Introduction

The theory of groups really began with GALOIS (1811- 1832) who demonstrated that polynomials are best understood by examining certain groups of permutations of their roots .since that time groups have arisen in almost every branch of mathematics.

One of the most important groups is the transformation groups and that is the subject of this thesis'

What is the transformation groups and what are their actions ?

Transformation groups can be defined as ((a collection of transformations which forms a group with composition as operation)).

To illustrate this : a map $f : x \rightarrow x$ from a set X to itself is called a transformation of the set X .It is not our interest to study the properties of a single transformation but rather to study families of transformations of a set X at the same time. In particular we are interested in studying sets of transformations of X which exhibit a group structure .

If G is the set of transformations of X which has a group structure then we call G a transformation group of X and X is called the G- set . We also say that G acts on X . If X has some additional mathematical structure then we usually require the transformations to preserve this structure.

As an example of transformation groups : consider the usual Euclidean plane. Of course ,it is a two dimensional set, but there is more to it , namely, a concept of distance between pairs of points ,one way to study the Euclidean plane is to concentrate on distance and find enough characteristics of this concept of distance to be able to describe of the Euclidean plane geometry in terms of distance.

Group theory abstracts this one level instead of looking at points in the plane and distance between points , group theory starts with transformations of the plane that preserve distance ,then studies the operations on transformations and relations between these transformations . The basic set for group theory is not the set of points in the plane , but the set of the transformations of the plane.

A transformation that preserves distance is usually called isometry . An isometry *T* of the Euclidean plane associates to each point *a* of the plane a point *Ta* of the plane (that is a function from the plane to itself) , and this association preserves distance in the sense that if *a* and *b* are two points of the plane , then the distance , *d (Ta ,Tb),* between the points *Ta* and Tb equals the distance *d (a,b)* between the points *a* and *b*.

Isometries are particularly important , because if a transformation preserves distance , then it will automatically preserve other geometric quantities like area and angel.

The set of all isometries of the plane has a structure of a group .There are lots of other collections of transformations that have this same structure , and any one of them would do as an example of a group. Of course not every collection of transformations will form a group . To be a group , the collection will have to have some minimal properties .

The first one is that it is closed under composition. If *T* is one transformation in the group ,and v is another transformation then you could first perform *T* , and then perform *V* . The result is that you will have performed the composition of *T* followed by *V* , often denoted by $V \circ T$ (an example will be given later in chapter3).

The second property is the identity element of the group which is the trivial transformation that does nothing and is denoted by *I* .

The third property is the inverse, for each isometry T of the plane there is an inverse T^{-1} which undoes what T does. For instance , if T translate the plane one unit to the right, then T^{-1} translates it back to the left one unit .The easiest way to characterize the inverse T^{-1} is to say that their composition is the identity transformation I .

The remaining properties of groups , associability , is obvious for transformation groups . And it is important to say that transformation groups usually are not commutative (an example will be given later) .

This is one example of transformation group and there are so many examples.

This thesis is mainly concerned with the study of some transformation groups and their actions by defining them and showing their properties and giving examples, restricting ourselves in three examples of transformation groups including the study of Lie group in its geometric structure and algebraic structure and a study about manifolds and symmetries . also there is a study of group representation , irreducible representation of a group and conjuagacy classes of a group (we will state a theorem relating the number of irreducible representation to the number of conjuagacy classes of the symmetric group of transformation) .

The main problem discussed by this thesis is how to use invariants in solving variants or classifying them .

This thesis is comprised of five chapters:

The first chapter is devoted to basic concepts which are of relevance to our thesis containing the definition of a group with satisfying examples, the second section is for permutation groups , the third section is for subgroups ,the forth section is for cossets and indices, the fifth section is for homomorphisms, the sixth section is for isomorphisms, and the seventh section is for cyclic groups.

The second chapter is devoted to the study of transformation groups ,and is consists of three sections , the first section is for the definition of transformation groups , the second section is for the properties of the transformation groups ,and the third section is for isometries .

The third chapter is devoted to the study of two more examples of transformation groups which is Lie groups in its geometric structure and algebraic structure and symmetric spaces

The forth chapter is devoted to the study of the representation theory of Lie groups, , irreducible representation of a group and conjuagacy classes of a group (we will state a theorem relating the number of irreducible representation to the number of conjuagacy classes of the symmetric group of transformation) .

The fifth chapter is devoted to the study of classification theory of and some examples of group actions as applicants in chemistry and physics.

And of course there is a summary and a list of references.

Chapter 1

Basic Concepts of Groups

In this chapter we define some notations and concepts and state some well-known results which will be used later on and we begin by the concept of groups.

Group theory is the abstraction of ideas that were common to a number of major areas which were being studied essentially simultaneously .The three main areas that were to give rise to group theory are :

1 - geometry at the beginning of the nineteenth century.

2- number theory at the end of eighteenth century.

3- the theory of algebraic equations at the end of eighteenth century leading to the study of permutations.

Geometry has been studied for a very long time so it is reasonable to ask what happened to geometry at the beginning of the nineteenth century that was to contribute to the rise of group concept. Geometry had begun to lose its metric character with projective and non-Euclidean geometries being studied .Also the movement to study geometry in n dimensions led to an abstraction in geometry itself. The difference between metric and incidence geometry comes from the work of Monge, his student Carnot and perhaps most importantly the work of Poncelet. Non-Euclidean geometry was studied by: Lamberet, Gauss, Lobachevski and Janos Bolyai among others.

Mobius in 1827 , although he was completely unaware of the group concept ,began to classify geometries using the fact that a particular geometry studies properties invariant under a particular group .Steiner in 1832 studies notions of synthetic geometry which were to eventually become part of the study of transformation groups.

In 1761 Euler studied modular arithmetic . In particular he examined the remainders of powers of a number modulo n. Although Euler‘s work is , of course ,not stated in group theoretic terms he does provide an example of the decomposition of an abelian group into cosets of a subgroup . He also powers a special case of the order of a subgroup being a divisor of the order of the group .

Gauss in 1801 was to take Euler works much further and gives a considerable amount of work on modular arithmetic which amounts to a fair amount of theory of abelian groups .He examined orders of elements and proved that there is a subgroup for every number dividing the order of a cyclic group . Gauss also examined other abelian groups .

Gauss examined the behavior of forms under transformations and substitutions .He partitions forms into classes and then defines a composition on the classes . Gauss proved that the order of composition of three forms is immaterial so, in modern language , the associative Law holds. In fact Gauss has a finite abelian group and later (in 1869) Schering, who edited Gauss's work, found a basis for this abelian group .

Permutations were first studied by Lagrange in his 1770 paper on the theory of algebraic equations . Lagrange's main object was to find out why cubic and quartic equations could be solved algebraically. In studying the cubic , for example, Lagrange assumes that the roots of a cubic equation are x', x "and x'". Then taking $1, w, w^2$ as the cube roots of unity , he examined the expression :

$$R = x' + wx'' + w^2x'''$$

and notes that it takes just two different values under the six permutations of the roots x' , x" , x'" .Although the beginnings of permutation group theory can be seen in this work, Lagrange never composes his permutations so in some sense never discusses groups at all.

The first person to claim that equations of degree 5 could be solved algebraically was Ruffini in 1799, he published a work whose purpose was to demonstrate the insolubility of the general quintic equation . Ruffini's work is based on that of Lagrange but Ruffini introduces groups of permutations . These he calls permutazione and explicitly uses the closure property(the associative law always holds for permutations) Ruffini divides his permutazione into types , namely simplitazione semplice which are cyclic groups in modern notation, and permuatzione composta which are non-cyclic groups . The permutazione composta Ruffini divides into three types which in today's notations are intransitive groups , transitive imprimitive groups and transitive primitive groups.

Ruffini's proof of the insolubility of the quintic has some gaps and, disappointed with the lack of reaction to his paper Ruffini published further proofs . In a paper of 1802 he shows that the group of permutations associated with an irreducible equation is transitive taking his understanding well beyond that of Lagrange.

Cauchy played a major role in developing the theory of permutations. His first paper on the subject was in 1815 but at this stage Cauchy is motivated by permutations of roots of equations .However , in 1844 Cauchy published a major work which steps up the theory of permutations as a subject in its own right. He introduces the notation of powers , positive and negative, of permutations (with the power 0 giving the identity permutation m.) ,defines the order of a permutation, introduces cycle notation and used the term systeme des substitutions conjuguees for a group . Cauchy calls two permutations similar if they have the same cycle structure and proves that this is the same as the permutations being conjugate.

Abel , in 1832 , gave the first accepted proof of the insolubility of the quintic , and he used the exiting ideas on permutations of roots but little new in the development of the group theory.

Galois in 1831 was the first really to understand that the algebraic solution of an equation was related to the structure of a group of permutations related to the equation . By 1832 Galois had discovered that special subgroups (now called normal subgroups) are fundamental. He calls the decomposition of a group into cosets of a subgroup a proper decomposition if the right and left coset decompositions coincide. Galois then shows that the non-abelian simple group of smallest order has order 60.

Galois's work was not known until Liouville published Galois's papers in 1846 .Liouville saw clearly the connection between Cauchy 's theory of permutations and Galois's work. However Liouville failed to grasp that the importance of Galois's work lay in the group concept.

Betti began in 1851 publishing a work relating permutation theory and the theory of equations . In fact Betti was the first to prove that Galois's group associated with an equation was in fact a group of permutations in the modern sense . Serret published an important work discussing Galois's work, still without seeing the significance of the group concept.

Jordan , however , in papers of 1865,1869 and 1870 shows that he realizes the significance of groups of permutations .He defines isomorphism of permutation groups and proves the Jordan-Holder theorem for permutation groups .Holder was to prove it in the context of abstract groups in 1889.

