Chapter 6

Universal Cycles and Thom Isomorphism

We also show the Thom isomorphism in Kasparov's KK-theory in a form stated without proof in the conspectus. As an application, we show Bott periodicity and a Thom isomorphism for algebras of Schwartz functions. As a special case, this applies to the theories kk for locally convex algebras considered by Cuntz.

Section (6.1): Bivarant K-Theory:

The classical proof of the Thom isomorphism in K-theory for a complex bundle $E \to X$ over a compact space X is based on the Thom elements λE (see [201]). These Thom elements are certain classes in the reduced K-theory of Thom space E^+ of the bundle. Using an orthogonal of the bundle, one may then deduce the Thom isomorphism from Bott periodicity.

We define the analogous smooth bivariant Thom classes in Cuntz's kk ([202]). These smooth bivariant Thom classes have certain multiplicativity properties which, combined with elementary formal properties of kk, yield a very simple proof of the Thom isomorphism for kk. As kk is universal for diffotopy invariant, half exact and stable functors, this yields the Thom isomorphism for any such functor. In particular, using the bivariant Chern-Connes character from [Cun97], we obtain the Thom isomorphism in bi- and monovariant periodic cyclic cohomology. The analogous result for split exact functors ([203]) is more difficult to obtain, and requires entirely different techniques.

We then proceed to give a proof of Bott periodicity and a version of the Thom isomorphism that was stated without proof in [204]. This is essentially based on the techniques of [205], but we use instead the connection formalism of Connes and Skandalis [206], and deduce Bott periodicity basically from the theory of the harmonic oscillator. We base our proof on the functoriality of KK for principal bundles.

We note that the Thom elements λE, and hence the smooth bivariant Thom classes in kk, may also be obtained as restrictions of the fibred versions of the Bott element used below in the proof of the Thom isomorphism in KK by replacing the operator and applying a Morita equivalence.

We recall the definition of kk-theory as given by Cuntz in [207], to which we refer for details.

Definition (6.1.1)[200]: *An exact sequence*

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow 0$$

of locally convex algebras will be called semisplit if it is split as a sequence of topological vector spaces. It will be called a split exact sequence, if it is split in the category of locally convex algebras.

The essential ingredients for the theory kk are as follows:

(a) The smooth compacts $\mathcal{K}$, which are the algebra of matrices with rapidly decreasing entries,

(b) the smooth Toeplitz algebra, which is as a vector space $\mathcal{K} \oplus \mathcal{C}^{\infty}(\mathbb{S})$, and a locally convex algebra with the multiplication it inherits from the C^*-Toeplitz algebra,

(c) the resulting Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}^{\infty} \longrightarrow \mathcal{C}^{\infty}(\mathbb{S}) \longrightarrow 0$$

(d) the cone extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C} \longrightarrow \mathbb{C} \longrightarrow 0$$

(e) the universal semisplit extension (cf. Definition (6.1.1)) of a locally convex algebra A

$$0 \longrightarrow JA \longrightarrow TA \longrightarrow A \longrightarrow 0$$

where TA is the locally convex tensor algebra of the vector space A, and the quotient map the map induced by universality from id_A, JA the kernel of this map. It also has a continuous linear split $A \to TA$ given by the inclusion. Using the universality of TA, one shows that this is in fact the initial object in the category of semisplit extensions of A. More generally:

Proposition (6.1.2)[200]: *For any semisplit extension of A' and morphism $\varphi : A \to A'$ (this is called a semisplit morphism extension) there are maps σ, ψ such that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & JA & \longrightarrow & TA & \longrightarrow & A & \longrightarrow & 0 \\ & & \sigma\downarrow & & \downarrow\psi & & \downarrow\varphi & & \\ 0 & \longrightarrow & B & \longrightarrow & C & \xrightarrow{s} & A' & \longrightarrow & 0 \end{array}$$

commutes.

The map ψ is the map induced from $s \circ \varphi$ on TA by universality, and it restricts by commutativity to $\sigma : JA \to B$.

Definition (6.1.3)[200]: *The map σ of the last definition is called the classifying map of the morphism extension.*

The (diffotopy class of a) classifying map of such a morphism extension does not depend on the split (join two different splits by the obvious linear path). More generally, we have classifying maps $J^n A \to B$ for an *n*-step extension. We define $T^n A$ and $J^n A$ by iteration of the functors J and T (this is independent of the order up to diffotopy).

Definition (6.1.4)[200]: *The groups $kk_m^{alg}(A, B)$ for locally convex algebras A and B are now defined as the direct limit*

$$\lim_{\rightarrow}(\mathrm{Hom}(J^{m+n}A, \mathcal{K} \otimes_\pi S^n B)/\sim),$$

and $kk_m^{\mathcal{L}^p}(A, B) := kk_m^{alg}(A, B \otimes_\pi \mathcal{L}^p)$. The class of a homomorphism φ will be denoted $\langle\varphi\rangle$.

Here $\sim$ denotes diffotopy, and the connecting maps Λ_n are given by sending $\varphi : J^{n+m}A \to \mathcal{K} \otimes_\pi S^n B$ to the classifying map of

$$\begin{array}{ccccccc} & & & & J^{n+m}A & & \\ & & & & \downarrow\varphi & & \\ 0 & \longrightarrow & \mathcal{K} \otimes_\pi S^{n+1}B & \longrightarrow & \mathcal{K} \otimes_\pi S^n B & \longrightarrow & 0. \end{array}$$

To simplify notation, we write kk for kk^{alg}; further we will suppress the tensor product by $\mathcal{K}$ whenever this does not essentially change the arguments.

We note that $kk^{\mathcal{L}^p}$ does not depend on p (as long as p is large enough). The further properties of the theory may be nicely developed using triangulated categories and postponing the stabilisation by $\mathcal{K}$ or $\mathcal{L}^p$ to the end. We refer to [208] for further details on this.

Denote by s the canonical map $J^m(A \otimes_\pi B) \to J^m(A) \otimes_\pi B$. We will define a multiplicative transformation

$$\tau_D : kk_*^{alg}(A, B) \to kk_*^{alg}(A \otimes D, B \otimes D)$$

as follows: For $\varphi : J^m A \to S^n B$, set

$$\tau_D\langle\varphi\rangle := \langle J^m(A \otimes_\pi D) \xrightarrow{s} J^m A \otimes_\pi D \xrightarrow{\varphi \otimes_\pi \mathrm{id}_D} S^n \otimes_\pi D\rangle.$$

To show that τ_D is well defined, it suffices to show $\tau_D\langle\varphi\rangle = \tau_D\langle\Lambda\varphi\rangle$, which will follow from

$$\langle \Lambda\big(J^m(A \otimes_\pi D)\big) \xrightarrow{s} J^m A \otimes_\pi D \xrightarrow{\varphi \otimes_\pi \mathrm{id}_D} S^n \otimes_\pi D \rangle$$

$$= \langle J^{m+1}(A \otimes_\pi D) \xrightarrow{s} J^{m+1} A \otimes_\pi D \xrightarrow{\Lambda\varphi \otimes_\pi \mathrm{id}_D} S^{n+1} B \otimes_\pi D \rangle .$$

The equality of these maps is obtained by the usual arguments, using the defining diagrams for both maps. The following easy proposition determines the action of τ_D on elements of kk^{alg} given by extensions:

Proposition (6.1.5)[200]: *If $\langle \varphi \rangle$ is the class in $kk_1(A, B)$ of the classifying map of a semisplit extension*

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

then $\tau_D \langle \varphi \rangle$ is the class of the classifying map of

$$0 \longrightarrow B \otimes_\pi D \longrightarrow C \otimes_\pi D \longrightarrow A \otimes_\pi D \longrightarrow 0.$$

An analogous formula holds for extensions of higher length.

Proof. This follows by uniqueness of classifying maps and commutativity of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J(A \otimes D) & \longrightarrow & T(A \otimes D) & \longrightarrow & A \otimes D & \longrightarrow & 0 \\
 & & \downarrow s & & \downarrow & & \mathrm{id}\downarrow & & \\
0 & \longrightarrow & JA \otimes D & \longrightarrow & TA \otimes D & \longrightarrow & A \otimes D & \longrightarrow & 0 \\
 & & \downarrow \varphi \otimes \mathrm{id}_D & & \downarrow & & \mathrm{id}\downarrow & & \\
0 & \longrightarrow & B \otimes D & \longrightarrow & C \otimes D & \longrightarrow & A \otimes D & \longrightarrow & 0
\end{array}$$

where the outer edge is defining the classifying map of the extension tensored by D, and is therefore diffotopic to $(\varphi \otimes \mathrm{id}_D) \circ s$.

Definition (6.1.6)[200]: *We define the cup product by*

$$kk_i(A, B) \times kk_j(A', B') \to kk_{i+j}(A \otimes A', B \otimes B')$$

$$(\alpha, \beta) \mapsto \alpha \cup \beta \coloneqq \tau_{A'}(\alpha) \cap \tau_B(\beta)$$

As an easy consequence of the above proposition, we may describe the cup product of the classes of two extensions

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0\,, 0 \longrightarrow B' \longrightarrow C' \longrightarrow A' \longrightarrow 0$$

by the class of the length two extension

$$0 \longrightarrow B \otimes_\pi B' \longrightarrow C \otimes_\pi B' \longrightarrow A \otimes_\pi C' \longrightarrow A \otimes_\pi A' \longrightarrow 0$$

We give two easy properties of the cup product:

Proposition (6.1.7)[200]: *The cup product is graded commutative and the following formula describes the relation between the cup and cap product*

$$(\alpha \cup \beta) \cap (\gamma \cup \delta) = (\alpha \cap \gamma) \cup (\beta \cap \delta),$$

where $\alpha \in kk(A,B), \beta \in kk(A',B'), \gamma \in kk(B,C)$ *and* $\delta \in kk(B',C')$.

Proof. Represent α by $\varphi : J^m A \to S^{\underline{m}} B$ and β by $\psi : J^n A' \to S^{\underline{n}} B'$. Then the commutativity of $\cup$ up to sign follows by the usual arguments from commutativity of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J^{m+n}(A \otimes A') & \longrightarrow & J^m(A \otimes J^n A') & \xrightarrow{J^m(\mathrm{id}_A \otimes \psi)} & J^m(A \otimes S^{\underline{n}} B') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^n(J^m A \otimes A') & \longrightarrow & J^m A \otimes J^n A' & \xrightarrow{J^m(\mathrm{id}_A \otimes \psi)} & J^m(A) \otimes S^{\underline{n}} B & \longrightarrow & 0 \\
 & & \downarrow J^n(\varphi \otimes A') & & \downarrow \varphi \otimes J^n A' & & \downarrow \varphi \otimes S^{\underline{n}} B & & \\
0 & \longrightarrow & J^n(S^{\underline{m}} B \otimes A') & \longrightarrow & S^{\underline{m}} B \otimes J^n A' & \xrightarrow[S^{\underline{m}} B \otimes \psi]{} & S^{\underline{m}} B \otimes S^{\underline{n}} B & \longrightarrow & 0
\end{array}$$

To prove the formula given above, we calculate

$$\begin{aligned}
(\alpha \cup \beta) \cap (\gamma \cup \delta) &= \tau_{A'}(\alpha) \cap \tau_B(\beta) \cap \tau_{B'}(\gamma) \cap \tau_C(\delta) \\
&= \tau_{A'}(\alpha) \cap \tau_{A'}(\gamma) \cap \tau_C(\beta) \cap \tau_C(\delta) \\
&\overset{*}{=} \tau_{A'}(\alpha \cap \gamma) \cap \tau_C(\beta \cap \delta) \\
&= (\alpha \cap \gamma) \cup (\beta \cap \delta),
\end{aligned}$$

where we use multiplicativity of τ at $*$.

In particular, the formula above shows that the cup product $\alpha \cup \beta$ of invertible elements is invertible with inverse $\pm(\alpha^{-1} \cup \beta^{-1})$ (invertibility with respect to the cap product).

This proof uses kk_m, the version of kk theory developed by Cuntz in [202], which is a bivariant K-theory on the category of m-algebras. It nicely reduces the computations of products almost entirely to K-theory, where one may use the classical results. We denote throughout this section by Σ the reduced suspension, i.e., the sphere with base point the north pole, and by H the tautological line bundle on the two-sphere. We first formulate the ingredients for this proof:

(i) The theory kk_m as developed in [202], and the fact that it has Bott periodicity ([202])

(ii) Phillips K-theory for Frechet-algebras (for this paragraph, à Frechet algebra is complex, locally multiplicatively convex algebra which is a Frechet space) from [209]

(iii) The comparison between $kk_m(\mathbb{C},\cdot)$ and Phillips K-theory from [202].

(iv) The multiplicative elements $\lambda_E \in \widetilde{K}^0(X^E, \infty) = K_0(C_0(E)\tilde{})$ for a bundle $E \to X$ (X^E denotes the Thom space of V)

Let us first comment on the relation between the first three items: The isomorphism in (iii) is constructed in [202], and it is easily seen to be compatible with outer products. One thus sees that the class $\langle H \rangle - 1$, i.e., the generator of $\widetilde{K}^0(\Sigma^2)\big(\approx K_0(S^2)\big)$ in topological K-theory, is mapped to a generator in $kk_m(\mathbb{C}, S^2)$, by using that the K-theory of the smooth subalgebra S^2 coincides with the reduced topological K-theory of Σ^2. Further, as everything commutes with outer products, the generator of $\widetilde{K}(\Sigma^{2n})$ yields (up to sign) the canonical generator y_{2n} of the group $kk(\mathbb{C}, S^{2n})$ for every n. We denote x_{2n} the inverse of y_{2n}.

Our goal is to introduce bivariant Thom elements. Throughout this section, all bundles $E \twoheadrightarrow X$ are taken to be smooth complex bundles over a manifold. We denote by $S(E)$ the smooth functions on the total space of the bundle that are Schwartz functions along the fibre (compare also [210]). We define $km(A) := kk_m(\mathbb{C}, A)$ for every locally convex algebra.

We recall from [211], chapter 4, and Atiyah [201], p. 100 that the Thom-element for a complex bundle over a compact space is a certain class $\lambda_E \in K\big(C_0(E)\big)$. Because algebraic K-theory is invariant under passage to smooth subalgebras, and k_m coincides with Phillips K-theory for Fréchet algebras, we may view λ_E as elements in $k_m\big(S(E)\big)$ by choosing a smooth representative of λ_E. The properties of the elements λ_E in topological K-theory then yield the following proposition.

Proposition (6.1.8)[200]: *For every smooth complex bundle $\pi : E \twoheadrightarrow X$ over a compact manifold, there is an element $\lambda_E \in k_m\big(S(E)\big)$ such that the following properties hold*

(i) If E is the trivial n-dimensional complex bundle, then $\lambda_E = \big[1_{S(X)}\big] \cup y_{2n} \in k_m\big(S(X) \otimes_\pi S(\mathbb{C}^n)\big)$.

(ii) $\lambda_E \cup \lambda_{E'} = \lambda_{E \times E'}$, where $E' \twoheadrightarrow Y$ is a bundle over a compact manifold Y and $E \times E'$ denotes the product bundle over $X \times Y$.

Using these elements, we define bivariant Thom elements:

Definition (6.1.9)[200]: *For every two smooth complex bundles $\pi : E \twoheadrightarrow X, \pi' : E' \twoheadrightarrow X$ over a compact manifold X we denote by $\varphi_{E,E'} : S(E \times E') \to S(E \oplus E')$ the map induced by the inclusion $E \oplus E' \hookrightarrow E \times E'$, and set*

$$t_{E,E'} \coloneqq \varphi_{E,E'*}\left(\tau_{S(E)}(\lambda_{E'})\right) \in kk_m\big(S(E), S(E \oplus E')\big).$$

The bivariant Thom elements have the following properties:

Lemma (6.1.10)[200]: *Let X be a compact manifold.*

(i) Let $\pi' : E' \twoheadrightarrow X$ be the complex n-dimensional trivial bundle. Then

$$t_{E,E'} = \tau_{S(E)}(y_{2n}) \in kk_m\big(S(E), S(E) \otimes_\pi S(\mathbb{R}^n)\big).$$

(ii) Let $\pi : E \twoheadrightarrow X, \pi' : E' \twoheadrightarrow X$ be two bundles. Then

$$t_{E,E'} \cap t_{E \oplus E',E''} = t_{E,E' \oplus E''}.$$

Proof. The first statement is straightforward. We abbreviate $\tau_{S(E)}$ by τ_E for a given bundle. Then we have

$$\varphi_{E,E'} * \left(\tau_{E \oplus E'}(\lambda_{E''})\right) = \left(\varphi_{E,E'} \otimes \mathrm{id}_{S(E'')}\right)_* \left(\tau_{E \times E'}(\lambda_{E''})\right) \tag{1}$$

and

$$\theta \coloneqq \varphi_{E \oplus E',E''} \circ \left(\varphi_{E,E'} \otimes \mathrm{id}_{S(E'')}\right) = \varphi_{E,E' \oplus E''} \circ \left(\mathrm{id}_{S(E)} \otimes \varphi_{E',E''}\right). \tag{2}$$

Thus we get

$$\begin{aligned}
t_{E,E'} \cap t_{E \oplus E',E''} &= \varphi_{E,E'*}\big(\tau_E(\lambda_{E'})\big) \cap \varphi_{E \oplus E',E''*}\left(\tau_{E \oplus E'}(\lambda_{E''})\right) \\
&= \varphi_{E \oplus E',E''*}\left(\tau_E(\lambda_{E'}) \cap \varphi_{E,E'*}\left(\tau_{E \oplus E'}(\lambda_{E''})\right)\right) \\
&= \theta_*\left(\tau_E(\lambda_{E'}) \cap \left(\tau_{E \times E'}(\lambda_{E''})\right)\right) \\
&= \theta_*\big(\tau_E(\lambda_{E'} \cup \lambda_{E''})\big)\big) \\
&= \theta_*\big(\tau_E(\lambda_{E' \times E''})\big)\big) \\
&= \varphi_{E,E' \oplus E''*}\left(\tau_E\big(\lambda_{E' \oplus E''}\big)\right) \\
&= t_{E,E' \oplus E''}
\end{aligned}$$

Theorem (6.1.11)[200]: (The smooth Thom isomorphism)**.** *Let X be a closed manifold. There is an invertible element in $kk_m\big(S(X), S(E)\big)$ for every complex bundle $E \mapsto X$.*

Proof. Let $E^{\perp}$ be a bundle such that $E \oplus E^{\perp}$ is trivial. We have by the above Lemma:

$$t_{X,E} \cap t_{E,E^{\perp}} = t_{X,E\oplus E^{\perp}} = \tau_{S(E)}(y_n),$$

for some n, thus $t_{X,E}$ is right invertible. So is $t_{E,E^{\perp}}$ by

$$t_{E,E^{\perp}} \cap t_{E\oplus E^{\perp},E} = t_{E,E^{\perp}\oplus E} = \tau_{S(E)}(y_m).$$

Thus $t_{X,E}$ is right invertible with invertible right inverse, thus invertible.

We fix throughout a real vector space V of dimension n with a positive definite scalar product $\langle\cdot\,|\,\cdot\rangle$, and write $\mathrm{Cliff}(V,\langle\cdot\,|\,\cdot\rangle)$ for the Clifford algebra of V, where we use the convention $v^2 = +\langle v|v\rangle$. We denote by $\mathbb{C}l(V)$ the complexification of $\mathrm{Cliff}(V)$, and abbreviate $\mathbb{C}l(\mathbb{R}^n) \coloneqq \mathbb{C}_n$. See [212] for further details on Clifford algebras. For $v \in V$, we denote by $\varepsilon(v) \in \mathbb{B}(\Lambda^*_{\mathbb{C}}V)$ the operator of exterior multiplication with v. We view $\Lambda^*_{\mathbb{C}}V$ as a Hilbert space with the canonical scalar product. Recall that the Clifford algebra is functorial with respect to linear maps preserving the scalar product, and $\mathrm{Cliff}(V)^{(i)}, i = 0, 1$, denotes the $(-1)^i$-eigenspaces of the map γ induced by $v \mapsto -v$; in particular, the exterior algebra is graded by γ.

Recall that the CAR algebra is the algebra generated by $\varepsilon_k, \bar{\varepsilon}_l$ subject to the relations

$$\{\varepsilon_k, \bar{\varepsilon}_l\} = \delta_{kl}, \quad \{\varepsilon_k, \varepsilon_l\} = 0 = \{\bar{\varepsilon}_k, \bar{\varepsilon}_l\},$$

where $\{a, b\} = ab + ba$ denotes the anticommutator.

