Sudan University of sciences and

Technology college of Graduate studies

The formal solution of constrained mechanical system

الحل المصاغ للأنظمة الميكانيكية المقيدة

A Thesis submitted in partial fulfillment for the Degree of M.SC in mathematics

By:

Anwar Mahgoub Awad Hamad

Supervisor by:

Dr. Emad Elden Abdallah Abdelrahim

2015

الآية

قال تعالى:-

(وما أوتيتم من العلم إلا قليلا)

صدق الله العظيم

الإسراء:٨٥

Dedication

I dedicate this thesis with love and respect to

- *My father and mother….*
- *My brother and sisters….*
- *My teachers……..*
- *My relatives and friends..*

Acknowledegment

First of all for most , I should offer my thanks, obedience and gratitude to Allah the greatest from whom .

I receive guidance and help.

I would like to express my deepest thanks, appreciation to my supervisor : Dr.Emad Eldeen Abdallah Abdelrahim.

I am very grateful to the department of mathematics of Sudan University of science and Technology, my thanks to everybody who helped me to do this research.

Last, I am grateful to my family for their fall support.

Abstract

In this work we discuss the scope of Lagrangian mechanics and constrained systems. We discuss the problem of central force two–bodies and its formal solution.

Also we study the rotations and rigid bodies by using Euler's equations, Lagrangian description and Legender transformations with some examples and applications.

الخلاصة

في هذا العمل ناقشنا نطاق ميكانيكا لاجرانج وأنظمة التقيد . ناقشنا مسالة قوة الطرد المركزية لجسمين وصيغة الحل لها . أيضا قمنا بدراسة التدوير والأجسام المتماسكة باستخدام معادلة اويلر، وصف لاجرانج وتحويلات لاجندر مع بعض الأمثلة والتطبيقات.

The contents

Subject	Number
الآية	2
Dedication	3
Acknowledgement	4
Abstract (English)	5
Abstract (Arabic)	6
The contents	7
Introduction	8
Chapter(1) **Hamiltonian and Lagrangian mechanics**	
Section(1.1): Analytical and Hamiltonian mechanics	9
Section(1.2):Lagrangian mechanics	29
Chapter (2) **The central force and small oscillations**	
Section(2.1):The central force two-body problem	*45*
Section(2.2):The small oscillations	*64*
Chapter (3) **Rotation and Legendre transformations**	
Section (3.1): Rotation and rigid bodies	*75*
Section(3.2): Legendre transformations	*99*
Chapter (4) **The Hamiltonian formulation and Infinitesimal canonical transformations**	
Section(4.1):The Hamiltonian formulation	*102*
Section(4.2): Infinitesimal canonical transformations	*119*
References	*129*

Introduction

Today , the relation between developments in physics and mathematics is a rather complicated one and there are more interrelation than one naively expects.

This research consider the mathematical structures of the analytical mechanics, and organized as follows:

In chapter (1) , we discuss the three celebrated laws of mechanics, which formulated by Newton . We exp lane the concepts of the Hamiltonian and Lagrangian mechanics and their mathematical structures with the correspondence physical for menologys with , some examples.

In chapter (2) , we start by studing the theory of conic section in a brief account . Also we discuss the central force tow–bodies problem and associated kepler's laws which derived by Newton . We discuss the small oscillations in several situations with some applications.

In chapter (3) , we study the Rotation, rotating coordinate systems, and the inertia tensor , then we apply them to discuss the rotations of rigid bodies . Also we discuss the Legender transformations with some applications .

In chapter (4) , we deal with the Hamiltonian formulation and Infintesimal canonical transformations with some examples and applications.

Chapter (1)

Hamiltonian and Lagrangian mechanics

Section(1.1): Analytical and Hamiltonian mechanics

Mechanics is the paradise of the mathematical sciences, because with it one comes to the fruits of mathematics Leonardo da Vinci Sir Isaac Newton was a Master of the Mint. He also formulated three celebrated laws of mechanics, which we can paraphrase as follows:

1. A particle not subject to any force moves on a straight line at constant speed.

2. In the presence of a force, the position of a particle obeys the equations of motion

$$m\ddot{x}_i = F_i(x, \dot{x}) \tag{1.1}$$

3. The force exerted by a particle on another is equal in magnitude, but op-posite in direction, to the force exerted by the other particle on the first. A "particle" is here thought of as an entity characterized by its mass m, its location in space, and by nothing else. The aim of Newton's mechanics is to predict the location at arbitrary times, given the position and velocity at some initial time. This is done by means of a solution of the differential equations above.

An overdot denotes differentiation with respect to the time parameter t (this notation, as well as Differential Calculus itself, was invented by Newton), and x_i may denote a vector in three-dimensional space. Sometimes it will be understood that we are dealing with a set of N particles, and moreover we often "suppress indices". Then the force $F_i(x, \dot{x})$ is a $3N$ component function of the $3N$ variables x_i and their $3N$ derivatives $\dot{x}_i$. Since the index notation may be a bit unfamiliar, let

me note that whenever indices occur in a formula, it is understood that they can take any of a specified set of integer values.

If $i \in \{1, 2, \ldots, n\}$ then eq. (1.1) stands for n separate equations. Throughout we employ Einstein's summation convention, which means that whenever a certain index occurs twice in a particular term, a sum over all its allowed values is understood, eg.

$$x_i y_i \equiv \sum_{i=1}^{n} x_i y_i = \sum_{j=1}^{n} x_j y_j = x_j y_j \tag{1.2}$$

It does not matter which letter is being used for a repeated index. To avoid confusion, the same index never occurs thrice or more in a single term. We will introduce index notation in a more sophisticated "tensorial" way, but for the time being this is all there is to it. By the way index notation is not always the best choice it does not, for instance, make use of any special properties of three dimensional space but it has the advantage that it can be used for everything, which is why I always use it.

In the following we will study analytical and Hamiltonian mechanics. What are we to think of Newton's laws? A physicist might follow Newton in using them to predict the position of the planets as they go around the sun, and will conclude that they are very meaningful. A mathematician might say that they do not say very much, only that the position of a particle is described by a set of ordinary differential equations. A philosopher might object that they say nothing at all the first and second law together seem to state that a particle moves in a straight line, unless it does something else, in which case we say that it is subject to a force. But the philosopher Kant valued Newton's laws highly, and tried to prove that they are somehow necessary consequences of the way our minds perceive the world, and have a status similar to Euclid's axioms in geometry. Kant overestimated Newton's laws, but they do have content as they stand. It is a highly non-trivial fact that a second order differential equation is being postulated, since this means that the position

and the velocity can be chosen arbitrarily at a given instant, but not the acceleration.

Analytical mechanics is at once more general and more special than Newton's theory. It is more general because it is more abstract. Its equations do not necessarily describe the positions of particles, but may be applied to much more general physical systems (such as field theories, including Einstein's general relativity theory). In the version we will study it is more special because only a restricted set of forces will be allowed in eq. (1.1). Let us see what kind of restrictions on the function F_i that are of physical interest. Newton's third law is already a restriction. It can be reformulated as the statement that the total momentum of a system composed of several particles is conserved:

$$\frac{dP_i}{dt} \equiv \frac{d}{dt}\sum_{particles} m\dot{x}_i = 0 \tag{1.3}$$

where the sum is over all the particles in the system. The existence of such a conserved vector is clearly an interesting fact. By the way this formulation is quite superior when we deal with time dependent masses, say with rockets. Now consider the function

$$E = T + V = \frac{m\dot{x}^2}{2} + V(x) \tag{1.4}$$

where V is some function of x, known as the potential energy. The function T is called the kinetic energy, while E itself is the energy of the system. Clearly

$$\dot{E} = \dot{x}_i \left(m\ddot{x}_i + \partial_i V(x)\right) \tag{1.5}$$

It follows that if the force is given by

$$F_i(x, \dot{x}) = F_i(x) = -\partial_i V(x) \tag{1.6}$$

then the energy of the system is conserved. Systems for which a conserved energy function exists are called conservative. In our example, and indeed in many interesting cases, the energy can be divided into kinetic and potential parts, and the equation of motion is given by

$$m\ddot{x}_i = -\partial_i V(x) \tag{1.7}$$

This move is typical of analytical mechanics, where vectors are usually derived from scalar functions.

Analytical mechanics devises methods to derive the differential equations de-scribing a given system, strategies for solving them, and ways of describing the solutions if they cannot be obtained in explicit form .We will tentatively restrict ourselves to conservative systems only. If you like this is a strengthening of the third law, and it is believed that all isolated systems in Nature are of this type.

What we are trying to do is to find some properties that all the Laws of Physics, and in particular all allowed equations of motion, have in common. Now the philosopher Leibnitz who was the other of the two inventors of Differential Calculus argued that we live in the best of all possible worlds. Is it evident from eq. (1.7) that this is so? Indeed it is, as was first realized half a century after the publication of Newton's Principia. The inspiration came from optics, and the laws of reflection and refraction. It is known that the angle of reflection is equal to the angle of incidence, and it was observed by the Greeks that this implies that light always travels on the shortest path available between two points A and B, subject to the restriction that it should be reflected against the surface. If the angle of reflection were to differ from the angle of incidence, the distance covered by light in going from A to B would be greater than it has to be. For refraction, we have Snell's Law. Any medium can be assigned an index of refraction n, and the angle of refraction is related to

$$n_1 \sin\theta_i = n_2 \sin\theta_r \tag{1.8}$$

Fermat noted that if

$$n = \frac{c}{v}, \tag{1.9}$$

where v is the velocity of light in the medium and c is a constant (independent of the medium), then Snell's law can be derived from what is now known as Fermat's principle, namely that the time taken for light to go from A to B is a minimum. More generally, whatever the medium, we can evaluate the integral

$$I = \int_{\gamma} n(x(s))ds \tag{1.10}$$

along an arbitrary path $\gamma(s)$ between A and B. Then the path actually taken by light, in going from A to B through a medium with a position dependent $n(x)$, is that specific path which results in the smallest possible value of the integral I. Fermat's principle unifies the laws of refraction and reflection, since it also implies the equality between the angles of incidence and reflection.

Is there a similar principle underlying mechanics? At least for systems obeying eq. (1.7), there is. Consider two points A and B, and suppose that a particle starts out at A at time $t = t_1$, and then moves along an arbitrary path from A to B with whatever speed that is consistent with the requirement that it should arrive at B at the time $t = t_2$. In mathematical terms we are dealing with a function $x(t)$ such that

$$x(t_1) = x_A \qquad x(t_2) = x_B \tag{1.11}$$

but otherwise arbitrary. For any such function $x(t)$ we can evaluate the integral

$$S[x(t)] = \int_{t_1}^{t_2} (T - V)dt = \int_{t_1}^{t_2} \left(\frac{m\dot{x}^2}{2} - V(x)\right) dt \tag{1.12}$$

S is known as the action. It is a functional, i.e. a function of a function the functional $S[x(t)]$ assigns a real number to any function $x(t)$. Note that $S[x(t)]$ is not a function of t, hence the square bracket notation. On the other hand it is a function of x_A, x_B, t_1, and t_2, but this is rarely written out explicitly.

The statement, to be verified in the next section, is that the action functional (1.12) has an extremum (not necessarily a minimum) for precisely that function $x(t)$ which obeys the differential equation (1.7). This is known as Hamilton's Principle, or with less than perfect historical accuracy as the Principle of Least Action.

Hamiltonian mechanics deals with those, and only those, equations of motion which can be derived from Hamilton's Principle, for some choice of the action functional.

This is a much more general class than that given by eq. (1.7), but it does exclude some cases of physical interest. Hamiltonian mechanics forms only a part of analytical mechanics namely that part that we will focus on.

Note once again what is going on. The original task of mechanics was to predict the trajectory of a particle, given a small set of data concerning its state at some intial time t. We claim that there exists another formulation of the problem, where we can deduce the trajectory given half as much data at each of two different times. So there seems to be a local, causal way of looking at things, and an at first sight quite different global, teleological viewpoint. The claim begins to look reasonable when we observe that the amount of "free data" in the two formulations are the same. Moreover, if the two times t_1 and t_2 approach each other infinitesimally closely, then what we are in effect specifying is the position and the velocity at time t_1, just as in the causal approach.

Why do principles like Fermat's and Hamilton's work? In both cases, we are extremizing a quantity evaluated along a path, and the path actually taken by matter in nature is the one which makes the quantity in question assume an extremal value. The point about extrema not only minima is that if the path is varied slightly away from the extremal path, to a path which differs to order ϵ from the extremal one, then the value of the path dependent quantity suffers a change which is of order ϵ squared. At an

extremum the first derivative vanishes. In the case of optics, we know that the description of light as a bundle of rays is valid only in the approximation where the wavelength of light is much less than the distance between A and B. In the wave theory, in a way, every path between A and B is allowed. If we vary the path slightly, the time taken by light to arrive from A to B changes, and this means that it arrives out of phase with the light arriving along the first path. If the wavelength is very small, phases from light arriving by different paths will be randomly distributed, and will cancel each other out through destructive interference. This argument fails precisely for the extremal paths: for them, neighbouring paths take approximately the same time, light from all neighbouring paths will arrive with the same phase, and constructive interference takes place. Thus, whenever the wavelength is negligibly small, it will appear that light always travels along extremal paths.

Only in the twentieth century was it realized that Hamilton's Principle works for the same reason that Fermat's Principle works. Classical mechanics is a kind of geometrical optics limit of a "wave mechanics" of matter, operating in configuration space. But that is another story.

Let us now verify the claim , namely that Newton's differential equations, for suitable choices of the dynamical system, are mathematically equivalent to the requirement that a certain functional of all possible paths of the particles should assume an extremum value at the actual trajectory. First we stare at the definition of the action functional:

$$S[x(t)] = \int_{t_1}^{t_2} \left(\frac{m\dot{x}^2}{2} - V\,(x) \right) dt \qquad (1.13)$$

How do we find the extrema of such a function of a function? Let us begin with some formal considerations. For a function f of an ordinary number x, it is easy enough to find the extrema. We consider how the function values $f(x)$ change as we change the number x:

$$\delta f(x) \equiv f(x + \delta x) - f(x) = \delta x \partial_x f(x) \qquad (1.14)$$

We assume that δx is so small that second order terms can be ignored. If the derivative is zero at the point x, the function has a minimum, or a maximum, or at least an inflection point there. For a function of several variables, the condition for an extremum (a minimum, a maximum, or a saddle point) is that

$$\delta f(x) = \sum_i \delta x_i \frac{\partial f}{\partial x_i}(x_1, \ldots, x_N) = 0 \qquad (1.15)$$

for arbitrary choices of the δx_i, which means that all the N partial derivatives have to vanish at the extremal points. Now a functional of a function $x(t)$ can be regarded as a function of an infinite number of variables, say of the Fourier coefficients of the original function. You can also regard t as a label of the infinite number of variables on which the functional depends a kind of continuous index and then what we have to do is to replace the sum in eq. (1.15) with an integral. Like this:

$$\delta S = S[x(t) + \delta x(t)] - S[x(t)] = \int_{t_1}^{t_2} \delta x(t) \frac{\delta S}{\delta x}(t) dt \qquad (1.16)$$

We assume that it is possible to bring δS to this form. Then the functional derivative of $S[x]$ will be defined as the very expression that occurs to the right in the integrand. The equations of motion, as obtained from Hamilton's Principle, then state that the functional derivative of the action is zero, since the form of the function $\delta x(t)$ is arbitrary.

It remains to be seen if we really can bring δ to this form if not, we would have to conclude that $S[x(t)]$ is "not differentiable". First of all, note that we are all the time evaluating the action between definite integration limits. Then the extremum, if it exists, will be given by that particular trajectory which starts at the point $x(t_1)$ at time t_1, and ends at the point $x(t_2)$ at time t_2, and for which the functional derivative vanishes. We can make this work for the action functional (1.13).

Imagine that we know its value for a particular function $x(t)$, and ask how this value changes if we evaluate it for a slightly different function

$$\tilde{x}(t) = x(t) + \delta x(t), \qquad \delta x(t) = \epsilon f(t) \tag{1.17}$$

where $f(t)$ is, for the time being, an arbitrary function while ϵ is an infinitesimally small constant. It is important for the following argument that $f(t)$ is arbitrary, or nearly so. That ϵ is "infinitesimally small" simply means that we will neglect terms of quadratic and higher orders in ϵ in the calculation which follows:

$$\delta S = S[\tilde{x}(t)] - S[x(t)]$$

$$= \int_{t_1}^{t_2} \left(\frac{m}{2} (\dot{x} + \delta \dot{x})^2 - V(x + \delta x) \right) - S[x(t)] dt$$

$$= \int_{t_1}^{t_2} \left(\frac{m}{2} \dot{x}^2 + m\dot{x}\, \delta\dot{x} - V(x) - \delta x \partial_x V(x) + o(\epsilon^2) \right) - S[x(t)] dt$$

$$= \int_{t_1}^{t_2} (m\dot{x}\, \delta\dot{x} - \delta x \partial_x V(x)) + o(\epsilon^2)\, dt \tag{1.18}$$

The action functional has an extremum at the particular function $x(t)$ for which this expression vanishes to first order in ϵ. What we want to see is what kind of restrictions this requirement sets on the function. To see this, we perform a partial integration

$$\delta S = \int_{t_1}^{t_2} \left(-\delta x \left(m\ddot{x} + \partial_x V(x) \right) + \frac{d}{dt} (m \delta x \dot{x}) \right) dt \tag{1.19}$$

Unfortunately this is not quite of the form (1.16), due to the presence of the total derivative in the integrand. Therefore we impose a restriction on the so far arbitrary function $f(t)$ that went into the definition of δx, so that

$$\delta x(t_1) = \delta x(t_2) = 0 \tag{1.20}$$

This is a way of saying that we are interested only in functions $x(t)$ that

have certain preassigned starting and end points at specified times. With this restriction, the total derivative in eq. (1.19) goes away. The first term has to vanish for all allowed choices of the functions $\delta x(t)$. After a moment's reflection, we see that this can happen only if the factor multiplying δx in the integrand is zero. Hence we have proved that the action functional has an extremum, among all possible functions obeying

$$x(t_1) = x_A \qquad x(t_2) = x_B \tag{1.21}$$

for those and only those functions which obey

$$m\ddot{x} = -\partial_x V(x) \tag{1.22}$$

at all intermediate points.

So we have proved, in this particular case at least, that Newton's equations of motion can be derived from the condition that a certain action functional shall have an extremum value. Note also that the restrictions that we had to set on the function $x(t)$, eqs. (1.21), make perfect sense. To obtain a definite trajectory it is not enough to impose the equations of motion. It is also necessary to set some initial values. For differential equations of second order, it is natural to make a choice of $x(0)$ and $\dot{x}(0)$. From the point of view of the action, it is natural to impose the value of $x(t)$ at two different times, which is the same amount of information. It should be noted though that whatever values of $x(0)$ and $\dot{x}(0)$ we choose there is always a unique solution for some range of t, while it is perfectly possible that the equation of motion is such that there is no solution, or several solutions, for a given pair of $x(t_1)$ and $x(t_2)$. The rest of this course is an elaboration of the contents of this section. If you have not understood everything perfectly yet there is still time.

It is one thing to be able to set up equations for a physical system, and perhaps to prove theorems to the effect that a solution always exists and is unique, given suitable initial conditions. Another issue of obvious interest is how to solve these equations, or at least how to extract

information from them. What precisely do we mean when we say that a differential equation is "soluble"? Consider, a first order differential equation for a single variable:

$$\dot{x} = f(x) \tag{1.23}$$

where f is some function. This can be solved by means of separation of variables:

$$dt = \frac{dx}{f(x)} \quad \Rightarrow \quad t(x) = \int_c^x \frac{dx'}{f(x')} \tag{1.24}$$

where c is a constant determined by the initial condition. If we do this integral, and then invert the resulting function $t(x)$ to obtain the function $x(t)$, we have solved the equation. We will regard eq. (1.24) as an implicit definition of $x(t)$, and eq. (1.23) is soluble in this sense. This is reasonable, since the manipulations required to extract $t(x)$ can be easily done on a computer, to any desired accuracy, even if we cannot express the integral in terms of elementary functions. But there are some limitations here: It may not be possible to invert the function $t(x)$ except for small times.

Next consider a second order equation, such as the equation of motion for a harmonic oscillator:

$$m\ddot{x} = -kx \tag{1.25}$$

This is a linear equation, and we know how to express the solution in terms of trigonometric functions, but our third example a pendulum of length l is already somewhat worse:

$$ml^2\ddot{\theta} = -gml\sin\theta \tag{1.26}$$

Let us therefore approach eq. (1.25) in a systematic fashion, which might yield results also for the pendulum. As a first step, note that any second order differential equation can be rewritten as a pair of coupled first order equations:

$$\dot{p} = -kx \qquad m\dot{x} = p \tag{1.27}$$

The second equation defines the new variable p. Unfortunately coupled first order equations are difficult to solve, except in the linear case when they can be decoupled through a Fourier transformation.

The number of degrees of freedom of a dynamical system is defined to be one half times the number of first order differential equations needed to describe the evolution. It will turn out that, for systems whose equations of motion are derivable from the action principle, the number of first order equations will always be even, so the number of degrees of freedom is always an integer for such systems. A system with n degrees of freedom will be described by a set of $2n$ in general coupled first order equations, and the difficulties one encounters in trying to solve them will rapidly become severe.

In the cases at hand, with one degree of freedom only, one uses the fact that these are conservative systems, which will enable us to reduce the problem to that of solving a single first order equation. For the harmonic oscillator the conserved quantity is

$$E = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} \tag{1.28}$$

The number E does not depend on t. Equivalently

$$\dot{x}^2 = \frac{2E}{m} - \frac{k}{m}x^2 \tag{1.29}$$

Taking a square root we are back to the situation we know, and we proceed as before:

$$dt = dx\sqrt{\frac{m}{2E - kx^2}} \quad \Leftrightarrow \quad t(x) = \int_c^x \sqrt{\frac{m}{2E - kx'^2}}\, dx' \tag{1.30}$$

Inverting the function defined by the integral, we find the solution $x(t)$. The answer is a trigonometric function, with two arbitrary constants E and c determining its phase and its amplitude. For our purposes the trigonometric function is defined by this procedure.

We can play the same trick with the non-linear equation for the pendulum, and we end up with

$$t(\theta) = \int_c^{\theta} \frac{d\theta'}{\sqrt{\frac{2}{ml^2}(E + gml\, cos\theta')}} \tag{1.31}$$

We integrate, and we invert. This defines the function $\theta(t)$. We could leave it at that, but since our example is a famous one, we manipulate the integral a bit further for the fun of it. Make the substitution

$$sin\frac{\theta'}{2} \equiv k\, sin\ \phi' \Rightarrow d\theta' = \frac{2k\, cos\ \phi' d\,\phi'}{\sqrt{1-k^2\, sin^2\,\phi'}} \tag{1.32}$$

The constant k is undetermined at this stage. The integral becomes

$$t(\theta) = \sqrt{\frac{l}{2g}} \int_c^{\phi\,(\theta)} \frac{2k\, cos\ \phi' d\,\phi'}{\sqrt{1-k^2\, sin^2\,\phi'}\sqrt{\frac{E}{gml}+1-2sin^2\frac{\theta'}{2}}}. \tag{1.33}$$

The integrand simplifies if we choose the constant k such that

$$2k^2 \equiv \frac{E}{gml} + 1\,. \tag{1.34}$$

One further substitution takes us to our desired standard form;

$$t(\theta) = \sqrt{\frac{lg}{g}} \int_c^{\phi\,(\theta)} \frac{d\,\phi'}{\sqrt{1-k^2\, sin^2\,\phi'}} = \quad |\ sin\ \phi' \equiv x'|$$

$$= \sqrt{\frac{l}{g}} \int_c^{x(\theta)} \frac{dx'}{\sqrt{\left(1-x'^2\right)\left(1-k^2x'^2\right)}}\,. \tag{1.35}$$

Just as eq. (1.30) can be taken as an implicit definition of a trigonometric function, this integral implicitly defines the function $\theta(t)$ as an elliptic function. If you compare it with the previous integral (1.30), you see that an elliptic function is a fairly natural generalization of a trigonometric, i.e. “circular”, function. Since elliptic functions turn up in many contexts they have been studied in depth by mathematicians.

Anyway, the above examples were some of the simplest examples of completely soluble dynamical systems. Just wait till we get to the insoluble ones.

Why did this work at all? The answer is that we had one degree of freedom, and one constant of the motion, namely E. This reduced the problem to that of solving a single uncoupled equation. This suggests a general strategy for solving the equations of motion for a system containing n degrees of freedom,

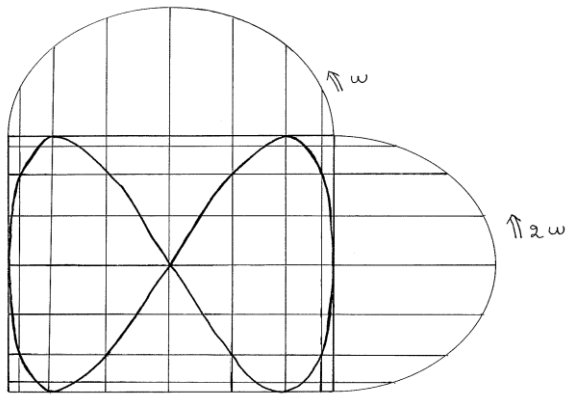


Figure (1.1)

A Lissajous figure, and how to draw it; $x = cos\ \omega t, y = sin\ 2\omega t$.

i.e. solving $2n$ coupled first order equations: One must find a set of n constants of the motion with suitable properties, so that the problem reduces to that of computing n integrals. This idea forms the core of Hamilton-Jacobi theory. It works sometimes, but not very often. As a result the notion of what it means to "solve" a set of differential equations evolved somewhat: a solution might consist, say, of a convergent power series in t. But frequently this strategy also fails. A typical Hamiltonian system will exhibit an amount of "chaotic" behaviour, and there may not exist any effective procedure to generate the long term behaviour of the solutions on a computer. What one has to do then is to find out which questions one can reasonably ask concerning such systems.