Klein proposed the Erlangen Program in 1872 which was the group theoretic classification of geometry . Groups were certainly becoming center stage in mathematics.

Perhaps the most remarkable development had come even before Betti's work . It was due to the English mathematician Cayley. As early as 1849 Cayley published a paper linking his ideas on permutations with Cauchy's . In 1854 Cayley wrote two papers which are remarkable for the insight they have of abstract groups .At that time the only known groups were groups of permutations and even this was a radically new area, yet Cayley defines an abstract group and gives a table to display the group multiplication . He gives the Cayley's table of some special permutation groups but, much more significantly for the introduction of the abstract group concept, he realized that matrices and quaternions were groups.

Cayley's papers of 1854 were so far ahead of their time that they had little impact. However when Cayley returned to the topic in 1878 with four papers on groups ,one of them called the theory of groups ,the time was right for the abstract group concept to move towards the center of mathematical investigation .Cayley proved ,among many other results , that every finite group can be represented as a group of permutations. Cayley's work prompted Holder, in 1893, to investigate groups of order p^3, pq^2, pq^r and p^4.

Frobinius and Netto carried the theory of groups forward . As far as the abstract concept is concerned , the next major contributor was Von Dyck. Von Dyck who had obtained his doctorate under Klein's supervision then became Klein's assistant. Von Dyck , with fundamental papers in 1882 and 1883 , constructed free groups and the definition of abstract groups in terms of generators and relations. so we can say that group theory is the most major theory of the 20th century.

1.1.1 Definition

A group is a set, say G , and a binary operation which we call $*$, satisfying the following properties:

1 – closure under $*$: if a , b are in G , then $a * b$ is also in G .This means that if we have two elements from G , and a $*$ b is not in G , G cannot be a group.

2 – associativity of $*$: If a , b , c are elements of G ,

$(a * b) * c = a * (b * c)$, as we have for addition and multiplication but not subtraction of natural numbers .

3 – existence of an identity element : There is an element in G , which we write e such that for any a in G

$$e * a = a * e = a \qquad (1.1)$$

We call e the identity element , which sometimes notate I , I, or E .

4 – existence of inverses : For any *a* in G , there exists some element *q* in G , such that :

$$q * a = a * q = e \qquad (1.2)$$

We usually write *q* as a^{-1}.

Any set with a binary operator that satisfies these four properties is a group. Technically , a group is a set and an operation , which can also be written as an ordered pair (G , ∗) , although it is common practice to speak of the group as the set G. It is important to note , however ,that one set can form two different groups under different operations.

Note that the identity element is unique.

We shall call the group *additive* if the operation is a kind of addition , in this case , it is standard to denote the operation by + , the identity element by 0 and the inverse of *a* in G by - *a* .

We shall call the group *multiplicative* if the operation is a kind of multiplication . In this case we often use ∗ or a dot , to denote the operation and write $a * b$ as *ab* for brevity . We often denote the identity by *e* or 1,and the inverse of *a* in G as a^{-1} . .

Note that in our group axioms above we don't assume *commutatively* (which means that if we have any *x and y* , $x * y = y * x$) a property were used to having when doing algebra on the real numbers . This property holds in some groups , but not in others .If it holds for a particular group , then we call that group *abelian* .

It is a convention that one only speaks of an additive group when the group is abelian .

The Order of a Group

We call the number of the elements of the group the order of the group and is denoted by | G | (also by *O (G)*).

Finite and Infinite Groups

We say a group is finite if it had a finite number of elements and if it doesn't have a finite number of elements we call it an infinite group.

The order of an element *a* within G is the first natural number n such that a^n is the identity . If no such n exists , then it is considered of infinite number , and all powers of *a* are different.

Examples:

Example 1.1.2

$(z_2,+)$

Z_2 = { o , 1 } is the set of remainders when dividing integers by 2 . There are only two such possible remainders , 0 and 1 . So in Z_2 , we have two elements {0 , 1 } . This set is called the set of integers modulo 2 .Note that an integer is equal to its remainder modulo 2 . Let us denote the operation of addition modulo 2 by "+ " , defined as the usual addition of integers , So is $(Z_2,+)$ a group ?

Let us go through the requirements :

1 – Closure : can be verified quickly by going through all possible cases , 0 + 0 =0 , 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 0 thus closure holds.

2 – Associativity : $a +(b + c) = (a + b)+c$ for instance 0+ (0 + 1) = 0 + 1 = 1 and (0 + 0) + 1 = 0 + 1 = 1 and we can go through all possible cases , So the associativity holds.

3 – inverses : 1 +1 =2 =0 modulo 2 , so 1 is the inverse of 1 , 0 + 0 = 0 ,so 0 is the inverse of 0. Since 0 and 1 are the only elements ,every element has an inverse .Thus inverses exist.

So we have shown that every property of group is satisfied ,so $(Z_2, +)$ is a group.

Example 1.1.3 :

($(Z_5)^*$, ×)

first $(Z_5)^*$ means Z_5 without zero , we have { 1,2,3,4}. Take x to be regular multiplication modulo 5.

Now we go through the requirements :

1 – Closure can be verified quickly by inspection : e.g 3 x 4 = 12 = 2 modulo 5 . the remaining cases can be done as well. Thus closure holds.

2 – Associativity : e.g 1 x(2 x 3) = 1x1 = 1, and (1 x 2) x 3 = 2 x 3 =1 so associativity holds.

3 – Identity : 1 X 1 = 1 , 1 X 2= 2 , 1 X 3 = 3 , 1 X 4 = 4 so 1 is the identity element for multiplication . Thus an identity exists.

4 – Inverses : 1 x 1 = 1 so 1 is the inverse of 1 , 2 x 3 = 6 = 1 so 2 and 3 are inverses of each other , and 4 x 4 = 16 = 1 , so 4 is the inverse of 4 , Thus inverses exists.

Therefore($(Z_5)^*$, ×) is a group.

Example 1.1.4

(Z , +)

The integers forms a group with the operation of addition +. Again we are going to show that the four axioms of a group is satisfied;

1 – Closure : We require that if a and b are integers , then $a + b$ is an integer . But this is true by definition, closure holds.

2 – Associativity : We require that if a , b and c are integers , then $(a + b) + c = a + (b + c)$ But again we know this is true from normal addition. Thus associativity holds.

3 – Identity : 0 is the identity since $0 + a = a + 0 = a$ for any integer a. Thus an identity exists .

4 – Inverses : a has an inverse $-a$, for $-a + a = a + -a = 0$ .for any integer a . Thus inverses exists .

So (Z , +) is a group.

Example 1.1.5

(Q , x)

Q is the set of all rational numbers , that is numbers that can be formed as the ratio of two integers , a is the set of all rational numbers , that is numbers that can be formed as the ratio of two integers , *a / b.*

(Q , X) is not a group , the closure , associative , and identity holds , but since 0 is in Q , the inverse of 0 would have to be 1/ 0 which has no meaning , 0 does not have an inverse, so (Q , x) is not a group.

1.2.1 Permutation Groups :

Permutations were first studied by Lagrange in his 1770 paper on the theory of algebraic equation, although the beginnings of permutation group theory can be seen in his work , he never composes his permutations .Ruffini introduces groups of permutations and Cauchy played a major role in developing the theory of permutations.

A permutation is a rearrangement of objects of the elements of a given set . Given a set with n elements , there are *n!* possible distinct ways of arranging all those elements .For instance, the elements of the set { 1 , 2 , 3 } can be arranged in the following six { 3*!* } ways:

(1,2,3) , (2,1,3) , (2,3,1) , (3,1,2) ,(1,3,2) , (3,2,1)

A permutation can be written also using a two level notation, which describes the rearrangement of the elements by actually showing which element moves where , for example

$$\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$$

denotes that the first element goes to the third position , the second to the first and the third to the second , this can be represented as (312) . The identity permutation ,which changes nothing is generally written simply as (1) ,but also may be written as (1)(2)(3) (in the above example)

Permutation are essentially bijective functions from a set onto itself, and they can be composed just like other functions are composed.

If permutation p1 moves 1 to 2 and p2 moves 2 to 3 ,then the composite of these moves 1 to 3. in the two level notation this can be written as:

say p1 is 1 2 3

2 3 1

and p2 is :

1 2 3

1 3 2

then their composition is :

1 2 3

3 2 1

The set of all permutations on a given set forms a group under composition , because the composition of two permutation yields another permutation, and the composite is associative. The identity permutation serves as the identity, and in order to invert a permutation, all that needs to be done is to swap the top and bottom rows ,and , for standard notation , to rearrange the columns so that the top row is correctly ordered, for instance, to invert

3 2 1

1 3 2

Swapping the rows gives

1 3 2

3 2 1

Ordering the columns gives

1 2 3

3 1 2

The set of all permutations on n objects i.e. { 1 ……n} is an important group called the symmetric group and is written as s_n, and has order of n*!* (n factorial) it is not an abelian group generally.

1.3.1 Subgroups

Inside groups there could be other smaller groups.

1.3.2 Definition

A subgroup is a subset of a group G which is also a group under the operation in G . More precisely, a subgroup of a group (G , .) is a group (H ,*) such that H is a subset of G and * is the restriction of . to H X H . this means that multiplication in H is the same as in G .This is usually written as H ⊆ G

1.3.3 Definition

A Proper subgroup of a group G is a subgroup H which is a proper subset of G i.e. H ≠ G. The trivial subgroup of any group is the subgroup { e } consisting of just the identity element.

1.3.4 Proposition:

Let G be a group with identity element e, and let H be a subset of G . Then H is a subgroup of G if and only if the following conditions hold:

I) $a b^{-1} \in H$ for all a , b ∈ H.