Proposition (6.1.12)[200]: *The $*$-algebra generated by $\varepsilon(V)$, where ε denotes the outer multiplication on the exterior algebra, yields a representation of the CAR algebra by setting*

$$\varepsilon_k \coloneqq \varepsilon(e_k) \;\; \textit{and} \;\; \bar{\varepsilon}_k \coloneqq (\varepsilon_k)^*$$

where $\{e_k\}$ is an orthonormal basis for V.

Further, we have the relation

$$\sum_k \bar{\varepsilon}_k \varepsilon_k = \sum_p (n-p)\mathrm{id}_{\Lambda^p_{\mathbb{C}}(V)}.$$

Proof. That the ε_k and $\bar{\varepsilon}_k$ fulfil the relations of the CAR algebra is straightforward to show. Further, if $x = e_{k_1} \wedge \cdots \wedge e_{k_p}$, then

$$\sum_k \bar{\varepsilon}_k \varepsilon_k(x) = \sum_k \bar{\varepsilon}_k \left(e_k \wedge e_{k_1} \wedge \cdots \wedge e_{k_p}\right)$$

$$= nx - \sum_{q=1}^{p} (-1)^{q-1} \delta_k^{k_q} e_k \wedge e_{k_1} \wedge \cdots \wedge \hat{e}_{k_q} \wedge \cdots \wedge e_{k_p}$$

$$= nx - \sum_{q=1}^{p} e_{k_1} \wedge \cdots \wedge e_{k_p}$$

$$= (n-p)x.$$

Proposition (6.1.13)[200]: *Let $\{e_k\}$ be a basis of V. Then the CAR algebra associated to this basis yields anticommuting representations of the Clifford algebras Cliff$(V, -\langle \cdot \mid \cdot \rangle)$ and Cliff$(V, \langle \cdot \mid \cdot \rangle)$ by setting*

$$\hat{\gamma}_k \coloneqq \varepsilon_k - \bar{\varepsilon}_k \quad \text{or} \quad \check{\gamma}_k \coloneqq \varepsilon_k + \bar{\varepsilon}_k.$$

Furthermore, we have the relations

$$\{\hat{\gamma}_k, \check{\gamma}_l\} = 0 \quad \text{and} \quad \delta^{kl} \hat{\gamma}_k \check{\gamma}_l = n - 2 \sum_k \bar{\varepsilon}_k \varepsilon_k \,.$$

Proof. Follows from a direct calculation.

This yields

Corollary (6.1.14)[200]: *There are two anticommuting representations c_+ and c_- of $\mathbb{C}l(V)$ on $\Lambda_{\mathbb{C}}^* V$ induced by*

$$c_+(v) \coloneqq \varepsilon(v) + \bar{\varepsilon}(v) \quad \text{and} \quad c_-(v) \coloneqq i\big(\varepsilon(v) - \bar{\varepsilon}(v)\big).$$

For details concerning KK-theory, we refer to [205]; the unbounded picture was developed in [213].

Let V be a euclidean space. We set $H \coloneqq \Omega_{\mathbb{C}}(V) \coloneqq L^2(V) \widehat{\otimes} \Lambda_{\mathbb{C}}^* V$ and $\coloneqq C_0(V) \widehat{\otimes} \Lambda_{\mathbb{C}}^* V$. Here we view H as graded into forms of even and odd degree, and B carries the grading coming from the canonical grading on the Clifford algebra. Note that $c_+(v) = \varepsilon(v) + \varepsilon(v)^*$ corresponds to multiplication on the left under the identification $\Lambda_{\mathbb{C}}^* V$ with $\mathbb{C}_n$. We denote c_+ and c_- the extensions of c_+ and c_- to representations of $A \coloneqq C_0(V) \widehat{\otimes} \mathbb{C}l(V)$ on H. We denote $f : V \to \mathbb{C}l(V)$ the inclusion of V into its complex Clifford algebra.

Definition (6.1.15)[200]: *We define the unbounded regular operators D_1^0 in H and D_2 in B by*

$$D(D_1^0) = C_c^\infty(V) \widehat{\otimes} \Lambda_{\mathbb{C}}^* V \text{ and } D(D_2) \coloneqq \{\omega \in B | f\omega \in B\}$$

and set

$$D_1^0 \coloneqq d + d^* = \hat{\gamma}^i \partial_i \quad and \quad D_2 \coloneqq M_f$$

where M_f denotes the operator of multiplication by f. Then D_1^0 is essentially self adjoint, and we denote by D_1 its closure.

Lemma (6.1.16)[200]: *(H, c_+, D_1) is an unbounded Kasparov $(C_0(V) \widehat{\otimes} \mathbb{C}l(V), \mathbb{C})$-module.*

Proof. Let $\mu \in C^\infty(V) \widehat{\otimes} \mathbb{C}l(V)$. If μ is just a smooth function h, $[D_1, c_+(h)] = \hat{\gamma}^i \partial_i(h)$, and is therefore bounded if h has bounded derivatives.

We may thus assume μ is not a scalar function, and write it as a product $\mu_1 \cdots \mu_k$, where the μ_i are forms of degree one. Hence

$$[D_1, c_l(\mu)] = \sum_{i=1}^{k} (-1)^{i-1} c_l(\mu_1 \cdots \mu_{i-1}) [D_1, c_l(\mu_i)] c_l(\mu_{i+1} \cdots \mu_k)\,,$$

and we may therefore reduce to the case where μ is of the form $f \cdot e$ for a smooth function f. We set $E \coloneqq \operatorname{span}(e)^\perp$ and decompose $H = \Omega_{\mathbb{C}}^*(E) \oplus \varepsilon(e)\big(\Omega_{\mathbb{C}}^*(E)\big)$. We may further assume that $e = e_1$, where e_1 is the first basis vector of $\mathbb{R}^n$, as the definition of $d + d^*$ is independent of the choice of a basis. With respect to this decomposition, $(d + d^*)$ and $c_l(\mu)$ decompose as

$$c_l(he_1) = c_l(h) c_l(e_1) = \begin{pmatrix} h & \\ & h \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & h \\ h & \end{pmatrix}$$

and

$$\begin{pmatrix} (d_E + d_E^*) & -\partial_1 \\ \partial_1 & -(d_E + d_E^*) \end{pmatrix},$$

where for a real vector space V, d_V denotes the complexified exterior derivative on $\Omega_{\mathbb{C}}^*(V)$. From this it is easily seen that the commutator is bounded.

For f a smooth function of compact support, $f(1 + D_1^2)^{-1} f^*$ is compact, being a pseudodifferential operator of negative order on a compact manifold. Thence $c_+(\mu)(1 + D_1^2)^{-1/2}$ is compact for all $\mu \in C_c^\infty(V) \widehat{\otimes} \mathbb{C}l(V)$, and the result extends to $C_0(V) \widehat{\otimes} \mathbb{C}l(V)$ by continuity.

Definition (6.1.17)[200]: *We set*

$$x_n \coloneqq [(H, c_+, D_1)] \in KK\big(S^n \widehat{\otimes} \mathbb{C}_n, \mathbb{C}\big) \text{ and } y_n \coloneqq [(B, 1, D_2)] \in KK\big(\mathbb{C}, S^n \widehat{\otimes} \mathbb{C}_n\big).$$

The elements x_n and y_n are multiplicative (up to sign, which we neglect):

$$x_n \cup x_m = x_{n+m} \quad \text{and} \quad y_n \cup y_m = y_{n+m}.$$

Hence they can be built up by iterated cup products of the one dimensional x_1 and y_1: More explicitly, on the Hilbert space $H_1 \coloneqq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, consider the unbounded, densely defined operator

$$\begin{pmatrix} & -\partial \\ \partial & \end{pmatrix} : (g,h) \mapsto (-\partial h, \partial g).$$

This corresponds exactly to the operator $d + d^*$ under the isomorphism

$$\varphi : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \widehat{\otimes} \Lambda^*_{\mathbb{C}} \mathbb{R}^n, \qquad (g,h) \mapsto g + hdx.$$

If we identify $C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1$ with $C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ with the standard odd grading, and let the first summand act by diagonal matrices, the second by off-diagonal ones, we obtain again a Kasparov module. Denote the coordinates on a k'th copy H_k of $H = L^2(\mathbb{R})$ by x_k, set

$$S_k \coloneqq 1 \widehat{\otimes} \cdots \widehat{\otimes} \begin{pmatrix} & u_k \\ u_k & \end{pmatrix} \widehat{\otimes} \cdots \widehat{\otimes} 1,$$

where the only entry not equal to 1 is in the k-th tensor-factor. Define

$$S \coloneqq \sum_{k=1}^{n} S_k \in \mathbb{B}\left(\bigotimes_{k=1}^{n} H_k \right).$$

Under the isomorphism

$$\bigotimes_{k=1}^{n} H_k \to L^2(\mathbb{R}^n) \widehat{\otimes} \Lambda^*_{\mathbb{C}} \mathbb{R}^n,$$

$$(g_1, h_1) \widehat{\otimes} \cdots \widehat{\otimes} (g_n, h_n) \mapsto\to (g_1 + h_1 dx^1) \cdots (g_n + h_n dx^n)$$

this operator is easily seen to coincide with $d + d^*$. The Clifford algebras are well known to be multiplicative (for the graded tensor product).

Similarly, we may decompose $C_0(\mathbb{R}^n) \widehat{\otimes} \mathbb{C}_n \approx \widehat{\otimes}_{k=1}^{n} C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1$, and hence get multiplicativity of the elements y_n and a decomposition of the (unbounded) operator

$$T \coloneqq \sum_{k=1}^{n} 1 \widehat{\otimes} \cdots \widehat{\otimes} \begin{pmatrix} & u_k \\ u_k & \end{pmatrix} \widehat{\otimes} \cdots \widehat{\otimes} 1,$$

where the u_k are the coordinate functions. Note the crucial difference in the sign conventions between the definition of $D_1 = S$ and $D_2 = T$.

Proposition (6.1.18)[200]: *The operators D_1 and $c_+(f)$ in H fulfil the following relations*

$$[D_1, c_+(f)] = \sum_p (-n + 2p)\mathrm{id}_{\Omega^p_{\mathbb{C}}(V)} \text{ and } \left(D_1 + c_+(f)\right)^2 = H + [D_1, c_+(f)],$$

where $H = -\Delta + x^2$.

Proof. Using the relations from Proposition (6.1.13) and Proposition (6.1.12), we get

$$\begin{aligned}
[D_1, c_+(f)] &= \hat{\gamma}^i \partial_i \left(x_j \check{\gamma}^j\right) + x_j \check{\gamma}^j \hat{\gamma}^i \partial_i \\
&= \delta_{i,j} \hat{\gamma}^i \check{\gamma}^j + x_j (\hat{\gamma}^i \check{\gamma}^j + \check{\gamma}^j \hat{\gamma}^i) \partial_i \\
&= \delta_{i,j} \hat{\gamma}^i \check{\gamma}^j \\
&= n - 2 \sum_i \bar{\varepsilon}_i \varepsilon_i \\
&= \sum_p (-n + 2p)\mathrm{id}_{\Omega^p_{\mathbb{C}}(V)}
\end{aligned}$$

We recall:

Lemma (6.1.19)[200]: *The Hermite functions* ψ_n *are an orthonormal basis of* $L^2(\mathbb{R})$ *such that*

$$\left(-\partial^2 + x^2 - (2n+1)\right)\psi_n = \psi_n.$$

Proposition (6.1.20)[200]: *If we define the unbounded operator* $\overline{D}$ *on* H *as the closure of the operator* $\left(d + d^* + c_+(x)\right)$ *with domain the smooth complex forms of compact support, then* $(H, 1, \overline{D})$ *is an unbounded cycle representing the product* $y_n \cap x_n$.

Proof. As for every h of compact support and $v \in \mathbb{C}_n$ the operator $h(\overline{D} - D_1)$ is bounded, $(H, c_+, \overline{D})$ is a spectral triple, being a bounded perturbation of (H, c_+, D_1). Further, $\overline{D}$ itself has compact resolvent by the above Lemma, hence $(H, 1, \overline{D})$ is a spectral triple.

Recall that $B \coloneqq C_0(\mathbb{R}^n) \widehat{\otimes} \mathbb{C}_n$, and denote by $\theta : B \widehat{\otimes}_{c_+} H \to H, \mu \dot{\otimes} \omega \mapsto c_+(\mu)\omega$ the canonical isomorphism. Then $\theta^{-1} q(D_1) \theta$ is a B-connexion for $q(D_1)$, as

$$\theta\left(\mu \dot{\otimes} q(D_1)\omega - (-1)^{\partial\mu} \theta^{-1} q(D_1) \theta\left(\mu \dot{\otimes} \omega\right)\right) = [c_+(\mu), q(D_1)](\omega)$$

is compact. Hence as $q(\overline{D})$ is a compact perturbation of $q(D_1)$:

$$\begin{aligned}
&\theta\left(\mu \dot{\otimes} q(D_1)\omega - (-1)^{\partial\mu} \theta^{-1} q(\overline{D}) \theta\left(\mu \dot{\otimes} \omega\right)\right) \\
&= [c_+(\mu), q(\overline{D})]\omega - \left((-1)^{\partial\mu}\left(q(\overline{D}) - q(D_1)\right) c_+(\mu)\right)\omega
\end{aligned}$$

is compact operators, showing that $q(\overline{D})$ is a connection for $q(D)$. It remains only to check the positivity condition for $q(\overline{D})$. The operator in y_n corresponds, under θ, to $C \coloneqq q\big(c_+(x)\big)$, and we set $T_t \coloneqq (1+t+\overline{D}^2)^{-1} \in \mathbb{K}(H)$. First of all, $\overline{D}\big[T_0^{1/4}, C\big]$ is compact, as

$$\overline{D}\big[T_0^{1/4}, C\big] = c\int_0^\infty \overline{D}T_t[C,\overline{D}^2]T_t\,\frac{dt}{t^{1/4}}$$

for a constant c; we have here applied the integral formula from [203]. Further

$$T_0^{1/4}[C, c_+(x)]T_0^{1/4} = T_0^{1/4}\frac{x^2}{\sqrt{1+x^2}}T_0^{1/4} \geq 0.$$

This shows that all the summands in the decomposition of the commutator are either compact or positive.

Theorem (6.1.21)[200]: *x_n and y_n yield KK-equivalences between $S^n \widehat{\otimes} \mathbb{C}_n$ and $\mathbb{C}$, more precisely, $x_n \cap y_n = 1_{S^n\widehat{\otimes}\mathbb{C}_n}$ and $y_n \cap x_n = 1_{\mathbb{C}}$.*

Proof. We will calculate $y_n \cap x_n = 1_{\mathbb{C}}$ and deduce $x_n \cap y_n = 1_{S^n\widehat{\otimes}\mathbb{C}_n}$ from this by the Atiyah rotation trick (cf. [214]). As the elements x_n and y_n are multiplicative and the Kasparov product associative, it suffices to prove the case $n = 1$.

By Proposition (6.1.20), all we have to do is calculate the Fredholm index of $\overline{D}^0 \coloneqq \partial + x$ on $L^2(\mathbb{R})$. Now $\mathrm{Ker}(D^*D) = \mathrm{Ker}(D)$ for any densely defined unbounded operator. Hence by Lemma (6.1.19)

$$\dim \mathrm{Ker}(\overline{D}^0) = \dim \mathrm{Ker}(-\partial^2 + x^2 - 1) = 1,$$

and $\dim \mathrm{Coker}(\overline{D}^0) = 0$, thus $\mathrm{ind}\big(\overline{D}^{(0)}\big) = 1$. This shows that $y_n \cap x_n = 1_{\mathbb{C}}$.

Now comes the rotation trick due to Atiyah, which deduces $x \cap y$ from $y \cap x$. Denote by σ the flip of $\big(C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1\big) \widehat{\otimes} (C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1)$.

Since $SO(n)$ acts on $C_0(\mathbb{R}^n) \otimes \mathbb{C}_n$, the complex unit i induces another isomorphism i_* of $\big(C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1\big) \widehat{\otimes} (C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1)$.

The $*$-automorphisms σ and i_* are determined by their action on a subset which generates (as an algebra) a dense $*$-subalgebra; hence, denoting the basis vectors of $\mathbb{R}^2$ by e_1 and e_2, they are given by

$$\sigma : f(x)e_1 \mapsto f(y)e_2$$
$$g(y)e_2 \mapsto g(x)e_1$$
$$i_* : f(x)e_1 \mapsto f(y)e_2$$
$$g(y)e_2 \mapsto -g(-x)e_1.$$

Therefore $i_* \circ (1 \otimes \rho) = \sigma$, where ρ denotes the automorphism of $C_0(\mathbb{R}) \widehat{\otimes} \mathbb{C}_1$ induced by $-\mathrm{id}_{\mathbb{R}}$.

Homotoping i to $\mathrm{id}_{\mathbb{C}}$, i.e., using that $SO(2) = S^1$ is connected, we see that σ is homotopic to $1 \otimes \rho$. Set $D \coloneqq C_0(\mathbb{R}^2) \widehat{\otimes} \mathbb{C}_2$. Then using that the cup-product coincides with the cap-product over $\mathbb{C}$, we obtain

$$\begin{aligned} x \cap y = y \cup x &= \tau_D^r(y) \cap \tau_D^l(x) \\ &= \tau_D^r(y) \cap \sigma_*\big(\tau_D^r(x)\big) \\ &= \tau_D^r(y) \cap (1 \otimes \rho)_*\big(\tau_D^r(x)\big) \\ &= (1 \otimes \rho)_*\big(\tau_D^r(y \cap x)\big) \\ &= (1 \otimes \rho)_*. \end{aligned}$$

Thence x has y as a left inverse by the first part of the proof, and it is right invertible also - consequently y must be this right inverse as well.

Remark (6.1.22)[200]: We have used that if D is a densely defined unbounded operator with adjoint D^*, then $\mathrm{Ker}(D^*) = \mathrm{Im}(D)^\perp \cap D(D^*)$, and therefore

$$\begin{aligned} \mathrm{Ker}(D^*D) &= \mathrm{Ker}(D) \cup \big(\mathrm{Ker}(D^*) \cap \mathrm{Im}(D)\big) \\ &= \mathrm{Ker}(D) \cup \big(\mathrm{Im}(D)^\perp \cap D(D^*) \cap \mathrm{Im}(D)\big) = \mathrm{Ker}(D). \end{aligned}$$

If V a vector space with a positive definite scalar product, then denote now by $\varepsilon_l(v)$ the exterior multiplication on the left by $v \in V$ on $\Lambda_{\mathbb{C}}^*(V)$, and the exterior multiplication on the right by $\varepsilon_r(v)$. We then also have their adjoints, accordingly denoted ε_l^* and ε_r^*.

By Corollary (6.1.14), we already have anticommuting representations of $\mathbb{C}l(V)$ on $\Lambda_{\mathbb{C}}^*(V)$ on the left by setting $c_+^l = \varepsilon_l + \varepsilon_l^*$ and $c_-^l = i(\varepsilon_l - \varepsilon_l^*)$ of $\mathbb{C}l(V)$.

Further, it is easily seen that under the identification of vector spaces $\Lambda_{\mathbb{C}}^* V \approx \mathbb{C}l(V)$ the natural action of $\mathbb{C}l(V)$ on itself by right multiplication c_+^r corresponds to $\varepsilon_r + \varepsilon_r^*$. And there is yet a fourth representation c_+^r of $\mathbb{C}l(V)$ on the right defined by $i(\varepsilon_r - \varepsilon_r^*)$.

The representations c_+^r and c_-^r again anticommute, and c_+^r commutes with c_+^l because they correspond to right and left multiplication on $\mathbb{C}l(V)$. The definition of the element y_n from above is thus actually based on the use of c_+^l for the operator and c_+^r for the right action. With this at hand, we may define another version of the Dirac and Bott elements. We denote by f_- the function $V \to \mathbb{B}(\Lambda_{\mathbb{C}}^* V)$ given by $v \mapsto c_-^l(v)$, and by $\varphi : C_0(V) \to \mathbb{B}(H)$ the action by multiplication.

Proposition/Definition (6.1.23)[200]: *If we define the unbounded operators D_1' and D_2' by*

$$D(D_1') \coloneqq C_c^\infty(V) \widehat{\otimes} \Lambda_{\mathbb{C}}^* V \text{ and } D(D_2') \coloneqq \{\omega \in B | f_{-\omega} \in B\}$$

and set

$$D_1' \coloneqq i\check{\gamma}^k \partial_k \text{ and } D_2' \coloneqq M_{f_-}$$

then we obtain Kasparov $\big(C_0(V), \mathbb{C}l(V)\big)$- and $\big(\mathbb{C}l(V), C_0(V)\big)$-modules (H, φ, D_1') and (B, c_+^l, D_2').

