Chapter 3

Categories of Fractions and Bivariant K-theory

Using elementary ideas from the theory of categories of fractions, we construct bivariant homology/cohomology groups $\mathrm{E}(A,B)$ for C^*-algebras, which satisfy general excision axioms, and are equal to Kasparov's groups $KK(A,B)$ for nuclear (or more generally K-nuclear) C^*-algebras. We determine its kk^{alg}-invariants using a natural extension for the Weyl algebra. Using analogous methods the kk^{alg}-invariants can be determined for many other algebras of similar type.

Section (3.1): Excision in KK-Theory:

This work is an investigation into the KK-theory of C^*-algebras introduced by Kasparov [80]; our approach is algebraic in nature and is more or less a continuation of the paper [81], in which KK-theory is characterized by means of some simple axioms.

Kasparov's theory can be described as a sort of calculus for computing K-theory groups of C^*-algebras, which generalizes constructions due to Atiyah and Singer in the index theory of elliptic operators [82,83]. A family of elliptic operators on a smooth, closed manifold M, parametrized by a compact space X, gives rise to an 'index map' $K(M) \to K(X)$ in Atiyah-Hirzebruch K-theory, or in other words a map $K\big(C(M)\big) \to K\big(C(X)\big)$ in C^*-algebra K-theory (see [84] or [85] for a survey of K-theory for C^*-algebras). Following and broadly extending ideas of Atiyah [82], Kasparov defines a notion of generalized elliptic operator for a pair of C^*-algebras A and B, which, in the case $A = C(M)$ and $B = C(X)$, includes the notion of a family of elliptic pseudodifferential operators on M, parametrized by X. There are natural notions of isomorphism and homotopy for these generalized elliptic operators, and the resulting set of equivalence classes, which is in a natural way an abelian group, is denoted $KK(A,B)$. The index construction of Atiyah and Singer extends to a pairing

$$K(A) \otimes KK(A,B) \to K(B). \tag{1}$$

The most important aspect of Kasparov's theory is the existence of an associative

'product map'

$$KK(A,B) \otimes KK(B,C) \to KK(A,C) \tag{2}$$

which is compatible with (1), so that a composition of index maps may be represented as the index map of a 'product' generalized elliptic operator. The formula for the product given in [80] is rather complicated (it is somewhat streamlined in [86]) but it is nevertheless possible to compute it in many circumstances, and in practice each such computation amounts to some sort of index formula. A survey of applications is given in [87].

Following ideas of Cuntz [84], the algebraic structure of Kasparov's theory is worked out in [81]. The basic idea is to regard (2) as the law of composition in an additive category, whose objects are (separable) C^*-algebras and whose morphisms are elements of the groups $KK(A,B)$ (there do exist elements which act as identity morphisms). This category then turns out to have a remarkably simple structure (see [81] or Theorem (3.1.25) below). It follows, for example, that there is a unique functorial pairing (2), up to normalization, and that associativity is 'automatic'.

Kasparov's point of view in [80] is that the groups $KK(A,B)$ constitute a sort of bivariant homology/cohomology theory for C^*-algebras. The motivation for this is that the group $KK(C,B)$ is naturally isomorphic to $K(B)$, whilst $KK(A,C)$ is the 'K-homology' of A (for example, if $A = C(X)$, and X is a finite complex, then $KK(A,C)$ is the even K-homology group of X). From this point of view, it is natural to study questions of excision in KK-theory. In the present context, this amounts to asking whether or not the sequences

$$KK(C,J) \xrightarrow{j_*} KK(C,A) \xrightarrow{q_*} KK(C,B) \tag{3}$$

and

$$KK(B,D) \xrightarrow{q^*} KK(A,D) \xrightarrow{j^*} KK(J,D) \tag{4}$$

associated with a short exact sequence of C^*-algebras

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} B \longrightarrow 0 \tag{5}$$

are exact in the middle. (From (3) and (4), the existence of long exact homology/ cohomology sequences associated with (5) follows from Bott periodicity for KK-theory

[80] and a standard argument adapted from algebraic topology [80]. By Bott periodicity the sequences are periodic: they are cyclic six term exact sequences.) Kasparov proves the exactness of (3) under the hypothesis that C is a nuclear C^*-algebra, and the exactness of (4) under the assumption that A is nuclear (see [80). Simpler proofs of somewhat more general results are given in [88]. In [89] the notion of 'K-nuclear' C^*-algebra is introduced and the excision problem is settled for this class.

Whilst from a certain point of view the class of K-nuclear C^*-algebras is rather large (certainly all commutative C^*-algebras are K-nuclear, and so KK-theory does give a homology/cohomology theory for compact metric spaces), the present state of knowledge is rather unsatisfactory. This is so for practical, as well as theoretical, reasons: see [90] for a situation in the K-theory of foliations where stronger excision results are needed.

Unfortunately it seems likely to most experts that excision in KK-theory does nol hold in general. The purpose of this paper is to construct (using the term rather loosely) a theory $E(A,B)$ which *does* have excision in general. The groups $E(A,B)$ are obtained from a category **E,** and so there is automatically a product of the form (2). Furthermore there is a natural transformation $KK(A,B) \to E(A,B)$, compatible with products, which is an isomorphism if A is K-nuclear. In fact, **E** can be regarded as a sort of 'universal' extension of the homology/cohomology theory KK from K-nuclear C^*-algebras to all (separable) C^*-algebras.

The construction of **E** is based on the notion of a category of fractions, familiar in category theory. Unfortunately the standard presentation of this, [91], is not quite adequate for our purposes. We give an elementary account of the necessary algebraic results, which slightly extends the discussion in [91]. We construct various homotopy categories of C^*-algebras. Much of the theory has been worked out by Rosenberg in [95] and we refer to this paper for a number of computations [92]. Finally, we assemble the results of the previous sections to define **E,** prove its excision properties (Theorem (3.1.23)), relate it to KK-theory (Theorem (3.1.26)), and characterize **E** as a 'universal' homology/cohomology theory (Theorem (3.1.27)).

An interesting and quite different approach to more or less the same problem of excision is given by Skandalis in [89], where subgroups $KK_{\mathrm{nuc}}(A,B) \subset KK(A,B)$ are defined, and shown to satisfy excision. The main distinction between $KK_{\mathrm{nuc}}(A,B)$ and $E(A,B)$ is that *all* generalized elliptic operators give rise to elements of $E(A,B)$, and so **E** 'contains' the elliptic operator calculus on C^*-algebra K-theory. Now, Skandalis has shown in [89] that certain identities involving products of specific elliptic operators, arising in differential topology, do not hold in KK-theory. A consequence is that KK-theory is inadequate as a tool to prove the corresponding relations among K-groups. However, it is *possible* that within **E**-theory these elliptic operator identities do hold. Thus from the point of view of applications it would be extremely interesting to develop a concrete realization of **E** (or a similar functor). This aspect of **E**-theory is discussed in a little more detail in [93].

Let **A** be a small, additive category (for our purposes it is convenient to include in the definition of additive category the existence of finite products and a zero object). If Σ is a set of morphisms in **A** then we shall denote by $\boldsymbol{A}[\Sigma^{-1}]$ the *additive* category obtained from **A** by inverting the morphisms in Σ. To be precise, $\boldsymbol{A}[\Sigma^{-1}]$ is an additive category, together with an additive functor $F: A \to \boldsymbol{A}[\Sigma^{-1}]$, such that:

(i) the objects of $\boldsymbol{A}[\Sigma^{-1}]$ are the objects of **A;**

(ii) for every object A of **A,** $F(A) = A$;

(iii) for every morphism σ in $\boldsymbol{A}, F(a)$ is an isomorphism; and

(iv) if $G: A \to B$ is any additive functor such that $G(\sigma)$ is an isomorphism for every σ in Σ then there exists a *unique* additive functor $\hat{G} : \boldsymbol{A}[\Sigma^{-1}] \to \mathbf{B}$ such that $G = \hat{G} \circ F$.

The existence of $\boldsymbol{A}[\Sigma^{-1}]$ for an arbitrary $\boldsymbol{A}$ and Σ is easy to establish (see [94] for example), whilst of course the uniqueness of $\boldsymbol{A}[\Sigma^{-1}]$ up to canonical isomorphism is guaranteed by condition (iv).

In order to simplify notation, if ϕ is a morphism in **A** then we shall denote the morphism $F(\phi)$ in $\boldsymbol{A}[\Sigma^{-1}]$ by $\bar{\phi}$.

Definition (3.1.1)[79]: (compare [91]). We shall say that Σ is an *admissible* set of morphisms if it is closed under composition of morphisms, contains all identity morphisms, and if the following four conditions involving morphisms in **A** are satisfied.

(L1) Given $\phi : A \to B$ and $\sigma : A \to A'$, with σ in Σ, there exists a commutative square

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\sigma"'] & B \arrow[d, dotted, "\tau"] \\ A' \arrow[r, dotted, "\phi'"'] & B' \end{tikzcd}$$

with τ in Σ.

(L2) Denote by Σ_L the class of morphisms in **A** which is generated (under composition) by Σ and all split monomorphisms in **A**. If a composition $\phi \circ \sigma$ is zero, where σ is an element of Σ, then there exists τ in Σ_L such that $\tau \circ \phi = 0$.

(R1) Given $\sigma: A \to A'$ and $\phi': B' \to A'$, with σ in Σ, there exists a commutative diagram

$$\begin{tikzcd} A' & B' \arrow[l, "\phi'"'] \\ A \arrow[u, "\sigma"] & B \arrow[l, dotted, "\phi"] \arrow[u, dotted, "\tau"'] \end{tikzcd}$$

with τ in Σ_L.

(R2) Denote by Σ_R the set of morphisms in **A** generated by Σ and all split epimorphisms in **A.** If a composition $\sigma\varphi$ is zero, with σ in Σ, then $\phi\tau = 0$ for some τ in Σ_R.

Remarks (3.1.2)[79]: (a) The above conditions are weakened versions of the conditions given in [91] for Σ to admit a 'calculus of left and right fractions' (unfortunately the C^*-algebra constructions of the next section do not quite fit into the framework of [91], and so the extra generality of Definition (3.1.1) is necessary). Following [91], we will obtain a concrete description of $\boldsymbol{A}[\Sigma^{-1}]$ for admissible sets Σ.

(b) The conditions (R1) and (R2) are of course simply the duals of (L1) and (L2). We shall consider in detail the consequences of (L1) and (L2); corresponding results concerning (R1) and (R2) follow by reversing arrows.

(c) Note that if Σ is generated under composition by a set Σ^0 and a set of isomorphisms in **A,** then in order to verify that Σ is admissible, it suffices to check (L1)-(R2) for morphisms σ in Σ^0, for then (L1)-(R2) hold for all σ in Σ by a simple induction argument.

(d) Assuming that Σ is admissible, the condition (L1) is satisfied not just for all σ in Σ but in fact for all σ in Σ_L. To show this, it suffices, as in the previous remark, to consider the case where σ is a split monomorphism, with a left inverse π. In (L1) we may then take $\phi' = \phi\pi$, and $\tau = 1_B$. We shall refer to this generalized condition simply as (L1) below. Similarly, condition (R1) is satisfied for all σ in Σ_R. Concerning Σ_L and Σ_R, we also note that the images in $\boldsymbol{A}[\Sigma^{-1}]$ of morphisms in Σ_L and Σ_R are left and right invertible, respectively.

We assume that Σ is admissible.

Definition (3.1.3)[79]: (compare [91]). Let A and B be objects in **A** and denote by $\mathcal{F}_B(A)$ the set of ordered pairs of morphisms (σ, ϕ) in **A** of the form

$$A \xrightarrow{\phi} B' \xleftarrow{\sigma} B$$

and with σ in Σ. We shall say that (σ_1, ϕ_1) and (σ_2, ϕ_2) are *related* if there exists a commutative diagram

$$\begin{tikzcd}
 & B' \arrow[dr, "\alpha_1"] & \\
A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & B \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_2"] & C \\
 & B_2' \arrow[ur, "\alpha_2"'] &
\end{tikzcd} \tag{6}$$

(meaning that $\alpha_1\phi_1 = \alpha_2\phi_1$ and $\alpha_1\sigma_1 = \alpha_2\sigma_1$), where $\alpha_1\sigma_1$ is in Σ_L.

Lemma (3.1.4)[79]: *The relation of Definition (3.1.3) is an equivalence relation.*

Proof. Symmetry and reflexivity are obvious. Suppose that $(\alpha_2\phi_2)$ is related to $(\sigma_3\varphi_3)$ by some diagram

$$\begin{tikzcd}
 & B_2' \arrow[dr, "\beta_2"] & \\
A \arrow[ur, "\phi_2"] \arrow[dr, "\phi_3"'] & B \arrow[u, "\sigma_2"'] \arrow[d, "\sigma_3"] & D. \\
 & B_3' \arrow[ur, "\beta_3"'] &
\end{tikzcd}$$

By (L1) there is a commuting square

$$\begin{tikzcd}
B \arrow[r, "\alpha_2\sigma_2"] \arrow[d, "\beta_2\sigma_2"'] & C \arrow[d, "\gamma"] \\
D \arrow[r, "\delta"'] & E
\end{tikzcd}$$

with γ in Σ_L, and hence $\gamma\alpha_2\sigma_2 = \delta\alpha_2\sigma_2 \in \Sigma_L$. By (L2), applied to $\phi = \gamma\alpha_2 - \delta\beta_2$ and $\sigma = \sigma_2$, there exists $\tau: E \to E_1$ in Σ_L such that $\tau\gamma\alpha_2 = \tau\delta\beta_2$. Then the diagram

$$\begin{tikzcd}
 & B_1' \arrow[dr, "\tau\gamma\alpha_1"] & \\
A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_3"'] & B \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_3"] & E_1. \\
 & B_3' \arrow[ur, "\tau\delta\beta_3"'] &
\end{tikzcd}$$

commutes (note that we need the morphism τ so that $\tau\gamma\alpha_1\phi_1 = \tau\delta\beta_3\phi_3$) and so $(\sigma_2\phi_1)$ is related to $(\sigma_3\phi_3)$.

Definition (3.1.5)[79]: Denote by $[\sigma, \phi]$ the equivalence class of (σ, ϕ), and denote by $F_B(A)$ the quotient of $\mathcal{F}_B(A)$ by this equivalence relation.

Lemma (3.1.6)[79]: (i) *For any finite collection* $\{[\sigma_1, \Psi_1], \dots, [\sigma_n, \Psi_n]\}$ of *elements of* $F_B(A)$*, there exists some* $\sigma : B \to B'$ *in* Σ *and morphisms* $\phi_1, \dots, \phi_n$ *in* **A** *such that* $[\sigma, \phi_i] = [\sigma_i, \Psi_i]$ *for* $i = 1, \dots, n$.

(ii) *The operation* $[\sigma, \phi_i] + [\sigma, \phi_2] = [\sigma, \phi_1 + \phi_2]$ *is well defined and makes* $F_B(A)$ *into an abelian group.*

Proof. (i) If $n = 1$ the result is trivial. Suppose then that there exist ϱ and $\theta_1, \dots, \theta_{n-1}$ such that $[\sigma_i, \Psi_i] = [\varrho, \theta_i]$ for $i = 1, \dots, n-1$. By (L1) there is a commuting diagram

$$\begin{tikzcd} B \arrow[r, "\varrho"] \arrow[d, "\sigma_n"'] & B' \arrow[d, "\tau"] \\ B'' \arrow[r, "\alpha"'] & B''' \end{tikzcd}$$

with $\tau \in \Sigma$, and hence $\sigma \in \Sigma$, where $\sigma = \tau\varrho = \alpha\sigma_n$. It is then easily checked that $[\sigma, \alpha\Psi_n] = [\sigma_n, \Psi_n]$ and $[\varrho, \theta_i] = [\sigma, \tau\theta_i]$ for $i = 1, \dots, n-1$.

(ii) We must show that if $[\sigma_1, \phi_1] = [\sigma_2, \phi_2]$ and $[\sigma_1, \Psi_1] = [\sigma_2, \Psi_2]$ then $[\sigma_1, \phi_1 + \Psi_1] = [\sigma_2, \phi_2 + \Psi_2]$ (this, together with part (i) will show that the operation is well defined on $F_B(A)$; it is clear that $F_B(A)$ will then be an abelian group). Given commuting diagrams

$$\begin{tikzcd} & B_1' \arrow[dr, "\alpha_1"] & \\ A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & B \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_2"] & C \\ & B_2' \arrow[ur, "\alpha_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B_1' \arrow[dr, "\beta_1"] & \\ A \arrow[ur, "\Psi_1"] \arrow[dr, "\Psi_2"'] & B \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_2"] & D \\ & B_2' \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

as in Definition (3.1.1), by (L1) there exists a commuting square

$$\begin{tikzcd} B \arrow[r, "\alpha_1\sigma_1"] \arrow[d, "\beta_1\sigma_1"'] & C \arrow[d, "\gamma"] \\ D \arrow[r, "\delta"'] & E \end{tikzcd}$$

with γ in Σ. By (L2) there exist $\tau_i : E \to E_i$ in Σ_L such that $\tau_i\gamma\alpha_i = \tau_i\delta\beta_i$, for $i = 1,2$, and applying (L1) to

$$\begin{tikzcd} E \arrow[r, "\tau_1"] \arrow[d, "\tau_2"'] & E_1 \\ E_2 & \end{tikzcd},$$

we obtain a single $\tau : E \to E'$ in Σ_L such that $\tau\gamma\alpha_i = \tau\delta\beta_i$ for $i = 1,2$. Then the commuting diagram

$$\begin{tikzcd}
 & B_1' \arrow[dr, "\tau\gamma\alpha_1"] & \\
A \arrow[ur, "\phi_1 + \Psi_1"] \arrow[dr, "\phi_2 + \Psi_2"'] & B \arrow[u, "\sigma_1"] \arrow[d, "\sigma_2"] & E' \\
 & B_2' \arrow[ur, "\tau\delta\alpha_2"'] &
\end{tikzcd}$$

shows that $[\sigma_1, \phi_1 + \Psi_1] = [\sigma_2, \phi_2 + \Psi_2]$.

We are of course thinking of $F_B(A)$ as a model for $A[\Sigma^{-1}](A, B)$**.** Observe that if $[\sigma_1, \phi_1] = [\sigma_2, \phi_2]$ in $F_B(A)$ then $\bar{\sigma}_1^{-1}\bar{\phi}_1 = \bar{\sigma}_2^{-1}\bar{\phi}_2$ in $A[\Sigma^{-1}](A, B)$**.** Indeed, given the diagram (6), we have $\bar{\alpha}_1\bar{\phi}_1 = \bar{\alpha}_2\bar{\phi}_2$ and $\bar{\alpha}_1 = \bar{\alpha}_2\bar{\sigma}_2\bar{\sigma}_1^{-1}$, so that $\bar{\alpha}_2\bar{\phi}_2 = \bar{\alpha}_2\bar{\sigma}_2\bar{\sigma}_1^{-1}\bar{\phi}_1$. But $\bar{\alpha}_2\bar{\sigma}_2$, and hence $\bar{\alpha}_2$, is a split monomorphism, and so from $\bar{\alpha}_2\bar{\phi}_2 = \bar{\alpha}_2\bar{\sigma}_2\bar{\sigma}_1^{-1}\bar{\phi}_1$, we obtain $\bar{\phi}_2 = \bar{\sigma}_2\bar{\sigma}_1^{-1}\bar{\phi}_1$, and hence $\bar{\sigma}_2^{-1}\bar{\phi}_2 = \bar{\sigma}_1^{-1}\bar{\phi}_1$. The following result is the converse of this computation:

Proposition (3.1.7)[79]: *Every element of $A[\Sigma^{-1}](A, B)$ is equal to one of the form $\bar{\sigma}^{-1}\bar{\phi}$ where $\sigma \in \Sigma$, and $\bar{\sigma}_1^{-1}\bar{\phi}_1 = \bar{\sigma}_2^{-1}\bar{\phi}_2$ if and only if $[\sigma_1, \phi_1] = [\sigma_2, \phi_2]$.*

Proof. An element of $A[\Sigma^{-1}](A, B)$ is a **Z**-linear combination of compositions of the form

$$\bar{\varrho}_k^{-1}\bar{\Psi}_k\bar{\varrho}_{k-1}^{-1}\bar{\Psi}_{k-1} \dots \bar{\varrho}_1^{-1}\bar{\Psi}_1,$$

where $\varrho_j \in \Sigma$ for $j = 1, \dots, k$. Using (L1) repeatedly, any such composition is equal to some $\bar{\varrho}^{-1}\bar{\Psi}$. hus any morphism from A to B may be written as a linear combination of morphisms $\bar{\sigma}_i^{-1}\bar{\phi}_i (i = 1, \dots, n)$. Using Lemma (3.1.6) and the remark preceding this proposition, these may be written as $\bar{\sigma}^{-1}\bar{\phi}_i (i = 1, \dots, n)$, and then $\sum_{i=1}^{n} m_i(\bar{\sigma}^{-1}\bar{\phi}_i) = \bar{\sigma}^{-1}\left(\overline{\sum m_\iota\phi_\iota}\right)$, *so* the first part of the proposition is proved.

Now we may make $A \to F_B(A)$ into a (contravariant) functor on **A** in the obvious way, and it follows easily from the group structure given in Lemma (3.1.6) that F_B is an additive functor from **A** to abelian groups. If $\varrho: A \to A'$ is in Σ then $F(\varrho) : F_B(A') \to F_B(A)$ is injective. Indeed, if $F(\varrho)([\sigma, \phi]) = 0$, so that there exists a commuting diagram

$$\begin{tikzcd}
 & B' \arrow[dr, "\alpha_1"] & \\
A \arrow[ur, "\phi\varrho"] \arrow[dr, "0"'] & B \arrow[u, "\sigma"'] \arrow[d, equal] & C \\
 & B \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

(with $\alpha_1\sigma$ in Σ_L), then by (L2), since $\alpha_1\phi\varrho = 0$, there exists $\tau \in \Sigma_L$ such that $\tau\alpha_1\phi = 0$. The diagram

$$\begin{tikzcd}
 & B' \arrow[dr, "\tau\alpha_1"] & \\
A \arrow[ur, "\phi"] \arrow[dr, "0"'] & B \arrow[u, "\sigma"'] \arrow[d, equal] & D \\
 & B \arrow[ur, "\tau\alpha_2"'] &
\end{tikzcd}$$

then shows that $[\sigma, \phi] = 0$. On the other hand, $F(\varrho)$ is surjective since it is easily seen that if $[\sigma, \phi] \in F_B(A)$ then $F(\varrho)([\tau\sigma, \phi']) = [\sigma, \phi]$, where $\tau \in \Sigma$ and ϕ' are chosen using (L1) so that the square

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\varrho"'] & B' \arrow[d, "\tau"] \\
A' \arrow[r, "\phi'"'] & B''
\end{tikzcd}$$

commutes. Thus F_B passes to a functor on $A[\Sigma^{-1}]$**.** The proof is completed by noting that under the homomorphism $F(\bar{\sigma}^{-1}\bar{\phi}) = F(\phi)F(\sigma)^{-1}$, the 'unit element' $[1, 1] \in F_B(B)$ is sent to $[\sigma, \phi] \in F_B(A)$, since $F(\sigma)([\sigma, 1]) = [1, 1]$ (by the above) and $F(\phi)([\sigma, 1]) = [\sigma, \phi]$.

