Chapter 1

Extensions of Abelian Categories and Minimal Projectives

We obtain bounds for $\mathrm{gl.dim}\, A \ltimes_l M$. in the case A is a semisimple ring. We prove the truth of the Strong No-Loops Conjecture for $A \ltimes_l M$. This resolution turns out to be a graded resolution if our algebra and module are graded. We apply this resolution to the study of the Ext-algebra of the algebra; namely, we present a new method for computing Yoneda products using the constructions of the resolutions. We also use our resolution to show a case of the "no loop" conjecture.

Section (1.1): Abelian Categories and the Strong No Loops Conjecture:

Let A be an associative unitary ring, M be an A-bimodule, and $T_A(M)$ be the tensor ring of M. Given a natural number, $l \geq 2$, we denote by $A \ltimes_l M$ the factor ring of $T_A(M)$ by the two-sided ideal J generated by $\otimes_A^l M$ and we call it the truncated extension of A by M of truncation index $l-1$. For $l=2$, the ring $A \ltimes_2 M$ coincides with the well known trivial extension ring of A by M; cf. [2]. Our purpose is to study the group $Ext^n_{A \ltimes_l M} At(X,Y)$, $n \in \mathbb{N} \cup \{0\}$, under the assumption that M is flat as left A-module and $Y(M+J)=O$. This last condition is always satisfied, if Y is a simple $(A \ltimes_l M)$- module. If X and Y are both simple $(A \ltimes_l M)$-modules, then we obtain

$$\mathrm{Ext}^n_{A \ltimes_l M} At(X,Y) \cong \bigoplus_{p+q=n} \mathrm{Ext}^P_A{\sim}(X \otimes_A (\otimes^{\phi(q)} M), Y)\,, n \in \mathbb{N} \cup \{0\}, \qquad (1)$$

where the function $\phi : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N} \cup \{0\}$ is defined by

$$\phi(X) = \begin{cases} l_S & \text{if} \quad X = 2S\,, \\ l_S + 1 & \text{, if} \quad X = 2S\,. \end{cases}$$

If A is a semisimple ring, then our formula implies the truth of the so-called "Strong No-Loops Conjecture" (d. [3]), which in our case asserts the following:

If 5 is a simple $(A \ltimes_l M)$-module with $Ext^n_{A \ltimes_l M}(S\,,S) \neq 0$, then proj.dim $S = \infty$.

We introduce the notion of truncated extension $\mathcal{D}(\mathcal{F},l)$ of an abelian category $\mathcal{D}$ by a right exact endofunctor $\mathcal{F}: \mathcal{D} \longrightarrow \mathcal{D}$. If $l = 2$, then $\mathcal{D}(\mathcal{F},l)$ coincides with the usual trivial extension of $\mathcal{D}$ by $\mathcal{F}$, cf. [2]. In Proposition (1.1.6) we show that if $\mathcal{D}$ has enough projectives, then also $\mathcal{D}(\mathcal{F},l)$ has. By using projective resolutions of objects of $\mathcal{D}(\mathcal{F},l)$, we define the contravariant functor $Ext^n_{\mathcal{D}(\mathcal{F},l)}(-\,,Y)$ to be, as usual, the right nth derived functor of $\mathrm{Hom}_{\mathcal{D}(\mathcal{F},l)}(-\,,Y)$. By using adjoint pairs of functors between $\mathcal{D}$ and $\mathcal{D}(\mathcal{F},l)$, we reduce the computation of $Ext^n_{\mathcal{D}(\mathcal{F},l)}(-\,,Y)$ to the computation of the nth cohomology group of a complex constructed entirely inside $\mathcal{D}$. We prefer to work in this general setting since the category of the right $(A \ltimes_l M)$-modules is isomorphic to the truncated

extension category Mod $A(\mathcal{F}, l)$, where Mod A denotes the category of right A-modules and $\mathcal{F}$ denotes the functor - $\oplus_A M$.

We define a faithful right exact functor Θ from $\mathcal{D}(\mathcal{F}, l)$ to the category Comp$(\mathcal{D})$ of complexes of $\mathcal{D}$ and prove, in Proposition (1.1.9), that any projective object of . $\mathcal{D}(\mathcal{F}, l)$ has acyclic image under Θ. We shall use this result in the proof of our Main Theorem, Theorem (1.1.18).

We deal with double and triple complexes constructed over Mod A. The theory developed is completely independent of the rest of the Section and we hope that it has its own interest. The main result, Proposition (1.1.17), is obtained by applying the theory of spectral sequences and deals with the computation of the total complex of a certain triple complex. We need this result in the next section in order to prove Theorem (1.1.18).

We assume that $\mathcal{D}$ is an abelian category with enough projectives and injectives and that $\mathcal{F}$ is an exact functor. In Theorem (1.1.18) we describe for certain objects of $\mathcal{D}(\mathcal{F}, l)$ the group $Ext^n_{\mathcal{D}(\mathcal{F},l)}(X, Y)$ as the pth cohomology group of a total complex of a bicomplex of $\mathcal{D}$. By using this description we obtain in Proposition (1.1.20) the previously mentioned formula (1) for $Ext^n_{\mathcal{D}(\mathcal{F},l)}(X, Y)$, where X and Y are simple objects of $\mathcal{D}(\mathcal{F}, l)$. This formula enables us to prove in Theorem (1.1.26) the validity of a generalized version of the Strong No-Loops Conjecture for truncated extensions.

We construct truncated extensions of categories of k-linear representations of pairs (Q, S), where Q is a quiver and S is a set of relations. Using Proposition (1.1.27), we calculate the bounds for gl.dim Λ, where Λ is the k-algebra corresponding to (Q, S).

Given an a belian category $\mathcal{D}$ and a covariant additive endofunctor $\mathcal{F}: \mathcal{D} \longrightarrow \mathcal{D}$, there is a well known "tensor extension" $\mathcal{D}(\mathcal{F})$ of $\mathcal{D}$ by $\mathcal{F}$, with objects the pairs (X, f), where $f: \mathcal{D}X \longrightarrow X$ is a morphism of $\mathcal{D}$ and where the set morphism from (X, f) to (Y, g) is the subset of the morphisms $a: X \longrightarrow Y$ of $\mathcal{D}$, satisfying the condition $g \circ \mathcal{F}a = a \circ f$. The composition in $\mathcal{D}(\mathcal{F})$ is the same as that in $\mathcal{D}$.

For any object (X, f) of $\mathcal{D}(\mathcal{F})$, we denote by f_0 the identity morphism id_x of X, by f_1, the morphism f and by $f_n: \mathcal{F}^n X \longrightarrow X, n \in \mathbb{N}$, the composition $f \circ \mathcal{F}f \circ \cdots \circ \mathcal{F}^{n-1}f$.

Given a natural number $l \geq 2$ we denote by $\mathcal{D}(\mathcal{F}, l)$ the full subcategory of $\mathcal{D}(\mathcal{F})$ with objects the (X, f) satisfying $f_l = O$.

Definition (1.1.1)[1]: The category $\mathcal{D}(\mathcal{F}, l)$ is said to be the truncated extension of $\mathcal{D}$ by $\mathcal{F}$ of truncation index $l - 1$.

It can be easily verified that if $\mathcal{F}$ is right exact, then $\mathcal{D}(\mathcal{F})$ and $\mathcal{D}(\mathcal{F}, l)$ are abelian categories.

In the following we assume always that $\mathcal{D}$ is an abelian category and $\mathcal{F}$ is a covariant, additive, right exact functor.

For $l = 2$, the truncated extension $\mathcal{D}(\mathcal{F}, 2)$ coincides with the trivial extension of $\mathcal{D}$ by $\mathcal{F}$, cf. [2].

We introduce the following functors between $\mathcal{D}(\mathcal{F})$ and $\mathcal{D}$:

(i) The functor $\mathcal{Z}: \mathcal{D} \to \mathcal{D}(\mathcal{F})$ is defined on the objects X of $\mathcal{D}$ by $\mathcal{Z}X = (X, O_x)$ and on the morphisms $a: X \to Y$ of $\mathcal{D}$ by $\mathcal{Z}a = a$. In the following we write just 0 for O_x

(ii) Our next functor $\mathcal{U}: \mathcal{D}(\mathcal{F}) \to \mathcal{D}$ is defined on the objects (X, f) of $\mathcal{D}(\mathcal{F})$ by $\mathcal{U}(X, f) = X$ and on the morphisms $a: (X, f) \to (Y, g)$ by $\mathcal{U}a = a$.

(iii) The functor $\mathcal{g}: \mathcal{D}(\mathcal{F}) \to \mathcal{D}$ is defined as follows:

Given an object (X, f) of $\mathcal{D}$, we consider the cokernel $\pi_{(x.j)}: X \to \mathrm{Coker} f$ of f and define $\mathcal{g}(X, f)$ to be equal to Coker f.

Given a morphism $a: (X, f) \to (Y, g)$ of $\mathcal{D}(\mathcal{F})$, there is a uniquely determined morphism $\xi: \mathrm{Coker}\, f \to \mathrm{Coker}\, g$ making the right square

$$\begin{array}{ccccc} \mathcal{F}X & \xrightarrow{f} & X & \xrightarrow{\pi_{(X.f)}} & \mathrm{Coker}\, f \\ \downarrow \mathcal{F}u & & \downarrow a & & \downarrow \xi \\ \mathcal{F}Y & \xrightarrow{g} & Y & \xrightarrow{\pi_{(Y.g)}} & \mathrm{Coker}\, g \end{array}$$

commutative since the left square is always commutative. We define $\mathcal{g}_a$ to be equal to ξ

The above defined functors can be also considered as functors between $\mathcal{D}$ and $\mathcal{D}(\mathcal{F}, l)$.

We shall denote these restricted functors again by $\mathcal{Z}, \mathcal{U}$, and $\mathcal{g}$.

(iv) There is a fourth functor $\mathcal{F}: \mathcal{D} \to \mathcal{D}(\mathcal{F}, l)$, which depends on the choice of the truncation index.

Given an object X of $\mathcal{D}$, we consider the object

$$TX = X \oplus \mathcal{F}X \oplus \cdots \oplus \mathcal{F}^{l-1}X$$

and the morphism $t(X): \mathcal{F}TX \to TX$ defined by the $l \times l$ matrix

$$t(x) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1_{\mathcal{F}x} & 0 & \cdots & 0 & 0 \\ 0 & 1_{\mathcal{F}^2 x} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{\mathcal{F}^{l-1} x} & 0 \end{pmatrix}$$

Since $t(X)_l = 0$, the pair $(TX, t(X))$ is an object of $\mathcal{D}(\mathcal{F}, l)$.

Given a morphism $a\colon X \to Y$ of $\mathcal{D}$ we consider the morphism $t(a)\colon TX \to TY$ defined by the $l \times l$ matrix

$$t(a) = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{F}_a & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \mathcal{F}_a^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \mathcal{F}_a^{l-1} \end{pmatrix}$$

Since $t(a) \circ t(X) = t(Y) \circ \mathcal{F}t(a)$, the morphism $t(a)$ is a morphism of $\mathcal{D}(\mathcal{F}, l)$.

Lemma (1.1.2)[1]: (i) The pairs $(\mathcal{Z}, \mathcal{g})$ and $(\mathcal{g}, \mathcal{U})$ form adjoint pairs of functors.

(ii) The composed functor $\mathcal{g}\mathcal{F}$ is naturally equiualent to the identity functor $\mathrm{Id}_{\mathcal{g}}$ on $\mathcal{D}$.

Proof. Compare [4, Lemma 2.1].

Remark (1.1.3)[1]: (i) Let $\epsilon\colon \mathrm{Id}_{\mathcal{D}(\mathcal{F},l)} \to \mathcal{Z}\mathcal{g}$ denote the unit of $(\mathcal{g}, \mathcal{Z})$. For any object (X, f), the morphism $\mathcal{U}\epsilon_{(x,f)}$ equals the canonical epimorphism $\pi_{(X,f)}\colon X \to \mathrm{Coker} f$

(ii) Let $a\colon \mathcal{F}\mathcal{U} \to \mathrm{Id}_{\mathcal{D}(\mathcal{F},l)}$ denote the counit of $(\mathcal{F}, \mathcal{U})$. For any object (X, f), the morphism $\alpha_{(x.f)}$ equals $f_0, f_1, \dots, f_{l-1}$ In particular $\alpha_{(x.f)}$ is an epimorphism for any object (X, f) of $\mathcal{D}(\mathcal{F}, l)$.

Lemma (1.1.4)[1]: (i) An object (X, f) of $\mathcal{D}(\mathcal{F}, l)$ is the zero object if and only if $\mathcal{g}(\mathcal{F}, f)$ is the zero object of $\mathcal{D}$,

(ii) If $p\colon (X, f) \to (Y, g)$ is an epimorphism with $\mathcal{g}_p = 0$, then p is the zero morphism

Proof. (i) Since $\mathcal{g}(\mathcal{F}, l)$ is zero, f has to be an epimorphism, The right exactness of $\mathcal{F}$ implies that f_n has to be also an epimorphism for any $n \in \mathbb{N}$. Since $f_l = 0$, the epimorphism f equals zero, Hence, X is the zero object of $\mathcal{D}$.

(ii) Since $\mathcal{g}_p = 0$ is an epimorphism, $\mathcal{g}(y, g)$ is the zero object of $\mathcal{D}$. Hence, (Y, g) is the zero object of $\mathcal{D}(\mathcal{F}, l)$ and $p = 0$.

We recall (cf. [2]) that an epimorphism $p\colon X \to Y$ is said to be essential, if for any $q\colon X \to Y$, with $p \circ q$ an epimorphism, it follows that q is also an epimorphism.

Lemma (1.1.5)[1]: For any object $(X,\ f)$ of $\mathcal{D}(\mathcal{F},l)$ the canonical epimorphism $\epsilon_{(x.f)}$ is essential.

Proof. Let p: $(X,f) \to (\mathrm{Coker}a,h)$ be a cokernel of a: $(Y,g) \to (X,f)$. The unit E induces the commutative diagram:

$$\begin{array}{ccc} (X,f) & \xrightarrow{\epsilon_{(x.f)}} & \mathcal{Z}\mathcal{g}(X,f) \\ p\downarrow & & \downarrow \mathcal{Z}\mathcal{g}_p \\ (\mathrm{Coker}a,h) & \xrightarrow{\epsilon_{(\mathrm{Coker}a,h)}} & \mathcal{Z}\mathcal{g}(\mathrm{Coker}a,h) \end{array}$$

Suppose that $\epsilon_{(x.f)} \circ a$ is an epimorphism. Since $\mathcal{g}_p \circ \epsilon_{(x.f)} \circ a = \epsilon_{(\mathrm{Coker}a,h)} \circ p \circ a = 0$, we have $\mathcal{Z}\mathcal{g}_p = 0$ and since $\mathcal{Z}$ is faithful, we have $\mathcal{g}_p = 0$. By Lemma (1.1.4), the morphism p has to be zero. Hence a is an epimorphism.

Proposition (1.1.6)[1]: (i) If P is a projective object of $\mathcal{D}$ then $\mathcal{F}P$ is a projective object of $\mathcal{D}(\mathcal{F},l)$.

(ii) An object $(X,\ f)$ of $\mathcal{D}(\mathcal{F},l)$ is projectu:e if and only if $\mathcal{g}(X,f)$ is projectiv*e* and $(X,\ f)$ is isomorphic to $\overline{\mathcal{F}\mathcal{g}}(X,f)$.

(iii) If $\mathcal{D}$ has enough projectioes, then also $\mathcal{D}(\mathcal{F},l)$ has enough projec lives.

Proof. (i) The functor $\mathcal{F}$ sends projectives to projectives, since it is left adjoint to the exact functor $\mathcal{U}$.

(ii) If $(X,\ f)$ is projective, then $\mathcal{g}(X,f)$ is also projective because $\mathcal{g}$ is left adjoint to the exact functor $\mathcal{Z}$. Hence, $\mathcal{F}\mathcal{g}(X,f)$ is projective and there is a morphism p: $\mathcal{F}\mathcal{g}(X,f) \to (X,\ f)$ making the diagram

$$\begin{array}{ccccc} & & \mathcal{F}\mathcal{g}(X,f) & & \\ & & \downarrow \epsilon_{\mathcal{F}\mathcal{g}(x.f)} & & \\ (X,f) & \xrightarrow{\epsilon_{(x.f)}} & \mathcal{Z}\mathcal{g}(X,f) & \cong & \mathcal{Z}\mathcal{g}\mathcal{F}\mathcal{g}(X,f) \end{array}$$

commutative, i.e $\epsilon_{(x.f)} \circ p = \epsilon_{\mathcal{F}\mathcal{g}(x.f)}$ Since p is an essential epimorphism which splits, it has to be an isomorphism.

(iii) Given an object $(X,\ f)$ of $\mathcal{D}(\mathcal{F},l)$, we consider the epimorphism $\mathcal{U}_{\epsilon_{(x.f)}}$: $\mathcal{U}(X,f) \to \mathcal{U}\mathcal{Z}\mathcal{g}(X,f)$, Since $\mathcal{D}$ has enough projectives, there is an epimorphism p: $P \to \mathcal{g}(X,\ f)$ with P projective. Let q: $P \to \mathcal{U}(X,f)$ be a lifting of p over $\mathcal{U}_{\epsilon_{(x.f)}}$ i.e., $\mathcal{U}_{\epsilon_{(x.f)}} \circ q = p$.

From $\epsilon_{(x.f)} \circ \alpha_{(x.f)} = \alpha_{\mathcal{Z}\mathcal{G}(x.f)} \circ \mathcal{F}\mathcal{U}_{\epsilon_{(x.f)}}$ we obtain $\epsilon_{(x.f)} \circ \alpha_{(x.f)} = \mathcal{F}q \circ \alpha_{\mathcal{Z}\mathcal{G}(x.f)} \circ \mathcal{F}\left(\mathcal{U}_{\epsilon_{(x.f)}} \circ q\right) = \alpha_{\mathcal{Z}\mathcal{G}(x.f)} \circ \mathcal{F}q$. Since this last morphism is an epimorphism and since $\epsilon_{(x.f)}$ is essential, $\alpha_{(x.f)} \circ \mathcal{F}q : \mathcal{F}P \to (X, f)$ has to be an epimorphism, Hence, assuming that $\mathcal{D}$ has enough projectives, we may define the right nth derived functors of $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ and $\mathrm{Hom}_{\mathcal{D}(\mathcal{F},l)}(-, (Y, g)), \forall n \in \mathbb{N}$.

