Sudan University of Science and Technology

Travelling Wave Solutions for the Modified Korteweg-De-Viress equation and the generalized Nonlinear Schrödinger equation

حلول الموجة المتنقلة لمعادلة كورتوي دي فريس المعدلة ومعادلة شرودنجر الغير خطية المعممة

A thesis Submitted in partial Fulfillment for the Degree of M.Sc. in Mathematics

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Dedication

This research is dedication with love and affection

To my parents,

My brothers and my sister

Acknowledgements

Weeping may endure for a night but joy comes in the morning. I thank the Almighty God for His unprecedented love and grace upon me. I would like to express my sincere gratitude to the Sudan (SUST) for graduate me this wonderful opportunity to do a high graduate studies (MSc) in mathematical science. A million thanks to Academy of engineering science for making Academy of engineering science a global reality. My loving Kindness goes to my parents brothers and my sister .I Would like to give especial appreciation to my supervisor Dr. Abdullah Habila Ali Kaitan for his steadfast love , guidance and support in making this research a success . I love you all and I pray that you guys will always be therefore me. On love!!

Abstract

The sine - Gordon equation appears in the propagation of fluxions in Josephson Junctions between two superconductors. It also appears in many scientific fields such as the notions, and dislocation in crystals where sin (u) is due to periodic structure of rows of atom. The term sin (u) is the Josephson current across on insulator between two superconductors.

In this thesis we formally drive exact travelling wave solutions for the modified Kortewey - De – Vriss (MKdv – sine - Gordon) equation. The proposed analysis depends mainly on a variable separated ODE method. Two distinct sets of exact solitary wave solutions, that posse's distinct physical structure, are formally derived for each equation. The derived solutions include other results and introduce entirely new solutions.

The capability of extended $\tanh - \coth$**,** $\sin e - \cos ine$ **and Exp – function methods as alternative approaches to obtain the analytic solution of different types of applied differential equations in engineering mathematics has been revealed.**

In this study, the generalized nonlinear Schrödinger (GNLS) equation is solved by different methods. To obtain the single – solution for the equation, the extended $\tanh - \coth$ **and sine – cosine method is used. Furthermore, for this nonlinear evolution equation the Exp – function method is applied to derive various travelling wave solution. Results show that while the first two procedures easily provide a concise solution, the Exp – function method provides powerful evolution equations in mathematical physics.**

الخلاصة

تظهر معادلة جيب جوردان عند إنتشار التدفق عند تقاطعات جوزيفسون بين إثنين من الموصلات الفائقة .

وتظهر أيضآ في العديد من المجالات العلمية مثل المفاهيم والإضطرابات في البلورات ؛ ويرجع ذلك لدورية $\sin u$ في صفوف من الزرة. الحد $\sin u$ هو تيار جوزيفسن عبر عازل بين إثنين من الموصلات الفائقة.

في هذا البحث قمنا باشتقاق حلول الموجات المتنقلة المضبوطة لمعادلة كورتوى دي فريس – جوردان للجيب المعدلة. التحليل المقترح يعتمد أساسآ على طريقة المعادلات التفاضلية من نوع فصل المتغيرات . تم إشتقاق مجموعتين من الحلول الموجية المنعزلة المضبوطة ذات بنيات فيزيائية مختلفة ؛ والحلول المشتقة تشمل نتائج أخرى وتقدم حلول جديدة كليآ.

تم إكتشاف كفاءة طريقة إمتداد $\tanh - \coth$ و $\sin e - \cos ine$ الدالة الأسية كمنهج بديل للحصول على الحلول التحليلية لأنواع مختلفة من المعادلات التفاضلية في الرياضيات الهندسية.

في هذة الدراسة تم حل معادلة شرودينجر الغير خطية المعممة بطرق مختلفة . وتم إستخدام طريق $\tanh - \coth$ $\sin e - \cos ine$ الممتدة للحصول على الحل المنفرد المحكم . إضافة لذلك تم تطبيق طريقة الدالة الإسية في هذة المعادلة غير الخطية المتطورة للحصول على حل موجي متنقل مختلف . أظهرت النتائج أن طريقة الدالة الأسية يزود بمفهوم رياضي قوي في حل المعادلات التفاضلية الغير خطية المتطورة في الرياضيات الفيزيائية عند القيام باول إجرائين ويزود بكل سهولة بحل مختصر.

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Chapter (1)

The Variable Separated ODE Method for Traveling Wave Solution:

Section (1.١): Introduction

It is well – known that the linear Boussinesq equation is a fourth order PDE given by [1], [3], [7].

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = 0 \qquad (1.1)$$

That includes the physical dispersion term. The sine – Gordon equation is a second order nonlinear PDE given by

$$u_{tt} - u_{xx} + \sin u = 0 \qquad (1.2)$$

This equation appeared first in the study of differential geometry of surfaces with Gaussian curvature K = -1. The sine – Gordon equation is completely integrable, infinite dimensional Hamiltonian system [1] and it can be solved by the inverse scattering method [4], [6], [7]. Integral means that there are a sufficiently large number of conserved quantities [1], [3]. The term sinu is the Josephson current across an insulator between two superconductors [7], [10], [20].

$$u_{tt} - u_{xx} + \sin u + \sin 2u = 0, \qquad (1.3)$$

Appears in many scientific applications However, the sinh – Gordon equation

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{1.4}$$

Appears in integrable quantum field theory, kink dynamic [1], [2], [3], [7], [8], [9], [16], and fluid dynamic.The sinh – Gordon equation is completely integral because it possesses similarity reduction to third Painleve equation [17], [21], [27], Moreover, the double sinh - Gordon equation

$$u_{tt} - u_{xx} + \sinh u + \sinh 2u = 0, \tag{1.5}$$

has lot of scientific applications as well. it is know that scaling for explicit solution for nonlinear evolution equation, by using different method many powerful method such as Backland transformation, inverse scatting method Hirota bilinear forms pseudo spectral method, the $\tanh - \sec h.$ method,[4],[5],[6],[11],[12], the sine – cosine method [10] and many other techniques were successfully used to investigate these types of equations practically there is no unified method that can be used to handle all type of nonlinear problems. In this chapter, the Boussinesq – Double sine Gordon (B – Sine - Gordon) equation, The Boussinesq – double Sinh – Gordon equation (Boussinesq – Sinh - Gordon) and the Boussinesq – Liouvill equation.(1.1) and (1.2) (BL -I) and (BL – II) Given by:

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sin u + \frac{3}{2}\sin 2u, \qquad (1.6)$$

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sinh u + \frac{3}{2}\sinh 2u, \qquad (1.7)$$

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^{u} + \frac{3}{4}\sinh e^{2u}, \qquad (1.8)$$

And

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^{-u} + \frac{3}{4}e^{-2u}, \qquad (1.9)$$

We aim to emphasize the power of the variable separated ODE method that will be employed here. This method is developed by Sirendaoreij et at in [9] used by fu et at in [2] and by Wazwaz,[13],[14]. It works effectively if the equations involve Sine, Cosine hyperbolic Sine and hyperbolic Cosine function. In what follows highlight the main features of the method as introduced in [9] where more details and examples can be found in [9] [2] [13] [14].

Section (1.2): The Variable Separated ODE Method:

The stated before, this method was developed by Sirendaoreij et at [9]. We first unite the independent variable x and t into one wav variable ξ = x – ct to carry out PDE into an equivalent ODE. The method depend mainly on assuming that u(ξ) satisfies an additional variable separated ODE given by

$$u' = \frac{du}{d\xi} = G(u) \qquad (1.10)$$

Where G(u) is a suitable function of Sine , Cosine hyperbolic Sine hyperbolic Cosine Differentiating (1.10) and equating the coefficient of like terms of the resulting equation with the ODE reduced from the PDE yield a system of algebraic equations than can be solved to determine the unknown parameter It is worth noting the variable separated ODE

(1.10) can be solved easily by using the method of separation of variable. It should be noted that the section of G (u) can be performed in more than one choice as will be seen later.

Section (1.3): The Boussinesq – double Sine–Gordon equation, [23].

A variable separated ODE method.

We first examine the Boussinesq – double Sine – Gordon equation [23].

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sin u + \frac{3}{2}\sin 2u, \tag{1.11}$$

That by using wave variable $\xi = x - ct$ can be converted to the ODE.

$$(c^2 - \alpha)u^{ll} + u^{(iv)} = \sin u + \frac{3}{2}\sin 2u \tag{1.12}$$

The Variable separated ODE $u^l(\xi) = G(u)$ can be selected in two ways as well be discussed.

First Selection of G (u):

We assume that $u(\xi)$ satisfies ODE

$$u' = \frac{du}{d\xi} = a\cos(\frac{u}{2}) \tag{1.13}$$

Where a is an arbitrary constant. Differentiating (1.13) with respect to ξ we obtain:

$$u''(\xi) = -\frac{a^2}{4}\sin u$$

$$u'''(\xi) = -\frac{a^2}{4}\cos u \cos\frac{u}{2} \tag{1.14}$$

$$u^{(iv)}(\xi) = \frac{3a}{32}\sin(2u) + \frac{a^2}{8}\sin u$$

Substituting Eq(1.13) and Eq(1.14) into Eq(1.12) gives:

$$(\frac{a^4}{8} - \frac{a^2}{4}(c^2 - \alpha))\sin u + \frac{3a^4}{32}\sin 2u = \sin u + \frac{3}{2}\sin 2u. \tag{1.15}$$

Equating the Coefficient of like tragicomic function on both sides gives:

$a = 2$,

c = $\sqrt{\alpha + 1}$, α ≥ -1 (1.16)

Eq.(1.13) is separable hence we get:

$$\frac{du}{\cos(\frac{u}{2})} = \frac{\beta}{c^2 - k} \tag{1.17}$$

Where by integrating both sides we find the following solution:

$$4\arctan h\left[\tan(\frac{u}{4})\right]=a(\xi-\xi_0) \tag{1.18}$$

$$4arc\coth\left[\tan(\frac{u}{4})\right]=a(\xi-\xi_0) \tag{1.19}$$

$$4\arctan h\left[\cot(\frac{u}{4})\right]=a(\xi-\xi_0) \tag{1.20}$$

$$4\,arc\ \coth\left[\cot(\frac{u}{4})\right]=a(\xi-\xi_0) \tag{1.21}$$

$$2\ln\ \tan\left[\frac{u}{4}+\frac{\pi}{4}\right]=a(\xi-\xi_0) \tag{1.22}$$

$$2\ln\left[\sec(\frac{u}{4})+\tan(\frac{u}{2})\right]=a(\xi-\xi_0) \tag{1.23}$$

at $u=0,,\quad \xi=\xi_0,$

Where ξ_o is constant of integration. It is important to note that the lost solution obtained above for u(x,t) is an implicit solution combining (1.16) with the previous result leads to the following implicit solitary wave solution.

$$u_1(x,t)=4\arctan\left[\tanh(\frac{1}{2}(x-\sqrt{\alpha+1}t-\xi_0))\right] \tag{1.24}$$

$$u_2(x,t)=4\arctan\left[\coth(\frac{1}{2}(x-\sqrt{\alpha+1}t-\xi_0))\right] \tag{1.25}$$

$$u_3(x,t)=4arc\cot\left[\tanh(\frac{1}{2}(x-\sqrt{\alpha+1}t-\xi_0))\right] \tag{1.26}$$

$$u_4(x,t)=4arc\cot\left[\coth(\frac{1}{2}(x-\sqrt{\alpha+1}t-\xi_0))\right] \tag{1.27}$$

$$u_5(x,t) = 4\arctan\left\lfloor \exp(x - \sqrt{\alpha+1}t - \xi_0)\right\rfloor - \pi \qquad (1.28)$$

Second selection of G (u):

We next assume that u (ξ) satisfies variable separated ODE

$$u^l(\xi) = \frac{du}{d\xi} = a\sin(\frac{u}{2}) \qquad (1.29)$$

Differentiating Eq(1.29) with respect to ξ we obtain:

$$u^{ll}(\xi) = \frac{a}{4}\sin u$$

$$u^{lll}(\xi) = \frac{a^3}{4}\cos u \sin\frac{u}{2} \qquad (1.30)$$

$$u^{(iv)}(\xi) = \frac{3a^4}{32}\sin(2u) - \frac{a^4}{8}\sin u$$

Substituting Eq(1.29) and Eq(1.30) into Eq(1.12) gives:

$$(\frac{a^2}{4}(c^2-\alpha) - \frac{a^4}{8})\sin u + \frac{3a^4}{32}\sin 2u = \sin u + \frac{3}{2}\sin 2u. \qquad (1.31)$$

Where by equaling the coefficients of like functions on both sides we obtain:

$$a = 2,$$

$$c = \sqrt{\alpha + 3}, \alpha \geq -3 \tag{1.32}$$

Eq.(1.29) is separable hence we set:

$$\frac{du}{\sin(\frac{u}{2})} = a \; d\xi \tag{1.33}$$

Where by integrating both sides we find the following solution:

$$2\ln\left[\tan(\frac{u}{4})\right] = a(\xi - \xi_0) \tag{1.34}$$

$$-2\ln\left[\cot(\frac{u}{2})\right] = a(\xi - \xi_0) \tag{1.35}$$

And

$$2\ln\left[\csc(\frac{u}{2}) - \cot(\frac{u}{2})\right] = a(\xi - \xi_0) \tag{1.36}$$

Where ξ_0 is constant of integration. It is important to note that the last solution obtained above for u(x, t) it is an implicit solution. Combining (1.32)with the pervious result leads to the following solitary wave solution

$$u_6(x,t) = 4\arctan\left\lfloor \exp(x - \sqrt{\alpha + 1}t - \xi_0) \right\rfloor \tag{1.37}$$

And

$$u_7(x,t) = 4arc\cot\left[\exp(-x-\sqrt{\alpha+1}t-\xi_0)\right] \qquad (1.38)$$

Section (1.4): The Boussinesq – double Sinh – Gordon equation [12], [13], [14], [15], [30]:

We next investigate the Boussinesq – double Sinh – Gordon equation (Boussinesq –Sinh – Gordon) equation [30].