II) $e \in H$

iii) $a^{-1} \in H$ for all a ∈ H.

Theorem 1.1 (Lagrange)

If H is a subgroup of the finite group G , then the order of H is devisor of the order of G.

To prove that a subgroup is a group , we need to only check for

1- closure

2- identity existence

3- inverse existence

We don’t need to check for associativity because this is given to us by the larger groups. If the group is finite ,then we don’t need to check for the third one because if the order of an element in the subgroup is n , then a n-1 is its inverse .

1.3.5 Example

The even numbers under addition form a subgroup of the integers under addition. But the odd numbers do not because 1 + 1 = 2, which is not odd, so closure is violated, and 0 is not odd , so they have no identity.

1.4.1 Cosets and Index

1.4.2 Definition:

Let H be a subgroup of the group G , and let $a \in G$. The set

$$aH = \{ x \in G \mid x = ah \text{ for some } h \text{ in } H \}$$

is called the left coset of H in G determined by a .

Similarly , the right coset of H in G determined by a is the set

$$Ha = \{ x \in G \mid x = ha \text{ for some } h \in H\}$$

1.4.3 . Definition

The number of left cosets of H in G is called the index of H in G , and is denoted by [G.H]

1.4.4 Proposition:

Let H be a subgroup of the group G , and let a , b be elements of G . Then the following conditions are equivalent

(1) $bH = aH$.

(2) $bH \subseteq aH$.

(3) b is in aH.

(4) $a^{-1} b$ is in H.

A result similar to the above proposition holds for right cosets. Let H be a subgroup of the group G ,and let a , b ϵ G . Then the following conditions are equivalent

(1) Ha = Hb.

(2) Ha $\subseteq$ Hb.

(3) a is in Hb.

(4) ab^{-1} is in H.

(5) ba^{-1} is in H.

(6) b is in Ha.

(7) Hb $\subseteq$ Ha.

The index of H in G could be defined as the number of right cosets of H in G, since there is a one-to-one correspondence between left cosets and right cosets.

1.4.5. Normal Subgroups

Definition : A subgroup H of the group G is called a normal subgroup if .

$$ghg^{-1} \in H \qquad (1.3)$$

1.5.1 Homomorphism's

A homomorphism is a structure – preserving map between two algebraic structures (such as groups , rings , or vector spaces).The word homomorphism comes from the Greek language (homos) meaning "same" and (morphe) meaning " shape " .

1.5.2 Definition : Homomorphism Between Groups :

Given two groups (G , $*$) and (H , .) ,a group homomorphism from (G , $*$) to (H , .) is a function

$$h : G \longrightarrow H \qquad (1.4)$$

such that ,for all u and v in G it holds that

$$h(u * v) = h(u) \cdot h(v) \qquad (1.5)$$

where the group operation on the left hand side of the equation is that of G and on the right hand side that of H .

As a result the group homomorphism maps the identity element in G to the identity element in H: f (eG) =eH.

1.5.3. Product of Homomorphism's is a Homomorphism

If G , H , and K are groups , T∈ Hom (G,H), and u ∈ Hom (H,K), then

TU ∈ Hom (G, K)

1.5.4 If T∈ Hom (G,H) , then GT∈ H .

1.5.5 Kernel of a Homomorphism

We define the kernel of h to be the set of elements in G which are mapped to the identity in H .

ker (h) = { u in G : h (u) = eH. (1.6)

note that a kernel of a homomorphism is a normal subgroup

1.5.6 Image of a Homomorphism

we define the image of h to be :

im (h) = { h (u) : u in G } (1.7)

The image of a homomorphism is a subgroup of H.

The homomorphism h is injective and is called a monomorphism if and only if Ker(h) = {eG }.

Isomorphism:

The word isomorphism comes from the Greek language (isos) meaning (equal) and (morphe) meaning (shape) i.e. a structure-preserving mappings.

1.6.1 Definition :

Given two groups (G , ∗) and (H , ι) , a group isomorphism from (G , ∗) to (H , ι) is a bijective group homomorphism from G to H i.e.

$$f : G \longrightarrow H \qquad (1.8)$$

such that for all u and v in G it holds that

$$f (u * v) = f (u) \iota f (v) \quad (1.9)$$

and we say that the two groups G and H are isomorphic.

This is written as : $(G , *) \quad \eta \quad (H , \iota)$

1.6.2 Properties

1 – the kernel of an isomorphism from $(G , *)$ to (H , ι) , is always { eG} , where eG is the identity of the group (H , ι)

2 – if (H , ι) is isomorphic to (H , ι) , and if G is abelian then so is H.

3 - If $(G ,*)$ is a group that is isomorphic to (H , ι) [where f is the isomorphism], then if a belongs to G and has order n , then so does f (a).

4 – If (H , ι) is a locally finite group that is isomorphic to (H , ι) , then (H , ι) is also locally finite.

An isomorphism from a group $(G , *)$ to itself is called an automorphism of this group .Thus it is a bijection $f : G \longrightarrow G$

such that:

$$f (u) * f(v) = f (u * v).$$

An automorphism always maps the identity to itself.

1.7.1 Cyclic Groups :

1.7.2 Definition :

A cyclic group is a group that can be generated by a single element ,in the sense that the group has an element *a* (called a generator of the group) such that ,when written multiplicatively, every element of the group is a power of *a*.

A group G is called cyclic if there exists an element in *a* in G such that

$$G = (a) = \{ a^n \mid n \text{ is an integer } \}$$

Since any group generated by an element in the group is a subgroup of that group, showing that the only subgroup of a group G is G itself suffices to show that G is cyclic.

For example : if G = { a_0 , a_1 , a_2 ,a_3 , a_4 ,a_5 } is a group, then $a_5 = a_0$ and G is cyclic. In fact ,G is essentially the same as the set { 0 , 1 , 2 , 3 , 4 , 5 } with addition modulo 6. For example , 1 + 2 = 3 (mod 6) corresponds to $a_1 . a_2 = a_3$, and 2 + 5 = 1 (mod 6) corresponds to $a_2 . a_5 = a_7 = a_1$, and so on .

Theorem 1.2 : Every cyclic group is abelian

$G = (a) = \{ a^n \mid n \in Z \}$.

if g_1 and g_2 are any two elements of G ,there exists integers r and s such that $g_1 = a_r$ and $g_2 = a_s$. Then

$$g_1 g_2 . = a_r a_s = a_{r+s} = a_{s+r} = a_s a_r = g_2 g_1,$$

So G is abelian.

1.7.3 Properties :

1 – If G is a cyclic group of order n ,then every subgroup of G is cyclic

2 – The order of any subgroup of G is a divisor of n .

3 – For each positive divisor k of n ,the group G has exactly one subgroup of order k .

4- If G is cyclic and T is a homomorphism of G ,then GT is cyclic.

1.7.4 Examples of Cyclic Groups :

1 – The integers under addition is a cyclic group , the number 1 is a generator ,also -1 is a generator.

2 – The set of complex numbers { 1 , - 1 , I , - I } under multiplication of complex numbers ,and the generator is I, also – I is a generator.

CHAPTER 2

Transformation Groups

This chapter consists of the definition of the transformation groups and the properties that any collection of transformations should have to be a group. The third section is for isometries as an example of transformation group.

2.1.1 Definition of Transformation :

A transformation is a mapping from a set to itself i.e. given a non-empty set X , Then

$$F : X \longrightarrow X$$

In other words a transformation is a general term of four specific ways to manipulate the shape of a point , a line , or surface . The original shape of the object is called the pre-image and the final shape and position of the object is the image under the transformation .

Types of transformations in math:

1- Translation .

2 - Reflection.

3- Rotation.

4- Dilation.

A composition of transformations means that two or more transformations will be performed on one object . For instance , We could perform a reflection and then a translation on the same point .

A collection of transformations can form a group called the transformation group .It is not true that every collection of transformations will form a group ,these collection of transformations should have some properties which is stated later on .

2.1.2 Definition

A collection of transformations which forms a group with composition as the operation. A dynamical system or, more generally , a topological group G together with a topological space X where each g in G gives rise to a homeomorphism of X in a continuous manner with respect to the algebraic structure of G

2 .1.3 Definition

An effective transformation group is a transformation group in which the identity element is the only element to leave all points fixed

2.2.1 Properties of a Transformation Group :

First composition :

The first one is that is closed under composition, if *T* is one transformation in the group , and *U* is another , then we could first perform *T* , and then perform *U* . The result is that we will have performed the composition of *T* followed by *U*, and this is often denoted $U \circ T$.

Suppose that *T* is the transformation that translates a point *a* one unit to the right .In terms of a coordinate system , *T* will translate the point $a = (a_1,a_2)$ to the point $Ta = (a_1 + 1 , a_2)$

Suppose also that *U* is another transformation that reflects a point across the diagonal line y = x . Then $U (a_1,a_2) = (a_2 , a_1)$. The composition, $U \circ T$ will first move *a* one unit right , then reflect it across the diagonal line y = x , so that $(U \circ T)(a_1, a_2) = U (a_1 + 1, a_2) = (a_2, a_1 + 1)$.

Second : the identity element.

The transformation that does nothing is called the identity transformation and we denote it by *I* . Thus for any point *a* in the plane

$Ia = a$

So we can say that when the identity transformation is composed with any other transformation, the other transformation is all that results, that is to each transformation *T* ,

$$I \circ T = T \ , \quad \text{and} \quad T \circ I = T. \qquad (2.1)$$

Third : inverses

For each transformation T there is an inverse transformation T^{-1} which undoes what T does. For instance, if T translates the plane one unit to the right , then T^{-1} translates it back to the left one unit. If T is a 45 –rotation clockwise about a fixed point , thenT^{-1} rotates the plane 45 counterclockwise about the same point .

