Chapter 5

First Alexandroff Decomposition Theorem for Topological MV-Algebra

Let $(X, \mathcal{F})$ be an Alexandroff space, let $A(\mathcal{F})$ be the Boolean subalgebra of 2^X generated by $\mathcal{F}$, let G be a Hausdorff commutative topological lattice group and let $\mathrm{rba}_{\mathcal{F}}(A(\mathcal{F}), G)$ denote the set of all order bounded $\mathcal{F}$-inner regular finitely additive set functions from $A(\mathcal{F})$ into G. Using some special properties of the elements of $\mathrm{rba}_{\mathcal{F}}(A(\mathcal{F}), G)$, we extend to this setting the first decomposition theorem of Alexandroff. We deduce from this a representation theorem for Hausdorff order continuous locally solid Dedekind complete ℓ-groups which admits a 0-neighbourhood base consisting of sublattices.

Section (5.1): The Alexandroff Decomposition Theorem:

In [168] proved an Alexandroff decomposition type theorem (see [148, 149, 150, 169]) for an order bounded inner regular measure μ on a Boolean algebra L with values in a Hausdorff topological lattice group with a strong assumption, i.e. G has a base of neighbourhoods of 0 consisting of sublattices (see also [151,152,153,170]).

In the present Section we prove that this decomposition theorem is a particular case of a general decomposition theorem (see Theorem (5.1.14)) which holds with a weaker assumption on G, i.e. the positive cone of G is closed, and also holds if L is replaced by a weaker structure, i.e. if L is a D-lattice (= lattice ordered effect algebra) and μ is an exhaustive modular measure on L.

We also prove a Hewitt-Yosida type decomposition theorem which extends another result of [168] (see also [171] and [172,156,157]).

We recall that effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 (see [154,155,173]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see [174], [175], [176]) and in Mathematical Economics (see [177], [178], [179] [158], [159], [160], [161]), in particular of orthomodular posets and MV-algebras. Therefore the study of measures on effect algebras allows to extend and unify two independent theories: non-commutative measure theory (study of measures on orthomodular lattices) and fuzzy measure theory (study of measures on MV-algebras).

We prove the main decomposition theorem and we compare it with the decomposition theorems of [162,163,164,168].

An effect algebra $(L,\oplus,0,1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(i) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(ii) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(iii) For every $a \in L$, there exists a unique $a\perp \in L$ such that $a \oplus a\perp$ is defined and $a \oplus a\perp = 1$.

(iv) If $a \oplus 1$ is defined, then $a = 0$.

In every effect algebra a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a.

Moreover we can define a binary relation on L by $a \leq b$ if and only if there exists $c \in L$ such that $c \oplus a = b$ and ≤ is a partial ordering in L, with 0 as the smallest element. We say that two elements a, b ∈ L are orthogonal, and we write $a \perp b$, if $a \oplus b$ exists. Then $a \perp b$ if and only if $a \leq b^{\perp}$. Moreover, for every $a \in L$, we have $b^{\perp} = 1 \ominus a$.

Effect algebras are a common generalization of orthomodular posets and MV-algebras. For a study, we refer to [165,166,180].

If $a_1, \ldots, a_n \in L$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that afinite subset $\{a_1, \ldots, a_n\}$ $of\ L\ is$ orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists. We say that a subset $A\ of\ L$ is orthogonal if every finite subset of A is orthogonal. If $A = \{a_\alpha : \alpha \in I\}$ is an orthogonal set in L, we set

$$\bigoplus_{\propto \in I} a_\alpha = \sup\left\{\bigoplus_{\propto \in I} a_\alpha : F \subseteq I \text{ finit}\right\}$$

provided that the right hand side exists.

If (L, ≤) is a lattice, we say that the effect algebra is a lattice ordered effect algebra or a D-lattice. In this case, we set $a \vartriangle b = (a \vee b) \ominus (a \wedge b)$ for $a, b \in L$.

If G is a group, a function $\mu : L \to G$ is said to be a measure if $a \perp b$ implies $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure if and only if $a \leq b$implies $\mu(b \ominus a) = \mu(b) - \mu(a)$.

A function $\mu : L \to G$ is said to be modular if, for every $a, b \in L, \mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

If G is a topological group, a measure $\mu : L \to G$ is said to be exhaustive if, for every orthogonal sequence $\{a_n\}\ in\ L, \mu(a_n) \to 0$. Moreover μ is said to be σ-additive if, for every orthogonal sequence $\{a_n\}$ such that $a = \oplus_n a_n$ exists,

$$\mu(a) = \sum_{n=1}^{\infty} \mu(a_n)$$

and μ is to be σ −order continuous $(\sigma - o.c.)$ if, for every decreasing sequence $\{a_n\}$ in L such that inf a_n = 0, $lim_n\, \mu(a_n) = 0$.By [181], a measure $\mu : L \to G$ is σ-additive if and only if it is $\sigma - o.c.$

By [181], if G is a topological Abelian group, every modular measure μ generates on L a $D - uniformity\ \mathcal{U}\ (\mu)$, i.e. a uniformity which makes the lattice operations, as well as $\ominus$ and $\oplus$, uniformly continuous. A base of $\mathcal{U}\ (\mu)$ is the Family consisting the sets

$$(a, b) \in L \times L : (\forall\, c \leq a\ b)(\mu(c) \in W),$$

where W is a neighbourhood of 0 in G. We write $\mathcal{U}\ (\mu) = 0$ if $\mathcal{U}$ (μ) is the trivial uniformity. By [182], the set of all D-uniformities on L is a distributive lattice with respect to the usual order between uniformities.

A group G is said to be an ordered group if there is an order relation $\leq\ on\ G$ with the following property: If $x, y \in G$ and $x \leq y$, then $x + z \leq y + z$ and $z + x \leq z + y$ for every $z \in G$. If $(G, \leq)$ is a lattice, we say that G is a lattice group or an -group. If G is a topological group and an -group and ∨,∧ are continuous, we say that G is a topological -group.

If G is an ℓ -group, we set, for every $x \in G$,

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0,$$

$$|x| = x^+ + x^-.$$

It is known (see e.g. [183]) that, for every $x \in G$, we have

$$x = x^+ - x^- \quad \text{and} \quad |x| = x \vee (-x).$$

In an ordered group G, a subset A of G is said to be order convex if, for every $x, y \in A$ with $x < y$, the interval $[x, y] = \{z \in G : x \leq z \leq y\}$ is contained in A and order bounded if A is contained in an interval of G.

We say that G is order complete if, for every non-empty majorized subset D of G, sup D exists in G and quasi-order complete if, for every majorized upward directed subset D of G, sup D exists in G.

It is known (see [183]) that every order complete ℓ-group is Abelian.

If $(G, \tau, +)$ is an ordered topological group, we say that G is locally order convex if the set of all order convex neighborhoods of 0 in G forms a base for τ and that G is compatible if G^+ = {x ∈ G : x ≥ 0} is closed in τ .

If G is an ℓ -group, a subset A of G is said to be solid if, for every $x \in G$ and $y \in A$ with $|x| \leq |y|$, we have $x \in A$. We say that a topological ℓ -group G is locally solid if there exists a base of neighbourhoods of 0 consisting of solid sets.

We need the following result ([184]).

Proposition (5.1.1)[167]: Let G be a topological ℓ -group. Then G is locally solid if and only if G is a locally order convex topological lattice.

Following [168], we say that an ordered group G has the property (oc) if: (oc) for every directed upward subset D of G such that $x = \sup D$ exists in G, we have $x \in \overline{D}$, and a topological ℓ -group G satisfies the condition (M) if: (M) there exists a base of neighbourhoods of 0 consisting of sublattices of G.

We need the following results ([168] and [185]).

Proposition (5.1.2)[167]: Let G be a Hausdorff order complete ℓ-group with the property (oc). If V is a sublattice of G and D is a non-empty majorized subset of V, then $x = sup\ D$ belongs to $\overline{V}$.

Proposition (5.1.3)[167]: Let G be an ordered topological group which is locally order convex. Then the following conditions are equivalent:

(a) G has the property (oc).

(b) For every increasing net $(x_i)_{i \in I}$ in G such that

x = $\sup_{i\in I} x_i$ exists in G, we have $x = lim_{i\in I} x_i$.

In what follows L is a D-lattice and G is a Hausdorff topological Abelian group. In this Section we suppose that G is ordered.

A measure $\mu : L \to G$ is said to be order bounded if $\mu(L)$ is order bounded in G. If $\mu \geq 0$, we say that μ is a positive measure. Denote by $B(L,G)$ the set of all order bounded G-valued modular measures on L. It is clear that $B(L,G)$ contains all positive G-valued modular measures on L and it is an ordered group with respect to the natural order between G-valued functions.

The aim of this Section is to prove that, if G is order complete, then $B(L,G)$ is an ℓ-group (See Corollary (5.1.7)).

Let $\mu : L \to G$ be a function. Then μ is a modular measure if and only if, for every $a, b \in L$, the following equality holds:

$$\mu(a) = \mu((a \vee b) \ominus b) + \mu(a \wedge b). \qquad (*)$$

Proof. If μ is a modular measure, then for every $a, b \in L$, we have μ((a ∨ b) * b)= μ(a ∨ b) − μ(b)= μ(a) − μ(a ∧ b), whence the assertion.

Conversely, if c and d are orthogonal elements in L, applying $(*)$ with $a = c \oplus d$ and $b = d$, we obtain $\mu(c \oplus d) = \mu(c) + \mu((c \oplus d) \wedge d) = \mu(c) + \mu(d)$. Hence μ is a measure and therefore, by $(*)$, we also obtain that μ is modular.

Lemma (5.1.4)[167]: Let $\mu : L \to G$ be a modular measure and $a, b \in L$. Then, for every $c, d \in L$ with $c \leq (a \vee b) \ominus b$ and $d \leq a \wedge b$, there exists $e \in L$ such that $e \leq a$ and $\mu(e) = \mu(c) + \mu(d)$.