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sinh u + \frac{3}{2}\sinh 2u, \qquad (1.39)$$

That can be converted to the ODE

$$(c^2-\alpha)u^{ll} + u^{(iv)} = \sinh u + \frac{3}{2}\sinh 2u, \qquad (1.40)$$

Upon using the wave variable ξ = x – ct the separable ODE $u^l(\xi) = G(u)$ can be selected in two forms as used before.

First selection of G(u):

We assume that u (ξ) satisfies the variable separated ODE given by:

$$u^l(\xi) = \frac{du}{d\xi} = acoch(\frac{u}{2}) \qquad (1.41)$$

Differentiating Eq(1.41) with respect to ξ we obtain

$$u^{ll}(\xi) = \frac{a^2}{4}\sinh u$$

$$u^{III}(\xi) = \frac{a^3}{4}\cosh u \cosh\frac{u}{2} \qquad (1.42)$$

$$u^{(iv)}(\xi) = \frac{3a^4}{32}\sinh u + \frac{a^4}{8}\sinh u$$

Substituting Eq(1.41) and Eq(1.42) into Eq(1.40) gives

$$(\frac{a^2}{4}(c^2 - \alpha) + \frac{a^4}{8})\sinh u = \frac{3a^4}{32}\sinh u + \frac{3}{2}\sinh 2u, \qquad (1.43)$$

Equating the coefficients from both sides' yields:

$$a = 2,$$

$$c = \sqrt{\alpha - 3}, \alpha \geq 1 \qquad (1.44)$$

Eq.(1.41) is separable hence we set:

$$\frac{du}{\cosh(\frac{u}{2})} = a \, d\xi \qquad (1.45)$$

Where by integrating both sides we find the following solution:

$$4\arctan\left[\tanh(\frac{u}{4})\right] = a(\xi - \xi_0) \qquad (1.46)$$

$$-4\arctan\left[\coth(\frac{u}{4})\right] = a(\xi - \xi_0) \qquad (1.47)$$

$$-4arc\cot\left[\tanh(\frac{u}{4})\right] = a(\xi - \xi_0) \qquad (1.48)$$

$$4arc\cot\left[\coth(\frac{u}{4})\right] = a(\xi - \xi_0) \qquad (1.49)$$

$$4\arctan\left[\exp(\frac{u}{2})\right] = a(\xi - \xi_0) \qquad (1.50)$$

$$-4arc\cot\left[\exp(\frac{u}{2})\right] = a(\xi - \xi_0) \qquad (1.51)$$

at $u = 0,$, $\xi = \xi_0,$

Where ξ_0 is constant of integration Combining Eq(1.44) with the previous result leads to the following solution

$$u_1(x,t) = 4\arctan h\left[\tan(-\frac{1}{2}(x - \sqrt{\alpha - 1}t - \xi_0))\right] \qquad (1.52)$$

$$u_2(x,t) = 4arc\coth\left[\tan(-\frac{1}{2}(x - \sqrt{\alpha - 1}t - \xi_0))\right] \qquad (1.53)$$

$$u_3(x,t) = 4\arctan h\left[\cot(-\frac{1}{2}(x - \sqrt{\alpha - 1}t - \xi_0))\right] \qquad (1.54)$$

$$u_4(x,t) = 4arc\coth\left[\cot(\frac{1}{2}(x - \sqrt{\alpha - 1}t - \xi_0))\right] \qquad (1.55)$$

$$u_5(x,t) = 2\ln\left[\tan(\frac{1}{2}(x - \sqrt{\alpha - 1}t - \xi_0))\right] \qquad (1.56)$$

$$u_6(x,t) = 2\ln\left[\cot(\frac{1}{2}(x - \sqrt{\alpha - 1}t - \xi_0))\right] \qquad (1.57)$$

Second Selection Of G(u):

We next assume that $u(\xi)$ satisfies the ODE given by:

$$u^l(\xi) = \frac{du}{d\xi} = a\sinh(\frac{u}{2}) \qquad (1.58)$$

Where a is parameter that will be determined? Differentiating Eq**(1.58)** with respect to ξ we obtain:

$$u^{ll}(\xi) = \frac{a^2}{4}\sinh u$$

$$u^{lll}(\xi) = \frac{a^3}{4}\cosh u \sinh\frac{u}{2} \quad (1.59)$$

$$u^{(iv)}(\xi) = \frac{3a^4}{32}\sinh 2u - \frac{a^4}{8}\sinh u$$

Substituting Eq(1.59) into Eq(1.40) gives.

$$(\frac{a^2}{4}(c^2-\alpha) - \frac{a^4}{8})\sinh u + \frac{3a^4}{32}\sinh 2u = \sinh u + \frac{3}{2}\sinh 2u, \quad (1.60)$$

That gives:

$$a = 2$$

c = $\sqrt{\alpha - 3}$, α ≥ -3 (1.61)

Eq.(1.58) is separable hence we set:

$$\frac{du}{\sinh(\frac{u}{2})} = a d\xi \quad (1.62)$$

Where by integrating both sides we find the following solution

$$2\ln\left[\tanh(\frac{u}{4})\right] = a(\xi - \xi_0) \quad (1.63)$$

$$-2\ln\left[\coth(\frac{u}{4})\right] = a(\xi-\xi_0) \quad (1.64)$$

And

$$2\ln\left[\sec h(\frac{u}{2}) - \coth(\frac{u}{2})\right] = a(\xi-\xi_0) \quad (1.65)$$

Where ξ_0 is constant of integration. To note that the last solution obtained above for u(x,t) is an implicit solution . Combining (1.61) with the previous results leads to the following explicit solitary wave solution

$$u_1(x,t) = 4\operatorname{arctan}h\left[\exp((x-\sqrt{\alpha+3}t-\xi_0))\right] \quad (1.66)$$

And

$$u_2(x,t) = 4arc\coth\left[\exp((x-\sqrt{\alpha+3}t-\xi_0))\right] \quad (1.67)$$

Section (1.5): The Boussinesq – Liouville equation:

We next consider the Boussinesq – Liouville (BL – I) equation

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^u + \frac{3}{4}e^{2u} \quad (1.68)$$

That can be converted to the ODE

$$(c^2-\alpha)u^{ll} + u^{(iv)} = e^u + \frac{3}{4}e^{2u}, \quad (1.69)$$

Upon using $\xi = x - ct$ the separable ODE $u^l(\xi) = G(u)$ can be selected as

$$u^l(\xi) = \frac{du}{d\xi} = be^{\frac{u}{2}} \quad (1.70)$$

Differentiating Eq(1.70) with respect to ξ we obtain

$$u^{II}(\xi) = \frac{b^2}{2} e^u .$$

$$u^{III}(\xi) = \frac{b^3}{2} e^{\frac{3u}{2}} \tag{1.71}$$

$$u^{(iv)}(\xi) = \frac{3b^4}{4} e^{2u}$$

Substituting Eq(1.70) and Eq(1.71) into Eq(1.68) gives

$$\frac{b^2}{2}(c^2-\alpha)e^u + \frac{3b^4}{4}e^{2u} = e^u + \frac{3}{4}e^{2u}. \tag{1.72}$$

Equating the coefficients from both sides' yields:

$$b = \pm 1$$

$$c = \sqrt{\alpha+2}\ ,\ \alpha \geq -2 \tag{1.73}$$

Eq.(1.70) is separable hence we set

$$e^{\frac{u}{2}}\, du = bd\,\xi \tag{1.74}$$

Where by integrating both sides we find the following solution:

$$2e^{2u} = b(\xi - \xi_0) \tag{1.75}$$

This gives the exact solution

$$u(x,t) = 2\ln\left[\left(\frac{1}{2}(x - \sqrt{\alpha+2}t - \xi_0)\right) \right] \tag{1.76}$$

at $u=0, \xi=\xi_0$

Section (1.6): The Boussinesq – Liouville equation II:

We next consider the Boussinesq – Liouville II (BL - II) equation

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = -e^{-u} - \frac{3}{4}e^{-2u} \qquad (1.77)$$

That can be converted to the ODE

$$(c^2 - \alpha)u^{ll} + u^{(iv)} = e^{-u} + \frac{3}{4}e^{-2u}, \qquad (1.78)$$

Upon using $\xi = x - ct$. The separable ODE $u^{l}(\xi)$ = G (u) can be selected as used before

$$u^{l}(\xi) = \frac{du}{d\xi} = be^{\frac{-u}{2}} \qquad (1.79)$$

Differentiating Eq(1.79) with respect to ξ we Obtain:

$$u^{lL}(\xi) = -\frac{b^2}{2} e^{-u}.$$

$$u^{lll}(\xi) = \frac{b^3}{2}e^{\frac{-3u}{2}} \qquad (1.80)$$

$$u^{(iv)}(\xi) = -\frac{3b}{4}e^{-2u}$$

Substituting Eq(1.79) and Eq(1.80) into Eq(1.77) gives

$$\frac{b^2}{2}(c^2-\alpha)e^{-u}+\frac{3b^4}{4}e^{-2u}=e^{-u}+\frac{3}{4}e^{-2u} \quad (1.81)$$

Equating the coefficients from both sides' yields and proceeding as we find:

$$b=\pm 1$$

c = $\sqrt{\alpha+2}$, α ≥ -2. (1.82)

Eq.(1.79) is separable hence we set

$$e^{\frac{-u}{2}}du=bd\xi \quad (1.83)$$

Where by integrating both sides we find the following solution:

$$-2e^{\frac{-u}{2}}=b(\xi-\xi_0), \quad (1.84)$$

This gives the exact solution.

$$u(x,t)=-2\ln\left[\left(\frac{1}{2}(x-\sqrt{\alpha+2}t-\xi_0)\right)\right] \quad (1.85)$$

Chapter (2)

Traveling Wave Solutions for Combined and Double Combined Sine and Cosine – Gordon equation:

Section (2.1): Introduction

The first appearance of the Sine - Gordon equation [30]

$$u_{tt} - u_{xx} + \sin u = 0 \qquad (2.1)$$

is this study of differential geometry of surfaces with Gaussian curvature K = -1. Eq. (1) gained it is significations because of the coalitional behaviors of a solution that arise from these equation also appears in the a propagation of flux one in Josephson [1],[2],[3],[5],[6],[8] between two superconductors , then in many sufficiency fields such as the motion of a rigid pendula attached wire [4], solid state physics nonlinear optic stability of fluid motions . It also appears in dislocations in crystals where Sinu is due to periodic structure of rows of atoms [4]. The term Sinu is the Josephson curry across an insulator between two superconductors [1], [2] the double sine – Gordon equation (2.3) appears in solution theory of **DNA** molecular [1],[31]. The sine – Gordon equation is integrable and can be solved by the inverse scatting method. Many powerful methods such as Backlund transformation [7], [19], inverse scattering method Hirota bilinear forms pseudo-spectral method the $\tanh - \sec h$ method

[9], [13], the sine – cosine method [14], and many others were successfully usedto investigate these types of equations. Practically, there is no unified method that can be used to handle all types of nonlinearity.

Traveling waves appear as solutions of mathematical physics models in a wide range of scientific applications that range from chemical reaction to water surface gravity waves,[18],[24],[25]. Since the significant work in **KdV** equation other equations have been used to characterize a broad range of fields such as nonlinear optics magnet fluid dynamics and ion acoustic waves in plasmas among several others [27],[28],[30].

Previous work [12], [13] the sine –Gordon and the double sine – Gordon equation, the sinh – Gordon equation and the double sinh – Gordon equations given by:

$$u_{tt} - ku_{xx} + 2\alpha \sin u = 0 \qquad (2.2)$$

$$u - ku + 2\alpha \sin u + \beta \sin(2u) = 0 \qquad (2.3)$$

$$u_{tt} - ku_{xx} + 2\alpha \sinh u + 0 \qquad (2.4)$$

And

$$u - ku + \alpha \sinh u + \beta \sinh(2u) = 0 \qquad (2.5)$$

Respectively were investigated by using the standard tanh method [9],[13].

As state before these equations were subjected to considerable amount of research work where a variety of methods, were present to drive traveling wave solution. Recently, Wazwaz, [15] examined the combined sinh – Gordon and the double combined sinh – cosh – Gordon equations.

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0 \quad (2.6)$$

And

$$u_{tt} - ku_{xx} + \alpha \sinh u + \alpha \cosh u + \beta \sinh(2u) + \beta \cosh(2u) = 0 \quad (2.7)$$

Where two reliable methods, namely the $\tanh$ method, and the variable separated ODE method were used:

In this work we aim to compute travelling wave solutions to two model problems namely the combined sine – cosine – Gordon equation and the double combined sine – cosine – Gordon equation given by

$$u_{tt} - ku_{xx} + \alpha \sin u + \beta \cos u = 0 \quad (2.8)$$

And

$$u_{tt} - ku_{xx} + \alpha \sin u + \alpha \cos u + \beta \sin(2u) + \beta \cos(2u) = 0 \quad (2.9)$$

Respectively we adopted for this work a strategy that depends on a variable separated ODE. The method was developed by Sirendaoreji et al. in [1] and used by fu et al. In, [2]. And by Wazwaze in [15], [16], [17]. The variable separated ODE method has been fully described. The method has established scientific value and reliability and the previous works emphasized its power for equations that involve sine, cosine hyperbolic sine, cosine and hyperbolic cosine function.

In what follows we present the key ideas of the variable separated ODE method as introduce in [1.7] where more examples can be found there.

Section (2.2): The Variable Separated ODE method:

We first unite the independent variable x and t [25], [27], [30]. Into one wave variable ξ = x – ct to carry out PDE in two independent variables

$$p(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) \tag{2.10}$$

Into an ODE

$$Q(u, u^l, u^{ll}, u^{lll}, \ldots) = 0 \tag{2.11}$$

That can be integrated as long as all terms contain deriva tive. Usually the integration constants are considered to be zero in view of the localized; however the non – zero constants can be used and handled as well.

The method depends mainly on assume that u (ξ) satisfies an additional variable separated ODE given by:

$$u^l = \frac{du}{d\xi} = G(u), \tag{2.12}$$

Where G(u), is a suitable function of sine , cosine hyperbolic sine hyperbolic cosine . Substituting (2.12) into the given equation yields a system of algebraic equation that can be solved to determine the unknown parameters. It is noting that the variable separated ODE (2.12) can be solved easily using the method of separation of variables.