The composition of a transformation and its inverse is the trivial transformation that does nothing .The inverse T^{-1} of a transformation T is characterized by the two equations:

$$T^{-1} \circ T = I \quad \text{and} \quad T \circ T^{-1} = \qquad (2.2)$$

Forth : associativity:

Associativity is obvious for transformation groups . If you have three transformations T , U and V, then the triple composition V O UO T can be found in either of two ways in terms of ordinary composition, either

compose V with the result of composing U with T or ,

(2) compose V ° U (which is the result of composing V with U) with T .

In other words ,composition satisfies the associative id

$$V \circ (U \circ T) = (V \circ U) \circ T. \qquad (2.3)$$

Composition is always an associative property.

Fifth : Commutivity:

Usually transformation groups aren’t commutative, that is don’t expect that

$$T \circ U = U \circ T \qquad (2.4)$$

For instance , with the example transformations T and U above, where T is the translation to right by one unit , Ta = $(a_1 + 1 , a_2)$, and U is the reflection across the diagonal line y = x so that

$$U (a_1, a_2) = (a_2 , a_1), \qquad (2.5)$$

we found that the composition U o T was given by the formula

$$(U \circ T)(a_1 , a_2)= (a_2, a_1 + 1), \qquad (2.6)$$

but we can show that the reverse composition T o U is given by the formula

$$(T \circ U)(a_1, a_2) = (a_2+1, a_1). \qquad (2.7)$$

These aren't equal, so $U \circ T$ does not equal $T \circ U$.

So transformation groups aren't usually commutative .

2.3.1 Isometries (as an Example of Transformation Group)

2.3.2 Definition

An isometry is a transformation that preserves distance. An isometry *T* of the Euclidean plane associates to each point *a* of the plane a point *Ta* of the plane (that is , a function from the plane to itself), and this association preserves distance in the sense that if *a* and *b* are two points of the plane ,then the distance d(*Ta, Tb*)between the points *Ta* and *Tb* equals the distance *d(a ,b)* between the points *a* and *b*.

There are several kinds of isometries of the plane , there are

1- translations

2- rotations

3- reflections

4- glide reflections

2.3.3 Definition of Translations:

Translation simply means moving ,in other words , a translation is a transformation that slides each point of a figure the same distance in the same direction. As an example of a translation , if we have a triangle ABC ,and each point (A,B,C) of the triangle is moved to (A^l ,B^l, C^l) ,following the rule 2 to the right and 1 up i.e. it is moved 2 inches to the right and 1 inch up. Notice that we can translate a pre-image to any combination of two of the four directions.

This translation is a transformation on the coordinate plane.

The transformation in this case would be

$$T (a , b) = (a + 2 , b + 1). \qquad (2.8)$$

Translation of Axes :

An equation corresponding to a set of points in the system of coordinate axes may be simplified by taking the set of points in some other suitable coordinate system, but all geometrical properties remain the same.

One such transformation type is , in which the new axes are transformed parallel to the original axes and origin is shifted to a new point.

A transformation of this type is called translation of axes .

2.3.4 Rotation :

Definition :

A rotation is a transformation that turns a figure about a fixed point. The fixed point about which a figure is rotated is called the center of rotation. A rotation is different from a translation , which has no fixed points .The rotation leave the distance between any two points unchanged after the transformation. It is important to know the frame of reference when considering rotations, as all rotations are described relative to a particular frame of reference. In general for any orthogonal transformation on a body in a coordinate system there is an inverse transformation which if applied to the frame of reference results in the body being at the same coordinates. For example in two dimensions rotating a body clockwise about a point keeping the axes fixed is equivalent to rotating the axes counterclockwise about the same point .

In two dimensions only a single angel is needed to specify a rotation – the angel of rotation. To calculate the rotation two methods can be used, either matrix algebra or complex numbers. In each the rotation is acting to rotate an object counterclockwise through an angel θ about the origin.

1-Matrix algebra :

To carry out a rotation using matrices the point (x , y) to be rotated is written as a vector ,then multiplied by a matrix calculated from the angle 0

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

where (x', y') are the co-ordinates of the point after rotation, and the formulae for x' and y' can be seen to be

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta.$$

The vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ have the same magnitude and are separated by an angle θ as expected.

2-Complex Numbers:

Points can also be rotated using complex numbers, as the set of all such numbers, the complex plane, is geometrically a two dimensional plane. the point (x, y) on the plane is represented by the complex number $z = x + iy$ This can be rotated through an angle θ by multiplying it by $e^{i\theta}$, then expanding the product using Euler's formula as follows:

$$\begin{aligned} e^{i\theta} z &= (\cos\theta + i\sin\theta)(x + iy) \\ &= (x\cos\theta + iy\cos\theta + ix\sin\theta - y\sin\theta) \\ &= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta) \\ &= x' + iy', \end{aligned}$$

which gives the same result as before,

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta.$$

Like complex numbers rotations in two dimensions are commutative, unlike in higher dimensions. They have only one degree of freedom, as such rotations are entirely determined by the angle of rotation.

3-Three Dimensions:

Rotations in ordinary three-dimensional space differ than those in two dimensions in a number of important ways. Rotations in three dimensions are generally not commutative, so the order in which rotations are applied is important. They have three degrees of freedom, the same as the number of dimensions.

2.3.5 Reflections:

A reflection (also called a flip) is a transformation that reflects every point on a shape to a point that is the same distance on the opposite of a fixed line – the mirror line .Reflections in two perpendicular axes produce a rotation of 180 degree (a half turn).

The two common reflections are : a horizontal reflection and a vertical reflection. The horizontal reflection flips across and vertical reflection flips up / down..

For example,

Reflection over the x – axis is $p(a, b) = (a, -b)$.

Reflection over the y – axis is $p(a, b) = (-a, b)$.

Reflection over a line $y = x$ is $p(a, b) = (b, a)$.

Reflections are everywhere …in mirrors, glass, and even in a lake. and it does not matter what direction the mirror – line goes, the reflected image is always the same size , it just faces the other way. so reflection is just like looking in a mirror.

In plane, to find the reflection of a point one drops a perpendicular from the point onto the line (plane) used for reflection , and continues it to the same distance on the other side . To find the reflection of a figure , one reflects each point of the figure.

For example the mirror image of the Latin letter p for a reflection with respect to a vertical axis would look q. Its image by reflection in a horizontal axis would look like b.

2.3.6 Glide Reflection :

It is the combination of a reflection in a line and a translation along that line .Reversing the order of combining gives the same result.

The combination of a reflection in a line and a translation in a perpendicular direction is a reflection in a parallel line. Thus the effect of a reflection combined with any translation is a glide reflection, with as special case just a reflection.

For example , there is an isometry consisting of the reflection on the x =axis, followed by translation of one unit parallel to it .In coordinates, it takes (x , y) to (x+1 , - y) . It fixes a system of parallel lines.

The isometry group generated by just a glide reflection is an infinite cyclic group.

Combining two equal glide reflections gives a pure translation with a translation vector that is twice that of the glide reflection , so the even powers of the glide reflection form a translation group.

In 3D the glide reflection is called a glide plane . It is a reflection in a plane combined with a translation parallel to the plane.

Observations :

1- Glide reflection changes the orientation: if a polygon is traversed clockwise, its image is traversed counterclockwise , and vice versa.
2- Reflection is an isometry (because when two or more transformations are combined to form a new transformation , the result is called a composition of transformations. Since translations and reflections are both isometries , a glide reflection is also an isometry: a glide reflection preserves distances.
3- Reflection preserves angles.
4- Reflection maps parallel lines onto parallel lines.
5- Unless the translation part of a glide reflection is trivial (denoted by a 0 vector) , the glide reflection has neither fixed points , nor fixed lines, save

the axis of reflection itself. If the translational part is trivial , the glide reflection becomes a common reflection and inherits all its properties.

The importance of the glide reflection lies in the fact that it is one of the four isometries of the plane.

A glide reflection is an opposite isometry, because translation is a direct isometry and a reflection is an opposite isometry ,so their composition is an opposite isometry.

The four isometries mentioned above ,(translations , rotations, reflections and glide reflection)are types of rigid motion of the plane, so we can define rigid motion as follows:

A rigid motion is any way of moving all the points in the plane such that :

1 – the relative distance between points stays the same and

2 – the relative position of the points stays the same.

Another common sort of transformation which does not preserve lengths are dilatations.

Dilatations happen when you make an image larger or smaller. Its shape and orientation remain the same , but its size and position change.

A dilation is a transformation that produces an image that is the same shape as the original , but is a different size.

A dilation stretches or shrinks the original figure .

The description of a dilation includes the scale factor (or ratio , which measures how much larger or smaller the image is) and the center of the dilation.

A dilation with a scale factor 2 means that the image is twice as large as the pre-image.

The center of the dilation is a fixed point on the plane about which all points are expanded or contracted . It is the only the invariant point under a dilation.

Like other transformation , prime notation is used to distinguish the image from the pre-image. The image always has a prime after the letter.

If the scale factor , denoted by K, is greater than 1 ,the image is an enlargement (a stretch).

If the scale factor is between 0 and 1 , the image is a reduction (a shrink)

Properties Preserved Under a Dilation :

1- Angle measures (remain the same).
2- Parallelism (parallel lines remain parallel).
3- Colinearity (points stay on the same lines).
4- Midpoint (midpoints remain the same in each figure).
5- Orientation (lettering order remains the same).

And it is very important to say that distance is not preserved.

Most dilation in coordinate geometry , use the origin (0 , 0) as the center of the dilation..

2.4.1 Symmetric Group :

The symmetric group on a set X is the group whose underlying set is the collection of all bijections from X to X (all one –to-one and onto functions from the set to itself) and whose group operation is that of function composition.