Proof. Set $s = (a \vee b) \ominus c$.

(i) First we prove that $c = (a \vee s) \ominus s$.

Since $c \leq (a \vee b) \ominus b$, we have $s \geq b$. Since $s \leq a \vee b$, we obtain $a \vee s = a \vee b$.

Then we have $s = (a \vee s) \ominus c$ and $(a \vee s) \ominus s = (a \vee s) \ominus ((a \vee s) \ominus c) = c$.

(ii) Now observe that $a \ominus (a \wedge s)^{\perp} d$ since, by $s \geq b$, we have $a \ominus (a \wedge s) \leq a \ominus (a \wedge b)$, whence $a \ominus (a \wedge s) \leq (a \wedge b)^{\perp} \leq d^{\perp}$. Hence set $e = (a \ominus (a \wedge s)) \oplus d$.

We have $e \leq (a \ominus (a \wedge b)) \oplus (a \wedge b) = a$ and, since μ is a modular measure, by (i) we obtain $\mu(e) = \mu(a \ominus (a \wedge s)) + \mu(d) = \mu((a \vee s) \ominus s) + \mu(d) = \mu(c) + \mu(d)$.

Let $\mu : L \to G$ be a modular measure. Set

$$\mu + (a) = \sup\{\mu(b): b \in L, \ b \leq a\}$$

for $a \in L$. Then $\mu +$ is a positive modular measure.

Proof. By Lemma (5.1.4), we have to prove that, for every $a, b \in L$,

$$\mu + (a) = \mu + ((a \vee b) \ominus b) + \mu + (a \wedge b). \qquad (*)$$

Let $p \leq a$. Since μ is a modular measure, we have $\mu(p) = \mu((p \vee b) \ominus b) + \mu(p \wedge b) \leq \mu + ((a \vee b) \ominus b) + \mu + (a \wedge b)$, whence $\mu + (a) \leq \mu + \big((a \vee b) \ominus b\big) + \mu + (a \wedge b)$. Now let $c \leq (a \vee b) \ominus b$ and $d \leq a \wedge b$. By Lemma (5.1.4), we can find $p \leq a$ such that $\mu(p) = \mu(c) + \mu(d)$. Then $\mu(c) + \mu(d) \leq \mu + (a)$, whence $\mu + ((a \vee b) \ominus b) + \mu + (a \wedge b) \leq \mu + (a)$.

Corollary (5.1.5)[167]: Suppose that G is order complete. Then the set $B(L, G)$ is an -group and, for every $\mu \in B(L, G), \mu \vee 0$ and $(-\mu) \vee 0$ are given, respectively, by the formulae:

And

$$\mu^+ (a) = sup\{\mu(b): b \in L, \ b \leq a\}$$

$$\mu^-(a) = -inf\{\mu(b): b \in L, \ b \leq a\}.$$

Proof. By [183], it is sufficient to prove that, for every $\mu \in B(L, G), \mu \vee 0$ exists in $B(L, G)$ and equals μ^+. By Proposition (5.1.6) , μ^+ is a modular measure and therefore it is order bounded since it is positive. Trivially μ^+ is an upper bound of μ and 0. Moreover, let λ ∈ $B(L, G)$ be an upper bound of μ and 0. If $a \in L$ and $b \leq a$, since λ is positive, we have $\lambda(a) \geq \lambda(b) \geq \mu(b)$, whence $\lambda(a) \geq \mu + (a)$. Hence $\mu + = \mu \vee 0$ in $B(L, G)$. The equality $\mu^- = (-\mu) \vee 0$ follows from [183].

The aim of this Section is to prove that Alexandroff and Hewitt–Yosida decomposition theorems proved in [168] for measures on Boolean algebras with a stronger assumption on G (see [168]) are particular cases of a general decomposition theorem which holds in general for modular measures on D-lattices and with a weaker assumption on G.

If $\mu : L \to G$ is a modular measure and $\mathcal{U}$ is a D-uniformity, we write:

(a) $\mu \ll \mathcal{U}$ *if* μ *is* $\mathcal{U}$ -continuous,

(b) $\mu \perp \mathcal{U}$ if, for every neighbourhood W of 0 in G and every neighbourhood U of 0 in $\mathcal{U}$, there exists $a \in L$ such that $\mu(b) \in W$ for every $b \leq a$ and $a^\perp \in U$.

If $\lambda : L \to G$ is another modular measure, $\mu \ll \lambda$ means $\mu \ll \mathcal{U}(\lambda)$ and $\mu \perp \lambda$ means $\mu \perp \mathcal{U}(\lambda)$.

In [186], the following decomposition theorem has been proved.

Theorem (5.1.6)[167]: Let $\mu : L \to G$ be an exhaustive modular measure and $\mathcal{U}$ a D-uniformity on L. Then there exist unique modular measures $\lambda, \nu : L \to G$ such that $\mu = \lambda + \nu$, $\lambda \ll \mathcal{U}$ and $\nu \perp \mathcal{U}$. Moreover λ and ν are exhaustive, $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$ and $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$.

Definition (5.1.7)[167]: A modular measure $\mu : L \to G$ is said to be purely non σ-additive if, for every σ-additive modular measure $\lambda : L \to G$ such that $\lambda \ll \mu$, we have $\lambda = 0$.

Definition (5.1.8)[167]: Let $\mu : L \to G$ be a function such that $\mu(0) = 0$ and $K \subseteq L$. We say that μ is K -inner regular if, for every $c \in L$ and every neighbourhood W of 0 in G, there exist $b \in K$ such that $b \leq c$ and $\mu(d) \in W$ whenever $d \in L$ and $d \leq c \ominus b$.

Recall that, for a modular measure $\mu : L \to G$, a base of neighbourhoods of 0 in $\mathcal{U}(\mu)$ is the family consisting of the sets $a \in L : (\forall b \leq a)(\mu(b) \in W)$, where W is a neighbourhood of 0 in G. Therefore it is clear that, if μ is a K -inner regular modular measure and λ is a modular measure such that $\lambda \ll \mu$, then λ is K -inner regular, too.

Definition (5.1.9)[167]: Let $K \subseteq L$ and $\mu : L \to G$ be a modular measure. We say that μ is K -smooth if, for every decreasing net $(a_i)_{i \in I}$ in K such that $\inf_{i \in I} a_i = 0$ in L, we have that a_i converges to 0 in $\mathcal{U}(\mu)$.

We say that a D-uniformity $\mathcal{U}$ is K -smooth if, for every decreasing net $(a_i)_{i \in I}$ in K such that $\inf_{i \in I} a_i = 0$ in L, $(a_i)_{i \in I}$ converges to 0 in $\mathcal{U}$. Then a modular measure $\mu : L \to G$ is K -smooth if and only if $\mathcal{U}(\mu)$ is K -smooth.

Definition (5.1.10)[167]: Let K be a subset of L and $\mu : L \to G$ be a modular measure. We say that μ is K -singular if, for every K -smooth modular measure $\lambda : L \to G$ such that $\lambda \ll \mu$, we have that $\lambda = 0$.

It is clear that, if μ is a K -singular (K -smooth, respectively) modular measure and λ is a modular measure such that $\lambda \ll \mu$, then λ is K -singular (K -smooth, respectively), too.

By [186], the following result holds.

Proposition (5.1.11)[167]: Let μ be a G-valued modular measure on L and U a D-uniformity on L. Then:

(a) $\mu \ll \mathcal{U}$ if and only if $\mathcal{U}(\mu) \leq \mathcal{U}$.

(b) $\mu \perp \mathcal{U}$ if and only if $\mathcal{U}(\mu) \wedge \mathcal{U} = 0$.

(c) μ is purely non σ-additive if and only if, for every σ-additive modular measure $\lambda : L \to G$, we have $\mu \perp \lambda$.

Lemma (5.1.12)[167]: Let $\mu : L \to G$ be an exhaustive modular measure and $\mathcal{U}$ a D-uniformity such that $\mathcal{U} \leq \mathcal{U}(\mu)$. Then there exists a modular $measure\ \nu : L \to G$ such that $\nu\ \mu$ and $\mathcal{U} = \mathcal{U}(\nu)$. Moreover, if G is a compatible ordered group and μ is positive, we can choose ν such that $0 \leq \nu \leq \mu$.

Proof. we can find two modular measures $\lambda, \nu : L \to G$ such that $\mu = \lambda + \nu, \nu \ll \mathcal{U}$ and $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$. Hence, by Proposition (5.1.13), $\nu \ll \mu$ and, as proved in [186], $\mathcal{U}$= $\mathcal{U}$ (ν).

Now suppose that μ is positive and G is compatible. In this case, from $\lambda(L) \cup \nu(L) \subseteq \mu(L)$, we obtain that λ and ν are positive, too, and therefore $\nu \leq \mu$.

Proposition (5.1.13)[167]: Let $\mu : L \to G$ be an exhaustive modular measure and $K \subseteq L$. Denote by $\mathcal{U}$ the supremum of all K -smooth D-uniformities on L. Then the following conditions are equivalent:

(a) μ is K -singular.

(b) μ ⊥ .

(c) For every K -smooth modular measure $\lambda : L \to G, \mu \perp \lambda$.

Proof. (a) ==⇒ b). By Lemma (5.1.14), we can find a modular measure $\nu : L \to G$ such that $\nu \ll \mu$ and $\mathcal{U}(\nu) = \mathcal{U}(\mu) \wedge \mathcal{U}$. Then ν is K -smooth, since $(\nu) \leq \mathcal{U}$. By (1), we obtain $\nu = 0$. Hence $\mathcal{U}(\mu) \wedge \mathcal{U} = 0$. By Proposition (5.1.13), we have that $\mu \perp \mathcal{U}$.