Section (2.3): The Combined Sine – Cosine - Gordon equation [26], [28], [29].

A combined sine – cosine Gordon equation

$$u_{tt} - ku_{xx} + \alpha \sin u + \beta \cos u = 0 \tag{2.13}$$

Can be converted to the ODE

$$(c^2 - k)u^{ll} + \alpha \sin u + \beta \cos u = 0 \tag{2.14}$$

Or equivalent

$$u^{ll} + \frac{\alpha}{(c^2 - k)} \sin u + \frac{\beta}{(c^2 - k)} \cos u = 0, k \neq c^2. \tag{2.15}$$

We next assume that $u(\xi)$ satisfies the variable separated ODE given by:

$$u^{l}(\xi) = \frac{du}{d\xi} = a \sin \frac{u}{2} + b \cos \frac{u}{2}, \tag{2.16}$$

Where a and b are parameters that will be determined. Differentiating Eq(2.16) with respect to ξ gives

$$u^{ll}(\xi) - \frac{a^2 - b^2}{4} \sin u - \frac{ab}{2} \cos u = 0. \tag{2.17}$$

Comparing Eq(2.15) with Eq(2.17) we obtain:

$$ab = \frac{2\beta}{k - c^2},\ ,$$

$$a^2 - b^2 = \frac{4\alpha}{k - c^2}, k \succ c^2, a \neq b, \tag{2.18}$$

So that

$$a = \frac{2\beta}{\sqrt{2\gamma(k - c^2)}}, k \succ c^2, a \succ \beta, \tag{2.19}$$

$$b = \frac{2\gamma}{\sqrt{2\gamma(k - c^2)}}, k \succ c^2, \alpha \succ \beta$$

Where γ is given by:

$$\gamma = \sqrt{\alpha^2 + \beta^2} - \alpha. \tag{2.20}$$

Eq.(2.16) is separable, hence we set

$$\frac{1}{a \sin\frac{u}{2} + b \cos\frac{u}{2}} du = d\xi, \tag{2.21}$$

Where by integrating both sides we find the following solutions:

$$\frac{4}{\sqrt{b^2 + a^2}} \operatorname{arctan} h\left(\frac{b \tan\frac{u}{4} - a}{\sqrt{b^2 + a^2}}\right) = \xi + \xi_0, \tag{2.22}$$

$$\frac{4}{\sqrt{b^2 + a^2}} \operatorname{arctan} h\left(\frac{b \cot\frac{u}{4} + a}{\sqrt{b^2 + a^2}}\right) = \xi + \xi_0, \tag{2.23}$$

$$\frac{4}{\sqrt{b^2+a^2}} arc\coth\left(\frac{b\tan\frac{u}{4}-a}{\sqrt{b^2+a^2}}\right)=\xi+\xi_0, \qquad (2.24)$$

Or

$$\frac{4}{\sqrt{b^2+a^2}} arc\coth\left(\frac{b\cot\frac{u}{4}+a}{\sqrt{b^2+a^2}}\right)=\xi+\xi_0, \qquad (2.25)$$

Where ξ_0 is constant of integration. Using Eq(2.19) given the exact

Solutions

$$u(x,t)=4\arctan\left(\frac{2\sqrt[4]{\beta^2+\alpha^2}}{\sqrt{2\gamma}}\tanh\left[\frac{\sqrt[4]{\alpha^2+\beta^2}}{2\sqrt{k-c^2}}((x-ct)+\xi_0)\right]+\frac{\beta}{\gamma}\right), k\succ c^2, \qquad (2.26)$$

$$u(x,t)=4arc\cot\left(\frac{2\sqrt[4]{\beta^2+\alpha^2}}{\sqrt{2\gamma}}\tanh\left[\frac{\sqrt[4]{\alpha^2+\beta^2}}{2\sqrt{k-c^2}}((x-ct)+\xi_0)\right]-\frac{\beta}{\gamma}\right), k\succ c^2, \qquad (2.27)$$

$$u(x,t)=4\arctan\left(\frac{2\sqrt[4]{\beta^2+\alpha^2}}{\sqrt{2\gamma}}\coth\left[\frac{\sqrt[4]{\alpha^2+\beta^2}}{2\sqrt{k-c^2}}((x-ct)+\xi_0)\right]+\frac{\beta}{\gamma}\right), k\succ c^2, \qquad (2.28)$$

$$u(x,t)=4arc\cot\left(\frac{2\sqrt[4]{\beta^2+\alpha^2}}{\sqrt{2\gamma}}\coth\left[\frac{\sqrt[4]{\alpha^2+\beta^2}}{2\sqrt{k-c^2}}((x-ct)+\xi_0)\right]-\frac{\beta}{\gamma}\right), k\succ c^2, \qquad (2.29)$$

Section (2.4): The double Combined Sine – Cosine -Gordon equation [22], [24].

We next consider double combined sine – cosine- Gordon equation

$$u_{tt} - ku_{xx} + \alpha \sin u + \alpha \cos u + \beta \sin(2u) + \beta \cos(2u) = 0 \quad (2.30)$$

That can be converted to the ODE:

$$u^{ll} + \frac{\alpha}{(c^2 - k)} \sin u + \frac{\beta}{(c^2 - k)} \cos u + \frac{\beta}{(c^2 - k)} \sin(2u) + \frac{\beta}{(c^2 - k)} \cos(2u) = 0, k \neq c^2. \quad (2.31)$$

We next assume that u(ξ) satisfies a variable separated ODE given by:

$$u^l(\xi) = \frac{du}{d\xi} = a + \cos u + r \sin u, \quad (2.32)$$

Where a , b and r are parameters that will be determined. Differentiating Eq(2.32) with respect to ξ give:

$$u^{ll}(\xi) + ab \sin u - ar \cos u + \frac{b^2 - a^2}{2} \sin(2u) - br \cos(2u) = 0, \quad (2.33)$$

Comparing Eq(2.31) with Eq(2.33) we obtain

$$ab = \frac{\alpha}{c^2 - k},$$

$$ar = \frac{\alpha}{k - c^2}, \quad (2.34)$$

$$br = \frac{\beta}{k - c^2},$$

$$\frac{b^2 - r^2}{2} = \frac{\beta}{c^2 - k},$$

So that:

$$a = \frac{\alpha}{\sqrt{\beta(c^2 - k)}},$$
$$b = \frac{\beta}{\sqrt{\beta(c^2 - k)}}, \quad (2.35)$$
$$r = \frac{\beta}{\sqrt{\beta(c^2 - k)}},$$

This means that $r = -b$. Eq.(2.32) is separable, hence we s

$$\frac{1}{a + b\cos u - b\sin u} du = d\xi, \quad (2.36)$$

Where by integrating both sides we find the following:

$$\frac{2}{\sqrt{a^2 - b^2}} \arctan\left(\frac{a-b}{\sqrt{a^2 - 2b^2}} \tan\frac{u}{2} - \frac{b}{\sqrt{a^2 - 2b^2}}\right) = \xi + \xi_0, \quad (2.37)$$

$$-\frac{2}{\sqrt{2b^2 - a^2}} \arctan h\left(\frac{a-b}{\sqrt{2b^2 - a^2}} \tan\frac{u}{2} - \frac{b}{\sqrt{2b^2 - a^2}}\right) = \xi + \xi_0,$$

$$\frac{2}{\sqrt{a^2 - b^2}} arc\cot\left(\frac{a-b}{\sqrt{a^2 - 2b^2}} \tan\frac{u}{2} - \frac{b}{\sqrt{a^2 - 2b^2}}\right)$$

(2.38)

(2.39)

$$-\frac{2}{\sqrt{b^2 - a^2}} arc\coth\left(\frac{a-b}{\sqrt{2b^2 - a^2}} \tan\frac{u}{2} - \frac{b}{\sqrt{2b^2 - a^2}}\right) = \xi + \xi_0, \quad (2.40)$$

Where ξ is constant of integration. This in turn gives the exact solutions.

$$u(x,t) = 2\arctan\left(\frac{\sqrt{\alpha^2 - 2\beta^2}}{\alpha - \beta}\left[\cot(\frac{1}{2}\frac{\sqrt{\alpha^2 - 2\beta^2}}{\sqrt{\beta(c^2 - k)}}((x - ct) + \xi_0)\right] + \frac{\beta}{\alpha - \beta}\right), \alpha^2 \succ 2\beta^2, \quad (2.41)$$

$$u(x,y) = 2\arctan\left[\frac{\sqrt{\alpha^2 - 2\beta^2}}{\alpha - \beta}\left(\cot(\frac{1}{2}\frac{\sqrt{\alpha^2 - 2\beta^2}}{\sqrt{\beta(c^2 - k)}}((x - ct) + \xi_0) + \frac{\beta}{\alpha - \beta}\right)\right], \alpha^2 \succ 2\beta^2 \quad (2.42)$$

$$u(x,t) = -2\arctan\left[\frac{\sqrt{\alpha^2 - 2\beta^2}}{\alpha - \beta}\left(\tanh(\frac{1}{2}\frac{\sqrt{\alpha^2} - 2\beta^2}{\sqrt{\beta(c^2 - k)}}(x - ct) + \xi_0)\right) - \frac{\beta}{\alpha - \beta}\right], 2\beta^2 \succ \alpha^2 \quad (2.43)$$

And

$$u(x,t) = -2\arctan\left(\frac{\sqrt{\alpha^2 - 2\beta^2}}{\alpha - \beta}\left[\coth\left(\frac{1}{2}\frac{\sqrt{\alpha^2 - 2\beta^2}}{\sqrt{\beta(c^2 - k)}}((x - ct) + \xi_0)\right] - \frac{\beta}{\alpha - \beta}\right), 2\beta^2 \succ \alpha^2, \quad (2.44)$$

Chapter (3)

The Traveling Wave solution for The MKdv – sine Gordon and the MKdv – sinh – Gordon equation:

Section (3.1): Introduction

The sine – Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0, \qquad (3.1)$$

Appeared first in study of differential geometry of surface [17], [24], with Gaussian curvature k = -1 . The sine – Gordon equation is a completed integrable infinite dimensional Hamiltonian system, [23]. And it can be solved by the invers scattering method. integrable means that there is a sufficiently large number of conserved quantities [7],[23] .This equation can be used in interpreting biological concepts such as DNA dynamic [1], [30].

The sine – Gordon equation (3.1) appears in the propagation of fluxions in Josepgson junctions [17], [18], [19], [20], [21], [22], between two superconductors. It also appears in many scientific fields such as the motion of a rigid pendula a attached to a stretched wire [19], [4]. Solid state physic optics stability of fluid motions and dislocations in crystals where sine u is due to periodic structure of rows of atoms [4], [19]. The term sin u is the Josphson current across an insulator between two superconductors [1], [2].

The sinh – Gordon equation

$$u_{tt} - u_{xx} + \sinh\ u = 0, \qquad (3.2)$$

Appears in integral quantum filed theory kink dynamics, and fluid dynamic [18], the sinh – Gordon equation is completely integrable because it possesses similarity reduction to third Painleve equation. The **Kdv** equation, [23].

$$u_t + \alpha u u_x + u_{xxx} = 0, \qquad (3.3)$$

And the modified **Kdv** equation

$$u_t + \alpha u^2 u_x + u_{xxx} = 0, \qquad (3.4)$$

Are the leading equations that paved the way for the development of the solitary wave theory? Kdv equations are classical paradigm of integrable non linear evolution equation that arises in many physical phenomena ion as ion – acoustic waves in plasma s and surface water waves.

It is well – know that searching for explicit solution for nonlinear evolution equation by using different methods is the goal for many researchers. Many powerful methods such as Backend transformation inverse scattering method Hirota bilinear forms pseudo spectral method the $\tanh - \sec h$ method, [9],[13] the sine – cosine method [14] and many other techniques were successfully used to investigate these types of equations practically.

There is no unified method that can be used to handle all types of nonlinear problems. in this work we aim to investigate the modified Kdv – sine – Gordon

(MKdv – sine – Gordon) equation and the

modified Kdv – sine – Gordon equation (MKdv – sine – Gordon) given by:

$$u_{xt} + \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} = \sin u, \tag{3.5}$$

And

$$u_{xt} - \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} = \sinh u, \tag{3.6}$$

Respectively. The MKdv - sine – Gordon equation describes nonlinear wave propagation in one – dimensional mono – atomic lattice in which the an harmonic potential competes with the dislocation potential and which can be solved by the inverse scattering transform method, [1] . Sirendaoreji and Jing, [1] investigate only the modified Kdv – sine – Gordon (MKdv – sine – Gordon) equation, (3.5).

The objectives of this work are two folds. The first goal is to extend the study presented in [1] to drive more exact solitary wave solution in addition to that obtained in [1]. Secondly we aim to investigate the modified Kdv – sinh – Gordon equation (MKdv – sinh – Gordon) (6) to formally sole this equation .

As stated before our approach is based on a variable separated ODE method developed by Sirendaoreji et al. in [1] used by fu et al. in [2] and by Wazwaz, [15], [16] . The method has scientific value and reliability. It works effectively if the equation involves sine, cosine, and hyperbolic cosine functions.

In what follows we highlight the main features of the method as introduced in [1] where more details and example can be found in [[1], [2], [15], [16].

Section (3.2): The Variable Separated ODE Method:

As stated before this method was developed by Sirendaoreji et al, [1]. We first unite the in developed variable x and t into one wave variable $\xi = x - ct$ to carry out a PDE into an equivalent ODE. The method depends mainly on assuming that $u(\xi)$ satisfies an additional variable separated ODE given by

$$u^{l} = \frac{du}{d\xi} = G(u), \qquad (3.7)$$

Where G(u) is a suitable function of sine , cosine, hyperbolic cosine . Differentiating (3.7) and equating the coefficients of like terms of the resulting equation with the ODE reduced from the PDE yields a system of algebraic equations that can be solved to determine the unknown parameters. It is worth noting that the variable separated ODE (3.7) can be solved easily by using the method of separation of variables. It should be noted that the selection of G (u) can be performed in more than one choice as will be seen later.

