{-1} y_0 + x_1}{2}\right)\right] = 1. \qquad (6)$$

Note Equation (6) and $y_0 \in P(y_0^*)$. We have $y_1^* = y_0^*$ and

$$y_0^*\left[V_0\left(\frac{V_0^{-1}y_0+x_1}{2}\right)\right]=1. \qquad (7)$$

Since $x_2 \in V_0^{-1}[A(y_0^*)]$, we get that $y_0^*\left[V_0x_2+V_0\left(\frac{V_0^{-1}y_0+x_1}{2}\right)\right]=2$, which implies that $\left\|V_0x_2+V_0\left(\frac{V_0^{-1}y_0+x_1}{2}\right)\right\|=2$. By Lemma (1.1.5) , we get that

$$\left\|x_2+\frac{V_0^{-1}y_0+x_1}{2}\right\|=2,$$

and there exists $x_2^* \in S_1(X^*)$ such that

$$x_2^*\left(x_2+\frac{V_0^{-1}y_0+x_1}{2}\right)=2,$$

by the Hahn-Banach theorem. Note that $|x_2^*(x_2)|, |V_0^{-1}y_0|, |x_2^*(x_1)| \le 1$. We have

$$x_2^*(V_0^{-1}y_0)=x_2^*(x_1)=x_2^*(x_2)=1,$$

and

$$x_2^*[V_0^{-1}y_0+(\lambda x_1+(1-\lambda)x_2)]=2.$$

Therefore, we get that $\|V_0^{-1}y_0+(\lambda x_1+(1-\lambda)x_2)\|=2$, which implies that

$$\|y_0+V_0(\lambda x_1+(1-\lambda)x_2)\|=2, \qquad (8)$$

by Lemma (1.1.5) .Then, from (8) and the similar argument we used to deduce (7), we can also obtain

$$y_0^*[V_0(\lambda x_1+(1-\lambda)x_2)]=y_1^*(y_0)=1,$$

that is $\lambda x_1+(1-\lambda)x_2 \in V_0^{-1}[A(y_0^*)]$. Thus $V_0^{-1}[A(y_0^*)]$ is convex and the proof is completed.

Lemma (1.1.7) [1]:

Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $y_0^* \in \mathcal{P}(Y^*)$, there exists $x_0^* \in S_1(X^*)$ such that

$$y_0^*(y) = \pm \Longrightarrow x_0^*(V_0^{-1}y) = y_0^*(y),$$

for any $y \in S_1(Y)$.

Proof:

If $y \in S_1(Y)$ and $y_0^*(y) = 1$, then $y \in A(y_0^*)$. By Lemma (1.1.6) , $V_0^{-1}[A(y_0^*)] \subseteq S_1(X)$ is convex and does not meet with the interior of $B_1(X)$. (It is evident that the interior of $B_1(X)$ is not empty). Therefore, by the Eidelheit Separation theorem, there exists $x_0^* \in S_1(X^*)$ such that

$$\sup\{x_0^*(\bar{x}) : \bar{x} \in B_1(X)\} \leq \inf\{x_0^*(x) : x \in V_0^{-1}[A(y_0^*)]\},$$

which implies that

$$1 \leq \inf\{x_0^*(x) : x \in V_0^{-1}[A(y_0^*)]\} \leq \inf\{\|x_0^*\| \cdot \|x\| : x \in V_0^{-1}[A(y_0^*)]\} = 1$$

that is $x_0^*(x) = 1$ for any $x \in V_0^{-1}[A(y_0^*)]$.

Furthermore, if $\tilde{y} \in S_1(Y)$ and $y_0^*(\tilde{y}) = -1$, then $-\tilde{y} \in A(y_0^*)$. Since $y_0^* \in \mathcal{P}(Y^*)$, there exists $y_0 \in P(y_0^*)$ ($\subseteq A(y_0^*)$), and we have that

$$2 \geq \|V_0^{-1}\tilde{y} - V_0^{-1}y_0\| = \|\tilde{y} - y_0\| \geq |y_0^*(\tilde{y} - y_0)| = 2,$$

that is $\|V_0^{-1}y_0 + (-V_0^{-1}\tilde{y})\| = 2$. By Lemma (1.1.5) , we have $\|y_0 + V_0(-V_0^{-1}\tilde{y})\| = 2$. Therefore, there exists $y_1^* \in S_1(Y^*)$ such that

$$y_1^*\big(y_0 + V_0(-V_0^{-1}\tilde{y})\big) = 2,$$

by the Hahn-Banach theorem. Then we have

$$y_1^*(y_0) = y_1^*\big(V_0(-V_0^{-1}\tilde{y})\big) = 1. \quad (9)$$

Note that Equation (9) and $y_0 \in \mathcal{P}(y_0^*)$. We have that $y_1^* = y_0^*$ and thus $y_0^*[V_0(-V_0^{-1}\tilde{y})] = 1$. By the conclusion in the previous part of this proof, we obtain immediately that $x_0^*(-V_0^{-1}\tilde{y}) = 1$, that is $x_0^*(V_0^{-1}\tilde{y}) = -1$. Thus the proof is completed.

We will give the definition of ‘‘sharp corner points’’. These points play an important role in our result concerning the isometric extension problem in Gâteaux differentiability space (in particular, separable spaces or reflexive spaces).

Definition (1.1.8) [1]:

Let Y be normed space. Then $y_0^* \in S_1(Y^*)$ is called a sharp corner point of $B_1(Y^*)$, if it satisfies the following conditions:

(i) For any $y \in S_1(Y)$ with $|y_0^*(y)| < 1$ and $\varepsilon > 0$, there exists $\tilde{y}_\varepsilon \in S_1(Y)$ such that

$$y_0^*(\tilde{y}_\varepsilon) = 1 \text{ and } \|\tilde{y}_\varepsilon \pm y\| \le 1 + |y_0^*(y)| + \varepsilon.$$

(ii) For any $y \in S_1(Y)$ with $0 < |y_0^*(y)| < 1$ and $\varepsilon > 0$, there exists $\bar{y}_\varepsilon \in S_1(Y)$ such that

$$y_0^*(\bar{y}_\varepsilon) = \frac{y_0^*(y)}{|y_0^*(y)|} \text{ and } \|\tilde{y}_\varepsilon - y\| \le 1 - |y_0^*(y)| + \varepsilon.$$

These sharp corner points of $B_1(Y^*)$ are denoted by S $\mathcal{C}(Y^*)$. Then we will give an important lemma as follows.

Lemma (1.1.9) [1]:

Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $y_0^* \in \mathcal{P}(Y^*) \cap$ S $\mathcal{C}(Y^*)$, then we have

$$x_0^*(V_0^{-1}y) = y_0^*(y) \quad \forall y \in S_1(Y).$$

where $x_0^* \in S_1(X^*)$ is the functional obtained in Lemma (1.1.7) .

Proof:

We take two steps to complete the proof:

a. $|y_0^*(y)| = |x_0^*(V_0^{-1}y)|$ for any $y \in S_1(Y)$.

Indeed, for any $y \in S_1(Y)$, we can assume that $|y_0^*(y)| < 1$ (otherwise we can immediately get (a) by Lemma (1.1.7) [1]). Note $y_0 \in$S $\mathcal{C}(Y^*)$ and Lemma (1.1.7) . For any $\varepsilon > 0$, there exists $\tilde{y}_\varepsilon \in S_1(Y)$ such that

$$x_0^*(V_0^{-1}\tilde{y}_\varepsilon) = y_0^*(\tilde{y}_\varepsilon) = 1,$$

and

$$1 \pm x_0^*(V_0^{-1}y) = |\pm 1 - x_0^*(V_0^{-1}y)| = \left|x_0^*\left(V_0^{-1}(\pm\tilde{y}_\varepsilon)\right) - x_0^*(V_0^{-1}y)\right|$$
$$\le \|V_0^{-1}(\pm\tilde{y}_\varepsilon) - V_0^{-1}y\| = \|\tilde{y}_\varepsilon \pm y\| \le 1 + |y_0^*(y)| + \varepsilon.$$

Since ε is arbitrary, we obtain that

$$|x_0^*(V_0^{-1}y)| \le |y_0^*(y)|, \quad \forall y \in S_1(Y).$$

If $|y_0^*(y)| = 0$, it is clear that $|x_0^*(V_0^{-1}y)| = 0$. Otherwise, note that $y \in$S $C(Y)$ and Lemma (1.1.7) . For any $\varepsilon > 0$, there exists $\bar{y}_\varepsilon \in S_1(Y)$ such that

$$|x_0^*(V_0^{-1}\bar{y}_\varepsilon)| = |y_0^*(\bar{y}_\varepsilon)| = 1,$$

and

$$\begin{aligned} 1 - |x_0^*(V_0^{-1}y)| &= |x_0^*(V_0^{-1}\bar{y}_\varepsilon)| - |x_0^*(V_0^{-1}y)| \\ &\le |x_0^*(V_0^{-1}\bar{y}_\varepsilon) - x_0^*(V_0^{-1}y)| \\ &\le \|V_0^{-1}\bar{y}_\varepsilon - V_0^{-1}y\| \\ &= \|\bar{y}_\varepsilon - y\| \le 1 - |y_0^*(y)| + \varepsilon. \end{aligned}$$

Therefore, we get that

$$|y_0^*(y)| \le |x_0^*(V_0^{-1}y)|, \quad \forall y \in S_1(Y)$$

and complete the first step.

b. $y_0^*(y) = x_0^*(V_0^{-1}y)$ for any $y \in S_1(Y)$.

Indeed, if $y_0^*(y) = 0$, then we have $x_0^*(V_0^{-1}y) = 0$ because of (a). Otherwise, note that $y_0^* \in$S $C(Y^*)$ and Lemma (1.1.7) [1]. For any $\varepsilon > 0$, there exists $\bar{y}_\varepsilon \in S_1(Y)$ such that

$$x_0^*(V_0^{-1}\bar{y}_\varepsilon) = y_0^*(\bar{y}_\varepsilon) = \frac{y_0^*(y)}{|y_0^*(y)|},$$

and

$$\begin{aligned} 1 = |y_0^*(\bar{y}_\varepsilon)| &= |x_0^*(V_0^{-1}\bar{y}_\varepsilon)| \le |x_0^*(V_0^{-1}y)| + |x_0^*(V_0^{-1}\bar{y}_\varepsilon) - x_0^*(V_0^{-1}y)| \\ &\le |y_0^*(y)| + |x_0^*(V_0^{-1}\bar{y}_\varepsilon - V_0^{-1}y)| \le |y_0^*(y)| + \|V_0^{-1}\bar{y}_\varepsilon - V_0^{-1}y\| \\ &= |y_0^*(y)| + \|\bar{y}_\varepsilon - y\| \le 1 + \varepsilon. \end{aligned}$$

We can get

$$0 \le |x_0^*(V_0^{-1}\bar{y}_\varepsilon) - x_0^*(V_0^{-1}y)| - (|x_0^*(V_0^{-1}\bar{y}_\varepsilon)| - |x_0^*(V_0^{-1}y)|)$$

that is

$$0 \le \left|\frac{y_0^*(y)}{|y_0^*(y)|} - x_0^*(V_0^{-1}y)\right| - \left(\left|\frac{y_0^*(y)}{|y_0^*(y)|} - x_0^*(V_0^{-1}y)\right|\right) \le \varepsilon.$$

Since ε is arbitrary, we have that $x_0^*(V_0^{-1}y)$ and $y_0^*(y)$ have the same sign because $y_0^*(\bar{y}_\varepsilon) = \frac{y_0^*(y)}{|y_0^*(y)|}$. The proof is completed.

Proposition (1.1.10) [1]:

Let Y be a strictly convex Banach space and $\dim \mathrm{Y} \geq 2$. Then we have that S $\mathcal{C}(Y) = \phi$.

Proof:

It is clear that if $y_0^* \in S_1(Y^*)$, there exists at most one element $y_0^* \in S_1(Y^*)$ such that $y_0^*(y_0) = 1$. Otherwise, if there exists $y_1 \in S_1(Y)$ such that $y_0 \neq y_1$ and $y_0^*(y_1) = 1$, then for any $\lambda \in (0,1)$, we have that

$$1 = y_0^*(\lambda y_0 + (1-\lambda)y_1) \leq \|y_0^*\| \cdot \|\lambda y_0 + (1-\lambda)y_1\| < 1,$$

which is impossible. Assume that S $\mathcal{C}(Y^*) \neq \phi$ and $y \in$S $\mathcal{C}(Y^*)$. Note that $\ker y \neq \{\theta\}$ since $\dim Y \geq 2$. For any $y \in S_1(Y) \cap \ker y_0^*, y \neq \theta$ and $\varepsilon > 0$, there exists unique $\tilde{y}$ such that

$$y_0^*(y) = 1 \quad \text{and} \quad \|y_0 \pm y\| \leq 1 + |y_0^*(y)| + \varepsilon = 1 + \varepsilon.$$

Since ε is arbitrary, we get that $\|y_0 \pm y\| \leq 1$ and

$$2 = \|y_0 + y + y_0 - y\| \leq \|y_0 + y\| + \|y_0 - y\| \leq 2,$$

that is

$$\|y_0 + y + y_0 - y\| = \|y_0 + y\| + \|y_0 - y\|.$$

Since Y is strictly convex, we get that $y_0 + y = y_0 - y$, which is impossible.

Proposition (1.1.11) [1]:

Let Y be a real Banach space. Then any smooth point of $S_1(Y^*)$ is not a sharp corner point.

Proof:

Suppose that f_0 is a smooth point of $S_1(Y^*)$. There is a unique $y_0^{**} \in S_1(Y^{**})$ such that $y_0^{**}(f_0) = 1$. If there does not exist $y_0 \in S_1(Y)$ such that $\mathrm{g}(y_0) = y_0^{**}(\mathrm{g})$ for any $\mathrm{g} \in Y^*$, that is, $A(f) = \phi$, f_0 is clearly not a sharp corner point.

If $y_0 \in S_1(Y)$ given above exists, we assume that f_0 is also a sharp corner point. For any $y \in S_1(Y)$ with $0 < f_0(y) < 1$ and $\varepsilon > 0$, we see that $\| y - y_0 \| \leq 1 - f_0(y) + \varepsilon$, that is,

$$\|y - y_0\| \leq 1 - f_0(y) = f_0(y_0) - f_0(y).$$

Note that $f_0(y_0) - f_0(y) \leq \| y - y_0 \|$. We have that

$$\|y - y_0\| = f_0(y_0) - f_0(y) = f_0(y_0 - y),$$

which implies that

$$f_0\left(\frac{y_0 - y}{\|y - y_0\|}\right) = 1.$$

However, it is impossible since $f_0 \in S_1(Y^*)$ is a smooth point.

Section (1.2): Gâteaux differentiability spaces

In this section, let us recall some results for Gâteaux differentiability space, separable space, Asplund generated space, and so on .

Definition (1.2.1) [1]:

A Banach space E is said to be a Gâteaux differentiability space (weak-Asplund space) if for any continuous convex function f on it, there exists a dense (dense G_δ) subset $E_0 \subseteq E$ such that f is Gâteaux differentiable at any $x_0 \in E_0$.

Proposition (1.2.2) [1]:

A Banach space E is a Gâteaux differentiability space if and only if any weak* compact convex subset of E^* is the weak* closed convex hull of its weak*-exposed points .

Proposition (1.2.3) [1]:

Let E and E_1 be Banach spaces. Suppose that $T : E \to E_1$ is linear and continuous. If E is a Gâteaux differentiability space and $T(E)$ is dense in E_1, then E_1 is also a Gâteaux differentiability space. In particular, if a Banach space F is the image of a Gâteaux differentiability space by a linear continuous mapping, then F is also a Gâteaux differentiability space.

Definition (1.2.4) [1]:

A Banach space E is called Asplund generated if there exists an Asplund space X and a linear continuous operator $T : X \to E$ such that $T(X)$ is dense in E.

Remark (1.2.5) [1]:

Recall that a Banach space E is called an Asplund space if for any continuous convex function f on it, there exists a dense G_δ subset $E_0 \subseteq E$ such that f is Fréchet differentiable at any $x_0 \in E_0$. Moreover, we have the following important facts:

(i) A Banach space E is an Asplund space if and only if E^* has the Radon-Nikodym property.

(ii) All the reflexive spaces [5] that is (Let X be anormed space and X^{**} = (X^*) *denote the second dual vector space of X. the Canonical map $X \to \hat{X}$ define by$\hat{X}(F) = F(X), F \in X^*$ gives an isometric linear isomorphism (embedding) from X into X^{**} the space X is called reflexive if this map is surjective) and $c_0(\Gamma)$ space (for any index set Γ) are Asplund spaces.

Proposition (1.2.6) [1]:

Any weakly compactly generated space is an Asplund generated space. Any subspace of an Asplund generated space is a weak-Asplund space.

Proposition (1.2.7) [1]:

Any separable Banach space is a weak-Asplund space. Moreover, if a Banach space E whose dual space E^* admits a strictly convex norm, then E is also a weak-Asplund space .

Definition (1.2.8) [1]:

Let Ω be a compact space. Then $t_0 \in \Omega$ is called a G_δ-point if there exists a countable collection of open subsets $\{G_n \subseteq \Omega : n \in \mathbb{N}\}$ such that $\{t_0\} = \bigcap_{n=1}^{\infty} G_n$. Ω is said to be scattered if any subset of Ω has an isolated point.

Proposition (1.2.9) [1]:

Let Ω be a compact space. Then $C(\Omega)$ is Asplund if and only if Ω is scattered .

Theorem (1.2.10) [1]:

Let X and Y be normed spaces. Suppose that V_0 is an isometry from $S_1(X)$ into $S_1(Y)$ and

$$\|V_0 x - |\lambda| V_0 y\| \leq \|x - |\lambda|\|, \ \ \forall x, y \in S_1(X), \lambda \in \mathbb{R}.$$

Then V_0 can be extended to an isometry on the whole space. Moreover, if V_0 is surjective, then V_0 can be linearly extended too.

Sketch of proof:

For integrating , we write the main idea of the proof as follows: Let

$$Vx = \begin{cases} \|x\| V_0\left(\frac{x}{\|x\|}\right), & x \neq \theta, \\ \theta, & x = \theta. \end{cases}$$

Then we have that $\|Vx - Vy\| \leq \|x - y\|$ for any $x, y \in S_1(Y)$ and $\|Vx - Vy\| = \|x - y\|$ if $\| x \| = \| y \|$, $x = \theta$ or $y = \theta$. Indeed, V is an isometry. Otherwise, there exist $x_0, y_0 \in X$ with $\|y_0\| > \|x_0\| > 0$ such that $\|Vx_0 - Vy_0\| < \|x - y\|$. We can take $z_0 \in X$ such that $\|z_0\| = \|y_0\|$ and $z_0 \in \overrightarrow{y_0 x_0}$ (the semi-line with the starting point y_0 and crossing x_0). Then we get the following inequality:

$$\|z_0 - y_0\| = \|z_0 - x_0\| + \|x_0 - y_0\| > \|Vz_0 - Vx_0\| + \|Vx_0 - Vy_0\| \geq \|Vz_0 - Vy_0\|,$$

which is impossible. If V_0 is surjective, we can also get a linear isometric extension by the Mazur-Ulam theorem.

We can now show the following.

Theorem (1.2.11) [1]:

Let X be a Banach space and Y be a Gâteaux differentiability space. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $\mathcal{P}(Y^*) \cap \mathrm{S}\ \ \mathcal{C}(Y^*)$ is weak*-dense in $\mathcal{P}(Y^*)$, then V_0 can be extended to a linear isometry on the whole space.

Proof:

For any $x_1, x_2 \in S_1(X)$ and $\lambda \in \mathbb{R}$, we have that

$$\|V_0 x_1 - |\lambda| V_0 x_2\| = \sup_{y^* \in S_1(Y^*)} |y^*(V_0 x_1 - |\lambda| V_0 x_2)|.$$

By Proposition (1.2.2), we get that

$$\|V_0 x_1 - |\lambda| V_0 x_2\| = \sup_{y^* \in S_1(Y^*)} |y_0^*(V_0 x_1 - |\lambda| V_0 x_2)|$$

$$= \sup_{y_0^* \in \mathcal{P}(Y^*) \cap \mathrm{AS}\ \ \mathcal{C}(Y^*)} |y_0^*(V_0 x_1 - |\lambda| V_0 x_2)|. \quad (10)$$

By Lemma (1.1.9) , for any $y_0 \in \mathcal{P}_0(Y^*)$, there exists $x_0^* \in S_1(X^*)$ (x_0^* is obtained in Lemma (1.1.7) such that

$$\begin{aligned}|y_0^*(V_0x_1 - |\lambda|V_0x_2)| &= |y_0^*(V_0x_1) - y_0^*(|\lambda|V_0x_2)| \\ &= |x_0^*(x_1) - x_0^*(|\lambda|x_2)| \\ &\leq \|x_1 - |\lambda|x_2\|. \qquad (11)\end{aligned}$$

Note Equations. (10) and (11). We get immediately that

$$\|V_0x_1 - |\lambda|V_0x_2\| \leq \|x_1 - |\lambda|x_2\|, \quad \forall x_1, x_2 \in S_1(X), \quad \lambda \in \mathbb{R},$$

and complete the proof because of Theorem (1.2.10).

Corollary (1.2.12) [1]:

Let X be a Banach space and Y be a separable Banach space (more generally, Y^* admits a strictly convex norm).

Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $\mathcal{P}(Y^*) \cap \mathrm{S}\ \mathcal{C}(Y^*)$ is weak*-dense in $\mathcal{P}(Y^*)$, then V_0 can be extended to a linear isometry on the whole space.

Corollary (1.2.13) [1]:

Let X be a Banach space and $Y = (\ell^1)$. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. Then V_0 can be extended to a linear isometry on the whole space.

Proof:

Note that Y is separable and Corollary (1.2.12). We only need to check that $\mathcal{P}(Y^*) \subseteq \mathrm{S}\ \mathcal{C}(Y^*)$. It is easy to see that

$$\mathcal{P}(Y^*) = \{\{\theta_n\}: \{\theta_n\} \in (\ell^\infty), \theta_n = \pm\mathbf{1}, n \in \mathbb{N}\}.$$

Let $y_0^* \in \mathcal{P}(Y^*)$ and $y \in S_1(Y)$ with $|y_0^*(y)| < 1$. If $y_0^* = \{\theta_n^0\}$ and $y = \{y(n)\}$, we can take $\tilde{y} = \{\tilde{y}(n)\}$ such that

$$\tilde{y}(n) = \theta_n^0|y(n)|, \quad \forall n \in \mathbb{N}.$$

Then we have that $\{\tilde{y}(n)\} \in S_1(Y), y_0^*(y) = 1$ and

$$\begin{aligned}\|\tilde{y} \pm y\| &= \sum_{n=1}^{\infty} |\tilde{y}(n) \pm y(n)| = \sum_{n=1}^{\infty} |\theta_n^0|y(n)| \pm y(n)| \\ &= \sum_{n=1}^{\infty} \big||y(n)| \pm \theta_n y(n)\big| = \sum_{n=1}^{\infty} |y(n)| \pm \sum_{n=1}^{\infty} \theta_n y(n)\end{aligned}$$

$$= \mathbf{1} \pm y_0^*(y) \leq \mathbf{1} + |y_0^*(y)|.$$

Moreover, if $y_0^*(y) \neq 0$, we can also take $\bar{y} = \frac{y_0^*(y)}{|y_0^*(y)|} \cdot \tilde{y}$ and have that

$$\|\bar{y} - y\| = \sum_{n=1}^{\infty} \left| \frac{y_0^*(y)}{|y_0^*(y)|} \cdot \theta_n^0 |y(n)| - y(n) \right|$$

$$= \sum_{n=1}^{\infty} \left| y(n) - \frac{y_0^*(y)}{|y_0^*(y)|} \cdot \theta_n^0 y(n) \right|$$

$$= \sum_{n=1}^{\infty} |y(n)| - \frac{y_0^*(y)}{|y_0^*(y)|} \sum_{n=1}^{\infty} \theta_n^0 y(n) = 1 - \frac{y_0^*(y)}{|y_0^*(y)|} y_0^*(1) = 1 - |y_0^*(y)|.$$

Then we complete the proof.

Corollary (1.2.14) [1]:

Let X be a Banach space and $Y = (c_0)$. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. Then V_0 can be extended to a linear isometry on the whole space.

Proof:

Note that Y is separable and Corollary (1.2.12) [1]. We only need to check that $\mathcal{P}(Y^*) \subseteq$S $\mathcal{C}(Y^*)$. It is easy to see that

$$p(Y^*) = \{\pm e_n^* : n \in \mathbb{N}\},$$

where $e_n^* = (0, \ldots, 0, \mathbf{1}, 0, \ldots) \in (\ell^1)$ for any $n \in \mathbb{N}$. Let $e_n^* \in \mathcal{P}(Y^*)$ and $y \in S_1(Y)$ with $|e_{n_0}^*(y)| < 1$. We can take $\tilde{y} = e_{n_0} \in S_1(Y)$. Then we have that

$$\|\tilde{y} \pm y\| = \left\| \{e_{n_0}(n) \pm y(n)\} \right\| = \sup_{n \in \mathbb{N}} \left| e_{n_0}(n) \pm y(n) \right|$$

$$\leq \mathbf{1} + |y(n_0)| = 1 + \left| e_{n_0}^*(y) \right|.$$

Moreover, if $e_{n_0}^*(y) \neq 0$, we can take

$$\bar{y} = y + \left(\frac{e_{n_0}^*(y)}{\left| e_{n_0}^*(y) \right|} - e_{n_0}^*(y) \right) e_{n_0} \in S_1(Y).$$

that is, $y = \{\bar{y}(n)\}$ with

$$\bar{y}(n) = \begin{cases} \dfrac{y(n_0)}{|y(n_0)|}, & \text{if } n = n_0, \\ y(n), & \text{if } n \neq n_0. \end{cases}$$

We can get that

$$\|\bar{y} - y\| = \sup_n |\bar{y}(n) - y(n)| = \left| \frac{y(n_0)}{|y(n_0)|} - y(n_0) \right|$$
$$= 1 - |y(n_0)| = 1 - |e_{n_0}^*(y)|.$$

Then we complete the proof.

Corollary (1.2.15) [1]:

Let X be a Banach space and $Y = C(K)$ (K is a compact metric space). Suppose that $Z \subseteq Y$ is a linear closed subspace, and there exists a dense subset $T \subseteq K$ such that all the ''peak functions'' whose peak is $t \in T$ are in Z. If V_0 is an isometric mapping from $S_1(X)$ onto $S_1(Z)$, then V_0 can be extended to a linear isometry on the whole space.

Proof:

Note that $C(K)$ is a separable Banach space and

$$\mathcal{P}(Y^*) = \{\pm\delta_k^* : k \in K\} \ \left(\delta_{k_0}^*(y) = y(k_0) \text{ for every } y = y(k) \in Y\right).$$

It is easy to see that

$$\{\pm\delta_t^* : t \in T\} \subseteq \mathcal{P}(Z^*)$$

and $\{\pm\delta_t^* : t \in T\}$ is weak*-dense in $\mathcal{P}(Z^*)$. By Corollary (1.2.12), we only need to prove that $\delta_{t_0}^* \in$S $\mathcal{C}(Z^*)$ for any $t_0 \in T$ (because it is similar to prove that $-\delta_{t_0}^* \in$S $\mathcal{C}(Z^*)$ for any $t_0 \in T$).