(b) ==⇒ (c). Let $\lambda : L \to G$ be a K -smooth modular measure. Then $(\lambda) \leq \mathcal{U}$ Hence we have $\mathcal{U}(\mu) \wedge \mathcal{U}(\lambda) = \mathcal{U}(\mu) \wedge \mathcal{U}(\lambda) \wedge \mathcal{U} = 0$ by (b) we obtain that $\mu \perp \lambda$.

(c) ==⇒ (a). If $\lambda : L \to G$ is a K -smooth modular measure such that $\lambda \ll \mu$, by (c) we have $\mathcal{U}(\lambda) = \mathcal{U}(\lambda) \wedge \mathcal{U}(\mu) = 0$, from which $\lambda = 0$.

Theorem (5.1.14)[167]: (Alexandroff decomposition theorem) Let $K \subseteq L$ and $\mu : L \to G$ be an

exhaustive modular measure. Then there exist unique exhaustive modular measures $\lambda, \nu : L \to G$ such that:

(a) $\mu = \lambda + \nu$.

(b) λ is K -smooth.

(c) ν is K -singular.

Moreover:

(d) If $H \subseteq L$ and μ is H -inner regular, then λ and ν are H -inner regular, too.

(e) If G is a compatible ordered group and μ is positive (resp. order bounded), then λ and ν are positive (resp. order bounded), too.

Proof. Denote by $\mathcal{U}$ the supremum of all K -smooth D-uniformities on L.

There exist unique modular measures $\lambda, \nu : L \to G$ such that $\mu = \lambda + \nu$, $\lambda \ll \mathcal{U}$ and $\nu \perp \mathcal{U}$. Moreover λ and ν are exhaustive, $\lambda(L) \cup \nu(L) \subseteq \mu(L)$ and $\mathcal{U}(\mu) = \mathcal{U}(\lambda) \vee \mathcal{U}(\nu)$. Therefore λ *is* K -smooth since, $\mathcal{U}(\lambda) \leq \mathcal{U}$ and ν is K -singular. Conversely, let $\lambda, \nu : L \to G$ be modular measures with properties (a), (b) and (c). Then, $\nu \perp \mathcal{U}$. Moreover we have $\mathcal{U}(\lambda') \leq \mathcal{U}$. and therefore by Proposition (5.1.13), $\lambda' \ll \mathcal{U}$. Hence the uniqueness follows.

(d) is trivial since $\lambda \ll \mu$ and $\nu \ll \mu$.

(e) follows from $\lambda(L) \cup \nu(L) \subseteq \overline{\mu(L)}$, since G is compatible and therefore both the positive cone and the order intervals of G are closed.

Theorem (5.1.15)[167]: (Hewitt-Yosida decomposition theorem) Let $\mu : L \to G$ be an exhaustive modular measure. Then there exist unique modular measures $\lambda : L \to G$ and $\nu : L \to G$ such that

(a) $\mu = \lambda + \nu$.

(b) λ is σ-additive.

(c) ν is purely non σ-additive.

Moreover:

(d) If $H \subseteq L$ and μ *is* H -inner regular, then λ and ν are H -inner regular, too.

(e) If G is a compatible ordered group and μ is positive (resp. order bounded), then λ and ν are positive (resp. order bounded), too.

Proof. (a), (b) and (c) have been proved in [186] as a consequence.

(d) and (e) can be proved.

Following [168], we give the following definitions:

(i) A subset K of L is a paving if 0 ∈ K and K is closed with respect to finite suprema.

(ii) A subset $K\ of\ L$ is said to be a δ-paving if K is a paving and every countable subset of K has an infimum in L which belongs to .

(iii) A Lindelof space is a pair (X, F), where X is a non-empty set, F is a δ-paving containing X in the Boolean algebra of all subsets of X and every covering of X consisting of complements of elements of F contains a countable subcovering.

If G is an ordered group, $\mu : L \to G\ is$ a positive modular measure and K is a paving, μ is said to be purely finitely additive (purely f.a.) if, for every σ-additive positive modular measure $\lambda \leq \mu$, we have λ = 0.

(i) μ is said to be K -smooth in the sense of de Lucia-Morales if, for every decreasing net $(a_i)_{i\in I}$ in K such that inf $a_i = 0$ in L, we have $\lim_i \mu(a_i) = 0$ in G.

(ii) μ is said to be K -singular in the sense of de Lucia-Morales if μ is K -inner regular and, for every positive K-smooth and K-inner regular modular measure λ such that $\lambda \leq \mu$, we have $\lambda = 0$.

(iii) If $\mu : L \to G$ is an order bounded measure, μ is said to be K -singular (resp. K -smooth) in the sense of de Lucia-Morales if μ^+ and μ^- are K -singular (resp. K -smooth). In a similar way, μ is said to be purely f.a. if μ^+ and μ^- are purely f.a..

In [168] the following decomposition theorems have been proved.

Theorem (5.1.16)[167]: Let G' be a Hausdorff Abelian -group which is order complete, locally order convex, has the property (oc) and satisfies the condition (M), A a Boolean algebra and $K \subseteq A$ a δ-paving. Then, for every order bounded K-inner regular measure $\mu : A \to G'$, there exist unique order bounded K-inner regular measures $\lambda, \mu : A \to G'$ such that:

(a) $\mu = \lambda + \nu$.

(b) $\lambda\ is\ K$ -smooth in the sense of de Lucia-Morales.

(c) $\nu\ is\ K$ -singular in the sense of de Lucia-Morales.

Theorem (5.1.17)[167]: Let $(X, \mathfrak{F})$ be a Lindelof space, $\mathcal{A}$ an algebra of subsets of X which contains $\mathfrak{F}$ and G' a Hausdorff Abelian -group which is order complete, locally order convex,

has the property (oc) and satisfies the condition (M). Then, for every $\mathfrak{F}$ -inner regular and order bounded measure $\mu : \mathcal{A} \to G'$, there exist unique $\mathfrak{F}$ -inner regular and order bounded measures $\lambda, \nu : \mathcal{A} \to G'$ such that $\mu = \lambda + \nu$, λ is σ-additive and ν is purely finitely additive. We want to prove that the decomposition Theorems (5.1.15) and (2.5.16) proved in [168] are particular cases of Theorem (5.1.16) and Theorem (5.1.17).

Proposition (5.1.18)[167]: Suppose that G is a locally order convex ℓ -group with the property (M). Then G is locally solid (and therefore it is compatible).

Proof. Let V be a convex neighbourhood of 0 in G. To prove that G is locally solid, it is sufficient to prove that V contains a solid neighbourhood of 0 in G. Set

$$W = \{ x \in G : [-|x|, |x|] \subseteq V .$$

It is clear that $\subseteq V$. Moreover W is solid since, if $x \in W$ and y is an element of G such that $|y| \leq |x|$, then $[-|y|, |y|] \subseteq [-|x|, |x|]$ and therefore $y \in W$.

It remains to prove that W is a neighbourhood of 0 in G.

Since G satisfies the condition (M), we can find a neighbourhood V' of 0 in G such that $V' \subseteq V$ and V' is a sublattice of G.

We prove that $V' \cap (-V') \subseteq W$, $f$rom which we obtain the assertion.

If $x \in V' \cap (-VV')$, since V' is a sublattice, we have that $|x| = x \vee (-x)$ and $-|x| = x \wedge (-x)$ belongs to V' and therefore to. Since V is convex, we obtain that $[-|x|, |x|] \subseteq V$ and therefore $x \in W$.

By Proposition (5.1.1), we obtain that G is compatible.

Proposition (5.1.19)[167]: Suppose that G is an order complete group with the property (oc). Then every order bounded modular measure $\mu : L \to G$ is exhaustive.

Proof. we have $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive. Then it is sufficient to prove the assertion in the case that μ is positive.

Let $\{a_n\}$ be an orthogonal sequence. For each $n \in N$, set

$$x_n = \sum_{i=1}^{n} \mu(a_i)$$

Since $\mu \geq 0$, $\{x_n\}$ is an increasing sequence in G and $x_n = \mu(\bigoplus_{i=1}^{n} a_i) \leq \mu(1)$.

Since G is order complete, there exists $x = \sup_n x_n$ in G. Since G has the property (oc), by Proposition (5.1.3) we have $x = \lim_n x_n$. Since $\mu(a_n) = x_n - x_{n-1}$ (with x_{-1}=0), we obtain $\lim_n \mu(a_n)$ =0.

For $\mu \in B(L, G)$, we set $|\mu| = \mu^+ + \mu^-$.

Proposition (5.1.20)[167]: Suppose that G is an order complete locally order convex group. Then, if $\mu \in B(L, G)$, $\mathcal{U}(|\mu|) = \mathcal{U}(\mu+) \vee \mathcal{U}(\mu-)$.

Proof. Recall that, for a positive modular measure λ, a base of $\mathcal{U}(\lambda)$ is the family consisting of the sets $(a, b) \in L \times L : \lambda(a \vartriangle b) \in$ W, where W is a neighbourhood of 0 in G. Moreover, by Corollary (5.1.7), μ^+ and μ^- are modular measures and therefore $|\mu|$ is a modular measure, too.

Let W be a neighbourhood of 0 in G.

(i) Since G is locally order convex, we can choose a convex neighbourhood V of 0 in G such that $V \subseteq W$ and therefore, since $0 \leq \mu^+ \leq |\mu|$ and $0 \leq \mu^- \leq |\mu|$, we have that $\mu^+(a \vartriangle b) \in W$ and $\mu^-(a \vartriangle b) \in W$ $whenever$ $|\mu|(a \vartriangle b) \in V$.

(ii) Since G is a topological group, we can choose a neighbourhood V' of 0 in G such that $V' + V' \subseteq W$. Then we have $|\mu|(a\, b) \in W$ whenever $\mu^+(a\ b) \in V'$ and $\mu^-(a\ b) \in V'$.