Section (3.3): The MKdv – sine – Gordon equation.

avariable separated ODE method:

We first examine the modified Kdv – sine – Gordon (MKdv – sine – Gordon) equation

$$u_{xt}+\frac{3}{2}u_x^2u_{xx}+u_{xxxx}=\sin u, \quad (3.8)$$

That by using the wave variable $\xi = x - ct$ can be covert to the ODE

$$-cu^{ll}+\frac{3}{2}(u^l)^2u^{ll}+u^{(iv)}=\sin u. \quad (3.9)$$

To achieve our gold we can select $u^l(\xi)=G(u)$ in two forms.

First Selection of G (u):

Following [1] we assume that $u(\xi)$ satisfies the variable separated ODE given by

$$u^l(\xi)=\frac{du}{d\xi}=a\cos\frac{u}{2}, \quad (3.10)$$

Where a is an arbitrary constant. Differentiating Eq(3.10) with respect to ξ we obtain:

$$u^{ll}(\xi)=-\frac{a^2}{4}\sin u,$$

$$u^{lll}=-\frac{a^2}{4}\cos\frac{u}{2}\cos u, \quad (3.11)$$

$$u^{(iv)}(\xi)=\frac{3a^4}{16}\sin u\cos u+\frac{a^4}{8}\sin u,.$$

Substituting Eq(3.10) and Eq(3.11) into Eq(3.9) gives:

$$(a^4-4ca^2+16)\sin u=0, \quad (3.12)$$

That gives

$$c=\frac{a^4+16}{4a^2}, \quad (3.13)$$

Where a is an arbitrary constant. The result Eq(3.13 is consistent with The result in [1]:

Eq.(3.10) is separable hence we set

$$\frac{du}{\cos\frac{u}{2}} = ad\xi \tag{3.14}$$

Where by integrating both sides we find the following solution:

$$4\arctan h\left[\tan(\frac{u}{4})\right] = a(\xi - \xi_0), \tag{3.15}$$

$$4arc\coth\left[\tan(\frac{u}{4})\right] = a(\xi - \xi_0), \tag{3.16}$$

$$4\arctan h\left[\cot(\frac{u}{4})\right] = a(\xi - \xi_0), \tag{3.17}$$

$$4arc\coth\left[\cot(\frac{u}{4})\right] = a(\xi - \xi_0), \tag{3.18}$$

$$2\ln\left[\sec(\frac{u}{2}) + \tan(\frac{u}{2})\right] = a(\xi - \xi_0), \tag{3.19}$$

Where ξ_0 is a constant of integration. It is important to note that the last solution obtained above for u(x,t) is an implicit . Combining Eq(3.13) with the previous result leads to the following explicit solitary wave solution:

$$u_1(x,t) = 4\arctan\left[\tanh(\frac{a}{4}(x - \frac{a^4+16}{4a^2}t - \xi_0))\right], \tag{3.20}$$

$$u_2(x,t) = 4\arctan\left[\coth(\frac{a}{4}(x - \frac{a^4+16}{4a^2}t - \xi_0))\right], \tag{3.21}$$

$$u_3(x,t) = 4arc\cot\left[\tanh(\frac{a}{4}(x - \frac{a^4+16}{4a^2}t - \xi_0))\right], \qquad (3.22)$$

$$u_4(x,t) = 4arc\cot\left[\coth(\frac{a}{4}(x - \frac{a^4+16}{4a^2}t - \xi_0))\right], \qquad (3.23)$$

$$u_5(x,t) = 4\arctan\left[\exp(\frac{a}{4}(x - \frac{a^4+16}{4a^2}t - \xi_0))\right] - \pi. \qquad (3.24)$$

Second Selection Of G(u):

Following [1], we assume that $u(\xi)$ satisfies the variable separated ODE given by:

$$u^{l}(\xi) = \frac{du}{d\xi} = a\sin\frac{u}{2}, \qquad (3.25)$$

Differentiating Eq(3.25) with respect to ξ we obtain

$$u^{ll}(\xi) = -\frac{a^2}{4}a\sin u \qquad (3.26)$$

Substituting Eq(3.25) and Eq(3.26) into Eq(3.9) gives:

. $$(a^4 - 4ca^2 - 16)\sin u = 0, \qquad (3.27)$$

That gives

$$c = \frac{a^2 - 16}{4a^2}, \qquad (3.28)$$

Eq.(3.25) is separable hence we set:

$$\frac{du}{\sin\frac{u}{2}} = ad\,\xi \qquad (3.29)$$

Where by integrating both sides we find the following solution:

$$2\ln\left[\tan(\frac{u}{4})\right] = a(\xi - \xi_0), \tag{3.30}$$

$$-2\ln\left[\tan(\frac{u}{4})\right] = a(\xi - \xi_0), \tag{3.31}$$

And

$$2\ln\left[\csc(\frac{u}{2}) - \cot(\frac{u}{2})\right] = a(\xi - \xi_0), \tag{3.32}$$

Where ξ_0 is a constant of integration It is important to note that the last solution obtained above for u(x,t) is an implicit . Combining Eq(3.28) with the previous result leads to the following solitary wave solution:

$$u_6(x,t) = 4\arctan\left[\exp(\frac{a}{2}(x - \frac{a^4-16}{4a^2}t - \xi_0))\right], \tag{3.33}$$

And

$$u_7(x,t) = 4arc\cot\left[\exp(-\frac{a}{2}(x - \frac{a^4-16}{4a^2}t - \xi_0))\right], \tag{3.34}$$

Section (3.4): The MKdv – sinh – Gordon equation

We next examine the modified Kdv – sinh – Gordon (MKdv – sinh– Gordon) equation

$$u_{xt} - \frac{3}{2}u_x u_{xx} + u_{xxxx} = \sinh\ u, \tag{3.35}$$

That by using the wave variable $\xi = x - ct$ can be covert to the ODE

$$-cu^{ll} - \frac{3}{2}(u^{l})^{2}u^{ll} + u^{(iv)} = \sinh u, \qquad (3.36)$$

We can select $u^{l}(\xi) = G(u)$ in two forms as used before. We believe that this equation has not been examined before.

First select of G(u):

We assume that $u(\xi)$ satisfies the variable separated ODE given by

$$u^{l}(\xi) = \frac{du}{d\xi} = a\cosh\frac{u}{2}, \qquad (3.37)$$

Differentiating Eq(3.37) with respect to ξ we obtain

$$u^{ll}(\xi) = \frac{a^{2}}{4}\sinh u$$

$$u^{lll}(\xi) = \frac{a^{3}}{4}\cosh u \cosh\frac{u}{2}, \qquad (3.38)$$

$$u^{iv}(\xi) = \frac{3a^{4}}{16}\sinh u \cosh u + \frac{a^{4}}{8}\sinh u$$

Substituting Eq(3.36) and Eq(3.37) into Eq(3.35) gives:

. $$(a^{4} + 4ca^{2} + 16)\sinh u = 0, \qquad (3.39)$$

That gives

$$c = -\frac{a^{4} + 16}{4a^{2}}, \qquad (3.40)$$

Eq.(3.37) is separable hence we set:

$$\frac{1}{\cosh(\frac{u}{2})}du = ad\xi, \qquad (3.41)$$

Where by integrating both sides we find the following solution:

$$4\arctan\left[\tanh(\frac{u}{4})\right] = a(\xi - \xi_0), \qquad (3.42)$$

$$-4\arctan\left[\coth(\frac{u}{4})\right] = a(\xi - \xi_0), \qquad (3.4٣)$$

$$-4arc\cot\left[\tanh(\frac{u}{4})\right] = a(\xi - \xi_0), \qquad (3.4٤)$$

$$4arc\cot\left[\coth(\frac{u}{4})\right] = a(\xi - \xi_0),$$

(3.45)

$$4\arctan\left[\exp(\frac{u}{2})\right] = a(\xi - \xi_0),$$

(3.46)

$$-4arc\cot\left[\exp(\frac{u}{2})\right] = a(\xi - \xi_0), \qquad (3.4٧)$$

Where ξ_0 is a constant of integration Combining Eq(3.4٠) with the previous results leads to the following solution:

$$u_1(x,t) = 4aec\tanh[\tan(\frac{a}{4}(x + \frac{a^2+16}{4a^2}t - \xi_0))] \qquad (3.4٨)$$

$$u_2(x,t) = 4aec\coth[\tan(-\frac{a}{4}(x - \frac{a^2+16}{4a^2}t - \xi_0))] \qquad (3.٤٩)$$

$$u_3(x,t) = 4aec\tanh[\cot(\frac{a}{4}(x + \frac{a^2+16}{4a^2}t - \xi_0))] \qquad (3.5٠)$$

$$u_4(x,t)=4aec\coth[\cot(\frac{a}{4}(x+\frac{a^2+16}{4a^2}t-\xi_0))] \qquad (3.5١)$$

$$u_5(x,t)=2\ln[\tan(\frac{a}{4}(x+\frac{a^2+16}{4a^2}t-\xi_0))] \qquad (3.5٢)$$

$$u_6(x,t)=2\ln[\cot(\frac{a}{4}(x+\frac{a^2+16}{4a^2}t-\xi_0))] \qquad (3.5٣$$

Second selection of G (u):

We next assume that u (ξ) satisfies the variable separated ODE given by:

$$u^{l}(\xi)=\frac{du}{d\xi}=a\sinh\frac{u}{2}, \qquad (3.5٤)$$

Where a is parameter that will determine Differentiating Eq(3.54) with respect to ξ we obtain:

$$u^{ll}(\xi)=\frac{a^2}{4}\sinh u$$

$$u^{lll}(\xi)=\frac{a^3}{4}\sinh\frac{u}{2}\cosh u, \qquad (3.5٥)$$

$$u^{iv}(\xi)=\frac{3a^4}{16}\sinh u\cosh u-\frac{a^4}{8}\sinh u$$

Substituting Eq(3.5٤) and Eq(3.5٥) into Eq(3.3٦) gives.

. $$(a^4-4ca^2-16)\sin u=0, \qquad (3.5٦)$$

This gives

$$c=\frac{a^2-16}{4a^2} \qquad (3.5٧)$$

Eq.(3.5٤) is separable hence we set:

$$\frac{1}{\sinh \frac{u}{2}} du = a d\xi , \qquad (3.5٨)$$

Where by integrating both sides we find the following solutions:

$$2 \ln\left[\tanh(\frac{u}{4}) \right] = a(\xi - \xi_0), \qquad (3.٥٩)$$

$$-2 \ln\left[\coth(\frac{u}{4}) \right] = a(\xi - \xi_0), \qquad (3.60)$$

And

$$2 \ln\left[\cosh(\frac{u}{2}) - \coth(\frac{u}{2}) \right] = a(\xi - \xi_0), \qquad (3.61)$$

Where ξ_0 is a constant of integration It is important to note that the last solution obtained above for u(x,t) is an implicit . Combining Eq(3.57) with the previous result leads to the following solitary wave solution

$$u_1(x,t) = 4 \text{ arctan } h[\exp(\frac{a}{4}(x - \frac{a^4 - 16}{4a^2} t - \xi_0))] \qquad (3.62)$$

And

$$u_2(\xi) = 4 arc \coth\left[\exp(\frac{a}{4}(x - \frac{a^4 - 16}{4a^2} t - \xi_0)) \right] \qquad (3.63)$$

Chapter (4)

Exact Traveling Wave Solutions for the Generalized Nonlinear Schrödinger Equation

Section (4.1): Introduction.

The nonlinear partial differential equation (NPDEs) are widely used to describe many important phenomena and dynamic process in physics chemistry biology , fluid dynamic , plasma , optical fibers and other areas of engineering[11],[12],[13] . Many efforts have been made to study NPDEs. One of the most exciting advances of nonlinear science and theoretical physics has been development of methods that look for exact solution for nonlinear evolution equations [16],[26]. The availability of dynamic computations such as Mathematic or maple has popularized direct seeking for exact solutions of nonlinear equations [5], [8], [11], [16].

Therefore, exact solution methods of nonlinear evolution equations have become more and more important resulting in methods like variation iteration method, Homotopy perturbation method,[6] .Exp – function method ,the sine – cosine method, the homogenous balance method, $\tanh - \sec h$ method and Extended $\tanh - \coth$ method,[30]. Most of exact solution has been obtained by these methods, including the solitary wave solution, shock wave solutions periodic wave solutions and the like.

In this chapter. We propose Extended $\tanh - \coth$, sine – cosine and Exp – function methods to obtain an exact single – solution and travelling wave solutions of the generalized nonlinear Schrödinger (GNLS) equation with a source [30]. In order to illustrate the effectiveness and convenience of these methods, we consider the GNLS equation in the form [13],[14],[30],

$$iu_t + au_{xx} + bu|u|^2 + icu_{xxx} + id(u|u|^2) = ke^{i(x(\xi)-wt)} \tag{4.1}$$

Where $\xi = \alpha(x - vt)$ is a real function and a, b, c, d, k, α, v and w are all real. The GNLS equation (1.1) plays an important role in many nonlinear sciences. It arises as an asymptotic limit for a slowly varying dispersive wave envelope in nonlinear medium. For example, its signification in optical solution communication plasma physics has been proved [5], [21], [24].

Furthermore the GNLS equation enjoys remarkable (e.g., bright and solution, lax pair Liouvile inerrability, inverse scattering transformation conservation laws Backland transformation, etc)[3],[8],[11].

The rest of this chapter is as follow: we provide in simple way the mathematical framework of Extended $\tanh - \coth$, sine – cosine and Exp – function methods, respectively. In order to illustrate the application of these methods, generalized nonlinear Schrödinger equation with a source in investigated, and several exact solutions, including Soliton like solutions and trigonometric functions solution, are obtained [30].

Section (4.2): tanh Method and Extended tanh method:

We now describe the tanh method for a given partial differential equation. This method was defined by Malfliet and Fan and Han.

Wazwaz summarized the main steps of using this method as follows, [10], [11], [12], [13], [14].