For any $\delta_{t_0}^* \in \mathcal{P}(Y^*), z \in S_1(Z)$ with $|\delta_{t_0}^*(z)| = |z(t_0)| \leq 1$, and $\varepsilon > 0$ (if $z(t_0) \neq 0$, we also assume that $\varepsilon < \frac{|z(t_0)|}{2}$), there exists an open neighborhood $G(t_0)$ of t_0 in K such that

$$|z(k) - z(t_0)| < \varepsilon, \ \forall k \in G(t_0). \qquad (12)$$

By Urysohn's Lemma [6] that is (A topological space X is normed iff for any two nonempty closed disjoint subsets A and B of X there's continuous map $f: X \to [0,1]$ such that $F(A) = \{0\}$ *and* $F(B) = \{1\}$ afunction F with this property is called Urysho nfunction). we can get $y(k) \in C(K)$ such that

$$y(t_0) = \mathbf{1},\ \ y(k) \equiv 0 \ \ \big(\forall k \in K\backslash G(t_0)\big)$$

and

$$0 \le y(k) \le 1,\ \ \forall k \in K.$$

Then we can make a ''peak function'' $\mathrm{p}_{t_0}(k) \in C(K)$ (whose peak is t_0 and $\mathrm{p}_{t_0}(t_0) = 1$), which is equal to 0 on $K\backslash G(t_0)$ and takes non-negative value on K. Let

$$\tilde{z}_\varepsilon(k) = \min\Big(y(k), \mathrm{p}_{t_0}(k)\Big).$$

It is easy to see that $\tilde{z}_\varepsilon(k)$ is also a ''peak function'' on K whose peak is t_0 and $0 \le \tilde{z}_\varepsilon(k) \le 1$, and thus $\tilde{z}_\varepsilon \in S_1(Z)$ by the hypotheses of Z. By (12), we have that $\tilde{z}_\varepsilon \pm z \in Z$ and

$$\begin{aligned}\|\bar{z}_\varepsilon \pm z\| &= \max\Big(\max_{k\in G(t_0)}|\tilde{z}_\varepsilon(k) \pm z(k)|, \max_{k\in K\backslash G(t_0)}|z(k)|\Big)\\ &\le \max\Big(\max_{k\in G(t_0)}|\tilde{z}_\varepsilon(k)| + \max_{k\in G(t_0)}|z(k)|, \max_{k\in K\backslash G(t_0)}|z(k)|\Big)\\ &\le 1 + (|z(t_0)| + \varepsilon) = 1 + \delta^*_{t_0}(z) + \varepsilon.\end{aligned}$$

Moreover, if $\delta^*_{t_0}(z) = z(t_0) \ne 0$, we first change above ''peak function'' $\mathrm{p}_{t_0}(k)$ into $\mathrm{p}_{t_0}(k)$ which may be very sharp in above neighborhood $G(t_0)$, and let it satisfy the following condition:

$$\bar{\mathrm{p}}_{t_0}(k) \le 1 - \frac{|z(k)| - |z(t_0)|}{1 - |z(t_0)|},\ \ \forall k \in G(t_0). \tag{13}$$

When we take

$$\bar{z}_\varepsilon = z + \left(\frac{\delta^*_{t_0}(z)}{\left|\delta^*_{t_0}(z)\right|} - \delta^*_{t_0}(z)\right)\bar{\mathrm{p}}_{t_0},$$

by the hypotheses of Z, we have that $\bar{z}_\varepsilon \in Z$ and

$$\bar{z}_\varepsilon(k) = \begin{cases} \dfrac{z(t_0)}{|z(t_0)|}, & \text{if } k = t_0;\\ z(k) + (1 - |z(t_0)|)\dfrac{z(t_0)}{|z(t_0)|}\bar{\mathrm{p}}_{t_0}(k), & \text{if } k \in G(t_0)\backslash\{t_0\};\\ z(k), & \text{if } k \in K\backslash G(t_0).\end{cases}$$

Note that both $z(k)$ and $(1-|z(t_0)|)\frac{z(t_0)}{|z(t_0)|}\bar{\mathrm{p}}_{t_0}(k)$ have the same sign because of (12). By (13), we obtain that

$$\left|z(k)+(1-|z(t_0)|)\frac{z(t_0)}{|z(t_0)|}\bar{\mathrm{p}}_{t_0}(k)\right| = |z(k)|+(1-|z(t_0)|)\bar{\mathrm{p}}_{t_0}(k)$$
$$\leq 1.$$

Then we have that $\bar{z}_\varepsilon \in S_1(Z), \bar{z}_\varepsilon - z \in Z$ and

$$\|\bar{z}_\varepsilon - z\| = \left\|\left(\frac{\delta^*_{t_0}(z)}{|\delta^*_{t_0}(z)|} - \delta^*_{t_0}(z)\right)\bar{\mathrm{p}}_{t_0}\right\| = 1-|\delta^*_{t_0}(z)|.$$

Then we complete the proof by Corollary (1.2.12).

Theorem (1.2.16) [1]:

Let X be a Banach space and $Y = C(\Omega)$ (Ω is a compact Hausdorff space). Suppose that there exists a dense subset $T \subseteq \Omega$ such that T contains all the G_δ-points of Ω. If a linear closed subspace $Z \subseteq Y$ contains all such ‘‘peak functions’’ whose peak is $t \in T$ and V_0 is an isometric mapping from $S_1(X)$ onto $S_1(Z)$, then V_0 can be extended to a linear isometry on the whole space.

Proof:

It is the case that $\{\pm\delta^*_t : t \in T\} \subseteq \mathcal{P}(Y^*)$ and $\delta^*_t \in$S $\mathcal{C}(Z^*)$ for any $t \in T$ by the similar arguments of Corollary (1.2.13). There exists $x^*_t \in S_1(X^*)$ such that

$$\delta^*_{t_0}(z) = x^*_t(V_0^{-1}z),\ \ \forall z \in S_1(Z),$$

by Lemma (1.1.9) . Note that $\bar{T} = \Omega$. We have

$$\begin{aligned}\|V_0x_1 - |\lambda|V_0x_2\| &= \sup_{\omega\in\Omega}|(V_0x_1)(\omega) - |\lambda|(V_0x_2)(\omega)| \\ &= \sup_{t\in T}|(V_0x_1)(t) - |\lambda|(V_0x_2)(t)| \\ &= \sup_{t\in T}|\delta^*_t(V_0x_1) - |\lambda|\delta^*_t(V_0x_2)| \\ &= \sup_{t\in T}|x^*_t(x_1) - |\lambda|x^*_t(x_2)| \\ &= \|x_1 - |\lambda|x_2\|.\ \ \forall x_1, x_2 \in S_1(X).\end{aligned}$$

Then we complete the proof by Theorem (1.2.10).

Theorem (1.2.17) [1]:

Let X be a Banach space and $Y = c_0(\Gamma), c(\Gamma)$ or $\ell^\infty(\Gamma)$ (Γ is an infinite index set). Suppose that $Z \subseteq Y$ is a linear closed subspace and $\{e_\gamma : \gamma \in \Gamma\} \subseteq Z$. If V_0 is a surjective isometry between $S_1(X)$ and $S_1(Z)$, then V_0 can be extended to a linear isometry on the whole space.

Proof:

Note that $\{\pm e_\gamma^* : \gamma \in \Gamma\} \subseteq \mathcal{P}(Y^*)$ where

$$e_{\gamma_0}^*(e_\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_0; \\ 0, & \text{if } \gamma \neq \gamma_0; \end{cases}$$

for any $\gamma \in \Gamma$. By the similar arguments of Corollary (1.2.15) [1], we have that $e_\gamma^* \in$S $\mathcal{C}^*(Z^*)$ for any $\gamma \in \Gamma$. Therefore there exists $x_\gamma^* \in S_1(X^*)$ such that

$$e_\gamma^* = x_\gamma^*(V_0^{-1}z), \ \ \forall z \in S_1(Z),$$

by Lemma (1.1.9). We can get that

$$\begin{aligned} \|V_0x_1 - |\lambda|V_0x_2\| &= \sup_{\gamma\in\Gamma} |(V_0x_1)(\gamma) - |\lambda|(V_0x_2)(\gamma)| \\ &= \sup_{\gamma\in\Gamma} \left| e_\gamma^*(V_0x_1) - |\lambda| e_\gamma^*(V_0x_2) \right| \\ &= \sup_{\gamma\in\Gamma} \left| x_\gamma^*(V_0x_1) - |\lambda| x_\gamma^*(V_0x_2) \right| \\ &\leq \|x_1 - |\lambda|x_2\|, \ \ \forall x_1, x_2 \in S_1(X). \end{aligned}$$

Then we complete the proof by Theorem (1.2.10).

Theorem (1.2.18) [1]:

Let X be a Banach space and Y be an Asplund generated space. Suppose that V_0 is an isometric mapping from the unit sphere $S_1(X)$ into $S_1(Y)$ which satisfies the following condition:

($*$) For any $x_1, x_2 \in S_1(X)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$,

$\|\lambda_1V_0x_1 + \lambda_2V_0x_2\| = 1 \Longrightarrow \lambda_1V_0x_1 + \lambda_2V_0x_2 \in V_0[S(X)]$.

Let $Z = \overline{\text{span}}\{V_0x : x \in S_1(X)\}$. Suppose that $\mathcal{P}(Z^*) \cap$S $\mathcal{C}(Z^*)$ is weak*-dense in $\mathcal{P}(Z^*)$. Then V_0 can be extended to a linear isometry on the whole space.

Proof:

We first prove that $S_1(Z) = V_0[S_1(X)]$. Note the condition $(*)$ and the equality

$$\sum_{k=1}^{n} \lambda_k V_0 x_k = \left\| \sum_{k=1}^{n-1} \lambda_k V_0 x_k \right\| \sum_{k=1}^{n-1} \frac{\lambda_k}{\left\| \sum_{k=1}^{n-1} \lambda_k V_0 x_k \right\|} V_0 x_k + \lambda_n V_0 x_n.$$

By induction, we get that

$$\left\| \sum_{k=1}^{n} \lambda_k V_0 x_k \right\| = 1 \Rightarrow \sum_{k=1}^{n} \lambda_k V_0 x_k \in V_0[S_1(X)];\ \ \forall x_k \in S_1(X), \lambda_k \in \mathbb{R} (1 \le k \le n), n \in \mathbb{N}.$$

Therefore, we have that

$$S_1(Z) = V_0[S_1(X)].$$

Note Proposition (1.2.6) and that Z is a closed subspace of Y. The conclusion is clear by Theorem (1.2.11) [1].

Chapter 2

Banach Space and α-Large Families

In this chapter we show the notion of $\alpha -$ large families of finite subsets of an infinite set is defined for every countable ordinal number a, extending the known notion of large families. The definition of α -large families is based on the transfinite hierarchy of the Schreier families S_α, $\alpha < \omega_1$. As an application based on those families we construct a reflexive space. $\mathfrak{x}^{\alpha}_{2^{N_0}}, \alpha < \omega_1$with density the continuum, such that every bounded non-norm convergent sequence $\{x_k\}_K$ has subsequence generating ℓ_1^∞ as spreading model.

Section (2.1): α-Large and a Transfinite Sequence of Compact Hereditary Families:

One of the most significant examples of Banach spaces is Tsirelson space, presented in the nineteen seventies. The main property of this space, is that it fails to contain a copy of c_0 or ℓ_p, answering in the negative a problem posed by Banach. It is still an open problem whether there exist Tsirelson type spaces in the non-separable setting. A version of this problem has recently been solved in the negative direction in, namely it was shown that spaces spanned by an uncountable basic sequence such that their norm satisfies an implicit formula, similar to the one of Tsirelson space, always contain a copy of c_0 or ℓ_p. To be more precise, if κ is an uncountable ordinal number, $\mathcal{B}$ is a hereditary and compact family of finite subsets of κ, $0 < \theta < 1$ is a real number, and $\|\cdot\|_{\theta,\mathcal{B}}$, is the unique norm defined on $c_{00}(\kappa)$ satisfying the following implicit formula

$$\|x\|_{\theta,\mathcal{B}} = \max\left\{\|x\|_\infty, \sup\left\{\theta\sum_{i=1}^{n}\|E_i x\|_{\theta,\mathcal{B}} : \{E_i\}_{i=1}^{d} \; is \; \mathcal{B} - \text{admissible}\right\}\right\}$$

then the completion of $\left(c_{00}(\kappa), \|\cdot\|_{\theta,\mathcal{B}}\right)$ contains a copy of c_0 or ℓ_p.

As it seems not possible to have a non-separable space, that strongly resembles Tsirelson space, a natural question is which properties of this space can be transferred to the non-separable setting. Besides being reflexive, one of the main properties of Tsirelson space, is that it admits only ℓ_1 as a spreading model, i.e. every bounded sequence without a norm convergent subsequence has a subsequence that generates a spreading model equivalent to the usual basis of ℓ_1. The main purpose is the construction of a non-separable reflexive Banach space $\mathfrak{X}_{2^{\aleph_0}}$, with the aforementioned property[2].

Theorem (2.1.1) [2]:

There exists a reflexive Banach space $\mathfrak{X}_{2^{\aleph_0}}$ generated by an unconditional basic sequence $\{e_\xi\}_{\xi<2^{\aleph_0}}$, admitting only ℓ_1 as a spreading model.

The construction of this space is based on the notion of α-large families, which is defined as follows. If A is an infinite set, $\mathcal{B}$ is a hereditary and compact family of finite subsets of A and α is a countable ordinal number, we say that $\mathcal{B}$ is α-large, if its restriction on every infinite subset of A, in a certain sense, contains a copy of S_α, the Schreier family of order α. Equivalently, if its restriction on every infinite subset of A, has Cantor-Bendixson index, greater than or equal to $\omega^\alpha + 1$. We prove the existence of such families on the cardinal number $2^{\aleph_0}$, by constructing for $\alpha < \omega_1, \mathcal{G}_\alpha$ an α-large, hereditary and compact family of finite subsets of $\{0, \mathbf{1}\}^{\mathbb{N}}$. We believe that these families are of independent interest, as they retain some of the most important properties of the families $S_\alpha, \alpha < \omega_1$. They are therefore a generalization of the Schreier families, defined on the continuum and a study of them is included here.

We define the notion of α-large families of finite subsets of an infinite set and a brief study of them is given [2].

We devoted to the construction of the families $\{\mathcal{G}_\alpha\}_{\alpha<\omega_1}$. Initially, using the Schreier family S_1 and diagonalization, we recursively define some auxiliary families $\mathrm{G}_\alpha, \alpha < \omega_1$, which are subsets of $[\{0,1\}^{\mathbb{N}}]^{<\omega} \times \{0,1\}^{\mathbb{N}}$. The construction method used, imposes strong Schreier like properties on the families $\mathcal{G}_\alpha$, which are in fact the projection of G_α, on the component $[\{0,1\}^{\mathbb{N}}]^{<\omega}$. Next, properties of these families, which are crucial for the proof of the main result are included, among others, the fact that for $\alpha < \omega_1, \mathcal{G}_\alpha$ is an α-large, compact and hereditary family of finite subsets of $\{0,1\}^{\mathbb{N}}$.Some additional results concerning the similarity of the $\mathcal{G}_\alpha$ to the $S_\alpha, \alpha < \omega_1$, are proven [2].

We concentrated on the construction of the space $\mathfrak{X}_{2^{\aleph_0}}$. The first step is the definition of a sequence of spaces $\{(X_n, \|\cdot\|_n)\}_n$, each one based on the family $\mathcal{G}_\alpha$. Inparticular, the norm of these spaces is defined on $c_{00}(2^{\aleph_0})$ in a similar manner as the norm of Schreier space is defined on $c_{00}(\mathbb{N})$ and they all have the unit vector basis $\{e_\xi\}_{\xi<2^{\aleph_0}}$ as an unconditional Schauder basis. For $n \in \mathbb{N}$, the main two properties of the space X_n are the following. Firstly, every subsequence of the basis admits only ℓ^1 as a spreading model and secondly the space X_n is c_0 saturated. Next, using the spaces $X_n, n \in \mathbb{N}$ and Tsirelson space T, a norm is defined on $c_{00}(2^{\aleph_0})$, in the following manner. For $x \in c_{00}(2^{\aleph_0})$, set

$$\|x\| = \left\| \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n e_n \right\|_T .$$

The completion of $c_{00}(2^{\aleph_0})$ with this norm is the desired space $\mathfrak{X}_{2^{\aleph_0}}$, which has the unit vector basis $\{e_\xi\}_{\xi<2^{\aleph_0}}$ as an unconditional Schauderbasis. The proof of the fact that this space admits only ℓ^1 as a spreading model, relies on the study of the behavior of the $\|\cdot\|_n$ norms on a normalized weakly null sequence $\{x_k\}_k$ in $\mathfrak{X}_{2^{\aleph_0}}$. Moreover, using the fact that the spaces X_n are c_0 saturated, we prove that every subspace of $\mathfrak{X}_{2^{\aleph_0}}$ contains a copy of a subspace of T, which yields that the space is reflexive [2].

We concerns the construction, for $\alpha < \omega_1$, of reflexive spaces $\mathfrak{X}_{2^{\aleph_0}}$ having an unconditional Schauderbasis with size $2^{\aleph_0}$, admitting ℓ_1^α as a unique spreading model. The construction method used is a variation of the one used for the space $\mathfrak{X}_{2^{\aleph_0}}$.

We introduce the notion of α-large families which concerns the complexity of a family $\mathcal{B}$ of finite subsets of a given infinite set A. This notion extends the well known concept of large families and it is defined using the transfinite hierarchy of the Schreier families $\{S_\alpha\}_{\alpha<\omega_1}$ After providing the definition of α-large families we also give a useful characterization linking this notion with the Cantor-Bendixson index of a compact and hereditary family of finite subsets of a given infinite set.

Let A be a set, $\mathcal{B}$ be a family of subsets of A, $\mathcal{B}$ be a subset of A and k be a natural number. We define

$$[B]^k = \{F \subseteq B:\ \# F = k\}$$

and

$$\mathcal{B} \upharpoonright B = \{F \in \mathcal{B}: F \subset B\}.$$

If $\mathcal{F}$ is a family of subsets of the natural numbers, L is an infinite subset of $\mathbb{N}$ and $\phi : \mathbb{N} \to L$ is the uniquely defined order preserving bijection, we define

$$\mathcal{F}[L] = \{\phi(F): F \in \mathcal{F}\}.$$

Definition (2.1.2) [2]:

Let A be an infinite set and $\mathcal{B}$ a family of finite subsets of A.

(i) We say that B is large, if for every $k \in \mathbb{N}$, and $\mathcal{B}$ infinite subset of A, we have that $[B]^k \cap \mathcal{B} \neq \phi$.

(ii) Given a countable ordinal number α, we say that $\mathcal{B}$ is α-large, if for every B infinite subset of A, there exists a one to one map $\phi : \mathbb{N} \to B$, such that $\phi(F) \in \mathcal{B}$, for every $F \in S_\alpha$.

Lemma (2.1.3) [2]:

Let $\mathcal{F}, \mathcal{G}$ be hereditary and compact families of finite subsets of the natural numbers, such that for every L infinite subset of the natural numbers, the Cantor-Bendixson index of $\mathcal{F} \upharpoonright L$, is strictly smaller than the Cantor-Bendixson index of $\mathcal{G} \upharpoonright L$. Then for every M infinite subset of the natural numbers, there exists L a further infinite subset of M, such that $\mathcal{F} \upharpoonright L \subseteq \mathcal{G} \upharpoonright L$.

Proposition (2.1.4) [2]:

Let A be an infinite set, $\mathcal{B}$ be a hereditary and compact family of finite subsets of A and α be a countable ordinal number. Then, the following assertions are equivalent:

(i) $\mathcal{B}$ is α-large.

(ii) For every B infinite subset of A, the Cantor-Bendixson index of $\mathcal{B} \upharpoonright B$ is greater than or equal to $\omega^\alpha + 1$.

Proof:

Given that (i) holds, (ii) is an immediate consequence of the fact that the Cantor-Bendixson index of S_α is equal to $\omega^\alpha + 1$ for every countable ordinal number α .

For the converse, we may clearly assume that $\mathcal{B}$ is a hereditary and compact family of finite subsets of the natural numbers. For a given countable ordinal α, if (ii) holds, we shall prove the following statement.

For every infinite subset of the natural numbers M, there exists L an infinite subset of M, such that $S_\alpha[L] \subset \mathcal{B}$.

The desired result evidently follows from the above. To prove this statement, we distinguish three cases.

Case (1):$\alpha = 1$:

Assume that for every infinite subset of the natural numbers M, the Cantor-Bendixson index of $\mathcal{B} \restriction M$ is infinite. This means that every such M contains as subsets elements of $\mathcal{B}$, of unbounded cardinality. Since $\mathcal{B}$ is hereditary, we conclude that it is large and therefore it also is 1-large.

Case (2): α is a limit ordinal number:

Then there is $\{\beta_k\}_k$ a strictly increasing sequence of ordinal numbers with $\sup_k \beta_k = \alpha$, such that $S_\alpha = \cup_k \{F \in S_{\beta_k} : \min F \geq k\}$.

Using Lemma (2.1.3) ,choose $L_1 \supset \cdots \supset L_k \supset \cdots$ infinite subsets of M, such that $S_{\beta_k} \restriction L_k \subset \mathcal{B}$, for all k.

Choose $L = \{\ell_1 < \cdots < \ell_k < \cdots\}$ an infinite subsets of M, with $\ell_m \in L_k$, for every $m \geq k$. It is not hard to check that $S_\alpha[L] \subset \mathcal{B}$.

Case (3): α is a successor ordinal number:

If $\alpha = \beta + 1$, then the following holds.

For every M infinite subset of the naturals and $n \in \mathbb{N}$, there exists L a further infinite subset of M, such that $(S_\beta * \mathcal{A}_n) \restriction L \subset \mathcal{B}$, where

$$S_\beta * \mathcal{A}_n = \left\{ \bigcup_{i=1}^{n} F_i \in S_\beta, i = 1, \dots, n \right\}.$$

The above statement follows form Lemma (2.1.3) and the fact that the Cantor-Bendixson index of $S_\beta * \mathcal{A}_n$ is equal to $\omega^\beta n + 1 < \omega^\alpha$.

Therefore, given M an infinite subset of the natural numbers, we may choose $L_1 \supset \cdots \supset L_n \supset \cdots$ infinite subsets of M such that $(S_\beta * \mathcal{A}_n) \restriction L_n \subset \mathcal{B}$.

Choose $L = \{\ell_1 < \cdots < \ell_n < \cdots\}$ an infinite subsets of M, with $\ell_m \in L_n$, for every $m \geq n$. Once more, it is not hard to check that $S_\alpha[L] \subset \mathcal{B}$.

In this section we define a transfinite sequence $\mathcal{G}_\alpha, \alpha < \omega_1$ of compact and hereditary families of finite subsets of $\{0,1\}^{\mathbb{N}}$ with each $\mathcal{G}_\alpha$ being α-large for $\alpha < \omega_1$. We shall first recursively define an auxiliary transfinite sequence $\{\mathrm{G}_\alpha\}_{\alpha<\omega_1}$ of subsets of $[\{0,1\}^{\mathbb{N}}]^{<\omega} \times \{0,1\}^{\mathbb{N}}$, which will then be used to define the $\mathcal{G}_\alpha$ for $\alpha < \omega_1$. We then prove the main properties of these families and we conclude this section by showing the $\mathcal{G}_\alpha$ have some similar properties to the Schreier families S_α.

For $\sigma = \{\sigma(i)\}_{i=1}^\infty$ and $\tau = \{\tau(i)\}_{i=1}^\infty$ in $\{0,1\}^{\mathbb{N}}$, we define $\sigma \wedge \tau$ and $|\sigma \wedge \tau|$ as follows:

(i) $\sigma \wedge \tau$ σ and $|\sigma \wedge \tau| = \infty$, if $\sigma = \tau$.

(ii) $\sigma \wedge \tau = \phi$ and $|\sigma \wedge \tau| = 0$, if $\sigma(1) \neq \tau(1)$.

(iii) $\sigma \wedge \tau = \{\sigma(i)\}_{i=1}^\infty$ and $|\sigma \wedge \tau| = \ell$, if $\sigma \neq \tau, \sigma(1) = \tau(1)$ and $\ell = \min\{i \in \mathbb{N}: \sigma(i+1) \neq \tau(i+1)\}$.

For $s = \{s(i)\}_{i=1}^k$ and $t = \{t(i)\}_{i=1}^\ell$ finite sequences of 0's and 1's, we say that s is an initial segment of t and write $s \sqsubseteq t$, if $k \leq \ell$ and $s(i) = t(i)$ for $i = 1, \dots, k$. We say that s is a proper initial segment of t and write $s \subsetneq t$, if $s \sqsubseteq t$ and $s \neq t$.

Definition (2.1.5) [2]:

We define G_α to be all pairs (F,σ), where $F = \{\tau_i\}_{i=1}^{d} \in [\{0,1\}^{\mathbb{N}}]^{<\omega}, d \in \mathbb{N}$ and $\sigma \in \{0,1\}^{\mathbb{N}}$, such that the following are satisfied:

(i) $\sigma \neq \tau_i$ for $i = 1, \dots, d$.

(ii) $\sigma \wedge \tau_1 \neq \phi$ and if $d > 1$, then $\sigma \wedge \tau_1 \subsetneq \sigma \wedge \tau_2 \subsetneq \cdots \subsetneq \sigma \wedge \tau_d$.

(iii) $d \leq |\sigma \wedge \tau_1|$.

Define $\widetilde{\min}(F,\sigma) = |\sigma \wedge \tau_1|$ and $\widetilde{\max}(F,\sigma) = |\sigma \wedge \tau_d|$.

Assume that α is a countable ordinal number, G_β have been defined for $\beta < \alpha$ and that for $(F,\sigma) \in G_\beta$, $\widetilde{\min}(F,\sigma)$ and $\widetilde{\max}(F,\sigma)$ have also been defined.

Definition (2.1.6) [2]:

Let $\beta < \alpha, (F_i,\sigma_i)_{i=1}^{d}, d \in \mathbb{N}$ be a finite sequence of elements of G_β and $\sigma \in \{0,1\}^{\mathbb{N}}$. We say that $(F_i,\sigma_i)_{i=1}^{d}$ is a skipped branching of σ in G_β, if the following are satisfied:

(i) The $F_i, i = 1, \dots, d$ are pairwise disjoint.

(ii) $\sigma \neq \sigma_i$ for $i = 1, \dots, d$.

(iii) $\sigma \wedge \sigma \neq \phi$ and if $d > 1$, then $\sigma \wedge \sigma_1 \subsetneq \sigma \wedge \sigma_2 \subsetneq \cdots \subsetneq \sigma \wedge \sigma_d$.

(iv) $|\sigma \wedge \sigma| < \widetilde{\min}(F_i,\sigma_i)$ for $i = 1, \dots, d-1$.

(v) $d \leq |\sigma \wedge \sigma_1|$.

Definition (2.1.7) [2]:

Let $\beta < \alpha, \sigma \in \{0,1\}^{\mathbb{N}}$ and $(F_i,\sigma)_{i=1}^{d}, d \in \mathbb{N}$ be a finite sequence of elements of G_β. We say that $(F_i,\sigma)_{i=1}^{d}$ is an attached branching of σ in G_β if the following are satisfied:

i. The $F_i = 1, \dots, d$ are pairwise disjoint.

ii. If $d > 1$, then $\widetilde{\max}(F_i,\sigma) < \widetilde{\min}(F_{i+1},\sigma)$, for $i = 1, \dots, d-1$.

iii. $d \leq \widetilde{\min}(F_1,\sigma)$.

We are now ready to define G $_{\alpha}$, distinguishing two cases.

Definition (2.1.8) [2]:

If α is a successor ordinal number with $\alpha = \beta + 1$, we define G $_{\alpha}$ to be all pairs (F, σ) where $F \in [\{0,1\}^{\mathbb{N}}]^{<\omega}$ and $\sigma \in \{0, \mathbf{1}\}^{\mathbb{N}}$, such that one of the following is satisfied:

(i) $(F, \sigma) \in$G $_{\beta}$.

(ii) There is $(F_i, \sigma_i)_{i=1}^{d}$ a skipped branching of σ in G $_{\beta}$ such that $F = \bigcup_{i=1}^{d} F_i$.

In this case we say that (F, σ) is skipped. Moreover set $\widetilde{\min}(F, \sigma) = |\sigma \wedge \sigma_1|$ and $\widetilde{\max}(F, \sigma) = |\sigma \wedge \sigma_d|$.

(iii) There is $(F_i, \sigma)_{i=1}^{d}$ an attached branching of σ in G $_{\beta}$ such that $F = \bigcup_{i=1}^{d} F_i$.

In this case we say that (F, σ) is attached. Moreover set $\widetilde{\min}(F, \sigma) = \widetilde{\min}(F_1, \sigma)$ and $\widetilde{\max}(F, \sigma) = \widetilde{\max}(F_d, \sigma)$.

If α is a limit ordinal number, fix $\{\beta_n\}_n$ a strictly increasing sequence of ordinal numbers with $\sup_n \beta_n = \alpha$.

We define

$$\mathcal{G}_{\alpha} = \bigcup_{n=1}^{\infty} \{(F, \sigma) \in \mathcal{G}_{\beta} : \widetilde{\min}(F, \sigma) \geq n\}.$$

Remark (2.1.9) [2]:

If α is a limit ordinal number, the sequence $\{\beta_n\}_n$ may be chosen in such a manner that the following are satisfied:

$$\mathcal{G}_{\alpha} = \bigcup_{n=1}^{\infty} \{(F, \sigma) \in \mathcal{G}_{\beta_n} : \widetilde{\min}(F, \sigma) \geq n\}$$

and

$$S_{\alpha} = \bigcup_{n=1}^{\infty} \{F \in S_{\beta_n} : \min F \geq n\}.$$

From now on, we shall assume that this is the case.

Remark (2.1.10) [2]:

Translating Definitions (2.1.5) , (2.1.6) , (2.1.7) and (2.1.8) one obtains the following:

(i) If $(F,\sigma) \in G_1$, then $\#F \leq \widetilde{\min}(F,\sigma)$.