Even in situations where one can solve the equations, things may not be altogether simple. Consider two harmonic oscillators, with the explicit solution

$$x = a\cos(\omega_1 t + \delta_1)\, y = b\cos(\omega_2 t + \delta_2) \qquad (1.36)$$

The trajectory in the x-y-plane is a Lissajous figure. Fig. (1.1) explains how to draw them; further examples are readily produced with a computer. If $\omega_1 = \omega_2$ the trajectory is an ellipse, with circles and straight lines as special cases. More generally, if there exist integers m and n such that

$$m\omega_1 = n\omega_2 \tag{1.37}$$

the trajectory is a closed curve. If there are no such integers the trajectory eventually fills a rectangle densely, and never closes on itself. Now put yourself into the position of an experimentalist trying to determine by means of measurements whether the trajectory will be closed or not. Another example of this type is a particle moving on a straight line on a plane, but confined to a quadratic box and bouncing elastically from its walls. Let us ask whether the trajectory is periodic or whether it will eventually come arbitrarily close to any point in the box. The answer depends on the initial condition for the direction of motion. If the angle between this direction and one of the walls is called α, the question is whether tan α is a rational number or not. Theoretically this is fine, but for someone who wants to decide the question by means of measurements of the initial velocity it is not.

Now we will discuss Phase space. It is worthwhile formalizing things a bit further. With the understanding that every set of ordinary differential equations can be written in first order form, we write down the general form of N coupled first order equations for N real variables z_i:

$$\dot{z}_i = f_i(z_1, \ldots, z_N;\ t), 1 \leq i \leq N \tag{1.38}$$

where the N functions f_i are arbitrary. We simplify things by assuming that there is no explicit dependence on time. We then have the equations that describe an autonomous dynamical system, namely

$$\dot{z}_i = f_i(z_1, \ldots, z_N) \tag{1.39}$$

There are theorems that guarantee the existence and uniqueness of such systems for some range of the parameter t. Thus

$$z_i = z_i(z_{01}, \ldots, z_{0N}, t) \tag{1.40}$$

where z_{0i} are the initial values of z_i.

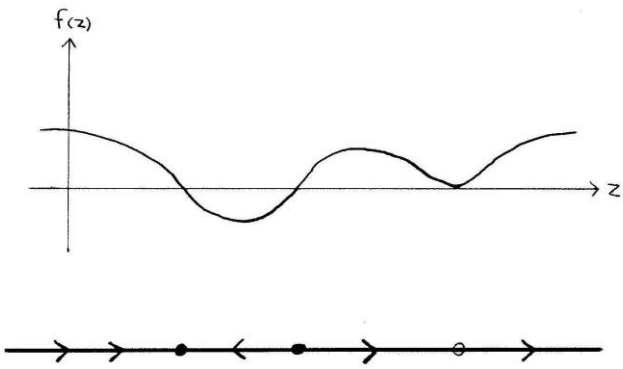
There is no guarantee that such solutions can be obtained in any explicit form. If we discretize the time variable a computer can easily generate approximative solutions, but it may be practically impossible to produce accurate solutions over long intervals of time.

We assume that the physical systems we are interested in as far as we attempt to describe them can be fully characterized by the N real numbers z_i. We imagine a space whose points are labelled in a one-to-one fashion by these numbers, and call it phase space.

The set of all possible states of a physical system is in one-to-one correspon-dence with the points of an N dimensional phase space. The time development of a system is uniquely determined by its position in phase space.

This is the first of several abstract spaces that we will encounter, and you must get used to the idea of abstract spaces.

A particle moving in space has a 6 dimensional phase space, because its position (3 numbers) and its velocity (3 numbers) at a given time determine its position at all times, given Newton's laws. Anything else can either be computed from these numbers this is true for its acceleration or else it can



Figure(1.2)

A one dimensional phase space, containing one stable and one unstable fixed point, as well as one fixed point which is structurally unstable.

be ignored this would be true for how it smells, if it does. The particle also has a mass, but this number is not included in phase space because it is given once and for all. Two particles moving in space have a 12 dimensional phase space, so high dimensional phase spaces are often encountered. We will have to picture them as best we may.

Now consider time evolution according to eq. (1.39). Because of the theorems I alluded to, we know that through any point z_0 there passes a unique curve $z_i(t)$, with a unique tangent vector $\dot{z}_i$. These curves never cross each other. When the system is at a definite point in phase space, it knows where it is going. The curves are called trajectories, and their tangent vectors define a vector field on phase space called the phase space flow. Imagine that we can see such a flow. Then there are some interesting things to be observed. We say that the flow has a fixed point wherever the tangent vectors vanish. If the system starts out at a fixed point at $t = 0$, it stays there forever. There is an important distinction to be made between stable and unstable fixed points. If you start out a system close to an unstable fixed point it starts to move away from it, while in the stable case it will stay close forever. The stable fixed point may be an attractor, in which case a system that starts out close to the

fixed point will start moving towards it. The region of phase space which is close enough for this to happen is called the basin of attraction for the attractor.

Consider a one dimensional phase space, with the first order system

$$\dot{z} = f(z) \tag{1.41}$$

For generic choices of the function f all fixed points are either stable attractors, or unstable repellors, but for special choices of f we can have fixed points that are approached by the flow only on one side. The latter are structurally unstable, in the sense that the smallest change in f will either turn them into pairs of attractors and repellors, or cause them to disappear altoghether.

In two dimensions there are more possibilities. We can have sources and

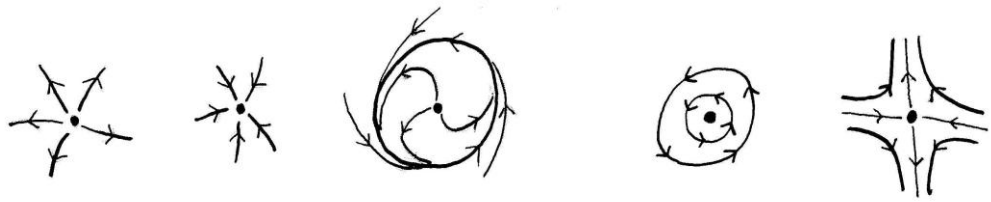


Figure (1.3)

Fixed points in a two dimensional phase space: a source, a sink, a limit cycle, an elliptic fixed point, and a hyperbolic fixed point.

sinks, as well as stable elliptic and unstable hyperbolic fixed points.

The phase space of the harmonic oscillator is R^2, and it contains one elliptic fixed point. It is elliptic because it is surrounded by closed trajectories, and hence it is stable. In the case of the pendulum phase space has a non-trivial topology: since the coordinate θ is a periodic angle phase space is the surface of an infinitely long cylinder. It contains two fixed points. One of them is elliptic, and the other the state where the pendulum is pointing upwards is hyperbolic. What is special about the hyperbolic fixed point is that there are two trajectories leading into it, and two leading out of it. The length of the tangent vectors $\dot{\theta}$ decrease as the

fixed point is approached. Taking the global structure of phase space into account we see that a trajectory leaving the fixed point is in fact identical to one of the incoming ones. Hence there are really only two special trajectories. A striking fact about them is that they divide phase space into regions with qualitatively different behaviour. One region where the trajectories go around the elliptic fixed point, and two regions where the trajectories go around the cylinder. For this reason the special trajectories are called separatrices, and the regions into which they divide phase space are called invariant sets by definition an invariant set in phase space is a region that one cannot leave by following the phase space flow.

It is very important that one see how to relate this abstract discussion of the phase space of the pendulum to known facts about real pendula.

It is not by accident that the phase space of the pendulum is free of sources and sinks. The reason is, that only elliptic or hyperbolic fixed points can occur in Hamiltonian mechanics. Real pendula tend to have some amount of dissipation present (because they are imperfectly isolated from the environment), and then the situation changes. Speaking of Hamiltonian systems it is worthwhile to point out that the example of the two harmonic oscillators in eq. (1.36) is less frivolous than it may appear. The phase space is four dimensional, but there are two conserved quantities

$$2E_1 = p_1^2 + \omega_1^2 x_1^2 \qquad 2E_2 = p_2^2 + \omega_2^2 x_2^2 \tag{1.42}$$

This means that any given trajectory will be confined to a two dimensional surface in phase space, labelled by E_1 and E_2. This surface is a torus, with topology $S^1 \times S^1$. In a sense to be made precise later, non-chaotic motion in a Hamiltonian system always takes place on a torus in phase space.

Finally we observe that we have the beginnings of a strategy to understand any given dynamical system. We begin by locating the fixed

points of the phase space flow. Then we try to determine the nature of these fixed points. If the equations are linear this is straightforward. If not, we can try linearization of the equations around the fixed points. There is a theorem we can lean on here:

The Hartman-Grobman theorem: The nature of the fixed points is unchanged by linearization, as long as the fixed points are isolated and as long as no elliptic fixed points occur.

The caveat in the statement will be explained presently.

Now consider the pendulum. Its phase space is a cylinder described by the coordinates (θ, p_θ). To see if the phase space flow has any fixed points, we set

$$\dot{\theta} = \frac{1}{ml^2} p_\theta = 0 \qquad \dot{p}_\theta = -gml \sin\theta = 0 \tag{1.43}$$

Hence there are fixed points at $(\theta, p_\theta) = (0, 0)$ and $(\pi, 0)$. Linearizing around them we find the former to be elliptic and the latter to be hyperbolic. If this remains true for the non-linear equations we can easily draw a qualitatively correct picture of the phase space flow. No integration is needed.

Were we justified in assuming that the fixed points are elliptic? To see what can go wrong, consider the non-linear equation

$$\ddot{x} + \epsilon x^2 \dot{x} + x = 0 \tag{1.44}$$

In the linearised case ($\epsilon = 0$) there is a single elliptic fixed point. In the non-linear system the flow will actually spiral in or out from the fixed point, depending on the sign of ϵ, so this is an example where the exceptions to the Hartman-Grobman theorem are important. But in the case of the "pure" pendulum we know that the non-linear system is Hamiltonian, and therefore sources and sinks cannot appear our analysis of the pendulum was therefore accurate.

Our tentative strategy works very well when the phase space is two dimensional, but if the dimension of phase space exceeds two things can get very complicated indeed. A famous example is the at first sight innocent looking Lorenz equations

$$\dot{z}_1 = -az_1 + az_2$$
$$\dot{z}_2 = bz_1 - z_2 - z_1z_3$$
$$\dot{z}_3 = -cz_3 + z_1z_2 \qquad (1.45)$$

They capture some aspects of thermal convection in a fluid. Lorenz was a metereologist interested in the long term accuracy of weather prediction, but he found himself unable to do long term prediction even in this simple model.The non-linear terms have a dramatic effect, and the slightest change in the initial data will cause the trajectory to go to completely different regions of the three dimensional phase space.

Section(1.2): Lagrangian mechanics

With the agreement that the action integral is an important object, we give a name also to its integrand, and call it the Lagrangian. In the examples that we considered so far, and in fact in most cases of interest, the Lagrangian is a function of a set of n variables q_i and their n first order derivatives $\dot{q}_i$:

$$S[q(t)] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\ dt \qquad (1.46)$$

We use "q" to denote the coordinates because the Lagrangian formalism is very general, and can be applied to all sorts of systems where the interpretation of the variables may differ from the interpretation of "x" as the position of some particle. The space on which q_i are the coordinates is called the configuration space. Its dimension is one half that of phase space. It is an intrinsic property of the physical system we are studying, and is a very useful concept. we should try to think as much as possible in terms of the configuration space itself, and not in terms of the particular

coordinates that we happen to use (the qs), since the latter can be changed by coordinate transformations. In fact one of the advantages of the Lagrangian formalism is that it is easy to perform coordinate transformations directly in the Lagrangian. We will see examples of this later on. Moreover there are situations such as that of a particle moving on a sphere when several coordinate systems are needed to cover the whole configuration space. The important thing is the sphere itself, not the coordinates that are being used to describe it. Which is not to say that coordinates are not useful in calculations they definitely are.

The pair $(q, \dot{q})$ determines a tangent vector (to some curve) at the point whose coordinate is q. Taken together q and $\dot{q}$ are coordinates on the tangent bundle of configuration space. The dimension of the tangent bundle equals that of phase space.

Now we will discuss the scope of Lagrangian mechanics. Among all those functions $q_i(t)$ for which $q_i(t_1)$ and $q_i(t_2)$ are equal to some arbitrarily prescribed values, the action functional has an extremum for precisely those functions $q_i(t)$ which obey the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{1.47}$$

provided such functions exist. This is straightforward to verify by means of the calculus of variations; indeed (suppressing indices)

$$\delta S = \int_{t_1}^{t_2} \left(\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}}\right) dt$$

$$= \int_{t_1}^{t_2} \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\delta q \frac{\partial L}{\partial \dot{q}}\right) dt \tag{1.48}$$

The total derivative term gives rise to a boundary term that vanishes because we are only varying functions whose values at t_1 and t_2 are kept fixed, so that δq is zero at the boundary. The Euler-Lagrange equations follow as advertized. The question is to what extent the equations of motion that actually occur in physics are of this form.

There are some that cannot be brought to quite this form by any means, including some of considerable physical interest; most of them involve dissipation of energy of some sort. An example is that of a white elephant sliding down a hillside covered with flowers. But then frictional forces are not fundamental forces. A complete description of the motion of the elephant would involve the motion of the atoms in the elephant and in the flowers, both being "heated" by friction. It is believed that all complete, fundamental equations are derivable from Hamilton's principle, and hence that they fall within the scope of Lagrangian mechanics or of quantum mechanics, which is structurally similar in this regard.

Generally speaking we expect Lagrangian mechanics to be applicable whenever there is no dissipation of energy. For many simple mechanical systems the Lagrangian equals the difference between the kinetic and the potential energy,

$$L(x, \dot{x}) = T(\dot{x}) - V(x) \tag{1.49}$$

For instance,

$$L = \frac{m\dot{x}^2}{2} - V(x) \Rightarrow \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{\partial V}{\partial x} - m\ddot{x} \tag{1.50}$$

Even in some situations where there is no conservation of energy, analytical mechanics applies. The simplest examples involve Lagrangians which depend explicitly on the time t. Dissipation is not involved because we keep careful track of the way that energy is entering or leaving the system.

Now for an example where the Lagrangian formalism is useful. Suppose we wish to describe a free particle in spherical polar coordinates

$$x = r\cos\phi\sin\theta \;,\; y = r\sin\phi\sin\theta \,, z = r\cos\theta \tag{1.51}$$

That is to say, we wish to derive the equations for $\ddot{r}$, $\ddot{\theta}$, and $\ddot{\phi}$. This requires an a mount of calculation, but the amount shrinks if we perform the change of variables directly in the Lagrangian:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}\left(r^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\, \dot{\phi}^{\,2}\right) \qquad (1.52)$$

Then we obtain the answer as the Euler-Lagrange equations from this Lagrangian. This is often the simplest way to perform a coordinate transformation even if the Lagrangian is not known, so that one first has to spend some time in deriving it.

A famous example for which $L \neq T - V$ is that of an electrically charged particle moving in an external electromagnetic field. This example is so important that we will give it in some detail. First of all, "external" signifies that we are dealing with an approximation, in which we ignore that the presence of the electrically charged particle will affect the electromagnetic field in which it moves. In many situations, this is an excellent approximation. The equations of motion to be derived are the Lorentz equations

$$m\ddot{x}_i = e\left(E_i(x,t) + \epsilon_{ijk}\dot{x}_j B_k(x,t)\right) \qquad (1.53)$$

The epsilon tensor occurring here may be unfamiliar . For the moment let me just say that the second term on the right hand side means the cross product of the velocity and the magnetic field. With this hint we should be able to follow the argument at least in outline, so we proceed. This example is more tricky than the previous ones, since the force depends not only on the position but also on the velocity of the particle (as well as explicitly on time, but this is no big deal). It turns out that in order to derive the Lorentz equation from a Lagrangian, we need not only one but four potentials, as follows:

$$E_i(x,t) = -\partial_i\, \phi\,(x,t) - \partial_t A_i(x,t) \quad B_i(x,t) = \epsilon_{ijk}\partial_j A_k(x,t) \qquad (1.54)$$

Here ϕ is known as the scalar potential and A_i as the vector potential. (They are both parts of a relativistic four vector.) It is possible to show that the following action yields the Lorentz equation when varied with respect to x:

$$S[x(t)] = \int \left(\frac{m\dot{x}^2}{2} + e\dot{x}_i A_i(x,t) - e\,\phi\,(x,t) \right) dt \qquad (1.55)$$

If we consider a time independent electric field with no magnetic field present, the Lorentz equation reduces to the more familiar form

$$m\ddot{x}_i \;=\; -e\partial_i\,\phi\,(x) \qquad (1.56)$$

This has the same form as Newton's Law of Gravity, if the potential is specified correctly. The reason why the full Lorentz equation is much more complicated has to do with the special relativity theory. The magnetic field is a relativistic complication. The relativistic version of Newton's of gravity is yet more complicated, and is given by Einstein's general relativity theory.

An important difference between gravity and electricity, also in the non relativistic case, is that particles couple to gravity through the mass, and all particles have mass while only some have electric charge. Moreover the mass serving as "charge" for gravitational forces is the same as the mass occurring on the left hand side of Newton's equations.

A strength of the Lagrangian formalism is the way it deals with constrained systems. An example of a constrained system is the pendulum with a rigid rod, which featured already in section (1.1) Deriving eq. (1.26) using a Cartesian coordinate system and Newton's methods is not trivial, since we have to figure out the constraint force acting in the direction of the rod. Using the Lagrangian formalism we simply write down the Lagrangian as $L = T - V$, change to polar coordinates, and quickly arrive at the Euler-Lagrange equation in the

form of eq. (1.26). For a more complicated system, the latter method wins hands down.

Another problem in the same vein is that of a particle constrained to move on the surface of a sphere

$$x^2 + y^2 + z^2 = 1 \tag{1.57}$$

The action is still given by

$$S[x, y, z] = \int \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt \tag{1.58}$$

This action is a functional of three functions which are constrained to obey the constraint (1.58), and we are going to extremize it only with respect to variations that obey the constraint. In this particular case there is an easy way to proceed. Using spherical polar coordinates the constraint becomes $r = 1$, while the angular coordinates can be varied freely. From eq. (1.52) we see immediately that the Lagrangian describing the motion in the angular directions is

$$L = \frac{m}{2}\left(\dot{\theta}^2 + sin^2\theta\, \dot{\phi}^2\right) \tag{1.59}$$

Of course this depended on the lucky accident that we have a good parametrization of the constraint surface. Correct results would also be obtained if we simply solve for

$$z = z(x, y) = \pm\sqrt{1 - x^2 - y^2} \tag{1.60}$$

and insert the result back into the action that describes the free particle, ie.

$$S[x, y] = \int \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}(x, y)^2)dt \tag{1.61}$$

Now we can vary x and y freely, except that they are not allowed to exceed one in absolute value.

There are some unavoidable weaknesses here. From eq. (1.61) it appears as if the configuration space were the unit disk in the plane, since x and y are not allowed to take values outside this disk. Or perhaps the configuration space is two copies of the unit disk, since there are two

branches of the square root? But the true configurations space is a sphere. What we see is a reflection of the known fact that it is impossible to cover a sphere with a single coordinate system our equations have only a "local" validity. This kind of difficulties will become more pronounced in the general problem we are heading for: Consider a Lagrangian L_0 defined on an n dimensional configuration space, with coordinates $q_1, \ldots, q_n$, and suppose that the system is confined to live in the $(n - m)$ dimensional submanifold defined by the m conditions

$$\Phi_I(q_1, \ldots, q_n) = 0 , \qquad 1 \leq I \leq m \tag{1.62}$$

Derive equations of motion consistent with this requirement. One way to do this is to solve for m of the $q_s, q_1, \ldots, q_m$ say, by means of the m conditions (1.61), and insert the result in the action. In general this will be a lot of hard work, and the difficulties we had with coordinatizing the sphere will recur with a vengeance.

The fact that the procedure avoids dealing with the constraint forces is a weakness too. If we try to design a pendulum in such a way that the approximation of a totally rigid rod holds for the kind of motion the pendulum will be subject to, we will want to know how strong the constraint force actually is. The method of Lagrange multipliers solves this problem, and at the same time has the advantage that the difficulties with coordinatizing the constraint surface are postponed to a later stage. The claim that we will verify is that extremizing the action

$$S[q] = \int L_0(q, \dot{q})\, dt \tag{1.63}$$

using only variations consistent with the constraints (1.62), is equivalent to extremizing the action

$$S[q, \lambda] = \int L_0(q, \dot{q}) + \lambda_1 \Phi_1(q) + \cdots + \lambda_m \Phi_m(q) dt \tag{1.64}$$

under arbitrary variations of the functions q and λ. The λs are the Lagrange multipliers, and are treated as new dynamical variables.

Indeed, when the action (1.63) is varied with respect to the λs we obtain the constraints (1.63) as equations of motion. When we vary with respect to the qs the resulting equations will contain the otherwise undetermined Lagrange multipliers, and it not obvious that these equations have anything to do with the problem we wanted to consider. But they do. Consider the analogous problem encountered in trying to find the extrema of an ordinary function $f(q)$ of the n variables q, subject to the m conditions $\Phi(q) = 0$. (Remember suppression of indices.) First suppose that we use the constraints to solve for m of the qs it will not matter which ones and call them y, leaving $n - m$ independent variables x. The extrema of $f(q)$ may be found through the equations

$$0 = \delta f = \delta x \partial_x f + \delta y \partial_y f \tag{1.65}$$

where, however, the variations δy are not independent variations, but have to be consistent with the constraints. In fact they are linear function of the δxs, given by the conditions

$$0 = \delta \Phi = \delta x \partial_x \Phi + \delta y \partial_y \Phi \tag{1.66}$$

This equation has to be solved for δy and the result inserted into eq. (1.64), which is therefore really an expression of the form $\delta x(\partial_x f + something\ else) = 0$. It does not imply $\partial_x f = 0$.

Since $\delta \Phi = 0$ for the variations we consider, nothing prevents us from rewriting eq. (1.64) in the form

$$0 = \delta f = \delta f + \lambda \delta \Phi = \delta x(\partial_x f + \lambda \partial_x \Phi) + \delta y\big(\partial_y f + \lambda \partial_y \Phi\big) \tag{1.67}$$

where the λs are arbitrary functions. The δys are still given in terms of the δxs, so it would seem at first sight that we cannot conclude that $\partial_x f + \lambda \partial_x \Phi = 0$, but and here comes the punch line in fact we can, provided we choose the so far arbitrary functions λ in such a way that $\partial_y f + \lambda \partial_y \Phi = 0$. Since the division of the qs into xs and ys was arbitrary, we see that the “restricted” way of finding the extremamaking

variations consistent with the constraints is equivalent to solving the $n + m$ equations

$$\Phi(q) = 0 \qquad \partial_q f + \lambda \partial_q \Phi = 0 \tag{1.68}$$

for q and λ. But these are precisely the equations that we obtain from the Lagrange multiplier method, in which we do not care about the constraints while varying the action. In all fairness though, we have not solved the equations, we have just derived them in a convenient way. As long as the constraints depend only on q (and not on $\dot{q}$) it is straightforward to generalize the argument from functions to functionals. From the action

$$S[q, \lambda] = S_0[q] + \int dt\, \lambda \Phi(q) \tag{1.69}$$

we rederive the constraints, together with the equations of motion

$$\frac{\delta S}{\delta q} = \frac{\delta S_0}{\delta q} + \lambda \partial_q \Phi = 0 \tag{1.70}$$

This is the analogue of the second equation (1.67). Written out, if $L = L(q, \dot{q})$ and if there is only one constraint, this is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \lambda \frac{\partial \Phi}{\partial q_i} \tag{1.71}$$

$$\Phi(q) = 0 \tag{1.72}$$

These equations have a simple interpretation. The constraint defines a surface in configuration space. We have modified the unconstrained systems by adding an extra force term $\lambda \partial_i \Phi$ to the equations. This force is directed along the gradient of the constraint function, which means that it acts in a direction orthogonal to the constraint surface. To ensure that the trajectory is confined to the surface we must choose the strength of the force (given by λ) in such a way that this is ensured. In other words, once we have solved these equations we know the strength of the constraint forces.

We have also refrained from committing us to any coordinate system adapted to the specific form of the constraint surface. That this is an

advantage becomes evident when we return to the problem of the particle on a sphere, starting from the Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda(x^2 + y^2 + z^2 - 1) \qquad (1.73)$$

The equations of motion (in these inertial coordinates) are

$$m\ddot{x} = 2\lambda x \qquad m\ddot{y} = 2\lambda y \qquad m\ddot{z} = 2\lambda z \qquad (1.74)$$

By inspection we see that there are three constants of the motion,

$$J_x = y\dot{z} - z\dot{y} \quad J_y = z\dot{x} - x\dot{z} \quad J_z = x\dot{y} - y\dot{x} \qquad (1.75)$$

At this point we go over to polar coordinates, using eqs. (1.51) with $r = 1$. The constants of the motion become

$$J_x = -\dot{\theta}\, sin\,\phi - \phi\, cos\,\phi\, \cos\theta \sin\theta$$

$$J_y = \dot{\theta} cos\,\phi - \phi\, sin\,\phi\, cos\theta sin\,\theta$$

$$J_z = \phi\, sin^2\theta \qquad (1.76)$$

It is possible to check directly, using the equations of motion for θ and ϕ, that these are constants of the motion but only J_z is "obviously" conserved. The coordinate system (θ, ϕ) somehow "hides" the others. By the way, the kinetic energy can be expressed as

$$T = \frac{m}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta\right) = \frac{m}{2}\left(\mathrm{J}_x^2 + \mathrm{J}_y^2 + \mathrm{J}_z^2\right) \qquad (1.77)$$

This is the angular momentum squared.

So far we have dealt only with holonomic constraints, that is constraints that involve the configuration space variables only. But consider a ball moving without friction across a plane. The configuration space has five dimensions: the position (x, y) of the center of mass, and three angular coordinates describing the orientation of the ball. Now suppose instead that the ball rolls without slipping. If we are given the position of the center of mass as a function of time then the motion of the ball is fully determined, which suggests that x and y are the "true" degrees of freedom, and that the constrained configuration space is two dimensional. But the situation is more complicated than that (and cannot

be described by holonomic constraints). It is impossible to solve for the angular coordinates in terms of x and y. Indeed from our experience with such things we know that the orientation of the ball at a given point depends on how it got there. Mathematically there is a constraint relating the velocity of the center of mass to the angular velocity; the ball spins in a way determined by the motion of its center of mass. Constraints that cannot be expressed as conditions on the configuration space are called anholonomic. The Lagrange multiplier method can be generalized so that it applies to this case as well, but we will not do so here.