(i) .Wazwaz first considered a general form of nonlinear equation:

$$N(u, u_t, u_x, u_{xx}, ...) = 0 \tag{4.2}$$

(ii) . To find the travelling wave solution of Eq.(3.1), he introduced the wave variable:

$$\xi = k(x + \lambda t), \tag{4.3}$$

So that:

$$u(x,t) = U(\xi), \tag{4.4}$$

Therefore Eq.(4.1) constructs ODE of form:

$$N(U, K\lambda U^{/}, KU^{/}, K^2 U^{//}, ...) = 0 \tag{4.5}$$

(iii) .If all terms of the resulting ODE contain derivation in ξ , then by integrating this equation, and by considering the constant of integration to be zero, we obtain simplified ODE.

(iv) . Introducing a new independent variable:

$$Y = \tanh(\xi) \qquad or \ (Y = \coth(\xi)) \tag{4.6}$$

Leads to a change in the derivatives:

$$\frac{d^2}{d\xi^2}\left(1-Y^2\right)\left(-2Y\frac{d}{dY}+\left(1-y^2\frac{d^2}{dy^2}\right)\right), \qquad (4.7)$$

$$\frac{d^3}{d\xi^3}\left(1-Y^2\right)\left((6Y^2-2)\frac{d}{dY}-6Y(1-Y^2)\frac{d^2}{dY^2}+\left((1-Y^2)^2\frac{d^3}{dy^3}\right)\right),$$

And the remaining derivatives may be derived similarly.

(v). Introduce the answer and then solution of $U(\xi)$ is in the form of:

(a) tanh Method:

$$U(\xi)=\sum_{p=0}^{m}a_pY^p=a_0+\dots+a_mY^m. \qquad (4.8)$$

(b) Extended tanh method:

$$U(\xi)=\sum_{p=-m}^{m}a_pY^p=a_{-m}Y^{-m}+\dots+a_0+\dots+a_mY^m. \qquad (4.9)$$

Where m is appositive integer which is unknown to be later determined and a_p are unknown constants.

(vi) .To determine the parameter m, we usually balance linear terms of the highest order in the resulting equation with the highest order nonlinear terms. With m, determined as described by balancing the coefficients with the same power in the resulting equation, a system of algebraic equations involving the a_p (p = -m… 0… m) k and λ is derived.

Section (4.3): Sine – cosine method

Wazwaz has summarized the main steps of using sine – cosine method, as listed below:

(i). Introducing the wave variable $\xi = x - ct$ in to the PDE the following function is obtained;

$$\phi\,(u\,,u_t,u_x,u_{tt},u_{xx},u_{xt},u_{xxx}\cdots) = 0 \qquad (4.10)$$

Where $u(x,t)$ is travelling wave solution. This allows the following changes:

$$u(x,t) = U(\xi), \qquad (4.11)$$

$$\frac{\partial}{\partial t} = -c\frac{d}{d\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2\frac{d^2}{d\xi^2},$$

$$\frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2},\ldots \qquad (4.12)$$

And so for the other derivates.Using Eqs.(4.3) and Eq(4.1), the nonlinear PDE Eq(4.1) is changed to a nonlinear ODE:

$$N\,(U\,,-cU^{I},c^2U^{II},U^{II},-cU^{I},U^{III}\ldots) = 0 \qquad (4.13)$$

(ii) If all terms of the resulting ODE contain derivative of ξ, then by intenerating this equation and considering the constant of integration zero, a simplified ODE is obtained.

(iii) By virtue of this solution the answer is introduced as:

$$U(\xi)=u(x,t)=\lambda \sin^{\beta}(\mu\xi), \qquad |\mu\xi| \prec \frac{\pi}{2} \tag{4.14}$$

Or

$$U(\xi)=u(x,t)=\lambda \cos^{\beta}(\mu\xi), \qquad |\mu\xi| \prec \frac{\pi}{2\mu} \tag{4.15}$$

Where λ, μ and β are parameters to be determined later. μ and C is the wave number and the wave speed, respectively. By sequence differentiating on the power of Esq.(3.5) and Eq(3.6) with respect to ξ the following equations are attained:

$$\begin{aligned}
&U(\xi)=\lambda \sin^{\beta}(\mu\xi),\\
&U^{n}(\xi)=\lambda^{n}\sin^{n\beta}(\mu\xi),\\
&(U^{n})_{\xi}=n\mu\beta\lambda^{n}\cos(\mu\xi)\sin^{n\beta-1}(\mu\xi),\\
&(U^{n})_{\xi\xi}=-n^{2}\mu^{2}\beta^{2}\lambda^{n}\sin^{n\beta}(\mu\xi)+n\mu^{2}\lambda^{n}\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi),\\
&(U^{n})_{\xi\xi\xi}=n\lambda^{n}\mu^{3}\beta(n\beta^{2}-3\beta+2)\sin^{n\beta-3}(\mu\xi)\cos^{3}(\mu\xi)\\
&\quad+\lambda^{n}\mu^{3}n\beta(3\beta n-2)\sin^{n\beta-1}(\mu\xi)\cos(\mu\xi),
\end{aligned} \tag{4.16}$$

And

$$\begin{aligned}
&U(\xi)=\lambda \cos^{\beta}(\mu\xi),\\
&U^{n}(\xi)=\lambda^{n}\cos^{n\beta}(\mu\xi),\\
&(U^{n})_{\xi}=-n\mu\beta\lambda^{n}\sin(\mu\xi)\cos^{n\beta-1}(\mu\xi),
\end{aligned} \tag{4.17}$$

$$(U^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi),$$

$$(U^n)_{\xi\xi\xi} = n\lambda^n\mu^3\beta(-n\beta^2+3n\beta-2)\cos^{n\beta-3}(\mu\xi)\sin^3(\mu\xi)$$

$$+\lambda^n\mu^3 n\beta(3\beta n-2)\sin^{n\beta-1}(\mu\xi)\sin(\mu\xi),$$

And so for the other derivatives:

(iv) After substituting Eq.(4.7) or Eq. (4.8) into the reduced Eq.(4.4) obtained above by equating the two sides of the equation a system of algebraic equations is obtained which can be solved with computerized Symbolic calculation.

Section (4.4): Summary of Exp – Function method.

In addition, this method was successfully applied to Kdv equation with variable coefficients high – dimensional nonlinear evolution equation Burgers and combine Kdv – m Kdv (Extended Kdv) equations etc.

In this section for a given PDE we commence by looking for an Exp – function type solution of the following form in terms of exp(ξ):

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)} = \frac{a_c \exp(c\xi)+\ldots+a_{-d}\exp(-d\xi)}{a_p \exp(p\xi)+\ldots+a_{-q}\exp(-q\xi)} \tag{4.18}$$

Where c, d, p and q are positive integers which are the unknowns to be later determined a_n and b_m are unknowns constant.

Alternatively, we can also assume that the solution can be expressed in this form:

$$U(\xi) = \sum_{j=1}^{n} a_j \varphi^j \qquad (4.19)$$

Where φ is the solution of the sub-equation $\varphi^{/} = r + p\varphi + q\varphi^2$

Section (4.5): Exact Solution Of GNLS Equation With a source

Using extended $\tanh$ method

To study the exact travelling wave solutions of the GNLS Eq.(4.1), we consider a plane wave transformation in this form:

$$u(x,t) = \Psi(\xi) e^{i|x(\xi) - wt|} \qquad (4.20)$$

Where $\Psi(\xi)$ is a real function for convenience let $\chi = \beta\xi + x_0$ where β and x_0 are real constants and $\xi = \alpha(x - vt)$. Then by replacing Eq. (4.1) and its appropriate derivative in Eq.(4.1) and separating the real and imaginary parts of the result we obtain the two following ordinary differential equations:

$$c\alpha^3\psi^{///} + (-\alpha v + 2a\beta\alpha^2 - 3c\alpha^3\beta^2)\psi^{/} 3d\alpha\psi^2\psi^{/} = 0 \qquad (4.21)$$

$$(\alpha^2\psi^{///} - 3c\psi^3\beta)\psi^{//} + (\alpha\phi v + w - a\alpha^2\beta^2 + c\alpha^3\beta^3)\psi + (b - d\alpha\beta)\psi^3 - k = 0 \qquad (4.22)$$

Integrating Eq.(4.2) once with respect to ξ yields:

$$c\alpha^2\psi^{//}(\xi) + p\psi(\xi) + d\psi^3(\xi) - c = 0 \qquad (4.23)$$

Where $p = -v + 2a\beta\alpha - 3c\alpha^2\beta^2$ and C is an integration constant. Since the same function $\psi(\xi)$ satisfies two Eqs.(4.3) and Eq(5.4) we obtain the following constraint condition:

$$\frac{(a\alpha^{2}-3c\psi^{3}\beta)}{c\alpha^{2}}=\frac{(\alpha\beta\, v+w-a\alpha^{2}\beta^{2}+c\alpha^{3}\beta^{3})}{p}=\frac{(b-d\alpha\beta)}{d}=\frac{k}{c} \qquad (4.24)$$

Our main purpose is solving Eq.(4.4). Considering the homogenous balance between $\psi''(\xi)$ and $\psi^{3}(\xi)$ in Eq.(4.4) yields m=1. We suppose that the solution

of Eq.(4.4) can be expressed by:

$$\psi(\xi)=\sum_{p=-1}^{1}a_{p}Y^{p}=a_{-1}Y^{-1}+a_{0}+Y^{1} \qquad (4.25)$$

Where $Y=\tanh(\xi)$ or $Y=\coth(\xi)$, substituting Eq. (4.6) into Eq. (4.4) and then setting the coefficients of all independent terms in Y, the following algebraic relations are obtained:

$$d\,b_{1}^{3}+2c\alpha^{2}b_{1}=0\ ,$$

$$3d\,b_{1}^{2}a_{0}=0$$

$$3d\,a_{1}b_{1}^{2}+3db_{1}a_{0}^{2}-2c\,\alpha^{2}b_{1}+2a\alpha\beta\,b_{1}-3c\alpha^{2}\beta^{2}b_{1}-vb_{1}=0 \qquad (4.26)$$

$$-c+2a\alpha\beta\,a_{0}-va_{0}-3c\alpha^{2}a_{0}+6da_{1}b_{1}a_{0}+da_{0}^{3}=0$$

$$-2c\alpha^{2}+2a\alpha\beta\,a_{1}+3d\,a_{1}a_{0}^{2}-3c\alpha^{2}\beta^{2}a_{1}+3d\,a_{1}^{2}\,b_{1}-va_{1}=0$$

$$3d\,a_{1}^{2}a_{0}=0\,,$$

$$2c\alpha^{2}+da_{1}^{3}=0\,,$$

The above equations are cumbersome to solve. Using a modern computer algebra system, say maple gives:

Case1:

$$a_1 = 0, a_0 = 0, b_1 = b_1, \alpha = \alpha, \beta = \beta, a = a, c = c \qquad (4.27)$$

$$d = -\frac{2c\alpha^2}{a_1^2}, v = -2c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2, c = 0$$

Case2:

$$a_1 = a_1, a_0 = 0, b_1 = 0, \alpha = \alpha, \beta = \beta, a = a, c = c \qquad (4.28)$$

$$d = -\frac{2c\alpha^2}{a_1^2}, v = -2c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2, c = 0$$

Case 3:

$$a_1 = -b_1, a_0 = 0, b_1 = b_1, \alpha = \alpha, \beta = \beta, a = a, c = c, d = -\frac{2c\alpha^2}{b_1^2} \qquad (4.29)$$

$$v = 4c\alpha^2 + 2a\alpha\beta - 3\alpha^2\beta^2, c = 0$$

Case 4:

$$a_1 = b_1, a_0 = 0, b_1 = b_1, \alpha = \alpha, \beta = \beta, a = a, c = c. \qquad (4.30)$$

$$d = -\frac{2c\alpha^2}{b_1^2}, v = -8c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2, c = 0.$$

Substituting Eq.(4.8) and Eq(4.11) into Eq.(4.6), we obtain the following exact solution for Eq.(4.1).

$$\psi_1(\xi) = b_1 \coth(\xi), \quad \xi = \alpha(x - (-2c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.31)$$

$$\psi_2(\xi) = a_1 \tanh(\xi), \quad \xi = \alpha(x - (-2c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.32)$$

$$\psi_3(\xi) = -b_1(\coth(\xi) - \coth(\xi)), \quad \xi = \alpha(x - (4c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.33)$$

$$\psi_4(\xi) = b_1(\coth(\ \xi\ ) + \coth(\ \xi\ )), \quad \xi = \alpha(x - (-8c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.34)$$

Or

$$u_1(\xi) = b_1\coth(\ \xi\ )\times e^{i(\beta\xi - wt + x_0)}, \quad \xi = \alpha(x - (-2c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.35)$$

$$u_2(\xi) = a_1\tanh(\ \xi\ )\times e^{i(\beta\xi - wt + x_0)}, \quad \xi = \alpha(x - (-2c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.36)$$

$$u_3(\xi) = -b_1(\tanh(\ \xi\ ) - \coth(\ \xi\ )) \times e^{i(\beta\xi - wt + x_0)}, \qquad (4.37)$$
$$\xi = \alpha(x - (4c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t).$$

$$u_4(\xi) = b_1(\tanh(\ \xi\ ) + \coth(\ \xi\ ))\times e^{i(\beta\xi - wt + x_0)},$$
$$\xi = \alpha(x - (-8c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.38)$$

Defining α and β as two imaginary numbers; the obtained solitary solution can be converted into periodic solution. Therefore we define:

$$\alpha = iA, \beta = iB$$

Where $i = \sqrt{-1}$ and then apply the following transformations:

$$\sinh(\ i\xi\ ) = i\sinh(\ \xi\ ), \quad \cosh(i\xi) = \cos(\xi), \quad \tanh(i\xi) = i\tan(\xi), \qquad (4.39)$$
$$\coth(i\xi) = -i\cot(\xi), \quad \mathrm{sech}(i\xi) = \sec(\xi), \quad \mathrm{csch}(i\xi) = -i\csc(\xi),$$

In Eq.(4.12) and Eq.(4.19), the results are:

$$\psi_1(x,t) = -ib_1\cot(\ Ax\ - (2cA^3 - 2aA^2B - 3cA^3B^2)t), \qquad (4.40)$$

$$\psi_2(x,t) = ia_1 \tan(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t), \qquad (4.41)$$

$$\psi_3(x,t) = -ib_1(\tan(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t) + \cot(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t)) \ldots (4.42)$$

$$\psi_4(x,t) = ib_1(\tan(Ax + (-8cA^3 + 2aA^2B + 3cA^3B^2)t) - \cot(Ax + (-8cA^3 + 2aA^2B + 3cA^3B^2)t)) \ldots (4.43)$$

Or

$$u_1(x,t) = -ib_1 \cot(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t) \times e^{i\left(BA(-x + (2aAB - 3cA^2B^2)t) - wt + x_0\right)},$$

$$u_2(x,t) = ia_1 \tan(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t) \times e^{i\left(BA(-x + (2aAB - 3cA^2B^2)t) - wt + x_0\right)},$$

$$u_3(x,t) = -ib(\tan(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t)) + \cot(Ax + (4cA^2B + 3cA^3B^2)t)). (4.44)$$
$$\times e^{i(BA(-x - (4cA^2 + 2a\alpha\beta + 3cA^2B^2)t) - wt + x_0)}$$

$$u_4(\xi) = b_1(\tanh(\xi) + \coth(\xi)) \times e^{i(\beta\xi - wt + x_0)}, \xi = \alpha(x - (-8c\alpha^2 + 2a\alpha\beta - 3c\alpha^2\beta^2)t). \qquad (4.45)$$

Using sine – cosine function methods

In this section, the sine – cosine method is applied to the GNLS equation solution.