We denote these functors by $Ext^n_{\mathcal{D}}(-, y)$ and $Ext^n_{\mathcal{D}(\mathcal{F},l)}(-, (Y, g))$, respectively.

Our aim is to give a formula for $Ext^n_{\mathcal{D}(\mathcal{F},l)}\big((X, f), \mathcal{Z}Y\big)$ using $t^P_{\mathcal{F}}\ (X', Y'), P \in \mathbb{N} \cup \{0\}$. Let

$$P\big((X, f)\big)_*: \cdots \longrightarrow \mathcal{F}P_{i-1} \xrightarrow{q_{i-1}} \mathcal{F}P_i \xrightarrow{q_i} \mathcal{F}P_{i+1} \to \cdots \to \mathcal{F}P_{-1} \xrightarrow{q_{-1}} \mathcal{F}P_0 \longrightarrow 0$$

be a deleted projective resolution of an object $(X,\ f)$.

By applying the functor $\mathrm{Hom}_{\mathcal{D}(\mathcal{F},l)}(-, \mathcal{Z}Y)$ on $P\big((X, f)\big)_*$ and using the fact that $(\mathcal{G}, \mathcal{Z})$ is an adjoint pair of functors and that $\mathcal{G}\mathcal{Z} = \mathrm{Id}_{\mathcal{D}}$ we see that $Ext^n_{\mathcal{D}(\mathcal{F},l)}\big((X, f), \mathcal{Z}Y\big)$ is isomorphic to the nth cohomology group of the complex

$$0 \to \mathrm{Hom}_{\mathcal{D}}(P_0, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}q_1, Y)} \mathrm{Hom}_{\mathcal{D}}(P_{-1}, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}q_2, Y)} \mathrm{Hom}_{\mathcal{D}}(P_{-2}, Y) \tag{2}$$

$$\to \cdots \to \mathrm{Hom}_{\mathcal{D}}(P_{i+1}, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}q_i, Y)} \mathrm{Hom}_{\mathcal{D}}(P_i, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}q_{i-1}, Y)} \mathrm{Hom}_{\mathcal{D}}(P_{i-1}, Y)$$

Hence, in order to compute $Ext^n_{\mathcal{D}(\mathcal{F},l)}\big((X, f), \mathcal{Z}Y\big)$, we must compute the nth cohomology group of the complex (2).

Example (1.1.7)[1]: Let A be unitary ring and Mod A be the category of the right A-modules. Given an $(A - A)$-bimodule M, we form the truncated extension $A \ltimes_l M$ of truncation index $l - 1$.

The category Mod $A \ltimes_l M$ of the right $(A \ltimes_l M)$-modules is isomorphic to the truncated extension Mod $A(- \otimes_A M, l)$.

In this section our aim is to relate the category $\mathcal{D}(\mathcal{F}, l)$ with the category of complexes Comp($\mathcal{D}$) over $\mathcal{D}$, via an additive, faithful functor from $\mathcal{D}(\mathcal{F}, l)$ to Comp($\mathcal{D}$).

For any object $(X,\ f)$ of $\mathcal{D}$, there is a short complex

$$\mathcal{F}^{l+1}X \xrightarrow{\mathcal{F}^l f_1} \mathcal{F}^l X \xrightarrow{\mathcal{F} f_{l-1}} \mathcal{F}\ X \xrightarrow{f_1} X\ , \tag{3}$$

since $f_l = f_{l-1} \circ \mathcal{F}^{l-1} f_1 = f_1 \circ \mathcal{F}\ f_{l-1}$ and $f_l = 0$.

This short complex (3) extends to the left by applying powers of $\mathcal{F}^l$. In this way, we obtain a complex with

$$X_{-n} = \begin{cases} \mathcal{F}^{(s-l)l+1}\, l_{l-1}, & \text{if } \; n = 2s \\ \mathcal{F}^{sl+1}\, l_1, & \text{if } n = 2s+1 \end{cases} \qquad \forall\, n \in \mathbb{N}$$

and

$$d_{-n} = \begin{cases} \mathcal{F}^{sl}\, X, & \text{if } n = 2s \\ \mathcal{F}^{sl+1}\, X, & \text{if } n = 2s+1 \end{cases} \qquad \forall\, n \in \mathbb{N}\,.$$

Furthermore, we define

$$X_n = \begin{cases} X\,, & \text{if } \; n = 0 \\ Coker\, f\,, & \text{if } \; n = 1 \\ 0 \;\;, & \text{if } \; n \geq 2 \end{cases} \qquad \forall\, n \in \mathbb{N} \cup \{0\}$$

and

$$d_n = \begin{cases} \epsilon_{(X,f)}\,, & \text{if } \; n = 0 \\ 0, & \text{if } \; n \geq 2 \end{cases} \qquad \forall\, n \in \mathbb{N} \cup \{0\}$$

Definition (1.1.8)[1]: The resulting complex $(X_n, d_n)_{n\in\mathbb{N}}$ obtained from $(X,\ f)$ is said to be the standard complex of $(X,\ f)$. In the following we shall denote this complex by $(X,\ f)_*$.

We shall use the notion of the standard deleted complex $d(X,f)_*$ of (X,f). This is the complex obtained from $(X,\ f)_*$ by replacing $X_1 = \text{Coker}\, f$ and $d_0 = \epsilon_{(X.f)}$ by the zero module and the zero homomorphism, respectively.

Given a morphism $\alpha: (X,\ f) \to (Y, g)$, there is a morphism $\alpha_* : (X,\ f) \to (Y, g)_*$ defined as

$$a_{-n} = \begin{cases} \mathcal{F}^{sl}\, \alpha, & \text{if } n = 2s \\ \mathcal{F}^{sl+1}\, \alpha, & \text{if } n = 2s+1 \end{cases} \qquad \forall\, n \in \mathbb{N}$$

and

$$a_n = \begin{cases} \alpha\,, & \text{if } \; n = 0 \\ \alpha_1\,, & \text{if } \; n = 1 \\ 0\;, & \text{if } \; n \geq 2 \end{cases} \qquad \forall\, n \in \mathbb{N} \cup \{0\}$$

where α_1 is the unique morphism making the diagram

$$\begin{array}{ccccccc} X_0 & = & X & \xrightarrow{\epsilon_{(X,f)}} & X_l & = & \text{Coker}\, f \\ \downarrow \alpha & & & & & & \downarrow \alpha_1 \\ Y_0 & = & Y & \xrightarrow{\epsilon_{(Y,g)}} & Y_l & = & \text{Coker}\, g \end{array}$$

commutative. Hence, there is an additive, faithful, right exact functor $\Theta: \mathcal{D}(\mathcal{F}, l) \to$

$\mathrm{Comp}(\mathcal{D})$ defined on the objects (X,f) by $\Theta(X,f)=(X,f)_*$ and on the morphisms $\alpha\colon (X,f)\to(Y,g)$ by $\Theta\alpha=\alpha_*\colon (X,f)_*\to(Y,g)_*$.

Proposition (1.1.9): For any object X of $\mathcal{D}$, the standard complex $\Theta\mathcal{F}$ is acyclic.

Proof. Let us consider the differential $d_{-n}\colon (TX)_{-n}\to (TX)_{-n+1}$, $n\in\mathbb{N}$. If $n=2s$, then d_{-n} is an $l\times l$ matrix $(\mathcal{T}_{ij})$, with $\mathcal{T}_{l,l-1}=1_{\mathcal{F}^{sl}X}$ and zero elsewhere. Hence, $\mathrm{Ker}d_{-n}=\mathcal{F}^{sl}[\mathcal{F}X\oplus\ \cdots\oplus\mathcal{F}^{l-1}X]$ and $\mathrm{Im}d_{-n}=\mathcal{F}^{sl}X$.

If $n=2s+1$, then d_{-n} is an $l\times l$ matrix (σ_{ij}), with$\sigma_{k+1,k}=1_{\mathcal{F}^{sl+k}X}$ for $l\le k\le l-1$ and zero elsewhere. Hence, $\mathrm{Ker}\ d_{-n}=\mathcal{F}^{(s+1)l}X$ and $\mathrm{Im}d_{-n}=\mathcal{F}^{sl}[\mathcal{F}X\oplus\cdots\oplus\mathcal{F}^{l-1}X]$.

Proposition (1.1.10)[1]: For any projective object $(X,\ f)$ of $\mathcal{D}(\mathcal{F},l)$, the standard complex $\Theta(X,f)$ is acyclic.

Proof. By Proposition (1.1.6)(ii), any projective object of $\mathcal{D}(\mathcal{F},l)$ is of the form $\mathcal{F}X$.

In the following, we shall use the functor Θ in order to estimate the group $Ext^n_{\mathcal{D}(\mathcal{F},l)}\big((X,f),\mathcal{Z}Y\big)$. For this reason we need to develop some facts on triple complexes.

The results exposed here are independent of the remaining parts of the section. We consider them as a tool for studying homological behavior of various algebraic systems. Let R be an associative ring with identity element and let Mod R the category of unitary right R-modules,

We recall (cf. [5]) that a triple complex of Mod R is a quadruple $(A,\ x,y,z)$, where $A=(A_{i,j,k})_{(i,j,k)\in\mathbb{Z}^3}$ is a family of R-modules, x = $x=(x_{i,j,k})_{(i,j,k)\in\mathbb{Z}^3}$, $y=(y_{i,j,k})_{(i,j,k)\in\mathbb{Z}^3}$ and $z=(z_{i,j,k})_{(i,j,k)\in\mathbb{Z}^3}$ are families of R-module homomorphisms

$$x_{i,j,k}\colon A_{i,j,k}x_{i,j,k}\to A_{i-1,j,k}\ ,y_{i,j,k}\colon A_{i,j,k}x_{i,j,k}\to A_{i\ ,j-1,k},$$
$$z_{i,j,k}\colon A_{i,j,k}x_{i,j,k}\to A_{i\ ,j,k-1}$$

satisfying the identities

$$x^2\ =0\,,\qquad y^2=0\,,\qquad z^2\ =0\,,$$
$$y\circ x+x\circ y=0\,,\qquad x\circ z+z\circ x=0\,,\qquad y\circ z+z\circ y=0$$

In the following we shall denote $(A,\ x,y,z)$ just by A_{***}.

Definition (1.1.11)[1]: A triple complex A_{***} is said to be a negative cone triple complex, if for any positive integer i,j, or k, the R-module $A_{i,j,k}$ equals the zero R-module.

Starting with a triple complex, there are various constructions leading to double or single complexes. In the next lines we shall exhibit some of these constructions for later use.

(i) The first construction yields a single complex and it is a classical on Given a negative cone triple complex A_{***} we consider its total complex $\mathrm{Tot}(A)_* = (T(A)_n, t(A)_n)_{n\in\mathbb{Z}}$ where $T(A)_n = \oplus_{i+j+k=n} A_{i,j,k}$ and where $t(A)_n$ is the canonical restriction of $x + y + Z$ to $T(A)_n$.

(ii) The second construction yields double complexes.

Given a negative cone triple complex A_{***}, we choose one of the families of its morphisms, say the family x, and we consider the double complex

$$\mathrm{Part}_x(A)_{**} = \left(P_x(A)_{n,m}\,, P_x(A)_{n,m}\,, q_x(A)_{n,m}\right)_{n,m\in\mathbb{Z}^2}$$

with $P_x(A)_{n,m} = \oplus_{j+k=m} A_{n,j,k}$ where $P_x(A)_{n,m}: P_x(A)_{n,m} \to P_x(A)_{n-1,m}$ is the canonical restncuon of x to $P_x(A)_{n,m}$ and $q_x(A)_{n,m}: P_x(A)_{n,m} \to P_x(A)_{n-1,m}$ is the canonical restriction of $y + Z$ to $P_x(A)_{n,m}$.

The double complex $\mathrm{Part}_x(A)_{**}$ is a third quadrant double complex.

Definition (1.1.12)[1]: We call $\mathrm{Part}_x(A)_{**}$ the partial double complex to the direction x.

By choosing one of the families y or z; we construct similarly the partial double complexes $\mathrm{Part}_y(A)_{**}$ and $\mathrm{Part}_z(A)_{**}$.

Lemma (1.1.13)[1]: Let A_{***} be a negative cone triple complex.

The total complex $\mathrm{Tot}\left(\mathrm{Part}_x(A)\right)_*$ of $\mathrm{Part}_x(A)_{**}$ coincides with the total complex $\mathrm{Tot}(A)_*$ of A_{***}.

The same holds also for the families x and z ,

(iii) The next construction yields also double complexes.

Given a triple complex A_{***}, we choose one of the families of its morphisms, say the family x, and a fixed integer $\alpha \in \mathbb{Z}$.For any pair $(j,k) \in \mathbb{Z}$, we consider the commutative diagram with exact rows,

$$\begin{array}{ccccccc}
0 & \to & \ker x_{\alpha,j,k} & \xrightarrow{l_{\alpha,j,k}} & A_{\alpha,j,k} & \xrightarrow{x_{\alpha,j,k}} & A_{\alpha-1,j,k} \\
 & & \downarrow \bar{y}_{\alpha,j,k} & & \downarrow y_{\alpha,j,k} & & \downarrow y_{\alpha-1,j,k} \\
0 & \to & \mathrm{Ker} x_{\alpha,j-1,k} & \xrightarrow{l_{\alpha,j-1,k}} & A_{\alpha,j-1,k} & \xrightarrow{-x_{\alpha,j-1,k}} & A_{\alpha-1,j-1,k}
\end{array}$$

where $\bar{y}_{\alpha,j,k}$ exists and is uniquely determined, because of the commutativity of the right-hand square.

A similar construction yields a uniquely determined homomorphism

$\bar{z}_{\alpha,j,k} : \mathrm{Ker} z_{\alpha,j,k} \rightarrow \mathrm{Ker} z_{\alpha,j,k-1}$ satisfying $z_{\alpha,j,k} \circ l_{\alpha,j,k} = l_{\alpha,j,k-1} \circ \bar{z}_{\alpha,j,k}$.

One sees easily that the triple

$$K[\alpha]_{y,z}(A)_{**} = \left(\ker x_{\alpha,j,k}, \bar{y}_{\alpha,j,k}, \bar{z}_{\alpha,j,k} \right)$$

is a double complex.

By choosing, instead of x, the family of y or z, we may also construct the double complexes $K[\alpha]_{x,z}(A)_{**}$ and $K[\alpha]_{x,y}(A)_{**}$.

If A_{***} is a negative cone triple complex, then $K[\alpha]_{y,z}(A)_{**}$, $K[\alpha]_{x,z}(A)_{**}$ and $K[\alpha]_{x,y}(A)_{**}$ are all third quadrant double com- plexes,

Remark (1.1.14)[1]: There is a well known analogue of the previous construction in the case of double complexes:

Given a double complex $A_{**} = (A, x, y)$, we choose one of the families of morphisms, say the family x and a fixed integer $\alpha \in \mathbb{Z}$ Now, the family of the kernels Ker $x_{\alpha,j}$ together with the induced family of morphisms $\bar{y}_{\alpha,j}$: Ker $x_{\alpha,j} \rightarrow$ Ker $x_{\alpha,j-1}$ forms a complex $K[\alpha]_y(A)_*$. By choosing the family y of A, we obtain $K[\alpha]_x(A)_*$.

Lemma (1.1.15)[1]: Let A_{***} be a negatire cone triple complex and

$$\mathrm{Part}_x(A)_{**} = \left(P_x(A)_{n,m}, P_x(A)_{n,m}, q_x(A)_{n,m}\right)_{n,m\in\mathbb{Z}^2}$$

its partial double complex to the direction x.

For any non-positive integer a, the complex $\mathrm{Tot}K[\alpha]_{y,z}(A)_*$ coincides with the complex $K[\alpha]_q$, $(\mathrm{Part}_x(A)_*$.

Proof. Let $(X_i, \chi_i)_{i\in\mathbb{Z}}$ denote the complex $\mathrm{Tot}K[\alpha]_{y,z}(A)_*$ .For any non-positive integer ω , it holds that $X_\omega = \oplus_{j+k=\omega}\mathrm{Ker}\, x_{\alpha,j,k}$ Moreover, $X_\omega : X_\omega \rightarrow X_{\omega-1}$ restricted on the direct summand Ker $x_{\alpha,j,k}$ equals

$$\begin{pmatrix} \bar{y}_{\alpha,j,k} \\ \bar{z}_{\alpha,j,k} \end{pmatrix}.$$

Let $(\Psi_i, \psi_i)_{i\in\mathbb{Z}}$ denote the complex $K[\alpha]_q$, $(\mathrm{Part}_x(A)_*$. Since Ψ_ω, is the kernel of $\mathrm{P}_x(A)_{\alpha,\omega}$: $\bigoplus_{j+k=\omega} A_{\alpha,j,k} \rightarrow \bigoplus_{j+k=\omega} A_{\alpha-1,j,k}$ and since $\mathrm{P}_x(A)_{\alpha,\omega}$ restncted on $A_{\alpha,j,k}$ equals $x_{\alpha,j,k}: A_{\alpha,j,k} \rightarrow A_{\alpha-1,j,k}$ we obtain

$$\Psi_\omega = \mathrm{X}_\omega.$$

The restriction of $\psi_\omega: \Psi_\omega \rightarrow \Psi_{\omega-1}$ on $\mathrm{Ker}\, x_{\alpha,i,j}$ coincides with

$$\begin{pmatrix} \bar{y}_{\alpha,j,k} \\ \bar{z}_{\alpha,j,k} \end{pmatrix},$$

because

$$\begin{pmatrix} y_{\alpha,j,k} \\ z_{\alpha,j,k} \end{pmatrix} \circ l_{\alpha,j,k} = \begin{pmatrix} l_{\alpha,j-1,k} & 0 \\ 0 & l_{\alpha,j,k-1} \end{pmatrix} \circ \begin{pmatrix} \bar{y}_{\alpha,j,k} \\ \bar{z}_{\alpha,j,k} \end{pmatrix}.$$

Hence $\psi_\omega = \chi_\omega$.