Proposition (3.1.8)[79]: *Every element of $A[\Sigma^{-1}](A, B)$ is equal to one of the form $\bar{\phi}\bar{\sigma}^{-1}$, where $\sigma \in \Sigma$, and $\bar{\phi}_1\bar{\sigma}_1^{-1} = \bar{\phi}_2\bar{\sigma}_2^{-1}$ if and only if there is a commuting diagram*

$$\begin{tikzcd}
 & A_1' \arrow[dr, "\alpha_1"] & \\
B \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & A \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_2"] & C \\
 & A_2' \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

with $\sigma_1\alpha_1 \in \Sigma_R$.

Proof. Apply Proposition (3.1.7) to the opposite category of **A.**

Having so described $A[\Sigma^{-1}]$, we can proceed to prove the excision result we require.

Theorem (3.1.9)[79]: Suppose that $A \xrightarrow{l} B \xrightarrow{\pi} C$ is a sequence of morphisms in **A** such that for every object D in **A** the sequences of abelian groups

$$\boldsymbol{A}(C,D) \to \boldsymbol{A}(B,D) \to \boldsymbol{A}(A,D)$$

and

$$\boldsymbol{A}(D,A) \to \boldsymbol{A}(D,B) \to \boldsymbol{A}(D,C)$$

are exact in the middle. Then the sequences

$$\boldsymbol{A}[\Sigma^{-1}](C,D) \to \boldsymbol{A}[\Sigma^{-1}](B,D) \to \boldsymbol{A}[\Sigma^{-1}](A,D)$$

and

$$\boldsymbol{A}[\Sigma^{-1}](D,A) \to \boldsymbol{A}[\Sigma^{-1}](D,B) \to \boldsymbol{A}[\Sigma^{-1}](D,C)$$

are also exact in the middle.

Proof. We shall consider just the contravariant sequence; the covariant one is dual. Let $\bar{\sigma}^{-1}\bar{\phi} \in \boldsymbol{A}[\Sigma^{-1}](B,D)$ and suppose that $\bar{\sigma}^{-1}\bar{\phi}\bar{l} = 0$. There is a diagram of the form

$$\begin{tikzcd}
 & D' \arrow[dr, "\alpha_1"] & \\
A \arrow[ur, "\varphi_1"] \arrow[dr, "0"'] & D \arrow[u, "\sigma"] \arrow[d, equal] & E \\
 & D \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

with $\alpha_1\sigma \in \Sigma_L$. We have that $(\alpha_1\phi)l = 0$, and so by the hypothesis of the theorem, $\alpha_1\phi = \Psi\pi$ for some $\Psi : C \to E$ in **A**. Now, $\bar{\alpha}_1$ has a left inverse $\beta : E \to D'$ in $\boldsymbol{A}[\Sigma^{-1}]$, and from the equality $\alpha_1\phi = \Psi\pi$ we obtain $\bar{\phi} = \beta\bar{\Psi}\bar{\pi}$. Therefore $\bar{\sigma}^{-1}\bar{\phi} = (\bar{\sigma}^{-1}\beta\bar{\Psi})\bar{\pi}$, and so $\bar{\sigma}^{-1}\bar{\phi}$ has the pre-image $\bar{\sigma}^{-1}\beta\bar{\Psi}$ in $\boldsymbol{A}[\Sigma^{-1}](C,D)$, and the sequence is exact at $\boldsymbol{A}[\Sigma^{-1}](B,D)$.

Denote by $\boldsymbol{C}^*$**-Alg** the category of *separable* C^*-algebras and $*$-homomorphisms. This is of course not a small category so that the results of the previous section will not immediately apply to the categories associated to $\boldsymbol{C}^*$**-Alg** that we are about to construct. However, $\boldsymbol{C}^*$**-Alg** does have a small, equivalent subcategory-take the set of C^*-algebras which are algebras of operators on a fixed separable, infinite-dimensional Hilbert space, for example-and it is sufficient for our purposes to work with it. Since they are of a completely straightforward

nature, we shall not bother to comment on the various details associated with this substitution. Denote by **H** the category of separable C^*-algebras and homotopy classes of $*$-homomorphisms (see for example [95]). We shall denote the homotopy class of a $*$-homomorphism f by $[f]$.

We may also form a stable homotopy category **S** using the suspension functor $S : \mathbf{H} + \mathbf{H}$, defined by

$$S\left(A \xrightarrow{[f]} B\right) = C_0(0,1) \otimes A \xrightarrow{[1\otimes f]} C_0(0,1) \otimes B.$$

The objects of **S** are pairs (A, m), where A is a separable C^*-algebra and $m \in \mathbf{Z}$ (we shall write A_m instead of (A, m)). The morphisms are given by:

$$S(A_m, B_n) = \lim_{\vec{K}} \mathbf{H}(S^{m+K}A, S^{n+K}B)\,.$$

Here $\mathbf{H}(S^{m+K}A, S^{n+K}B)$ is mapped into $\mathbf{H}(S^{m+K+1}A, S^{n+K+1}B)$ by suspension (we consider only those K such that $K + m \geq 0$ and $K + n \geq 0$). Given a $*$-homomorphism $f: S^{m+K}A \to S^{n+K}B$, *we* shall denote the corresponding morphism in $S(A_m, B_n)$ by $\{f\}$. As is the case with stable homotopy of spaces, **S** is an additive category (compare [95]).

Now, denote by $\mathcal{K}$ the C^*-algebra of compact operators on a separable Hilbert space. The following facts about $\mathcal{K}$ are well known and easily derived.

Lemma (3.1.10)[79]: (i) *The C^*-algebras $\mathcal{K} \otimes \mathcal{K}$ and $\mathcal{K}$ are isomorphic, and any two $*$-isomorphisms are homotopic.*

(ii) *Any two rank one projections in $\mathcal{K}$ are homotopic.*

(iii) *If $p \in \mathcal{K}$ is a rank-one projection then the $*$-homomorphism $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ given by $k \to k \otimes p$ is homotopic to a $*$-isomorphism.*

Define functors T: $\mathbf{H} \to \mathbf{H}$ and T: $\mathbf{S} \to \mathbf{S}$ by

$$T(A) = A \otimes \mathcal{K}, \qquad T([f]) = [f \otimes 1]$$

and

$$T(A_n) = (A \otimes \mathcal{K})_n, \qquad T(\{f\}) = \{f \otimes 1\}.$$

Fixing a rank-one projection $p \in \mathcal{K}$ and a $*$-isomorphism $h : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$, define natural transformations $\eta : 1 \to T$ and $\mu : T^2 \to T$ by means of the $*$-homomorphisms

$$e_A: A \to A \otimes \mathcal{K}, \qquad e_A(a) = a \otimes p,$$

for η and

$$m_A: A \otimes \mathcal{K} \otimes \mathcal{K} \to A \otimes \mathcal{K}, m_A(a \otimes k_1 \otimes k_2) = a \otimes h(k_1 \otimes k_2),$$

for μ. Using Lemma (3.1.10) we obtain the following result: $\mathcal{K}C^*$

Lemma (3.1.11)[79]: (i) *The natural transformation μ is an isomorphism, with inverse $T\eta$.*

(ii) *Both η and μ commute with T in the sense that $T\eta = \eta T$ and $T\mu = \mu T$ as natural transformations.*

(By $T\eta$ *we* mean the natural transformation $T \to T^2$ given by $(T\eta)_A = T(\eta_A)$, and by ηT *we* mean the natural transformation $T \to T^2$ given by $(T\eta)_A = \eta_{T(A)}$. The natural transformations μT and $T\mu$ are defined similarly.)

Using T, η and μ, define new categories **TH** and **TS** as follows (we shall spell out the construction for **TH;** the case of **TS** is the same). The objects of **TH** are the objects of **H,** and

$$\mathrm{TH}(A, B) = \mathrm{H}(A, TB).$$

Given $\phi : A \to TB$ and $\Psi : B \to TC$ their composition is given by

$$a \xrightarrow{\phi} TB \xrightarrow{T(\Psi)} T^2C \xrightarrow{\mu_C} TC.$$

It is easily checked that this composition law gives a category and that $F: \mathrm{H} \to \mathrm{TH}$ given by

$$F\left(A \xrightarrow{\phi} B\right) = A \xrightarrow{\eta_B \phi} TB$$

is a functor. (In fact, by Lemma (3.1.11), T, η and μ form an idempotent monad, and **TH** is the Kleisli category of this monad-see [94] for the terminology.)

The main feature of **TH** (and similarly for **TS**) is that the functor $F: \mathrm{H} \to \mathrm{TH}$ is *stable,* that is, $F([e_A]) : A \to A \otimes \mathcal{K}$ is an isomorphism for every A. Indeed, the identity $*$-homomorphism on $A \otimes \mathcal{K}$ determines a morphism in **TH** from $A \otimes \mathcal{K}$ to A which is inverse to $F([e_A])$. Furthermore, if $G : \mathrm{H} \to \mathbf{X}$ is any stable functor then there is a unique functor $\hat{G} : \mathrm{TH} \to \mathbf{X}$ such that $G = \hat{G} \circ F$, namely

$$\hat{G}\left(A \xrightarrow{[F]} B\right) = G(A) \xrightarrow{G([F])} G(B \otimes \mathcal{K}) \xrightarrow{G([e_B])^{-1}} G(B).$$

(Thus **TH** is obtained from **H** by inverting the morphisms $[e_A]$.)

The group structure on $\mathrm{TS}(A, B) = S(A, B \otimes \mathcal{K})$ makes **TS** an additive category

and $F: \mathrm{S} \to \mathrm{TS}$ an additive functor.

We are mainly interested in the category **TH**, in the sense that we are interested in functors on $\boldsymbol{C}^*$**-Alg** which are homotopy invariant and stable.

Definition (3.1.12)[79]: Let $f: A \to B$ be a $*$-homomorphism. The *mapping cone* off is the C^*-algebra

$$C_f = \{a \oplus g \in A \oplus C_0[0,1) \otimes B | f(a) = g(a)\}.$$

(We identify $C_0[0,1) \otimes B$ with the C^*-algebra of continuous maps from $[0,1]$ to B which vanish at 1. Note that following standard C^*-algebra usage, '$\oplus$' denotes Cartesian product-the product in C^*-Alg.)

There is a natural map $p : C_f \to A$ given by $p(a \oplus g) = a$. As far as we are concerned, the principal fact about C_f is the following result.

Theorem (3.1.13)[79]: *For any object D_n of* **TS** *and any $m \in \mathbf{Z}$ the sequences*

$$\mathrm{TS}(B_m, D_n) \xrightarrow{\{f\}^*} \mathrm{TS}(A_m, D_n) \xrightarrow{\{p\}^*} \mathrm{TS}\left(\left(C_f\right)_m, D_n\right)$$

and

$$\mathrm{TS}\left(D_n, \left(C_f\right)_m\right) \xrightarrow{\{p\}_*} \mathrm{TS}(D_n, A_m) \xrightarrow{\{f\}_*} \mathrm{TS}(D_n, B_m)$$

are exact in the middle.

These sequences are nothing more than pieces of the Puppe exact sequences of algebraic topology. For a proof, see [95]. (Actually, [95] deals with **S**, not *TS***,** and so it is necessary to make the simple observation that the mapping cones for $f: A + B$ and $f \otimes 1: A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ are related by $C_f \otimes \mathcal{K} \cong C_{f \otimes 1}$.)

Definition (3.1.14)[79]: (i) Denote by Σ^{00} the class of morphisms $\phi : A_m \to A'_{m'}$ in **S** of the form $\phi = \{j\}$, where $j : \Sigma^{m+N} A \to \Sigma^{m'+N} A'$ is an injection of $\Sigma^{m+N} A$ onto an ideal of $\Sigma^{m'+N} A'$, such that the quotient C^*-algebra $\Sigma^{m'+N} A' / \Sigma^{m+N} A$ is contractible.

(ii) Denote by Σ^0 the set of morphisms σ in **TS** of the form $\sigma = F(\sigma_0)$, where $F: \mathbf{S} \to \mathbf{TS}$ is the canonical functor, and $\sigma_0 \in \Sigma^{00}$.

(iii) Denote by Σ the class of morphisms in **TS** generated under composition by Σ^0 and the class of all isomorphisms in **TS.**

We will show that Σ is admissible, in the sense of Definition (3.1.1). The following is one of two technical C^*-algebra results needed (Lemma (3.1.17) being the other).

Lemma (3.1.15)[79]: Let A be an ideal in A' and let B be a stable C^*-algebra (that is, $B \cong B \otimes \mathcal{K}$). Any $*$-homomorphism $f_0 : A \to B$ is homotopic to a $*$-homomorphism $f_1 : A \to B$ which extends to a $*$-homomorphism from A' into the multiplier algebra $M(B)$.

Proof. By Lemma (3.1.10) we may assume that B is of the form $B' \otimes \mathcal{K}$ for some B', and that f_0 is of the form

$$A \xrightarrow{e_A} A \otimes \mathcal{K} \xrightarrow{f_0' \otimes 1} B' \otimes \mathcal{K}.$$

Since e, extends to a map from A' into $M(A \otimes \mathcal{K})$, it suffices to show that $f_0' \otimes 1 : A \otimes \mathcal{K} \to B' \otimes \mathcal{K}$ is homotopic to a map which extends to multiplier algebras. Let $v_1 \in M(A \otimes \mathcal{K})$ be the isometry constructed in Lemma (3.1.4) of [81]. Then $\mathrm{Ad}(v_1)$ maps $M(A \otimes \mathcal{K})$ into $M(\tilde{A} \otimes \mathcal{K}) \subset M(A \otimes \mathcal{K})$, where a denotes $\tilde{A}$ with a unit adjoined (if A has a unit we could set $\tilde{A} = A$, although the result we are trying to prove is trivial in this case anyway). Now, $f_0' \otimes 1$ extends to a map from $M(\tilde{A} \otimes \mathcal{K})$ to $M(A \otimes \mathcal{K})$ (see [81]) and so it suffices to show that $\mathrm{Ad}(v_1) : A \otimes \mathcal{K} \to A \otimes \mathcal{K}$ is homotopic to the identity, for then we can define f_1 to be the composition

$$A \xrightarrow{e_A} A \otimes \mathcal{K} \xrightarrow{\mathrm{Ad}(v_1)} A \otimes \mathcal{K} \xrightarrow{f_0' \otimes 1} B' \otimes \mathcal{K}.$$

But according to [81], $\mathrm{Ad}(v_1)$ is homotopic to $1 \otimes \mathrm{Ad}(w_0)$ for some isometry w_0 and then by Lemma (3.1.10), $1 \otimes \mathrm{Ad}(w_0) : A \otimes \mathcal{K} \to A \otimes \mathcal{K}$ is homotopic to the identity map.

Proposition (3.1.16)[79]: *The class* Σ *satisfies property* (L1) *of Definition (3.1.1)*.

Proof. Let $\phi: A_m \to B_n$ be a morphism in **TS,** represented by some $*$-homomorphism

$$f: S^{m+M}A \to S^{n+M}B \otimes \mathcal{K},$$

and let $\sigma = F(\sigma_0)$ be a morphism in Σ^0, where $\sigma_0: A_m \to A'_{m'}$ is a morphism in Σ^{00}, represented by some $*$-homomorphism

$$j: S^{m+N}A \to S^{m'+N}A'$$

as in Definition (3.1.14). By taking suspensions of f or j we may assume that $N = M$ (note that a suspension of j is still an inclusion of an ideal, with contractible quotient). By Lemma (3.1.15) we may assume that f extends to a $*$-homomorphism from $S^{m'+N}A'$ to $M(S^{n+N}B \otimes \mathcal{K})$, which we shall call f. Let D be the subalgebra of $M(S^{n+N}B \otimes \mathcal{K}) \oplus S^{m'+N}A'/S^{m+N}A$ generated by $S^{n+N}B \otimes \mathcal{K} \oplus 0$ and elements of the form $\tilde{f}(x) \oplus \dot{x}$, where $x \in S^{m'+N}A'$ and $\dot{x}$ denotes the image of x in the quotient algebra (this subalgebra is automatically closed, and

so is a C^*-algebra). Under the natural inclusion $i: S^{n+N}B \otimes \mathcal{K} \to D, S^{n+N}B \otimes \mathcal{K}$ is an ideal in D; the quotient is isomorphic to $S^{m'+N}A'/S^{m+N}A$, and is therefore contractible. The map i thus gives rise to a morphism $\{i\} : (B \otimes \mathcal{K})_n \to D_{-N}$, in **S** which is an element of Σ^{00}; the map $f': S^{m'+N}A' \to D$ defined by $x \mapsto f(x) \oplus \dot{x}$ gives rise to a morphism $\{f'\} : A'_{m'} \to D_{-N}$ in **S**. The standard map $e : B \to B \otimes \mathcal{K}$ gives rise to a morphism $\{e\} : B_n \to (B \otimes \mathcal{K})_n$, in **S** which is invertible in **TS.** Since the diagram

$$\begin{tikzcd} A_m \arrow[r, "\phi"] \arrow[d, "\sigma"'] & B_n \arrow[d, "F(\{i\})F(\{e\})"] \\ A'_{m'} \arrow[r, "F(\{f'\})"'] & D_{-N} \end{tikzcd}$$

in **TS** is easily seen to commute (using Lemma (3.1.10)), we are done.

Lemma (3.1.17)[79]: *If $f: B' \to A'$ is any $*$-homomorphism then there is a homotopy equivalence $h : B'' \to B'$ such that the composition $f \circ h : B'' \to A$ is homotopic to a surjective $*$-homomorphism.*

Proof. Take B'' to be the free product $B' * (C_0[0,1) \otimes A)$ (this is the coproduct in $\boldsymbol{C^*}$**-Alg**), and let $h = 1 * 0 : B'' \to B'$. Then $f \circ h: B'' \to A$ is equal to $f * 0: B'' \to A$, which is homotopic to the surjection $f * e : B'' \to A$, where $e : C_0[0,1) \otimes A \to A$ is evaluation at $0 \in [0,1)$.

Proposition (3.1.18)[79]: *The class* Σ *satisfies property* (R1) *of Definition (3.1.1)*.

Proof. Let $\phi' : B'_{n'} \to A'_{m'}$ be represented by some $*$-homomorphism

$$f: S^{n'+M}B' \to S^{m'+M}A' \otimes \mathcal{K}$$

and let $\sigma = F(\sigma_0) : A_m \to A'_{m'}$ be a morphism in Σ^0, where $\sigma \in \Sigma^{00}$ is represented by a $*$-homomorphism

$$j: S^{m+N}A \to S^{m'+N}A'$$

as in Definition (3.1.14). As in Proposition (3.1.16), we may assume that $M = N$. By Lemma (3.1.17) there exists a homotopy equivalence $h : B'' + S^{n'+N}B'$ such that $f \circ h$ is homotopic to a surjective $*$-homomorphism $g : B'' \to S^{m'+N}A' \otimes \mathcal{K}$. Let B be the ideal

$$\begin{tikzcd}
A'_{m'} & & B'_{n'} \arrow[ll, "\phi'"'] \\
 & & B''_{-N} \arrow[u, "\cong"', "F(\{h\})"] \\
A_m \arrow[uu, "\sigma"] & (A \otimes \mathcal{K})_m \arrow[l, "F(\{e\})^{-1}"] & B_{-N} \arrow[l, "F(\{g|B\})"] \arrow[u, "F(\{i\})"']
\end{tikzcd}$$

$g^{-1}[S^{m+N}A \otimes \mathcal{K}]$ of B''. The quotient B''/B is $*$-isomorphic to $S^{m'+N}A' \otimes \mathcal{K}/S^{m+N}A \otimes \mathcal{K}$ and is therefore contractible. The diagram in **TS**

commutes, where $i: B \to B''$ is the inclusion map, and so we are done.

In order to establish that the properties (L2) and (R2) hold, we need to recall some basic facts concerning mapping cones. For the simple proof of the following lemma, compare for example [95]. Note that given a $*$-homomorphism $f: A \to B$ there is a natural map $s : SB \to C_f$ given by $s(g) = 0 \oplus g$.

Lemma (3.1.19)[79]: *Given a $*$-homomorphism of C^*-algebras $h : X \to Y$ the compositions*

$$C_h \xrightarrow{p} X \xrightarrow{h} Y$$

and

$$SX \xrightarrow{Sh} SY \xrightarrow{S} C_h$$

are homotopic to zero. If $h : X \to Y$ is homotopic to zero then $[p] : C_h \to X$ is a split epimorphism in **H** *and $[s] : SY \to C_h$ is a split monomorphism.*

Proposition (3.1.20)[79]: *The class Σ satisfies property* (L2) *of Definition (3.1.1)*.

Proof. Let $\phi : A'_{m'} \to B_n$ be represented by $f: S^{m'+M}A' \to S^{n+M}B \otimes \mathcal{K}$ and let $\sigma = F(\sigma_0) : A_m \to A'_{m'}$ be a morphism in Σ given by $j : S^{m+N}A \to S^{m'+N}A'$ as in Definition (3.1.14). We may assume that $M = N$. Furthermore, if $\sigma\phi = 0$, then by choosing N large enough we may assume that the composition fj is homotopic to zero. It follows from Lemma (3.1.19) that the map $s: S^{n+1+N}B \otimes \mathcal{K} \to C_{fj}$ is a split monomorphism in **H**, and so the map $F(\{s\}): (B \otimes \mathcal{K})_n \to \left(C_{fj}\right)_{-N-1}$ is a split monomorphism in **TS**. The C^*-algebra C_{fj} is an ideal in C_f (via the

embedding $a \oplus g \to a \oplus g$); the quotient is isomorphic to $S^{m'+N}A'/S^{M+N}A$, and is therefore contractible. Also, by Lemma (3.1.19) the composition

$$S^{m'+1+N}A' \to S^{n+1+N}B \otimes \mathcal{K} \to C_f$$

is homotopic to zero. It follows that $\tau \circ \phi = 0$ in **TS,** where τ is the composition

$$B_n \xrightarrow{F(\{e\})} (B \otimes \mathcal{K})_n \xrightarrow{F(\{s\})} \left(C_{fj}\right)_{-N-1} \xrightarrow{F(\{i\})} \left(C_f\right)_{-N-1}.$$

Since $F(\{e\})$ is an isomorphism, $F(\{s\})$ is a split monomorphism, and $F(\{i\}) \in \Sigma^0$, we have that $\tau \in \Sigma_{\mathrm{L}}$ as required.

Proposition (3.1.21)[79]: *The class satisfies property* (R2) *of Definition (3.1.1)*.

Proof. Suppose that $\sigma\phi = 0$ in **TS,** where $\phi : A_m \to B_n$ is represented by $f: S^{m+M}A \to S^{n+M}B \otimes \mathcal{K}$ and $\sigma \in \Sigma, \sigma: B_n \to B'_{n'}$, is given by $j: S^{n+N}B \to S^{n'+N}B'$. We may assume that $M = N$ and that the composition

$$S^{m+N}A \xrightarrow{f} S^{n+N}B \otimes \mathcal{K} \xrightarrow{j\otimes 1} S^{n'+N}B \otimes \mathcal{K}$$

is homotopic to zero. By Lemma (3.1.19) the morphism $[p] : \left(C_{(j\otimes 1)f}\right)_{-N} \to A_m$ is a split epimorphism in **H,** and hence $F(\{p\}) : \left(C_{(j\otimes 1)f}\right)_{-N} \to A_m$ is a split epimorphism in **TS.** On the other hand, the composition

$$C_f - C_{(j\otimes 1)f} \to S^{m+N} \xrightarrow{f} S^{n+N}B \otimes \mathcal{K}$$

is homotopic to zero (by Lemma (3.1.19) again). Also, C_f is included in $C_{(j\otimes 1)f}$ as an ideal; the quotient is isomorphic to $\left(S^{n'+1+N}B' \otimes \mathcal{K}\right)/(S^{n+1+N}B' \otimes \mathcal{K})$, and is hence contractible. Thus if τ is the composition

$$\left(C_f\right)_{-N} \xrightarrow{F(\{i\})} \left(C_{(j\otimes 1)f}\right)_{-N} \xrightarrow{F(\{p\})} A_m$$

then $\tau \in \Sigma_R$ and $\phi\tau = 0$.