Substituting Eq.(4.6) and Eq(3.8) with n = 1 into Eq.(4.4) and rewriting the equation in terms of cosine function gives.

$$c\alpha^2\lambda\beta^2\mu^2\cos(\mu\xi)^{\beta-2} - c\alpha^2\beta^2\lambda\mu^2\cos(\mu\xi)^{\beta} - c\alpha^2\lambda\beta\mu^2\cos(\mu\xi)^{\beta-2} - \lambda v\cos(\mu\xi)^{\beta}$$
$$+ 2\lambda a\alpha\beta\cos(\mu\xi)^{\beta} - 3\lambda c\alpha^2\beta^2\cos(\mu\xi)^{\beta} + d\lambda^3\cos(\mu\xi)^{3\beta} - c = 0 \qquad (4.46)$$

Balancing the terms of the cosine function in Eq.(4.30), we have

$$3\beta = \beta - 2 \Rightarrow \beta = -1$$

Substituting Eq.(4.31) into Eq.(4.4) and equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equation:

$$\cos^{0}(\mu\xi):-c=0 \quad (4.47)$$

$$\cos^{-1}(\mu\xi):-\lambda v+2\lambda a\alpha\beta-3\lambda c\alpha^{2}\beta^{2}-c\alpha^{2}\lambda\mu^{2}=0 \quad (4.48)$$

$$\cos^{-3}(\mu\xi):2c\alpha^{2}\lambda\mu^{2}+d\lambda^{3}=0 \quad (4.49)$$

Solving Eq(4.32) , Eq(4.33) andEq(4.34) equations by Maple we obtain:

$$c=0, v=2a\alpha\beta-3c\alpha^{2}\beta^{2}-c\alpha^{2}\mu^{2}, \lambda=\lambda,$$
$$d=\frac{-2c\alpha^{2}\mu^{2}}{\lambda^{2}}, \mu=\mu, c=c, a=a, \beta=\beta, \alpha=\alpha. \quad (4.50)$$

The results in Eq.(4.35) can be easily obtained if we use the sine method, Eq.(4.5) as well.

Combining Eq.(4.35) and Eq(4.6) the following exact solution will be obtained:

$$\psi(x,t)=\frac{\lambda}{\cos(\mu\alpha(x-(2a\alpha\beta-3\alpha^{2}\beta^{2}-c\alpha^{2}\mu^{2})t))}, \quad (4.51)$$

$$u(x,t)=\frac{\lambda e^{i(\beta\alpha(x-(2a\alpha\beta-3c\alpha^{2}\beta^{2}-c\alpha^{2}\mu^{2})t)-wt+x)}}{\cos(\mu\alpha(x-(2a\alpha\beta-3\alpha^{2}\beta^{2}-c\alpha^{2}\mu^{2})t))}, \quad (4.52a)$$

Eq.(4.37) satisfies Eq.(4.1).In addition using Eq.(4.20) and Eq(4.21) into Eq. (4.37) we have:

$$u(x,t) = \lambda \operatorname{sec}h(\mu A(x + (2aAB + 3cA^2B^2 - cA^2\mu^2)t)) \times e^{i(AB(-x+(-2aAB-3cA^2B^2+cA^2\mu^2)t)-wt+x_0)}$$

(4.52a) this is the exact kink – shaped solitary wave solution of GNLS equation, [12].

Section (4.6): Using Exp –Function method [30].

In this section the exp – Function method is applied to solve the GNLS equation. To determine the value of c and p, we balance the linear term of the highest order with the highest order $\psi^{ll}(\xi)$ nonlinear term in Eq. (4.1). We have:

$$\psi^{ll}(\xi) = \frac{c_1 \exp[(3p+c)\xi] + \dots}{c_2 \exp[4p\xi] + \dots}, \qquad (4.53)$$

$$\psi^3(\xi) = \frac{c_3 \exp[3c\xi] + \dots}{c_4 \exp[3p\xi] + \dots} \times \frac{\exp(p\xi)}{\exp(p\xi)} = \frac{c_3 \exp[(3c+p)\xi] + \dots}{c_4 \exp[4p\xi] + \dots}, \qquad (4.54)$$

Where c_i is determined coefficient only for simplicity balancing the highest order of Exp – function in Esq. (4.38) and (4.39).gives.

$$3p + c = p + 3c \qquad (4.55)$$

So

$$p = c, \qquad (4.56)$$

Similarly to determine value of d and q the linear term of lowest order in Eq. (4.1) is balanced:

And

$$\psi^{ll}(\xi) = \frac{...+d_1 \exp[-(3q+d)\xi]}{...+d_2 \exp[-4q\xi]}, \quad (4.57)$$

And

$$\psi^3(\xi) = \frac{...+d_3 \exp[-(3d+q)\xi]}{...+d_4 \exp[-4d\xi]}, \quad (4.58)$$

Where d_i is determined coefficient only for simplicity, Balancing the highest order of Exp – Function in Eq.(4.43) and Eq(4.42).we have

$$-(3q+d) = -(q+3d) \quad (4.59)$$

So

$$q = d \quad (4.60)$$

Case 1: p = c =1, d = q =1

According to case 1: Eq.(4.1) reduces to

$$\psi(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \quad (4.61)$$

Substituting Eq.(4.46) into Eq(4.4) and using Maple we have

$$\frac{1}{A} \sum_{j=-3}^{3} c_j e^{j\xi} = 0 \quad (4.62)$$

Where

$$A = (b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi))^3$$

And c_n are coefficients of $\exp(n\xi)$. The coefficients of $\exp(n\xi)$ must be zero, therefore we have

$$c_{-3} = -cb_{-1}^3 + 2a\alpha\beta a_{-1}b_{-1}^2 - 3c\alpha^2\beta^2 a_{-1}b_{-1}^2 + da_{-1}^3 - va_{-1}b_{-1}^2 = 0$$

$$c_2 = 2a\alpha\beta a_0 b_1^2 + c\alpha^2 a_0 b_{-1}^2 - va_{-1}b_0 b_{-1} - 3c\alpha^2\beta^2 a_0 b_{-1}^2 + 4a\alpha\beta a_{-1}b_0 b_{-1}$$
$$- c\alpha^2 a_{-1}b_0 b_{-1} - va_0 b_{-1}^2 + 3da_{-1}^2 a_0 - 3cb_0 b_{-1}^2 - 6c\alpha^2\beta^2 a_{-1}b_0 b_{-1} = 0$$

$$c_0 = 4a\alpha\beta a_1 b_0 b_{-1} - cb_0^3 + 2\alpha\beta a_0 b_0^2 + 4a\alpha\beta a_{-1}b_1 b_1 - 2va_0 b_1 b_{-1} - va_0 b_0^2$$
$$+3c\alpha^2 a_0 b_{-1}b_0 - 6c\alpha^2\beta^2 a_1 b_0 b_{-1} - 6c\alpha^2\beta^2 a_0 b_1 b_{-1} + da_0^3 - 2va_1 b_0 b_{-1} - 6cb_1 b_0 b_{-1}$$
$$+3c\alpha^2 a_{-1}b_1 b_0 - 2va_{-1}b_1 b_0 - 6c\alpha^2 a_0 b_1 b_{-1} + 4a\alpha\beta a_0 b_1 b_{-1} - 3c\alpha^2\beta^2 a_0 b_0^2 - 6c\alpha^2\beta^2 a_1 b_0 + 6da_1 a_{-1}a_0 = 0$$

$$c_2 = c\alpha^2 a_0 b_1^2 - 2va_1 b_1 b_0 + 2a\alpha\beta a_0 b_1^2 - 3c\alpha^2\beta^2 a_0 b_1^1 - c\alpha^2 a_0 b_0^2$$
$$- c\alpha^2 a_1 b_1 b_0 + 3da_1^2 a_0 - va_0 b_1^2 - 3cb_1^2 b_0 - 6c\alpha^2\beta^2 a_1 b_1 b_0$$
$$+ 4a\alpha\beta a_1 b_0 b_1 = 0$$

$$c_3 = 2a\alpha\beta a_1 b_1 - 3c\alpha^2\beta^2 a_1 b_1 + da_1^3 - cb_1^3 - va_1 b_1^2 = 0, \tag{4.63}$$

$$a_1 = a_1,\ a_0 = a_0,\ a_{-1} = \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_1^2},\ b_1 = b_1,\ b_0 = b_0,\ b_{-1} = \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1},\ a = a,$$
$$c = \frac{b_1(-a_0 b_1 + a_1 b_0)}{\alpha^2 a_1(a_1 b_0 + a_0 b_1)},\quad d = \frac{-b_1^3 c b_0}{a_1^2(a_1 b_0 + a_0 b_1)},\quad \alpha = \alpha,\quad \beta = \beta,$$
$$v = \frac{2a\alpha\beta a_1^2 b_0 + 2a\alpha\beta a_1 a_0 b_1 + 3b_1^2 c\beta^2 a_0 - 3b_1 c\beta^2 a_1 b_0 - 2cb_1 b_0 a_1 - a_0 b_1^2 c}{a_1(a_1 b_0 + a_0 b_1)}, \tag{4.64}$$

Where $a_0, a_1, b_0,$ and $b_1,$ are arbitrary constant parameters which are determine according to the boundary initial condition.

Substituting this result into Eq(4.46), we obtain the following generalized solitary solutions of Eq.(4.4).

$$\psi(\xi)=\frac{a_1\exp(\xi)+a_0+\dfrac{b_0(a_1b_0+a_0b_1)}{8b_1^2}\exp(-\xi)}{b_1\exp(\xi)+b_0+\dfrac{b_0(a_1b_0+a_0b_1)}{8a_1b_1}\exp(-\xi)}, \qquad (4.65)$$

Or

$$u(\xi)=\frac{a_1\exp(\xi)+a_0+\dfrac{b_0(a_1b_0+a_0b_1)}{8b_1^2}\exp(-\xi)}{b_1\exp(\xi)+b_0+\dfrac{b_0(a_1b_0+a_0b_1)}{8a_1b_1}\exp(-\xi)}\times e^{i(\beta\xi-wt+x_0)}, \qquad (4.66)$$

Where

$$\xi=\alpha(x-\frac{2a\alpha\beta a_1^2b_0+2a\alpha\beta a_1a_0b_1+3b_1^2c\beta^2a_0-3b_1c\beta^2a_1b_0-2cb_1b_0a_1-a_0b_1^2c}{a_1(a_1b_0+a_0b_1)}t).$$

Case: (1.2).

$$a_1=\frac{a_{-1}b_1}{b_{-1}},\quad a_0=\frac{a_{-1}(-b_0^2+8b_1b_{-1})}{b_0b_{-1}},\quad a_{-1}=a_{-1},\ b_1=b_1,\ b_0=b_0,\ b_{-1}=b_{-1},\ a=a,$$

$$\alpha=\alpha,\ \beta=\beta,\quad c=\frac{c(-b_0^2+4b_1b_{-1})}{4\alpha^2b_0a_{-1}},\qquad d=\frac{-b_1^2b_0^2c}{8a_{-1}^3b_1},$$

$$v=\frac{-b_0^1c-8b_1b_{-1}c-6c\beta^2b_0^2+24c\beta^2b_1b_{-1}+16a\alpha\beta a_{-1}b_1}{8a_{-1}b_1} \qquad (4.67)$$

Substituting Eq.(4.53) into Eq(4.46) gives the following solution:

$$\psi(\xi)=\frac{\dfrac{a_{-1}b_1}{b_1}e^{\xi}+\dfrac{a_{-1}(-b_0^2+8+b_{-1}b_1)+a_{-1}e^{-\xi}}{8b_1^2}}{b_1e^{\xi}+b_0+b_{-1}e^{-\xi}}, \qquad (4.68)$$

Or

$$u(\xi) = \frac{\frac{a_{-1}b_1}{b_1}e^{\xi} + \frac{a_{-1}(-b_0^2 + 8 + b_{-1}b_1) + a_{-1}e^{-\xi}}{8b_1^2}}{b_1 e^{\xi} + b_0 + b_{-1}e^{-\xi}} \times e^{i(\beta\xi - wt + x_0)}, \tag{4.69}$$

Where

$$\xi = \alpha\left(x - \frac{-b_0^2 c - 8b_{-1}b_1 c - 6cb_0^2\beta^2 + 24c\beta^2 b_1 b_{-1} + 16a\alpha\beta b_1 a_{-1}}{8a_{-1}b_1}t\right).$$

This is exact solutions of generalized nonlinear Schrödinger (GNLS) equation.