To check that the symmetric group on a set x is indeed a group , it is necessary to verify the group axioms of associativity , identity , and inverses.

1 – associativity :

The operation of function composition is always associative.

2- identity :

The trivial bijection that assigns each element of X to itself serves as an identity for the group.

3 – inverses :

Every bijection has an inverse function that undoes its action, and thus each element of a symmetric group does have an inverse.

The symmetric group of degree n is the symmetric group on the set $X = \{ 1, 2, ..., n \}$.The elements of the symmetric group on a set x are the permutations of x .

2.4.2 Definition :

A permutation of a set X is a function from X into X which is both one-to-one and onto.

The symmetric group on a set X is denoted by S_X ,or Sym(X) among others (no need to mention them here).

If X is the set $\{ 1, 2, ..., n \}$, then the symmetric group on X is also denoted S_n or Sym(n).

2.4.3 Definition :

The subgroup of S_n consisting of the even permutations of n letters is the alternating group A_n on n letters.

Symmetric groups on infinite sets behave quite differently than symmetric groups on finite sets.

The symmetric group on a set of n has order $n!$. Symmetric groups are considered as examples of non-abelian groups , but It is abelian if and only if $n \leq 2$. For $n = 0$ and $n = 1$ (the empty set and the singleton set) the symmetric group is trivial (note that this agrees with $0! = 1! = 1$), and in these cases the alternating group equals the symmetric group, rather than being an index two subgroup. The group S_n is solvable if and only if $n \leq 4$.

2.4.4 Properties :

Symmetric groups are Coxeter groups and reflection groups.(a Coxeter group, named after H.S.M. Coxeter, is an abstract group that admits a formal description in terms of mirror symmetries. Indeed, the finite Coxeter groups are precisely the finite Euclidean reflection groups; the symmetry groups of regular polyhedral are an example). They can be realized as a group of reflections with respect to hyper planes $x_i = x_j, 1 \le i < j \le n$. Braid groups B_n admit symmetric groups S_n as quotient groups.

2.4.5 Cayley's Theorem:

Every group *G* is isomorphic to a subgroup of the symmetric group on the elements of *G*, as a group acts on itself faithfully by (left or right) multiplication.

Proof :

Where *g* is any element of *G*, consider the function

$$f_g : G \to G$$

, defined by $f_g(x) = g{*}x$. By the existence of inverses, this function has a two-sided inverse, $f_{g^{-1}}$. So multiplication by *g* acts as a bijective function. Thus, f_g is a permutation of *G*, and so is a member of Sym(*G*).

The group operation in a symmetric group is function composition, denoted by the symbol o or simply by juxtaposition of the permutations. The composition $f \circ g$ of permutations *f* and *g*, pronounced "f after g", maps any element *x* of *X* to *f*(*g*(*x*)). Concretely, let

$$f = (1\ 3)(4\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

and

$$g = (1\ 2\ 5)(3\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}.$$

Applying *f* after *g* maps 1 first to 2 and then 2 to itself; 2 to 5 and then to 4; 3 to 4 and then to 5, and so on. So composing *f* and *g* gives

$$fg = f \circ g = (1\ 2\ 4)(3\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}.$$

A cycle of length L =*k*·*m*, taken to the *k*-th power, will decompose into *k* cycles of length *m*:

For example (*k*=2, *m*=3),

$(1\ 2\ 3\ 4\ 5\ 6)^2 = (1\ 3\ 5)(2\ 4\ 6).$

2.4.6 Transposition

A transposition is a permutation which exchanges two elements and keeps all others fixed; for example (1 3) is a transposition. Every permutation can be written as a product of transpositions; for instance, the permutation *g* from above can be written as *g* = (1 5)(1 2)(3 4). Since *g* can be written as a product of an odd number of transpositions, it is then called an odd permutation, whereas *f* is an even permutation.

The representation of a permutation as a product of transpositions is not unique; however, the number of transpositions needed to represent a given permutation is either always even or always odd. There are several short proofs of the invariance of this parity of a permutation.

The product of two even permutations is even, the product of two odd permutations is even, and all other products are odd. Thus we can define the sign of a permutation:

$$\operatorname{sgn}(f) = \begin{cases} +1, & \text{if } f \text{ is even} \\ -1, & \text{if } f \text{ is odd.} \end{cases}$$

With this definition,

$$\operatorname{sgn} : S_n \to \{+1, -1\}$$

is a group homomorphism ({+1, –1} is a group under multiplication, where +1 is *e*, the neutral element). The kernel of this homomorphism, i.e. the set of all even permutations, is called the alternating group A_n. It is a normal subgroup of S_n, and for $n \geq 2$ it has $n!\,/\,2$ elements. The group S_n is the semi direct product of A_n and any subgroup generated by a single transposition.

Furthermore, every permutation can be written as a product of *adjacent transpositions,* that is, transpositions of the form $(a\ a+1)$. For instance, the permutation *g* from above can also be written as

g = (4 5)(3 4)(4 5)(1 2)(2 3)(3 4)(4 5). The representation of a permutation as a product of adjacent transpositions is also not unique.

2.4.7 Cycles

A cycle of *length* k is a permutation f for which there exists an element x in $\{1,...,n\}$ such that x, $f(x)$, $f^2(x)$, ..., $f^k(x) = x$ are the only elements moved by f; it is required that $k \geq 2$ since with $k = 1$ the element x itself would not be moved either. The permutation h defined by

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix}$$

is a cycle of length three, since $h(1) = 4$, $h(4) = 3$ and $h(3) = 1$, leaving 2 and 5 untouched. We denote such a cycle by (1 4 3). The order of a cycle is equal to its length. Cycles of length two are transpositions. Two cycles are *disjoint* if they move disjoint subsets of elements. Disjoint cycles commute, e.g. in S_6 we have (3 1 4)(2 5 6) = (2 5 6)(3 1 4). Every element of S_n can be written as a product of disjoint cycles; this representation is unique up to the order of the factors.

Special elements:

Certain elements of the symmetric group of $\{1,2, ..., n\}$ are of particular interest (these can be generalized to the symmetric group of any finite totally ordered set, but not to that of an unordered set).

The order reversing permutation is the one given by**:**

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

In geometry, The symmetry group of an object (image, signal, etc.) is the group of all isometries under which it is invariant with composition as the operation. It is a subgroup of the isometry group of the space concerned.

The "objects" may be geometric figures, images and patterns. The definition can be made more precise by specifying what is meant by image or pattern, e.g., a function of position with values in a set of colors. For symmetry of physical objects, one may also want to take physical composition into account The group of isometries of space induces a group action on objects in it.

The symmetry group is sometimes also called full symmetry group in order to emphasize that it includes the orientation-reversing isometries (like reflections, glide reflections and improper rotations) under which the figure is invariant. The subgroup of orientation-preserving isometries (i.e. translations, rotations, and compositions of these) which leave the figure invariant is called its proper symmetry group. The proper symmetry group of an object is equal to its full symmetry group if and only if the object is chiral (Chiral objects have no superimposable mirror images and form a pair of enantiomers. (and thus there are no orientation-reversing isometries under which it is invariant).

Any symmetry group whose elements have a common fixed point, which is true for all finite symmetry groups and also for the symmetry groups of bounded figures, can be represented as a subgroup of orthogonal group O(n) by choosing the origin to be a fixed point. The proper symmetry group is a subgroup of the special orthogonal group SO(n) then, and therefore also called rotation group of the figure.

Discrete symmetry groups comes in three types:

(1) finite point groups, which include only rotations, reflections, inversion and not inversion - they are in fact just the finite subgroups of O(n).

(2) infinite lattice groups, which include only translations, and (3) infinite space groups which combines elements of both previous types, and may also include extra transformations like screw axis and glide reflection.

There are also *continuous* symmetry groups, which contain rotations of arbitrarily small angles or translations of arbitrarily small distances. The group of all symmetries of a sphere O(3) is an example of this, and in general such *continuous symmetry groups are studied as Lie groups. With a* categorization of subgroups of the Euclidean group corresponds a categorization of symmetry groups.

Two geometric figures are considered to be of the same symmetry type if their symmetry groups are conjugate subgroups of the Euclidean group $E(n)$ (the isometry group of R^n), where two subgroups H_1, H_2 of a group G are *conjugate*, if there exists $g \in G$ such that $H_1=g^{-1}H_2g$.

For example:

- two 3D figures have mirror symmetry, but with respect to different mirror planes.
- two 3D figures have 3-fold rotational symmetry, but with respect to different axes.
- two 2D patterns have translational symmetry, each in one direction; the two translation vectors have the same length but a different direction.

When considering isometry groups, one may restrict oneself to those where for all points the set of images under the isometries is topologically closed. This excludes for example in 1D the group of translations by a rational number. A "figure" with this symmetry group is non-drawable and up to arbitrarily fine detail homogeneous, without being really homogeneous.

In wider contexts, a symmetry group may be any kind of transformation group, or automorphism group(In set theory, an automorphism of a set *X* is an arbitrary permutation of the elements of *X*. The automorphism group of *X* is also called the symmetric group on *X*).. Once we know what kind of mathematical structure we are concerned with, we should be able to pinpoint what mappings preserve the structure. Conversely, specifying the symmetry can define the structure, or at least clarify what we mean by an invariant, geometric language in which to discuss it; this is one way of looking at the Erlangen programmer.

For example, automorphism groups of certain models of finite geometries are not "symmetry groups" in the usual sense, although they preserve symmetry. They do this by preserving *families* of point-sets rather than point-sets (or "objects") themselves.

The low-degree symmetric groups have simpler structure and exceptional structure and often must be treated separately.

