Sudan University of Science and Technology

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Estimating the Numberal Eigenvalues and Banach – Mazur Rotation Problem with Strong Proximality

تقدير القيم الذاتية العددية و مسألة دوران باناخ – مازور مع التقريب القوي

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Dedication

To my great father .lovely mother , brothers , sister and my husband .

Acknowledgments

Firstly, I thank the AL Mighty Allah to gave me health, strength and patience to complete this work. I would like to express my deep gratitude to my supervisor Prof Shawgy Hussein Abdalla for this guidance , support valuable comment and advice. I would like to express my sincere thanks to my father ,my mother and my husband for moral support.

Abstract

We extend previous results on operators, in Hilbert space. The method employs complex analysis and a new finite dimensional reduction, allowing us to avoid using the existing theory of determinants in Banach space, which would require strong restriction. We show that the spaces, and all infinite dimensional subspaces of their quotient spaces do not admit equivalent almost transitive renormings. We study product integrability of functions with values in unital Banach algebras. The product integrals are understood in the sense of Kurzweil, Mcshane or Riemann. We investigate a variation of the transitivity problem for proximinality properties of subspace and intersection properties of balls in Banach spaces .

الخلاصة

مددنا نتائج سابقة على المؤثرات في فضاء هلبرت، الطريقة تستخدم التحليل المركب واختزال البعد المنتهي الجديد ويسمح لنا إلى تفادي استخدام نظرية الوجود للمحددات في فضاء باناخ والذي يتطلب قصرا قويا . أوضحنا أن الفضاءات وكل الفضاءات الجزئية لا نهائية البعد لفضاءات القسمة لها لا تسمح إعادة النظائم المتعدية دائما مكافئة. تمت دراسة تكاملية الضرب للدوال مع القيم في جبر باناخ الأحادي. تكاملات الضرب تم استيعابها في حالة كيرزيل وماشان أو ريمان ، تقصينا التغيير لمسالة التعدية لأجل خصائص التقريب للفضاء الجزئي وخصائص التقاطع للكرات في فضاءات باناخ.

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Chapter 1

Eigenvalues of linear operators on Banach space

Let L_0be bounded operator on a Banach space, and consider a perturbation $L = L_0 + K$, where $K$is compact. This work is concerned with obtaining bounds on the number of eigenvalues of L in subset of the complement of the essential spectrum of L_0, intrems of the approximation numbers of the perturbing operator K. Our result can be considered as wide generalizations of classical results on the distribution of eigenvalues of compact operators, which correspond to the case $L_0 = 0$. An example is constructed showing that there are some essential difference in the possible distribution of eigenvalues of operators in general Banach space, compared to the Hilbert space case.

Estimating the number of eigenvalues of linear operators on Banach spaces.

Section(1.1) : Eigenvalues as zeros of analytic functions

The study of the distribution of eigenvalues of compact operators on a Banach space is a classical and well-developed subject. Of primary concern is the problem of relating sum ability properties of some sequence of singular numbers (like the approximation numbers or Weyl-numbers) of a compact operator L to the summability properties of its sequence of eigenvalues. For instance, a result of König , which generalizes the classical Wey estimate for Hilbert space operators, says that

$$\sum_j |\lambda_j(L)|^p \leq 2(2e)^{\frac{p}{2}} \sum_j \alpha_j^p\,(L), \qquad p > 0,$$

Where $\lambda_j(L)$ and $\alpha_j\ (L)$ denote the non-zero eigenvalues and the approximation numbers of L, respectively. An immediate consequence of this estimate is a bound on the number of eigenvalues $n_L(s)$ of L outside the closed disk $B_s = \{\lambda \in C \mid |\lambda| \leq s\}$, namely

$$n_L(s) \leq \frac{2(2e)^{\frac{p}{2}}}{s^p} \sum_j \alpha_j^p\,(L) \qquad (1)$$

Our goal in the present section is to prove bounds analogous to (1) for non-compact operators $L = L_0 + K$, where L_0 is a bounded operator and K is a compact operator on a complex Banach space X. In such a case, Weyl's theorem on preservation of the essential spectrum implies that

$$\sigma_{ess}(L) = \sigma_{ess}(L_0) \subset \sigma(L_0) \subset B_{\|L_0\|},$$

So that, for any $s > \|L_0\|$, the part of the spectrum of L_0 outside B_s consists of a finite number of eigenvalues of finite algebraic multiplicity. We wish to express this fact quantitatively by explicitly bounding the number of eigenvalues $n_L(s)$in B_s^c. For example, one of our results is

$$n_L(s) \leq C(p) \cdot \frac{s}{(s - \|L_0\|)^{p+1}} \sum_j \alpha_j^p\,(L) \qquad s > \|L_0\| \qquad (2)$$

Note that in the very special case $L_0 = 0\,(so\ that\ L = K)$, (2) reduces to the classical result (1), up to the value of a multiplicative constant.

While, as mentioned above, the distribution of eigenvalues of compact operators on Banach spaces is very well-studied, the same cannot be said for the type of generalization considered here. Indeed, essentially all results that we are aware of only concern the case where $L = L_0 + K$ is a Hilbert space operator . We only mention the classics and some more recent works. In particular, we only cite works about general $\text{non} - \text{selfadjoint}$ operators.) As we discuss below, the methods employed in the Hilbert space setting cannot be directly extended to Banach spaces, and therefore some essentially new ideas are required, and these are developed here.

While we will pursue the same approach as mentioned in the previous paragraph, a key technical innovation of the present work is that we will not rely on the known determinant theory for Banach space operators instead, we will use a finite-dimensional reduction argument to construct the required holomorphic function,

whose zeros in a certain domain $\Omega \subset \mathbb{C}$ coincide with the eigenvalues of L in this domain, using only (generalized) determinants of finite-rank operators. In this way we are able to avoid the strong assumptions on K required for directly employing in finite-dimensional determinant theory. This enables us to obtain results in which the only assumption on K is that it is approximable by finite rank operators, i.e. that its approximation numbers $\alpha_j(K)$ tend to zero (but not assuming anything about their summability). In particular, this means that our results are new even when specializing to the case of Hilbert space operators. When the approximation numbers are (p-)summable, our bounds take a particularly simple form.

In the next section we will gather some preliminary results concerning approximation numbers and determinants of finite rank operators. we will construct a holomorphic function whose zeros coincide with the eigenvalues of L, and we will prove our eigenvalue estimates. we will provide some remarks concerning the sharpness of our results, including a comparison with previously obtained results in Hilbert spaces and an example which shows that there is an essential difference in the distribution of eigenvalues between the Hilbert space case and the general Banach space case considered here.

Let $(X, \|.\|_X)$ be a complex Banach space and let $B(X)$ and $F(X)$ denote the classes of bounded and finite rank operators on X, respectively. The operator norm of $L \in B(X)$ will be denoted by$\|L\|$. We defne the nth approximation number of $L \in B(X)$ as

$$\alpha_n(L) := inf\{\|L - F\| : F \in \mathcal{F}(X), rank(F) < n\}, n \in \mathbb{N}$$

If $\alpha_n(L) \to 0$ for $n \to \infty$, then L is a compact operator on X. On the other hand, in some Banach spaces not every compact operator can be approximated by finite rank operators, as has been shown by Enflo.

We recall the following properties of the approximation numbers For $K, L, M \in \boldsymbol{B}(X)$ and $n, m \in \mathbb{N}$

(i) $\|L\| = \alpha_1(L) \geq \alpha_2(L) \geq \cdots \geq 0,$

$(ii)\ \alpha_{n+m-1}(K + L) \leq \alpha_n(K) + \alpha_m(L),$

$(iii)\ \alpha_1(KLM) \leq \|K\|\alpha_n(L)\|M\|,$

$(iv)\ \alpha_1(L) = 0\ if\ rank(L) < n.$

Let $K \in \boldsymbol{B}(X)$ be a compact operator and let $\lambda_1(K), \lambda_2(K), \ldots$ denote its non-zero eigenvalues, ordered such that $|\lambda_1(K)| \geq |\lambda_2(K)| \geq \cdots > 0$ and counted according to their algebraic multiplicity, where the algebraic multiplicity is defined as the rank of the Riesz projection of K with respect to the considered eigenvalue. The following estimate is due to König for $p \in (0, \infty)$

$$\sum_j |\lambda_1(K)|^p \leq 2(2e)^{\frac{p}{2}} \sum_j \alpha_j^p(K). \qquad (3)$$

Let $F \in \mathcal{F}(X)$. For $n \in \mathbb{N}$ the n-regularized determinant of $\mathbb{1} - F$, where $\mathbb{1}$ denotes the identity operator on X, is defined in terms of the (finite number of) eigenvalues of F, as follows:

$$\det_n(\mathbb{1} - F) := \prod_k \left[(1 - \lambda_k(F)) \exp\left(\sum_{j=1}^{n-1} \frac{\lambda_k^j(F)}{j} \right) \right]$$

Here we use the standard convention that $\sum_{j=1}^{0} \quad (\ldots) := 0$.

As a first simple but important property of regularized determinants let us note that $\det_n(\mathbb{1} - F) \neq 0$ iff $\mathbb{1} - F$ is invertible in $\boldsymbol{B}(X)$.In the following, we will gather some less obvious properties. To this end, let us denote the extended complex plane by $\hat{\mathbb{C}}$ and for a subspace Y of X let us set

$$\mathcal{F}(X; Y) := \{F \in \mathcal{F}(X) : ran(F) \subset Y\}.$$

We note that Y is an invariant subspace of $F \in \mathcal{F}(X; Y)$ and that the non-zero eigen-values of F and F_Y(the restriction of F to Y) coincide. In particular,

$$\det_n(\mathbb{1} - F) = \det_n(\mathbb{1}_Y - F_Y).$$

Proposition (1.1.1)[1]:

Let $G \subset \hat{\mathbb{C}}$be open and let $Y \subset X$ be a finite-dimensional subspace. Suppose that $F : G \to \mathcal{F}(X;Y)$ is analytic. Then the map $\det_n(\mathbb{1} - F(\lambda))$ is analytic on G as well.

Proof:

We would like to use the fact that for Hilbert space operators the analyticity of the regularized determinant has been proven . To this end, for every $\lambda \in G$ we denote by $F_Y(\lambda)$ the restriction of $F(\lambda)$to Y. From the discussion preceding the proposition we know that $\det_n\big(\mathbb{1} - F(\lambda)\big) = \det_n(\mathbb{1}_Y - F_Y)$. Choose a norm$\|.\|_Y$on Y such that $\mathcal{H} = (Y, \|.\|_Y)$ is a Hilbert space and let

$J : \mathcal{H} \to (Y, \|.\|_Y)$denote the canonical isomorphism. Since the eigenvalues of $F_Y(\lambda)$ and $J^{-1}F_Y(\lambda)J$ coincide (including multiplicity), we obtain that

$\det_n(\mathbb{1}_Y - F_Y(\lambda)) = \det_n(\mathbb{1}_{\mathcal{H}}J^{-1}F_Y(\lambda)J)$. It remains to note that

$(\mathbb{1}_{\mathcal{H}}J^{-1}F_Y(\lambda)J) \in \mathcal{F}(H)$is analytic and of finite rank.

Remark (1.1.2)[1]:

The assumption that the ranges of all operators $F(\lambda)$ are contained in a single space Y is certainly not necessary. However, it is sufficient for our purposes and, as we have seen above, it allows for a very easy proof. For completeness we should note that, without this assumption, the analyticity$\lambda \longmapsto \det_n(\mathbb{1} - F(\lambda))$

Proposition (1.1.3)[1]:

Let $p \in (0, \infty)$ and $F \in \mathcal{F}(X)$. Then there exists a constant Γ_p, depending only on p, such that

$$\left|\det_{[p]}(\mathbb{1} - F)\right| \leq \exp\left(2(2e)^{\frac{p}{2}}\Gamma_p \sum_j \alpha_j^p(F)\right) \qquad (4)$$

Where$[p] = \min\{n \in \mathbb{N}: n \geq p\}$.

Proof:

There exists a constant $\Gamma_p > 0$ such that for $\lambda \in \mathbb{C}$:

$$\left|(1-\lambda)\exp\left(\sum_{j=1}^{[p]-1}\frac{\lambda^j}{j}\right)\right| \leq \exp\left(\Gamma_p|\lambda|^p\right)$$

This implies that

$$\left|\det_{[p]}(\mathbb{1} - F)\right| \leq \exp\left(\Gamma_p\sum_j\left|\lambda_j(F)\right|^p\right)$$

Now apply estimate (3)

Remark (1.1.4)[1]:

A short calculation shows that $\Gamma_p \leq 1/p \; if \; p \leq 1$. Moreover, for integer-valued $p \geq 2$ we have $\Gamma_p \leq (p-1)/p$ if $p \neq 3$ and $\Gamma_p \leq 1$,

We recall that the essential spectrum of $L \in \boldsymbol{B}(X)$ is defined as

$\sigma_{ess}(L) = \{\lambda \in \mathbb{C} : \lambda - L$ is not a Fredholm operator$\}$. (5)

Here an operator is called Fredholm if it has closed range and both its kernel and cokernel are finite-dimensional. Moreover, the discrete spectrum of $L, \sigma_d(L)$, consists of all isolated eigenvalues of L of finite algebraic multiplicity. The elements of the discrete spectrum will be called discrete eigenvalues. They can accumulate only at the essential spectrum.

By Weyl's theorem, the essential spectrum is invariant under a compact perturbation, so if $L_1 \in \boldsymbol{B}(X)$ and $L_2 - L_1$ is compact, then $\sigma_{ess}(L_2) = \sigma_{ess}(L_1)$. The discrete spectra of L_2 and L_1certainly need not coincide. In the following, assuming that the difference $L_2 - L_1$ is of finite rank, we will identify the discrete eigenvalues of L_2 outside the spectrum of L_1 with the zeros of a certain holomorphic function.

Proposition (1.1.5)[1]:

Let $L_1, L_2 \in \boldsymbol{B}(X)$ and suppose that $L_2 - L_1 \in \mathcal{F}(X)$. Let U denote the unbounded component of $\mathbb{C} \setminus \sigma(L_1)$. For$\lambda \in U$ and $p \in (0, \infty)$ define

$$d_p^{L_2,L_1} := \det_{[p]}(\mathbb{1} - (L_2 - L_1)(\lambda - L_1)^{-1}) \qquad (6)$$

Then the following hold:

(i) $d_p^{L_2,L_1}$ *is analytic on* U,

(ii) $\log\left|d_p^{L_2,L_1}(\lambda)\right| =$

$2(2e)^{p/2}\Gamma_p \sum_k \alpha_k^p\,(L_2 - L_1)(\lambda - L_1)^{-1}$, *where* Γ_p *is as in Proposition* (1.1.3),

$(iii)\lambda_0 U$ *is a zero of* $d_p^{L_2,L_1}$ *of order* k *if and only if it is a discrete*

eigenvalue of L_2 of algebraic multiplicity k, we call $d_p^{L_2,L_1}$ the pth perturbation determinant of $L_2$by L_1.

Proof :

For $\lambda \in U$, we set $F(\lambda) := (L_2 - L_1)(\lambda - L_1)^{-1}$. Then $F(\lambda)$ is analytic and of finite rank and $\mathrm{ran}(F(\lambda)) \subset ran(L_2 - L_1)$ for all $\lambda \in U$, it is clear that $\lambda_0 \in U$ is a zero of $d_p^{L_2,L_1}$iff 1is an eigenvalue of $F(\lambda_0)$. We now show that this is the case iffλ_0 is an eigenvalue of L_2. Indeed, if 1 is an eigenvalue of $F(\lambda_0)$ then there exists $x \in X, x \neq 0$, with $F(\lambda_0)x = x$, that is $(L_2 - L_1)(\lambda_0 - L_1)^{-1}x = x$, so setting $y = (\lambda_0 - L_1)^{-1}x$ we have $(L_2 - L_1)y = (\lambda_0 - L_1)y$, that is $L_{2y} = \lambda_{0y}$, so $\lambda_0$is an eigenvalue of L_2. Conversely, if $\lambda_0 \in \mathbb{C}\setminus\sigma(L_1)$ is an eigenvalue of L_2, then we have $L_{2y} = \lambda_{0y}$ for some

$\mathcal{Y} \in X, y \neq 0$. Thus setting $x = (\lambda_0 - L_1)\mathcal{Y}$ we obtain

$$F(\lambda_0)x = (L_2 - L_1)(\lambda_0 - L_1)^{-1}x = (L_2 - L_1)y = (\lambda_0 - L_1)y = x,$$

So that indeed is an eigenvalue of $F(\lambda_0)$.

That all eigenvalues of L_2 in U are discrete follows from the fact that the spectrum of L_2 in the unbounded component of $\mathbb{C}\backslash\sigma_{ess}(L_2)$ is purely discrete and from the fact that

$$U\ \mathbb{C}\backslash\sigma(L_1) \subset \mathbb{C}\backslash\sigma_{ess}(L_1) = \mathbb{C}\backslash\sigma_{ess}(L_2),$$

Which shows that U is a subset of this unbounded component.

It remains to show that the multiplicities of λ_0 as a zero of $d_p^{L_2,L_1}$ and as an eigenvalue of L_2 coincide. For that purpose, let us first note that it is no restriction to assume that$\lambda_0 \neq 0$. Now we denote the Riesz projection of L_2 with respect to $\lambda_0 \in \sigma(L_2) \cap U$ by P, and we set $T = L_2P\ and\ T^{\perp} = L_2(\mathbb{1} - P)$. Note that T is of finite rank, with $\sigma(T) = \{\lambda_0\}$, and that $\lambda_0 \in \sigma(T^{\perp})$. In particular, there exists a ball B around λ_0 such that $0 \notin B$ and such that $\lambda - L_1$ and $\lambda - T^{\perp}$ are invertible for all $\lambda \in B$. Now a short computation, using $TT^{\perp} = T^{\perp}T = 0$ and

$L_2 = T + T^{\perp}$, shows that for $\lambda \in B$

$$\mathbb{1} - (L_2 - L_1)(\lambda - L_1)^{-1} = (\mathbb{1} - \lambda^{-1}T)(\mathbb{1} - (T^{\perp} - L_1)(\lambda - L_1)^{-1}).$$

Hence, following, there exists a holomorphic function $C_{L_2,L_1,p}(\lambda)$ such that $d_p^{L_2,L_1}(\lambda) = det_1(\mathbb{1} - \lambda^{-1}T)det_{[p]}(\mathbb{1} - (T^{\perp} - L_1)(\lambda - L_1)^{-1}) \exp\big(C_{L_2,L_1,p}(\lambda)\big)$.

The operator $(\mathbb{1} - (T^{\perp} - L_1)(\lambda - L_1)^{-1}) = (\lambda - T^{\perp})(\lambda - L_1)^{-1})$ is invertible for $\lambda \in B$, so we see that the multiplicity of λ_0 as a zero of $d_p^{L_2,L_1}$ coincides with its multiplicity as a zero of $det_1(\mathbb{1} - \lambda^{-1}T) = (1 - \lambda^{-1}\lambda_0)^{rank\,(P)}$ But the rank of P coincides with the algebraic multiplicity of λ_0 as an eigenvalue of L_2.

Let $L_0 \in \boldsymbol{B}(X)$ and let K be a compact operator on X. We assume that K is the uniform limit of finite rank operators, i.e.

$$\lim_{n\to\infty} \alpha_n\,(K) = 0.$$

In the following, we will be interested in the discrete spectrum of the operator

$$L := L_0 + K.$$

Let $\Omega \subset \hat{\mathbb{C}}$ denote a connected, open set with $\infty \in \Omega$ and such that

$$\bar{\Omega} \cap \sigma(L_0) = \emptyset, \qquad (7)$$

Which implies that

$$S(\Omega) := \sup_{\lambda \in \Omega} \|(\lambda - L_0)^{-1}\| < \infty$$

Remark (1.1.6)[1]:

We note that the resolvent $R : \Omega \to \boldsymbol{B}(X), \boldsymbol{R}(\lambda) = (\lambda - L_0)^{-1}$, is analytic on $\Omega$with $R(\infty) = 0$.

Theorem (1.1.7)[1]:

Let $p \in (0, \infty)$ and $N \in \mathbb{N}_0$ such that $\alpha_{N+1}(K) < 1/S(\Omega)$. Then there exists a bounded holomorphic function $d : \Omega \to \mathbb{C}$ (depending on $p, N, L$and L_0) with the following properties:

$(i) d(\infty) = 1,$

$(ii) for\ all\ \lambda \in \Omega\ we\ have$

$$|d(\lambda)| \leq \exp\left(\frac{C_p \|(\lambda - L_0)^{-1}\|^p}{(1 - \alpha_{N+1}(K)\|(\lambda - L_0)^{-1}\|)^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)^p\right)\right)$$

$$\leq \exp\left(\frac{C_p S(\Omega)^p}{\left(1 - \alpha_{N+1}(K) S(\Omega)\right)^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)^p\right)\right)$$

Where

$$C_p = 2(2e)^{p/2}\Gamma_p, \qquad (8)$$

With Γ_p as in Proposition (1.1.3).

(iii) $\lambda_0 \in \Omega$ is a zero of d of order m if and only if it is a discrete eigenvalue of L of algebraic multiplicity m.

The proof of this theorem consists of two steps: First, we will use a finite dimensional reduction argument to construct a family of holomorphic functions which satisfy point (i) and (iii) of the theorem (and which 'almost' satisfy estimate (ii)). In the second step, we will use an approximation argument involving Montel's theorem to construct the function d.

To begin, let us $fix\ p > 0$ and let $N \in \mathbb{N}_0$ be chosen such that

$\alpha_{N+1}(K) < 1/S(\Omega)$. Then for $\eta \in (0, \frac{1}{S(\Omega)} - \alpha_{N+1}(K))$ there exists $F \in \mathcal{F}(X)$ of rank at most N such that

$$\|K - F\| < \alpha_{N+1}(K) + \eta$$

And so for all $\lambda \in \Omega$ we can estimate

$$\begin{aligned}\|(K - F)(\lambda - L_0)^{-1}\| &\leq \|(K - F)\|\|(\lambda - L_0)^{-1}\| \\ &\leq (\alpha_{N+1}(K) + \eta)\|(\lambda - L_0)^{-1}\| \leq (\alpha_{N+1}(K) + \eta)S(\Omega) < 1\end{aligned}$$

In particular, the operator $\mathbb{1} - (K - F)(\lambda - L_0)^{-1}$ is invertible and

$$\|[\mathbb{1} - (K - F)(\lambda - L_0)^{-1}]^{-1}\| \leq (1 - (\alpha_{N+1}(K) + \eta)\|(\lambda - L_0)^{-1}\|)^{-1} \quad (9)$$

Therefore, for $\lambda \in \Omega\backslash\{\infty\}$ the operator

$$\lambda - (L - F) = [\mathbb{1} - (K - F)(\lambda - L_0)^{-1}](\lambda - L_0)^{-1} \qquad (10)$$

is invertible, as the product of two invertible operators, and so $\Omega \subset \hat{\mathbb{C}}\backslash\sigma(L - F)$. It follows that the perturbation determinant

$$d_F(\lambda) := d_p^{L,L-F}(\lambda) = det_{[p]}(\mathbb{1} - F[\lambda - (L - F)^{-1}]) \qquad (11)$$

Is well-defined and analytic on Ω, and we have $d_F(\infty) = 1$. From Proposition(1.1.7)(with $L_2 = L, L_1 = L - F$) we further know that $\lambda_0 \in \Omega$ is a zero

of d_F if and only if it is a discrete eigenvalue of L of the same multiplicity. Proposition (1.1.7) also implies that

$$|d_F(\lambda)| \leq \exp\left(2(2e)^{p/2}\Gamma_p \sum_{j=1}^{N}\left(\alpha_j^p F[\lambda-(L-F)^{-1}]\right)\right)$$

Let us estimate the approximation numbers on the right-hand side of the previous inequality: Using (9) and (10) we obtain

$$\begin{aligned}\alpha_j\ (F[\lambda-(L-F)^{-1}]) &= \alpha_j\ (F(\lambda-L_0)^{-1}[\mathbb{1}-(K\ -\ F)(\lambda-L_0)^{-1}]^{-1}) \\ &\leq \alpha_j\ (F(\lambda-L_0)^{-1})\|[\mathbb{1}-(K\ -\ F)(\lambda-L_0)^{-1}]^{-1}\| \\ &\leq \frac{\alpha_j\ (F(\lambda-L_0)^{-1})}{1-(\alpha_{N+1}(K)+\eta)\|(\lambda-L_0)^{-1}\|}\end{aligned}$$

We continue, using that $\alpha_j\ (A\ +\ B) \leq \alpha_j\ (A)\ + \|B\|$,

$$\begin{aligned}\alpha_j\ (F(\lambda-L_0)^{-1}) &= \alpha_j\ \left((F-K)(\lambda-L_0)^{-1}+K(\lambda-L_0)^{-1}\right) \\ &\leq \|(F-K)(\lambda-L_0)^{-1}\| + \alpha_j\ (K(\lambda-L_0)^{-1}) \\ &\leq \|(\lambda-L_0)^{-1}\| + \left(\|F-K\|\alpha_j\ (K)\right) \\ &\leq \|(\lambda-L_0)^{-1}\| + \left(\alpha_{N+1}(K)+\eta+\alpha_j\ (K)\right)\end{aligned}$$

Therefore

$$\alpha_j\ (F[\lambda-(L-F)]^{-1}) \leq \frac{\|(\lambda-L_0)^{-1}\|\left(\alpha_{N+1}(K)+\eta+\alpha_j\ (K)\right)}{1-(\alpha_{N+1}(K)+\eta)\|(\lambda-L_0)^{-1}\|}$$

And so

$$\sum_{j=1}^{N}\alpha_j^p\ (F[\lambda-(L-F)]^{-1}) \leq \frac{\|(\lambda-L_0)^{-1}\|\sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\eta+\alpha_j\ (K)\right)^p}{(1-(\alpha_{N+1}(K)+\eta)\|(\lambda-L_0)^{-1}\|)^p}$$

Finally, we obtain the following upper bound on the function d_F: For all $\lambda \in \Omega$

$$|d_F(\lambda)| \leq \exp\left(2(2e)^{p/2}\Gamma_p \frac{\|(\lambda - L_0)^{-1}\|^p \sum_{j=1}^{N}\left(\alpha_{N+1}(K) + \eta + \alpha_j\ (K)\right)^p}{(1 - (\alpha_{N+1}(K) + \eta)\|(\lambda - L_0)^{-1}\|)^p}\right)$$

Let us collect all our results up to this point in the following lemma.

Lemma (1.1.8):

Let $N \in \mathbb{N}_0$ be such that $\alpha_{N+1}(K) < 1/S(\Omega)$ and fix some

$\eta \in (0, \frac{1}{S(\Omega)} - \alpha_{N+1}(K))$. Then there exists F of rank at most N such that the holomorphic function $d_F : \Omega \to \mathbb{C}$ defined by (11) satisfies (12) and $d_F(\infty) = 1$. In addition, $\lambda_0 \in \Omega$is a zero of d_F of order m if and only if it is a discrete eigenvalue of L of algebraic multiplicity m.

We conclude the proof of Theorem(1.1.7) with the following limiting argument: Choose $\mathbb{N}_0 \in N$ such that $\alpha_{N+1}(K) < 1/S(\Omega)$. Let$l_0 \in \mathbb{N}$ denote the smallest integer such that $\frac{1}{l_0} < \frac{1}{S(\Omega)} - \alpha_{N_0+1}(K)$. Then by the previous lemma for every

$l \geq l_0$there exists an operator F_l of rank at most N_0 such that the holomorphic function d_{F_l} on Ω defined by (11) satisfies

$$|d_F(\lambda)| \leq \exp\left(2(2e)^{p/2}\Gamma_p \frac{\|(\lambda - L_0)^{-1}\|^p \sum_{j=1}^{N}\left(\alpha_{N+1}(K) + \frac{1}{l} + \alpha_j\ (K)\right)^p}{\left(1 - \left(\alpha_{N+1}(K) + \frac{1}{l}\right)\|(\lambda - L_0)^{-1}\|\right)^p}\right)$$

$$\leq \exp\left(2(2e)^{p/2}\Gamma_p \frac{S(\Omega)^p \sum_{j=1}^{N}\left(\alpha_{N+1}(K) + \frac{1}{l_0} + \alpha_j\ (K)\right)^p}{\left(1 - \left(\alpha_{N+1}(K) + \frac{1}{l_0}\right)S(\Omega)^p\right)}\right)$$

For all $\lambda \in \Omega$. The right-hand side of this inequality is a uniform bound for the sequence of holomorphic functions $\left(d_{F_l}\right)_{l \geq l_0}$. Using Montel's theorem there exists a locally uniformly convergent subsequence. Calling the local uniform limit of this subsequence d, let us check that this function satisfies all conditions of Theorem

(1.1.7): First of all, it is clear that d satisfies the estimate stated under point (ii). This uniform bound on d also implies that d is holomorphic at infinity. The local uniform convergence of d_{F_l} and the fact that $d_{F_l}(\infty) = 1$ imply that also

$d(\infty) = 1$. Finally, Hurwitz' theorem and the fact that the zero sets of all functions d_{F_l}coincide with $\sigma_d(L) \cap \Omega$ imply the assertion concerning the zero set of d. This completes the proof of Theorem (1.1.7)

Remark (1.1.9)[1]:

We note that if the approximation numbers $\{\alpha_j(K)\}$ are p-summable for some $p \in (0, \infty)$, then we can do without assumption (7) and prove that there exists a holomorphic function $\tilde{d} : U \to \mathbb{C}$, defined on the entire unbounded component U of $\mathbb{C} \setminus \sigma(L_0)$, which satis?es points (i) and (iii) of Theorem (1.1.7) and the inequality

$$\log\left|\tilde{d}(\lambda)\right| \leq C_p \|(L_0 - \lambda)^{-1}\|^p \sum_{j=1}^{\infty} \alpha_j^p\,(K) \qquad \lambda \in U$$

We are not going to use this result in the present section, so let us just provide a rough sketch of proof: First, one approximates the set U with sets U_n which satisfy (7), then one applies Theorem (1.1.7) to obtain a sequence of holomorphic functions d_n defined on U_n and finally one uses Hurwitz' theorem (twice) to obtain the desired function $\tilde{d}$.

Section (1.2):Estimating the number of eigenvalues with sharpness result

We repeat our assumptions from the previous section: Let $L_0 \in \boldsymbol{B}(X)$ and let K be a compact operator on X, which is the uniform limit of finite rank operators. Set $L := L_0 + K$. We wish to estimate the number of eigenvalues of L in a domain Ω which is 'away' from the spectrum of L_0. To quantify the notion of 'away' we recall the definition of the ε −pseudospectrum of a linear operator L_0:

$$\sigma_\varepsilon(L_0) := \{\lambda \in \mathbb{C} : \|(\lambda - L_0)^{-1}\| \geq \varepsilon^{-1}\} \qquad (13)$$

In this section we will assume that $\Omega \subset \hat{\mathbb{C}}$is an open and simply connected set satisfying $\infty \in \Omega$, with

$$\bar{\Omega} \cap \sigma_{\varepsilon}(L_0) = \emptyset \qquad (14)$$

For some $\varepsilon > 0$. We note that (14) is just another way to express the condition

$$S(\Omega) := \sup_{\lambda \in \Omega} \|(\lambda - L_0)^{-1}\| < \frac{1}{\varepsilon} \qquad (15)$$

The ε −pseudospectrum of linear operators has been studied extensively in the last two decades, both from an analytical and a numerical perspective.