Remark (6.1.24)[200]: One easily checks that, for example $c_+^l(v) = (\varepsilon_r - \varepsilon_r^*)(v) \circ \gamma$, where γ denotes again the grading operator; hence all these actions are all distinct. It is also straightforward to see that the element D_1 from the first definition, for example, corresponds to the left handed Dirac operator for the scalar product $-\langle \cdot \mid \cdot \rangle$.

We discuss how principal *G*-bundles induce a functor from the category KK_G to KK (more precisely, RKK).

We assume that all fibre bundles $p : E \twoheadrightarrow X$ are locally trivial; the projection is then open (for principal bundles p is always open.) Also, if the structure group and base are (locally) compact, the corresponding principal bundle is (locally) compact. In fact, it is not hard to show the stronger statement that the projection is closed when the fibre is compact and all spaces are locally compact and all groups are compact.

For any locally compact group G and right G-spaces F, F' we denote by $C_G(F, F')$ the continuous equivariant maps from F to F' that vanish at infinity. For a bundle $E \twoheadrightarrow X$ over a topological space we denote by $\Gamma(E)$ the continuous sections vanishing at infinity.

When speaking of principal bundles, we will always assume that the action of the appropriate group on the principal bundle is on the right. As usual, a topological space with continuous action of a topological group G will be called a G-space.

Let X be a topological space, $P \twoheadrightarrow X$ a principal G-bundle for a group G and Y a left G-space. Then

$$\Gamma(P \times_G Y) \cong C_G(P, Y)$$

where Y is viewed as a right G-space. In particular

$$\Gamma\big(P \times_G C(Y)\big) \cong C(P \times_G Y)$$

where G acts on $C(Y)$ by $(g.f)(y) \coloneqq f(g^{-1}y)$ for all $g \in G, y \in Y$ and $f \in C(Y)$. In particular, $\Gamma\big(P \times_G C(\mathbb{R}^n)\big) \cong C(E)$, where $E \twoheadrightarrow X$ is a vector bundle associated to the principal G-bundle $P \twoheadrightarrow X$ with fibre a G-space $\mathbb{R}^n$.

In general, given a topological group G, a topological left G-space F and topological principal G-bundle $P \twoheadrightarrow X$ over a topological space X, we set $F_P \coloneqq \Gamma(P \times_G F)$. Then $(P \times_G \cdot) = \cdot_P$ is actually a functor from the category of left G-spaces and G-maps to the category of "$C(X)$"-objects in the target category; if $\varphi : A \to B$ is a $*$-homomorphism, we denote by φ_P the map $A_P \to B_P$.

A B-Hilbert module E is called a G-Hilbert module, if B is a G-algebra and the group G acts on E continuously through B-linear operators such that for all $g \in G, \xi, \eta \in E, b \in B$:

$$g(\xi b) = (g\xi)(gb) \text{ and } \langle g\xi | g\eta \rangle = g\langle \xi | \eta \rangle.$$

E_P is then a Hilbert B_P-module by setting for all $\tilde{\xi}, \tilde{\eta} \in C_G(P, E) \approx \Gamma(P \times_G E)$

$$\langle \tilde{\xi} | \tilde{\eta} \rangle(p) \coloneqq \langle \tilde{\xi}(p) | \tilde{\eta}(p) \rangle,$$

which is clearly an element of $C_G(P, B)$.

Definition (6.1.25)[200]: *A $C(X)$-algebra, where X is a locally compact space, is a C^*-algebra A with a homomorphism $C(X) \to Z\big(M(A)\big)$ into the center of the multiplier algebra of A that is nongenerate, i.e. $C(X)A = A$.*

Note that $C(X)A = A$ is equivalent to $C(X)A$ is dense in A, by Proposition 1.8 in [215]. Every Hilbert A-module E over a $C(X)$-algebra inherits a $C(X)$- action through bounded operators because E can be viewed as an $M(A)$- module; every $T \in \mathbb{B}(E)$ is then automatically $C(X)$-linear.

Proposition (6.1.26)[200]: *Let E be a G-Hilbert module over a G-C^*-algebra $B, P \twoheadrightarrow X$ a principal G-bundle. Then*

$$\mathbb{B}_{B_P}(E_P) \cong \Gamma_b\big(P \times_G (\mathbb{B}_B(E), str)\big)$$

where $\mathbb{B}_B(E)$ carries the strict operator topology and the sections are bounded in norm. Further

$$\mathbb{B}_{B_P}(E_P) \cong \Gamma_b\left(P \times_G \left(\mathbb{B}_B(E), \|\cdot\|_{op}\right)\right).$$

The spectrum of an operator is the closure of the union of the spectra in the fibres.

Proof. This follows easily by using the description of sections as equivariant maps and local trivialisations.

We suppose for the rest that all spaces are second countable.

Proposition/Definition (6.1.27)[200]: *Let $P \twoheadrightarrow X$ be a principal G-bundle over X. For every Kasparov module $x = (E, \varphi, F)$ we define a new Kasparov module (using Proposition (6.1.29))*

$$x_P \coloneqq (E_P, \varphi_P, F_P).$$

Proof. Obvious by Proposition (6.1.26).

The definition of $RKK_X(A, B) \coloneqq RKK(X; A, B)$ is given in [216].

Proposition (6.1.28)[200]: $._P$ *yields a functor from KK_G to KK, more precisely with values in RKK_X. In particular: $x_P \cap y_P = (x \cap y)_P$.*

Proof. The only nontrivial part remaining is compatibility with the product. Using an equivariant product for two given modules, it is easily seen that the fibration of the product is a product of the fibrations of the modules using Proposition (6.1.26).

We give a proof of a version of the Thom isomorphism that was stated without proof in [216]. It says that the two algebras one can naturally associate to a bundle – the functions on the total space and sections of the cliffordified bundle – are naturally the same in KK-theory.

Theorem (6.1.29)[200]: *If $E \twoheadrightarrow X$ is a vector bundle, then the associated algebras $\Gamma\big(\mathbb{C}l(E)\big)$ and $C(E)$ are isomorphic in KK, i.e., admit a KK-equivalence.*

If E is a spinc-bundle, then $\Gamma(E)$ and $C(X)$ are KK-equivalent (through an element of KK^1 in odd dimension).

Proof. We choose a hermitian metric on E and write E as the associated bundle to an $O(n)$-principal bundle, $E = P \times_{O(n)} \mathbb{R}^n$. The complex Clifford bundle does not depend on the choice of hermitian metric and decomposes as $\mathbb{C}l(E) = P \times_{O(n)} \mathbb{C}_n$.

Let $[x_n] \in KK_{O(n)}\big(C(\mathbb{R}^n) \widehat{\otimes} \mathbb{C}_n, \mathbb{C}\big)$ be the class of the Bott element and $[y_n]$ its inverse. Then equipping a second copy of $\mathbb{C}_n$ with the same $O(n)$-action, we can define a $\big(C(\mathbb{R}^n) \widehat{\otimes} \mathbb{C}_{2n}, \mathbb{C}_n\big)$-module $x \coloneqq \tau^r_{\mathbb{C}_n}(x_n)$. This yields modules

$$x_P \coloneqq \left(\tau^r_{\mathbb{C}_n}(x_n)\right) P \text{ and } y_P \coloneqq \left(\tau^r_{\mathbb{C}_n}(y_n)\right) P$$

that are inverse to each other by Proposition (6.1.28), and hence

$$\Gamma\big(\mathbb{C}l(E)\big) \sim \Gamma\big(\big(P \times_G \mathbb{C}(\mathbb{R}^n)\big) \widehat{\otimes}_{C(X)} \Gamma\big(\mathbb{C}l(E \oplus E)\big)$$

are KK-equivalent. The bundle $\mathbb{C}l(E \oplus E)$ has a canonical complex structure, is thus $spin^c$, and hence Morita equivalent to $C(X)$ (by [217], 2.11 Theorem); Morita invariance of KK now proves the theorem.

Section (6.2): Homological Invariants of Locally Convex Algebras:

The subject of this Section are homology-type invariants of locally convex algebras. It analyses to what extend locally convex bimodules are universal for such homology theories.

Let us first recall the situation in the setting of (separable) C^*-algebras. On this category, we have at our disposal Kasparov's bifunctor KK. When KK is restricted to the first or second variable, one obtains K-homology or K-theory, respectively. The elements of KK may be described in essentially three ways: The extension picture, the module picture and the quasihomomorphism picture. The extension picture yields, when specialised to the first variable, the Ext-theory of Brown, Douglas and Fillmore ([219]), which is based on earlier work of Busby ([220]); here classes in $\mathrm{Ext}(A, \mathbb{C})$ are described as extensions of A by the compact operators. In the module picture, classes in $KK(A, B)$ are represented by bounded operators acting on a Hilbert B-module, which carries a left A-action ([221]); in the so-called Baaj-Julg picture, the operator is replaced by a regular operator on the Hilbert module. In the quasihomomorphism picture, all information is contained in a pair of homomorphisms, both from A into an algebra containing $\mathbb{K}\otimes B$ as an ideal, with difference in $\mathbb{K}\otimes B$ ([222]/[223]).

In each of these pictures, there is a corresponding description of the Kasparov product $KK(A, B) \times KK(B, C) \to KK(A, C)$. In the extension picture, it corresponds to a splice, or Yoneda product, of extensions ([224]). In the module picture it can be defined rather explicitly using certain operators M and N, whose existence is guaranteed by Kasparov's technical

theorem. The proof of this theorem is based on properties of the category of C^*-algebras; so is the proof of the fundamental theorem in ([222]), which is used in the definition of the product in the quasihomomorphism picture. The most calculable description of the product is the one in the module picture, where it is a generalisation of the Atiyah-Singer sharp product.

It is well known that Kasparov's bivariant K-functor is universal among certain functors. If H is a split exact, homotopy invariant and stable functor, then every class in $KK(A,B)$ induces a homomorphism $H(A) \to H(B)$. Furthermore, it can be shown that this induction process is compatible with Kasparov products: if we are given classes $x \in KK(A,B)$ and $y \in KK(B,C)$, then the homomorphism induced by the Kasparov product of x and y is the composition of the homomorphisms induced separately by x and y.

In the locally convex setting, the situation is different. We still have at our disposal a bivariant K-theory, namely the theory kk constructed by Cuntz ([225]/[226]). The definition of kk is based on linearly split extensions of locally convex algebras of arbitrary length, and the composition corresponds again to the Yoneda product of extensions. However, elements of kk still appear naturally in form of what we will call locally convex Kasparov modules. For example, a spectral triple on a compact spinc-manifold M satisfies additional summability conditions, and therefore defines a locally convex (A,B)-Kasparov module, where $A := C^\infty(M)$ and I is some ideal in the algebra bounded operators on a Hilbert space. To this object, one may associate a quasihomomorphism φ of locally convex algebras, which in turn induces a homomorphism $H(\varphi): H(A) \to H(B)$ for every split exact functor. Already at this stage, there are some unresolved issues. If we take $H := kk(\cdot, B)$, then it is not clear in how far elements of the form $kk(\varphi, B)$, where φ is the quasihomomorphism associated to a locally convex (A,B)-Kasparov module x, exhaust the group $kk(A,B)$. In [227], it suggested that this is not the case. However, as we noted above, elements of kk can be represented as such locally convex bimodules, and we show that they suffice to obtain the isomorphisms under split exact, diffotopy invariant L^p-stable functors we are interested in.

We therefore develop a complete framework of locally convex Kasparov modules, together with equivalence relations between them, and study how they induce morphisms under split exact functors on the category of locally convex algebras. We show that smooth

versions of Kasparov's Dirac and dual-Dirac elements appear naturally as locally convex Kasparov modules.

The crucial, and most difficult question now is to what extent the above compatibility of the induction process with Kasparov products still holds in this more general context. Essentially, the fact that in the C^*-setting any two composable classes in KK have a product which is again represented by a Kasparov product, corresponds to the fact that the splice of two extensions of length, say two, may again be represented by a length two extension; the analogous property for a functor on locally convex algebras is very likely false.

It is thus necessary to find sufficient conditions for a locally convex Kasparov module to represent the composition $H(\psi) \circ H(\varphi)$ of two homomorphisms induced under a functor H by quasihomomorphisms coming from locally convex Kasparov modules. There are two closely related approaches to this problem. The first consists in finding operators M and N, as in Kasparov's original description of the product, that satisfy the correct algebraic relations with respect to locally convex algebras involved in the construction of the product. This approach is feasible at least to some extent, but it turns out to be very technical.

The second approach, which we will follow here, is based on a new interpretation of the Connes-Skandalis connection formalism ([228]). Already in the C^*-setting, it yields a genuine description of the Kasparov product, based only on Kasparov's version of the theorem of Voiculescu ([229]). The essential idea is that the existence of a product for two (locally convex) Kasparov modules may be interpreted as the existence of an extension of a Kasparov module. Just like the connection formalism by Connes and Skandalis, this extension condition only yields a sufficient criterion for a cycle to represent a product; the existence of such an extension remains a consequence of Kasparov's technical theorem, whose validity is restricted to the C^*-setting. However, the existence of the aforementioned extension is easier to show than the existence of the operators M and N. In the case where the first cycle is represented by a class in K-theory, we show that it is always possible to construct such an extension, and thus show the existence of a product in complete generality. This fact yields Bott periodicity. Using the formalism of locally convex Kasparov modules, we are also able to prove a smooth version of the Thom isomorphism theorem for the algebra of fibrewise

Schwartz functions on a smooth vector bundle over a compact smooth manifold for every split exact, $\mathcal{L}^p$-stable, diffotopy invariant functor.

The formalism of locally convex Kasparov modules is further useful in order to define explicitly classes in kk, for example those related to pseudodifferential operators. This is quite difficult in general due to the very abstract definition of kk. Our considerations also show that kk is very likely not an analogue of Kasparov's KK-theory, but rather of the E-theory of Connes and Higson [230].

In the sequel, we will work on the category $\mathcal{LC}$ of locally convex algebras. By a locally convex algebra we will mean a complete locally convex vector space that is at the same time a topological algebra, i.e., the multiplication is continuous (and associative). This means that for every continuous seminorm p on $\mathcal{A}$ there is a continuous seminorm q on A such that for all $a, b \in A$

$$p(ab) \leq q(a)q(b).$$

Recall that for two locally convex topological vector spaces V and W, their projective tensor product is defined as the algebraic tensor product $V \odot W$ of V and W with the locally convex structure induced from the family of seminorms defined for all $x \in V \odot W$ by

$$p \odot q(x) := \inf\left\{\sum_{i=1}^{n} p(v_i)q(w_i) | x = \sum_{i=1}^{n} v_i \odot w_i\right\},$$

where p and q run through systems of seminorms defining the topologies of V and W (see [231]). The completed projective tensor product of complete locally convex spaces as above will be denoted $V \otimes_\pi W$, the extension of $p \odot q$ to $V \odot_\pi W$ will be denoted by $p \otimes q$. The topology on $V \otimes_\pi W$ is equal to the completion of the final topology for the canonical map $V \times W \to V \odot W$.

The injective tensor product of locally convex spaces is defined in [232], or again in the original [231]. For complete locally convex spaces V and W, the completion of $V \odot W$ in this topology will be denoted $V \otimes_\varepsilon W$; if V and W are Banach spaces, $V \otimes_\varepsilon W$ is a Banach space with a certain norm $\|\cdot\|_\varepsilon$. Recall that for complete subspaces $V' \leq V$ and $W' \leq W, \otimes_\varepsilon$, , in contrary to $\otimes_\pi$, does have the property that $V' \otimes_\varepsilon \hookrightarrow V \otimes_\varepsilon W$ (see [232] Proposition 43.7).

For C^* -algebras A and B, the minimal tensor product $A \otimes_* B$ is obtained by taking the tensor product of the Hilbert spaces from a faithful representation of A and B and pulling back (and completing) the induced norms on $A \odot B$ (see [233] for details, e.g., the independence of the presentation, nuclearity, exactness and so on); we denote the obtained C^*-norm by $\|\cdot\|_*$. The maximal tensor product $A \otimes_{\max} B$ is obtained as the completion in the norm $\|\cdot\|_{\max}$ given by the supremum over all norms coming from $*$-representations of A and B. If B is nuclear we will often not distinguish between these two C^*-tensor products.

We denote by $\mathcal{Z}$ the locally convex algebra of differentiable functions from $[0,1]$ all of whose derivatives vanish at the endpoints, by C those that are zero in 1, and by S those that are zero in both endpoints; ; we define S^n as the n-fold tensor product of Sn with itself, similarly for C^n and $\mathcal{Z}^n$. We further get functors mapping a locally convex algebra $\mathcal{A}$ to $\mathcal{Z}\mathcal{A}$, $C\mathcal{A}$ and $S\mathcal{A}$ as in the C^*-setting. We write $ev_t^{\mathcal{A}}: Z\mathcal{A} \to \mathcal{A}$ for the evaluations in $t \in I$.

Definition (6.2.1)[218]: Let $\varphi_0, \varphi_1: \mathcal{A} \to \mathcal{B}$ be homomorphisms of locally convex algebras. A diffotopy between φ_0 and φ_1 is by definition a homomorphism $\Phi: \mathcal{A} \to \mathcal{Z}\mathcal{B}$ such that $ev_i^{\mathcal{B}} \circ \Phi = \varphi_i$ for $i = 0,1$.

We note from [234] that this is not the same as a family of morphisms that is pointwise differentiable.

Remark (6.2.2)[218]: The well known decomposition for Hilbert spaces into matrices with respect to a projection carries over to locally convex algebras as follows:

Let $a \in \mathcal{A}$ and p an idempotent in some larger algebra containing $\mathcal{A}$. Set

$$a_1 := pap, \quad a_2 := pa(1-p), \quad a_3 := (1-p)ap, \quad a_4 := (1-p)a(1-p),$$

which entails $a_1 + a_2 + a_3 + a_4 = a$. As all of the subalgebras $p\mathcal{A}p, (1-p)\mathcal{A}(1-p), (1-p)\mathcal{A}p$, and $p\mathcal{A}(1-p)$ have empty intersection, A is isomorphic to the inner direct sum of them (as a locally convex space). If we write the a as a 2×2-matrix

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

then the multiplication on $\mathcal{A}$ corresponds to the multiplication of matrices. This is usually called a Morita context (see e.g. [235]); we have reserved this expression for a more analytic version of a context below.

We collect some general statements concerning functors and exact sequences of locally convex algebras.

Definition (6.2.3)[218]: An exact sequence

$$0 \to C \to \mathcal{B} \to \mathcal{A} \to 0$$

of locally convex algebras will be called semisplit if it is split as a sequence of topological vector spaces. It will be called a split exact sequence, if it is split in the category of locally convex algebras. It will be called a double split sequence, if there are two splits for the quotient map.

Definition (6.2.4)[218]: A functor H on the category of locally convex algebras with values in the category of abelian groups will be called split exact, if for every split exact sequence

$$0 \to C \to \mathcal{B} \to \mathcal{A} \to 0$$

of locally convex algebras the sequence

$$0 \to H(C) \to H(\mathcal{B}) \to H(\mathcal{A}) \to 0$$

is exact.

Remark (6.2.5)[218]: If H is split exact, then the sequence obtained by applying H automatically splits.

Definition (6.2.6)[218]: Let H be a functor from locally convex algebras to abelian groups, and J a locally convex algebra containing an idempotent p. Then H is called

(a) J-stable if $H(\iota)$ is an isomorphism, denoted $\theta_{\mathcal{A}}^{J}$, for every locally convex algebra $\mathcal{A}$, where $\iota: \mathcal{A} \to \mathcal{A} \otimes_{\pi} J, a \mapsto a \otimes p$.

(b) invariant under inner automorphisms if $H(Ad_U) = id_{H(\mathcal{A})}$ for every invertible $U \in \mathcal{A}$,

(c) pointwise diffotopy invariant, if for every family $\varphi_t: \mathcal{A} \to \mathcal{B}$ of morphisms of locally convex algebras such that $\varphi_t(a)$ is smooth for all $a \in \mathcal{A}, H(\varphi_0) = H(\varphi_1)$.

(d) diffotopy invariant if for every homomorphism $\varphi: \mathcal{A} \to \mathcal{B}$ we have $H(ev_0) \circ H(\varphi) = H(ev_1) \circ H(\varphi)$.

Remark (6.2.7)[218]: If P is a class of idempotents in J that are all conjugate to another, in the sense that J is an ideal in a larger algebra $\hat{J}$ such that for any two minimal projections p and q there is an invertible element $u \in \hat{J}$ with $u^{-1}pu = q$, then the definition does not depend on the choice of idempotent in P if we suppose that H is invariant under conjugation by such u

(of course it suffices that conjugation by u is continuous $\mathcal{B} \to \mathcal{B}$, so one can weaken the ideal condition).

For example, the minimal projections in the compacts or the p-summable operators have this property.