Definition (3.1.22)[79]: Let **E** be the full subcategory of the category of fractions $\mathrm{TS}[\Sigma^{-1}]$ (with Σ as in Definition (3.1.14)) whose objects are the objects A_0 in **TS.**

We shall identify the objects A_0 of **E** with the class of separable C^*-algebras A. There is of course an obvious functor from ***C**-Alg** to **E**, mapping A_0 to A.

We obtain the following result.

Theorem (3.1.23)[79]: *If* $0 \to J \to^{j} A \to^{q} B \to 0$ *is an exact sequence of separable* C^**-algebras then* for any separable C^*-algebra C the sequences of abelian groups

$$E(B,C) \xrightarrow{q^*} E(A,C) \xrightarrow{j^*} E(J,C)$$

and

$$E(C,J) \xrightarrow{j_*} E(C,A) \xrightarrow{q_*} E(C,B)$$

are exact in the middle.

Proof. Since Σ is an admissible class in **TS** (by Propositions (3.1.16), (3.1.18), (3.1.20) and (3.1.21)) it follows from Theorems (3.1.9) and (3.1.13) that these sequences would be exact if $j : J \to A$ were replaced with $p : C_q \to A$. There is an inclusion $i : J + C_q$, of J as an ideal in C_q, namely $i(a) = a \oplus 0$, such that the $p \circ i = j: J \to A$, and so it suffices to show that i maps to an isomorphism in $\mathbf{E}$. But the quotient C_q/J is isomorphic to $C_0[0,1) \otimes B$ and is hence contractible. Therefore, by definition of Σ, i is invertible in $\mathbf{E}$.

Remark (3.1.24)[79]: Theorem (3.1.23) asserts that the functors $\mathbf{E}(C,.)$ and $\mathbf{E}(.,C)$ on C^***-Alg** are *half-exact.* They are in addition stable and homotopy invariant and so by Cuntz's Bott Periodicity Theorem [84], there is a natural six term exact sequence

$$\begin{tikzcd}
 & \mathbf{E}(C,A) \arrow[r, "q*"] & \mathbf{E}(C,B) \arrow[dr, "\partial"] & \\
\mathbf{E}(C,J) \arrow[ur, "j*"] & & & \mathbf{E}(C,SJ) \arrow[d, "S_{j*}"] \\
 & \mathbf{E}(C,SB) \arrow[ul, "\partial"] & \mathbf{E}(C,SA) \arrow[l, "Sq_*"] &
\end{tikzcd}$$

as well as a similar sequence for $\mathrm{E}(.,C)$ (these are natural with respect to change of the exact sequences, as well as natural in C).

Now, denote by KK the category obtained from Kasparov's KK-theory scribed in the Introduction. The functor C^*-Alg $\to$ KK is characterized in follows.

Theorem (3.1.25)[79]: Let A be an additive category and let F: C^*-Alg $\to A$ be a functor with the following properties:

(i) F is a homotopy functor;

(ii) F is stable; and

(iii) *if* $0 \to J \to^{j} \underset{\to_p}{\overset{\leftarrow s}{}} B \to 0$ is a split exact sequence of C^*-algebras then $F(A)$ is the direct sum of $F(J)$ *and* $F(B)$ *via the maps* $F(j)$ *and* $F(s)$.

There exists a unique functor $\tilde{F}: KK \to A$ such that F is the composition C^*-Alg $\to KK \to^{\tilde{F}} A$. (Furthermore, the functor C^*-Alg $\to KK$ itself has properties (i), (ii) and (iii).)

There is a suspension functor $S : KK \to KK$,

$$S\left(A \xrightarrow{\phi} B\right) = SA \xrightarrow{S\phi} SB,$$

which is compatible with the suspension functor on C^*-Alg (see for example [81], where S is denoted $1_{C_0(0,1)} \boxtimes -$). It follows from the Bott Periodicity Theorem in KK-theory [80] that $S : KK \to KK$ is full and faithful. Using this, it is a simple matter to compare KK and E.

Theorem (3.1.26)[79]: *There is a unique (additive) functor* $\mathbf{KK} \to \mathbf{E}$ *such that the diagram*

$$\begin{tikzcd} & C^* - \boldsymbol{Alg} \arrow[dl] \arrow[dr] & \\ \mathbf{KK} \arrow[rr] & & \mathbf{E} \end{tikzcd}$$

commutes. If A is a K-nuclear C^*-algebra then for any B the homomorphism $KK(A,B) \to \mathrm{E}(A,B)$ *is* an isomorphism.

Proof. The existence and uniqueness of a functor from KK to E follows from Theorem (3.1.25) and Remark (3.124), since the six term exact sequences for a split exact sequence degenerate into split exact sequences of E-groups, and these imply that E satisfies condition (iii) of Theorem (3.1.25), whilst the other conditions are obviously satisfied. If A is *any* C^*-algebra then we may define a functor K_A from S into abelian groups as follows. On objects we set

$$K_A(B_m) = \varinjlim_{N} KK(S^N A, S^{m+N} B), \tag{7}$$

where the direct limit is taken using the suspension functor. A morphism $\{f\} : B_m \to C_n$ in S induces homomorphisms $KK(S^N A, S^{m+N} B) \to KK(S^N A, S^{n+N} C)$, for large enough N, compatible with suspension, and so we obtain a homomorphism $K_A(\{f\}): K_A(B_m) \to K_A(C_n)$.

Since **KK** is stable it follows that K_A passes to a functor from **TS** to abelian groups (which we will also call K_A). Suppose now that A is K-nuclear (see [89). Then all suspensions $S^m A$ of A are K-nuclear ([89]), and the functors $KK(S^m A, -)$ on C^*-Alg are half-exact by [89]. It follows from the resulting long exact sequence that if $0 \to B \to B' \to B'/B \to 0$ is a short exact sequence of C^*-algebras with B'/B contractible then the map $KK(S^m A, B) \to KK(S^m A, B')$ is an isomorphism. Therefore the functor K_A passes to a functor on $\mathbf{TS}[\Sigma^{-1}]$ (with Σ as in Definition (3.1.14)) and so by restricting this we obtain a functor K_A on the category **E**. Since the suspension functor $S : \mathbf{KK} \to \mathbf{KK}$ is fully faithful, the direct limit in (7) is superfluous for objects in **E**: we have $K_A(B) = \mathbf{KK}(A, B)$. Denote by E_A the functor $\mathbf{E}(A, -)$ on **E** or **KK**. The functor $\mathbf{KK} \to \mathbf{E}$ gives a natural transformation $\tau : K_A \to E_A$, and we may define a natural transformation $\sigma : E_A \to K_A$ (considering E_A and K_A as functors on **E** or **KK**) by

$$\sigma_A(\phi) = K_A(\phi)(1_A) \in KK(A, B),$$

where $1_A \in E_A(A) = E(A, A)$ is the identity morphism. It follows from the Yoneda Lemma that σ and τ are mutual inverses.

We shall conclude by outlining a characterization of E, analogous to the characterization of **KK** in Theorem (3.1.25).

Theorem (3.1.27)[79]: *Let F be any functor from $\boldsymbol{C}^*$-**Alg** to an additive category $\boldsymbol{A}$ which satisfies the following conditions:*

(i) *F is a homotopy functor;*

(ii) *F is stable; and*

(iii) *if $0 \to J \to A \to B \to 0$ is any short exact sequence of C^*-algebras, and C is any other C^*-algebra, then the sequences of abelian groups*

$$A(B, C) \xrightarrow{q^*} A(A, C) \xrightarrow{j^*} A(J, C)$$

and

$$A(C, J) \xrightarrow{j_*} A(C, A) \xrightarrow{q_*} A(C, B)$$

are exact in the middle.

There exists a unique functor $\tilde{\mathrm{F}}: \mathrm{E} + \mathrm{A}$ such that F is equal to the composition C^*-Alg $\to$ $\mathbf{E} \to^{\mathrm{F}} \boldsymbol{A}$.

By Theorem (3.1.23) and the properties of **TS,** the functor $\boldsymbol{C}^*$**-Alg** $\to$ **E** satisfies properties (i), (ii) and (iii). Thus **E** is characterized by Theorem (3.1.27) as the universal stable cohomology theory on $\boldsymbol{C}^*$**-Alg.** In distinction to Theorem (3.1.25), it is *not* obvious that such a universal theory exists.

Proof. By Cuntz's Bott Periodicity Theorem [84], there are natural isomorphisms $A\big(F(S^2A), F(B)\big) \cong A\big(F(A), F(B)\big) \cong A\big(F(A), F(S^2B)\big)$ (where we regard $A\big(F(A), F(B)\big)$, etc., as a bifunctor on $\boldsymbol{C}^*$**-Alg**). By property (i), we may regard F as a functor on **H,** and by means of these isomorphisms we may construct from F a functor on S. By property (ii) we obtain from this a functor on **TS.** Property (iii) implies that associated to any short exact sequence of C^*-algebras,

$$0 \to J \xrightarrow{j} A \xrightarrow{Q} B \to 0,$$

there are six term exact sequences in the C and D variables of $A\big(F(C), F(D)\big)$. If B is contractible then these exact sequences degenerate to isomorphisms

$$j^* : A\big(F(A), F(D)\big) \to A\big(F(J), F(D)\big)$$

and

$$j_* : A\big(F(C), F(J)\big) \to A\big(F(C), F(A)\big).$$

It follows that $f(j) : F(J) \to F(A)$ is an isomorphism, and so $F(\sigma)$ is an isomorphism for any σ in the class Σ of Definition (3.1.14). Thus F passes to a functor on $\mathrm{TS}[\Sigma^{-1}]$, and hence to a functor $\tilde{F}$ on **E.** Uniqueness follows from the fact that **E** is generated by the image of **KK** $\to$ **E** (on which $\tilde{F}$ is unique by Theorem (3.1.25)) together with the inverses of certain morphisms in this image.

As an application, using the approach of [81] it is straightforward to show the existence of an associative pairing

$$E(A_1, B_1) \otimes E(A_2, B_2) \to E(A_1 \otimes A_2, B_1 \otimes B_2),$$

as long as A_1 and B_1 (or A_2 and B_2) are nuclear, or more generally K-nuclear. In the absence of some nuclearity condition the situation is unclear, the reason being that in general the tensor product

$$0 \to C \otimes j \to C \otimes A \to C \otimes B \to 0$$

of an exact sequence with an auxiliary C^*-algebra C is not necessarily exact.

Section (3.2): The Weyl Algebra:

In this section we develop a bivariant topological K-theory for arbitrary locally convex algebras. This covers a very wide class of examples and even the case of algebras over the complex numbers without a priori topology. In fact, every complex algebra A with a countable basis can be considered as a locally convex algebra with the fine topology defined by the collection of all seminorms on A.

The construction of the bivariant theory that we present here is fundamentally the same as the one in [97] where we had constructed such a theory for the category of locally convex algebras with submultiplicative seminorms. As before it is based on extensions of higher length and "classifying maps" for such extensions. We use the opportunity however to streamline the approach in [97]. Motivated by the thesis of A. Thom [98] we define the theory as noncommutative stable homotopy, i.e. as an inductive limit over suspensions of both variables (using a noncommutative suspen- sion for the first variable), rather than an inductive limit over inverse Bott maps as in [97]. This simplifies some of the arguments and also clarifies the fact that Bott periodicity becomes automatic once we stabilize by (smooth) compact operators in the second variable. As pointed out in [98], a natural framework for these constructions is the one of triangulated categories.

Just as for the theory kk in [97] we obtain a bivariant Chern-Connes character from the new bivariant K-theory $kk^{\mathrm{alg}}(A,B)$ to bivariant periodic cyclic homology $HP_*(A,B)$. In contrast to [97] however we loose part of the control over the coefficients $kk^{\mathrm{alg}}(\mathbb{C},\mathbb{C})$ of the theory. Recall that for the theory kk in [97] we had $kk(\mathbb{C},\mathbb{C}) = \mathbb{Z}$.

For the new theory the coefficient ring $R = kk^{\mathrm{alg}}(\mathbb{C},\mathbb{C})$ is a unital ring admitting a unital homomorphism into the field of complex numbers (via the identification $\mathbb{C} = HP_*(\mathbb{C},\mathbb{C})$). In particular, $R \neq 0$. We leave the determination of R open for the time being. All the abelian groups $kk^{\mathrm{alg}}(A,B)$ are modules over R and the product in kk^{alg} is R-linear. kk^{alg} has the usual properties which allow to compute $kk^{\mathrm{alg}}(A,B)$ in many cases as R-modules.

A main motivating example for our theory is the Weyl algebra W which is the unital complex algebra generated by two elements x, y satisfying the commutation relation $xy - yx = 1$. It is the example of an algebra which cannot be topologized using submultiplicative seminorms. It is easy to see that the only submultiplicative semi-norm on W is 0. We construct an extension W' of W which can be used to compute the kk^{alg}-invariants for W (as R-modules). The result is that W is isomorphic to $\mathbb{C}$ in the category kk^{alg}. Along the same lines kk^{alg}-isomorphisms can be determined for many other algebras of similar type.

Generally speaking, the information on a given locally convex algebra obtained through kk^{alg} is similar to the one obtained from periodic cyclic homology. In the first case, we obtain R-modules, in the second case vector spaces over $\mathbb{C}$. Thus torsion or divisibility phenomena cannot be controlled in both cases. Therefore, in the last section, we describe another bivariant K-theory kk which is defined on the category of locally convex algebras but which restricts to the theory kk of [97] on the category of m-algebras. In particular, we get for the coefficients of this theory $kk(\mathbb{C},\mathbb{C}) = \mathbb{Z}$ and the invariants given by the theory are $\mathbb{Z}$-modules and therefore contain more information. The drawback of the theory is the fact that it admits long exact sequences only for certain extensions and that, in general, all computations in this setting become rather technical and complicated. However, the fundamental extension that we constructed for the Weyl algebra fits into the framework of this theory, so that we can compute the kk-invariants for the Weyl algebra at least partially. We give some examples of applications to the structure of the Weyl algebra that could not be obtained from periodic cyclic homology.

Recall the definition of the projective tensor product by Grothendieck, [99], [100]. For two locally convex vector spaces V and W the projective topology on the tensor product $V \otimes W$ is given by the family of seminorms $p \otimes q$, where p is a continuous seminorm on V and q a continuous seminorm on W. Here $p \otimes q$ is defined by

$$p \otimes q(z) = \inf\left\{\sum_{i=1}^{n} p(a_i)q(b_i) : z = \sum_{i=1}^{n} a_i \otimes b_i\,, a_i \in V, b_i \in W\right\}$$

for $\in V \otimes W$. We denote by $V \widehat{\otimes} W$ the completion of $V \otimes W$ with respect to this family of seminorms.

By a locally convex algebra we mean an algebra over $\mathbb{C}$ equipped with a complete locally convex topology such that the multiplication $A \times A \to A$ is (jointly) continuous.

This means that, for every continuous seminorm α on A, there is another continuous seminorm α' such that

$$\alpha(xy) \leq \alpha'(x)\alpha'(y)$$

for all $x, y \in A$. Equivalently, the multiplication map induces a continuous linear map $A \widehat{\otimes} A \to A$ from the completed projective tensor product $A \widehat{\otimes} A$.

It is convenient to group the continuous seminorms on a locally convex algebra into sequences of the form $(\alpha_1, \alpha_2, \ldots)$ such that $\alpha_i(xy) \leq \alpha_{i+1}(x)\alpha_{i+1}(y)$ for all i. We call such a sequence a submultiplicative sequence of seminorms. Thus, a locally convex algebra is a

complex algebra with a locally convex structure determined by a family of submultiplicative sequences of seminorms.

The class of locally convex algebras is very vast. For instance it contains all algebras over $\mathbb{C}$ generated algebraically by a countable family of generators if we equip them with the "fine" locally convex topology. The fine topology on a complex vector space V is given by the family of all seminorms on V. For completeness we prove the following proposition, cf. also [101].

Proposition (3.2.1)[96]: Every algebra A over $\mathbb{C}$ with a countable basis is a locally convex algebra with the fine topology.

Proof. It is easy to see that A is complete with the fine topology. Let $(e_i)_{i\in\mathbb{N}}$ be a basis of A and α a seminorm on A. Put $b_{ij} = \alpha(e_i e_j)$. Then, given $x = \sum \lambda_i e_i$ and $y = \sum \mu_j e_j$, we have

$$\alpha(xy) \leq \sum |\lambda_i \mu_j|\, b_{ij}$$

Define inductively $\beta = \max\{\sqrt{b_{11}}, 1\}$ and

$$\beta_j = \left(\max\left\{\frac{b_{kj}}{\beta_k}, \frac{b_{jk}}{\beta_k} : k < j\right\} \cup \{\sqrt{b_{11}}, 1\}\right)$$

Then $b_{kj} \leq \beta_j \beta_k$ and $b_{jk} \leq \beta_j \beta_k$ for all $k \leq j$. Therefore, defining a seminorm β on A by

$$\beta\left(\sum \lambda_i e_i\right) = \sum |\lambda_i|\, \beta_i$$

we get that $\alpha(xy) \leq \sum |\lambda_i \mu_j|\, \beta_i \beta_j = \beta(x)\beta(y)$ for all x, y in A.

For locally convex algebras A and , the projective tensor product $A \widehat{\otimes} B$ is again a locally convex algebra.

As in [97] we will use the name m-algebra for a locally convex algebra A where the topology is given by a family of submultiplicative seminorms, i.e. seminorms α such that

$$\alpha(xy) \leq \alpha(x)\alpha(y)$$

for all $x, y \in A$. The projective tensor product of two m-algebras is again an m-algebra. Most of the standard algebras that we use in our constructions are m-algebras. In particular we will use the algebras described in the following subsections.

Let $[a, b]$ be an interval in $\mathbb{R}$. We denote by $\mathbb{C}[a, b]$ the algebra of complex-valued C^∞-functions f on $[a, b]$, all of whose derivatives vanish in a and in b (while f itself may take arbitrary values in a and b).

Also the subalgebras $\mathbb{C}(a,b]$, $\mathbb{C}[a,b)$ and $\mathbb{C}(a,b)$ of $\mathbb{C}[a,b]$, which, by definition consist of functions f , that vanish in a, in b, or in a and b, respectively, will play an important role.

The topology on these algebras is the usual Fréchet topology, which is defined by the following family of submultiplicative norms p_n:

$$p_n(f) = \|f\| + \|f'\| + \frac{1}{2}\|f''\| + \cdots + \frac{1}{n!}\left\|f^{(n)}\right\|$$

Here of course $\|g\| = \sup\{|g(t)|: t \in [a,b]\}$ and $f^{(n)}$ denotes the n-th derivative of f.

We note that $\mathbb{C}[a,b]$ is nuclear in the sense of Grothendieck [99] and that, for any complete locally convex space V, the space $\mathbb{C}[a,b] \widehat{\otimes} V$ is isomorphic to the space of C^∞-functions on $[a,b]$ with values in V, whose derivatives vanish in both endpoints,[100]. Exactly the same comments apply to the algebra $C^\infty(M)$ of smooth functions on a compact smooth manifold where an m-algebra topology is given by seminorms which are defined locally like the p_n above.

Given a locally convex algebra A, we write $A[a,b], A[a,b)$ and $A(a,b)$ for the locally convex algebras $A \widehat{\otimes} \mathbb{C}[a,b]$, $A \widehat{\otimes} \mathbb{C}[a,b)$ and $A \widehat{\otimes} \mathbb{C}(a,b)$ (their elements are A-valued C^∞-functions whose derivatives vanish at the endpoints).

Definition (3.2.2)[96]: Two continuous homomorphisms $\alpha, \beta: A \to B$ between locally convex algebras are called differentiably homotopic, or diffotopic, if there is a family $\varphi_t: A \to B$, $t \in [0,1]$, of continuous homomorphisms, such that $\varphi_0 = \alpha$, $\varphi_1 = \beta$ and such that the map $t \mapsto \varphi_t(x)$ is infinitely often differentiable for each $x \in A$. An equivalent condition is that there should be a continuous homomorphism $\varphi: A \to C^\infty([0,1]) \widehat{\otimes} B$ such that $\varphi(x)(0) = \alpha(x)$, $\varphi(x)(1) = \beta(x)$ for each $x \in A$.

Let $h: [0,1] \to [0,1]$ be a monotone and bijective C^∞-map, whose restriction to $(0,1)$ gives a diffeomorphism $(0,1) \to (0,1)$ and whose derivatives in 0 and 1 all vanish. Replacing φ_t by $\psi_t = \varphi_{h(t)}$ one sees that α and β are diffotopic if and only if there is a continuous homomorphism $\psi: A \to \mathbb{C}[0,1] \widehat{\otimes} B$ such that $\psi(x)(0) = \alpha(x)$, $\psi(x)(1) = \beta(x)$, $x \in A$. This shows in particular that diffotopy is an equivalence relation.

A locally convex algebra A is called contractible, if its identity map is diffotopic to 0, i.e. if there is a continuous homomorphism $A \to A(0,1]$ such that the composition with the evaluation map $A(0,1] \to A$ at 1 is the identity map of A. The simplest examples of contractible algebras are the algebras $\mathbb{C}(0,1]$ and the algebra $t\mathbb{C}[t]$ of polynomials without

constant term. The required map $A \to A(0,1]$ maps in both cases the generator t to the function $s \to st$ for $s \in (0,1]$.

Below we will also apply the notion of diffotopy to algebras A over $\mathbb{C}$ which we regard as a locally convex space with the topology given by all possible semi-norms on A (the fine topology). In this case $\mathbb{C}[0,1] \widehat{\otimes} A$ is simply the algebraic tensor product $\mathbb{C}[0,1] \otimes A$.

The m-algebra $\mathcal{K}$of "smooth compact operators" consists of all $\mathbb{N} \times \mathbb{N}$-matrices (a_{ij}) with rapidly decreasing matrix elements $a_{ij} \in \mathbb{C}$, $i,j = 0,1,2\ldots$. The topology on $\mathcal{K}$ is given by the family of norms p_n , $n = 0,1,2\ldots$, which are defined by

$$p_n\big((a_{ij})\big) = \sum_{i,j} |1+i|^n\,|1+j|^n |a_{ij}|$$

Thus, $\mathcal{K}$ is isomorphic to the projective tensor product $s \widehat{\otimes} s$, where s denotes the space of rapidly decreasing sequences $a = (a_i)_{i\in\mathbb{N}}$. The topology on s is determined by the seminorms $\alpha_n((a_i)) = \sum |1+i|^n\,|a_i|$. It is equipped with the natural bilinear form $\langle a|b\rangle = \sum a_i b_i$. In this description, the matrix product in $\mathcal{K}$ is given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 \otimes \langle b_1|a_2\rangle b_2$ using the canonical bilinear form. This shows that the p_n are submultiplicative. Since s is isomorphic, as a locally convex space, to the Schwartz space $\mathcal{S}(\mathbb{R})$ which in turn is isomorphic to $\mathbb{C}(0,1)$ we see that $\mathcal{K} \cong \mathbb{C}(0,1) \widehat{\otimes} \mathbb{C}(0,1)$. It is important to note that these identifications can be chosen to be compatible with the canonical bilinear forms on s, $\mathcal{S}(\mathbb{R})$ and $\mathbb{C}(0,1)$, respectively, and therefore also with the multiplication on $\mathcal{K} \cong s \widehat{\otimes} s$.