Our general result will provide estimates on the number $\mathcal{N}_L(\Omega')$ of discrete eigenvalues of L (counting algebraic multiplicity) in subsets $\Omega' \subset \Omega$. We denote by $\phi: \Omega \to \mathbb{D}$ a conformal mapping of Ω to the open unit disk $\mathbb{D}$, which satisfies $\phi(\infty) = 0$, whose existence is assured by Riemann's Mapping Theorem. We define

$$r_{\Omega}(\Omega') := \sup_{\lambda \in \Omega} |\phi(z)|$$

Note that $0 \leq r_{\Omega}(\Omega') \leq 1$, that $r_{\Omega}(\Omega') = 0 \; iff \; (\Omega') = \{\infty\}$, and that the values of r_{Ω} do not depend on the choice of the conformal mappingϕ, since all such mappings differ only by a multiplicative constant of norm 1.

Theorem (1.2.1)[1]:

Let $p \in (0, \infty)$ and let $\Omega \subset \hat{\mathbb{C}}$ be open and simply connected with $\infty \in \Omega$. Moreover, suppose that Ω satisfies (14) for some $\varepsilon > 0$. Then for any $\Omega' \subset \Omega$ with $0 < r_{\Omega}(\Omega') < 1$ the following hold:

(i) $If N \in \mathbb{N}_0 \; is \; such \; that \; \alpha_{N+1}(K) < \varepsilon, then$

$$\mathcal{N}_L(\Omega') \leq \frac{C_p}{(\varepsilon - \alpha_{N+1}(K))^p \log\left(\frac{1}{r_{\Omega}(\Omega')}\right)} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p$$

(ii) $If \; \{\alpha_j(K)\} \in l^p(\mathbb{N}), then$

$$\mathcal{N}_L(\Omega') \leq \frac{C_p}{\varepsilon^p \log\left(\frac{1}{r_\Omega(\Omega')}\right)} \sum_{j=1}^{N} \alpha_j(K)$$

In both cases, C_p is as defined in (8).

Proof:

Note that from Jensen's identity we know that for a bounded holomorphic function h on $\mathbb{D}$ with $|h(0)| = 1$ we have

$$\int_0^1 \frac{n(h;s)}{s} ds \leq \log\|h\|_\infty,$$

Where $n(h;s)$ denotes the number of zeros of $h$in B_s. From here we can deduce that for $0 < r < 1$

$$n(h;r)\log\frac{1}{r} = \int_r^1 \frac{n(h;s)}{s} ds \leq \int_0^1 \frac{n(h;s)}{s} ds \leq \log\|h\|_\infty, \qquad (16)$$

We will apply this result to the function $h = d \circ \phi^{-1}$, where $d : \Omega \to \mathbb{C}$ is the holomorphic function from Theorem (1.1.7) Note that by part (iii) of that theorem, every eigenvalue of L in Ω' corresponds to a zero of d, hence to a zero of h in $\phi(\Omega')$, which is a subset of the disk of radius $r = r_\Omega(\Omega')$ around 0. Therefore

$$\mathcal{N}_L(\Omega') = \#\{w \in \phi(\Omega') : h(w) = 0\} \leq n\big(h, r_\Omega(\Omega')\big) \qquad (17)$$

By part (ii) of Theorem (1.1.7) and by (15), we have

$$\log\|h\|_\infty \leq \frac{C_p}{(\varepsilon - \alpha_{N+1}(K))^p} \sum_{j=1}^{N} \big(\alpha_{N+1}(K) + \alpha_j(K)\big)^p$$

Which together with (16) and (17) implies (i).

To obtain (ii) from (i), we distinguish between the cases $0 < p < 1$ and $p \geq 1$,respectively. If $0 < p < 1$, we can use the inequality $(a+b)^p \leq a^p + b^p (a, b \geq 0)$to obtain

$$\sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p \leq \left(\sum_{j=1}^{N}\alpha_{N+1}(K)+\sum_{j=1}^{N}\alpha_j(K)\right) \qquad (18)$$

In case that $p \geq 1$, the Minkowski inequality gives

$$\sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p \leq \left(\sum_{j=1}^{N}\alpha_{N+1}^p(K)\right)^{1/p}+\left(\sum_{j=1}^{N}\alpha_j^p(K)\right)^{1/p} \qquad (19)$$

We note that since $j \longmapsto \alpha_j(K)$ is non-increasing and $\{\alpha_j(K)\} \in l^p(N)$, we have

$$\sum_{j=1}^{N}\alpha_{N+1}^p(K) = N\alpha_{N+1}^p(K) \to 0 \qquad (N \to \infty)$$

Indeed, we have

$$(2j).\alpha_{2j}^p(K) = 2\sum_{m=j+1}^{2j}\alpha_{2j}^p(K) \leq 2\sum_{m=j+1}^{2j}\alpha_m^p(K) \to 0 \quad (j \to \infty)$$

And in a similar manner one can show that $(2j+1)\alpha_{2j}^p(K) \to 0$ for$(j \to \infty)$. Therefore, (18) and (19) imply that for all $p > 0$

$$\lim_{N\to\infty}\sum_{j=1}^{N}\left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p \leq \sum_{j=1}^{\infty}\alpha_j^p(K)$$

So that taking $N \to \infty$ in (i) gives (ii).

While the above result is very general, applying it to bound the number of eigenvalues in specific sets requires computing the quantity $r_\Omega(\Omega')$, which is generally hard. We will here deal with the special but very important case of estimating the number of eigenvalues outside a disk, that is we take

$\|L_0\| < t < s$and

$$\Omega = B_t^c, \Omega' = B_s^c.$$

Then Ω is simply connected, with$\infty \in \Omega$, and for $\lambda \in \Omega$we have

$$\|(\lambda - L_0)^{-1}\| = |\lambda|^{-1}\|(\mathbb{1} - \lambda^{-1}L_0)^{-1}\| \leq |\lambda|^{-1}(\mathbb{1} - \lambda^{-1}\|L_0\|)^{-1} < (t - \|L_0\|)^{-1}$$

Which shows that (14) holds with $\varepsilon = t - \|L_0\|$. The conformal mapping $\phi: \Omega \to \mathbb{D}$is given by $\phi(w) = \frac{t}{w}$, so that$r_\Omega(\Omega') = \frac{t}{s}$. Therefore, denoting the number of eigenvalues (counted with multiplicities) of L in B_s^c by $n_L(s)$, i.e. $n_L(s) := \mathcal{N}_L(B_s^c)$, Theorem (1.2.1) implies that if

$$\|L_0\| + \alpha_{N+1}(K) < t < s \qquad (20)$$

Then

$$n_L(s) \leq \frac{C_p}{\log\left(\frac{s}{t}\right)\left[t - \left(\|L_0\| + \alpha_{N+1}(K)\right)\right]^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p \qquad (21)$$

To optimize the bound we should take t satisfying (20) so as to minimize the right-hand side of (21). That is we need to maximize the function

$$f(t) = \log\left(\frac{s}{t}\right)[t - a]^p,$$

Where

$$a = \|L_0\| + \alpha_{N+1}(K),$$

in the interval (a, s) – note that this function vanishes at the endpoints and is positive in the interior of this interval, so that its maximum is obtained in the interior. We can find it by setting$f'(t) = 0$, where

$$f'(t) = -\frac{1}{t}[t - a]^p + p\log\left(\frac{s}{t}\right)[t - a]^{p-1}.$$

Denoting by $W(x)$ the Lambert W −function $W: [0, \infty) \to [0, \infty)$, which is defined by

$$W(x)e^{W(x)} = x,$$

A short computation gives

$$f'(t^*) = 0 \Leftrightarrow 1 - \frac{a}{t^*} = p\log\left(\frac{s}{t^*}\right) \Leftrightarrow \frac{a}{pt^*}e^{\frac{a}{pt^*}} = \frac{a}{ps}e^{\frac{1}{p}} \Leftrightarrow \frac{a}{pt^*} = W\left(\frac{a}{ps}e^{\frac{1}{p}}\right) \Leftrightarrow t^* = \frac{a}{pW\left(\frac{a}{ps}e^{\frac{1}{p}}\right)}$$

Thus

$$\max_{t\in[a,s]} \log\left(\frac{s}{t}\right)\ [t - a]^p = f(t^*) = \log\left(\frac{ps}{a}W\left(\frac{a}{ps}e^{\frac{1}{p}}\right)\right)\left[\frac{a}{pW\left(\frac{a}{ps}e^{\frac{1}{p}}\right)} - a\right]^p$$

$$= \left[\frac{1}{pW\left(\frac{1}{p}e^{\frac{1}{p}}\cdot\frac{a}{s}\right)} - 1\right]^{p+1} W\left(\frac{1}{p}e^{\frac{1}{p}}\cdot\frac{a}{s}\right)\cdot\frac{a^p}{s^p}\cdot s^p$$

Therefore, defining $\Phi_p : (0,1) \to \mathbb{R}$ by

$$\Phi_p(x) = \frac{\left[W\left(\frac{1}{p}e^{\frac{1}{p}}\cdot x\right)\right]^p}{\left[\frac{1}{p} - W\left(\frac{1}{p}e^{\frac{1}{p}}\cdot x\right)\right]^{p+1}\cdot x^p} \qquad (22)$$

We obtain the following result.

Theorem (1.2. 2)[1]:

Let $p \in (0,\infty)$ and s $>\|L_0\|$:

(i) *If* $N \in \mathbb{N}_0$ *is such that* $\alpha_{N+1}(K) < s - \|L_0\|$, *then*

$$n_L(s) \le \frac{C_p}{s_P}\cdot\Phi_p\left(\frac{\|L_0\| + \alpha_{N+1}(K)}{s}\right)\sum_{j=1}^{N}\left(\alpha_{N+1}(K) + \alpha_j(K)\right)^p \qquad (23)$$

(ii) If$\{\alpha_j(K)\epsilon l^p(N)\}$, then

$$n_L(s) \le \frac{C_p}{s_P}\cdot\Phi_p\left(\frac{\|L_0\|+\alpha_{N+1}(K)}{s}\right)\sum_{j=1}^{\infty}\alpha_j^p(k) \qquad (24)$$

In both cases, C_p is as given in (1.1.11).

Here (ii) is obtained from (i) by taking $N \to \infty$, as in Theorem (1.2.1).

The previous theorem can be regarded as a broad generalization of the classical eigen-value estimates for compact operators. Indeed, if L is compact, i.e. L_0= 0, we obtain from (24) that

$$n_L(s) \leq \frac{peC_p}{s_P} \cdot \sum_j \alpha_j^p(L) \qquad (25)$$

where we used the fact that, as a calculation shows, $\varphi_p(0) = lim_{x\to 0}\varphi_p(x) = pe$. This inequality, up to a constant, recovers the classical results. Estimate (23) seems to be new even in case that $L_0 = 0$.

Concerning the asymptotic behavior (for $S \to \|L_0\|$) of the right-hand sides of (23) and (24), one can show that $\varphi_p(x) \sim (1-x)^{-(p+1)}\ for\ x \to 1^-$, which, for instance, in the summable case implies that

$$n_L(s) = O\left(\frac{s}{s - \|L_0\|^{p+1}}\right). \qquad (26)$$

The following corollary makes (26) more precise, and gives bounds on$n_L(s)$ which do not involve the function φ_p and which are only slightly weaker than those of Theorem (1.2.2).

Corollary (1.2.3)[1]:

Let$p \in (0,\infty)\ and\ s > \|L_0\|$.

(i)$If\ N \in N_0\ is\ such\ that\ \alpha_{N+1}(K) < s - \|L_0\|$, then

$$n_L(s) \leq \frac{c_p(p+1)^{p+1}}{p^p} \cdot \frac{s}{\left[s - \left(\|L_0\| + \alpha_{N+1}(K)\right)\right]^{p+1}} \sum_{j=1}^{\infty} \left(\alpha_{N+1}(K) + \alpha_j\right)^p \quad (27)$$

(ii) $If\ \{\alpha_j(K)\} \in l^p(N)$, then

$$n_L(s) \leq \frac{c_p(p+1)^{p+1}}{p^p} \cdot \frac{s}{\left(s - (\|L_0\|)\right)^{p+1}} \sum_{j=1}^{\infty} \alpha_j^p(K) \quad (28)$$

In both cases, $c_p$is as defined in (1.1.11).

Note that (28) is equivalent to the inequality (2) presented in the Introduction (setting $C(p) = C_p \frac{(p+1)^{p+1}}{P^p}$.)

Proof :

The corollary is a direct consequence of Theorem (1.2.2)and the estimate

$$\Phi_p(x) \leq \frac{(p+1)^{p+1}}{P^p} \cdot \frac{1}{(1-x)^{p+1}} \quad ,0 < x < 1 \qquad (29)$$

To prove the last estimate, we define

$$g: (0,1) \to \left(\frac{0,1}{p}\right), \qquad g(x) = W\left(\frac{1}{p} e^{\frac{1}{p}} \cdot x\right)$$

And

$$h(x) := (1-x)^{P+1}\Phi_p(x) = p^{P+1}\left(\frac{g(x)}{x}\right)^p \left(\frac{1-x}{1-pg(x)}\right)^{p+1} ,x \in (0,1),$$

See (22). We show below that h is monotonically increasing in (0,1), so in particular

$$h(x) \leq \lim_{y \to 1-} h(y) = \frac{(p+1)^{p+1}}{P^p} \qquad (30)$$

Where in the computation of the limit we used l'Hôpital's rule and the fact that $g(1) = 1/p$.The validity of (29) is an immediate consequence of estimate (30). To show that h is monotonically increasing, we use the fact that$W' = \frac{1}{x} \cdot \frac{W(x)}{W(x)+1}$, and so

$$g'(x) = \frac{1}{x} \cdot \frac{g(x)}{g(x)+1}$$

And differentiate $h(x)$, obtaining

$$h'(x) = p^{P+1} \cdot \frac{(1-x)^p g(x)^p}{x^{p+1}\big(1-pg(x)\big)^{p+2}} \cdot (p(x+p)g(x) - (p+1)x)$$

Thus we have $h'(x) > 0$ for all $x \in (0,1)$ if and only if

$$f(x) := \frac{p}{p+1}(x+p) - \frac{x}{g(x)} > 0$$

For all $x \in (0,1)$. But $\lim_{x\to 1-} f(x) = 0$, and f is strictly monotonically decreasing since $f'(x) := \frac{p}{p+1} - \frac{x}{g(x)+1} < 0$ for all $x \in (0,1)$. Therefore $f(x) > 0$ and hence $h'(x) > 0$ for all $x \in (0,1)$.

We now express our results in terms of bounds on sums of powers of eigenvalues of $L = L_0 + K$ outside the disk of radius $\|L_0\|$. Besides the intrinsic interest in such a formulation, it will be convenient for discussing issues related to the sharpness of the results obtained.

By integration by parts one has

$$q\int_{\|L_0\|}^{\infty} n_L(s)(s-\|L_0\|)^{q-1}\,ds = \sum_{\lambda\in\sigma_d(L),|\lambda|>\|L_0\|} (|\lambda| - \|L_0\|)^q, \qquad q>0 \qquad (31)$$

Where in the sum each eigenvalue is counted according to its algebraic multiplicity. Using (28), and the fact that $n_L(s) = 0$ for $s > \|L_0\| + \|K\| \geq \|L\|$, we obtain

$$\begin{aligned}\sum_{\lambda\in\sigma_d(L),|\lambda|>\|L_0\|} (|\lambda| - \|L_0\|)^q &= q\int_{\|L_0\|}^{\infty} n_L(s)(s-\|L_0\|)^{q-1}\,ds \\ &\leq q\frac{C_p(p+1)^{p+1}}{p^p}\sum_j \alpha_j^p(K) q\int_{\|L_0\|}^{\|L_0\|+\|K\|} \frac{s}{(s-\|L_0\|)^{p+2-q}}\,ds \\ &= q\frac{C_p(p+1)^{p+1}}{p^p}\left[\frac{1}{q-p-1}\|L_0\| + \frac{1}{q-p}\|K\|\right]\|K\|^{q-p-1} \\ &\cdot \sum_j \alpha_j^p(K) \qquad (32)\end{aligned}$$

Where the finiteness of the integral, hence the validity of the inequality, requires

$q > p + 1 \; if \; L_0 \neq 0, and \; q > p \; if \; L_0 = 0.$

We thus have the following facts, where we distinguish between the cases $L_0 = 0$ (which implies that L is compact) and $L_0 \neq 0$.

Corollary (1.2.4)[1]:

Let $L_0, K \in \boldsymbol{B}(X) \; and \; L := L_0 + K.$

(i) If $L_0 \neq 0$, then for any $p > 0, q > p + 1$

$$\{\alpha_j(K)\} \in l^p(\mathbb{N}) \Rightarrow \sum_{\lambda \in \sigma_d(L), |\lambda| > \|L_0\|} (|\lambda| - \|L_0\|)^q < \infty \qquad (33)$$

(ii) If $L_0 = 0$, then for every $p > 0, q > p$

$$\{\alpha_j(K)\} \in l^p(\mathbb{N}) \Rightarrow \sum_{\lambda \in \sigma_d(L)} (|\lambda|)^q < \infty \qquad (34)$$

Noting the difference in the condition on the exponent q between the case $L_0 \neq 0 (q > p + 1)$ and the case $L_0 = 0 \, (q > p)$, it is natural to ask whether this refects a real difference in the possible distribution of eigenvalues in the two cases, or a limitation of our methods of proof. That is, we seek to determine to what extent the results we have obtained are sharp. We therefore ask:

(i) What is the infimum$qB(p)$ $[q_B^0(p)]$ of all exponents q such that the implication (33) (34) is valid for all Banach spaces X, all $L_0 \in \boldsymbol{B}(X)$ and all compact $K \in \boldsymbol{B}(X)$ with $\{\alpha_j(K)\} \in l^p(\mathbb{N})$?

(ii) Is the above infimum a minimum?

Let us first consider the case $L_0 = 0$: Here, it is well-known that for all $p > 0$ we have $q_B^0(p) = p$ and that the infimum is a minimum, as follows from König's result (3) above. So we see that for this case we almost recover the optimal exponent. What about the case $L_0 \neq 0$? Our results imply

$$max(1, p) = q_B(p) \leq p + 1,$$

Where the lower bound follows from (35) below, while the upper bound follows from Corollary (1.2.4). Otherwise we do not know much about the value $q_B(p)$, nor whether the infimum is a minimum.

Let us note that if we restrict ourselves to Hilbert spaces, and define the constant $q_H(p)$ analogously, then it is known that

$$q_H(p) = max(1, p) \qquad (35)$$

and that the infimum is again a minimum. Thus for the case of general L_0, L on a Hilbert space the situation is the same as for $L_0 = 0$ on a general Banach space as long as $p \geq 1$, but quite different for p smaller than one.

The question is thus whether the fact that the results concerning the exponent $q_B(p)$ that we obtain are weaker than the known results for Hilbert space operators is due to non-sharpness of our results, or rather to a real difference between what can happen in Hilbert spaces and in general Banach spaces, respectively. If the latter is the case, then this must be demonstrated by constructing appropriate examples. While we do not have an answer to the above question, we do have an example which shows that eigenvalues of perturbations can behave in a 'worse' way in general Banach spaces than in Hilbert spaces. Indeed, below we will construct an example with $X = l^1(\mathbb{N})$, where $L - L_0$ is of finite rank, and where

$$\sum_{\lambda \in \sigma_d(L), |\lambda| > \|L_0\|} (|\lambda| - \|L_0\|) = \infty$$

Note that for a finite rank perturbation on a Hilbert space the above sum will always be finite, as follows from the considerations above and the fact that the approximation numbers of finite rank operators are p-summable for every $p > 0$.

Example (1.2.5)[1]:

It is well-known that there exist holomorphic functions h on the unit disk, with uniformly bounded Taylor coeffcients, such that

$$\sum_{w\in\mathbb{D},h(w)=0} (1-|w|) = \infty \qquad (36)$$

Where each zero is counted according to its order. Let us fix such a (normalized) function

$$h(w) = 1 - \sum_{k=1}^{\infty} b_k w^k,$$

With $\{b_k\} \in l^\infty(\mathbb{N})$. Now we choose $X = l^1(\mathbb{N})$ and let L_0 denote the shift operator on $l^1(\mathbb{N})$, i.e.

$$L_0\delta_n = \delta_{n+1}, \qquad n \in (\mathbb{N})$$

Where $\{\delta_n\}$ denotes the canonical Schauder basis of $l^1(\mathbb{N})$. Clearly,

$\|L_0\| = 1$. Next, we define a rank one operator K on $l^1(\mathbb{N})$ by

$$Kf = \langle f, b\rangle\delta_1,$$

Where$\langle .,. \rangle$ denotes the dual pairing between $l^1$and l^∞, and we set $L = L_0 + K$. For $|\lambda| > 1$ we then have that $\lambda \in \sigma_d(L)$ iff

$$det_1(\mathbb{1} - K(\lambda - L_0)^{-1}) = 0.$$

It is not difficult to see that, setting $w = \lambda^{-1}$,

$$det_1(\mathbb{1} - K(\lambda - L_0)^{-1}) = 1 - \langle(\lambda - L_0)^{-1}\delta_1, b\rangle = 1 - w\sum_{k=0}^{\infty}\langle L_0^k\delta_1, b\rangle w^k$$

$$= 1 - \sum_{k=0}^{\infty} b_k w^k = h(w),$$

From (36) we thus obtain that

$$\sum_{\lambda\in\sigma_d(L),|\lambda|>\|L_0\|} (|\lambda| - \|L_0\|) = \sum_{w\in\mathbb{D},h(w)=0} \frac{1-|w|}{|w|} = \infty\,.$$

Chapter 2

Maximal norms in Banach spaces

We show that the space $L_p,\ 1 < p < \infty,\ p \neq 2$ have continuum different renaming's with 1-unconditional bases each with a different maximal isometry group, and that every symmetric space other than L_2 has at least a countable number of each renaming. On the other hand we show that the spaces $L_p,\ 1 < p < \infty,\ p \neq 2$,have continuum different renormings each with an isometry group which is not contained in any maximal bounded sub group of the group of isomorphism of L_2.

Section (2.1): Almost and convex transivity

The long-standing Banach-Mazur rotation problem asks whether every separable Banach space with a transitive group of linear surjective isometrics' is isometrically isomorphic to a Hilbert space. This problem has attracted a lot of attention in the literature and there are several related open problems. In particular it is not known whether a separable Banach space with a transitive group of isometries is isomorphic to a Hilbert space, and, until now, it was unknown if such a space could be isomorphic to $\ell_2$for some $p \neq 2$. We show that this is impossible and we provide further restrictions on classes of spaces that admit transitive or almost transitive equivalent norms (a norm on X is called almost transitive if the orbit under the group of isometries of any element x in the unit sphere of X is norm dense in the unit sphere of X).

It is well-known that spaces $L_p[0,1]$ for $1 < p < \infty$ are almost transitive. In 1993 Deville, Godefroy and Zizler asked whether every super-reflexive space admits an equivalent almost transitive norm. Recently Ferenczi and Rosendal answered this question negatively by exhibiting a complex super-reflexive HI space which does not admit an equivalent almost transitive renorming.

The first result of this section is that the spaces $\ell_2, 1 < p < \infty, p \neq 2$, and all infinite-dimensional subspaces of their quotient spaces do not admit equivalent

almost transitive renaming's. We also prove that all infinite-dimensional subspaces of Asymptotic-ℓ_2 spaces for $1 \leq p < \infty, p \neq 2$, fail to admit equivalent almost transitive norms. Moreover we give an example of a super-reflexive space which does not contain either an Asymptotic-ℓ_2 space or a subspace which admits an almost transitive norm.

We combine our results with known results about the structure of subspaces of $L_p, 2 < p < \infty$, and we obtain that if X is a subspace of $L_p, 2 < p < \infty$, or, more generally, of any non-commutative L_p-space for $2 < p < \infty$, such that every subspace of X admits an equivalent almost transitive norm, then X is isomorphic to a Hilbert space. The same result is also true for subspaces of the Schatten class $S^P(\ell_2)\ for\ 1 < p < \infty, p \neq 2$,This suggests the following question.

Problem (2.1.1)[2]:

Suppose that every subspace of a Banach space X admits an equivalent almost transitive renorming. Is X isomorphic to a Hilbert space?

As additional information related to this problem, we prove that if such a space X is isomorphic to a stable Banach space then X is ℓ_2-saturated (Corollary (2.1.22) and Remark (2. 1.23).

Our results are a consequence of a new property of almost transitive spaces with a Schauder basis. Namely we prove that in such spaces the unit vector basis of ℓ_2^2belongs to the two-dimensional asymptotic structure. We also obtain some information about the asymptotic structure in higher dimensions. Our method relies on an application of the classical Dvoretzky theorem and it enables us to give up to date the most significant restrictions on isomorphic classes of spaces which admit a transitive or almost transitive norm. In particular we obtain estimates for the power types of the upper and lower envelopes of X and of p and q in (p, q)-estimates of X (Corollaries (2.1.15) and (2.1.18). From this we obtain a version of Krivine's theorem for spaces with asymptotic unconditional structure and a subspace which admits an almost transitive norm (Theorem (2.1.15)). Another consequence is a characterization

of subspaces of Orlicz sequence spaces which contain a subspace which admits an equivalent almost transitive norm (Theorem (2.1.20).

The second part of the section is devoted to the study of maximal renormings on Banach spaces. For a Banach space $(X, \|\cdot\|)$ we denote by $\mathrm{Isom}(X, \|\cdot\|)$ the group of all linear surjective isometries of X. We say that a Banach space $(X, \|\cdot\|)$ is maximal if whenever$|||\cdot|||$ is an equivalent norm on Xsuch that $Isom(X, \|\cdot\|) \subset Isom(X, |||\cdot|||)$,then $Isom(X, \|\cdot\|) \subset Isom(X, |||\cdot|||)$. This notion was introduced by Pelczy´nski and Rolewicz and has been extensively studied, a survey Rolewicz proved that all non-hilbertian 1-symmetric spaces are maximal.

The study of isometry groups of renormings of X is equivalent to the study of bounded subgroups of the group $GL(X)$ of all isomorphisms from X onto X. Indeed, if G is a bounded subgroup of $GL(X)$ we can define an equivalent norm k $\|\cdot\|_G$on X by

$$\|\cdot\|_G = \sup_{g \in G} \|gx\|$$

Then G is a subgroup of $Isom(X, \|\cdot\|_G)$. Thus Wood's problem asks whether for every space $X, GL(X)$ has a maximal bounded subgroup.

In 2013 Ferenczi and Rosendal answered Wood's problem negatively by exhibiting a complex super-reflexive space and a real reflexive space, both without a maximal bounded subgroup of the isomorphism group.

In 2006 Wood asked, what he called a more natural question, whether for every Banach space there exists an equivalent maximal renorming whose isometry group contains the original isometry group, i.e. whether every bounded subgroup of $GL(X)$ is contained in a maximal bounded sub-group of $GL(X)$. As elaborated by Wood and Ferenczi and Rosendal this question is related to the Dixmier's unitarisability problem whether every countable group all of whose bounded representations on a Hilbert space are unitarisable is amenable.

There are several known groups with bounded representations which are not unitarizable; however it is unknown whether isometry groups of renormings of

Hilbert space induced by these representations are maximal or even contained in a maximal isometry group of another equivalent renorming. The answer to this question would elucidate the following portion of the Banach-Mazur problem.

Problem (2.1.2)[2]:

Suppose that $\|.\|$ is an equivalent maximal norm on a Hilbert space $\mathcal{H}$. Is $|||\cdot|||$ necessarily Euclidean?

In this section we show that even in Banach spaces which have a maximal bounded subgroup of $GL(X)$ there can also exist bounded subgroups of $GL(X)$which are not contained in any maximal bounded subgroup of $GL(X)$. We prove that $\ell_p, 1 < p < \infty, p \neq 2$, the 2-convexified Tsirelson space $T^{(2)}$, and the space U with a universal unconditional basis, have a continuum of renormings none of whose isometry groups is contained in any maximal bounded subgroup of $GL(X)$. We note that $T^{(2)}$ is a weak Hilbert space. We do not know whether $T^{(2)}$or general weak Hilbert spaces, other thanℓ_p, have a maximal bounded subgroup of $GL(X)$.

We also study maximal bounded subgroups of $GL(X)$ for Banach spaces X with 1-unconditional bases. We prove that $\ell_p, 1 < p < \infty, p \neq 2$, and U have continuum different renormings with 1-unconditional bases each with a different maximal isometry group, and that every symmetric space other than ℓ_p has at least a countable number of such renormings. As mentioned above it is unknown whether the Hilbert space has a unique, up to conjugacy, maximal bounded subgroup of $GL(X)$. Motivated by our results we ask the following question.

Question (2.1.3)[2]:

Does there exist a separable Banach space X with a unique, up to conjugacy, maximal bounded subgroup of $GL(X)$? If yes, does X have to be isomorphic to a Hilbert space?

Throughout this section X and Y will denote real or complex infinite dimensional Banach spaces and the term subspace always means a closed infinite-dimensional linear subspace.

If a Banach space X has a Schauder basis, a normalized Schauder basis will be denoted (e_i) and its biorthogonal sequence will be denoted (e_i^*). For $n \geq 1, P_n^X$ will denote the basis projection from X onto the linear span of $e_1, \ldots, e_n$. The support of $x \in X$ is defined by $supp\ x = \{i \in \mathbb{N} : e_i^*(x) \neq 0\}$. If $x, \mathcal{Y} \in X$ we write $x < \mathcal{Y}$ if $max\ supp\ x < min\ supp\ \mathcal{Y}$ and $x > N$ if $supp\ x \subset [N + 1, \infty)$. We say that

a sequence (x_i) of vectors is a normalized block basis if $x_1 < x_2 < x_3 < \ldots$ and $\|x_i\| = 1\ (i \geq 1)$.

Lemma (2.1.4)[2]:

Suppose that X has a Schauder basis and contains a subspace Y which is almost transitive. Given $\delta > 0, \mathcal{Y}_0 \in Y$, and $N \in \mathbb{N}$ there exists $\mathcal{Y} \in Y$, with $\|P_N^X(\mathcal{Y})\| < \delta$, such that for all scalars a, b, we have

$$(1)(1 - \delta)(|a|^2\|\mathcal{Y}_0\|^2 + |b|^2)^{\frac{1}{2}} = \|a\mathcal{Y}_0 + b\mathcal{Y}\ \ \| \leq (1 + \delta)(|a|^2\|\mathcal{Y}_0\|^2 + |b|^2)^{1/2}.$$

Proof:

By compactness there exists $n := n(N, \delta)$ such that if $(\mathcal{Y}_i)_{i=1}^n \subset B_Y$ then there exist $1 \leq i \leq j \leq n$ such that $\left\|P_N^X(\mathcal{Y}_j - \mathcal{Y}_i)\right\| < \delta$. By Dvoretzky's theorem and almost transitivity of, there exist $(\mathcal{Y}_i)_{i=1}^n \subset S_Y$ such that for all scalars $a, b_1, \ldots, b_n$, we have

$$(1 - \delta)\left(|a|^2\|\mathcal{Y}_0\|^2 + \sum_{i=1}^n |b|^2\right)^{\frac{1}{2}} \leq \left\|a\mathcal{Y}_0 + \sum_{i=1}^n b_i\mathcal{Y}_i\right\| \leq (1 + \delta)(|a|^2\|\mathcal{Y}_0\|^2 + \sum_{i=1}^n |b|^2)^{1/2}.$$

Choose $1 \leq i \leq j \leq n$ such that $\|P_N^X(\mathcal{Y}_j - \mathcal{Y}_i)\| < \delta$ and set $\mathcal{Y} = (1/\sqrt{2})(\mathcal{Y}_j - \mathcal{Y}_i)$.