Definition (1.1.16)[1]: A negative cone triple complex A_{***} is said to be exact to the direction x, if for any fixed pair (n, m) of non-positive integers the complex $A_{m,n} = (A_{i,m,n}, x_{i,m,n})_{i\in\mathbb{Z}}$ is exact.

One may give analogous definitions for the directions y or z, There is also a similar definition for third quadrant double complexes, where only two directions are involved.

Proposition (1.1.17)[1]: Let $A_{***} = (A,\ x, y,\ z)$ be a negative cone triple complex. If A_{***} is exact to the direction x, then the nth cohomology group $\mathcal{H}^n((\mathrm{Tot}(A)_*))$ is isomorphic to the nth cohomology group $\mathcal{H}^n\big(\mathrm{Tot}K[0]_{y,z}(A)_*\big)$.

Proof. Because of Lemma (1.1.12), $\mathcal{H}^n(\mathrm{Tot}(A)_*) \cong \mathcal{H}^n(\mathrm{Tot}(\mathrm{part}_x(A)\,)_*)$. Since A_{***} is exact to the direction x, the third quadrant double complex $\mathrm{part}_x(A)_{**}$ is exact to the direction P_x. So, the usual spectral sequence of $\mathrm{part}_x(A)_{**}$ collapses and $\mathcal{H}^n(\mathrm{Tot}(\mathrm{part}_x(A)\,)_*) \cong \mathcal{H}^n\big(K[0]_{P_x}(\mathrm{part}_x(A)\,)_*\big)$, cf. [6]. Because of Lemma (1.1.14), we have

$$\mathcal{H}^{\mathrm{n}}\big(\mathrm{K}[0]_{\mathrm{P_x}}(\mathrm{part_x}(\mathrm{A})\,)_*\big) \cong \mathcal{H}^{\mathrm{n}}\big(\mathrm{TotK}[0]_{\mathrm{y,z}}(\mathrm{A})_*\big)\,.$$

In the remaining part of the section we make the following three assumptions:

(i) The functor $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{D}$ is exact.

(ij) The category $\mathcal{D}$ has enough projectioes.

(iii) The category $\mathcal{D}$ has enough injectiues.

Our aim in this section is the computation of $\mathrm{Ext}^{n}_{\mathcal{D}(\mathcal{F},l)}\big((X,f),\mathcal{Z}Y\big)$ by using information from $\mathcal{D}$ and $\mathcal{F}$.

Let Y be an object of $\mathcal{D}$ and $(X,\ f)$ be an object of $A \ltimes_l M$. Let

$$I_*\ (Y): I_0 \xrightarrow{p_0} I_{-1} \xrightarrow{p_1} I_{-2} \to \cdots \to I_k \xrightarrow{p_k} I_{k-1} \to \cdots$$

be a deleted injective resolution of Y and

$$P(X,f)_* :\ \cdots \to \mathcal{F}P_{i-1} \xrightarrow{q_{i-1}} \mathcal{F}P_i \xrightarrow{q_i} \mathcal{F}P_{i+1} \longrightarrow \cdots \longrightarrow \mathcal{F}P_{-1} \xrightarrow{q-1} \mathcal{F}P_0 \longrightarrow 0$$

be a deleted projective resolution of (X,f).

Let $d(X,f)_*$ be the deleted standard complex of $(X,\ f)$.

Theorem (1.1.18)[1]: Let W_{**} be the third quadrant double complex $\mathrm{Hom}_{\mathcal{D}}\big(d(X,f)_*\ , I_*\ (Y)\big)$.

The group $\mathrm{Ext}^{n}_{\mathcal{D}(\mathcal{F},l)}\big((X,f),\mathcal{Z}Y\big)$ is isomorphic to the nth cohomology group $\mathcal{H}^n(\mathrm{Tot}(W)_*)$, $\forall n \in \mathbb{N} \cup \{0\}$.

Proof. Applying on $P(X,f)_*$ the functor $\Theta: \mathcal{D}(\mathcal{F},l) \to \mathrm{Comp}(\mathcal{D})$, we obtain a third quadrant double complex $S_{**} = (S,t,r)$, where the rows t are the deleted standard complexes of $\mathcal{F}P_i$ and where the columns r consist of the images of $P(X,f)_*$ under $\mathcal{F}^m$, with $m = sl$ or $sl + 1, s \geq 0$.

All rows of S_{**} are exact, since they are deleted standard complexes of projective objects of $\mathcal{D}(\mathcal{F},l)$ and all columns of S_{**} are exact because of the assumption that $\mathcal{F}$ is exact.

Let $A_{***} = (A,\ x,y,z)$ be the negative cone triple complex $\mathrm{Hom}_{\mathcal{D}}\big(S_{**}\ , I_*\ (Y)\big)$, i.e., $x_{i,i,k} = \mathrm{Hom}_{\mathcal{D}}\big(t_{i-1,j}, I_k\big)$, $y_{i,j,k} = \mathrm{Hom}_{\mathcal{D}}\big(r_{i\ ,j-1}, I_k\big)$, and $z_{i,i,k}\ \mathrm{Hom}_{\mathcal{D}}\big(S_{i\ ,j}, P_k\big)$,where i,j,k are non-positive integers. This triple complex is exact to both direction x and direction Y because $I_*\ (Y)$ is an injective resolution. Hence, by Proposition (1.1.17) we obtain

$$\mathcal{H}^n\left(\mathrm{Tot}\big(K[0]_{y,z}(A)\ \big)_*\right) \cong \mathcal{H}^n\big(\mathrm{Tot}K[0]_{y,z}(A)_*\big)\ \cong \mathcal{H}^n\left(\mathrm{Tot}\big(K[0]_{x,z}(A)\ \big)_*\right). \quad (4)$$

We shall estimate separately the cohomology groups

$$\mathcal{H}^n\left(\mathrm{Tot}\big(K[0]_{y,z}(A)\ \big)_*\right) \text{ and } \mathcal{H}^n\left(\mathrm{Tot}\big(K[0]_{x,z}(A)\ \big)_*\right)$$

We start with the third quadrant double complex

$$K[0]_{y,z}(A)_{**}\big(\mathrm{Ker}\ x_{0,j,k}\ , \bar{y}_{0,j,k}\ , \bar{z}_{0,j,k}\big)\ .$$

From the exactness of the commutative diagram

$$\begin{array}{ccccccc} \mathcal{F}TP_j & \xrightarrow{l_{-1,j}} & TP_j & \xrightarrow{\epsilon_{\mathcal{F}P_j}} & P_j & \longrightarrow & 0 \\ \downarrow \epsilon_{\mathcal{F}q_j} & & \downarrow q_j & & \downarrow c_{q_j} & & \\ \mathcal{F}TP_{j+1} & \xrightarrow{l_{-1,j+1}} & TP_{j+1} & \xrightarrow{\epsilon_{\mathcal{F}P_{j+1}}} & P_{j+1} & \longrightarrow & 0 \end{array}$$

we obtain

$$\mathrm{Ker}\, x_{0,j,k} = \mathrm{Hom}_{\mathcal{D}}(P_j, I_j), \bar{y}_{0,j,k} = \mathrm{Hom}_{\mathcal{D}}(\mathcal{g}q_j, I_j)$$

and $\bar{z}_{0,j,k} = \mathrm{Hom}_{\mathcal{D}}(P_j, \rho_j)$.

The double complex $K[0]_{y,z}(A)_{**}$ is exact to the direction $\bar{z}$, because for any fixed non-positive integer j, the complex $(\mathrm{Ker}\, x_{0,j,k}, \bar{z}_{0,j,k})$ can be obtained from $I_*(Y)$ by applying the exact functor $\mathrm{Hom}_{\mathcal{D}}(P_j, -)$. So, the usual spectral sequence collapses and $\mathcal{H}^n\left(\mathrm{Tot}(K[0]_{y,z}(A))_*\right)$ is isomorphic to the nth cohomology group of the complex K_* formed from the kernels of $\bar{z}_{0,j,0}$ and the induced by $\bar{y}_{0,j,0}$ homomorphisms. But then K_* coincides, up to sign of the homomorphisms, with the complex (2) yielding $\mathrm{Ext}^n_{\mathcal{D}(\mathcal{F},l)}((X,f), \mathcal{Z}Y)$.

We leave it for the reader to verify that the third quadrant double complex $(K[0]_{x,z}(A))_{**}$ is isomorphic to W_{**}. Hence,

$$\mathrm{Ext}^n_{\mathcal{D}(\mathcal{F},l)}((X,f), \mathcal{Z}Y) \cong \mathcal{H}^n(\mathrm{Tot}(W)_*).$$

Remark (1.1.19)[1]: The theory developed so far is related to the theory of relative homological algebras of the 1960s, cf. [7, 8].

Since the functor $\mathcal{U}$ is faithful, the adjoint pair $(\mathcal{F}, \mathcal{U})$ enables us to lift, via $\mathcal{F}$, projective classes of epimorphisms of $\mathcal{D}$ to projective classes of epimorphisms of $\mathcal{D}(\mathcal{F}, l)$ cf. [8; 9, 10, 5]. If the projective class of epimorphisms in !if is the class of splitting epimorphisms, then the relative projectives in $\mathcal{D}(\mathcal{F}, l)$ are exactly the direct summands of objects of the form $\mathcal{F}\mathcal{Z}$, where $\mathcal{Z}$ is an object of $\mathcal{D}$. Hence, any object (X, f) of $\mathcal{D}(\mathcal{F}, l)$ possesses a relative projective resolution of the form

$$T_* = T_*(X,f): \cdots \xrightarrow{t_n} T_n(X,f) \xrightarrow{t_{(n-1)}} T_{n-1}(X,f) \xrightarrow{t_{(n-2)}} \cdots \xrightarrow{t_0} T_0(X,f) \longrightarrow 0$$

unique up to homotopy, where $T_1(X, f)$ is direct summand of some $\mathcal{F}\mathcal{Z}$. So, one can form relative derived functors and in particular the relative right derived functors of $\mathrm{Hom}_{\mathcal{D}(\mathcal{F},l)}(-, (Y, g))$ denoted by $\mathrm{Ext}^n_{\mathcal{D}(\mathcal{F},l)}(-, (Y, g))$

There is a functorial way to construct a relative projective resolution of a given object (X, f) of $\mathcal{D}(\mathcal{F}, l)$, such that $\mathcal{g}T_* = \Theta$. Using this fact and since $(\mathcal{g}, \mathcal{Z})$ is an adjoint pair of functors, one see that

$$\mathrm{Ext}^n_{\mathcal{D}(\mathcal{F},l)}(-, (X, f), \mathcal{Z}Y) = \mathcal{H}^n(\mathrm{Hom}_{\mathcal{D}}(\Theta(X, f), Y))$$

and that

$$W_{**} = \mathrm{Hom}_{\mathcal{D}(\mathcal{F},l)}(T_*(X, f), \mathcal{Z}I_*(Y))$$

So, the homology of W_{**} in the E)-direction is $\mathrm{Ext}^*_{\mathcal{D}(\mathcal{F},l)}((X, f), \mathcal{Z}I_*(Y))$ and in the other direction is $\mathrm{Ext}^*_{\mathcal{D}}(\Theta(X, f), Y)$.

Let $\emptyset: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the function introduced in this section.

Proposition (1.1.20)[1]: For any two objects $X, Y$of $\mathcal{D}$ if it holds that

$$\mathrm{Ext}^n_{\mathcal{D}(\mathcal{F},l)}(X\mathcal{Z}, \mathcal{Z}ZY) \cong \bigoplus_{p+q=n} \mathrm{Ext}^n_{\mathcal{D}}\left(\mathcal{F}^{\emptyset(q)}X, Y\right) \quad , \forall n \in \mathbb{N}$$

Proof. The deleted standard complex of $\mathcal{Z}X = (X, 0)$ has zero differential. Hence, the differential w of the third quadrant double complex

$$W_{**} = (W, w, v) = \mathrm{Hom}_{\mathcal{D}}(\, d(\, X, 0)_*\, , I_*(Y)\,)$$

equals zero.

For the nth cohomology group we have

$$\mathcal{H}^n(Tot(W)_*) = [Kerv_{-n,0}] \bigoplus_{p+q=n, p\geq 0} [Keru_{_q,_p}/Imv_{_q,_p+1}]$$

Now,

$\mathrm{Kerv}_{-n,0} \cong \mathrm{Hom}_{\mathcal{D}}\left(\mathcal{F}^{\emptyset(q)}X, Y\right)$ and

$$\mathrm{Keru}_{_q,_p}/\mathrm{Imv}_{_q,_p+1} \cong \mathrm{Ext}^p_{\mathcal{D}}\left(\mathcal{F}^{\emptyset(q)}X, Y\right)$$

Remark (1.1.21)[1]: In the trivial extension case we have $l = 2$ and the function $\emptyset$ is the identity function on $\mathbb{N} \cup \{0\}$. Hence,

$$\mathrm{Ext}^n_{\mathcal{D}(\mathcal{F},l)}(\mathcal{Z}X, \mathcal{Z}Y) \cong \bigoplus_{p+q=n} \mathrm{Ext}^n_{\mathcal{D}}(\mathcal{F}^q X, Y) \ .$$

This is exactly the formula obtained by I. Palmef ahd J. E. Roos; ct. [11] for trivial extension rings.

Proposition (1.1.22)[1]: Let $\mathcal{D}(\mathcal{F}, l)$ be a truncated extension of $\mathcal{D}$ by an exact Junctor $\mathcal{F}$.

(i) If $\mathcal{F}^{m} = 0, m \in \mathbb{N}$, and if there is some $p \in \mathbb{N}$ with $\mathrm{Ext}^{j}_{\mathcal{D}}(X, Y) = 0$ for all $j \geq p + 1$ and all objects X, Y of $\mathcal{D}$, then $\mathrm{Ext}^{n}_{\mathcal{D}(\mathcal{F},l)}(X\mathcal{Z}, \mathcal{Z}Y) = 0$ for all $n \in \mathbb{N}, n \geq p$, satisfying $\emptyset(n - p) \geq m$.

(ii) If $\mathcal{F}$ is not nilpotent, then for any $n \in \mathbb{N}$ there is some $\mathcal{Z}X$, such that the Junctor $\mathrm{Ext}^{n}_{\mathcal{D}(\mathcal{F},l)}(\mathcal{Z}X, -)$ is non-zero.

Proof. (i) It is a direct implication of Proposition (1.1.20).

(ii) For any $n \in \mathbb{N}$, there is an object X with $\mathcal{F}^{\emptyset(q)}X \neq 0$. Hence, the group $\mathrm{Hom}_{\mathcal{D}}\left(\mathcal{F}^{\emptyset(n)}X, \mathcal{F}^{\emptyset(n)}Y\right)$ is not trivial. Since this group is a direct summand of $\mathrm{Ext}^{n}_{\mathcal{D}(\mathcal{F},l)}\left(X\mathcal{Z}, \mathcal{Z}\mathcal{F}^{\emptyset(n)}Y\right) \neq 0$ we are done.

A non-zero object X of an abelian category $\mathscr{p}$ is said to be simple, if there is no non-zero monomorphism $\alpha\colon Y \to X$.

For the simple objects of $\mathcal{D}(\mathcal{F}, l)$ there is the following characterization:

Lemma (1.1.23)[1]: A non-zero object (Y, g) of $\mathcal{D}(\mathcal{F}, l)$ is simple if and only if $(Y, g) = \mathcal{Z}X$ with X a simple object of $\mathcal{D}$.

Proof. Left for the reader.

Using this characterization of simple objects, we derive formula (1).

We also reformulate Proposition (1.1.22) in the case of finite-dimensional algebras:

Proposition (1.1.24)[1]: Let A be a finite-dimensional algebra over a field k and M a finite-dimensional $(A - A)$-bimodule, which is projective from the left side.Let $A \ltimes_{l} M$ be the truncated extension of A by M of truncation index $l - 1$.

(i) If the functor $\mathcal{F} = - \otimes_{A} M$ is nilpotent, where $m = \min\{r \in \mathbb{N}: \mathcal{F}^{r} = 0\}$ and if $\mathrm{gl.dim}A = p$, then $p \leq \mathrm{gl.dim}A \ltimes_{l} M \leq t - 1$, where

$$T = \min\{n \in \mathbb{N}: n \geq p, \emptyset(n - p) \geq m\},$$

(ii) If the functor $\mathcal{F} = - \otimes_{A} M$ is not nilpotent or if $\mathrm{gl.dim}A = \infty$, then $\mathrm{gl.dim}A \ltimes_{l} M = \infty$.

Proof. A finite-dimensional algebra Λ has finite global dimension if and only if there is some fixed natural number p with $\mathrm{Ext}^{p}_{\Lambda}(S, T) = 0$ for any two simple Λ-modules S and T; cf. [12]. But any simple $(A \ltimes_{l} M)$- module is of the form $\mathcal{Z}X$, where X is a simple A-

module.

Remark (1.1.25)[1]: Let Λ be a basic finite-dimensional k-algebra with Gabriel quiver Q, where k is an algebraically closed field. It is well known (d. [13] that $\Lambda \cong \mathrm{kQ/I}$, where I is an admissible ideal.

Let A denote the semisimple algebra $\Lambda/\mathrm{rad}\,\Lambda$ and J the ideal of kQ generated by the arrows of Q.

If $I = J^l, l \geq 2$, then Λ is truncated extension of A by the $(A-A)$ –bimodule J/J^2.

In this case our formula (1) for the semisimple Λ-module $K = \Lambda/rad\,\Lambda$ reduces to $\mathrm{Ext}^n_\Lambda(K, K) \cong \mathrm{Hom}_A(K \otimes_A (\otimes^{\emptyset(n)} J/J^2), K)$, which is isomor phic to the k-dual of $J^{\emptyset(n)}/J^{\emptyset(n)+1}$. This is a special case of the formula for $\mathrm{Ext}^n_\Lambda(K, K)$ obtained by K. Bongartz (cf. [14]) for an arbitrary finite dimensional k-algebra $A \cong kQ / I$.