From (i) and (ii), we obtain the assertion.

Proposition (5.1.21)[167]: Suppose that G is an order complete locally order convex ℓ-group with the properties (oc) and (M). Then, for any $\mu \in B(L, G)$, $\mathcal{U}(\mu) = \mathcal{U}(|\mu|)$.

Proof. Let W ,V be neighbourhoods of 0 in G such that V is convex and $V - V \subseteq W$. Since, by Proposition (5.1.22), $\mathcal{U}(|\mu|) = \mathcal{U}(\mu^+) \vee \mathcal{U}(\mu^-)$, we can find a neighbourhood V_0 of 0 in G such that, for every $a, b \in L, \mu^+(a \vartriangle b) \in V$ and $\mu^-(a \vartriangle b) \in V$ whenever $|\mu|(a \vartriangle b) \in V0$. Since $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are monotone and V is convex, we have that, for every $a, b \in L$ and $c \leq a \vartriangle b, \mu(c) \in W$whenever $|\mu|(a \vartriangle b) \in V_0$. Therefore $\mathcal{U}(\mu) \leq \mathcal{U}(|\mu|)$.

Moreover, since G satisfies the condition (M), we can find aneighbourhood W_0 of 0 in G such that W_0 is a sublattice of G and $\overline{W_0} \subseteq W$. Therefore, the condition $\mu(c) \in W_0$ for every $c \leq a \vartriangle$ b implies $\mu^+(a \vartriangle b) = \sup\{\mu(c) : c \leq a \vartriangle b\} \in \overline{W_0} \subseteq W$. Hence

$\mathcal{U}(\mu^+) \leq \mathcal{U}(\mu)$. Since $\mu^-(a) = sup\{-\mu(b): b \leq a\}$ and G has a base of symmetric neighbourhoods of 0, we obtain in a similar way that $\mathcal{U}(\mu^-) \leq \mathcal{U}(\mu)$. Hence $\mathcal{U}(|\mu|) \leq \mathcal{U}(\mu)$. In the next result we use the fact that, by [182], the exhaustive Duniformities on L form a Boolean algebra.

Proposition (5.1.22)[167]: Let $K \subseteq L$. If λ and μ are G-valued exhaustive K -singular (resp. purely non σ-additive) modular measures on L, then $-\mu$ and $\lambda + \mu$ are K -singular (resp. purely non σ-additive).

Proof. It is clear that $\mathcal{U}(\mu) = \mathcal{U}(-\mu)$ Then, if μ is K -singular (resp. purely non σ-additive), $-\mu$ has the same property. Moreover, since $\mathcal{U}(\lambda + \mu) \leq \mathcal{U}(\mu) \vee \mathcal{U}(\lambda)$, if ν is a modular measure, by [182] we have

$$\mathcal{U}(\lambda + \mu) \wedge \mathcal{U}(\nu) \leq (\mathcal{U}(\lambda) \vee \mathcal{U}(\mu)) \wedge \mathcal{U}(\nu) = (\mathcal{U}(\lambda) \wedge \mathcal{U}(\nu)) \vee (\mathcal{U}(\mu) \wedge \mathcal{U}(\nu)).$$

we can find modular measures $\nu_1, \nu_2 : L \to G$ such that $\nu_1 \ll \lambda, \nu_2 \ll \mu$,

$\mathcal{U}(\nu_1) = \mathcal{U}(\lambda) \wedge \mathcal{U}(\nu)$ and $\mathcal{U}(\nu_2) = \mathcal{U}(\mu) \wedge U(\nu)$. Therefore

$$\mathcal{U}(\lambda + \mu) \wedge \mathcal{U}(\nu) \leq \mathcal{U}(\nu_1) \vee \mathcal{U}(\nu_2).$$

(i) Suppose that λ and μ are K -singular and ν is K -smooth. Then we have that ν_1 and ν_2 are K-smooth since $\mathcal{U}(\nu_1) \leq U\mathcal{U}(\nu)$ and $\mathcal{U}(\nu_2) \leq \mathcal{U}(\nu)$ and therefore $\nu_1 = \nu_2 = 0$. Hence we obtain $\mathcal{U}(\lambda + \mu) \wedge \mathcal{U}(\nu) = 0$. we obtain that $\lambda + \mu$ is K -singular.

(ii) Now suppose that λ and μ are purely non σ-additive and ν is σ-additive. Then, since $\mathcal{U}(\nu_1) \leq \mathcal{U}(\nu)$ and $\mathcal{U}(\nu_2) \leq \mathcal{U}(\nu)$, we have, by [181], that ν_1 and ν_2 are σ-additive, too. Therefore we have $\nu_1 = \nu_2 = 0$ and then, as before, $\mathcal{U}(\lambda + \mu) \wedge \mathcal{U}(\nu) = 0$. we obtain that $\lambda + \mu$ is purely non σ-additive.

Proposition (5.1.23)[167]: Suppose that G is locally order convex. Let $\mu : L \to G$ be a positive modular measure and $K \subseteq L$. Then:

(a) If μ is K -inner regular, then μ is K -singular in the sense of de Lucia Morales if and only if μ is K -singular.

(b) μ is K -smooth in the sense of de Lucia-Morales if and only if μ is K -smooth.

(c) μ is purely non σ-additive if and only if μ is purely f.a..

Moreover, if G is an order-complete ℓ-group with the properties (oc) and (M), then the previous equivalences hold for any order bounded modular measure $\mu : L \to G$.

Proof. (i) First suppose that μ is positive.

In this case, a base of neighbourhoods of 0 in $\mathcal{U}$ (μ) is the family consisting of the sets $\{a \in L : \mu(a) \in W\}$, where W is a neighbourhood of 0 in G. Then, if is a net in L, we have that $a_i \to 0$ in $\mathcal{U}$ (μ) if and only if $\mu(a_i) \to 0$ in G. Therefore it is clear that the equivalence in (b) holds.

Moreover, since G is locally order convex, we have that, if $\lambda : L \to G$ is a modular measure such that $\lambda \leq \mu$, then $\lambda \ll \mu$. Therefore we obtain that:

(a) If μ is K -inner regular and -singular, then μ is also K -singular in the sense K of de Lucia-Morales.

(b) If μ is purely non σ-additive, then μ is purely f.a..

Conversely, if $\lambda : L \to G$ is a modular measure such that $\lambda \ll \mu$, we can find a modular measure $\nu : L \to G$ such that $0 \leq \nu \leq \mu$ and $\mathcal{U}(\nu) = \mathcal{U}(\lambda)$. Hence, if λ is K -smooth (resp. σ-additive), ν *is* K -smooth (resp. σ-additive), too. Therefore in (a) and in (c) the equivalence holds.

(ii) Now suppose that G is an order-complete ℓ -group with the properties (oc) and (M), and remove the assumption that μ is positive. Recall that, μ is exhaustive. Moreover, we have $\mathcal{U}(\mu) = \mathcal{U}(|\mu|)$. Therefore $|\mu|$ is exhaustive, too, and then μ^+ and μ^- are exhaustive since $\mu^+ \leq |\mu|$ and $\mu^- \leq |\mu|$. In a similar way, if μ *is* K −inner regular, we can obtain that μ^+ and μ^- are K -inner regular, too.

(a) and (c): Since $\mu^+ \leq |\mu|$ and $\mu^- \leq |\mu|$, we have that, if μ *is* K -singular (resp. purely non σ-additive), then $|\mu|, \mu^+$ and μ^- are K -singular (resp. purely non σ-additive), too, and therefore by (i) K -singular in the sense of de Lucia-Morales (resp. purely f.a.). Conversely, if μ *is* K -singular in the sense of de Lucia-Morales (resp. purely f.a.), we have by (i) that μ^+ and μ^- are K –singular (resp. purely non σ-additive). $|\mu| = \mu^+ - \mu^-$ is K -singular (resp. purely non σ-additive). μ *is* K -singular (resp. purely non σ-additive), too.

(b): It is clear that, if μ is K -smooth, then μ is also K -smooth in the sense of de Lucia-Morales since μ is continuous with respect to $\mathcal{U}(\mu)$. Conversely, suppose that μ is K -smooth in the sense of de Lucia-Morales. Since $|\mu| = \mu^+ + \mu^-$, we have that $|\mu|$ is K -smooth in the sense of de Lucia-Morales and therefore K -smooth by (i).

Remark (5.1.24)[167]: In [187] the following decomposition theorem has been proved: Suppose that G is a quasi order-complete locally order convex group with the property (oc), L' is an effect algebra and K, H are pavings in L'. Then, for every positive H -inner regular measure $\mu : L' \to G'$, there exist two positive H -inner regular measures $\lambda, \nu : L' \to G'$ such that

(a) $\mu = \lambda + \nu$.

(b) λ *is* K -smooth.

(c) ν is K -singular.

Moreover, if $G'$is order-complete and L' is a Boolean algebra, the decomposition is unique.

If L' is a D-lattice and μ is modular, this decomposition of μ is not a consequence, since the assumptions of [187] do not imply that G is compatible, as the next example shows. Nevertheless, with a similar proof as in [187] and using the results, it is possible to prove that in this case λ and ν are modular, too, and, if G' is order-complete, the decomposition is unique as in the Boolean case.

The next example has been suggested by Hans Weber.

Example (5.1.26)[167]: Denote by τ the usual topology in R and by $\leq$ the usual order in R.Set $C = \{xQ : x \geq 1\}$. For $a, b \in R$, define $a \preccurlyeq b$ if and only if $b - a \in C$.

We see that $(R, \subseteq, \tau)$ is a quasi order-complete locally order-convex group with the property (oc), but it is not compatible.

It is clear that $(R, \preccurlyeq, \tau)$ is not compatible and, since $a \preccurlyeq b$ implies $a \leq b$, it is locally order convex.