Sym(0) and Sym(1)

The symmetric groups on the empty set and the singleton set are trivial, which corresponds to 0! = 1! = 1. In this case the alternating group agrees with the symmetric group, rather than being an index 2 subgroup, and the sign map is trivial.

Sym(2)

The symmetric group on two points consists of exactly two elements: the identity and the permutation swapping the two points. It is a cyclic group and so abelian. In Galois theory, this corresponds to the fact that the quadratic formula gives a direct solution to the general quadratic polynomial after extracting only a single root.

Sym(3)

is isomorphic to the dihedral group of order 6, the group of reflection and rotation symmetries of an equilateral triangle, since these symmetries permute the three vertices of the triangle. Cycles of length two correspond to reflections, and cycles of length three are rotations.

Sym(4)

The group S_4 is isomorphic to proper rotations of the cube; the isomorphism from the cube group to Sym(4) is given by the permutation action on the four diagonals of the cube.

Sym(5)

Sym(5) is the first non-solvable symmetric group. Sym(5) is one of the three non-solvable groups of order 120 up to isomorphism. Sym(5) is the Galois group of the general quintic equation, and the fact that Sym(5) is not a solvable group translates into the non-existence of a general formula to solve quintic polynomials by radicals.

Sym(6)

Sym(6), unlike other symmetric groups, has an outer automorphism

2.5.1 Linear Transformation

One of the most important transformation groups is the group of *linear transformations* of *n*-dimensional space.

Definition :

Let *V* and *W* be vector spaces over the same field *K*. A function $f : V \to W$ is said to be a *linear map* if for any two vectors *x* and *y* in *V* and any scalar ***a*** in *K*, the following two conditions are satisfied:

$$f(x+y) = f(x) + f(y) \qquad \text{additivity} \qquad (2.8)$$

$$f(ax) = af(x) \qquad \text{homogeneity of degree 1.} \qquad (2,9)$$

This is equivalent to requiring that for any vectors $x_1, ..., x_m$ and scalars $a_1, ..., a_m$, the equality

$$f(a_1x_1 + \cdots + a_mx_m) = a_1f(x_1) + \cdots + a_mf(x_m) \qquad (2.10)$$

holds.

It immediately follows from the definition that $f(0) = 0$.

In other words , a linear map (also called a linear transformation, linear function or linear operator) is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication.

The expression "linear operator" is commonly used for linear maps from a vector space to itself (endomorphism's).

In the language of abstract algebra, a linear map is a set of vector spaces,

Images and kernels

There are some fundamental concepts underlying linear transformations, such as the *kernel* and the *image* of a linear transformation, which are analogous to the *zeros* and *range* of a function.

2.5.2 Kernel

The *kernel* of a linear transformation T: V →W is the set of all vectors in V which are mapped to the zero vector in W, i.e.,

$$\ker T = \{v \in V \mid T\mathbf{v} = \mathbf{0}\}$$

The kernel of a transform T: V →W is always a subspace of V. The dimension of a transform or a matrix is called the *nullity*..

2.5.3 Image

The *image* of a linear transformation T:V→W is the set of all vectors in W which were mapped from vectors in V. For example with the trivial mapping T:V→W such that Tx=0, the image would be 0.

More formally, we say that the image of a transformation T:V→W is the set:

$$\operatorname{im} T = \{w \in W \mid w = T\mathbf{v} \text{ and } \mathbf{v} \in V\}$$

2.5.4 Cokernel:

The cokernel is defined as :

$$\operatorname{coker} f := W/f(V) = W/\operatorname{im}(f).$$

This is the *dual* notion to the kernel: just as the kernel is a *sub*space of the *domain*, the co-kernel is the *quotient* space .

2.5.5 Rank-Nullity Theorem

Let *V* and *W* be vector spaces over a field *K*., and let T: V →W be a linear transformation. Assuming the dimension of T: V →W is finite, then

Dim (V) = dim (Ker (T)) + dim (Im (T)) where Dim (V) is the dimension of *V*, Ker is the kernel, and Im is the image. Note that dim (Ker (T)) is called the nullity of T and dim (Im (T))is called the rank of T.

Examples

- The identity map and zero map are linear.
- The map $x \mapsto cx$, where *c* is a constant, is linear.
- For real numbers, the map $x \mapsto x^2$ is not linear.
- For real numbers, the map $x \mapsto x + 1$ is not linear.

If *A* is a real $m \times n$ matrix, then *A* defines a linear map from R^n to R^m by sending the column vector $x \in R^n$ to the column vector $Ax \in R^m$. Conversely, any linear map between finite-dimensional vector spaces can be represented in this manner.

2.5.6 Matrices

If *V* and *W* are finite-dimensional, and one has chosen bases in those spaces, then every linear map from *V* to *W* can be represented as a matrix. This is useful because it allows concrete calculations. Conversely, matrices yield examples of linear maps: if *A* is a real *m*-by-*n* matrix, then the rule $f(x) = Ax$ describes a linear map $R^n \to R^m$

Let $\{v_1, \cdots, v_n\}$ be a basis for *V*. Then every vector *v* in *V* is uniquely determined by the coefficients $c_1,\ldots,c_n$ in

$$c_1v_1 + \cdots + c_nv_n.$$

If $f : V \to W$ is a linear map,

$$f(c_1v_1 + \cdots + c_nv_n) = c_1f(v_1) + \cdots + c_nf(v_n), \quad (2.11)$$

which implies that the function f is entirely determined by the values of $f(v_1), \cdots, f(v_n)$.

Now let $\{w_1, \ldots, w_m\}$ be a basis for *W*. Then we can represent the values of each $f(v_j)$ as

$$f(v_j) = a_{1j}w_1 + \cdots + a_{mj}w_m. \qquad (2.12)$$

Thus, the function *f* is entirely determined by the values of $a_{i,j}$.

If we put these values into an *m*-by-*n* matrix *M*, then we can conveniently use it to compute the value of *f* for any vector in *V*. For if we place the values of $c_1,\ldots,c_n$ in an n-by-1 matrix *C*, we have *MC* = the m-by-1 matrix whose i.th element is the coordinate of f(v) which belongs to the base w_i.

A single linear map may be represented by many matrices. This is because the values of the elements of the matrix depend on the bases that are chosen.

2.5.7 Examples of Linear Transformation Matrices

Some special cases of linear transformations of two-dimensional space R^2 are illuminating:

- rotation by 90 degrees counterclockwise:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- rotation by θ degrees counterclockwise:

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- reflection against the *x* axis:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- reflection against the *y* axis:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- scaling by 2 in all directions:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- horizontal shear mapping:

$$A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

- squeezing:

$$A = \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix}$$

- projection onto the *y* axis:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

2.5.8 Properties :

1 - The composition of linear maps is linear: if $f : V \to W$ and $g : W \to Z$ are linear, then so is their composition $g \circ f : V \to Z$. It follows from this that the class of all vector spaces over a given field *K*, together with *K*-linear maps as morphisms, forms a category.

2 - The inverse of a linear map, when defined, is again a linear map.

3 - If $f_1 : V \to W$ and $f_2 : V \to W$ are linear, then so is their sum $f_1 + f_2$ (which is defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$).

4 -If $f : V \to W$ is linear and *a* is an element of the ground field *K*, then the map *af*, defined by $(af)(x) = a\,(f(x))$, is also linear.

Thus the set $L(V,W)$ of linear maps from *V* to *W* itself forms a vector space over *K*, sometimes denoted Hom(*V*,*W*). Furthermore, in the case that *V*=*W*, this vector space (denoted End(*V*)) is an associative under composition of maps, since the composition of two linear maps is again a linear map, and the composition of maps is always associative.

5 - Given again the finite-dimensional case, if bases have been chosen, then the composition of linear maps corresponds to the matrix multiplication, the addition of linear maps corresponds to the matrix addition, and the multiplication of linear maps with scalars corresponds to the multiplication of matrices with scalars.

2.5.9 Classification:

Here are some important classifications that do not require any additional structure on the vector space.

Let V and W denote vector spaces over a field, F. Let $T:V \to W$ be a linear map.

1-T is said to be *injective* or a *monomorphism* if any of the following equivalent conditions are true:

T is one-to-one as a map of sets.

Ker $T = 0_V$

*T is monic or left-cancellable, which is to say, for any vector space U and any pair of linear maps $R:U \to V$ and $S:U \to V$, the equation $TR=TS$ implies $R=S$.

*T is left-invertible, which is to say there exists a linear map $S:W \to V$ such that ST is the identity map on V.

*- T is said to be *surjective* or an *epimorphism* if any of the following equivalent conditions are true:

*T is onto as a map of sets.

*coker $T = 0$

T is epic or right-cancellable, which is to say, for any vector space U and any pair of linear maps $R:W \to U$ and $S:W \to U$, the equation $RT=ST$ implies $R=S$.

T is right-invertible, which is to say there exists a linear map $S:W \to V$ such that TS is the identity map on V.

T is said to be an *isomorphism* if it is both left- and right-invertible. This is equivalent to T being both one-to-one and onto (a bijection of sets) or also to T being both epic and monic, and so being a bimorphism.

Applications

A specific application of linear maps is for geometric transformations, such as those performed in computer graphics, where the translation, rotation and scaling of 2D or 3D objects is performed by the use of a transformation matrix.

Another application of these transformations is in compiler optimizations of nested-loop code, and in parallelizing compiler techniques. However, not every linear transformation has an inverse; for instance, projections don't. So, in order to have inverses, we only consider invertible transformations. They correspond to nonsingular matrices, that is, matrices with nonzero determinant. Thus, the nonsingular n by n matrices form a group of transformations on the vector space R^n. This group is called a *general linear group*, and it is denoted $\mathrm{GL}_n(R)$.