We then have the following properties:

Proposition (6.2.8)[218]: On the category of locally convex algebras, if H is a functor with values in abelian groups, then

(i) if H is split exact and φ and ψ are orthogonal homomorphisms, $H(\varphi + \psi) = H(\varphi) + H(\psi)$

(ii) if H is M_n stable for some n, then H is M_m-stable for all $m \in \mathbb{N}$

(iii) if H is M_2-stable, then it is invariant under inner automorphisms

(iv) if H is $\mathcal{L}^p$-stable, then the natural map $M_n(\mathcal{A}) \hookrightarrow \mathcal{A} \otimes_\pi \mathcal{L}^p$ induces an isomorphisms

(v) $\mathcal{L}^p$-stability implies M_n-stability

(vi) $H(\iota) = n\theta^{M_n}$ if H is M_n-stable and additive, with $\iota: \mathcal{A} \to M_n(\mathcal{A}), a \mapsto 1_n \otimes a$, a the canonical inclusion; similarly, $H(\iota) = n\theta^{\mathcal{L}^p}$ if we view $M_n(A)$ as a sitting inside $\mathcal{A} \otimes_\pi \mathcal{L}^p$.

Proof.

(i) This follows as H preserves direct sums:

$$H(A \oplus B) = H(A) \oplus H(B).$$

Therefore, $H(\Delta_A) = \Delta_{H(A)}$, where we denote by $\Delta: A \to A \oplus A, a \mapsto (a, a)$ the diagonal. The result follows from $\varphi + \psi = \varphi \oplus \psi \circ \Delta_A$ (compare, for example, [236], Proposition 3.1.2).

(iii) is also proved in [236]: If $U \in \mathcal{A}$ is invertible, then $\begin{pmatrix} U & \\ & 1 \end{pmatrix}$ defines an automorphism of $\mathcal{A} \otimes M_2$. As the two natural inclusions of $\mathcal{A}$ in $\mathcal{A} \otimes M_2$ are the same under any M_2-stable functor, U acts trivially.

If $\mathcal{H}$ is a separable Hilbert-space, then factoring the natural inclusion $\iota: M_n \to \mathcal{L}^p(\mathcal{H})$ as the stabilisation $M_n \to M_n \otimes_\pi \mathcal{L}^p(\mathcal{H})$ followed by the canonical identification $\mathcal{L}^p(\mathcal{H}^n) \approx \mathcal{L}^p(\mathcal{H})$- which both induce isomorphisms - we get (iv) in the scalar case; tensoring by $\mathcal{A}$ gives the general statement.

(v) follows now from commutativity of

$$\begin{tikzcd} H(\mathcal{A}) \arrow[d, "\theta_{\mathcal{A}}^{M_n}"'] \arrow[dr, "\theta_{\mathcal{A}}^{\mathcal{L}^p}"] & \\ H\big(M_n(\mathcal{A})\big) \arrow[r, "H(\iota\otimes \mathrm{id}_A)"'] & H(\mathcal{L}^p\otimes_\pi \mathcal{A}) \end{tikzcd}$$

The first part of (vi) follows from (iii), and entails the second.

We restate the following fundamental result from [236], Theorem 4.2.1, in the form appropriate for later use (using Proposition (6.2.8)-(v)):

Theorem (6.2.9)[218]: Every functor from the category of locally convex algebras to the category of abelian groups which is split exact and $\mathcal{L}^p$-stable for some $p \geq 2$ is diffotopy invariant.

The following definition of Morita context is basically from [226], but we carry it over to arbitrary functors. We also add some isomorphisms to the definitions in [226] in order to make the existence of a Morita-context a weaker condition than being isomorphic.

Definition (6.2.10)[218]: Let A and B be locally convex algebras. Then a Morita-context from A to B is given by data $(\varphi, D, \psi, \xi_i, \eta_i)$, where D is a locally convex algebra, $\varphi: A \to D, \psi: B \to D$ are isomorphisms onto subalgebras of D, and sequences η_i, ξ_i in D such that

(i) $\eta_i \varphi(A) \xi_j \in \psi(B)$ for all i, j

(ii) $\big(\eta_i \varphi(a) \xi_j\big)_{ij} \in \mathcal{K} \otimes_\pi \psi(B)$

(iii) $\sum \xi_i, \eta_i\, \varphi(a) = \varphi(a)$ for all $a \in A$ (convergence in $\varphi(A)$).

With this definition:

(a) If $\varphi: A \to B$ is an isomorphism, then we get aMorita context $(\varphi^+, B^+, \cdot^+)$ from A to B, where B^+ is the unitization of $B, b \mapsto b^+$ the canonical embedding, and $\varphi^+ = \varphi \circ \cdot^+$

(b) In particular, there is now a canonical Morita-context from A to A, for any locally convex algebra A

(c) If $B \subseteq \mathcal{K}\otimes_\pi C$ is a subalgebra, and we call a corner in B a subalgebra $A \subseteq B$ of the form $\sum_{i=0}^{k} e_{0i}\, B \sum_{i=0}^{k} e_{i0}$, then there is a trivial context from A to B

(d) If B is row and column-stable ($e_{0i}B, Be_{i0}$ for all i), then there is a context from B to A, where B is a subalgebra of $\mathcal{K}\otimes_\pi A$.

Definition (6.2.11)[218]: Let H be a functor on the category of locally convex algebras. Then we define $\theta: A \to \mathcal{K}\otimes_\pi B, a \mapsto \left(\psi^{-1}(\eta_i\varphi(a)\xi_i)\right)_{ij}$ and $H(\varphi, D, \psi, \xi_i, \eta_i) \coloneqq H(\theta)$.

Note that for $a, a' \in A$ we have

$$\left(\psi^{-1}(\eta_i\varphi(a)\xi_i)\right)_{ij}\left(\psi^{-1}(\eta_k\varphi(a')\xi_l)\right)_{kl} = \left(\psi^{-1}\left(\eta_i\varphi(a)\sum_m \xi_m, \eta_m\,\varphi(a')\xi_i\right)\right)_{ij}$$

which equals $\theta(aa')$ by (iii) in the definition of a Morita context. Hence θ is indeed a homomorphism.

Definition (6.2.12)[218]: A Morita-bicontext between locally convex algebras A and B is given by two Morita-contexts from A to B and B to A respectively, of the form $(\varphi, D, \psi, \xi_i^A, \eta_i^A)$ and $(\psi, D, \varphi, \xi_i^B, \eta_i^B)$ such that

(i) $\varphi(A)\xi_i^A\xi_i^B \subseteq \varphi(A), \eta_i^B\eta_i^A\varphi(A) \subseteq \varphi(A)$ (left compatibility)

(ii) $\psi(B)\xi_j^B\xi_j^A \subseteq \psi(B), \eta_i^A\eta_j^B\psi(B) \subseteq \psi(B)$ (right compatibility).

Theorem (6.2.13)[218]: Given two Morita contexts as in the above definition, and a diffotopy invariant, $\mathcal{K}$-stable functor H, we have

$H(\psi, D, \varphi, \xi_i^B, \eta_i^B) \circ H(\varphi, D, \psi, \xi_i^A, \eta_i^A) = \mathrm{id}_{H(A)}$ if they are left compatible

$H(\varphi, D, \psi, \xi_i^B, \eta_i^B) \circ H(\psi, D, \varphi, \xi_i^B, \eta_i^B) = \mathrm{id}_{H(B)}$ if they are right compatible

Proof. The proof is an adaptation of the one from [226] Lemma 7.2. Denote the isomorphism $H(A) \to H(\mathcal{K}\otimes_\pi A)$ given by $\mathcal{K}$-stability by ε_A. Then more precisely, we have to show that

$$H(\varphi, D, \psi, \xi_i^B, \eta_i^B) \circ \varepsilon_B^{-1} \circ H(\varphi, D, \psi, \xi_i^A, \eta_i^A)$$

is invertible. We suppose left compatibility; then denoting θ^A and θ^B the maps $A \to \mathcal{K}\otimes_\pi B$ and $B \to \mathcal{K}\otimes_\pi A$ determined by the two contexts, and multiplying by $\varepsilon_{\mathcal{K}\otimes_\pi A}$ on the left, we see that it suffices to show that the composition $(\mathcal{K}\otimes\theta_B) \circ \theta_A$ induces an invertible map under H. Now this is the map

$$a \mapsto \left(\varphi^{-1}\left(\eta_i^B(\eta_k^A\varphi(a)\xi_l^A)\xi_j^B\right)\right)_{iklj} \tag{3}$$

and it is diffotopic to the stabilisation as follows. The $L \times L$ matrix with entries $\varphi^{-1}\left(\hat{\eta}_\alpha(t)\varphi(a)\hat{\xi}_\beta(t)\right), \alpha, \beta \in \mathbb{N}^2 \cup \{0\}$ with

$$\begin{array}{ll} \hat{\xi}_0(t) = \cos(t)\,1 & \hat{\xi}_{il}(t) = \sin(t)\,\xi_i^B\xi_l^A \\ \hat{\eta}_0(t) = \cos(t)\,1 & \hat{\eta}_{kj}(t) = \sin(t)\,\eta_k^B\eta_j^A \end{array}$$

yields a diffotopy of the map in equation (3).

The notion of quasihomomorphisms was introduced in [223], and used for example in [236] to induce elements in kk. We modify it according to our needs.

Definition (6.2.14)[218]: Let $\mathcal{A}$ and $\mathcal{B}$ be locally convex algebras, $\widehat{\mathcal{B}}$ an algebra containing $\mathcal{B}$ as a subalgebra. Then an $\mathcal{LC}$-quasihomomorphism from $\mathcal{A}$ to $\mathcal{B}$ (relative to $\widehat{\mathcal{B}}$) is given by a pair $(\alpha, \bar{\alpha})$ of morphisms $\mathcal{A} \to \widehat{\mathcal{B}}$ such that the maps

(a) $\mathcal{A} \to \widehat{\mathcal{B}}, a \mapsto \alpha(a) - \bar{\alpha}(a)$,

(b) $\mathcal{A} \times \mathcal{B} \to \widehat{\mathcal{B}}, (a, b) \mapsto \alpha(a)b$ and

(c) $\mathcal{B} \times \mathcal{A} \to \widehat{\mathcal{B}}, (b, a) \mapsto b\alpha(a)$

are all three actually $\mathcal{B}$-valued and continuous (with respect to the topology on $\mathcal{A}$ and $\mathcal{B}$). We denote such a quasihomomorphism by $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$.

If $\varphi: C \to \mathcal{A}$ is a homomorphism of locally convex algebras, then $(\alpha \circ \varphi, \bar{\alpha} \circ \varphi): C \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ defines a quasihomomorphism which we will denote $(\alpha, \bar{\alpha}) \circ \varphi$.

Split exact functors have additional functoriality properties with respect to quasihomomorphisms. We first define the objects involved in the construction.

Lemma (6.2.15)[218]: Let $\mathcal{A}$ be a locally convex algebra, $\widehat{\mathcal{B}}$ an algebra and $\mathcal{B}$ a locally convex subalgebra of $\widehat{\mathcal{B}}$, and $\alpha: \mathcal{A} \to \widehat{\mathcal{B}}$ a homomorphism of algebras such that the maps $\mathcal{A} \times \mathcal{B} \to \widehat{\mathcal{B}}, (a, b) \mapsto \alpha(a)b$ and $\mathcal{B} \times \mathcal{A} \to \widehat{\mathcal{B}}, (b, a) \mapsto b\alpha(a)$ are actually $\mathcal{B}$-valued and continuous (with respect to the topology on $\mathcal{A}$ and $\mathcal{B}$). Then the sum of topological vector space $D \coloneqq \mathcal{A} \oplus \mathcal{B}$, equipped with the multiplication

$$(a, b)(a', b') \coloneqq (a', \alpha(a)b' + b\alpha(a') + bb')$$

is a locally convex algebra D_α. $\mathcal{B}$ identifies to an ideal in D_α that carries the subspace topology via the inclusion $\iota_{\mathcal{B}}: \mathcal{B} \to D_\alpha, b \mapsto (0, b)$ and $\mathcal{A}$ is a quotient of D_α

Further D_α fits into a split short exact sequence

$$0 \to \mathcal{B} \to D_\alpha \to \mathcal{A} \to 0$$

with split $a' \coloneqq \mathrm{id}_{\mathcal{A}} \oplus 0$.

Proof. By hypothesis, the multiplication is continuous. Further, we may identify D_α with the subalgebra

$$\left\{(a, x) \in \mathcal{A} \times \widehat{\mathcal{B}} | x - \alpha(a) \in B\right\}$$

of $\mathcal{A} \times \widehat{\mathcal{B}}$, and hence the multiplication is associative.

Remark (6.2.16)[218]: If $\mathcal{A}, \widehat{\mathcal{B}}, \mathcal{B}$ and α come from an $\mathcal{LC}$-quasihomomorphism $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$, then $\bar{\alpha}' \coloneqq \mathrm{id}_{\mathcal{A}} \oplus (\alpha, \bar{\alpha}): \mathcal{A} \to D_\alpha$ defines another split of the above extension. Further we see that it is actually possible to suppose that $\widehat{\mathcal{B}}$ is a locally convex algebra and $\alpha, \bar{\alpha}$ are morphisms of locally convex algebras. However, the elements we will use do not naturally have this form.

We will usually identify $\mathcal{B}$ with its image in D and write α and $\bar{\alpha}$ instead of $\alpha', \bar{\alpha}'$.

Definition (6.2.17)[218]: Let $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ be an $\mathcal{LC}$-quasihomomorphism. Then for every split exact functor H on the category of locally convex algebras the composition $H(\iota_{\mathcal{B}})^{-1} \circ \left(H(\alpha) - H(\bar{\alpha}')\right)$ defines a group homomorphism $H(\mathcal{A}) \to H(\mathcal{B})$, denoted $H(\alpha, \bar{\alpha})$ and called the homomorphism induced by the quasihomomorphism $(\alpha, \bar{\alpha})$; if there is no risk of confusion, we write $H(\alpha, \bar{\alpha}) = (\alpha, \bar{\alpha})_*$.

Lemma (6.2.18)[218]: If $(\varphi_1, \varphi_2, \varphi_3)$ is a morphism of double split extensions, i.e.:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B} \arrow[r, "\iota"] \arrow[d, "\varphi_1"'] & D - \pi \arrow[r] \arrow[d, "\varphi_2"'] & \mathcal{A} \arrow[r] \arrow[d, "\varphi_3"] \arrow[l, bend right, "\alpha"'] \arrow[dl, bend left, "\bar{\alpha}"] \arrow[dl, bend right, "\beta"'] & 0 \\
0 \arrow[r] & \mathcal{B}' \arrow[r, "\iota'"] & D' - \pi' \arrow[r] & \mathcal{A}' \arrow[r] \arrow[l, bend left, "\bar{\beta}"] & 0
\end{tikzcd} \tag{4}$$

commutes in the usual sense and $\varphi_2 \circ \alpha = \beta \circ \varphi_3, \varphi_2 \circ \bar{\alpha} = \bar{\beta} \circ \varphi_3$, then for every split exact functor H

$$H(\varphi_1) \circ H(\alpha, \bar{\alpha}) = H\big(\beta, \bar{\beta}\big) \circ H(\varphi_3).$$

Proof. As $H(\iota')$ is injective and

$$\begin{aligned} H(\iota') \circ H(\varphi_1) H(\alpha, \bar{\alpha}) &= H(\varphi_2) \circ \big(H(\alpha) H(\bar{\alpha})\big) \\ &= (H(\beta) H(\bar{\beta})) H(\varphi_3) \\ &= H(\iota') H(\beta, \bar{\beta}) H(\varphi_3) \end{aligned}$$

the result follows.

Remark (6.2.19)[218]: It suffices to suppose that we are given a commutative diagram such as the one in (4) that commutes after applying H.

Remark (6.2.20)[218]: In the setting of C^*-algebras, one usually applies a slightly different construction (e.g. [236]). If we fix a C.-quasihomomorphism $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \mathcal{M}(\mathcal{B}) \trianglerighteq \mathcal{B}$ and denote by $\bar{D}_\alpha$ the subalgebra of $\mathcal{A} \oplus_{C^*} \mathcal{M}(\mathcal{B})$ generated by $\mathcal{B}$ and $\big(a, \alpha(a)\big)$, then there is a unique morphism θ yielding a commutative diagram

$$\begin{tikzcd}
 & & \mathrm{id}_{\mathcal{A}} \oplus^0 & & \\
0 \arrow[r] & \mathcal{B} \arrow[r] \arrow[d, equal] & D_\alpha \arrow[r] \arrow[d, dashed, "\theta"] & \mathcal{A} \arrow[r] \arrow[d, equal] \arrow[l, bend right] & 0 \\
0 \arrow[r] & \mathcal{B} \arrow[r] & \bar{D}_\alpha \arrow[r] & \mathcal{A} \arrow[r] \arrow[l, bend left] & 0 \\
 & & \mathrm{id}_{\mathcal{A}} \oplus^\alpha & &
\end{tikzcd}$$

That is, because the left hand side commutes, θ has to fix $\mathcal{B}$, and as the right hand side commutes, $(a, 0) \mapsto \big(a, \alpha(a)\big)$. It is easily calculated that the morphis thus determined actually is a$*$-homomorphism (if we equip D_α with the obvious involution). Therefore the construction coincides with the one in the C^*-setting.

Proposition (6.2.21)[218]: Let $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ be an $\mathcal{L}C$-quasihomomorphism, H a split exact functor on the category of locally convex algebras with values in abelian groups. Then the following properties hold:

(i) $H\big((\alpha,\bar{\alpha})\circ\varphi\big)=H(\alpha,\bar{\alpha})\circ H(\varphi)$ for every morphism $\varphi\colon C\to\mathcal{A}$,

(ii) $H\big(\psi\circ(\alpha,\bar{\alpha})\big)=H(\psi)\circ H(\alpha,\bar{\alpha})$ for every homomorphism $\psi\colon\widehat{\mathcal{B}}\to\hat{C}$ of algebras, such that there is a locally convex subalgebra $C\subseteq\hat{C}$ such that $(\psi\circ\alpha,\psi\circ\bar{\alpha})\colon\mathcal{A}\rightrightarrows\hat{C}\trianglerighteq C$ is a quasihomomorphism,

(iii) $H(\alpha,\bar{\alpha})=-H(\alpha,\bar{\alpha})$,

(iv) if $\alpha-\bar{\alpha}$ is a homomorphism orthogonal to $\bar{\alpha}$, then $H(\alpha,\bar{\alpha})=H(\alpha-\bar{\alpha})$.

Proof. For (i), just note that $\varphi\oplus\mathrm{id}_{\mathcal{B}}\colon C\oplus\mathcal{B}\to\mathcal{A}\oplus\mathcal{B}$ is actually a morphism of locally convex algebras $D_{\alpha\circ\varphi}\to D_{\alpha}$ such that $(\mathrm{id}_{\mathcal{B}},\varphi\oplus\mathrm{id}_{\mathcal{B}},\varphi)$ is a morphism of the double split exact sequences associated to $D_{\alpha\circ\varphi}$ and D_{α}, and apply Lemma (6.2.18).

Similar reasoning applies to the morphism $\psi,\psi\oplus\mathrm{id}_{\mathcal{A}},\mathrm{id}_{\mathcal{A}}$ and proves (ii).

For (iii), use that with $\varphi_2\colon D_{\alpha}\to D_{\bar{\alpha}}(a,b)\mapsto(a,\alpha(a)-\bar{\alpha}(a)+b)$ we have a morphism $\big(\mathrm{id}_{\mathcal{B}_*},\varphi_{2_*}-\mathrm{id}_{\mathcal{A}_*}\big)$ after applying H and use Remark (6.2.19).

(v) follows from Proposition (6.2.8)(i).