Let us return to Newton's Third Law. It amounts to a restriction on the kind of forces that are allowed in the second law, and implies that there exist a set of constants of the motion, namely the momenta. (The terminology is a little unfortunate, since we will soon introduce something called "canonical momenta". They are indeed identical with the conserved momenta in simple cases, but logically there need be no connection.) Constants of the motion are useful when trying to solve the equations of motion, and Emmy Noether proved a theorem explaining when and why they exist. We present the proof for a Lagrangian of the general form $L = L(q, \dot{q})$, and afterwards we discuss a simple example. Let us say at the outset that the argument is quite subtle. Consider first an arbitrary variation of the action. According to eq. (1.48) the result is

$$\delta S = \int_{t_1}^{t_2} \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) + \left[\delta q \frac{\partial L}{\partial \dot{q}} \right]_{t_1}^{t_2} dt \tag{1.78}$$

In deriving the equations of motion the variations $\delta q(t)$ were restricted in such a way that the boundary terms vanish. This time we do something different. The variations are left unrestricted, but we assume that the function $q(t)$ that we vary around obeys the Euler-Lagrange equations. Then the only nonvanishing term is the boundary term, and

$$\delta S = \epsilon\big(Q(t_2) - Q(t_1)\big), \qquad \epsilon Q(t) \equiv \delta q \frac{\partial L}{\partial \dot{q}}. \tag{1.79}$$

Here and in the following ϵ is the constant occuring in $\delta q = \epsilon f$, where f is an arbitrary function of t. The point is to ensure that there is nothing infinitesimal about Q.

So far nothing has been assumed about the variations. Now suppose that, for the given Lagrangian, there exists a set of variations δq of some specified form

$$\delta q = \delta q(q, \dot{q}) \tag{1.80}$$

such that for these special variations

$$\delta S = 0 \tag{1.81}$$

It is understood that the Lagrangian is such that eq. (1.80) holds as an identity, regardless of the choice of $q(t)$, for the special variations δq. (Note that, given a Lagrangian, it is not always the case that such variations exist. But sometimes they do.) Next comes the crux of the argument. Consider variations of the particular kind that makes eq. (1.80) hold as an identity so that $\delta q = \epsilon f$ is a known function and restrict attention to $q(t)$s that obey the equations of motion.

With both these restrictions in force, we can combine eqs. (1.80) and (1.78) to conclude that

$$0 = \delta S = \epsilon\big(Q(t_2) - Q(t_1)\big) \tag{1.82}$$

The times t_1 and t_2 are arbitrary, and therefore we can conclude that $Q = Q(q, \dot{q})$ is a constant of the motion.

What this theorem does for us is to transform the problem of looking for constants of the motion to the problem of looking for variations under which the variation of the action is identically zero. Before we turn to examples we generalize the argument slightly, and state the theorem properly. Thus, suppose that there exists a special form of δq, such that

$$\delta S = \int_{t_2}^{t_1} \frac{d}{dt} \Lambda(q, \dot{q}) dt \tag{1.83}$$

Here Λ can be any function the important and unusual thing is that the integrand is a total time derivative. Then the quantity Q, defined by

$$\epsilon Q(q,\dot{q}) \;=\; \delta q_i \frac{\partial L}{\partial \dot{q}_i} - \Lambda(q,\dot{q}) \tag{1.84}$$

is a constant of the motion. This is easy to see along the lines we followed above.

The theorem can now be stated as follows:

Noether's theorem: To any variation for which δS takes the form (1.82), there corresponds a constant of the motion given by eq. (1.83).

We will have to investigate whether Lagrangians can be found for which such variations exist, otherwise the theorem is empty. Fortunately it is by no means empty, indeed eventually we will see that all useful constants of the motion arise in this way.

For now, one example but one that has many symmetries will have to suffice. Consider a free particle described by

$$L \;= \frac{m}{2}\dot{x}_i\dot{x}_i \tag{1.85}$$

Since only $\dot{x}$ appears in the Lagrangian, we can choose

$$\delta x_i = \epsilon_i \tag{1.86}$$

where ϵ_i is independent of time. Then the variation of the action is automatically zero, Noether's theorem applies, and we obtain a vector's worth of conserved charges

$$P_i \;= \frac{\partial L}{\partial \dot{x}_i} = \; m\dot{x}_i \tag{1.87}$$

We use the letter P rather than Q because this is the familiar conserved momentum vector whose presence is postulated in Newton's Third Law.

Another set of three conserved charges can be found easily, since

$$\delta x_i = \epsilon_{ijk}\epsilon_j x_k \Rightarrow \; \delta S \;=\; 0 \tag{1.88}$$

Here ϵ_i is again independent of t, and ϵ_{ijk} is the totally anti-symmetric epsilon tensor. Noether's theorem now implies the existence of another conserved vector, namely

$$L_i = \epsilon_{ijk} x_j \dot{x}_k \,. \tag{1.89}$$

This is the angular momentum vector.

We know that there is at least one more conserved quantity, namely the kinetic energy. Actually there are several, but the story now becomes a bit more complicated because we have to deal with variations for which the variation of the Lagrangian is a total derivative, as in eq. (1.83), rather than zero. Thus

$$\delta x_i = \epsilon \dot{x}_i \Rightarrow \delta S = \int \frac{d}{dt}\left(\frac{\epsilon m}{2}\dot{x}^2\right) dt \tag{1.90}$$

Using eq. (1.83) we obtain the constant of the motion

$$E = \frac{m}{2}\dot{x}_i \dot{x}_i \tag{1.91}$$

This is the conserved energy of the particle. There is yet another conserved quantity that differs from the others in being an explicit function of time but its total time derivative vanishes since it also depends on the time dependent dynamical variables. Thus

$$\delta x_i = -\epsilon_i t \Rightarrow \delta S = \int \frac{d}{dt}(-m\epsilon_i x_i)\, dt \tag{1.92}$$

Eq. (1.83) gives the conserved charge

$$Q_i = m x_i - t m \dot{x}_i \tag{1.93}$$

and it is easy to check that its total time derivative vanishes as a consequence of the equations of motion. Our analysis of the free particle ends here, but we will return to it in a moment, to show that the conserved quantities have a clear physical meaing.

What does it all mean? What does it mean for an action $S[q(t)]$ to admit variations $\delta q(t)$ leaving the action unaffected? To see this, select a solution $q(t)$ of the equations of motion. We know that this gives an

extremum of the action. Then consider $q'(t) = q(t) + \delta q(t)$, where the variation is of the special kind that leaves the value of the action unchanged. Obviously then $S[q'(t)] = S[q(t)]$, so that the extremum is not an isolated point in the space of all qs, but rather occurs for a set of qs that can be reached from each other by means of iteration of the special variation $\delta q(t)$, In other words, given a particular solution of the equations of motion, we can get a whole set of new solutions if we apply the special variation, without going through the work of solving the equations of motion again. This leads to an important definition: A symmetry transformation is any transformation of the space of functions $q(t)$ having the property that it maps solutions of the equations of motion to other solutions.

This is not a property of the individual solutions, but of the set of all solutions. The special variations occurring in the statement of Noether's theorem are examples of symmetry transformations. Given the converse of the statement that we proved, namely that any constant of the motion gives rise to a special variation of the kind considered by Noether, we observe that any constant of the motion arises because of the presence of a symmetry.

Let us interpret the symmetry transformations that we found for the free particle, beginning with eq. (1.85). This is clearly a translation in space. Therefore momentum conservation is a consequence of translation invariance. It is immediate that we can iterate the infinitesimal translations used in Noether's theorem to obtain finite translations, and the statement is that given a solution to the equations of motion all trajectories that can be obtained by translating this solution are solutions, too. To be definite, given that $(vt, 0, 0)$ is a solution for constant $v, (a + vt, b, c)$ is a solution too, for all real values of (a, b, c). Translation invariance acquires more content when used in the fashion of

Newton's third law, which we can restate as "the action for a set of particles has translation symmetry". For free particles this is automatic. When interactions between two particles are added, the law becomes a restriction on the kind of potentials that are admitted in

$$L = \frac{m_1}{2}\dot{x}_2^1 + \frac{m_2}{2}\dot{x}_2^2 - V\,(x_1, x_2) \qquad (1.94)$$

Indeed invariance under (1.85) requires that $V(x_1, x_2) = V(x_1 - x_2)$, which is a strong restriction.

Eq. (1.87) expresses the fact that the Lagrangian has rotation symmetry, while eq. (1.89) is an infinitesimal translation in time: Given a solution $x(t)$, the function

$$x'(t) = x(t + t_0) = x(t) + t_0\dot{x}(t) + o(t_0^2) \qquad (1.95)$$

is a solution too. So we can make the elegant summary that conservation of momentum, angular momentum and energy are consequences of symmetries under translations and rotations in space, together with translations in time. Eq. (1.89) expresses invariance under "boosts", since it changes all velocities by a constant amount. The free particle is exceptional because we can reach any solution by a symmetry transformation, starting from any given solution.

Having said all this, it is not true that every symmetry gives rise to a constant of the motion. Discrete symmetries like reflections, that do not arise by iterating an infinitesimal symmetry, are counterexamples . To sum up, symmetries are important from two quite different points of view. Given the equations they facilitate the search for solutions, but they also facilitate the search for the correct equations (if we believe that they should exhibit a certain symmetry). Noether's theorem is a tool for discovering symmetries, as well as for deducing their corresponding constants of the motion.

Chapter (2)

The central force and small oscillations

Section(2.1):The central force two-body problem

We start this section by studing the conic section . The theory of conic sections was one of the crowning achievements of the Greeks. Their results will be important in the gravitational two body problem, but the theory is no longer as well known as it deserves to be, so here is a brief account.

By definition a conic section is the intersection of a circular cone with a plane. The straight lines running through the apex of the cone are called its generators and we will consider a cone that extends in both directions from its apex. If you like, it is the set of one dimensional subspaces in a three dimensional vector space. Generically, the plane will intersect the cone in such a way that every generator crosses the plane once, or in such a way that exactly two of the generators miss the plane. Apollonius proved that the intersection is an ellipse in the first case, and a hyperbola in the second. There is a borderline case when exactly one generator is missing. Then the intersection is a parabola. We ignore the uninteresting case when the plane goes through the apex of the cone.

This is all very easy if we use the machinery of analytic geometry. For simplicity, choose a cone with circular base, symmetry axis orthogonal to the base, and opening angle 90 degrees. It consists of all points obeying

$$x^2 + y^2 - z^2 = 0 \tag{2.1}$$

Without loss of generality, the plane can be described by

$$cx + z = d\,. \tag{2.2}$$

Inserting the solution for z in the equation it is easy to see that the intersection is either an ellipse, a hyperbola, or a parabola provided you

recognize their equations, as I assume. The section is a circle if $c = 0$ and a parabola if $c = \pm 1$.

It is an interesting exercise to prove this in the style of Apollonius. Let the cone have arbitrary opening angle. Take the case when the plane intersects every generator once, let us say in the upper half of the cone. Place two spheres inside the cone, one above and one below the plane, and let them grow until each touches the plane in a point and the cone in a circle. (See Fig.(2.1).) This clearly defines the spheres uniquely. Denote the points by F_1 and F_2, and the circles by C_1 and C_2. Now consider a point P in the intersection of the cone

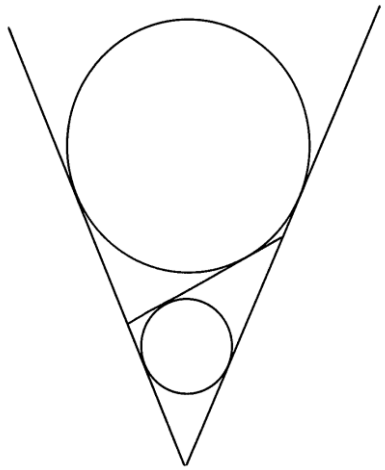


Figure (2.1).

A vertical cross section through Apollonius' proof but to understand the proof you have to think in three dimensions.

and the plane. The generator passing through P intersects the circles C_1 and C_2 in the points Q_1 and Q_2. Now the trick is to prove that the distance PF_1 equals the distance PQ_1, and similarly the distance PF_2 equals the distance PQ_2. This is true because the distances measure the lengths of two tangents to the sphere, meeting at the same point. It then follows that the sum of the distances PF_1 and PF_2 is constant and equal to the length of the segment of the generator between the circles C_1 and C_2, independently of which point P on the intersection we choose. This

property defines the ellipse. This is the proof that the intersection between the cone and the plane is an ellipse with its foci at F_1 and F_2. If you are unable to see this, consult an old fashioned geometry research.

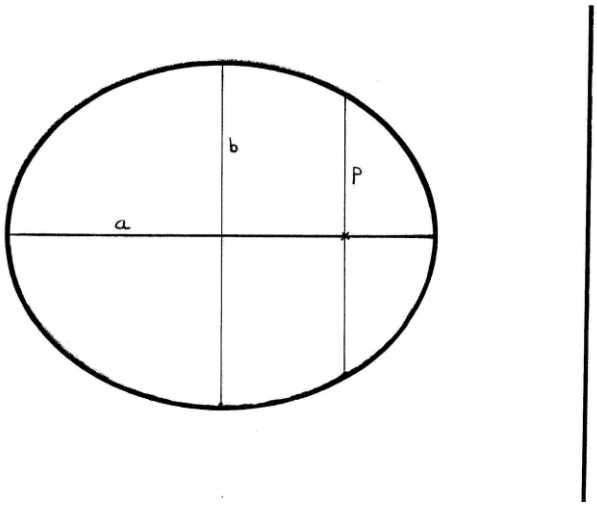


Figure (2.2).

An ellipse, with eccentricity 0.58, semi-major axis a, semi-minor axis b, and latus rectum p. The directrix appears on the right.

For our purposes it is convenient to define the ellipse somewhat differently. An ellipse of eccentricity $e < 1$ can be defined as the set of points whose distance from a given point, called a focus, is e times the distance to a straight line, called a directrix . For the circle $e = 0$, and the directrix is at infinity.

The latus rectum of an ellipse is a chord through the focus parallel to the directrix, and has length $2p$. Now place the origin of a coordinate system at that focus, with the x-axis pointing towards the directrix. The distance of a point on the ellipse to the focus is

$$r = e(\text{distance to the directrix}) = e\left(\frac{p}{e} - x\right) \qquad (2.3)$$

(To see this, note that the distance from the focus to the directrix is $\frac{p}{e}$.)

Otherwise expressed

$$r = p - er\cos\phi \Leftrightarrow \frac{p}{r} = 1 + e\cos\phi\ , \qquad (2.4)$$

where $\phi = 0$ gives the point closest to the directrix. For a general point on the ellipse we find

$$x^2 + y^2 = r^2 = (p - ex)^2 \Leftrightarrow \frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (2.5)$$

Where

$$a \equiv \frac{P}{1-e^2} \qquad b^2 = Pa = (1 - e^2)a^2. \qquad (2.6)$$

The major axis of the ellipse has length $2a$, and the minor axis has length $2b$. Finally the distance between the center and the focus equals ea. To see this we set $\phi = 0$ and $\phi = \pi$ in eq. (2.4), and calculate

$$\frac{r(\pi)-r(0)}{2} = \frac{P}{2}\left(\frac{1}{1-e} - \frac{1}{1+e}\right) = ea. \qquad (2.7)$$

Similar treatments can be given for the hyperbola, for which $e > 1$, and for the parabola, for which $e = 1$.

In our study of the two-body problem we will find it interesting to relate an ellipse centered at a focus to an ellipse centered at the origin. The latter is described in Cartesian coordinates by the complex trajectory

$$w(t) = a \cos t + ib \sin t\,. \qquad (2.8)$$

The parameter t must not be confused with the angle ϕ between the radius vector and the x-axis. Surprisingly, if we square this ellipse we obtain an ellipse with its focus at the origin:

$$z(t) = w^2 = \frac{a^2 + b^2}{2}\cos 2t + iab \sin 2t + \frac{a^2 - b^2}{2}$$
$$\equiv A \cos 2t + iB \sin 2t + EA, \qquad (2.9)$$

Where

$$EA = \sqrt{A^2 - B^2}\ , \qquad (2.10)$$

E is the eccentricity of an ellipse with semi-major axis A and semi-minor axis B, and consequently EA is the distance between its focus and its center. This is again an ellipse, but centered at one of its foci and traversed twice as the original ellipse is traversed once.

This trick was introduced by Karl Bohlin, working at the Pulkovo observatory in Russia in 1911. His point was that the transformation $w = \sqrt{z}$ is not analytic at the origin. This enabled him to deal with collisions between point particles, thought of as limiting cases of elliptical orbits whose eccentricity At the collision the particle presumably reverses its direction, but this is a rather singular occurence. In terms of the variable w it is an undramatic event.

Now we will discuss the central force two-body problem. Johannes Kepler spent his life pondering the observations of the solar system made by Tycho Brahe, and found that the motion of the planets around the sun follows three simple rules:

1. A planet moves along an ellipse with the sun in one of the foci.
2. The radius vector covers equal areas in equal times.
3. The square of the period of all the planets is proportional to the cube of their major axes.

To appreciate Kepler's work fully, note that there are important facts about the solar system (such as what the distances are) that do not follow simple rules. Moreover the observational data gave the planetary orbits projected on a sphere centered at a point which itself moves along an ellipse around the sun, so it was not obvious that they admitted of a simple description at all.

Newton derived Kepler's laws from his own Laws, with the necessary assumption that the force between the planets and the sun is directed along the radius vector (the force is central) and is inversely proportional to the square of the distance. This remains the number one success story of physics, so we should be clear about why this is so. Naively Kepler's laws may seem simpler than Newton's, but this is not so, for at least two reasons. One is that Newton's laws unify a large body of phenomena, from the motion of planets to the falling of stones close to the Earth. The

other reason is that improved observations reveal that Kepler's laws are not quite exact, and the corrections can be worked out mathematically from Newton's laws.

For Mercury (which is hard to observe) the eccentricity $e = 0.21$, for the Earth $e = 0.02$, and for Mars $e = 0.09$. Kepler's main concern was with Mars. If $e = 0$ the ellipse becomes a circle. Unbound motion through the solar system is described by hyperbolas with $e > 1$, but Kepler did not know this.

We want to derive Kepler's laws. As an approximation we assume that it suffices to treat the planets independently of each other. Moreover we assume that the precise shapes of the sun and the planets are unimportant and that they can be approximated as being pointlike. (In the Principia, Newton proved from properties of the inverse square law that this approximation is exact for spherical bodies.) Later on, we can go back to these assumptions and see if we can relax them this will give the corrections referred to above. So we have decided that the configuration space of our problem has six dimensions, spanned by the positions of the sun (X) and one planet (x_p), and we try the Lagrangian

$$L = \frac{M}{2}\dot{X}_i\dot{X}_i + \frac{m_p}{2}\dot{x}_{p_i}\dot{x}_{p_i} - V(X, x_p). \tag{2.11}$$

Depending on the form of the function V we may have to exclude the points $X = x_p$ from the configuration space we insist that the function V takes finite values only as a function on configuration space. Anyway this will give six coupled second order differential equations. In general they will not admit any simple solutions. Kepler's work implies that the solutions should be simple, so we try to build some symmetries into the problem. We aim for at least six conserved quantitities, one for each degree of freedom, since this should result in a soluble problem. Conservation of momentum and energy is already postulated, so we need

two more. The answer is rotational symmetry. It is not an objection that an ellipse is not symmetric under rotations. All we need is that if a particular ellipse is a solution, then any ellipse which can be obtained from it by means of rotations is a solution too, even if there is no planet moving along it due to the choice of initial conditions. It might seem that rotational symmetry is overdoing it, since it will yield three conserved quantitities, but for reasons fully explained by Hamilton-Jacobi theory only two of these are really useful. With translational and rotational symmetry in place, we find

$$V\,(X, x_p\,) \;=\; V\,(X \;-\; x_p\,) \;=\; V\,(|X \;-\; x_p\,|)\,. \qquad (2.12)$$

This works for any function V of one variable. Next we introduce coordinates $x_i \;=\; x_{p_i} - X_i$ which are invariant under translations, together with coordinates describing the center of mass. Then the center of mass coordinates decouple, and their equations can be solved and set aside. There remains a Lagrangian for a one-body problem, involving only three degrees of freedom:

$$L = \frac{m}{2}\dot{x}_i{}^2 - V(|x|) \qquad (2.13)$$

Here m is the reduced mass, almost equal to the mass of the planet since the sun is very heavy in comparison. The coordinate x_i vanishes at the center of mass of the system, which is well inside the sun, and can be approximately identified with the center of the sun. This maneouvre should be familiar from elementary mechanics, I just want to emphasize that it is translational symmetry in action.

Rotational symmetry implies the existence of a conserved vector

$$L_i = m\varepsilon_{ijk}x_j\dot{x}_k \qquad (2.14)$$

This vector is orthogonal to both x and $x^{\cdot}$, which means that the motion is confined to a plane orthogonal to the angular momentum vector. We are down to a two dimensional configuration space. To take maximal

advantage of spherical symmetry we introduce spherical polar coordinates, chosen so that the plane containing the orbit is at $\theta = \pi/2$. The Lagrangian simplifies to

$$L = \frac{m}{2}\left(\dot{x}^2 + r^2\,\dot{\phi}^{\,2}\right) - V(r). \tag{2.15}$$

We have used the constant direction of the angular momentum vector. But its magnitude is constant too. This happens because the Lagrangian (2.15) is invariant under translations in the angle ϕ. Using Noether's theorem we find the constant of the motion

$$L = \frac{\partial l}{\partial\,\dot{\phi}} = mr^2\,\dot{\phi}\,. \tag{2.16}$$

This equation is of considerable interest in itself. It says that

$$\dot{A} = \frac{r^2\,\dot{\phi}}{2} = \frac{1}{2m} = constant, \tag{2.17}$$

where $\dot{A}$ is the area covered by the radius vector per unit time. But this is Kepler's Second Law, which therefore holds for all central forces. We are on the right track. Together with Kepler's second law, energy conservation is enough to solve the problem. Using eq. (2.16) the conserved energy is

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\phi}^{\,2}\right) + V(r) = \frac{m\dot{r}^2}{2} + \frac{l^2}{2mr^2} + V(r) = constant. \tag{2.18}$$

This gives the formal solution

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - V(r)\right)}}\,. \tag{2.19}$$

It may or may not, depending on our choice of $V\,(r)$, be possible to do the integral in terms of elementary functions, but anyway this equation determines the function $r(t)$, and hence solves the problem. To find $\phi\,(t)$ we combine eqs. (2.16) and (2.19), and get

$$d\,\phi = \frac{ldt}{mr^2} = \frac{ldr}{r^2\sqrt{2m(E - \frac{l^2}{2mr^2} - V(r))}}\,. \tag{2.20}$$

This equation determines $\phi\ (r(t))$, and the central force problem is thereby fully solved at the formal level. If we are only interested in the form of the orbits, and not the time development, eq. (2.20) is all we need it will give us $\phi\ (r)$, and after inversion $r(\ \phi\ )$, which is the equation for the form of the orbit.

In the following we will study existence and stability of circular orbits there is a special kind of solution that we can look for directly, with no great expense of effort, namely circular orbits. From the expression (2.18) we see that the two body central force problem has been reduced to one dimensional motion in the effective potential

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r). \qquad (2.21)$$

A simple case is $V\ (r)\ =\ 0$, i.e. no force at all. In the effective one dimensional problem this corresponds to a repulsive V_{eff} the particle comes in from infinity, reaches a minimum value of r, and then disappears to infinity again. If we want a bound orbit the potential $V\ (r)$ must be attractive.

A circular orbit is one for which $r^{\cdot}\ =\ 0$ identically, which means that the particle is sitting at the bottom of the effective potential if it does have a bottom. The radius r of the circular orbit must obey

$$V'_{eff} = 0\ . \qquad (2.22)$$

If this happens at a local maximum of V_{eff} the solution is unstable, and unlikely to be realized in Nature. The orbit is stable under small perturbations if and only if

$$V''_{eff} > 0, \qquad (2.23)$$

at the value of r for which $V'_{eff} = 0$.

We look into these equations for the special choice

$$V(r) = -kr^{\beta} \qquad \Longrightarrow \qquad V_{eff} = \frac{l^2}{2mr^2} - kr^{\beta}\ , \qquad (2.24)$$

with α arbitrary. The radius of the circular orbit is found to be

$$r = \left(-\frac{l^2}{\beta k m}\right)^{\frac{1}{\beta+2}} \tag{2.25}$$

This makes sense only if $l^2 \neq 0$ a question of initial conditions and $\beta k < 0$. (The case $\beta = -2$ is obviously special.) Then the second derivative will be positive if and only if

$$\beta > -2\,. \tag{2.26}$$

Thus stability of the circular orbit requires that the force does not fall off too quickly with distance.

There is still the question whether small departures from the circular orbit will give rise to ellipses, or to something more complicated. This is really a question about the ratio between the time it takes for the planet to complete a full revolution in ϕ, to the time it takes to complete a full oscillation in r. If the orbit is an ellipse centered at a focus these times must be equal, and this is likely to happen only for a very special $V(r)$. For general bounded motion the amount by which the perihelion precesses during one period of the radial motion follows from eq. (2.20). It is

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{L dr}{r^2\sqrt{2m(E - V_{eff}(r))}}\,. \tag{2.27}$$

For the planets, Kepler's first law requires that $\Delta\phi = 2\pi$.

A final comment: if we do choose $\alpha = -1$ we have the problem that the energy is unbounded from below. We now see that this problem cannot be too serious, because

$$V_{eff}(r) = \frac{l^2}{2mr^2} - \frac{k}{r} \tag{2.28}$$

is in fact bounded from below whenever $l \neq 0$. The case when $l = 0$ is indeed troublesome from a physical point of view: the two bodies will collide, and we do not have a prescription for what is to happen after the

collision. The case of coinciding particles is not included in our configuration space.