We present now a generalized version of the Strong No-Loops Conjecture for truncated extensions.

Theorem (1.1.26)[1]: Let $\mathcal{D}(\mathcal{F}, l)$ be a truncated extension, with $\mathcal{F}$ exact and let X be a simple object of $\mathcal{D}$.

If $Ext^1_{\mathcal{D}}(X, X) = 0$ and $Ext^1_{\mathcal{D}(\mathcal{F},l)}(\mathcal{Z}X, \mathcal{Z}X) \neq 0$ the $Ext^n_{\mathcal{D}(\mathcal{F},l)}(\mathcal{Z}X, \mathcal{Z}X) \neq 0$ for any $n \in \mathbb{N}$.

Proof. Because of Proposition (1.1.20), the hypothesis of the theorem implies that the group $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}X, X)$ is non-trivial. Since X is simple, there is a (non-zero) epimorphism $\chi: \mathcal{F}X \to X$ and since $\mathcal{F}$ is right exact, the composition $\chi \circ \mathcal{F}\chi \circ \cdots \circ \mathcal{F}^{n-1}\chi$ is a non-zero epimorphism for any $n \in \mathbb{N}$, i.e., the group $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}^n X, X)$ is non-trivial. In particular, the direct summand $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}^{\emptyset(n)} X, X)$ of $Ext^n_{\mathcal{D}(\mathcal{F},l)}(\mathcal{Z}X, \mathcal{Z}X)$ is always non zero.

Corollary (1.1.27)[1]: Let $A \ltimes_l M$ be a truncated extension of a semisimple ring A and S a simple $(A \ltimes_l M)$-module. If $Ext^n_{A \ltimes_1 M}(S, S) \neq 0$, then $Ext^n_{A \ltimes_l M}(S, S) \neq 0$, for any $n \in \mathbb{N}$.

There is a natural way to construct truncated extensions. Given a quiver Q with a set of relations S, we consider the group $G(Q, S)$ of automor phisms of (Q, S). Any element $g \in G(Q, S)$ induces an endofunctor $\mathcal{F}_g$ of the category $\mathcal{D} = \mathrm{Rep}(Q, S)$ of the k-linear representations of (Q, S), which is of course an equivalence. Our first example is based on this construction.

Example (1.1.28)[1]: Let us consider the pair (Q, S), where Q is the quiver below and $S = \phi$.

$$Q = \begin{tikzcd} 1 \arrow[rr, "a"] & & 2 \arrow[dl, "b"] \\ & 3 \arrow[ul, "c"] & \end{tikzcd}$$

Let g be the element of $G(Q, S)$ mapping the vertices of Q as follows:

$1 \to 3, 2 \to 1$, and $3 \to 2$. In this case, the action of g on the vertices determines uniquely the action of g on the arrows.

The quiver Q' of the truncated extension $\mathcal{D}(\mathcal{F}, l)$ of $\mathcal{D}$ by the functor $\mathcal{F}_g$ equals

$$Q' = \begin{tikzcd} 1 \arrow[rr, shift left, "a"] \arrow[rr, shift right, leftarrow, "e"'] \arrow[dr, shift left, "c"] & & 2 \arrow[dl, shift left, "b"] \\ & 3 \arrow[ul, shift left, "d"] \arrow[ur, shift left, "f"] & \end{tikzcd}$$

The set of relations S_1 of $\mathcal{D}(\mathcal{F}, l)$ equals $T \cup U_l$, where $T = \{ae - de, bf - ea, cd - fb\}$ is fixed for any I and where

$$U_l = \begin{cases} \{fe, ed, df\}, & \text{for } l = 2 \\ \{dfe, fed, edf\}, & \text{for } l = 3 \\ \{edfe, dfed, fedf\}, & \text{for } l = 4 \end{cases}$$

and so on.

Let I_l denote the ideal of the path algebra kQ' generated by S_l. By proposition (1.1.24)(ii), the global dimension of $\Lambda \cong kQ'/I_l$ is always infinite, since $\mathcal{F}_g$ is an equivalence.

Example (1.1.29)[1]; Let us consider the pair (Q, S), where Q is the quiver below and $S = \phi$.

$$Q = \underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xrightarrow{b} \underset{3}{\bullet} \xrightarrow{c} \underset{4}{\bullet}$$

Let $\mathcal{D}$ be the category $\mathrm{Rep}(Q, S)$ of the k-linear representations of (Q, S).

We define an exact functor $\mathcal{F}$: $\mathcal{D} \to \mathcal{D}$ as follows:

If $X = X_2 \xrightarrow{f_b} X_2 \xrightarrow{f_b} X_3 \xrightarrow{f_c} X_4$ is an object of $\mathcal{D}$, then its image under $\mathcal{F}$ is defined to be $\mathcal{F}X = 0 \xrightarrow{f_b} X_1 \xrightarrow{f_b} X_2 \xrightarrow{f_c} X_3$. If $X = (x_1,\ x_2,\ x_3, x_4)$ is a morphism from X to Y, where $Y = Y_2 \xrightarrow{f_b} Y_2 \xrightarrow{f_b} Y_3 \xrightarrow{f_c} Y_4$, then its image under $\mathcal{F}$ is defined to be $\mathcal{F}_X = (0, x_1,\ x_2,\ x_3)$: $\mathcal{F}X \longrightarrow \mathcal{F}Y$.

The quiver Q' of the truncated extension $\mathcal{D}(\mathcal{F}, l)$ of $\mathcal{D}$ by the functor $\mathcal{F}$ equals

$$Q' = \underset{1}{\bullet} \underset{d}{\overset{a}{\rightrightarrows}} \underset{2}{\bullet} \underset{e}{\overset{b}{\rightrightarrows}} \underset{3}{\bullet} \underset{f}{\overset{c}{\rightrightarrows}} \underset{4}{\bullet}$$

The set of relations S_1 of $\mathcal{D}(\mathcal{F}, l)$ equals $T \cup U_l$, where $T = \{ae - db,\ bf - ec\}$ is fixed for any I and where

$$U_l = \begin{cases} \{de, ef\}, & \text{for } l = 2 \\ \{def\}, & \text{for } l = 3 \\ \emptyset, & \text{for } l \geq 4. \end{cases}$$

Let I_l denote the ideal of the path algebra kQ' generated by Sf. By Proposition (1.1.24)(i), the global dimension of $\Lambda \cong kQ'/I_l$ is finite for any $l \in \mathbb{N} \cup \{0\}$.

Moreover, using Proposition (1.1.24)(i), we see that the global dimension of the truncated extension A is bounded as

$$1 \leq \text{gl.dim } \Lambda \leq \begin{cases} 4\,, & \text{for } l = 2 \\ 3\,, & \text{for } l = 3 \\ 2\,, & \text{for } l \geq 4\,. \end{cases}$$

In fact the global dimension of A equals

$$\text{gl.dim } \Lambda = \begin{cases} 3\,, & \text{for } l = 2 \\ 2\,, & \text{for } l = 3 \\ 2\,, & \text{for } l \geq 4\,. \end{cases}$$

Example (1.1.30)[1]: Let us consider the pair (Q, S), where Q is the quiver of the previous example and where the set of relations is $S = \{ab, bc\}$.

Let $\mathcal{D}$ be the category $\text{Rep}(Q, S)$ of the k-linear representations of (Q, S).

We define an exact functor $\mathcal{F}$: $\mathcal{D} \to \mathcal{D}$ with the data as in the revious example. The quiver Q' of the truncated extension $\mathcal{D}(\mathcal{F}, l)$ is the same as that of the previous example.

The set of relations S_l of $\mathcal{D}(\mathcal{F}, l)$ equals $S \cup T \cup U_l$ where T and U_l are also as in the previous example.

Let I_l denote the ideal of the path algebra kQ' generated by S_l . By Proposition (1.1.24)(i), the global dimension of $\Lambda \cong kQ'/I_l$ is finite for any $l \in \mathbb{N} \cup \{0\}$.

Here, $\mathrm{gl.dim}\,\Lambda$ is bounded as

$$3 \leq \mathrm{gl.dim}\ \Lambda\ \leq \begin{cases} 6, & \text{for } l = 2 \\ 5, & \text{for } l = 3 \\ 4, & \text{for } l \geq 4. \end{cases}$$

In fact , $\mathrm{gl.dim}\,\Lambda =\ 3$ for any $l\ \geq 2$.

Section (1.2): Resolutions of Minimal Projective:

In the study of homological properties of rings and modules, projective resolutions are a basic tool. Such resolutions occur naturally in commutative ring theory, the representation theory of finite dimensional algebras, group representation theory, algebraic geometry, and algebraic topology [16, 17, 18, 19, 20]. On the other hand, with the introduction of computers, computational and algorithmic techniques have grown in importance [21, 22]. Both theoretical and practical results are needed. This Section present a new method of constructing projective resolutions in a broad setting which has both theoretical and computational implications. In particular, in the graded and finite dimensional cases, our results provide a recursive procedure for computing minimal projective resolutions.

The class of algebras studied in this Section consists of quotients of path algebras. We fix a field K for the remainder of this paper. If Q is a finite directed graph, which we call a quiver, then the path algebra, KQ, is the K-algebra with K-basis consisting of finite directed paths in Q. Thus, elements of KQ consist of K-linear combinations of paths in Q. The multiplicative structure on basis elements p and q is defined by concatenation pq if the terminus of p equals the origin of q, and by 0 otherwise. We view the vertices as paths of length 0 with multiplication given as follows. If v and w are vertices and p is a path, we let v. w be v if $v = w$ and 0 otherwise. We let $v \cdot p = p$ if v is the origin of p and 0 otherwise, and we define $p \cdot \omega$ similarly. The multiplication on paths is extended linearly to arbitrary elements of KQ. Note that the free associative K-algebra on n noncommuting variables is isomorphic to the path algebra KQ where Q has one vertex and n loops. We let Q_0 denote the vertex set of Q.

Let Q be a quiver and I be a (two-sided) ideal in the path algebra KQ. Let Λ denote KQ/I for the remainder of the introduction. The algebras in this class include all affine (that is, finitely generated) associative K-algebras. Every finite dimensional K-algebra is Morita equivalent to an algebra in this class if K is algebraically closed. Furthermore, this class includes graded K-algebras $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \ldots$ where Λ_0 is a product of a finite number of

copies of K, each Λ_i is a finite dimensional K-vector space and Λ is generated in degrees 0 and 1; that is, for $i, j \geq 0$, $\Lambda_i \Lambda_j = \Lambda_{i+j}$.

Let M be a $\Lambda = KQ/I$-module. Let $F \to M \to 0$ be an exact sequence of KQ-modules with $F = \coprod_{v \in Q_0} vKQ$. A main theme of the Section is the construction of a filtration of F by KQ-submodules which contains all the information needed to construct the Λ-projective resolution of M, the Λ-syzygies and the Yoneda product of extensions of Λ-modules. In particular, we find a filtration

$$\ldots \subset F^n \subset F^{n-1} \subset \ldots \subset F^1 \subset F^0;$$

such that $.F = F^0, \quad M = F^0 / F^1$ and

$$\ldots F^n / F^n I \to F^{n-1} / F^{n-1} I \to \ldots \to F^1 / F^1 I \to F^0 / F^0 I \to M \to 0$$

is a Λ-projective resolution of M with the maps induced by the inclusions of the filtration. For the basic construction, we do not assume that Λ is finite dimensional, or even noetherian. Furthermore, we do not assume that the Λ-module M is finitely generated. For our minimality results, M will be finitely generated and Λ either finite dimensional or graded.

We provide a recursive formula to compute F^n as a KQ-submodule of F^{n-1} from the previously obtained $F^{n-1} \subset F^{n-2}$. To explicitly find F^n from our formula, one must write an intersection of certain submodules of a projective KQ-module as a direct sum of cyclic submodules. A method for finding the generators of these cyclic submodules employs the theory of right Grobner bases, and will appear elsewhere.

Our construction resembles earlier resolutions of Bongartz, Butler, Eilenberg, Eilenberg-Nagao-Nakayama and Gruenberg [23, 24, 25]. Their resolutions are almost never minimal in the finite dimensional case, and deal only with resolutions of semisimple modules. We recall their resolution. Let J denote the ideal of KQ generated by the arrows of Q. Furthermore, assume that $J^N \subseteq I \subseteq J^2$ for some positive integer $N \geq 2$. Then we have the filtration

$$\ldots \subset J\,I^n \subset I^n \ldots \subset I^3 \subset J\,I^2 \subset I^2 \subset J\,I \subset I \subset J \subset KQ.$$

Note that J/I is the Jacobson radical of Λ and that $\Lambda/(J/I)$ is isomorphic to KQ/J. One gets the following Λ-projective resolution of KQ/J

$$\ldots I^n / I^{n+1} \to J\,I^{n-1} / J\,I^n \to \ldots \to I / I^2 \to J / J\,I \to KQ / I \to KQ / J \to 0,$$

where the maps are induced by the inclusions.

We give a general construction of a projective resolution of an arbitrary Λ-module M, where Λ is a quotient of a path algebra. We show that if Λ is a right noetherian algebra and M is a finitely generated Λ-module, then the resolution is finitely generated. If Λ is graded and M is a graded module, we show how to modify the construction to obtain a graded projective resolution.

We provide algorithmic techniques to adjust the construction to obtain minimal projective resolutions in both the finite dimensional and the graded cases. The section ends with explicit computations of syzygies and Ext-groups.

If Λ_0 denotes Λ modulo its radical in the finite dimensional case, or, Λ modulo its graded radical in the graded case, then we study the algebraic structure of

$$E(\Lambda) = \coprod_{n \geq 0} Ext_{\Lambda}^{n}(\Lambda_0, \Lambda_0).$$

Furthermore, if M is either a finite dimensional Λ-module or a graded Λ module, we investigate the $E(\Lambda)$-module structure of $(M) = \coprod_{n \geq 0}(M, \Lambda_0)$. A major result of the Section is that this module structure is included in the information obtained in the construction of the resolution. In particular, one need not "lift maps" to find the Yoneda products.

We apply our techniques to prove one case of the No Loop Conjecture. Namely, we prove that if Λ is a finite dimensional K-algebra and a is a loop at the vertex v such that a^n is the first power of a belonging to the ideal I but a^n is not in $J\,I + I\,J$, then $Ext_{\Lambda}^{n}(S,S) \neq (0)$,for all $n \geq 1$, where S is the simple Λ-module corresponding to the vertex v.

We investigate the inuence of the characteristic of the ground field K on the structure of projective resolutions. Other than some new examples, we show that if the global dimension of Λ is bounded by 2 in one characteristic, then the global dimension will be finite in all characteristics. We also provide an example of an algebra that has infinite global dimension in only one characteristic.

Finally, we note that all modules will be right modules unless otherwise stated. We also introduce some terminology. We say that an element x in the path algebra KQ is right uniform, if $x \neq 0$ and there is a vertex υ such that $x\upsilon = x$. Note that if $x \neq 0$ is an element of KQ, then $x = \sum_{\upsilon \in Q_0} x\upsilon$. Hence, every nonzero element of KQ is a sum of right uniform elements. From a different point of view, $KQ = \coprod_{\upsilon \in Q_0} KQ\upsilon$ as left modules. Hence, every nonzero element is a sum of right uniform elements in a unique way. An element is right

uniform if and only if it is nonzero and a linear combination of paths ending at a single vertex. Finally, note that if x is a right uniform element with $x\upsilon = x$ for some $\upsilon \in Q_0$, then xKQ is a right projective KQ-module isomorphic to υKQ.

Let Q be a finite quiver, and let $R = KQ$ denote the path algebra of Q over a field K. Let I be a two-sided ideal in R such that $I \subseteq J^2$, where J denotes the ideal of R generated by the arrows of the quiver Q. Let $\Lambda = R/I$ be the quotient algebra, and let M be a right Λ-module. In this section we construct, in an algorithmic way, a projective resolution (P, δ) of M over Λ. This resolution need not be finitely generated in general, but it is when Λ is noetherian and is finitely generated as a right ideal in R. In particular, if I is an admissible ideal of R, that is, $J^N \subseteq I \subseteq J^2$ for some N >1, then Λ is a finite dimensional K-algebra and the resolution (P, δ) becomes a finitely generated resolution. We also show how, in this case, we can adjust P in an algorithmic way to obtain a minimal projective resolution of M_Λ. Moreover, if Λ is graded by the natural grading induced from the length grading on R, then the resolution constructed for a graded Λ-module is also graded.

We shall use the following well-known properties of the path algebra $R = KQ$:

(a) for every x in R, the R-module xR is projective, and,

(b) for each R-submodule Y of $\coprod_i x_i R$ with x_i in R, we have $Y = \coprod_j y_i R$ for some y_j in $\coprod_{i=1} x_i R$ (of course, if Y is finitely generated, then we can write $Y = \coprod_{j=1}^{i} y_i R$ for some finite set $\{y_1, \ldots, y_t\}$ in $\coprod_{i=1} x_i R$, $[G]$. We now introduce the notation that will be needed in defining the resolution (P, δ) of M, and, throughout this Section.