Lemma (3.2.3)[96]: There are isomorphisms $s \cong \mathcal{S}(\mathbb{R}) \cong \mathbb{C}(0,1)$ which respect the canonical bilinear forms on these spaces (where the bilinear form on $\mathcal{S}(\mathbb{R})$ and $\mathbb{C}(0,1)$ is given by $\langle f|g\rangle = \int fg$).

Proof. The isomorphism between s and $\mathcal{S}(\mathbb{R})$ is given by expansion of $f \in \mathcal{S}(\mathbb{R})$ with respect to the orthonormal basis of Hermite polynomials, while an isomorphism between $\mathcal{S}(\mathbb{R})$ and $\mathbb{C}(-\pi/2, \pi/2)$ can for instance be obtained by composition with the function arctan, cf. e.g. [102].

We call a homomorphism $\mathcal{K} \to \mathcal{K}$ canonical if it is induced by a continuous linear map $s \to s$ respecting the bilinear form. The bilinear form preserving isomorphism $s \cong \mathbb{C}(0,1)$ and the induced isomorphism $\mathcal{K} \cong \mathbb{C}(0,1) \widehat{\otimes} \mathbb{C}(0,1)$ can be used to prove the following lemma.

Lemma (3.2.4)[96]: All canonical homomorphisms $\mathcal{K} \to \mathcal{K}$ are diffotopic.

Proof. Let α be such a canonical homomorphism which is induced by the bilinear form preserving map β. It is clear that there exist continuous bilinear form preserving linear maps $\beta_1, \beta_2 : \mathbb{C}(0,1) \to \mathbb{C}(0,1)$ such that β_1 is diffotopic to the identity while β_2 is diffotopic to β (through bilinear form preserving maps) and such that $\langle \beta_1(x) | \beta_2(y) \rangle = 0$ for all x, y in $\mathbb{C}(0,1)$. Put

$$\varphi_t(x) = \cos(t)\beta_1(x) + \sin(t)\beta_2(x)$$

Then φ_t preserves the bilinear form for all t and induces a diffotopy between a canonical homomorphism diffotopic to id and one diffotopic to α. There is an obvious bilinear form preserving isomorphism $s \cong s \widehat{\otimes} s$ which induces an isomorphism $\theta : \mathcal{K} \to \mathcal{K} \widehat{\otimes} \mathcal{K}$. By the previous lemma, this isomorphism is diffotopic to the inclusion map $\iota : \mathcal{K} \to \mathcal{K} \widehat{\otimes} \mathcal{K}$ that maps x to $e_{00} \otimes x$ (where e_{00} is the matrix with elements a^{ij} , for which $a^{ij} = 1$, whenever $i = j = 0$, and $a^{ij} = 0$ otherwise).

The elements of the algebra $C^\infty S^1$ can be written as Laurent series in the generator z (defined by $z(t) = t, t \in S^1 \subset \mathbb{C}$). The coefficients of these series are rapidly decreasing. Thus

$$C^\infty(S^1) = \left\{ \sum_{k \in \mathbb{Z}} a_k z^k : \sum_{k \in \mathbb{Z}} |a_k||k|^n < \infty \text{ for all } n \in \mathbb{N} \right\}$$

whence a natural isomorphism with the space $s = s(\mathbb{Z})$ of rapidly decreasing sequences. Submultiplicative norms that determine the topology are given by

$$q_n\left(\sum a_k z^k\right) = \sum |1 + k|^n \, |a_k|$$

This topology on $C^\infty S^1$ is obviously the same as the one used for algebras of differentiable functions.

As a topological vector space, the smooth Toeplitz algebra $\mathcal{T}$ may be defined as the direct sum $\mathcal{T} = \mathcal{K} \oplus C^\infty(S^1)$. To describe the multiplication in $\mathcal{T}$, we let $\mathcal{T}$ act on $s = s(\mathbb{N})$. The elements of $\mathcal{K}$ act in the natural way while the linear combinations $\sum a_k z^k$ act by truncated convolution: For $a = \sum_{k \in \mathbb{Z}} a_k z^k$ and $\xi = (\xi_i)_{i \in \mathbb{N}} \in s$ we define $a\xi$ by

$$(a\xi)_i = \sum_{k+j=i} a_k \xi_j \quad i \in \mathbb{N}$$

In this way we represent $\mathcal{T}$ as a subspace of $\mathrm{End}(s)$ and define the multiplication on $\mathcal{T}$ as the one inherited from $\mathrm{End}(s)$. We denote by v and v^* the images of z and z^{-1} in $\mathcal{T} \subset \mathrm{End}(s)$. Then v and v^* are just the right and left shift operators on the sequence space s. We have $v^*v = 1$ and $1 - vv^* = e$ where e is the projection onto the first basis vector of s.

If p_n are the norms on $\mathcal{K}$ defined and $q_n$the norms on $C^\infty(S^1)$ defined above, it is easy to see that each norm $p_n \oplus q_n$ is submultiplicative on $\mathcal{T} = \mathcal{K} \oplus C^\infty(S^1)$ with the multiplication introduced above. Obviously, K is a closed ideal in $\mathcal{T}$and the quotient $\mathcal{T}/\mathcal{K}$ is $C^\infty(S^1)$.

Proposition (3.2.5)[96]: $\mathcal{T}$ is the universal unital m-algebra generated by two elements v and v^* satisfying the relation $v^*v = 1$ such that the assignment $z^k \mapsto v^k$ and $z^{-k} \mapsto (v^*)^k$ for $k \in \mathbb{N}$, extends to a continuous linear map $C^\infty(S^1) \to \mathcal{T}$.

For the proof see [97].

The smooth Toeplitz algebra (without this specific locally convex topology) had been used before also by Karoubi, [103]).

Let V be a complete locally convex space. Consider the algebraic tensor algebra

$$T_{alg}V = V \oplus V \otimes V \oplus V^{\otimes^3} \oplus \dots$$

with the usual product given by concatenation of tensors. There is a canonical linear map $\sigma: V \to T_{alg}V$ mapping V into the first direct summand. We equip $T_{alg}V$ with the locally convex topology given by the family of all seminorms of the form $\alpha \circ \varphi$, where φ is any homomorphism from $T_{alg}V$ into a locally convex algebra B such that $\varphi \circ \sigma$ is continuous on V , and α is a continuous seminorm on B. Equivalently, we equip $T_{alg}V$ with the locally convex topology determined by all submultiplicative sequences of seminorms $(\alpha_1, \alpha_2, \dots)$ where all α_i are continuous on $\subset T_{alg}V$. We further denote by TV the completion of $T_{alg}V$ with respect to this locally convex structure.

A more concrete description of the locally convex structure on TV has been determined by Valqui, [104], who used this free extension to generalize the proof of excision in periodic cyclic homology from m-algebras to general locally convex algebras.

- TV is a locally convex algebra.
- Every continuous linear map $s: V \to B$ into a locally convex algebra B induces a continuous homomorphism $\gamma_s: TV \to B$ defined by

$$\gamma_s(x_1 \otimes \ldots \otimes x_n) = s(x_1)s(x_2) \ldots s(x_n)$$

In particular, TV is contractible.

- If A is a locally convex algebra, then the canonical homomorphism $TA \to A$ induced by the identity map of A is continuous.

For any locally convex algebra A we have the natural extension

$$0 \to JA \to TA \xrightarrow{\pi} A \to 0. \qquad (8)$$

Here π maps a tensor $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $x_1 x_2 \cdots x_n \in A$ and JA is defined as Ker π. This extension is (uni)versal in the sense that, given any extension $0 \to I \to E \to B \to 0$ of a locally convex algebra B, admitting a continuous linear splitting, and any continuous homomorphism $\alpha: A \to B$, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & JA & \to & TA & \to & A & \to & 0 \\ & & \downarrow \gamma & & \downarrow \tau & & \downarrow \alpha & & \\ 0 & \to & I & \to & E & \to & B & \to & 0 \end{array} \qquad (9)$$

The map $\tau: TA \to E$ is obtained by choosing a continuous linear splitting $s: B \to E$ in the given extension and mapping $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $s'(x_1)s'(x_2) \cdots s'(x_n) \in E$, where $s' := s \circ \alpha$. Then γ is the restriction of τ.

Choosing $0 \to JB \to TB \to B \to 0$ in place of the second extension $0 \to I \to E \to B \to 0$, we see that $A \to JA$ is a functor, i.e. any homomorphism $\alpha: A \to B$ induces a homomorphism $J(\alpha): JA \to JB$.

Definition (3.2.6)[96]: The map $\gamma: JA \to I$ in the commutative diagram (9) is called the classifying map (associated to the extension together with the homomorphism α). In the special case where $B = A$ and $\alpha = \mathrm{id}$, we simply call γ the classifying map for the extension $0 \to I \to E \to A \to 0$.

If s and s¯ are two different continuous linear splittings for a given extension, then so is $s_t = ts + (1-t)\bar{s}$ for each t in $[0,1]$. This differentiable family of splittings induces a diffotopy between the classifying maps defined from s or $\bar{s}$. Therefore the classifying map is unique up to homotopy.

Classifying maps can also be defined for linearly split extensions of higher length of the form

$$0 \to I \to E_1 \to E_2 \to \cdots \to E_n \to A \to 0$$

We consider such an extension as a complex, denoting the arrows (boundary maps) by π_i and we say that it is linearly split if there is a continuous linear map s of degree -1 such that $s\pi + \pi s = \mathrm{id}$. Every such splitting s induces a commutative diagram of the form

$$\begin{array}{ccccccccccccc} 0 & \to & I & \to & E_1 & \to & E_2 & \to & \cdots & \to & E_n & \to & A & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \cdots & & \uparrow & & \uparrow & & \\ 0 & \to & J^n A & \to & T\,(J^{n-1}A) & \to & T\,(J^{n-2}A) & \to & \cdots & \to & TA & \to & A & \to & 0 \end{array}$$

The leftmost vertical arrow in this diagram is the classifying map for this n-step extension. It depends on s only up to diffotopy.

Definition (3.2.7)[96]; Let A and B be locally convex algebras. For any continuous homomorphism $\varphi: A \to B$, we denote by $\langle\varphi\rangle$the equivalence class of φ with respect to diffotopy and we set

$$\langle A, B\rangle = \{\langle\varphi\rangle | \varphi \text{ is a continuous homomorphism } A \to B\}$$

Note that we work with diffotopy rather than with continuous homotopy since we want to construct a character into cyclic theory and since cyclic theory is invariant only under differentiable homotopies.

We denote by $\mathbb{C}(0,1)^n \cong \mathbb{C}(0,1) \widehat{\otimes} \ldots \widehat{\otimes}\, \mathbb{C}(0,1)$ the algebra of infinitely differentiable functions on the n-cube $[0,1]^n$ whose derivatives vanish on the boundary. For every locally convex algebra B and every $n \geq 1$, there is a canonical map $\varphi: B(0,1)^n \oplus B(0,1)^n \to B(0,1)^n$, which is unique up to diffotopy. For any second locally convex algebra A, this makes $\langle A, B(0,1)^n\rangle$ into an abelian group in the usual way by putting $\langle\alpha\rangle + \langle\beta\rangle = \varphi \circ (\alpha \oplus \beta)$.

Moreover, for each k in $\mathbb{N}$, there is a canonical map

$$\langle J^k A, \mathcal{K} \widehat{\otimes} B(0,1)^k\rangle \to \langle J^{k+1}A, \mathcal{K} \widehat{\otimes} B(0,1)^{k+1}\rangle$$

mapping the diffotopy class of α to the diffotopy class of α', where α' is defined by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J^{k+1}A & \to & TJ^kA & \to & J^kA & \to & 0 \\ & & \downarrow \alpha' & & \downarrow \tau & & \downarrow \alpha & & \\ 0 & \to & B'(0,1)^{k+1} & \to & B'(0,1)^k[0,1) & \to & B'(0,1)^k & \to & 0 \end{array}$$

where $B' = \mathcal{K} \widehat{\otimes} B$.

Definition (3.2.8)[96]: Let A and B be locally convex algebras. We define

$$kk^{\mathrm{alg}}(A,B) = \lim_{\vec{k}} \langle J^k A, \mathcal{K} \widehat{\otimes} B(0,1)^k \rangle$$

Remark (3.2.9)[96]: An alternative and basically equivalent definition of kk^{alg} could be based on equivalence classes of extensions of arbitrary length, rather than on their classifying maps. Such an alternative approach has been discussed, in the case of C^*-algebras, in [98].

We can also define abelian groups kk^{alg} for each n in $\mathbb{Z}$ by

$$kk^{\mathrm{alg}}(A,B) = \lim_{\vec{k}} \langle J^{k-n} A, \mathcal{K} \widehat{\otimes} B(0,1)^k \rangle$$

where the inductive limit is taken over all $k \in \mathbb{N}$ such that $k - n \geq 0$. Thus, for n in $\mathbb{N}$,

$$kk_n^{\mathrm{alg}}(A,B) = kk^{\mathrm{alg}}(J^n A, B), \quad kk_{-n}^{\mathrm{alg}}(A,B) = kk^{\mathrm{alg}}(A, B(0,1)^n)$$

There is a natural product $kk^{\mathrm{alg}}(A_1,A_2) \times kk^{\mathrm{alg}}(A_2,A_3) \to kk^{\mathrm{alg}}(A_1,A_3)$ defined in the following way. Assume that two elements a in $kk^{\mathrm{alg}}(A_1,A_2)$ and b in $kk^{\mathrm{alg}}(A_2,A_3)$ are represented by $\alpha: J^n A_1 \to \mathcal{K} \widehat{\otimes} A_2(0,1)^n$ and $\beta: J^m A_2 \to \mathcal{K} \widehat{\otimes} A_3(0,1)^m$, respectively. We define their product $a \cdot b$ in $kk^{\mathrm{alg}}(A_1,A_2)$ by the following composition of maps

$$J^m(J^n A_1) \xrightarrow{J^m(\alpha)} J^m\big(\mathcal{K} \widehat{\otimes} A_2(0,1)^n\big) \to \mathcal{K} \widehat{\otimes} (J^m A_2)(0,1)^n \xrightarrow{\beta(0,1)^n} \mathcal{K} \widehat{\otimes} A_3(0,1)^{n+m}$$

The arrow in the middle is obtained from the natural map $J(D \widehat{\otimes} E) \to J(D) \widehat{\otimes} E$ (choosing $E = \mathcal{K}(0,1)$) that exists for any two locally convex algebras D and E, while the maps $J^m(\alpha)$ and $\beta(0,1)^n$ are the natural maps induced by α and β, respectively. The product is well defined (i.e. does not depend on the choice of representatives for a and b in the inductive limits defining kk^{alg}) because of the following lemma

Lemma (3.2.10)[96]: (cf. [98]). For any locally convex algebra D we denote by $\rho_D: JD \to D(0,1)$ the classifying map for the suspension extension

$$0 \to D(0,1) \to D[0,1) \to D \to 0.$$

(a) Let

$$0 \to I \to E \xrightarrow{\pi} A \to 0$$

be a linearly split extension of locally convex algebras and α its classifying map. Then the two maps $J^2 A \to I(0,1)$ defined by the compositions $J^2 A \xrightarrow{J(\alpha)} JI \xrightarrow{\rho_I} I(0,1)$ and $J^2 A \xrightarrow{\rho_{JA}} (JA)(0,1) \xrightarrow{\alpha(0,1)} I(0,1)$ are diffotopic.

(b) The natural map $\rho_{JA}: J^2A \to (JA)(0,1)$ is diffotopic to the composition $J^2A \to \big(JA(0,1)\big) \to (JA)(0,1)$

Proof. (a) The two compositions are classifying maps for the two 2-step extensions in the first and last row of the following diagram

$$\begin{array}{ccccccccccc}
0 & \to & I(0,1) & \to & I[0,1) & \to & E & \to & A & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I(0,1) & \to & E[0,1) & \to & Z_\pi & \to & A & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I(0,1) & \to & E(0,1) & \to & A[0,1) & \to & A & \to & 0
\end{array}$$

where $Z_\pi$is the mapping cylinder. They are diffotopic since they also are classifying maps for the extension in the middle row.

(b) The first map is the classifying map for the first row in the following commutative diagram

$$\begin{array}{ccccccccccc}
0 & \to & (JA)(0,1) & \to & (TA)(0,1) & \to & A[0,1) & \to & A & \to & 0 \\
 & & \uparrow \varphi & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & J\big(A(0,1)\big) & \to & T\big(A(0,1)\big) & \to & A[0,1) & \to & A & \to & 0
\end{array}$$

while the second map is the composition of the classifying map for the second row composed with the first vertical arrow φ in the diagram - thus also a classifying map for the first row.

Note that every continuous homomorphism $\alpha: A \to B$ induces an element $kk(\alpha)$ in $kk^{\mathrm{alg}}(A,B)$ and that $(\beta \circ \alpha) = kk(\alpha) \cdot kk(\beta)$. We denote by 1_A the element $kk(\mathrm{id}_A)$ induced by the identity homomorphism of A. It is a unit in the ring $kk^{\mathrm{alg}}(A,A)$.

More generally, of course, we obtain a bilinear product $kk_s^{\mathrm{alg}}(A_1,A_2) \times kk_t^{\mathrm{alg}}(A_2,A_3) \to kk_{s+t}^{\mathrm{alg}}(A_1,A_3)$. Every n-step extension

$$0 \to I \to E_1 \to E_2 \to \ldots \to E_n \to A \to 0$$

defines via its classifying map an element in $kk_{-n}^{\mathrm{alg}}(A,I)$. Given two such extensions of length n and m, the product of the corresponding elements in kk_{-n}^{alg} and kk_{-m}^{alg} is represented by the classifying map for the Yoneda product (concatenation) of the two extensions.

Remark (3.2.11)[96]: The definition that we gave for kk^{alg} furnishes an abelian group also without the tensor product by $\mathcal{K}$ in the second variable and the (bilinear) product can be defined in exactly the same way. We refer to [98] for a thorough discussion of the theory obtained without stabilization or only stabilization by finite matrices on the right hand side,

in the case of C^*-algebras. If we stabilize however by $\mathcal{K}$ on the right hand side we have a second possibility to define the addition in $kk^{\mathrm{alg}}(A,B)$ (this second definition was used in [97]). For two continuous homomorphisms $\alpha,\beta: D \to \mathcal{K} \widehat{\otimes} E$ we define the direct sum $\alpha \oplus \beta$ as

$$\alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : D \to M_2(\mathcal{K} \widehat{\otimes} E) \cong \mathcal{K} \widehat{\otimes} E$$

With the addition defined by $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha \oplus \beta\rangle$ the set $\langle D, \mathcal{K} \widehat{\otimes} E\rangle$ of diffotopy classes of homomorphisms from D to $\mathcal{K} \widehat{\otimes} E$ is an abelian semigroup with 0-element $\langle 0\rangle$. It is easy to see that this definition of addition coincides with the one given above in the case where $E = \mathcal{K} \widehat{\otimes} B(0,1)^n$ for $n \geq 1$.

In the case where we stabilize by $\mathcal{K}$on the right hand side, we can also use the following "basic lemma" from [97] to show that the product is well defined.

Lemma (3.2.12)[96]: Assume given a commutative diagram of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I & \to & A_{01} & \to & A_{02} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{10} & \to & A_{11} & \to & A_{12} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{20} & \to & A_{21} & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all the rows and columns represent extensions of locally convex algebras with continuous linear splittings.

Let γ_+ and γ_- denote the classifying maps $J^2B \to I$ for the two extensions of length 2

$$0 \to I \to A_{10} \to A_{21} \to B \to 0$$

$$0 \to I \to A_{01} \to A_{12} \to B \to 0$$

associated with the two edges of the diagram. Then $\gamma_+ \oplus \gamma_-: J^2B \to M_2(I)$ is diffotopic to 0.

Proof. We assume first that the extension $0 \to I \to A_{20} \to A_{21} \to B \to 0$ is trivial (i.e. admits a homomorphism splitting) and show that then $\gamma_+ \sim 0$ ($\gamma_- = 0$ is obvious in this case). By definition, γ_+ is constructed as a classifying map from the following diagram

$$\begin{array}{ccccccccc} 0 & \to & I & \to & A_{01} & \to & A_{02} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow \alpha & & \\ 0 & \to & J^2B & \to & TJB & \to & JB & \to & 0 \end{array}$$

where α is the classifying map for the extension

$$0 \to A_{02} \to A_{12} \to B \to 0$$

The commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A_{01} & \to & A_{11} & \to & A_{21} & \to & 0 \\ & & \downarrow \pi & & \downarrow & & \downarrow & & \\ 0 & \to & A_{02} & \to & A_{12} & \to & B & \to & 0 \end{array}$$

with the splitting homomorphism $\beta: B \to A_{21}$ shows that $\alpha: JB \to A_{02}$ can be lifted to a homomorphism $\alpha': JB \to A_{01}$ such that $\pi \circ \alpha' = \alpha$. In fact, choose α' as the classifying map for

$$\begin{array}{ccccccccc} 0 & \to & A_{01} & \to & A_{11} & \to & A_{21} & \to & 0 \\ & & & & & & \uparrow \beta & & \\ & & & & & & B & & \end{array}$$

It follows that $\gamma_+ \sim 0$.

Returning now to the general case, consider the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I & \to & I & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{01} + A_{10} & \to & A_{11} & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{02} + A_{20} & \to & A_{11}/I & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

From the first step it follows that the classifying map for the (linearly split) extension

$$0 \to I \to A_{01} + A_{10} \to A_{11}/I \to B \to 0$$

is diffotopic to 0.

An obvious rotation argument shows that the composition of the classifying map $J^2B \to I$ for the extension

$$\begin{array}{ccccccccc} 0 & \to & I & \to & A_{01} + A_{10} & \to & A_{02} \oplus A_{20} & \to & 0 \\ & & & & & & \uparrow & & \\ & & & & & & JB & & \end{array}$$

with the inclusion $I \to M^2(I)$ is diffotopic to the map $x \mapsto \begin{pmatrix} \gamma_+(x) & 0 \\ 0 & \gamma_-(x) \end{pmatrix}$ where $\gamma_+, \gamma_- : J^2B \to I$ are the classifying maps associated with the diagrams

$$\begin{array}{ccccccccc} 0 & \to & I & \to & A_{01} & \to & A_{02} & \to & 0 \\ & & & & & & \uparrow \eta_+ & & \\ & & & & & & JB & & \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & I & \to & A_{10} & \to & A_{20} & \to & 0 \\ & & & & & & \uparrow \eta_- & & \\ & & & & & & JB & & \end{array}$$

and η_+, η_- are classifying maps for the extensions $0 \to A_{02} \to A_{11} \to B \to 0$ and $0 \to A_{20} \to A_{11} \to B \to 0$.

This lemma is much stronger than Lemma (3.2.10) in the case where the right hand side is stabilized by $\mathcal{K}$. It also shows that the product of the elements of kk^{alg} induced by the classifying maps for the extensions $0 \to I \to A_{01} \to A_{02} \to 0$ and $0 \to A_{02} \to A_{12} \to B \to 0$ is the negative of the product of the elements of kk_{-1}^{alg} induced by the classifying maps for the extensions $0 \to I \to A_{10} \to A_{20} \to 0$ and $0 \to A_{20} \to A_{21} \to B \to 0$.

kk_*^{alg} is best viewed as an additive category whose objects are locally convex algebras and whose (graded) morphism sets between objects A and B are $kk_*^{\mathrm{alg}}(A, B)$. The map $\alpha \to kk(\alpha)$ is a functor from the category of locally convex algebras to the category k_*^{alg} .