What force laws are consistent with Kepler's First Law? Hooke's law does give elliptical orbits. This is most easily seen by transforming the Lagrangian back to Cartesian coordinates:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\phi}^{\,2}\right) - kr^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - k(x^2 + y^2). \qquad (2.29)$$

This is two harmonic oscillators of equal frequencies, and the corresponding Lissajous figures are indeed ellipses. They are not the right kind of ellipses however, since they are centered at the origin. Let us call them Hooke ellipses. They can be related to Kepler ellipses (centered at a focus). If

$$w = |w|e^{i\,\phi} = a\cos t + ib\sin t \qquad (2.30)$$

is a Hooke ellipse, then

$$z = w^2 = |w|^2 e^{2i\,\phi} = \frac{a^2-b^2}{2} + \frac{a^2+b^2}{2}\cos 2t + ib\sin 2t\,. \qquad (2.31)$$

This is indeed an ellipse centered at a focus. For the Hooke ellipse $w(t)$ we know that the force law is

$$\ddot{w} = -w \;\Rightarrow\; |\dot{w}\,|^2 + |w|^2 = 2\varepsilon. \qquad (2.32)$$

Moreover Kepler's second law holds, so that

$$|w|^{\,2}\frac{d\,\phi}{dt} = constant\,. \qquad (2.33)$$

This relates the parameter t to the angle ϕ between the radius vector and the major axis. The idea now is to introduce a new time τ , related to t in such a way that Kepler's second law holds also for the ellipse we get when we square the Hooke ellipse. Thus, remembering that the phase of Z is $2\,\phi$, we require

$$2|Z|^2\frac{d\,\phi}{d\tau} = constant\,. \qquad (2.34)$$

A suitable choice of the two constants gives the desired relation

$$\frac{d\tau}{dt} = \frac{|z|^2}{|w|^2} = |w|^2 \;\Leftrightarrow\; \frac{d}{d\tau} = \frac{1}{|w|^2}\frac{d}{dt}\,. \qquad (2.35)$$

At this point then we have obtained a function $Z(\tau)$ that describes motion in accordance with Kepler's First and Second Laws. It only remains to investigate what differential equation it obeys. But using eq. (2.32) this is a simple calculation:

$$\frac{d^2z}{dT^2} = \frac{1}{|w|^2}\frac{d}{dt}\left(\frac{1}{|w|^2}\frac{dw^2}{dt}\right) = \frac{2}{|w|^2}\frac{d}{dt}\left(\frac{\dot{w}}{\bar{w}}\right) = -4\epsilon\frac{z}{|z|^3} \tag{2.36}$$

whereϵ is the constant energy of the Hooke ellipse. This is precisely Newton's force law for gravity. So we conclude that Kepler's First and Second Laws together imply the inverse square law, with the potential

$$V(r) = -\frac{k}{r} \tag{2.37}$$

There is no other solution. The argument is water tight because every Kepler ellipse can be obtained from a Hooke ellipse using Bohlin's trick, and a Hooke ellipse arises only in the harmonic oscillator potential.

To confirm our conclusion let us go back to eq. (2.20), which gives a formal solution for the form of the orbit. We choose eq. (2.37) for $V\ (r)$, and we also perform the substitution

$$u = \frac{1}{r} \implies du = -\frac{dr}{r^2} \tag{2.38}$$

The result is

$$\mathrm{d}\,\phi = -\frac{\mathrm{Ldu}}{\sqrt{2\mathrm{mE}-\mathrm{L}^2\mathrm{u}^2+2\mathrm{mku}}} \tag{2.39}$$

This defines u as a trigonometric function of ϕ, with the energy $E < 0$ and the phase ϕ_0 as undetermined integation constants. In fact

$$\phi = \phi_0 - \arccos\frac{\frac{\mathrm{L}^2\mathrm{u}}{\mathrm{mk}}-1}{\sqrt{1+\frac{2\mathrm{EL}^2}{\mathrm{mk}^2}}} \tag{2.40}$$

Inverting this, and cleaning up the answer a little, we obtain

$$\frac{L^2}{mkr} = \frac{L^2u}{mk} = 1 + \sqrt{1+\frac{2EL^2}{mk^2}}\cos\left(\phi - \phi_0\right) \tag{2.41}$$

The constant ϕ_0 is the value of the coordinate ϕ for which the planet is at its perihelion, that is when it is closest to the sun. Comparing to eq. (2.4) we read off that the eccentricity of the ellipse is

$$e = \sqrt{1 + \frac{2EL^2}{mk^2}} \quad . \tag{2.42}$$

The semi-major axis of the ellipse is

$$a = \frac{p}{1-e^2} = \frac{l^2}{mk}\frac{mk^2}{2|E|l^2} = \frac{k}{2|E|} . \tag{2.43}$$

The solution remains valid also for $E > 0$, in which case it describes a hyperbola with $e > 1$. Physically this is an unbound trajectory, like that of a spaceship heading for the stars.

Now we will discuss Kepler's Third Law. Kepler's Third Law awaits proof. Here is a simple one: since the areal velocity is constant, the period T is simply related to the area of the ellipse. For the special case we are looking at eq. (2.7) .

$$\frac{1}{2m}T = A = \pi ab. \tag{2.44}$$

Remembering that $b^2 = ap$, and using eq. (2.21) to identify the latus rectum p, gives

$$T^2 = \frac{4\pi^2 m^2 a^3 p}{l^2} = 4\pi^2 \frac{m}{k} a^3 \ . \tag{2.45}$$

This is Kepler's Third Law. We see that it holds only to the extent that we can regard the reduced masses m of all the planets to be the same.

Here is a more involved proof, using the full force of our solution (2.19). We rewrite it using our expressions for a and e:

$$dt = \sqrt{\frac{ma}{k}}\frac{rdr}{\sqrt{-\frac{l^2}{2m|E|}+2ar-r^2}} = \sqrt{\frac{ma}{k}}\frac{rdr}{\sqrt{a^2e^2-(r-a)^2}} . \tag{2.46}$$

We can do the integral if we can find a substitution that simplifies the integral

$$\int^r \frac{rdr}{\sqrt{1-(r-1)^2}} , \tag{2.47}$$

and such a substitution is readily found. Reinserting the constants we set

$$r = a + ae\cos\sigma \tag{2.48}$$

Now we can do the integral. With a suitable choice of the integration constant we obtain.

$$t = \sqrt{\frac{ma^3}{k}}(\sigma + e\sin\sigma). \tag{2.49}$$

Taken together, eqs. (2.28-2.29) provide a parametric representation of the orbit, and we can read off its period

$$T = 2\pi\sqrt{\frac{ma^3}{k}}. \tag{2.50}$$

This is Kepler's Third Law once again.

In the following study self-similarity and the virial theorem. Kepler's Third Law says that, given a solution, one can simply enlarge it to get another solution provided one also slows down the rate at which things are happening. It is really a consequence of mechanical similarity or self-similarity, a kind of symmetry not covered by Noether's theorem. It arises as follows. Take the Lagrangian

$$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q), \tag{2.51}$$

where for once we do not use the dot notation because we will soon have two different time parameters to reckon with. Assume that the potential is homogeneous of degree β, meaning that there exists a real number β such that for any real non-zero number λ

$$V(\lambda q) = \lambda^{\beta} V(q). \tag{2.52}$$

There could be several variables q_i. For simplicity I write only q. Let us also change the time scale, and define a new function $q'$by

$$q(t) \to q'(t') = \lambda q(t),\ t' = \lambda^{\frac{2-\beta}{2}} t. \tag{2.53}$$

It follows that

$$\frac{dq'}{dt'} = \frac{dt}{dt'}\frac{d}{dt}\big(\lambda q(t)\big) = \lambda^{\frac{\beta}{2}}\frac{dq}{dt}. \tag{2.54}$$

We can now check that our rescalings represent a symmetry because, under this transformation,

$$L\left(q,\frac{dq}{dt}\right) \longrightarrow \; L\left(q',\frac{dq'}{d\tau}\right) = \lambda^{\beta} L\left(q,\frac{dq}{dt}\right). \qquad (2.55)$$

This has the effect of changing the value of the action with a constant factor, and it follows that $q'(t')$ is an extremum of $S[q'(t')]$ if $q(t)$ is an extremum of $S[q(t)]$. In this sense it is a symmetry of the action. The harmonic oscillator has a potential $V \sim q2$, homogeneous with $\beta = 2$. Scaling symmetry is present with $t' = t$. Given a solution $q(t)$ there is another solution that is a blown up version of this, with amplitude a factor of λ larger. Because $t = t'$ the period of the oscillations are unaffected by the scaling, and we see without looking at any explicit solutions that the period of the oscillations are independent of their amplitudes. Galilei first made this observation while celebrating mass in the cathedral of Pisa. Newton's law of gravity uses a homogeneous potential with $\beta = -1$, so similarity holds with $t \to t' = \lambda^{\frac{3}{2}} t$. Two ellipses with the same shape (and the planetary orbits are all close to circular) will therefore have their periods and their axes related by

$$R = R' = \lambda R \qquad T \longrightarrow T' = \lambda^{\frac{3}{2}} T \Longrightarrow \frac{T'^2}{T^2} = \frac{R'^3}{R^3}. \qquad (2.56)$$

This is Kepler's Third Law for the third time.

Another dramatic theorem can be proved for self-similar systems. It is called the Virial Theorem, and relates the time averages of the kinetic and potential energies to each other. If it exists, the time average of a function $f(t)$ is defined by

$$\langle f \rangle = \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} f(t')dt' \qquad (2.57)$$

For the argument to follow it is important that the time average of the derivative of a bounded function is zero, i.e.

$$\langle \frac{df}{dt} \rangle = \lim_{t\to\infty} \frac{1}{t}\big(f(t) - f(t_0)\big) = 0 \qquad (2.58)$$

whenever $f(t) < \infty$ for all t.

We are ready to study the time average of the kinetic energy, given the assumptions that the system obeys Newton's law

$$m\ddot{x}_i = -\partial_i V(x), \tag{2.59}$$

that the potential is homogeneous of degree β, that the motion is bounded in space, and that the velocities are everywhere finite. On the other hand we are not restricting the index i. It could run between $1 \leq i \leq 3N$, in which case we are actually studying an N-body problem; this could be a cluster of galaxies under the tentative assumption that the cluster is a bound system, or it could be 10^{23} atoms confined in a box. Regardless of the number of variables Euler's theorem on homogeneous functions states that

$$V(\lambda x) = \lambda^{\beta} V(x) \Rightarrow x_i \partial_i V(x) = \beta V(x). \tag{2.60}$$

Then

$$2\langle T\rangle = \langle m\dot{x}^2\rangle = \langle \frac{d}{dt}(m x_i \dot{x}_i - m x_i \ddot{x}_i\rangle = -\langle x_i m \ddot{x}_i\rangle$$
$$= \langle x_i \partial_i V(x)\rangle = \beta\langle V(x)\rangle . \tag{2.61}$$

This is the conclusion we were after.

For bounded motion in homogeneous potentials

$$2\langle T\rangle = \beta\langle V\rangle, \tag{2.62}$$

where β is the degree of homogeneity of $V(x)$.

For the inverse square law the virial theorem implies that

$$\langle 2T + V\rangle = 0 \implies \langle E\rangle = \langle E\rangle = \langle T + V\rangle = -\langle T\rangle \leq 0 . \tag{2.63}$$

This is the familiar fact that motion bounded by gravity can take place only if the total energy is negative. For the harmonic oscillator we deduce that the time average $\langle T\rangle$ and $\langle V\rangle$ are equal.The virial theorem has been used by astronomers to estimate the masses of clusters of stars and clusters of galaxies, assuming that they are gravitationally bound. This

led to the first evidence for dark matter. The calculation in eq. (2.61) is of interest even for non-potential forces, if we break it off after the first line:

$$2\langle T\rangle = -\langle x_i f_i\rangle \tag{2.64}$$

If the forces are the constraint forces keeping an ideal gas contained inside a box, we can use this relation to deduce the ideal gas law. We turn the sum into an integral, recall the definition of the pressure P as force per unit area, and apply Gauss' law to the result:

$$2\langle T\rangle = P\int x_i dA_i = P\int \partial_i x_i\, dV = 3PV \tag{2.65}$$

If we are willing to identify $\langle T\rangle$ with (a factor times) the temperature T we have Boyle's Law,

$$PV = RT \tag{2.66}$$

The three-body problem three masses interacting according to Newton's Law of Gravity is not soluble in the sense that the two-body problem is. The number of conserved quantities is the same in both problems, and for the nine degrees of freedom in the three-body problem this is not enough. But the three-body problem is very important, and in fact motivated many of the developments that we will come to later on.

A natural first step is to look for special exact solutions, which may be used as starting points for perturbation theory, or in other ways. An interesting example was found by Lagrange. Let us begin by assuming that the motion takes place in a plane, and use complex numbers $z_i(t)$ to denote the trajectories. The equations are

$$\begin{aligned}
\ddot{z}_1 &= -m_2\frac{z_1-z_2}{|z_1-z_2|^3} - m_3\frac{z_1-z_3}{|z_1-z_3|^3}\\
\ddot{z}_2 &= -m_3\frac{z_2-z_3}{|z_2-z_3|^3} - m_1\frac{z_2-z_1}{|z_2-z_1|^3}\\
\ddot{z}_3 &= -m_1\frac{z_3-z_1}{|z_3-z_1|^3} - m_2\frac{z_3-z_2}{|z_3-z_2|^2}
\end{aligned} \tag{2.67}$$

We assume that the center of mass is at rest,

$$m_1 z_1 + m_2 z_2 + m_3 z_3 = 0 \tag{2.68}$$

The particles form a triangle, with sides represented by

$$w_1 = z_3 - z_2\,,\ w_2 = z_1 - z_3\,, \qquad w_3 = z_2 - z_1 \tag{2.69}$$

It turns out to be convenient to rewrite the equations of motion in terms of these variables. A small calculation shows them to take the form

$$\ddot{w}_1 = -m\frac{w_1}{|w_1|^3} + m_1 a$$

$$\ddot{w}_2 = -m\frac{w_2}{|w_2|^3} + m_2 a \tag{2.70}$$

$$\ddot{w}_3 = -m\frac{w_3}{|w_3|^3} + m_3 a \tag{2.71}$$

where $m = m_1 + m_2 + m_3$ and

$$a = \frac{w_1}{|w_1|^3} + \frac{w_2}{|w_2|^3} + \frac{w_3}{|w_3|^3} \tag{2.72}$$

This time we are looking for a special solution, not at the general case. So let us assume that the triangle is an equilateral one,

$$w_2 = e^{\frac{2\pi i}{3}} w_1 \ , \quad w_3 = e^{\frac{4\pi i}{3}} w_1 \tag{2.73}$$

Then $a = 0$, and the only equation we need to solve is

$$\ddot{w}_1 = -m\frac{w_1}{|w_1|^3} \tag{2.74}$$

This we know how to do. To interpret the solution, solve for

$$mz_1 = m_3 w_2 - m_2 w_3 \Longrightarrow m^2|z_1|^2 = ({m_2}^2 + m_2 m_3 + {m_3}^2)|w_1|^2 \tag{2.75}$$

and so on. A small calculation then shows that

$$\ddot{z}_1 = -\frac{({m_2}^2 + m_2 m_3 + {m_3}^2)^{\frac{3}{2}}}{m^2}\frac{z_1}{|z_1|^3} \tag{2.76}$$

and similarly for the other two particles. Hence the particles are all being accelerated towards their common center of mass, with “effective masses” that take an unexpected form. Each individual particle travels on an ellipse, but the tree of them do so in unison, in such a way that they always span an equilateral triangle.

A special case of this solution is of considerable physical interest. Let one

of the particles have negligibly small mass. Then the remaining pair trace out the same orbits that they would follow in the absence of the third member.

Nevertheless the three particles span an equilateral triangle. This is the origin of the two Lagrange points on the orbit of a planet, where small bodies may sit. To draw this conclusion we should also investigate whether the exact solution is stable under small perturbations. This turns out to be the case. The Lagrange points we have found are called L_4 and L_5, since there is another set of three equilibria on the axis through the two bodies although they are of less interest since they are unstable.

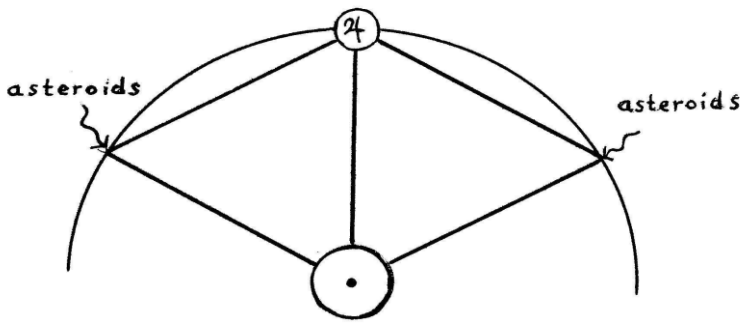


Figure (2.3).

The two stable Lagrange points in Jupiter's orbit. The Greek and Trojan asteroids lie within roughly one astronomical unit from the Lagrange points.

About a thousand asteroids have been found close to the Lagrange points L_4 and L_5 on the orbit of Jupiter. They are known as the Greek and Trojan asteroids (with names taken from Homeros). It has been observed that the Earth's Lagrange points are suitable places where an alien civilisation could place a satellite surveying the Earth; however, when the STEREO spacecrafts passed through (in 2009) they found nothing of the sort. But what can we say about the three-body problem in general? To celebrate the sixtieth anniversary of King Oscar II of Sweden and Norway a large prize was offered for a solution to the following

problem: “For a system of arbitrarily many mass points that attract each other according to Newton’s laws, assuming that no two points ever collide, give the coordinates of the individual points for all time as the sum of a uniformly convergent series whose terms are made up of known functions.” The prize was awarded to Henri Poincar´e, who did not solve the problem but whose contribution laid the foundations of the modern theory of possibly chaotic dynamical systems. For the three-body problem a solution was in fact found by Karl F. Sundman in 1912. He did express a generic solution as a uniformly convergent power series in $t^{\frac{1}{3}}$. The catch is that the series converges very slowly. It is estimated that, in order to get useful information, one would have to sum the first $10^{8000000}$ terms. Hence the interest in the exact general solution dwindled from that point in.

With the advent of the computer it has become possible to follow a large number of solutions to the three body problem on the screen, with no special effort. The zoo of solutions include ones where the third body escapes from the system, leaving the remaining pair more tightly bound than before.

Section (2.2): The small oscillations

Now we will discuss Small oscillations . An important class of equations that we can actually solve are the linear ones. Their importance stems from the fact that departures from equilibrium are described by equations that are linear to first order near a minimum, most smooth potentials look like a collection of harmonic oscillators. When left alone they are too simple to be interesting, but once we couple them to external forces many delightful things happen. Or harmful things, depending on your point of view. Either way the subject is of interest to engineers and physicists alike. It is perhaps worth remarking that once we allow the external

forces to depend on time the phase space picture is affected in a significant way the phase space trajectories are now allowed to cross themselves, because the external conditions may have changed by the time the trajectory returns to the initial point.

We begin by Forced oscillations with a single degree of freedom obeying the equation

$$m\ddot{x} + kx = 0 \Leftrightarrow \ddot{x} + w^2 x = 0 \ , \ w = \sqrt{\frac{k}{m}} \qquad (2.77)$$

The assumption that the string constant k is positive was slipped in when we wrote $\omega_2 = \frac{k}{m}$. The general solution is

$$x(t) = a_1 \cos \omega t + a_2 \sin \omega t = Acos\,(\omega t + \delta)$$
$$= Re\left[\alpha e^{iwt}\right] \qquad (2.78)$$

where

$$A = \sqrt{{a_1}^2 + {a_2}^2}, \ tan\,\delta = \frac{a_2}{a_1} \qquad (2.79)$$

and α is a complex constant carrying information about both the amplitude A and the phase δ. It is in fact highly convenient to write the solution as the real part of a complex solution. This is a trick that works because the equation is linear: two solutions can be added, and the result is still a solution. It does not matter if we take the real part before or after the addition but care must be exercised if we multiply two solutions together.

The total energy of the oscillator is

$$E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 = \frac{m}{2}w^2 A^2 \qquad (2.80)$$

It is manifestly independent of time. And the subject is exhausted. To make matters more interesting we introduce an external force, and consider the forced oscillator

$$m\ddot{x} + kx = f(t) \Leftrightarrow \ddot{x} + w^2 x = \frac{1}{m} f(t) \qquad (2.81)$$

To solve this equation, with some specified function $F(t)$, we appeal to the general theory of ordinary differential equations: it is enough to find one particular solution and then add the general solution of the homogeneous equation (the one with $F = 0$). It goes without saying that some forces are more important than others. An interesting example is the periodically varying force

$$F(t) = f\, cos\,(\Omega t + \phi) \tag{2.82}$$

which we will solve under the assumption that $\Omega \neq \omega$. For a particular solution we try the Ansatz

$$x_{part}(t) = B\, cos\,(\Omega t + \phi) \tag{2.83}$$

Plugging this into the equation will determine B. Adding the general homogeneous solution we find the noteworthy general solution

$$x(t) = A cos\,(\omega t + \delta) + \frac{f}{m(\omega^2 - \Omega^2)}\, cos\,(\Omega t + \phi) \tag{2.84}$$

The remarkable thing is that the amplitude will get very large if the system is driven by a periodic force whose frequency is close to the natural frequency of the system. This phenomenon is called resonance. In fact, it may well be that resonance drives the system out of the regime in which the harmonic oscillator approximation is valid. The solution is a superposition of two harmonics with different frequencies. Close to resonance the frequencies of the two harmonics almost coincide, and we will observe the phenomenon of beats. Recall that

$$cos\,\omega_1 t + cos\,\omega_2 t = 2\, cos\,\frac{(\omega_1 - \omega_2)t}{2}\, cos\,\frac{(\omega_1 + \omega_2)t}{2} \tag{2.85}$$

If $\omega_1 \approx \omega_2$ this can be regarded as a vibration with frequency $\omega \approx \omega_1 \approx \omega_2$, but with an amplitude modulated by a sine-wave of very low frequency. Our case is a bit more complicated because of the differing amplitudes and phases of the harmonics that we superpose. Let us assume that $\Omega = \omega + \epsilon$, and rewrite the general solution as the real part of the complex amplitude

$$x(t) = \alpha e^{i\omega t} + \beta e^{i(\omega+\epsilon)t} = (\alpha + \beta e^{i\epsilon t})e^{i\omega t} \qquad (2.86)$$

Provided ϵ is small compared to ω it makes sense to look at this as if the system were oscillating at frequency ω, but modulated by a slowly time varying amplitude whose square is given by

$$|\alpha + \beta e^{i\epsilon t}|^2 = |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| \cos(\epsilon t + a\, phase) \qquad (2.87)$$

Thus the amplitude varies between $|\beta| - |\alpha|$ and $|\beta| + |\alpha|$. If ϵ is small we see beats. To deal with a force of a quite arbitrary form we call on the complex numbers to do a trick. Observe first that

$$\ddot{x} + \omega^2 x = \frac{d}{dt}(x^{\cdot} + i\omega x) - i\omega(x^{\cdot} + i\omega x) \qquad (2.88)$$

Hence we define the complex variable

$$z = x^{\cdot} + i\omega x \qquad (2.89)$$

in terms of which the original equation becomes the first order equation

$$z^{\cdot} - i\omega z = \frac{1}{m}F(t) \qquad (2.90)$$

Since the variable is now complex this is actually a pair of real first order equations, so we are not violating any rules concerning how to count the degrees of freedom. Once we have solved for $z(t)$ we recover $x(t)$ through

$$x(t) = \frac{1}{\omega} Im[z(t)] \qquad (2.91)$$

Solving the first order system is straightforward. A particular solution to eq. (2.90) can be written in integral form. Adding the general solution of the homogeneous equation we obtain

$$z(t) = e^{i\omega t}(z_0 + \frac{1}{m}\int_{-\infty}^{t} F(t')e^{-i\omega t'}dt') \qquad (2.92)$$

The particular solution is known as Duhamel's integral. Let us take a look at the energy budget. Assume that $z(-\infty) = 0$, so that the system starts out at rest. The energy we end up with is then

$$E(\infty) = \frac{m}{2}|z(\infty)|^2 = \frac{1}{2m}\left|\int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt\right|^2 \qquad (2.93)$$

There is a net transfer of energy if the external force has a Fourier component corresponding to the intrinsic frequency of the oscillator. If the force acts for a short time as compared with the natural time scale in the problem, which is set by ω the exponential in the integrand can be ignored, and the energy imparted to the system is the kinetic energy associated to the momentum $\int F dt$.

In the following study damped and forced oscillations. Friction is a particular kind of coupling to the environment. In many situations the strength of the coupling grows with velocity. Examples include particles encountering air resistance, or more generally particles moving in a viscous medium. Hence we try the equation

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \Leftrightarrow \ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0\,, \omega_0^2 = \frac{k}{m}, \lambda = \frac{\gamma}{2m} \qquad (2.94)$$

This is the damped oscillator.

As an aside, note that the damped oscillator is realised in electical circuit theory. A capacitor leads to a voltage difference $VC = q/C$, where q is the charge and C the capacitance. Across an inductor there is a voltage difference $V_L = L\dot{I}$, where $I = \dot{q}$ is the current and L the inductance. Coupling the capacitor, the inductor, and a resistor with resistance R in a series we obtain the equation

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0\,. \qquad (2.95)$$

Thus $\omega_0 = 1/\sqrt{LC}$. There are numerous useful analogies between electrical, acoustical, and mechanical systems but this is by the way.