Then (1) follows from (2).

Remark (2.1.5)[2]:

For a Banachspace X, let

$$FR(X) := \{1 \leq r \leq \infty : \ell_r \text{ is finitely representable in } X\}.$$

Suppose that $r \in FR(Y)$, where Y is as in Lemma (2.1.3) The proof yields (after the obvious modifications) the same conclusion except (1) should be replaced by

$$(3)(1-\delta)(|a|^r\|\mathcal{Y}_0\|^r + |b|^r)^{\frac{1}{r}} \leq \|a\mathcal{Y}_0 + b\mathcal{Y}\| \leq (1+\delta)(|a|^r\|\mathcal{Y}_0\|^r + |b|^r)^{\frac{1}{r}}.$$

with the obvious modification for $r = \infty$In particular, by the Maurey-Pisier theorem, this holds for all $r \in [p_Y, 2] \cup \{q_Y\}$, there

$$p_Y := sup\{1 \leq p \leq 2 : Y \text{ has type } p\}$$

and

$$q_Y := inf\{2 \leq q \leq \infty : Y \text{ has cotype } q\}.$$

Theorem (2.1.6)[2]:

Suppose that X has a Schauder basis and contains a subspace Y which is almost transitive. Let $r \in FR(Y)$. Then, given $\varepsilon > 0$ and any sequence (a_i) of nonzero scalars, there exists a normalized block basis (x_i) in X such that, for all $m \geq 1$ and all scalars b, we have

$$(\mathbf{4})(1-\varepsilon)\left(\sum_{k=1}^{m}|a_k|^r + |b|^r\right)^{1/r} \leq \left\|\sum_{k=1}^{m} a_k x_k + b_{x_{m+1}}\right\|$$
$$\leq (1+\varepsilon)\left(\sum_{k=1}^{m}|a_k|^r + |b|^r\right)^{1/r}$$

Proof:

First we prove the result for $r = 2$. Let $(\delta_i)_{i=1}^{\infty}$be a (suffciently small) positive decreasing sequence. We construct (x_i) and an auxiliary sequence $(\mathcal{Y}_i) \subset Y$ iteratively. Let $\mathcal{Y}_1 \in S_Y$be chosen arbitrarily. Chose a finitely supported vector $x_1 \in S_x$such that $\|\mathcal{Y}_1 - x_1\| < \delta_1$. Let (ε_i) be a strictly increasing sequence satisfying $0 < \varepsilon_i < \varepsilon/3$ for all i. Suppose$m \geq 1$ and that $\mathcal{Y}_i \in Y$ and finitely

supported$x_i \in S_X$ $(1 \leq i \leq m)$ have been chosen such that $x_1 < x_2 < \cdots < x_m, \|x_i - \mathcal{Y}_i\| < \delta_i$, and, for all $1 \leq j \leq m$ and scalars b,

$$(\mathbf{5})\left(1 - \varepsilon_j\right)\left(\sum_{k=1}^{j-1}|a_k|^2 + |b|^2\right)^{1/2} \leq \left\|\sum_{k=1}^{j-1} a_k\mathcal{Y}_k + b_{\mathcal{Y}_j}\right\|$$
$$\leq \left(1 + \varepsilon_j\right)\left(\sum_{k=1}^{j-1}|a_k|^2 + |b|^2\right)^{1/2}$$

Let $N := max\ supp(x_m)$ and let $\delta > 0$ be suffciently small. By Lemma (2.1.4) applied to $\mathcal{Y}_0 = \sum_{k=1}^{m} a_k\mathcal{Y}_k$,there exists $\mathcal{Y}_{m+1} \in Y$uch that $\|P_N^X(\mathcal{Y}_{m+1})\| < \delta$ and, for all scalars a, b,

$$(6)\ (1 - \delta)(|a|^2\|\mathcal{Y}_0\|^2 + |b|^2)^{\frac{1}{2}} \leq \|a\mathcal{Y}_0 + b\mathcal{Y}_{m+1}\|$$
$$\leq (1 + \delta)(|a|^2\|\mathcal{Y}_0\|^2 + |b|^2)^{1/2}.$$

In particular, for all scalars b, we have

$$(1 - \varepsilon_m)(1 - \delta)\left(\sum_{k=1}^{m}|a_k|^2 + |b|^2\right)^{1/2} \leq \left\|\sum_{k=1}^{m} a_k\mathcal{Y}_k + b\mathcal{Y}_{m+1}\right\|$$
$$\leq (1 + \varepsilon_m)(1 + \delta)\left(\sum_{k=1}^{m}|a_k|^2 + |b|^2\right)^{1/2}$$

and hence (5) holds for $n = m + 1$ provided $\delta(1 + \varepsilon_m) < \varepsilon_{m+1} - \varepsilon_m$, which we may assume. Set $\tilde{x}_{m+1} := P_{N_1}^X(\mathcal{Y}_{m+1}) - P_N^X(\mathcal{Y}_{m+1})$, where $N_1$is chosen suffciently large to ensure that $\left\|\mathcal{Y}_{m+1} - P_{N_1}^X(\mathcal{Y}_{m+1})\right\| < \delta$. Then $x_m < \tilde{x}_{m+1}$ and

$$\|\tilde{x}_{m+1} - \mathcal{Y}_{m+1}\| \leq \left\|\mathcal{Y}_{m+1} - P_{N_1}^X(\mathcal{Y}_{m+1})\right\| + \|P_N^X(\mathcal{Y}_{m+1})\| < 2\delta$$

By (6) $|1 - \|\mathcal{Y}_{m+1}\||| \leq \delta$, and hence $|1 - \|\tilde{x}_{m+1}\||| \leq 3\delta$ Let

$x_{m+1} = \tilde{x}_{m+1}/\|\tilde{x}_{m+1}\|$ Then $\|x_{m+1}\| = 1$ and

$$\|x_{m+1} - \mathcal{Y}_{m+1}\| \leq \|x_{m+1} - \tilde{x}_{m+1}\| + \|\tilde{x}_{m+1} - \mathcal{Y}_{m+1}\| < 3\delta + 2\delta = 5\delta$$

Nence$\|x_{m+1} - \mathcal{Y}_{m+1}\| < \delta_{m+1}$ provided $5\delta < \delta_{m+1}$, which we may assume. This completes the proof of the inductive step.

Finally, provided$(\sum_{i=1}^{\infty}|a_i|\delta_i) < |a_i|\varepsilon/3$, which we may assume,(4) follows from (5) by an easy triangle inequality calculation and the fact that

$\|x_i - \mathcal{Y}_i\| < \delta_i$for all $i \geq 1$

The proof for $r \in FR(Y)$ is very similar except (3) is used instead of (1).

Since the behaviour of block bases in $\ell_p$and c_0 is well understood, as an immediate consequence we obtain

Theorem (2.1.7)[2]:

No subspace of $\ell_p, 1 \leq p < \infty, p \neq 2$, or of c_0 admits an almost transitive renorming.

Proof:

Let Y be a subspace of X so that$Y$ admits an equivalent almost transitive norm $|||\cdot|||$It is well-known that any equivalent norm on a subspace may be extended to an equivalent norm on the whole space, So $|||\cdot|||$extends to an equivalent norm

$|||\cdot|||$on$X$. By Theorem (2.1.6) for any $\varepsilon > 0$ there exists a normalized block basis (x_k) in X such that for all $n \in \mathbb{N}$,

$$(1-\varepsilon)n^{1/2} \leq |||\sum_{k=1}^{n} x_k||| \leq (1+\varepsilon)n^{1/2}$$

It is well known that when $X = \ell_p, 1 \leq p < \infty$, every block basis is isometrically equivalent to the standard basis of ℓ_p, Since $|||\cdot|||$is C-equivalent to

$\|\cdot\|_{\ell_p}, n$is arbitrary and $p \neq 2$, we obtain a contradiction. The proof for$c_0$is similar.

Remark (2.1.8)[2]:

F. Cabello-Sanchez proved that almost transitive Banach spaces which are either Asplund or have the Radon-Nikodym property are actually super-reflexive, and thus, in particular, it was known that $\ell_1$and c_0 and their subspaces do not admit an equivalent almost transitive norm. We note however that there do exist spaces without the Radon-Nikodym property which are almost transitive, e.g. it is well known that $L_1[0,1]$ is almost transitive. Also Lusky proved that every separable Banach space $(X, \|\cdot\|)$ is complemented in a separable almost transitive space $(X, \|\cdot\|)$, its norm being an extension of the norm on X.

Remark (2.1.9)[2]:

It is instructive to observe that Theorem (2.1.5), when applied to the Haar basis of $L_p[0,1]$, does not contradict the fact that $L_p[0,1]$ is almost transitive. This is simply because the unit vector basis of ℓ_2 is $(1+\varepsilon)$ $-$equivalent to a block basis of the Haar basis. We thank the anonymous referee for this remark.

Remark (2.1.10)[2]:

It is clear from the proof of Theorem (2.1.5)that if we consider the corresponding infinite asymptotic game in X then the vector player has a winning strategy. More precisely,suppose that $\varepsilon > 0$ and (a_n) are fixed.Then $\forall n_1 \exists x_1 > n_1\ s.t. \forall n_2 \exists x_2 > n_2 \ldots$ such that the outcome (x_i) satisfies (4).

We recall the notion of asymptotic structure introduced by Maurey, Milman, and Tomczak-Jaegermann. A basis $(b_i)_{i=1}^{n}$ of unit vectors for an n $-$dimensional normed

space belongs to $\{X,(e_i)\}_n$ if, given $\varepsilon > 0$, the second player has a winning strategy in the asymptotic game to produce a sequence $(x_i)_{i=1}^n$ that is $(1+\varepsilon)$-equivalent to $(b_i)_{i=1}^n$. Precisely, $\forall m_1 \exists x_1 > m_1\ s.t. \forall m_2 \exists x_2 > m_2 \ldots$

|Such that there exist c and C, with $0 < c \leq C$ and $C/c < 1+\varepsilon$,

Such that for all scalars $(a_i)_{i=1}^n$, we have

$$c\left\|\sum_{i=1}^{n} a_i b_i\right\| \leq \left\|\sum_{i=1}^{n} a_i x_i\right\| \leq C\left\|\sum_{i=1}^{n} a_i b_i\right\|$$

Theorem (2.1.11)[2]:

Suppose that X has a Schauder basis and contains a subspace Ywhich is almost transitive. Let $r \in FR(Y)$. Then, for all $n \geq 2$ and for all nonzero scalars $a_1,\ldots,a_{n-1}$, there exists $(b_i)_{i=1}^n \in \{X,(e_i)\}n$ such that for all scalars λ and

for all $1 \leq k < n$, we have

$$(8)\ \left\|\sum_{i=1}^{n} a_i b_i + \lambda b_{k+1}\right\| = \left(\sum_{i=1}^{k} |a_i|^r |\lambda|^r\right)^{1/r}$$

In particular, the unit vector basis of ℓ_r^2belongs to $\{X,(e_i)\}_2$

Proof:

By a theorem of Knaust, Odell, and Schlumprecht given $\varepsilon_n \downarrow 0$, the basis (e_i) may be blocked as (E_k), where $E_k = span\{e_i : m_k \leq i < m_k + 1\}$ so that, for all $n \geq 1$, if $m_n \leq x_1 < x_2 < \cdots < x_n$ is a skipped sequence of

unit vectors with respect to the blocking (E_k) $(i.e., if\ i < j$ then there exists k

such that $x_i < m_k \leq m_{k+1} \leq x_j$), then $(x_i)_{i=1}^n$is $(1+\varepsilon_n)$-equivalent to some

$(b_i)_{i=1}^n \in \{X,(e_i)\}_n$. The result now follows from the proof of Theorem (2.1.5)

Remark (2.1.12)[2]:

Theorem (2.1.11) remains valid for the asymptotic structure associated to the collection $\boldsymbol{B}^{\mathbf{0}}(X)$ of subspaces of X of finite codimension, and in that case it is not necessary to assume that X has a basis.

Recall that X is an asymptotic$-\ell_p$ space $(1 \leq p < \infty)$ in the sense of [30] if there exists $C > 0$ such that for all $n \geq 1$, for all $(b_i)_{i=1}^n \in \{X, (e_i)\}_n$, and for all scalars $(a_i)_{i=1}^n$, we have

$$(9)\ \frac{1}{C}\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{n} a_i b_i\right\| \leq C\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}$$

and that X is an asymptotic ℓ_p −space if (in place of (9)) we have

$$(10)\ \frac{1}{C}\max_{1 \leq i \leq n}|a_i| \leq \left\|\sum_{i=1}^{n} a_i b_i\right\| \leq C\max_{1 \leq i \leq n}|a_i|$$

There exists $C > 0$ such that for all $n \leq x_1 < \cdots < x_n$,we

$$\frac{1}{C}\left(\sum_{i=1}^{n}\|x_k\|^p\right)^{1/p} \leq \left\|\sum_{k=1}^{n} x_k\right\| \leq C\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p}$$

With the obvious modification for $p = \infty$. The latter narrower class of asymptotic-ℓ_p space contains, e.g., the p-convexified Tsirelson space $T^{(p)}$for $1 \leq p < \infty$, while T^*, the dual of the Tsirelson space T $(= T^{(1)})$, is an asymptotic-ℓ_∞ space.

Theorem (2.1.13)[2]:

Suppose X is an Asymptotic-ℓ_p space,$1 \leq p \leq \infty, p \neq 2$. Then no subspace Y of X admits an equivalent almost transitive norm.

Proof:

Suppose X contained such a subspace$Y$.By Theorem (2.1.11), for all$n \geq 1$there

exists $(b_i)_{i=1}^n \in \{X,(e_i)\}_n$such that $\|\sum_{i=1}^n b_i\| = \sqrt{n}$, But this contradicts (9) (or (10) if $p = \infty$) when n is sufficiently large.

For $1 < p < \infty$, let C_p denote the class of Banach spaces X which are isomorphic to a subspace of an $\ell_p - sum$ of finite-dimensional normed spaces. It is known that if $X \in C_p$ and Y is isomorphic to a subspace of a quotient space of X, then $Y \in C_p$. In particular, C_p contains every infinite-dimensional subspace of a quotient space of ℓ_p.

$Corollary$ (2.1.14)[2]:

Let $1 < p < \infty, p \neq 2$. If $X \in C_p$ then X does not admit an equivalent almost transitive norm. In particular, no infinite-dimensional subspace of a quotient space of ℓ_p admits an equivalent almost transitive norm.

Proof:

$X$is isomorphic to a subspace of $Z_p := (\sum_{n=1}^{\infty} \oplus \ell_\infty^n)_p$since every finite-

dimensional normed space is 2-isomorphic to a subspace of ℓ_∞^n provided n is suffciently large. $Z_p$is an Asymptotic ℓ_p-space with respect its natural basis (e_i). By Theorem (2.1.13), does not admit an equivalent almost transitive norm.

Recall that the upper envelope is the norm $r_X$on c_{00}given by

$$r_X((a_i)) := \sup\left\{\left\|\sum_{i=1}^n a_i b_i\right\| : n \geq 1, (b_i)_{i=1}^n \in \{X,(e_i)\}_n\right\},$$

and the lower envelope is the function $g_X$given by c_{00}

$$g_X((a_i)) := \inf\left\{\left\|\sum_{i=1}^n a_i b_i\right\| : n \geq 1, (b_i)_{i=1}^n \in \{X,(e_i)\}_n\right\},$$

Corollary (2.1.15)[2]:

Suppose that X has a Schauder basis $(e_i)$and contains a subspace Y which is

almost transitive. Then

$$g_X((a_i)) \leq \left(\sum_{i=1}^{\infty} |a_i|^{q_Y}\right)^{1/q_Y} \leq \left(\sum_{i=1}^{\infty} |a_i|^{p_Y}\right)^{1/p_Y} \leq r_X(a_i)$$

with the obvious modification if $q_Y = 8$.

Theorem (2.1.16)[2]:

Suppose that X has an unconditional basis (e_i). If X contains an almost transitive subspace Y, then there exist $p \in [p_X, p_Y]$ and $q \in [q_X, q_Y]$ such that, for $r = p$ and $r = q$ and for all $n \geq 1$ and $\varepsilon > 0$, there exist disjointly supported vectors $(x_i)_{i=1}^n \subset X$such that $(x_i)_{i=1}^n$ is $(1 + \varepsilon)$ −equivalent to the unit vector basis ofℓ_r^n. In particular, if $Y = X$, then $p = p_X$ and $q = q_X$.

Proof:

Sari defined $\{X, (e_i)\}^d$as the collection of all finite sequences $(w_i)_{i=1}^n$

Such that, for some $m \geq n$ there exist $(b_i)_{i=1}^m \in \{X, (e_i)\}_m$and a partition

$\{A_1, \ldots, A_n\}\ of\ \{1, \ldots, m\}$ such that $w_i = \sum_{j \in A_i} \alpha_j\, b_j$ for some scalars $(\alpha_j)_{i=1}^m$

And $\|w_i\| = 1$.Thus,$(w_i)_{i=1}^n$is a normalized basis for a block subspace of the asymptotic space with basis (b_i).

Recall that the upper disjoint-envelope function is the norm r_X^don c_{00} given by:

$$r_X^d\left((\alpha_j)\right) := \sup\left\{\left\|\sum_{i=1}^{n} a_i w_i\right\| : n \geq 1, (w_i)_{i=1}^n \in \{X, (e_i)\}^d\right\}$$

The lower disjoint-envelope function $g_X^d$is defined similarly with supremum replaced by infimum. Corollary (2.1.14) gives

$$g_X^d\left((\alpha_j)\right) \leq \left(\sum_{i=1}^{\infty}|a_i|^{q_Y}\right)^{1/q_Y} \leq \left(\sum_{i=1}^{\infty}|a_i|^{p_Y}\right)^{1/p_Y} \leq r_X^d(a_i)$$

In particular, r_X^dhas power type p and g_X^d has power type q for some $p \in [1, p_Y]$

$q \in [q_Y, \infty]$ It follows from that$\{X, (e_i)\}$contains the unit vector basis of $\ell_p^n$and of ℓ_q^n for all $n \geq 1$, which implies,

for $r = p$ and $r = q$, the existence of disjointly supported vectors $(x_i)_{i=1}^n \subset X$

such that $(x_i)_{i=1}^n$ is $(1 + \varepsilon)$-equivalent to the unit vector basis of ℓ_r^n.

Remark (2.1.17)[2]:

Theorem (2.1.15) holds also (with the same proof) under the weaker assumption that X has asymptotic unconditional structure. for the definition of this notion.

Recall that a basis satisfies (p, q)-estimates, where $1 < q \leq p < \infty$, if there exists $C > 0$ such that

$$\frac{1}{C}\left(\sum_{i=1}^{n}\|x_k\|^p\right)^{1/p} \leq \left\|\sum_{k=1}^{n} x_k\right\| \leq C\left(\sum_{k=1}^{n}\|x_k\|^q\right)^{1/q}$$

Whenever $x_1 < x_2 < \cdots < x_n$.Theorem (2.1.5) has the following immediate consequence.

Corollary (2.1.18)[2]:

Suppose that a Banach space X with a Schauder basis (e_i) contains a subspace Y which admits an equivalent almost transitive norm. If (ei) satisfies (p, q) −estimates, then $q \leq 2 \leq p$.

Proof:

Let $|||\cdot|||$ be the equivalent almost transitive norm on Y. Then (as in The-orem (2.1.6) $|||\cdot|||$ extends to an equivalent norm $|||\cdot|||$on X. Clearly, (e_i) satisfies (p, q)-estimates under$|||\cdot|||$.

A natural question, in the light of Theorem (2.1.12), is whether every super-reflexive space which does not admit an almost transitive norm must contain an asymptotic-ℓ_p space? The next result answers this question negatively.

Corollary (2.1.19)[2]:

There exist super-reflexive spaces which do not contain either an Asymptotic-ℓ_p space or a subspace which admits an almost transitive norm.

Proof:

The spaces $S_{q,r}(\log_2(x+1))$ $(1 < q < r < \infty)$ constructed are super-reflexive and satisfy (r,q)-estimates. Hence, if $1 < q < r < 2$ or $2 < q < r < \infty, S_{q,r}(\log_2(x+1))$ does not contain any subspace which admits an almost transitive norm. Moreover, $S_{q,r}(\log_2(x+1))$ is complementably minimal, has a subsymmetric basis, and does not contain a copy of any ℓ_p space, which is easily seen to preclude the containment of an Asymptotic-ℓ_p space.

Theorem (2.1.20)[2]:

Suppose that X is an Orlicz sequence space ℓ_M (where M is an Orlicz function). Then X contains a subspace Y which admits an almost transitive norm if and only if X contains a subspace isomorphic to ℓ_2.

Proof:

Corollary (2.1.17) implies that the Matuszewska-Orlicz indices of M satisfy $\alpha_M \leq 2 \leq \beta_M$, which in turn implies by a theorem of Lindenstrauss and Tzafriri thatℓ_M contains a subspace isomorphic to ℓ_2. Conversely, if X contains a subspace Y isomorphic to ℓ_2 then Y admits an equivalent transitive norm.

We say that a Banach space X is convex tran-sitive if for any x in the unit sphere of X, $conv\{T\,x : T \in Isom(X, \|\cdot\|)\}$ is equal to However in super-reflexive Banach spaces convex transitivity is equivalent to almost transitivity. A long list of additional related results is summarized

Corollary (2.1.21)[2]:

For $1 < p < \infty, p \neq 2,$ no infinite-dimensional subspace of a quotient space of ℓ_p admits a convex transitive renorming.

It is well known that the spaces $L_p[0,1], 1 < p < \infty$, with the original norm are almost transitive. Next we consider their subspaces which admit an almost transitive renorming.

Corollary (2.1.22)[2]:

Let X be a subspace of $L_p[0,1], 2 < p < \infty,$ which admits an equivalent convex transitive norm. Then X contains a subspace isomorphic to ℓ_2.

Proof :

either$X$ contains a subspace isomorphic to ℓ_2 or X is isomorphic to a subspace of ℓ_p. By Corollary (2.1.21), the latter case would contradict the fact that X admits a convex transitive norm, which proves the corollary.

Corollary (2.1.23)[2]:

Let X be a subspace of $L_p[0,1], 2 < p < \infty$, or, more generally, of any non-commutative L_p-space for $2 < p < \infty$, so that every subspace Y of X admits an equivalent convex transitive norm, then X is isomorphic to ℓ_2.

Proof:

In the commutative case, and in the non-commutative case, either X is isomorphic to ℓ_2 or X contains a subspace Y isomorphic to ℓ_p. By Corollary (2.1.2), in the latter case Y does not admit an equivalent convex transitive norm, which proves the corollary.

Corollary (2.1.24)[2]:

Let X be a subspace of the Schatten class $S^p(\ell_2), 1 < p < \infty, p \neq 2,$ so that every

subspace Y of X admits an equivalent convex transitive norm, then X is isomorphic to ℓ_2.

Proof:

The proof is the same as that of Corollary (2.1.24), except that we use to conclude that either X is isomorphic to ℓ_2 or X contains a subspace Y isomorphic to ℓ_p, which, by Corollary(2.1.20), gives the conclusion of the corollary.

Corollary (2.1.25)[2]:

Let X be a subspace of $L_p[0,1], 1 < p < 2,$ such that every subspace Y of X admits an equivalent convex transitive norm. Then, for all $1 \leq r < 2, X$ is isomorphic to a subspace of $L_r[0,1]$ and every subspace of X contains almost isometric copies of ℓ_2.

Proof:

By Corollary (2.1.21), X does not contain a copy of ℓ_r for any $1 \leq r < 2$. Thus, by a theorem of Rosenthal, X is contained in $L_r[0,1]$ for all $p \leq r < 2$. The latter implies by a theorem of Aldous that every subspace of X contains isomorphic (even almost isometric [24]) copies of ℓ_2.

Section(2.2): Maximal Bounded Subgroup of the isomorphism group and the isometry group not contained in it

Maximal isometry groups of equivalent renormings of a Banach space X are exactly maximal bounded subgroups of the group $GL(X)$ of isomorphisms from X onto X. Thus all results in this section can be stated equivalently in the terminology of bounded subgroups of $GL(X)$. We choose the terminology of isometry groups of renormings of X since our arguments rely heavily on Rosenthal's characterization of isometry groups of a general class of Banach spaces, which we recall below.

A Banach space X with a normalized 1-unconditional basis $\{e_\gamma\}_{\gamma\in\Gamma}$ is called impure if there exist a$\alpha \neq \beta$ in G so that (e_α, e_β) is isometrically equivalent to the

usual basis of 2-dimensional $\ell_2^2$and for all $x, x' \in span(e_\alpha, e_\beta)$ with

$\|x\| = \|x'\|$and for all $\mathcal{Y} \in span\{e_\Upsilon : \Upsilon \neq \alpha, \beta\}$ we have$\|x + y\| = \|x' + y\|$. Otherwise the space X is called pure. For convenience, we will also say that $\{e_\Upsilon\}_{\Upsilon\in\Gamma}$is pure (resp. impure) if $(X, \{e_\Upsilon\}_{\Upsilon\in\Gamma})$ is pure (resp. impure).

Definition (2.2.1)[2]:

Let X be a Banach space and Y be a subspace of X. Y is said to be well-embedded in X if there exists a subspace Z of X so that $X = Y + Z$ and for all $y, y' \in Y, z \in Z$, if $\|y\| = \|y'\|$then$\|y + z\| = \|y' + z\|$.

$Y$is called a well-embedded Hilbert space if Y is well-embedded and Euclidean. $Y$is called a Hilbert component of X if Y is a maximal well-embedded Hilbert subspace (a similar concept in complex Banach spaces was introduced by Kalton and Wood).

If X is space with a 1-unconditional basis $E = P\{e_\Upsilon\}_{\Upsilon\in\Gamma}$ and $(H_\Upsilon)_{\Upsilon\in\Gamma}$ are Hilbert spaces all of dimension at least 2, then $Z = (\sum_{\Gamma} \oplus (H_\Upsilon)_E$ is called a functional hilbertian sum.

Rosenthal proved the following useful fact:

Theorem (2.2.2)[2]:

If Y is a well-embedded Hilbert subspace of X, then there exists a Hilbert component of X containing$Y$.

we will use extensively is the following:

Theorem (2.2.3)[2]:

Let X be a pure space with a 1-unconditional basis $E = \{e_\Upsilon\}_{\Upsilon\in\Gamma}$ and $\{H_\Upsilon\}_{\Upsilon\in\Gamma}$ be Hilbert spaces all of dimension at least 2, and let $Z = (\sum_\Gamma \oplus H_\Upsilon)_E$ be the corresponding functional hilbertian sum. Let $P(Z)$ denote the set of all bijections s $\sigma: \Gamma \to \Gamma$ so that

$(a)\{e_{\sigma(\Upsilon)}\}_{\Upsilon\in\Gamma}$ *is isometrically equivalent to* $\{e_\Upsilon\}_{\Upsilon\in\Gamma}$ *, and*

$(b) H_{\sigma(\Upsilon)}$ *is isometric to* $H_{(\Upsilon)}$ *for all* $\Upsilon \in \Gamma$.

Then $T : Z \to Z$ is a surjective isometry if and only if there exist $\sigma \in P\,(Z)$ and surjective linear isometries $T_\Upsilon : H_\Upsilon \to H_{\sigma(\Upsilon)}$, for all $\Upsilon \in \Gamma$, so that for all

$z = (z_\Upsilon)_{\Upsilon \in \Gamma}$ in Z, and for all $\Upsilon \in \Gamma$,

$$(11)\ (Tz)_{\sigma(\Upsilon)} = T_\Upsilon(z_\Upsilon).$$

In particular, if $T \in Isom(Z)$ and H is Hilbert component of Z., then $T(H)$ is a

is a Hilbert component of Z

Theorem (2.2.3) is valid for both real and complex spaces. For separable complex Banach spaces it was proved earlier by Fleming and Jamison.

As a consequence of we obtain a condition when maximal isometry groups of functional hilbertian sums are conjugate to each other.

Proposition (2.2.4)[2]:

Suppose $(Z, \|\cdot\|)$ has two renormings $\|\cdot\|_1$and $\|\cdot\|_2$ such that

$(Z, \|\cdot\|_1)$is isometric to a functional hilbertian sum, $Z_1 = \left(\sum_{\Gamma_1} \oplus H_\Upsilon\right)_{E_1}$, and

$(Z, \|\cdot\|_2)$is isometric to a functional hilbertian sum, $Z_2 = \left(\sum_{\Gamma_2} \oplus H_\Upsilon\right)_{E_2}$, where

E_1*and* $E_2$are pure. Suppose $G_1 := Isom(Z, \|\cdot\|_1)$ and $G_2 := Isom(Z, \|\cdot\|_2)$are conjugate in the isomorphism group of $(Z, \|\cdot\|)$ and are maximal. Then there exists a bijection $\rho: \Gamma_1 \to \Gamma_2$ such that H_Υ is isometric to $H_{\rho(\Upsilon)}$ for all $\Upsilon \in \Gamma_1$.

Proof:

Let $G_1 = T^{-1} G_2 T$ for some isomorphism $T$of $(Z, \|\cdot\|)$. Define

$$\|z\|_3 := \sup_{g \in G_1} \|g(z)\| \qquad (z \in Z)$$

Clearly,$\|\cdot\|_3$ *is* G_1 −invariant and is equivalent to $\|\cdot\|$. Since $(Z, \|\cdot\|_1)$ is isometric to

the functional hilbertian sum Z_1, and since $G_1$is its isometry group, it follows that $(Z, \|\cdot\|_3)$ is isometric to a functional hilbertian sum $Y_1 = \left(\sum_{\Gamma_1} \oplus H_\Upsilon\right)_{F_1}$, . Moreover, by maximality of G_1, we have that $G_1 = Isom(Z, \|\cdot\|_3)$. In particular, since$F_1$ is pure, it follows that F1 is also pure, for otherwise the isometry group of $(Z, \|\cdot\|_3)$ would strictly contain G_1. On the other hand,

$$\|z\|_3 := \sup_{g \in G_2} \|T^{-1} g T(z)\|,$$

Which implies likewise that $(Z, \|\cdot\|_3)$ is isometric to a functional Hilbert sum $Y_2 = \left(\sum_{\Gamma_2} \oplus H_\Upsilon\right)_{F_2}$, where F_2 is pure. The existence of the bijection ρ now follows from a uniqueness theorem of Rosenthal for pure functional hilbertian sums.