Case2. $p = c, d = q = 2$

As mentioned above the value of c and d can be freely chosen, we set p = c = 2 and d = q = 2, then the trial function, Eq.(4.1), is:

$$\psi(\xi) = \frac{a_2\exp(2\xi) + a_1\exp(\xi) + a_0 + a_{-1}\exp(-\xi) + a_{-2}\exp(-2\xi)}{b_2\exp(2\xi) + b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}, \tag{4.70}$$

By the same operation illustrated above we, obtain:

Case (2.1):

$$a_2 = -\frac{cb_0^2}{8a_{-2}^2},\ a_1 = 0,\ a_0 = \frac{b_0(a_{-2}^3 + c)}{a_{-2}^2},\ a_{-1} = 0,\ a_{-1} = a_{-2},\ b_2 = -\frac{cb_0^2}{8a_{-2}^3},\ b_1 = 0,\ b_0 = b_0,\ b_{-1} = 0,$$

$$b_{-2} = 1,\ a = a,\ c = \frac{2a_{-2}^3 + c}{a_{-2}^2},\ \alpha = \alpha,\ \beta = \beta,\ d = d,$$

$$v = \frac{--4c + 4a_{-2}^3 + 6\beta a_{-2}^3 + 3\beta^2 c + 8a\alpha\beta a_{-2}}{4a_{-2}} \tag{4.71}$$

Substituting Eq.(4.57) into Eq(4.56) gives the following solution:

$$\psi(\xi)=\frac{-\frac{cb_0^2}{8a_{-2}^2}e^{2\xi}-\frac{b_0(a_0^2+c)+}{a_{-2}^2}a_{-2}e^{-2\xi}}{\frac{-cb_0^2}{8a_{-2}^3}e^{2\xi}+b_0+e^{-2\xi}}, \qquad (4.72)$$

Or

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$$u(\xi)=\frac{-\frac{cb_0^2}{8a_{-2}^2}e^{2\xi}-\frac{b_0(a_0^2+c)+}{a_{-2}^2}a_{-2}e^{-2\xi}}{\frac{-cb_0^2}{8a_{-2}^3}e^{2\xi}+b_0+e^{-2\xi}}\times e^{i(\beta\xi-wt+x_0)}, \qquad (4.73)$$

Where

$$\xi=\alpha(x-\frac{-4c+4a_{-2}^3+6ca_{-2}^3\beta^2+3c\beta^2+8a\alpha\beta a_{-2}}{4a_{-2}}t).$$

Case: (2.2):

$$a_2=\frac{b_0(a_0b_{-2}+a_{-2}b_0)}{8b_{-2}^2},\ a_1=0,\ a_0=a_0,\ u_{xx}\ a_{-2}=a_{-2},\ b_2=\frac{b_0(a_0b_{-2}+a_{-2}b_0)}{8a_{-2}b_{-2}},\ b_1=0,\ b_0=b_0,$$

$$b_{-1}=0,\ b_{-2}=b_{-2},\ \alpha=\alpha,\ \beta=\beta,\ a=a,$$

$$c=-\frac{cb(a_0b_{-2}-a_{-2}b_0)}{4a_{-2}\alpha^2(a_0b_{-2}+a_{-2}b_0)},\ d=\frac{cb_{-2}^3b_0}{a_{-2}^2(a_1b_{-2}+a_{-2}b_0)},$$

$$v=\frac{-4a_0b_{-2}^2c-8cb_0a_{-2}+3cb_{-2}^2\beta^2a_0-3cb_{-2}\beta^2a_{-2}b_0+8a\alpha\beta a_0b_{-2}a_{-2}+8a\alpha\beta a_{-2}^2b_0}{4a_{-2}(a_0b_{-2}+a_{-2}b_0)} \qquad (4.74)$$

Substituting Eq.(4.60) into Eq(4.56) gives the following solution:

$$\psi(\xi)=\frac{\frac{b_0(a_1b_0+a_0b_1)}{8b_{-2}^2}e^{2\xi}+a_0+a_{-2}e^{-2\xi}}{\frac{b_0(a_1b_0+a_0b_1)}{8a_1b_1}e^{2\xi}+b_0+b_{-2}e^{-2\xi}}, \qquad (4.75)$$

Or

$$u(\xi)=\frac{\dfrac{b_0(a_1b_0+a_0b_1)}{8b_{-2}^2}e^{2\xi}+a_0+a_{-2}e^{-2\xi}}{\dfrac{b_0(a_1b_0+a_0b_1)}{8a_1b_1}e^{2\xi}+b_0+b_{-2}e^{-2\xi}}\times e^{i(\beta\xi=wt+x_0)}, \tag{4.76}$$

$$\xi=\alpha\left(x-\frac{-4cb_{-2}^2-8b_{-2}b_0ca_{-2}+3cb_{-2}^2\beta^2a_0-3c\beta^2b_0b_{-2}a_{-2}+8a\alpha\beta a_0b_{-2}a_{-2}+8a\alpha\beta a_{-2}^2b_0}{8a_{-2}(a_0b_{-2}+a_{-2}b_0)}t\right).$$

Case: (2.3).

$$a_2=a_2,\; a_1=0,\; a_0=\frac{b_0(a_2^3+c)}{a_2^2},\; a_{-1}=0,\; a_{-2}=-\frac{cb_0^2}{8a_2^2},\; b_1=0,\; b_2=1,\; b_0=b_0,$$

$$b_{-1}=0,\; b_{-2}=-\frac{cb_0^2}{8a_2^3},\; a=a,\; \alpha=\alpha,\; \beta=\beta,\; c=-\frac{2a_2^3+c}{4\alpha a_2},\; d=d,$$

$$v=\frac{8a\alpha\beta a_2-4c+4a_2^3+6\beta^2a_2^3+3\beta^2c}{4a_2} \tag{4.77}$$

Substituting Eq.(4.63) into Eq(4.56) gives the following solution:

$$\psi(\xi)=\frac{a_2e^{2\xi}-\dfrac{b_0(a_2^3+c)+}{a_2^2}-\dfrac{cb_0^2}{8a_2^2}e^{-2\xi}}{e^{2\xi}+b_0-\dfrac{cb_0^2}{8a_2^3}e^{-2\xi}}, \tag{4.78}$$

Or

$$u(\xi)=\frac{a_2e^{2\xi}-\dfrac{b_0(a_2^3+c)+}{a_2^2}-\dfrac{cb_0^2}{8a_2^2}e^{-2\xi}}{e^{2\xi}+b_0-\dfrac{cb_0^2}{8a_2^3}e^{-2\xi}}\times e^{i(\beta\xi-wt+x_0)}, \tag{4.79}$$

Where

$$\xi = \alpha(x - \frac{8a\alpha\beta a_2 - 4c + 4a_2^3 + 6a_2^3\beta^2 + 3c\beta^2}{4a_2}t).$$

Case (2.4):

$$a_2 = a_2,\ a_1 = a_1,\ a_0 = 0,\ a_{-1} = \frac{3a_1^3}{8a_2^2},\quad a_{-2} = \frac{9a_1^4}{64a_2^3},\ b_2 = \frac{2b_0a_2^2}{3a_1^2},\ b_1 = 0,\ b_0 = b_0,$$

$$b_{-1} = 0,\ b_{-2} = \frac{3b_0a_1^2}{32a_2^2},\quad \alpha = \alpha,\ \beta = \beta,\ a = a,\ d = \frac{2cb_0^3a_2^3}{9a_1^6},$$

$$v = \frac{7cb_0a_2 + 12\alpha\beta a_1^2 + 6cb_0a_2\beta^2}{6a_1^2} \tag{4.80}$$

Substituting Eq.(4.69) into Eq(4.56) gives the following solution:

$$\psi(\xi) = \frac{a_2e^{2\xi} + a_1e^{\xi} + \frac{3a_1^3}{8a_2^2}e^{-\xi} + \frac{9a_1^4}{8a_2^2}e^{-2\xi}}{\frac{2b_0a_2^2}{3a_1^2}e^{2\xi} + b_0 - \frac{3b_0a_1^2}{32a_2^2}e^{-2\xi}},$$

Or

$$u(\xi) = \frac{a_2e^{2\xi} + a_1e^{\xi} + \frac{3a_1^3}{8a_2^2}e^{-\xi} + \frac{9a_1^4}{8a_2^2}e^{-2\xi}}{\frac{2b_0a_2^2}{3a_1^2}e^{2\xi} + b_0 - \frac{3b_0a_1^2}{32a_2^2}e^{-2\xi}} \times e^{i(\beta\xi - wt + x_0)},$$

Where

$$\xi = \alpha(x - \frac{7cb_1a_2 + 12a\alpha\beta a_1^2 + 6cb_0a_0\beta^2}{6a_1^2}t)$$

Case: (2.5):

$$a_2=a_2,\ a_1=a_1,\ a_0=\frac{(4da_2^3+3cb_2^3)da_2^2a_1^2}{4(2da_2^2+cb_2^3)^2},\ a_{-1}=-\frac{ca_2b_2^3a_1^3d}{8(2da_2^3+cb_2^3)^2},\ a_{-2}=\frac{a_2^3c^2b_2^6a_1^4d^2}{64(2da_2^3+cb_2^3)^4},$$
$$b_2=b_2,\ b_1=0,\ b_0=-\frac{(4da_2^3+cb_2^3)da_2a_1^2b_2}{4(2da_2^3+cb_2^3)^2},\ b_{-1}=0,\ b_{-2}=\frac{a_2^2c^2b_2^7a_1^4d^2}{64(2da_2^3+cb_2^3)^4},\ \alpha=\alpha,\ a=a,$$
$$\beta=\beta,\quad c=-\frac{2da_2^3+cb_2^3}{\alpha^2a_2b_2^2},\ d=d,\ v=\frac{da_2^3-cb_2^3+2a\alpha\beta a_2b_2^2+6\beta da_2^3+3\beta^2cb_2^3}{b_2^2a_2}, \tag{4.81}$$

Substituting Eq.(4.72) into Eq(4.56) gives the following solution:

$$\psi(\xi)=\frac{\left(a_2e^{2\xi}+a_1e^{\xi}+\frac{(4da_2^3+3cb_2^3)da_2^2a_1^2}{4(2da_2^3+cb_2^3)^2}-\frac{ca_2b_2^3a_1^3d}{8(2da_2^3+cb_2^3)^2}e^{-\xi}+\frac{a_2^3c^2b_2^6a_1^4d^2}{64(2da_2^3+cb_2^3)^4}e^{-2\xi}\right)}{\left(b_2e^{2\xi}-\frac{da_2a_1^2b_2(4da_2^3+cb_2^3)}{4(2da_2^3+cb_2^3)^2}+\frac{c^2b_2^7a_2^2a_1^4d^2}{64(2da_2^3+cb_2^3)^4}e^{-2\xi}\right)}..(4.82)$$

or

$$u(\xi)=\left[\frac{\left(a_2e^{2\xi}+a_1e^{\xi}+\frac{(4da_2^3+3cb_2^3)da_2^2a_1^2}{4(2da_2^3+cb_2^3)^2}-\frac{ca_2b_2^3a_1^3d}{8(2da_2^3+cb_2^3)^2}e^{-\xi}+\frac{a_2^3c^2b_2^6a_1^4d^2}{64(2da_2^3+cb_2^3)^4}e^{-2\xi}\right)}{\left(b_2e^{2\xi}-\frac{da_2a_1^2b_2(4da_2^3+cb_2^3)}{4(2da_2^3+cb_2^3)^2}+\frac{c^2b_2^7a_2^2a_1^4d^2}{64(2da_2^3+cb_2^3)^4}e^{-2\xi}\right)}.\right]\times e^{i(\beta\xi-wt+x_0)}.(4.83)$$

Where

$$\xi=\alpha\left(x-\frac{da_2^3-cb_2^3+2a\alpha\beta a_2b_2^2+6da_2^3\beta^2+3\beta cb_2^3}{a_2b_2^2}t\right)$$

There are exact solutions of generalized nonlinear Schrödinger (GNLS) equation.

Case (2.6): $p=c=3, d=q=1$

As mentioned above the values of c and d can selected without limits. Setting $p = c = 3,$

And $d = q = 1$ then the trial function, Eq.(4.1),changes to

$$\psi(\xi) = \frac{a_3 e^{(3\xi)} + a_2 e^{(2\xi)} + a_1 e^{(\xi)} + a_0 + a_{-1} e^{(-\xi)}}{b_3 e^{(3\xi)} + b_2 e^{(2\xi)} + b_1 e^{(\xi)} + b_0 + b_{-1} e^{(-\xi)}} \qquad (4.83)$$

By the same approach as demonstrated above we obtain:

Case :(3.1):

$$a_3 = \frac{b_2(a_2 b_1 + a_1 b_2)}{8 b_1^2},\ a_2 = a_2,\ a_1 = a_1,\ a_0 = 0,\ a_{-1} = 0,\ b_3 = \frac{b_2(a_2 b_1 + a_1 b_2)}{8 a_1 b_1},\ b_2 = b_2,$$

$$b_1 = b_1,\ b_0 = 0,\ a = a,\ c = \frac{c b_1 (a_2 b_1 - a_1 b_2)}{\alpha^2 a_1 (a_2 b_1 + a_1 b_2)},\ d = -\frac{b_1^3 c b_2}{a_1^2 (a_2 b_1 + a_1 b_2)},\ \beta = \beta,\ \alpha = \alpha,$$

Substituting Eq.(4.76) into Eq(4.75) gives the following solution:

$$\psi(\xi) = \frac{\frac{b_2(a_2 b_1 + a_1 b_2)}{8 b_1^2} e^{3\xi} + a_2 e^{(2\xi)} + a_1 e^{\xi}}{\frac{b_2(a_2 b_1 + a_1 b_2)}{8 a_1 b_1} e^{3\xi} + b_2 e^{(2\xi)} + b_1 e^{\xi}},$$

Or

$$u(\xi) = \frac{\frac{b_2(a_2 b_1 + a_1 b_2)}{8 b_1^2} e^{3\xi} + a_2 e^{(2\xi)} + a_1 e^{\xi}}{\frac{b_2(a_2 b_1 + a_1 b_2)}{8 a_1 b_1} e^{3\xi} + b_2 e^{(2\xi)} + b_1 e^{\xi}} \times e^{i(\beta\xi - wt + x_0)}, \qquad (4.\ 84)$$