Now observe that, if $D \subseteq R$ is a majorized set with respect to $\preccurlyeq$, then D has a maximal element with respect to $\preccurlyeq$, otherwise we can construct a sequence $\{d_n\}$ in D such that, for each $n \in N$, $d_n \geq d_0 + n$, a contradiction with the assumption that D is majorized.

Therefore, if D is a majorized directed upward subset of R and m is a maximal element of D with respect to $\preccurlyeq$, then we have that $m = max\ D$ with respect to $\preccurlyeq$.

Now it is clear that $(R, \preccurlyeq, \tau)$ is quasi order-complete and has the property (oc).

Section (5.2): Topological ℓ-Groups:

In recent years there appeared some articles containing results for measures with values in a Hausdorff o.c. locally solid Dedekind complete ℓ-group which admits a 0-neighbourhood base consisting of sublattices (see [189–190], [191]). It is natural to ask whether these ℓ-groups can be characterized. It turns out that such ℓ-groups can be embedded in a product $\mathbb{R}^A$ endowed with a topology which agrees on order bounded sets with the product topology (Theorem (5.2.9)). As a consequence, results for measures with values in these groups can be reduced to the real-valued case.

We first present a representation theorem for complete MV-algebras endowed with a Hausdorff o.c. locally convex topology which admits a 0-neighbourhood base consisting of sublattices (Theorem (5.2.5)). This is the main tool to characterize the class of locally solid ℓ-groups mentioned above (Theorem (5.2.9)). Finally we briefly show how the main result of [190], the group-valued version [190, Theorem 5.1] of Alexandroff's decomposition theorem [192, Theorems 1 and 3 of §13], can be deduced from Alexandroff's original real-valued version with the aid of Theorem (5.2.9). In the same way also the Second Decomposition Theorem [189, p. 119], [189, Theorem 4.8] and the main result of [193], the group-valued version [193, Theorem 4.4] of Alexandroff's decomposition theorem [192, Theorem 5 of §13] can immediately be reduced to the real-valued case with the aid of Theorem (5.2.9).

Let $L = (L, +, ' ; 0, 1)$ be an *MV-algebra*, i.e. $(L, +)$ is a commutative semigroup with $0, 1 \in L$ and $\ell : L \to L$ satisfying

(L1) $x + 1 = 1$, (L2) $x'' = x$,

(L3) $0' = 1$, (L4) $(x' + y)' + y = (x + y')' + x$.

Any MV-algebra is a distributive lattice where the lattice operations are defined by

$$x \vee y = (x + y')' + x \quad \text{and} \quad x \wedge y = (x' \vee y')'.$$

It follows that $x \leq y$ iff $x' + y = 1$. One defines $y - x := (x + y')'$ for $x \leq y$ and $x \Delta y := (x \vee y) - (x \wedge y)$ for any $x, y \in L$.

The standard examples of MV-algebras are intervals of an ℓ -group:

Example (5.2.1)[188]: Let $(G,\oplus,\leq)$ be an ℓ-group and $0 \leq e \in G$. Then the order interval $[0,e] \coloneqq \{x \in G: 0 \leq x \leq e\}$ with the truncated sum $x + y := (x \oplus y) \wedge e$ and $x' \coloneqq e - x$ becomes an MV-algebra.

Mundici [194] proved that *every* MV-algebra is of this type.

A *topological MV-algebra* is an MV-algebra endowed with a topology such that $+$ and $'$ (and therefore $\wedge,\vee,\Delta$) are continuous.

Proposition (5.2.2)[188]: *The topology of a topological MV-algebra is uniquely determined by its* 0*-neighbourhood system.*

Proof. Let (L,τ) be a topological MV-algebra, $a \in L$ and $f,g : L \to L$ defined by $f(x) = a - (x \wedge a)$ and $g(x) = x - (x \wedge a)$. Since f and g are continuous and $f(a) = g(a) = 0$, for any 0-neighbourhood U the set $U(a) \coloneqq f^{-1}(U) \cap g^{-1}(U) = \{x \in L: a - (x \wedge a), x - (x \wedge a) \in U\}$ is a neighbourhood of a. These sets form even a neighbourhood base of a in (L,τ): Let V be a neighbourhood of a. Choose a 0-neighbourhood U such that $(a\Delta U) + U \subseteq V$. Then $(a) \subseteq V$. In fact, for $x \in U(a)$ we have $= \big(a - (a - a \wedge x)\big) + (x - x \wedge a) \in (a\Delta U) + U \subseteq V$.

A topology on a lattice is called *locally convex* if every point of the lattice has a neighbourhood base consisting of convex sets, where a subset U of the lattice is convex if $a,b \in U$ and $a \leq b$ implies $[a,b] \subseteq U$. If u is a uniformity on L such that $+$ and $'$ (and therefore $\wedge,\vee,\Delta$) are uniformly continuous, we call (L,u) a *uniform MV-algebra*; the topology induced by u is then locally convex (see [195, Proposition 1.1.6]). As observed in [196, Theorem 2.1] (and [197, Theorem 4.1.1]) the uniformity of a uniform MV-algebra (L,u) is uniquely determined by its 0-neighbourhood system $\mathcal{F}$; moreover $\mathcal{F}$ satisfies

$$\forall F \in \mathcal{F}, \exists G \in \mathcal{F}, \forall x,y \in G, \qquad [0, x+y] \subseteq F. \qquad (*)$$

Vice versa, any filter $\mathcal{F}$ on L satisfying $(*)$ is the 0-neighbourhood system for a uniformity u such that (L,u) is a uniform MV-algebra.

Proposition (5.2.3)[188]: *For a topological MV-algebra (L,τ) the following conditions are equivalent*:

(a) τ *is locally convex.*

(b) 0 *has a neighbourhood base consisting of convex sets.*

(c) *There is a* (*unique*) *uniformity* u *on* L *inducing the topology* τ *such that* (L, u) *is a uniform MV-algebra.*

Proof. Obviously (a) implies (b). If (b) is satisfied, the 0-neighbourhood system $\mathcal{F}$ satisfies $(*)$. As observed above, there is a (unique) uniformity u such that (L, u) is a uniform MV-algebra and $\mathcal{F}$ is its 0-neighbourhood system. By Proposition (5.2.2) τ agrees with the topology induced by u. $(c) \Rightarrow (a)$: By [195, Proposition 1.1.6] any uniformity on a lattice making the lattice operations uniformly continuous induces a locally convex topology.

To formulate the main tool in this Section, Theorem (5.2.4), we still need some definitions. A topology on a lattice is *order continuous* (o.c.) if order convergence of a monotone net implies topological convergence. A topological MV-algebra (L, τ) is o.c. iff $L \ni x_\alpha \downarrow 0$ implies $x_\alpha \to 0(\tau)$. An example of an o.c. locally convex topological MV-algebra is $\mathcal{L}_\infty$, the real unit interval with the operations defined in Example (5.2.1). One denotes with $\mathcal{L}_{n+1} := \{i/n : i = 1, \ldots, n\}$ the (discrete) MV-subalgebra of $\mathcal{L}_\infty$ consisting of $n + 1$ elements, $n \in \mathbb{N}$. The *center* $C(L)$ of L is the set of all complemented elements of L.

Theorem (5.2.4)[188]: *([196,197]) Let* (L, τ) *be an o.c. locally convex Hausdorff topological MV-algebra and* L *be lattice complete. Then there is an algebraic and topological isomorphism from* L *onto a product* $(L_0, \tau_0) \times \prod_{\alpha \in A} \mathcal{L}_{n_\alpha}$, *where* $2 \leq n_\alpha \leq \infty, (L_0, \tau_0)$ *is a connected o.c. locally convex Hausdorff topological MV-algebra and* $C(L_0)$ *is atomless.*

Because of Proposition (5.2.3) this theorem is equivalent to the decomposition theorem for uniform MV-algebras formulated in [197, Section 4.2] and in [196, Section 6] (observing that a lattice complete o.c. MV-algebra is by [195, Corollary 4.7] as uniform space complete). The main result of this section, Theorem (5.2.5), is based on Theorem (5.2.4).

Theorem (5.2.5)[188]: *For a Hausdorff topological MV-algebra* (L, τ) *the following conditions are equivalent*:

(i) $(L, \leq)$ *is a complete lattice,* (L, τ) *is o.c. and locally convex and has a* 0*-neighbourhood base consisting of sublattices of* $(L, \leq)$.

(ii) *There exists an algebraic and topological isomorphism from* L *onto a product* $\prod_{\alpha \in A} \mathcal{L}_{n_\alpha}$ *where* $2 \leq n_\alpha \leq \infty$.

(iii) (L, τ) *is compact.*

Proof. (i) $\Rightarrow$ (ii): We use the isomorphism $L \simeq (L_0, \tau_0) \times \prod_{\alpha \in A} \mathcal{L}_{n_\alpha}$ of Theorem (5.2.4). It is enough to show that $L_0 = \{0\}$. It is known that $C(L_0)$ is a Boolean algebra which is order closed in L_0, i.e. $x \in L_0$ and $C(L_0) \ni x_\alpha \downarrow x$ implies $x \in C(L_0)$. Therefore $C(L_0)$ is lattice complete and τ_0 induces an o.c. FN-topology on $C(L_0)$. Since moreover $C(L_0)$ is atomless, it follows by [198, Theorems 2.8 and 6.1] that every 0-neighbourhood in $(C(L_0), \tau_0)$ contains a finite subset F such that $\sup F = e_0 := \max C(L_0) = \max L_0$. Since (L_0, τ_0) has a 0-neighbourhood base consisting of sublattices, it follows that e_0 is contained in every 0-neighbourhood of (L_0, τ_0). As (L_0, τ_0) is Hausdorff, we obtain $e_0 = 0$, i.e. $L_0 = \{0\}$.

(ii) $\Rightarrow$ (i) is obvious. For (ii) $\Leftrightarrow$ (iii) see [196, Section 6] or [197, Corollary 4.2.7(a)].