We are now ready to describe a countable number of different equivalent maximal norms on Banach spaces with 1-symmetric bases, which are not isomorphic toℓ_2. Henceforth we shall say that a basis E for a Banach space X is non-hilbertian if X is not isomorphic to a Hilbert space.

Theorem (2.2.5)[2]:

Let X be a pure Banach space with a non-hilbertian 1-symmetric

basis$E = \{e_k\}_{k=1}^{\infty}$, let $n \in \mathbb{N}, n \geq 2$, and $Z_n = Z_n(X) = (\sum_{k=1}^{\infty} \oplus H_k)_E$, where, for all $k \in \mathbb{N}, H_k$is isometric to ℓ_2^n.Then Z_n is isomorphic to X and the isometry

group of Z_n is maximal.

Moreover, if $n \neq m$ then $Isom(Z_n)$ *and* $Isom(Z_m)$ are not conjugate to each other in the isomorphism group of X.

Proof:

It is easily seen that Z_n is isomorphic to the direct sum of n copies of X and hence isomorphic to X itself since X has a symmetric basis.

By Theorem (2.2.3), all isometries of Z_n have form (2.1.14), and, since the basis is 1-

symmetric and all H_k are isometric to each other, the set $P(Z_n)$ is equal to the set of all bijections of $\mathbb{N}$.

Suppose that Z_n has a renorming $\tilde{Z}_n = (Z_n, |||\cdot|||)$ so that $Isom(\tilde{Z}_n) \supseteq Isom(Z_n)$. Then the 1-unconditional basis of $Z_n$is also 1-unconditional in $\tilde{Z}_n$, and for all $k \in \mathbb{N}$the subspace H_k is well-complemented in $\tilde{Z}_n$. By theorem (2.2.2), for each $k \in \mathbb{N}$, there exists a Hilbert component of $\tilde{Z}_n$containing H_k. Thus every Hilbert component of $\tilde{Z}_n$ has dimension greater than or equal to n, and $\mathbb{N}$ can be split into subsets$\{A_j\}_{j\in J}$, so that every Hilbert component of $\tilde{Z}_n$is given by

$$\tilde{H}_j = \left(\sum_{k\in A_j} \oplus H_k\right)_2$$

and

$$\tilde{Z}_n = \left(\sum_{j\in J} \oplus \tilde{H}_j\right)_{\{\tilde{e}_j\}_{j\in J}}$$

for some 1-unconditional basis$\{\tilde{e}_j\}_{j\in J}$. Since Z_n, and thus also $\tilde{Z}_n$, is not isomorphic to ℓ_2, J is not finite.

Therefore, by Theorem (2.2.3), all isometriesof $\tilde{Z}_n$.

Suppose that there exists $k_0 \in \mathbb{N}$so that $H_{k_0}$is strictly contained in a component $\tilde{H}_{j_0}$of $\tilde{Z}_n$. Since $\tilde{H}_{j_0}$ is strictly larger than H_{j_0}, there exists $k'_0 \neq k_0$ so that $Hk'_0 \subset \tilde{H}_{j_0}$, Let $j_1, j_2 \in J$ be distinct indices in J, both different from j_0

And let $k_1, k_2 \in \mathbb{N}$be such that $H_{k_1} \subset \tilde{H}_{j_1}$and $H_{k_2} \subset \tilde{H}_{j_2}$, Let $\sigma: \mathbb{N} \to \mathbb{N}$ be bijection such that $\sigma(k_0) = k_1$ and $\sigma(k'_0) = k_2$. Let $T : Z_n \to Z_n$ be defined for all$z = (z_k)_{k\in\mathbb{N}} \in Z_n$by

$$(Tz)_{\sigma(k)} = z_k$$

By Theorem (2.2.3), $T \in Isom(Z_n)$, and thus, by assumption

$T \in Isom(\tilde{Z}_n)$. However $T(\tilde{H}_{j_0}) \cap (\tilde{H}_{j_1}) \neq \emptyset$ component of $\tilde{Z}_n$, which contradicts the fact that $T \in Isom(\tilde{Z}_n)$. Hence every Hilbert component of $\tilde{Z}_n$ is a Hilbert component of $\tilde{Z}_n$and, since $P(Z_n)$ contained all bijections of $\mathbb{N}$,

$P(\tilde{Z}_n) \subseteq P(Z_n)$. Hence $Isom(\tilde{Z}_n) \subseteq Isom(Z_n)$, and thus $Isom(Z_n)$ is maximal.

The 'moreover' sentence follows from Proposition (2.2.4)

Remark (2.2.6)[2]:

Theorem (2.2.5) applies in particular to the space $S(T^{(2)})$, the symmetrization of the 2-convexified Tsirelson space. Indeed, it is known that $S(T^{(2)})$ does not contain ℓ_2, and it is easy to verify that for all $k, l \in \mathbb{N}$ $\|e_k + e_i\| S(T^{(2)}) = 1$, and thus the standard basis of $S(T^{(2)})$ is pure. It is clear that the isometry groups of renormings described in Theorem (2. 2.5) are not almost transitive.

It is known that any symmetric weak Hilbert space is Hilbertian, but in some sense the space $S(T^{(2)})$ is very close to a weak Hilbert space.

We do not know whether or not the space $S(T^{(2)})$ admits an almost transitive renorming, or whether there exists any non-hilbertian symmetric space which admits an almost transitive renorming.

Theorem (2.2.7)[2]:

Let X be a pure Banach space with a non-hilbertian 1-symmetric

basis $E = \{e_k\}_{k=1}^{\infty}$, and let $Z = \left(\sum_{k=1}^{\infty} \oplus \ell_2^k\right)_E$. Let J be any subset of $\mathbb{N}$, with

$\min J \geq 2$andlet $\mathbb{N} \in \bigcup_{j \in J} A_j$, and where$A_j$ are disjoint infinite subsets of $\mathbb{N}$. For

$k \in A_j$, Let$H_k = \ell_2^j$ and let

$$Z_J = Z_J(E) = \left(\sum_{k=1}^{\infty} \oplus H_k\right)_E .$$

Then Z_J is isomorphic to Z, Isom(Z_J) is maximal, and, if $J \neq J'$ then Isom(Z_J) and Isom($Z_{J'}$) are not conjugate in the group of isomorphisms of Z. Hence there are continuum di?erent (pairwise non-conjugate) maximal isometry groups of renormings of Z.

Proof:

It is well-known that if $\{k_i\}_{k=1}^{\infty}$ is any unbounded sequence of positive integers then $\left(\sum_{k=1}^{\infty} \oplus \ell_2^{k_j}\right)_E$ is 4-isomorphic to Z. In particular, Z, in particular,Z_J is 4-isomorphic to Z for al J

Let $P(Z_J)$ be the set defined in Theorem (2.2.3) Then, by the symmetry of E,

$P(Z_J)$consists of all bijections $\sigma : \mathbb{N} \to \mathbb{N}$, so that, for all $j \in J, \sigma(A_j) = A_j$.

By Theorem (2.2.3), Isom(Z_J) consists of all maps $T : Z_J \to Z_J$ so that there exist $\sigma \to P(Z_J)$ and surjective linear isometries $T_k : H_k \to H_{\sigma(k)}$, for all $k \in \mathbb{N}$, so that

for all $z = (z_k)_{k\in\mathbb{N}} \in Z_J$, where $z_k \in H_k$, and for all $k \in \mathbb{N}$,

$$(12)(T_z)_{\sigma(k)} = T_k(z_k).$$

As in Theorem (2. 2.5), let $Z_j = \left(\sum_{k\in A_j} \oplus \ell_2^j\right)_E$. Then $T \in Isom(Z_J)$ if and only

if there exist surjective linear isometries$S_j: Z_j \to Z_j, j \in J$so that for all

$z = (\tilde{z}_j)_{j\in J} \in Z_J$, where $\tilde{z}_j \in Z_j$, and for all $j \in J$,

$$\widetilde{(Tz)} = S_j(\tilde{z}_j)$$

Thus

$$Isom(Z_J) = \sum_{j \in J} \oplus \ Isom(Z_j).$$

The proof that $Isom(Z_J)$ is maximal is essentially the same as the proof that $Isom(Z_n)$ is maximal.

The fact that if $J \neq J'$ then $Isom(Z_J)$ *and* $Isom(Z_{J'})$ are not conjugate to each other in the isomorphism group of Z follows from Proposition (2.2.4).

Remark (2.2.8)[2]:

As Z is a separable Banach space the collection of equivalent norms on Z has cardinality c. Hence Theorem (2.2.7) implies that the cardinality of any maximal collection of pairwise non-conjugate maximal bounded subgroups of $GL(Z)$ is exactly equal to C.

Let E_p be the standard unit vector basis of ℓ_p. It is a well-known consequence of the fact that $\ell_p$is a prime space that ℓ_p is isomorphic to$Z_p = \left(\sum_{k=1}^{\infty} \oplus \ \ell_2^k\right)_{E_p}$

For$1 < p < \infty$. Hence we obtain the following consequence of Theorem (2.2.7)

Theorem (2.2.9)[2]:

For $1 < p < \infty, p \neq 2, \ell_p$ admits a continuum of renormings whose isometry groups are maximal and are not pairwise conjugate in the isomorphism group of ℓ_p.

The latter theorem may be generalized as follows.

Proposition (2.2.10)[2]:

Let X be a space with a non-hilbertian symmetric basis E such that

i. $X(X)$ (i.e., the E-sum of infinitely many copies of X) is isomorphic to X, and
ii. Xcontains uniformly complemented and uniformly isomorphic copies ofℓ_2^n.

Then X is isomorphic to$Z = \left(\sum_{k=1}^{\infty} \oplus \ L_2^k\right)_E$and hence X has a continuum of renormings whose isometry groups are maximal and are not pairwise conjugate in

the isomorphism group of X.

Proof:

The hypotheses imply that Z is isomorphic to a complemented subspace of X. Since Z is isomorphic to its square Z $\oplus$ Z it follows from the Pelczyn´ski decomposition method that Z is isomorphic to X.

Remark (2.2.11)[2]:

We note that there exist symmetric spaces X which are not iso-morphic to l_p and so that X(X) is isomorphic to X.

Let S be the collection of all partitions $B = (B_k)_{k=1}^{\infty}$ of N into finite sets of bounded size, i.e

$$NB := max|B_k| < \infty.$$

We define a partial order $\leq$ $on\ S\ by\ B \leq B^{\sim}$ if B is a refinement of$B^{\sim}$, and let $S_B = \{B^{\sim} \in S : B \leq B^{\sim}\}$

The following properties are easily proved:

i. (S,≤) has cardinality of the continuum,
ii. $(S, \leq)$ contains order-isomorphic copies of every countable ordinal,
iii. $(S_B, \leq)$ is order-isomorphic to $(S, \leq)$ with $B \leftrightarrow (\{k\})_{k=1}^{\infty}$in this isomorphism.

Let $E_p = \{e_k\}_{k=1}^{\infty}$be the standard basis for ℓp, where $1 < p < \infty, p = 2$, and, for $k \geq 1$, let Hk be isometric to $l_2^{2^k}$. We define a space Y as follows:

$$Y := (\sum_{k=1}^{\infty} \oplus H_k)_{E_p} \qquad (14)$$

Y is isometric to a renorming ofl_p. Let $(x_i)_{i=1}^{\infty} \subset l_p$be the basis of $l_p$which is sent to the natural basis of Yunder the isometry.

For each $B \in S$, consider the specific renorming of $l_p$given by

$$Y_B := \left(\sum_{k=1}^{\infty} \oplus H_k^B \right)_{E_P},$$

Where$H_k^B = \left(\sum_{i \in B_k} \oplus H_i\right)_{l_2}$, for which (xi) is the basis ofl_p corresponding to the natural basis ofY_B. This is possible because the condition $NB := max_k |B_k| < \infty$ guarantees that the norm of $Y_B$is equivalent to the norm of Y. In fact, we have

$$\|x\|_Y \leq \|x\|_{Y_B} \leq N_B^{1/p-1/2} \|x\|_Y \quad (x \in l_p, 1 < p < 2)$$

and

$$N_B^{1/p-1/2} \|x\|_Y \leq \|x\|_{Y_B} \leq \|x\|_Y \quad (x \in l_p, 2 < p < \infty).$$

We identify $Y_B$with this particular renorming of l_p in which the basis (xi) of $l_p$corresponds to the natural basis of Y_B. Note that $Y_B$is not a maximal renorming of $l_p$since

$$Isom(Y_B) \subsetneq (Isom(Y_{B^\sim})$$

for all B $\in S_B \backslash \{B\}$.

Proposition(2.2.12)[2]:

Let $B, B^\sim \in S$. If $Isom(Y_B) and\ Isom(Y_{B^\sim})$ are conjugate in the isomorphism group of ℓp then B = B.

Proof:

This does not follow from Proposition(2.2.3) because the maximality hypothesis is not satisfied. Note that $n_k^B := dimH_k^B = \sum_{i \in B_k} 2^i$. By the uniqueness of binary representations the map k → $n_k^B$is one-to-one.

Since $Isom(Y_B)\ and\ Isom(Y_{B^\sim})$ are conjugate to each other it follows that the $Isom(Y_B) invariant\ and\ Isom(Y_{B^\sim}) invariant\ subspaces\ of\ l_p$ have the same dimensions. By Theorem (2.2.2)there is an n-dimensional $Isom(Y_B)$-invariant subspace of ℓp if and only if $n = \sum_{k \in A} n_k^B$ for some finite $A \subset N$, and similarly

for $Isom(Y_{B^{\sim}})$. Thus, by uniqueness of the binary representation, $B = B^{\sim}$.

Proposition (2.2.13)[2]:

Let $B \in S$ and let $(l_p, \|.\|_0)$ be a renorming of $l_p$so that $Isom(l_p, \|.\|_0) \supseteq Isom(Y_B)$. Then there exists $B \in S_B$ so that

$$Isom(l_p, \|.\|_0) = Isom(Y_B).$$

In particular, $Isom(l_p, \|.\|_0)$ is not maximal. Conversely, every$B^{\sim} \in S_B$ determines such a renorming.

Proof:

By Theorem(2.2.2), it is clear that for every $B^{\sim} \in S_B, Isom(Y_{B^{\sim}}) \supseteq Isom(Y_B)$, which is the last sentence of the theorem.

Now suppose $that\ Isom(l_p, \|.\|_0) \supseteq Isom(Y_B)$. Thus, $(l_p, \|.\|_0)$ is a functional hilbertian sum

$$(l_p, \|.\|_0) = \left(\sum_{K=1}^{\infty} \oplus H_k^B\right)_E$$

for some (possibly impure) 1-unconditional basis E. Moreover, $(x_i)_{i=1}^{\infty}$ is the basis

of ℓp corresponding to the natural basis of $(\sum_{k=1}^{\infty} \oplus H_k^B)_E$ Arguing as in the proof of Theorem(2.2.4), N can be partitioned into disjoint subsets $\{A_j\}_{j \in J}$, so that every

Hilbert component of $(l_p, \|.\|_0)$ is given by

$$H^{\sim}{}_j = \left(\sum_{K \in A_j} \oplus H_k^B\right)_2$$

and

$$(l_p, \|.\|_0) = \left(\sum_{j\in J} H^{\sim}{}_j\right)_{E_0}$$

for some pure 1-unconditional basisE_0. Let

$B^{\sim}{}_j = \cup_{k\in A_j} B_k$(j≥1)

Since $\|.\|_0$is equivalent to $\|.\|_{Y_B}$it follows that $maxj \geq 1 \left|B^{\sim}{}_j\right| < \infty$. Let $B^{\sim} = \left(B^{\sim}{}_j\right)_{j=1}^{\infty}$. Then$B^{\sim} \in S_B$, and (since E_0 is pure) Theorem(2.2.2) gives

$$Isom(l_p, \|.\|_O) = Isom(Y_{B^{\sim}}).$$

since $Y_B$is not maximal for any B$\in S$, it followsthat $(l_p, \|.\|_0)$ is not maximal .

Theorem(2.2.14)[2]:

$For\ 1 < p < \infty, p = 2, l_p$has a continuum of renormings none of whose isometry groups is contained in any maximal bounded subgroup of the iso-morphism group ofl_p. Moreover, these isometry groups are not pairwise conjugate in (l_p) .

Remark (2.2.15)[2]:

Symmetry of the standard basis of ℓp is not used in the proof of Theorem (2.2.14). In particular, the theorem holds for any space Z with the following properties:

i: Z has an unconditional basis E such that no subsequence of E is equivalent

to the unit vector basis of l_2 ; ;

ii: Z is isomorphic to $(\sum_{n=1}^{\infty}\oplus H_n)_E$ for every collection $(H_n)_{n=1}^{\infty}$ of finite-dimensional Hilbert spaces.

These properties are satisfied for example by the space $(\sum_{n=1}^{\infty}\oplus l_2^n)_E$, $where\ E$ is symmetric, pure, and non-hilbertian .

Proposition (2.2.16)[2]:

$T^{(2)}$admits a continuum of renormings none of whose isometry groups is contained in any maximal bounded subgroup of the isomorphism group of$T^{(2)}$. Moreover, these isometry groups are not pairwise conjugate in GL($T^{(2)}$) .

Proof:

The standard basis $\{e_i\}_{i=1}^{\infty}$ $of\ T^{(2)}$is pure since $\|e_i + e_j\|_{T^{(2)}} = 1$for all $i = j$. For each $J \subseteq \{2n: n \geq 1\}$, let $\{n_K^J\}_{k\geq 1}$ be the arrangement of $J \cup \{2n-1: n \geq 1\}$ as an increasing sequence, and let $m_K^J := 2^{n_K^J}$. Note that for all $J\ and\ k \geq 1$, we have

$m_k^J \leq 2^{2k-1}$ (15)

Let$A^J = \cup_k \geq 1\{i : m_K^J \leq i < 2m_K^J\}$. Then the growth condition (15) ensures that the subsequences $\{e_i\}_{i\in A^J}$ and $\{m_k^J\}_{k\geq 1}$ of the basis $\{e_i\}_{i\geq 1}$ are in fact both equivalent to the whole sequence $\{e_i\}_{i\geq 1}$.

Note that from the definition of the $T^{(2)}$norm we have that $\{e_i: m_k^J \leq i < 2m_k^J\}$

is 2-equivalent to the unit vector basis of $l_2^{m_k^J}$. Let $F_k^J := span(e_i: m_k^J \leq i < 2m_k^J)$.

Suppose $x_k \in F_K^J (k \geq 1)$Then ,

$$\frac{1}{\sqrt{3}}\left\|\sum_{k=1}^{\infty}\|x_k\|e_{m_K^J}\right\|_{T^{(2)}} \leq \left\|\sum_{k=1}^{\infty} x_k\right\|_{T^{(2)}} \leq \sqrt{18}\left\|\sum_{k=1}^{\infty}\|x_k\|\, e_{m_K^J}\right\|_{T^{(2)}} .$$

Since$\|.\|_2$ and$\|.\|_{T^{(2)}}$ are 2-equivalent on $F_K^J$and since $\left\{e_{m_K^J}\right\}_{k\geq 1}$ is equivalent to $\{e_i\}_{i\geq 1}$ there exists a constant $C > 0$ such that

$$\frac{1}{c}\left\|\sum_{k=1}^{\infty}\|x_k\|2^{e_k}\right\|_{T^{(2)}} \leq \left\|\sum_{k=1}^{\infty} x_k\right\|_{T^{(2)}} \leq C\left\|\sum_{k=1}^{\infty}\|x_k\|2^{e_k}\right\|_{T^{(2)}} .$$

But this implies that the natural basis of $X_J := \left(\sum_{k=1}^{\infty} \oplus\, l_2^{m_K^J}\right)_{T^{(2)}}$ is equivalent to $\{e_i\}_{i\in A^J}$and hence also equivalent to $\{e_i\}$. In particular, $X_J$is isomorphic to$T^{(2)}$ and therefore may be regarded as a renorming of$T^{(2)}$. Since$T^{(2)}$ does not contain an isomorphic copy ofl_2, arguing as in Proposition (2.2.13)$Isom(X_J)$ is not contained in any maximal isometry group, and, arguing as in Proposition (2.2.12), if J ≠ J′ then $Isom(X_J)$ and $Isom(X_{J'})$ are not conjugate in the isomorphism group of $T^{(2)}$.

We do not know whether or not $T^{(2)}$ admits an equivalent maximal norm.

Let U be the space with a universal unconditional basis constructed by Pelczy´nski. We finish the paper by observing that both Proposition (2.2.10) and Theorem (2.2.14) hold for$U$.

Theorem (2.2.17)[2]:

The space U with a universal unconditional basis has two continua of renormings whose isometry groups are not pairwise conjugate in the isomorphism group of U such that the renormings of the first continuum are maximal and for the renormings of the second continuum no isometry group is contained in any maximal bounded subgroup of $GL(U)$.

Proof. It is known that U has a symmetric basis $E = (e_i)_{i=1}^{\infty}$. By renorming U, we may assume that E is pure. To see this, let B be the unit ball of any norm on U for which $E = (e_i)_{i=1}^{\infty}$ is a normalized 1-symmetric basis. Let

$$B_1 = \overline{conv}\{B \cup \{e_i \pm e_j : i \neq j\}\}.$$

Then B_1 is the unit ball for an equivalent norm $\|\cdot\|$on U such that $(e_i)_{i=1}^{\infty}$ is a 1-symmetric basis satisfying

$$\|e_i\| = \|e_i \pm e_j\| = 1 (i \neq j).$$

In particular, for all $i \neq j, (e_i, e_j)$ is not isometric to the unit basis ofℓ_2^2, so E is pure. The universality property of U implies that U (U) is isomorphic to U and that U is

isomorphic to every space of the form

$$Z = \left(\sum_{n=1}^{\infty} \oplus H_n \right)_E$$

Where $(H_n)_{n=1}^{\infty}$ is any collection of finite-dimensional Hilbert spaces. From this it follows that conditions of Remark 4.4 are satisfied, and thus Theorem (2.2.14) is valid for U.

U (with the symmetric basis E) satisfies the hypotheses of Proposition (2.2.10) and hence its conclusion holds. However, we prefer to give a more direct argument which also shows that the analogue of Proposition (2.2.13) (with $E_p$replaced by E in the definition of Y_B) holds for U . So consider a renorming $(\tilde{Z}, \|\cdot\|_0)$ of any space $Z$of the form as above, such that $Isom(\tilde{Z}, \|\cdot\|_0) \supseteq Isom(Z)$. Then arguing as in the proof of Theorem (2.2.5), the Hilbert components of $\tilde{Z}$ are spans of unions of Hilbert components of Z, that is,ℕcan be partitioned into disjoint subsets $\{N_j\}_{j\in J}$,so that every Hilbert component of $\tilde{Z}$ is given by

$$\tilde{H}_j = \left(\sum_{k\in N_j} \oplus H_k \right)_2$$

and

$$\tilde{Z} = \left(\sum_{j\in J} \oplus \tilde{H}_j \right)_{\{\tilde{e}_j\}_{j\in J}}$$

for some pure 1-unconditional basis $\{\tilde{e}_j\}_{j\in J}$. Since the symmetric basis E is not equivalent to the unit vector basis if ℓ_2 it follows that the constant of equivalence between $(e_i)_{i=1}^{n}$ and the standard basis of ℓ_2^nbecomes unbounded as $n \to \infty$. But this implies that the sets N_j have uniformly bounded size, i.e.

$$\max_{j\in J} |N_j| < \infty,$$

for otherwise the norms of Z and $\bar{Z}$ would not be equivalent.

Chapter 3

Product integration in Banach algebras

We introduce new concepts of strong Kurzweil and McShane product integrals, and investigate their properties. We also provide necessary and sufficient conditions for product integrability of functions with countably many discontinuities.

Section (3.1): Product integral and their strong verisons

The concept of product integration goes back to V. Volterra who studied linear systems of differential equations of the form

$$W'(t) = A(t)W(t), t \in [a,b],$$

$$W(a) = \boldsymbol{I}, \qquad (1)$$

where $\boldsymbol{I}$ is the identity matrix, $A : [a,b] \to R^{n\times n}$ is a given continuous function and $W : [a,b] \to R^{n\times n}$ is the unknown function. To find the solution (whose existence and uniqueness is easy to prove), Volterra considered products of the form

$$\big(I + A(\xi_m)(t_m - t_{m-1})(I + A(\xi_{m-1})(t_{m-1} - t_{m-2})\big) \cdots \big(I + A(\xi_1)(t_1 - t_0)\big), (2)$$

where $a = t_0 < t_1 < \ldots < t_m = b$ and $\xi_i \in [t_{i-1}, t_i], i \in \{1,\ldots,m\}$, is a tagged partition of $[a,b]$. As the lengths of the subintervals $[t_{i-1}, t_i]$ approach zero, the value of the product (2) tends to a matrix which is called the product integral of A over $[a,b]$; let us denote it by $\prod_a^b (I + A(t)\, dt)$. . Now, if we consider the indefinite product integral$W(t) = \prod_a^t (I + A(s)ds)$, it can be shown that W is the solution of (1).

Volterra observed that this procedure still works if A is no longer continuous but merely Riemann integrable; in this case, the indefinite product integral satisfies

$$W(t) = I + \int_a^t A(s)W(s)ds \quad , \qquad t \in [a,b],$$

where the integral on the right-hand side is the Riemann integral. Consequently, we have $W'(t) = A(t)W(t)$ almost everywhere in $[a,b]$. Later, the concept of product integral was extended to Lebesgue integrable functions A. On the other hand, P.R. Masani realized that the definition of the product integral makes sense if A takes values in an arbitrary unital Banach algebra. For example, if X is a Banach space and A has values in the Banach algebra $L(X)$ of all bounded linear operators on X, then the product integral of A provides the unique solution of the operator equation (1).

It is well known that the Henstock–Kurzweil integral (also known as the gauge integral) generalizes the integrals of Riemann, Lebesgue, and Newton. Hence, it seems natural to replace the original Riemann-type definition of product integral by a gauge-type definition; this idea was pursued by J. Jarník, J. Kurzweil and Š. Schwabik who also introduced the related concept of McShane product integral . These authors developed a complete theory of Kurzweil and McShane product integrals for matrix-valued functions, but also realized that the proofs of their main results are not applicable in infinite dimension.

The aim of this chapter is to provide a satisfactory theory of Kurzweil and McShane product integration in infinite-dimensional Banach algebras. We show that in infinite dimension, the two product integrals cannot be expected to have exactly the same properties as in finite dimension; for example, the indefinite product integrals need not be differentiable almost everywhere. On the other hand, we introduce the con-cepts of strong Kurzweil and McShane product integrals, whose properties are completely analogous to the finite-dimensional product integrals. In infinite dimension, the class of strongly Kurzweil/McShane product integrable functions is a proper subset of the class of Kurzweil/McShane product integrable functions, but is still wide enough for

practical purposes. For example, a function with countably many discontinuities is strongly Kurzweil product integrable if and only if it is Kurzweil product integrable.

The section is organized as follows. summarizes some preliminaries, namely the definitions of the Kurzweil and McShane integrals and their strong counterparts, and the properties of the exponential and logarithm functions in Banach algebras. We recall the definitions of the Kurzweil and McShane product integrals, introduce their strong versions, and develop some of their basic properties. The section is devoted to the properties of the indefinite strong product integrals, which are then used to show that strong McShane product integrability coincides with Bochner integrability. We develop necessary and sufficient conditions for product integrability of functions with countably many discontinuities.

For applications of product integrals to differential equations in the real and complex domain, probability theory, dynamic equations on time scales.

Let us recall some definitions of integrability for vector-valued functions.

A tagged partition of an interval $[a, b]$ is a collection of point–interval pairs $D = (\xi_i, [t_{i-1}, t_i])_{i=1}^{m}$, where $a = t_0 < t_1 < \cdots < tm = b$ and $\xi_i \in [t_{i-1}, t_i]$ for every $i \in \{1, \ldots, m\}$. We refer to $t_0, \ldots, t_m$ as the division points of D, while $\xi_1, \ldots, \xi_m$ are the tags of D. If we relax the assumption $\xi_i \in [t_{i-1}, t_i]$ and replace it by $\xi_i \in [a, b]$, then the

collection D is called a free tagged partition.

Given a function $\delta : [a, b] \to R^+$ (called a gauge on $[a, b]$), a free tagged partition is called δ-fine if

$$[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)^{''}, \qquad i \in \{1, \ldots, m\}.$$

Let X be a Banach space. A function $f: [a, b] \to X$ is called Henstock–Kurzweil integrable if there is a vector $S_f \in X$ with the following property: To

each $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to R^+$ such that

$$\left\| \sum_{i=1}^{m} f(\xi_i)(t_i - t_{i-1}) - S_f \right\| < \varepsilon \qquad (3)$$

for every δ-fine tagged partition of $[a,b]$. In this case, S_f is called the Henstock–Kurzweil integral of f over $[a,b]$, and is denoted by $\int_b^a f(t)dt$.

If (3) holds for all δ-fine free tagged partitions of $[a,b]$, then f is called McShane integrable over $[a,b]$. The McShane integral S_f will again be denoted by $\int_b^a f(t)dt$.

The definition of Riemann integrability is obtained from the definition of Henstock–Kurzweil integrability if the gauge δ is assumed to be constant on $[a,b]$. In this case, the integral $\int_b^a f(t)dt$ .is called the Riemann (or Graves) integral.

A function $f : [a,b] \to X$ is called strongly Henstock–Kurzweil integrable if there is a function $F : [a,b] \to X$ with the following property: To each $\varepsilon > 0$, there is a gauge $\delta: [a,b] \to R^+$ such that

$$\sum_{i=1}^{m} \|f(\xi_i)(t_i - t_{i-1}) - (F(t_i) - F(t_{i-1})\| < \varepsilon \qquad (4)$$

for every δ-fine tagged partition of $[a,b]$. In this case, we define the strong Henstock–Kurzweil integral of f over $[a,b]$ as $\int_a^b f(t)dt = F(b) - F(a)$. This integral is also known as the Henstock–Lebesgue integral or variational Henstock integral.

If (4) holds for all δ-fine free tagged partitions of $[a,b]$, then f is called strongly McShane integrable over $[a,b]$. The strong McShane integral is again defined as $\int_a^b f(t)dt = F(b) - F(a)$. Some authors refer to this integral as the variational McShane integral.

For the basic properties of these integrals. We also need the concept of Bochner integrability in particular the following facts:

i. A strongly measurable function $f : [a,b] \to X$ is Bochner integrable if and only if the function $\|f\|$ is Lebesgue integrable.
ii. A function $f : [a,b] \to X$ is Bochner integrable if and only if there exists an absolutely continuous function $F : [a,b] \to X$ such that $F'\acute{I} = f$ almost everywhere in $[a,b]$; in that case, we have $\int_a^b f = F(b) - F(a)$.
iii. A function $f : [a,b] \to X$ is strongly McShane integrable if and only if it is Bochner integrable; in this case, the values of the integrals coincide .

Note that in infinite-dimensional spaces, Riemann integrability does not necessarily imply Bochner inte-grability .

we assume that X is a complete normed algebra with a unit element I whose norm is 1, it is a unital Banach algebra.