(ii) If $(F,\sigma) \in G_{\beta+1}$ and $(F_i,\sigma_i)_{i=1}^{d}$ is a skipped branching of σ in G_β such that $F = \bigcup_{i=1}^{d} F_i$, then we have that $d \leq \widetilde{\min}(F,\sigma)$.

(iii) If $(F,\sigma) \in G_{\beta+1}$ and $(F_i,\sigma)_{i=1}^{d}$ is an attached branching of σ in G_β such that $F = \bigcup_{i=1}^{d} F_i$, then we have that $d \leq \widetilde{\min}(F,\sigma)$.

We now proceed to prove some key properties of the families G_β.

Lemma (2.1.11) [2]:

Let $\sigma, \sigma', \tau \in \{0,1\}^{\mathbb{N}}$, not all equal. The following are equivalent:

(i) $\sigma \wedge \tau \subsetneq \sigma \wedge \sigma'$.

(ii) $\sigma \wedge \tau = \sigma' \wedge \tau$.

Proof:

Assume that (i) holds. We have that $\tau(j) = \sigma(j) = \sigma'(j)$, for $j = 1, \dots, |\sigma \wedge \tau|$. Whereas, for $j = |\sigma \wedge \tau| + \mathbf{1}$, we have that $\tau(j) \neq \sigma(j) = \sigma'(j)$. Therefore, $|\sigma' \wedge \tau| = |\sigma \wedge \tau|$, which means that $\sigma \wedge \tau = \sigma' \wedge \tau$.

The inverse is proved similarly.

Lemma (2.1.12) [2]:

Let α be a countable ordinal number and $(F,\sigma) \in G_\alpha$. Then there exist τ_m, τ_M in F such that the following are satisfied:

(i) $\widetilde{\min}(F,\sigma) = |\sigma \wedge \sigma_m|$ and $\widetilde{\max}(F,\sigma) = |\sigma \wedge \sigma_M|$.

(ii) For $\tau \in F$ we have that $\sigma \wedge \tau_m \sqsubseteq \sigma \wedge \tau \sqsubseteq \sigma \wedge \tau_M$.

Moreover, if α is a successor ordinal number with $\alpha = \beta + 1$ the following hold:

(iii) If (F,σ) is skipped and $(F_i,\sigma_i)_{i=1}^{d}$ is a skipped branching of σ in G $_\beta$ such that $F=\bigcup_{i=1}^{d}F_i$, then for $i=1,\dots,d$ and $\tau\in F_i$, we have that $\sigma\wedge\sigma_i=\sigma\wedge\tau$.

(iv) If (F,σ) is attached and $(F_i,\sigma_i)_{i=1}^{d}$ is an attached branching of σ in G $_\beta$ such that $F=\bigcup_{i=1}^{d}F_i$, then for $1\le i<j\le d$ and $\tau_1\in F_i,\tau_2\in F_j$, we have that $\sigma\wedge\tau_1\subsetneq\sigma\wedge\tau_2$.

Proof:

We prove this lemma by transfinite induction. For $\alpha=1$ the desired result follows immediately from the definition of G $_1$. Assume now that α is a countable ordinal number and that the statement holds for every $(F,\sigma)\in$G $_\beta$, for every $\beta<\alpha$. If α is a limit ordinal number, then the result follows trivially from the inductive assumption and the definition of G $_\alpha$. Assume therefore that $\alpha=\beta+1$ and let $(F,\sigma)\in$G $_\alpha$.

We treat first the case when (F,σ) is skipped. Let $(F_i,\sigma_i)_{i=1}^{d}$ be a skipped branching of σ in G $_\beta$, such that $F=\bigcup_{i=1}^{d}F_i$.

We first prove part (iii), i.e. for $\tau\in F_i$, we have that $\sigma\wedge\sigma_i=\sigma\wedge\tau, i=1,\dots,d$.

By the inductive assumption, there exists $\tau_m^i\in F_i$ such that $\widetilde{\min}(F_i,\sigma_i)=|\tau_1\wedge\tau_m^i|$ and for every $\tau\in F_i$, we have that $\sigma_i\wedge\tau_m^i\sqsubseteq\sigma_i\wedge\tau$.

Since, by definition, $|\sigma\wedge\sigma_i|<\widetilde{\min}(F_i,\sigma_i)=|\sigma_1\wedge\tau_m^i|\le|\sigma_i\wedge\tau|$, it follows that $\sigma\wedge\sigma_i\subsetneq\sigma_i\wedge\tau$ and by Lemma (2.1.11) $\sigma\wedge\sigma_i=\sigma\wedge\tau$.

Choosing any $\tau_m\in F_1$ and $\tau_M\in F_d$, it is easy to see that (i) and (ii) are satisfied.

Assume now that (F,σ) is attached. Let $(F_i,\sigma)_{i=1}^{d}$ be an attached branching of σ in G $_\beta$, such that $F=\bigcup_{i=1}^{d}F_i$.

By the inductive assumption, there exist $\tau_m^i,\tau_M^i\in F_i$ such that $\widetilde{\min}(F_i,\sigma)=|\sigma\wedge\tau_m^i|,\widetilde{\max}(F_i,\sigma)=|\sigma\wedge\tau_M^i|$ and for every $\tau\in F_i$ we have that $\sigma\wedge\tau_m^i\sqsubseteq\sigma\wedge\tau_M^i$.

We will show that for $1 \leq i < j \leq d$, we have that $\sigma \wedge \tau_M^i \subsetneq \sigma \wedge \tau_m^j$. This proves both (iv) and that $\tau_m = \tau_m^1, \tau_M = \tau_M^d$ have the desired properties.

However, this follows immediately from the fact that $|\sigma \wedge \tau_M^i| = \widetilde{\max}(F_i, \sigma) < \widetilde{\min}(F_j, \sigma) = |\sigma \wedge \tau_m^j|$.

The following result is an immediate consequence of Lemma (2.1.12) .

Corollary (2.1.13) [2]:

Let α be a countable ordinal number and $(F, \sigma) \in$G $_\alpha$. Then the following hold:

i. $\widetilde{\min}(F, \sigma) = \min\{|\sigma \wedge \tau|: \tau \in F\}$.
ii. $\widetilde{\max}(F, \sigma) = \max\{|\sigma \wedge \tau|: \tau \in F\}$.

Corollary (2.1.14) [2]:

Let α be a countable ordinal number and $(F, \sigma) \in$G $_\alpha$, such that $\#F \geq 2$. Then

$$\widetilde{\min}(F, \sigma) \leq \min\{|\tau_1, \tau_2|: \tau_1, \tau_2 \in F \text{ with } \tau_1 \neq \tau_2\}.$$

Proof:

Let $\tau_1 \neq \tau_2$ be in F. By Lemma (2.1.12), there exists $\tau_m \in F$, such that $\widetilde{\min}(F, \sigma) = |\sigma \wedge \tau_m|$ and $\sigma \wedge \tau_m \sqsubseteq \sigma \wedge \tau_1$ as well as $\sigma \wedge \tau_m \sqsubseteq \sigma \wedge \tau_2$. It follows that $\sigma \wedge \tau_m \sqsubseteq \tau_1 \wedge \tau_2$. We conclude that min $(F, \sigma) \leq |\tau_1 \wedge \tau_2|$.

Lemma (2.1.15) [2]:

Let α be a countable ordinal number and $(F, \sigma) \in$G $_\alpha$, such that $\#F \geq 2$. Then there exists $\sigma^1 \in \{0,1\}^{\mathbb{N}}$, such that $(F, \sigma') \in$G $_\alpha$ and

$$\widetilde{\min}(F, \sigma') \leq \min\{|\tau_1, \tau_2|: \tau_1, \tau_2 \in F \text{ with } \tau_1 \neq \tau_2\}.$$

Proof:

We prove this lemma by transfinite induction on α. Assume that $\alpha = 1, (F, \sigma) \in$G $_1$, such that $\#F \geq 2$ and $F = \{\tau_i\}_{i=1}^d, d \geq 2$ such that the assumptions of Definition (2.1.5) are satisfied. Then $\sigma \wedge \tau_1 \subsetneq \sigma \wedge \tau_2$ and by Lemma (2.1.11) we have that $\sigma \wedge \tau_1 = \tau_1 \wedge \tau_2$. We conclude that $\widetilde{\min}(F, \sigma) = |\sigma \wedge \tau_1| = |\tau_1 \wedge \tau_2|$. Corollary (2.1.14) yields that $\widetilde{\min}(F, \sigma) = \min\{|\tau_1 \wedge \tau_2| : \tau_1, \tau_2 \in F \text{ with } \tau_1 \neq \tau_2\}$ and hence, the desired σ' is σ itself.

Assume now that α is a countable ordinal number and that the conclusion holds for every $\beta < \alpha$.

If α is a limit ordinal number, choose $\{\beta_n\}_n$ a strictly increasing sequence of ordinal numbers with $\sup_n \beta_n = \alpha$, such that the assumptions of Definition (2.1.8) are satisfied. Let $(F, \sigma) \in G_\beta$ with $\#F \leq 2$. Then there is $n \in \mathbb{N}$ such that $(F, \sigma) \in G_{\beta_n}$ and $\widetilde{\min}(F, \sigma) \geq n$. Corollary (2.1.14) yields the following:

$$\min\{|\tau_1, \tau_2| : \tau_1, \tau_2 \in F \text{ with } \tau_1 \neq \tau_2\} \geq n. \qquad (1)$$

By the inductive assumption, there exists $\sigma' \in (F, \sigma') \in G_{\beta_n}$ and $\widetilde{\min}(F, \sigma) \leq \min\{|\tau_1, \tau_2| : \tau_1, \tau_2 \in F \text{ with } \tau_1 \neq \tau_2\}$.. By (2) we have that $\widetilde{\min}(F, \sigma') \in G_\alpha$.

Assume now that α is a successor ordinal number with $\alpha = \beta + 1$ and let
$(F, \sigma) \in G_\alpha$ with $\#F \geq 2$. If $(F, \sigma) \in G_\beta$, then the inductive assumption yields the desired result. If this is not the case, then (F, σ) is either skipped, or attached. If it is attached, then there is $(F_i, \sigma_i)_{i=1}^d$ an attached branching of σ, such that $F = \bigcup_{i=1}^d F_i$. If $d = 1$, then $(F, \sigma_1) \in G_\beta$ and by the inductive assumption we are done. Otherwise, choose $\tau_1 \in F_1, \tau_2 \in F_2$. Lemma (2.1.12) (iii) yields that $\sigma \wedge \tau_1 = \sigma \wedge \sigma_1 \subsetneq \sigma \wedge \sigma_2 = \sigma \wedge \tau_2$ and by Lemma (2.1.11) we have that $\sigma \wedge \tau_1 = \tau_1 \wedge \tau_2$. We conclude that $\widetilde{\min}(F, \sigma) = |\sigma \wedge \sigma_1| = |\sigma \wedge \tau_1| = |\tau_1 \wedge \tau_2|$ and therefore, applying Corollary (2.1.14) we have that σ is the desired σ'.

If on the other hand (F, σ) is attached, using similar reasoning, Lemma (2.1.12) (iv) and Corollary (2.1.3) , we conclude the desired result.

Corollary (2.1.16) [2]:

Let $\{(F_k, \sigma_k)\}_k$ be a sequence in $\bigcup_{\beta<\omega_1} G_\beta$ with $\{\widetilde{\min}(F_k, \sigma_k)\}_k$ tending to infinity. Then, if F is an accumulation point of $\{F_k\}_k$, we have that $\#F \leq 1$.

Proof:

Let F be an accumulation point of $\{F_k\}_k$, and assume that there are $\tau_1 \neq \tau_2$ in F. Then there exists L an infinite subset of the natural numbers, such that $\tau_1, \tau_2 \in F_k$, for every $k \in L$. Corollary (2.1.4) yields that $|\tau_1 \wedge \tau_2| \geq \widetilde{\min}(F_k, \sigma_k)$, for all $k \in L$. We conclude that $|\tau_1 \wedge \tau_2| = \infty$,i.e. $\tau_1 = \tau_2$, a contradiction.

The following two lemmas will both be useful in the sequel.

Lemma (2.1.17) [2]:

Let α be a countable ordinal number and $(F, \sigma) \in G_\beta$. Let also $\sigma' \in \{0,1\}^{\mathbb{N}}$, such that $\sigma' \wedge \tau = \sigma \wedge \tau$ for all $\tau \in F$. Then the following hold:

(i) $(F, \sigma') \in G_\alpha$.

(ii) $\widetilde{\min}(F, \sigma') = \widetilde{\min}(F, \sigma)$ *and* $\widetilde{\max}(F, \sigma') = \widetilde{\max}(F, \sigma)$.

Proof:

We prove this lemma by transfinite induction. The case $\alpha = 1$ follows easily from the definition of G_1. Assume now that the result holds for every $\beta < \alpha$. The case where α is a limit ordinal number is trivial, assume therefore that $\alpha = \beta + 1$ and let $(F, \sigma) \in G_\alpha \in \{0,1\}^{\mathbb{N}}$ such that the assumptions of the lemma are satisfied. Notice that it is enough to show that (i) is true, since part (ii) of the conclusion follows immediately from (i) and Corollary (2.1.13).

We treat first the case when (F, σ) is skipped, i.e. there exists $(F_i, \sigma_i)_{i=1}^d$ a skipped branching of σ in G_β, with $F = \bigcup_{i=1}^d F_i$. To show that $(F, \sigma') \in G_\alpha$, it suffices to show that $(F_i, \sigma_i)_{i=1}^d$ is a skipped branching of σ'.

Notice that it is enough to show that $\sigma \wedge \sigma_i = \sigma' \wedge \sigma_i$ for $i = 1, \dots, d$, which, by Lemma (2.1.11), is equivalent to $\sigma \wedge \sigma_i \subsetneq \sigma \wedge \sigma'$ for $i = 1, \dots, d$.

Fix $1 \leq i \leq d$ and chose $\tau \in F_i$. Lemma (2.1.12) (iii) yields that $\sigma \wedge \sigma_i = \sigma \wedge \tau = \sigma' \wedge \tau$. Once more, Lemma (2.1.11) [2] yields that $\sigma \wedge \sigma_i = \sigma \wedge \tau \subsetneq \sigma \wedge \sigma'$.

Assume now that (F, σ) is attached, i.e., there exists $(F_i, \sigma')_{i=1}^{d}$ an attached branching of σ in G_β, with $F = \bigcup_{i=1}^{d} F_i$. Since, by the inductive assumption, the conclusion holds for the $(F_i, \sigma), i = 1, \dots, d, \sigma'$ it is straightforward to check that $(F_i, \sigma')_{i=1}^{d}$ an attached branching of σ' in G_β and therefore $(F, \sigma') \in G_\alpha$.

Lemma (2.1.18) [2]:

Let $(F, \sigma) \in \bigcup_{\beta < \omega_1} G_\beta$ and $\sigma' \in \{0,1\}^{\mathbb{N}}$ such that $\sigma \wedge \tau \subsetneq \sigma' \wedge \tau$ for all $\tau \in F$. Then, if $\alpha = \min\{\beta: (F, \sigma) \in G_\beta\}$, α is not a limit ordinal number and the following hold:

(i) If $\alpha = 1$, then $\# F = 1$.
(ii) If $\alpha = \beta + 1$, then there exists $\sigma'' \in \{0,1\}^{\mathbb{N}}$ with $(F, \sigma'') \in G_\beta$.

Proof:

The fact that α is not a limit ordinal number follows trivially from Definition (2.1.8) . The case $\alpha = 1$ is easy, we shall therefore only prove the case $\alpha = \beta + 1$. Since $(F, \sigma) \notin G_\beta$, it is either skipped or attached.

Assume first that there is $(F_i, \sigma_i)_{i=1}^{d}$ a skipped branching of σ in G_β with $F = \bigcup_{i=1}^{d} F_i$. If $d = 1$, then $\sigma'' = \sigma_1$ is evidently the desired element of $\{0,1\}^{\mathbb{N}}$. We will therefore prove that $d = 1$. Towards a contradiction, assume that $d \geq 2$ and choose $\tau_1 \in F_1, \tau_2 \in F_2$.

Lemma (2.1.12) (iii) yields that $\sigma \wedge \tau_1 = \sigma \wedge \sigma_1 \subsetneq \sigma \wedge \sigma_2 = \sigma \wedge \tau_2$. By the assumption, $\sigma \wedge \tau_1 \subsetneq \sigma' \wedge \tau_1$ and using Lemma (2.1.11) we conclude that $\sigma \wedge \tau_1 = \sigma \wedge \sigma'$. Similarly, we conclude that $\sigma \wedge \tau_2 = \sigma \wedge \sigma'$. We have shown that $\sigma \wedge \sigma' \subsetneq \sigma \wedge \sigma'$, which is absurd.

If (F,σ) is attached, then using similar arguments and Lemma (2.1.12) (iv), one can prove the desired result.

Proposition (2.1.19) [2]:

Let α be a countable ordinal number, $(F,\sigma) \in \mathrm{G}_{\beta}$ and G be a non-empty subset of F. Then $(G,\sigma) \in \mathrm{G}_{\beta}$.

Proof:

We proceed by transfinite induction. For $\alpha = 1$ the result easily follows from the definition of G_1. Assume that the statement is true for every $\beta < \alpha$. The case when α is a limit ordinal number is an easy consequence of the inductive assumption and Corollary (2.1.13) . Assume therefore that $\alpha = \beta + 1$ and let (F,σ) be in G_{α} and $G \subset F$.

Consider first the case, when (F,σ) is skipped and $(F_i)_{i=1}^{d}$ be a skipped branching of σ in G_{β}, such that $F = \bigcup_{i=1}^{d} F_i$.

Set $\{i_1 < \cdots < i_p\} = \{i \in \{1,\ldots,d\}: G \cap F_i \neq \phi\}$ and $G_j = G \cap F_{i_j}$ for $j = 1,\ldots,p$. By the inductive assumption, $\left(G_j, \sigma_{i_j}\right)$ is in G_{β} for $j = 1,\ldots,p$ and, evidently, it is enough to show that $\left(G_j, \sigma_{i_j}\right)_{j=1}^{p}$ is a skipped branching of σ. Obviously, assumptions (i), (ii) and (iii) from Definition (2.1.6) are satisfied.

Corollary (2.1.13) yields that $\widetilde{\min}\left(F_{i_j}, \sigma_{i_j}\right) \leq \widetilde{\min}\left(G_j, \sigma_{i_j}\right)$ and hence (iv) is satisfied. Moreover $p \leq d \leq |\sigma \wedge \sigma_1| \leq \left|\sigma \wedge \sigma_{i_1}\right|$, which means that (v) is also satisfied.

If on the other hand (F,σ) is attached, using similar reasoning and Corollary (2.1.13), the desired result can be easily proven.

We are now ready to define the families $\mathcal{G}_{\alpha}$, for $\alpha < \omega_1$ and prove their main properties.

Definition (2.1.20) [2]:

For a countable ordinal number α we define

$\mathcal{G}_{\alpha} = \{F \subset \{0,1\}^{\mathbb{N}}$: there exists $\sigma \in \{0,1\}^{\mathbb{N}}$ with $(F, \sigma \in)\ \mathrm{G}_{\alpha}\} \cup \{\phi\}$.

Proposition (2.1.21) [2]:

Let α be a countable ordinal number. Then G_{α} is α-large. In particular, for every B infinite subset of $\{0,1\}^{\mathbb{N}}$ there exists a one to one map $\phi : \mathbb{N} \to B$ with $\phi(F) \in \mathcal{G}_{\alpha}$ for ever $F \in S_{\alpha}$y and $\alpha < \omega_1$.

Proof:

Let B be an infinite subset of $\{0,1\}^{\mathbb{N}}$ Choose $\{\tau_k\}_k$ pairwise disjoint elements of B and $\sigma \in \{0,1\}^{\mathbb{N}}$, with $\lim_{k} \tau_k = \sigma$, such that $\sigma \wedge \tau_k \subsetneq \sigma \wedge \tau_{k+1}$ for all $k \in \mathbb{N}$. Define $\phi : \mathbb{N} \to B$, with $\phi(k) = \tau_k$.

We shall inductively prove that for every $\alpha < \omega_1$ and $F \in S_{\alpha}$, the following hold:

(i) $(\phi(F), \sigma) \in G_{\alpha}$.

(ii) $\widetilde{\min}(\phi(F), \sigma) = |\sigma \wedge \tau_{\min F}|$ and $\widetilde{\max}(\phi(F), \sigma) = |\sigma \wedge \tau_{\max F}|$.

The case $\alpha = 1$ can be easily derived from the definition of G_1. Assume now that α is a countable ordinal number and that the statement is true for every $F \in S_{\beta}$ and $\beta < \alpha$.

We treat first the case when α is a limit ordinal number. Choose $\{\beta_n\}_n$ a strictly increasing sequence of ordinal numbers with $\sup_n \beta_n = \alpha$, such that

$$G_{\alpha} = \bigcup_{n=1}^{\infty} \{(G, \sigma') \in G_{\beta_n} : \widetilde{\min}(G, \sigma') \geq n\}$$

as well as

$$S_{\alpha} = \bigcup_{n=1}^{\infty} \{F \in S_{\beta_n} : \min F \geq n\}.$$

Then, if $F \in S_{\alpha}$, there exists $n \in \mathbb{N}$ with $F \in S_{\beta_n}$ and $\min F \geq n$. The inductive assumption yields that $(\phi(F), \sigma) \in G_{\beta_n}$ and $\widetilde{\min}(\phi(F), \sigma) = |\sigma \wedge \tau_{\min F}| \geq \min F \geq n$. We conclude that $(\phi(F), \sigma) \in G_{\alpha}$ and, of course$\widetilde{\min}(\phi(F), \sigma) = |\sigma \wedge \tau_{\min F}|$.

Assume now that $\alpha = \beta + 1$ and let $F \in S_\alpha$. Then there exist $\min F \le F_1 < \cdots < F_d$ in S_β with $F = \bigcup_{i=1}^{d} F_i$.

The inductive assumption yields that $(\phi(F_i), \sigma)_{i=1}^{d}$ is an attached branching of σ in G_β and hence $(\phi(F), \sigma) \in G_\alpha$.

Moreover, $\widetilde{\min}(\phi(F), \sigma) = \widetilde{\min}(\phi(F_1), \sigma) = |\sigma \wedge \tau_{\min F_1}| = |\sigma \wedge \tau_{\min F}|$. Similarly, we conclude that $\widetilde{\max}(\phi(F), \sigma) = |\sigma \wedge \tau_{\max F}|$.

subset of A, such that the Cantor-Bendixson index of $\mathcal{G}_\alpha \upharpoonright B$ is equal to $\omega^\alpha + 1$ for all $\alpha < \omega_1$. Since we do not make use of this fact, we omit

The result concerning the families $\mathcal{G}_\alpha$, $\alpha < \omega_1$ is the following.

Theorem (2.1.22) [2]:

Let α be a countable ordinal number. Then $\mathcal{G}_\alpha$ is an α-large, hereditary and compact family of finite subsets of $\{0,1\}^{\mathbb{N}}$.

Proof:

All we need to prove, is that $\mathcal{G}_\alpha$ is compact and we do so by transfinite induction. Let us first treat the case $\alpha = 1$ and assume F is in the closure of $\mathcal{G}_1$.

If F is finite, since $\mathcal{G}_1$ is hereditary, then $F \in \mathcal{G}_1$. It is therefore sufficient to show that F cannot be infinite. Since $\mathcal{G}_1$ is hereditary, we may assume that F is countable and let $\{\tau_i : i \in \mathbb{N}\}$ be an enumeration of F.

We conclude, that setting $F_k = \{\tau_i : i = 1, \dots, k\}$, then $F_k \in \mathcal{G}_1$ and $\# F_k = k$. Choose $\{\sigma_k\}_k$ a sequence in $\{0,1\}^{\mathbb{N}}$ such that $(F_k, \sigma_k) \in G_1$ for all k.

We yield that $k \le \widetilde{\min}(F_k, \sigma_k)$ for all k. On the other hand, by Corollary (2.1.14) we have that $\widetilde{\min}(F_k, \sigma_k) \le |\tau_1 \wedge \tau_2|$. We conclude that $k \le |\tau_1 \wedge \tau_2|$ for all $k \in \mathbb{N}$, which is obviously not possible.

Assuming now that α is a countable ordinal number such that $\mathcal{G}_\beta$ is compact for every $\beta < \alpha$, we will show that the same is true for $\mathcal{G}_\alpha$.

We treat first the case in which α is a limit ordinal number. Fix $\{\beta_n\}_n$ a strictly increasing sequence of ordinal numbers with $\sup_n \beta_n = \alpha$ such that

$$\mathcal{G}_\alpha = \bigcup_{n=1}^{\infty} \{(F, \sigma) \in \mathcal{G}_{\beta_n} : \widetilde{\min}(F, \sigma) \geq n\}.$$

Let F be in the closure of $\mathcal{G}_\alpha$. As previously, if F is finite then it is in $\mathcal{G}_\alpha$ and it is therefore enough to show that F cannot be infinite. Once more, we may assume that $F = \{\tau_i : i \in \mathbb{N}\}$. Setting $F_k = \{\tau_1, \dots, \tau_k\}$, we have that $F_k \in \mathcal{G}_\alpha$, therefore there exists $\{\sigma_k\}_k$, with $(F_k, \sigma_k) \in \mathrm{G}_\alpha$.

Using Corollary (2.1.14) we have that $\widetilde{\min}(F_k, \sigma_k) \leq |\tau_1 \wedge \tau_2| = d$. In other words, $(F_k, \sigma_k) \in \mathrm{G}_{\beta_{n_k}}$, with $n_k \leq d$ for all k. Passing, if necessary, to a subsequence, we have that $(F_k, \sigma_k) \in \mathrm{G}_{\beta_{n_0}}$, for all k. We conclude that $F \in \mathcal{G}_{\beta_{n_0}}$, in other words $\mathcal{G}_{\beta_{n_0}}$ is not compact, which is absurd.

Assume now that $\alpha = \beta + 1$. Let F be in the closure of $\mathcal{G}_\alpha$. As previously, it is enough to show that F cannot be infinite. Once more, we may assume that $F = \{\tau_i : i \in \mathbb{N}\}$.

Set $F_k = \{\tau_i : i = 1, \dots, k\}$, for all k. Then $F_k \in \mathcal{G}_\alpha$, i.e. there exists σ_k such that $(F_k, \sigma_k) \in \mathrm{G}_\alpha$. Setting $d = |\tau_1 \wedge \tau_2|$, Corollary (2.1.15), yields the following:

$$\widetilde{\min}(F_k, \sigma_k) \leq d \text{ for all } k. \qquad (2)$$

By Definition (2.1.4), Remark (2.1.10) and (2), for every $k \in \mathbb{N}$, there exist $\{F_j^k\}_{j=1}^{m_k}$ pairwise disjoint sets in $\mathcal{G}_\alpha$, with $F_k = \bigcup_{j=1}^{m_k} F_j^k$ and $m_k \leq d$. Passing to a subsequence, we may assume that $m_k = m$, for all k.

By the compactness of $\mathcal{G}_\alpha$, we may pass to a further subsequence and find $G_1, G_2, \dots, G_m \in \mathcal{G}_\beta$, such that $\lim_k F_j^k = G_j$, for $j = 1, \dots, m$.

We conclude that $F = \lim_k F_k = \lim_k \left(\bigcup_{j=1}^{m} F_j^k\right) = \bigcup_{j=1}^{m} G_j$. Since $\bigcup_{j=1}^{m} G_j$ is a finite set, this cannot be the case.

Although the initial motivation behind the definition of the $\mathcal{G}_\alpha$ families was the construction of a nonseparable reflexive space with ℓ_1 as a unique spreading model, we believe that they are of independent interest, as they retain many of the properties of the families S_α. They are therefore a version of these families, defined on the Cantor set $\{0,1\}^{\mathbb{N}}$. We present a few more properties the $\mathcal{G}_\alpha$ have in common with the S_α.

Lemma (2.1.23) [2]:

Let $\alpha < \beta$ be countable ordinal numbers. Then there exists $n \in \mathbb{N}$ such that

$$\{(F,\sigma) \in G_\alpha : \widetilde{\min}(F,\sigma) \geq n\} \subset \mathrm{G}_\beta.$$

Proof:

Fix α a countable ordinal number. We prove this proposition by means of transfinite induction, starting with $\beta = \alpha + 1$. In this case the result follows from the definition of G_β, for $n = 1$.

Assume now that β is a countable ordinal number with $\alpha < \beta$, such that the statement holds for every $\alpha < \gamma < \beta$. If $\beta = \gamma + 1$, by the inductive assumption, there exists $n \in \mathbb{N}$, such that $\{(F,\sigma) \in \mathrm{G}_\alpha : \widetilde{\min}(F,\sigma) \geq n\} \subset \mathrm{G}_\gamma$. Evidently, we also have that $\{(F,\sigma) \in \mathrm{G}_\alpha : \widetilde{\min}(F,\sigma) \geq n\} \subset \mathrm{G}_\beta$.