To find the solution of the damped oscillator we try the Ansatz $x = e^{i\omega t}$. Inserting this, and cancelling the exponential, leads to the equation

$$-\omega^2 + 2i\lambda\omega + \omega_0^2 = 0 \Leftrightarrow \omega = i\lambda \pm \sqrt{\omega_0^2 - \lambda^2} \qquad (2.96)$$

Hence the general solution is

$$x(t) = a_1 e^{-\lambda t + i\sqrt{\omega_0^2 - \lambda^2}} + a_2 e^{-\lambda t - i\sqrt{\omega_0^2 - \lambda^2}} \tag{2.97}$$

There are two qualitatively distinct cases to consider. If the damping is weak, that is if $\omega_0^2 > \lambda^2$, the general solution is a damped oscillation

$$x(t) = e^{-\lambda t}\left(a_1 e^{iw_{eff}t} + a_2 e^{-iw_{eff}t}\right) \tag{2.98}$$

Note that the effective frequency w_{eff} is always smaller than the "bare" frequencyω_0. If, on the other hand, the damping is strong enough so that $\omega_0^2 < \lambda 2$ the general solution decays exponentially without oscillations. It is instructive to consider the critical case $\omega_0^2 = \lambda^2$. In the limit we obtain the solution $ae^{-\lambda t}$, but this cannot be the general solution since it contains only one integration constant. In fact the general solution is

$$x(t) = (a + bt)e^{-\lambda t} \tag{2.99}$$

This must be the general solution because it contains two integration constants.

Finally we come to the forced and damped oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = F(t) = f\cos(\Omega t) \tag{2.100}$$

where we restrict ourselves to the important case of a periodic external force. We rewrite this in its complex formulation as

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f}{m}e^{i\Omega t} \tag{2.101}$$

make the Ansatz

$$x_{part}(t) = Be^{i\Omega t} \tag{2.102}$$

insert in the equation, cancel the exponential, and solve for B. The result is

$$(-\Omega^2 + 2i\lambda\Omega + \omega_0^2)B = \frac{f}{m} \Leftrightarrow B = \frac{1}{m}\frac{f}{\omega_0^2 - \Omega^2 + 2i\lambda\Omega} \tag{2.103}$$

The amplitude is the absolute value of B. In fact

$$B = be^{i\delta}, b = \frac{1}{m}\frac{f}{\sqrt{\left(\omega 20_0^2 - \Omega\right)^2 + 4\lambda^2\Omega^2}}, \tan\delta = \frac{2\lambda\Omega}{\Omega^2 - \Omega_0^2} \tag{2.104}$$

The real solution is then

$$x(t) = b\cos(\Omega t + \delta) \qquad (2.105)$$

Of research we should add the general homogeneous solution, but since this decays to zero we ignore it. When the force has acted for some time the transient part will be effectively zero, and a steady state described by the above solution sets in.

Because of the damping the amplitude no longer goes to infinity at resonance, but it still has a pronounced maximum. The phase δ shows that the system does not oscillate in phase with the external force. When is very small so is δ, meaning that the oscillations follow the external force with no phase shift. The sign of δ is negative, but its tangent switches sign at $\Omega = \omega 0$. Indeed δ always lies between 0 and $-\pi$, meaning that the system always lags behind the force. For very large the phase shift is close to $-\pi$, because the accelerations are large, and the acceleration of an oscillator is 180^0 out of phase with its displacement.

We should look at the energy budget of the damped oscillator. We begin with the damped but free oscillator ($F = 0$). Then the time derivative of the energy function is

$$\dot{E} = \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2\right) = \dot{x}(m\ddot{x} + kx) = -\gamma\dot{x}^2 = -2m\lambda\dot{x}^2 \qquad (2.106)$$

Now consider the forced and damped oscillator in steady state. It is still losing energy to the frictional forces at exactly this rate. This energy must then be supplied as work by the external force, which is something we may want to know about. Inserting the solution (2.105) we obtain the energy supplied by the external force per unit time as

$$\dot{E} = 2m\lambda b^2\Omega^2 \sin^2(\Omega t + \delta) \qquad (2.107)$$

For most purposes it will be enough to know the time average of this quantity. Since the amplitude b depends on this time average is a function of the frequency . Making use of a relation from (2.104) we obtain

$$I(\Omega) = \langle \dot{E} \rangle = \frac{f^2\lambda}{m} \frac{\Omega^2}{(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2} \tag{2.108}$$

This has a maximum at $\Omega = \omega_0$. Let us suppose that $\Omega = \omega_0 + \epsilon$, and that both ϵ and the damping λ are small compared to ω_0 .In this approximation we can replace $I(\Omega)$ with the function

$$I(\epsilon) = \frac{f^2\lambda}{4m} \frac{1}{\epsilon^2 + \lambda^2} \tag{2.109}$$

This is the famous Lorentzian line shape function, giving the intrinsic width of spectral absorption (and emission) lines although the dominating effect when astronomers observe them is the broadening due to the Doppler shift, since the atoms are in thermal motion.

We see that the width of the Lorentzian line shape function grows with the damping λ. On the other hand the total area under the curve is (almost) independent of λ, as follows from the calculation

$$\int_0^\infty I(\Omega)d\Omega = \int_{\omega_0}^\infty I(\epsilon)d\epsilon \approx \int_{-\infty}^\infty I(\epsilon)d\epsilon = \frac{f}{4m}\int_{-\infty}^\infty \frac{1}{1+\frac{\epsilon^2}{\lambda^2}} \frac{d\epsilon}{\lambda} = \frac{f\pi}{4m} \tag{2.110}$$

Let us now consider N degrees of freedom. The variables are x_i, and the equations to be solved are

$$m_{ij}\ddot{x}_j + \gamma_{ij}\dot{x}_j + k_{ij}x_j = F_i(t),\ 1 \leq i,j \leq N \tag{2.111}$$

When $\gamma_{ij} = 0$ they follow from the Lagrangian

$$L = \frac{1}{2}m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}k_{ij}x_ix_j + x_iF_i \tag{2.112}$$

and it follows that m_{ij} and k_{ij} are symmetric when considered as matrices. It is less obvious that

$$\gamma_{ij} = \gamma_{ji} \tag{2.113}$$

Nevertheless it is true, because of a subtle argument from statistical physics that we take on trust here.

To start with we now try to solve the equations for a set of harmonic oscillators that interact among themselves, but not with the outside world. That is

$$m_{ij}x_j + k_{ij}x_j = 0 \tag{2.114}$$

We know that the general solution must contain $2N$ integration constants, or equivalently N complex integration constants. So we try the Ansatz

$$xi = \alpha_i e^{i\omega t} \tag{2.115}$$

Inserting this into the equation, and cancelling the exponential, gives the matrix equation

$$\left(-\omega^2 m_{ij} + k_{ij}\right)\alpha_j = 0 \tag{2.116}$$

There is a non-zero solution for the vector α_i if and only if the determinant of the matrix vanishes, that is if and only if

$$\left|k_{ij} - \omega^2 m_{ij}\right| = 0 \tag{2.117}$$

This is an eigenvalue equation for ω. It is also a polynomial equation of order N, which means that it will have N not necessarily distinct roots. On physical grounds all the roots must be real and positive. If not there would be exponentially growing or decaying solutions, and the system would be unstable.

The conclusion so far is that we have found N in general distinct frequencies, called normal frequencies. The next step is to actually solve eq. (2.116) for the eigenvector α_i. It will be determined only up to an overall complex constant. Doing so for each normal frequency we end up with N particular solutions, each coming with one undetermined complex number. Adding all these particular solutions together gives the general solution to our problem.

A simple example of a system admitting two normal modes is given by two identical pendula connected by an elastic spring.

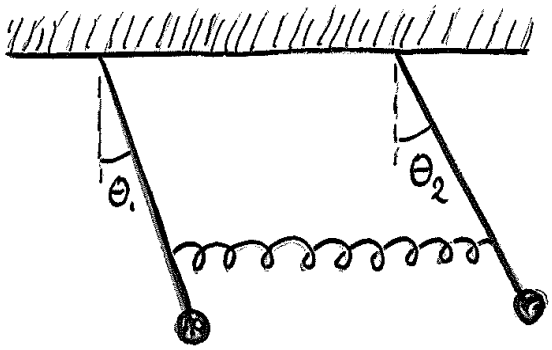


Figure (2.4)

Two identical pendula connected by an elastic spring.

Assuming that the oscillations are small we use the Lagrangian

$$L = T - V = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 - \theta_1^2 - \theta_2^2 - k\left(\theta_1 - \theta_2\right)^2\right) \qquad (2.118)$$

To find the normal modes we must diagonalise T and V. This is achieved by

$$q_1 = \frac{1}{\sqrt{2}}(\theta_1 + \theta_2),\ q_2 = \frac{1}{\sqrt{2}}(\theta_1 - \theta_2) \qquad (2.119)$$

The Lagrangian becomes

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - \omega^2 q_2^2),\ \omega^2 = 1 + 2k \qquad (2.120)$$

The normal modes have a simple interpretation. If $q_2 = 0$ the two pendula move in phase with the original frequency (equal to 1), if $q_1 = 0$ the pendula move in opposite phase with a frequency increased by the spring. As an afterthought we observe that, in this case, the two normal modes could have been identified from the outset using only symmetry considerations.

The general solution for θ_1, θ_2 is thus a sum of two harmonicsand if the difference in frequency between the two is small we should be able to observe the interesting phenomenon of beats, as in eq. (2.87).

When we couple a set of harmonic oscillators to an external force it is the normal modes that count. As the simplest example, consider

$$\ddot{x}_1 + \omega_0^2 x_1 + k(x_1 - x_2) = f\, cos\Omega$$

$$\ddot{x}_2 + \omega_0^2 - k(x_1 - x_2) = 0 \qquad (2.121)$$

In terms of the normal modes $q_1 = x_1 + x_2, q_2 = x_1 - x_2$ this becomes

$$\ddot{q}_1 + \omega_0^2 q_1 = f\, cos\Omega t\; \ddot{q}_2 + (\omega_0^2 + 2k)q_2 = f\, cos\Omega t \qquad (2.122)$$

Each normal mode gives rise to its own resonance frequency. The mathematical problem has been reduced to that of studying two or in general N independent harmonic oscillators.

Finally, let us consider eq. (2.114) for N degrees of freedom. We can decouple the degrees of freedom in three steps: First we diagonalise the symmetric matrix k_{ij}, using rotations. Its eigenvalues must be positive . In the next step we rescale its eigenvectors so that the diagonalised matrix becomes the identity matrix. These coordinate changes are conveniently described on the Lagrangian level,

$$L = \frac{1}{2}\sum_{i,j}\left(\dot{x}_i k_{ij}\dot{x}_j - x_i m_{ij} y_j\right) = \frac{1}{2}\sum_{i,j}\left(\dot{y}_i \lambda_i \delta_{ij}\dot{y}_j - y_i m'_{ij} y_j\right)$$

$$= \frac{1}{2}\sum_{i,j}\left(\dot{z}_i \delta_{ij}\dot{z}_j - z_i \frac{1}{\sqrt{\lambda_i}} m'_{ij} \frac{1}{\sqrt{\lambda_j}} z_j\right) \qquad (2.123)$$

For once we did not use the Einstein summation convention, since it would not work here. Note that m'_{ij} is not the same matrix as m_{ij}, since the coordinate system was rotated. In the final step we define the symmetrix matrix

$$m''_{ij} = \frac{1}{\sqrt{\lambda_i}} m'_{ij} \frac{1}{\sqrt{\lambda_j}} \qquad (2.124)$$

we diagonalise it, and arrive at the Lagrangian

$$L = \frac{1}{2}\sum_i (\dot{u}_i \dot{u}_i - \omega_i^2 u_i u_i) \qquad (2.125)$$

The normal modes $u_i$are now decoupled, and the equations of motion trivial to solve.

Chapter (3)

Rotation and Legendre transformations

Section (3.1): Rotation and rigid bodies

Most people have played with spinning tops, and know that their dynamics is very rich. The subject is of considerable practical importance, say to spacecraft designers. To understand it we must understand the Lie group of rotations in three dimensional space, and this is where we begin the story.

Now we will discuss rotations . In the plane, the mathematics of rotations is fairly trivial. Any rotation takes place around a fixed point, and is uniquely characterized by its angle of rotation ϕ . Its action on the coordinates is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.1}$$

What does this mean? Actually it can mean two quite different things: A passive coordinate transformation changing the coordinates used to describe a given point, or an active rotation moving a given point to another point described by different values of the coordinates. These are two very different operations, although the formulæ describing them are mostly identical. One must stay awake during calculations.

This attended to, we observe that rotations form a group. A group is a set of objects g_1, g_2, etc, together with a rule for combining them, so that $g_1 \circ g_2 = g_3$ is again a member of the set. This rule must be associative. One of the elements is the identity element e, and has the property that $e \circ g = g \circ e = g$ for any other element g, and finally every element g possesses an inverse g^{-1}such that $g \circ g^{-1} = g^{-1} \circ g = e$. In general it may or may not be true that $g_1 \circ g_2 = g_2 \circ g_1$. For the two dimensional

rotation group the elements can be written $g(\phi)$, so the number of elements are continuously infinite. The set of all group elements thus forms an abstract space, called the group manifold. In this case the group manifold is a circle, coordinatized by ϕ. It is also a Lie group. This means that the continuous parameters of a product is given by an analytic function of the parameters of its factors. Thus

$$g(\phi_1) \circ g(\phi_2) = g\left(\phi_3(\phi_1, \phi_2)\right) \tag{3.2}$$

where ϕ_3 is an analytic function of its arguments. In our example

$$\phi_3 = \phi_1 + \phi_2 \tag{3.3}$$

which is certainly analytic. The surprising thing about Lie groups is that they can be almost completely understood through a careful study of their properties in a small neighbourhood of the identity element. All rotation groups are Lie groups.

Rotations in three dimensional space are hard to understand, in the first place because rotations do not commute in general. Rotations can be represented by matrices acting on vectors,

$$x_i' = R_{ij}x_j \tag{3.4}$$

By definition this is a rotation if the lengths of all vectors are preserved,

$$x_i'x_i' = x_ix_i \tag{3.5}$$

Hence the matrix R_{ij} must be subject to some restrictions. On the other hand it is immediate from the definition that rotations form a group. To see how the matrix is restricted, we observe that scalar products must be preserved too, and then we perform a little calculation:

$$x_i'y_i' = R_{ki}x_iR_{kj}y_j = x_iR_{ki}R_{kj}y_i = x_i\delta_{ij}y_i = x_iy_i \tag{3.6}$$

Here δ_{ij} is the Kronecker delta. Since the vectors are arbitrary this is equivalent to

$$\delta_{kl}R_{ki}R_{lj} = R_{ki}R_{kj} = j_{ij} \tag{3.7}$$

In matrix notation this is

$$R^T R = RR^T = 1 \tag{3.8}$$

The group properties can be now be checked on the level of matrices. Matrix multiplication is associative, and moreover

$$R_1 R_2 = R_3 \Rightarrow R_3^T R_3 = (R_1 R_2)^T R_1 R_2 = R_2^T R_1^T R_1 R_2 = 1 \tag{3.9}$$

This confirms that R3 is a group element. We observe that

$$1 = det\, RR^T = det\, R\, det\, R^T = (det\, R)^2 \Rightarrow det\, R = \pm 1 \tag{3.10}$$

The group can be restricted by insisting that $det\, R = 1$. This restriction can be formulated in terms of the epsilon tensor:

$$R_{im} R_{jn} R_{kp} \epsilon_{mnp} = \epsilon_{ijk}\, det\, R = \epsilon_{ijk} \tag{3.11}$$

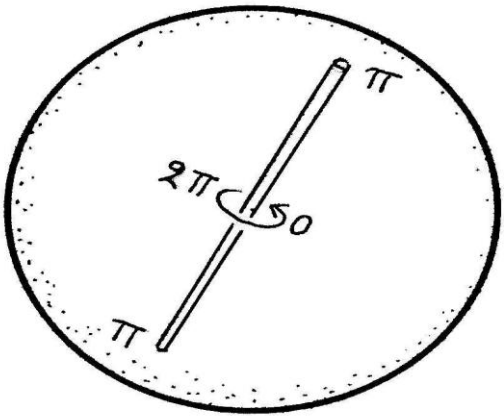


Figure (3.1)

To coordinatize the set of all rotations (the group manifold of the rotation group), choose a rotation axis and a number along it. Note that the two endpoints of the axis represent the same rotation.

where the definition of the determinant was used. In N dimensions the restricted rotation group is called the special orthogonal group, denoted $SO(N)$. "Special" refers to the restriction that the determinant equals one. If matrices with determinant -1 are admitted the group is called $O(N)$, and includes reflections.

Euler's theorem states that any rotation in 3-space is a rotation around a fixed axis. To prove it, note that an orthogonal matrix is unitary, and the eigenvalues of a unitary matrix always take the form

$$\lambda = e^{i\phi} \tag{3.12}$$

for some angle ϕ. One can prove this by diagonalizing the unitary matrix. But an orthogonal matrix is a real matrix too, which means that

$$\det(R - \lambda 1) = 0 \Rightarrow \left(\det(R - \lambda 1)\right)^* = \det(R - \lambda_1)^* \tag{3.13}$$

Hence complex eigenvalues, if they occur, must occur in pairs because their complex conjugates are eigenvalues too. Since the number of eigenvalues of an $SO(3)$ matrix is odd, one of them must be real, and in fact it must equal one because the determinant does. The corresponding eigenvector is the fixed axis of rotation. Note that rotations in even dimensions (like two dimensions) work differently. Note also that rotations can be hard to grasp: if the rotation axes of R_1 and R_2 are known, what is the rotation axis of $R_1 R_2$?

The group manifold of $SO(3)$, that is to say the set of all rotations, is easy to visualize. The set of all rotation axes can be identified with the surface of a sphere, or more precisely with the pairs of antipodal points where the axis cuts the sphere. An arbitrary rotation is determined by its axis and an angle ϕ, hence we can think of the set of all rotations as a solid ball with the identity matrix at its center, with ϕ as a radial coordinate, and with antipodal points on its surface identified because of the periodicity of ϕ. It sounds simple, but there are some subtleties. The topology of this group manifold is such that there are closed curves, starting and ending at the identity element, that cannot be shrunk to zero in a continuous way. There is a famous trick one can play with a pair of scissors sliding along a belt, to verify that this property has tangible consequences.

We will need a coordinate system on the group manifold, and we choose the Euler angles for this purpose. From our present perspective it is a little difficult to make them appear natural. We introduce them by brute force, as follows: Define

$$R_{\Psi} = \begin{pmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$R_{\phi} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.14)$$

Compute

$$R(\psi, \theta, \phi) = R_{\psi} R_{\theta} R\varphi_{\phi} =$$

$$\begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \qquad (3.15)$$

In the absence of the argument that makes this construction appear natural, how do we know that this is correct in the sense that an arbitrary rotation matrix can be expressed in terms of the Euler angles? Recall that every orthogonal matrix can be thought of as three orthonormal column vectors. By inspection we see that ψ and θ can be chosen so that the third column agrees with any unit vector, and further inspection of the remaining columns shows that they are restricted only to the extent needed for them to fill out a right handed orthonormal triad. All this is true provided that

$$0 \leq \phi < 2\pi \ , \ 0 \leq \psi < 2\pi \ , \ 0 \leq \theta \leq \pi \qquad (3.16)$$

We accept this, and now we have a coordinate system on the group manifold $SO(3)$ available whenever we need one. In writing eq. (3.15) we proved that an arbitrary rotation can be effected by first rotating around the z-axis, then around the x-axis, and finally around the z-axis again. There are $3 \times 2 \times 2 = 12$ different ways of choosing the axes

here, leading to 12 different ways of parametrizing an arbitrary rotation matrix. This has the consequence that whenever Euler angles are encountered in the literature, one must check which of the 12 possible definitions that was used.

A final point: I advertized that Lie groups can be understood through astudy of what they look like close to the origin. To see how for the orthogonal groups, note that an orthogonal matrix can be written in the form

$$R = e^{A} \equiv 1 + A + \frac{1}{2!}A^2 + \cdots \tag{3.17}$$

This is an orthogonal matrix if and only if the matrix A is anti-symmetric,

$$AT = -A \tag{3.18}$$

We can define a special class of curves in the group by

$$R(t) = e^{tA} \Rightarrow R(0) = 1 \tag{3.19}$$

We would like to claim that every rotation group element can be written as e^{tA}, for some choice of anti-symmetric matrix A. This is actually so for any $SO(3)$ rotation, but not for the additional reflections present in $O(3)$. To see how it works, we begin by using the definition (3.17) to conclude that

$$\exp\left[\mathrm{t}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos \mathrm{t} & \sin \mathrm{t} \\ -\sin \mathrm{t} & \cos \mathrm{t} \end{pmatrix} \tag{3.20}$$

So the statement is true for any two dimensional rotation matrix with unit determinant. But we can use Euler's theorem to reduce the three dimensional case to the two dimensional one we simply adapt our coordinates so that one of the axes points along the eigenvector of the given but otherwise arbitrary rotation matrix.

We can now see by means of a second order Taylor expansion what the noncommutativity means in terms of what goes on close to the identity element:

$$R_1(t_1)R_2(t_2){R_1}^{-1}(t_1){R_2}^{-1}(t_2)$$

$$\approx \left(1 + t_1A_1 + \frac{{t_1}^2}{2}{A_1}^2\right)\left(1 + t_2A_2 + \frac{{t_2}^2}{2}{A_2}^2\right)\left(1 - t_1A_1 + \frac{{t_1}^2}{2}{A_1}^2\right)\left(1 - t_2A_2 + \frac{{t_2}^2}{2}{A_2}^2\right)$$

$$\approx 1 + t_1t_2(A_1A_2 - A_2A_1) \tag{3.21}$$

The group elements commute only if the commutator $[A_1, A_2]$ vanishes. Anti-symmetric matrices can be added and multiplied with real numbers, while staying anti-symmetric. Therefore they form a vector space, known as the tangent space of the group manifold at the identity element. This vector space is also known as the Lie algebra of the group, because the commutator $[A_1, A_2]$ provides us with a peculiar way to obtain a third "vector" from any given pair of "vectors". For $SO(3)$ any anti-symmetric matrix can be expressed as a linear combination of the three matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(3.22)

They obey

$$[A_1, A_2] = A_3 \qquad [A_2, A_3] = A_1 \qquad [A_3, A_1] = A_2$$

(3.23)

They form a basis for the Lie algebra and would perhaps look more familiar had we multiplied the Lie algebra elements A_i with an imaginary factor of i, to make them Hermitian matrices. The idea here (a grand one.) is that it works backwards too from a Lie algebra of commutators one can reconstruct a Lie group. In index notation an arbitrary element of the $SO(3)$ Lie algebra can be written as

$$A_{ij} = \epsilon_{ijk} j \omega_k \tag{3.24}$$

where ω_i is a vector. If a physical system is subject to a time dependent rotation $R(t)$ such that

$$\dot{R}_{ij}(0) = \epsilon_{ijk}\omega_k \tag{3.25}$$

then ω_i is referred to as the angular velocity vector at time $t = 0$. In space it is pointing along Euler's fixed axis of rotation. If we return to the picture of the group manifold as a solid ball with antipodal points on its surface identified, we recognize the angular velocity vectors as a set of tangent vectors sitting at the identity element, that is at the center of the ball.

In the following study rotating coordinate systems . Newton's Laws take a simple form only in inertial coordinate systems, that is in coordinate systems adapted to the straight lines referred to in his First Law (or to Absolute Space, as Newton himself expressed things). If the coordinate system is not inertial more work is needed it would not have been easy to figure out the laws if all experiments had been carried out on roundabouts. Let $X_i(t)$ denote the trajectory of a particle relative to an inertial system, and let $x_i(t)$ denote the same trajectory described relative to a rotating coordinate system. Assume that the coordinates agree at $t = 0$. Then there exists at-dependent rotation matrix R such that

$$X_i(t) = R_{ij}(t)x_j(t) = \left(\delta_{ij} + \epsilon_{ikj}w_k(t) + o(t^2)\right)x_j(t) \tag{3.26}$$

At time $t = 0$ the inertial and rotating coordinate systems agree, and we assume that the angular velocity vector ω_i is constant. The time derivatives are related by

$$\dot{X}_i = R_{ij}\dot{x}_j + \dot{R}_{ij}x_j \tag{3.27}$$

At $t = 0$ (a point in time picked by convention) our assumption gives

$$\dot{X}_i = \dot{x}_i + \epsilon_{ikj}\omega_k x_j \tag{3.28}$$

The dot-notation can get confusing at this point, so it may be helpful to write this as an operator relation

$$\dot{X}_i = D_{tij}x_j = \left(\delta_{ij}\frac{d}{dt} + \epsilon_{ikj}w_k\right)x_j \qquad (3.29)$$

If we differentiate twice we obtain Newton's second law in the form

$$F_i = m\,\ddot{X}_i = mD_{tij}D_{tjk}x_k =$$

$$m\ddot{x}_i + 2m\epsilon_{ijk}\omega_j\dot{x}_k + m\epsilon_{ijk}\epsilon_{kmn}\omega_j\omega_m x_n \qquad (3.30)$$

By the $\epsilon - \delta$-identity, or equivalently by means of the formula for repeated cross products, this is

$$m\ddot{x}_i = F_i - 2m\epsilon_{ijk}\omega_j\dot{x}_k + m(\omega^2\delta_{ij} - \omega_i\omega_j)x_j \qquad (3.31)$$

This is what Newton's second law looks like on the roundabout. The "extra" terms on the right hand side are known as inertial or fictitious forces but they feel real enough. The third term on the right hand side of eq. (3.31) is known as the cen-trifugal force. It is responsible for the repulsive part of the effective potential (2.18) in the two body problem. The second term is the Coriolis force. It is perpendicular both to the velocity $\dot{x}_i$ and to the angular velocity ω_i. To see that such a term must be there, consider a free particle moving out from the center on a frictionless rotating disk, and anchor the coordinate system to the disk. Alternatively, consider the Foucault pendulum somewhere on earth. Choose acoordinate system with a vertical z-axis, and let the pendulum perform small oscillations in the x-y-plane. We ignore all terms of second order in the angular velocity i of the earth. In particular we ignore the centrifugal force, but the angular velocity enters the equations through the Coriolis term:

$$\ddot{x} = -k^2x + 2\Omega_z\dot{y} \qquad \ddot{y} = -k^2y - 2\Omega_z\,\dot{x} \qquad (3.32)$$

Here $\Omega z = |\Omega|\,sin\,\theta$, and θ is the latitude of the pendulum. To solve the equations, introduce the complex variable $w = x + iy$ and make the Ansatz $w = e^{\lambda t}$:

$$\ddot{w} + k^2w + 2i\Omega_z\dot{w} = 0 \Rightarrow \lambda^2 + k^2 + 2i\Omega_z\lambda = 0\,. \qquad (3.33)$$

The solutions are

$$\lambda = -i\Omega_z \pm i\sqrt{k^2 + {\Omega_z}^2} \approx -i\Omega_z \pm i\left(k + \frac{{\Omega_z}^2}{2k}\right) \approx -i\Omega_z \pm ik \qquad (3.34)$$

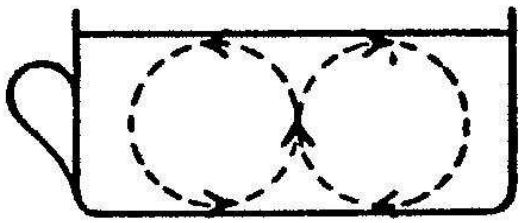


Figure (3.2)

This picture illustrates Einstein's article on Beer's Law, so presumably this is what the great man's tea cup looked like. Therefore

$$w = e^{-i\Omega_z t}\left(c_1 e^{ikt} + c_2 e^{-ikt}\right) \qquad (3.35)$$

At the equator, where $z = 0$, we can arrange the initial conditions so that the pendulum swings in a plane with $y = 0$ (say), but at non-zero latitudes the plane will turn at a rate given by $sin\,\theta$ times the rate of rotation of the earth. At the Poles it will make one revolution in 24 hours. It is interesting to see how the fictitious forces shape the face of the earth. They explain the erosion of river beds, as shown by Albert Einstein. He began by looking at a simpler problem: Take a cup of tea, with some tea leaves in it, and set the tea into rapid rotation by means of a spoon. You will see the tea leaves gather into a little clump at the bottom, just at the center of the cup. This happens because the flow velocity of the tea is zero at the edges of the cup, and rises gradually from zero as one moves away from the edges. Therefore the angular velocity will be greater at the tea surface, and lower close to the bottom. Therefore the outwards directed centrifugal force will be greater at the surface, and as a result a vertical velocity profile will be set up in the cup. Therefore the tea leaves staying close to the bottom due to their specific weight will be carried to the center of the cup, as was to be explained.