Choose a family $\{f_i^0\}_{i \in A}$ of elements of R such that the projective Λ-module $\coprod_{i \in A} f_i^0 R$ / $\coprod_{i \in A} f_i^0 I$ maps onto M. Without loss of generality we choose the family to consist of vertices in R (repetitions allowed). We have

$$0 \to \Omega_R^1(M) \to \coprod_{i \in A} f_i^0 R \to M \to 0,$$

and, we then choose a set $\{f_i^{1*}\}$ of elements of $\coprod_{i \in A} f_i^0 R$ such that $\Omega_R^1(M) = \coprod_{i \in A} f_i^{l*} R$. Discard all the elements f_i^{l*} that are in $\coprod_{i \in A} f_i^0 I$ and denote by $\{f_i^1\}$ those f^{1*}, s that are not elements of $\coprod_{i \in A} f_i^0 I$. Assume that we have constructed families of elements of $\coprod_{i \in A} f_i^0 R : \{f_i^k\}_i$ for each $k = 0, \ldots, n$. We now construct the family $\{f_i^{n+1}\}_i$ as follows. We consider the intersection $(\coprod_i f_i^n R) \cap \left(\coprod_j f_j^{n-1} I\right)$. We stop if the intersection is zero, and we

set it equal to some $\coprod_l f_l^{n+l*}R$ otherwise. Discard all the elements of the form f^{n+1*} that are in $\coprod_i f_i^n I$, and denote the remaining ones by $\{f_i^{n+1}\}_i$. If each element of the form f^{n+1*} is in $\coprod_i f_i^n I$, we again stop at this stage of the construction. Note that we may assume that for each n, each element f_i^n can be chosen to be right uniform, that is, there is a vertex v (dependent on f_i^n) such that $f_i^n \upsilon = f_i^n$. An element of R is uniform if it is a linear combination of paths in R, all starting at one vertex, and, all ending at one vertex. We also note that, for each n > 0, we have a representation of f_k^n in $\coprod_i f_i^{n-1}R$ as follows:

$$f_k^n = \sum_i f_i^{n-1} h_{i,k}^{n-1,n}$$

for scalars $h_{i,k}^{n-1,n}$ in R. Note that for each k, all but a finite number of $h_{i,k}^{n-1,n}$ are zero. It is convenient to encode this information in the matrix $(h_{i,k}^{n-1,n})$. Furthermore, since the f_k^n, s and the f_i^{n-1}, s are right uniform, it follows that each $h_{i,k}^{n-1,n}$ is uniform.

Setting $F^n = \coprod_i f_i^n R$ $F^n = \coprod_i f_i^n R$, from our construction, we have the following filtration of the right projective R-module F^0:

$$\ldots \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0.$$

Definition (1.2.1)[15]: For each $n \geq 0$ let $P_n = \coprod_i f_i^n R / \coprod_i f_i^n I$, and let $\delta^n : P_n \to P_{n-1}$ be the homomorphism induced by the inclusion $\coprod_i f_i^n R \subset \coprod_i f_j^{n-1} R$. We also define the matrix $(h_{i,k}^{n-1,n})$ where $\bar{h}$ denotes the image in Λ of the element h in R.

Note that the boundary maps δ^n are, in fact, determined by multiplication by the matrix $(\bar{h}^{n-1,n})$, which gives a formula for the coordinates.

We can now state our first result.

Theorem (1.2.2)[15]: $(P,\delta): \ldots \to P_n \xrightarrow{\delta^n} P_{n-1} \to \ldots \to P_1 \xrightarrow{\delta^1} P_0 \to M \to 0$ is a projective resolution of M over Λ.

Proof. It is clear that for each $n \geq 0$, the modules P_n are projective Λ -modules. From the following commutative diagram with exact rows

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \coprod_i f_i^0 I \arrow[r, equal] \arrow[d] & \coprod_i f_i^0 I \arrow[d] & & \\
0 \arrow[r] & \Omega_R^1(M) \arrow[r] \arrow[d] & \coprod_i f_i^0 R \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_\Lambda^1(M) \arrow[r] \arrow[d] & \coprod_i f_i^0 R / \coprod_i f_i^0 I \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

it follows that we have exactness at P_0. We now show that for each $n > 0$ we have $\delta^n \delta^{n+1} = 0$. We must show that $(\bar{h}_{i,k}^{n-1,n})(\bar{h}_{k,l}^{n,n+1})$ is the zero matrix, or, equivalently that for each i and l, the sum $\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1}$ is in I. But $\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1})$ is an element of $\coprod f_i^{n-1} R$, and we also have

$$\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1}) = \sum_k (\sum_i f_i^{n-1} h_{i,k}^{n-1,n}) h_{k,l}^{n,n+1} = \sum_k f_i^n h_{i,k}^{n,n+1} = f_l^{n+1}$$

which lies in $\coprod f_i^{n-1} R$. We infer from the uniqueness of the representations as elements of direct sums that for each i and l, the element $\sum_k h_{i,k}^{n-1,n} h_{k,l}^{n,n+1}$ is in I.

It remains to show that for each n, $\operatorname{Ker} \delta^n \subseteq \operatorname{Im} \delta^{n+1}$. Let $(\bar{x}_k)_k$ be in the kernel of δ^n. Therefore, $\sum_k h_{i,k}^{n-1,n} \bar{x}_k = \bar{0}$ for all i, or, equivalently $\sum_k h_{i,k}^{n-1,n} x_k$ is in the ideal I for all i. On the other hand,

$$\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k) = \sum_k (\sum_i f_i^{n-1} h_{i,k}^{n-1,n}) x_k = \sum_k f_k^n x_k$$

is an element of $\coprod_k f_k^n R$ and, since $\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k)$ is also in $\coprod f_i^{n-1} I$, we have that the element $\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k)$ is in $\coprod f_j^{n+1*} R$. Therefore, we can rewrite it as

$$\sum_i f_i^{n-1}(\sum_k h_{i,k}^{n-1,n} x_k) = \sum_j f_j^{n-1}\gamma_j + u,$$

where γ_j is an element in R and u is an element of $\coprod f_k^n I$. We claim that we have $\delta^{n+1}((\bar{\gamma}_j)_j) = (\bar{x}_k)_k$, where $\bar{\gamma}_j$ denotes the image in Λ of the element γ_j in R. To prove this we have

$$\delta^{n+1}((\bar{\gamma}_j)_j) = (\bar{h}_{k,j}^{n,n+1})(\bar{\gamma}_j)_j = (\sum_j \bar{h}_{k,j}^{n,n+1}\bar{\gamma}_j)_k$$

But, we have

$$\sum_k f_k^n(\sum_j h_{k,j}^{n,n+1}\gamma_j) = \sum_j f_j^{n+1}\gamma_j = \sum_k f_k^n x_k$$

modulo $\coprod f_k^n I$. Hence, we infer that for each k we get $\sum_j h_{k,j}^{n,n+1}\gamma_j = x_k$ modulo I. This proves that Ker $\delta^n \subseteq \operatorname{Im}\delta^{n+1}$ and the proof is complete.

We show next that the resolution constructed above is a finitely generated resolution if we assume in addition that Λ is noetherian and I is finitely generated as a right ideal in R.

Theorem (1.2.3)[15]: Assume that Λ is noetherian and that I is finitely generated as a right ideal of R = KQ. Let M_Λ be finitely generated. Then the resolution (P,δ) of M_Λ is finitely generated.

Proof. First observe that we may choose $f_1^0,\ldots, f_k^0$ in R such that $\coprod_{i=1}^k f_i^0 R / \coprod_{i=1}^k f_i^0 RI$ maps onto M. To prove the theorem, it is enough to show that, for each n > 0, the direct sums $\coprod_{i=1}^k f_i^{n*}R$ are finite. We prove this first for n = 1. We have the exact sequence of R-modules,

$$0 \to \coprod_{i=1}^k f_i^0 I \to \Omega_R^1(M) \to \Omega_\Lambda^1(M) \to 0,$$

and, since both ends are finitely generated, then so is the middle term. But $\Omega_R^1\left(\Omega_\Lambda^{n-1}(M)\right) = \coprod_i f_i^{n*} R$ hence this sum must be finite. We show now by induction, that, for each n > 1 we have $\Omega_R^1\left(\Omega_\Lambda^{n-1}(M)\right) = \coprod_i f_i^{n*}R$ and that they are all finitely generated. (Here by $\Omega_\Lambda^k(M)$ we mean the kernel of the map $\coprod_i f_i^{k-1}R / \coprod_i f_i^{k-1}I \to \Omega_\Lambda^{k-1}(M)$,if $n \geq 2$ we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \left(\coprod_i f_i^{n-2} I\right) \cap \left(\coprod_j f_j^{n-1} R\right) & \longrightarrow & \coprod_j f_j^{n-1} R & \longrightarrow & \Omega_R^{n-1}(M) & \longrightarrow & 0
\end{array}$$

$$0 \longrightarrow \coprod_i f_i^{n-2} I \longrightarrow \Omega_R^1\left(\Omega_\Lambda^{n-2}(M)\right) \longrightarrow \Omega_\Lambda^{n-1}(M) \longrightarrow 0$$

which shows that $\Omega_R^1\left(\Omega_\Lambda^{n-1}(M)\right) = \coprod_i f_i^{n*} R$, and the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \coprod_j f_j^{n-1} I & = & \coprod_j f_j^{n-1} I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_R^1\left(\Omega_\Lambda^{n-1}(M)\right) & \longrightarrow & \coprod_j f_j^{n-1} R & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\
 & & & & & & \| & & \\
0 & \longrightarrow & \Omega_\Lambda^n(M) & \longrightarrow & \coprod_j f_j^{n-1} R / \coprod_j f_j^{n-1} I & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and, from the left vertical exact sequence, an easy induction argument shows that $\Omega_R^1(\Omega_\Lambda^{n-1}(M)$ is finitely generated. Thus the sum $\coprod_i f_i^{n*} R$ is finite. The proof of the theorem is now complete.

Remark (1.2.4)[15]: If I is an admissible ideal in R, then I is finitely generated as a right ideal in R and $\Lambda = R/I$ is a finite dimensional K-algebra, hence noetherian. Therefore, as a corollary of the above the resolution is finitely generated for any finitely generated Λ-module M_Λ when I is an admissible ideal.

The path algebra $R = KQ$ has a natural grading $R = \coprod_i (KQ)_i$ where, for each i, $(KQ)_i$ denotes the K-vector space spanned by the paths of Q of length i. Each $(KQ)_i$ is endowed with an obvious $(KQ)_0 - (KQ)_0$-bimodule structure, and, $J = \coprod_{i \geq 1} (KQ)_i$ is the graded radical of R. If $I \subseteq J^2$ is a two-sided ideal generated by homogeneous elements, then $\Lambda = KQ/I$ has

an induced grading. In this case we say that Λ is length graded. By a graded Λ-module M, we will always mean a graded Λ-module $M = \coprod_{i \in Z} M_i$ such that, $M_i = (0)$ for sufficiently small i, and, each M_i is a finite dimensional K-vector space. In particular, Λ is a graded Λ-module. Given a graded module, it has a projective cover in the category of graded modules and degree zero maps, and, its kernel is again a graded module in our sense. Note also, that as a graded algebra, Λ is generated in degrees 0 and 1.

Proposition (1.2.5)[15]: Let M_Λ be a graded Λ-module. Then, the projective resolution (P, δ) of Definition (1.2.1) can be chosen to be a graded resolution of M_Λ.

Proof. Since M_Λ is graded, we now take $\coprod f_i^0 R \to M \to 0$ as a degree 0 homomorphism with the f_i^0's homogeneous elements of R in the appropriate degrees. We have a sequence of R-modules

$$0 \to \Omega_R^1 M \to \coprod_i f_i^0 R \to M \to 0$$

which is exact in grR. Since $\Omega_R^1(M)$ is a graded submodule, we may take $\coprod_i f_i^{l*} R \to \Omega_\Lambda^l(M)$ with the elements f_i^{1*} right uniform homogeneous elements of $\coprod f_i^0 R$. Since I is a homogeneous ideal of R, it follows that I is also a homogeneous right ideal of R. Thus $\coprod f_0 R$ is also a graded submodule of $\coprod f^0 R$. Therefore, we have that $f^{2*} R = (\coprod f^1 R)(\coprod f^0 R)$ is also a graded submodule of $\coprod f^0 R$. We inductively construct a chain of graded R-submodules of $\coprod f^0 R$:

$$\ldots \subset \coprod f^{\,n} R \subset \coprod f^{\,n-1} R \subset \ldots \subset \coprod f^{\,0} R$$

The result follows by taking quotients modulo I.

We give an example showing that the projective resolution constructed in the previous section need not to be minimal when I is an admissible ideal. However, when I is an admissible ideal, we prove that the elements $\{f^n\}$ can be chosen such that the resolution is minimal. We also compare this resolution with the Bongartz-Butler-Gruenberg resolution in [23].

We start with an example showing that the resolution constructed in Definition (1.2.1) need not be minimal.

Example (1.2.6)[15]: Let R be the path algebra of the following quiver:

$$\begin{tikzcd}
 & 2 \arrow[dr, "b"] & & \\
1 \arrow[ur, "a"] \arrow[dr, "c"'] & & 4 \arrow[r, shift left, "e"] \arrow[r, shift right, "f"'] & 5 \\
 & 3 \arrow[ur, "d"'] & &
\end{tikzcd}$$

and let I be the ideal of K generated by $ab-cd$, bf and de. Let $\Lambda = R/I$, and let S_1 be the simple Λ-module corresponding to the vertex v_1. The ideal I is a 7-dimensional vector space with a basis given by the elements $ab-cd$, $abe-cde$, $abf-cdf$, abf, cdf, bf and de. We construct now a projective resolution of S_1 over Λ using the resolution described in Definition (1.2.1).

We can take $f^0 = v_1$, so we have $0 \to \Omega_R^1 S_1 \to v_1 R \to S_1 \to 0$ and we can decompose $\Omega_R^1 S_1 = f_1^1 R \coprod f_2^1 R$, where $f_1^1 = a$ and $f_2^1 = c$. Next we note that $\coprod f^{2*} R\,(aR \coprod bR) \cap \mathcal{V}_1 I = \mathcal{V}_1 I$, and a K-basis of $v_1 I$ is the set $\{ab-cd, abe-cde, abf-cdf, abf, cdf\}$. We decompose $v_1 I$ as

$$\mathcal{V}_1 I = (ab-cd)R \coprod abfR \coprod cdef.$$

But abf and cdf are in $\coprod f^1 I = aI \coprod cI$. So $f^2 = ab-cd$ and we have $f^2 R = (ab-cd)R$. Now we compute f^{3*}. We have that $\coprod f^{3*} R\,(ab-cd)R \cap (aI \coprod cI)$, and it easy to check that $\coprod f^{3*} R = (0)$, so that we obtain the following projective Λ-resolution of S_1, which turns out to be minimal:

$$0 \to (ab-cd)R/(ab-cd)I \to aR/aI \coprod cR/cI \to \mathcal{V}_1 R/\mathcal{V}_1 I \to S_1 \to 0$$

We remark that we could have decomposed $v_1 I$ also in the following way: $\mathcal{V}_1 I = cdfR \coprod (ab-cd)R \coprod cdeR$ and $cdeR$ is contained in $I \coprod cI$, but cdf and $(ab-cd)$ are not in $aI \coprod cI$. So we can write $f_1^2 = cdf$ and $f_2^2 = (ab-cd)$. We continue and get $\coprod f^{3*} R = (cdfR \coprod (ab-cd)R) \cap (aI \coprod cI) = abfR$. Finally, we get the following Λ-projective resolution of S_1, which is clearly not minimal:

$$0 \to abfR/abfI \to cdfR/cdfR \coprod (ab-cd)R/(ab-cd)I$$

$$\to aR/aI \coprod cR/cI \to v_1R/v_1I \to S_1 \to 0.$$

The next result shows, as in the above example, that we can always choose the elements $\{f^n\}$ in such a way that we obtain a minimal projective resolution of a finitely generated Λ-module when I is an admissible ideal.

Assume now that I is an admissible ideal, hence Λ is finite dimensional. Choose $\{f_i^0\}$ such that $\coprod_i f_i^0 R / \coprod_i f_i^0 RI$ is a projective cover of M_Λ. We have the following.

Theorem (1.2.7)[15]: In the resolution $(P,\delta) = \left(\coprod f_i^n R/f_i^n I, \left(\bar{h}^{n-1,n}\right)\right)$ the elements $\{f_j^n\}$ can be chosen in such a way that, for each n, no proper K-linear combination of a subset of $\{f_j^n\}$ is in $\coprod f^{n-1}I + f^{n*}J$. Moreover, there is a decomposition

$$\coprod f^{n*}R = \left(\coprod f_i^n R\right) \coprod \left(\coprod f_i^{n'} R\right)$$

where the elements $f_i^{n'}$ can be chosen to be in $\coprod f^{n-1}I$.

Proof. For each $n \geq 2$ we have the decomposition

(i) $\coprod f^{n*}R = (\coprod f^{n-1}R) \cap (\coprod f^{n-1}R)$

Step 1: We show first that we can adjust the decomposition (i) to obtain a decomposition of the type

(ii) $\coprod f^{n*}R = (f_1^n R \coprod \ldots \coprod f_t^n R) \coprod \left(f_{t+1}^{n'} R \coprod \ldots \coprod f_k^{n'} R\right)$

where each $f_i^{n'}$ is in $\coprod f^{n-1}I + f^{n*}J$, and, no proper K-linear combinations of a subset of $\{f_i^n\}$ is in $\coprod f^{n-1}I + f^{n*}J$. To prove this claim, start with the decomposition (i). If $x = \alpha_1 f_1^n + \ldots + \alpha_s f_s^n$ is a K-linear combination, where, say $\alpha_1 \neq 0$, then we have $f_1^n R \coprod .. \coprod f_t^n R = xRf_2^n R \coprod \ldots \coprod f_t^n R$. Thus, if x is in $\coprod f^{n-1}I + f^{n*}J$, we adjust our initial decomposition to the decomposition

$$\coprod f^{2*}R = \left(f_2^n R \coprod f_t^n R\right) \coprod \left(f_{t+1}^{n'} R \coprod \ldots \coprod f_k^{n'} R\right)$$

thus becoming one of the $f^{n'}$'s. The element x may be assumed to be right uniform. We continue this process and the claim is proved.

Step 2: By the first step, we may assume that we have a decomposition of the type

$$\coprod f^{2*}R = \left(f_1^n R \coprod f_t^n R\right) \coprod \left(f_{t+1}^{n'} R \coprod \ldots \coprod f_k^{n'} R\right)$$

where each of the $f^{n'}$'s is in $\coprod f^{n-1}I + f^{n*}J$. We show now that we can further adjust this decomposition in such a way that each $f^{n'}$ is in fact in $\coprod_i f_i^{n'} I$.