If β is a limit ordinal number, fix $\{\beta_k\}_k$ a strictly increasing sequence of ordinal numbers, such that $\beta = \lim_k \beta_k$ and

$$G_\alpha = \bigcup_k \{(F,\sigma) \in G_{\beta_k} : \widetilde{\min}(F,\sigma) \geq k\}.$$

Choose $k_0 \in \mathbb{N}$ with $\alpha < \beta_{k_0}$. By the inductive assumption, there exists $m \in \mathbb{N}$, such that $\{(F,\sigma) \in \mathrm{G}_\alpha : \widetilde{\min}(F,\sigma) \geq m\} \subset \mathrm{G}_{\beta_{k_0}}$. Setting $n = \max\{k_0, m\}$, we have the desired result.

Lemma (2.1.24) [2]:

Let $\alpha < \beta$ be countable ordinal numbers. Then there exists $n \in \mathbb{N} \cup \{0\}$ such that G $_{\alpha} \subset$ G $_{\beta+n}$.

Proof:

Fix β a countable ordinal number. We proceed by transfinite induction on α. In the case $\alpha = 1$, it is easily checked that G $_1 \subset$ G $_{\beta}$. Assume now that α is a countable ordinal with $\alpha < \beta$, such that the statement holds for every $\gamma < \alpha$. If $\alpha = \gamma + 1$, then by the inductive assumption there exists $n \in \mathbb{N} \cup \{0\}$ with G $_{\gamma} \subset$ G $_{\beta+n}$. We conclude that G $_{\alpha} \subset$G $_{\beta+(n+1)}$. If α is a limit ordinal, fix $\{\alpha_k\}_k$ a strictly increasing sequence of ordinal numbers, such that $\alpha = \lim_k \alpha_k$ and

$$G_\alpha = \bigcup_k \{(F,\sigma) \in G_{\alpha_k} : \widetilde{\min}(F,\sigma) \geq k\}.$$

Lemma (2.1.23) yields that there exists $m \in \mathbb{N}$ with $\{(F,\sigma) \in$G $_{\alpha} : \widetilde{\min}(F,\sigma) \geq m\} \subset$ G $_{\beta}$. The inductive assumption, yields that for $k = 1, \dots, m-1$, there exists $n_k \in \mathbb{N} \cup \{0\}$ with G $_{\alpha_k} \subset$ G $_{\beta+n_k}$. Setting $n = \{m, n_1, \dots, n_{m-1}\}$, it can be easily checked that G $_{\alpha} \subset$ G $_{\beta+n}$.

Proposition (2.1.25) [2]:

Let $\alpha < \beta$ be countable ordinal numbers. Then there exists $n \in \mathbb{N}$ such that

$$\{F \in \mathcal{G}_\alpha : \# F \geq 2 \text{ and } \min\{|\tau_1 \wedge \tau_2| : \tau_1, \tau_2 \in F, \tau_1 \neq \tau_2\} \geq n\} \subset \mathcal{G}_\beta.$$

Proof:

Let $\alpha < \beta$ be countable ordinal numbers. Choose $n \in \mathbb{N}$ such that the conclusion of Lemma (2.1.23) is satisfied. We show that this n is the desired natural number. Le $F \in \mathcal{G}_\alpha$ with $\# F \geq 2$ and $\min\{|\tau_1 \wedge \tau_2| : \tau_1, \tau_2 \in F, \tau_1 \neq \tau_2\} \geq n$. Then there exists $\sigma \in \{0,1\}^{\mathbb{N}}$ with $(F,\sigma) \in$G $_{\alpha}$. Lemma (2.1.15) yields that there exists $\sigma' \in \{0,1\}^{\mathbb{N}}$ such that $(F,\sigma') \in$ G $_{\alpha}$ and $\widetilde{\min}(F,\sigma') \geq n$. By the choice of n, we have that $(F,\sigma') \in$ G $_{\beta}$, i.e. $F \in \mathcal{G}_\alpha$.

The following proposition is an obvious conclusion of Lemma (2.1.24) .

Proposition (2.1.26) [2]:

Let $\alpha < \beta$ be countable ordinal numbers. Then there exists $n \in \mathbb{N} \cup \{0\}$ such that $\mathcal{G}_\alpha \subset \mathcal{G}_{\beta+n}$.

Section (2.2): The space$\mathfrak{X}_{2^{\aleph_0}}$ and Spaces Adimitting Spreading Model

In this section we define the space$\mathfrak{X}_{2^{\aleph_0}}$ and prove that it is reflexive, has an unconditional Schauder basis of length the continuum and that it admits only ℓ_1 as a spreading model. In the beginning we define a sequence of non-separable spaces $X_n, n \in \mathbb{N}$. Each one is defined using the family $\mathcal{G}_n$ in a similar manner as the Schreier family S_1 is used to define the space . Then the construction of $\mathfrak{X}_{2^{\aleph_0}}$ is presented, which combines the spaces X_n and Tsirelson space, using a method appeared the end the properties of the space $\mathfrak{X}_{2^{\aleph_0}}$ are deduced by directly using the structure of the families $\mathcal{G}_n$.

Before proceeding to the definition of the spaces X_n and $\mathfrak{X}_{2^{\aleph_0}}$, let us first recall the notion of ℓ_1^α spreading models.

Definition (2.2.1) [2]:

Let $\{x_k\}_k$ be a sequence in a Banach space and α be a countable ordinal number. We say that $\{x_k\}_k$ generates an ℓ_1^α spreading model, if there exists a constant $c > 0$ such that for every $F \in S_\alpha$ and every real numbers $\{\lambda_k\}_{k\in F}$ the following holds:

$$\left\|\sum_{k\in F} \lambda_k x_k\right\| \geq c \sum_{k\in F} |\lambda_k|.$$

Let us from now on fix a one to one and onto map $\tau \to \xi_\tau$ from $\{0,1\}^{\mathbb{N}}$ to the cardinal number $2^{\aleph_0}$.

Definition (2.2.2) [2]:

For $n \in \mathbb{N}$ define a norm on $c_{00}(2^{\aleph_0})$ in the following manner:

(i) For $n \in \mathbb{N}$, we may identify an $F \in \mathcal{G}_n$ with a linear functional $F : c_{00}(2^{\aleph_0}) \to \mathbb{R}$ in the following manner. For $x = \sum_{\xi<2^{\aleph_0}} \lambda_\xi c_\xi \in c_{00}(2^{\aleph_0})$

$$F(x) = \sum_{\tau\in F} \lambda_{\xi_\tau}.$$

(ii) For $x \in c_{00}(2^{\aleph_0})$ define

$$\|x\|_n = \sup\{|F(x)| : F \in \mathcal{G}_n\}.$$

Set X_n to be the completion of $(c_{00}(2^{\aleph_0}), \|\cdot\|_n)$.

Proposition (2.2.3) [2]:

Let $n \in \mathbb{N}$. Then the following hold:

(i) The space X_n is c_0 saturated.

(ii) The unit vector basis $\{e_\xi\}_{\xi<2^{\aleph_0}}$ is a normalized, suppression unconditional and weakly null basis of X_n, with the length of the continuum.

(iii) Any subsequence of the unit vector basis admits only ℓ_1 as a spreading model.

By T we denote Tsirelson space as defined and by t$\{e_n\}_n$ we denote its usual basis. We are now ready to define the space $\mathfrak{X}_{2^{\aleph_0}}$, using the spaces X_n, Tsirelson space T and a method appeared .

Definition (2.2.4) [2]:

Define the following norm on $c_{00}(2^{\aleph_0})$. $F \in c_{00}(2^{\aleph_0})$

$$\|x\| = \left\| \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n e_n \right\|_T .$$

Set $\mathfrak{X}_{2^{\aleph_0}}$ to be the completion of $(c_{00}(2^{\aleph_0}), \|\cdot\|)$.

Set $\lambda = \left\| \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \right\|_T$ and for $\xi < 2^{\aleph_0}, \tilde{e}_\xi = \frac{1}{\lambda} e_\xi$. Since $\{e_\xi\}_{\xi<2^{\aleph_0}}$ is normalized and suppression unconditional in X_n, and $\{e_n\}_n$ is 1-unconditional in T, we conclude that $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$ is a normalized suppression unconditional basis of $\mathfrak{X}_{2^{\aleph_0}}$.

For $n \in \mathbb{N}$ define $P_n : \mathfrak{X}_{2^{\aleph_0}} \to X_n$ with $P_n x = \frac{1}{2^n} x$. Evidently P_n is well defined and $\|P_n\| \leq 1$, for all $n \in \mathbb{N}$.

The main result is the following, which is a combination of Proposition (2.2.15) and Corollary (2.2.17) , which will be presented in the sequel.

Theorem (2.2.5) [2]:

The space $\mathfrak{X}_{2^{\aleph_0}}$ is a non-separable reflexive space with a suppression unconditional Schauder basis with the length of the continuum, having the following property. Every normalized weakly null sequence in $\mathfrak{X}_{2^{\aleph_0}}$ has a subsequence that generates an ℓ_1^n spreading model, for every $n \in \mathbb{N}$.

Lemma (2.2.6) [2]:

Let $\{\tilde{e}_{\xi_k}\}_k$ be a subsequence of the basis $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$ of $\mathfrak{X}_{2^{\aleph_0}}$. Then it has a subsequence that generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Proof:

Set $B = \{\tau: \xi_\tau = \xi_k \text{ for sme } k \in \mathbb{N}\}$. By Proposition (2.1.21) [2] there exists a one to one map $\phi : \mathbb{N} \to B$ such that $\phi(F) \in \mathcal{G}_n$ for every $F \in S_n$ and $n \in \mathbb{N}$.

Pass to L an infinite subset of the natural numbers such that the map $\tilde{\phi} : L \to 2^{\aleph_0}$ with $\tilde{\phi}(j) = \xi_{\phi(j)}$ is strictly increasing. We will show that $\left\{\tilde{e}_{\xi_{\phi(j)}}\right\}_{j\in L}$ admits an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

By unconditionality, it is enough to show that there are positive constants c_n such that for every $n \in \mathbb{N}$, $F \in S_n, F \subset L$ and $\{t_j\}_{j\in F}$ positive real numbers, we have that

$$\left\|\sum_{j\in F} t_j \tilde{e}_{\xi_{\phi(j)}}\right\| \geq c_n \sum_{j\in F} t_j.$$

By definition, we have that $\left\|\sum_{j\in F} t_j \tilde{e}_{\xi_{\phi(j)}}\right\| \geq \frac{\lambda}{2^n}\left\|\sum_{j\in F} t_j e_{\xi_{\phi(j)}}\right\|_n$ and by the choice of ϕ, we have that $\phi(F) \in \mathcal{G}_n$. Hence, $\phi(F)\left(\sum_{j\in F} t_j e_{\xi_{\phi(j)}}\right) = \sum_{j\in F} t_j$ which yields that $\left\|\sum_{j\in F} t_j e_{\xi_{\phi(j)}}\right\|_n = \sum_{j\in F} t_j$.

We finally conclude that $\left\|\sum_{j\in F} t_j \tilde{e}_{\xi_{\phi(j)}}\right\| \geq \frac{\lambda}{2^n}\sum_{j\in F} t_j$.

Proposition (2.2.7) [2]:

Let $\{x_k\}_k$ be a normalized, disjointly supported block sequence of $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$, such that $\limsup_k \|x_k\|_\infty > 0$. Then $\{x_k\}_k$ has a subsequence that generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Proof:

By unconditionality, it is quite clear, that by passing, if necessary, to a subsequence of $\{x_k\}_k$, there exist $\varepsilon > 0$ and $\{\tilde{e}_{\xi_k}\}_k$ a subsequence of $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$, such that for any $\lambda_1, \dots, \lambda_m$ real numbers, one has that

$$\left\|\sum_{k=1}^{m} \lambda_k x_k\right\| > \varepsilon \left\|\sum_{k=1}^{m} \lambda_k \tilde{e}_{\xi_k}\right\|.$$

Lemma (2.2.6) yields the desired result.

Proposition (2.2.8) [2]:

Let $\{x_k\}_k$ be a normalized block sequence in $\mathfrak{X}_{2^{\aleph_0}}$, such that $\lim_k \|P_n x_n\|_n = 0$, for all $n \in \mathbb{N}$. Then $\{x_k\}_k$ has a subsequence equivalent to a block sequence in T. In particular, $\{x_k\}_k$ has a subsequence that generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Proof:

Using a sliding hump argument, it is easy to see, that passing, if necessary, to a subsequence of $\{x_k\}_k$, there exists $\{I_k\}_k$ increasing intervals of the natural numbers, such that if we set $y_k = \sum_{n\in I_k} \frac{1}{2^n} \|x_k\|_n e_n$, then $\{x_k\}_k$ is equivalent to $\{y_k\}_k$.

Lemma (2.2.9) [2]:

Let $\{x_k\}_k$ be a normalized, disjointly supported block sequence of $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$, such that the following holds. There exist $c > 0, n_0 \in \mathbb{N}, (F_k, \sigma_k) \in G_{n_0}$ for $k \in \mathbb{N}$ and $\sigma \in \{0,1\}^{\mathbb{N}}$ satisfying the following:

(i) $|F_k(x_k)| > c$ for all $k \in \mathbb{N}$.

(iii) The F_k are pairwise disjoint.
(iv) $\sigma \neq \sigma_k$ for all $k \in \mathbb{N}$.
(v) $\sigma \wedge \sigma_k \subsetneq \sigma \wedge \sigma_{k+1}$ for all $k \in \mathbb{N}$.
(vi) $|\sigma \wedge \sigma_k| < \widetilde{\min}(x_k)$ for all $k \in \mathbb{N}$.

Then $\{x_k\}_k$ generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Proof:

By changing the signs of the x_k, we may assume that $F_k(x_k) > c$ for all $n \in \mathbb{N}$.

Arguing in a similar manner as in the proof of Proposition (2.1.23) [2] one can inductively prove that for every $n \in \mathbb{N}$ and $G \in S_n$ the following hold:

(a) $(\cup_{k\in G} F_k, \sigma) \in G_{n_0+n}$.

(b) $\widetilde{\min}(\cup_{k\in G} F_k, \sigma) = |\sigma \wedge \sigma_{\min G}|$ and $\widetilde{\max}(\cup_{k\in G} F_k, \sigma) = |\sigma \wedge \sigma_{\max G}|$.

Since $\{x_k\}_k$ is unconditional, it is enough find positive constants $c_n > 0$, such that fixing $G \in S_n$ and $\{\lambda_k\}_{k\in G}$ non-negative reals, we have the following:

$$\left\|\sum_{k\in G} \lambda_k x_k\right\| > c_n \sum_{k\in G} \lambda_k.$$

Properties (a) and (b), yield that $F = \cup_{k\in G} F_k \in \mathcal{G}_{\mathrm{n_0+n}}$. This means the following:

$$\left\|\sum_{k\in G} \lambda_k x_k\right\| \geq \left\|P_{n_o+n}\left(\sum_{k\in G} \lambda_k x_k\right)\right\|_{n_o+n}$$

$$= \frac{2}{2^{n_0+n}} \left\|\sum_{k\in G} \lambda_k x_k\right\|_{n_0+n}$$

$$> \frac{2c}{2^{n_0+n}} \sum_{k \in G} \lambda_k.$$

Lemma (2.2.10) [2]:

Let $\{x_k\}_k$ be a normalized, disjointly supported block sequence of $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$, such that the following holds. There exist $c > 0, n_0 \in \mathbb{N}, \sigma \in \{1,0\}^{\mathbb{N}}$, a sequence $\{F_k\}_k$ in $\mathcal{G}_{n_0}$satisfying the following:

(i) $|F_k(x_k)| > c$ for all $k \in \mathbb{N}$.
(ii) The set F_k are pairwise disjoint.
(iii) $(F_k, \sigma_k) \in \mathrm{G}_{\,n_0}$ for all $k \in \mathbb{N}$.
(iv) $\widetilde{\max}(F_k, \sigma) < \widetilde{\min}(F_{k+1}, \sigma)$ for all $k \in \mathbb{N}$.

Then $\{x_k\}_k$ generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Lemma (2.2.11) [2]:

Let $\{x_k\}_k$ be a sequence in $\mathfrak{X}_{2^{\aleph_0}}$ and $n \in \mathbb{N}$ such that $\lim_k \|P_n x_k\|_n = 0$. Then for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ the following holds:

$$|F(x_k)| < \varepsilon \quad \text{for every } F \in \mathcal{G}_n.$$

Proof:

Fix $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$, such that $\|P_n x_k\|_n = \frac{1}{2^n} \|x_k\|_n < \frac{1}{2^n} \varepsilon$, for every $k \geq k_0$. By definition of the norm $\|\cdot\|_n$, this means the following:

$$|F(x_k)| < \varepsilon \quad \text{for every } F \in \mathcal{G}_n.$$

Lemma (2.2.12) [2]:

Let $\{x_k\}_k$ be a normalized, disjointly supported block sequence of $\{\tilde{e}_\xi\}_{\xi<2^{\aleph_0}}$, such that $\lim_k \|x_k\|_\infty = 0$ and there exists $n \in \mathbb{N}$ such that $\limsup_k \|P_n x_k\|_n > 0$. Assume moreover, that if $n_0 = \min\left\{n : \limsup_k \|P_n x_k\|_n > 0\right\}$, there exist $c > 0, \sigma \in \{0,1\}^{\mathbb{N}}$ and $\{F_k\}_k$ a sequence in $\mathcal{G}_{n_0}$ satisfying the following:

(i) $|F_k(x_k)| > c$ for all $k \in \mathbb{N}$.

(ii) The set F_k are pairwise disjoint.
(iii) $(F_k, \sigma_k) \in G_{n_0}$ for all $k \in \mathbb{N}$.

Then $\{x_k\}_k$ has a subsequence that generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Proof:

We shall prove that for every k_0, m natural numbers, there exist $k \geq k_0$ and $G_k \subset F_k$ such that $|G_k(x_k)| > 2/c$ and $\widetilde{\min}(G_k, \sigma) > m$.

If the above statement is true, we may clearly choose $\{G_k\}_k$ in $\mathcal{G}_{n_0}$ satisfying the assumptions of Lemma (2.2.10) , which will complete the proof.

We assume that $n_0 \geq 2$, as the case $n_0 = 1$ uses similar arguments and the fact that $\lim_k \|x_k\|_\infty = 0$. Fix $k_0, m \in \mathbb{N}$. By Lemma (2.2.11), choose $k \geq k_0$, such that the following holds:

$$|F(x_k)| < \frac{c}{2m} \quad \text{for every } F \in \mathcal{G}_{n_0-1}. \tag{3}$$

We distinguish two cases.

Case (1):

There is $\left(F_i^k, \sigma_i^k\right)_{i=1}^d$ a skipped branching of σ in G_{n_0-1} with $\cup_{i=1}^d F_i^k$.

Case (2):

There is $\left(F_i^k, \sigma\right)_{i=1}^d$ an attached branching of σ in G_{n_0-1} with $\cup_{i=1}^d F_i^k$.

In either case, by Proposition (2.1.19) we have that if we set $G_k = \cup_{i=m+1}^d F_i^k$, then $(G_k, \sigma) \in G_{n_0}$. Moreover, (3) yields that $|G_k(x_k)| > c/2$.

All that remains, is to show that $\widetilde{\min}(G_k, \sigma) > m$.

If we are in case (1), then $\widetilde{\min}(G_k, \sigma) = |\sigma \wedge \sigma_{m+1}^k|$. By Definition (2.1.6) we have that $|\sigma \wedge \sigma_i^k| < |\sigma \wedge \sigma_{i+1}^k|$ for $i = 1, \dots, m$, which of course yields that $|\sigma \wedge \sigma_{m+1}^k| > m$.

If, on the other hand, we are in case (2), then $\widetilde{\min}(G_k, \sigma) = \widetilde{\min}(F_{m+1}^k, \sigma)$. By Definition (2.1.7) we have that $\widetilde{\min}(F_{m+1}^k, \sigma) > m$.

Lemma (2.2.13) [2]:

Let $\{x_k\}_k$ be a normalized, disjointly supported block sequence of $\{\tilde{e}_\xi\}_{\xi < 2^{\aleph_0}}$, such that there exists $n \in \mathbb{N}$ such that $\limsup_k \|P_n x_k\|_{n_n} > 0$. Then, passing if necessary, to a subsequence, there exist $c > 0$ and $(F_k, \sigma_k) \in G_n$ satisfying the following:

(i) The set F_k are pairwise disjoint.
(ii) $|F_k(x_k)| > c$ for all $k \in \mathbb{N}$.

Proof:

Pass to a subsequence of $\{x_k\}_k$ and choose $\varepsilon > 0$, such that the following holds:

$$\|P_n x_k\|_n = \frac{1}{2^n} \|x_k\|_n > \varepsilon, \text{ for all } k \in \mathbb{N}.$$

By the definition of the norm $\|\cdot\|_n$, there exist $(F_k, \sigma_k) \in G_n$ with $|F_k(x_k)| > 2^n \varepsilon$, for all $k \in \mathbb{N}$. By virtue of Proposition (2.1.19) and the fact that $\{x_k\}_k$ is disjointly supported, we may assume that the F_k are pairwise disjoint. Setting $c = 2^n \varepsilon$ finishes the proof.

Proposition (2.2.14) [2]:

Let $\{x_k\}_k$ be a normalized, disjointly supported block sequence of $\{\tilde{e}_\xi\}_{\xi < 2^{\aleph_0}}$, such that $\lim_k \|x_k\|_\infty = 0$ and there exists $n \in \mathbb{N}$ such that $\limsup_k \|P_n x_k\|_n > 0$. Then $\{x_k\}_k$ has a subsequence that generates an ℓ_1^n spreading model for every $n \in \mathbb{N}$.

Proof:

Set $n_0 = \min\left\{n: \limsup_k \|P_n x_k\|_n > 0\right\}$ and as in the proof of Lemma (2.2.12) let us assume that $n_0 \geq 2$. Apply Lemmas (2.2.13) and (2.2.11) , pass to a subsequence of $\{x_k\}_k$ and find $c > 0, (F_k, \sigma_k) \in G_{n_0}$, such that the following are satisfied:

(i) The set F_k are pairwise disjoint.

(ii) $|F_k(x_k)| > c$ for all $k \in \mathbb{N}$.

(iii) $|F_k(x_k)| < c/4$ for all $k \in \mathbb{N}$ and $F \in \mathcal{G}_{n_0-1}$.

Passing to a further subsequence, choose $\sigma \in \{0,1\}^{\mathbb{N}}$ such that $\lim_k \sigma_k = \sigma$. We distinguish two cases.

Case (1):

$$\lim_k \max\{|G(x_k)| : G \subset F_k \text{ with } (G,\sigma) \in G_{n_0}\} = 0.$$

Case (2):

$$\limsup_k \max\{|G(x_k)| : G \subset F_k \text{ with } (G,\sigma) \in G_{n_0}\} > 0.$$

Let us first treat case (1). Pass once more to a subsequence of $\{x_k\}_k$, satisfying the following:

(a) $\max\{|G(x_k)| : G \subset F_k \text{ with } (G,\sigma) \in G_{n_0}\} < c/4$ for all $k \in \mathbb{N}$.

(b) $\sigma \neq \sigma_k$, for every $k \in \mathbb{N}$.

(c) $\sigma \wedge \sigma_k \subsetneq \sigma \wedge \sigma_{k+1}$ for all $k \in \mathbb{N}$.

We shall prove the following. For every k, there exists $G_k \subset F_k$, such that the following hold:

(d) $|G_k(x_k)| > c/2$.

(e) $|\sigma \wedge \sigma_k| < \widetilde{\min}(G_k, \sigma_k)$.

Combining (b), (c), (d) and (e), we conclude that the assumptions of Lemma (2.2.9) are satisfied, which proves the desired result, in case (1).

Set $G_k'' = \{\tau \in F_k : \sigma_k \wedge \tau = \sigma \wedge \tau\}$. Proposition (2.1.19) and Lemma (2.1.18) yield that $(G_k'', \sigma_k) \in \mathrm{G}_{n_0}$. Setting $F_k'' = F_k \backslash G_k''$, property (a) yields that $|F_k''(x_k)| > 3c/4$.

Set $G_k' = \{\tau \in F_k' : \sigma_k \wedge \tau \subsetneq \sigma \wedge \tau\}$. Once more, Proposition (2.1.19) yields that $(G_k', \sigma_k) \in \mathrm{G}_{n_0}$, however Lemma (2.1.18) yields $G_k' \in \mathcal{G}_{n_0 - 1}$ and therefore, by (iii) we have that $|G_k'(x_k)| < c/4$.

Set $G_k = F_k' \backslash G_k'$. Then we have that $|G_k(x_k)| > c/2$, i.e. (d) holds.

We will show that (e) also holds. By Corollary (2.1.13) , there exists $\tau \in G_k$, with $\widetilde{\min}(G_k, \sigma_k) = |\sigma_k \wedge \tau|$. Since $\tau \notin G_k''$, we have that $|\sigma_k \wedge \tau| \neq |\sigma \wedge \tau|$.

We will show that $|\sigma \wedge \tau| < |\sigma_k \wedge \tau|$. Assume that this is not the case, i.e. $|\sigma_k \wedge \tau| < |\sigma \wedge \tau|$. In other words, $\sigma_k \wedge \tau \subsetneq \sigma \wedge \tau$. This means that $\tau \in G_k'$ a contradiction.

We conclude that $\wedge \tau \subsetneq \sigma_k \wedge \tau$. Lemma (2.1.11) yields that $\sigma \wedge \tau = \sigma_k \wedge \sigma$. Applying Lemma (2.1.11) once more, we conclude that $\sigma \wedge \tau_k \subsetneq \sigma_k \wedge \tau$, i.e. $|\sigma \wedge \tau_k| < |\sigma_k \wedge \tau| = \widetilde{\min}(G_k, \sigma_k)$, which completes the proof for case (1).

It only remains to treat case (2). Observe, that in this case, we may easily pass to a subsequence of $\{x_k\}_k$, satisfying the assumptions of Lemma (2.2.12) . This completes the proof.

Combining Propositions (2.2.7) , (2.2.8) and (2.2.12) , one obtains the following.

Proposition (2.2.15) [2]:

Let $\{x_k\}_k$ be a normalized weakly null sequence in $\mathfrak{X}_{2^{\aleph_0}}$. Then $\{x_k\}_k$ has a subsequence that generates an $n \in \mathbb{N}$ spreading model for every $n \in \mathbb{N}$.

Proposition (2.2.16) [2]:

The space $\mathfrak{X}_{2^{\aleph_0}}$ is saturated with subspaces of Tsirelson space.

Proof:

It is an immediate consequence of Proposition (2.2.15) that $\mathfrak{X}_{2^{\aleph_0}}$ does not contain a copy of c_0. By Proposition (2.2.3), the spaces X_n are c_0 saturated and therefore, the operators $P_n : \mathfrak{X}_{2^{\aleph_0}} \to X_n$, are strictly singular.

We conclude, that in any infinite dimensional subspace Y of $\mathfrak{X}_{2^{\aleph_0}}$, $n_0 \in \mathbb{N}$ and $\varepsilon > 0$, there exists $x \in Y$ with $\|x\| = 1$ and $\|P_n x\|_n < \varepsilon$ for $n = 1, \ldots, n_0$. One may easily construct a normalized sequence in Y, satisfying the assumption of Proposition (2.2.8), which completes the proof.

In particular, the previous result yields that neither c$_0$ nor ℓ_1 embed into $\mathfrak{X}_{2^{\aleph_0}}$. Using James' well known theorem for spaces [7] that is(Abanach Space B is reflexive if and only if every continuous liner functional on B altains it is Maxmum on the closed unit ball in B) with an unconditional basis, we conclude the following.

Corollary (2.2.17) [2]:

The space $\mathfrak{X}_{2^{\aleph_0}}$ is reflexive.

Definition (2.2.18) [2]:

Let α be a countable ordinal number. Define $\|\cdot\|_{T_\alpha}$ to be the unique norm on $c_{00}(\mathbb{N})$ that satisfies the following implicit formula, for every $x \in c_{00}(\mathbb{N})$:

$$\|x\|_{T_\alpha} = \max\left\{\|x\|_\infty, \frac{1}{2}\sup\sum_{i=1}^{d}\|E_i x\|_{T_\alpha}\right\},$$

where the supremum is taken over all $E_1 < \cdots < E_d$ subsets of the natural numbers with $\{\min E_i : i = 1, \ldots, d\} \in S_\alpha$.

Define the Tsirelson space of order α, denoted by T_α, to be the completion of $c_{00}(\mathbb{N})$ with the aforementioned norm.

The space T_α is reflexive and the unit vector basis $\{e_n\}_n$, forms a 1-unconditional basis for T_α. Moreover, every normalized weakly null sequence in T_α, has a subsequence that generates an ℓ_1^α spreading model.