Proposition (3.2.13)[96]: Let A be a locally convex algebra. There is an invertible element in $kk^{\mathrm{alg}}(JA, A(0,1))$ giving rise to an isomorphism $kk^{\mathrm{alg}}(JA, B) \cong kk^{\mathrm{alg}}(A(0,1), \mathrm{B})$.

Proof. The natural map $J(A(0,1)) \to (JA)(0,1)$ can be viewed as defining an element in $kk^{\mathrm{alg}}(A(0,1), JA)$. The classifying map for the suspension extension

$$0 \to A(0,1) \to A[0,1) \to A \to 0$$

can be viewed as defining an element in $kk^{\mathrm{alg}}(JA, A(0,1))$. The products of these two elements give the identity elements in $kk^{\mathrm{alg}}(JA, JA)$ and $kk^{\mathrm{alg}}(A(0,1), A(0,))$. As a consequence we see that

$$kk^{\mathrm{alg}}(A(0,1), B(0,1)) \cong kk^{\mathrm{alg}}(J\,A, B(0,1)) \cong kk^{\mathrm{alg}}(A, B)$$

Let $\alpha: A \to B$ be a continuous homomorphism between locally convex algebras. The mapping cone $C_\alpha \subset A \oplus B[0,1)$ is defined to be the set of pairs $(x, f) \in A \oplus B[0,1)$ such that $f(0) = \alpha(x)$.

Lemma (3.2.14)[96]: Let D be a locally convex algebra and $\alpha: A \to B$ a continuous homomorphism between locally convex algebras

(a) The sequence

$$kk_*^{\mathrm{alg}}(D, C_\alpha) \xrightarrow{\cdot kk(\pi)} kk_*^{\mathrm{alg}}(D, A) \xrightarrow{\cdot kk(\alpha)} kk_*^{\mathrm{alg}}(D, B)$$

is exact. Here $\pi: C_\alpha \to A$ denotes the projection onto the first summand and $.kk(\pi)$ multiplication on the right by $kk(\pi)$.

(b) The sequence in (a) can be extended to an exact sequence

$$\xrightarrow{\cdot kk(\pi(0,1))} kk_*^{\mathrm{alg}}\big(D, A(0,1)\big) \xrightarrow{\cdot kk(\alpha(0,1))} kk_*^{\mathrm{alg}}\big(D, B(0,1)\big)$$
$$\to kk_*^{\mathrm{alg}}(D, C_\alpha) \xrightarrow{\cdot kk(\pi)} kk_*^{\mathrm{alg}}(D, A) \xrightarrow{\cdot kk(\alpha)} kk_*^{\mathrm{alg}}(D, B)$$

Proof. (a) follows directly from the definition of the mapping cone. In fact, let $\alpha': X \to Y$ be any continuous homomorphism between locally convex algebras and φ a homomorphism from D to X . Then $\alpha \circ \varphi$ is diffotopic to 0 if and only if there is a homomorphism $\tilde{\varphi}: D \to C_{\alpha'}$ such that $\varphi = \pi \circ \tilde{\varphi}$.

(b) follows by a standard argument from the iteration of the construction under (a). One uses the fact that the mapping cone C_π for the projection $\pi: C_\alpha \to A$ is diffotopic to $B(0,1)$ and the following commutative diagram

$$\begin{array}{ccc} C_\iota & \xrightarrow{\pi'} & C_\alpha \\ \downarrow & & \| \\ B(0,1) & \xrightarrow{\iota} & C_\alpha \end{array}$$

In this diagram ι is the inclusion of $B(0,1)$ into the second component of $C_\alpha$and the first vertical arrow is a diffotopy equivalence (it maps $f \in B(0,1)$ to $(\iota f, 0) \in C\pi \subset C\alpha \oplus A[0,1))$.

Similarly, the mapping cone C_ι for $\iota: B(0,1) \to C_\alpha$ is contained in $A(0,1) \oplus B([0,1) \times [0,1))$. The projection $C_\iota \to A(0,1)$ is a diffotopy equivalence making the following diagram commute

$$\begin{array}{ccc} C_\iota & \longrightarrow & B(0,1) \\ \downarrow & & \| \\ A(0,1) & \xrightarrow{\alpha(0,1)} & B(0,1 \end{array}$$

Lemma (3.2.15)[96]: Let $\alpha: A \to B$ and D be as in Lemma (3.2.12)

(a) The sequence

$$kk_*^{\mathrm{alg}}(C_\alpha, D) \xleftarrow{kk(\pi)\cdot} kk_*^{\mathrm{alg}}(A, D) \xleftarrow{kk(\alpha)\cdot} kk_*^{\mathrm{alg}}(B, D)$$

is exact.

(b) The sequence in (a) can be extended to a long exact sequence of the form

$$\xleftarrow{kk(\pi(0,1))\cdot} kk_*^{\mathrm{alg}}(A(0,1), D) \xleftarrow{kk(\alpha(0,1))\cdot} kk_*^{\mathrm{alg}}(B(0,1), D)$$

$$\leftarrow kk_*^{\mathrm{alg}}(C_\alpha, D) \xleftarrow{kk(\pi)\cdot} kk_*^{\mathrm{alg}}(A, D) \xleftarrow{kk(\alpha)\cdot} kk_*^{\mathrm{alg}}(B, D)$$

Proof. Let $\varphi: J^n A \to \mathcal{K} \widehat{\otimes} D(0,1)^n$ represent an element z of $kk^{\mathrm{alg}}(A, D)$ such that $\varphi \circ J^n(\pi)$ is diffotopic to 0. To simplify notation, we write D' for $\mathcal{K} \widehat{\otimes} D(0,1)^n$. The fact that $\varphi \circ J^n(\pi)$ is diffotopic to 0 means that there is a homomorphism γ such that the following diagram is commutative

$$\begin{array}{ccc} D'[0,1) & \to & D' \\ \uparrow \gamma & & \uparrow \varphi \\ J^n C_\alpha & \xrightarrow{J^n(\pi)} & J^n A \end{array}$$

The restriction of γ defines a map γ' such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \to & D'(0,1) & \to & D'[0,1) & \to & D' & \to & 0 \\ & & \uparrow \gamma' & & \uparrow \gamma & & \uparrow \varphi & & \\ 0 & \to & J^n(B(0,1)) & \to & J^n C_\alpha & \xrightarrow{J^n(\pi)} & J^n A & \to & 0 \end{array}$$

Let $\kappa_B: J^{n+1} B \to J^n(B(0,1))$ denote the natural map and let z' denote the element of $kk^{\mathrm{alg}}(B, D)$ defined by $' \circ \kappa_B$.

Define ψ by the commutative diagram

$$\begin{array}{ccc} & & J^n(A(0,1)) \\ & & \downarrow \mu_t \\ C_{J^n(\pi)} & \subset & J^n A(-1,0] \oplus J^n(C_\alpha) \\ \downarrow \varphi & & \downarrow \varphi(-1,0] \oplus \gamma \\ D'(-1,1) & \subset & D'(-1,0] \oplus D'[0,1) \end{array}$$

There is a differentiable family of homomorphisms $\mu_t: J^n(A(0,1)) \to C_{J^n(\pi)}$ such that μ_0 is the canonical map $J^n(A(0,1)) \to (J^n A)(0,1)$ composed with the natural inclusion

$$(J^n A)(0,1) \cong (J^n A)(-1,0) \to C_{J^n(\pi)}$$

while μ_1 is the composition of $J^n(\alpha(0,1))$ with the map $J^n(B(0,1)) \to J^n(C_\alpha)$ induced by the inclusion $B(0,1) \to C_\alpha$ (we use here a straightforward modification of the standard homotopies for maps $A(0,1) \to C_\pi$ which also appear in the proof of 5.1 (b).

Let now finally $\kappa_A: J^{n+1}A \to J^n(B(0,1))$ denote the natural map. Then $\psi \circ \mu_t \circ \kappa_A$ defines a diffotopy between the image of φ in $\langle J^{n+1}A, \mathcal{K} \widehat{\otimes} D(0,1)^{n+1} \rangle$ and the composition $\gamma' \circ \kappa_B \circ J^n(\alpha)$.

Thus $z = kk(\alpha) \cdot z'$ as desired.

Proposition (3.2.16)[96]: Let $0 \to I \to A \xrightarrow{q} B \to 0$ be a linearly split extension and $e: I \to C_q$ the inclusion map defined by $e: x \mapsto (x, 0) \in C_q \subset A \oplus B[0,1)$. Then $kk(e)$ is an invertible element in $kk_0^{\mathrm{alg}}(I, C_q)$.

Proof. We show that the inverse is given by the element u of $kk_0^{\mathrm{alg}}(I, C_q)$ determined by the classifying map $JC_q \to I$ for the extension

$$0 \to I(0,1) \to A[0,1) \to C_q \to 0$$

The commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I_Q(0,1) & \to & I_q[0,1) & \to & C_q & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow e & & \\ 0 & \to & I(0,1) & \to & I[0,1) & \to & I & \to & 0 \end{array}$$

with the obvious maps shows that $kk(e) \cdot u = 1I$.

On the other hand, let $\Delta: A[0,1) \to A[0,1)[0,1)$ be the diagonal map defined by $(\Delta f)(x,y) = f(|(x,y)|)$ for $(x,y) \in [0,1] \times [0,1]$. Let also $\psi: A[0,1)[0,1) \to C_q[0,1)$ be the map induced by the obvious quotient map $[0,1) \to C_q$.

Defining e' as $\psi \circ \Delta$ we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C_q(0,1) & \to & C_q[0,1) & \to & C_q & \to & 0 \\ & & \uparrow e(0,1) & & \uparrow e' & & || & & \\ 0 & \to & I(0,1) & \to & A[0,1) & \to & C_q & \to & 0 \end{array}$$

which shows that $u \cdot kk(e) = 1_{C_q}$.

Theorem (3.2.17)[96]: Let D be any locally convex algebra. Every extension admitting a continuous linear section

$$E: 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$$

induces exact sequences in $kk^{\mathrm{alg}}(D,\cdot)$ and $kk^{\mathrm{alg}}(\cdot, D)$ of the following form:

$$\cdots \to kk_1^{\mathrm{alg}}(D,A) \xrightarrow{\cdot kk(q)} kk_1^{\mathrm{alg}}(D,B)$$

$$\to kk_0^{\mathrm{alg}}(D,I) \xrightarrow{\cdot kk(i)} kk_0^{\mathrm{alg}}(D,A) \xrightarrow{\cdot kk(q)} kk_*^{\mathrm{alg}}(D,B) \to \cdots$$

and

$$\cdots \leftarrow kk_{-1}^{\mathrm{alg}}(D,A) \xleftarrow{\cdot kk(q)} kk_{-1}^{\mathrm{alg}}(D,B)$$

$$\leftarrow kk_0^{\mathrm{alg}}(D,I) \xleftarrow{\cdot kk(i)} kk_0^{\mathrm{alg}}(D,A) \xleftarrow{\cdot kk(q)} kk_*^{\mathrm{alg}}(D,B) \leftarrow \cdots$$

The given extension E defines a classifying map $JB \to I$ and thus an element of $kk_{-1}^{\mathrm{alg}}(B,I)$, which we denote by $kk(E)$. The boundary maps are (up to a sign) given by right and left multiplication, respectively, by this class $kk(E)$.

Proof. The long exact sequences follow from a combination of Lemmas (3.2.12), (3.2.13) and Proposition (3.2.14). To determine the boundary maps, consider the mapping cylinder $Z_q \subset A \oplus B[0,1]$ consisting of all pairs $(x,f) \in A \oplus B[0,1]$ such that $f(0) = q(x)$.

The commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & B(0,1) & \to & B[0,1) & \to & B & \to & 0 \\
 & & \downarrow & & & & || & & \\
0 & \to & C_q & \to & Z_q & \to & B & \to & 0 \\
 & & \uparrow e & & \uparrow & & || & & \\
0 & \to & I & \to & A & \to & B & \to & 0
\end{array}$$

shows that $kk(E) \cdot kk(e)$ equals $kk(j)$ in $kk_0^{\mathrm{alg}}(B(0,1), C_q) \cong kk_1^{\mathrm{alg}}(B, C_q)$ where $j: B(0,1) \to C_q$ is the natural inclusion. By construction, the boundary map in Lemmas (3.2.12), (3.2.13) is multiplication by $kk(j)$. On the other hand, the identification between I and C_q in Proposition (3.2.14) is given by multiplication with $kk(e)$.

A locally convex algebra C is called k-contractible if it is isomorphic in the category kk^{alg} to 0, or equivalently if $kk_0^{\mathrm{alg}}(C,C) = 0$ (which implies that $kk_n^{\mathrm{alg}}(A,C) = kk_n^{\mathrm{alg}}(C,A) = 0$ for all n and A). Of course, everycontactible algebra is k-contractible.

Corollary (3.2.18)[96]: Let $E: 0 \to I \to C \to B \to 0$ be a linearly split extension where C is k-contractible. Then the element $k(E) \in kk_{-1}^{\mathrm{alg}}(B, I)$ is invertible, i.e. there is $u \in kk_{-1}^{\mathrm{alg}}(B, I)$ such that $kk(E) \cdot u = 1_B$ and $u \cdot kk(E) = 1_I$.

More generally, if γ is the classifying map for a linearly split n-step extension

$$0 \to I \to C_1 \to \ldots \to C_n \to B \to 0$$

where $C_1, \ldots, C_n$ are k-contractible, then the element

$$kk(\gamma) \in kk_0^{\mathrm{alg}}(J^n(B), I) \cong kk_{-n}^{\mathrm{alg}}(B, I)$$

is invertible.

Proof. Since every n-step extension is a Yoneda product of 1-step extensions and $kk(\gamma)$ is the product of the kk^{alg}-elements corresponding to the corresponding classifying maps, it suffices to prove the assertion for the first case of one-step extensions. In this case the product by $kk(E)$ gives by Lemma (3.2.12) the boundary map $kk_1^{\mathrm{alg}}(D, B) \to kk_0^{\mathrm{alg}}(D, I)$ associated to the extension E for all D, thus also for $D = I$. Since, C is k-contractible, this boundary map must be an isomorphism. Therefore there is an element $u \in kk_1^{\mathrm{alg}}(I, B)$ such that $\cdot\, kk(E) = 1_I$.

Similary the boundary map$kk_1^{\mathrm{alg}}(I, B) \to kk_0^{\mathrm{alg}}(B, B)$, given by left multip-lication by $kk(E)$ in Lemma (3.2.13) is an isomorphism. Therefore there is an element $u' \in kk_1^{\mathrm{alg}}(I, B)$ such that $kk(E) \cdot u' = 1_B$

Let $\bar{\alpha}$ and α be two homomorphisms $A \to E$ between locally convex algebras.

Assume that B is a closed subalgebra of E such that $\alpha(x) - \bar{\alpha}(x) \in B$ and $\alpha(x)B \subset B$, $B\alpha(x) \subset B$ for all $x \in A$. We call such a pair $(\alpha, \bar{\alpha})$, a quasi-homomorphism from A to B. We construct an element $kk(\alpha, \bar{\alpha})$ of $kk^{\mathrm{alg}}(A, B)$ in the following way. Define $\alpha', \bar{\alpha}': A \to A \oplus E$ by $\alpha'(x) = (x, \alpha(x))$, $\bar{\alpha}' = (x, \alpha'(x))$ and E' as the subalgebra of $E \oplus A$ generated by all elements $\alpha'(x)$, $x \in A$ and by $0 \oplus B$. Let

$$D = \{f \in E'[0,1] \exists x \in A, f(0) = \alpha'(x), f(1) = \bar{\alpha}'(x)\}$$

Then the classifying map for the linearly split extension

$$0 \to B(0,1) \to D \to A \to 0$$

defines the element $kk(\alpha, \bar{\alpha})$ of $kk^{\mathrm{alg}}(A, B)$.

This construction is obviously invariant under diffotopy: If α and $\bar{\alpha}$ are two homomorphisms $A \to E[0,1]$ such that $\alpha(x) - \bar{\alpha}(x) \in B[0,1]$ and $\alpha(x)B[0,1] \subset B[0,1]$, $B[0,1]\alpha(x) \subset B[0,1]$ for all $x \in A$ then $kk(\alpha_t, \bar{\alpha}_t)$ is constant.

Remark (3.2.19)[96]: It suffices to assume that $\alpha(x) - \bar{\alpha}(x) \in B$ and $\alpha(x)B \subset B, B\alpha(x) \subset B$ for all x in a generating subset G of A.

Lemma (3.2.20)[96]: Let $\varphi = \alpha - \bar{\alpha}$. If $\varphi(x)\bar{\alpha}(y) = \bar{\alpha}(y)\phi(x) = 0$ for all $x, y, \in A$, then φ is a homomorphism and $kk(\alpha, \bar{\alpha}) = kk(\varphi)$.

Proof. The first assertion is obvious and the second one follows from the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B(0,1) & \to & D & \to & A & \to & 0 \\ & & \uparrow \psi(0,1) & & \uparrow \psi & & \| & & \\ 0 & \to & A(0,1) & \to & A[0,1) & \to & A & \to & 0 \end{array}$$

where $\psi(f)(t) = \bar{\alpha}(f(0)) + \varphi(f(t))$.

Proposition (3.2.21)[96]: Let E be a locally convex algebra containing $\mathcal{K} \widehat{\otimes} A$ as a closed ideal and $\alpha, \bar{\alpha}: A \to E[0,1]$ a pair of homomorphisms such that $\alpha(x) - \bar{\alpha}(x) \in \mathcal{K} \widehat{\otimes} A[0,1)$ for all x in A.

Assume further that $\varphi = \alpha_0 - \bar{\alpha}_0$ is a homomorphism, satisfies the assumptions of Lemma (3.2.17) and is diffotopic to the natural inclusion map $\iota: A \to \mathcal{K} \widehat{\otimes} A$. Then A is k-contractible in the sense of (3.2.18).

Proof. We have $0 = kk(\alpha_1, \bar{\alpha}_1) = kk(\alpha_0, \bar{\alpha}_0) = kk(\iota)$. On the other hand $kk(\iota)$ is an invertible element in $kk^{\text{alg}}(A, \mathcal{K} \widehat{\otimes} A)$.

We discuss in this section a rather weak form of Morita equivalence which is however sufficient for our purposes in later sections.

Definition (3.2.22)[96]: Let A and B be locally convex algebras. A Morita context from A to B is a locally convex algebra E containing A and B as subalgebras and containing two sequences (ξ_i) and (η_j) in E satisfying

(a) $\eta_j A \xi_i \subset B$ for all i, j.

(b) the family $(\eta_j a \xi_i)$ is rapidly decreasing for each $a \in A$ (i.e. $\alpha(\eta_j a \xi_i)$ is rapidly decreasing in i and j for any continuous seminorm α on B).

(c) $(\sum \xi_i \eta_i)a = a$ for all $a \in A$ (i.e the partial sums converge in A)

Given a Morita context, we can define a continuous homomorphism $A \to \mathcal{K} \widehat{\otimes} B$ by

$$a \mapsto (\eta_i a \xi_j)$$

where we write the elements of $\mathcal{K} \widehat{\otimes} B$ as $\mathbb{N} \times \mathbb{N}$-matrices. Thus, in particular, we obtain an element of $kk^{\mathrm{alg}}(A, B)$ which we denote by $kk\left((\xi_i), (\eta_j)\right)$.

Lemma (3.2.23)[96]: Every Morita context $\left((\xi_i), (\eta_j)\right)$ from A to B induces an element $kk\left((\xi_i), (\eta_j)\right)$ of $kk^{\mathrm{alg}}(A, B)$. If $((\xi_l'), (\eta_k'))$ is a Morita context from B to A realized in the same locally convex algebra E and if in addition $A\xi_i\xi_l' \subset A$, $\eta_k'\eta_j A \subset A$, for all i, j, k, l, then

$$kk((\xi_i), (\eta_j)) \cdot kk((\xi_l'), (\eta_k')) = 1_A$$

If also $B\xi_l'\xi_i \subset B, \eta_k\eta_j' B \subset B$ for all i, j, k, l, then A and B are kk^{alg}-equivalent.

Proof. The composition of the two induced maps

$$A \to \mathcal{K} \widehat{\otimes} B \to \mathcal{K} \widehat{\otimes} \mathcal{K} \widehat{\otimes} A \cong \mathcal{K} \widehat{\otimes} A$$

maps $a \in A$ to the $\mathbb{N}^2 \times \mathbb{N}^2$ –matrix

$$(\eta_k'\eta_j a \xi_i \xi_l')$$

This map is diffotopic to the canonical inclusion of A into $\mathcal{K} \widehat{\otimes} A$ via the homotopy that maps $a \in A$ to the $L \times L$-matrix $(\hat{\eta}_\alpha(t) a \hat{\xi}_\beta(t))$ where $L = \mathbb{N}^2 \cup \{0\}$ and

$$\hat{\xi}_0(t) = \cos t 1 \quad \hat{\xi}_{il}(t) = \sin t\, \xi_i \xi_l'$$
$$\hat{\eta}_0(t) = \cos t 1 \quad \hat{\eta}_{kj}(t) = \sin t\, \eta_k' \eta_j$$

The conditions $A\xi_i\xi_l' \subset A$, $\eta_k'\eta_j A \subset A$ ensure that the elements $(\hat{\eta}_0(t) a \hat{\xi}_{il}(t))$ and $(\hat{\eta}_{kj}(t) a \hat{\xi}_0(t))$ are in A.

Remark (3.2.24)[96]: A typical example of two algebras that are Morita equivalent in the sense of Lemma (3.2.20) are algebras A and B where B is a subalgebra of $\mathcal{K} \widehat{\otimes} A$ such that $(e_{00} \otimes 1) B (e_{00} \otimes 1) = e_{00} \otimes A$ and such that $B(e_{j0} \otimes 1) \subset B$ and $(e_{0j} \otimes 1) B \subset B$ for all j. In fact, we can choose $\xi_0 = \eta_0 = e_{00} \otimes 1$ (the other ξ_i, η_j all being 0) and $\xi_l' = e_{l0} \otimes 1, \eta_k' = e_{k0} \otimes 1$ in this case.

A situation closely related to Morita invariance arises in the following way. Let V be a complete locally convex space equipped with a continuous non-zero bilinear form $\langle \mid \rangle$. Then $\mathcal{M} = V \widehat{\otimes} V$ becomes a locally convex algebra. Assume now that $\psi: V \to s$ is a continuous

linear map that preserves the bilinear form. It induces a continuous homomorphism $\alpha_\psi: \mathcal{M} \to \mathcal{K}$. We have the following

Lemma (3.2.25)[96]: The map $\alpha_\psi \otimes \mathrm{id}: \mathcal{M} \widehat{\otimes} A \to \mathcal{K} \widehat{\otimes} A$ induces a kk^{alg} -equivalence for each locally convex algebra A.

Its inverse is given by the natural inclusion $A \to \mathcal{M} \widehat{\otimes} A$ mapping $x \in A$ to $e \otimes x$ where e is a one-dimensional projection in $\mathcal{M}$.