Lemma (3.1.1)[3].

For every $n \in \mathrm{N}$ and all $x_1,\ldots,x_n,y_1,\ldots,y_n \in X$, we have

$$x_n,\ldots,x_1 - y_n \cdots y_1 = \sum_{j=1}^{n} x_n \cdots x_{j+1}(x_j - y_j)y_{j-1} \cdots y_1 \qquad (5)$$

Recall that in a unital Banach algebra X, we may introduce the exponential and logarithm function as follows:

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = I + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \quad x \in X,$$

$$log\, x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x - I)^n}{n}, \qquad \|x - I\| < 1.$$

These functions have similar properties as in the familiar case when

$X = R^{n\times n}$, in particular:

i. The exponential and logarithm are continuous functions.
ii. For every x ∈ X, expx is an invertible element and its inverse is exp(−x).
iii. If x, y ∈ X are such that xy = yx, then exp(x + y) = expx expy and $\log xy = logx + logy$ if all three logarithms are defined.
iv. $exp(logx) = x\ if\ \|x - I\| < 1, and\ log(expx) = x\ if\ \|x\| < log2.$
v. We have the estimates

$$\|exp\ x\| \leq exp\|x\|, \tag{6}$$

$$\|exp\ x - exp\ y\| \leq \|x - y\| \exp(\max(\|x\|, \|y\|)), \tag{7}$$

$$\|log\ x - log\ y\| \leq \frac{\|x - y\|}{1 - max(\|x - I\|, \|y - I\|)}. \tag{8}$$

The first one follows immediately from the definition of the exponential function

$$\|(x-I)^n - (y-I)^n\| = \left\|\sum_{k=1}^{n}(x - I)^{n-k}(x-y)(y-I)^{k-1}\right\|$$
$$\leq \|x-y\| \sum_{k=1}^{n} \|x-I\|^{n-k}\|y-I\|^{k-1}$$
$$\leq \|x-y\| \cdot n \cdot max\ \|x-I\|, \|y-I\|^{n-1},$$

and therefore

$$\|\ log\ x - log\ y\| = \left\|\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x - I)^n - (y - I)^n}{n}\right\|$$
$$\leq \sum_{n=1}^{\infty}\frac{(x - I)^n - (y - I)^n}{n}$$
$$\leq \|x - y\| \sum_{n=1}^{\infty} \max\ (\|x - I\|, \|y - I\|)^{n-1}$$
$$= \frac{\|x - y\|}{1 - max(\|x - I\|, \|y - I\|)}.$$

We start with the definitions of the Riemann, McShane and Kurzweil product integrals $\prod_a^b(I + A(t)\,dt)$. Let us make the following agreement: If $m \in N$ and $x_1, \ldots, x_m \in X$, then the symbol $\prod_{i=m}^{1} x_i$ stands for the product $x_m x_{m-1} \cdots x_1$.

Definition (3.1.2)[3]

A function $A : [a,b] \to X$ is called Kurzweil product integrable, if there exists an invertible element $P_A \in X$ with the following property: For each $\varepsilon > 0$, there exists a gauge $\delta: [a,b] \to R^+$ such that

$$\left\| \prod_{i=m}^{1} (I + A(\xi_i)(t_i - t_{i-1})) - P_A \right\| < \varepsilon \qquad (9)$$

for all δ-fine tagged partitions of $[a,b]$. In this case, P_A is called the Kurzweil product integral of A and will be denoted by$\prod_a^b(I + A(t)\,dt)$. This integral is also known under the name Perron product integral.

If (9) holds for all δ-fine free tagged partitions of $[a,b]$, then A is called McShane product integrable over $[a,b]$. The McShane product integral P_A will again be denoted by$\prod_a^b(I + A(t)\,dt)$.

The definition of Riemann product integrability is obtained from the definition of Kurzweil product integrability if the gauge $\overset{a}{\delta}$ is assumed to be constant on $[a,b]$. In this case, the integral $\prod_a^b(I + A(t)\,dt)$is called the Riemann product integral.

Remark (3.1.2)[3]:

Let us mention two basic properties which are common to all three types of product integrals:

If $\prod_a^b(\boldsymbol{I} + A(t)dt)$ exists, and if $c \in (a,b)$, then $\prod_a^c(\boldsymbol{I} + A(t)dt)$ and $\prod_c^b(\boldsymbol{I} + A(t)dt)$ exist as well, and

$$\prod_a^b (I + A(t)dt) = \prod_c^b (I + A(t)dt) \prod_a^c (I + A(t)dt).$$

Conversely, if the product integrals on the right-hand side exist for a certain $c \in (a, b)$, then the product integral on the left-hand side exists as well and the equality holds.

i. If $\prod_a^b (I + A(t)dt)$ exists, then the functions $t \mapsto \prod_a^t (I + A(s)ds)$ and $t \mapsto \prod_t^b (I + A(s)ds)$ are continuous on $[a, b]$.

More information on Riemann product integrals, including the proofs of the two properties from Re-mark (3.1.3). Kurzweil product integrals where they were called "Perron product integrals" and denoted by $(PP) \int_a^b (I + A(t)dt)$.

McShane product integrals where they were referred to as "Bochner product integrals"; In these sources, the Kurzweil and McShane product integrals are studied in the special cases $X = R^{n\times n}$ or $X = L(Y)$, where Y is a Banach space; however, the proofs of the two properties from Remark (3.1.3) remain valid in all unital Banach algebras.

Remark (3.1.4)[3]:

Next, we summarize some more specific properties of the Riemann, McShane and Kurzweil product integrals:

i. Riemann or McShane product integrability implies Kurzweil product integrability; this follows immedi-ately from the definitions.
ii. A function is Riemann product integrable if and only if it is Riemann integrable.
iii. In Definition (3.1.2), we were assuming that P_A is an invertible element of X. For the Riemann product integral, it turns out that this assumption is not necessary, i.e., if the products in (3.1.10) have a limit P_A, then it is always invertible. On the other hand, this is no longer true for Kurzweil or McShane product integrals.
iv. For the Kurzweil product integral, we have the following Hake-type theorem: Assume

that $\prod_a^b(I + A(s)ds)$ exists for all $t \in [a,b)$. If $limt_{t\to b-}\prod_a^t(\boldsymbol{I} + A(s)ds)$ exists and is invertible, then $\prod_a^b(\boldsymbol{I} + A(s)ds)$ exists as well and is equal to the limit.

J. Jarník and J. Kurzweil pointed out that in the finite-dimensional case $X = R^{n\times n}$, the indefinite Kurzweil product integral $W(t) = \prod_a^t(\boldsymbol{I} + A(s)ds)$ has the following property: For every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{m}\|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon$$

for every δ-fine tagged partition of $[a,b]$.

This property plays a key role in their proof that $W'(t) = A(t)W(t)$ almost everywhere in $[a,b]$. However, the original proof of the above-mentioned property is no longer applicable in infinite dimension which holds if and only if the dimension is finite . Moreover, one can easily construct examples of functions whose indefinite Kurzweil product integral are not differentiable almost everywhere.

A similar situation is known from the Henstock–Kurzweil integration theory; in infinite dimension, the indefinite Henstock–Kurzweil integral need not be differentiable almost everywhere, but one can overcome this problem by working with the strong Henstock–Kurzweil integral. These facts provide a motivation for introducing the strong Kurzweil and McShane product integrals as follows.

Definition (3.1.5)[3]:

A function $A : [a,b] \to X$ is called strongly Kurzweil product integrable if there is a function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, both W and W^{-1} are bounded, and for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon \qquad (10)$$

for every δ-fine tagged partition of $[a, b]$. In this case, we define the strong Kurzweil product integral as $\prod_a^b(I + A(t)dt) = W(b)W(a)^{-1}$.

If (10) holds for all δ-fine free tagged partitions of $[a, b]$, then A is called strongly McShane product inte-grable over $[a, b]$. The strong McShane product integral is again defined as $\prod_a^b(I + A(t)dt) = W(b)W(a)^{-1}$.

As we will see later in the section, the properties of strong product integrals are quite similar to the properties of finite-dimensional product integrals.

Note that if A is strongly product integrable, then the function W from Definition (3.1.5) is not unique (any function obtained from W by multiplying it from the right by an arbitrary invertible element of X has the same properties as W). However, the next theorem shows that strong product integrability implies ordinary product integrability, and if W is an arbitrary function satisfying the conditions from Definition (3.1.5), then $W(b)W(a)^{-1}$ is the ordinary product integral of A; this justifies the correctness of Definition (3.1.5), and also explains why we use the same symbol for product integrals and strong product integrals. (On the other hand, product integrability does not necessarily imply strong product integrability.

Theorem (3.1.6)[3]:

If $A : [a, b] \to X$ is strongly Kurzweil/McShane product integrable, it is also Kurzweil/McShane product integrable and the values of the integrals coincide.

Proof:-

Let us prove the statement concerning Kurzweil product integrals; the proof of the McShane counter-part is a straightforward modification. Consider the function W from Definition (3.1.5). There exists a constant $M > 0$ such that $\|W(t)\| \leq M$ and $\|W(t)^{-1}\| \leq M$ for all $t \in [a, b]$. Take an arbitrary

$\varepsilon \in (0, \frac{1}{M^2})$. There exists a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon$$

for every δ-fine tagged partition of $[a,b]$. Consequently,

$$\sum_{i=1}^{m} \|W(t_i)^{-1}(I + A(\xi_i)(t_i - t_{i-1}))W(t_{i-1}) - I\| < M^2\varepsilon < 1$$

If $y_1, \ldots, y_m \in X$ are such that $\sum_{i=1}^{m} \|y_i\| \leq 1$,

$$\left\|(I + y_m) \cdots (I + y_1) - I - \sum_{i=1}^{m} y_i\right\| \leq \left(\sum_{i=1}^{m} \|y_i\|\right)^2.$$

By letting

$$y_i = W(t_i)^{-1}(I + A(\xi_i)(t_i - t_{i-1}))W(t_{i-1}) - I, i \in \{1, \ldots, m\}$$

we get

$$\left\|W(t_m)^{-1}\left(\prod_{i=m}^{1}(I + A(\xi_i)(t_i - t_{i-1}))\right)W(t_0) - I\right\|$$

$$= \left\|\prod_{i=m}^{1} W(t_i)^{-1}(I + A(\xi_i)(t_i - t_{i-1}))W(t_{i-1}) - I\right\|$$

$$= \|(I + y_m) \cdots (I + y_1) - I\| \leq \sum_{i=1}^{m} \|y_i\| + \left(\sum_{i=1}^{m} \|y_i\|\right)^2$$

$$< M^2\varepsilon + M^4\varepsilon^2.$$

It follows that

$$\prod_{i=m}^{1} \|(I + A(\xi_i)(t_i - t_{i-1})) - W(b)W(a)^{-1}\|$$

$$= \left\|\prod_{i=m}^{1}(I + A(\xi_i)(t_i - t_{i-1})) - W(t_m)W(t_0)^{-1}\right\| < M^4\varepsilon + M^6\varepsilon^2$$

for every δ-fine tagged partition of $[a, b]$, which proves that the Kurzweil product integral exists and equals $W(b)W(a)^{-1}$.

Remark (3.1.7)[3]:

The strong Kurzweil and McShane product integrals have the following properties:

i. If the strong integral $\prod_a^b(I + A(t)dt)$ exists, and if $c \in (a, b)$, then the strong integrals $\prod_a^c(I + A(t)dt)$and $\prod_c^b(I + A(t)dt)$ exist as well, and

$$\prod_a^b(I + A(t)dt) = \prod_c^b(I + A(t)dt) \prod_a^c(I + A(t)dt) \quad . \qquad (11)$$

Indeed, strong product integrability on subintervals is a direct consequence of Definition (1.1.3), while the relation (11) follows from Theorem (3.1.6) and Remark (3.1.3)

ii. If A is strongly product integrable, then (10) holds with $W(t) = \prod_a^t(I + A(s)ds)$, i.e., it reduces to

$$\sum_{i=1}^{m} \left\| I + A(\xi_i)(t_i - t_{i-1}) - \prod_{t_{i-1}}^{t_i}(I + A(s)ds) \right\| < \varepsilon$$

The reason is that for every $i \in \{1, \ldots, m\}$, A is strongly product integrable on $[t_{i-1}, t_i]$, and it follows from Theorem (3.1.6) that $\prod_{t_{i-1}}^{t_i}((I + A(s)\, ds) = W(t_i)W(t_{i-1})^{-1}$.

iii. If the strong integral$\prod_c^b(I + A(t)dt)$exists, then the functions $t \mapsto \prod_a^t(I + A(s)ds)$ and $t \mapsto \prod_t^b(I + A(s)ds)$ are continuous on $[a, b]$. This is a direct consequence of Theorem (3.1.6) and Remark (3.1.3).

According to the previous two facts, the function W from Definition (3.1.5) is necessarily continuous.

iv. If $X = R^{n\times n}$ and A is Kurzweil/McShane product integrable, then it is also strongly Kurzweil/McShane product integrable .

Next, we recall the definitions of the exponential product integrals and introduce their strong counter-parts. The following two definitions are motivated by the fact that

$$exp\ (A(\xi_i)(t_i - t_{i-1}) \ = I + A(\xi_i)(t_i - t_{i-1}) + O(\ \|A(\xi_i)\|^2\ (t_i - t_{i-1})^2)\ ,$$

and it seems plausible that the higher-order terms do not contribute to the value of the product integral.

Definition (3.1.8)[3]:.

A function $A : [a,b] \to X$ is called Kurzweil exponentially product integrable, if there exists an invertible element $P_A \in X$ with the following property: For each $\varepsilon > 0$, there exists a gauge $\delta : [a,b] \to R^+$ such that

$$\left\| \prod_{i=m}^{1} exp\ (A(\xi_i)(t_i - t_{i-1})) - P_A \right\| < \varepsilon \qquad (12)$$

for all δ-fine tagged partitions of $[a,b]$. In this case, P_A is called the Kurzweil exponential product integral of A and will be denoted by $\prod_a^b exp(A(t)dt)$.

If (3.1.13) holds for all δ-fine free tagged partitions of $[a,b]$, then A is called McShane exponentially product integrable over $[a,b]$. The McShane exponential product integral P_A will again be denoted by $\prod_a^b exp(A(t)dt)$.

The definition of Riemann product integrability is obtained from the definition of Kurzweil exponential product integrability if the gauge δ is assumed to be constant on $[a,b]$. In this case, the integral $\prod_a^b exp(A(t)dt)$is called the Riemann exponential product integral.

Definition (3.1.9)[3]:

A function $A : [a,b] \to X$ is called strongly Kurzweil exponentially product integrable if there is a function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, both W and W^{-1} are bounded, and for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{m} \| exp\,(A(\xi_i)(t_i - t_{i-1})) \; - W(t_i)W(t_{i-1})^{-1} \| \; < \; \varepsilon \qquad (13)$$

for every δ-fine tagged partition of $[a, b]$. In this case, we define the strong Kurzweil exponential product integral as $\prod_a^b exp(A(t)dt) \; = \; W(b)W(a)^{-1}$.

If (13) holds for all δ-fine free tagged partitions of $[a, b]$, then A is called strongly McShane exponentially product integrable over $[a, b]$. The strong McShane exponential product integral is again defined as $\prod_a^b exp(A(t)dt) \; = \; W(b)W(a)-1$

Remark (3.1.10)[3]:

It is not our intention to develop a systematic theory of exponential product integrals; let us collect only some basic facts that will be needed later.

i. If A is Riemann integrable, then the Riemann product integral $\prod_a^b exp(A(t)dt)$ exists and equals $\prod_a^b (1 + A(t)dt)$
ii. If A is Bochner integrable, then the McShane product integrals $\prod_a^b exp(A(t)dt)$ and $\prod_a^b (1 + A(t)dt)$ exist and are equal to each other. Moreover, the corresponding indefinitenb product integral

$$W(t) \; = \; I \; + \int_a^t A(s)W(s)\, d_s \quad , \qquad t \in [a, b],$$

where the integral on the right-hand side is the Bochner integral; for a proof of this fact (the original proof for $X = R^{n\times n}$ remains valid in unital Banach algebras).

i. If A is strongly Kurzweil/McShane exponentially product integrable, then it is also Kurzweil/McShane exponentially product integrable and the values of the integrals coincide. The proof is the same as the proof of Theorem (3.1.6) it is enough to replace all terms of the form $\boldsymbol{I} + A(\xi_{\boldsymbol{i}})(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1})$ by $exp(A(\xi_{\boldsymbol{i}})(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1})$.

ii. If $X = R^{n\times n}$ and A is Kurzweil/McShane exponentially product integrable, then it is also strongly Kurzweil/McShane exponentially product integrable.

Lemma (3.1.11)[3]:.

For every function $A : [a,b] \to X$ and every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - exp\,(A(\xi_i)(t_i - t_{i-1}))\| < \varepsilon$$

for every δ-fine free tagged partition of $[a,b]$.

Proof:-

Take a gauge $\delta : [a,b] \to R^+$ such that

$$\|A(\xi)\|\,\delta(\xi) < \frac{1}{2}, \qquad \|A(\xi)\|^2\delta(\xi) < \frac{\varepsilon}{2(b-a)e}$$

for all $\xi \in [a,b]$. Then, for every δ-fine free tagged partition of $[a,b]$, we get

$$\|I + A(\xi_i)(t_i - t_{i-1}) - \exp\,(A(\xi_i)(t_i - t_{i-1})\,\|$$

$$\leq \|A(\xi_i)\|^2\,(t_i - t_{i-1})^2 \exp\big(\|A(\xi_i)\|(t_i - t_{i-1})\big)$$

$$\leq \delta(\xi_i)\|A(\xi_i)\|^2(t_i - t_{i-1}) \exp(2\delta(\xi_i)\|A(\xi_i)\|)$$

$$< \frac{\varepsilon(t_i - t_{i-1})}{b-a},$$

and consequently

$$\sum_{i=1}^{m} \big\|I + A(\xi_i)(t_i - t_{i-1}) - exp\big(A(\xi_i)(t_i - t_{i-1})\big)\big\|$$

$$< \sum_{i=1}^{m} \frac{\varepsilon(t_i - t_{i-1})}{b-a} = \varepsilon.$$

Corollary (3.1.12)[3]:

A function $A : [a, b] \to X$ is strongly Kurzweil/McShane product integrable if and only if A is strongly Kurzweil/McShane exponentially product integrable. In this case, the values of the product integral and exponential product integral coincide.

In general, product integrals are much harder to calculate than ordinary integrals. However, if the values of $A: [a, b] \to X$ commute with each other, it turns out that

$$\prod_a^b (I + A(t)dt) = \exp\left(\int_a^b A(t)\, dt\right)$$

whenever at least one of the integrals exists. For Riemann-type integrals, this theorem was proved by Masani. The corresponding statement for Kurzweil-type integrals and $X = R^{n\times n}$ can be . Here we focus on strong Kurzweil and McShane product integrals.

Theorem (3.1.13)[3]:

Consider a function $A : [a, b] \to X$ such that

$$A(t_1)A(t_2) = A(t_2)A(t_1)\ for\ all\ t_1, t_2 \in [a, b]. \qquad (14)$$

If A is strongly Henstock–Kurzweil/McShane integrable, then it is strongly Kurzweil/McShane product in-tegrable. In this case, we have

$$\prod_a^b (I + A(t)dt) = \exp\left(\int_a^b A(t)dt\right)$$

We prove the statement concerning the Kurzweil integrals; the McShane version can be obtained in an analogous way. Clearly, it is enough to prove that A is strongly Kurzweil exponentially product integrable.

Let $W(t) = exp(\int_a^b A(s)ds), t \in [a, b]. If\ [x, y] \subset [a, b]$, then

Proof:-

$W(y)W(x)^{-1} = exp\left(\int_a^y A(s)\, ds\right)exp\left(\int_a^x a\, A(s)\, ds\right)^{-1} =$
$exp\left(\int_a^y A(s)\, ds\right)exp\left(_\int_a^x a\, A(s)\, ds\right)$

$$\exp\left(\int_a^y A(s)\, ds - \int_a^x A(s)\, ds\right) = exp\int_x^y A(s)\, ds$$

because assumption (3.1.15) implies that $\int_a^y A(s)ds$ and $\int_a^x A(s)ds$ commute. (Think of the integrals as limits of integral sums, which obviously commute.)

Let $M = sup_{t\in[a,b]} \left\| \int_a^t A(s)ds \right\|$. Then we have

$$\left\| \int_x^y A(s)ds \right\| = \left\| \int_a^y A(s)ds - \int_a^x A(s)ds \right\| \leq 2M$$

Whhenever $[x, y] \subset [a, b]$.

Take an arbitrary $\varepsilon > 0$. There exists a gauge $\delta : [a, b] \to R^+$ such that

$$\sum_{i=1}^{m} \left\| A(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} A(s)\, ds \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a, b]$. Without loss of generality, we can assume the gauge is chosen so that $2\|A(\xi)\|\delta(\xi) \leq 1$ for all $\xi \in [a, b]$.

Using the estimate (7), we get

$$\sum_{i=1}^{m} \left\| \exp\left((A(\xi_i)(t_i - t_{i-1})\right) W(t_i)W(t_{i-1})^{-1} \right\|$$

$$= \sum_{i=1}^{m} \exp\left(A(\xi_i)(t_i - t_{i-1})\right) - \exp\left(\int_{t_{i-1}}^{t_i} A(s)ds\right)$$

$$\leq \sum_{i=1}^{m} \left\| A(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} A(s)ds \right\| exp\left(max\left(\|A(\xi_i)\|(t_i - t_{i-1}), \right.\right.$$

$$\left.\left. \left\| \int_{t_{i-1}}^{t_i} A(s)ds \right\| \right) \right)$$

$$\sum_{i=1}^{m} \left\| A(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} A(s)ds \right\| exp\,(max(1, 2M))$$

$$< exp\,(max(1, 2M))\varepsilon$$

for every δ-fine tagged partition of $[a, b]$. This proves that A is strongly Kurzweil product integrable and

$$\prod_a^b exp(A(t)\,dt) = W(b)W(a)^{-1} = exp(\int_a^b A(t)\,dt).$$

Theorem (3.1.14)[3]:

Consider a function $A : [a, b] \to X$ such that (14) holds. If A is strongly Kurzweil/McShane product integrable, then it is strongly Kurzweil–Henstock/McShane integrable. In this case, we have

$$\prod_a^b I + A(t)\,dt = exp\left(\int_a^b A(t)\,dt \right).$$

Proof:

We prove the statement concerning the Kurzweil integrals; the McShane counterpart can be ob-tained in an analogous way. We use the fact that strong product integrability implies strong exponential product integrability. Let $W(t) = \prod_a^t exp(A(s)\,ds)$, $t \in [a, b]$.

Denote $M = sup_{t \in [a,b]} \|W(t)^{-1}\|$ Since W is (uniformly) continuous, there exists a $\Delta > 0$ such that $[x, y] \subset [a, b]$ and $\|x - y\| < \Delta$ implies

$$\|W(x) - W(y)\| < \frac{1}{2M}$$

Without loss of generality, we can assume that $b - a < \Delta$ (otherwise, we can split $[a, b]$ into subintervals whose length is smaller than Δ). Now, if $[x, y] \subset [a, b]$, we have

$$\left\| \prod_x^y exp\,(A(s)ds) - I \right\| = \|W(y)W(x)^{-1} - \mathrm{I}\| \leq \|W(y) - W(x)\| \cdot \|W(y)^{-1}\|$$

$$< \frac{1}{2},$$

$$\left\| \left(\prod_x^y exp\,(A(s)ds) \right)^{-1} - I \right\| = \|W(x)W(y)^{-1} - I\|$$

$$\leq \|W(x) - W(y)\| \cdot \|W(y)^{-1}\| < \frac{1}{2}$$

$Let\ F(t) = log(\prod_a^t exp(A(s)\,ds))., then$

$F(y) - F(x) =$

$$log\left(\prod_a^y exp(A(s)\,ds)\right) - log\left(\prod_a^x exp(A(s)\,ds)\right)$$

$$= log\left(\prod_a^y exp(A(s)\,ds)\right) + log\left(\prod_a^x exp(A(s)\,ds)\right)^{-1}$$

$$= log\left(\left(\prod_a^y exp(A(s)\,ds)\right) log\left(\prod_a^x exp(A(s)\,ds)\right)^{-1}\right)$$

$$= log\left(\prod_x^y exp(A(s)\,ds)\right)$$

because assumption (14) implies that $\prod_a^y exp(A(s)\,ds)$ and

$(\prod_a^x exp(A(s)\, ds))^{-1}$ commute. (Think of the product integrals as the limits of products from Definition (3.1.8) these products obviously commute.)

Take an arbitrary $\varepsilon > 0$. There exists a gauge $\delta: [a, b] \to R^+$ such that

$$\sum_{i=1}^{m} \left\| exp\, A(\xi_i)(t_i - t_{i-1}) \; - \prod_{t_{i-1}}^{t_i} exp\, _A(s)\, ds \right\| < \varepsilon$$

for every δ-fine tagged partition of $[a, b]$. Without loss of generality, assume the gauge δ is chosen so that

$$2\, \|A(\xi)\, \|\delta(\xi)\, exp\, (2\|A(\xi)\, \|\delta(\xi)) \; \leq \frac{1}{2}, \qquad \xi \in [a, b],$$

which in turn means that

$$\begin{aligned} &\left\| exp\, \left(A(\xi_{\boldsymbol{i}})(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1})\right) \; - \; I \right\| \\ &\qquad \leq \; \|A(\xi_{\boldsymbol{i}})\|\, (t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1})\, exp\, (\|A(\xi_{\boldsymbol{i}})\|(ti \; - \; ti - 1)) \\ &\qquad < \; _\|A(\xi_{\boldsymbol{i}})\|2\delta(\xi_{\boldsymbol{i}})\, exp\, (\|A(\xi_{\boldsymbol{i}})\|2\delta(\xi_{\boldsymbol{i}})\,) \; \leq \; 12, \end{aligned}$$

and also

$$\|A(\xi_{\boldsymbol{i}})\|(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1}) < \; \|A(\xi_{\boldsymbol{i}})\|\, 2\delta(\xi_{\boldsymbol{i}}) \leq \frac{1}{2} < log2.$$

Using the estimate (8), we get

$$\sum_{i=1}^{m} \|A(\xi_{\boldsymbol{i}})(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1}) \; - \; F(t_i) \; - \; F(t_{\boldsymbol{i}-1})\|$$

$$= \sum_{i=1}^{m} \left\| log\, (exp\, (A(\xi_i)(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1}))) - \; log \left(\prod_{t_{\boldsymbol{i}-1}}^{t_{\boldsymbol{i}}} exp\, _A(s)\, ds \right) \right\|$$

$$\leq \sum_{i=1}^{m} \frac{\left\| exp(A((\xi_i)(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1})) \; - \; \left(\prod_{t_{\boldsymbol{i}-1}}^{t_{\boldsymbol{i}}} exp\, _A(s)\, ds\, \right) \right\|}{1 \; - \; max(\, \left\| exp\left(A(\xi_i)(t_{\boldsymbol{i}} - t_{\boldsymbol{i}-1})\right) - \; I \right\|, \left\| \prod_{t_{\boldsymbol{i}-1}}^{t_{\boldsymbol{i}}} \exp\, (A(s)\, ds) \; - \; I \right\|)}$$

$$\leq 2\sum_{i=1}^{m} exp\big(A(\xi_i)(t_i - t_{i-1})\big) - \prod_{t_{i-1}}^{t_i} \exp\,(A(s)\,ds) < 2\varepsilon,$$

i.e., A is strongly Henstock–Kurzweil integrable and $\int_a^b A(t)dt = F(b) - F(a) = log(\prod_a^b exp(A(s)\,ds))$.

Our next goal is to show that every Bochner integrable function is strongly McShane product integrable. In the following lemma, the symbol μ stands for the Lebesgue measure in R.

Lemma (3.1.15)[3]:.

Assume that $A : [a,b] \to X$ is Bochner integrable. Then for every $\varepsilon > 0$, there is a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{r}\sum_{j=1}^{s} \left\| exp\left(A(t_i)\mu(J_i \cap L_j)\right) - exp(A(s_j)\mu(J_i \cap L_j))\right\| < \varepsilon$$

for each pair of δ-fine free tagged partitions $(t_i, J_i)_{i=1}^{r}$ and $(s_j, L_j)_{j=1}^{s}$.

Proof:-

Let $M = exp(1 + \int_a^b \|A(t)\| dt)$. Take an arbitrary $\varepsilon>0$. Every Bochner integrable function is strongly McShane integrable, and therefore satisfies the so-called condition S^*M i.e., there is a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{r}\sum_{j=1}^{s} \|A(t_i) - A(s_j)\|\, \mu(J_i \cap L_j) < \frac{\varepsilon}{M}$$

for each pair of δ-fine free tagged partitions $(t_i, J_i)_{i=1}^{r}$ and $(s_j, L_j)_{j=1}^{s}$. Since $\|A\|$ is Bochner and therefore also McShane integrable, we can assume that δ is chosen in such a way that

$$\left| \sum_{i=1}^{r} \|A(t_i)\|\, \mu(J_i) - \int_a^b \|A(t)\|\, dt \right| < 1$$

for every δ-fine free tagged partition $(t_i, J_i)_{i=1}^{r}$ of $[a,b]$. Consequently, using the estimate (7), we get

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\left\| exp\left(A(t_i)\mu(J_i \cap L_j)\right) - exp\,(A(s_j)\mu(J_i \cap L_j)\right\|$$

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{s}\| A(t_i) - A(s_j)\|\,\mu(J_i \cap L_j)\exp\,(\max\,(\|A(t_i)\|\,\mu(J_i \cap L_j), \|A(s_j)\|\,\mu(J_i \cap L_j)$$

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{s}\| A(t_i) - A(s_j)\|\,\mu(J_i \cap L_j)\exp\,(\max\,(\sum_{i=1}^{r}\|A(t_i)\|\,\mu(J_i)\ \ , \sum_{j=1}^{s}\|A(s_j)\|\,\mu(L_j)))$$

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{s}\| A(t_i) - A(s_j)\|\,\mu(J_i \cap L_j)\exp\,(1 + \int_a^b \|A(t)\|\,dt) < \varepsilon$$

Theorem (3.1.16)[3]:.

If $A : [a,b] \to X$ is Bochner integrable, then it is strongly McShane product integrable.

Proof:

By Corollary (3.1.12) it is enough to prove that A is strongly McShane exponentially product integrable. From Remark (1.1.10) we already know that A is McShane exponentially product integrable.

Let $W(t) = \prod_a^t exp(A(s)ds), t \in [a,b]$. Denote $M = exp(1 + \int_a^b A(t)dt)$. Take an arbitrary $\varepsilon > 0$ and let $\delta : [a,b] \to R^+$ be the corresponding gauge from Lemma (3.1.15). Since $\|A\|$ is Bochner integrable and therefore also McShanen integrable, we can assume that δ is chosen is such a way that if $(\eta_i, [u_{i-1}, u_i])_{i=1}^{m}$ is a δ-fine free tagged partition of $[a,b]$, then

$$\left|\sum_{i=1}^{m}\|A(\eta_i)\|\,_(u_i - u_{i-1}) - \int_a^b \|A(t)\|\,dt\right| < 1. \qquad (15)$$

Consider an arbitrary δ-fine free tagged partition $(\xi_{\boldsymbol{i}}, [t_{\boldsymbol{i}-1}, t_{\boldsymbol{i}}])_{\boldsymbol{i}=1}^{r}$ of $[a, b]$.