Proposition (6.2.22)[218]:

Let

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I^1{}_1 \arrow[r, "\varphi^1{}_{12}"] \arrow[d, "\varphi^{12}{}_1"] & I^1{}_2 \arrow[r, "\varphi^1{}_{23}"] \arrow[d, "\varphi^{12}{}_2"] & I^1{}_3 \arrow[r] \arrow[d, "\varphi^{12}{}_3"] \arrow[l, bend right, "\bar{\sigma}^1{}_{23}"'] \arrow[l, bend left, "\sigma^1{}_{23}"] & 0 \\
0 \arrow[r] & I^2{}_1 \arrow[r, "\varphi^2{}_{12}"] \arrow[d, "\varphi^{23}{}_1"] \arrow[u, bend left, "\sigma^{23}{}_1"] \arrow[u, bend right, "\bar{\sigma}^{23}{}_1"'] & I^2{}_2 \arrow[r, "\varphi^2{}_{23}"] \arrow[d, "\varphi^{23}{}_2"] \arrow[u, bend left, "\sigma^{23}{}_2"] \arrow[u, bend right, "\bar{\sigma}^{23}{}_2"'] & I^2{}_3 \arrow[r] \arrow[d, "\varphi^{23}{}_3"] \arrow[u, bend left, "\bar{\sigma}^{23}{}_3"] \arrow[u, bend right, "\sigma^{23}{}_3"'] \arrow[l, bend right, "\bar{\sigma}^2{}_{23}"'] \arrow[l, bend left, "\sigma^2{}_{23}"] & 0 \\
0 \arrow[r] & I^3{}_1 \arrow[r, "\varphi^3{}_{12}"] \arrow[d] & I^3{}_2 \arrow[r, "\varphi^3{}_{23}"] \arrow[d] & I^3{}_3 \arrow[r] \arrow[d] \arrow[l, bend right, "\bar{\sigma}^3{}_{23}"'] \arrow[l, bend left, "\sigma^3{}_{23}"] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a diagram that is row- and column-wise given by double-split short exact sequences of locally convex algebras and morphisms. We suppose that all squares involving only morphisms of type φ commute, and that $\varphi^2{}_{12} \circ \sigma^{23}{}_1 = \sigma^{23}{}_2 \circ \varphi^3{}_{12}, \sigma^2{}_{23} \circ \varphi^1{}_{23} = \varphi^{12}{}_2 \circ \sigma^1{}_{23}, \sigma^{23}{}_2 \circ \sigma^3{}_{23} = \sigma^2{}_{23} \circ \sigma^{23}{}_3$ and the same relations with all σ replaced by $\bar{\sigma}$; finally $\bar{\sigma}^{23}{}_2 \circ \sigma^3{}_{23} = \sigma^2{}_{23} \circ \bar{\sigma}^{23}{}_3$ and $\sigma^{23}{}_2 \circ \bar{\sigma}^3{}_{23} = \bar{\sigma}^2{}_{23} \circ \sigma^{23}{}_3$. Then for every split exact functor H we have

$$H(\sigma^{23}{}_1, \bar{\sigma}^{23}{}_1) \circ H(\sigma^3{}_{23}, \bar{\sigma}^3{}_{23}) = H(\sigma^1{}_{23}, \bar{\sigma}^1{}_{23}) \circ H(\sigma^{23}{}_3, \bar{\sigma}^{23}{}_3),$$

in other words, the composition of the morphisms induced by the outer edges in the diagram are the same.

Proof. As

$$H\left(\varphi^2{}_{12}\right) \circ H\left(\varphi^{12}{}_1\right) = H\left(\varphi^{12}{}_2\right) \circ H\left(\varphi^1{}_{12}\right)$$

and both sides are injective, it suffices to show

$$H\left(\varphi^2{}_{12}\right) \circ H\left(\varphi^{12}{}_1\right) \circ H(\sigma^{23}{}_1, \bar{\sigma}^{23}{}_1) \circ H(\sigma^3{}_{23}, \bar{\sigma}^3{}_{23})$$
$$= H\left(\varphi^{12}{}_2\right) \circ H\left(\varphi^1{}_{12}\right) \circ H(\sigma^1{}_{23}, \bar{\sigma}^1{}_{23}) \circ H(\sigma^{23}{}_3, \bar{\sigma}^{23}{}_3).$$

This follows from the hypotheses by an easy diagram chase.

Definition (6.2.23)[218]: We will call a quasihomomorphism $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ *a* quasihomomorphism in standard form if $\widehat{\mathcal{B}}$ is isomorphic to D_α.

Given a quasihomomorphism in standard form as above and a supplementary locally convex algebra C, we define the quasihomomorphisms $(\alpha, \bar{\alpha}) \otimes C \coloneqq (\alpha \otimes C, \bar{\alpha} \otimes C)$, and similarly $C \otimes (\alpha, \bar{\alpha})$.

If A and B are locally convex algebras, we define

$$\sigma_{\mathcal{A},\mathcal{B}}: \mathcal{A} \otimes_\pi \mathcal{B} \to \mathcal{B} \otimes_\pi \mathcal{A}, a \otimes b \mapsto b \otimes a.$$

Note that as the projective tensor product preserves split exactness, tensoring by C yields again a quasihomomorphism, which is furthermore again in standard form.

If we start with an arbitrary quasihomomorphism $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$, then we may replace $\widehat{\mathcal{B}}$ by D_α to obtain a quasihomomorphism in standard form.

Now let $(\alpha_i, \bar{\alpha}_i): \mathcal{A}_i \rightrightarrows \widehat{\mathcal{B}}_i \trianglerighteq \mathcal{B}_i$ be two $\mathcal{L}C$-quasihomomorphisms. Then we have associated exact sequences $E_i \coloneqq (0 \to \mathcal{B}_i \to D_i \to \mathcal{A}_i \to 0)$ as in Lemma (6.2.15) and double splits $\alpha_i', \bar{\alpha}_i'$. As the projective tensor product preserves exactness of linearly split extensions, we obtain double split exact sequences containing $E_1 \otimes_\pi \mathcal{B}_2, E_1 \otimes_\pi D_2,\ E_1 \otimes_\pi \mathcal{A}_2$, and similarly $\mathcal{B}_1 \otimes_\pi E_2, D_1 \otimes_\pi E_2$ and $\mathcal{A}_1 \otimes_\pi E_2$ that fit into a diagram as in the above Lemma, and obviously satisfy the commutativity conditions. Identifying $D_1 \otimes \hat{C}$ with the algebra associated to the quasihomomorphism $(\alpha_1, \bar{\alpha}_1): \otimes C: \mathcal{A}_1 \otimes_\pi C \rightrightarrows \mathcal{B}_1 \otimes_\pi C \mathcal{B}_1 \otimes_\pi C$ for any locally convex algebra C, and similarly for the others, we have proved most of the following

Proposition (6.2.24)[218]:

$$H\left(\mathcal{B}_1 \otimes (\alpha_2, \bar{\alpha}_2)\right) \circ H\left((\alpha_1, \bar{\alpha}_1) \otimes \mathcal{A}_2\right) = H\left((\alpha_1, \bar{\alpha}_1) \otimes \mathcal{B}_2\right) \circ H\left(\mathcal{A}_1 \otimes (\alpha_2, \bar{\alpha}_2)\right),$$

or, in other words, the obvious outer product of quasihomomorphisms is compatible with the induction process. In particular

$$H\mathcal{B}_1\otimes(\varphi)\circ H\big((\alpha_1,\bar{\alpha}_1)\otimes\mathcal{A}_2\big)=H\big((\alpha_1,\bar{\alpha}_1)\otimes\mathcal{B}_2\big)\circ H\big(\mathcal{A}_1\otimes(\varphi)\big),$$

for every homomorphism $\varphi\colon\mathcal{A}_2\to\mathcal{B}_2$.

If $(\alpha,\bar{\alpha})\colon\mathcal{A}\rightrightarrows\widehat{\mathcal{B}}\trianglerighteq\mathcal{B}$ is a quasihomomorphism in standard form and C a supplementary locally convex algebra, then

$$H\big((\alpha,\bar{\alpha})\otimes C\big)=H\big(\sigma_{C,\mathcal{B}}\big)\circ H\big(C\otimes(\alpha,\bar{\alpha})\big)\circ H\big(\sigma_{\mathcal{A},C}\big).$$

Proof. The last statement follows by identifying $D_{\alpha\otimes C}\approx D_\alpha\otimes_\pi C$, applying Lemma (6.2.18) to

$$\begin{tikzcd}
 & & & \mathcal{A}\otimes_\pi C \arrow[dl, bend right, "\alpha\otimes C"'] \arrow[dl, bend left, "\bar{\alpha}\otimes C"] \arrow[dd, "\sigma_{\mathcal{A},C}"'] & \\
0 \arrow[r] & \mathcal{B}\otimes_\pi C \arrow[r] \arrow[dd, "\sigma_{\mathcal{B},C}"'] & D_{\alpha\otimes C} \arrow[r] \arrow[dd, "\sigma_{\alpha,C}"'] & \mathcal{A}\otimes_\pi C \arrow[r] & 0 \\
 & & & C\otimes_\pi\mathcal{A} \arrow[dl, bend right, "C\otimes\alpha"'] \arrow[dl, bend left, "C\otimes\bar{\alpha}"] & \\
0 \arrow[r] & C\otimes_\pi\mathcal{B} \arrow[r] & D_{C\otimes\alpha} \arrow[r] & C\otimes_\pi\mathcal{A} \arrow[r] & 0
\end{tikzcd}$$

and multiplying the result by $\sigma_{\mathcal{B},C_*^{-1}}=\sigma_{C,\mathcal{B}_*}$ on the left.

In particular, for suitably nice ideals and algebras, it is easy to see that for "unbounded $(\mathcal{A}_i,\mathcal{B}_i)$ cycles" $[D_i]$ we get $Qh([D_1]\otimes_\pi\mathcal{B}_2)=Qh([D_1])\otimes_\pi\mathcal{B}_2$. Hence we may apply the above to obtain

$$H(\mathcal{B}_1\otimes_\pi[D_2])\circ H([D_1]\otimes_\pi\mathcal{A}_2)=H([D_1]\otimes_\pi\mathcal{B}_2)\circ H(\mathcal{A}_1\otimes_\pi[D_2]).$$

We will make use of the following result, which shows that the induction process is automatically compatible with stabilisation for stable functors, later on:

Proposition (6.2.25)[218]: Let H be a J-stable, split exact functor. Then for every quasihomomorphism $(\alpha,\bar{\alpha})$ from $\mathcal{A}$ to $\mathcal{B}$ the following diagram commutes:

$$\begin{tikzcd}
H(\mathcal{A}) \arrow[r, "H(\alpha,\bar{\alpha})"] \arrow[d, "\theta_{\mathcal{A}}^J"'] & H(\mathcal{B}) \arrow[d, "\theta_{\mathcal{B}}^J"] \\
H(\mathcal{A}\otimes_\pi J) \arrow[r, "H\big((\alpha,\bar{\alpha})\otimes J\big)"'] & H(\mathcal{B}\otimes_\pi J)
\end{tikzcd}$$

Proof. Identify $D_{\alpha\otimes J} \approx D_\alpha \otimes_\pi J$ and apply Lemma (6.2.18) to

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B} \arrow[r] \arrow[d, "\iota_{\mathcal{B}}"'] & D_\alpha \arrow[r] \arrow[d, "\iota_{D_\alpha}"'] & \mathcal{A} \arrow[r] \arrow[d, "\iota_{\mathcal{A}}"] \arrow[l, bend right, "\bar{\alpha}"] & 0 \\
0 \arrow[r] & \mathcal{B}\otimes_\pi J \arrow[r] & D_\alpha \otimes_\pi J \arrow[r] & \mathcal{A}\otimes_\pi J \arrow[r] \arrow[l, bend right, "\bar{\alpha}\otimes J"] & 0
\end{tikzcd}$$

where the maps ι are inducing the stabilisation isomorphism under H and make the diagram commute.

Let $\widehat{\mathcal{B}}$ be an algebra containing $Z\mathcal{B}$ as an ideal. Set $x: [0,1] \to \mathbb{R}, t \mapsto t$, and $C_i\mathcal{B} := \{f \in Z\mathcal{B} | f(i) = 0\}$. Then $(x - i)C_i\mathcal{B} = C_i\mathcal{B}$. Hence, if $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq Z\mathcal{B}$ is a quasihomomorphism, we get quasihomomorphisms

$$(\alpha_i, \bar{\alpha}_i): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}}/C_i\mathcal{B} \trianglerighteq \mathcal{B}.$$

Definition (6.2.26)[218]: A diffotopy is a quasihomomorphism $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq Z\mathcal{B}$ We have:

Lemma (6.2.27)[218]: If H is a diffotopy invariant, split exact functor and $(\alpha, \bar{\alpha}): \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq Z\mathcal{B}$ a diffotopy with $Z\mathcal{B}$ and ideal in $\widehat{\mathcal{B}}$, then

$$H(\alpha_0, \bar{\alpha}_0) = H(\alpha_1, \bar{\alpha}_1).$$

Proof. Denote $\pi_i: \widehat{\mathcal{B}}/C_i\mathcal{B}$ the quotient maps, and apply Lemma (6.2.18) to

$$\begin{tikzcd}
0 \arrow[r] & Z\mathcal{B} \arrow[r] \arrow[d, "\pi_i"'] & D_\alpha \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{A}}\otimes_{\pi_i}"'] & \mathcal{A} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{A}}"] \arrow[l, bend right, "\bar{\alpha}'"] & 0 \\
0 \arrow[r] & \mathcal{B} \arrow[r] & D_{\alpha_i} \arrow[r] & \mathcal{A} \arrow[r] \arrow[l, bend right, "\bar{\alpha}_i'"] & 0.
\end{tikzcd}$$

In this section, $K(\mathcal{A})$ denotes the algebraic K-theory of an algebra $\mathcal{A}$,$\cup$ the outer product $K(\mathcal{A})\otimes K(\mathcal{B}) \to K(\mathcal{A}\otimes_\pi \mathcal{B})$. The notion of invariance under inner automorphisms was given in Definition (6.2.6).

Proposition (6.2.28)[218]: Let H be a split exact, M_2-stable functor from locally convex algebras to abelian groups. For every locally convex algebras $\mathcal{A}$ and $\mathcal{B}$ there is a product:

$$K(\mathcal{A}) \otimes H(\mathcal{B}) \to H(A\otimes_\pi B), \qquad x \otimes b \mapsto x \cdot b.$$

If $A = \mathbb{C}$ and $x = 1 \in K(\mathcal{A})$, then

$$x \cdot b = b. \tag{5}$$

Furthermore, if $f: \mathcal{A}_1 \to \mathcal{A}_2, g: \mathcal{B}_1 \to \mathcal{B}_2$ are homomorphisms, $x_1 \in K(\mathcal{A}_1), b_1 \in H(\mathcal{B}_1)$:

$$f_*(x_1) \cdot g_*(b_1) = (f\otimes g)_*(x_1 \cdot b_1). \tag{6}$$

If C is another locally convex algebra, $x \in K(\mathcal{A}), y \in K(\mathcal{B})$ and $c \in H(C)$

$$(x \cup y) \cdot c = x \cdot (y \cdot c).$$

Proof. By Proposition (6.2.8), H is invariant under inner automorphisms and M_n-stable for all n.

Assume first that $\mathcal{A}$ is unital. Let p be an idempotent in $M_n(\mathcal{A})$. Then define a morphism

$$\varphi_p^{\mathcal{B}}: \mathcal{B} \to M_n(\mathcal{A}\otimes_\pi \mathcal{B}), b \mapsto p\otimes b.$$

This yields (by the universal property of the enveloping group) a homomorphism of groups

$$F_{\mathcal{A}}^{\mathcal{B}}: K(\mathcal{A}) \to \mathrm{Hom}\big(H(\mathcal{B}), H(\mathcal{A}\otimes_\pi \mathcal{B})\big), [p] \mapsto H\big(\varphi_p^{\mathcal{B}}\big),$$

which defines the product $x \cdot b \coloneqq F_{\mathcal{A}}^{\mathcal{B}}(x)(b)$. The compatibility with unital morphisms is straightforward. If C is another unital algebra, then for all $x \in K(\mathcal{A}), y \in K(\mathcal{B})$ we have

$$F_{\mathcal{A}}^{\mathcal{B}\otimes_\pi C}(x) \circ F_{\mathcal{B}}^{C}(y) = F_{\mathcal{A}\otimes_\pi \mathcal{B}}^{C}(x \cup y)$$

directly from the definitions.

To treat the nonunital case, it suffices to apply the following Lemma.

Lemma (6.229)[218]: If

$$0 \longrightarrow J \longrightarrow \mathcal{A}_1 \xrightarrow{\pi} \mathcal{A}_2 \longrightarrow 0$$

is a split exact sequence of locally convex algebras with $\mathcal{A}_1$ and $\mathcal{A}_2$ unital, then $K(J)H(\mathcal{B}) \subseteq H(J \otimes B)$.

Proof. Let $x \in K(J) \subseteq K(\mathcal{A}_1)$, then

$$(\pi \otimes \mathrm{id})_*(x \cdot b) = \pi_*(x) \cdot b = 0.$$

Remark (6.2.30)[218]: Properties (6.2.3) and (6.2.4) classify the pairing, because if $p \in M_n(\mathcal{A}), f: \mathbb{C} \to M_n(\mathcal{A})$ the homomorphism $\lambda \mapsto \lambda p$, then

$$[p] \cdot b = [f(1)] \cdot b = H(f \otimes \mathrm{id}_{\mathcal{B}})(1 \cdot b) = H(f \otimes \mathrm{id}_{\mathcal{B}})(b).$$

It remains to compare the two ways a given quasihomomorphism acts on H:

Proposition (6.2.31)[218]: Let $(\alpha, \bar{\alpha}): \mathbb{C} \rightrightarrows \hat{\mathcal{A}} \trianglerighteq \mathcal{A}$ be a quasihomomorphism and H a split exact, M_2-stable functor. Then for every locally convex algebra $\mathcal{B}$

(i) the map $H(\mathcal{B}) \to H(\mathcal{A} \otimes_\pi \mathcal{B})$ given as multiplication with $K(\alpha, \bar{\alpha})(1) \in K(\mathcal{A})$ and

(ii) the map $H\big((\alpha, \bar{\alpha}) \otimes \mathcal{B}\big) = H(\iota)^{-1} \circ \big(H(\alpha \otimes \mathrm{id}_{\mathcal{B}}) - H(\bar{\alpha} \otimes \mathrm{id}_{\mathcal{B}})\big)$ from Definition (6.2.17) coincide.

Proof. We may suppose, replacing by D_α, that $\hat{\mathcal{A}}$ is locally convex and $\mathcal{A}$ a closed ideal in it. Then for $\mathcal{B} = \mathbb{C}$ this follows directly from the definitions. The general case follows from the scalar case by applying it to the functor $H^{\mathcal{B}} := H(\cdot \otimes \mathcal{B})$.

This shows that quasihomomorphisms from $\mathbb{C}$ to B are determined by their action on K-theory:

Lemma (6.2.32)[218]: Let $(\alpha_i, \bar{\alpha}_i): \mathbb{C} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ be two quasihomomorphisms. Then $H(\alpha_1, \bar{\alpha}_1) = H(\alpha_2, \bar{\alpha}_2)$ for all split exact M_2 stable functors if and only if the equality holds for the specific functor K.

We also have the following useful

Lemma (6.2.33)[218]: Let $(\alpha, \bar{\alpha}): \mathbb{C} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ be a quasihomomorphism. Then there is a quasihomomorphism $(\beta, \bar{\beta}): \mathbb{C} \rightrightarrows M_2(\mathcal{B}^+) \trianglerighteq M_2(\mathcal{B})$ such that

$$H(\theta_B) \circ H(\alpha, \bar{\alpha}) = H(\alpha', \bar{\alpha}')$$

for every split exact M_2-stable functor, where $\theta_{\mathcal{B}}: \mathcal{B} \to M_2(\mathcal{B})$ denotes the stabilisation.

More precisely, there is a quasihomomorphism $(\beta, \bar{\beta}): \mathbb{C} \rightrightarrows M_2(\mathcal{B}^+) \trianglerighteq M_2(\mathcal{B})$ that is "stably inner equivalent" to $(\alpha, \bar{\alpha})$.

Proof. We identify a homomorphism $\varphi: \mathbb{C} \to \widehat{\mathcal{B}}$ with the idempotent $\varphi(1)$. By Proposition (6.2.8), H is invariant under inner automorphisms. Assume further, eventually replacing by D_α, that $\widehat{\mathcal{B}}$ is a locally convex algebra, that $\mathcal{B}$ is an ideal in $\widehat{\mathcal{B}}$ and that $\widehat{\mathcal{B}}$ is unital. We identify $\mathcal{B}^+$ with the subalgebra of $\widehat{\mathcal{B}}$ generated by $\mathcal{B}$ and 1. Set $p \coloneqq \alpha(1), \bar{p} \coloneqq \bar{\alpha}(1)$. If

$$q \coloneqq \begin{pmatrix} p & \\ & 1-p \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} \bar{p} & \\ & 1-p \end{pmatrix} \quad \text{and} \quad V \coloneqq \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix},$$

then the quasihomomorphisms $(p \oplus 0, \bar{p} \oplus 0)$ and $(q, \bar{q})$ induce the same maps

$$VqV = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \in M_2(\mathcal{B}^+) \text{and } V\bar{q}V = V(\bar{q} - q)V + VqV \in M_2(\mathcal{B}^+).$$

We may thus set $(\alpha', \bar{\alpha}') \coloneqq (VqV, V\bar{q}V)$.