Proof. Let $\xi \in s$ and $\eta \in V$ be vectors such that $\langle \xi | \xi \rangle = \langle \eta | \eta \rangle = 1$. Consider the family of maps $\psi_t: V \to V \widehat{\otimes} s \oplus V \widehat{\otimes} s$ defined by

$$\psi_t(x) \ = \ (\cos t(x \otimes \xi), \sin t(\eta \otimes \psi(x)))$$

for $t \in [0, \pi/2]$. It induces a differentiable family of continuous homomorphisms $\alpha_{\psi_t}: \mathcal{M} \to M_2(\mathcal{M} \widehat{\otimes} \mathcal{K})$ which is for $t = 0$ the natural inclusion of $\mathcal{M}$into $\mathcal{M} \widehat{\otimes} \mathcal{K}$ (in the upper left corner of M_2) and for $t = \pi/2$, α_ψ composed with the natural inclusion of $\mathcal{K}$ into $\mathcal{M} \widehat{\otimes} \mathcal{K}$ (in the lower right corner of M_2).

Recall that we defined the smooth Toeplitz algebra $\mathcal{T}$ as the universal unital m- algebra generated by two elements v and v^* satisfying the relation $v^* v = 1$ such that the canonical linear map $C^\infty S^1 \to \mathcal{T}$ becomes continuous. It fits into an extension

$$0 \ \to \ K \ \to \ \mathcal{T} \ \to C^\infty S^1 \to \ 0$$

We denote by e the idempotent $1 - v^* v$ in .

Lemma (3.2.26)[96]: (compare [105]) There are unique continuous homomorphisms $\varphi, \varphi' :$ $\mathcal{T} \to \mathcal{T} \widehat{\otimes} \mathcal{T}$, such that

$$\varphi(v) = v(1-e) \otimes 1 + e \otimes v \qquad \varphi(v^*) = (1-e)v^* \otimes 1 + e \otimes v^*$$
$$\varphi'(v) = v(1-e) \otimes 1 + e \otimes 1 \qquad \varphi'(v^*) = (1-e)v^* \otimes 1 + e \otimes 1$$

These two homomorphisms are diffotopic through a diffotopy $\psi_t: \mathcal{T} \to \mathcal{T} \widehat{\otimes} \mathcal{T}$, $t \in [0, \pi/2]$, for which $\psi_t(x) - \varphi(x) \in \mathcal{K} \widehat{\otimes} \mathcal{T}$ holds for all $t \in [0, \pi/2]$, $x \in \mathcal{T}$

Proof. We show, that φ and φ' are both diffotopic to ψ, where

$$\psi(v) = v \otimes 1 \qquad \psi(v^*) = v^* \otimes 1$$

We write $e_{ij}$for the matrix units in $\mathcal{K} \subset \mathcal{T}$. With this notation we write linear combinations $e_{ij} \otimes x$, $0 \le i, j \le n-1$, $x \in \mathcal{T}$ as $n \times n$-matrices with entries in $\mathcal{T}$. For $t \in [0, \pi/2]$ we put

$$u_t = v^2 v^{*2} \otimes 1 + \begin{pmatrix} e_{00} + \cos t\, vv^* & \sin t\, v \\ -\sin t\, v^* & \cos t\, 1 \end{pmatrix}$$

$$u_t' = v^2 v^{*2} \otimes 1 + \begin{pmatrix} \cos t\, 1 & \sin t\, 1 \\ -\sin t\, 1 & \cos t\, 1 \end{pmatrix}$$

Then u_t and u_t' are obviously invertible elements in $\mathcal{T} \widehat{\otimes} \mathcal{T}$ and lie in fact in the subalgebra generated by $\mathcal{K} \widehat{\otimes} \mathcal{T}$ and $1 \widehat{\otimes} \mathcal{T}$. By the characterization of $\mathcal{T}$as a universal algebra in Proposition (3.2.5) there exist continuous homomorphisms $\varphi_t, \varphi_t' : \mathcal{T} \to \mathcal{T} \widehat{\otimes} \mathcal{T}$, such that

$$\varphi_t(v) = u_t(v \otimes 1) \qquad \varphi_t(v^*) = (v^* \otimes 1)u_t^{-1}$$

$$\varphi_t'(v) = u_t'(v \otimes 1) \qquad \varphi_t'(v^*) = (v^* \otimes 1)u_t'^{-1}$$

For the verification of the necessary growth conditions we refer to [97].

We may consider the algebra $\mathbb{C}(0,1)$ as the subalgebra of $\mathrm{C}^\infty \mathrm{S}^1$ of functions vanishing, together with all derivatives at the point 1. Denote by $\mathcal{T}_0$ the preimage of $\mathbb{C}(0,1)$ in the extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C^\infty S^1 \to 0$$

so that we obtain an extension

$$0 \to \mathcal{K} \to \mathcal{T}_0 \to \mathbb{C}(0,1) \to 0$$

Consider also the algebra $\mathcal{T}_0' \supset \mathcal{T}_0$ defined as the kernel of the natural map $\mathcal{T} \to \mathbb{C}$ mapping v and v^* to 1. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & \mathcal{T}_0' & \to & D & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{K} & \to & \mathcal{T}_0 & \to & \mathbb{C}(0,1) & \to & 0 \end{array}$$

where $D = \{f \in C^\infty(S^1) | f\,(1) = 0\}$ and the last vertical arrow is a diffotopy equivalence shows that $kk^{\mathrm{alg}}(\mathcal{T}_0, \mathcal{T}_0) \cong kk^{\mathrm{alg}}(\mathcal{T}_0, \mathcal{T}_0')$. Moreover, the split extension

$$0 \to \mathcal{T}_0' \to \mathcal{T} \to \mathbb{C} \to 0$$

shows that $kk^{\mathrm{alg}}(\mathcal{T}_0, \mathcal{T}_0') \cong kk^{\mathrm{alg}}(\mathcal{T}_0, \mathcal{T}_0) \oplus kk^{\mathrm{alg}}(\mathcal{T}_0, \mathbb{C})$. Tensoring everything with an arbitrary locally convex algebra A shows that the natural inclusion map $\mathcal{T}_0 \widehat{\otimes} A \to \mathcal{T} \widehat{\otimes} A$ induces an injective map $kk_0^{\mathrm{alg}}(\mathcal{T}_0 \widehat{\otimes} A, \mathcal{T}_0 \widehat{\otimes} A) \to kk_0^{\mathrm{alg}}(\mathcal{T}_0 \widehat{\otimes} A, \mathcal{T} \widehat{\otimes} A)$ for each locally convex algebra A.

Proposition (3.2.27)[96]: For each locally convex algebra A, the algebra $\mathcal{T}_0 \widehat{\otimes} A$ is k-contractible (i.e. isomorphic to 0 in the category kk^{alg}).

Proof. It suffices to show that the natural inclusion map $kk_0^{\mathrm{alg}}(\mathcal{T}_0 \widehat{\otimes} A, \mathcal{T}_0 \widehat{\otimes} A) \to kk_0^{\mathrm{alg}}(\mathcal{T}_0 \widehat{\otimes} A, \mathcal{T} \widehat{\otimes} A)$ is 0. Denote by $\bar{\varphi}$ the continuous homomorphism $\mathcal{T} \to \mathcal{T}_0 \widehat{\otimes} \mathcal{T}$ defined by

$$\bar{\varphi}(v) = v(1-e) \otimes 1 \qquad \bar{\varphi}(v^*) = (1-e)v^* \otimes 1$$

Using the diffotopy ψ_t from the previous lemma, we consider the restriction of the pair $(\psi_t, \bar{\varphi})$ to homomorphisms from $\mathcal{T}_0$ to $\widehat{\otimes} \mathcal{T}$. It induces for all t the same element $kk(\psi_t, \bar{\varphi})$ in $kk^{\mathrm{alg}}(\mathcal{T}_0, \mathcal{K} \widehat{\otimes} \mathcal{T}) \cong kk^{\mathrm{alg}}(\mathcal{T}_0, \mathcal{T})$. However, for $t = 0$ (where $\psi_0 = \varphi'$) we have $\psi_0 - \bar{\varphi} = \iota$ where ι is the natural inclusion map $\mathcal{T}_0 \to \mathcal{K} \widehat{\otimes} \mathcal{T}$, while for $t = 1$ (where $\psi_1 = \varphi'$) we have $\psi_1 - \bar{\varphi} = 0$ on $\mathcal{T}_0$.

The same argument applies if we tensor $\mathcal{T}_0$by A and all homomorphisms by id_A .

Theorem (3.2.28)[96]: (Bott periodicity) The extension

$$E: 0 \to \mathcal{K} \widehat{\otimes} A \to \mathcal{T}_0 \widehat{\otimes} \to A(0,1) \to 0$$

defines an invertible element ε in $kk_{-1}^{\mathrm{alg}}(A(0,1), A) \cong kk_{-2}^{\mathrm{alg}}(A, A)$. As a consequence $kk_n^{\mathrm{alg}}(A, B) \cong kk_{n-2}^{\mathrm{alg}}(A, B)$ for all A, B and n.

Proof: This follows from Corollary (3.2.16) and Proposition (3.2.23).

Remark (3.2.29)[96]: The fact that the inverse Bott map $\varepsilon: J^2 A \to \mathcal{K} \widehat{\otimes} A$ associated as a classifying map to the extension

$$0 \to \mathcal{K} \widehat{\otimes} A \to \mathcal{T}_0 \widehat{\otimes} A \to \mathbb{C}[0,1) \widehat{\otimes} A \to A \to 0$$

defines an invertible element, shows that the definition of kk^{alg} that we use in this article gives the same result as the analogue of the formally different definition given in [97]:

$$kk_*^{\mathrm{alg}}(A, B) = \lim_{\vec{k}} \langle J^{2k+*} A, \mathcal{K} \widehat{\otimes} B \rangle$$

where the inductive limit is taken using the ε-map from $J^{k+2} A$ to $\mathcal{K} \widehat{\otimes} J^k A$. In fact, there are obvious natural maps from the theory defined that way into kk^{alg} defined as above and vice versa, which are inverse to each other.

Note also that conceivable different versions of the Toeplitz algebra would not work for the proof of Bott periodicity above. A natural candidate for instance would be the enveloping m-algebra of the unital algebra generated algebraically by two elements v and v^* satisfying the relation $v^* v = 1$. This algebra would not work since the quotient by the natural ideal $'\mathcal{K}$

generated by $e = 1 - vv^*$, which is a different (smaller than $\mathcal{K}$) completion of the algebra M_∞ of finite matrices of arbitrary size, is too small to admit a homomorphism from $\mathbb{C}(0,1)$ into it. This is closely related to the fact, studied by Thom in [98], that stabilization in the definition kk^{alg} on the right hand side by algebras smaller than $\mathcal{K}$ leads to theories which are not periodic, such as connective K-theory.

Theorem (3.2.30)[96]: Let D be any locally convex algebra. Every extension admitting a continuous linear section

$$E: 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$$

induces exact sequences in $kk^{\mathrm{alg}}(D,\cdot)$ and $kk^{\mathrm{alg}}(D,\cdot)$ of the following form:

$$\begin{array}{ccccc} kk_0^{\mathrm{alg}}(D,I) & \xrightarrow{\cdot kk(i)} & kk_0^{\mathrm{alg}}(D,A) & \xrightarrow{\cdot kk(q)} & kk_0^{\mathrm{alg}}(D,B) \\ \uparrow & & & & \downarrow \\ kk_1^{\mathrm{alg}}(D,B) & \xleftarrow{\cdot kk(q)} & kk_1^{\mathrm{alg}}(D,A) & \xleftarrow{\cdot kk(i)} & kk_1^{\mathrm{alg}}(D,I) \end{array} \tag{10}$$

and

$$\begin{array}{ccccc} kk_0^{\mathrm{alg}}(I,D) & \xleftarrow{kk(i)\cdot} & kk_0^{\mathrm{alg}}(A,D) & \xleftarrow{kk(q)\cdot} & kk_0^{\mathrm{alg}}(B,D) \\ \uparrow & & & & \uparrow \\ kk_1^{\mathrm{alg}}(B,D) & \xrightarrow{kk(q)\cdot} & kk_1^{\mathrm{alg}}(A,D) & \xrightarrow{kk(i)\cdot} & kk_1^{\mathrm{alg}}(I,D) \end{array} \tag{11}$$

The given extension E defines a classifying map $JB \to I$ and thus an element of $kk_1^{\mathrm{alg}}(B,I)$, which we denote by $kk(E)$. The vertical arrows in (10) and (11) are (up to a sign) given by right and left multiplication, respectively, by this class $kk(E)$.

With every pair of locally convex algebras A, B we can associate the bivariant cyclic homology groups $HP_*(A,B)$, $\star = 0,1$ which are in fact complex vextor spaces. The bivariant theory HP_* has properties analogous to those of kk_*^{alg}. In particular it is diffotopy invariant in both variables, invariant under tensoring one of the arguments by $\mathcal{K}$ and, most importantly, it has six-term exact sequences exactly like those in 8.5, replacing there kk^{alg} by HP, [106], [104], [107], [108] where the boundary maps are given by an element in HP_1 determined by the given extension.

Therefore the standard extensions

$$0 \to JA \to TA \to A \to 0$$

$$0 \to A(0,1) \to A[0,1) \to A \to 0$$

determine isomorphisms $HP_1(A,JA) \cong HP_0(A,A), HP_1(A(0,1),A) \cong HP_1(A,A)$ and invertible elements $\alpha_A \in HP_1(A,JA)$ and $\beta_A \in HP_1(A(0,1),A)$. By iteration we have canonical invertible elements $\alpha_A^n \in HP_n(A,J^nA)$ and $\beta_A^n \in HP_n(A(0,1)^n,A)$.

Let moreover γ^n be the element of $HP_0(J^nA,A(0,1)^n)$ induced by the canonical homomorphism $J^nA \to A(0,1)^n$. The following relations are then easily checked:

$$\alpha_A^n \cdot \gamma^n \cdot \beta_A^n = 1_A \qquad \gamma^n \cdot \beta_A^n \cdot \alpha_A^n = 1_{J^nA} \quad \beta_A^n \cdot \alpha_A^n \cdot \gamma^n = 1_{A(0,1)}$$

where 1_B denotes the identity element of $HP_0(B,B)$ (induced by the identity map of B).

Theorem (3.2.31)[96]: There are natural maps $ch: kk^{\mathrm{alg}}(A,B) \to HP_s(A,B)$ such that

$$ch(\varphi \cdot \psi) = ch(\varphi) \cdot ch(\psi)$$

in $HP_{s+t}(A,C)$ for $\varphi \in kk^{\mathrm{alg}}(A,B)$ and $\psi \in kk^{\mathrm{alg}}(B,C)$ and such that $ch(1_A) = 1_A$ for each A. Here s and t are in $\mathbb{Z}$ and HP_s is considered to be 2-periodic in s.

Proof. Ignoring first the stabilization by $\mathcal{K}$ assume that $\varphi \in kk^{\mathrm{alg}}(A,B)$ is represented by a homomorphism $\eta: J^{n+s}A \to B(0,1)^n$. We then define

$$ch(\varphi) = \alpha_A^{n+s} HP_0(\eta) \beta_B^n$$

The relation $\alpha_A^n \cdot \gamma^n \cdot \beta_A^n = 1_A$ mentioned above generalizes to $\alpha_A^n \cdot \theta^n \cdot \beta_A^n = \theta$ if θ is an element of $HP_0(A,B)$ determined by a homomorphism $A \to B$ and θ^n is the element of $HP_0(J^nA,A(0,1)^n)$ determined by the homomorphism $J^nA \to B(0,1)^n$ induced by θ. Since the inductive limit in the definition of kk_s^{alg} is taken exactly with respect to the passage from homomorphisms $A \to B$ to homomorphisms $J^nA \to B(0,1)^n$, this shows that ch is well defined. The compatibility with the product follows from Lemma (3.2.10).

Finally, the natural inclusion $B \to \mathcal{K} \widehat{\otimes} B$ induces an invertible element ι in $HP_0(B, \mathcal{K} \widehat{\otimes} B)$. The formula for ch then becomes

$$ch(\varphi) = \alpha_A^{n+s} HP_0(\eta) \beta_B^n \iota^{-1}$$

if φ is represented by the map $\eta: J^{n+s}A \to \mathcal{K} \widehat{\otimes} B(0,1)^n$. The stabilization by $\mathcal{K}$ does not lead to any essential change in the arguments for well-definedness and multiplicativity.

It is important to note that the Bott periodicity element $\varepsilon \in kk_2^{\mathrm{alg}}(A,A)$ is not mapped to 1 under ch, but rather to $2\pi i$, cf. [97].

Proposition (3.2.32)[96]: There is a bilinear graded commutative exterior product

$$kk_i^{\mathrm{alg}}(A_1, B_1) \times kk_j^{\mathrm{alg}}(A_2, B_2) \to kk_{i+j}^{\mathrm{alg}}\big(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2\big)$$

denoted by $(\alpha, \beta) \mapsto \alpha \times \beta$.

Proof. There are natural maps $J\,(A_1 \widehat{\otimes} A_2) \to JA_1 \widehat{\otimes} A_2$ and $J\,(A_1 \widehat{\otimes} A_2) \to A_1 \widehat{\otimes} JA_2$. For all $s, t \in \mathbb{N}$ there is a natural map $J^{s+t}\,(A_1 \widehat{\otimes} A_2) \to J^s A_1 \widehat{\otimes} J^t A_2$.

The exterior product × and the interior product are compatible in the following way:

Lemma (3.2.33)[96]: Let $\alpha_1 \in kk_0^{\mathrm{alg}}(A_1, B_1)$, $\alpha_2 \in kk_0^{\mathrm{alg}}(A_2, B_2)$, $\beta_1 \in kk_0^{\mathrm{alg}}(B_1, C_1)$, $\beta_2 \in kk_0^{\mathrm{alg}}(B_2, C_2)$. Then

$$(\alpha_1 \times \alpha_2) \cdot (\beta_1 \times \beta_2) = (\alpha_1 \cdot \beta_1) \times (\alpha_2 \cdot \beta_2)$$

On the coefficient ring $R := kk_0^{\mathrm{alg}}(\mathbb{C}, \mathbb{C})$ the exterior and interior products obviously coincide. With this product R is a commutative unital ring. $kk_i^{\mathrm{alg}}(A, B)$ is a module over R for all A,B and i and by Lemma (3.2.31) the composition product

$$kk_i^{\mathrm{alg}}(A, B) \times kk_j^{\mathrm{alg}}(B, C) \to kk_{i+j}^{\mathrm{alg}}(A, C)$$

is R-linear.

By Theorem (3.2.29), there is a unital ring homomorphism $ch: R \to \mathbb{C} = HP_0(\mathbb{C}, \mathbb{C})$. In particular R is not zero!

The Weyl algebra W is the unital algebra generated by two elements x, y satisfying the relation $[x, y] = 1$. A fundamental reference for the properties of this algebra is [109]. It is easy to see that W can not carry any submultiplicative seminorm. In fact, the completion of W with respect to such a seminorm would be a unital Banach algebra containing two elements x, y such that $xy - yx = 1$. This relation shows that for the (non-empty!) spectra we have $Sp(xy) = Sp(yx) + 1$. On the other hand we would have $Sp(xy) \cup \{0\} = Sp(yx) \cup \{0\}$. This shows that the spectrum of xy would have to be unbounded which is impossible in a Banach algebra.

As explained above however, W is a locally convex algebra with the fine topology and we will from now on always assume that W carries this topology.

To construct the fundamental extension, let W'' denote the algebra with two generators x', y' satisfying the relations

$$(x'y' - y'x')y' = y' \qquad x'(x'y' - y'x') = x'$$

We equip W' first with the fine locally convex topology (later we will also consider other locally convex structures on W').

Let $f = x'y' - y'x' - 1$ in the algebra $\widetilde{W'}$ with unit adjoined. This element will play an important role in the following computations. The relations $fy' = x'f = 0$ imply that

$$fx'y' = x'y'f = f + f^2$$

Lemma (3.2.34)[96]: We have

$$x'^m(x'y') = (m + x'y')x'^m\,, \quad [x', x'y'] = x'$$

for each $m \in \mathbb{N}$.

Proof. We have $x'(x'y') = x'(y'x' + 1) = (1 + x'y')x'$ whence the second equation. The first identity follows from this by the derivation rule.

Let $\pi: W' \to W$ be the natural map and I the kernel of π. Then I is the ideal generated in W' by all elements of the form fa and , for $a \in W'$.

We consider natural representations of W and W' as operators on the spaces of (arbitrary) functions with compact support $V_{\mathbb{R}} = \mathcal{F}_c([0,\infty))$ and $V_{\mathbb{N}} = c_0(\mathbb{N})$ where $\mathbb{N} = \{0,1,2,\dots\}$ (the support condition is in fact inessential). We define representations $\rho': W' \to \mathrm{End}\, V_{\mathbb{R}}$ and $\rho: W \to \mathrm{End} V_{\mathbb{N}}$ by

$$\rho'(x') = \sqrt{t}\, U_{-1} \quad \rho'(y') = U_1\sqrt{t}$$

and

$$\rho(x) = \sqrt{N}\, U_{-1} \quad \rho(y) = U_1\sqrt{N}$$

where U_1 and U_{-1} are right and left translation:

$$(U_1\xi)(s) = \xi(s-1) \qquad (U_{-1}\xi)(s) = \xi(s+1)$$

and $\sqrt{t}, \sqrt{N}$ are the square roots of the "number operators" in End $V_{\mathbb{R}}$ and End $V_{\mathbb{N}}$ defined by

$$(\sqrt{t}\xi)(s) = \sqrt{s}\xi(s) \qquad (\sqrt{N}\,\xi)(n) = \sqrt{n}\xi(n)$$

for $\xi \in V_{\mathbb{R}}, V_{\mathbb{N}}$.

It is clear that $\rho(x)$, $\rho(y)$, and $\rho(x)$, $\rho(y)$ satisfy the relations of W' and W, respectively. In fact ρ is nothing but the standard representation of the Heisenberg commutation relation by creation and annihilation operators used in quantum mechanics.

Lemma (3.2.35)[96]: Every element w in I has a unique representation as a finite sum of the form

$$w = \sum_{k,l} w_{kl} \quad k, l = 0,1,2, \ldots$$

with$w_{kl} = y'^k\, P_{kl}(x'y') f x'^l l$ where f is defined as before Lemma (3.2.32), P_{kl} is a polynomial in $\mathbb{C}[t]$ in general and P_{00} is in $t\mathbb{C}[t]$.

Moreover, the representation ρ' is faithful on I

Proof. Every element of I is a linear combination of elements of the form af, fb or afb with a, b in $'$. We know that $fy' = x'f = 0$ and, from Lemma (3.2.32),

$$fx'^n\, y'^k = f\big((n-1) + x'y'\big)x'^{n-1}\, y'^{k-1}$$

$$= \begin{cases} (n-1) + x'y'(n-2+x'y') \cdots (n-k+x'y') f x'^{n-k} & n \geq k \\ 0 & n < k \end{cases}$$

and similarly

$$x'^n\, y'^k f = y'^{(k-n)} P(x'y') f$$

for a certain polynomial P . This shows that w can be written as claimed.

To show that ρ' is faithful on I note that the elements w_{kl} are mapped under ρ' to $z_{kl} = U_1^k Q_{kl} P_{kl}(t) \rho'(f) U_{-1}^l$where $Q_{kl}$is a non-zero product of functions of the form $\sqrt{t+n}, n \in \mathbb{N}$, that depends only on k and l (in fact it is not difficult to write it down explicitly) and $\rho'(f)$ is multiplication by the function $\hat{f}$ given by

$$\hat{f}(t) = \begin{cases} 1-t & t \leq 1 \\ 0 & t \geq 1 \end{cases}$$

It is clear that the operator $\sum z_{kl}$ in End ($V_{\mathbb{R}}$) is 0 only if all P_{kl} are 0.