Let G be an additive written commutative ℓ-group. Observe that any Dedekind complete ℓ-group is automatically commutative. A subset U of G is *solid* if $x \in G, y \in U, |x| \leq |y|$ imply $x \in U$. G is called *locally solid* if G is endowed with a group topology which has a 0-neighbourhood base consisting of solid subsets. Such a topology is locally convex and makes the lattice operations uniformly continuous. We will determine all o.c. Hausdorff locally solid Dedekind complete ℓ-groups which admit a 0-neighbourhood base consisting of sublattices.

Proposition (5.2.6)[188]: *Let (G, τ) be a locally solid ℓ-group which admits a* 0*-neighbourhood base consisting of sublattices. Then G has a* 0*-neighbourhood base consisting of solid sublattices of G. If τ is o.c., then G has a* 0*-neighbourhood base consisting of order closed solid sublattices of G.*

Proof. Let U be a 0-neighbourhood. We may assume that U is solid and closed in (G, τ). Let V and W be 0-neighbourhoods such that V is a lattice, W is solid and $W \subseteq V \subseteq U$. Then $W^* := \{x \in G: |x| \leq \sup F \text{ for some finite set } F \subseteq W\}$ is a solid sublattice of G contained in U. Moreover W^* is a 0-neighbourhood since $W \subseteq W^*$. Also the closure $\overline{W}^*$ of W^* is a solid lattice contained in U. If τ is o.c., then $\overline{W}^* \ni x_\alpha \uparrow x \in G$ implies $x \in \overline{W}^*$. This implies that $\overline{W}^*$ is order closed.

Example (5.2.7)[188]: Let A be a nonempty set and G an ℓ-subgroup of $\mathbb{R}^A$ such that $f \in G$ and $B \subseteq A$ implies $f \cdot \chi_B \in G$ (where χ_B denotes the characteristic function of B). For $s \in [0, +\infty]^A$ let $U(s) := \{f \in G: |f| \leq s\}$.

(a) If τ is an o.c. Hausdorff locally solid group topology on G which has a 0-neighbourhood base consisting of sublattices, then there exists a set $S \subseteq (0, +\infty]^A$ such that the sets $U(s), s \in S$, form a 0-neighbourhood base in (G, τ) and S has the following properties:

(i) $\forall s_1, s_2 \in S, \exists s \in S, 2s \leq \inf\{s_1, s_2\}$,

(ii) $\inf S = 0$,

(iii) for every $f \in G$ and $s \in S$ the set $\{\alpha \in A: s(\alpha) \leq f(\alpha)\}$ is finite.

(b) Vice versa, if a set $S \subseteq (0, +\infty]^A$ has the properties (i), (ii), (iii), then sets $U(s), s \in S$, form a 0-neighbourhood base for an o.c. Hausdorff locally solid group topology τ on G. Moreover, τ agrees on order bounded subsets of G with the product topology of $\mathbb{R}^A$.

Proof. (a) By Proposition (5.2.6) every 0-neighbourhood of (G, τ) contains a 0-neighbourhood U which is an order closed solid sublattice of G. Let $s(\alpha) \coloneqq \sup\{f(\alpha): f \in U\}$. We show that $U = U(s)$: The inclusion $\subseteq$ is obvious. To prove $\supseteq$ let $f \in U(s)$. Then $\{g \wedge |f|: 0 \leq g \in U\}$ is an upwards directed subset of U with supremum $|f|$. Therefore $|f| \in U$, hence $f \in U$. Next we show that $U(s) = U(\bar{s})$ for some strictly positive function s: If $s(\alpha) > 0$ put $s(\alpha) \coloneqq \bar{s}(\alpha)$. If $s(\alpha) = 0$ then $G_\alpha \coloneqq \{f(\alpha): f \in G\}$ is a discrete subgroup of $\mathbb{R}$ since τ is o.c. and $U(s)$ is a 0-neighbourhood in (G, τ). Choose $0 < \bar{s}(\alpha) < \min\{x \in G_\alpha: x > 0\}$. Then $U(s) = U(\bar{s})$.

Let S be the set of all functions $s \in (0, +\infty]^A$ such that $U(s)$ is a 0-neighbourhood in (G, τ). Since $U(s), s \in S$, form a 0-neighbourhood base in (G, τ) as we have shown, condition (i) is obviously satisfied. Since τ is Hausdorff, $\inf S = 0$. To verify (iii) suppose that there are $f \in G, s \in S$ and a sequence (α_n) of different elements of A such that $s(\alpha_n) \leq f(\alpha_n)$. Then $G \ni 2f \cdot \chi_{\{\alpha_i: i \geq n\}} \downarrow 0$, hence $2f \cdot \chi_{\{\alpha_i: i \geq k\}} \in U(s)$ for some $k \in \mathbb{N}$, thus $2f(\alpha_k) \leq s(\alpha_k)$, a contradiction.

(b) Obviously, by (i), there is a unique locally solid group topology τ such that the sets $U(s), s \in S$, form a 0-neighbourhood base in (G, τ). By (ii), τ is Hausdorff. It now follows from (iii) that τ agrees on order bounded subsets
of G with the product topology of $\mathbb{R}^A$, in particular, τ is o.c.

Theorem (5.2.8)[188]: *Let G be a Dedekind complete ℓ-group with an order unit and τ an o.c. Hausdorff locally solid group topology on G which admits a 0-neighbourhood base consisting of sublattices. Then there is a family $(R_\alpha)_{\alpha \in A}$, where R_α is a discrete subgroup of $(\mathbb{R}, +)$ or*

$R_\alpha = \mathbb{R}$, *such that* G *is* (*as* ℓ*-group*) *isomorphic to* $\prod_{\alpha\in A} R_\alpha \cap l_\infty(A)$. *Here* $l_\infty(A)$ *denotes the space of all real-valued bounded functions on* A. *The topology on* $\prod_{\alpha\in A} R_\alpha \cap l_\infty(A)$ *corresponding to* τ *is described in Example (5.2.7).*

Proof. Let e be an order unit of G. Then [0, e] is a topological MV-algebra. By Theorem (5.2.5) we can identify $[0, e]$ with a product of the type $\prod_{\alpha\in A} \mathcal{L}_{n_\alpha}$ where $2 \le n_\alpha \le \infty$. Let $R_\alpha = \mathbb{R}$ if $n_\alpha = \infty$, and $R_\alpha = \mathbb{Z} \cdot \frac{1}{n_\alpha - 1}$ if $n < \infty$. Then $G = \prod_{\alpha\in A} R_\alpha \cap l_\infty(A)$.

Now we also consider ℓ-groups without an order unit.

Theorem (5.2.9)[188]: *Let* G *be a Dedekind complete* ℓ*-group and* τ *an o.c. Hausdorff locally solid group topology on* G *which admits a* 0*-neighbourhood base consisting of sublattices. Then* G *is* (*as* ℓ*-group*) *isomorphic to a solid* ℓ*-subgroup of a product* $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}$. *The topology on* $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}$ *corresponding to* τ *is described in Example (5.2.7).*

Proof. Let E be the set of all positive elements $e \in G$ such that $[0, e]$ is isomorphic to $\mathcal{L}_\infty$ or to $\mathcal{L}_2$, and let A be a maximal orthogonal subset of E. We first show that A is also a maximal orthogonal subset of $G\backslash\{0\}$: Suppose that $u \in G\backslash\{0\}$ and $|u| \wedge a = 0$ for any $a \in A$. Since by Theorem (5.2.5) the interval $[0, |u|]$ is isomorphic to a product $\prod_{\gamma\in\Gamma} \mathcal{L}_\gamma$, there exists an element $e \in E \cap [0, |u|]$, thus $e \wedge a = 0$ for any $a \in A$, a contradiction to the maximality of A. For $a \in A$ we denote by P_a the projection onto the band B_a generated by a, i.e. $P_a(x) = y$ if $x = y + z, y \in B_a$ and $z \in B_a^\perp$. Then $x \mapsto \left(P_a(x)\right)_{a\in A}$ defines an ℓ-group isomorphism from G onto a solid subgroup of $\prod_{a\in A} B_a$ (see e.g. [199, Preliminaries]). Finally observe that $B_a \simeq \mathbb{R}$ if $[0, a] \simeq \mathcal{L}_\infty$ and $B_a \simeq \mathbb{Z}$ if $[0, a] \simeq \mathcal{L}_2$. _

As mentioned the preceding theorem can be used to reduce e.g. the decomposition theorem [190, Theorem 5.1] to the original real-valued version of Alexandroff's decomposition theorem [192, Theorems 1 and 3 of §13]. We use the terminology of [190]. Let G be given as in Theorem (5.2.9), let $(X, \mathcal{F})$ be an Alexandroff space and $A(\mathcal{F})$ the algebra generated by the "closed" sets $\mathcal{F}$. Then, by [190, Theorem 5.1], any order-bounded $\mathcal{F}$-inner regular measure $\mu : A(\mathcal{F}) \to G$ has a decomposition $\mu = \xi + \eta$ where $\xi, \eta : A(\mathcal{F}) \to G$ are order-bounded $\mathcal{F}$-inner regular measures, ξ is $\mathcal{F}$-smooth and $\eta(E) = 0$ for any $E \in A(\mathcal{F})$ contained in an A-compact subset of X. Moreover, under an additional "topological"