Given a countable ordinal number α, we shall construct $\{\mathcal{G}_n^{\alpha}\}_n$ un an increasing sequence of families of finite subsets of $[0,1]^{\mathbb{N}}$, strongly related to $\{\mathcal{G}_n\}_n$. As before, we first define some auxiliary families $\mathrm{G}_n^{\alpha} n \in \mathbb{N}$.

Definition (2.2.19) [2]:

We define G_n^{α} to be all pairs (F,σ), where $F=\{\tau_i\}_{i=1}^{d} \in [\{0,\mathbf{1}\}^{\mathbb{N}}]^{<\omega}, d \in \mathbb{N}$ and $\sigma \in \{0,\mathbf{1}\}^{\mathbb{N}}$, such that the following are satisfied:

(i) $\sigma \neq \tau_i$ for $i=1,\dots,d$.

(ii) $\sigma \wedge \tau_1 \neq \phi$ if $d>1$, then $\sigma \wedge \tau_1 \subsetneq \sigma \wedge \tau_2 \subsetneq \cdots \subsetneq \sigma \wedge \tau_d$.

(iii) $\{|\sigma \wedge \tau_i| : i=1,\dots,d\} \in S_{\alpha}$.

Define $\widetilde{\min}(F,\sigma)=|\sigma \wedge \tau_1|$ and $\widetilde{\max}(F,\sigma)=|\sigma \wedge \tau_d|$.

Assume that $n \in \mathbb{N}$, G_k^{α} have been defined for $k \leq n$ and that for $(F,\sigma) \in \mathrm{G}_n^{\alpha}$, $\widetilde{\min}(F,\sigma)$ and $\widetilde{\max}(F,\sigma)$ have also been defined.

Definition (2.2.20) [2]:

Let $(F_i,\sigma_i)_{i=1}^{d}, d \in \mathbb{N}$ be a finite sequence of elements of G_n^{α} and $\sigma \in [0,1]^{\mathbb{N}}$.

We say that $(F_i,\sigma_i)_{i=1}^{d}$ is a skipped branching of σ in G_n^{α}, if the following are satisfied:

(i) The $F_i, i=1,\dots,d$ are pariwise disjoint.
(ii) $\sigma \neq \tau_i$ for $i=1,\dots,d$.
(iii) $\sigma \wedge \tau_1 \neq \phi$ if $d>1$, then $\sigma \wedge \tau_1 \subsetneq \sigma \wedge \tau_2 \subsetneq \cdots \subsetneq \sigma \wedge \tau_d$.
(iv) $|\sigma \wedge \tau_i| < \widetilde{\min}(F_i,\sigma_i)$ for $i=1,\dots,d$.
(v) $\{|\sigma \wedge \tau_i| : i=1,\dots,d\} \in S_{\alpha}$.

Definition (2.2.21) [2]:

Let $\sigma \in [0,1]^{\mathbb{N}}$ and $(F_i,\sigma)_{i=1}^{d}, d \in \mathbb{N}$ be a finite sequence of elements of G_n^{α}.

We say that $(F_i,\sigma)_{i=1}^{d}$, is an attached branching of σ in G_n^{α} if the following are satisfied:

(i) The $F_i, i = 1, \dots, d$ are pariwise disjoint.

(ii) If $d > 1$, then $\widetilde{\max}(F_i, \sigma) < \widetilde{\min}(F_{i+1}, \sigma)$, for $i = 1, \dots, d - 1$.

(iii) $\{\widetilde{\min}(F_i, \sigma)\ i = 1, \dots, d\} \in S_\alpha$.

We are now ready to define G_{n+1}^{α}.

Definition (2.2.22) [2]:

We define G_{n+1}^{α} to be all pairs (F, σ), where $F \in [\{0,1\}^{\mathbb{N}}]^{<\omega}$ and $\sigma \in \{0,1\}^{\mathbb{N}}$, such that one of the following is satisfied:

(i)$(F, \sigma) \in G_n^{\alpha}$.

(ii) There is $(F_i, \sigma_i)_{i=1}^{d}$ a skipped branching of σ in G_n^{α} such that $F = \bigcup_{i=1}^{d} F_i$.

In this case we say that (F, σ) is skipped. Moreover set $\widetilde{\min}(F, \sigma) = |\sigma \wedge \sigma_1|$ and $\widetilde{\max}(F, \sigma) = |\sigma \wedge \sigma_d|$.

(iii) There is $(F_i, \sigma)_{i=1}^{d}$ an attached branching of σ in G_n^{α} such that $= \bigcup_{i=1}^{d} F_i$.

In this case we say that (F, σ) is attached. Moreover set $\widetilde{\min}(F, \sigma) = \widetilde{\min}(F_1, \sigma)$ and $\widetilde{\max}(F, \sigma) = \widetilde{\max}(F_d, \sigma)$.

Definition (2.2.23) [2]:

For a countable ordinal number α and $n \in \mathbb{N}$ we define

$G_n^{\alpha} = \{F \subset \{0,1\}^{\mathbb{N}}$: there exists $\sigma \in \{0,1\}^{\mathbb{N}}$ with $(F, \sigma) \in G_n^{\alpha}\} \cup \{\phi\}$.

Proposition (2.2.24) [2]:

Let α be a countable ordinal number. Then for every B infinite subset of $\{0,1\}^{\mathbb{N}}$ there exists a one to one map $\phi : \mathbb{N} \to B$ with $\phi(F) \in \mathcal{G}_n^{\alpha}$ for every $F \in S_\alpha^n$ and $n \in \mathbb{N}$.

Theorem (2.1.23)] takes the following form and the proof uses the compactness of S_α and Corollary (2.1.19) .

Theorem (2.2.25) [2]:

Let α be a countable ordinal number and $n \in \mathbb{N}$. Then $\mathcal{G}_n^\alpha$ is an α-large, hereditary and compact family of finite subsets of $[0,1]^{\mathbb{N}}$.

In order to define the desired space $\mathfrak{X}_{2^{\aleph_0}}^\alpha$, one takes the same steps as in the previous section. All proofs are identical.

Definition (2.2.26) [2]:

For α a countable ordinal number and $n \in \mathbb{N}$ define a norm on $c_{00}(2^{\aleph_0})$ in the following manner:

(i) For $n \in \mathbb{N}$, we may identify an $F \in \mathcal{G}_n^\alpha$ with a linear functional $F: c_{00}(2^{\aleph_0}) \to \mathbb{R}$ in the following manner. For $x = \sum_{\xi < 2^{\aleph_0}} \lambda_\xi e_\xi \in c_{00}(2^{\aleph_0})$

$$F(x) = \sum_{\xi < 2^{\aleph_0}} \lambda_{\xi_\tau}.$$

(ii) For $x \in c_{00}(2^{\aleph_0})$ define

$$\|x\|_n^\alpha = \sup\{|F(x)|: F \in \mathcal{G}_n^\alpha\}.$$

Set X_n^α to be the completion of $(c_{00}(2^{\aleph_0}), |\cdot|_n^\alpha)$.

Definition (2.2.27) [2]:

Define the following norm on $c_{00}(2^{\aleph_0})$. For $x \in c_{00}(2^{\aleph_0})$

$$\|x\| = \left\| \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|_n^\alpha e_n \right\|_{T_\alpha}.$$

Set $\mathfrak{X}_{2^{\aleph_0}}^\alpha$ to be the completion of $(\mathrm{c}_{00}(2^{\aleph_0}), |\cdot|_{\mathrm{n}}^{\alpha})$.

Theorem (2.2.28) [2]:

The space $\mathfrak{X}_{2^{\aleph_0}}^\alpha$ is a non-separable reflexive space with a suppression unconditional Schauder basis with the length of the continuum, having the following property. Every normalized weakly null sequence in $\mathfrak{X}_{2^{\aleph_0}}^\alpha$ has a subsequence that generates an ℓ_1^α spreading model.

Chapter 3

Polynomials on Banach spaces

In this chapter we study Banach spaces of traces of real poly-nominal on $\mathbb{R}^n$ to compact subsets equipped with supremum norms .

Recall that the Banach-Mazur distance between two k-dimensional real Banach spaces E, F is defined as

$$d_{BM}(E,F) := \inf\{\|u\| \cdot \|u^{-1}\|\},$$

where the infimum is taken over all isomorphisms $u: E \to F$. We say that E and F are equivalent if they are isometrically isomorphic (i.e., $d_{BM}(E.F) = 1$). Then $\ln d_{BM}$ determines a metric on the set $\mathcal{B}_k$ of equivalence classes of isometrically isomorphic k-dimensional Banach spaces (called the Banach-Mazur compactum).It is known that $\mathcal{B}_k$ is compact of d_{BM}-“diameter” $\sim k$.

Let C(K) be the Banach space of real continuous functions on a compact Hausdorff space K equipped with the supremum norm. Let $\mathrm{F} \subset \mathrm{C(K)}$ be a filtered subalgebra with filtration $\{0\} \subset \mathrm{F}_0 \subseteq \mathrm{F}_1 \subseteq \cdots \subseteq \mathrm{F}_\mathrm{d} \subseteq \cdots \subseteq \mathrm{F}$ (that is, $\mathrm{F} = \bigcup_{\mathrm{d} \in \mathbb{Z}_+} \mathrm{F}_\mathrm{i}$ and $\mathrm{F}_\mathrm{i} \cdot \mathrm{F}_\mathrm{j} \subset \mathrm{F}_{\mathrm{i+j}}$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{Z}_+$) such that $\mathrm{n}_\mathrm{d} := \dim \mathrm{F}_\mathrm{d} < \infty$ for all d. In what follows we assume that F_0 contains constant functions on K.

Theorem (3.1) [3]:

Suppose there are $c \in \mathbb{R}$ and $\{p_d\}_{d \in \mathbb{N}} \subset \mathbb{N}$ such that

$$\frac{\ln n_{d \cdot p_d}}{p_d} \le c \quad \text{for all } d \in \mathbb{N}. \tag{1}$$

Then there exist linear injective maps $i_d : F_d \hookrightarrow \ell^\infty_{n_d \cdot p_d}$ such that

$$d_{BM}\big(F_d, i_d(F_d)\big) \le e^c, \;\; d \in \mathbb{N}.$$

Proof :

Since $\dim F_i = n_i, i \in \mathbb{N}$, and evaluations δ_z at points $z \in K$ determine bounded linear functionals on F_i, the Hahn-Banach theorem implies easily that span $\{\delta_z\}_{z \in K} = F_i^*$. Moreover, $\|\delta_z\|_{F_i^*} = 1$ for all $z \in K$ and the closed unit ball of F_i^* is the balanced convex hull of the set $\{\delta_z\}_{z \in K}$. Let $\{f_{1_i}, \dots, f_{n_i i}\} \subset F_i$ be an Auerbach basis with the dual basis

$\{\delta_{z_{1_i}}, \dots, \delta_{z_{n_i i}}\} \subset F_i^*$, that is, $f_{ki}\left(\delta_{z_{1_i}}\right) := f_{ki}(z_{li}) = \delta_k$ (the Kronecker-delta) and $\|f_{ki}\|_K = 1$ for all k. (Its construction is similar to that of the fundamental Lagrange interpolation polynomials for $F_i = \mathcal{P}_i^n \backslash_K$,

Now, we use a "method of E. Landau".

By the definition, for each $g \in F_i$ we have $g(z) = \sum_{k=1}^{n_i} f_{ki}(z) g(z_{ki})\,, z \in K$. Hence, $\|g\|_K \leq n_i \|g\|_{\{z_{1_i}, \dots, z_{n_i i}\}}$. Applying the latter inequality to $g = f^{p_d}\,, f \in F_d$, containing in $F_i, i:\ d \cdot p_d$, and using condition (1) we get for $A_d := \{z_{1_i}, \dots, z_{n_i i}\} \subset K$

$$\|f\|_K = (\|g\|_K)^{\frac{1}{pd}} \leq (n_d \cdot pd)^{\frac{1}{pd}} \cdot \left(\|g\|_{A_d}\right)^{\frac{1}{pd}} \leq e^c \cdot \|f\|_{A_d}.$$

Thus, restriction $F_d \longmapsto F_d \backslash_{A_d}$ determines the required map $i_d : F_d \hookrightarrow \ell_{n_d \cdot p_d}^{\infty}$.

As a corollary we obtain:

Corollary (3.2) [3]:

Suppose $\{n_d\}_{d \in \mathbb{N}}$ grows at most polynomially in d, that is,

$$\exists k, \hat{c} \in \mathbb{R}_+ \quad \text{such that } \forall d \;\; n_d \leq \hat{c} d^k. \qquad (2)$$

Then for each natural number $s \geq 3$ there exist linear injective maps $i_{d,s} : F_d \hookrightarrow \ell_{N,d,s}^{\infty}$, where $N_{d,s} := \left\lfloor \hat{c} d^k \cdot s^k \cdot \left(\lfloor \ln(\hat{c} d^k) \rfloor + 1 \right)^k \right\rfloor$, such that

$$d_{BM}\left(F_d, i_{d,s}(F_d)\right) \leq (e s^k)^{\frac{1}{s}}, \quad k \in \mathbb{N}.$$

Let $\mathcal{F}_{\hat{c},k}$ be the family of all possible filtered algebras F on compact Hausdorff spaces K satisfying condition (2) [3]. By $\mathcal{B}_{\hat{c},k,\bar{n}_d} \subset \mathcal{B}_{\bar{n}_d}$ we denote the closure in $\mathcal{B}_{\bar{n}_d}$ of the set formed by all subspaces F_d of algebras $F \in \mathcal{F}_{\hat{c},k}$ having a fixed dimension $\bar{n}_d \in \mathbb{N}$.

Corollary (3.2) allows to estimate the metric entropy of $\mathcal{B}_{\hat{c},k,\bar{n}_d}$. Recall that for a compact subset $S \subset \mathcal{B}_{\bar{n}_d}$ its ε-entropy ($\varepsilon > 0$) is defined as $H(S, \varepsilon) := \ln(S, d_{BM}, 1 + \varepsilon)$, where $N(S, d_{BM}, 1 + \varepsilon)$ is the smallest number of open d_{BM}-"balls" of radius $1 + \varepsilon$ that cover S.

Proof :

We set $p_d := s \cdot (\lfloor \ln(\hat{c}d^k) \rfloor + 1), d \in \mathbb{N}$. Then the condition of the corollary implies

$$\frac{\ln n_{d \cdot p_d}}{pd} \leq \frac{\ln(\hat{c}d^k) + k \ln p_d}{pd} \leq \frac{1}{s} + \frac{k \ln s}{s} =: c.$$

Thus the result follows from Theorem (3.1)

Corollary (3.3) [3]:

For $k \geq 1$ there exists a numerical constant C such that for each $\varepsilon \in (0, \frac{1}{2}]$

$$H(\mathcal{B}_{\hat{c},k,\bar{n}_d}, \varepsilon) \leq (CK \cdot \ln(k+1))^k \cdot (\hat{c}d^k)^2 \cdot (\ln(\hat{c}d^k) + 1)^{k+1} \cdot \left(\frac{1}{\varepsilon}\right)^k \cdot \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{k+1}$$

Let $\mathcal{P}_d^n$ be the space of real polynomials on $\mathbb{R}^n$ of degree at most d. For a compact subset $K \subset \mathbb{R}^n$ by $\mathcal{P}_d^n \backslash_K$ we denote the trace space of restrictions of polynomials in $\mathcal{P}_d^n$ to K equipped with the supremum norm. Applying Corollary (3.2) to algebra $\mathcal{P}^n \backslash_K := \bigcup_{d \geq 0} \mathcal{P}_d^n \backslash_K$ we obtain:

A. There exist linear injective maps $i_{d,K}: \mathcal{P}_d^n \backslash_K \hookrightarrow \ell_{N,d,n}^{\infty}$, where

$$N_{d,n} := \lfloor e^{2n} \cdot (n+2)^{2n} \cdot d^m \cdot (2n + 1 + \lfloor n \ln d \rfloor)^n \rfloor, \quad (3)$$

such that

$$d_{BM}\left(\mathcal{P}_d^n \backslash_K, i_{d,K}(\mathcal{P}_d^n \backslash_K)\right) \leq (e \cdot (n+2)^2)^{\frac{1}{n+2}} \ (< 2.903). \quad (4)$$

Indeed,

$$\tilde{N}_{d,n} := \dim \mathcal{P}_d^n \backslash_K \leq \binom{d+n}{n} < \left(\frac{e \cdot (d+n)}{n}\right)^n \leq \left(\frac{e \cdot (1+n)}{n}\right)^n \cdot d^n < e^{2n} \cdot d^n. \quad (5)$$

Hence, Corollary (3.2) with $c = e^{2n}, k := n$ and $s := (n+2)^2$ implies the required result.

If K is $\mathcal{P}^n$-determining (i.e., no nonzero polynomial vanish on K), then $\widetilde{N}_{d,n} = \binom{d+n}{n}$ and so for some constant c(n) (depending on n only) we have

$$\widetilde{N}_{d,n} < N_{d,n} \le c(n) \cdot \widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\right)^n. \qquad (6)$$

Hence, $V_{d,n} \coloneqq i_{d,K}(\mathcal{P}^n_d \backslash_K)$ is a "large" subspace of $\ell^\infty_{N_{d,n}}$. Therefore from (A) applied to $V_{d,n}(K)$ we obtain:

B. There is a constant $c_1(n)$ (depending on n only) such that for each $\mathcal{P}^n$-determining compact set $K \subset \mathbb{R}^n$ there exists an m-dimensional subspace $F \subset \mathcal{P}^n_d \backslash_K$ with

$$m \coloneqq \dim F > c_1(n) \cdot \left(\widetilde{N}_{d,n}\right)^{\frac{1}{2}} \ \ and \ \ d_{BM}(F, \ell^\infty_m) \le 3. \qquad (7)$$

In turn, if $\hat{d} \in \mathbb{N}$ is such that $N_{\hat{d},n} \le c_1(n) \cdot \left(\widetilde{N}_{d,n}\right)^{\frac{1}{2}}$, then due to property (A) for each $\mathcal{P}^n$-determining compact set $K' \subset \mathbb{R}^n$ there exists a e $\widetilde{N}_{\hat{d},n}$-dimensional subspace $F_{\hat{d},n,K'} \subset F$ such that

$$d_{BM}\left(F_{\hat{d},n,K'}, \mathcal{P}^n_{\hat{d}} \backslash_{K'}\right) < 9. \qquad (8)$$

Further, the dual space $\left(V^n_d(K)\right)^*$ of $V^n_d(K)$ is the quotient space of $\ell^1_{N_{d,n}}$. In particular, the closed ball of $\left(V^n_d(K)\right)^*$ contains at most $c(n) \cdot \widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\right)^n$ extreme points, see (6). Thus the balls of $\left(V^n_d(K)\right)^*$ and $V^n_d(K)$ are "quite different" as convex bodies. This is also expressed in the following property (similar to the celebrated John ellipsoid theorem [8] that is The John ellipsoid $E(K)$ of a convex body $K \subset \mathbf{R}^n$ is B If and only if $B \subseteq K$ and there exists an Integer $m \ge n$ and, for $i = 1, ..., m$, Real numbe $c_i > 0$ and Unit vector $u_i \in \mathbf{S}^{n-1} \cap \partial K$ such that

$$\sum_{i=1}^{m} c_i u_i = 0$$

and, for all $x \in \mathbf{R}^n$

$$x = \sum_{i=1}^{m} c_i (x \cdot u_i) u_i.$$

but with an extra logarithmic factor) which is a consequence of property (A) .

C. There is a constant $c_2(n)$ (depending on n only) such that for all $\mathcal{P}^n$-determining compact sets $K_1, K_2 \subset \mathbb{R}^n$

$$d_{BM}\left(\mathcal{P}_d^n\backslash_{K_1}, \left(\mathcal{P}_d^n\backslash_{K_2}\right)^*\right) \leq c_2(n) \cdot \left(\widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\right)\right)^{\frac{1}{2}}. \qquad (9)$$

A stronger inequality is valid if we replace $\left(\mathcal{P}_d^n\backslash_{K_2}\right)^*$ above by $\ell^1_{\widetilde{N}_{d,n}}$,

Remark (3.4) [3]:

Property (C) has the following geometric interpretation. By definition, $\left(\mathcal{P}_d^n\backslash_{K_2}\right)^*$ is an $\widetilde{N}_{d,n}$-dimensional real Banach space generated by evaluation functionals δ_x at points $x \in K_2$ with the closed unit ball being the balanced convex hull of the set $\{\delta_x\}_{x \in K_2}$. Thus K_2 admits a natural isometric embedding into the unit sphere of $\left(\mathcal{P}_d^n\backslash_{K_2}\right)^*$. Moreover, the Banach space of linear maps $\left(\mathcal{P}_d^n\backslash_{K_2}\right)^* \to \mathcal{P}_d^n\backslash_{K_1}$ equipped with the operator norm is isometrically isomorphic to the Banach space of real polynomial maps $p: \mathbb{R}^n \to \mathcal{P}_d^n\backslash_{K_1}$ of degree at most d (i.e., $f^* \circ p \in \mathcal{P}_d^n$ for all $f^* \in \left(\left(\mathcal{P}_d^n\backslash_{K_1}\right)^*\right)$ with norm $\|p\| := \sup_{x \in K_2} \|p(x)\|_{\mathcal{P}_d^n\backslash_{K_1}}$. Thus property (C) is equivalent to the following one:

C′. There exists a polynomial map $\mathrm{p} : \mathbb{R}^n \to \mathcal{P}_d^n\backslash_{K_1}$ of degree at most d such that the balanced convex hull of $p(K_2)$ contains the closed unit ball of $\mathcal{P}_d^n\backslash_{K_1}$and is contained in the closed ball of radius $c_2(n) \cdot \left(\widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\right)\right)^{\frac{1}{2}}$ of this space (both centered at 0).

Our next property, a consequence of Corollary (3.3) and (5), estimates the metric entropy of the closure of the set $\widetilde{\mathcal{P}}_{d,n} \subset \mathcal{B}_{\widetilde{N}_{d,n}}$ formed by all $\widetilde{N}_{d,n}$-dimensional spaces $\mathcal{P}_d^n\backslash_K$ with $\mathcal{P}^n$-determining compact subsets $K \subset \mathbb{R}^n$.

D. There exists a numerical constant $c > 0$ such that for each $\varepsilon \in (0, \frac{1}{2}]$,

$$H\left(\mathrm{cl}\left(\tilde{\mathcal{P}}_{d,n}\right),\varepsilon\right) \le (cn^2 \cdot \ln(n+1))^n \cdot d^{2n} \cdot (1+\ln d)^{n+1} \cdot \left(\frac{1}{\varepsilon}\right)^n$$

$$\cdot \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{n+1}. \qquad (10)$$

Remark (3.5) [3]:

The above estimate shows that $\tilde{\mathcal{P}}_{d,n}$ with sufficiently large d and n is much less massive than $\mathcal{B}_{\tilde{N}_{d,n}}$. Indeed, as follows

$$H\left(\mathcal{B}_{\tilde{N}_{d,n}},\varepsilon\right) \sim \left(\frac{1}{\varepsilon}\right)^{\frac{\tilde{N}_{d,n}-1}{2}} \quad \text{as} \quad \varepsilon \to 0^+$$

(here the equivalence depends on d and n as well). On the other hand, it implies that for any $\varepsilon > 0$,

$$0 < \liminf_{\tilde{N}_{d,n}\to\infty} \frac{\ln H\left(\mathcal{B}_{\tilde{N}_{d,n}},\varepsilon\right)}{\tilde{N}_{d,n}} \le \limsup_{\tilde{N}_{d,n}\to\infty} \frac{\ln H\left(\mathcal{B}_{\tilde{N}_{d,n}},\varepsilon\right)}{\tilde{N}_{d,n}} < \infty.$$

It might be of interest to find sharp asymptotics of $H\left(\mathrm{cl}\left(\tilde{\mathcal{P}}_{d,n}\right),\varepsilon\right)$ as $\varepsilon \to 0^+$ and $d \to \infty$, and to compute (up to a constant depending on n) d_{BM}-"diameter" of $\tilde{\mathcal{P}}_{d,n}$.

Similar results are valid for K being a compact subset of a real algebraic variety $X \subset \mathbb{R}^n$ of dimension $m < n$ such that if a polynomial vanishes on K, then it vanishes on X as well. In this case there are positive constants $cX, \tilde{c}X$ depending on X only such that $\tilde{c}Xd^m \le \dim \mathcal{P}_d^n\backslash_K \le cXd^m$. For instance, Corollary (3.2) with $c = cX, k := m$ and $s := (m+2)^2$ implies that $\mathcal{P}_d^n\backslash_K$ is linearly embedded into $\ell_{N_{d,X}}^\infty$, where $N_{d,X} := \lfloor cXd^m \cdot (m+2)^{2m} \cdot (\lfloor \ln(cXd^m) \rfloor + 1)^m \rfloor$, with distortion < 2.903. We leave the details.

Lemma (3.6) [3]:

Let $S_{\bar{n}_d} \subset \mathcal{B}_{\bar{n}_d}$ be the subset formed by all $\bar{n}_d$-dimensional subspaces of $\ell^{\infty}_{N_{d,s}}$. Consider $0 < \xi < \frac{1}{\bar{n}_d}$ and let $R = \frac{1+\xi\bar{n}_d}{1-\xi\bar{n}_d}$. Then $S_{\bar{n}_d}$ admits an R-net T_R of cardinality at most $\left(1+\frac{2}{\xi}\right)^{N_{d,s}\cdot\bar{n}_d}$.

Now given $\varepsilon \in (0,\frac{1}{2}]$ we choose $s = \lfloor s_\varepsilon \rfloor$ with s_ε satisfying $(es_\varepsilon^k)^{\frac{1}{s_\varepsilon}} = \sqrt[4]{1+\varepsilon}$ and ξ such that $R = R_\varepsilon = \sqrt[4]{1+\varepsilon}$. Then according to Corollary (3.2) and Lemma (3.6) , $\mathrm{dist}_{BM}\left(T_{R_\varepsilon}, \mathcal{B}_{\hat{c},k,\bar{n}_d}\right) < \sqrt{1+\varepsilon}$. For each $p \in T_{R_\varepsilon}$ we choose $q_p \in \mathcal{B}_{\hat{c},k,\bar{n}_d}$ such that $d_{BM}(p, q_p) < \sqrt{1+\varepsilon}$. Then the multiplicative triangle inequality for d_{BM} implies that open d_{BM}-"balls" of radius $1+\varepsilon$ centered at points $q_p, p \in T_{R_\varepsilon}$, cover $\mathcal{B}_{\hat{c},k,\bar{n}_d}$. Hence,

$$N\left(\mathcal{B}_{\hat{c},k,\bar{n}_d}, d_{BM}, 1+\varepsilon\right) \leq \text{card}\, T_{R_\varepsilon} \leq \left(1+\frac{2}{\xi}\right)^{N_{d,s}\cdot\bar{n}_d}. \tag{11}$$

Next, the function $\varphi(x) = \ln(ex^k)^{\frac{1}{x}}$ decreases for $x \in \left[e^{\frac{k-1}{k}}, \infty\right)$ and $\lim_{x\to\infty} \varphi(x) = 0$. Its inverse φ^{-1} on this interval has domain $\left(0, e^{-\frac{k-1}{k}}\right]$, increases and is easily seen (using that $\varphi \circ \varphi^{-1} = \mathrm{id}$) to satisfy

$$\varphi^{-1}(x) \leq \frac{3k}{x} \cdot \ln\left(\frac{3k}{x}\right), \quad x \in \left(0, e^{-\frac{k-1}{k}}\right].$$

Since $\frac{1}{4}\ln(1+\varepsilon) < e^{-\frac{k-1}{k}}$ for $\varepsilon \in \left(0, \frac{1}{2}\right]$, the required s_ε exists and the previous inequality implies that

$$s_\varepsilon \leq \frac{12k}{\ln(1+\varepsilon)} \cdot \ln\left(\frac{12k}{\ln(1+\varepsilon)}\right). \tag{12}$$

Further, we have

$$\frac{1}{\xi} = \frac{\bar{n}_d(1+R_\varepsilon)}{R_\varepsilon - 1} = \frac{\bar{n}_d\left(\sqrt[4]{1+\varepsilon}+1\right)}{\sqrt[4]{1+\varepsilon}-1}$$
$$= \frac{\bar{n}_d\left(\sqrt[4]{1+\varepsilon}+1\right)^2 \cdot \left(\sqrt[4]{1+\varepsilon}+1\right)}{\varepsilon}. \tag{13}$$

From (11), (12), (13) invoking the definition of $N_{d,s}$ we obtain

$$\ln N\left(\mathcal{B}_{\hat{c},k,\bar{n}_d}, d_{BM}, 1+\varepsilon\right) \le \bar{n}_d \hat{c} d^k (\ln(\hat{c} d^k) + 1)^k \ln\left(\frac{21\bar{n}_d}{\varepsilon}\right)\left(\frac{12k}{\ln(1+\varepsilon)} \ln\left(\frac{12k}{\ln(1+\varepsilon)}\right)\right)^k.$$

Using that $\bar{n}_d \le \hat{c} d^k$ and the inequality $\frac{2}{3} \cdot \varepsilon \le \ln(1+\varepsilon), \varepsilon \in \left(0, \frac{1}{2}\right]$, we get the required estimate.