Let $y = f_j^{n'}$ be such that $f_j^{n'}$ is not in $\coprod f^{n-1}I$. We can write $y = a' + b'$ where a' is in $\coprod f^{n-1}I$ and b' is in $\coprod f^{n*}J$. But $\coprod f^{n-1}I$ is contained in $\coprod f^{n*}R = (\coprod f^{n-1}R) \cap (\coprod f^{n-2}I)$, so we can write $a' = ya - q$ and $b' = yb + q$ for some a in R, b in J and some q in $\Pi_{f^{n*} \neq y} f^{n*}R$.

We get $y = y(a+b)$ and, since y is right uniform with terminus w, we have $(a+b) = \omega$, so $a = b - \omega$. Let $z = (\omega + b)(\omega + b^2)(\omega + b^3)\ldots(\omega + b^{2^n})$. We multiply $a' = ya - q$ in $\coprod f^{n-1}I$ by z on the right and we obtain $y(\omega + b^{2^{n+1}}) - qz$ in $\coprod f^{n-1}I$ or $y - yb^{2^{n+1}} - qz$ in $\coprod f^{n-1}I$. Since I is admissible, for large enough n, $b^{2^{n+1}}$ is in I, so $yb^{2^{n+1}}$ is in $yI \subset \coprod f^{n-1}I$, so $y - qz$ is in $\coprod f^{n-1}I$.

We claim now that $(y - qz)R \coprod \left(\coprod_{f^{n*} \neq y} f^{n*}R\right) = \coprod f^{n*}R$. To show this we first observe that $\coprod f^{n*}R = (y - qz)R + \left(\coprod_{f^{n*} \neq y} f^{n*}R\right)$. It is obvious that the sum is direct.

In this way it is clear that we can adjust our decomposition, and that we may assume that each of the $f_i^{n'}$ is in fact in $\coprod f^{n-1}I$. The case where $n = 1$ is similar.

We now give the analogous result for the graded (not necessarily noetherian) case.

Theorem (1.2.8)[15]: Assume Λ is length graded and let M be a graded Λ-module. Then the resolution $(P, \delta) = \left(\coprod f_i^n R / \coprod_i f_i^n I, \left(\bar{h}^{n-1,n}\right)\right)$ can be chosen to be a graded resolution in such a way that for each n, the f_i^{n*}'s are homogeneous elements (and hence the f_i^n's), with no proper K-linear combination of a subset of $\{f_i^n\}$ is in $\coprod f^{n-1}I + \coprod f^{n*}J$. Moreover, there is a decomposition

$$\coprod f^{n*}R = \left(\coprod f^n R\right) \coprod \left(\coprod f^{n'} R\right)$$

where the elements $f^{n'}$ can be chosen to be homogeneous elements in $\coprod f^{2*}I$.

Proof. By Proposition (1.2.5) we begin with a graded resolution of M. For each n $n \geq 2$, we have a decomposition

(i) $\coprod f^{n*}R = (\coprod f^{n-1}R) \cap (\coprod f^{n-2}I)$

with the f^{n*}'s homogeneous.

Step 1: We show first that we may adjust the decomposition (iii), to obtain a decomposition of the type

(ii) $\coprod f^{n*}R = (\coprod f^{n}R)\coprod(\coprod f^{n'}R)$

where each $f^{n'}$ is a homogeneous element in $\coprod f^{n-1}R + \coprod f^{n*}J$, and, no proper K-linear combination of a subset of $\{f^{n}\}$ is in $\coprod f^{n-1}I + \coprod f^{n*}J$. For each degree, there are only a finite number of f^{n*}'s in that degree, since each homogeneous component of M and Λ is finite dimensional. Fixing a degree, we obtain

(iii') $f^{n*}R = (\coprod f_1^{n}R \coprod \dots \coprod f_t^{n}R)\coprod(\coprod f_{t+1}^{n'}R \coprod \dots \coprod f_k^{n'}R)$

and we proceed, degree by degree, as in the proof of step 1 of Theorem (1.2.7).

Step 2: By the first step, we may assume that we have a collection of decompositions of the type

$$\coprod f^{n*}R = \left(\coprod f_1^{n}R \coprod \dots \coprod f_t^{n}R\right)\coprod\left(\coprod f_{t+1}^{n'}R \coprod \dots \coprod f_k^{n'}R\right)$$

where all f^{n*}, f_i^{n} and $f_j^{n'}$ are homogeneous in the same degree, and, where each of the $f_j^{n'}$'s is in $\coprod f^{n-1}I + \coprod f^{n*}J$. We show now that we can adjust these decompositions in such a way that, each $f_j^{n'}$ is in fact in $\coprod f^{n-1}I$. Let $y = f_j^{n'}$ be such that y is not in $\coprod f^{n-1}I$, and of degree k. We can write $y = a' + b'$ where a' is in $\coprod f^{n-1}I$, b' is in $\coprod f^{n*}J$, and both are homogeneous of degree k. Since $f^{n*}R$ is contained in $\coprod f^{n-1}I$, we can write $a' = ya - q$ for some q in $(\Pi_{f^{n*} \neq y} f^{n*}R)$, homogeneous of degree s. Note that a must be a homogeneous element of $R_0 = (KQ)_0$. By right uniformity $ya = y$. Thus $y - q$ is in $\coprod f^{n-1}I$ and we also have $(y-q)R\coprod(\coprod_{f^{n*} \neq y} f^{n*}R) = \coprod f^{n*}R$.

This completes the proof.

We can now show that the adjusted resolution is minimal.

Theorem (1.2.9)[15]: Let M be a Λ-module and let $(P) = \left(\coprod f^{n}R/\coprod f^{n}I, (\bar{h}^{n-1,n})\right)$ be the projective resolution of M as in Theorem (1.2.2), where the representatives $\{f^{n}\}$ are chosen

in such a way, that for each n, no proper K-linear combination of a subset of $\{f^n\}$ lies in $\coprod f^{n-1}I + \coprod f^{n*}J$. Then, the resolution (P) is minimal.

Proof. It is enough to show that for each n, the entries $h^{n-1,n}$ are in J, where $f^n = \sum f^{n-1}h^{n-1,n}$. We prove this for each n, the case $n = 1$ being obvious. Assume that for some $n > 1$ we have a representative written as $f_j^n = f_1^{n-1}h_1 + \ldots + f_t^{n-1}h_t$, where not all h are in J. Using the fact that the f^{n-1}'s are right uniform elements of R, we get an expression

$$f_j^n = f_1^{n-1}\alpha_1 + \ldots + f_t^{n-1}\alpha_t + f_1^{n-1}r_1 + \ldots + f_t^{n-1}r_t$$

where α_i is in K and r_i is in J for $i = 1,\ldots,t$ (not all necessarily nonzero). Let $x = \sum_{i=1}^{t} f_i^{n-1}\alpha_i$.

Then we can write x as $x = f_j^n - \sum_{i=1}^{t} f_i^{n-1}r_i$ in $\coprod f^{n-2}I + \coprod f^{n-1}J$, which is a contradiction to the choice of the elements $\{f^{n-1}\}$.

A projective resolution of $\Lambda/\underline{r}$ and the groups $DExt_\Lambda^n(\Lambda/\underline{r}, \Lambda/\underline{r})$ are given in [23], where $D = Hom_k(,K)$ is the usual duality. We now give some information on the elements f^{n-1}'s, which enables us to give a connection between the Bongartz- Butler-Gruenberg resolution in [23] and our resolution.

Proposition (1.2.10)[15]: Let M be a Λ-module and, for each $n \geq 0$, choose $\{f^n\}$ in such a way that the resolution $\left(\coprod f^{n-1}R / \coprod f^n I, \bar{h}\right)$ is minimal as in Theorem (1.2.9).

(a) Let $n \geq 1$ and write $f^n = f_1^{n-1}r_1 + \ldots + f_t^{n-1}r_t$ with r_i in R. Then, the elements f^n can be chosen (adjusted) in such a way that each of the elements $r_1,\ldots,r_t$ are in J/I.

(b) The elements f^{n+1} are in $(\coprod f^n J) \cap (\coprod f^{n+1}I)$ and

$$\left(\coprod f^n J\right)\left(\coprod f^{n+1}I\right) \subset \begin{cases} \left(\coprod f^0 J I^m\right) \cap \left(\coprod f^0 I^m J\right) & \text{if } n = 2m, \\ \left(\coprod f^0 J I^{m+1}\right) \cap \left(\coprod f^0 I^m J\right) & \text{if } n = 2m+1, \end{cases}$$

(c) $\Omega_\Lambda^{n-1}(M)/\Omega_\Lambda^{n-1}(M)\underline{r} \cong \coprod f^n R / \left((\coprod f^n R) \cap (\coprod f^{n+1}I) + \coprod f^n J\right)$ and

$$\coprod f^{n+1}I + \coprod f^n J \subset \begin{cases} \coprod f^0 J I^m + \coprod f^0 I^m J & \text{if } n = 2m, \\ \coprod f^0 I^{m+1} + \coprod f^0 J I^m J & \text{if } n = 2m+1, \end{cases}$$

(d) $DExt^n_\Lambda(M,\Lambda/\underline{r}) \cong \coprod f^n R\,/\big((\coprod f^n R) \cap (\coprod f^{n-1} I) + \coprod f^n J\big)$

Proof. (a) It is clear that not all the r_i's can be in I. Write, say,

$$f_j^n = f_1^{n-1} r_1 + \ldots + f_k^{n-1} r_k + f_{k+1}^{n-1} r_{k+1} + \ldots + f_s^{n-1} h_s$$

where $r_1,\ldots,r_k$ are not in I, but $r_{k+1},\ldots,r_s$ are all in I.

Let $x_1 = f_1^{n-1} r_1 + \ldots + f_k^{n-1} r_k$. Then it is easy to check that $\coprod f_i^{n*} R = x_1 R \coprod (f_2^{n*} R \coprod \ldots \coprod f^{n*} R)$, since $x_1 = f_1^n - u$, where u is in $\coprod f^{n-1} I \subset (\coprod (f^{n-1} R) \cap (\coprod f^{n-2} I) = \coprod f^{n*} R)$. Furthermore, x_1 is not in $\coprod f^{n-1} I + \coprod f^{n*} J$ $\coprod f^{n-1} I + \coprod f^{n*} J$; otherwise, f_1^n is. So we can replace f_1^n with x_1. Continue this process. The fact that the coefficients are in J follows from the minimality of the projective resolution.

(b) We have that $f^{n+1} = \sum f^n h^{n,n+1}$, where $h^{n,n+1}$ is in J, since the resolution is minimal. Therefore, f^{n+1} is in $\sum f^n J$. Moreover,

$$f^{n+1} = \sum f^n h^{n,n+1} = \sum f^{n-1} h^{n-1,n+1} h^{n,n+1}$$

where the last sum is in $\coprod f^{n-1} I$, since we have a resolution over $\Lambda = R/I$. It follows immediately from this that f^{n+1} is in $(\coprod f^n J) \cap (\coprod f^{n-1} I)$.

We saw above that f^i is in $\coprod f^{i-2} I$ for $i \geq 2$. This implies that f^{2m} is in $\coprod f^0 I^m$ and that f^{2m+1} is in $\coprod f^1 I^m \subset \coprod f^0 J I^m$. The last claim in (b) follows immediately from this.

(c) and (d) The second claim of (c) follows in a similar fashion to (b). Since the projective resolution of M is minimal and $\Lambda/\underline{r}$ is semisimple, it follows immediately that

$$DExt^n_\Lambda(M,\Lambda/\underline{r}) \cong Tor^\Lambda_n(M,\Lambda/\underline{r}) \coprod f^n R \,/ \coprod f^n J$$

Furthermore, the minimality of the resolution, and the above imply that $\coprod f^n J = \coprod f^n J + (\coprod f^n J) \cap (\coprod f^{n-1} I)$. The result follows.

It is shown in [Bo] that

$$DExt^n_\Lambda(\Lambda/\underline{r},\Lambda/\underline{r}) \cong \begin{cases} (I^m \cap JI^{m-1} J)/(JI^m + I^m J) & for\ n = 2m \geq 2 \\ (JI^m \cap I^m J)/(I^{m+1} + JI^m J) & for\ n = 2m+1 \geq 1 \end{cases}$$

We see that the formulas for $DExt^n_\Lambda(\Lambda/\underline{r},\Lambda/\underline{r})$ and for $DExt^n_\Lambda(M,\Lambda/\underline{r})$ in Proposition (1.2.10) are very similar, when M is any finitely generated Λ-module.

Let $\Lambda = R/I$ where R = KQ and $I \subseteq J^2$ is an admissible ideal of R, or a length homogeneous ideal. Let $\Lambda_0 = R/I$. Note that Λ_0 is a semisimple Λ- module. We recall that

the Ext-algebra of Λ is the graded K-algebra $E(\Lambda) = \coprod_{n\geq 0} Ext^n_\Lambda(\Lambda, \Lambda_0)$ with the obvious addition and with multiplication given by the Yoneda product. Given a right Λ-module M, we have a graded left $E(\Lambda)$-module $E(M) = \coprod_{n\geq 0} Ext^n_\Lambda(M, \Lambda_0)$. In this section we show how to use the minimal projective resolutions, to effectively describe the $E(\Lambda)$ saction on $E(M)$ (and thus, if $M = \Lambda_0$, the multiplicative structure of $E(\Lambda)$ itself), in an algorithmic way, by working at the level of the path algebra KQ.

We start by recalling the following interpretation of the Yoneda product, which will be used in this section. First, we observe that if M is a finitely generated Λ-module, in the admissible case, or M is graded in the length homogeneous case, we may consider a minimal Λ-resolution of M:

$$\ldots \to p_n \xrightarrow{\delta^n} p_{n-1} \to \ldots \to p_0 \to M \to 0$$

This resolution is minimal in the sense that, for each n > 0, we have $\operatorname{Im}\delta^n \subseteq P_{n-1}\underline{r}$ where $\underline{r}$ is the Jacobson radical of Λ in the case where I is admissible, and $\underline{r}$ is the graded radical of Λ in the graded case which need not be finite dimensional. Since $Ext^n_\Lambda(M, \Lambda_0)$ is the cohomology of the complex

$$0 \to Hom_\Lambda(p_0, \Lambda_0) \xrightarrow{\delta_*} Hom_\Lambda(p_1, \Lambda_0) \xrightarrow{\delta_*} \ldots$$

and, since Λ_0 is semisimple, the boundary maps of this complex are all zero. It follows that, for each $n \geq 0$, we have $Ext^n_\Lambda(M, \Lambda_0) = Hom_\Lambda(p_n, \Lambda_0)$. Now let $\hat{f}$ be $Ext^n_\Lambda(M, \Lambda_0)$ and let $\hat{g}$ be in $Ext^m_\Lambda(\Lambda_0, \Lambda_0)$. We have the following commutative diagram with exact rows (5)

$$\begin{tikzcd}
P^M_{n+m} \arrow[r] \arrow[d, "l_m"] & \ldots \arrow[r] & P^M_{n+1} \arrow[r] \arrow[d, "l_1"] & P^M_n \arrow[d, "l_0"] \arrow[dr, "\hat{f}"] & & \\
P^\Lambda_m \arrow[r] \arrow[d, "\hat{g}"] & \ldots \arrow[r] & P^\Lambda_1 \arrow[r] & P^\Lambda_0 \arrow[r] & \Lambda_0 \\
\Lambda_0 & & & &
\end{tikzcd}$$

where the top row is part of a minimal projective resolution of M, the bottom sequence is part of a minimal projective resolution of Λ_0, and the vertical maps $l_0, l_1, \ldots, l_m$ are successive listings (not necessarily unique) of $\hat{f}$. Then we have the following description of the $E(\Lambda)$ action on E(M): $\hat{g} * \hat{f} =$ the composition $\hat{g} \circ l_m$.

It is well known that this action is well defined. We also remark that, since $P_n^M = \coprod_i (f_i{}^n R / f_i{}^n I)$, we have a dual basis of $Ext_\Lambda^n(M, \Lambda_0) = Hom_\Lambda(P_n^M, \Lambda_0)$ consisting of the maps $\{\hat{f}_i{}^n\}$ defined in the obvious way. In order to describe this action at the level of the path algebra KQ, we construct a commutative diagram of R-modules where, the first and the third rows are minimal projective Λ-resolutions of $\Omega_\Lambda^n M$ and, Λ_0 respectively, and the back face of the parallelepiped is the front face modulo the ideal I:

$$\begin{tikzcd}
 & P_{n+r}^M \arrow[r] & P_{n+r-1}^M \arrow[r] \arrow[d] & \cdots \arrow[r] & P_n^M \arrow[r] \arrow[d] & \Omega_\Lambda^n(M) \arrow[r] \arrow[dd, equal] & 0 \\
\coprod f^{n+r}R \arrow[ur] \arrow[r, hook] \arrow[dd] \arrow[d] & \coprod f^{n+r-1}R \arrow[ur] \arrow[r, hook] \arrow[dd] & \cdots \arrow[r] & \coprod f^n R \arrow[ur] \arrow[r] \arrow[dd] & \Omega_\Lambda^n M \arrow[r] \arrow[dd] & 0 \arrow[d] \\
 & P_r^\Lambda \arrow[r] & P_{r-1}^{\breve{\Lambda}} \arrow[r] & \cdots \arrow[r] & \breve{\Lambda} \arrow[r] & \Lambda_0 \arrow[r] & 0 \\
\coprod g^r R \arrow[r, hook] \arrow[d] & \coprod g^{r-1}R \arrow[ur] \arrow[r, hook] & \cdots \arrow[r] & R \arrow[ur] \arrow[r] & \Lambda_0 \arrow[r] \arrow[ur, equal] & 0 \\
R \arrow[ur] & \Lambda_0 \arrow[ur] & & & & & \\
 & & & & & & \hat{f}_j^n \\
 & & & & & & \hat{g}_i^r
\end{tikzcd}$$

We denote by g_i the map $\coprod g_i R \to R$ as well as its image modulo I, $\coprod g_i \Lambda \to \Lambda_0$.