We will now introduce abstract Kasparov modules, an intermediate step between Kasparov-modules and quasihomomorphisms. There are several versions of this notion of abstract Kasparov module, compare [236] and [227]. In the latter, constructions for abstract Kasparov modules (in the C^*-setting) are done by simply lifting an abstract Kasparov module back to a classical one, and then transferring the construction back to the abstract setting (compare for example [227], Lemma 8.33). For the product, this leads to the notion of double Kasparov module, and as noted again in [227], the conditions in the non-C^*-setting get extremely technical, and it is not clear how useful this construction is for bornological algebras.

We will describe the product in a different manner below, similar to the approach we sketched for KK-theory. Again, we have to modify the definitions to make them applicable to our settings.

Definition (6.2.34)[218]: Let $\mathcal{A}, \mathcal{B}$ be locally convex algebras, $\widehat{\mathcal{B}}$ a unital algebra and $\mathcal{B} \subseteq \widehat{\mathcal{B}}$. An abstract Kasparov module from $\mathcal{A}$ to $\mathcal{B}$ with respect to $\widehat{\mathcal{B}}$ is a triple $(\alpha, \bar{\alpha}, U)$ such that $U \in \widehat{\mathcal{B}}$ is invertible, $\alpha, \bar{\alpha}: \mathcal{A} \to \widehat{\mathcal{B}}$ are two homomorphisms and the map

$$\mathcal{A} \to \widehat{\mathcal{B}}, a \mapsto \alpha(a) - U^{-1}\hat{\alpha}(a)U$$

is actually $\mathcal{B}$-valued and continuous. We then define the associated quasihomomorphism $Qh(\alpha, \bar{\alpha}, U) \coloneqq (\alpha, U^{-1}\bar{\alpha}U)$.

We proceed to define an appropriate notion of locally convex bimodule:

Definition (6.2.35)[218]: Let $\mathcal{A}$ and $\mathcal{B}$ be locally convex algebras. An $\mathcal{L}C$-Kasparov module from $\mathcal{A}$ to $\mathcal{B}$ is given by an algebra $\widehat{\mathcal{B}}$ containing $\mathcal{B}$, elements $\varepsilon, F \in \widehat{\mathcal{B}}$ and an algebra morphism $\varphi: \mathcal{A} \to \widehat{\mathcal{B}}$n such that $\varepsilon^2 = 1, F\varepsilon = -\varepsilon F, [\varphi(a), \varepsilon] = 0$ and such that the maps

(i) $\mathcal{B} \to \widehat{\mathcal{B}}, b \mapsto bF, Fb, \varepsilon b, b\varepsilon$

(ii) $\mathcal{A} \otimes \mathcal{B} \to \widehat{\mathcal{B}}, (a, b) \mapsto \varphi(a), b\varphi(a)$

(iii) $\mathcal{A} \to \widehat{\mathcal{B}}, a \mapsto \varphi(a)(1 - F^2), (1 - F^2)\varphi(a)$

(iv) $\mathcal{A} \to \widehat{\mathcal{B}}, [\varphi(a), F]$

are actually $\mathcal{B}$-valued and continuous. We then call ε a grading and denote the $\mathcal{L}C$-Kasparov $(\mathcal{A}, \mathcal{B})$-module by $\left(\widehat{\mathcal{B}}, \varphi, F\right)$.

Note that it suffices to suppose the map $a \mapsto \varphi(a)(1 - F^2)$ be continuous $\mathcal{B}$-valued and $(1 - F^2)\varphi(a) \in \mathcal{B}$, continuity of the second in a then follows; similarly for the maps $a \mapsto \varphi(a)b$ and $a \mapsto b\varphi(a)$ in (ii). There is also an obvious notion of representation of $\mathcal{L}C$-Kasparov modules which we will use in the sequel. We also identify $\mathcal{L}C$-Kasparov modules that are isomorphic in the obvious sense.

Proposition (6.2.36)[218]: For a given $\mathcal{L}C$-Kasparov $(\mathcal{A}, \mathcal{B})$-module $\left(\widehat{\mathcal{B}}, \varphi, F\right)$ with grading ε we set $W_F \coloneqq \varepsilon(1 - F^2) + P_\varepsilon F + p_\varepsilon^\perp F(2 - F^2)$ where $P_\varepsilon \coloneqq 1/2(1 + \varepsilon)$ and $P_\varepsilon^\perp \coloneqq 1 - P_\varepsilon$. This defines an associated abstract Kasparov module $AKM\left(\widehat{\mathcal{B}}, \varphi, F\right) \coloneqq (P_\varepsilon\varphi, P_\varepsilon^\perp\varphi, W_F)$.

Proof. W_F is it's own inverse:

$$W_F^2 = (1 - F^2)^2 + (P_\varepsilon + P_\varepsilon^\perp)F^2(2 - F^2) = 1,$$

and modulo a continuous $\mathcal{B}$-valued map we have

$$P_\varepsilon\varphi(a)W_F - W_F P_\varepsilon^\perp\varphi(a) = P_\varepsilon[\varphi(a), F] - \varphi(a)(1 - F^2) \in J. \tag{7}$$

As follows from the hypothesis, multiplication by W_F is continuous on $\mathcal{B}$, thus the result follows by multiplying equation 5 by W_F on the left.

We may thus give the following

Definition (6.2.37)[218]: For every $\mathcal{LC}$-Kasparov module $\left(\hat{B}, \varphi, F\right)$ we define

$$Qh\left(\widehat{\mathcal{B}}, \varphi, F\right) \coloneqq Qh\left(AKM\left(\widehat{\mathcal{B}}, \varphi, F\right)\right),$$

and if H is a split exact functor

$$H\left(\widehat{\mathcal{B}}, \varphi, F\right) \coloneqq H\left(Qh\left(\widehat{\mathcal{B}}, \varphi, F\right)\right) = H\left(Qh\left(AKM\left(\widehat{\mathcal{B}}, \varphi, F\right)\right)\right).$$

The point of this definition is the following: Using the usual transformation $F \longmapsto U_F$ (see below) is adequate in the setting of p-summable Fredholm modules, but it is not, for example, on smooth functions.

Proposition (6.2.38)[218]: Let (E, φ, F) be a C^*.-Kasparov (A, B)-module with grading ε such that $F = F^*$. and $\|F\| \leq 1$, set $U_F := \varepsilon\sqrt{1 - F^2} + F$. Then the C^*-quasihomomorphisms

$$(P_\varepsilon \alpha, U_F P_\varepsilon^\perp \alpha U_F) \text{ and } (P_\varepsilon \alpha, W_F P_\varepsilon^\perp W_F)$$

are equivalent.

Proof. Set

$$U_F^{'} := \begin{pmatrix} F & \sqrt{1 - F^2} \\ \sqrt{1 - F^2} & -F \end{pmatrix}, W_F^{'} := \begin{pmatrix} F & (1 - FF^*) \\ (1 - F^*F) & (F^*F - 2)F^* \end{pmatrix}.$$

Then $(E \oplus E^{op}, \varphi \oplus 0, U_F^{'})$ and $(E \oplus E^{op}, \varphi \oplus 0, W_F^{'})$ are perturbations of the same module, hence equivalent. A straightforward calculation shows that the quasihomomorphism induced by $(E \oplus E^{op}, \varphi \oplus 0, U_F^{'})$ is equivalent to $(P_\varepsilon \alpha, U_F P_\varepsilon^\perp \alpha U_F)$; similarly for W_F and $W_F^{'}$. One may also see this by remarking that U_F and W_F can be obtained from passing to the Fredholm picture, i.e., regrouping $E \oplus E^{op}$ into even and odd parts by applying a permutation matrix. The conjugation of $U_F^{'}$ and $W_F^{'}$ by this permutation matrix yields operators with off diagonal entries U_F, U_F^* and W_F, W_F^* , respectively.

There is an obvious sum operation on abstract Kasparov modules, given by

$$(\alpha, \bar{\alpha}, U) + (\alpha^{'}, \bar{\alpha}^{'} U^{'}) := (\alpha \oplus \alpha^{'}, \bar{\alpha} \oplus \bar{\alpha}^{'}, U \oplus U^{'}).$$

Further every homomorphism $\varphi \in \mathcal{LC}(\mathcal{A}, \mathcal{B})$ yields a canonical abstract Kasparov module $(\mathcal{B}, \varphi, 0)$ and also an operation φ^* on abstract Kasparov modules in the obvious way. The same statements hold for Kasparov modules. In particular, if $\psi\,; \mathcal{A}^{'} \to \mathcal{A}$ is a homomorphism and $(\widehat{\mathcal{B}}, \varphi, F)$ an $\mathcal{LC}$-Kasparov module, then we write $\psi^*(\widehat{\mathcal{B}}, \varphi, F) := (\widehat{\mathcal{B}}, \varphi \circ \psi, F)$.

Definition (6.2.39)[218]: Two $\mathcal{LC}$ -Kasparov modules $(\widehat{\mathcal{B}}_0, \varphi_0, F_0), (\widehat{\mathcal{B}}_1, \varphi_1, F_1)$ between locally convex algebras $\mathcal{A}$ and $\mathcal{B}$ are called

(a) $\mathcal{B}$-perturbations of one another if $\widehat{\mathcal{B}}_0 = \widehat{\mathcal{B}}_1, \varphi_0 = \varphi_1$ and $a \longmapsto \varphi(a)(F - F^{'}), (F - F^{'})\varphi(a)$ is $\mathcal{B}$-valued and continuous

(b) Diffotopic, if there is an $\mathcal{LC}$-Kasparov $(\mathcal{A}Z\mathcal{B})$-module $(\widehat{\mathcal{B}}, \varphi, F)$ such that $Z\mathcal{B}$ is an ideal in $\widehat{\mathcal{B}}$, and for $i = 1,2$

$$(\pi_i)_*(\widehat{\mathcal{B}}, \varphi, F) := \left(\widehat{\mathcal{B}}/\mathcal{C}_i\,\mathcal{B}, \pi_i \circ \varphi, \pi_i(F)\right) = (\widehat{\mathcal{B}}_i, \varphi_i, F_i)$$

where $\pi_i\colon \widehat{\mathcal{B}} \to \widehat{\mathcal{B}}/\mathcal{C}_i\,\mathcal{B}$ is the quotient map.

Proposition (6.2.40)[218]: $\mathcal{B}$-perturbation is a weaker equivalence relation than diffotopy, and the transformation Qh from $\mathcal{LC}$-Kasparov modules to $\mathcal{LC}$-quasihomomorphisms preserves diffotopy.

Proof. Let $(\widehat{\mathcal{B}}, \varphi, F), (\widehat{\mathcal{B}}, \varphi, G)$ be $\mathcal{B}$-perturbations of one another. Let $g \in \mathcal{C}$ with $g(1) = 1$, set $f := 1 - g$, define $\tilde{F} := fF + gG \in Z\mathcal{B}$. Then multiplication by $\tilde{F}$ on $Z\mathcal{B}$ is continuous because multiplication by F and G and elements of Z is. Further

$$a(\tilde{F}^2 - 1) = a\left(f^2(T^2 - 1) + g^2(T^2 - 1) + fg(T^2 - 1) + (T^2 - 1)\right)$$
$$+ fg\left((T - T')T' + (T' - T)T\right)$$

which is continuous in a. The rest is obvious.

The next example shows that the induction process is correctly normalized.

Example (6.2.41)[218]: Let $(\widehat{\mathcal{B}}, \varphi, F)$ be an $\mathcal{LC}$-Kasparov $(\mathbb{C}, \mathcal{B})$-module with grading ε, where $\mathcal{B}$ is some locally convex algebra, such that φ is unital and $\bar{S}S = 1$, where $F = \begin{pmatrix} & \bar{S} \\ S & \end{pmatrix}$ with respect to P_ε. Then we get

$$W_F = \begin{pmatrix} 0 & \bar{S} \\ S & S\bar{S} - 1 \end{pmatrix}, \quad (P_\varepsilon^\perp \varphi)^{W_F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - S\bar{S} \end{pmatrix}.$$

Hence the map $(P_\varepsilon^\perp \varphi)^{W_F} - P_\varepsilon \varphi$ is a homomorphism orthogonal to $P_\varepsilon \varphi$, namely the map

$$\mathbb{C} \to \mathcal{B}, \lambda \longmapsto \lambda \begin{pmatrix} 0 & \\ & 1 - S\bar{S} \end{pmatrix}.$$

Thus for every split exact functor H

$$H(\widehat{\mathcal{B}}, \varphi, F) = -H(1 - S\bar{S})$$

by Proposition (6.2.21)-(iv) and (iii).

If, on the other hand, $S\bar{S} = 1$, we obtain the homomorphism

$$P_\varepsilon \varphi (P_\varepsilon^\perp \varphi)^{W_F}\colon \lambda \mapsto \lambda(1 - S\bar{S}) \oplus 0,$$

and by Proposition (6.2.21) (iv)

$$H\left(\widehat{\mathcal{B}},\varphi,F\right)=H(1-S\bar{S}).$$

We also deduce the second from the first by a rotation in matrices and using diffotopy invariance.

The following is an abstract variant of [221], Theorem (6.2.5):

Proposition (6.2.42)[218]: Let $(\mathbb{B}(\mathcal{H}),\varphi,F)$ be a $(\mathbb{C},\mathcal{L}^p)$-module, and H a split exact, diffotopy invariant and $\mathcal{L}^p$-stable functor. Then $H(Qh(\mathbb{B}(\mathcal{H}),\varphi,F).A)\otimes_\pi\mathcal{A})=n\theta$ for every locally convex algebra $\mathcal{A}$, where $\theta\colon H(\mathcal{A})\to H(\otimes_\pi\mathcal{L}^p)$ denotes the stabilisation map, and n is the Fredholm index of F viewed as an operator on $\varphi(1)\mathcal{H}$.

Proof. It suffices to prove the case with $A=C$, the general case follows by applying the result to the functor $H(\cdot\otimes_\pi A)$. Setting $P:=\varphi(1)$ and replacing by the module

$$\left(PB(H)P,P_\varphi,PFP\right),$$

we may assume that φ is unital (because $H(P^\perp B(H)P^\perp\varphi,P^\perp\varphi,P^\perp FP^\perp)=0$). With respect to the grading, $F=\begin{pmatrix} & S\\ T & \end{pmatrix}$. By hypothesis, T is Fredholm and hence has closed cokernel.

Assume that T has negative index. Without loss of generality, it is injectif, and we can define the bounded operator T' as T^{-1} on $\mathrm{Im}(T)$ and zero on $\mathrm{Im}(T)^\perp$. We have

$$(T'-S)T=1-ST\in L^p,$$

and as T is invertible modulo $L^p, T'-S\in L^p$ follows. Therefore the module $\left(\mathbb{B}(\mathcal{H}),\varphi,\begin{pmatrix} & T'\\ T & \end{pmatrix}\right)$ is equivalent to the former one. Now the above example shows that

$$H(B(H),\varphi,F)=-H(1-TT')=-n\theta$$

by Proposition (6.2.8) (vi).

In case that T has negative index, we reduce to the former case by passing to the module $\left(\hat{B},\varphi,-F\right)$ with grading $-\varepsilon$.

We now construct the smooth analogue of the Bott elements. We assume in this section that the dimension n is even. The point is that otherwise we would have to work with graded functors, as certain gradings would not be inner.

We recall that the C^*-Kasparov $(C, S^n \otimes C_n)$-module y_n was defined by Kasparov ([221], see also [237]) as $\left(S^n \otimes C_n, 1, q(D_2)\right)$, where D_2 denotes the operator of multiplication by the inclusion $R^n \hookrightarrow C_n$ and 1 denotes the action of $\mathbb{C}$ given by scalar multiplication. In [221], the grading is the one coming from the natural grading of the Clifford algebras. We may assume that the algebra $S^n \otimes C_n$ is trivially graded and the grading on the Hilbert module $S^n \otimes C_n$ is given by left multiplication with the volume element. This does not change the class in KK-theory, because it corresponds to applying a graded Morita context. In fact, more generally:

If A is a C^*-algebra with an inner grading κ_U such that $U^2 = 1$, then there is a graded Morita context between A and A with the trivial grading.

The context is given by letting (A, κ_U) (with grading) act on the Hilbert (A, id_A)-module (trivially graded) A with grading induced by left multiplication with U. Taking Kasparov products with the corresponding KK-equivalence transforms the grading exactly to the ones used above. Note that this does not signify forgetting the grading (in fact, x_n is trivial in KK if we just view it as ungraded).

As we need to define an $\mathcal{LC}$-Kasparov $(\mathbb{C}, \mathrm{S}^{\mathrm{n}} \otimes \mathbb{C}_{\mathrm{n}})$-module, the operator $\mathrm{q}(\mathrm{D}_2)$ used in the C^*-setting is inadequate, as $1 - \mathrm{q}(\mathrm{D}_2)^2 = (1 + \mathrm{D}_2^2)^{-1}$ is not a Schwartz function. The idea, in dimension 1, is that it suffices to replace the inclusion $\mathrm{f}\colon \mathbb{R} \to \mathbb{C}_1$ by the function $\mathrm{t} \to \left(0, \mathrm{t}\mathrm{e}^{\mathrm{t}^2}\right)$, or any other odd real-valued function h growing sufficiently fast at infinity; in arbitrary dimension we define $\mathcal{LC}$-Kasparov $(\mathbb{C}, \mathrm{S}^{\mathrm{n}} \otimes \mathbb{C}_{\mathrm{n}})$-modules

$$\mathrm{y}_{\mathrm{n}}^{\infty} \coloneqq \left(\mathrm{C}_{\mathrm{b}}(\mathbb{R}^{\mathrm{n}}) \otimes \mathbb{C}_{\mathrm{n}}, 1, \mathrm{q}\left(\mathrm{h}(\mathrm{D}_2)\right)\right).$$

We then have the following weak multiplicativity property (in contrast to the classical setting)

Lemma (6.2.43)[218]: Let H be a split exact, diffotopy invariant M_2-stable functor, $\mathrm{m}, \mathrm{n} \in 2\mathbb{N}$. Denote $\mathrm{D}_{\mathrm{n},2}$ the operator D_2 in dimension n. Then

$$\begin{aligned}
&\mathrm{H}\left((\mathrm{C}_{\mathrm{b}}\mathbb{R}^{\mathrm{m}+\mathrm{n}}) \otimes \mathbb{C}_{\mathrm{m}+\mathrm{n}}, 1, \mathrm{q}\left(\mathrm{h}\left(\mathrm{D}_{\mathrm{m},2}\right) + \mathrm{h}\left(\left(\mathrm{D}_{\mathrm{n},2}\right)\right)\right)\right) \\
&\quad = \mathrm{H}(\mathrm{y}_{\mathrm{m}+\mathrm{n}}^{\infty}) \\
&\quad = \mathrm{H}\left(\mathrm{y}_{\mathrm{m}}^{\infty} \otimes (\mathrm{S}(\mathbb{R}^{\mathrm{n}}) \otimes \mathbb{C}_{\mathrm{n}})\right) \circ \mathrm{H}(\mathrm{y}_{\mathrm{n}}^{\infty}).
\end{aligned}$$

Proof. The first equality follows by factoring over K-theory or from a diffotopy using the operator

$$q\left(e^{t\|y\|^2}h\big(c_+(x)\big)+e^{t\|x\|^2}h\big(c_+(y)\big)\right),$$

where $x \in R^m, y \in R^n$.

To see the second equality, note that in K-theory the class of y_{m+n}^{∞} is the outer product of y_m^{∞} and y_n^{∞} . Further, $H(y_{m+n}^{\infty})$ coincides with the action of the K-theory class $[y_{m+n}^{\infty}]$ by multiplication, as defined in Proposition (6.2.28). As the pairing is compatible with outer products (Proposition (6.2.28)), we get the above statement.

Note also that the operators $q\big(h(D_2)\big)$ are equivariant with respect to the $O(n)$-action; this is not true for the operators built up from the one-dimensional ones, which do however have a representative given by an equivariant module according to the above Lemma.

We come now to the notion of spectral triple, which is central in noncommutative geometry. Recall that $[T, D]$ is called bounded, where D is a regular self adjoint operator on a Hilbert space, and T a bounded operator, if T preserves the domain of D and $[\mathrm{T}, \mathrm{D}]$ extends to an element of $\mathbb{B}(\mathcal{H})$.

Definition (6.2.44)[218]: A spectral triple is given by an involutive algebra A, a representation $\varphi : \mathfrak{A} \to \mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ and a selfadjoint (unbounded) operator D on $\mathcal{H}$ such that for all $\mathrm{a} \in \mathfrak{A}$

$$[\varphi(a), D] \in B(H), \varphi(a)(1+D^2)^{-1/2} \in K(H).$$

If $\mathcal{H}$ is graded and φ is even with respect to the induced grading on $\mathbb{B}(\mathcal{H})$, and D is odd, then the spectral triple is called even. If $\mathcal{H}$ is ungraded, the spectral triple is called an odd spectral triple.