Chapter 3

Lie Groups and Symmetric Spaces

This chapter is devoted to Lie groups and symmetric spaces .

We begin this chapter with some important definitions that are related to the study of Lie group and symmetric spaces.

Topological Spaces

A topological space is a set of points, along with a set of neighborhoods for each point, that satisfy a set of axioms relating points and neighborhoods. There are many equivalent definitions, there is a neighborhood definition , open set definition , and others .Here is the open set definition.

3.1.1 Definition :

A *topological space* is a set X together with a collection of subsets of X, called open sets and satisfying the following axioms:

1. The empty set and X itself are open.
2. Any union of open sets is open.
3. The intersection of any finite number of open sets is open.

3.1.2 Manifolds

A manifold is a topological space that resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighborhood that is homeomorphic to the Euclidean space of dimension n. Lines and circles, but not figure eights , are one-dimensional manifolds. Two-dimensional manifolds are also called surfaces. Examples include the plane, the sphere, and the torus, which can all be realized in three dimensions, but also the Klein bottle and real projective plane which cannot.

Although near each point, a manifold resembles Euclidean space, globally a manifold might not. For example, the surface of the sphere is not an Euclidean space, but in a region it can be charted by means of geographic maps: map projections of the region into the Euclidean plane. When a region appears in two neighboring maps (in the context of manifolds they are called *charts*), the two representations do not coincide exactly and a transformation is needed to pass from one to the other, called a *transition map*.

The concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be described and understood in terms of the relatively well-understood properties of Euclidean space. Manifolds naturally arise as solution sets of systems of equations and as graphs of functions. Manifolds may have additional features. One important class of manifolds is the class of differentiable manifolds. This differentiable structure allows calculus to be done on manifolds. A Riemannian metric on a manifold allows distances and angles to be measured.

The spherical Earth is navigated using flat maps or charts, collected in an atlas. Similarly, a differentiable manifold can be described using mathematical maps, called *coordinate charts*, collected in a mathematical *atlas*. It is not generally possible to describe a manifold with just one chart, because the global structure of the manifold is different from the simple structure of the charts. For example, no single flat map can represent the entire Earth without separation of adjacent features across the map's boundaries or duplication of coverage. When a manifold is constructed from multiple overlapping charts, the regions where they overlap carry information essential to understanding the global structure.

3.1.3 Charts

A coordinate map, a coordinate chart, or simply a chart, of a manifold is an invertible map between a subset of the manifold and a simple space such that both the map and its inverse preserve the desired structure. For a topological manifold, the simple space is some Euclidean space R^n and interest focuses on the topological structure. This structure is preserved by homeomorphisms, invertible maps that are continuous in both directions.

In the case of a differentiable manifold, a set of charts called an atlas allows us to do calculus on manifolds. Polar coordinates, for example, form a chart for the plane R^2 minus the positive *x*-axis and the origin.

3.1.4 Atlases

The description of most manifolds requires more than one chart (a single chart is adequate for only the simplest manifolds). A specific collection of charts which covers a manifold is called an atlas. An atlas is not unique as all manifolds can be covered multiple ways using different combinations of charts. Two atlases are said to be C^k equivalent if their union is also a C^k atlas.

The atlas containing all possible charts consistent with a given atlas is called the maximal atlas (i.e. an equivalence class containing that given atlas (under the already defined equivalence relation given in the previous paragraph)). Unlike an ordinary atlas, the maximal atlas of a given manifold is unique. Though it is useful for definitions, it is an abstract object and not used directly (e.g. in calculations).

3.1.5 Transition maps

Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. If two charts overlap, parts of them represent the same region of the manifold, just as a map of Europe and a map of Asia may both contain Moscow. Given two overlapping charts, a transition function can be defined which goes from an open ball in R^n to the manifold and then back to another (or perhaps the same) open ball in R^n. The resultant map, like the map T in the circle example above, is called a change of coordinates, a coordinate transformation, a transition function, or a transition map.

3.1.6 Additional Structure

An atlas can also be used to define additional structure on the manifold. The structure is first defined on each chart separately. If all the transition maps are compatible with this structure, the structure transfers to the manifold.

This is the standard way differentiable manifolds are defined. If the transition functions of an atlas for a topological manifold preserve the natural differential structure of R^n (that is, if they are diffeomorphisms), the differential structure transfers to the manifold and turns it into a differentiable manifold. Complex manifolds are introduced in an analogous way by requiring that the transition functions of an atlas are holomorphic functions. The structure on the manifold depends on the atlas, but sometimes different atlases can be said to give rise to the same structure. Such atlases are called compatible

3.2.1 Differentiable Manifolds

For most applications a special kind of topological manifold, a differentiable manifold, is used. If the local charts on a manifold are compatible in a certain sense, one can define directions, tangent spaces, and differentiable functions on that manifold. In particular it is possible to use calculus on a differentiable manifold. Each point of an *n*-dimensional differentiable manifold has a tangent space. This is an *n*-dimensional Euclidean space consisting of the tangent vectors of the curves through the point.

There are a number of different types of differentiable manifolds, depending on the precise differentiability requirements on the transition functions. Some common examples include the following.

- A differentiable manifold is a topological manifold equipped with an equivalence class of atlases whose transition maps are all differentiable. In broader terms, a C^k-manifold is a topological manifold with an atlas whose transition maps are all *k*-times continuously differentiable.

- A smooth manifold or C^∞-manifold is a differentiable manifold for which all the transition maps are smooth. That is, derivatives of all orders exist; so it is a C^k-manifold for all k. An equivalence class of such atlases is said to be a smooth structure.

3.3.1 Lie Group

A Lie group is a group that is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure.(so Lie groups lie at the intersection of two fundamental fields of mathematics: algebra and geometry) . Lie groups are named after Sophus Lie, who laid the foundations of the theory of continuous transformation groups.

Lie groups are smooth manifolds and as such can be studied using differential calculus, in contrast with the case of more general topological groups. One of the key ideas in the theory of Lie groups is to replace the *global* object, the group, with its *local* or linearized version, which Lie himself called its "infinitesimal group" and which has since become known as its Lie algebra.

3.3.2 Definition

A real Lie group is a group that is also a finite-dimensional real smooth manifold, and in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$$\mu : G \times G \to G \quad \mu(x, y) = xy$$

means that μ is a smooth mapping of the product manifold $G \times G$ into G. These two requirements can be combined to the single requirement that the mapping

$$(x, y) \mapsto x^{-1}y$$

be a smooth mapping of the product manifold into G.

3.3.3 First Examples

The 2×2 **real invertible matrices** form a group under multiplication, denoted by **GL(2, R)**:

$$\mathrm{GL}(2, \mathbf{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = ad - bc \neq 0 \right\}.$$

This is a four-dimensional noncompact real Lie group. This group is disconnected; it has two connected components corresponding to the positive and negative values of the determinant.

- The rotation matrices form a subgroup of GL(2, R), denoted by SO(2, R). It is a Lie group in its own right: specifically, a one-dimensional compact connected Lie group which is diffeomorphic to the circle. Using the rotation angle φ as a parameter, this group can be parameterized as follows:

$$\mathrm{SO}(2,\mathbf{R}) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : \varphi \in \mathbf{R}/2\pi\mathbf{Z} \right\}.$$

Addition of the angles corresponds to multiplication of the elements of SO(2, R), and taking the opposite angle corresponds to inversion. Thus both multiplication and inversion are differentiable maps.

- The orthogonal group also forms an interesting example of a Lie group.

All of the previous examples of Lie groups fall within the class of classical groups.

More Examples of Lie Groups

Lie groups occur in abundance throughout mathematics and physics. Matrix groups or algebraic groups are (roughly) groups of matrices (for example, orthogonal and symplectic groups), and these give most of the more common examples of Lie groups.

Examples with a Specific Number of Dimensions

The **circle group** S^1 consisting of angles mod 2π under addition or, alternatively, the complex numbers with absolute value 1 under multiplication. This is a one-dimensional compact connected abelian Lie group.

- The 3-sphere S^3 forms a Lie group by identification with the set of quaternions of unit norm, called versors. The only other spheres that admit the structure of a Lie group are the 0-sphere S^0 (real numbers with absolute value 1) and the circle S^1 (complex numbers with absolute value 1). For example, for even $n > 1$, S^n is not a Lie group because it does not admit a nonvanishing vector field and so *a fortiori* cannot be parallelizable as a differentiable manifold. Of the spheres only S^0, S^1, S^3, and S^7 are parallelizable. The latter carries the structure of a Lie quasigroup (a nonassociative group), which can be identified with the set of unit octonions.

- The (3-dimensional) metaplectic group is a double cover of SL(2, R) playing an important role in the theory of modular forms. It is a connected Lie group that cannot be faithfully represented by matrices of finite size, i.e., a nonlinear group.
- The Heisenberg group is a connected nilpotent Lie group of dimension 3, playing a key role in quantum mechanics.
- The Lorentz group is a 6 dimensional Lie group of linear isometries of the Minkowski space.
- The Poincaré group is a 10 dimensional Lie group of affine isometries of the Minkowski space.
- The group U(1)×SU(2)×SU(3) is a Lie group of dimension 1+3+8=12 that is the gauge group of the Standard Model in particle physics. The dimensions of the factors correspond to the 1 photon + 3 vector bosons + 8 gluons of the standard model
- The exceptional Lie groups of types G_2, F_4, E_6, E_7, E_8 have dimensions 14, 52, 78, 133, and 248. Along with the A-B-C-D series of simple Lie groups, the exceptional groups complete the list of simple Lie groups. There is also a Lie group named $E_{7½}$ of dimension 190, but it is not a *simple* Lie group.

Examples with *n* Dimensions

Euclidean space R^n with ordinary vector addition as the group operation becomes an n-dimensional noncompact **abelian** Lie group.