For every $\boldsymbol{i} \in \{1, \ldots, k\}$, A is Bochner integrable and therefore also exponentially McShane product integrable on $[t_{\boldsymbol{i}-1}, t_{\boldsymbol{i}}]$. Hence, there exists a δ-fine free tagged partition $(\xi_j^i, [t_{j-1}^i, t_j^i])_{j=1}^{l_i}$ of $[t_{\boldsymbol{i}-1}, t_{\boldsymbol{i}}]$ such that

$$\left\| \prod_{j=l^i}^{1} exp\ A\ (\xi_j^i\ )\Delta t_j^i - \prod_{t_{\boldsymbol{i}-1}}^{t_{\boldsymbol{i}}} exp\ (A(t)dt) \right\| < \frac{\varepsilon}{k}, \qquad i \in \{1, \ldots, k\},$$

where $\Delta t_j^i = t_j^i - t_{j-1}^i$

Note that the collections $(\xi_{\boldsymbol{i}}, [t_{\boldsymbol{i}-1}, t_j^i])$ and$(\xi_j^i, [t_{j-1}^i, t_j^i])$, where $\boldsymbol{i} \in \{1, \ldots, k\}$ and $\boldsymbol{j} \in \{1, \ldots, \boldsymbol{l^i}\}$, are δ-fine free tagged partitions of $[a, b]$.

Using the identity (5), we obtain the estimate

$$\prod_{j=l^i}^{1} exp\ (A(\xi_i)\Delta t_j^i) - \prod_{j=l^i}^{1} exp\ _A\ (\xi_j^i\ )\Delta t_j^i$$

$$\leq \sum_{j=1}^{l^i} \left(\prod_{p=l^i}^{j+1} exp\ (\|A(\xi_i)\|\ \Delta t_p^i \right) \ \left\| exp\ (A(\xi_i)\Delta t_j^i \exp\left(A(\xi_j^i)\Delta t_j^i\right) \right\|$$

$$\left(\prod_{\mathrm{p=j-1}}^{1} \exp\ (\|\mathrm{A}\ (\xi_{\mathrm{p}}^{\mathrm{i}})\|\Delta \mathrm{t}_{\mathrm{p}}^{\mathrm{i}}\ ) \right)$$

$$\leq\ exp\left(\sum_{p=1}^{l^i} A\ \left(\xi_p^i\right)\Delta t_p^i \right) \exp\left(\sum_{p=1}^{l^i} \|A(\xi_i)\|\ \Delta t_p^i \right)$$

$$\sum_{j=1}^{l^i} \left\| exp\left(A(\xi_i)\Delta t_j^i\right) -\ exp\ (A\ \left(\xi_j^i\right)\ \Delta t_j^i \right\|$$

$$\leq\ exp\left(\sum_{i=1}^{k} \sum_{p=1}^{l^i} \|A(\xi_i)\|\ \Delta t_p^i \right)\ exp\left(\sum_{i=1}^{k} \sum_{p=1}^{l^i} \left(A\ \left(\xi_p^i\right)\Delta t_p^i\right) \right)$$

$$\sum_{j=1}^{l^i} \left\| exp\ (A(\xi_i)\Delta t_j^i) - \exp\ (A\left(\xi_j^i\right)\Delta t_j^i\ ) \right\|$$

$$\leq\ M^2\ \sum_{j=1}^{l^i} \left\| exp\ (A(\xi_i)\Delta t_j^i) - \exp\ (A\left(\xi_j^i\right)\Delta t_j^i\ ) \right\|,$$

where the last inequality is a consequence of (3.1.16). According to the conclusion of Lemma (3.1.15) we get

$$\sum_{i=1}^{k} \left\| \prod_{j=l^i}^{1} exp\ (A(\xi_i)\Delta t_j^i) - \prod_{j=l^i}^{1} exp\ (A\left(\xi_j^i\right)\Delta t_j^i) \right\| \leq\ M^2$$

$$< M^2 \sum_{i=1}^{k} \sum_{j=1}^{l^i} \left\| exp\ (A(\xi_i)\Delta t_j^i) - exp\ (A\left(\xi_j^i\right)\Delta t_j^i) \right\| < M^2 \varepsilon$$

By combining the previous estimate with (3.1.9), we obtain

$$\sum_{i=1}^{k} \| exp\ (A(\xi i)(t_i\ -\ t_{i-1}))\ - W(t_i)W(t_{i-1})^{-1}\ \|$$

$$= \sum_{i=1}^{k} \left\| \prod_{j=l^i}^{1} exp\ (A(\xi_i)\Delta t_j^i) - \prod_{t_{i-1}}^{t_i} exp\ _A(t)\ dt \right\|$$

$$\leq \left\| \sum_{i=1}^{k} \prod_{j=l^i}^{1} exp\ (A(\xi_i)\Delta t_j^i) - \prod_{j=l^i}^{1} exp\ (A\left(\xi_j^i\right)\Delta t_j^i) \right\|$$

$$+ \sum_{i=1}^{k} \left\| \prod_{j=l^i}^{1} exp\ (A\left(\xi_j^i\right)\Delta t_j^i)\ \ - \prod_{t_{i-1}}^{t_i} exp\ (A(t)dt) \right\|\ \ < \varepsilon\ (M^2\ +\ 1),$$

which proves that A is strongly McShane exponentially product integrable.

Section (3.2): Indefinite strong product integrals and functions with countbly many discontinuities

We investigate the properties of the indefinite strong Kurzweil and McShane product integrals. Also, we prove that strong McShane product integrability is equivalent to Bochner integrability. The proofs of Theorems

(3.2.1), (3.2.6), and (3.2.7) are fairly straightforward adaptations of the proofs of Theorems (3.1.2), (3.2.4), and (3.2.5), but we provide them here for reader's convenience.

Theorem (3.2.1)[3]:

Assume that $A : [a,b] \to X$ is strongly Kurzweil product integrable and $W(t) = \prod_a^t (I + A(s)ds)$, $t \in [a,b]$. Then there exists a set $Z \subset [a,b]$ of measure zero with the following property: For every $\varepsilon > 0$ and $t \in [a,b] \backslash Z$, there is a $\delta(t) > 0$ such that

$$\|I + A(t)(y - x) - W(y)W(x)^{-1}\| \leq \varepsilon(y - x) \qquad (17)$$

whenever $[x,y] \subset (t - \delta(t), t + \delta(t)) \cap [a,b]$.

Proof:

Let $Z \subset [a,b]$ be the set of all $t \in [a,b]$ for which (3.1.18) does not hold. For every $t \in Z$, there exist $\eta(t) > 0$ and sequences $\{x_l(t)\}_{l=1}$ and $\{y_l(t)\}_{l=1}$ such that

$$x_l(t) \leq t \leq y_l(t), \lim_{l\to\infty}(y_l(t) - x_l(t)) = 0,$$

$$\|I + A(t)(y_l(t) - x_l(t)) - W(y_l(t))W(x_l(t))^{-1}\| \geq \eta(t)\,(y_l(t) - x_l(t)). \qquad (18)$$

For every $n \in N$, let $Z_n = \{t \in [a,b];\ \eta(t) \geq 1/n\}$. Clearly, $Z = \bigcup_{n=1}^{\infty} Z_n$. Assume that Z has outer Lebesgue measure $\mu^*(Z) > 0$. Hence, there is an $r \in N$ such that $\mu^*(Z_r) > 0$. Consider an arbitrary gauge $\delta : [a,b] \to R^*$. For every $t \in Z_r$, find $l_0(t) \in N$ such that

$$t - \delta(t) < x_l(t) \leq t \leq y_l(t) < t + \delta(t)$$

for all $l \geq l_0(t)$. The collection of intervals $\{[x_l(t), y_l(t)];\ t \in Z_r, l \geq l_0(t)\}$ is a Vitali cover of the set Z_r. By the Vitali covering theorem, it contains a finite subsystem of intervals $([\xi_j, \eta_j])_{j=1}$

for which $\tau_j - \delta(\tau_j) < \xi_j \leq \tau_j \leq \eta_j < \tau_j + \delta(\tau j), \tau_j \in Z_r$,

$$j \in \{1, 2, \ldots, s\},$$

$$\eta_j \leq \xi_j + 1, \quad j \in \{1, 2, \ldots, s - 1\},$$

$$\mu^*\left(Z_r \setminus \bigcup_{j=1}^{s} [\xi_j, \eta_j]\right) < \frac{1}{2}\mu^*(Z_r).$$

Consequently,

$$\sum_{j=1}^{s} (\eta_j - \xi_j) \geq \mu^*\left(Z_r \cap \bigcup_{j=1}^{s} [\xi_j, \eta_j]\right) > \frac{1}{2}\mu^*(Z_r).$$

This inequality together with (18) yields

$$\sum_{j=1}^{s} \|I + A(\tau_j)(\eta_j - \xi_j) - W(\eta_j)W(\xi_j)^{-1}\| \geq \frac{1}{r}\sum_{j=1}^{s}(\eta_j - \xi_j) > \frac{1}{2r}\mu^*(Z_r) \ .$$

Since the expression on the right-hand side is a constant independent on the choice of the gauge δ, we get a contradiction with the assumption that A is strongly Kurzweil product integrable.

Theorem (3.2.1)[3]:

Assume that $A : [a, b] \to X$ is strongly Kurzweil product integrable and $W(t) = \prod_a^t (I + A(s)ds)$, $t \in [a, b]$. Then $W'(t) = A(t)W(t)$ almost everywhere on $[a, b]$.

Proof:

Consider an arbitrary $\varepsilon > 0$. By Theorem (3.2.1) there exists a $Z \subset [a, b]$ of measure zero such that for every $t \in [a, b] \setminus Z$ there is $\delta(t) > 0$ such that

$$\|I + A(t)(y - t) - W(y)W(t)^{-1}\| \leq \varepsilon(y - t)$$

whenever $y \in (t, t + \delta(t))$. Dividing by $(y - t)$, we get

$$A(t) - \frac{W(y) - W(t)}{y - t} W(t)^{-1} \leq \varepsilon,$$

and consequently

$$\left\| A(t)W(t) - \frac{W(y) - W(t)}{y - t} \right\| \leq \varepsilon \|W(t)\|,$$

i.e., the right-sided derivative of W at the point t exists and equals $A(t)W(t)$. A similar reasoning shows that also the left-sided derivative of W at t exists and equals $A(t)W(t)$.

Theorem (3.2.3)[3]:.

Every strongly Kurzweil product integrable function is strongly measurable.

Proof:-

If $A : [a, b] \to X$ is strongly Kurzweil product integrable, then the indefinite Kurzweil product Integral $W(t) = \prod_a^t (I + A(s)ds)$, $t \in [a, b]$, satisfies $W' = AW$ almost everywhere on $[a, b]$. W and W^{-1} are continuous, and therefore strongly measurable. W' is also strongly measurable because it is the derivative of a strongly measurable function (a verification of this fact, which is analogous to the proof for real-valued functions. Now, $A = W'W^{-1}$ is almost everywhere equal to a product of two strongly measurable functions, and therefore A is also strongly measurable.

Example (3.2.4)[3]:

Consider the space X of all bounded real functions defined on $[0, 1]$ equipped with the supre-mum norm. For two functions $f, g \in X, let\ fg \in X$ be the function obtained by pointwise multiplication of f and g. Clearly, X is a unital Banach algebra.

Let $A : [0,1] \to X$ be given by $A(t) = \chi_{[0,t]}$. L.M. Graves that A is nowhere continuous, but Riemann integrable. R. Gordon noted that A is not strongly measurable. From the viewpoint of product integration theory, it is useful to

observe the following facts:

(i) A is Riemann product integrable (because it is Riemann integrable) and therefore also Kurzweil product integrable.

(ii) A is neither Bochner integrable nor strongly Kurzweil/McShane product integrable (because it is not strongly measurable).

(iii) A is McShane product integrable.

function $t \to \chi_{[t,1]}$) and $\int_0^1 A(t)dt = S$, where $S(\tau) = 1-\tau$ for $\tau \in [0, 1]$. For a given let us verify the last claim. First, we show that A is McShane integrable which is concerned with a similar

$\varepsilon > 0$, let $\delta(t) = \varepsilon$ for all $t \in [0, 1]$. Now, consider an arbitrary δ-fine free tagged partition $(\xi_i, [t_{i-1}, t_i])_{i=1}^m$ of [0, 1]. We have

$$\left\| \sum_{i=1}^{m} A(\xi i)(t_i - t_{i-1}) - S \right\|$$

$$= \sup_{\tau \in [0,1]} \left| \sum_{i=1}^{m} A(\xi i)(\tau)(t_i - t_{i-1}) - S(\tau) \right|$$

$$= \sup_{\tau \in [0,1]} \left| \sum_{i;\, \xi_i \geq \tau} (t_i - t_{i-1}) - (1 - \tau) \right|$$

Note that$\xi_i \geq \tau$ impies $t_{i-1} > \xi_i - \varepsilon \geq \tau - \varepsilon$, and therefore

$$\sum_{i;\, \xi_i \geq \tau} (t_i - t_{i-1}) < 1 - (\tau - \varepsilon) = 1 - \tau + \varepsilon.$$

A lower bound for the same sum is obtained by finding an upper bound for $\sum_{i;\, \xi_i \geq \tau}(t_i - t_{i-1})$.Now $\xi_i \geq \tau$ implies $t_{i-1} > \xi_i - \varepsilon \geq \tau + \varepsilon$.

It follows that $\sum_{i;\, \xi_i \geq \tau}(t_i - t_{i-1}) < \tau + \varepsilon$, and consequently

$$\sum_{i;\xi_i \geq \tau} (t_i - t_{i-1}) = 1 - \sum_{i;\, \xi_i < \tau} (t_i - t_{i-1}) > 1 - (\tau + \varepsilon) = 1 - \tau - \varepsilon.$$

This proves

$$\sum_{i=1}^{m} A(\xi_i)(t_i - t_{i-1}) - S = \sup_{\tau\in[0,1]} \sum_{i;\xi_i\geq\tau} (t_i - t_{i-1}) - (1-\tau) \leq \varepsilon, (19)$$

or $\int_0^1 A(t)dt = S$.

Next, we claim that A is McShane product integrable with $\prod_0^1 (I + A(t)dt) = expS$. Take an arbitrary $\varepsilon > 0$ and consider the corresponding gauge $\delta : [0,1] \to R^*$ from Lemma (3.1.10) Without loss of generality, assume that $\delta(\xi) \leq \varepsilon$ for all $\xi \in [0,1]$. For every δ-fine free tagged partition $(\xi_i, [t_{i-1}, t_i])_{i=1}^{m}$ of [0, 1], we get

$$\left\| \prod_{i=m}^{1} (I + A(\xi_i)(t_i - t_{i-1})) - exp\ S \right\|$$

$$\leq \left\| \prod_{i=m}^{1} (I + A(\xi_i)(t_i - t_{i-1})) - \prod_{i=m}^{1} exp\ (A(\xi_i)(t_i - t_{i-1})) \right\|$$

$$+ \left\| \prod_{i=m}^{1} exp\ (A(\xi_i)(t_i - t_{i-1})) - exp\ S \right\|$$

An estimate for the first term can be obtained using identity (5):

$$\prod_{i=m}^{1} (I + A(\xi_i)(t_i - t_{i-1})) - \prod_{i=m}^{1} exp(A(\xi_i)(t_i - t_{i-1}))$$

$$\leq \sum_{i=1}^{m} \left(\prod_{j=m}^{i+1} 1 + \|A(\xi_j)\|(t_j - t_{j-1}) \right) \|I + A(\xi_i)(t_i - t_{i-1})$$

$$- exp\ (A(\xi_i)(t_i - t_{i-1}))\|$$

$$\cdot \prod_{j=i=1}^{1} exp\ (\|A(\xi_j)\|(t_j - t_{j-1})\ )$$

$$\leq \; exp \sum_{j=1}^{m} \|A(\xi_j)\|(t_j$$

$$- t_{j-1}) \; \sum_{i=1} \|I \; + \; A(\xi_i)(t_i - t_{i-1})$$

$$- \; exp \; (A(\xi_i)(t_i - t_{i-1}))\| \; \leq \; (exp \; 1)\varepsilon.$$

To estimate the second term, we use inequality (7) together with (19), as well as the fact that X is a commutative algebra:

$$\prod_{i=m}^{1} exp(\; I \; + \; A(\xi_i)(t_i - t_{i-1})) - \exp S$$

$$= \; \left\| exp\left(\sum_{i=1}^{m} A(\xi_i)(t_i - t_{i-1})\right) - \; exp \; S \right\|$$

$$\leq \sum_{i=1}^{m} \|A(\xi_i)(t_i - t_{i-1}) - S \; \| \; exp\left(max\left(\left\| \sum_{i=1}^{m} A(\xi_i)(t_i - t_{i-1}) \right\|, \|S\| \right)\right)$$

$$\leq \; \varepsilon \; exp \; 1$$

This confirms that the McShane product integral $\prod_0^1 (I + A(t)dt)$ exists and equals $\exp S$.

Definition (3.2.5)[3]:

A function $W : [a, b] \rightarrow X$ is said to satisfy the strong Luzin condition on $[a, b]$ if for every $\varepsilon > 0$ and $Z \subset [a, b]$ of measure zero, there exists a function $\delta : Z \rightarrow R^*$ such that

$$\sum_{j=1}^{m} \|W(v_j) \; - W(u_j)\| \; < \; \varepsilon$$

for every collection of point–interval pairs $(\tau_j, [u_j, v_j])_{j=1}^{m}$ with $[u_j, v_j] \subset [a, b], \tau_j \in Z$, and $\tau_j - \delta(\tau j) < u_j \leq \tau_j \leq v_j < \tau_j + \delta(\tau_j)$ for all $j \in \{1,2, \ldots, m\}$.

We remark that every function which satisfies the strong Luzin condition is necessarily continuous.

Theorem (3.2.6)[3]:

If $A : [a,b] \to X$ *is strongly Kurzweil product integrable, then the indefinite product integral* $W(t) = \prod_a^t (I + A(s)ds), t \in [a,b]$, *satisfies the strong Luzin condition on* $[a,b]$.

Proof:-

Let $M = sup_{t \in [a,b]}$ ë$W(t)$ë. Take arbitrary $\varepsilon > 0$ and $Z \subset [a,b]$ of measure zero. For every $i \in N, let\, Z_i = \{t \in Z;\ i - 1 \leq$ ë$A(t)$ë $< i\}$. Since Z_i has measure zero, it can be enclosed in an open set of an arbitrarily small measure. Hence, there exists a function $\delta_i : Z_i \to R^+$ such that

$$\mu\left(\bigcup_{t \in Z_i} (t - \delta_i(t), t + \delta_i(t))\right) \leq \frac{\varepsilon}{i2^i}$$

Using the strong product integrability of A, we find a gauge $\delta : [a,b] \to R^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon \qquad (20)$$

for every δ-fine tagged partition of $[a,b]$. Without loss of generality, we can assume that $\delta(t) \leq \delta_i(t)$ whenever $t \in Z_i$. Take an arbitrary collection of point–interval pairs $(\tau_j, [u_j, v_j])_{j=1}^{m}$ satisfying $[u_j, v_j] \subset [a,b], \tau_j \in Z$, and $\tau_j - \delta(\tau_j) < u_j \leq \tau_j \leq v_j < \tau_j + \delta(\tau_j)$ for all $j \in \{1,2,\ldots,m\}$. Then

$$\sum_{j=1}^{m} \|A(\tau_j)\| (v_j - u_j) = \sum_{i=1} \sum_{j;\, \tau_j \in Z_i} \|A(\tau_j)\| (v_j - u_j) \leq \sum_{i=1}^{\infty} i \frac{\varepsilon}{i2^i} = \varepsilon.$$

It follows from (20) that

$$\sum_{j=1}^{m}\|I - W(v_j)W(u_j)^{-1}\|$$

$$\leq \sum_{j=1}^{m}\|I + A(\tau_j)(v_j - u_j) - W(v_j)W(u_j)^{-1}\|$$

$$+ \sum_{j=1}^{m}\|A(\tau_j)\|(v_j - u_j) < 2\varepsilon,$$

and therefore

$$\sum_{j=1}^{m}\|W(v_j) - W(u_j)\| \leq \sum_{j=1}^{m}\|W(v_j)W(u_j)^{-1} - I\| \cdot \|W(u_j)\| \leq 2M\varepsilon$$

which confirms that W satisfies the strong Luzin condition.

Theorem (3.2.7)[3]:

Consider a function $A : [a,b] \to X$. Assume there is a function $W : [a,b] \to X$ which satisfies the strong Luzin condition, $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ for every $t \in [a,b]\setminus Z$, where $\mu(Z) = 0$. Then A is strongly Kurzweil product integrable and $\prod_a^b(I + A(t)dt) = W(b)W(a)^{-1}$.

Proof:-

The strong Luzin condition implies that W is continuous. Hence, W^{-1} is continuous as well. Let $M = sup_{t\in[a,b]} W(t)^{-1}$. Take an arbitrary $\varepsilon > 0$.

For every $t \in [a,b]\setminus Z$, there exists a $\Delta > 0$ such that

$$\|W(y) - W(t) - W'(t)(y - t\|) \leq \varepsilon(y - t), y \in [t, t+\Delta) \cap [a,b],$$

$$\|W(t) - W(x) - W'(t)(t - x)\| \leq \varepsilon(t - x), x \in (t - \Delta, t] \cap [a,b].$$

Consequently,

$$\|W(y) - W(x) - W'(t)(y - x)\|$$
$$\leq \|W(y) - W(t) - W'(t)(y - t)\|$$
$$+ \|W(t) - W(x) - W'(t)(t - x)\| \leq \varepsilon(y - x)$$

whenever $x, y \in [a, b]$ and $t - \Delta < x \leq t \leq y < t + \Delta$. This means that

$$\lim_{\substack{x,y\to t,\\ x\neq y, t\in[x,y]}} \frac{W(y) - W(x)}{y - x} = W'(t),$$

and thus

$$\lim_{\substack{x,y\to t,\\ x\neq y, t\in[x,y]}} \frac{W(y)W(x)^{-1} - I}{y - x} = \lim_{\substack{x,y\to t,\\ x\neq y, t\in[x,y]}} \left(\frac{W(y) - W(x)}{y - x} W(x)^{-1}\right)$$
$$= W'(t)W(t)^{-1} = A(t).$$

It follows that for every $t \in [a, b] \setminus Z$ and $\varepsilon > 0$, there exists a $\delta(t) > 0$ such that

$$\left\|\frac{W(y)W(x)^{-1} - I}{y - x} - A(t)\right\| < \frac{\varepsilon}{b - a}$$

whenever$x, y \in [a, b], t - \delta(t) < x \leq t \leq y < t + \delta(t)$, and $x < y$. Hence, we also have

$$\|W(y)W(x)^{-1} - I - A(t)(y - x)\| < \frac{\varepsilon(y - x)}{b - a}$$

if $x, y \in [a, b]$ and $t - \delta(t) < x \leq t \leq y < t + \delta(t)$.

Next, we extend the domain of δ to the whole interval $[a, b]$ by taking the function $\delta : Z \to R^+$ from Definition (3.2.5) Moreover, using the same argument as in the proof of Theorem (3.2.6) we can assume that δ is chosen in such a way that

$$\sum_{j=1}^{l} \|A(\tau_j)\| (v_j - u_j) < \varepsilon$$

for every collection of point–interval pairs $(\tau_j, [u_j, v_j])_{j=1}^{i}$ satisfying $[u_j, v_j] \subset [a, b], \tau_j \in Z$, and $\tau_j - \delta(\tau_j) < u_j \leq \tau_j \leq v_j < \tau_j + \delta(\tau_j)$ for all $j \in \{1,2,\ldots,l\}$.

Now, for every δ-fine tagged partition of $[a, b]$, we obtain

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\|$$

$$\leq \sum_{i;\, \xi_i \in Z} A(\xi_i)(t_i - t_{i-1}) + \sum_{i;\, \xi_i \in Z} \|W(t_i) - W(t_{i-1})\| \cdot \|W(t_{i-1})^{-1}\|$$

$$+ \sum_{i;\, \xi_i \in [a,b] \setminus Z} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \varepsilon + \varepsilon M$$

$$+ \sum_{i;\, \xi_i \in [a,b] \setminus Z} \frac{\varepsilon(t_i - t_{i-1})}{b - a} \leq \varepsilon(M + 2) .$$

Hence, A is strongly Kurzweil product integrable and $\prod_a^b (I + A(t)dt) = W(b)W(a)^{-1}$.

As an immediate consequence of Theorems (3.2.2), (3.2.6) and (3.2.7), we obtain the following characterization of strongly Kurzweil product integrable functions.

Corollary (3.2.8)[3]:

For every function $A : [a, b] \to X$, the following conditions are equivalent:

i. A is strongly Kurzweil product integrable.
ii. There is a function $W : [a, b] \to X$ which satisfies the strong Luzin condition, $W(t)^{-1}$ exists for all $t \in [a, b]$, and $W'(t) = A(t)W(t)$ for every $t \in [a, b] \setminus Z$, where $\mu(Z) = 0$.

Remark (3.2.9)[3]:

From Corollary (3.2.8) we deduce that the strong Kurzweil product integral has the following properties:

i. In Remark (3.1.7) we already noticed that strong product integrability on $[a,b]$ implies strong product integrability on every subinterval. Now, consider an arbitrary $c \in (a,b)$. If $A : [a,b] \to X$ is strongly Kurzweil product integrable on $[a,c]$ and $[c,b]$, then A is strongly Kurzweil product integrable on $[a,b]$.

To see this, it is enough to let

$$W(t) = \begin{cases} \prod_a^t (I + A(s)ds), & t \in [a,c], \\ \prod_c^t (I + A(s)\,ds) \prod_a^c (I + A(s)\,ds), & t \in (c,b], \end{cases}$$

and apply Corollary (3.2.8)

ii. If $A_1, A_2 : [a,b] \to X$ are such that $A_1 = A_2$ almost everywhere and $A1$ is strongly Kurzweil product integrable, then $A2$ is strongly Kurzweil product integrable and $\prod_a^b (I + A_1(t)dt) = \prod_a^b (I + A_1(t)dt)$.

iii. We have the following Hake-type theorem: Assume that the strong product integral $\prod_a^t (I + A(s)ds)$ exists for all $t \in [a,b)$. If $lim_{t \to b-} \prod_a^t (I + A(s)ds)$ exists and is invertible, then $\prod_a^b (I + A(s)ds)$ exists as well and is equal to the limit.

In practice, it can be difficult to verify whether a given function W satisfies the strong Luzin condition. The next theorem shows that if the relation $W'(t) = A(t)W(t)$ holds on $[a,b] \setminus Z$ with a countable set Z, then it is enough to assume that W is continuous.

Theorem (3.1.10)[3]:

Consider a function $A : [a,b] \to X$. Assume there is a continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ for all $t \in [a,b] \setminus Z$, where Z is countable. Then A is strongly Kurzweil product integrable and $\prod_a^b (I + A(t)\,dt) = W(b)W(a)^{-1}$.

Proof:-

Denote $Z = \{z_1, z_2, \ldots\}$. Take an arbitrary $\varepsilon > 0$. For every $n \in N$, let $\delta(zn) > 0$ be such that

$$\|A(zn)\|\,(y - x) < \frac{\varepsilon}{2n}$$

and

$$\|W(y)W(x)^{-1} - I\| < \frac{\varepsilon}{2n}$$

for every interval $[x, y] \subset (zn - \delta(zn), zn + \delta(zn)) \cap [a, b]$. (Note that $(x, y) \to W(y)W(x)^{-1}$ is a product of two continuous functions, and therefore a continuous function on $[a, b] \times [a, b]$, which equals $\boldsymbol{I}$ *when* $x = y$.) Next, consider an arbitrary $t \in [a, b] \backslash Z$. As in the proof of the previous theorem, there exists a $\delta(t) > 0$

such that

$$\|W(y)W(x)^{-1} - I - A(t)(y - x)\| < \frac{\varepsilon(y - x)}{b - a}$$

whenever $[x, y] \subset (t - \delta(t), t + \delta(t)) \cap [a, b]$.

The function δ is now defined on $[a, b]$. For an arbitrary δ-fine tagged partition of $[a, b]$, we get

$$\sum_{i=1}^{m} \|W(t_i)W(t_{i-1})^{-1} - I - A(\xi_i)(t_i - t_{i-1})\|$$

$$= \sum_{\substack{i \in \{1,\ldots,m\}, \\ \xi_i \notin Z}} \|W(t_i)W(t_{i-1})^{-1} - I - A(\xi_i)(t_i - t_{i-1})\|$$

$$+ \sum_{\substack{i \in \{1,\ldots,m\}, \\ \xi_i \in Z}} \|W(t_i)W(t_{i-1})^{-1} - I - A(\xi_i)(t_i - t_{i-1})\|$$

$$< \varepsilon + \sum_{\substack{i\in\{1,\dots,m\},\\ \xi_i\in Z}} (\|W(t_i)W(t_{i-1})^{-1} - I\| + \|A(\xi_i)\|(t_i - t_{i-1}))$$

$$\leq \varepsilon + 2\sum_{n=1}^{\infty} \frac{\varepsilon}{2n} = 3\varepsilon.$$

Note that both W and W^{-1} are continuous, and therefore bounded. Hence, A is strongly Kurzweil product integrable and$\prod_a^b(I + A(t)dt) = W(b)W(a)^{-1}$.

Example (3.2.11)[3]:

If $A(t) = A$ for all $t \in [a, b]$, then the strong Kurzweil product integral $\prod_a^b(I + A(t)dt)$ exists and equals $exp((b - a)A)$; to see this, it is enough to apply Theorem (3.2.10) with $W(t) = exp((t - a)A)$. According to Remark (3.2.9) the same result holds if $A(t) = A$ almost everywhere on $[a, b]$.

We now turn to the properties of the strong McShane product integrals. The proof of the following theorem is a simple adaptation of the proof.

Theorem (3.2.12)[3]:

If A is strongly McShane product integrable, then the indefinite product integral $W(t) = \prod_a^t(I + A(s)ds)$, $t \in [a, b]$, is absolutely continuous on $[a, b]$.