Definition (6.2.45)[218]: A spectral triple is called p-summable if $\varphi(\mathrm{a})(1+\mathrm{D}^2)^{-1/2} \in \mathcal{L}^{\mathrm{p}}$, and finitely summable if it is p-summable for some $\mathrm{p} \in \mathbb{N}$.

Because $(1+D)^{-1} = R_i(D)R_{-i}(D)$, we see that it is equivalent to demand $a(1+D^2)^{-1/2} \in L^p$ or $aR_i(D) \in L^p$. In [Con08], the topology on the algebra $\mathfrak{A}$ is reconstructed from the axioms. We go the other way, defining a spectral triple as an "unbounded $\mathcal{LC}$-Kasparov $(\mathcal{A}, \mathcal{L}^{\mathrm{p}})$-module" represented on some Hilbert space:

Definition (6.2.46)[218]: A continuous p-summable spectral triple is a spectral triple over a locally convex algebra $\mathcal{A}$ such that $[\mathrm{D}, \varphi(\mathrm{a})]$ is continuous if viewed as a function of a with values in the bounded operators, and $\varphi(\mathrm{a})(1+\mathrm{D}^2)^{-1/2}, (1+\mathrm{D}^2)^{-1/2}\varphi(\mathrm{a})$ are continuous as functions of a with values in $\mathcal{L}^{\mathrm{p}}$.

We will need a consequence of the Baaj-Julg integral formula for the proof of Proposition (6.2.49) below. We set $\mathrm{q}(\mathrm{x}) := \frac{\mathrm{x}}{\sqrt{1+\mathrm{x}^2}}$ for all $\mathrm{x} \in \mathbb{R} \cdot$ q is called the Woroniwicz transform - just like the Cayley-transform, it associates a bounded operator to an unbounded one, hence it can be used to define a functional calculus for unbounded operators ([239]). We have the Baaj-Julg formula from [240]

Lemma (6.2.47)[218]: Let D be a regular self adjoint unbounded operator on a Hilbert B-module E. Then as an integral in $B(E)$

$$(1+D^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty (1+t+D^2)^{-1}\frac{dt}{\sqrt{t}}.$$

Hence for every $\xi \in E$

$$D(1+D)^{-1/2}\xi = \frac{1}{\pi}\int_0^\infty D(1+t+D^2)^{-1}\frac{dt}{\sqrt{t}}.$$

Lemma (6.2.48)[218]: Let E be a Hilbert module, D a regular self-adjoint operator, $a \in \mathbb{B}(E)$. If $[D,a](1+D^2)^{\frac{\varepsilon-1}{2}}$ is bounded for some $\varepsilon > 0$, then

$$[q(D),a] = \frac{1}{\pi}\left(\int_0^\infty (1+t)(1+t+D^2)^{-1}[D,a](1+D^2)^{-1}\frac{dt}{\sqrt{t}} + \int_0^\infty D(1+t+D^2)^{-1}[a,D]D(1+D^2)^{-1}\frac{dt}{\sqrt{t}}\right),$$

as a uniformly converging integral of bounded operators.

Here the hypothesis that $[D,a](1+D^2)^{\frac{\varepsilon-1}{2}}$ is bounded means that a preserves the domain of D and that $[D,a](1+D^2)^{\frac{\varepsilon-1}{2}}$ is an adjointable operator on $\mathcal{D}\big((1+\mathrm{D}^2)^{\varepsilon/2}\big)$.

Proof. Set $\mathrm{T_t} := (1+\mathrm{t}+\mathrm{D}^2)^{-1}$, let $\mathrm{a} \in \mathrm{A}$ be homogeneous. Then we have

$$\begin{aligned}
[DT_t, a] &= DT_t a - (-1)^{\partial a \partial D} a D T_t \\
&= DT_t a T_t^{-1} T_t - (-1)^{\partial a \partial D} T_t^{-1} T_t a D T_t \\
&= (1+t)\big(DT_t a T_t - (-1)^{\partial a \partial D} T_t a D T_t\big) + DT_t a D^2 T_t - (-1)^{\partial a \partial D} D^2 T_t a D T_t)
\end{aligned}$$

$$= (1+t)(T_t[D,a]T_t) + (DT_t[a,D]DT_t).$$

Estimating the norms, we see that for every t

$$\|T_t[D,a]T_t\| \leq \|T_t\| \left\|[D,a](1+D^2)^{\frac{\varepsilon-1}{2}}\right\| \left\|(1+D^2)^{\frac{1-\varepsilon}{2}}T_t\right\| \leq C(1+t)^{-\frac{1+\varepsilon}{2}},$$

for a positive constant C, and similarly for the second term. Thus

$$[q(D),a] = \frac{1}{\pi}\int_0^\infty \big((1+t)(T_t[D,a]T_t) + (DT_t[a,D]DT_t)\big)\frac{dt}{\sqrt{t}}$$

as an integral with values in the bounded operators.

Note that a linear function $\varphi: \mathcal{A} \to \mathcal{L}^{\mathrm{p}}$ is continuous iff $\mathrm{a} \longmapsto \varphi(\mathrm{a})^*\varphi(\mathrm{a})$ is continuous into $\mathcal{L}^{\mathrm{p}/2}$ at 0. Hence for a continuous spectral triple $\mathrm{a} \longmapsto (1+\mathrm{D}^2)^{-1/4}\varphi(\mathrm{a})$ is also continuous (with values in $\mathcal{L}^{2\mathrm{p}}$). We have:

Proposition (6.2.49)[218]: If (H,φ,D) is a continuous p-summable spectral triple over a locally convex algebra A such that the multiplication $A \times A \to A$ is surjective and open, then $\big(B(H),\varphi,q(D)\big)$ is an LC-Kasparov (A,L^{2p})-module.

Proof. We set $T_t := (1+t+D^2)^{-1}$. As A carries the identification topology for the product, it suffices to show that $(a,b) \longmapsto [q(D),ab]$ is continuous on $A \times A$. Note first that for fixed $t \in R_{\geq 0}$

$$a \longmapsto T_t^{1/4}a = T_t^{1/4}(1+D^2)^{1/4}(1+D^2)^{-1/4}a$$

is a product of multiplier of $\mathcal{L}^{2p}$ with a continuous function $\mathcal{A} \to \mathcal{L}^{2p}$, hence continuous. Now we use Lemma (6.2.48) to develop

$$[q(D),ab] = \frac{1}{\pi}\left(\int (1+t)T_t a[D,b]T_t\frac{dt}{\sqrt{t}} + \int DT_t a[D,b]DT_t\frac{dt}{\sqrt{t}} + \int DT_t[D,a]bDT_t\frac{dt}{\sqrt{t}}\right).$$

These are norm-convergent integrals in $\mathcal{L}^{2p}$; for example:

$$\|DT_t a[D,b]DT_t\|_{\mathcal{L}^p} \leq \left\|DT_t^{1/4}\right\|\left\|T_t^{1/4}a\right\|_{\mathcal{L}^{2p}}\|D,b\|\left\|DT_t^{1/2}\right\|\left\|T_t^{1/2}\right\|$$

$$\leq \frac{C}{(1+t)^{3/4}}$$

for a constant C. Furthermore, the integrands are continuous functions of a and b, hence the result follows from dominated convergence.

We assume that the dimension n is even in order that the grading on the Clifford algebras be inner.

Recall again from [221] or [237] that x_n denotes the C^*-Kasparov $(S^n \otimes \mathbb{C}_n, \mathbb{C})$-module defined by letting $S^n \otimes \mathbb{C}_n$ act on L^2-forms on $\mathbb{R}^n$ and with operator the Dirac operator. The grading on $S^n \otimes \mathbb{C}_n$ is again inner; and we may therefore pass again to a C^*-Kasparov $(S^n \otimes \mathbb{C}_n, \mathbb{C})$-module x_n', where $S^n \otimes \mathbb{C}_n$ is viewed as graded. Denote $\iota: S^n \otimes \mathbb{C}_n \to S^n \otimes \mathbb{C}_n$ the inclusion.

Proposition (6.2.50)[218]: The restriction $x_n^\infty := \iota^*(x_n')$, to $S^n \otimes \mathbb{C}_n$, of x_n' is an $\mathcal{LC}$-Kasparov $(S^n \otimes \mathbb{C}_n, \mathcal{L}^{2(n+1)})$-module.

Proof. This follows from Proposition (6.2.49) once we have checked the hypotheses. That the commutators are bounded holds even for all smooth functions with bounded derivatives, and continuity of $f \longmapsto [f, D]$ is obvious. It suffices to show $f(1+D^2)^{-1/2} \in \mathcal{L}^{(n,+)}$ for scalar f; for every function of compact support, the operator $f(1+D^2)^{-1}f^*$ is an element of $\mathcal{L}^{(n+1)/2}$, being a pseudodifferential operator of order -1 on a compact manifold (see for example [241], Theorem (6.2.1)), hence $f(1+D)^{-1/2} \in \mathcal{L}^{n+1}$. In order to extend this to S, let h be a smooth function of compact support K that is constant 1 on the unit-cube; for $\alpha \in \mathbb{Z}^n$, we denote by t_α the operator of translation by α- an isometry when restricted to L^2 - and set $K_\alpha := t_\alpha(K)$. Define $H := \sum_\alpha t_\alpha h$, and $g := h/H$; then $\sum_\alpha t_\alpha(g) = 1$. As the Dirac operator commutes with translations, for every smooth scalar f and $\alpha \in \mathbb{Z}^n$:

$$\left\|(1+D)^{-1/2} t_\alpha(g) f\right\|_{n,+} \leq \left\|(1+D)^{-1/2} \circ t_\alpha \circ g \circ t_\alpha\right\|_{n,+} \|P_\alpha f\|_{op}$$

$$= \left\|(1+D)^{-1/2} g\right\|_{n,+} \operatorname{esssup}(P_\alpha f) = \left\|(1+D)^{-1/2} g\right\|_{n,+} \left\|f|_{K_\alpha}\right\|_\infty,$$

where $P_\alpha = \chi(K_\alpha)$ is the projection determined by the support of $t_\alpha(g)$. Now if f is rapidly decreasing, then it is easily seen that

$$\sum_\alpha \left\|f|_{K_\alpha}\right\|_\infty \leq C\|(1+x)^k\|_\infty$$

for an appropriate k and a constant C, and therefore

$$\|(1+D^2)^{-1}\|_{n,+} \leq C\left\|(1+D)^{-1/2} g\right\|_{n,+} \|(1+x)^k f\|_\infty,$$

which shows continuity of $f \longmapsto f(1+D^2)^{-1/2}$.

Alternatively, we may use the integral kernels of the operators and the following fact of independent interest from [242]: If $K(x,y) = f(x)g(x-y)$, then $\|T_K\|_p \leq \|f\|_p \|g\|_p$, where T_K denotes the operator with kernel K.

For brevity, we will call two quasihomomorphisms $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$ composable, if the target of the first is the domain of the second. We say they have a product for a given class of split exact functors if there is a quasihomomorphism $(\gamma, \bar{\gamma})$ such that

$$H(\beta, \bar{\beta}) \circ H(\alpha, \bar{\alpha}) = (\gamma, \bar{\gamma})$$

for every functor in the class.

We use the analogous wordings for abstract Kasparov modules and $\mathcal{LC}$-Kasparov modules. Recall that for a given quasihomomorphism $(\alpha, \bar{\alpha}) : \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$, the algebra $\mathcal{D}_\alpha$ is defined as the topological vector space $\mathcal{A} \bigoplus \mathcal{B}$ with a twisted multiplication, and that it comes equipped with a canonical inclusion ι of $\mathcal{B}$ into $\mathcal{D}_\alpha$.

Proposition (6.2.51)[218]: Let $(\widehat{\mathcal{B}}, \varphi_1, F_1)$ be a Kasparov $(\mathcal{A}, \mathcal{B})$-module, $(\hat{\mathcal{C}}, \varphi_2, F_2)$ a Kasparov $(\mathcal{B}, \mathcal{C})$-module, $(\alpha, \bar{\alpha}) := Qh(\widehat{\mathcal{B}}, \varphi_1, F_1)$, $\mathcal{D}_\alpha$ and ι the to α associated algebra and inclusion. Let H be a split exact functor. If there is a Kasparov (D_α, C)-module $(\hat{C}', \varphi_2', F_2')$ such that $H(\hat{C}, \varphi_2, F_2) = H(\hat{C}', \varphi_2', F_2') \circ H(\iota)$, then

$$H(\hat{C}, \varphi_2, F_2) \circ (\widehat{\mathcal{B}}, \varphi_1, F_1) = H\left(\alpha^*(\hat{C}, \varphi_2, F_2)\right) - H\left(\alpha^*(\hat{C}, \varphi_2, F_2)\right).$$

In particular, $(\widehat{\mathcal{B}}, \varphi_1, F_2)$ and $(\hat{\mathcal{C}}, \varphi_2, F_2)$ then have a product with respect to the class of split exact, M_2-stable functors.

Proof. This follows from

$$H(\iota) \circ H(\widehat{\mathcal{B}}, \varphi_1, F_1) = \alpha_* - \bar{\alpha}_*.$$

If H is M_2-stable, we may use an obvious diagonal sum to present the latter.

Corollary (6.2.52)[218]: Every two $\mathcal{LC}$-quasihomomorphisms $(\alpha, \bar{\alpha}) : \mathbb{C} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ and $(\beta, \bar{\beta}) : \mathcal{B} \rightrightarrows \hat{\mathcal{C}} \trianglerighteq \mathcal{C}$ have (up to M_2-stabilisation) a product with respect to split exact M_2-stable functors.

Proof. By Lemma (6.2.33) we can choose $(\alpha', \overline{\alpha}') : \mathbb{C} \rightrightarrows M_2(\mathcal{B}^+) \trianglerighteq M_2(\mathcal{B})$ such that $H(\theta_B) \circ H(\alpha, \bar{\alpha}) = H(\alpha', \bar{\alpha}')$, where θ_B denotes the stabilisation by M_2-matrices. Denoting by $M_2(\varphi)$ the inflation to matrices of a homomorphism φ, we get from Proposition (6.2.25):

$$H\left(\left(\beta,\bar{\beta}\right)\right)\circ H(\alpha,\bar{\alpha})=H(\theta_C)^{-1}\circ H\left(M_2(\beta),M_2\left(\bar{\beta}\right)\right)\circ H(\alpha',\bar{\alpha}').$$

Now $H\left(M_2(\beta^+),M_2\left(\bar{\beta}^+\right)\right)$ extends $H\left(M_2(\beta),M_2\left(\bar{\beta}\right)\right)$.

Let $n\in 2\mathbb{N}$. Recall that x_n^∞ and y_n^∞ are defined.

Theorem (6.2.53)[218]: Let H be a split exact, $\mathcal{L}^p$-stable functor. Then $H(x_n^\infty)$ and $H(y_n^\infty)$ are inverse to one another.

More generally, for any locally convex algebra $\mathcal{A}$:

$$H(\mathcal{A})\approx\ H\big((S(\mathbb{R}^n)\otimes\mathbb{C}_n)\otimes_\pi\mathcal{A}\big).$$

Proof. By Theorem (6.2.9), H is diffotopy invariant. We first compute $H(x_n^\infty)\circ H(y_n^\infty)$; denote by $\left(\beta,\bar{\beta}\right)$ the $\mathcal{LC}$-quasihomomorphism associated to x_n^∞ and by $\left(\beta,\bar{\beta}\right)$ the $\mathcal{LC}$-quasihomomorphism associated to y_n^∞. Then $\left(\beta,\bar{\beta}\right)$ and $(\alpha,\bar{\alpha})$ have a product by Corollary (6.2.50).

By Proposition (6.2.42), it suffices to calculate the index of the corresponding operator. Now the product-quasihomomorphism defines a class in KK-theory. It has index one by Proposition (6.2.38), Remark (6.2.20), the description of products in [243] and Bott periodicity in KK. Thus

$$H(\alpha,\bar{\alpha})\circ H\left(\beta,\bar{\beta}\right)=1. \tag{8}$$

We now show that $H(x_n^\infty)$ also has a left inverse. We proceed to calculate the product with the morphism induced by

$$\mathcal{L}^p\otimes\left(\beta,\bar{\beta}\right):\ \mathcal{L}^p\rightrightarrows\mathcal{L}^p\otimes_\pi D_\beta\trianglerighteq\mathcal{L}^p\otimes_\pi S^n\otimes\mathbb{C}_n.$$

as defined in Definition (6.2.23).

To unclutter notation, set $\mathsf{S}:=S^n\otimes\mathbb{C}_n,\ p:=\ 2(n+1)$. Note that $\sigma\mathsf{S},\mathsf{S}$ (the shift $\mu\otimes v\longmapsto v\otimes\mu$) is diffotopic to $\mathrm{id}_{\mathsf{S}}\ \otimes\theta$, where ¥è is the tensor product of $f(x)\longmapsto f(-x)$ with the automorphism of $\mathbb{C}_n$ determined by $-\mathrm{id}_{\mathbb{R}}$. We get, using Proposition (6.2.24)

$$\rho:=H\left(\left(\beta,\bar{\beta}\right)\otimes\mathcal{L}^p\right)\circ H(x_n^\infty)=H(\mathsf{S}\otimes(\alpha,\bar{\alpha}))\circ H\left(\left(\beta,\bar{\beta}\right)\otimes\mathsf{S}\ \ \right).$$

Switching the factors in the tensor product (see again Proposition (6.2.24), we have

$$\rho=H(\sigma_{\mathcal{L}^p},\mathsf{S})\circ H\big((\alpha,\bar{\alpha})\otimes\mathsf{S}\ \ \big)\circ H(\sigma\mathsf{S},\mathsf{S}\ \ )\circ H\left(\left(\beta,\bar{\beta}\right)\otimes\mathsf{S}\ \ \right).$$

Now we apply the diffotopy to replace $H(\sigma\mathsf{S}\ \ ,\mathsf{S}\ \ )$:

$$\rho = H(\sigma_{\mathcal{L}^p}, \mathrm{S}) \circ H\big((\alpha, \bar{\alpha}) \otimes \mathrm{S}\ \big) \circ H(\mathrm{S}\ \ \otimes \theta) \circ H\left(\left(\beta, \bar{\beta}\right) \otimes \mathrm{S}\ \ \right).$$

Another application of Proposition (6.2.24) yields

$$\rho = H(\sigma_{\mathcal{L}^p}, \mathrm{S}) \circ H\big((\mathcal{L}^p \otimes \theta) \circ H\big((\alpha, \bar{\alpha}) \otimes \mathrm{S}\ \ \big) \circ H\left(\left(\beta, \bar{\beta}\right) \otimes \mathrm{S}\ \ \right)$$

As we have shown above that $H(\alpha, \bar{\alpha}) \circ\ H\left(\beta, \bar{\beta}\right) = 1$ for any functor, we may apply this to H^{S} $:= H(\cdot \otimes_{\pi} \mathrm{S})$ above to deduce that ρ is invertible. Because $H(x_n^{\infty})$ has right inverse $H(y_n^{\infty})$ by 6, and is left invertible, because ρ is invertible, $H(x_n^{\infty})^{-1} = H(y_n^{\infty})$.

If $\mathcal{A}$ is any locally convex algebra, we may apply the result to $H(\cdot \otimes_{\pi} \mathcal{A})$ to obtain the result in general.

Remark (6.2.54)[218]: If one does not want to refer to Bott periodicity in KK, one can proceed as follows:

Denote $D_{\beta} \subseteq S \widehat{\otimes} \mathbb{C}_n \oplus \mathbb{C}$ the algebra associated to $\left(\beta, \bar{\beta}\right)$ (Lemma (6.2.15)); then D_{β} acts in a natural way on $\mathcal{H}$ for it may be viewed as a subalgebra of $\mathcal{C}_b^{\infty}(\mathbb{R}^n) \otimes \mathbb{C}_n$.

Let $\bar{F}_1 := q(\bar{D}_1) \in \mathbb{B}(\mathcal{H})$, where $\bar{D}_1$ is the closure of $d + d^* + c_+(x)$. Then $(\mathcal{H}, c_+, \bar{D}_1)$ is a continuous spectral triple, and $\bar{D}_1$ has summable resolvent because we have the eigenbasis of Hermite polynomials (see also [237]). Furthermore, $(D_1 - \ \bar{D}_1)\varphi(a)$ is bounded for every $a \in S^n \otimes \mathbb{C}_n$, therefore $q(\bar{D}_1)$ and $\bar{F}_1$ are summable perturbations of one another. Consequently they induce the same morphisms under H (Proposition (6.2.40)), and we may instead calculate the product using the operator $\bar{F}_1$ (Lemma (6.2.27)). Set $(\alpha, \bar{\alpha}) := Qh(\mathcal{H}, c_+, \bar{F}_1)$.