This shows at the same time the injectivity of ρ' and the uniqueness of the P_{kl}.

Lemma (3.2.36)[96]: The ideal I is Morita equivalent in the sense of Lemma (3.2.20) to the algebra $t(1-t)\mathbb{C}[t]$ of polynomials vanishing in 0 and 1. As a consequence I is k-equivalent to $\mathbb{C}(0,1)$.

Proof. According to Lemma (3.2.33) there is an inclusion map $t(1-t)\mathbb{C}[t] \to I$ mapping $(1-t)P$, with $P \in t\mathbb{C}[t]$, to $fP(x'y')$.

To construct the Morita context from I to $t(1-t)\mathbb{C}[t]$, think of W' as represented in End $V_{\mathbb{R}}$. Using the notation of Definition (3.2.19) we choose $\xi_i = \xi' = U_1^i \chi$ and $\eta_j = \eta' = \chi U_{-1}^j$where χ is the multiplication operator by the characteristic function for the interval $(0,1]$.

From the extension $0 \to (1-t)\mathbb{C}[t] \to \mathbb{C}[t] \to \mathbb{C} \to 0$ where $t\mathbb{C}[t]$ is contractible (see subsection Proposition (3.2.1)), it follows that $(1-t)t\mathbb{C}[t]$ is isomorphic in the category kk^{alg} to $\mathbb{C}(0,1)$.

Lemma (3.2.37)[96]: The representation ρ' is faithful and we have a commutative diagram

$$\begin{array}{ccc} W' & \overset{\rho'}{\to} & \mathrm{End}V_{\mathbb{R}} \\ \downarrow \pi & & \downarrow \eta \\ W & \overset{\rho}{\to} & \mathrm{End}V_{\mathbb{N}} \end{array}$$

where η is restriction to the invariant subspace of $V_{\mathbb{R}}$consisting of functions with finite support in $\mathbb{N}$ (which is isomorphic to $V_{\mathbb{N}}$).

Proof. The commutativity of the diagram is obvious. Since W is simple it shows that $\mathrm{Ker}\,\rho' \subset I = \mathrm{Ker}\,\pi$. But ρ' is faithful on I by (3.2.30).

Using the notation of Lemma (3.2.21) we define invertible elements u_t in $\mathcal{T}$ by

$$u_t = v^2 v^{*2} + \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

where the 2×2-matrix represents a linear combination of the matrix units e_{ij} , $i,j = 0,1$ in $\mathcal{K} \subset \mathcal{T}$. We can define a differentiable family φ_t, $t \in [0,\pi/2]$ of homomorphisms $W \to W' \widehat{\otimes} \mathcal{T}$ by

$$\varphi_t(x') = x' \otimes v^* u_t^{-1} \quad \varphi_t(y') = y' \otimes u_t v$$

and another homomorphism $\bar{\varphi}$ defined by

$$\bar{\varphi}(x) = x \otimes (1-e)v^* \quad \varphi_t(y') = y' \otimes v(1-e)$$

For each $t \in [0,\pi/2]$ and each $z \in W'$ we have that

$$\varphi t(z) - \varphi(z) \in W \otimes \mathcal{K} \subset W \otimes T$$

We therefore obtain induced elements $kk(\varphi_t, \bar{\varphi})$ in $kk^{\mathrm{alg}}(W', W' \widehat{\otimes} \mathcal{K})$ which do, by homotopy invariance, not depend on t.

Lemma (3.2.38)[96]: The linear space B generated by all elements of the form

$$y'^n c x'^m \otimes e_{nm} \ , \quad n,m = 0,1,2\ldots$$

where c is a polynomial in $x'y'$ in general, and a polynomial without constant term for $n = m = 0$, is a subalgebra of $W' \widehat{\otimes} \mathcal{K}$.

For each $z \in W'$ we have

$$\varphi_0(z) - \bar{\varphi}(z) \in B \qquad \varphi_0(z)B \subset B \qquad B_{\varphi_0}(z) \subset B$$

Proof. The first assertion follows from the fact that $x'^n y'^n$ is a polynomial without constant term in $x'y'$ according to Lemma (3.2.32).

According to Remark (3.2.19) it suffices to check the second assertion for $z = x'$ and $z = y'$ where it follows again immediately from Lemma (3.2.32).

Lemma (3.2.39)[96]: Let $f_0 = 1$ and $f_n \in \mathbb{C}[t]$, $n \geq 1$ denote the polynomial

$$f_n(t) = t(t+1)(t+2)\ldots(t+n-1)$$

and let $\mathcal{K}_0 \subset \mathcal{K}$ denote the algebra of finite matrices.

The map $y'^n c x'^m \otimes e_{nm} \mapsto c f_m \otimes e_{nm}$ defines an injective homomorphism $B \to \mathbb{C}[t] \widehat{\otimes} \mathcal{K}_0$. The image of B under this injection consists of all linear combinations of elements $c f_m \otimes e_{nm}$ with $c \in \mathbb{C}[t]$ in general, and $c \in t\mathbb{C}[t]$ for $n = m = 0$.

Proof. The identity $x'^n y'^n = f_n(x'y')$ from Lemma (3.2.32), shows that the map in question is a homomorphism. Injectivity follows from 11.4. The rest is obvious.

Theorem (3.2.40)[96]: The algebra W' is k-contractible, i.e. $kk^{\mathrm{alg}}(W', W') = 0$.

Proof. We show first that B is k-contractible. Denote by B_0 the kernel of the homomorphism $B \to \mathcal{K}_0$ induced by the evaluation $t \mapsto 1$ from $\mathbb{C}[t] \widehat{\otimes} \mathcal{K}_0$ to $\mathcal{K}_0$. The image $N \subset \mathcal{K}_0$ of B under this map consists of all linear combinations of elements e_{n0}, $n \geq 1$ and is therefore nilpotent of order 2 ($N^2 = 0$) and thus also contractible. We obtain an extension

$$0 \to B_0 \to B \to N \to 0$$

$B_0$is obviously Morita equivalent to $t\mathbb{C}[t]$ in the sense of (3.2.20) (see Remark (3.2.24)) and therefore also k-contractible. It follows that B is k-contractible.

As a consequence, with the notation above, $kk(\varphi_{\pi/2}, \bar{\varphi}) = kk(\varphi_0, \bar{\varphi}) = 0$ since these morphisms are products of the corresponding elements in $kk^{\mathrm{alg}}(W', B)$ by the element $kk(j)$ induced by the inclusion $B \to W' \widehat{\otimes} \mathcal{K}$ and since B is k-contractible. On the other hand $\varphi_{\pi/2}, -\bar{\varphi}$ satisfies the conditions of Lemma (3.2.17) and is a homomorphism equal to the canonical embedding $W' \to W' \widehat{\otimes} \mathcal{K}$. It follows that $kk^{\mathrm{alg}}(W', W') = 0$.

Note that exactly the same argument shows that W' is isomorphic to 0 in HP_*, i.e. $HP_*(W', W') = 0$.

Theorem (3.2.41)[96]: The Weyl algebra W is isomorphic to $\mathbb{C}$in kk^{alg}.

Proof. Since W' is k-contractible, the element $kk(E)$ in $kk_0^{\mathrm{alg}}(W,I)$ associated with the extension $0 \to I \to W' \to W \to 0$ is invertible,

$$kk_1^{\mathrm{alg}}(W,I) \cong kk_1^{\mathrm{alg}}\big(W,\mathbb{C}(0,1)\big) \cong kk_0^{\mathrm{alg}}(W,\mathbb{C})$$

It follows also from this result that W is isomorphic, in the category HP_* defined by bivariant periodic cyclic homology, to $\mathbb{C}$ (invertible elements are mapped by the character ch to invertible elements). The Hochschild and cyclic homology of W however have been known for a long time, cf. [110].

From the result above we can also deduce the kk^{alg}- invariants for the higher dimensional versions W_n of W. By definition W_n is generated by elements $x_1,\dots,x_n$ and $y_1,\dots,y_n$ such that $[x_i,y_j]=\delta_{ij}1$ and $[x_i,x_j]=0$, $[y_i,y_j]=0$ for all i,j. Thus

$$W_n = W \widehat{\otimes} \dots \widehat{\otimes}\ W\ = W \otimes \dots \otimes W\ .$$

The n-fold exterior product of the invertible element $kk(E)$ in $kk_1^{\mathrm{alg}}(W,I) \approx kk^{\mathrm{alg}}(W,\mathbb{C})$ constructed above, by itself, gives an invertible element in $kk^{\mathrm{alg}}(W_n,\mathbb{C})$.

It is represented by the n-step extension

$$0 \to I^{\otimes n} \to I^{\otimes n-1} \otimes W' \dots \to W' \otimes W^{\otimes n-1} \to W^{\otimes n} \to 0$$

Using similar extensions one may also compute the kk^{alg}-invariants for further algebras of similar type. Consider for instance the algebra $\mathcal{D}$ of differential operators generated by the Fréchet-algebra $S(\mathbb{R})$ of Schwartz functions together with the differentiation operator D, satisfying the relation $[D,f]=f'$ for $f \in S(\mathbb{R})$. We make it into a locally convex algebra by defining a seminorm α on $\mathcal{D}$ to be continuous, if the map $S(\mathbb{R}) \ni\mapsto \alpha(fD^n)$ is continuous for each n. Using a similar extension as for W we can show that $kk_0^{\mathrm{alg}}(\mathbb{C},\mathcal{D})=0$, $kk_1^{\mathrm{alg}}(\mathbb{C},\mathcal{D})=R$.

We construct a different locally convex topology on W' in such a way that the completion $\bar{I}$ of the ideal I becomes an m-algebra (in fact I will be Morita equivalent to $C^\infty(0,1)$ with the canonical m-algebra structure). This gives an extension

$$0 \to \bar{I} \to \bar{W}' \to W \to 0$$

where the ideal is an m-algebra. This extension will be used in section 15 to obtain more precise information on the K-theoretic invariants of .

Recall from Lemma (3.2.32) the formulas

$$x'^m\,(x'y') = (m+x'y')x'^m\,, \quad (x'y')y'^m = y'^m(m+x'y')$$

Putting as above $f = (x'y' - y'x') - 1$ in $\widetilde{W}'$, we obtain

$$x'^k\, y'^l f = x'^{k-1}\, y'^{l-1}((l-1) + x'y')f = x'^{k-1}y'(l-1)(l+f)f$$

By induction (on l) we obtain

$$x'^k y'^l f = \begin{cases} y'^{l-k}(l+f)(l-1+f)\dots(l+1-k+f)f & l \geq k \\ 0 & l < k \end{cases}$$

and, similarly

$$f x'^k y'^l = \begin{cases} (k+f)(k-1+f)\dots(k+1-l+f)f x'^{k-l} & k \geq l \\ 0 & k < l \end{cases} \qquad (12)$$

Lemma (3.2.42)[96]: (a) $f x'^k y'^l f = 0$ if $k = l$

(b) $f\, x'^{n+1} y'^{n+1} f\ = y'^n f = (n+1+f) f x'^n y'^n f$

(c) $f\, x'^n y'^n f f\ = (n+f)(n-1+f)\dots(2+f)(1+f)f^2$

Proof. Using the relations $[x', y'] = 1 + f$ and $x'f = fy' = 0$ we obtain

$$x'^k y'^l = x'^{k-1}y'x'y'^{l-1} + x'^{k-1}y'x'y'^{l-1} = x'^{k-1}y'^l x' + l x'^{k-1}y'^{l-1} + x'^{k-1}y'^{l-1}f$$

Multiplying this by f on both sides and using $x'f = 0$ gives (a) and (b) (note also that f commutes with $x'^n y'^n$ by Lemma (3.2.32). (c) follows from (b) by induction.

By Lemma (3.2.33) every element of I has a unique representation of the form

$$\sum y'^k z_{kl} x'^l$$

where $z_{kl} \in f\mathbb{C}[x'y']$. We write I_0 for the ideal $f\mathbb{C}[x'y']$in $\mathbb{C}[x'y']$ and consider this algebra from now on as a subalgebra of $C^\infty[0,1]$ by mapping $x'y'$ to the function t and f to the function $t - 1$ (this corresponds to the representation of W' constructed).

We decompose W' as $W' = I \oplus W$ using the linear splitting $s: W \ni y^n x^m \to y'^n(1 + f\,)x'^m \in W'$. Every element of W' can then be uniquely represented in the form

$$\sum y'^k z_{kl} x'^l$$

with, this time, $z_{kl} \in I_1$, where I_1 is the subalgebra $(1+f)\mathbb{C}[x'y']$ of $C^\infty[0,1]$ (I_0 consist in this identification of the polynomial functions on [0, 1], that vanish at 1). Here, the elements of the direct summand W correspond to sums where all z_{kl} are equal to $1 + f = x'y' - y'x'$.

Let β be a submultiplicative seminorm on I_1such that $\beta(1) = 1$ and $\beta(1+f) \leq 2$ (this is the case for the restrictions of the canonical submultiplicative seminorms on $C^\infty[0,1]$ to I_1).

We consider seminorms β_φ defined, using such a seminorm β and a monotone increasing function $\varphi: \mathbb{N} \to \mathbb{R}_+$, by

$$\beta_\varphi\left(\sum y'^k z_{kl} x'^l\right) = \sum \varphi(k)\,\varphi(l)\beta(z_{kl})$$

Let now $, b \in I$. They can be written uniquely as $= \sum y'^k f z_{kl} x'^l$, $b = \sum y'^i f w_{ij} x'^j$ with $z_{kl}, w_{ij} \in I_1$. We have

$$y'^k f z_{kl} x'^l y'^l f w_{ij} x'^j = \begin{cases} y'^k z_{kl} w_{ij} f^2 (l+f) \dots (2+f)(1+f) x'^j & l = i \\ 0 & l \neq i \end{cases}$$

It follows that

$$\begin{aligned} \beta_\varphi\left(y'^k f z_{kl} x'^l y'^i f w_{ij} x'^j\right) &= \varphi(k)\varphi(j)\beta(z_{kl} f w_{ij} f^2 (l+f) \dots (1+f)) \\ &\leq \varphi(k)\beta(z_{kl}) 4(l+2) \dots (1+2)\beta(w_{ij})\phi(j) \\ &\leq \phi(k)\beta(z_{kl}) 2(l+2)!\,\beta(w_{ij})\varphi(j) \end{aligned}$$

if $l = i$ (and 0 otherwise). This last expression can be estimated by

$$\varphi(k)\beta(z_{kl})\varphi(l)\varphi(i)\beta(w_{ij})\varphi(j)$$

if $\varphi(l)^2 \geq 2(l+2)!$.

We thus obtain: If $\varphi(l) \geq \sqrt{2(l+2)!}$, then β_φ is submultiplicative on I .

Let now $a = \sum y'^k f z_{kl} x'^l \in I$ and $b = \sum y'^l (1+f) x'^j \in s(W)$. We have from equation (12) at the beginning of this section

$$y'^k f z_{kl} x'^l y'^l (1+f) x'^j = \begin{cases} y'^k z_{kl} f (l+f) \dots (l+1-i+f) x'^j & l \geq i \\ 0 & l < i \end{cases}$$

whence

$$\beta_\varphi(y'^k f z_{kl} x'^l y'^l (1+f) x'^j) \leq \varphi(k)\beta(z_{kl}) 4(l+1) \dots (l+2-i)\varphi(l-i+j)$$

(provided that $l \geq i$).

This, in turn, can be estimated by

$$\varphi(k)\beta(z_{kl}) 4(l+1)\,!\,\varphi(2l)\psi(j)$$

if $\psi(j) \geq \sup_{l \in \mathbb{N}} \frac{\varphi(l+j)}{\varphi(2l)}$

Setting $\varphi'(l) = 4(l+1)!\,\varphi(2l)$ this takes the form $\varphi(k)\beta(z_{kl})\varphi'(l)\psi(j)$. There is an analogous estimate for products of the form $y'^l (1+f) x'^j y'^k z_{kl} f x'^l$. As a consequence we got putting

$$\alpha_\psi\left(\sum_{i,j} y'^l(1+f)x'^j\right) = \sum \psi(i)\psi(j)$$

with ψ as above, the inequalities

$$\beta_\varphi(ab) \leq \beta_{\varphi'}(a)\alpha_\psi(b)$$

$$\beta_\varphi(ab) \leq \alpha_\psi(b)\beta_{\varphi'}(a)$$

for $a \in I, b \in s(W)$.

Let now finally $b_1 = s(c_1), b_2 = s(c_2) \in s(W)$.

Then $b_1 b_2 = s(c_1 c_2\ ) + \omega(c_1, c_2)$, where $\omega(a,b) = s(ab) - s(a)s(b)$. We can choose ψ' such that

$$\alpha_\psi(s(c_1c_2)) \ + \ \beta_\varphi(\omega(c_1,c_2)) \ \leq \ \alpha_{\varphi'}(b_1)\alpha_{\varphi'}(b_2)$$

Let then α, α' on $W' = I + s(W)$ be given by

$$\alpha(a+b) = \beta_\varphi(a) + \alpha_\psi(b)$$

$$\alpha'(a+b) = \beta_{\varphi'}(a) + \alpha_{\varphi'}(b)$$

It follows that $\alpha(zw) \leq \alpha'(z)\alpha'(w)$ for all $, w \in W'$. Note also that, for suitable ψ, α_ψ dominates any given seminorm on $s(W)$.

In conclusion we have proved the following theorem

Theorem (3.2.43)[96]: There is a locally convex structure on W' given by a family of semi-norms (α_φ, ψ) satisfying the following conditions:

(a) the multiplication on W' is continuous
(b) as a locally convex space W' is the direct sum of I and W where W carries the fine locally convex topology and the restrictions of all seminorms to I are submultiplicative

(a) the locally convex structure on I_1 (defined as after Lemma (3.2.37)) inherited from W' is the usual Fréchet topology on $C^\infty[0,1] \supset I_1$.

The completion $\overline{W}'$ of W' gives rise to a linearly split extension of locally convex algebras

$$0 \ \rightarrow \ \bar{I} \ \rightarrow \overline{W}' \rightarrow \ W \ \rightarrow \ 0$$

where I_1 is an m-algebra. Moreover, the completion of $I_1 \subset W'$ in this topology is isomorphic to $C^\infty[0,1]$.

Remark (3.2.44)[96]: To define the topology with these properties on W', it suffices to take on I only seminorms β_φ where φ is obtained from $\varphi_0(k) := \sqrt{2(k+1)!}$ By iteration of the operation $\varphi'(k) = 4(k+1)!\,\varphi(2k)$.

Many computations of the K-theoretic invariants for C^*-algebras carry over to the locally convex setting. In this section we discuss the case of crossed products of locally convex algebras by $\mathbb{Z}$. Let A be a unital locally convex algebra (the non-unital case can be treated by adjoining a unit) and α a continuous automorphism of A. The crossed product $A \rtimes_\alpha \mathbb{Z}$ is defined as the complete unital locally convex algebra generated by A together with an invertible element u satisfying $uxu^{-1} = \alpha(x)$ for all x in A (thus for the topology of $A \rtimes_\alpha \mathbb{Z}$ we allow all seminorms on the algebra generated algebraically by A and u, u^{-1} for which the map $A \ni \mapsto au^n$ is continuous for each $n \in \mathbb{Z}$).

To describe the kk^{alg}-invariants for $A \rtimes_\alpha \mathbb{Z}$ we follow the argument outlined in [105].

We denote by $\mathcal{T}_\alpha$ the closed subalgebra of $\mathcal{T} \widehat{\otimes} A \rtimes_\alpha \mathbb{Z}$ generated by $1 \otimes A$ and $v \otimes u, v^* \otimes u^{-1}$. Since the subalgebra of $C^\infty(S^1) \widehat{\otimes} (A \rtimes_\alpha \mathbb{Z})$ generated by $1 \otimes A$ and $z \otimes u, z^{-1} \otimes u^{-1}$ is isomorphic to $A \rtimes_\alpha \mathbb{Z}$ (z the canonical generator of $C^\infty(S^1)$), we obtain, by restriction, a continuous homomorphism $\mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z}$ whose kernel is obviously dense in the ideal $\mathcal{K} \widehat{\otimes} A$ of $\mathcal{T}_\alpha$. Thus, we have a linearly split extension

$$0 \to \mathcal{K} \widehat{\otimes} A \to \mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z} \to 0$$

There is a natural inclusion map $j: A \to \mathcal{T}_\alpha$ defined by $j(x) = 1 \otimes x$ and a quasihomomorphism $(id, \mathrm{Ad}(v \otimes 1))$ from $\mathcal{T}_\alpha$ to $\mathcal{K} \widehat{\otimes} A$ defining elements $kk(j)$ in $kk_0^{\mathrm{alg}}(A, \mathcal{T}_\alpha)$ and $\tau = kk(id, \mathrm{Ad}(v \otimes 1))$ in$kk_0^{\mathrm{alg}}\big(\mathcal{T}_\alpha \mathcal{K} \widehat{\otimes} A\big) \cong kk_0^{\mathrm{alg}}(\mathcal{T}_\alpha A)$ here $\mathrm{Ad}(v \otimes 1)$ is defined by $\mathrm{Ad}(v \otimes 1)(x) = (v \otimes 1)x(v^* \otimes 1)$.

Proposition (3.2.45)[96]: The elements $kk(j)$ and τ are inverse to each other.

Proof. It is clear that $(j) \cdot \tau = 1_A$. To prove that $\tau \cdot kk(j) = 1_{\mathcal{T}_\alpha}$ let ψ be as in the proof of (3.2.26). Let k be the homomorphism from $\mathcal{T}_\alpha$ to $\mathcal{T} \widehat{\otimes} \mathcal{T}_\alpha$ which is given by the restriction of $\psi \otimes \mathrm{id}_{A \rtimes \mathbb{Z}}$ to $\mathcal{T}_\alpha$. The product $\tau \cdot kk(j)$ is given by $kk(k, Ad(v \otimes 1 \otimes 1)k)$. By 8.1, the homomorphism $k: \mathcal{T}_\alpha \to \mathcal{T} \widehat{\otimes} \mathcal{T}_\alpha$ is diffotopic to the sum of $\mathrm{Ad}(v \otimes 1 \otimes 1)k$ and a homomorphism k' defined by $k'(1 \otimes a) = e \otimes 1 \otimes a$, for a in A, and $k'(v \otimes u\,) = e \otimes v \otimes u$. Therefore $\cdot\, kk(j) = kk(k') = 1_{\mathcal{T}_\alpha}$.

Proposition (3.2.46)[96]: Let κ be the inclusion of A into $\mathcal{K}\widehat{\otimes}A$ and s the inclusion of $\mathcal{K}\widehat{\otimes}A$ into $\mathcal{T}_\alpha$. Then the element $kk(s\kappa)\cdot\tau$ of $kk^{\mathrm{alg}}(A,A)$ is equal to $1_A - kk(\alpha^{-1})$.