Notation. Let, $\left(\coprod f_i^n R / \coprod f_i^n I, \left(\overline{h}^{n-1,n}\right)\right)$ be a minimal projective resolution of M over Λ, and let $\left(\coprod g_i^n R / \coprod g_i^n I, \left(\overline{k}^{n-1,n}\right)\right)$ be a minimal projective resolution of Λ_0 over Λ. Recall that the entries of the matrices $(\overline{h}^{n-1,n})$ and $(\overline{k}^{n-1,n})$ are given by the expressions

$$f_i^n = \sum_j f_j^{n-1} h_{j,i}^{n-1,n} \quad \text{and} \quad g_i^n = \sum_j g_j^{n-1} k_{j,i}^{n-1,n}$$

Recall also that for each $n \geq 0$ we have

$$\left(\coprod f^{n-1}R\right)\cap\left(\coprod f^{n-2}I\right)=\left(\coprod f^{n}R\right)\coprod\left(\coprod f^{n'}R\right)$$

where each $f^{n'}$ is in $\coprod f^{n-1}I$. We have similar statements and notation involving the g's, and, note that there are no $g^{0'}$'s and g^{1}'s appearing. We have the following lemma which is valid over an arbitrary ring S.

Lemma (1.2.11)[15]: Let A, B, C and D be S-modules satisfying $B\subseteq C$ and $(A+C)\cap D=(0)$. Then we have

$$(A\coprod B\coprod D)\cap(C\coprod D)\subseteq(A\cap C)\coprod B\coprod D$$

Proof. We first observe that we have the inclusion $(A\coprod B\coprod C)\cap(C\coprod D)\subseteq[(A\coprod B)\cap C]\coprod D$ since, if $x=a+b+d=c+d'$, then $a+b-c=d'-d$ is in D, since b is also in C, we get that $d=d'$ and a + b = c. Therefore, x is in $[(A\coprod B)\cap C]+D$. But $D\cap(A+C)=(0)$ and $B\subseteq C$ so this sum is direct. The lemma follows now immediately since $B\subseteq C$ implies the well-known equality $(A\coprod B)\cap C=(A\cap C)\coprod B$.

The following rather technical proposition is crucial to our description.

Proposition (1.2.12)[15]: (a) For each $n\geq 0$, the matrix $(h^{n,n+1})$ has entries in $\coprod g^{1}R$.

(b) For each $r\geq 2$ and $n\geq 0$, the product of matrices $(h^{n,n+1})\ldots(h^{n+r-1,n+r})$ has entries in $\coprod g^{1}R\coprod\left(\prod_{2\leq i\leq r}g^{i'}R\right)$.

Note that for simplicity we are using the following notation: the matrix $(\mathrm{h}^{\mathrm{n,n+1}})$ actually denotes the matrix $\left(h_{j,i}^{n,n+1}\right)$ where the entries $h_{j,i}^{n,n+1}$ are obtained by expanding the elements of the form f^{n+1} in terms of the elements of the form f^{n}.

We now claim that we have the following inclusion:

Note that, since $(\coprod q^{r-1}R)\cap(\coprod q^{r-2}I)=(\coprod g^{r}R)\coprod\left(\coprod g^{r'}R\right)$, the claim will indeed imply that the entries of our product of matrices are in the required sum. To prove this claim, let $A=\coprod g^{r-1}R$, $B=\coprod g^{r-1'}R$, $C=\coprod g^{r-2}I$, and $D=\coprod_{2\leq i\leq r-2g^{i'}}R$. With this notation we must prove that we have an inclusion:

$$(A\coprod B\coprod D)\cap(C\coprod D)\subseteq(A\cap C)\coprod B\coprod D.$$

This is precisely the statement of Lemma (1.2.11) once one shows that $B\subseteq C$ and that $(A+C)\cap D=(0)$. It is clear that $B\subseteq C$. We also have that A and C are included in $\coprod g^{r-2}R$ and $D\cap(\coprod g^{r-2}R)=(0)$; hence $D\cap(A+C)=(0)$. The proof is now complete.

Definition (1.2.13)[15]: It follows from the previous proposition that each entry of the matrix product $(h^{n,n+1}) \dots (h^{n+r-1,n+r})$ is an element of the form $\sum g^r l^{r,n,n+r} + \sum_{2\le i\le r} g^{i'} s^{i,n,n+r}$. Let $(l^{r,n,n+r})$ be the corresponding matrix. (Note that the number of rows of this matrix is the number of elements in the family $\{g^r\}$, and that the number of columns is the number of elements in the family $\{f^{n+r}\}$.)

Proposition (1.2.14)[15]: For each $n \ge 0$ and $r \ge 0$, the following diagram of R-module commutes modulo I:

$$\begin{array}{ccc} \coprod f^{n+r}R & \xrightarrow{(h^{n+r-1,n+r})} & \coprod f^{n+r-1}R \\ \downarrow (l^{r,n,n+r}) & & \downarrow (l^{r-1,n,n+r-1}) \\ \coprod g^r R & \xrightarrow{(k^{r-1,r})} & \coprod g^{r-1}R \end{array}$$

Proof*:* We shall use a Sweedler-type notation in order to avoid multiple indices. Note that by Proposition (1.2.12) we have that the entries of the product of r consecutive $(h^{n+i,n+i+1})$-matrices have the form $X = \sum g^r l^{r,n,n+r} + \sum_{2\le i\le r} g^{i'} s^{i,n,n+r}$. We have the following:

$$X = \sum g^r l^{r,n,n+r} + \sum g^{r'} s^{r,n,n+r} + \sum_{2\le i\le r-1} g^{i'} s^{i,n,n+r}$$

$$= \sum \left(\sum g^{r-1} k^{r-1,r}\right) l^{r,n,n+r} + \sum \left(\sum g^{r-1} a^{r-1,r}\right) s^{r,n,n+r} + \sum_{2\le i\le r} g^{i'} s^{i,n,n+r}$$

$$= \sum g^{r-1} \left(\sum k^{r-1,r} l^{r,n,n+r} + a^{r-1,r} s^{r,n,n+r}\right) + \sum_{2\le i\le r-1} g^{i'} s^{i,n,n+r},$$

where each $a^{r-1,r}$ is in the ideal I.

On the other hand, by expressing the matrix product as

$$[(h^{n,n+1}) \dots (h^{n+r-2,n+r-1})] \cdot (h^{n+r-1,n+r}),$$

we see that each entry of this product is an entry of the product

$$\left(\sum g^{r-1} l^{r-1,n,n+r-1} + \sum_{2\le i\le r-1} g^{i'} s^{i,n,n+r-1}\right) \cdot (h^{n+r-1,n+r}).$$

Therefore, it has the form

$$\sum g^{r-1} \left(\sum l^{r-1,n,n+r-1}\, h^{n+r-1,n+r}\right) + \sum_{2\le i\le r-1} g^{i'} \left(\sum s^{i,n,n+r-1}\, h^{n+r-1,n+r}\right).$$

From the uniqueness of the direct sum decomposition we obtain

$$\sum g^{r-1}\left(\sum l^{r-1,n,n+r-l}h^{n+r-1,n+r}\right)=\sum g^{r-1}\left(\sum k^{r-1,r}l^{r,n,n+r}+a^{r-1,r}s^{r,n,n+r}\right)$$

where we recall that each $a^{r-1,r}$ is in I. It is immediate that the above equality implies the proposition.

We are ready to prove the main result of this section.

Theorem (1.2.15)[15]: Let $\hat{f}_j^n$ be in $\mathrm{Ext}_\Lambda^n(M,\Lambda_0)$ and let $\hat{g}_i^r$ be in $\mathrm{Ext}_\Lambda^r(\Lambda_0,\Lambda_0)$ be two basis elements. Then, the element $\hat{g}_i^r * \widehat{f}_j^n$ of $\mathrm{Ext}_\Lambda^{n+r}(M,\Lambda_0)$ is given by

$$\hat{g}_i^r * \hat{f}_j^n=\sum_t \tilde{l}_{i,j,t}^{r,n,n+r}\hat{f}_t^{n+r}$$

where $\tilde{l}$ is the image in $\Lambda/\underline{r}$ of the element l of R.

Proof: We infer from the previous proposition that for each $n\geq 0$, and $r>1$, we have a commutative diagram of Λ-modules:

$$\begin{array}{ccc}
\coprod \bar{f}^{n+r}\Lambda & \xrightarrow{(\bar{h}^{n+r-1,n+r})} & \coprod \bar{f}^{n+r-1}\Lambda \\
\downarrow (l^{r,n,n+r}) & & \downarrow \left(\bar{l}^{r-1,n,n+r-1}\right) \\
\coprod \bar{g}^{r}\Lambda & \xrightarrow{(\bar{k}^{r-1,r})} & \coprod \bar{g}^{r-1}\Lambda
\end{array}$$

This shows, as promised, that we have constructed a diagram (5) as in the beginning of this section, and, that the product $\hat{g}_i^r * \hat{f}_j^n$ is in fact the composition $\hat{g}_i^r \circ \left(\bar{l}^{r,n,n+r}\right)$. The theorem now follows.

This section is devoted to applying the minimal projective resolution to the no loop conjecture. In particular, we show a special case of this conjecture.

Let K, Q, and R be as above, and let I be an admissible ideal in R. As before, denote R/I by Λ. The no loop conjecture says the following. If S is a simple Λ-module such that $\mathrm{Ext}_\Lambda^1(S,S)\neq(0)$, then $\mathrm{pd}_\Lambda S=\infty$. In [27], Igusa proved that the global dimension of Λ must be infinite, whenever there exists a simple Λ-module S with $\mathrm{Ext}_\Lambda^1(S,S)\neq(0)$. (See also [28].)

First we prove a small technical lemma.

Lemma (1.2.16)[15]: Suppose a is an arrow in Q, where $a: v\to v$ for some vertex v in Q. Let S be the simple Λ-module corresponding to this vertex, and assume that the minimal resolution of S has been constructed finding the elements f^n, $f^{n'}$ and $h^{n-1,n}$. The residue class of a^n in $vI/v(JI+IJ)$ is nonzero if and only if

$$a^n = \sum_i f_i^2 r_i + \sum_i f_i^{2'} s_i,$$

where the coefficients r_i and s_i are R, and where some r_i is not in J.

Proof: Assume that $a^n = \sum_i f_i^2 r_i + \sum_i f_i^{2'} s_i$ for r_i and s_i in R, where one of the r_i's is not in J. Assume that the residue class of a^n in $vI = v(JI + IJ)$ is zero. Since each $f_i^{2'}$ is in $\coprod f^1 I$, it is in vJI. Hence the set $\{f_i^2\}$ is linearly dependent in $vI/v(JI + IJ)$. This is a contradiction to the choice of f_i^2 (Theorem (1.2.9)), so that the residue class of a^n in $vI/v(JI + IJ)$ is nonzero.

Conversely, assume that $a^n = \sum_i f_i^2 r_i + \sum_i f_i^{2'} s_i$ for some r_i and s_i in R, where all r_i are in J. Then clearly the residue class of a^n in $vI/v(JI + IJ)$ is zero.

Now we show that if S is a simple Λ-module corresponding to a vertex with a loop a where a^n is in I and a^n has a nonzero residue class in $vI/v(JI + IJ)$, then $\mathrm{pd}_\Lambda S = \infty$.

Proposition (1.2.17)[15]: Suppose a is an arrow in Q, where $a: v \to v$ for some vertex v in Q. Let S be the simple Λ-module corresponding to this vertex, and assume that a^n is in I for some $n \geq 2$ and that the residue class of a^n in $vI/v(JI + IJ)$ is nonzero. Then $\mathrm{Ext}_\Lambda^i(S, S) \neq (0)$ for all $i \geq 1$. In particular, $\mathrm{pd}_\Lambda S = \infty$.

Proof: Note that the hypothesis implies that a^{n-1} is not in I. It is well-known that $\mathrm{Ext}_\Lambda^i(S, S) \neq (0)$ for $i = 1,2$.

By the above lemma, we have that $a^n = \sum_i f_i^2 r_i^2 + \sum_i f_i^{2'} s_i^2$ for some r_i^2 and s_i^2 in R, where some r_i^2 is not in J. Let $z_2 = \sum_i f_i^{2'} s_i^2$. Then $b_2 = a^n - z_2$ is in $\coprod f^2 R$. We want to show that $b_2 a$ is in $\coprod f^2 R \cap \coprod f^1 I$. Since a^n is in I and z_2 is in $\coprod f^{2'} R$, the element z_2 is in $\coprod f^1 R$ and, therefore $b_2 a$ is in $\coprod f^2 R \cap \coprod f^1 I$.

Assume that $b_2 a$ is in $\coprod f^2 I$. Then there exist u_i in I such that $\sum_i f_i^2 r_i^2 a = \sum_i f_i^{2'} u_i$. Hence, $r_i^2 a$ is in I for all i. There exists an i_0 such that $r_{i_0}^2 = cv + h$ for c in K^* and h in J. Then

$$c^{-1}\left(v + (c^{-1}h)^{2^m}\right) \dots (v + (c^{-1}h)^2)\left(v - (c^{-1}h)\right)(cv + h)a = a - (c^{-1}h)^{2^{m+1}}$$

is in I. Since I is admissible and h is in J, for some large m, we get that a is in I. This is a contradiction, which shows that $b_2 a$ is not in $\coprod f^2 I$ and, $\mathrm{Ext}_\Lambda^3(S, \Lambda/\underline{\mathrm{r}}) \neq (0)$.

By the above, we can write $b_2 a$ as

$$b_2 a = \sum_i f_i^3 r_i^3 + \sum_i f_i^{3'} s_i^3$$

for some elements r_i^3 and s_i^3 in R. Expanding the two sides in the elements f^2, we get

$$\sum_i f_i^2 r_i^2 a = \sum_{t,i} f_t^2 h_{t,i}^{2,3} r_i^3 + \sum_{t,i} f_t^2 \alpha_{t,i}^3 s_i^3,$$

where each α_i^3 is in I. This implies that

$$r_t^2 a = \sum_i h_{t,i}^{2,3} r_i^3 + \sum_i \alpha_{t,i}^3 s_i^3.$$

There exists some t such that $r_t^2 a$ is not in J^2. If r_i^3 is in J for all i, then the right hand side is in J^2. Therefore, we infer that not all r_i^3 are in J. Since $b_2 a$ ends in v, there are also some f_i^3 such that $f_i^3 v \neq 0$. Hence, $\mathrm{Ext}_\Lambda^3(S,S) \neq (0)$. Let $z_3 = \sum_i f_i^{3'} s_i^3$ and $b_3 = b_2 a - z_3$.

Assume now that we have shown that

(a) $b_{2m+1} = b_{2m} a - z_{2m+1} = \sum_i f_i^{2m+1} r_i^{2m+1}$, where not all r_i^{2m+1} are in J,

(b) $b_{2m} = \sum_i f_i^{2m} r_i^{2m}$, where not all r_i^{2m} are in J,

(c) z_{2m+1} is in $\coprod f^{2m+1'} R$.

We want to show that $b_{2m+1} a^{n-1}$ represents some nonzero element in $\mathrm{Ext}_\Lambda^{2m+2}(S,S)$.

The element $b_{2m+1} a^{n-1}$ is clearly in $\coprod f^{2m+1} R$. Since b_{2m} is in $\coprod f^{2m} R$ and z_{2m+1} is in $\coprod f^{2m} I$, we have that $b_{2m+1} a^{n-1} = b_{2m} a^n + z_{2m+1} a^{n-1}$ is in $\coprod f^{2m} I$. Therefore, $b_{2m+1} a^{n-1}$ is in $\coprod f^{2m+1} R \cap \coprod f^{2m} I$.

Assume that $b_{2m+1} a^{n-1}$is in $\coprod f^{2m+1} I$, that is, $b_{2m+1} a^{n-1} = \sum f_i^{2m+1} u_i^{2m+1}$, where u_i^{2m+1} is in I for all i. Expanding the elements in terms of the elements f^{2m} we get

$$\begin{aligned}\sum_i f_i^{2m} r_i^{2m} a^n &= \sum_i f_i^{2m+1} u_i^{2m+1} z_{2m+1} a^{n-1} \\ &= \sum_i f_t^{2m} h_{t,i}^{2m,2m+1} u_i^{2m+1} \sum_i f_t^{2m} \beta_t^{2m+1} a^{n-1}\end{aligned}$$

where β_t^{2m+1} is in I for all t. For each t, we have that

$$r_t^{2m} a^n = \sum_i h_{t,i}^{2m,2m+1} u_i^{2m+1} + \beta_t^{2m+1} a^{n-1}.$$

The right hand side is in $JI + IJ$ for all t. For some t, the element r_t^{2m} is not in J. Using that I is an admissible ideal and similar arguments as before, we conclude that a^n is in $JI + IJ$. This is a contradiction, and therefore $b_{2m+1}a^{n-1}$ is not in

$$\coprod f^{2m+1}I \text{ and } \mathrm{Ext}_\Lambda^{2m+2}(S, \Lambda, \underline{r}) \neq (0).$$

By the above, $b_{2m+1}a^{n-1}$ has a representation of the form

$$b_{2m+1}a^{n-1} = \sum_i f_i^{2m+2} r_i^{2m+2} + \sum_i f_i^{2m+2'} s_i^{2m+2}.$$

Expanding in terms of the elements f^{2m}, we get

$$\sum_i f_i^{2m} r_i^{2m} a^n = \sum_i f_i^{2m+2} r_i^{2m+2} + \sum_i f_i^{2m+2'} s_i^{2m+2} + z_{2m+1}a^{n-1}$$

$$= \sum f_t^{2m} h^{2m,2m+1} h^{2m+1,2m+2} r_i^{2m+2} + \sum f_t^{2m+1} \alpha_{l,i}^{2m+2} s_i^{2m+2} + z_{2m+1}a^{n-1}$$

$$= \sum f_t^{2m} h^{2m,2m+1} h^{2m+1,2m+2} r_i^{2m+2} + \sum f_t^{2m} h^{2m,2m+1} \alpha_{l,i}^{2m+2} s_i^{2m+2} + z_{2m+1}a^{n-1}$$

for some $\alpha_{l,i}^{2m+2}$ in I. We have that z_{2m+1} is in $\coprod f^{2m}I$, so that, if r_i^{2m+2} is in J for all i, then $r_i^{2m}a^n$ is in $JI + IJ$ for all i. Since there exists $r_{i_0}^{2m}$ which is not in J, we show as before that a^n is in $JI + IJ$. This is a contradiction, and consequently, not all r_i^{2m+2} are in J, and $\mathrm{Ext}_\Lambda^{2m+2}(S, S) \neq (0)$. Let $z_{2m+2} = \sum_i f^{2m+2'} s_i^{2m+2}$ and $b_{2m+2} = b_{2m+1}a^{n-1} - z_{2m+2}$. We can now move on to the following step.