Proof:-

Let $M = sup_{t\in[a,b]} \|W(t)\|$. Take an arbitrary $\varepsilon > 0$. There is a gauge $\delta : [a, b] \to R^+$ such that

$$\sum_{i=1}^{m} \|I + A(\xi_i)(t_i - t_{i-1}) - W(t_i)W(t_{i-1})^{-1}\| < \frac{\varepsilon}{2M}$$

for every δ-fine free tagged partition of $[a, b]$. We fix one of these δ-fine free tagged partitions and let

$$\sigma = \frac{\varepsilon}{2M(max_{i=1,\dots,m}\|A(\xi_i)\| + 1)}.$$

Consider an arbitrary collection of non-overlapping intervals $([u_j, v_j])_{j=1}^{r}$in [a, b] in [a,b], where $\sum_{j=1}^{r}(v_j - u_j) < \sigma$.. By subdividing the intervals $[u_j, v_j]$ if necessary, it can be assumed that for every $j \in \{1,\dots,r\}$, we have $[u_j, v_j] \subset [\, t_{i-1}, t_i]$for a certain $i \in \{1,\dots,m\}$; let $\tau_j = \xi_i$. Then

$$\sum_{j=1}^{r}\| W(v_j) - W(u_j)\| \leq \sum_{j=1}^{r}\|I - W(v_j)W^{-1}(u_j)\| \cdot \|W(u_j)\|$$

$$\leq M\sum_{j=1}^{r} \| I + A(\tau_j)(v_j - u_j) - W(v_j)W^{-1}(u_j)\|$$

$$+ M \max_{i=1,\dots,q} \|A(\xi_i)\| \sum_{j=1}^{r}(v_j - u_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves that W is absolutely continuous.

Lemma (3.2.13)[3]:

Consider a function $A : [a,b] \to X$ and suppose there is an absolutely continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W\acute{I}(t) = A(t)W(t)$ almost everywhere on $[a,b]$. Then A is Bochner integrable.

Proof:

W is continuous, and therefore strongly measurable. Consequently, its derivative W', which is defined almost everywhere on $[a, b]$ and can be extended to $[a, b]$ in an arbitrary way, is also strongly measurable. Now, $A = W'W^{-1}$ is almost everywhere equal to a product of two strongly measurable functions, and therefore A is also strongly measurable. Next, note that AW is Bochner integrable (since it has an absolutely continuous primitive W). Hence, $\|AW\|$ is Lebesgue integrable. Because $W^{-1}$is continuous, it follows that $\|W^{-1}\|$ is bounded and measurable on $[a, b]$. Consequently, $\|A\| = \|AWW^{-1}\| \leq$

$\|AW\| \cdot \|W^{-1}\|$ is Lebesgue integrable, which in turn means that A is Bochner integrable.

Theorem (3.2.14)[3]:

For every function $A : [a, b] \to X$, the following conditions are equivalent:

i. A is strongly McShane product integrable.
ii. There is an absolutely continuous function $W : [a, b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a, b]$, and $W'(t) = A(t)W(t)$ for every $t \in [a, b] \setminus Z$, where $\mu(Z) = 0$.
iii. A is Bochner integrable.
iv. A is strongly measurable and $\|A\|$ is Kurzweil product integrable.

Proof:

The implication (i)⇒ (ii) follows from Theorems (3.2.12) and (3.2.13), implication (ii)⇒(iii) from Lemma (3.2.13), and implication (iii) ⇒ (i) from Theorem (1.1.16). Let us prove (iv)⇒(iii): Because $\|A\|$ is real-valued and Kurzweil product integrable, it is strongly Kurzweil product integrable. By Theorem (3.1.14), $\|A\|$ is Henstock–Kurzweil integrable. For nonnegative functions, Henstock–Kurzweil integrability implies Lebesgue integrability. Taking into account that A is strongly measurable, we deduce that A is Bochner integrable.

Finally, we prove the implication (iii) ⇒ (iv): Bochner integrability of A implies that A is strongly measurable, and also that $\|A\|$ is Lebesgue integrable, which in turn means that $\|A\|$ is Kurzweil product integrable.

Example (3.2.15)[3]:

If $A(t) = A$ almost everywhere on $[a, b]$, we know from Example (3.2.11) that the strong Kurzweil product integral $\prod_a^b (I + A(t)\, dt)$ exists and equals $\exp((b - a)A)$. For the same reason, the Kurzweil product integral $\prod_a^b (I + \|A(t)\| dt)$ exists and equals $exp((b - a)\|A\|)$. Hence, by Theorem (3.2.14), the strong

McShane product integral $\prod_a^b (I + A(t)\, dt)$exists as well.

A simple consequence is that for step functions with finitely many steps, the strong Kurzweil and strong McShane product integrals always exist and are easy to calculate. Indeed, if there is a partition $a = t_0 < t_1 < \cdots < t_m = b$ and $A(t) = A_i \in X$ for all $t \in (t_{i-1}, t_i)$, then

$$\prod_a^b (I + A(t)\, dt) = \prod_{i=m}^{1} \prod_{t_{i-1}}^{t_i} (I + A(t)\, dt) = \prod_{i=m}^{1} exp\, A_i(t_i - t_{i-1}) \ .$$

Our next goal is to obtain necessary and sufficient conditions for product integrability of functions with countably many discontinuities. This class includes every right regulated or left regulated function because these functions have only countably many discontinuities.

Theorem (3.2.16)[3]:

If $A : [a, b] \to X$ has countably many discontinuities, then the following conditions are equivalent:

i. A is Riemann product integrable.
ii. A is bounded.

If any of these conditions is satisfied, there is a Lipschitz-continuous function $W : [a, b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a, b]$, and $W'(t) = A(t)W(t)$ for all $t \in [a, b] \setminus Z$, where $Z \subset [a, b]$ is countable.

Proof:-

Recall that A is Riemann product integrable if and only if A is Riemann integrable. The implication (i) ⇒ (ii) follows from the fact that every Riemann integrable function is bounded. On the other hand, every bounded function which is almost everywhere continuous is Riemann integrable; this verifies the implication (ii) ⇒ (i). To prove the final statement, let

$$W(t) = \prod_{a}^{t}(I + A(s)\,ds)\ , \qquad t \in [a,b].$$

Then

$$W(t) = I + \int_{a}^{t} A(s)W(s)\,ds \quad , \qquad t \in [a,b],$$

where the integral on the right-hand side is the Riemann integral. The set Z of all discontinuities of A is countable, and we have $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$. Both A and W are bounded; let $L = sup_{t\in[a,b]} \|A(t)W(t)\|$. Consequently,

$$\|W(y) - W(x)\| = \left\|\int_{x}^{y} A(s)W(s)\,ds\right\| \leq L|y - x|$$

whenever $x, y \in [a,b]$; this proves that W is Lipschitz-continuous.

Theorem (3.2.17)[3]:

Assume that $A : [a,b] \to X$ has countably many discontinuities and is Kurzweil product integrable. Let

$$W(t) = \prod_{a}^{t}(I + A(s)\,ds) \quad , \qquad t \in [a,b].$$

Then $W^*(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$, where $Z \subset [a,b]$ is countable.

Consider an arbitrary $t \in (a,b)$ such that A is continuous at t. There exists a $\delta > 0$ such that A is bounded on $[t - \delta, t + \delta] \subset [a,b]$. Consequently, A is Riemann product integrable on $[t - \delta, t + \delta]$. Also, we have

$$W(s) = \prod_{t-\delta}^{s}(I + A(u)\,du) \prod_{a}^{t-\delta}(I + A(u)\,du) \quad , s \in [t - \delta, t + \delta].$$

The first product integral is understood in Riemann's sense; hence, at every

point s where A is continuous, $W'(s)$ exists and equals .

$$W'(s) = A(s)\prod_{t-\delta}^{s}(I + A(u)\,du)\prod_{a}^{t-\delta}(I + A(u)\,du).$$

In particular, for $s = t$ we get

$$W'(t) = A(t)\prod_{t-\delta}^{t}(I + A(u)\,du)\prod_{a}^{t-\delta}(I + A(u)\,du) = A(t)W(t).$$

This proves that $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$, where $Z \subset [a,b]$ is countable.

Theorem (3.2.18)[3]:

If $A : [a,b] \to X$ has countably many discontinuities, then the following conditions are equivalent:

i. A is strongly Kurzweil product integrable.
ii. A is Kurzweil product integrable.
iii. There is a continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, W is dif-ferentiable on $[a,b]$ with the possible exception of a countable set Z, and $W'(t) = A(t)W(t)$ for all $t \in [a,b]\backslash Z$.

Proof:-

The implication (i) ⇒ (ii) follows from Theorem (3.1.6), the implication (ii) ⇒ (iii) from Theorem (3.2.17), and (iii) ⇒ (i) is a consequence of Theorem (3.2.10).

Theorem (3.2.19)[3]:

If $A : [a,b] \to X$ has countably many discontinuities, then the following conditions are equivalent:

i. A is Bochner integrable.
ii. A is strongly McShane product integrable.

iii. There is an absolutely continuous function $W : [a,b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a,b]$, and $W'(t) = A(t)W(t)$ for all $t \in [a,b] \setminus Z$, where $Z \subset [a,b]$ is countable.

Proof:-

The equivalence (i) ⇔ (ii) follows from Theorem (3.2.14).

If A is strongly McShane product integrable, consider the indefinite McShane integral

$$W(t) = \prod_{a}^{t} (I + A(s)\, ds) \quad , t \in [a,b].$$

It follows from Theorem (3.2.7) that $W' = AW$ on $[a,b] \setminus Z$, where Z is countable. Also, W is absolutely continuous by Theorem (2.2.12); this proves the implication (ii) ⇒ (iii). Finally, the implication (iii)⇒(i) follows from Lemma (3.2.13).

Chapter 4

Intersection Properties of Balls in spaces

For instance we show that if $Z \subseteq Y \subseteq X$, where Z is a finite co-dimensional subspace of X which is strongly proximinal in Y and Y is an M-ideal in X, then Z is strongly proximinal in X. Towards this, we show that a finite co- dimensional subspace Y of X is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly in X^{**}. We also show that in an abstract L_1–space the notion of strongly subdifferentiable point and quasi-polyhedral point coincide. We also give an example to show that M-ideals need not be ball proximinal. Moreover, we show that in an L_1-predua space, M-ideals are ball proximinal.

Section (4.1) :Strong proximinality inBanach spaces

We consider only Banach spaces over the real field R and all subspaces we consider are assumed to be closed. For a Banach space X; B_X, S_X and $B[x, r]$ denote the closed unit ball, the unit sphere and the closed ball with center at x and radius r respectively. We consider every Banach space X, under the canonical embedding, as a subspace of X^{**}.

Let K be a non-empty closed subset of a Banach space X. For $x \in X$, let $d(x, K) = inf\{\|x - k\| : k \in K\}$ and $P_K(x) = \{k \in K : d(x, K) = \|x - k\|\}$. The set K is said to be proximinal in X if $P_K(x) \neq \emptyset$ for all $x \in X$. A subspace Y of X is said to be ball proximinal in X if for every $x \in X, P_{B_Y}(x) = \emptyset$.

Godefroy and Indumathi introduced a stronger version of proximinality called 'strong proximi-nality'

Definition (4.1.1)[4]:

A proximinal subspace Y of a Banach space X is said to be *strongly proximinal* in X if for every $x \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that $P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y$, where $P_Y(x, \delta) = \{y \in Y :$

$\|x - y\| < d(x,Y) + \delta\}$.

Franchetti and Payá introduced the notion of strong subdifferentiability in Banach spaces which in turn characterizes strongly proximinal hyperplanes.

Definition (4.1.2)[4]:

The norm of a Banach space X is said to be *strongly subdifferentiable* (in short SSD) at $x \in X$ if the one sided limit

$$d^{+}(x)(y) := \lim_{t \to 0^{+}} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $y \in B_X$. In this case, x is said to be an SSD point of X. If each $x \in S_X$ is an SSD-point of X, then the norm of X is said to be SSD.

The following result by Godefroy and Indumathi connects SSD-points with strongly proximinal subspaces of co-dimension one.

Theorem (4.1.3)[4]:

Let X be a Banach space. Then, for $f \in X^*$, ker (f) is strongly proximinal in X if and only if f is an SSD-point of X^*.

In the case of finite co-dimensional strongly proximinal subspaces, we recall the following result.

Theorem (4.1.4)[4]:

Let Y be a finite co-dimensional subspace of a Banach space X. If Y is strongly proximinal in X, then $Y^{\perp}$ is contained in the set of all SSD-points of X^*.

The following notion of a quasi-polyhedral point, Amir and Deutsch, is stronger than the notion of an SSD-point.

Definition (4.1.5)[4]:

A vector x in a Banach space X is said to be a *quasi-polyhedral* (in short QP)

point of X if there exists a $\delta > 0$ such that $J_{X^*}(z) \subseteq J_{X^*}(x)$ for $\|z - x\| < \delta$ and $\|z\| = \|x\|$, where $J_{X^*}(x) = \{f \in B_{X^*}: f(x) = \|x\|\}$.

Godefroy and Indumathi proved that a QP-point is also an SSD-point but the converse need not be true.

Theorem (4.1.6)[4]:

Let Y be a finite co-dimensional subspace of a Banach space X such that $Y^{\perp}$ is contained in the set of all QP-points of X^*. Then Y is strongly proximinal in X.

We now recall the notion of an M-ideal in a Banach space which is stronger than proximinality (in fact, stronger than strong proximinality).

Definition (4.1.7)[4]:

Let X be a Banach space.

i. A linear projection P on X is said to be an *M-projection* (L-projection) if $\|x\| = max\{\|Px\|, \|x - Px\|\}$ ($\|x\| = \|Px\| + \|x - Px\|$) for all $x \in X$. A function P: $X \to X$ is said to be a semi L-projection if $P^2 = P, P(\lambda x + P(z)) = \lambda P(x) + P(z)$ for all $\lambda \in R$, $x, z \in X$ and $\|x\| = \|P(x)\| + \|x - P(x)\|$ for all $x \in X$.
ii. A subspace Y of X is said to be an *M-summand* (*L-summand*) in X if it is the range of an M-pro-jection (L-projection). A subspace Y of X is said to be a semi *L-summand* if it is the range of a semi L-projection.
iii. A subspace Y of X is said to be an *M-ideal* (*semi M-ideal*) in X if $Y^{\perp}$ is an *L-summand* (semi L-sum-mand) in X^*.
iv. A subspace Y of X is said to be an *ideal* in X if $Y^{\perp}$ is the kernel of a norm one projection on X^*.

It is well-known that each Banach space is an ideal in its bidual.

We next recall some of the intersection properties of balls which are closely related to the proximinality properties.

Definition (4.1.8)[4]:

(a) Let $n \in \mathbb{N}$. A subspace Y of a Banach space X is said to have *the* (*strong*) *n-ball property* if, given n closed balls $\{B[a_i, r_i]\}_{i=1}^{n}$ in X such that $\bigcap_{i=1}^{n} B[a_i, r_i] \neq \emptyset$ and $Y \cap B[a_i, r_i] = \emptyset$ for all $i = 1, \ldots, n$, then $Y \cap (\bigcap_{i=1}^{n} B[a_i, r_i + \varepsilon]) \neq \emptyset$ for all $(\varepsilon \geq 0)$ $\varepsilon > 0$.

(b) A subspace Y of a Banach space X is said to have *the* (*strong*) $1\frac{1}{2}$- *ball property* if the conditions $x \in X$, $y \in Y$, $Y \cap B[x, r] \neq \emptyset$ and $\|x - y\| \leq r + s$ $(r, s > 0)$ imply that $Y \cap B[x, r + \varepsilon] \cap B[y, s + \varepsilon] \neq \emptyset$ for all $(\varepsilon \geq 0)$ $\varepsilon > 0$.

One of the interesting problems in approximation theory is the transitivity of various degrees of proximi-nality and intersection properties of balls. Precisely, let (P) be any one of the properties proximinality, strong proximinality, $1\frac{1}{2}$-*ball* property or 2-ball property and let Y and Z be subspaces of X with $Z \subseteq Y \subseteq X$ such that Z has property (P) in Y and Y has property (P) in X. Then is it necessary that Z has property (P) in X? The motivation for the study of transitivity problem where Pollul established the transitivity of proximinality for finite co-dimensional subspaces of c_0. Dutta and Narayana proved the transitivity of strong proximinality for finite co-dimensional subspaces of $C(K)$, and Payá and Yost proved the transitivity of 2-ball property. More results regarding the transitivity problem for the property (P).

On the other hand, it is also known that most of the properties listed above as (P), in general, are not transitive.

Motivated by these, since each M-ideal satisfies property (P), our main theme in this section is to discuss the following problem, which is a variation of the above mentioned transitivity problem.

Problem (4.1.9)[4]:

Let X, Y, Z be Banach spaces such that $Z \subseteq Y \subseteq X$ and Y be an M-ideal

in X. If (P) is a property which is shared by all M-ideals and if Z has property (P) in Y, does it follow that Z has property (P) in X?

The solution to Problem (4.1.9) is known to be positive when property (P) is the n-ball property but the problem is still open when property (P) is strong proximinality.

we prove that Problem (4.1.9) has an affirmative answer when (P) is strong proximinality and Z is of finite co-dimension in X. In order to prove this, we first prove that a finite co-dimensional proximinal subspace Y of a Banach space X is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

we also consider the following problem.

For an SSD-point fof X^*, there always exists a Hahn–Banach extension of f to X^{**} which is an SSD-point of X^{***}, namely the canonical image of f in X^{***}, But it is not known whether each Hahn–Banach extension of f to X^{**} is again an SSD-point of X^{***}. Coming to a more general set up, we consider the following problem.

Problem (4.1.10)[4]:

If Y is a subspace of a Banach space X and f $\in Y^*$ is an SSD-point of Y^*, then can we say that all Hahn–Banach extensions of f are SSD-points of X^*?

We show that the answer to Problem (4.1.10) is negative in general and is affirmative if the subspace Y is an M-ideal in X.

We now recall that a Banach space X is said to be an L_1-*predual space* if X^* is isometric to $L_1(\mu)$ for some positive measure μ.

we also prove that the converse of Theorem (4.1.4) and Theorem (4.1.6) are true for L_1-predual spaces.

we discuss the intersection properties of balls in Banach spaces. We restrict ourselves to the $1\frac{1}{2}$-ball property and semi M-ideals. We give an affirmative answer to Problem (4.1.9) when (P) is the n-ball property, where $n = 1\frac{1}{2}, 2$.

Corollary claims that M-ideals are ball proximinal subspaces. We disprove this by giving a counterexample and we also prove that in an L_1-predual space, M-ideals are ball proximinal.

We give an example to show that the strong proximinality assumption on a subspace is not sufficient to guarantee that any proximinal subspace of it is also proximinal in the bigger space. We also discuss some examples regarding intersection properties of balls.

In this section, we discuss Problem (4.1.9) with property (P) being strong proximinality and then we consider Problem (4.1.10). Moreover, we characterize finite co-dimensional strongly proximinal subspaces of an L_1-predual space.

It is observed that there exists a proximinal subspace of c_0, which is not proximinal in ℓ_∞. Since c_0 is an M-ideal in ℓ_∞, this example shows that Problem (4.1.9) does not have an affirmative answer when (P) is proximinality.

We now prove that for a subspace Y of a Banach space X, strongly proximinal subspace of Y continue to be strongly proximinal in X under a stronger assumption on the subspace Y.

By standard continuity and convexity arguments we can see that if φ is a lattice norm on R^2 with $(0,1)$ as an extreme point of its unit ball, then for each $\beta \in R_+, \varphi(\cdot,\beta)$ is an increasing function on R_+ and for any sequence (α_n) and for an element α in $R_+, \varphi(\alpha_n,\beta) \to \varphi(\alpha,\beta)$ implies $\alpha_n \to \alpha$.

Proposition (4.1.11)[4]:

Let $\mathrm{X} = \mathrm{Y} \oplus \mathrm{Z}$ and let φ be a lattice norm on R^2 with $(0,1)$ as an extreme point of its unit ball. Suppose, for $\mathrm{x} \in \mathrm{X}$, $\|\mathrm{x}\| = \varphi(\|\mathrm{y}\|, \|\mathrm{z}\|)$, where $\mathrm{x} = \mathrm{y} + \mathrm{z}$

with y ∈ Y and z ∈ Z. If W is a strongly proximinal subspace of Y, then W is a strongly proximinal subspace of X.

Proof:-

Let $x \in X$ and let $x = y + z$ with $y \in Y$ and $z \in Z$. If W is proximinal in Y, then the proximinality of W in X follows from the fact that $PW(y) \subseteq PW(x)$. We note that the convergence assumption on φ is not used yet.

Now let W be strongly proximinal in Y. Clearly, $d(x, W) = \varphi(d(y, W), \|z\|)$. Let (w_n) be a sequence in W such that $\|x - w_n\| \to d(x, W)$. Then, by the assumption on φ, $\|y - w_n\| \to d(y, W)$ and hence, by the strong proximinality of W in Y, $d(w_n, P_W(y)) \to 0$. Since $P_W(y) \subseteq P_W(x), d(w_n, P_W(x)) \to 0$ and hence the strong proximinality of W in X follows.

As an immediate consequence of Proposition (4.1.11), it follows that if Y is an L-summand in X, then any strongly proximinal subspace W of Y is strongly proximinal in X. When Y is an M-summand in X, the proof of Proposition (4.1.11) shows that W is proximinal in X if it is so in Y, but this proposition does not give any con-clusion regarding the strong proximinality of W in X even if W is strongly proximinal in Y as the convergence assumption on φ need not be satisfied in this case. So we consider this case separately as our next result.

For a Banach space X, let $C(X)$ denote the class of non-empty, bounded and closed subsets of X. Then the Hausdorff metric on $C(X)$ is given by

$$h(A, B) = max \left\{ \sup_{x \in A} d(x, B) \ , \sup_{\in B} z \, d(z, A) \right\} \ for \ A, B \in C(X).$$

Proposition (4.1.12)[4]:

Let X be a Banach space and Y be an M-summand in X. If W is strongly proximinal in Y, then W is strongly proximinal in X.

Proof:-

Let W be strongly proximinal in Y. Clearly, W is proximinal in X. Let $x \in X$ and let $x = y + z$ with $y \in Y$ and $z \in Z$. Then it follows that $d(x, W) = max\{d(y, W), \|z\|\}$ and $P_W(y) \subseteq P_W(x). Let\ \varepsilon > 0$. Suppose $\|z\| > d(y, W)$. Then $P_W(x) = B[y, \|z\|] \cap W$ and $PW(x, \eta) = B(y, \|z\| + \eta) \cap W$ for all $\eta > 0$. Since $\|z\| > d(y, W)$, there exists a $\delta > 0$ such that for $u \in Y$ with $\|u - y\| < 2\delta$ and for $\beta > 0$ with $|\beta - \|z\|| < 2\delta$, we get

$$h\,(B\,(y, \|z\|) \cap W, B(u, \beta) \cap W) < \varepsilon, \qquad (1)$$

where h is the Hausdorff metric on $C(Y)$. Now, by putting $u = y$ and $\beta = \|z\| + \delta$ in (1), we get $h(B(y, \|z\|) \cap W, B(y, \|z\| + \delta) \cap W) < \varepsilon$. Thus $B(y, \|z\| + \delta) \cap W \subseteq (B(y, \|z\|) \cap W) + \varepsilon B_X$ and hence $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$.

Now suppose $\|z\| \leq d(y, W)$. Then $P_W(x) = P_W(y)$ and $P_W(x, \delta) = P_W(y, \delta)$. Since W is strongly proximinal in Y, there exists a $\delta > 0$ such that $P_W(y, \delta) \subseteq P_W(y) + \varepsilon B_Y$. Thus $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_Y$ and hence the result follows.

We now recall some notation in order to state a characterization of finite co-dimensional strongly proximinal subspaces in Banach spaces.

Let X be a Banach space and let $\{f_1, \ldots, f_n\}$ be a set of linearly independent functionals in X^*. Let $M_1 = M_1^* = \|f_1\|, J_X(f_1) = \{x \in B_X : f_1(x) = \|f_1\|\}\ and\ J_{X^{**}}(f_1) = \{x^{**} \in B_{X^{**}} : x^{**}(f_1) = \|f_1\|\}$.

Now suppose, for an $i \in \{2, \ldots, n\}, J_X(f_1, \ldots, f_{i-1})$ is defined and is a non-empty set. Then define

$$Mi = \sup\{f_i(x) : x \in J_X(f_1, \ldots, f_{i-1})\},$$

$$M_i^* = sup\,\{x**(fi) : x^{**} \in J_{X**}(f_1, \ldots, f_{i-1})\},$$

$$J_X(f_1, \ldots, f_i) = \{x \in J_X(f_1, \ldots, f_{i-1}) : f_i(x) = M_i\},$$

$$J_{X^{**}}(f_1,\ldots,f_i) = \{x^{**} \in J_{X^{**}}(f_1,\ldots,f_i) : x^{**}(f_i) = M_i^*\}.$$

For $\varepsilon > 0$, let $J_X(f_1,\varepsilon) = \{x \in B_X : f_1(x) > \|f\| - \varepsilon\}$.

$For\ i = 2,\ldots,n$, define

$$J_X(f_1,\ldots,f_i,\varepsilon) = \{x \in J_X(f_1,\ldots,f_i,\varepsilon) : f_i(x) > M_i - \varepsilon\}.$$

Using a weak*-compactness argument, one can see that $J_{X^{**}}(f_1,\ldots,f_i) \neq \emptyset$ for $i = 1,\ldots,n$. it is proved that if Y is a finite co-dimensional proximinal subspace of X, then $J_X(f_1,\ldots,f_i) \neq \emptyset$ for $i = 1,\ldots,n$ and for every basis $\{f_i,\ldots,f_n\}$ of $Y^{\perp}$.

Throughout this section, we use the following characterization of finite co-dimensional strongly proximinal subspace.

Theorem (4.1.13)[4]:

Let Y be a finite co-dimensional proximinal subspace of a Banach space X. Then Y is strongly proximinal in X if and only if for any basis $\{f_i,\ldots,f_n\}$ of $Y^{\perp}$,

$$\lim_{\varepsilon\to 0}[sup\ \{d\ (x,JX(f_1,\ldots,f_i)) : x \in J_X(f_i,\ldots,f_n,\varepsilon)\}] = 0$$

for $1 \leq i \leq n$.

In other words, a necessary and sufficient condition for the strong proximinality of a finite co-dimensional subspace Y of X is: if $\{f_i,\ldots,f_n\}$ is a basis of $Y^{\perp}$ and $i \in \{1,\ldots,n\}$, then, for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $d(x, J_X(f_1,\ldots,f_i)) < \varepsilon$ whenever $x \in J_X(f_1,\ldots,f_i),\delta\varepsilon)$.

We now exhibit some relations between the notations defined above.

Proposition (4.1.14)[4]:

Let Y be a finite co-dimensional strongly proximinal subspace of a Banach space X and let $\{f_1,\ldots,f_n\} \subseteq S_{Y^{\perp}}$ be a basis of $Y^{\perp}$. For $1 \leq i \leq n$, let $M_i, M_i^*, J_X(f_1,\ldots,f_i)$ and $J_{X^{**}}(f_1,\ldots,f_i)$ be defined as above. Then, for $1 \leq i \leq$

n,

i. $M_i = M_i^*$,

ii. $J_{X^{**}}(f_1,\ldots,f_i) = \overline{J_X(f_1,f_2)}^{w^*}{}^*$.

Proof:-

i. Clearly, $M_1 = M_1^*$ and $M_2 \leq M_2^*$. Let $i \in \{1,\ldots,n\}$. Now suppose that $M_j = M_j^*$ for $1 \leq j \leq i$. Then $M_{i+1} \leq M_{i+1}^*$. Since $J_{X^{**}}(f_1,\ldots,f_i)$ is weak*-compact, f_{i+1} attains its supremum over $J_{X^{**}}(f_1,\ldots,f_i)$ at some element $x_0^{**} \in J_{X^{**}}(f_1,\ldots,f_i)$. Let (x_α) be a net in B_X such that $x_\alpha \to x^{**}$ in weak*-sense. Since $x_0^{**} \in J_{X^{**}}(f_1,\ldots,f_i), x_0^{**}(f_i) = M_j^* = M_j$ for $1 \leq j \leq i$. Hence, for $1 \leq j \leq, f_i(x_\alpha) \to M_j$. Since Y is strongly proximinal in X, by Theorem (4.1.13), it follows that $d(x_\alpha, J_X(f_1,\ldots,f_i)) \to 0$. Now let (z_α) be a net in $J_X(f_1,\ldots,f_i)$ such that $\|x_\alpha - z_\alpha\| \to 0$. Then $z_\alpha \to x_0^{**}$ in weak*-sense. Since $f_{i+1}(z_\alpha) \to x_0^{**}(f_{i+1}) = M_{i+1}^*$. we get $M_{i+1}^* = lim_\alpha f_{i+1}(z_\alpha) \leq M_{i+1}$. Now the result follows by induction.

ii. Since f_1 is an SSD-point of $X^*, \overline{J_X(f_1,f_2)}^{w^*} = J_{X^{**}}(f_1)$. Clearly, $J_X(f_1,f_2)^{w^*} \subseteq J_{X^{**}}(f_1,f_2)$. Let $\phi \in J_{X^{**}}(f_1,f_2$ and choose a net (x_α)in B_X such that $x_\alpha \to \phi$ in weak*-sense. Since $f_1(x_\alpha) \to \phi(f_1), d(x_\alpha, J_X(f_1)) \to 0$. Choose a net (y_α) in $J_X(f_1)$ such that $\|x_\alpha - y_\alpha\| \to 0$. Hence $y_\alpha \to \phi$ in weak*-sense. Since $f_2(y_\alpha) \to \phi(f_2) = M_2, d(y_\alpha, J_X(f_1,f_2)) \to 0$. Hence there exists a net $(z_\alpha) \subseteq J_X(f_1,f_2)$ such that $\|y_\alpha - z_\alpha\| \to 0$, which in turn implies that $z_\alpha \to \phi$ in weak*-sense. i.e., $\overline{J_X(f_1,f_2)}^{w^*} = J_{X^{**}}(f_1,f_2))$. By a

similar argument, we can prove (b) for $i > 2$.

Lemma (4.1.15)[4]:

Let Y be a finite co-dimensional strongly proximinal subspace of a Banach space

X. Let $\{f_1, \ldots, f_n\} \subset S_{Y^\perp}$ be a basis of $Y^\perp$. Then, for $x \in B_X$ and $1 \leq i \leq n, d(x, J_X(f_1, \ldots, f_i)) = d(x, J_{X^{**}}(f_1, \ldots, f_i))$.

Proof:-

If $n = 1$,

Since no new ideas are required for $n > 2$, we only prove the case $n = 2$. Hence we have to show that for

$$x \in B_X, d(x, J_X(f_1, f_2)) = d(x, J_{X^{**}}(f_1, f_2)).$$

Let $d = d(x, J_{X^{**}}(f_1, f_2))$. Since $J_{X^{**}}(f_1, f_2)$ is weak*-compact, it is proximinal in X^{**}. Choose $\phi \in J_{X^{**}}(f_1, f_2)$ such that $\|x - \phi\| = d$.

Since Y is strongly proximinal in X, for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $d(x, J_X(f_1, f_2)) < \varepsilon$ whenever $x \in J_X(f_1, f_2, \delta_\varepsilon)$.

Now let $\varepsilon > 0$ be arbitrary. Choose an $\varepsilon' > 0$ such that $0 < \varepsilon' < \min\{\delta_\varepsilon^{T\varphi}/2^2, \frac{\varepsilon}{2(d+1)}\}$. Let $E = span\{x, \phi\} \subseteq X^{**}$ and $F = span\{f_1, f_2\} \subseteq X^*$. Then, by principle of local reflexivity, there exists a bounded linear map $T : E \to X$ such that $T(x) = x, (1 - \varepsilon') \leq \|T(z^{**})\| \leq (1 + \varepsilon')$ *if* $z^{**} \in S_E$ and $f_i(T(z^{**})) = z^{**}(f_i)$ for $i = 1,2$.