Where

$$\xi = \alpha\left(x - \frac{3b_1^2\beta^2 a_2 - 3cb_1\,\beta^2 a_1 b_2 - a_2 b_1^2 c - 2ca_1 b_2 b_1 + 2a\alpha\beta a_1 a_2 b_1 + 2a\alpha\beta a_1^2 b_2}{a_1(a_2 b_1 + a_1 b_2)}t\right)$$

Case :(3.2):

$$a_3 = \frac{b_1(a_1 b_{-1} + a_{-1} b_1)}{8b_{-1}^2},\ a_2 = 0,\ a_1 = a_1,\ a_0 = 0,\ a_{-1} = a_{-1},\ b_3 = \frac{b_1(a_1 b_{-1} + a_{-1} b_1)}{8a_{-1} b_{-1}},$$

$$a_2 = 0,\ b_1 = b_1,\ b_0 = 0,\ b_{-1} = b_{-1},\ a = a,\ c = -\frac{cb_{-1}(a_1 b_{-1} + a_{-1} b_1)}{4a_{-1}(b_{-1} a_1 + b_1 a_{-1})},$$

$$d = -\frac{cb_{-1}^3 b_1}{a_{-1}^2(b_{-1} a_1 + b_1 a_{-1})},\ \alpha = \alpha,\ \beta = \beta,\ \beta = \beta,$$

$$v = \frac{3cb_1^2\beta^2 a_2 - 3cb_1\beta^2 b_2 a_1 - a_2 b_1^2 c - 2cb_2 b_1 a_1 + 2a\alpha\beta a_1 a_2 b_1 + 2a\alpha\beta a_1^2 b_2}{a_1(a_2 b_1 + a_1 b_2)} \quad (4.85)$$

Substituting Eq.(4.79) into Eq(4.75) gives the following solution:

$$\psi(\xi) = \frac{\dfrac{b_1(a_1 b_{-1} + a_{-1} b_1)}{8b_{-1}^2} e^{(3\xi)} + a_1 e^{(2\xi)} + a_{-1} e^{(-\xi)}}{\dfrac{b_1(a_1 b_{-1} + a_{-1} b_1)}{8a_{-1} b_{-1}} e^{(3\xi)} + b_1 e^{(2\xi)} + b_{-1} e^{(-\xi)}}, \quad (4.86)$$

Or

$$u(\xi) = \frac{\dfrac{b_1(a_1 b_{-1} + a_{-1} b_1)}{8b_{-1}^2} e^{(3\xi)} + a_1 e^{(2\xi)} + a_{-1} e^{(-\xi)}}{\dfrac{b_1(a_1 b_{-1} + a_{-1} b_1)}{8a_{-1} b_{-1}} e^{(3\xi)} + b_1 e^{(2\xi)} + b_{-1} e^{(-\xi)}} \times e^{i(\beta\xi - wt + x_0)}, \quad (4.87)$$

Where

$$\xi = \alpha\left(x - \frac{-4a_1cb_{-1}^2 - 8cb_1\,a_{-1}b_{-1} + 3cb_{-1}^2\beta^2 a_1 - 3cb_{-1}\beta^2 b_1 a_{-1} + 8a\alpha\beta a_{-1}b_{-1}a_1 + 8d\alpha\beta a_{-1}^2 b_1}{4a_{-1}(b_{-1}a_1 + a_{-1}b_1)}t\right),\ldots(5.81)$$

Case (3.1) and (3.2) are the same as case (2.4) and (2.2) respectably.

Therefore we conclude the Eq.(4.75) is equal toEq.(4.56):

Case 4: p = c = 3, d = q = 3

In this case we consider p = c = 3 and d = c = 3. Then the trial function Eq. (4.1) is presented as:

$$\psi(\xi) = \frac{a_3e^{(3\xi)} + a_2e^{(2\xi)} + a_1e^{(\xi)} + a_0 + a_{-1}e^{(-\xi)} + a_{-2}e^{(-2\xi)} + a_{-3}e^{(-3\xi)}}{b_3e^{(3\xi)} + b_2e^{(2\xi)} + b_1e^{\xi} + b_0 + b_{-1}e^{(-\xi)} + b_{-2}e^{(-2\xi)} + b_{-3}e^{(-3\xi)}}, \qquad (4.89)$$

By the same approach illustrated above we obtain:

Case: (4.1):

$$a_3 = a_3, a_2 = 0, a_1 = 0, a_0 a_0, a_{-1} = 0, a_{-2} = 0, a_3 = \frac{b_0(a_3b_0 + a_0b_3)}{8b_3^2}, b_3 = b_2 = 0, b_1 = 0$$

$$b_0 = b_0,\ b_{-1} = 0,\ b_{-2} = 0,\ b_{-3} = \frac{b_0(a_3b_0 + a_0b_3)}{8a_3b_3},\ a = a,\ c = \frac{cb_3(-a_0b_3 + a_3b_0)}{9\alpha^2 a_3(a_3b_0 + a_0b_3)},$$

$$d = -\frac{b_3^3 cb_0}{a_3^2(a_3b_0 + a_0b_3)},\ \alpha = \alpha,\ \beta = \beta,$$

$$v = \frac{6cb_0a_3b_3 + 3cb_3^2a_0 - cb_3^2\beta^2 a_0 + cb_3\beta^2 b_0a_3 - 6a\alpha\beta b_0a_3^2 - 6a_0b_3a\alpha\beta a_3}{3a_3(b_0a_3 + a_0b_3)}, \ldots\ldots\ldots.(5.83)$$

Substituting Eq.(4.83) into Eq(4.82) gives the following solution:

$$\psi(\xi) = \frac{a_3 e^{3\xi} + a_0 + \dfrac{b_0(a_3 b_0 + a_0 b_3)}{8a_2^2} e^{-3\xi}}{b_3 e^{2\xi} + b_0 + \dfrac{b_0(a_3 b_0 + a_0 b_3)}{8a_3 b_3} e^{-3\xi}}, \qquad (4.90)$$

Or

$$u(\xi) = \frac{a_3 e^{3\xi} + a_0 + \dfrac{b_0(a_3 b_0 + a_0 b_3)}{8a_2^2} e^{-3\xi}}{b_3 e^{2\xi} + b_0 + \dfrac{b_0(a_3 b_0 + a_0 b_3)}{8a_3 b_3} e^{-3\xi}} \times e^{i(\beta\xi - wt + x_0)}, \qquad (4.92)$$

Where

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$$\xi = \alpha\left(x - \frac{6cb_0a_3b_3 + 3cb_3^2a_0 - b_3^2c\beta^2a_0 + cb_3\beta^2b_0a_3 - 6a\alpha\beta b_0a_3^2 - 6a_0b_3a\alpha\beta a_3}{3a_3(b_0a_3 + a_0b_3)} t\right), ...(5.86) \qquad (4.93)$$

Case: (4.2):

$$a_3 = -\frac{a_{-1}^3}{216a_{-3}^2},\ a_2 = 0,\ a_1 = -\frac{a_{-1}^2}{6a_{-3}},\ a_0 = 0,\ a_{-1} = a_{-1},\ a_{-2} = 0,\ a_{-3} = a_{-3},$$

$$b_3 = \frac{2c\alpha^2 a_{-1}^3}{81ca_{-3}^2},\ b_2 = 0,\ b_0 = 0,\ b_{-1} = 0,\ b_{-2} = 0,\ b_{-3} = -\frac{16c\alpha^2 a_{-3}}{3c},\ a = a,\ a = a,$$

$$d = -\frac{512c^2\alpha^6}{27c^2},\ \alpha = \alpha,\ \beta = \beta,$$

Substituting Eq.(4.٩٣) into Eq(4.82) gives the following solution:

$$\psi(\xi) = \frac{-\dfrac{a_{-1}^3}{216a_{-3}^2} e^{3\xi} - \dfrac{a_{-1}^2}{6a_{-3}} e^{\xi} + a_{-1}e^{-\xi} + a_{-3}e^{-3\xi}}{\dfrac{2c\alpha^2 a_{-1}^3}{81a_{-3}^2} e^{3\xi} - \dfrac{16c\alpha^2 a_{-3}}{3c} e^{-3\xi}} \times e^{i(\beta\xi - wt + x_0)}, \qquad (4.94)$$

Or

$$u(\xi)=\frac{-\frac{a_{-1}^{3}}{216a_{-3}^{2}}e^{3\xi}-\frac{a_{-1}^{2}}{6a_{-3}}e^{\xi}+a_{-1}e^{-\xi}+a_{-3}e^{-3\xi}}{\frac{2c\alpha^{2}a_{-1}^{3}}{81a_{-3}^{2}}e^{3\xi}-\frac{16c\alpha^{2}a_{-3}}{3c}e^{-3\xi}}\times e^{i(\beta\xi-wt+x_0)}, \qquad (4.95)$$

Where

$$\xi=\alpha\left(x-\alpha\left(-2a\beta-6c\alpha+3c\alpha\beta^{2}\right)t\right)$$

Case: (4.3):

$$a_2=-\frac{a_1^2(-4c^2\alpha^4a_0^2-Cc\alpha^2a_0b_0+3c^2b_0^2)}{16c^2\alpha^4a_0^3},\ a_1=a_1,\ a_0=a_0,\ a_{-1}=0,\ a_{-2}=0,\ a_{-3}=0,$$

$$b_3=\frac{a_1^3b_0c(c^2\alpha^4a_0^2+2Cc\alpha^2a_0b_0+c^2b_0^2)}{32c^3\alpha^6a_0^6},\ b_2=\frac{a_1^2b_0(4c^2\alpha^4a_0^2+7Cc\alpha^2a_0b_0+3c^2b_0^2)}{16c^2\alpha^4a_0^4},\ b_1=0,$$

$$b_0=b_0,\ b_{-1}=0,\ b_{-2}=0,\ b_{-3}=0,\ a\ =a\ ,\ a\ =a\ ,$$

$$d=-\frac{b_0^2(cb_0+ca_0\alpha^2)}{2b_0^3},\ \alpha\ =\alpha,\ \beta\ =\beta,$$

$$v=-\frac{c\alpha^2a_0+6c\alpha^2\beta^2a_0-4a\alpha\beta a_0+3cb_0}{2a_0}, \qquad (4.96)$$

Substituting Eq.(4.93) into Eq(4.82) gives the following solution:

$$\psi(\xi)=\frac{\left(\frac{b_0a_1^3C(c^2\alpha^4a_0^2+2C\alpha^2b_0a_0+C^2b_0^2)}{4(2da_2^3+cb_2^3)^2}e^{3\xi}-\frac{a_1^2(-4c^2\alpha^4a_0^2-C\alpha^2a_0b_0+3C^2b_0^2)}{8(2da_2^3+cb_2^3)^2}e^{2\xi}+a_1e^{\xi}+a_0\right)}{\left(\frac{a_1^3b_{0_0}^2C(c\alpha^4a_0^2+2Ccb_0a_0+C^2b_0^2)}{4(2da_2^3+cb_2^3)^2}e^{3\xi}-\frac{b_0a_1^2(4c^2\alpha^4a_0^2+7Ccb_0a_0+3C^2b_0^2}{64(2da_2^3+cb_2^3)^4}e^{2\xi}+a_0\right)}..(4.97) \qquad (4.97)$$

$$u(\xi)=\left[\frac{\frac{b_0a_1^3C(c^2\alpha^4a_0^2+2Cc\alpha^2a_0b_0+C^2b_0^2)}{4(2da_2^3+cb_2^3)^2}e^{3\xi}-\frac{a_1^2(-4c^2\alpha^4a_0^2-Cc\alpha^2a_0b_0+3Cb_0^2)}{8(2da_2^3+cb_2^3)^2}e^{2\xi}+a_1e^{\xi}+a_0}{\frac{(a_1^3b_0^2C(c\alpha^4a_0^2+2Ccb_0a_0+C^2b_0^2)}{4(2da_2^3+cb_2^3)^2}e^{3\xi}-\frac{b_0a_1^2(4c^2\alpha^4a_0^2+7Ccb_0a_0+3Cb_0^2}{64(2da_2^3+cb_2^3)^4}e^{2\xi}+a_0}\right]\times e^{i(\beta\xi-wt+x_0)} \qquad (4.98)$$

Or

Where

$$\xi = \alpha(x - \frac{c\alpha^2 a_0 + 6c\alpha^2\beta^2 a_0 - 4a\alpha\beta a_0 + 3cb_0}{2a_0}t)$$

There are other exact solutions of generalized nonlinear Schrödinger (GNLS) equation.

Case 5 $p = c = 4, d = q = 2$

Finally we consider p = c 4 and d = q = 2 then the trial function, Eq.(4.1) changes to.

$$\psi(\xi) = \frac{a_4 e^{(4\xi)} + a_3 e^{(3\xi)} + a_2 e^{2(\xi)} + a_1 e^{(\xi)} + a_0 + a_{-1} e^{(-\xi)} + a_{-2} e^{(-2\xi)}}{b_4 e^{(4\xi)} + b_3 e^{(3\xi)} + b_2 e^{(2\xi)} + b_1 e^{(\xi)} + b_0 + b_{-1} e^{(-\xi)} + b_{-2} e^{(-2\xi)}} \tag{4.99}$$

We rewrite Eq.(4.92) in the following form:

$$\psi(\xi) = \frac{a_4 e^{(3\xi)} + a_3 e^{(2\xi)} + a_2 e^{(\xi)} + a_1 + a_0 e^{(-\xi)} + a_{-1} e^{(-2\xi)} + a_{-2} e^{(-3\xi)}}{b_4 e^{(3\xi)} + b_3 e^{(2\xi)} + b_2 e^{(\xi)} + b_1 + b_0 e^{(-\xi)} + b_{-1} e^{(-2\xi)} + b_{-2} e^{(-3\xi)}} \tag{4.100}$$

This has the same form as Eq.(4.93). Therefore, Case 5 is equivalent to Case 4:

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