A very similar story can be told about rivers, if there is a river bend. The vertical velocity profile will carry fast moving surface water to the outer river bank, and will cause erosion there. The latter will therefore suffer from more severe erosion than the inner bank, and the net result is that the whole river will start to move outwards it is meandering. The Coriolis force is also of importance to rivers, provided the rivers are large and do not bend too much. In fact it causes Beer's Law, an observation due to geographers according to which large rivers on the Northern hemisphere undermine their right banks, and large rivers on the Southern hemisphere undermine their left banks. Beer's law does not apply to small and bending rivers, where the centrifugal force dominates.

Even more dramatically, fix a coordinate system in the earth so that one of its axes point at the sun. It will make one revolution per year, and create a centrifugal force just balancing the gravitational attraction felt by the sun. This enables the sun to stand still. Now let the coordinate system follow the earth in its daily rotation. As a result a huge centrifugal force will act on the sun, completely overwhelming the gravitational force. The Coriolis force steps in to the rescue and prevents the sun from disappearing into outer space.

Now we will discuss the inertia tensor . A rigid body is defined as a number of particles at fixed distances from each other, or equivalently as a continuous mass distribution whose shape is fixed. In the latter case the mass points m are replaced by mass elements ρdV . The center of mass of the body can move, and its overall orientation can change by means of a rotation. Any rotation. Hence the configuration space of a rigid body is $R^3 \times SO(3)$. This is not only 6 dimensional, it has a non-trivial topology since the topology of the group manifold $SO(3)$ is non-trivial.

We introduce an internal coordinate system whose origin is fixed at some point O within the body. These coordinates are called x_i. We also

introduce an inertial coordinate system so that O has coordinates X_i. The velocity of any given point x_i within the body with respect to the inertial coordinate system is denoted v_i, and the velocity of the point O is denoted V_i. The body rotates around O with angular velocity i, but the internal coordinate system (with origin O) is not rotating. Make sure to note that the velocity v_i is not the time derivative of x_i, and that we will not be bothered by fictitious forces. The various quantities we introduced are related by

$$v_i = V_i + \dot{x}_i = V_i + \epsilon_{ijk}\Omega_k x_j \tag{3.36}$$

To what extent does it matter where the origin O is placed? If we translate it so that

$$x_i \rightarrow x_i' = x_i + a_i \tag{3.37}$$

we see that the instantaneous velocities transform according to

$$v_i = V_i + \epsilon_{ikj}\Omega_k x_j = V_i - \epsilon_{ikj}\Omega_k a_j + \epsilon_{ikj}\Omega_k x_j'$$
$$= V_i' + \epsilon_{ikj}\Omega_k x_j' \tag{3.38}$$

We observe that the angular velocity is unchanged, so we can talk of the angular velocity of the body without caring about the point relative to which we compute it. The kinetic energy of a rigid body is

$$T = \sum \frac{m}{2}\left(V_i + \epsilon_{ikj}\Omega_k x_j\right)^2$$
$$= \sum\left(\frac{m}{2}V^2 + m\epsilon_{ikj}V_i\Omega_k x_j + \frac{m}{2}(\epsilon_{ikj}\Omega_k x_j)^2\right) \tag{3.39}$$

(The sum runs over all the particles in the body, even though no explicit indices have been placed on them.) This can be rewritten in terms of the total mass M as

$$T = \frac{M}{2}V^2 + \frac{1}{2}\sum m\left(x^2\delta_{ij} - x_i x_j\right)\Omega_i\Omega_j + \epsilon_{ikj}V_i\Omega_k \sum m x_j \tag{3.40}$$

If O, the origin within the body, is placed at the center of mass the last term vanishes. Of research it also vanishes if the origin coincides with a

point that is fixed in space. Now define the inertia tensor with respect to O,

$$I_{ij} = \sum m\left(x^2\delta_{ij} - x_i x_j\right) \tag{3.41}$$

We observe that the angular momentum with respect to O is

$$L_i = \sum \epsilon_{ijk} x_j m \dot{x}_k = \sum \epsilon_{ijk}\epsilon_{kmn} m x_j \Omega_m x_n = I_{ij}\Omega_j \tag{3.42}$$

Unlike the angular velocity, both the angular momentum and the inertia tensor change if we shift the position of O. Let us assume that O is placed at the center of mass. Then

$$T = \frac{1}{2}MV^2 + \frac{1}{2}I_{ij}\Omega_i\Omega_j = \frac{1}{2}M^{-1}P_iP_i + \frac{1}{2}I_{ij}^{-1}L_iL_j \tag{3.43}$$

The inertia tensor describes the resistance of the body to changes of its rotation, just as the mass describes its resistance to changes of its translational state. But the former is a tensor, not a scalar, and hence much harder to grasp. Written out, the inertia tensor is

$$I = \begin{pmatrix} \sum m(x_2^2 + x_3^2) & -\sum m x_1 x_2 & -\sum m x_1 x_3 \\ -\sum m x_2 x_1 & \sum m(x_1^2 + x_3^2) & -\sum m x_2 x_3 \\ -\sum m x_3 x_1 & -\sum m x_3 x_2 & \sum m(x_1^2 x_2^2) \end{pmatrix} \tag{3.44}$$

This is a symmetric matrix, hence it can be diagonalized by rotations. This means that we can rotate the internal coordinate system so that

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{3.45}$$

The eigenvalues are known as moments of inertia. The eigenvectors are known as principal axes of the body, and are orthogonal to each other.

The moments of inertia are all positive since, for an arbitrary vector n_i,

$$n_i I_{ij} n_j = \sum m(n^2x^2 - (n_i x_i)^2) \geq 0 \tag{3.46}$$

Indeed the individual terms are all positive. A vanishing moment of inertia can occur only if all the particles lie on a line (because it would be necessary that $(n \cdot x)2 = n^2x^2$ for each individual particle recall that the notation suppresses the sum over all particles in the body, but in this

case all the individual terms are positive or zero.) Once we have adapted the coordinates so that the inertia tensor is diagonal we see immediately that

$$I_1 + I_2 = \sum m(x_1^2 + x_2^2 + 2x_3^2) \geq \sum m(x_1^2 + x_2^2) = I_3 \qquad (3.47)$$

Hence no moment of inertia can exceed the sum of the other two. Equality happens if and only if $x_3 = 0$ for all the particles, that is for a planar body. Some general facts about the inertia tensor can be stated.

• If the body is symmetric under reflection in a plane, two of the principal axes lie in it. Note that this theorem is enough to identify the principal axes of an ellipsoid.The body can also be symmetric under rotations around some axis through an angle which is some fixed fraction of 2π.

• If the body has a symmetry axis of order higher than two this axis must be a principal axis, and the other two moments of inertia are equal (because the corresponding eigenvectors cannot be unique).

A body with two equal moments of inertia is called a symmetrical top. If there are more than three higher order symmetry axes this is true for a cube, say it follows that all the eigenvalues are equal, so in fact every axis is a principal axis.

The shape of the body is reflected in the inertia tensor. In the gravitational N-body problem Newton proved that if all bodies are spherical, they can be regarded as mass points. Now we see that if a body is rigid, it can be regarded as an ellipsoid, and nothing else matters as far as its response to arbitrary forces is concerned.

The inertia tensor depends on the point with respect to which it is computed: Steiner's theorem: For a body of total mass M, let I_{ij} be its inertia tensor relative to the center of mass, and I'_{ij} its inertia tensor

relative to a point translated from the center of mass by the vector a_i. Then

$$I'_{ij} = I_{ij} + M\left(a^2\delta_{ij} - a_i a_j\right) \tag{3.48}$$

Roughly speaking it is easiest to rotate the body around its center of mass. The proof is easy, once we recall that

$$\sum m a_i x_i = a_i \sum m x_i = 0 \tag{3.49}$$

where x_i is position relative to the center of mass.

It is instructive to prove that any inertia tensor can be reproduced by placing four particles of equal mass at appropriate distances from each other. To do so, let the coordinates of the particles be

$$(x_i, y_i, z_i)\,,\ 1 \le i \le 4 \tag{3.50}$$

This permits us to define three vectors $x,\ \ y,\ \ z$ in R^4. We define a fourth auxiliary vector whose components are equal,

$$v^T = (1, 1, 1, 1) \tag{3.51}$$

That is to say that we are looking at the matrix

$$[x\ y\ z\ v] = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} \tag{3.52}$$

where the row index labels the four particles. Now we insist that the inertia tensor be diagonal and that the center of mass is at the origin. This translates into the six conditions

$$x \cdot y = x \cdot z = y \cdot z = v \cdot x = v \cdot y = v \cdot z = 0 \tag{3.53}$$

In other words the four vectors must be mutually orthogonal, which is easily arranged in R^4. It remains to check that we can reproduce the most general diagonal inertia tensor.

Now we will discuss Euler's equations. The following picture emerges: Associated with any rigid body there are three kinds of axes in space, none of which coincide in general. Its principal axes, that are associated

with its shape and with a little practice can be literally seen. Then its angular velocity vector, also visible to the practiced eye. But the axis that matters most to the motion of the body is its invisible angular momentum vector, because this is the axis that is conserved in force free motion. The visible axes will rotate around it as if by magic.

When its center of mass is fixed, the equation of motion for a rigid body subject to a torque τ_i is

$$\dot{L}_i = \sum m\epsilon_{ijk}x_jF_k = \tau_i \tag{3.54}$$

where both the angular momentum and the torque are computed with respect to the center of mass. The equation is complicated by the fact that the inertia tensor is time dependent,

$$\dot{L}_i = I_{ij}\,\dot{\Omega}_j + \dot{I}_{ij}\Omega_j \tag{3.55}$$

This is awkward, whether there is a torque or no.

For the rest of this section we confine ourselves to torque free motion, with the center of mass kept fixed. (Even if we are in the gravitational field of the earth it is possible to manufacture tops that are suspended at their centers of mass, and which therefore realize torque free motion.) Theoretically, the guiding idea will be to formulate the equations of motion with respect to acoordinate system fixed within the body, because then the inertia tensor istime independent. The direction of the angular momentum vector will change with time though, because we are in a rotating coordinate system. We denote it by J_i. We proceed in analogy with eq. (3.28), that is to say we assume that the intertial and body fixed coordinate systems agree at $t = 0$, and get for torque free motion

$$\dot{J}_i + \epsilon_{ikj}\Omega_kJ_j = I_{ij}\dot{\Omega}_j + \epsilon_{ikj}\Omega_kJ_j = 0 \tag{3.56}$$

Being moderately intelligent we adapt the rotating coordinate system so that its axes coincide with the principal axes, and arrive at Euler's equations

$$I_1\Omega_1 + (I_3 - I_2)\Omega_2\Omega_3 = 0$$
$$I_2\Omega_2 + (I_1 - I_3)\Omega_3\Omega_1 = 0 \qquad (3.57)$$
$$I_3\Omega_3 + (I_2 - I_1)\Omega_1\Omega_2 = 0$$

These equations can be solved exactly in terms of elliptic functions, which means that the spinning top in the absence of torque is an integrable system.

Much of the relevant information can be easily extracted. By inspection we see that the body can execute constant rotation around any principal axis, i.e. a solution is $\Omega_1 = \omega_0 = constant, \Omega_2 = \Omega_3 = 0$. Is this solution stable? Set

$$\Omega_1 = \omega_0 + \omega_1 \quad \Omega_2 = \omega_2 \quad \Omega_3 = \omega_3 \qquad (3.58)$$

where the ω_i time dependent and assumed to be small. Then expand the equations to first order in ω_i:

$$I_1\dot{\omega}_1 = 0 \qquad I_2\ddot{\omega}_2 + \frac{(I_1-I_3)(I_1-I_2)}{I_3}\omega_0^2\omega_3 = 0$$

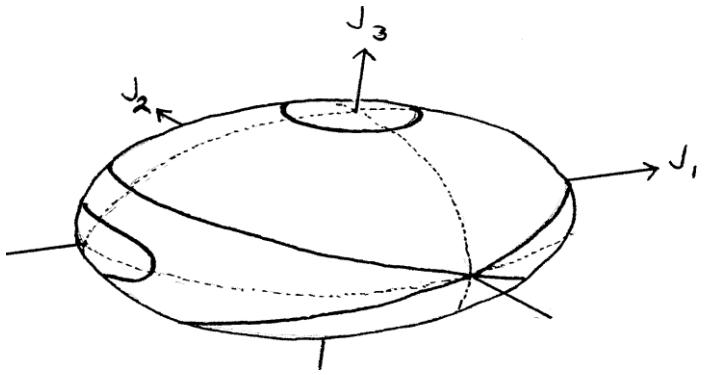
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_0\omega_3 = 0 \Longrightarrow$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_0\omega_2 = 0 \qquad I_3\ddot{\omega}_3 + \frac{(I_2-I_1)(I_3-I_1)}{I_2}\omega_0^2\omega_3 = 0 \qquad (3.59)$$

The perturbation will grow, and the solution will be unstable, unless

$$I_1 > I_2, I_3 \quad or \quad I_1 < I_2, I_3 \qquad (3.60)$$

We conclude that rotation around the largest and smallest of the principal axes is stable, rotation around the remaining principal axis is unstable. This can be seen more elegantly. The energy surface is the ellipsoid



Figure(3.3)

The inertia ellipsoid, and its intersections (as curves) with spheres of three different sizes.

$$\frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3} = 2E \tag{3.61}$$

The angular momentum vector itself is changing (in this coordinate system), but its magnitude remains constant:

$$J_1^2 + J_2^2 + J_3^2 = constant \tag{3.62}$$

This is a sphere, intersecting the energy surface along one dimensional curves. The system must move along them. It is seen that there are elliptic fixed points at the major and minor axes, and hyperbolic fixed points at the middle axis. Although it says nothing about the speed of the motion, this analysis does say more than the perturbative calculation since it describes all solutions exactly. Let us think a little bit more about this. The Euler equations define a dynamical system whose trajectories are confined to a three dimensional space, for which we can use the coordinates J_1, J_2, J_3. A solution of the Euler equations is a curve in this three dimensional space, and curves in a three dimensional space can be very unruly indeed. But the Euler equations are special because of the existence of two well behaved constants of the motion. Each of them defines a set of easy-to-describe surfaces that fill phase space, and any solution must lie where two such surfaces intersect. Hence the solutions are also easy to describe. But this is in fact a quite special property of the Euler equations. The Lorenz equations (1.45) behave quite differently: for them there are no well behaved constants of the motion, the solutions are unruly, and the system is chaotic.

We still lack a complete description of the motion in space. The Poinsot construction provides this. It describes how the angular velocity vector moves, relative to the body, and relative to absolute space. Using inertial coordinates we define the ellipsoid of inertia

$$X_i I_{ij} X_j = 2E \tag{3.63}$$

Think of it as a massless shell surrounding the body. If we compute the inertia tensor with respect to Absolute Space we will find that it moves. The numerical value $2E$ is at the moment just a convention, but since the kinetic energy E is constant the convention guarantees that the angular velocity vector Ω_i moves on this ellipsoid. There it traces out a curve known as the polhode (meaning "axis path"). In absolute space it traces out a curve known as the herpolhode. The angular velocity vector also obeys

$$L_i \Omega_i = 2E \tag{3.64}$$

Since the angular momentum vector is fixed in absolute space we can assume that it points along the X_3-axis, and then Ω_3 is a constant, which means that the herpolhode is a plane curve. The plane to which it is confined is known as the invariable plane, and is orthogonal to L_i. Now the normal vector of the ellipsoid of inertia is

$$n_i(X) = 2I_{ij} X_j \tag{3.65}$$

Evaluated at the point Ω_i this is

$$n_i(\Omega) = 2I_{ij}\Omega_j = 2L_i \tag{3.66}$$

This coincides with the normal of the invariable plane. Because of eq. (3.64) the distance from the center of the ellipsoid of inertia to the invariable plane is constant, and having placed the latter appropriately we conclude that the ellipsoid of inertia rolls on the invariable plane. It rolls without slipping because the point of tangency lies on the instantaneous rotation axis, so its velocity equals zero. Hence the polhode rolls without slipping on the herpolhode. The polhode is always a closed curve, but the herpolhode need not be when the point of tangency has made one full revolution on the ellipsoid, the body will have turned through some angle around the X_3-axis. Hence there are two frequencies involved, and if they

are not commensurable in the sense of eq. (1.37) the herpolhode never closes. This is quite reminiscent of the Lissajous figures.

Things simplify for a symmetrical top because then both the polhode and the herpolhode are circles. Moreover the symmetry axis of the top coincides with a principal axis of the ellipsoid of inertia, which means that the tip of the symmetry axis also traces out a circle around L_i . This motion is called regular precession, and is not to be confused with the precession of a top placed in a gravitational field, or with the precession of the equinoxes caused by the spinning earth. Note that Euler's equations (3.57) for a symmetrical top are easily integrated in terms of trigonometric functions.

The Earth is a symmetrical top, with $I_1 \neq I_2 = I_3$ and

$$\frac{I_1 - I_3}{I_1} \approx \frac{1}{305} \tag{3.67}$$

This is the ellipticity of the Earth. Moreover its rotation axis differs slightly from its geometrical symmetry axis. Judging from Euler's equations, with $\omega_0 = 2\pi/day$, we expect the rotation axis to move at the rate of one revolution per 305 days, and the polhode to be a circle surrounding the geometrical North Pole. Although small the polhode is only about 15 meters across such a motion is indeed observed, and is known as the Chandler wobble. However, the polhode is not a circle, and there is a period of close to 14 months as well as an annual periodicity. The latter is presumably caused by metereological disturbances, while the longer period is due to the effect we have discussed. The discrepancy with our prediction arises because the Earth is not perfectly rigid, and can be explained if the Earth has an elasticity approximating that of steel. Poincar´e produced arguments showing that a fluid core inside a rigid shell need not invalidate the argument. Incidentally, from this example one can see how difficult it would be to devise an experimental test of

rigid body dynamics. What experiments can do is to estimate the importance of friction, the departure from rigidity, and so on, and to guide mathematical modelling of such effects.

Now we will discuss the Lagrangian description . Evidently the Lagrangian of the rigid body must be $L = T - V$, but going back to our expression (3.43) for the kinetic energy of the body we are momentarily confused. We recognize neither q nor $\dot{q}$. They are still there though. Recall that the configuration space is the group $SO(3)$, whose tangent vectors are Lie algebra elements. An arbitrary rotation can be obtained by exponentiating an anti-symmetric matrix, of the general form

$$A_{ij} = \epsilon_{ikj}\Omega_k \Leftrightarrow A = \Omega_1 A_1 + \Omega_2 A_2 + \Omega_3 A_3 \qquad (3.68)$$

where we use the basis introduced in eqs. (3.22) and Ω_i is the angular velocity vector. Furthermore

$$R = e^{-tA} \Rightarrow A = -\dot{R}\,R^{-1} \qquad (3.69)$$

A minus sign was used because we are now considering an active rotation of the body, rather than a passive change to a rotating coordinate system. Once we have A we can read off the angular velocity vector by comparing to eq. (3.68). And at the expense of a minor amount of work we can express A in terms of the Euler angles . This is how we will express theLagrangian of the top in terms of coordinates. The calculation goes as follows:

$$-A = \dot{R}R^{-1} = \frac{d}{dt}\left(R_\psi R_\theta R_\emptyset\right)\left(R_\psi R_\theta R_\emptyset\right)^{-1}$$

$$= \dot{R}_\psi R_\psi^{-1} + R_\psi \dot{R}_\theta R_\theta^{-1} + R_\psi R_\theta \dot{R}_\emptyset R_\emptyset^{-1} R_\theta^{-1} R_\emptyset^{-1}$$

$$= -\dot{\psi}A_3 - \dot{\theta}(\cos\psi A_1 + \sin\psi A_2) - \dot{\phi}\,(\sin\theta(\sin\psi A_1 - \cos\psi A_2) + \cos\theta A_3) \qquad (3.70)$$

In the final step we fell back on the explicit formulæ (3.14). Comparing to eq. (3.68) we read off that

$$\Omega_1 = \dot{\theta}\, cos\, \psi + \dot{\phi}\, sin\, \theta\, sin\, \psi$$
$$\Omega_2 = \dot{\theta}\, sin\, \psi - \dot{\phi}\, sin\, \theta\, cos\, \psi$$
$$\Omega_3 = \dot{\psi} + \dot{\phi}\, cos\, \theta \qquad (3.71)$$

This we can insert in a Lagrangian.

We will choose a Lagrangian to describe a symmetrical top with one point fixed, subject to a gravitational force. The restriction to a symmetrical top implies that two of the eigenvalues of the inertia tensor are equal, say $I_1 = I_2$; we make this restriction for the pragmatic reason that the general case is not soluble. To connect the Euler angles to the orientation of the top we place the origin at the fixed point, and let the identity matrix correspond to a top whose symmetry axis is vertical. When the symmetry axis is inclined the angle ϕ describes its rotation around the vertical, the angle θ its inclination, and the angle ψ the rotation of the body around the symmetry axis. Hence these angles are referred to as the angle of precession, nutation, and spin, respectively (or yaw, pitch, and roll, if you are a spacecraft designer). In these coordinates the gravitational potential energy is

$$V = V(\theta) = Mgl\, cos\, \theta \qquad (3.72)$$

Now we can write the Lagrangian. We obtain

$$L = \frac{I_1}{2}(\Omega_1^2 + \Omega_2^2) + \frac{I_3}{2}\Omega_3^2 - V(\theta)$$
$$= \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta\right) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\, \cos\theta)^2 - Mgl\cos\theta \qquad (3.73)$$

For the symmetrical top neither ϕ nor ψ appear in the Lagrangian, which means that energy included we will get three constants of the motion, enough to solve the equations of motion explicitly. To be precise, we find the constants of motion

$$L_z = \frac{\partial L}{\partial \dot{\phi}} = I_1\, \dot{\phi}\, sin^2\theta + I_3 \cos\theta\left(\dot{\psi} + \dot{\phi}\, \cos\theta\right) \qquad (3.74)$$

$$L_3 = \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) \tag{3.75}$$

They are the angular momentum along the vertical axis (in absolute space) and along the symmetry axis of the top, respectively. We can solve these equations for the velocities:

$$\dot{\psi} = \phi\cos\theta = \frac{L_3}{I_3} \tag{3.76}$$

$$\dot{\phi} = \frac{L_2 - L_3\cos\theta}{I_1 sin^2\theta} \tag{3.77}$$

Next we turn to the energy, which is

$$E = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + Mgl\cos\theta$$

$$= \frac{I_1}{2}\dot{\theta}^2 + \frac{(L_2 - L_3\cos\theta)^2}{2I_1 sin^2\theta} + Mgl\cos\theta + \frac{L_3^2}{2I_3} \tag{3.78}$$

This leads to a first order differential equation which can be explicitly solved in terms of elliptic functions. Inserting the result successively in eqs. (3.77) and (3.76) will lead us to the complete solution for the heavy symmetrical top. As usual one can go a long way with qualitative arguments. Eq. (3.78), which governs the nutation of the top, can be described as one dimensional motion in an effective potential. It can be written on the form

$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{sin^2\theta} + \beta\cos\theta \tag{3.79}$$

where

$$\alpha = \frac{2EI_3 - L_3^2}{I_1 I_3}, \qquad b = \frac{L_2}{I_1}, \qquad a = \frac{L_3}{I_1} \tag{3.80}$$

$$\beta = \frac{2Mgl}{I_1} > 0 \tag{3.81}$$

If we introduce the variable $u = cos\,\theta$ the effective potential becomes a cubic polynomial,

$$\dot{u}^2 = \beta u^3 - (\alpha + a^2)u^2 + \left(2ab - \beta\right)u + \alpha - b2^2 \tag{3.82}$$

The velocity $\dot{\theta}$ can change sign only when the right hand side vanishes. The polynomial on the right hand side goes from $-\infty$ to ∞, and

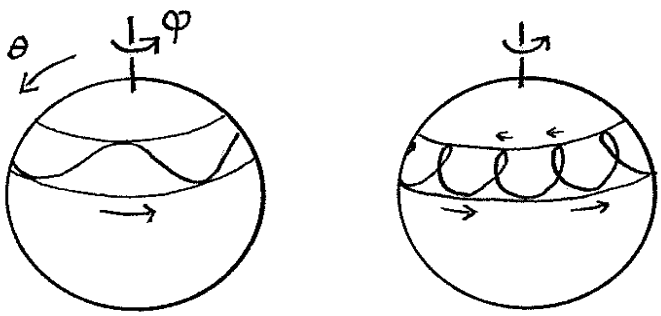


Figure (3.4)

Precession might possibly turn retrograde.

$$u = \pm 1 \Rightarrow \quad \dot{u}^2 = -(b \mp a)^2 \qquad (3.83)$$

On physical grounds $\dot{u}^2$must be positive somewhere within the physically relevant interval $-1 \leq u \leq 1$. Except for the special case $b = a$ it then follows that the cubic polynomial has two roots in the physically relevant interval. This observation is enough to reveal the key qualitative features of the nutation of the top clearly its axis is nutating between a maximal and a minimal value of θ.