Now let $x_1 = \frac{T\phi}{\|T\phi\|}$. Then

$$\|x - x_1\| \leq \|x - T_\phi\| + \left\|T_\phi - \frac{T_\phi}{\|T_\phi\|}\right\|$$

$$= \|T(x - \phi)\| + |1 - \|T_\phi\||$$

$$\leq (1 + \varepsilon')d + \varepsilon'$$

$$= d + \varepsilon'(1 + d) < d + \frac{\varepsilon}{2}$$

and for $i = 1, 2$; by Proposition (4.1.14)(i), we have

$$fi(x_1) = fi\left(\frac{T_\phi}{\|T_\phi\|}\right) \geq \frac{M_i^*}{1 + \varepsilon'} = \frac{M_i}{1 + \varepsilon'} = M_i - \frac{M_i\varepsilon'}{1 + \varepsilon'} > M_i - \varepsilon'$$
$$> M_i - \delta_\varepsilon/2^2 .$$

Thus $x_1 \in J_X(f_1, f_2, \delta\varepsilon/2^2)$ and $d(x_1, J_{X^{**}}(f_1, f_2)) \leq d(x_1, J_X(f_1, f_2)) < \varepsilon/2^2$. Let $\phi_1 \in J_{X^{**}}(f_1, f_2)$ be such that $\|x1 - \phi_1\| < \varepsilon/2^2$. Then, again by principle of local reflexivity, there exists an element $x2 \in B_X$ such that $\|x1 - x_2\| < \varepsilon/2^2$ and $f_i(x_2) > M_i - \delta\varepsilon/2^3$.

Proceeding inductively, we obtain a sequence (x_n) in B_X such that $\|x_n - x_{n-1}\| < \varepsilon/2^n$ and $f_i(x_n) > M_i - \delta\varepsilon/2^{n+1}$ for all $n \in$ N and $i = 1, 2$. Without loss of generality, we assume that $\delta\,\varepsilon/2^n \to 0$.

Clearly, (x_n) is a Cauchy sequence and hence there exists an element $z \in B_X$ such that $z = lim_{n\to\infty} x_n$. Now $f_i(z) = M_i$ for $i = 1, 2$ and hence $z \in J_X(f_1, f_2)$. Also $\|x - x_n\| \leq d + \varepsilon/2 + \ldots + \varepsilon/2^n$ $for\ all\ n \in N$. Now, letting $n \to \infty$, it follows that $\|x - z\| \leq d + \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $z \in J_X(f_1, f_2)$, $d(x, J_X(f_1, f_2) \leq d = d(x, J_{X^{**}}(f_1, f_2)$ and hence the result follows.

Proposition (4.1.16)[4]:

Let X be a Banach space and f $\in$ X*. Then f is an SSD-point of X* if and only if f is an SSD-point of X***.

Remark (4.1.17)[4]:.

If Y is a finite co-dimensional subspace of a Banach space X, then $dim(Y^\perp) = dim(X^{**}/Y^{\perp\perp})$ and therefore dimension of $Y^\perp$ in X^* equals the dimension of $Y^{\perp\perp\perp}$ in X^{***}.

Now, by combining Theorem (4.1.3) and Proposition (4.1.18), it follows that

a hyperplane Y in a Banach space X is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Our next result generalizes this to finite co-dimensional subspaces.

Theorem (4.1.18)[4]:

If Y is a finite co-dimensional proximinal subspace of a Banach space X, then Y is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

Proof:-

Suppose that Y is strongly proximinal in X. Let $\{f_1,\dots,f_n\} \subset S_{Y^{\perp\perp\perp}}$ be a basis of $Y^{\perp\perp\perp}$. As $Y^{\perp}$ is finite dimensional, $Y^{\perp\perp\perp} = Y^{\perp}$. Thus $\{f_1,\dots,f_n\}$ is also a basis of $Y^{\perp}$.

Now let $i \in \{1,\dots,n\}$ and let $\varepsilon > 0$. Since Y is strongly proximinal in X, there exists a $\delta > 0$ such that $d(x, J_X(f_1,\dots,f_i)) < \varepsilon$ whenever $x \in J_X(f_1,\dots,f_i,\delta)$. Then, for $x^{**} \in J_{X^{**}}(f_1,\dots,f_i,\delta)$, $x^{**}(f_j) > M_j - \delta$ for $1 \le j \le i$. Let (x_α) be a net in B_X such that $x_\alpha \to x^{**}$ in weak*-sense. Now, without loss of generality, we assume that $f_j(x_\alpha) > M_j - \delta$ for all α and for $1 \le j \le i$. Hence there exists an element $z_\alpha \in J_X(f_1,\dots,f_i)$ such that $\|x_\alpha - z_\alpha\| < \varepsilon$. Passing to a subnet of (z_α), if necessary, we may assume that $z_\alpha \to \phi$ in weak*-sense for some $\phi \in J_{X^{**}}(f_1,\dots,f_i)$. Thus $(x_\alpha - z_\alpha) \to (x^{**} - \phi)$ in the weak*-sense. Then

$$\|x^{**} - \phi\| \le \varliminf_{\alpha} \|x_\alpha - z_\alpha\| < \varepsilon$$

Therefore $d(x^{**}, J_{X^{**}}(f_1,\dots,f_i)) \le \|x^{**} - \phi\| < \varepsilon$. Hence, by Theorem (3.1.13), $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

Conversely, suppose that $Y^{\perp\perp}$ is a strongly proximinal subspace of X^{**}. Let $\{f_1,\dots,f_n\} \subset S_{Y_\perp}$ be a basis of $Y^{\perp}$ and let $\varepsilon > 0$. Since $Y^{\perp\perp\perp} = Y^{\perp}$, $\{f_1,\dots,f_n\}$ is also a basis of $Y^{\perp\perp\perp}$. Let $i \in \{1,\dots,n\}$. Clearly,

$J_X(f_1,\dots,f_i,\delta) \subseteq J_{X^{**}}(f_1,\dots,f_i,\delta)$. Since $Y^{\perp\perp}$ is strongly proximinal in X^{**}, there exists a $\delta > 0$ such that $d(x^{**}, J_{X^{**}}(f_1,\dots,f_i)) < \varepsilon$ whenever $x^{**} \in J_{X^{**}}(f_1,\dots,f_i,\delta)$. Then, for $x \in J_X(f_1,\dots,f_i,\delta)$, by Lemma (4.1.17), $d(x, J_X(f_1,\dots,f_i)) = d(x, J_{X^{**}}(f_1,\dots,f_i)) < \varepsilon$ and this completes the proof.

We now give an example to show that the strong proximinality need not be transitive. Before going to the proof, we now recall a characterization of SSD-points of ℓ_∞.

Theorem (4.2.19)[4]:

An element $x \in \ell_\infty$ is an SSD-point of ℓ_∞ if and only if $\sup\{|x(n)| : |x(n)| \neq \|x\|\} < \|x\|$.

Example (4.2.20)[4]:

There exist two subspaces Z and Y of finite co-dimension in ℓ_1 such that Z is strongly proximinal in Y and Y is strongly proximinal in ℓ_1, but Z is not strongly proximinal in ℓ_1.

Proof:

Let $f = (0, 1, 1, \dots)$ and $g = (1, -\frac{1}{2}, -\frac{1}{3}, \dots)$. Then, f and g are SSD-points of ℓ_∞ and hence, f and g are QP-points of ℓ_∞. Let $Z = ker(f) \cap ker(g)$ and $Y = ker(f)$. Since f is a QP-point of ℓ_∞, Y is strongly proximinal in ℓ_1. Also, since g attains its norm on Y and g is a QP-point of ℓ_∞, $g|Y$ is a QP-point of Y^*. Hence $Z = \ker(g|Y)$ is strongly proximinal in Y. Since, by Theorem (4.1.14), $f + g \in Z^{\perp}$ is not an SSD-point of ℓ_∞, it follows from Theorem (4.1.4) that Z is not strongly proximinal in ℓ_1.

Our next theorem shows that for an M-ideal Y in a Banach space X, a strongly proximinal subspace of Y having finite co-dimension in X remains to be strongly proximinal in X.

Theorem (4.1.21)[4]:

Let X be a Banach space and Z be a finite co-dimensional proximinal subspace of X. Let Y be an M-ideal in X and Z ⊆ Y ⊆ X. If Z is strongly proximinal in Y, then Z is strongly proximinal in X.

Proof:-

Let Z be strongly proximinal in Y. Then, by Theorem (4.1.20), it follows that $Z^{\perp\perp}$ is strongly proximinal in $Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is an M-summand in X^{**}, by Proposition (4.1.12), $Z^{\perp\perp}$ is strongly proximinal in X^{**}. Then, again by Theorem (4.1.20), Z is strongly proximinal in X. 2

We do not know whether we can replace the M-ideal assumption in Theorem (4.1.13) by the semi M-ideal assumption. The idea used in the proof of Theorem (4.1.13) will not be useful in the semi M-ideal case as the bidual of a semi M-ideal is again a semi M-ideal,

Remark (4.1.22)[4]:

We do not know whether the finite co-dimensionality assumption on Y in Theorem (4.1.13) is necessary. The answer is not known even if the strong proximinality in Theorem (4.1.13) is replaced by proximinality.

For a subspace Y of a Banach space X, one can ask about the strong subdifferentiability of Hahn–Banach extensions of an SSD-point of Y^*. To begin with, we give an example to show that all the Hahn–Banach extensions of an SSD-point of Y^* need not be SSD-points of X^*.

Example (4.1.23)[4]:.

There exist a subspace Y of ℓ_1 and an SSD-point of Y^* such that one of its Hahn–Banach extensions is not an SSD-point of ℓ_∞.

Proof:-

Let f, g, Z and Y be as in Example (4.1.22). Since $g|Y$ is an SSD-point of Y^* and $f + g$ is a Hahn–Banach extension of $g|Y$, the conclusion follows from Example (4.1.22).

We now prove that for an M-ideal Y in a Banach space X, the Hahn–Banach extension of an SSD-point of Y^* to X is an SSD-point of X^*.

Proposition (4.1.24)[4]:.

Let Y be a semi L-summand in a Banach space X and let $y \in Y$ be an SSD-*point of Y. Then y is also an* SSD-*point of X.*

Proof:-

Let $P : X \to X$ be a semi L-projection with range Y. Then

$$d^+(y)(x) = d^+(y)(Px) + \|x - Px\|.$$

Now the conclusion follows from the following equation.

$$\frac{\|y + tx\| - 1}{t} - d^+(y)(x)$$

$$= \|Px\| \left(\frac{\|y + t\|Px\| \left\| \frac{Px}{\|Px\|} \right\| - 1}{\|Px\| t} - d^+(y) \frac{Px}{\|Px\|}. \right)$$

for an M-ideal Y in $X, X^* = Y^* \otimes_1 Y^\perp$, the following corollary is immediate from Proposition (4.1.26).

Corollary (4.1.25)[4]:

If Y is an M-ideal in a Banach space X and $f \in Y^*$ is an SSD-point of Y^*, then the unique Hahn–Banach extension of f to X is also an SSD-point of X^*.

Since a QP-point is an SSD-point and also since the converse need not be true, it is natural to ask about the class of Banach spaces where the notions of

an SSD-point and a QP-point coincide. We now show that for a positive measure μ, these two notions coincide in $L_1(\mu)$.

Proposition (4.1.26)[4]:

For a positive measure μ, an SSD-point of $L_1(\mu)$ is also a QP-point of $L_1(\mu)$.

Proof:-

Let $f \in L_1(\mu)$ be an SSD-point. Since $L_1(\mu)$ is an L-summand in its bidual, f is an SSD-point of $L_1(\mu)^{**} = C(K)^{*}$ (up to an isometry) for some compact Hausdorff space K. f is a QP-point of $L_1(\mu)^{**}$ and hence f is a QP-point of $L_1(\mu)$.

Now it follows from the proof of Example (4.1.22) that the sum of two SSD-points in a Banach space need not be an SSD-point. But in our next result, we prove that the sum of two SSD-points of $L_1(\mu)$ is an SSD-point of $L_1(\mu)$.

Corollary (4.1.27)[4]:

For a positive measure μ, sum of two SSD-points of $L_1(\mu)$ is an SSD-point of $L_1(\mu)$.

Proof:-

Let f and g be two SSD-points of $L_1(\mu)$. Since $L_1(\mu)$ is an L-summand in its bidual, by Proposi-tion (4.1.24), f and g are SSD-points of $L_1(\mu)^{**} = C(K)^{*}$ (up to an isometry) for some compact Hausdorff space K. Since SSD-points of $C(K)^{*}$ are precisely the finitely supported measures, $f + g$ is an SSD-points of $C(K)^{*} = L_1(\mu)^{**}$. Hence $f + g$ is an SSD-point of $L_1(\mu)$.

Our next result characterizes finite co-dimensional strongly proximinal subspaces of L_1-predual spaces. The following result also shows that the converse of Theorem (4.1.4) and Theorem (4.1.6) are true in L_1-predual spaces.

Proposition (4.1.28)[4]:

Let X be an L_1-predual space and Y be a finite co-dimensional proximinal subspace of X. Then the following are equivalent

i. Y is strongly proximinal in X.
ii. $Y^{\perp} \subseteq \{x^* \in X^{*:}\ x^*$ is an SSD-point of $X^*\}$.
iii. $Y^{\perp} \subseteq \{x^* \in X^* : x^*$ is a QP-point of $X^*\}$.

Proof:-

The implication (i) ⇒ (ii) follows from Theorem (4.1.3) and the implication (ii) ⇒ (i) follows from Proposition (4.1.26) and Theorem (4.1.6). Finally, (ii) ⟺ (iii) follows from Proposition (4.1.26).

If Y is a finite co-dimensional strongly proximinal subspace of a Banach space X, then, by Theorem (4.1.4), Y is the intersection of finitely many strongly proximinal hyperplanes. Our next result shows that the converse of this is true in L_1-predual spaces.

Corollary (4.1.29)[4]:

Let X be an L_1-predual space and let $Y_1, \ldots, Y_n$ be strongly proximinal subspaces of finite co-dimension in X. Then $\bigcap_{i=1}^{n} Y_i$ is strongly proximinal in X.

Proof:-.

Let $Y = \bigcap_{i=1}^{n} Y_i$. For $1 \leq i \leq m$, let $f_{i,1}, \ldots, f_{i,n_i}$ be SSD-points of X^* such that $Y_i = \bigcap_{k=1}^{n} \ker(f_{i,k})$. Thus $Y = \bigcap_{i,k} ker(f_{i,k})$ and hence, by Corollary (4.1.29), $Y^{\perp} = span\{f_{i,k} : 1 \leq i \leq m, 1 \leq k \leq ni\} \subseteq \{f \in X^{*:}\ f\ is\ an\ SSD-point\ of\ X^*\}$. Hence, by Proposition (4.1.26), Y is strongly proximinal in X.

Section (4.2): The Intersection Properties of balls in Banach spaces with example

In this section, we consider Problem (4.1.9) with property (P) being $1\frac{1}{2}$-ball property or 2-ball property. Moreover, we prove that in an L_1-predual space, M-ideals are ball proximinal.

We now prove a variation of the transitivity problem for n-ball property with $n \in$ N.

Lemma (4.2.1)[4]:

Let Y be an M-summand in a Banach space X and Z be a subspace of Y. Let n ∈ N.

i. If Z has the (strong) n-ball property in Y, then Z has the (strong) n-ball property in X.
ii. If Z has the (strong) $\frac{1}{2}$-ball l property in Y, then Z has the (strong) 12-ball property in X.

Proof:-

(i)Let Z have the n-ball property in Y. Let $\varepsilon > 0$ and $\{B[x_i, r_i]\}1\boldsymbol{i} \leq \boldsymbol{n}$ be a family of n balls in X such that

$$B[x_i, r_i] \cap Z \neq \emptyset \; for \; all \; i = 1, \ldots, n \; and \; \bigcap_{i=1}^{n} B[x_i, r_i] \neq \emptyset.$$

Let $x \in \bigcap_{i=1}^{n} B[x_i, r_i]$ and $P : X \to X$ be an M-projection with range Y. Then $Px \in \bigcap_{i=1}^{n} B[x_i, r_i]$ and $B[x_i, r_i] \cap Z \neq \emptyset$. Then, by the n-ball property of Z in Y, there exists an element $z \in Z \cap (\bigcap_{i=1}^{n} B[x_i, r_i + \varepsilon])$. Hence $\|z - x\| \leq max\{\|z - Px_{\boldsymbol{i}}\|, \|x_{\boldsymbol{i}} - Px_{\boldsymbol{i}}\|\} \leq r_{\boldsymbol{i}} + \varepsilon \, for \, 1 \leq \boldsymbol{i} \leq n$. Thus Z has the n-ball property in X.

If Z has the strong n-ball property in Y, then the strong n-ball property of Z in X follows by taking $\varepsilon = 0$ in the above proof.

A similar proof also works for (*b*).

Our next result is an analogue of Theorem (4.1.20) in the context of *n*-ball property with $n = 1\frac{1}{2}, 2$.

Lemma (4.2.2)[4]:

Let Y be a subspace of a Banach space X and let $n = 1\frac{1}{2}, 2$. Then Y has the n-ball property in X if and only if $Y^{\perp\perp}$ has the n-ball property in X^{**}.

Proof:-

Suppose that Y has the 2-ball property in X. Then Y is a semi M-ideal in X and hence $Y^{\perp}$ is a semi L-summand in X^*. Then, $Y^{\perp\perp}$ is a semi M-ideal in X^{**} and hence $Y^{\perp\perp}$ has the 2-ball property in X^{**}.

Conversely, suppose that $Y^{\perp\perp}$ has the 2-ball property in X^{**}. Let $\varepsilon > 0$ and let $\{B[x_i, r_i]\}_{i=1,2}$ be two balls in X such that $B[x_i, r_i] \cap Y = \emptyset \; for \; i = 1,2$ and $B[x_1, r_1] \cap B[x_2, r_2] = \emptyset$.

Since $Y^{\perp\perp}$ is a weak*-closed subspace of $X^{**}, Y^{\perp\perp}$ has the strong 2-ball property in X^{**}. Hence there exists an element $x^{**} \in Y^{\perp\perp}$ such that $\|x^{**} - x_i\| \leq r_i \; for \; i = 1,2$.

Let $E = span\{x_1, x_2, x^{**}\}$ and $r = max\{r_1, r_2\}$. there exists a bounded linear map $T_\varepsilon : E \to X$ such that $T_\varepsilon(z) = z$ for

$z \in E \cap X, T_\varepsilon(E \cap Y^{\perp\perp}) \subset Y$ and $\|T_\varepsilon\| \leq 1 + \frac{\varepsilon}{r}$. Now take $z = T_\varepsilon(x^{**})$. Then $z \in Y$ and $\|z - x_i\| \leq r_i + \varepsilon \; for \; i = 1,2$. Hence Y has the 2-ball property in X.

Corollary (4.2.3)[4]:

Let Y be a semi M-ideal in a Banach space X. Then Y is a semi M-ideal in X^{**} if and only if Y is an M-ideal in Y^{**}.

Proof:-

Suppose Y is a semi M-ideal in X^{**}. Then Y is a semi M-ideal in $Y^{\perp\perp} = Y^{**}$ and hence, Y is an M-ideal in Y^{**}.

Conversely, suppose that Y is an M-ideal in Y^{**}. Since Y is a semi M-ideal in X, by Lemma (4.2.2), $Y^{\perp\perp}$ is a semi M-ideal in X^{**}. Then, Y is a semi M-ideal in X^{**}.

Theorem (4.2.4)[4]:.

Let Z and Y be subspaces of a Banach space X such that $Z \subseteq Y \subseteq X$ and Y is an M-ideal in X. Let $n = 1\frac{1}{2}, 2$. If Z has the n-ball property in Y, then Z has the n-ball property in X.

Proof:-

Case 1: $n = 1\frac{1}{2}$.

Since $Z \subseteq Y \subseteq X, Z^{\perp\perp} \subseteq Y^{\perp\perp} \subseteq X^{**}$. Then, by Lemma (4.2.2), $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in $Y^{\perp\perp}$ and by Lemma (4.2.1), $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in X^{**}. Then, by Lemma (4.2.2), Z has the $1\frac{1}{2}$-ball property in X.

Case 2: $n = 2$.

Since $Z \subseteq Y \subseteq X, Z^{\perp\perp} \subseteq Y^{\perp\perp} \subseteq X^{**}$. Then, by Lemma (4.2.2), $Z^{\perp\perp}$ is a semi M-ideal in $Y^{\perp\perp}$ and by Lemma (4.2.1), $Z^{\perp\perp}$ is a semi M-ideal in X^{**}. Then, by Lemma (4.2.2), Z is a semi M-ideal in X.

Remark (4.1.5)[4]:.

We do not know the analogue of Theorem (4.2.4) in the context of the strong $1\frac{1}{2}$-ball property and the strong 2-ball property.

It is proved that a subspace has the strong $1\frac{1}{2}$-ball property if and only if it is ball proximinal and has the $1\frac{1}{2}$-ball property. It is incorrectly assumed that M-ideals have the strong $1\frac{1}{2}$-ball property, which is not the case. However, it is well-known that M-ideals have the1_2-ball property and therefore that an M-ideal is ball proximinal if and only if it has the strong $1\frac{1}{2}$-ball property.

We now give a class of Banach spaces where M-ideals are ball proximinal.

Definition (4.2.6)[4]:

Let X be a Banach space and $n \in$ N. Then X has the $n.2.I.P.$ if any pairwise intersecting family of n balls in X actually intersect.

It is well-known that a Banach space is an L_1-predual space if and only if it has the $4.2.I.P$.

Theorem (4.2.7)[4]:

If X has the 3.2.I.P., then every M-ideal in X satisfies the strong 3-ball property. In partic-ular, an M-ideal in an L_1-predual space has the strong 3-ball property.

$$\bigcap_{i=1}^{n} B[x_i, r_i + \varepsilon]$$

Let Y be an M-ideal in X and let $\{B[x_i, r_i]\}_{i=1}^{3}$ be a family of 3 closed balls satisfying $B[x_i, r_i] \cap Y \neq \emptyset$ for $1 \leq i \leq 3$ and $\bigcap_{i=1}^{3} B[x_i, r_i] \neq \emptyset$.

Also, let $\varepsilon > 0$. Since Y is an M-ideal in X, there exists an element $y_0 \in Y$ such that $y_0 \in \bigcap_{i=1}^{3} B[x_i, r_i + \varepsilon]$. Now fix an $i \in \{1,2,3\}$. Then $\{B[x_i, r_i] : 1 \leq j \leq 3, j \neq i\} \cup \{B[y_0, \varepsilon]\}$ is a pair-wise intersecting family of 3 closed balls in X. Since X has the $3.2.I.P.$, the intersection of these three balls is non-empty. Since Y is an M-ideal in X, there exists an element $y_i \in Y$ such that

$$\|y_i - x_j\| \leq r_j + \frac{\varepsilon}{6} \quad for\ 1 \leq j \leq 3 \text{ and } j \neq i \quad \text{and}$$

$$\|y_i - x_0\| \leq \varepsilon + \frac{\varepsilon}{6}.$$

Let $y = \frac{1}{3}\sum_{i=1}^{3} y$. Then, for $1 \leq \boldsymbol{j} \leq 3$, we get

$$\|y - y_0\| \leq 2\varepsilon \quad and$$

$$\|y - x_j\| \leq \frac{1}{3}\left(\sum_{\substack{1\leq i\leq 3\\ i\neq j}} \|y_i - x_j\| + \|y_i - x_j\|\right)$$

$$\leq \frac{1}{3}\left(2\left(r_j + \frac{\varepsilon}{6}\right) + \|y_i - y_0\| + \|y_0 - x_j\|\right)$$

$$\leq r_j + \frac{5}{6}\varepsilon.$$

Now let $z_0 = y_0$ and $z_1 = y$. Suppose we have constructed $z_1, \ldots, z_m$ such that

$$\|z_k - z_{k-1}\| \leq 2\left(\frac{5}{6}\right)^{k-1}\varepsilon \qquad for\ 1 \leq k \leq m\ and$$

$$z_k - x_j \leq r_j + \left(\frac{5}{6}\right)^{k}\varepsilon \qquad for\ 1 \leq k \leq m\ and\ 1 \leq j \leq 3$$

Now fix an $\boldsymbol{i} \in \{1, 2, 3\}$. Then $\{B[x_j, \boldsymbol{r_j}] : 1 \leq \boldsymbol{j} \leq 3, \boldsymbol{j} = \boldsymbol{i}\} \cup \{B[z_{\boldsymbol{m}}, (\frac{5}{6})^{\boldsymbol{m}}\varepsilon]\}$ is a pairwise intersecting family of 3 closed balls in X. Then, by arguing as above, there exists an element $z_{\boldsymbol{m},\boldsymbol{i}} \in Y$ such that

$$\|z_{m,i} - x_j\| \leq r_j + \frac{1}{6}\left(\frac{5}{6}\right)^{m}\varepsilon \quad for\ 1 \leq j \leq 3\ and\ j_ = i\ and$$

$$\|z_{m,i} - z_m\| \leq \left(\frac{5}{6}\right)^{m}\varepsilon + \frac{1}{6}\left(\frac{5}{6}\right)^{m}\varepsilon$$

Now let $z_{\boldsymbol{m}} = \frac{1}{3}\sum_{\boldsymbol{i}=1}^{1} z_{\boldsymbol{m},\boldsymbol{i}}$. Then, for $1 \leq \boldsymbol{j} \leq 3$, we get

$$z_{m+1} - z_m \leq 2\left(\frac{5}{6}\right)^m \varepsilon \text{ and } \|z_{m+1} - x_j\| \leq r_j + \left(\frac{5}{6}\right)^{m+1} \varepsilon.$$

Thus, by induction, there exists a Cauchy sequence (z_m) in Y such that

$$\|z_m - x_j\| \leq r_j + \left(\frac{5}{6}\right)^m \varepsilon \quad for\ 1 \leq j \leq 3.$$

Now let $z = \lim_{m\to\infty} z_m$. Then $z \in \bigcap_{j=1}^{3} B[x_j, r_j] \cap Y$ and hence the theorem follows.

Proof:-

Let $k \in [0,1] \setminus \{0,1,\frac{1}{2},\frac{1}{3},\ldots\}$. Let $\mu, v \in C[0,1]^*$ be defined as $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{\frac{1}{n}}$ and $v = \frac{1}{2}(\delta_0 - \delta_k)$. Then $\|\mu\| = \|v\| = 1$. Now take $Z = \ker(\mu) \cap \ker(v)$ and $Y = \ker(v)$. Since $\operatorname{supp}(v)$ is finite, $\ker(v)$ is strongly proximinal in $C[0,1]$. Since $1 \in \ker(v)$ and $\mu(1) = 1$, $\mu|_{\ker(v)}$ is a norm-attaining functional on $\ker(v)$. Hence $\ker(\mu) \cap \ker(v) = \ker(\mu|_{\ker(v)})$ is a proximinal subspace of $\ker(v)$. Since v is not absolutely continuous with respect to μ on $\operatorname{supp}(\mu)$, by (9), $\ker(\mu) \cap \ker(v)$ is not proximinal in $C[0,1]$.

Our next example is a variant of Example (4.2.9). In fact, it shows that the notion of strong proximinality need not pass through ideals.

Example (4.2.10)[4]:

There exist two subspaces Z and Y of finite co-dimension in $C[0,1]$ such that Z is strongly proximinal in Y and Y is an ideal in $C[0,1]$, but Z is not proximinal in $C[0,1]$.

Proof.

Let μ, v and k be as in the proof of Example (4.2.9). Take $Z = \ker(\mu) \cap \ker(v)$ and $Y = \ker(\mu)$. Choose a continuous function $g:[0,1] \to [-1,1]$ such that $g(_n) = g(0) = 1$ for $n \geq 2$ and $g(1) = g(k) = -1$. Then $g \in \ker(\mu)$ and $v(g)$

$= 1$. Since $v|_{\ker}(\mu)$ attains its norm over $\ker(\mu)$, $\ker(\mu) \cap \ker(v) = \ker(v|\ker(\mu))$ is proximinal in $\ker(\mu)$.

Let $\lambda = -\sum_{n=2}^{\infty} \frac{1}{2^2} \delta_{\frac{1}{n}}$. Then $\ker(\mu) = ker(\lambda - \delta_1)\ and$ and $\|\lambda\| \leq 1$ and hence, $\ker(\mu)$ is an L_1-predual space. Then, $\ker(\mu)$ is an ideal in $C[0, 1]$. Since v is not absolutely continuous with respect to μ on $\text{supp}(\mu)$, $\ker(\mu) \cap \ker(v)$ is not proximinal in $C[0, 1]$.

Our next example shows that the semi M-ideals may not pass through L-summands.

Example (4.1.11)[4]:.

There exist a Banach space X which is an L-summand in X^{**} and a semi M-ideal Y in X such that Y is not a semi M-ideal in X^{**}.

Proof:-

Take $X = \ell_1$. Then X is an L-summand in its bidual. For the constant sequence $1 \in \ell_\infty, Y = ker(1)$ is a semi M-ideal in ℓ_1. But $\ker(1)$ is not a semi M-ideal in $(\ell_\infty)^*$. For, if $\ker(1)$ is a semi M-ideal in $(\ell_\infty)^*$, then $\ker(1)$ is a semi M-ideal in $ker(1)^{\perp\perp}$. Then, $\ker(1)$ is an M-ideal in $ker(1)^{\perp\perp}$. Since, a non-reflexive subspace which is an M-ideal in its bidual contains a subspace isomorphic to c_0, $\ker(1)$ is reflexive.

But this is a contradiction as ℓ_1 cannot have an infinite dimensional reflexive space. Hence $\ker(1)$ is not a semi M-ideal in $(\ell_\infty)^*$.

Corollary (4.2.8)[4]:

If X has the 3.2.I.P., then every M-ideal in X is ball proximinal. In particular, M-ideals in L_1-predual spaces are ball proximinal

Our first example shows that the strong proximinality assumption on a subspace is not sufficient to guarantee that any proximinal subspace of it is

also proximinal in the bigger space.

Example (4.2.9)[4]:

There exist two subspaces Z and Y of finite co-dimension in $C[0,1]$ such that Z is proximinal in Y and Y is strongly proximinal in $C[0,1]$, but Z is not proximinal in $C[0,1]$.

List of Symbols

Symbol	Page
Ess: essential	2
Inf :infimum	3
Detn: determinant	4
ran: range	4
min : minimum	6
sup: supremum	9
l^p : Lebesgue space	16
Max : maximum	18
l^1 : Lebesgue space	23
l^∞ : Lebesgue space	24
L_p : Lebesgue space	25
l_2 : Hilbert space	25
S^p :Schatten class	26
Isom : Isomorphism	27
Supp : support	29
$\oplus$: Direct sum	36
Conv : convex	39
Dim : dimension	50
ssD : strongly Subdiffentiable	100
Qp: quasi –polyhedral	100
Ker :kernel	112
$\otimes$: tensor product	114

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