As $(\tilde{\alpha}, \tilde{\bar{\alpha}})$ extends to D_{β}, we see that $\left(\beta, \bar{\beta}\right)$ and $(\tilde{\alpha}, \tilde{\bar{\alpha}})$ have a product with respect to H (Proposition (6.2.51)). By Proposition (6.2.42), it suffices to calculate the index of the corresponding operator. Now one may calculate, using the basis of Hermite polynomials, the product directly.

To calculate the product the other way around, one performs the rotation argument as in the proof above. Even though it is not clear that ρ in the above proof has a product, the morphism

$$H(\sigma_{\mathcal{L}^p}, \mathrm{S}\ ) \circ H(\mathcal{L}^p \otimes \theta) \circ H(\alpha, \bar{\alpha}) \otimes \mathrm{S}\ \ \big) \circ H\left(\left(\beta, \bar{\beta}\right) \otimes \mathrm{S},\right.$$

obtained at the end then has a product by the first half of the proof, and we are done.

Remark (6.2.55)[218]: After the first part of the proof, i.e., when we have shown$(x_n^\infty)_* \circ (y_n^\infty)_* = \theta_*$, where θ is the canonical map $\mathbb{C} \to \mathcal{L}^{2(n+1)}$, then we may actually set

$$z := (y_n^\infty)_* \circ \theta_*^{-1} \circ (x_n^\infty)_*$$

and deduce that z is an idempotent morphism from $H(S)$ to itself.

If we are willing to use $kk^{\mathcal{L}^p}$, then it is therefore possible to use a "shortcut" and avoid the rotation trick as follows. We know that the coefficients $kk^{\mathcal{L}^p}(\mathbb{C},\mathbb{C})$ are $\mathbb{Z}$ (see [236]) and that $kk^{\mathcal{L}^p}$ satisfies Bott periodicity and is M_2 stable, and hence $kk^{\mathcal{L}^p}\big(S(\mathbb{R}^n), S(\mathbb{R}^n)\big) = \mathbb{Z}$. Consequently

$$z := kk(\mathbb{C}, y_n^\infty) \circ \theta^{-1} \circ kk(\mathbb{C}, x_n^\infty)(1),$$

where 1 denotes the unit in $kk\big(S(\mathbb{R}^n), S(\mathbb{R}^n)\big)$, is an idempotent in $\mathbb{Z}$. If it was zero, it would be zero on all of $\mathbb{Z} = kk\big(S(\mathbb{R}^n), S(\mathbb{R}^n)\big)$, and we would get a contradiction by

$$1 = \big(\theta^{-1} \circ kk(\mathbb{C}, x_n^\infty) \circ \theta^{-1} \circ kk(\mathbb{C}, y_n^\infty)\big)^2(1) = 0.$$

However, this argument does not carry over to the Thom isomorphism, and already fails for the theory kk^{alg} (of [225]) stabilized only by the smooth compact operators (one may adopt the above arguments to kk^{alg}).

Definition (6.2.56)[218]: If $\mathcal{A}$ is a locally convex algebra and X a closed manifold, then we call $\mathcal{A}$ a $C^\infty(X)$-algebra if there is a homomorphism $\mu_A : C^\infty(X) \otimes_\pi \mathcal{A} \to \mathcal{A}$ that endows $\mathcal{A}$ with the structure of a left $C^\infty(X)$-module.

Definition (6.2.57)[218]: Let $\mathcal{A}$ and $\mathcal{B}$ be locally convex $C^\infty(X)$-algebras, $\widehat{\mathcal{B}}$ an algebra that carries a left $C^\infty(X)$-module structure. An $\mathcal{LC}$-quasihomomorphism $(\alpha, \bar{\alpha}) : \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ is called $C^\infty(X)$-linear, if α and $\bar{\alpha}$ are $C^\infty(X)$-linear.

With this definition, we have the following

Lemma (6.2.58)[218]: Let $(\alpha, \bar{\alpha}) : \mathcal{A} \rightrightarrows \widehat{\mathcal{B}} \trianglerighteq \mathcal{B}$ be a $C^\infty(X)$-linear $\mathcal{LC}$-quasihomomorphism. Then for every split exact functor

$$H(\alpha, \bar{\alpha}) \circ H(\mu_{\mathcal{A}}) = H(\mu_{\mathcal{B}}) \circ H\big(C^\infty(X) \otimes (\alpha, \bar{\alpha})\big).$$

We suppose throughout this section that E is an oriented bundle of even dimension.

For any bundle $E \twoheadrightarrow X$ we denote by $\mathcal{E}_E$ the canonical Hilbert $C_0(X)$-module of longitudinal differential forms on E that are locally continuous functions in the X-direction

with values in the L^2-forms along the fibres. We denote by $h \in L^2(\mathbb{R}^n)$ the function defined as $h(\xi) := \exp\left(-\frac{\|\xi\|^2}{2}\right)$. For every bundle, we get an inclusion

$$\mathcal{E}_E \hookrightarrow \mathcal{E}_{E\oplus E^\perp}, \omega \longmapsto \omega \otimes h.$$

Thus we get an embedding

$$\mathbb{K}_{C_0(X)}(\mathcal{E}_E) \subseteq \mathbb{K}_{C_0(X)}\left(\mathcal{E}_{E\oplus E^\perp}\right) \approx C_0\left(X, \mathbb{K}\left(L^2(\mathbb{R}^{p+q}) \otimes \mathbb{C}_{p+q}\right)\right). \tag{9}$$

We also denote by D_E the longitudinal Dirac operator on $\mathcal{E}_E$.

In order to avoid the choice of an algebra of Schwartz sections, we assume for the rest of this paragraph that the base space is a compact manifold. By $S(E)$ we denote the smooth functions on E that are Schwartz functions along the fibres, and by $\Gamma^\infty(E)$ the smooth sections of E. We denote S_E the space of smooth sections of the longitudinal Clifford bundle of E that are Schwartz in the direction of the fibres. We also denote $c_+ := \varepsilon + \varepsilon^*$ the fibrewise action of $S(E)$ on $\mathcal{E}_E$.

Remark (6.2.59)[218]: If $E = P \times_{O(p)} \mathbb{R}^p$ is a decomposition of E as an associated bundle for a principal $O(p)$-bundle P over X obtained by choosing a Riemannian metric, then

$$S_E = \Gamma^\infty\left(P \times_{O(p)} \left(S(\mathbb{R}^p) \otimes \mathbb{C}_p\right)\right),$$

and S_E is thus the analogue of the function space used in [237] and [221]. Using the identification from equation (7), we get the following

Definition (6.2.60)[218]: We define x_E^∞ as the $\mathcal{LC}$-Kasparov $(S_X, S_E \otimes_\pi \mathcal{L}^p)$-module

$$\left(\mathbb{B}_{C(X)}(\mathcal{E}_E), c_+, F_E := q(D_E)\right),$$

and set $(\alpha, \bar{\alpha}) := Qh(x_E^\infty)$. We denote y_E^∞ the $\mathcal{LC}$-Kasparov (S_X, S_E)-module obtained by taking y_n^∞ fibrewise, and denote $(\beta, \bar{\beta})$ the associated quasihomomorphism.

We will use the following factorisation result in the proof of the Thom isomorphism:

Lemma (6.2.61)[218]: Let $E, E' \twoheadrightarrow X$ be two smooth bundles over a compact manifold X. If $\varphi_{E,E'} : S_{E\times E'} \to S_{E\oplus E'}$ is the restriction of forms on $E \times E'$ to $E \oplus E'$, then there is a quasihomomorphism $(\alpha', \bar{\alpha})$ such that

$$H\left(\mathcal{L}^p \otimes \varphi_{X,E}\right) \circ H\left((\alpha, \bar{\alpha}) \otimes S_{E'}\right) = H(\alpha', \bar{\alpha}') \circ H\left(\varphi_{E,E'}\right).$$

Proof. This follows from Proposition (6.2.21)-(i), because $(\alpha, \bar{\alpha}) \otimes S_{E'}$ extends to a quasihomomorphism of C^*-algebras whose composition with the extension of $\mathcal{L}^p \otimes \varphi_{X,E}$ factors over the extension of $\varphi_{E,E'}$.

Theorem (6.2.62)[218]: Let H be a split exact, $\mathcal{L}^p$ stable functor, X a closed mani-fold and $E \twoheadrightarrow X$ an orientable real vector bundle of even rank n; let $\mathcal{A}$ be a supplementary locally convex algebra. Then there is an isomorphism

$$H(\mathcal{A} \otimes_\pi S_E) \approx H(\mathcal{A} \otimes_\pi S_E)\,.$$

If E is a spin^c-bundle, then

$$H\big(\mathcal{A} \otimes_\pi S(X)\big) \approx H\big(\mathcal{A} \otimes_\pi S(E)\big).$$

Proof. Using the functor $H^{\mathcal{A}} := H(\cdot \otimes_\pi \mathcal{A})$, we may reduce the case for a general algebra $\mathcal{A}$ to the case $\mathcal{A} = \mathbb{C}$. We also recall that H is diffotopy invariant by Theorem (6.2.9).

Let $B \in K(\mathrm{S}_{\ E})$ denote the class obtained by restricting y_E^∞ to $\mathbb{C}$. Then B acts as multiplication as in $F_E^X(B): H(S_X) \to (S_X \otimes_\pi S_E)$ by Proposition (6.2.28). Combining this with the multiplication $S_X \otimes_\pi S_E \to S_E$, we obtain a map

$$\mu(B): H(S_X) \to H(S_E).$$

We have

i) $H\big(\beta, \bar{\beta}\big) = \mu(B)$

ii) $H(\alpha, \bar{\alpha}) \circ \mu(B) = H\big(K(\alpha, \bar{\alpha})(B)\big)$

iii) $K(\alpha, \bar{\alpha})(B) = 1$

Hence $H(\alpha, \bar{\alpha}) \circ H\big(\beta, \bar{\beta}\big) = 1$ by Lemma (6.2.58):

$$\begin{aligned} H(\alpha, \bar{\alpha}) \circ H\big(\beta, \bar{\beta}\big) &= H(\alpha, \bar{\alpha}) \circ H(\mu_{\mathrm{S}}\, E) \circ F_E^X(B) \\ &\overset{58}{=} H(\mu\, S_X \circ H\big((\alpha, \bar{\alpha}) \otimes S_X) \circ F_E^X(B) \\ &= id_{H(S_E)} \end{aligned}$$

where we apply Proposition (6.2.28) and (6.2.31) to see the last inequality.

Using again the rotation trick, but this time fibrewise, we get that $H(\alpha, \bar{\alpha})$ is invertible. More in detail:

$$\begin{aligned} H\left(\mathcal{L}^p \otimes \big(\beta, \bar{\beta}\big)\right) \circ H(\alpha, \bar{\alpha}) &= H\big(\mathcal{L}^p \varphi_{X,E_*}(S_X \otimes B)\big) \circ H(\alpha, \bar{\alpha}) \\ &= H\big(\mathcal{L}^p \otimes \varphi_{X,E}\big) \circ H((\mathcal{L}^p \otimes_\pi S_X) \otimes B) \circ H(\alpha, \bar{\alpha}) \end{aligned}$$

$$\overset{23}{=} H\left(\mathcal{L}^p \otimes \varphi_{X,E}\right) \circ H\left((\alpha,\bar{\alpha}) \otimes S_E\right) \circ H(S_E \otimes B)$$
$$\overset{61}{=} H(\alpha,\bar{\alpha}) \circ H\left(\varphi_{E,E}\right) \circ H(S_E \otimes B)$$
$$= H(\alpha',\bar{\alpha}') \circ H\left(\varphi_{E,E}\left(S_E \otimes B\right)\right)$$
$$= H(\alpha',\bar{\alpha}') \circ H\left(\varphi_{E,E}(B \otimes S_E)\right)$$
$$\overset{61}{=} H\left(\mathcal{L}^p \otimes \varphi_{X,E}\right) \circ H^{S_E}\left((\alpha,\bar{\alpha})\right) \circ H^{S_E}(B)$$

Thus again $H(\alpha,\bar{\alpha})$ is left and right invertible, hence invertible with inverse $H\left(\beta,\bar{\beta}\right)$.

For the second part of the theorem, one applies the Morita context coming from the spinc structure.

Remark (6.2.63)[218]: As in the case of Bott periodicity, one may also calculate the product more directly: Let $(\alpha,\bar{\alpha}) := Qh(x_n^\infty)$ and $\left(\beta,\bar{\beta}\right) := Qh(y_n^\infty)$. Then there are obvious fibred versions $(\gamma,\bar{\gamma}) := \left((\alpha,\bar{\alpha}) \otimes \mathbb{C}_n\right)_{P^\infty}$ and $\left(\delta,\bar{\delta}\right) := \left(\left(\beta,\bar{\beta}\right) \otimes \mathbb{C}_n\right)_{P^\infty}$ of these quasihomomorphisms obtained as in the C^*-case. As in the proof of smooth Bott periodicity, $\left(\delta,\bar{\delta}\right)$ may be replaced with $\left(\tilde{\delta},\bar{\tilde{\delta}}\right)$ which is the fibred version of the quasihomomorphism associated to the $\mathcal{LC}$-Kasparov module obtained as before by replacing the operator in the Dirac element by $\bar{D}$. Then $\left(\tilde{\delta},\bar{\tilde{\delta}}\right)$ extends to $\left(D_{\alpha\otimes \mathrm{id}_{\mathbb{C}_n}}\right)_{P^\infty}$. Hence $(\gamma,\bar{\gamma})$ and $\left(\tilde{\delta},\bar{\tilde{\delta}}\right)$ have a product. As the equivariant index of their product over a point is one, we get the result.

Using a fibred version of the rotation argument, we see that $\left(\tilde{\delta},\bar{\tilde{\delta}}\right)$ and $(\gamma,\bar{\gamma})$ are inverse to each other.

We now sketch the possible applications of the results proved in this paper. Details will appear elsewhere.

(i) Much effort has gone into developing an analogue of Kasparov's bivariant K-functor for more general algebras, see for example the papers by Cuntz concerning the functor kk, as well as those of Weidner ([244]), or with a more moderate scope Phillips monovariant K-theory for Frechet algebras ([245]). The Bott periodicity Theorem (6.2.53) can be used to develop bivariant K-theories on categories more general than C^*-algebras. The recipe is similar to the techiques used by Higson in [246] to construct E-theory. However, it seems necessary to approach the construction from a more "homotopic" viewpoint, and to avoid

stabilisations. Essentially, one proceeds as follows: One forms the stable diffotopy category of the category of locally convex algebra, as put forward in [247]. One then inverts a certain class of morphisms, in order to obtain a split exact functor; stabilizing this functor by an appropriate operator ideal yields a stable homotopy invariant split exact functor. We show that, as a corollary of Theorem (6.2.53), this is the universal split exact, homotopy invariant and stable functor. It seems very unlikely at the moment that this functor coincides with Cuntz's kk, and thus that kk is not the universal split exact functor, as one might expect if it was the analogue of Kasparov's KK.

(ii) The Connes-Thom isomorphism for locally convex algebras: This important result due to Connes ([248]) relates the K-theory of a crossed product of a C^*-algebras $A \rtimes_\alpha \mathbb{R}^n$ by $\mathbb{R}^n$ from the K-theory of A. The result was later proved elegantly by Fack and Skandalis in the bivariant setting ([249]). Using the techniques developed so far and the Bott periodicity Theorem (6.2.53), it is possible to prove an analogous result for a smooth version of such crossed products.

(iii) Further, there here are applications to index theory and pseudodifferential operators. The passsage from locally convex algebras to double split extensions used above may be used to associate elements in bivariant K-theory groups to pseudodifferential operators. Using the Thom isomorphism (Theorem (6.2.62)), it is now possible to state an index theorem for any split exact functor with certain properties. In particular, this includes the program outlined recently by Cuntz which analyses index-theory in the setting of his functor kk.

Corollary (6.2.63)[250]: Let H be a split exact, $\mathcal{L}^p$ stable functor, X_n closed mani-fold and $E_n \twoheadrightarrow X_n$ orientable real vector bundles of even ranks n; let $\mathcal{A}$ be a supplementary locally convex algebra. Then there is an isomorphism

$$H(\mathcal{A} \otimes_\pi S_{E_n}) \approx H(\mathcal{A} \otimes_\pi S_{E_n}) .$$

If E_n are a spins-bundles, then

$$H\big(\mathcal{A} \otimes_\pi S(X_n)\big) \approx H\big(\mathcal{A} \otimes_\pi S(E_n)\big).$$

Proof. Using the functor $H^{\mathcal{A}} := H(\cdot \otimes_\pi \mathcal{A})$, we may reduce the case for a general algebra $\mathcal{A}$ to the case $\mathcal{A} = \mathbb{C}$. We also recall that H is diffotopy invariant by Theorem (6.2.9).

Let $B \in K(S_{E_n})(\mathrm{S}_{\ E})$ denote the class obtained by restricting $y_{E_n}^{\infty}$ to $\mathbb{C}$. Then B acts as multiplication as in $F_{E_n}^{X_n}(B): H(S_{X_n}) \to (S_{X_n} \otimes_\pi S_{E_n})$ by Proposition (6.2.28). Combining this with the multiplication $S_{X_n} \otimes_\pi S_{E_n} \to S_{E_n}$, we obtain a map

$$\mu(B): H(S_{X_n}) \to H(S_{E_n}).$$

We have

iv) $H(\alpha_n + \varepsilon_n, \bar{\alpha}_n + \bar{\varepsilon}_n) = \mu(B)$

v) $H(\alpha_n, \bar{\alpha}_n) \circ \mu(B) = H\big(K(\alpha_n, \bar{\alpha}_n)(B)\big)$

vi) $K(\alpha_n, \bar{\alpha}_n)(B) = 1$

Hence $H(\alpha_n, \bar{\alpha}_n) \circ H(\alpha_n + \varepsilon_n, \bar{\alpha}_n + \bar{\varepsilon}_n) = 1$ by Lemma (6.2.58):

$$\begin{aligned} H(\alpha_n, \bar{\alpha}_n) \circ H(\alpha_n + \varepsilon_n, \bar{\alpha}_n + \bar{\varepsilon}_n) &= H(\alpha_n, \bar{\alpha}_n) \circ H\left(\mu_{S_{E_n}}\right) \circ F_{E_n}^{X_n}(B) \\ &\overset{58}{=} H(\mu\, S_{X_n} \circ H\big((\alpha_n, \bar{\alpha}_n) \otimes S_{X_n}\big) \circ F_{E_n}^{X_n}(B) \\ &= \mathrm{id}_{H(S_{E_n})} \end{aligned}$$

where we apply Proposition (6.2.28) and (6.2.31) to see the last inequality.

Using again the rotation trick, but this time fibrewise, we get that $H(\alpha_n, \bar{\alpha}_n)$ is invertible. More in detail:

$$\begin{aligned} H\big(\mathcal{L}^p \otimes (\alpha_n + \varepsilon_n, \bar{\alpha}_n + \bar{\varepsilon}_n)\big) \circ H(\alpha_n, \bar{\alpha}_n) &= H\left(\mathcal{L}^p \varphi_{X_n, E_{n_*}}(S_{X_n} \otimes B)\right) \circ H(\alpha_n, \bar{\alpha}_n) \\ &= H\big(\mathcal{L}^p \otimes \varphi_{X_n, E_n}\big) \circ H((\mathcal{L}^p \otimes_\pi S_{X_n}) \otimes B) \circ H(\alpha_n, \bar{\alpha}_n) \\ &\overset{23}{=} H\big(\mathcal{L}^p \otimes \varphi_{X_n, E_n}\big) \circ H\big((\alpha_n, \bar{\alpha}_n) \otimes S_{E_n}\big) \circ H(S_{E_n} \otimes B) \\ &\overset{61}{=} H(\alpha_n, \bar{\alpha}_n) \circ H\big(\varphi_{E_n, E_n}\big) \circ H(S_{E_n} \otimes B) \\ &= H(\alpha_n', \bar{\alpha}_n') \circ H\big(\varphi_{E_n, E_n}\left(S_{E_n} \otimes B\right)\big) \\ &= H(\alpha_n', \bar{\alpha}_n') \circ H\big(\varphi_{E_n, E_n}(B \otimes S_{E_n})\big) \\ &\overset{61}{=} H\big(\mathcal{L}^p \otimes \varphi_{X_n, E_n}\big) \circ H^{S_E}\big((\alpha_n, \bar{\alpha}_n)\big) \circ H^{S_{E_n}}(B) \end{aligned}$$

Thus again $H(\alpha, \bar{\alpha})$ is left and right invertible, hence invertible with inverse $H(\alpha_n + \varepsilon_n, \bar{\alpha}_n + \bar{\varepsilon}_n)$.

For the second part of the theorem, one applies the Morita context coming from the spinc structure.