condition on $(X, \mathcal{F})$ the decomposition is unique. To reduce this theorem to the real-valued case we may assume by Theorem 3.4 that G is a solid ℓ-subgroup of a product $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}$ endowed with a topology described in Example (5.2.7) and A_1, A_2 are disjoint. Write $\mu = (\mu_\alpha)_{\alpha \in A}$ where $A = A_1 \cup A_2$. For $\alpha \in A$, let $\mu_\alpha = \xi_\alpha + \eta_\alpha$ be the real-valued Alexandroff decomposition according to [192, Theorem 1 of §13 on p. 614], $\xi \coloneqq (\xi_\alpha)$ and $\eta \coloneqq (\eta_\alpha)$; in particular we have $\xi_\alpha^+ \leq \mu_\alpha^+$ and $\xi_\alpha^- \leq \mu_\alpha^-$ for all $\alpha \in A$, hence $|\xi| \leq |\mu|$. Then $\mu = \xi + \eta$ is the G-valued decomposition as stated above. To see that ξ and η are actually G-valued, we first consider μ_α for $\alpha \in A_2$, i.e. the $\mathbb{Z}$-valued case. Let $\alpha \in A_2$ and $\mu_\alpha \neq 0$. Then μ_α has a finite range and therefore can be written as $\mu_\alpha = \sum_{i=1}^n c_n \delta_n$ where $c_i \in \mathbb{Z} \backslash \{0\}$ and δ_i are different $\{0, 1\}$-valued measures. With μ_α also δ_i are $\mathcal{F}$-inner regular. If δ is a $\{0, 1\}$-valued measure on $A(\mathcal{F})$ and $\delta(E) \neq 0$ for some $E \in A(\mathcal{F})$ contained in an A-compact subset K of X, then for any decreasing net $\left(F_\gamma\right)_{\gamma \in \Gamma}$ in $\mathcal{F}$ with $F_\gamma \downarrow \emptyset$ we have $F_{\gamma_0} \cap K = \emptyset$ for some $\gamma_0 \in \Gamma$, hence $\delta\left(F_\gamma\right) = \delta\left(F_\gamma \cap K\right) = 0$ for all $\gamma \geq \gamma_0$, thus δ is $\mathcal{F}$-smooth. Therefore, with $I = \left\{i \in \{1, \ldots, n\}: \delta_i \text{ is } \mathcal{F}\text{-smooth}\right\}$, $\xi_\alpha = \sum_{i \in I} c_i \delta_i$ and $\eta_\alpha = \mu_\alpha - \xi_\alpha$, the decomposition $\mu_\alpha = \xi_\alpha + \eta_\alpha$ is the Alexandroff decomposition of μ_α according to [192, Theorem 1 of §13]. This shows that, for $\alpha \in A_2, \xi_\alpha$ and η_α have values in $\mathbb{Z}$ and therefore ξ and η have values in $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}$. Since G is solid in $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}, |\mu|$ is G-valued and $|\xi| \leq |\mu|$, it follows that ξ and η are even G-valued. Under the additional "topological" condition on $(X, \mathcal{F})$ given in [190, Theorem 5.1] the uniqueness of the decomposition $\mu = \xi + \eta$ immediately follows from the uniqueness of the decompositions $\mu_\alpha = \xi_\alpha + \eta_\alpha, \alpha \in A$, in the real-valued case (see [192, Theorem 1 of §13]).

The same technique can be applied to reduce [193, Theorem 4.4] and [189, the Second Decomposition Theorem on p. 119 and Theorem 4.8] to the real-valued case with the aid of Theorem (5.2.9).

Corollary (5.2.10)[250]: *The topology of a sequence of a topological MV-algebra is uniquely determined by its* 0*-neighbourhood system.*

Proof. Let $\left(L, \tau_j\right)$ be a sequence of a topological MV-algebras, $\{a_i\}_{i=1}^n \in L$ and $f, g : L \to L$ defined by $f(x_r) = \{a_i\}_{i=1}^n - (x_r \wedge \{a_i\}_{i=1}^n)$ and $g(x_r) = x_r - (x_r \wedge \{a_i\}_{i=1}^n)$. Since f and g are continuous and $f(\{a_i\}_{i=1}^n) = g(\{a_i\}_{i=1}^n) = 0$, for any 0-neighbourhood U the set

$U(\{a_i\}_{i=1}^n) \coloneqq f^{-1}(U) \cap g^{-1}(U) = \{x_r \in L: \{a_i\}_{i=1}^n - (x_r \wedge \{a_i\}_{i=1}^n), x_r - (x_r \wedge \{a_i\}_{i=1}^n) \in U\}$ is a neighbourhood of $\{a_i\}_{i=1}^n$. These sets form even a neighbourhood base of $\{a_i\}_{i=1}^n$ in (L, τ_j): Let V be a neighbourhood of $\{a_i\}_{i=1}^n$. Choose a 0-neighbourhood U such that $(\{a_i\}_{i=1}^n \Delta U) + U \subseteq V$. Then $(\{a_i\}_{i=1}^n) \subseteq V$. In fact, for $x_r \in U(\{a_i\}_{i=1}^n)$ we have $= \left(\{a_i\}_{i=1}^n - (\{a_i\}_{i=1}^n - \{a_i\}_{i=1}^n \wedge x_r)\right) + (x_r - x_r \wedge \{a_i\}_{i=1}^n) \in (\{a_i\}_{i=1}^n \Delta U) + U \subseteq V$.

If $\{a_i\}_{i=1}^n, \{(a+\epsilon)_i\}_{i=1}^n \in U$ and $\epsilon \geq 0$ implies $[\{a_i\}_{i=1}^n, \{(a+\epsilon)_i\}_{i=1}^n] \subseteq U$. If u_i is a sequence of a uniformity on L such that $+$ and $'$ (and therefore $\wedge, \vee, \Delta$) are uniformly continuous (see [194]), we call (L, u_i) a sequence of a *uniform MV-algebras*; the topology induced by u_i is then locally convex (see [195]). As observed in [196] (and [197]) the uniformity of a uniform MV-algebra (L, u_i) is uniquely determined by its 0-neighbourhood system $\mathcal{F}$; moreover $\mathcal{F}$ satisfies

$$\forall F \in \mathcal{F}, \exists G \in \mathcal{F}, \forall x_r, x_{r+1} \in G, \qquad [0, x_r + x_{r+1}] \subseteq F. \qquad (*)$$

Vice versa, any filter $\mathcal{F}$ on L satisfying $(*)$ is the 0-neighbourhood system for a uniformity u_i such that (L, u_i) is a sequence of a uniform MV-algebras.

Corollary (5.2.11)[250]: For a sequence of a topological MV-algebras (L, τ_j) the following conditions are equivalent:

(a) τ_j is locally convex.

(b) 0 has a neighbourhood base consisting of convex sets.

(c) There is a (unique) sequence of uniformity u_i on L inducing the sequence of the topology τ_j such that (L, u_i) is a sequence of a uniform MV-algebras.

Proof. Obviously (a) implies (b). If (b) is satisfied, the 0-neighbourhood system $\mathcal{F}$ satisfies $(*)$. As observed above, there is a sequence of uniformity u_i such that (L, u_i) is a sequence of a uniform MV-algebras and $\mathcal{F}$ is its 0-neighbourhood system. By Example (5.2.7) τ_j agrees with the topology induced by u_i. (c) $\Rightarrow$ (a): By [195] any uniformity on a lattice making the lattice operations uniformly continuous induces a locally convex topology.

A sequence of a topological MV-algebras (L, τ_j) are o.c. iff $L \ni (x_r)_{\alpha^2-1} \downarrow 0$ implies $(x_r)_{\alpha^2-1} \to 0(\tau_j)$. An example of an o.c. locally convex topological MV-algebra is $\mathcal{L}_\infty$, the real unit interval with the operations defined in Example (5.2.1). One denotes with $\mathcal{L}_{n+1} \coloneqq$

$\{i/n: i = 1, \ldots, n\}$ the (discrete) MV-subalgebra of $\mathcal{L}_\infty$ consisting of $n+1$ elements, $n \in \mathbb{N}$. The *center* $C(L)$ of L is the set of all complemented elements of L.

Corollary (5.2.12)[250]: For a sequence of a Hausdorff topological MV-algebras $(\mathrm{L}, \tau_\mathrm{j})$ the following conditions are equivalent:

(i) $(\mathrm{L}, \leq)$ is a complete lattice, (L, τ_j) is a sequence of o.c. and locally convex and has a 0-neighbourhood base consisting of sublattices of $(L, \leq)$.

(ii) There exists an algebraic and topological isomorphism from L onto a product $\prod_{(\alpha^2-1)\in A} L_{n_{\alpha^2-1}}$ where $2 \leq n_{\alpha^2-1} \leq \infty$.

(iii) (L, τ_j) are compact.

Proof. (i) $\Rightarrow$ (ii): We use the isomorphism $L \simeq (L_0, \tau_0) \times \prod_{(\alpha^2-1)\in A} \mathcal{L}_{n_{\alpha^2-1}}$ of Theorem ((5.2.4)). It is enough to show that $L_0 = \{0\}$. It is known that $C(L_0)$ is a Boolean algebra which is order closed in L_0, i.e. $x_r \in L_0$ and $C(L_0) \ni (x_r)_{\alpha^2-1} \downarrow x_r$ implies $x_r \in C(L_0)$. Therefore $C(L_0)$ is lattice complete and τ_0 induces an o.c. FN-topology on $C(L_0)$. Since moreover $C(L_0)$ is atomless, it follows by [198] that every 0-neighbourhood in $(C(L_0), \tau_0)$ contains a finite subset F such that $\sup F = e_0 \coloneqq \max C(L_0) = \max L_0$. Since (L_0, τ_0) has a 0-neighbourhood base consisting of sublattices, it follows that e_0 is contained in every 0-neighbourhood of (L_0, τ_0). As (L_0, τ_0) is Hausdorff, we obtain $e_0 = 0$, i.e. $L_0 = \{0\}$.

(ii) $\Rightarrow$ (i) is obvious. For (ii) $\Leftrightarrow$ (iii) see [196] or [197].

Corollary (5.2.13)[250]: Let (G, τ_j) be a sequence of a locally solid ℓ-group which admits a 0-neighbourhood base consisting of sublattices. Then G has a 0-neighbourhood base consisting of solid sublattices of G. If τ_j is a sequence of o.c., then G has a 0-neighbourhood base consisting of order closed solid sublattices of G.