For the precession, described by eq. (3.77), an interesting qualitative questionis whether the derivative $\dot{\phi}$ changes its sign as θ evolves. The answer depends on the relative size of the initial data L_z and L_3; see Fig.(3.4) for the possibilities.

To see why there is precession in the first place, take a top whose point of contact with the ground is fixed but which is otherwise not subject to forces. Let it rotate around its angular momentum vector. Then grab hold of it, trying to increase the angle its axis makes with the vertical axis. If you pull hard enough in a downwards direction you will succeed to some extent, but there will be no z-component of the torque, and hence L_z must remain constant. The top manages to keep L_z constant because it begins to precess, thus adding an extra precessional contribution to L_z.

These considerations are applicable to the Earth, which is an oblate symmetrical top subject to tidal forces, primarily from the Moon. The

monotonic precession of the Earth was known to the Greeks. It has a period of 26 000 years which incidentally means that the position of the Sun relative to the Zodiac has drifted noticeably since the Greeks determined it, with no apparent consequences to astrology. The nutation of the Earth was reported by Bradley in the 18th century, after first observing it for a complete period, 18.7 years. Alesson, perhaps, for modern astronomers. It is a small effect, but significantly greater than the Chandler wobble.

Section (3.2) : Legendre transformations

Now we will discuss Legendre transformations. As is known, it is frequently a good idea to perform a transformation of some functions that one may be working on. The Fourier transform is an example of this idea. It turns out that the kind of manipulations that one may wish to perform on the original function are easier to do on its Fourier transform, and once this has been done one may revert to the original kind of function by means of an inverse Fourier transform.

Another example of this is the Legendre transformation of a function $f = f(x)$. We will use it to pass from the Lagrangian to the Hamiltonian formulation of the equations of motion, and later on to obtain an interesting class of "canonical" transformations of phase space. The Legendre transformation is also used to great effect in thermodynamics. In this interlude we discuss it from a purely mathematical point of view, assuming for simplicity that the function depends on one variable only.

We will assume that the function to be transformed is twice differentiable, and obeys

$$\frac{d^2 f}{dx^2} > 0 \tag{3.84}$$

This is not strictly necessary but convenient for our purposes.

A function obeying this condition is said to be convex. The Legendre transform of the convex function f is a function g defined as

$$g(p) = max_x\big(xp - f(x)\big) \tag{3.85}$$

The maximum that we ask for is achieved if and only if

$$p = \frac{df}{dx} \tag{3.86}$$

Precisely because we imposed condition (3.84) the derivative of f is a monotoneously increasing function of x, which means that eq. (3.86) can be solved to give the unique solution $x = x(p)$. Inserting this into the right hand side of eq. (3.85) defines the function $g(p)$ uniquely.

A function and its Legendre transform are related by

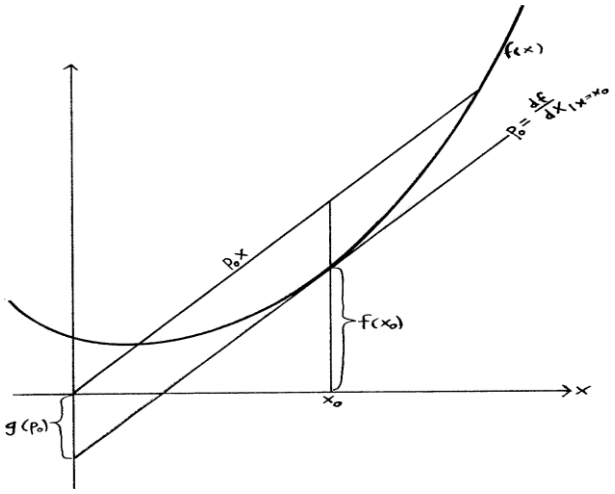


Figure (3.5)

The Legendre transformation of a function $f(x)$. The connection to eqs. (3.85) and (3.87) should be evident.

$$f(x) + g(p) = xp \tag{3.87}$$

If eq. (3.86) holds this equation defines the Legendre transform $g(p)$ once the original function $f(x)$ is known. But the equation is fully symmetric between f and g, which means that if

$$x = \frac{dg}{dp} \tag{3.88}$$

then eq. (3.87) defines $f(x)$ as the Legendre transform of the function $g(p)$. To make sure that this claim is valid one must show that the function g is convex because the function f is. But this is so because

$$\frac{d^2g}{dp^2} = \frac{dx}{dp} = \left(\frac{dp}{dx}\right)^{-1} = \left(\frac{d^2f}{dx^2}\right)^{-1} > 0 \qquad (3.89)$$

Hence eq. (3.88) can be solved for $p = p(x)$. We may conclude that all the information in the function f is present in its Legendre transform g, and also that unlike the Fourier transformation, say the Legendre transformation is its own inverse. Generalisation to an arbitrary number of variables is quite straightforward.

There is an interesting geometric interpretation of the Legendre transformation, and I hope this can be deciphered from Fig. (3.5) : to find out the value of the function g at p_0, start by drawing the straight line p_0x, and then use the definition of the Legendre transform to find $g(p_0)$ as an intercept of a tangent of the curve with the ordinate axis. If the original function describes a curve by giving the set of points on the curve, the Legendre transform describes it by giving the set of lines that are tangent to it. This is a rather deep idea, but one that belongs to another story.

Chapter (4)

The Hamiltonian formulation and Infinitesimal canonical transformations

Section (4.1): The Hamiltonian formulation

Any set of ordinary differential equations can be written in first order form, provided we introduce enough extra variables. The Hamiltonian formulation is a special way of doing this to the Euler-Lagrange equations, and reveals that the latter have a very special form. The central features of mechanics those that classical and quantum mechanics have in common are brought out veryclearly by the Hamiltonian formulation.

Now we will discuss Hamilton's equations and Hamiltonian flows For a Lagrangian of the form $L(q, \dot{q})$, which is general enough for our purposes, the Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{4.1}$$

These are second order ODEs. One obvious way to turn them into first order equations is to define the canonical momenta

$$pi \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{4.2}$$

The expression on the right hand side will be equal to $m\dot{q}_i$ in simple cases, andin general it will be some function of $\dot{q}$ and q. We already know that the right hand side is of some importance; it occurs when one sets boundary conditions in the variational principle, and in connection with Noether's theorem. See eq. (1.79). So the canonical momenta are the extra variables to be used in turning the Euler-Lagrange equations into first order form. The use of "q" for generalized coordinates and "p" for their momenta goes back to Jacobi, and was solidified by Whittaker in a famous textbook. Eqs. (4.1) now take the first order form

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{4.3}$$

But we need equations for $\dot{q}$ too. To this end we assume that eqs. (4.2) can be inverted, that is

$$p_i = p_i(q, \dot{q}) \Leftrightarrow \dot{q}_i = \dot{q}_i(q, p) \tag{4.4}$$

Now we do have a first order system equivalent to the original Euler-Lagrange equations. As a matter of fact it can happen that it is impossible to solve eqs. (4.2) for the velocities; exactly how to deal with every possibility was explained by Dirac when he developed his theory of constrained Hamiltonian systems.

Although this is interesting, from now on we assume for simplicity that the equations are invertible. Now form the function

$$H(q, p) = \dot{q}_i p_i - L(q, \dot{q}) \tag{4.5}$$

Thus H and L are Legendre transformations of each other, in the sense of chapter 3. It is a partial Legendre transformation, since only the $\dot{q}$ are involved while the coordinates q are left untouched. We see that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{\partial H}{\partial q_i} \tag{4.6}$$

These are Hamilton's equations, and the function H is known as the Hamilto-nian. The variables p_i are known as the canonical momenta. By means of the Legendre transformation we have traded the coordinate $\dot{q}$ for the coordinate p.

In practice, the Legendre transformation is usually easy to perform. We know that

$$L = \frac{m}{2}\dot{q}^2 \quad \Leftrightarrow \quad H = \frac{1}{2m}p^2 \tag{4.7}$$

An apparently more complicated case, involving many degrees of freedom, is

$$L = \frac{1}{2}M_{ij}(q)\dot{q}_i\dot{q}_j - V(q) \tag{4.8}$$

Here the matrix M depends on the configuration space variables. Using matrix notation the Hamiltonian is immediately found to be

$$H = \frac{1}{2} p_j M_{ij}^{-1} p_j + V(q) \tag{4.9}$$

We only have to check the matrix M is invertible, everywhere in configuration space. This in fact amounts to checking that the Lagrangian is a convex function of the variables $\dot{q}_i$.

Our phase space is spanned by the $2n$ variables q_i, p_i. If we compare to the general phase spaces discussed in section (1.2) we see that this is already a restriction: the phase space of a Hamiltonian system is always even dimensional. But there is another and more dramatic difference. In section(1.2) time evolution was described by the equations $\dot{z}_i = f_i(z)$, so that the general case is obtained by choosing $2n$ independent functions f_i. In the Hamiltonian case the time evolution is determined by a single function $H(q,p)$. This is indeed a very strong restriction, but it is one that Nature seems to respect for all her fundamental equations.

There are consequences. One of them is that

$$\dot{H} = \dot{q}_i \frac{\partial H}{\partial q_i} + \dot{p}_i \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = 0 \tag{4.10}$$

Hence the Hamiltonian is always a conserved quantity, unless it depends explicitly on time. The set of points in phase space obeying

$$H(q,p) = E \tag{4.11}$$

is called the energy surface. Time evolution takes place within the energy surface. In two dimensions (for one degree of freedom) the energy "surface" coincides with the one dimensional trajectories; hence Hamiltonian systems with one degree of freedom are always soluble.

Next let us imagine time evolution as the flow of a fluid. The flow lines are defined by the little arrows in phase space. The fluid is incompressible if, for any fixed finite volume in phase space, the amount of fluid going out through its surface equals the amount that is going in.

By Gauss' theorem the difference between them is equal to the integral of the divergence of the flow over the volume. Since this must vanish for every volume we conclude that the flow is that of an incompressible fluid if and only if its divergence vanishes.

For Hamiltonian time evolution this is indeed so:

$$div\dot{z} = \nabla \cdot \dot{z} = \partial_{q_i}\, \dot{q}_i(q,p) \; + \; \partial_{p_i}\, \dot{p}_i(q,p) = \; \partial_{q_i}\partial_{p_i}H \; + \; \partial_{p_i}(-\partial qiH) \; = 0 \quad (4.12)$$

This conclusion is worth stating as a theorem.

Liouville's theorem: In Hamiltonian mechanics the phase space flow preserves volume. Liouville's theorem is the origin of the claim made in section (1.2) that sources and sinks do not occur in Hamiltonian systems. In two dimensions the only fixed points that occur are either elliptic or hyperbolic.

On reflection one sees that the behaviour allowed by Liouville's theorem can still be very complex. Already in two dimensions the shape of a small piece of the phase space fluid passing close to a hyperbolic fixed point will be squeezed in one direction and stretched in another. In the end the original volume element can acquire a very involved shape, and for an observer with limited resolution it may in fact seem as if it has been smeared all over the energy surface, even though on microscopic scales it does preserve its volume.

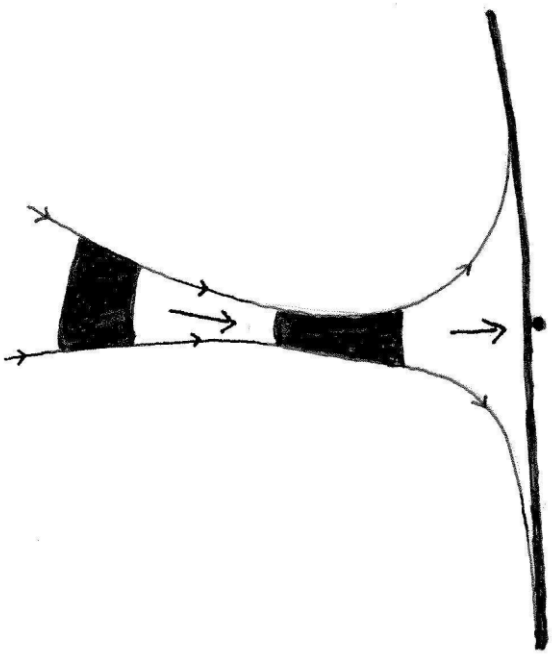


Figure (4.1)

Liouville's theorem in action: a volume element in phase space preserves its volume, but may become very distorted here because it approaches a hyperbolic fixed point.

In the following study kets and bras and all that. There is an intriguing geometrical structure behind the Hamiltonian equations of motion. To appreciate it, we begin by recalling the "kets and bras" notation in quantum mechanics. In quantum mechanics the state space is a vector space.

Vectors are called kets and denoted by $|\psi_i\rangle$. They form a vector space because one can take linear combinations of kets: if $|\psi_1\rangle$ and $|\psi_2\rangle$ are kets and a_1, a_2 are numbers then $|\psi_3\rangle$is a ket too, where

$$|\psi_3\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle \qquad (4.13)$$

In quantum mechanics the numbers a_1, a_2 are complex, but from now on we assume that all such numbers are real. It is important to realize that, in quantum mechanics, one never takes the scalar product of two vectors that is to say of two kets. What one does is to introduce another vector space whose vectors are called bras, and denoted by $\langle\phi|$. By

definition a bra is a linear function from the vector space of kets to the real numbers. Thus, given a bra $\langle\, \phi\, |$,

$$\langle\, \phi\, |\psi\rangle = a \tag{4.14}$$

is defined for all kets $|\psi\rangle$. Dirac calls this a "bracket", which explains the names "kets and bras". (It is believed that Dirac did not know that "bra" already had a meaning in English.) An added complication in quantum mechanics is that the vector spaces are often infinite dimensional function spaces, but we do not bother with this here. We do need to check that the bras form a linear space. But this is so because the map to the real numbers is linear,

$$(a_1\langle\, \phi_1| + a_2\langle\, \phi_2|)|\psi\rangle = a_1\langle\, \phi_1|\psi\rangle + a_2\langle\langle\, \phi_2|\psi\rangle \tag{4.15}$$

Hence$\langle\, \phi_3| = a_1\langle\, \phi_1| + a_2\, \langle\, \phi_2|$ is a bra.

At this point we have two vector spaces, and a priori they are different. The equation $\langle\psi|=|\psi\rangle$ makes no sense at all. Apples and pears are never equal. However, there is a final twist to the story, because there does exist a one-to-one correspondence between the kets and the bras. If you have a ket written out as a column vector with respect to some basis, there is a unique bra obtained by transposing the column to a row (and taking the complex conjugates of all the numbers, but this is irrelevant here since all our components are real).

This is an extra piece of structure, giving rise to a one-to-one correspondence $\langle\psi| \leftrightarrow |\psi\rangle$. It means that we can associate a unique number to any ket, namely the real number $\langle\psi|\psi\rangle$. This number is the "length squared" of the ket $|\psi\rangle$.

How can we make the distinction between bras and kets in the index notation? The answer is simple. Let us denote all the original vectors, the kets, by x^μ. We are using Greek rather than Latin indices, and will

continue to do so whenever it is our intention to use the indices in a correct "tensorial" way (that is, in the way I am just going to explain). When the indices are only used to label a set of objects we continue to use Latin indices. The important thing is that we place the index on a ket "upstairs". Such a vector is called contravariant. A bra will be denoted by u_μ, that is to say with its index "downstairs", and is called a covariant vector. This should represent a linear map from the space of all kets to the real numbers, and we define this map as

$$u(x) = u_\mu x^\mu = a \tag{4.16}$$

where Einstein's summation convention is understood. It is important to note that in the upstairs-downstairs notation, the expressions $x^\mu y^\mu$ and $u_\mu v_\mu$ are simply not allowed, so that we cannot define the length squared of a vector x^μ as $x^\mu x^\mu$. It is forbidden. There is no notion of length squared until we have introduced further structure. With these new rules, many of the formulæ. We still want to use them however, so we have to modify them in a suitable way.

Let us agree that x_μ simply stands for the components of a ket relative to some basis in the ket vector space. That is to say that

$$|\psi\rangle = |e_\mu\rangle x^\mu \tag{4.17}$$

where the vectors $|e_\mu\rangle$ form a basis (and must not be confused with the components u_μ of a bra vector). A matrix operating on the ket vector space will be written as a "mixed" tensor A^μ_ν. Then the equations

$$y^\mu = A^\mu_\nu x^\nu \quad and \quad v_\mu = u_\nu B^\nu_\mu \tag{4.18}$$

do make sense given that we always use the Einstein summation convention for repeated indices in "upstairs-downstairs" position. As a matter of fact these equations make two different kinds of sense. The vector

$$|\boldsymbol{\psi}'\rangle = |\boldsymbol{e_\mu}\rangle \boldsymbol{y^\mu} = |e_\mu\rangle A^\mu_\nu x^\nu = A|\psi\rangle \tag{4.19}$$

is a new vector, obtained by transforming the old vector $|\boldsymbol{\psi}\rangle$. This is an active transformation of the vector. On the other hand the vector

$$|\psi\rangle = |e_\mu\rangle x^\mu = |e_\sigma\rangle (A^{-1})^\sigma_\mu A^\mu_\nu x^\nu = |e'_\mu\rangle y^\mu \tag{4.20}$$

is just the original vector $|\boldsymbol{\psi}\rangle$ in a new coordinate system. This is a passive transformation of the vector.

In a passive transformation the components and the basis vectors transform in opposite ways. This is an important idea.We will insist that, whatever linear transformation we do of the bra and ket vector spaces, the numbers $\boldsymbol{u_\mu x^\mu}$ should remain unchanged. We can arrange this by insisting that whenever we transform the kets, we also transform the bras in the opposite way. That is to say

$$x_\mu \rightarrow A^\mu_\nu x^\mu \Leftrightarrow u_\mu \rightarrow u_\mu (A^{-1})^\nu_\mu \tag{4.21}$$

This is part of the origin of the names "contravariant" and "covariant" vectors they transform in opposite directions, to ensure that $\boldsymbol{u_\mu x^\mu}$ remains unchanged.

We can go on to define tensors with more than one index in the same way. A tensor with k indices running from 1 to n is defined as a collection of kn components transforming in specific way under changes of basis. Examples include

$$T^{\mu\nu\sigma} \rightarrow A^\mu_\alpha A^\nu_\beta A^\sigma_\gamma T^{\alpha\beta\gamma}, S_{\mu\nu\sigma} \rightarrow S_{\alpha\beta\gamma} (A^{-1})^\alpha_\mu (A^{-1})^\beta_\nu (A^{-1})^\gamma_\sigma \tag{4.22}$$

Students are usually disturbed by the fact that the components of a tensor with more than two indices cannot be displayed as a matrix, but really the definition does not require this. Still something is missing. To do so here we need to set up a one-to-one correspondence between the set of x^μ and the set of u_μ, and then define the analogue of $\langle\psi|\psi\rangle$.

The Kronecker delta is the key. We write it as a covariant tensor with both indices downstairs as $\delta_{\mu\nu}$. In fact we can be a bit more general. We

introduce a metric tensor. By definition this is any covariant symmetric tensor with two indices downstairs,

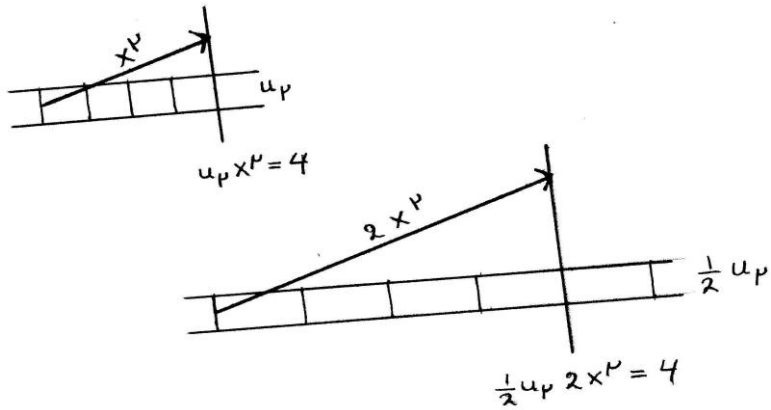
$$g_{\mu\nu} = g_{\nu\mu} \tag{4.23}$$

which has a contravariant inverse $g_{\mu\nu}$, in the sense that

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu} \tag{4.24}$$

Here the left hand side defines an operation which acts as the identity on both our vector spaces,

$$\delta^{\mu}_{\nu} x^{\nu} = x^{\mu} \; and \; u_{\nu}\delta^{\nu}_{\mu} = u_{\mu} \tag{4.25}$$



Figure(4.2)

Bras and kets: kets are arrows, bras collections of parallel planes (or "measuring tapes"). Their lengths are undefined but one can multiply them with real numbers, and the quantity $u_{\mu}x^{\mu}$ has a clear meaning.

Thus $\delta_{\mu\nu}$ is a special example of a metric tensor, while δ^{μ}_{ν} is an operator that acts on kets even though, when you write them out as matrices, they look the same.

Now we have the means to set up a one-to-one correspondence between bras and kets:

$$x^{\mu} \leftrightarrow x_{\mu} = g_{\mu\nu}x^{\nu} \Leftrightarrow u_{\mu} \leftrightarrow u^{\mu} = g^{\mu\nu}u_{\nu} \tag{4.26}$$

We define the length squared of a vector x^{μ} as $x^{\mu}g_{\mu\nu}x^{\nu}$. If $g_{\mu\nu} = \delta_{\mu\nu}$ we recover the naive definition. Which defined those special matrices that correspond to rotations, that is to say those matrices that transform the

vectors in such a way that their lengths are preserved. It is not hard to rewrite this equation in such a way that they are allowed by our new conventions. The result is

$$R_{\mu}^{\sigma} R_{\nu}^{\rho} g_{\sigma\rho} = g_{\mu\nu} \tag{4.27}$$

If the metric is the Kronecker delta, this equation says that the rotation is done by means of an orthogonal matrix. In this way we recover the full content of section (3.1).

There is a simple geometrical picture of kets and bras that may be helpful. Represent the kets as arrows pointing from the origin. If the ket x^{μ} is multiplied with , the length of the arrow in the picture doubles. Represent the bras with measuring tapes through the origin, or more precisely as a set of parallel hyperplanes with constant spacing. The bras transform oppositely to the kets, so if the ket is multiplied by 2 the bra must be multiplied with $1/2$. The number $u_{\mu} x^{\mu}$ should stay unchanged under this transformation. If we interpret this number as the number of level surfaces pierced by the arrow, we see that multiplying the ket with $1/2$ means that the spacing between the level surfaces doubles and we have arrived at a consistent geometrical picture.

The true length of x^{μ} is not defined yet. You might think that it could be defined by choosing some special measuring tape, rotating this tape until it is aligned with the arrow, and afterwards compute $u_{\mu} x^{\mu}$. But this assumes that the measuring tape is unchanged by the rotation, and the catch is that "rotation" is defined as a transformation that preserves the lengths of all arrows, or the spacing of all measuring tapes. We are caught in a circular definition, and forced to introduce some further structure a metric before we can define length.

The reason why we did not bother about these things in chapter 3 is that there was simply no need for them. Because the metric tensor was so

simple equal to the Kronecker delta the one-to-one correspondence between bras and kets was so obvious that we simply identified them from the outset, without comment. But the more careful treatment here is useful for various generalisations. In particular it is useful to understand the Hamiltonian formulation of the equations of motion. There may even be a moral here. Forbidding certain things (in this case summing over repeated indices in any way we please) sometimes gives much more freedom.

Now we will discuss the symplectic form. To begin with we assume that phase space is a vector space. More involved cases such as the phase space of a rigid body can wait. If we have a single vector x^{μ}, how can we assign a number to it? We know the answer. We introduce a metric tensor, and define

$$||x||^{2} = x^{\mu} g_{\mu\nu} x^{\nu} \tag{4.28}$$

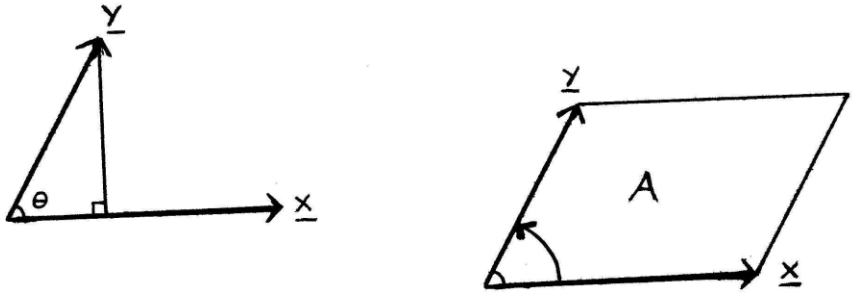
The simple choice $g_{\mu\nu} = \delta_{\mu\nu}$ is especially interesting, but in principle any choice of $g_{\mu\nu}$ is allowed, as long as it is invertible and symmetric. Once we have a metric we can also assign a real number to any pair of vectors, namely the scalar product

$$y^{\mu} g_{\mu\nu} x^{\nu} = \big||x|\big|\big||y|\big| cos\, \theta \tag{4.29}$$

But geometrically it is obvious that there is another number that we can assign to the pair, namely the (oriented) area A that they span. Let us adapt our coordinates so that the two vectors have only two non-zero components each. Then A is given by a determinant

$$A = \begin{vmatrix} x^{1} & y^{1} \\ x^{2} & y^{2} \end{vmatrix} \tag{4.30}$$

We can write this as



Figure(4.3)

Two ways of associating a number to a pair of vectors: the angle they subtend, and the area they span.

$$A = y^{\mu}\omega_{\mu\nu}x^{\nu} \tag{4.31}$$

where

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.32}$$

Like the scalar product the area is a bilinear function of the vectors. The next step is to "liberate" eq. (4.31) from its origins. Any anti-symmetric and invertible tensor can serve as a symplectic form on a vector space, just as any symmetric invertible tensor can serve as a metric. In other words eq. (4.31) will be taken to be a meaningful number associated to any pair of vectors, regardless of the dimension of the vector space, provided only that the symplectic form obeys

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{4.33}$$

and provided that there exists an inverse matrix $\omega^{\mu\nu}$ so that

$$\omega^{\mu\sigma}\omega_{\sigma\nu} = \delta^{\mu}_{\nu} \tag{4.34}$$

The latter equation is analogous to equation (4.24) for the metric. But it happens that an anti-symmetric $N \times N$ matrix has determinant zero if N is odd. If the determinant is zero its inverse cannot exist. Hence symplectic forms exist only on even dimensional vector spaces, while metrics exist on vector spaces of all dimensions. (This fact gave us

Euler's theorem in section(3.1) the antisymmetric matrix defining an infinitesimal rotation in 3-space necessarily has a zero eigenvalue, and hence the rotation it generates has a fixed axis.)