Assume now that we have shown that

(a) $b_{2m} = b_{2m-1}a^{n-1} - z_{2m} = \sum_i f_i^{2m} r_i^{2m}$, where not all r_i^{2m} are in J,

(b) $b_{2m-1} = \sum_i f_i^{2m-1} r_i^{2m-1}$, where not all r_i^{2m-1} are in J,

(c) z_{2m} is in $\coprod f^{2m'} R$.

We want to show that $b_{2m}a$ represents some nonzero element in $\mathrm{Ext}_\Lambda^{2m+1}(S, S)$.

The element $b_{2m}a$ is clearly in $\coprod f^{2m}R$. Since b_{2m-1} is in $\coprod f^{2m-1}R$, z_{2m} is in $\coprod f^{2m-1}I$ and $b_{2m}a = b_{2m-1}a^n - z_{2m}a$, we have that $b_{2m}a$ is in $\coprod f^{2m}R \cap \coprod f^{2m-1}I$.

Assume that $b_{2m}aa$ is in $\coprod f^{2m}I$. Then $b_{2m}a = \sum_i f_i^{2m} r_i^{2m}\, a = \sum_i f_i^{2m} u_i^{2m}$ with u_i^{2m} in I. Therefore, $r_i^{2m}a$ is in I for all i, and, as before, this would give the contradiction that a is in I. Thus $b_{2m}a$ is not in $\coprod f^{2m}I$ and $\mathrm{Ext}_\Lambda^{2m+1}(S, \Lambda/\underline{r}) \neq (0)$.

By the above, the element $b_{2m}a$ has a representation of the form

$$b_{2m}a = \sum_i f_t^{2m+1} r_i^{2m+1} + \sum_i f^{2m+1'} s_i^{2m+1}.$$

Expanding in terms of the elements f^{2m}, we get

$$\sum_i f_i^{2m} r_i^{2m} a = \sum f_i^{2m+1} r_i^{2m+1} + \sum_i f_i^{2m+1'} s_i^{2m+1}$$

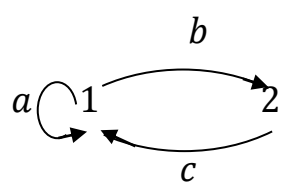
$$= \sum_t f_t^{2m} h_{t,i}^{2m,2m+1} r_i^{2m+1} + \sum_t f_t^{2m} \alpha_{t,i}^{2m+1} s_i^{2m+1}$$

where $\alpha_{t,i}^{2m+1}$ is in I. If all r_i^{2m+1} are in J, then $r_i^{2m}a$ is in J^2 for all i. As before, this is a contradiction, and therefore, there exists some r_i^{2m+1} not in J. Using the same arguments as before, it follows that $\mathrm{Ext}_\Lambda^{2m+1}(S,S) \neq (0)$. Combining what we have shown so far, the proof of the proposition is complete.

We point out that it follows from our proof of Proposition (1.2.17) that the residue class of a^{rn} is nonzero in $JI^{r-1}J \cap I^r/(JI^r + I^rJ)$ (which is isomorphic to $\mathrm{Tor}_{2r}^\Lambda(\Lambda/\underline{r}, \Lambda/\underline{r})$ by [23]), and that the residue class of a^{rn+1} is nonzero in $JI^r \cap I^rJ/(JI^rJ + I^{r+1})$ (which in turn is isomorphic to $\mathrm{Tor}_{2r+1}^\Lambda(\Lambda/\underline{r}, \Lambda/\underline{r})$).

However, when $\mathrm{Ext}_\Lambda^1(S,S) \neq (0)$, then $\mathrm{Ext}_\Lambda^i(S,S)$ is not in general nonzero for all $i \geq 1$. The following example was pointed out to us by Dieter Happel [29].

Let Q be the quiver given by



Let I be the ideal in KQ generated by $(a^2 - bc, cab, cb)$. Denote by S the simpleΛ-module corresponding to the vertex 1. As with all simple modules corresponding to a vertex with a loop, $\mathrm{Ext}_\Lambda^i(S,S) \neq (0)$ for $i = 1,2$. However, $\mathrm{Ext}_\Lambda^3(S,S) = (0)$, and, since $\Omega_\Lambda^4(S) = S$, then $\mathrm{Ext}_\Lambda^i(S,S) = (0)$ if $i \equiv 3$ mod 4, and nonzero otherwise for all $i \geq 1$.

Let Q be a finite quiver and let $\mathrm{I} = \langle \rho_1, \dots, \rho_n \rangle$ be an admissible ideal of KQ, where $\{\rho_1, \dots, \rho_n\}$is a minimal set of generators of the two-sided ideal I. Assume that for each $i =$

$1, \dots, n$, the element ρ_i is a linear combination of paths in Q with coefficients $+1$ or -1. Let $\Lambda = KQ/I$ be the corresponding finite dimensional algebra. We know that if Λ is a monomial algebra, then the global dimension of Λ (more generally, the projective dimension of each simple Λ-module) is independent of the characteristic of K (see [30]). If Λ is the incidence algebra of a partially ordered set, the global dimension of Λ, although always finite, can vary with the characteristic of K, unless gldim $\Lambda \leq 2$ in which case it is again characteristic independent (see [31, 32]). In this section, we give examples which show that the global dimension can uctuate rather wildly according to the characteristic of the field. The proofs of these examples can be easily done using the minimal projective resolution constructed in the second section. We start, however, by showing that, if the global dimension is two in one characteristic, then it cannot be infinite in another. More specifically, we prove the following.

Theorem (1.2.18)[15]: Let Q be a finite quiver and let E and F be two fields. Let $\rho_1, \dots, \rho_n$ be linear combinations of paths in Q, where the coefficients are ± 1, and let I_E (I_F respectively) denote the two-sided ideal of EQ (FQ respectively) generated by the elements $\rho_1, \dots, \rho_n$. Assume that I_E and I_F are both admissible ideals, and, that gldim $EQ/I_E \leq 2$. Then gldim $FQ/I_F < \infty$.

Proof. Any artin algebra of global dimension n has at least one simple module with injective dimension equal to $n-1$ by an argument similar to the one in [34]. The theorem follows from the remarks above by induction on the number of nonisomorphic simple modules.

Lemma (1.2.19)[15]: Let Λ be an artin algebra and let S_Λ be a simple Λ-module such that $\mathrm{id}_\Lambda S \leq 1$. Let e be the primitive idempotent corresponding to S. Then,

(i) $\mathrm{gldim}(1-e)\Lambda(1e-)\mathrm{gldim}$,

(ii) if $\mathrm{gldim} = \infty$, then $\mathrm{gldim}(1-e)(1-e) = \infty$. (Compare with [33, Proposition (1.2.10)].)

Proof: Since $\mathrm{id}_\Lambda S_\Lambda \leq 1$, we have $\mathrm{pd}_{\Lambda^{\mathrm{op}}} DS \leq 1$, where D is the usual duality, and we can use [34] to infer (i). Assume now that $\mathrm{gldim}\,\Lambda = \infty$. Then there is a Λ-module M_Λ such that $\mathrm{pd}\, M_\Lambda = \infty$. Let $N = \Omega_\Lambda^2 M$ —the second $syzygy$ of M. By [35], since $\mathrm{id} S \leq 1$, we have that, if

$$\mathcal{P}: \dots P_n \to \cdots \to P_0 \to N_\Lambda \to 0$$

is a minimal projective resolution of N, then no P_i has a summand isomorphic to $e\Lambda$. Thus, $\mathrm{Hom}_\Lambda\big((1-e)\Lambda, \mathcal{P}\big)$ is a minimal projective resolution of the $(1-e)\Lambda(1-e)$-module $\mathrm{Hom}_\Lambda\big((1-e)\Lambda, N\big)$. Hence, $\mathrm{gldim}(1-e)\Lambda(1-\ e) = \infty$.

The next result is well-known.

Lemma (1.2.20)[15]: Let $\Lambda = KQ/I$ be a finite dimensional quotient of the path algebra KQ by an admissible ideal I. Let $\rho_1, \dots, \rho_n$ be a minimal set of generators of I in the sense that I cannot be generated by a proper subset of $\{\rho_1, \dots, \rho_n\}$. For a vertex v of Q the following are equivalent:

(i) $\mathrm{id}S_v \leq 1$.

(ii) $\rho_i v = 0$ for each $i = 1, \dots, n$.

Definition (1.2.21)[15]: Let Q be a finite quiver and let I be an admissible ideal of KQ. Let $\rho_1, \dots, \rho_n$ be a minimal set of generators of I and assume that $\rho_i v = 0$ for each i. It is clear that there are no loops at the vertex v since, otherwise, we would have that $\mathrm{Ext}^2_{KQ/I}(S_v, S_v) \neq (0)$, contradicting the fact that the injective dimension of S_v is less or equal to 1. We construct a new quiver Q^* as follows: the vertices of Q^* are all the vertices of Q different from v. The arrows of Q^* are obtained from the arrows of Q in the following way: each arrow of Q whose origin and terminus are both different from v, is also an arrow of Q^*, and, for each path ab in Q:

$$\underset{u}{\bullet} \xrightarrow{a} \underset{v}{\bullet} \xrightarrow{b} \underset{w}{\bullet}$$

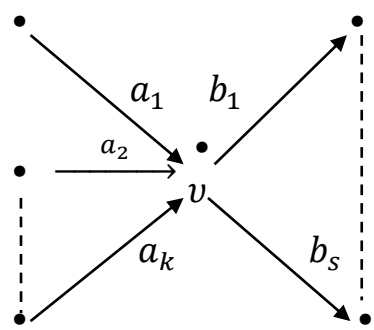
such that ab is not in I, we form a "new" arrow $C(a,b)$ in Q^*

$$\underset{u}{\bullet} \xrightarrow{C(a,b)} \underset{w}{\bullet}$$

The corresponding two-sided ideal I^* of KQ^* is defined in the following manner: if ρ_i is such that $v\rho_i = 0$, and if ρ_i is not a path of the form ab:

$$\bullet \xrightarrow{a} \underset{v}{\bullet} \xrightarrow{b} \bullet,$$

then we let ρ_i^* be the relation obtained from ρ_i, by replacing each occurrence of a path of the form $\bullet \xrightarrow{a} \underset{v}{\bullet} \xrightarrow{b} \bullet$ by the arrow $C(a,b)$. Assume now that ρ_i is a relation satisfying $v\rho_i \neq 0$. Let $a_1, \dots, a_k$ denote all the arrows into the vertex v, and let



$b_1, \dots, b_s$ denote all the arrows starting at v. We let ρ_{ij}^* be the relation obtained from $a_j\rho_i$ by replacing each path a_jb_l not belonging to I, that appears in $a_j\rho_i$, by the arrow $C(a_j, b_l)$. We now define I^* to be the two-sided ideal of KQ^* generated by all the elements of type ρ^*.

Proposition (1.2.22)[15]: There is a K-algebra isomorphism

$$KQ^*/I^* \simeq (1-e)KQ/I(1-e),$$

where e is the primitive idempotent corresponding to the vertex v.

Proof: Let $\phi: KQ^*/I^* \to (1-e)KQ/I(1-e)$ be $\phi(w) = \overline{w}$ for each vertex w of $Q^*, \phi(a) = \bar{a}$ for each arrow a of Q^* inherited from Q, and also $\phi\big(C(a,b)\big) = \overline{ab}$ for each arrow of Q^* of the form $C(a,b)$ (here $\bar{x}$ means the image in KQ/I of the element x of KQ). It is clear that ϕ induces a K-algebra homomorphism which is onto, and, it is not hard to show that $\mathrm{Ker}\phi = I^*$.

Remark (1.2.23)[15]; The generating set $\{\rho_i^*, \rho_{ik}^*\}$ of I^* need not be minimal. Indeed, it could happen that one of the ρ_{ik}^*'s can be written in terms of the remaining ρ^*'s. However, no element of the form ρ_i^* can be written in terms of the remaining ρ_i^*'s and ρ_{ik}^*'s. This means

that, by dropping if necessary some of the elements of the form ρ_{ik}^{*}, we get a minimal generating set of I^{*}, and, by construction the coefficients involved are again ± 1.

Finally, we should point out that I^{*} need not be admissible, for instance, one of the minimal relations may be a sum of $C(a,b)$ and linear combinations of other paths. In this case, it is easy to see that KQ^{*}/I^{*} is isomorphic to an algebra KQ^{**}/I^{**}, where Q^{**} is obtained from Q^{*} by dropping the arrow $C(a,b)$, and I^{**} is obtained from I^{*} in the obvious way, that is, by replacing every occurrence of $C(a,b)$ by the given linear combination.

The following examples illustrate how global dimension can be affected by the characteristic of the ground field.

Example (1.2.24)[15]: Let Q be the following quiver:

$$\begin{tikzcd}
1 \arrow[r, "a"] & 2 \arrow[rr, bend left, "b_1"] \arrow[rr, bend right, "b_n"'] & \vdots & 3 \arrow[r, bend left, "c_1"] \arrow[r, bend right, "c_2"'] & 4
\end{tikzcd}$$

where $n \geq 2$, and, let $I = \langle \sum_{i=1}^{n} ab_i, b_i c_1 + b_n c_2 \text{, for } i = 1, \dots, n_i \rangle$. Let $\Lambda = KQ/I$ and let S_1 be the simple Λ-module corresponding to the vertex 1.

We claim the following:

$$\mathrm{pd}_{\Lambda} S_1 = \begin{cases} 3, & \text{if char } k | n, \\ 2, & \text{if char } k \nmid n \end{cases}$$

and

$$\mathrm{gldim}\, \Lambda = \begin{cases} 3, & \text{if char } k | n, \\ 2, & \text{if char } k \nmid n. \end{cases}$$

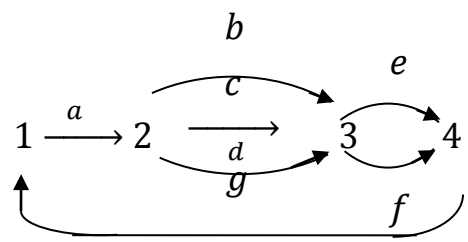
Proof: It is easy to see that $f^0 = v_1$ and $f^1 = a$. We can write $v_1 I \cap aR = v_1 I = \coprod_i f_i^{2*} R\ R$ and we can easily see that $f^2 = \sum_{i=1}^{n} ab_i$, since for each $i, ab_i c_1 + ab_n c_2$ is an element of $aI = f^1 I$. Next, in computing the intersection $f^2 R \cap f^1 I$, we must solve the equation

$$\sum_{i=1}^{n} (ab_i c_1 + ab_n c_2)\alpha_i = \left(\sum_{i=1}^{n} ab_i c_1\right)\beta + \left(\sum_{i=1}^{n} ab_i c_2\right)\gamma,$$

where $\alpha_1, \dots, \alpha_n, \beta, \gamma$ are in the field K. By identifying corresponding coefficients, we see that for each $i = 1, \dots, n, \alpha_i = \beta, \sum_{i=1}^{n} \alpha_i = \gamma$, and also that $\gamma = 0$. This implies that $n\beta = 0$, and

$f^2R \cap f^1I = (\sum_{i=1}^{n} ab_ic_1\beta)R$. If $\operatorname{char} K \nmid n$, then $\beta = 0$. Thus $f^2R \cap f^1I = (0)$, and hence $\operatorname{pd} S_1 = 2$. If $\operatorname{char} K | n$, then we can take $\beta = 1$; we also see that $\sum_{i=1}^{n} ab_ic_1$ is not in f^2I, since c_1 is not in I. In this case, we have $f^3 = \sum_{i=1}^{n} ab_ic_1$. It is easy to show that $f^3R \cap f^2I = (0)$. The remaining statements are obvious.

Example (1.2.25)[15]: A small adjustment of the previous example yields the following. Let Q be the quiver given by



and let $I = \langle ab + ac + ad, be + df, ce + df, de + df, eg, ga \rangle$. Let $\Lambda = KQ/I$ and let S_i be the simple Λ-module corresponding to the vertex i of Q. It is easy to see that Λ is finite dimensional. An easy calculation shows that Λ has finite global dimension if and only if the characteristic of $K \neq 3$. In characteristic diffierent from 3 we have that $\operatorname{pd}_\Lambda S_1 = 2, \operatorname{pd}_\Lambda S_2 = 5, \operatorname{pd}_\Lambda S_3 = 4$ and $\operatorname{pd}_\Lambda S_4 = 33$. In characteristic equal to 3 all simple Λ-modules have infinite projective dimension. In fact, we have the following initial segments of the minimal projective resolutions of the simple Λ-modules [36,26].

In characteristic 3:

$$0 \to S_1 \to P_4 \to P_3 \to P_2 \xrightarrow{a} P_1 \to S_1 \to 0,$$
$$0 \to S_1^2 \to P_4^3 \to P_3^3 \to P_2 \to S_2 \to 0,$$
$$0 \to S_1 \to P_4^2 \to P_3 \to S_4 \to 0,$$
$$0 \to S_1 \to P_4 \to S_4 \to 0.$$

In characteristic different from 3:

$$0 \to P_3 \xrightarrow{b+c+d} P_2 \xrightarrow{a} P_1 \to S_1 \to 0$$

and as above $\Omega_\Lambda^3(S_2) \simeq S_1^2, \Omega_\Lambda^2(S_3) \simeq S_1$ and $\Omega_\Lambda(S_4) \simeq S_1$.