Chapter 4

Countable Infinite Numbers of Complex Structures on the Banach Spaces

In this chapter we give examples of real Banach spaces with exactly infinite countably many complex structures and with $\omega_1$many complex structures.

Section (4.1): Construction and Complex Structures of The Space $\mathfrak{X}_{\omega_1}(c)$:

A real Banach space X is said to admit a complex structure when there exists a linear operator I on X such that $I^2 = -Id$. This turns X into a $\mathbb{C}$-linear space by declaring a new law for the scalar multiplication:

$$(\lambda + i\mu) \cdot x = \lambda x + \mu I(x) \quad (\lambda, \mu \in \mathbb{R}).$$

Equipped with the equivalent norm

$$\|x\| = \sup_{0 \le \theta \le 2n} \|\cos\theta x + \sin\theta\, Ix\|,$$

we obtain a complex Banach space which will be denoted by X^I. The space X^I is the complex structure of X associated to the operator I, which is often referred itself as a complex structure for X.

When the space X is already a complex Banach space, the operator $Ix = ix$ is a complex structure on $X_{\mathbb{R}}$ (i.e., X seen as a real space) which generates X. Recall that for a complex Banach space X its complex conjugate $\overline{X}$ is defined to be the space X equipped with the new scalar multiplication $\lambda. x = \bar{\lambda} x$.

Two complex structures I and J on a real Banach space X are equivalent if there exists a real automorphism T on X such that $TI = JT$. This is equivalent to saying that the spaces X^I and X^J are $\mathbb{C}$-linearly isomorphic. To see this, simply observe that the relation $TI = JT$ actually means that the operator T is $\mathbb{C}$-linear as defined from X^I to X^J .

We note that a complex structure I on a real Banach space X is an automorphism whose inverse is $-I$, which is itself another complex structure on X. In fact, the complex space X^{-I} is the complex conjugate space of X^I . Clearly the spaces X^I and X^{-I} are always $\mathbb{R}$-linearly isometric. On the other hand, J. Bourgain and N.J. Kalton constructed examples of complex Banach spaces not isomorphic to their corresponding complex conjugates, hence these space admit at least two different complex structures. The Bourgain example is an ℓ_2 sum of finite dimensional spaces whose distance to their conjugates tends to infinity. The Kalton example is a twisted sum of two Hilbert spaces, i.e., X has a closed subspace E such that E and X/E are Hilbertian, while X itself is not isomorphic to a Hilbert space. More recently R. Anisca constructed a complex weak Hilbert space not isomorphic to its complex conjugate.

Complex structures do not always exist on Banach spaces. The first example in the literature was the James space, proved by J. Dieudonne' . Other examples of spaces without complex structures are the uniformly convex space constructed by S. Szarek and the hereditary indecomposable space of W. T. Gowers and B. Maurey. W. T. Gowers and B. Maurey and S.A. Argyros, K. Beanland and T. Raikoftsalis also constructed a space with unconditional basis but without complex structures, the second is a weak Hilbert space. In general these spaces have few operators. For example, every operator on the Gowers-Maurey space is a strictly singular perturbation of a multiple of the identity and this forbids complex structures: suppose that T is an operator on this space such that $T^2 = -Id$ and write $T = \lambda Id + S$ with S a strictly singular operator. It follows that $(\lambda^2 + 1)Id$ is strictly singular and of course this is impossible.

More examples of Banach spaces without complex structures were constructed by P. Koszmider, M. Martı'n and J. Mer´ı . In fact, they introduced the notion of extremely non-complex Banach space: A real Banach space X is extremely non-complex if every bounded linear operator $T: X \to X$ satisfies the norm equality $\|Id + T^2\| = 1 + \|T\|^2$. Among their examples of extremely non complex spaces are $C(K)$ spaces with few operators (e.g. when every bounded linear operator T on $C(K)$ is of the form $T = gId + S$ where $g \in C(K)$ and S is a weakly compact operator on $C(K)$), a $C(K)$ space containing a complemented isomorphic copy of ℓ_∞ (thus having a richer space of operators than the first one mentioned) and an extremely non complex space not isomorphic to any $C(K)$ space.

Going back to the problem of uniqueness of complex structures, Kalton proved that spaces whose complexification is a primary space have at most one complex structure (this result may be found in V. Ferenczi and E. Galego). In particular, the classical spaces c_0, ℓ_p $(1 \leq p \leq \infty)$, $L_p[0,1]$ $(1 \leq p \leq \infty)$, and $C[0,1]$ have a unique complex structure.

We have mentioned before examples of Banach spaces with at least two different complex structures. In fact, V. Ferenczi constructed a space $X(\mathbb{C})$ such that the complex structure $X(\mathbb{C})^J$ associated to some operator J and its conjugate are the only complex structures on $X(\mathbb{C})$ up to isomorphism. Furthermore, every $\mathbb{R}$-linear operator T on $X(\mathbb{C})$ is of the form $T = \lambda Id + \mu^J + S$, where λ, μ are reals and S is strictly singular. Ferenczi also proved that the space $X(\mathbb{C})^n$ has exactly $n+1$ complex structures for every positive integer n. Going to the extreme, R. Anisca gave examples of subspaces of $L_p (1 \leq p < 2)$ which admit continuum many non-isomorphic complex structures.

The question remains about finding examples of Banach spaces with exactly infinite countably many different complex structures. A first natural approach to solve this problem is to construct an infinite sum of copies of $X(\mathbb{C})$, and in order to control the number of complex structures to take a regular sum, for instance, $\ell_1(X(\mathbb{C}))$. It follows that every $\mathbb{R}$-linear bounded operator T on $\ell_1(X(\mathbb{C}))$ is of the form $T = \lambda(T) + S$, where $\lambda(T)$ is the scalar part of T, i.e., an infinite matrix of operators on $X(\mathbb{C})$ of the form $\lambda_{i,j} Id + \mu_{i,j} J$, and S is an infinite matrix of strictly singular operators on $X(\mathbb{C})$. It is easy to prove that if T is a complex structure then $\lambda(T)$ is also a complex structure. Recall from that two complex structures whose difference is strictly singular must be equivalent. Unfortunately, the operator S in the representation of T is not necessarily strictly singular, and this makes very difficult to understand the complex structures on $\ell_1(X(\mathbb{C}))$.

It is necessary to consider a more “rigid” sum of copies of spaces like $X(\mathbb{C})$. We found this interesting property in the space $\mathfrak{X}_{\omega_1}$ constructed by S.A. Argyros, J. Lopez-Abad and S. Todorcevic. Based on that construction we present a separable reflexive Banach space $\mathfrak{X}_{\omega^2}(\mathbb{C})$ with exactly infinite countably many different complex structures which admits an infinite dimensional Schauder decomposition $\mathfrak{X}_{\omega^2}(\mathbb{C}) = \oplus_k \mathfrak{X}_k$ for which every $\mathbb{R}$-linear operator T on $\mathfrak{X}_{\omega^2}(\mathbb{C})$ can be written as $T = DT + S$, where S is strictly singular, $D_T \backslash_{\mathfrak{X}_k} = \lambda_k Id_{\mathfrak{X}_k}$ $(\lambda_k \in \mathbb{C})$ and $(\lambda_k)_k$ is a convergent sequence.

This construction also shows the existence of continuum many examples of Banach spaces with the property of having exactly ω complex structures and the existence of a Banach space with exactly ω_1 complex structures.

We construct a complex Banach space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ with a bimonotone transfinite Schauder basis $(e_\alpha)_{\alpha<\omega_1}$, such that every complex structure I on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is of the form $I = D + S$, where D is a suitable diagonal operator and S is strictly singular.

By a bimonotone transfinite Schauder basis we mean that $\mathfrak{X}_{\omega_1}(\mathbb{C}) = \overline{\text{span}}(e_\alpha)_{\alpha<\omega_1}$ and such that for every interval I of ω_1 the naturally defined map on the linear span of $(e_\alpha)_{\alpha<\omega_1}$

$$\sum_{\alpha<\omega_1} \lambda_\alpha e_\alpha \longmapsto \sum_{\alpha\in I} \lambda_\alpha e_\alpha$$

extends to a bounded projection $P_1: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_I = \overline{\text{span}}_{\mathbb{C}}(e_\alpha)_{\alpha\in I}$ with norm equal to 1.

Basically $\mathfrak{X}_{\omega_1}(\mathbb{C})$ corresponds to the complex version of the space $\mathfrak{X}_{\omega_1}$ constructed in modifying the construction in a way that its $\mathbb{R}$-linear operators have similar structural properties to the operators in the original space $\mathfrak{X}_{\omega_1}$ (i.e. the operators are strictly singular perturbation of a complex diagonal operator).

Recall that ω and ω_1 denotes the least infinite cardinal number and the least uncountable cardinal number, respectively. Given ordinals γ, ξ we write $\gamma + \xi, \gamma \cdot \xi, \gamma^\xi$ for the usual arithmetic operations . For an ordinal γ we denote by $\Lambda(\gamma)$ the set of limit ordinals $< \gamma$. Denote by $c_{00}(\omega_1, \mathbb{C})$ the vector space of all functions $x: \omega_1 \to \mathbb{C}$ such that the set $\text{supp}\, x = \{\alpha < \omega_1 : x(\alpha) \neq 0\}$ is finite and by $(e_\alpha)_{\alpha<\omega_1}$ its canonical Hamel basis. For a vector $x \in c_{00}(\omega_1, \mathbb{C})$ ran x will denote the minimal interval containing supp x. Given two subsets E_1, E_2 of ω_1 we say that

$E_1 < E_2$ if $\max E_1 < \min E_2$. Then for $x, y \in c_{00}(\omega_1, \mathbb{C}) x < y$ means that $\operatorname{supp} x < \operatorname{supp} y$. For a vector $x \in c_{00}(\omega_1, \mathbb{C})$ and a subset E of ω_1 we denote by E_x(or P_{E_x}) the restriction of x on E or simply the function $x\chi E$. Finally in some cases we shall denote elements of $c_{00}(\omega_1, \mathbb{C})$ as $f, g, h, \dots$ and its canonical Hamel basis as $(e_\alpha^*)_{\alpha<\omega_1}$ meaning that we refer to these elements as being functionals in the norming set.

The space $\mathfrak{X}_{\omega_1}$ shall be defined as the completion of $c_{00}(\omega_1, \mathbb{C})$ equipped with a norm given by a norming set $\kappa_{\omega_1}(\mathbb{C}) \subseteq c_{00}(\omega_1, \mathbb{C})$. This means that the norm for every $x \in c_{00}(\omega_1, \mathbb{C})$ is defined as $\sup\{|\phi(x)| = \left|\sum_{\alpha<\omega_1} \phi(\alpha)x(\alpha)\right| : \phi \in \kappa_{\omega_1}(\mathbb{C})\}$. The norm of this space can also be defined inductively.

We start by fixing two fast increasing sequences (m_j) and (n_j) that are going to be used in the rest of this work. The sequences are defined recursively as follows:

(i) $m_1 = 2$ and $m_{j+1} = m_j^4$;

(ii) $n_1 = 4$ and $n_{j+1} = (4n_j)^{s_j}$, where $s_j = \log_2 m_{j+1}^3$.

Let $\kappa_{\omega_1}(\mathbb{C})$ be the minimal subset of $c_{00}(\omega_1, \mathbb{C})$ such that

(a) It contains every $e_\alpha^*, \alpha < \omega_1$. It satisfies that for every $\phi \in \kappa_{\omega_1}(\mathbb{C})$ and for every complex number $\theta = \lambda + i\mu$ with λ and μ rationals and $|\theta| \leq 1, \theta\phi \in \kappa_{\omega_1}(\mathbb{C})$. It is closed under restriction to intervals of ω_1.

(b) For every $\{\phi_i, : i = 1, \dots, n_{2j}\} \subseteq \kappa_{\omega_1}(\mathbb{C})$ such that $\phi_1 < \cdots < \phi_{n_{2j}}$, the combination

$$\phi = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \phi_i \in \kappa_{\omega_1}(\mathbb{C}).$$

In this case we say that ϕ is the result of an (m_{2j}^{-1}, n_{2j})-operation.

(c) For every special sequence $(\phi_1, \dots, \phi_{n_{2j+1}})$ the combination

$$\phi = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} \phi_i \in \kappa_{\omega_1}(\mathbb{C}).$$

In this case we say that ϕ is a special functional and that ϕ is the result of an $\left(m_{2j+1}^{-1}, n_{2j+1}\right)$-operation.

(d) It is rationally convex.

Define a norm on c by setting

$$\|x\| = \sup\left\{\left|\sum_{\alpha<\omega_1} \phi(\alpha)x(\alpha)\right| : \phi \in \kappa_{\omega_1}(\mathbb{C})\right\}.$$

The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is defined as the completion of $(c_{00}(\omega_1, \mathbb{C}), \|\cdot\|)$.

This definition of the norming set $\kappa_{\omega_1}(\mathbb{C})$ is similar to others . We add the property of being closed under products with rational complex numbers of the unit ball. This, together with property (b) above, guarantees the existence of some type of sequences [4] in the same way they are constructed for $\mathfrak{X}_{\omega_1}$. It follows that the norm is also defined by

$$\|x\| = \sup\left\{\phi(x) = \sum_{\alpha<\omega_1} \phi(\alpha)x(\alpha) : \phi \in \kappa_{\omega_1}(\mathbb{C}), \phi(x) \in \mathbb{R}\right\}.$$

We also have the following implicit formula for the norm:

$$\|x\| = \max\left\{\|x\|_\infty, \sup_j \sup \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|, E_1 < E_2 < \cdots < E_{n_{2j}}\right\}$$

$$\vee \sup\left\{\frac{1}{m_{2j+1}}\left|\sum_{i=1}^{n_{2j+1}} \phi_i(Ex)\right| : (\phi_i)_{i=1}^{n_{j+1}} \text{ is } n_{2j+1} - \text{special}, E \text{ interval}\right\}.$$

It follows from the definition of the norming set that the canonical Hamel basis $(e_\alpha)_{\alpha<\omega_1}$ is a transfinite bimonotone Schauder basis of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. In fact, by Property (b) for every interval I of ω_1 the projection P_I has norm 1:

$$\|P_I x\| = \sup_{f \in \kappa_{\omega_1}(\mathbb{C})} |f P_I x| = \sup_{f \in \kappa_{\omega_1}(\mathbb{C})} |P_I f x| \leq \|x\|$$

Moreover, we have that the basis $(e_\alpha)_{\alpha<\omega_1}$ is boundedly complete and shrinking, the proof is the obvious modification to the one for $\mathfrak{X}_{\omega_1}$. In consequence $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is reflexive.

Proposition (4.1.1) [4]:

$$\overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*} = B_{\mathfrak{X}^*_{\omega_1}(\mathbb{C})}.$$

Proof:

Recall that the set $\kappa_{\omega_1}(\mathbb{C})$ is by definition rational convex. We notice that $\overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$ is actually a convex set. Indeed let $f, g \in \overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$ and $t \in (0,1)$. Suppose that $f_n \xrightarrow{\omega^*} f, g_n \xrightarrow{\omega^*} g$ and $t_n \to t$, where $f_n, g_n \in \kappa_{\omega_1}(\mathbb{C})$ and $t_n \in \mathbb{Q} \cap (0,1)$ for every $n \in \mathbb{N}$. then $tf + (1-t)g \in \overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$ because

$$t_n f_n + (1-t_n) g_n \xrightarrow{\omega^*} tf + (1-t)g.$$

In the same manner we can prove that $\mathfrak{X}^*_{\omega_1}(\mathbb{C})$ is balanced i.e., $\lambda \mathfrak{X}^*_{\omega_1}(\mathbb{C}) \subseteq \mathfrak{X}^*_{\omega_1}(\mathbb{C})$ for every $|\lambda| \le 1$. To prove the Proposition suppose that there exists $f \in B_{\mathfrak{X}^*_{\omega_1}(\mathbb{C})} \backslash \overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$. It follows by a standard separation argument that there exists $x \in \mathfrak{X}_{\omega_1}(\mathbb{C})$ such that

$$|f(x)| > \sup\{|g(x)| : g \in \kappa_{\omega_1}(\mathbb{C})\}$$

which is absurd.

Let $I \subseteq \omega_1$ be an interval of ordinals, we denote by $\mathfrak{X}_I(\mathbb{C})$ the closed subspace of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ generated by $\{e_\alpha\}_{\alpha \in I}$. For every ordinal $\gamma < \omega_1$ we write $\mathfrak{X}_\gamma(\mathbb{C}) = \mathfrak{X}_{[0,1)}(\mathbb{C})$. Notice that $\mathfrak{X}_I(\mathbb{C})$ is a 1-complemented subspace of $\mathfrak{X}_{\omega_1}(\mathbb{C})$: the restriction to coordinates in I is a projection of norm 1 onto $\mathfrak{X}_I(\mathbb{C})$. We denote this projection by P_I and by $P^I = (Id - P_I)$ the corresponding projection onto the complement space $(Id - P_I)\mathfrak{X}_{\omega_1}(\mathbb{C})$, which we denote $\mathfrak{X}^I(\mathbb{C})$.

A transfinite sequence $(y_\alpha)_{\alpha<\gamma}$ is called a block sequence when $y_\alpha < y_\beta$ for all $\alpha < \beta < \gamma$. Given a block sequence $(y_\alpha)_{\alpha<\gamma}$ a block subsequence of $(y_\alpha)_{\alpha<\gamma}$ is a block sequence $(x_\beta)_{\beta<\xi}$ in the span of $(y_\alpha)_{\alpha<\gamma}$. A real block subsequence of $(y_\alpha)_{\alpha<\gamma}$ is a block subsequence in the real span of $(y_\alpha)_{\alpha<\gamma}$. A sequence $(x_n)_{n\in\mathbb{N}}$ is a block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ when it is a block subsequence of $(e_\alpha)_{\alpha<\omega_1}$.

Theorem (4.1.2) [4]:

Let $T: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$, that is, T is a bounded $\mathbb{R}$-linear operator such that $T^2 = -Id$. Then there exists a bounded diagonal operator $D_T: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$, which is another complex structure, such that $T - D_T$ is strictly singular. Moreover $D_T = \sum_{j=1}^{k} \epsilon_j \, iP_I$, for some signs $(\epsilon_j)_{j=1}^{k}$ and ordinal intervals $I_1 < I_2 < \cdots < I_k$ whose extremes are limit ordinals and such that $\omega_1 = \bigcup_{j=1}^{k} I_j$.

Proof :

Let $\mathrm{T} : \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator which is a complex structure and $\mathrm{D_T}$ be the diagonal bounded operator associated to it. It only remains to prove that $\mathrm{T} - \mathrm{D_T}$ is strictly singular. And this follows directly from Proposition (4.1.3) , because by definition $\lim_n (\mathrm{T} - \mathrm{D_T})_{y_n}$ for every R.I.S. $(y_n)_n$ on $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

We come back to the study of the complex structures on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Denote by $\mathfrak{D}$ the family of complex structures $\mathrm{D_T}$ on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ as in Theorem (4.1.2) , i.e., $\mathrm{D_T} = \sum_{j=1}^{k} \epsilon_j i P_{I_j}$ where $(\epsilon_j)_{j=1}^{k}$ are signs and $I_1 < I_2 < \cdots < I_k$ are ordinal intervals whose extremes are limit ordinals and such that $\omega_1 = \bigcup_{j=1}^{k} I_j$. Notice that $\mathfrak{D}$ has cardinality ω_1.

Recall that two spaces are said to be incomparable if neither of them embed into the other.

Step (I):

There exists a family $\mathfrak{J}$ of semi normalized block subsequences of $(e^{\alpha})_{\alpha<\omega_1}$, called $R.I.S.$ (Rapidly Increasing Sequences), such that every normalized block sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has a real block subsequence in $\mathfrak{J}$.

Recall that a Banach space X is hereditarily indecomposable (or H.I) if no (closed) subspace of X can be written as the direct sum of infinite-dimensional subspaces. Equivalently, for any two subspaces Y, Z of X and $\epsilon > 0$, there exist $y \in Y, z \in Z$ such that $\|y\| = \|z\| = 1$ and $\|y - z\| < \epsilon$.

Step (II):

For every normalized block sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, the subspace $\overline{\operatorname{span}}_{\mathbb{R}}(x_n)_{n\in\mathbb{N}}$ is a real H. I. space.

Step (III):

Let $(x_n)_{n\in\mathbb{N}}$ be a $R.I.S$ and $T\colon \overline{\operatorname{span}}_{\mathbb{C}}(x_n)_{n\in\mathbb{N}} \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then $\lim_{n\to\infty} d(Tx_n, \mathbb{C}x_n) = 0$.

The proof of Step (I), (II) and (III) are given [4].

Step (IV):

Let $(\mathrm{x_n})_{\mathrm{n}\in\mathbb{N}}$ be a R. I. S and T: $\overline{\operatorname{span}}_{\mathbb{C}}(\mathrm{x_n})_{\mathrm{n}\in\mathbb{N}} \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then the sequence $\lambda_{\mathrm{T}}\colon \mathbb{N} \to \mathbb{C}$ defined by $\mathrm{d}(\mathrm{Tx_n}, \mathbb{C}\mathrm{x_n}) = \|\mathrm{Tx_n} - \lambda_{\mathrm{T}}(\mathrm{n})\mathrm{x_n}\|$ is convergent.

Proof of Step (IV):

First we note that the sequence $\left(\lambda_{\mathrm{T}}(\mathrm{n})\right)_{\mathrm{n}}$ is bounded. Then consider $(\alpha_{\mathrm{n}})_{\mathrm{n}}$ and $(\beta_{\mathrm{n}})_{\mathrm{n}}$ two strictly increasing sequences of positive integers and suppose that $\lambda_T(\alpha_n) \longrightarrow \lambda_1$ and $\lambda_T(\beta_n) \longrightarrow \lambda_2$, when $n \longrightarrow \infty$. Going to a subsequence we can assume that $x_{\alpha_n} < x_{\beta_n} < x_{\alpha_{n+1}}$ for every $n \in \mathbb{N}$.

Fix $\epsilon > 0$. Using the result of the Step (III), we have that $\lim_{n\to\infty}\left\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\right\| = 0$.By passing to a subsequence if necessary, assume

$$\left\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\right\| \leq \frac{\epsilon}{2^n 6},$$

for every $n \in \mathbb{N}$. Hence, for every $w = \sum_n a_n x_{\alpha_n} \in \operatorname{span}_{\mathbb{R}}\left(x_{\alpha_n}\right)_n$ with $\|w\| \leq 1$ we have

$$\|Tw - \lambda_1 w\| \leq \sum_n |a_n| \left\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\right\|$$

$$\leq \epsilon/3,$$

because $(e_\alpha)_{\alpha<\omega_1}$ is a bimonotone transfinite basis. In the same way, we can assume that for every $w \in \operatorname{span}_{\mathbb{R}}\left(x_{\beta_m}\right)_m$ with $\|w\| \leq 1, \|Tw -$

$\lambda_2 w\| \leq \epsilon/3$. By Step (II) we have that $\overline{\text{span}}_{\mathbb{R}}(x_{\alpha_n})_n \cup (x_{\beta_m})_m$, is real-H.I. Then there exist unit vectors $w_1 \in \text{span}_{\mathbb{R}}(x_{\alpha_n})_n$ and $w_2 \in \text{span}_{\mathbb{R}}(x_{\beta_m})_m$, such that $\|w_1 - w_2\| \leq \frac{\epsilon}{3}\|T\|$. Therefore,

$$\|\lambda_1 w_1 - \lambda_2 w_2\| \leq \|Tw_1 - \lambda_1 w_1\| + \|Tw_1 - Tw_2\| + \|Tw_2 - \lambda_2 w_2\| \leq \epsilon.$$

By other side

$$\|\lambda_1 w_1 - \lambda_2 w_2\| \geq \|(\lambda_1 - \lambda_2)w_1\| - \|\lambda_2(w_1 - w_2)\| = |\lambda_1 - \lambda_2| - |\lambda_2|\epsilon.$$

In consequence, $|\lambda_1 - \lambda_2| \leq (|\lambda_2|)\epsilon$. Since ϵ was arbitrary, it follows that $\lambda_2 - \lambda_1$.

Let $T : \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. There is a canonical way to associate a bounded diagonal operator D_T (with respect to the basis $(e_\gamma)_{\gamma<\omega_1}$) such that $T - D_T$ is strictly singular: Fix $\alpha \in \Lambda(\omega_1)$ a limit ordinal, and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be two $R.I.S.$ such that $\sup_n \max \text{supp}_{x_n} = \sup_n \max \text{supp}_{y_n} = \alpha + \omega$. By a property of $\mathfrak{J}$ we can mix the sequences $(x_n)_n, (y_n)_n$ in order to form a new $R.I.S.$ $(z_n)_{n\in\mathbb{N}}$ such that $z_{2k} \in \{x_n\}_{n\in\mathbb{N}}$ and $z_{2k-1} \in \{y_n\}_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$. Then it follows from Step (IV) that the sequences defined by the formulas $d(Tx_n, \mathbb{C}x_n) = \|Tx_n - \lambda_T(n)x_n\|$ and $d(Ty_n, \mathbb{C}y_n) = \|Ty_n - \mu(n)y_n\|$ are convergent, and by the mixing argument, they must have the same limit. Hence for each $\alpha \in \Lambda(\omega_1)$ there exists a unique complex number $\xi_T(\alpha)$ such that

$$\lim_{n\to\infty} \|Tw_n - \xi_T(\alpha)w_n\| = 0$$

for every $R.I.S.$ $(w_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}_{I_\alpha}$, where we write I_α to denote the ordinal interval $[\alpha, \alpha + \omega)$. We proceed to define a diagonal linear operator D_T on the (linear) decomposition of $\text{span}(e_\alpha)_{\alpha<\omega_1}$

$$\text{span}(e_\alpha)_{\alpha<\omega_1} = \bigoplus_{\alpha \in \Lambda(\omega_1)} \text{span}(x_\beta)_{\beta\in I_\alpha}$$

by setting $D_T(e_\beta) = \xi_T(\alpha)_{e_\beta}$ when $\beta \in I_\alpha$.

Observe in addition that this sequence $(\xi_T(\alpha))_{\alpha\in\Lambda(\omega_1)}$ is convergent. That is, for every strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in Λ

(ω_1), the corresponding subsequence $\left(\xi_T(\alpha_n)\right)_{n\in\mathbb{N}}$ is convergent. In fact, for every $n \in \mathbb{N}$, let $(y_n^k)_{k\in\mathbb{N}}$ be a $R.I.S.$ in $\mathfrak{X}_{I_{\alpha_n}}$.Then we can take a $R.I.S.$ $\left(y_n^{k_n}\right)_{k\in\mathbb{N}}$ such that $\left\|Ty_n^{k_n} - \xi_T(\alpha_n + \omega)y_n^{k_n}\right\| < 1/n$. It follows by Step (IV) there exists $\lambda \in \mathbb{C}$ such that $\lim_n \left\|Ty_n^{k_n} - \lambda y_n^{k_n}\right\| = 0$. This implies that $\lim_n \xi_T(\alpha_n + \omega) = \lambda$.

In general this operator D_T defines a bounded operator on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. The proof is the same that uses that certain James like space of a mixed Tsirelson space is finitely interval representable in every normalized transfinite block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. For the case of complex structures we have a simpler proof (see Proposition (4.1.1).

Proposition (4.1.3) [4]:

Let A be a subset of ordinals contained in ω_1 and $X = \overline{\operatorname{span}}_{\mathbb{C}}(e_\alpha)_{\alpha\in A}$. Let $T : X \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then T is strictly singular if and only if for every $R.I.S.$ $(y_n)_{n\in\mathbb{N}}$ on X, $\lim_n Ty_n = 0$.

Proof:

The proposition is trivial when the set A is finite, then we assume that A is infinite. Suppose that T is strictly singular. Let $(y_n)_{n\in\mathbb{N}}$ be a R.I.S. on X such that $\lim_n Ty_n \neq 0$, then by Step (IV) there is $\lambda \neq 0$ with $\lim_n \|Ty_n - \lambda y_n\| = 0$. Take $0 < \epsilon < |\lambda|$. By passing to a subsequence if necessary, we assume that $\left\|(T - \lambda \mathrm{Id})\backslash_{\overline{\operatorname{span}}\,(y_n)_n}\right\| < \epsilon$. This implies that $T\backslash_{\overline{\operatorname{span}}\,(y_n)_n}$ is an isomorphism which is a contradiction.