Just as we may wish to define metrics on curved spaces, not only on vector spaces, so we may wish to define symplectic forms on more general phase spaces. A symplectic form on an arbitrary phase space will be represented by an invertible and anti-symmetric tensor $\omega_{\mu\nu}$, but in principle its components may depend on the particular point of phase space where it sits. This cannot happen in a quite arbitrary fashion though, in fact the third and final requirement on a symplectic form is

$$\partial_\mu \omega_{\nu\sigma} + \partial_\sigma \omega_{\mu\nu} + \partial_\nu \omega_{\sigma\mu} = 0 \tag{4.35}$$

We will mostly be interested in the case when the components of $\omega_{\mu\nu}$are constant, and then this extra requirement is trivial. For now we only remark that the equation is recognisable as one of Maxwell's equations in electrodynamics. There it guarantees the existence of the vector potential. We will see that a similar object arises in Hamiltonian mechanics.

The most common situation is that the symplectic form is constant and "block diagonal", that is to say it has the form

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{4.36}$$

On a vector space, one can always choose the coordinates in such a way that any symplectic form takes this form. In fact there is a stronger statement known as Darboux' theorem: On any phase space with topology R^{2n}, coordinates can always be introduced so that the symplectic form takes this standard form. Although we do not need to go

into it now, it is interesting that metric tensors behave in a completely different way.

Because the symplectic form has an inverse, it can be used to relate contravariant vectors with covariant ones in a unique manner, just as a metric can. The archetypical contravariant vector is the little arrow $\dot{z}^{\mu}$ that gives the time evolution of a system, while the archetypical covariant vector is the gradient of a function, say $\partial_{\mu}H$. Let us relate them:

$$z^{\mu} = \omega^{\mu\nu}\partial_{\nu}H \tag{4.37}$$

Let us further suppose that the symplectic form takes the standard form (4.36), and let us adapt the description of the coordinates according to

$$z^{\mu} = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \\ . \\ . \\ . \\ z^{2n-1} \\ z^{2n} \end{pmatrix} = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \\ . \\ . \\ . \\ q_n \\ p_n \end{pmatrix} \tag{4.38}$$

Remembering that we agreed to be careless about the position of Latin indices we then find that

$$\dot{z}^{\mu} = \omega^{\mu\nu}\partial_{\nu}H \Leftrightarrow \begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} \tag{4.39}$$

This is indeed Hamilton's equations of motion hereby revealed as a topic in symplectic geometry.

Now we will discuss the algebraic structure of mechanics . In Hamiltonian mechanics a single function $H(z) = H(q,p)$ on phase space is used to specify time evolution. This is reminiscent of quantum mechanics.

Let us recall the basic facts about the latter. Time evolution is specified by choosing a definite Hermitian operator $\widehat{H}$. In the Heisenberg

picture the time evolution of an arbitrary operator is determined by the equation

$$\frac{d\hat{A}}{dt} = i\hat{h}[\hat{A}, \hat{H}] \tag{4.40}$$

Here the bracket represents a commutator. It obeys the algebraic relations

$$[a_1 \hat{A}_1 + a_2\hat{A}_2, \hat{B}] = a_1[\hat{A}_1, \hat{B}] + a_2[\hat{A}_2, \hat{B}] \tag{4.41}$$

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}] \tag{4.42}$$

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{C}, [\hat{A}, \hat{B}]] + [\hat{B}, [\hat{C}, \hat{A}]] = 0 \tag{4.43}$$

where a_1, $a_2$are arbitrary numbers. These relations also characterize the Lie bracket occurring in the study of Lie groups ; the last relation is known as the Jacobi identity.

In classical mechanics all functions commute, so at first sight there does not seem to be a classical analogue of the commutator. But there is one. Using the symplectic form we define the Poisson bracket of two arbitrary phase space functions as

$$\{A, B\} \equiv \partial_\mu A \omega^{\mu\nu} \partial_\nu B \tag{4.44}$$

By inspection we see that the Poisson bracket obeys

$$\{a_1 A_1 + a_2 A_2, B\} = a_1\{A_1, B\} + a_2\{A_2, B\} \tag{4.45}$$

$$\{A, B\} = -\{B, A\} \tag{4.46}$$

It is less obvious, but true that it also obeys the Jacobi identity

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0 \tag{4.47}$$

Granted this, the analogy to the quantum mechanical commutator is clear: phase space comes equipped with a Lie bracket. Eq. (4.39) can be written in the form

$$\dot{z}^\mu = \omega^{\mu\nu}\partial_\nu H = \{z^\mu, H\} \tag{4.48}$$

The time derivative of an arbitrary phase space function is

$$\dot{A} = \{A, H\} \tag{4.49}$$

The classical and the quantum mechanical Hamiltonians are analogous too.

These observations led Dirac, in the course of a Sunday walk in Cambridge, to the belief that any classical system can be "quantized" by setting up a correspondence between functions on phase space and operators on a (complex) linear space, and by replacing the Poisson brackets with commutators according to the rule

$$\{A, B\} \rightarrow \frac{1}{i\hbar}\left[\ \hat{A}, \hat{B}\ \right] \tag{4.50}$$

On the whole, although various complications do arise, Dirac's idea has provedto be correct.

If the symplectic form takes the standard form (4.36) we can use eq. (4.38) to split the $2n$ coordinates z^{μ} into n pairs (q_i, p_i). Then the Poisson bracket reads

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{4.51}$$

(Yes, as always a sum over repeated indices is understood.) Moreover eq. (4.48) becomes

$$\dot{q} \ = \ \{q, H\} \qquad\qquad \dot{p} \ = \ \{p, H\} \tag{4.52}$$

This form of Hamilton's equations is frequently seen.

For the explicit evaluation of Poisson brackets we look carefully at the definition, and observe that

$$\{A, F\} \ = \ \partial_{\nu} A \omega^{\nu\mu} \partial_{\mu} F \ = \ (\partial_q A \partial_p \ - \ \partial_p A \partial_q) F \tag{4.53}$$

Hence we can think of $\{A, \}$ as a differential operator acting on phase space functions. It obeys Leibniz' rule

$$\{A, BC\} \ = \ B\{A, C\} \ + \ \{A, B\}C \tag{4.54}$$

In this sense which is a very important one it acts like a kind of derivative. Using this the evaluation of any Poisson bracket can ultimately be made starting from the fundamental Poisson brackets

$$\{q_i, p_j\} \ = \ \delta_{ij}\ , \{q_i, q_j\} \ = \ \{p_i, p_j\} \ = \ 0 \tag{4.55}$$

We can now confirm that the Poisson bracket "acts like a derivative" in the sense that

$$\{q_i, A\} = \partial_{p_i} A \qquad \{p_i, A\} = -\partial_{q_i} A \tag{4.56}$$

where $A = A(q,p)$ is any function on phase space. Note that the quantum mechanical commutator behaves in a similar way, except that in quantum mechanics we have to care about the order in which we place things. The ordering on the right hand side of eq. (4.54) would be the correct one to use there, while ordering does not matter in classical mechanics.

In the following study the sphere as a phase space. A set of equations that at first sight cannot be cast into Hamiltonian form is Euler's equations for a spinning top, eqs. (3.57), for the simple reason that phase space is three dimensional. This is a bit deceptive though. Recall that $\mathrm{J}^2 = \mathrm{J}_1^2 + \mathrm{J}_2^2 + \mathrm{J}_3^2$ is constant. Let us therefore define angular coordinates θ, ϕ through

$$J_1 = J\cos\phi\,\sin\theta\,,\quad J_2 = J\sin\phi\,\sin\theta\,,\; J_3 = J\cos\theta \tag{4.57}$$

If we write out Euler's equations in terms of $Ji = \Omega_i/Ii$ we find, after a calculation, that they become

$$\dot{\theta} = J\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\cos\phi\,\sin\phi\,\sin\theta$$

$$\dot{\phi} = \mathrm{J}\left(\frac{1}{\mathrm{I}_1} - \frac{1}{\mathrm{I}_3}\right)\cos^2\phi\,\cos\theta - \frac{1}{\mathrm{J}}\left(\frac{1}{\mathrm{I}_3} - \frac{1}{\mathrm{I}_2}\right)\sin^2\phi\,\cos\theta \tag{4.58}$$

Now stare at this. Then define a function on the sphere,

$$H(\theta,\phi) = \frac{\mathrm{J}_1^2}{2I_1} + \frac{\mathrm{J}_2^2}{2I_2} + \frac{\mathrm{J}_3^2}{2I_3} \tag{4.59}$$

We also define the symplectic form

$$\omega_{\mu\nu} = J\begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} \quad \Leftrightarrow \quad \omega^{\mu\nu} = \frac{1}{\mathrm{J}\sin\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.60}$$

(The indexing here is such that $\omega_{\theta\phi} = J \sin\theta$.) A calculation now verifies that Euler's equations for fixed angular momentum squared take the Hamiltonian form

$$\dot{\theta} = \omega^{\theta\phi}\partial_{\phi}H\,, \quad \dot{\phi} = \omega^{\phi\theta}\partial_{\theta}H \tag{4.61}$$

Moreover it is easily checked that we have the Poisson bracket algebra

$$\{J_1,J_2\} = -J_3\,, \quad \{J_2,J_3\} = -J_1\,, \quad \{J_3,J_1\} = -J_2 \tag{4.62}$$

This follows directly from the definition (4.44).

This is an example where the phase space is a sphere, and where no natural split of the coordinates ($\theta and\ \phi$) into a canonical pair (q,p) exists. But according to Darboux' theorem canonical coordinates must exist on any open region of the sphere. Indeed it is easy to check that

$$q = \cos\theta\,,\ p = \phi \tag{4.63}$$

form a canonical pair. Moreover the area element on the sphere becomes, in these coordinates

$$dA = J \sin\theta d\theta d\phi = Jdqdp \tag{4.64}$$

Hence the canonical coordinates are the ones you would use if you want a map of the sphere to display areas correctly.

Closer examination of this example reveals that generalisation to any evendimensional sphere may be non-trivial. The point is that we cannot use any anti-symmetric non-degenerate tensor as a symplectic form. According to the definition it must also solve eq.(4.35). In the two-dimensional case this is a trivial point, but as it turns out there is simply no solution for any higher dimensional sphere. Other, non-spherical, examples may be given in higher dimensions however.

Section(4.2): Infinitesimal canonical transformations

We have yet to discuss the classical analogue of unitary transformations. Unitary transformations preserve commutators, so we define canonical

transfor-mations as transformations that preserve Poisson brackets. This is the same thing as saying that canonical transformations leave the symplectic form invariant, and is analogous to the definition of rotations as those transformations that leave a specific metric invariant, eq. (4.27). As always, we can regard the transformations as active, giving rise to a movement in phase space (or Hilbert space), or as passive transformations from one coordinate description to another.

Both viewpoints are useful, but for the general discussion the active viewpoint is the better. Anyway, if we are given a transformation on phase space

$$q \rightarrow q' = q'(q,p),\ p \rightarrow p' = p'(q,p) \tag{4.65}$$

then phase space functions are transformed according to

$$A'(q',p') = A(q,p) \tag{4.66}$$

("The new function takes the same value at the new point as the old function does at the old point.") The transformation is canonical if

$$\{A,B\} = C \Rightarrow \{A',B'\} = C' \tag{4.67}$$

We begin by looking for infinitesimal canonical transformations. Let us express the function A' as a function of q and p, and consider

$$A'(q,p) = A(q,p) + \delta A(q,p) \tag{4.68}$$

This is a definition of the function δA, which is assumed to be small so that terms of second order in δA can be ignored in the calculations we will do. Choose a function $F = F(q,p)$ on phase space, and set

$$\delta A = \epsilon\{A,F\} \tag{4.69}$$

Computing to first order in ϱ, and using the Jacobi identity, we find that

$$\begin{aligned}\{A',B'\} &= \{A,B\} + \epsilon(\{\{A,F\},B\} + \{A,\{B,F\}\}) \\ &= C - \epsilon\{F,\{A,B\}\} = C + \epsilon\{C,F\} = C + \delta C = C'\end{aligned} \tag{4.70}$$

(to that order). This works for any phase space function F. In principle it is possible to integrate eq. (4.69) to obtain a finite canonical

transformation the equation has the same form as Hamilton's equations of motion, and the only catch may be that it will be difficult to do the integration explicitly.

Again, note the similarity to quantum mechanics: in quantum mechanics any Hermitian operator generates a unitary transformation, in classical mechanics any phase space function generates a canonical transformation. At the same time there is an interesting difference between canonical and unitary transformations: there are many more phase space functions than Hermitian operators, hence there are many more canonical transformations. This has to do with the fact that the unitary transformations also preserve a scalar product, and in this sense they are analogous to rotations. If the space has a finite dimension, there are indeed rather "few" rotations, but the set of canonical transformations is always infinite dimensional because the space of all functions has infinite dimensions. (The discerning reader may think that the comparison is unfair, since rotations take place in a vector space, while the kind of transformation we now allow are more general than that. However, it can be shown that the set of transformations leaving a given metric tensor invariant is at most as large as the set of translations and rotations in a vector space, and indeed often smaller than that. So the objection has no force.)

Let us revisit the discussion of Noether's. We used it already to motivate the definition of the canonical momenta, but there is more to say about it. Let us choose spatial rotation, eq. (1.88), as an example of a transformation that we will want to make. It is easy to see that

$$L_i = \epsilon_{ijk} x_j p_k \Rightarrow \quad \{x_i, \epsilon_j L_j\} = \epsilon_{ijk}\epsilon_j x_k = \delta x_i \tag{4.71}$$

The Noether charge generates the transformation via the Poisson bracket. More is true. The ϵ-tensor obeys the identity

$$\epsilon_{ijk}\epsilon_{kmn} + \epsilon_{ink}\epsilon_{kjm} + \epsilon_{imk}\epsilon_{knj} = 0 \tag{4.72}$$

By means of this identity it is easy to show that

$$\{L_i, L_j\} = \epsilon_{ijk}L_k \tag{4.73}$$

The educated way of describing this result is to say that the rotational Noether charges form a Poisson bracket representation of the Lie algebra of the rotation group. In quantum mechanics, we have a commutator representation of the same Lie algebra.

It should come as no surprise to hear that the results concerning rotations and their Noether charges can be generalized to any Noether charges. Let us just mention time translation invariance, eq. (1.89). Here the Noether charge is the Hamiltonian itself. In perfect analogy to our discussion of rotations, we find that

$$\delta x_i = \epsilon\dot{x}_i = \{x_i, \epsilon H\} \tag{4.74}$$

Again the Noether charge generates the transformation to which it owes its existence.

Now we will discuss the symplectic one-form .To proceed further with canonical transformations we need to develop some more machinery. First a little discussion of vector fields. A contravariant vector field (with indices upstairs) is a an assignment of a little arrow to each point in a space. These arrows can always be regarded as the tangent vectors of a set of curves the flowlines of the vector field. Hence the general definition: “A dynamical system is a vector field on a manifold”. This is a mathematical way of saying that we are dealing with equations of the form $z_\mu = f_\mu(z)$, for some functions f_μ. In vector spaces we thought of covariant vectors (with indices downstairs) as assigning a set of parallel planes, as shown in Fig. 4.2. So we would expect that a covariant vector field is a set of level surfaces of some function. But this is simply not true in general. It is true if and only if the covariant vector field is the gradient

of some function, $u\mu = \partial\mu f$. And it can be shown that this is so if and only if

$$\partial_\mu u_\nu - \partial_\nu u_\mu = 0 \tag{4.75}$$

If this condition fails the level planes that exist at each point do not fit together to form level surfaces that extend over all space. Actually this phenomenon is well known from the theory of curve integrals. A covariant vector field is just what we need in the integrand of such an integral. Suppose that a curve γ is defined explicitly by the functions $z^\mu = z^\mu(t)$, where t is some parameter along the curve. Then

$$\int_\gamma u_\mu(x) dz^\mu = \int_{t_1}^{t_2} \frac{dz^\mu}{dt} x_\mu\big(x(t)\big) dt \tag{4.76}$$

It is known that this integral is independent of the path and defines a function $f(x(t))$, if and only if condition (4.75) holds. But the curve integral is well defined regardless of whether this is true or not. This makes sense: in a vector space a covariant vector is a map from the set of all arrows to the real numbers, and in general a covariant vector field is a map from the set of all flowlines to the real numbers.

This leads to a useful piece of notation. If the coordinates on our space are denoted by z^μ and if u_μ is a covariant vector field, then we define the differential one-form u as

$$u = dz^\mu u_\mu \tag{4.77}$$

That is, the coordinate differerentials dz^μ are chosen as the basis in which we expand the covariant vector. The gradient of a function f defines a one-form of a special kind, namely

$$df = dx^\mu \partial_\mu f \tag{4.78}$$

The notation is useful since it gives the correct behaviour of $u_\mu(x)$ under coordinate transformations. Let $x^{\mu'} = x^{\mu'}(x)$. Then

$$u = dx^\mu u_\mu = dx^{\mu'} \frac{\partial x^\nu}{\partial x^{\mu'}} u_\nu = dx^{\mu'} u'_\mu \Longrightarrow u'_\mu(x') = \frac{\partial x^\nu}{\partial x^{\mu'}} u_\nu(x) \tag{4.79}$$

Notice that everything happens automatically.

We brought this up because there is a symplectic one-form lurking behind the symplectic two-form $\omega_{\mu\nu}$. To see why we come back to eq. (4.35), which was part of the definition of the latter. If the components of $\omega_{\mu\nu}$ are constants it obviously holds, but in general it is something that must be checked. The same equation occurs in electrodynamics when written in tensor form. It is this equation that implies that the vector potential exists:

$$\partial_\mu \omega_{\nu\sigma} + \partial_\sigma \omega_{\mu\nu} + \partial_\nu \omega_{\sigma\mu} = 0 \Leftrightarrow \omega_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu \qquad (4.80)$$

Here θ_μ is some covariant vector field. In symplectic geometry it is known as the symplectic one-form. (If you like, "one-form" is simply another name for "covariant vector field".)

When electrodynamics is written in tensor notation one finds that the electric and magnetic fields can be written as a two-form

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_i \\ E_i & \epsilon_{ijk} B_k \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (4.81)$$

Eq. (4.80) are part of Maxwell's equations; they guarantee the existence of the vector potential, which is a one-form. There is also a second tensor equation relating the two-form to the electric current, but this has no analogue in symplectic geometry.

What does the symplectic one-form look like if we use our standard coordinates (q_i, p_i) on phase space, so that the symplectic two-form takes the form (4.36)? There is no unique answer to this question since the previous discussion implies that if

$$\theta'_\mu = \theta_\mu + \partial_\mu A \qquad (4.82)$$

then $\theta'_\mu$and θ_μ give rise to the same symplectic two-form. However, a possible choice of symplectic one-form is

$$\theta = p_i dq_i \tag{4.83}$$

This will be our standard choice. To make sure that you see how it works, consider a two dimensional phase space with

$$z^\mu = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \theta_\mu = (\ p\ , 0\ ) \tag{4.84}$$

It is simple to verify that

$$\omega = \begin{pmatrix} 0 & \partial_q\theta_2 - \partial_p\theta_1 \\ \partial_p\theta_1 - \partial_q\theta_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \{q\ , p\ \} = 1 \tag{4.85}$$

More generally, using eq. (4.38) we recover our standard symplectic two-form (4.36) in any dimension.

In the following study general transformation theory. So far we have discussed infinitesimal canonical transformations only, and we must now turn to finite ones. We will give the general transformation theory for the special case of two dimensional phase spaces only; however, every detail of the following arguments does have a natural and straightforward generalization to the higher dimensional cases, so the loss of generality is only apparent.

First of all we adopt a practice introduced by Whittaker, who denoted the transformed coordinates by capital letters rather than primes. Thus in the active interpretation the point labelled by (q, p) is transformed to the point labelled (Q, P),

$$Q = Q(q, p) \qquad P = P(q, p) \tag{4.86}$$

This is a canonical transformation if the Poisson brackets obey

$$\{Q, P\} = \{q, p\} \tag{4.87}$$

It is more convenient to study the transformation of the symplectic one-form. The transformation is canonical if it leaves the symplectic one-form unchanged. In fact we can be a little more general, because if we add a gradient to the symplectic one-form the symplectic two-form, and

hence the Poisson brackets, are unchanged. Hence the transformation is canonical if and only if

$$p_i dq_i = P_i dQ_i + dF \tag{4.88}$$

where F is some function $F = F(Q, P)$.

Let us assume that the canonical transformation we are about to construct is such that q and Q together also can serve as coordinates on phase space. Denote the function F by $F_1 = F_1(q, Q)$. Then we derive that

$$\left(p - \partial_q F_1\right)dq - \left(P + \partial_Q F_1\right)dQ = 0 \tag{4.89}$$

Since q and Q serve as coordinates since dq and dQ are linearly independent one-forms we can conclude from this that

$$p = \partial_q F_1(q, Q) P = -\partial_Q F_1(q, Q) \tag{4.90}$$

We assume that these equations are invertible, that is to say that they can beused to derive the explicit formulas

$$\begin{cases} \mathrm{Q} = \mathrm{Q}(\mathrm{q}\,, \mathrm{p}) \\ P = P(q\,, p) \end{cases} \Leftrightarrow \begin{cases} q = q(Q\,, P) \\ p = p(Q\,, P) \end{cases} \tag{4.91}$$

This is the canonical transformation generated by the generating function $F_1(q, Q)$.

You may have wondered how we knew that we could use q and Q as coordinates on phase space. The answer is that we did not know, but in a concrete situation we would have noticed whether anything was wrong once we tried to invert eqs.(4.90) in order to obtain the explicit canonical transformation. If you think about it, it is clear that there cannot exist a generating function $F_1(q, Q)$ generating the simplest canonical transformation we can think of, namely the identity transformation

$$Q = q \qquad\qquad P = p \tag{4.92}$$

Evidently the pair q and Q does not coordinatize phase space. To circumvent this difficulty we define a new generating function by means of a Legendre transformation from the first:

$$F_2 = F_1(q,Q) + QP \tag{4.93}$$

The sign conventions here are a little odd. Still the relation between the functions F_2 and F_1 is recognisably a Legendre transformation, and it follows by virtue of eqs. (4.90) that F_2 is a function of q and P only. If we assume that q and P together coordinatize phase space which is certainly true for the identity transformation we can use F_2 to generate canonical transformations by repeating the previous logic:

$$\begin{aligned} pdq &= d(PQ) - QdP + F_1 = -QdP + dF_2 \\ &= -QdP + \partial_q F_2 dq + \partial_P F_2 dP \end{aligned} \tag{4.94}$$

Comparing coefficients in front of the linearly independent one-forms, in this case dq and dP, we conclude that

$$p = \partial_q F_2 \qquad\qquad Q = \partial_P F_2 \tag{4.95}$$

Again we assume that these equations can be inverted to yield eqs.(4.91) in explicit form. The identity transformation is generated by

$$F_2(q,P) = qP \tag{4.96}$$

We can go on to define the generating functions $F_3 = F_3(p,Q)$ and $F_4 = F_4(p,P)$ by means of further Legendre transformations, in the obvious way.

We have recovered the conclusion from section(4.2), namely that every function on phase space generates a canonical transformation, but now we have a recipe for how to find such transformations in finite not only infinitesimal form. Hamilton's equations retain their form under all canonical transformations, with a new Hamiltonian defined as the function

$$K(Q,P) = H(q,p) = H(q(Q,P),p(Q,P)) \tag{4.97}$$

By the way, notice that the mathematical manipulations we are doing resemble those of thermodynamics. In fact there is a close relation between symplectic geometry and thermodynamics, although in the latter

case the situation is a little different in thermodynamics one always starts out in an odd dimensional space, with coordinates U, (T,S), (P,V), and so on. In phase space there is no analogue of the "extra" coordinate U.

The setup can be generalized somewhat, by allowing the generating transformations to depend on time. Hamilton's equations will again retain their form, but the numerical value of the Hamiltonian will change. In fact, at least on the level of Gedanken calculations, it is possible to choose a time dependent generating function ensuring that the transformed Hamiltonian vanishes, so that the resulting equations of motion solve themselves. This is the subject of Hamilton-Jacobi theory, but we will not go into this.

References

[1] A. Dahan Dalmedico et al.: Chaos et d´eterminisme, ´Editions du Seuil 1992.

[2] C. Lanczos: The Variational Principles of Mechanics, Dover 1986.

[3] Ernst Mach: Die Mechanik in ihrer Entwickling, many editions (avoid the ninth.).

[4] Julian Barbour: The End of Time, Weidenfeld and Nicolson 1999.

[5] I. Percival and D. Richards: Introduction to Dynamics, Cambridge 1982.

[6] L. D. Landau and E. M. Lifshitz: Mechanics; Pergamon 1960.

[7] Max Born, Cause, Purpose and Economy in Natural Laws, in: Physics in My Generation, Springer 1969.

[8] V. I. Arnold: Mathematical Methods of Classical Mechanics, Springer 1978.