Proof. Let U be a 0-neighbourhood. We may assume that U is solid and closed in the sequence (G, τ_j). Let V and W be 0-neighbourhoods such that V is a lattice, W is solid and $W \subseteq V \subseteq U$. Then $W^* \coloneqq \{x_r \in G: |x_r| \leq \sup F \text{ for some finite set } F \subseteq W\}$ is a solid sublattice of G contained in U. Moreover W^* is a 0-neighbourhood since $W \subseteq W^*$. Also the closure $\overline{W}^*$ of W^* is a solid lattice contained in U. If τ_j is a sequence of an o.c., then $\overline{W}^* \ni (x_r)_{\alpha^2-1} \uparrow x_r \in G$ implies $x_r \in \overline{W}^*$. This implies that $\overline{W}^*$ is order closed.

Corollary (5.2.14)[250]: Let A be a nonempty set and G an ℓ-subgroup of $\mathbb{R}^A$ such that $f \in G$ and $B \subseteq A$ implies $f \cdot \chi_B \in G$ (where χ_B denotes the characteristic function of B). For $s_{n-1} \in [0,+\infty]^A$ let $U(s_{n-1}) \coloneqq \{f \in G: |f| \leq s_{n-1}\}$.

(a) If τ_j is a sequence of an o.c. Hausdorff locally solid group topology on G which has a 0-neighbourhood base consisting of sublattices, then there exists a set $S \subseteq (0,+\infty]^A$ such that the sets $U(s_{n-1}), s_{n-1} \in S$, form a 0-neighbourhood base in the sequence (G,τ_j) and S has the following properties:

(i) $\forall s_n, s_{n+1} \in S, \exists s_{n-1} \in S, 2s_{n-1} \leq \inf\{s_n, s_{n+1}\}$,

(ii) $\inf S = 0$,

(iii) for every $f \in G$ and $s_{n-1} \in S$ the set $\{(\alpha^2-1) \in A: s_{n-1}(\alpha^2-1) \leq f(\alpha^2-1)\}$ is finite.

(b) Vice versa, if a set $S \subseteq (0,+\infty]^A$ has the properties (i), (ii), (iii), then sets $U(s_{n-1}), s_{n-1} \in S$, form a 0-neighbourhood base for a sequence of an o.c. Hausdorff locally solid group topology τ_j on G. Moreover, τ_j agrees on order bounded subsets of G with the product topology of $\mathbb{R}^A$.

Proof. (a) By Corollary (5.2.13) every 0-neighbourhood of the sequence (G,τ_j) contains a 0-neighbourhood U which is an order closed solid sublattice of G. Let $s_{n-1}(\alpha^2-1) \coloneqq \sup\{f(\alpha^2-1): f \in U\}$. We show that $U = U(s_{n-1})$: The inclusion $\subseteq$ is obvious. To prove $\supseteq$ let $f \in U(s_{n-1})$. Then $\{g \wedge |f|: 0 \leq g \in U\}$ is an upwards directed subset of U with supremum $|f|$. Therefore $|f| \in U$, hence $f \in U$. Next we show that $U(s_{n-1}) = U(\bar{s}_{n-1})$ for some strictly positive function s_{n-1}: If $s_{n-1}(\alpha^2-1) > 0$ put $\bar{s}_{n-1}(\alpha^2-1) \coloneqq s_{n-1}(\alpha^2-1)$. If $s_{n-1}(\alpha^2-1) = 0$ then $G_{\alpha^2-1} \coloneqq \{f(\alpha^2-1): f \in G\}$ is a discrete subgroup of $\mathbb{R}$ since τ_j is a sequence of an o.c. and $U(s_{n-1})$ is a 0-neighbourhood in the sequence (G,τ_j).

Choose $0 < \bar{s}_{n-1}(\alpha^2-1) < \min\{x_r \in G_{\alpha^2-1}: x_r > 0\}$. Then $U(s_{n-1}) = U(\bar{s}_{n-1})$.

Let S be the set of all functions $s_{n-1} \in (0,+\infty]^A$ such that $U(s_{n-1})$ is a 0-neighbourhood in the sequence (G,τ_j). Since $U(s_{n-1}), s_{n-1} \in S$, form a 0-neighbourhood base in (G,τ_j) as we have shown, condition (i) is obviously satisfied. Since the sequence τ_j is Hausdorff, $\inf S = 0$. To verify (iii) suppose that there are $f \in G, s_{n-1} \in S$ and a sequence

$((\alpha^2-1)_n)$ of different elements of A such that $s_{n-1}((\alpha^2-1)_n) \leq f((\alpha^2-1)_n)$. Then $G \ni 2f \cdot \chi_{\{(\alpha^2-1)_i : i \geq n\}} \downarrow 0$, hence $2f \cdot \chi_{\{(\alpha^2-1)_i : i \geq k\}} \in U(s_{n-1})$ for some $k \in \mathbb{N}$, thus $2f((\alpha^2-1)_k) \leq s_{n-1}((\alpha^2-1)_k)$, a contradiction.

(b) Obviously, by (i), there is a sequence of a unique locally solid group topology τ_j such that the sets $U(s_{n-1}), s_{n-1} \in S$, form a 0-neighbourhood base in the sequence (G, τ_j). By (ii),the sequence τ_j is Hausdorff. It now follows from (iii) that τ_j agrees on order bounded subsets of G with the product topology of $\mathbb{R}^A$, in particular,the sequence τ_j is an o.c.

Corollary (5.2.15)[250]: Let G be a Dedekind complete ℓ-group with an order unit and τ_j a sequence an o.c. Hausdorff locally solid group topology on G which admits a 0-neighbourhood base consisting of sublattices. Then there is a family $(R_{\alpha^2-1})_{(\alpha^2-1)\in A}$, where R_{α^2-1} is a discrete subgroup of $(\mathbb{R}, +)$ or $R_{\alpha^2-1} = \mathbb{R}$, such that G is (as ℓ-group) isomorphic to $\prod_{(\alpha^2-1)\in A} R_{(\alpha^2-1)} \cap l_\infty(A)$. Here $l_\infty(A)$ denotes the space of all real-valued bounded functions on A. The topology on $\prod_{(\alpha^2-1)\in A} R_{(\alpha^2-1)} \cap l_\infty(A)$ corresponding to the sequence τ_j is described in Corollary (5.2.14).

Proof. Let e be an order unit of G. Then $[0, e]$ is a topological MV-algebra. By Corollary (5.2.12) we can identify $[0, e]$ with a product of the type $\prod_{(\alpha^2-1)\in A} \mathcal{L}_{n_{(\alpha^2-1)}}$ where $2 \leq n_{(\alpha^2-1)} \leq \infty$. Let $R_{(\alpha^2-1)} = \mathbb{R}$ if $n_{(\alpha^2-1)} = \infty$, and $R_{(\alpha^2-1)} = \mathbb{Z} \cdot \frac{1}{n_{(\alpha^2-1)}-1}$ if $n < \infty$. Then $G = \prod_{(\alpha^2-1)\in A} R_{(\alpha^2-1)} \cap l_\infty(A)$.

Now we also consider ℓ-groups without an order unit.

Corollary (5.2.16)[250]: Let G be a Dedekind complete ℓ-group and the sequence τ_j an o.c. Hausdorff locally solid group topology on G which admits a 0-neighbourhood base consisting of sublattices. Then G is (as ℓ-group) isomorphic to a solid ℓ-subgroup of a product $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}$. The topology on $\mathbb{R}^{A_1} \times \mathbb{Z}^{A_2}$ corresponding to the sequence τ_j is described in Corollary (5.2.14).

Proof. Let E be the set of all positive elements $e \in G$ such that $[0, e]$ is isomorphic to $\mathcal{L}_\infty$ or to $\mathcal{L}_2$, and let A be a maximal orthogonal subset of E. We first show that A is also a maximal orthogonal subset of $G\backslash\{0\}$: Suppose that the sequence $u_i \in G\backslash\{0\}$ and $|u_i| \wedge \{a_i\}_{i=1}^n = 0$ for any $\{a_i\}_{i=1}^n \in A$. Since by Corollary (5.2.12) the interval $[0, |u_i|]$ is isomorphic to a product

$\prod_{\gamma\in\Gamma}\mathcal{L}_\gamma$, there exists an element $e \in E \cap [0, |u_i|]$, thus $e \wedge \{a_i\}_{i=1}^n = 0$ for any $\{a_i\}_{i=1}^n \in A$, a contradiction to the maximality of A. For $\{a_i\}_{i=1}^n \in A$ we denote by $P_{\{a_i\}_{i=1}^n}$ the projection onto the band $B_{\{a_i\}_{i=1}^n}$ generated by $\{a_i\}_{i=1}^n$, i.e. $P_{\{a_i\}_{i=1}^n}(x_r) = x_{r+1}$ if $x_r = x_{r+1} + x_{r+2}, x_{r+1} \in B_{\{a_i\}_{i=1}^n}$ and $x_{r+2} \in B^{\perp}_{\{a_i\}_{i=1}^n}$. Then $x_r \mapsto \left(P_{\{a_i\}_{i=1}^n}(x_r)\right)_{\{a_i\}_{i=1}^n \in A}$ defines an ℓ-group isomorphism from G onto a solid subgroup of $\prod_{\{a_i\}_{i=1}^n \in A} B_{\{a_i\}_{i=1}^n}$ (see e.g. [199, Preliminaries]). Finally observe that $B_{\{a_i\}_{i=1}^n} \simeq \mathbb{R}$ if $[0, \{a_i\}_{i=1}^n] \simeq \mathcal{L}_\infty$ and $B_{\{a_i\}_{i=1}^n} \simeq \mathbb{Z}$ if $[0, \{a_i\}_{i=1}^n] \simeq \mathcal{L}_2$.