Conversely, suppose that for every R.I.S. $(y_n)_n$ on X, $\lim_n Ty_n = 0$. Assume that T is not strictly singular. Then there is a block sequence subspace $Y = \overline{\operatorname{span}}\,(y_n)_{n\in\mathbb{N}}$ of X such that T restricted to Y is an isomorphism. By Step (I) we can assume that the sequence $(y_n)_n$ is already a R.I.S. on X. Then $\inf_n \|Ty_n\| > 0$. And we obtain a contradiction.

Given $Y \subseteq \mathfrak{X}_{\omega_1}(\mathbb{C})$ we denote by ιY the canonical inclusion of Y into $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

Corollary (4.1.4) [4]:

Let $\alpha \in \Lambda(\omega_1)$ and $T: \mathfrak{X}_{I_\alpha}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then there exists (unique) $\xi_T(\alpha) \in \mathbb{C}$ such that $T - \xi_T(\alpha)_{\iota \mathfrak{X}_{I_\alpha}(\mathbb{C})}$ is strictly singular.

Proof:

Let $\xi_T(\alpha)$ be the (unique) complex number such that $\lim\|Ty_n - \xi_T(\alpha)y_n\| = 0$ for every R.I.S. $(y_n)_n$ on $\mathfrak{X}_{I_\alpha}(\mathbb{C})$. Then by the previous Proposition $T - \xi_T\,(\alpha)_{\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}}$ is strictly singular.

Corollary (4.1.5) [4]:

Let $\alpha \in \Lambda(\omega_1)$ and $R: \mathfrak{X}_{I_\alpha}(\mathbb{C}) \to \mathfrak{X}^{I_\alpha}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then R is strictly singular.

Proof*:*

By the previous result, $\iota_{\mathfrak{X}^{I_\alpha}(\mathbb{C})}\, R = \lambda_\alpha \iota_{\mathfrak{X}^{I_\alpha}(\mathbb{C})} + S$ with S strictly singular. Then projecting by P^{I_α} we obtain $R = P^{I_\alpha} \circ \iota_{\mathfrak{X}^{I_\alpha}(\mathbb{C})} R = P^{I_\alpha} S$ which is strictly singular.

Proposition (4.1.6) [4]:

Let T be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then the linear operator D_T is a bounded complex structure.

Proof:

Let T be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and D_T the corresponding diagonal operator defined above. Fix $\alpha \in \Lambda(\omega_1)$. We shall prove that $\xi_T(\alpha)^2 = -1$. In fact,

$$T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} = P_{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + P^{I_\alpha} \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$$

$$= P_{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_1$$

where S_1 is strictly singular. This implies $P_{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} = \xi_T(\alpha) Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_2: \mathfrak{X}_{I_\alpha}(\mathbb{C}) \to \mathfrak{X}_{I_\alpha}(\mathbb{C})$ with S_2 strictly singular. Now computing:

$$\left(P_{I_\alpha}T\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}\right)\circ\left(P_{I_\alpha}T\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}\right)=P_{I_\alpha}T\circ P_{I_\alpha}T\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$$
$$=P_{I_\alpha}T\circ(Id-P^{I_\alpha})T\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$$
$$=P_{I_\alpha}T^2\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}-P_{I_\alpha}T\underline{P^{I_\alpha}T\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}}$$
$$=-Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}+S_3$$

where S_3 is strictly singular because the underlined operator is strictly singular. Hence we have that $(\xi_T(\alpha)^2+1)Id_{\mathfrak{X}_{I_\alpha}}$ is strictly singular. Which allow us to conclude that $\xi_T(\alpha)^2=-1$. The continuity of D_T is then guaranteed by the convergence of $\left(\xi_T(\alpha)\right)_{\alpha\in\Lambda\omega_1}$. In deed, we have that there exist ordinal intervals $I_1<I_2<\cdots<I_k$ with $\omega_1=\bigcup_{j=1}^{k}I_j$ and such that $D_T=\sum_{j=1}^{k}\epsilon_j iP_{I_j}$, for some signs $\left(\epsilon_j\right)_{j=1}^{n}$.

Corollary (4.1.7) [4]:

The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has ω_1 many complex structures up to isomorphism. Moreover any two non-isomorphic complex structures are incomparable.

Proof:

Let J be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. By Theorem (4.1.2) we have that $J-D_J$ is a strictly singular operator and $D_J\in\mathfrak{D}$. Recall that two complex structures whose difference is strictly singular must be equivalent. Then J is equivalent to D_j.

To complete the proof it is enough to show that given two different elements of $\mathfrak{D}$ they define non equivalent complex structures. Moreover, we prove that one structure does not embed into the other. Fix $J\neq K\in\mathfrak{D}$. Then there exists an ordinal interval $I_\alpha=[\alpha,\alpha+\omega)$ such that, without loss of generality, $J|\mathfrak{X}_{I_\alpha}=iId|_{\mathfrak{X}_{I_\alpha}}$ and $K|\mathfrak{X}_{I_\alpha}=-iId|_{\mathfrak{X}_{I_\alpha}}$. Suppose that there exists $T:\mathfrak{X}_{\omega_1}(\mathbb{C})^J\to\mathfrak{X}_{\omega_1}(\mathbb{C})^K$ an isomorphic embedding. Then T is in particular a $\mathbb{R}$-linear operator such that $TJ=KT$. We write using Corollary (4.1.4) , $T\backslash_{\mathfrak{X}_{I_\alpha}}=\xi_T(\alpha)_{\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}}+S$ with S strictly singular. Then $\xi_T(\alpha)J\backslash_{\mathfrak{X}_{I_\alpha}}-\xi_T(\alpha)K\backslash_{\mathfrak{X}_{I_\alpha}}=S_1$ where S_1 is strictly singular. In particular for each $x\in\mathfrak{X}_{I_\alpha},S_1x=2\xi_T(\alpha)ix$. It follows from the fact that $\mathfrak{X}_{I_\alpha}$ is infinite dimensional that $\xi_T(\alpha) = 0$. Hence $T\backslash_{\mathfrak{X}_{I_\alpha}} = S$, but this a contradiction because T is an isomorphic embedding.

The next corollary offers uncountably many examples of Banach spaces with exactly countably many complex structures.

Corollary (4.1.8) [4]:

The space $\mathfrak{X}_\gamma(\mathbb{C})$ has ω complex structures up to isomorphism for every limit ordinal $\omega_2 \leq \gamma < \omega_1$.

Proof:

Let J be a complex structure on $\mathfrak{X}_\gamma(\mathbb{C})$. We extend J to a complex structure defined in the whole space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ by setting $T = JP_1 + iP^I$, where $I = [0, \gamma)$. It follows that $T = D_T + S$ for an strictly singular operator S and a diagonal operator D_T like in Theorem (4.1.2) . Notice that $D_T x = ix$ for every $x \in \mathfrak{X}^I$, otherwise there would be a limit ordinal α such that $S\backslash_{\mathfrak{X}_{I_\alpha}} = 2iId\backslash_{\mathfrak{X}_{I_\alpha}}$. Hence $JP_I = D_T P_I + S$. Which implies that J has the form $J = \sum_{j=1}^{k} \epsilon_j i P_{I_j} + S_1$ where S_1 is strictly singular on $\mathfrak{X}_{\omega_1}(\mathbb{C}), (\epsilon_j)_{j=1}^{k}$, are signs and $I_1 < I_2 < \cdots < I_k$ are ordinal intervals whose extremes are limit ordinals and such that $\gamma = \bigcup_{j=1}^{k} I_j$. Now the rest of the proof is identical to the proof of the previous corollary. In particular, all the non-isomorphic complex structures on $\mathfrak{X}_\gamma(\mathbb{C})$ are incomparable.

We *also* have, using the same proof of the previous corollary, that for every increasing sequence of limit ordinals $A = (\alpha_n)_n$, the space $\mathfrak{X}_A = \oplus_n \mathfrak{X}_{I_{\alpha_n}}(\mathbb{C})$, where $I_{\alpha_n} = [\alpha_n, \alpha_n + \omega)$, has exactly infinite countably many different complex structures. Hence there exists a family, with the cardinality of the continuum, of Banach spaces such that every space in it has exactly ω complex structures.

Section (4.2): Observations

It is easy to check that subspaces of even codimension of a real Banach space with complex structure also admit complex structure. An interesting property of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is that any of its real hyperplanes (and thus every real subspace of odd codimension) do not admit complex structure.

Proposition (4.2.1) [4]:

The real hyperplanes of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ do not admit complex structure.

Proof:

By the results of V. Ferenczi and E. Galego it is sufficient to prove that the ideal of all $\mathbb{R}$-linear strictly singular operators on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has the lifting property, that is, for any $\mathbb{R}$-linear isomorphism on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that $T^2 + Id$ is strictly singular, there exists a strictly singular operator S such that $(T - S)^2 = -Id$. The proof now follows.

One open problem in the theory of complex structure is to know if the existence of more regularity in the space guarantees that it admits unique complex structure.

The purpose of this section is to give a proof for the results in the Step (I), (II) and (III). Several proofs are very similar to the corresponding ones .

First we clarify the definition of the norming set by defining what being a special sequence means. All the definitions we present in this part are the corresponding translation for the complex case.

Recall that $[\omega_1]^2 = \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$.

Definition (4.2.2) [4]:

A function $\varrho : [\omega_1]^2 \to \omega$ such that

$(i)\varrho(\alpha,\gamma) \le \max\{\varrho(\alpha,\beta),\varrho(\beta,\gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.

(ii) $\varrho(\alpha,\beta) \le \max\{\varrho(\alpha,\gamma),\varrho(\beta,\gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.

(iii) The set $\{\alpha < \beta : \varrho(\alpha,\beta) \le n\}$ is finite for all $\beta < \omega_1$ and $n \in \mathbb{N}$ is called a ϱ-function.

The existence of ϱ-functions is due to Todorcevic. Let us fix a ϱ-function $\varrho : [\omega_1]^2 \to \omega$ and all the following work relies on that particular choice of ϱ.

Definition (4.2.3) [4]:

Let F be a finite subset of ω1 and p ∈ ℕ , we write

$$\rho F = \rho\varrho(F) = \max_{\alpha,\beta\in F} \varrho(\alpha,\beta).$$

$\overline{F}^{p} = \{\alpha \leq \max F : \text{there is } \beta \in F \text{ such hat } \alpha \leq \beta \text{ and } \varrho(\alpha,\beta) \leq p\}$

We denote by $\mathbb{Q}_s(\omega_1,\mathbb{C})$ the set of finite sequences $(\phi_1, w_1, p_1, \dots, \phi_d, w_d, p_d)$ such that

(i) For all $i \leq d, \phi_i \in c_{00}(\omega_1,\mathbb{C})$ and for all $\alpha < \omega_1$ the real and the imaginary part of $\phi(\alpha)$ are rationals.

(ii) $(w_i)_{i=1}^{d}, (p_i)_{i=1}^{d} \in \mathbb{N}^d$ are strictly increasing sequences.

(iii) $p_i \geq \rho_{(\cup_{k=1}^{i} \operatorname{supp} \phi_k)}$ for every $i \leq d$.

Let $\mathbb{Q}_s(\mathbb{C})$ be the set of finite sequences $(\phi_1, w_1, p_1, \phi_2, w_2, p_2, \dots, \phi_d, w_d, p_d)$ satisfying properties (i), (ii) above and for every $i \leq d, \phi_i \in c_{00}(\omega_1,\mathbb{C})$. Then $\mathbb{Q}_s(\mathbb{C})$ is a countable set while $\mathbb{Q}_s(\omega_1,\mathbb{C})$ has cardinality ω_1. Fix a one to one function $\sigma: \mathbb{Q}_s(\mathbb{C}) \to \{2j : j \text{ is odd}\}$ such that

$$\sigma(\phi_1, w_1, p_1, \dots, \phi_d, w_d, p_d) > \max\left\{p_d^2, \frac{1}{\epsilon^2}, \max \operatorname{supp} \phi_d\right\}$$

where $\epsilon = \min\{|\phi_k(e_\alpha)| : \alpha \in \operatorname{supp} \phi_d, k = 1, \dots, d\}$. Given a finite subset F of ω_1, we denote by $\pi_F : \{1, 2, \dots, \#F\} \to F$ the natural order preserving map, i.e. π_F is the increasing numeration of F.

Given $\Phi = (\phi_1, w_1, p_1, \dots, \phi_d, w_d, p_d) \in \mathbb{Q}_s(\mathbb{C})$, we set

$$G_\Phi = \overline{\bigcup_{i=1}^{d} \operatorname{supp} \phi_i}^{pd} .$$

Consider the family $\pi_{G_\Phi}(\Phi) = (\pi_G(\phi_1), w_1, p_1, \pi_G(\phi_2), w_2, p_2, \dots, \pi_G(\phi_d), w_d, p_d)$ where

$$\pi_G(\phi_1)(n) = \begin{cases} \phi_k\left(\pi_{G_\Phi}(n)\right), & \text{if } n \in G_\Phi, \\ 0, & \text{otherwise.} \end{cases}$$

Finally $\sigma_p : \mathbb{Q}_s(\omega_1, \mathbb{C}) \to \{2j : j \text{ odd}\}$ is defined by $\sigma_p(\Phi) = \sigma(\pi_G(\Phi))$.

Definition (4.2.4) [4]:

A sequence $\Phi = \left(\phi_1, \phi_2, \ldots, \phi_{n_{2j+1}}\right)$ of functionals of $\kappa_{\omega_1}(\mathbb{C})$ is called a $2j + 1$ special sequence if

(SS.1) $\text{supp}\,\phi_1 < \text{supp}\,\phi_2 < \cdots < \text{supp}\,\phi_{n_{2j+1}}$. For each $k \leq n_{2j+1}, \phi_k$ is of type I, $w(\phi_k) = m_{2j_k}$ with j_1 even and $m_{2j_1} > n_{2j+1}^2$.

(SS.2) There exists a strictly increasing sequence $\left(p_1^\Phi, p_2^\Phi, \ldots, p_{2_{j+1}-1}^\Phi\right)$ of naturals numbers such that for all $1 \leq i \leq n_{2_{j+1}} - 1$ we have that $w(\phi_{i+1}) = \sigma_{\sigma_\varrho}(\Phi_i)$ where

$$\Phi_i = \left(\phi_1, w(\phi_1), p_1^\Phi, \phi_2, w(\phi_2), p_2^\Phi, \ldots, \phi_i, w(\phi_i), p_i^\Phi\right)$$

Special sequences in separable examples with one to one codings are in general simpler: they are of the form $(\phi_1, w(\phi_1), \ldots, \phi_k, w(\phi_k))$. Their main feature is that if $(\phi_1, w(\phi_1), \ldots, \phi_k, w(\phi_k))$ and $(\psi_1, w(\psi_1), \ldots, \psi_k, w(\psi_k))$ are two of them, there exists $i_0 \leq \min\{k, l\}$ with the property that

$$(\phi_i, w(\phi_i)) = (\psi_i, w(\psi_i)) \quad \text{for all } i \leq i_0 \tag{1}$$

$$\{w(\phi_i): i_0 \leq i \leq k\} \cap \{w(\psi_i): i_0 \leq i \leq l\} = \phi \tag{2}$$

In non-separable spaces, one to one codings are obviously impossible, and (1), (2) are no longer true. Fortunately, there is a similar feature to (1), (2) called the tree-like interference of a pair of special sequences Let $\Phi = \left(\phi_1, \ldots, \phi_{n_{2j+1}}\right)$ and $\psi = \left(\psi_1, \ldots, \psi_{n_{2j+1}}\right)$ be two $2_j + 1$-special sequences, then there exist two numbers $0 \leq k_{\Phi,\psi} \leq \lambda_{\Phi,\psi} \leq n_{2_{j+1}}$ such that the following conditions hold:

(TP.1) For all $i \leq \lambda_{\Phi,\psi}, w(\phi_i) = w(\psi_i)$ and $p_i^\Phi = p_i^\psi$.

(TP.2) For all $i < k_{\Phi,\psi}, \phi_i = \psi_i$.

(TP.3) For all $k_{\Phi,\psi} < i < \lambda_{\Phi,\psi}$

$$\text{supp}\,\phi_i \cap \overline{\text{supp}\,\psi_1 \cup \ldots \cup \text{supp}\,\lambda_{\Phi,\psi} - 1}^{p\lambda_{\Phi,\psi}-1} = \phi$$

and

$$\operatorname{supp}\psi_i \cap \overline{\operatorname{supp}\phi_1 \cup \ldots \cup \operatorname{supp}\phi_{\Phi,\psi} - 1}^{p\lambda_{\Phi,\psi}-1} = \phi$$

(TP.4) $\left\{w(\phi_i): \lambda_{\Phi,\psi} < i \leq n_{2_{j+1}}\right\} \cap \left\{w(\psi_i): i \leq n_{2_{j+1}}\right\} = \phi$ and $\left\{w(\psi_i):\ \lambda_{\Phi,\psi} < i \leq n_{2_{j+1}}\right\} \cap \left\{w(\phi_i): i \leq n_{2_{j+1}}\right\} = \phi$.

For the proof of Step (I) we shall construct a family of block sequences on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ commonly called rapidly increasing sequences (R. I. S.). These sequences are very useful because one has good estimates of upper bounds on $|f(x)|$ for $f \in \kappa_{\omega_1}(\mathbb{C})$ and x averages of R. I. S.

For the construction of the family $\mathfrak{J}$ the only difference from the general theory is that our interest now is to study bounded $\mathbb{R}$-linear operators on the complex space $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Hence, all the construction of $R.I.S.$ in a particular block sequence $(x_n)_{n\in\mathbb{N}}$ must be on its real linear span. We point out here that there are no problems with this, because all the combinations of the vectors $(x_n)_{n\in\mathbb{N}}$ to obtain $R.I.S.$ use rational scalars.

Definition (4.2.5) [4]:

$(R.I.S.)$. We say that a block sequence $(x_k)_k$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is a (C,ϵ)-$R.I.S.$, $C,\epsilon > 0$, when there exists a strictly increasing sequence of natural numbers $(j_k)_k$ such that:

(i) $\|x_k\| \leq C$;

(ii) $|\operatorname{supp} x_k| \leq m_{j_{k+1}}\epsilon$;

(iii) For all the functionals ϕ of $\kappa_{\omega_1}(\mathbb{C})$ of type I, with $\omega(\phi) < m_{j_k}$, $|\phi(x_k)| \leq \frac{C}{\omega(\phi)}$.

The following remark is immediately consequence of this definition.

Remark (4.2.6) [4]:

Let $\epsilon' < \epsilon$. Every (C,ϵ)-$R.I.S.$ has a subsequence which is a (C,ϵ')-$R.I.S.$

And for every strictly increasing sequence of ordinals $(\alpha_n)_n$ and every $\epsilon > 0$, $\left(e_{\alpha_n}\right)_n$ is a $(1,\epsilon)$-$R.I.S.$

Remark (4.2.7) [4]:

Let $(x_n)_n$ and $(y_n)_n$ be two (C,ϵ)-$R.I.S.$ such that $\sup_n \max \operatorname{supp} x_n = \sup_n \max \operatorname{supp} y_n$. Then there exists a (C,ϵ)-$R.I.S.$ such that $z_{2n-1} \in \{x_k\}_{k\in\mathbb{N}}$ and $z_{2n} \in \{y_k\}_{k\in\mathbb{N}}$.

Proof:

Suppose that $(t_k)_k$ and $(s_k)_k$ are increasing sequences of positive integers satisfying the definition of $R.I.S.$ for $(x_k)_k$ and $(y_k)_k$ respectively. We construct $(z_k)_k$ as follows. Let $z_1 = x_1$ and $j_1 = t_1$.Pick s_{k_1} such that $x_1 < y_{s_{k_1}}$ and $t_2 < s_{k_1}$. Then we define $j_2 = s_{k_1}$ and $z_2 = y_{s_{k_1}}$. Notice that

(i) $\|z_1\| \leq C$;

(ii) $|\operatorname{supp} z_1| \leq m_{t_2}\epsilon \leq m_{s_{k_1}}\epsilon = m_{j_2}\epsilon$;

(iii) For all the functionals ϕ of $\kappa_{\omega_1}(\mathbb{C})$ of type I, with $\omega(\phi) < m_{j_1}$, $|\phi(z_1)| \leq \frac{C}{\omega(\phi)}$.

Continuing with this process we obtain the desired sequence.

Theorem (4.2.8) [4]:

Let $(x_k)_k$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\epsilon > 0$. Then there exists a normalized block subsequence $(y_k)_k$ in span $\operatorname{span}_{\mathbb{R}}\{x_k\}$ which is a $(3,\epsilon)$-$R.I.S.$

For the proof of Theorem (4.2.8) we first construct a simpler type of sequence.

Definition (4.2.9) [4]:

Let X be a Banach space, $C \geq 1$ and $k \in \mathbb{N}$. A normalized vector y is called a $C - \ell_1^k$-average of X, when there exist a block sequence $(x_1, \ldots, x_k)$ such that

(a) $y = (x_1 + \ldots + x_k)/k$;

(b) $\|x_i\| \leq C$, for all $i = 1, \ldots, k$.

In the next result we want to emphasize that this special type of sequence are really constructed on the real structure of the space $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

Theorem (4.2.10) [4]:

For every normalized block sequence (x_n) of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every integer k, there exist $z_1 < \cdots < z_k$ in $\underset{\mathbb{R}}{\operatorname{span}}(x_n)$, such that $(z_1 + \cdots + z_k)/k$ is a $2 - \ell_1^k$-average.

Proof:

The proof is standard. Suppose that the result is false. Let j and n be natural numbers with

$$2^n > m_{2_j}$$

$$n_{2_j} > k^n.$$

Let $\mathrm{N} = \mathrm{k}^{\mathrm{n}}$ and $\mathrm{x} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{x_i}$. For each $1 \leq \mathrm{i} \leq$ n and every $1 \leq \mathrm{j} \leq \mathrm{k}^{\mathrm{n-i}}$, we define,

$$x(i,j) = \sum_{t=(j-1)k^i+1}^{jk^i} x_t.$$

Hence, $\mathrm{x}(0,\mathrm{j}) = \mathrm{x_j}$ and $\mathrm{x}(\mathrm{n},1) = \mathrm{x}$.

It is proved by induction on i that $\|x(i,j)\| \leq 2^{-i}k^i$, for all i,j. In particular, $\|x\| = \|x(n,1)\| \leq 2^{-n}k^n = 2^{-n}N$. Then by Property (1). of definition in the norming set

$$\|x\| \geq \frac{1}{m_{2_j}} \sum_{t=1}^{n_{2_j}} \|x_t\| = \frac{n_{2_j}}{m_{2_j}} > \frac{N}{m_{2_j}}.$$

Hence,

$$2^{-n}N > \frac{N}{m_{2_j}}$$

$$m_{2_j} > 2^n,$$

which is a contradiction.

Finally, for the construction of $R.I.S.$ we observe these simple facts

(i) If y is a $C-\ell_1^{n_{2j}}$-average of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\phi \in \kappa_{\omega_1}(\mathbb{C})$ has weight $\omega(\phi) < m_j$, then $|\phi(y)| \leq \frac{3C}{2\omega(\phi)}$;

(ii) Let $(x_k)_k$ be a block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that there exists a strictly increasing sequence of positive integers $(j_k)_k$ and $\epsilon > 0$ satisfying:

(a) Each x_k is a $2-\ell_1^{n_{j_k}}$-average;

(b) $|\text{supp}\, x_k| < \epsilon m_{j_{k+1}}$.

Then $(x_k)_k$ is a $(3,\epsilon)$-$R.I.S.$

To prove Step (II) and (III) we need a crucial result called the basic inequality which is very important to find good estimations for the norm of certain combinations of $R.I.S.$ in $\mathfrak{X}_{\omega_1}(\mathbb{C})$. First we need to introduce the mixed Tsirelson spaces.

The mixed Tsirelson space $T\left[\left(m_j^{-1}, n_j\right)_j\right]$ is defined by considering the completion of $c_{00}(\omega,\mathbb{C})$ under the norm $\|\cdot\|_0$ given by the following implicit formula

$$\|x\|_0 = \max\left\{\|x\|_\infty, \sup_j \sup \frac{1}{m_j}\sum_{i=1}^{n_j}\left\|E_j x\right\|_0\right\}.$$

The supremum inside the formula is taken over all the sequences $E_1 < \ldots < E_{n_j}$ of subsets of ω. Notice that in this space the canonical Hamel basis $(e_n)_n < \omega$ of $c_{00}(\omega,\mathbb{C})$ is 1-subsymmetric and 1-unconditional basis.

We can give an alternative definition for the norm of $T\left[\left(m_j^{-1}, n_j\right)_j\right]$ by defining the following norming set. Let $W\left[\left(m_j^{-1}, n_j\right)_j\right] \subseteq c_{00}(\omega,\mathbb{C})$ the minimal set of $c_{00}(\omega,\mathbb{C})$ satisfying the following properties:

(a) For every $\alpha < \omega, e_\alpha^* \in W\left[\left(m_j^{-1}, n_j\right)_j\right]$. If $\phi \in W\left[\left(m_j^{-1}, n_j\right)_j\right]$ and $\theta = \lambda + i\mu$ is a complex number with λ and μ rationals and $|\theta| \leq 1, \theta\phi \in W\left[\left(m_j^{-1}, n_j\right)_j\right]$;

(b) For every $\phi \in W\left[\left(m_j^{-1}, n_j\right)_j\right]$ and $E \subseteq \omega, E\phi \in W\left[\left(m_j^{-1}, n_j\right)_j\right]$;

(c) For every $j \in \mathbb{N}$ and $\phi_1 < \ldots < \phi_{n_j}$ in $W\left[\left(m_j^{-1}, n_j\right)_j\right]$, $(1/m_j)\sum_{i=1}^{n_j} \phi_i \in W\left[\left(m_j^{-1}, n_j\right)_j\right]$;

(d) $W\left[\left(m_j^{-1}, n_j\right)_j\right]$ is closed under convex rationals combinations.

Theorem (4.2.11) [4]: (Basic Inequality for $\boldsymbol{R.I.S.}$):

Let $(x_n)_n$ be a (C, ϵ) $R.I.S.$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $(b_k)_k \in c_{00}(\mathbb{C}, \mathbb{N})$. Suppose that for some $j_0 \in \mathbb{N}$ we have that for every $f \in \kappa_{\omega_1}(\mathbb{C})$ with weight $w(f) = m_{j_0}$ and for every interval E of ω_1,

$$\left|f\left(\sum_{k\in E} b_k\right)\right| \leq C\left(\max_{k\in E}|b_k| + \epsilon\sum_{k\in E}|b_k|\right).$$

Then for every $\mathrm{f} \in \kappa_{\omega_1}(\mathbb{C})$ of type I, there exist $g_1, g_2 \in c_{00}(\mathbb{C}, \mathbb{N})$ such that

$$\left|f\left(\sum_{k\in E} b_k\right)\right| \leq C(g_1 + g_2)\left(\sum_{k\in E}|b_k|\, e_k\right),$$

where $g_1 = h_1$ or $g_1 = e_t^* + h_1, t \notin \operatorname{supp} h_1$ and $h_1 \in W\left[\left(m_j^{-1}, 4n_j\right)_j\right]$ such that $h_1 \in \operatorname{conv}_{\mathbb{Q}}\left\{h \in W\left[\left(m_j^{-1}, 4n_j\right)_j\right]\right\}$ and m_j does not appear as a weight of a node

in the tree analysis of h_1, and $\|g_2\|_\infty \leq \epsilon$.