Proof. j is the sum of $\mathrm{Ad}(v\otimes 1)$ and $s\kappa$. On $\mathcal{T}_\alpha$

$$\mathrm{Ad}(v\otimes 1) = \mathrm{Ad}(1\otimes u^{-1})\mathrm{Ad}(v\otimes u) = \hat{\alpha}^{-1}\mathrm{Ad}(v\otimes u)$$

where $\hat{\alpha}$ is defined on $\mathcal{T}_\alpha$ by $\hat{\alpha}(1\otimes a) = 1\otimes\alpha(a)$ and $\hat{\alpha}(v\otimes u) = v\otimes u$. Since $\mathrm{Ad}(v\otimes u)$ is inner on $\mathcal{T}_\alpha$, one has $kk(\mathrm{Ad}(v\otimes u)) = 1_{\mathcal{T}_\alpha}$ (by a standard argument, inner endomorphisms are diffotopic to id after stabilization by 2×2-matrices). Therefore $kk(\mathrm{Ad}(v\otimes 1))\cdot\tau = kk(\alpha^{-1})$ on A and $1_A = kk(j)\cdot\tau = kk(s\kappa)\cdot\tau + kk(\mathrm{Ad}(v\otimes 1))\cdot\tau$.

Theorem (3.2.47)[96]: For every locally convex algebra D there is an exact sequence

$$\begin{array}{ccccc} kk_0^{\mathrm{alg}}(D,A) & \xrightarrow{(1_A - kk(\alpha^{-1}))} & kk_0^{\mathrm{alg}}(D,A) & \xrightarrow{\cdot kk(q)} & kk_0^{\mathrm{alg}}(D, A\rtimes_\alpha\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ kk_1^{\mathrm{alg}}(D, A\rtimes_\alpha\mathbb{Z}) & \xleftarrow{\cdot kk(q)} & kk_1^{\mathrm{alg}}(D,A) & \xleftarrow{(1_A - kk(\alpha^{-1}))} & kk_1^{\mathrm{alg}}(D,I) \end{array}$$

and a similar exact sequence in the first variable.

Proof. This is the long exact sequence associated with the extension

$$0\to\mathcal{K}\widehat{\otimes}A\to\mathcal{T}_\alpha\to A\rtimes_\alpha\mathbb{Z}\to 0$$

after identifying $kk^{\mathrm{alg}}(D,\mathcal{T}_\alpha)$ with $kk^{\mathrm{alg}}(D,A)$ using the previous two propositions.

Consider the unital algebra $A(D,u)$ generated by two elements D and u, where u is invertible, satisfying the commutation relation $[D,u]u^{-1} = 1$. This relation is used by Connes as one of the simplest examples of a spectral triple, see for instance [111]. The relation can be rewritten as $uDu^{-1} = D - 1$. Thus $A(D,u)$ is the crossed product $\mathbb{C}[D]\rtimes_\alpha\mathbb{Z}$ where $\mathbb{Z}$ acts on the polynomial ring $\mathbb{C}[D]$ by the automorphism determined by $\alpha(D) = D - 1$. Obviously $kk(\alpha) = 1$. Therefore the Pimsner-Voiculescu long exact sequence gives for instance for the K–theory $kk_0^{\mathrm{alg}}(\mathbb{C},A(D,u))\cong R$ and $kk_1^{\mathrm{alg}}(\mathbb{C},A(D,u))\cong R$. By the way, this algebra is isomorphic to the algebra generated by two elements x and z, where z is invertible, satisfying the relation $xz - zx = 1$.

There are many examples of algebras for which a computation of the kk^{alg}–invariants should be possible using similar methods. Consider for instance the algebra $B(D,u)$ generated by two elements u, D, where u and D are both invertible and satisfy the relation $[D,u]u^{-1} = 1$. One finds that $kk_0^{\mathrm{alg}}(\mathbb{C},A(D,u))\cong R$ and $kk_1^{\mathrm{alg}}(\mathbb{C},A(D,u))\cong R$.

Another crossed product construction that has been considered for instance in [112] is the crossed product by a derivation. Assume that A is a locally convex algebra and δ a continuous derivation of A. Define $A \ltimes_\delta \mathbb{N}$ as the algebra generated by A and an element D satisfying the relation $[D, a] = \delta(a)$ for each a ∈ A equipped with the finest locally convex topology such that the map $A \ni a \mapsto D^n a \in A \ltimes \delta$ is continuous for each n. In this way $A \ltimes_\delta \mathbb{N}$ becomes a locally convex algebra. The Weyl algebra is a special case of this construction with $A = \mathbb{C}[x]$. It is an obvious conjecture that $A \ltimes_\delta \mathbb{N}$ should be isomorphic to A in kk^{alg}. The corresponding result in periodic cyclic homology is known, see [113].

We define a variant of bivariant K-theory for locally convex algebras which reduces to the theory developed in [97] when applied to m-algebras. In particular, in this theory, we can control the coefficients and we have $kk(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$. This allows to obtain more precise information in many cases.

Recall that we defined in [97], for a locally convex vector space V, the tensor m-algebra $\hat{T}V$ (denoted simply TV in [97]) as the completion of the algebraic tensor algebra

$$TV = V \oplus V \otimes V \oplus V^{\oplus 3} \oplus \dots$$

with respect to the family $\{\hat{p}\}$ of seminorms, which are given on this direct sum as

$$\hat{p} = p + p \otimes p + p^{\otimes 3} + \cdots$$

where p runs through all continuous seminorms on V. The seminorms $\hat{p}$ are sub-multiplicative for the multiplication on TV. The completion m-algebra.

$\hat{T}V$ therefore is an.

If B is an m-algebra, then the natural homomorphism $TB \to B$ extends to a continuous homomorphism $\hat{T}B \to B$ and we obtain the free extension

$$0 \to \hat{J}B \to \hat{T}B \to B \to 0$$

in the category of m- algebras. It comes equipped with a natural continuous linear splitting $: B \to \hat{T}B$.

Let us now say that an extension

$$0 \to I \to E \overset{s}{\underset{\curvearrowleft}{\to}} D \to 0$$

of arbitrary locally convex algebras equipped with a continuous linear splitting s is weakly admissible (or that s is weakly admissible), if, for any continuous homomorphism $\varphi: B \to D$ from an m-algebra B, the natural homomorphism $\tau: TB \to E$ given by $b_1 \otimes \dots \otimes b_n \mapsto$

$s\varphi(b_1)s\varphi(b_2)\ldots s\varphi(b_n)$ extends to a continuous homomorphism and thus to a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & \hat{J}B & \to & \hat{T}B & \to & B & \to & 0 \\ & & \downarrow & & \downarrow \tau & & \downarrow \varphi & & \\ 0 & \to & I & \to & E & \to & D & \to & 0 \end{array}$$

We say that $0 \to I \to E \overset{s}{\underset{\smile}{\to}} D \to 0$ is admissible (or that s is admissible) if, for any m-algebra , the extension

$$0 \to I \widehat{\otimes} C \to E \widehat{\otimes} C \xrightarrow[\smile]{s\otimes \mathrm{id}_C} D \widehat{\otimes} C \to 0$$

is weakly admissible.

Lemma (3.2.48)[96]: Let $\sigma: B \to TB$ be the canonical linear map and, given $x, y \in B$, let $\omega(x,y) = \sigma(xy) - \sigma(x)\sigma(y)$. Let β be a submultiplicative seminorm on B and α a seminorm on TB which is submultiplicative on JB and satisfies

$$\alpha(\sigma(x)) \leq \beta(x)\,, \alpha(\omega(x,y)) \leq \beta(x)\beta(y)\,, \qquad \alpha(\sigma(x)\omega(y,z)) \leq \beta(x)\beta(y)\beta(z)$$

for all $x, y\,, z$ in B. Then $\alpha \leq \widehat{2\beta}$ for the canonical submultiplicative seminorm $\widehat{2\beta}$ on TB associated to the seminorm 2β.

Proof. We use the decomposition of TB in terms of curvature forms for the standard linear splitting $\sigma: B \to TB$.

Given an algebra B, we denote by ΩB the universal algebra generated by $x \in B$ with relations of B and symbols $dx, x \in B$, where dx is linear in x and satisfies $d(xy) = xd(y) + d(x)y$. We do not impose $d1 = 0$, i.e., if B has a unit, $d1 \neq 0$. ΩB is a direct sum of subspaces $\Omega^n B$ generated by linear combinations of $x_0 dx_1 \ldots dx_n$, and $dx_1 \ldots dx_n, x_j \in B$. This decomposition makes ΩB into a graded algebra. As usual, we write $\deg(\omega) = n$ if $\omega \in \Omega^n B$.

As a vector space, for $n \geq 1$,

$$\Omega^n B \cong \tilde{B} \otimes B^{\otimes n} \cong B^{\otimes(n+1)} \oplus B^{\otimes n} \tag{13}$$

(where $\tilde{B}$ is B with a unit adjoined, and $1 \otimes x_1 \otimes \cdots \otimes x_n$ corresponds to $dx_1 \cdots dx_n$). The operator d is defined on ΩB by

$$d(x_0 dx_1 \cdots dx_n) \;=\; dx_0 dx_1 \cdots dx_n$$
$$d(dx_1 \cdots dx_n) = 0$$

The connection between TB and ΩB is based on the following deformation

(1) The following defines an associative product (Fedosov product) on ΩB:

$$\omega_1 \circ \omega_2 = \omega_1\omega_2 + (-1)^{\deg(\omega_1)} d\omega_1 d\omega_2$$

where $\omega_1\omega_2$ is the ordinary product of ω_1, ω_2 in ΩA. Clearly, the even forms $\Omega^{ev}A$ form a subalgebra of $(\Omega A,\circ)$ for $\circ$.

(2) Let $\sigma: A \to TA$ be the canonical linear inclusion of A into the tensor algebra over A and, for $x, y \in A, \omega(x, y) = \sigma(xy) - \sigma x \sigma y$. Then the map

$$\alpha: x_0 dx_1 dx_2 \ldots dx_{2n-1} dx_{2n} \mapsto \sigma(x_0)\omega(x_1, x_2) \ldots \omega(x_{2n-1}, x_{2n})$$

defines an isomorphism of algebras $(\Omega^{ev}B,\circ) \xrightarrow{\alpha} TB$.

(3) JB is exactly the image of the space of even forms of degree > 0 under the map in (10). Define now a seminorm $\bar{\beta}$ on $TB \cong \Omega^{ev}B$ by

$$\bar{\beta} = \beta_0 + \beta_2 + \beta_4 \ldots$$

where β_{2n} is defined on $\Omega^{2n}B \cong B^{\otimes 2n} \bigoplus B^{\otimes(2n+1)}$ by the projective tensor powers of β. Thus, for instance

$$\bar{\beta}(\sigma(x_0)\omega(x_1, x_2) \ldots \omega(x_{2n-1}, x_{2n})) = \beta(x_0)\beta(x_1) \ldots \beta(x_{2n})$$

Since α is submultiplicative on products of several ω's, it is clear that $\alpha \leq \bar{\beta}$.

Let now $\omega = \omega(x_1, x_2) \ldots \omega(x_{2n-1}, x_{2n})$ be an element in Ω^{2n} that does not start with a $\sigma(x)$. Then, for $b \in B$, $\bar{\beta}(\sigma(b)\omega) = \beta(b)\bar{\beta}(\omega)$ and

$$\bar{\beta}(\sigma(b)\sigma(x_0)\omega = \bar{\beta}(\omega(b, x_0) + \sigma(bx_0)\omega) \leq 2\beta(b)\beta(x_0)\bar{\beta}(\omega) = 2\beta(b)\bar{\beta}(\sigma(x_0)\omega)$$

since β is submultiplicative. This shows that for arbitrary ω' in TB, we have

$$\bar{\beta}(\sigma(b)\omega') \leq 2\beta(b)\bar{\beta}(\omega').$$

Thus, in particular, by induction

$$\bar{\beta}(\sigma(x_0)\sigma(x_1) \ldots \sigma(x_n)) \leq 2^n \beta(x_0)\beta(x_1) \ldots \beta(x_n)$$

Remark (3.2.49)[96]: Let A be any locally convex algebra. If we denote by $\bar{T}A$ the completion of the tensor algebra TA with respect to all seminorms $\bar{\beta}$, defined as in the previous proof, for all continuous seminorms β on A, then $\bar{T}A$ is a locally convex algebra and the extension

$$0 \to \bar{J}A \to \bar{T}A \to A \to 0$$

is admissible.

Lemma (3.2.50)[96]: Assume that $0 \to I \to E \to D \to 0$ is an extension of locally convex algebras with a continuous linear splitting. If I is an m-algebra then the extension is admissible (with the given or with any other linear splitting).

The classifying map $\hat{J}D \to I$ is unique up to homotopy in such an extension.

Proof. Obviously it suffices to show that any such extension is weakly admissible. If $\varphi: B \to D$ is a continuous homomorphism from an m-algebra B and μ is a continuous seminorm on E which is submultiplicative on I, then the pulled back seminorm $\alpha = \mu \circ \tau$ in the diagram

$$\begin{array}{ccccccccc} 0 & \to & JB & \to & TB & \to & B & \to & 0 \\ & & \downarrow & & \downarrow \tau & & \downarrow \varphi & & \\ 0 & \to & I & \to & E & \to & D & \to & 0 \end{array}$$

satisfies the conditions of Lemma (3.2.48) with respect to a continuous submulti-plicative seminorm β on B. The assertion then follows from Lemma (3.2.48)

$$\begin{array}{ccccccccc} 0 & \to & JA & \to & TA & \to & A & \to & 0 \\ & & \downarrow & & \downarrow \tau & & \downarrow \beta & & \\ 0 & \to & I & \to & E & \to & D & \to & 0 \end{array} \qquad (14)$$

Given a locally convex algebra A, we equip the tensor algebra TA with the coarsest locally convex topology that makes all natural maps τ continuous in where the second row is an arbitrary admissible extension and $\beta: A \to D$ a continuous homomorphism between locally convex algebras. We denote the completions of TA and JA with respect to this locally convex structure by $\hat{T}A$ and $\hat{J}A$. Thus the vertical arrows in (14) extend to continuous homomorphisms from the completions $\hat{T}A$ and $\hat{J}A$ and $0 \to \hat{J}A \to \hat{T}A \to A \to 0$ is a universal admissible extension. Note however that, in general, the classifying map $\hat{J}A \to I$ depends on the continuous linear splitting.

The characteristic defining properties of $A \mapsto \hat{T}A$ are

(a) If A is an m-algebra, then $\hat{T}A$ is the m-algebra tensor algebra from [97].

(b) The map $A \mapsto \hat{T}A$ is functorial.

(c) For every m-algebra C, the natural map $T(A \otimes C) \to TA \otimes C$ extends to a continuous map $\hat{T}(A \widehat{\otimes} C) \to \hat{T}A \widehat{\otimes} C$.

Obviously, if B is an m-algebra, then, by universality, $\hat{T}B$ coincides with the tensor m-algebra described above and $\hat{T}B$ and $\hat{J}B$ are m-algebras. Thus, by induction, $\hat{J}^n B$ is an m-algebra for all n.

Definition (3.2.51)[96]: Let A and B be locally convex algebras. We define

$$kk(A,B) = \lim_{\vec{k}} \langle \hat{J}^k A, \mathcal{K} \widehat{\otimes} B(0,1)^k \rangle$$

We also define, for n in $\mathbb{N}$,

$$kk_n\,(A,B) \;=\; kk(\hat{J}^n\, A,B) \qquad kk_{-n}\,(A,B) \;=\; kk(A,B(0,1)^n)$$

The product $kk_n \times kk_m \to kk_{n+m}$ can be defined exactly as for kk^{alg}. Lemma (3.2.10) carries over, if the extension is admissible and J is replaced by $\hat{J}$, showing that the product is well-defined.

As remarked above in (3.2.48), this definition of kk, even though formally slightly different, reduces to the one given in [97] in the case where A and B are m-algebras. In fact, there are obvious maps both ways between the groups defined here and in [97], respectively. It is obvious that these maps are inverse to each other. Thus, the theory kk is an extension of the theory defined in [97] from the category of m-algebras to the category of locally convex algebras. Note that the definition of $\hat{J}A$ above is set up exactly in such a way as to make the theory functorial for maps from an m-algebra B into an arbitrary locally convex algebra A.

In particular, we have the following important and nontrivial fact, cf. [97].

Theorem (3.2.52)[96]: For every Banach algebra B, $kk_*(\mathbb{C},B)$ coincides with the usual topological K-theory $K_*(B)$. In particular, $kk_n(\mathbb{C},\mathbb{C}) = \mathbb{Z}$, for n even, and $kk_n(\mathbb{C},\mathbb{C}) = 0$ for n odd.

The arguments used above to prove the mapping cone exact sequences and the resulting long exact sequences carry over to prove the analogous statements for the theory kk, restricting however to admissible extensions. The crucial fact is that the extensions used in those arguments are admissible. In particular, if $0 \to I \to E \to B \to 0$ is any admissible extension then $0 \to I(0,1) \to E(0,1) \to B(0,1) \to 0$ is admissible and the suspension extension $0 \to A(0,1) \to A[0,1) \to A \to 0$ is admissible for any locally convex algebra A. We do not know if the Toeplitz extension $0 \to \mathcal{K} \widehat{\otimes} A \to \mathcal{T}_0 \otimes A \to A(0,1) \to 0$ is admissible for arbitrary A. Therefore the proof of Bott periodicity for $kk_n(A,B)$ carries over only if A or B is an m-algebra.

Thus $kk_n(A,B)$ is Bott-periodic, if A or B is an m-algebra and we obtain

Theorem (3.2.53)[96]: Let D be an m-algebra. Every extension of locally convex algebras

$$E: 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$$

admitting an admissible linear section, induces exact sequences in $kk(D,\cdot)$ and $kk(\cdot,D)$ of the following form:

$$\begin{array}{ccccc} kk_0(D,I) & \xrightarrow{\cdot kk(i)} & kk_0(D,A) & \xrightarrow{\cdot kk(q)} & kk_0(D,B) \\ \uparrow & & & & \downarrow \\ kk_1(D,B) & \xleftarrow{\cdot kk(q)} & kk_1(D,A) & \xleftarrow{\cdot kk(i)} & kk_1(D,I) \end{array} \tag{15}$$

and

$$\begin{array}{ccccc} kk_0(I,D) & \xleftarrow{kk(i)\cdot} & kk_0(A,D) & \xleftarrow{kk(q)\cdot} & kk_0(B,D) \\ \uparrow & & & & \uparrow \\ kk_1(B,D) & \xrightarrow{kk(q)\cdot} & kk_1(A,D) & \xrightarrow{kk(i)\cdot} & kk_1(I,D) \end{array} \tag{16}$$

The admissible linear section for E defines a classifying map $\hat{J}B \to I$ and thus an element of $kk_1(B,I)$, which we denote by $kk(E)$. The vertical arrows in (15) and (16) are (up to a sign) given by right and left multiplication, respectively, by this class $kk(E)$.

The natural continuous homomorphism $JA \to \hat{J}A$ induces a map $kk(A,B) \to kk^{\mathrm{alg}}(A,B)$. As a consequence of (3.2.31), the map $\mathbb{Z} = kk(\mathbb{C},\mathbb{C}) \to kk^{\mathrm{alg}}(\mathbb{C},\mathbb{C}) = R$ is injective (1 is mapped to $1_A \in HP_0(\mathbb{C},\mathbb{C})$ under the composition of this map and the Chern character ch).

Let us now apply these results to the Weyl algebra. Theorem (3.2.41) shows that we can complete the extension

$$0 \to I \to W' \to W \to 0$$

studied to a linearly split extension of the form

$$0 \to \bar{I} \to \bar{W}' \to W \to 0$$

where $\bar{I}$ is an m-algebra. In fact, the same argument as for I in Lemma (3.2.37) shows that $\bar{I}$ is Morita equivalent in the sense of Lemma (3.2.23) to the closure of $t(1-t)\mathbb{C}[t]$ in $C^\infty[0,1]$, which in turn is diffotopy equivalent to $\mathbb{C}(0,1)$.

The completed extension is admissible and we obtain a long exact sequence in kk, which can be compared to the long exact sequences in kk^{alg} and HP_* in the following commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \to & kk_0(D,\bar{I}) & \to & kk_0(D,\bar{W}') & \to & kk_0(D,W) & \xrightarrow{\delta} & kk_1(D,\bar{I}) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & kk_0^{\mathrm{alg}}(D,\bar{I}) & \to & kk_0^{\mathrm{alg}}(D,\bar{W}') & \to & kk_0^{\mathrm{alg}}(D,W) & \to & kk_1^{\mathrm{alg}}(D,\bar{I}) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & HP_0(D,\bar{I}) & \to & HP_0(D,\bar{W}') & \to & HP_0(D,W) & \to & HP_1(D,\bar{I}) & \to & \cdots \end{array}$$

For $D = \mathbb{C}$, we have $kk_*^{\mathrm{alg}}(\mathbb{C}, \overline{W}') = HP_*(\mathbb{C}, \overline{W}') = 0$ and $kk_0(\mathbb{C}, \bar{I}) = 0$, $kk_0(\mathbb{C}, \bar{I}) = \mathbb{Z}$. We think that it can be shown that $kk(\mathbb{C}, \overline{W}') = 0$, but the proof along the lines of the argument would be very technical, because the necessary continuity conditions are hard to check. At any rate it is very easy to see that the map $(\mathbb{C}, \bar{I}) \to kk(\mathbb{C}, \overline{W}')$ is 0.

However the commutative diagram above contains already enough information for some applications. It shows that $kk_0(\mathbb{C}, W) = G \oplus Z$, where $G \cong kk(\mathbb{C}, \overline{W}') = \ker \delta$ (which is presumably 0) and δ maps the $\mathbb{Z}$-summand isomorphically onto $\mathbb{Z} = kk_1(\mathbb{C}, \bar{I})$. Moreover, the $\mathbb{Z}$-summand is also the image of $\mathbb{Z} = kk_0(\mathbb{C}, \mathbb{C})$ under the natural map $\mathbb{C} \to W$ mapping a complex number to the corresponding multiple of 1. This follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}(0,1) & \to & \mathbb{C}(0,1] & \to & \mathbb{C} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{I} & \to & W' & \to & W & \to & 0 \end{array}$$

(where the map $\mathbb{C}(0,1] \to W'$ maps the generator t of $\mathbb{C}(0,1]$ to $x'y'$ in W') and the induced commutative diagram in the long exact sequences for $kk_*(\mathbb{C},\cdot)$. From this we can deduce for instance that the algebras $M_k(W)$ and $M_l(W)$ of matrices over W can not be isomorphic if $k \neq l$. This could also be shown using ring theoretic methods or from the fact that the algebraic K-group $K_0(W)$ is $\mathbb{Z}$ which follows from [114] , cf. also [115]. The result that we get here is however stronger, since the invariants that we use are diffotopy invariants. Thus, for instance, we can also show that for any homomorphism $\varphi: M_k(\mathbb{C}) \to \mathcal{K} \widehat{\otimes} W$ the image $\varphi(1_k)$ of the unit in $M_k(\mathbb{C})$ can be diffotopic to the unit 1_l of $M_l(W) \subset \mathcal{K} \widehat{\otimes} W$, only if l is a multiple of k.

As another example it follows that $M_k(W \widehat{\otimes} \mathcal{O}_\infty)$ can be isomorphic to $M_l(W \widehat{\otimes} \mathcal{O}_\infty)$ only if $k = l$ (using the fact that $kk_1\big(\mathbb{C}, \bar{I} \widehat{\otimes} \mathcal{O}_\infty\big) = \mathbb{Z}$), or that $W \widehat{\otimes} \mathcal{O}_n$ can be isomorphic to $W \widehat{\otimes} \mathcal{O}_n$ only if $n = m$, for the C^*-algebras $\mathcal{O}_\infty$ and $\mathcal{O}_n$considered in [116].