Proposition (4.2.12) [4]:

Let $f \in \kappa_{\omega_1}(\mathbb{C})$ or $f \in W\left[\left(m_j^{-1}, 4n_j\right)_j\right]$ be of type I. Consider $j \in \mathbb{N}$ and $l \in \left[\frac{n_j}{m_j}, n_j\right]$. Then for every set $F \subseteq c_{00}(\omega_1, \mathbb{C})$ of cardinality l,

$$\left|f\left(\frac{1}{l}\sum_{\alpha\in F} e_\alpha\right)\right| \leq \begin{cases} \dfrac{1}{w(f)m_j}, & \text{if } w(f) < m_j, \\ \dfrac{2}{w(f)}, & \text{if } w(f) \geq m_j, \end{cases}$$

If the tree analysis of f does not contain nodes of weight m_j, then

$$\left|f\left(\frac{1}{l}\sum_{\alpha\in F} e_\alpha\right)\right| \le \frac{2}{m_j^3}$$

Proposition (4.2.13) [4]:

Let $(x_k)_k$ be a (C,ϵ)- $R.I.S.$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ with $\epsilon \le \frac{1}{n_j}, l \in \left[\frac{n_j}{m_j}, n_j\right]$ and let $f \in \kappa_{\omega_1}(\mathbb{C})$ be of type I. Then,

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \le \begin{cases} \dfrac{3C}{w(f)m_j}, & \text{if } w(f) < m_j, \\ \dfrac{C}{w(f)} + \dfrac{2C}{n_j}, & \text{if } w(f) \ge m_j, \end{cases}$$

Consequentely, if $(x_k)_{k=1}^{l}$ is a normalized (C,ϵ)-$R.I.S.$ with $\epsilon \le \frac{1}{n_{2_j}}, l \in \left[\frac{n_{2_j}}{m_{2_j}}, n_{2_j}\right]$, then

$$\frac{1}{m_{2_j}} \le \left\|\frac{1}{l}\sum_{k=1}^{l} x_k\right\| \le \frac{2C}{m_{2_j}}.$$

Proof:

Let $(x_k)_k$ be a (C,ϵ)- $R.I.S.$ and take $b = \left(\frac{1}{l}, \dots, \frac{1}{l}, 0,0, \dots\right) \in c_{00}(\mathbb{N},\mathbb{C})$. It follows from the basic inequality that for every $f \in \kappa_{\omega_1}(\mathbb{C})$ of type I, there exist $h_1 \in W\left[\left(m_j^{-1}, 4n_j\right)_j\right]$ with $\omega(h_1) = \omega(f), t \in \mathbb{N}$ and $g_2 \in c_{00}(\mathbb{N},\mathbb{C})$ with $\|g\|_\infty \le \epsilon$ such that

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \le C(e_t^* + h_1 + g_2)\left(\frac{1}{l}\sum_{k=1}^{l} e_k\right).$$

Moreover,

$$\left|g_2\left(\frac{1}{l}\sum_{k=1}^{l} e_k\right)\right| \le \|g\|_\infty \left\|\frac{1}{l}\sum_{k\in E} e_k\right\|_1 \le \epsilon \le \frac{1}{n_j}.$$

Now by the estimatives on the auxiliary space $T\left[\left(m_j^{-1}, 4n_j\right)_j\right]$of the Proposition (4.2.12) , we have

(i) If $\omega(f) < m_j$,

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \le C\left(\frac{1}{l}+\frac{2}{\omega(f)m_j}+\frac{1}{n_j}\right)$$

$$\le C\left(\frac{m_j}{n_j}+\frac{2}{\omega(f)m_j}+\frac{1}{n_j}\right)$$

$$\le \frac{3C}{\omega(f)m_j}$$

(ii) If $\omega(f) \ge m_j$,

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \le C\left(\frac{1}{l}+\frac{C}{\omega(f)}+\frac{1}{n_j}\right)$$

$$\le \frac{C}{\omega(f)}+\frac{2C}{n_j}$$

And notice

(iii) $\frac{3C}{\omega(f)m_{2_j}} \le \frac{2C}{m_{2_j}}$, if $\omega(f) < m_{2_j}$,

(iv) $\frac{C}{\omega(f)}+\frac{2C}{n_{2_j}} \le \frac{C}{m_{2_j}}+\frac{C}{m_{2_j}}=\frac{2C}{m_{2_j}}$, if $\omega(f) \ge m_{2_j}$.

We conclude from the fact that $\kappa_{\omega_1}(\mathbb{C})$ is the norming set:

$$\left\|(1/l)\sum_{k=1}^{l} x_k\right\| \le 2C/m_{2_j}.$$

For the proof the second part of the theorem, let $(x_k)_{k=1}^{l}$ be a normalized (C,ϵ)- $R.I.S.$ with $\epsilon \le \frac{1}{n_{2_j}}, l \in \left[\frac{n_{2_j}}{m_{2_j}}, n_{2_j}\right]$. For every $k \le l$, we consider $x_k^* \in \kappa_{\omega_1}(\mathbb{C})$, such that $x_k^*(x_k)=1$ and $x_k^* \subseteq \operatorname{ran} x_k$, then $x^* = \frac{1}{m_{2_j}}\sum_{k=1}^{l} x_k^* \in \kappa_{\omega_1}(\mathbb{C})$ and $x^*\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)=\frac{1}{m_{2_j}}$. Hence, $\frac{1}{m_{2_j}} \le \left\|\frac{1}{l}\sum_{k=1}^{l} x_k\right\|$.

Proof of step (II):

Now we introduce another type of sequences in order to construct the conditional frame in $\mathfrak{X}_{\omega_1}(\mathbb{C})$. In fact, this space has no unconditional basic sequence.

Definition (4.2.14) [4]:

A pair (x, ϕ) with $x \in \mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\phi \in \kappa_{\omega_1}(\mathbb{C})$, is called a (C, j)-exact pair when:

(a) $\|x\| \leq C, \omega(\phi) = m_j$ and $\phi(x) = 1$.

(b) For each $\psi \in \kappa_{\omega_1}(\mathbb{C})$ of type I and $\omega(x) = m_i, i \neq j$, we have

$$|\psi(x)| \leq \begin{cases} \dfrac{2C}{m_i}, & \text{if } i < j, \\ \dfrac{C}{m_j^2}, & \text{if } i > j. \end{cases}$$

Proposition (4.2.15) [4]:

Let $(x_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then for every $j \in \mathbb{N}$, there exist (x, ϕ) such that $x \in \operatorname{span}_{\mathbb{R}}(x_n), \phi \in \kappa_{\omega_1}(\mathbb{C})$ and (x, ϕ) is a $(6,2j)$-exact pair.

Proof:

Fix $(x_n)_n$ a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and a positive integer j. By the Proposition (4.2.8) there exists $(y_n)_n$ a normalized $(3, 1/n_{2_j})$-$R.I.S.$ in $\operatorname{span}_{\mathbb{R}}(x_n)$. For every $1 \leq i \leq n_{2_j}$ and $\epsilon > 0$, we take $\phi_i \in \kappa_{\omega_1}(\mathbb{C})$ such that $\phi_i(y_i) > 1 - \epsilon$, and $\phi_i < \phi_{i+1}$. Let $x = \left(m_{2_j}/n_{2_j}\right) \sum_{i=1}^{n_{2_j}} y_i$ and $\phi = \left(1/n_{2_j}\right) \sum_{i=1}^{n_{2_j}} \phi_i \in \kappa_{\omega_1}(\mathbb{C})$. By perturbating x by a rational coefficient on the support of some y_i we may assume that then $\phi(x) = 1$ and using Proposition (4.2.9) we conclude that (x, ϕ) is a $(6,2j)$-exact pair.

Definition (4.2.16) [4]:

Let $j \in \mathbb{N}$. A sequence $(x_1, \phi_1, \dots, x_{2j+1}, \phi_{n_{2j+1}})$ is called a $(1, j)$-dependent sequence when:

(DS.1) $\operatorname{supp} x_1 \cup \operatorname{supp} \phi_1 < \cdots < \operatorname{supp} x_{n_{2j+1}} \cup \operatorname{supp} \phi_{n_{2j+1}}$.

(DS.2) The sequence $\Phi = \left(\phi_1, \ldots, \phi_{n_{2j+1}}\right)$ is a $2j+1$-special sequence.

(DS.3) (x_i, ϕ_i) is a $(6,2j)$-exact pair. $\#\operatorname{supp} x_i \leq m_{2_{j+1}}/n_{2_{j+1}}^2$ for every $i \leq i \leq n_{2_{j+1}}$.

(DS.4) For every $(2j+1)$-special sequence $\psi = \left(\psi_1, \ldots, \psi_{n_{2j+1}}\right)$ we have that

$$\bigcup_{k_{\Phi,\Psi}<i<\lambda_{\Phi,\Psi}} \operatorname{supp} x_i \cap \bigcup_{k_{\Phi,\Psi}<i<\lambda_{\Phi,\Psi}} \operatorname{supp} \psi_i = \phi,$$

where $k_{\Phi,\Psi}, \lambda_{\Phi,\Psi}$ are numbers introduced in Definition (4.2.4) [4].

Proposition (4.2.17) [4]:

For every normalized block sequence $(y_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every natural number j there exists a $(1,j)$-dependent sequence $(x_1, \emptyset_1, \ldots, x_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ such that x_i is in the $\mathbb{R}$-span of $(y_n)_n$ for every $i = 1, \ldots, n_{2j+1}$.

Proof:

Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $j \in \mathbb{N}$. We construct the sequence $(x_1, \emptyset_1, \ldots, x_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ inductively. First using Proposition (4.2.15) we choose a $(6,2j_1)$-exact pair $(x_1, \emptyset_1)$ such that j_1 is even, $m_{2j_1} > n_{2j_1}$ and $x \in \operatorname{span}_{\mathbb{R}}(y_n)_n$. Assume that we have constructed $(x_1, \emptyset_1, \ldots, x_{l-1}, \emptyset_{l-1})$ such that there exists $(p_1, \ldots, p_{l-1})$ satisfying

(i) $\operatorname{supp} x_1 \cup \operatorname{supp} \phi_1 < \cdots < \operatorname{supp} x_{l-1} \cup \operatorname{supp} \phi_{l-1}$, where $x_i \in \operatorname{span}_{\mathbb{R}}(y_n)_n$ and $(x_i, \emptyset_i)$ is a $(6,2j_1)$-exact pair.

(ii) $For 1 < i \leq l-1, w(\emptyset_i) = \sigma_\varrho(\emptyset_1, w(\emptyset_1), p_1, \ldots, \emptyset_{i-1}, w(\emptyset_{i-1}), p_{i-1})$.

(iii) For $1 < i \leq l-1$, $p_i \geq \max\{p_{i-1}, pF_i\}$, where $F_i = \bigcup_{k=1}^{i} \operatorname{supp} \phi_k \cup \operatorname{supp} x_k$.

To complete the inductive construction choose

$$p_{l-1} \geq \max\{p_{l-2}, pF_{i-1}\# \operatorname{supp} x_{l-1}\}$$

and $2j_l = \sigma_\varrho(\emptyset_1, w(\emptyset_1), p_1, \dots, \emptyset_{l-1}, w(\emptyset_{l-1}), p_{l-1})$. Hence take a $(6,2j_1)$-exact pair $(x_l, \emptyset_l)$ such that $x_l \in \operatorname{span}_{\mathbb{R}}(y_n)_n$ and $\operatorname{supp} x_{l-1} \cup \operatorname{supp} \phi_{l-1} < \dots < \operatorname{supp} x_l \cup \operatorname{supp} \phi_l$. Notice that properties (DS.1), (DS.2) and (DS.3) are clear by definition of the sequence and (DS.4) follows from (iii) and .

Modifying a little the previous argument we obtain the following:

Proposition (4.2.18) [4]:

For every two normalized block sequences $(y_n)_n$ and $(z_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every $j \in \mathbb{N}$ there exists a $(1,j)$-dependent sequence $(x_1, \emptyset_1, \dots, x_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ such that $x_{2l-1} \in \operatorname{span}_{\mathbb{R}}(y_n)$ and $x_{2l-1} \in \operatorname{span}_{\mathbb{R}}(z_n)$ for every $l = 1, \dots, n_{2j+1}$.

Another consequence of the basic inequality is the following proposition.

Proposition (4.2.19) [4]:

Let $(x_1, \emptyset_1, \dots, x_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ be a $(1,j)$-dependent sequence. Then:

(i) $\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} x_i \right\| \geq \frac{1}{m_{2j+1}}$;

(ii) $\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} x_i \right\| \geq \frac{1}{m_{2j}^3}$.

Proposition (4.2.20) [4]:

Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then the closure of the real span of $(y_n)_n$ is $H.I$.

Proof:

Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Fix $\epsilon > 0$ and two block subsequences $(z_n)_n$ and $(w_n)_n$ in $\operatorname{span}_{\mathbb{R}}(y_n)$. Take an integer j such that $m_{2j+1}\epsilon > 1$. By Proposition (4.2.18) there exist a $(1,j)$-dependent sequence $(x_1, \emptyset_1, \dots, x_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ such that $x_{2i-1} \in \operatorname{span}_{\mathbb{R}}(z_n)$ and $x_{2i} \in \operatorname{span}_{\mathbb{R}}(w_n)$. We define $z = (1/n_{2j+1}) \sum_{i=1(odd)}^{n_{2j+1}} x_i$ and $w = (1/n_{2j+1}) \sum_{i=1(even)}^{n_{2j+1}} x_i$. Notice that $z \in \operatorname{span}_{\mathbb{R}}(z_n)$ and $w \in$

$\operatorname{span}_{\mathbb{R}}(w_n)$. Then by Proposition (4.2.1) we get $\|z+w\| \geq (1/m_{2j+1})$ and $\|z-w\| \geq 1/m_{2j+1}^2$. Hence $\|z-w\| \leq \epsilon\|z+w\|$.

Definition (4.2.21) [4]:

A sequence $(z_1, \emptyset_1, \dots, z_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ is called a $(0, j)$-dependent sequence when it satisfies the following conditions:

(i) (0DS.1) The $\Phi = \phi_1, \dots, \phi_{n_{2j+1}}$-special sequence and $\phi_i(z_k) = 0$ for every $1 \leq i, k \leq n_{2j+1}$.

(ii)(0DS.2) There exists $\{\psi_1, \dots, \psi_{n_{2j+1}}\} \subseteq \kappa_{\omega_1}(\mathbb{C})$ such that $w(\psi_i) = w(\phi_i), \#\operatorname{supp} z_i \leq w(\phi_{i+1})/n_{2j+1}^2$ and (z_i, ψ_i) is a $(6, 2j_1)$-exact pair for every $1 \leq i \leq n_{2j+1}$.

(iii) (0DS.3) If $H = (h_1, \dots, h_{n_{2j+1}})$ is an arbitrary 2_{j+1}-special sequence, then

$$\left(\bigcup_{k,\Phi,H<i<\lambda\Phi,H} \operatorname{supp} z_i\right) \cap \left(\bigcup_{k,\Phi,H<i<\lambda\Phi,H} \operatorname{supp} h_i\right) = \phi.$$

Proposition (4.2.22) [4]:

For every $(0, j)$-dependent sequence $(x_1, \emptyset_1, \dots, x_{n_{2j+1}}, \emptyset_{n_{2j+1}})$ we have that

$$\left\|\frac{1}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} x_k\right\| \leq \frac{1}{m_{2j+1}^2}.$$

Proposition (4.2.23) [4]:

Let $(y_n)_n$ be a (C, ϵ)-$R.I.S.$, $Y = \overline{\operatorname{span}}_{\mathbb{C}}(y_n)$, and $T: Y \longrightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ on $\mathbb{R}$-linear bounded operator. Then $\lim_{n\to\infty} d(Ty_n \mathbb{C} y_n) = 0$.

Proof:

Suppose that $\lim_{n\to\infty} d(Ty_n, \mathbb{C}y_n) \neq 0$. Then there exists an infinite subset $B \subseteq \mathbb{N}$ such that $\inf_{n\in B} d(Ty_n \mathbb{C} y_n) > 0$. We shall show that for every $\epsilon > 0$ there exists $y \in Y$ such that $\|y\| < \epsilon\|Ty\|$ and this is a contradiction.

Claim (1):

There exists a limit ordinal $\gamma_0, A \subseteq \mathbb{N}$ infinite and $\delta > 0$ such that

$$\inf_{n\in A} d(P\gamma_0 T y_n, \mathbb{C}y_n) > \delta$$

To prove this claim we observe that

$$\gamma_0 = \min\left\{\gamma < \omega_1 : \exists A \in [\mathbb{N}]^\infty \inf_{n\in A} d\left(P_\gamma T y_n, \mathbb{C}y_n\right) > 0\right\}$$

is a limit ordinal. In fact, by the assumption the set on the right side is not empty. And if γ_0 is not limit, then we have $\gamma_0 = \beta + 1$. The sequence $(y_n)_n$ is weakly null (because $(e_\alpha)_\alpha$ is shrinking) and then

$$\lim_{n\to\infty} e^*_{\beta+1} T y_n = 0$$

And for large n and every $\lambda \in \mathbb{C}$

$$\left\|P_\beta T y_n - \lambda y_n\right\| \geq \left\|P_{\beta+1} T y_n - \lambda y_n\right\| - \left\|e^*_{\beta+1} T y_n\right\|$$

$$\geq \delta - \left|e^*_{\beta+1} T y_n\right| \geq \delta/2,$$

which is a contradiction.

Claim (2):

Fix γ_0 and $A \subseteq \mathbb{N}$ as in Claim (1). Then there exist a sequence $n_2 < n_3 < \cdots$ in A, a sequence of functionals $f_2, f_3, \ldots$ in $\kappa_{\omega_1}(\mathbb{C})$ and a sequence of ordinals $\gamma_1 < \gamma_2 < \cdots < \gamma_0$ such that

(i) $d\left(P_{[\gamma k, \gamma k+1]} T y_{nk+1}, \mathbb{C}y_{nk}\right) \geq \delta/2;$

(iii) $f_k T_{yn_k} \geq \delta/2;$

(iv) $f_k(y_{nk}) = 0;$

(v) $\operatorname{ran} f_k \subseteq \operatorname{ran} T y_{n_k};$

(vi) $\operatorname{supp} f_k \cap \operatorname{supp} y_{n_m} = \phi$ when $m \neq k$.

To prove this claim, let $\xi = \sup\max\operatorname{supp} y_n$. We analyze the three possibilities for ξ:

Case (a):$\boldsymbol{\xi < \gamma_0}$:

Let $n = \min A$ and choose $\xi < \gamma_1 < \gamma_0$ such that

$$\left\|P_{\gamma_0} T y_{n_1} - P_{\gamma_1} T y_{n_1}\right\| < \delta/2,$$

hence, $d(P_{\gamma_1} T y_{n_1}, \mathbb{C}y_{n_1}) > \delta/2$. By minimality of γ_0 we have

$$\inf_{n\in A} d\left(P_{\gamma_1} T y_n, \mathbb{C}y_n\right) = 0,$$

then we can choose $n_2 > n_1$ in A such that $d(P_{\gamma_1}Ty_{n_2}, \mathbb{C}y_{n_2}) < \delta/2$ and this implies that

$$d\left(\left(P_{\gamma_0} - P_{\gamma_1}\right)Ty_{n_2}, \mathbb{C}y_{n_2}\right) > \delta/2.$$

Approximating the vector $\left(P_{\gamma_0} - P_{\gamma_1}\right)Ty_{n_2}$ choose $\gamma_0 > \gamma_2 > \gamma_1$ such that $\left\|\left(P_{\gamma_0} - P_{\gamma_1}\right) \times Ty_{n_2}\right\|$ is so small in order to guarantee that

$$d\left(P_{[\gamma_1,\gamma_2]}Ty_{n_2}, \mathbb{C}y_{n_2}\right) \geq \delta/2.$$

Using the complex Hahn-Banach theorem, there exists $g_2 \in B_{\mathfrak{X}^*_{\omega_1}(\mathbb{C})}$ such that

(A) $g_2\left(P_{[\gamma_1,\gamma_2]}Ty_{n_2}\right) > \delta/2;$

(B) $g_2\left(y_{n_2}\right) = 0,$

and by Proposition (4.2.1) we can choose $h_2 \in \kappa_{\omega_1}(\mathbb{C})$ such that $h_2\left(\left(P_{[\gamma_1,\gamma_2]}Ty_{n_2}\right)\right) > \delta/2$ and $h_2\left(y_{n_2}\right)$ is arbitrarily small. Replacing h_2 by $\alpha h_2 + \beta k_2$ where $|\alpha| + |\beta| = 1$, $k_2\left(y_{n_2}\right)$ is close enough to 1, and $k_2 \in \kappa_{\omega_1}(\mathbb{C})$ we may assume that $h_2\left(y_{n_2}\right) = 0$.

Let $f_2 = h_2 P_{[\gamma_1,\gamma_2]\cap \operatorname{ran} Ty_{n_2}} \in \kappa_{\omega_1}(\mathbb{C})$. Again by minimality of γ_0, there exists $n_3 > n_2$ in A such that $d(P_{\gamma_2}Ty_{n_3}, \mathbb{C}y_{n_3}) < \delta/2$ and we can choose $\gamma_0 > \gamma_3 > \gamma_2$ satisfying

$$\left(P_{[\gamma_2,\gamma_3]}Ty_{n_3}, \mathbb{C}y_{n_3}\right) > \delta/2.$$

Again by Hahn-Banach and by Proposition (4.1.1) there exists a functional $h_3 \in \kappa_{\omega_1}(\mathbb{C})$ such that

(C) $h_3\left(P_{[\gamma_2,\gamma_3]}Ty_{n_3}\right) > \delta/2;$

(D) $h_3\left(y_{n_3}\right) = 0,$

then we define $f_3 = h_3 P_{[\gamma_2,\gamma_3]\cap \operatorname{ran} Ty_{n_3}} \in \kappa_{\omega_1}(\mathbb{C})$. The previous argument gives us the way to construct the sequences of Claim (2). Properties (1)-(5) are easy to check, in particular property (5) is true because $\min \operatorname{supp} f_k > \xi > \max \operatorname{supp} y_{n_l}$ for every positive integers k, l.

Case (b):$\boldsymbol{\xi > \gamma_0}$:

In this case we start by picking $n_1 \in A$ such that $\min \operatorname{supp} y_{n_1} > \gamma_0$.

Then we repeat exactly the same argument that in Case (a).

Case (c): $\xi = \gamma_0$:

We basically repeat the same argument of the Case (a) with the additional care of maintaining property (vi) true. That is, each time we choose the ordinal γ_{k+1} (with $\gamma_0 > \gamma_{k+1} > \gamma_k$) we take it such that $\gamma_{k+1} > \max \operatorname{supp} y_{n_{k+1}}$.

Claim (3):

There exists a $(0, \mathrm{j})$- dependent sequence $(\mathrm{z}_1, \phi_1, \dots, \mathrm{z}_{\mathrm{n}_{2\mathrm{j}+1}})$ such that

(E) $z_i \in X$ for every $1 \leq i \leq n_{2j+1}$;

(F) $\operatorname{ran} \phi_k \subseteq \operatorname{ran} Ty_k$ and $\phi_k(Tz_k) > \delta/2$.

Let j with $m_{2j+1} > 24/\epsilon\delta$. Choose j_1 even such that $m_{2j_1} > n_{2j+1}^2$ (see definition of special sequence) and $F_1 \subseteq A$ with $\#F_1 = n_{2j_1}$ such that $(y_{n_k})_{k \in F_1}$ is a $(3, 1/n_{2j+1}^2)$-$R.I.S$. Then define

$$\phi_1 = \frac{1}{m_{2j_1}} \sum_{i \in F_1} f_i \in \kappa_{\omega_1}(\mathbb{C}) \quad \text{and} \quad z_1 = \frac{m_{2j_1}}{n_{2j_1}} \sum_{k \in F_1} y_k$$

observe that $w(\phi_1) = m_{2j_1}, \phi_1(Tz_1) = \frac{1}{n_{2j_1}} \sum_{i \in F_1} f_i \left(\sum_{k \in F_1} Ty_k\right) > \delta/2$ and $\phi_1(z_1) = \frac{1}{n_{2j_1}} \sum_{i \in F_1} f_i \left(\sum_{k \in F_1} \ \right) = 0$. Select

$$p_1 \geq \max\{p_\varrho(\operatorname{supp} z_1 \cup \operatorname{supp} Tz_1 \cup \operatorname{supp} \phi_1), n_{2j+1}^2 \# \operatorname{supp} z_1\},$$

denote $2_{j2} = \sigma_\varrho(\phi_1, m_{2j_1}, p_1)$. Then take $F_2 \subseteq A$ with $\#F_2 = n_{2j_2}$ and $F_2 > F_1$ such that $(y_k)_{k \in F_2}$ is $(3, 1/n_{2j_2}^2)$-$R.I.S.$ and define

$$\phi_2 = \frac{1}{m_{2j_2}} \sum_{i \in F_2} f_i \in \kappa_{\omega_1}(\mathbb{C}) \quad \text{and} \quad z_2 = \frac{m_{2j_2}}{n_{2j_2}} \sum_{k \in F_2} y_k$$

So we have $\phi_1 < \phi_2, \phi_2(\mathrm{Tz}_2) > \delta$ and $\phi_2(\mathrm{z}_1) = \phi_2(\mathrm{z}_2) = 0$. Pick

$$p_2 \geq \max\{p_1, p_\varrho(\operatorname{supp} z_1 \cup \operatorname{supp} z_2 \cup \operatorname{supp} Tz_1 \cup \operatorname{supp} Tz_2 \cup \operatorname{supp} \phi_1 \cup \operatorname{supp} \phi_2), n_{2j+1}^2 \# \operatorname{supp} z_2\}$$

and set $2j_3 = \sigma_\varrho(\phi_1, m_{2j_1}, p_1, \phi_2, m_{2j_2}, p_2)$. Continuing with this procedure we form a sequence $(z_1, \phi_1, \dots, z_{n_{2j+1}}, \phi_{n_{2j+1}})$. Now we check that this is a $(0, j)$-dependent sequence.

Property (0DS.1) is clear, because of the construction of the functionals their weights satisfies $w(\phi_{i+1}) = m_{\sigma_\varrho}(\Phi_i)$ where $\Phi_i = (\phi_1, w(\phi_1), p_1, \dots, \phi_i, w(\phi_i), p_i)$.

Property (0DS.2) We proceed to the construction of the sequence $\{\psi_1, \dots, \psi_{n_{2j+1}}\}$ in $\kappa_{\omega_1}(\mathbb{C})$ such that (z_i, ψ_i) is a $(6, 2j_i)$-exact pair and $w(\psi_i) = w(\phi_i)$ for every $1 \le i \le n_{2_{j+1}}$. The other condition $\#\operatorname{supp} z_i \le w(\phi_{i+1})/n_{2j+1}^2$is already obtained by the construction of the weights. For each z_i there exists a subset $F_i \subseteq A$ with $\#F_i = n_{2_{j+1}}$, such that $z_i = (m_{2j_i}/n_{2j_i}) \sum_{k \in F_i} y_{n_k}$ where $(y_{n_k})_{k \in F_i}$ is a $(3, 1/n_{2j+1}^2)$ $R.I.S.$ Now we follow the same arguments as in Proposition (4.2.15). For every $k \in F_i$ we take $f_{n_k} \in \kappa_{\omega_1}(\mathbb{C})$ such that $f_{n_k}(y_{n_k}) = 1$ and $f_{n_k} < f_{n_{k+1}}$. Then $\psi_i = (1/m_{2j_i}) \sum_{k \in F_i} f_{n_k} \in \kappa_{\omega_1}(\mathbb{C})$ and (z_i, ϕ_i) is a $(6, 2j_i)$-exact pair.

Property (0DS.3) Let $H = (h_1, \dots, h_{n_{2j+1}})$ be an arbitrary $2j+1$-special sequence. We consider two cases: (a) Suppose that $\max \operatorname{supp} z_k \le \max \operatorname{supp} \phi_k$ for every $1 \le k \le n_{2j+1}$. Then $\operatorname{supp} z_k \subseteq \operatorname{supp} \overline{\phi_{\lambda_{\Phi,H^{-1}}}}^{p\lambda_{\Phi,H^{-1}}}$ for every $\kappa, \Phi, H < k < \lambda_{\Phi,H}$. Then for the second part of (TP.3) we obtain the desired result. (b) Suppose that $\max \operatorname{supp} \phi_k \le \max \operatorname{supp} z_k$ for every $1 \le k \le n_{2j+1}$. Then $\operatorname{supp} \phi_k \subseteq \operatorname{supp} \overline{z_{\lambda_{\Phi,H^{-1}}}}^{p\lambda_{\Phi,H^{-1}}}$ for every $\kappa\Phi, H < k < \lambda_{\Phi,H}$, and the result follows from the first part of (TP3).

Fix a $(0, j)$-dependent sequence as obtained in the previous claim, and define

$$z = (1/n_{2j+1}) \sum_{k=1}^{n_{2j+1}} z_k \quad and \quad \phi = (1/m_{2j+1}) \sum_{k=1}^{n_{2j+1}} \phi_k.$$

Then $\phi(Tz) = (1/n_{2j+1}) \sum_{k=1}^{n_{2j+1}} \phi_k(Tz) \ge \delta/m_{2j+1}$ and $\|z\| \le 12/m_{2j+1}^2$. Hence, $\|Tz\| \ge \delta/m_{2j+1} \|z\|/12 > \epsilon\|z\|$, and this completes the proof.

List of Symbols

Symbols	Page
d_{DM}: Banach Space- Mazur distance	**56**
Dim : dimension	**58**
Cl : closure	**61**
Card : cardinality	**62**
Dist :distant	**62**
ℓ_p **:** lebesque space	**66**
$\oplus$ **:** direct sum	**67**
Supp : support	**68**
R.I.S : Rapidly Increasing Sequences	**71**
H.I: Hereditarily indecomposable	**71**
D.S: Dependent Sequence	**88**
ℓ^p**:** Hilbert space	**2**
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Inf: infimum	**7**
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Ker: kernel	**10**
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