- The Euclidean group E(n, R) is the Lie group of all Euclidean motions, i.e., isometric affine maps, of n-dimensional Euclidean space R^n.
- The orthogonal group O(n, R), consisting of all $n \times n$ orthogonal matrices with real entries is an $n(n - 1)/2$-dimensional Lie group. This group is disconnected, but it has a connected subgroup SO(n, R) of the same dimension consisting of orthogonal matrices of determinant 1, called the special orthogonal group (for $n = 3$, the rotation group SO(3)).
- The unitary group U(n) consisting of $n \times n$ unitary matrices (with complex entries) is a compact connected Lie group of dimension n^2. Unitary matrices of determinant 1 form a closed connected subgroup of dimension $n^2 - 1$ denoted SU(n), the special unitary group.
- Spin groups are double covers of the special orthogonal groups, used for studying fermions in quantum field theory (among other things).
- The group GL(n, R) of invertible matrices (under matrix multiplication) is a Lie group of dimension n^2, called the general linear group. It has a closed connected subgroup SL(n, R), the special linear group, consisting of matrices of determinant 1 which is also a Lie group.
- The symplectic group Sp($2n$, R) consists of all $2n \times 2n$ matrices preserving a *symplectic form* on R^{2n}. It is a connected Lie group of dimension $2n^2 + n$.
- The group of invertible upper triangular n by n matrices is a solvable Lie group of dimension $n(n + 1)/2$. (cf. Borel subgroup)
- The A-series, B-series, C-series and D-series, whose elements are denoted by A_n, B_n, C_n, and D_n, are infinite families of simple Lie groups

3.3.4 Constructions

There are several standard ways to form new Lie groups from old ones:

- The product of two Lie groups is a Lie group.
- Any topologically closed subgroup of a Lie group is a Lie group. This is known as Cartan's theorem.
- The quotient of a Lie group by a closed normal subgroup is a Lie group.
- The universal cover of a connected Lie group is a Lie group. For example, the group R is the universal cover of the circle group S^1. In fact any covering of a differentiable manifold is also a differentiable manifold, but by specifying *universal* cover, one guarantees a group structure (compatible with its other structures).

3.4.1 Lie Group Actions on Manifolds

The importance of Lie groups stems primarily from their actions on manifolds.

3.4.2 Definition

Let G be a Lie group and M a smooth manifold. A left action of G on M is a map $G \times M \rightarrow M$, often written as $(g, p) \rightarrow g.p$ that satisfies

$$g_1 . (g_2 . p) = (g_1 . g_2) . p;$$

$$e . p = p$$

A right action is defined analogously as a map $M \times G \rightarrow M$ with (composition working in the reverse order :

$$(p . g_1) . g_2 = p. \; g_1 . g_2);$$

$$p . e = p$$

A manifold M endowed with a specific G-action is called a (left or right) G-space.

A right action can always be converted to a left action by the trick of defining $g . p$ to be $p . g^{-1}$; thus any results about left actions can be translated into results about right actions, and vice versa.

We will usually focus our attention on left actions, because their group law has the property that multiplication of group elements corresponds to composition of functions. However, there are some circumstances in which right actions arise naturally; we will see several such actions later in this chapter.

Let us introduce some basic terminology regarding Lie group actions. Let : $G \times M \longrightarrow M$ be a left action of a Lie group G on a smooth manifold M. (The definitions for right actions are analogous.)

The action is said to be smooth if it is smooth as a map from G x M into M, that is, if $\theta_g(p)$ depends smoothly on (g . p). If this is the case, then for each g ϵ G, the map

$$\theta_g : M \longrightarrow M$$

is an isomorphism , with inverse θ_g^{-1} .

(here is another notation because Sometimes it is useful to give a name to an action, such as $\theta : G \times M \longrightarrow M$, with the action of a group element g on a point p usually written $\theta_g(p)$. In terms of this notation, the conditions for a left action read

$$\theta_{g1} \circ \theta_{g2} = \theta_{g1g2};$$

$$\theta_e = Id_M ;$$

while for a right action the first equation is replaced by

$$\theta_{g1} \circ \theta_{g2} = \theta_{g2g1}$$

For left actions, we will generally use the notations g . p and θ_g (p) interchangeably. The latter notation contains a bit more information, and is useful when it is important to specify the specific action under consideration, while the former is often more convenient when the action is understood. For right actions, the notation p. g is generally preferred because of the way composition works)

1- For any p ϵ M, the orbit of p under the action is the set

$$G . p = \{g.\ p : g \in G\}$$

; the set of all images of p under elements of G.

2-The action is transitive if for any two points p, q ϵ M, there is a group element g such that g . p = q , or equivalently if the orbit of any point is all of M.

Given p ϵ M, the isotropy group of p, denoted by G_p , is the set of elements g ϵ G that fix p

$$G_p = \{ g \in G : g . p = p \}$$

3 - The action is said to be free if the only element of G that fixes any element of M is the identity : g . p = p for some p ϵ M implies g = e. This is equivalent to the requirement that $G_p = \{e\}$ for every p ϵ M.

4 - The action is said to be proper if the map G x M → M x M given by (g , p) → (g . p , p) is a proper map (i.e., the preimage of any compact set is compact). (Note that this is not the same as requiring that the map G x M →M defining the action be a proper map.)

3.4.3 Examples (Lie Group Actions).

(a) The natural action of GL(n; R) on R^n is the left action given by matrix multiplication: (A,x) → A_X, considering x ϵ R^n as a column matrix. This is an action because matrix multiplication is associative: (AB)x = A(Bx). It is smooth because the components of Ax depend polynomially on the matrix entries of A and the components of x. Because any nonzero vector can be taken to any other by a linear transformation, there are exactly two orbits: {0} and $R^n \setminus \{0\}$.

(b) The restriction of the natural action to O(n) X R^n → R^n defines a smooth left action of O(n) on R^n. In this case, the orbits are the origin and the spheres centered at the origin. To see why, note that any orthogonal linear transformation preserves norms, so O(n) takes the sphere of radius R to itself; on the other hand, any vector of length R can be taken to any other by an orthogonal matrix. (If v and $v^/$ are such vectors, complete v/|v| and $v^/\backslash|v^/|$ to orthonormal bases and let A and $A^/$be the orthogonal matrices whose columns are these orthonormal bases; then it is easy to check that $A^/A^{-1}$ takes v to $v^/$.)

(c) Further restricting the natural action to O(n) x $S^{n-1} \rightarrow S^{n-1}$, we obtain a transitive action of O(n) on S^{n-1}. It is smooth because $S^{n-1}$is an embeded sub manifold of R^n.

(d) The natural action of O(n) restricts to an action of SO(n) on S^{n-1}. When n = 1, this action is trivial because SO(1) is the trivial group consisting of the matrix (1) alone. But when n > 1, SO(n) acts transitively on S^{n-1}. To see this, it suffices to show that for any v ϵ S^n there is a matrix A ϵ SO(n) taking the first standard basis vector $e_1$to v. Since O(n) acts transitively, there is a matrix A ϵ O(n) taking e_1 to v. Either det A = 1, in which case A ϵ SO(n), or det A = −1, in which case the matrix obtained by multiplying the last column of A by −1 is in SO(n) and still takes e_1 to v.

(e) Any representation of a Lie group G on a finite-dimensional vector space V is a smooth action of G on V .

(f) Any Lie group G acts smoothly, freely, and transitively on itself by left or right translation. More generally, if H is a Lie subgroup of G, then the restriction of the multiplication map to H x G $\rightarrow$ G defines a smooth, free (but generally not transitive) left action of H on G; similarly, restriction to G x H $\rightarrow$ G defines a free right action of H on G .

(g) An action of a discrete group Γ on a manifold M is smooth if and only if for each g ϵ Γ, the map p $\rightarrow$ g . p is a smooth map from M to itself. Thus, for example, Z^n acts smoothly on the left on R^n by translation: $(m^1,\ldots, m^n) . (x^1,\ldots, x^n) = (m^1+ x^1,\ldots, m^n+ x^n)$.

3.4.4 Homogeneous Spaces

One of the most interesting kinds of group action is that in which a group acts transitively. A smooth manifold endowed with a transitive smooth action by a Lie group G is called a homogeneous G-space, or a homogeneous space or homogeneous manifold if it is not important to specify the group. Here are some important examples of homogeneous spaces.

(a) The natural action of O(n) on S^{n-1} is transitive so is the natural action of SO(n) on S^{n-1} when $n \geq 2$. Thus for $n \geq 2$, $S^{n-1}$is a homogeneous space of either O(n) or SO(n).

(b) Let E(n) denote the subgroup of GL(n + 1; R) consisting of matrices of the form

$$\left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in O(\text{n}), \text{b} \in R^n \right\}$$

where b is considered as an nx1 column matrix. It is straightforward to check that E(n) is an embedded Lie subgroup. If $S \subseteq R^{n+1}$ denotes the affine subspace defined by $x^{n+1} = 1$, then a simple computation shows that E(n) takes S to itself. Identifying S with R^n in the obvious way, this induces an action of E(n) on R^n, in which the matrix $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ sends x to Ax+ b. It is not hard to prove that these are precisely the transformations that preserve the Euclidean inner product. For this reason, E(n) is called the Euclidean group Because any point in R^n can be taken to any other by a translation, E(n) acts transitively on R^n, so R^n is a homogeneous E(n)-space.

(c) The group SL(2; R) acts smoothly and transitively on the upper half-plane H = {z ∈ C : Im z > 0} by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . Z = \frac{az + b}{cz + d} .$$

The resulting complex-analytic transformations of H are called Mobius transformations.

(d) If G is any Lie group and H is a closed Lie subgroup, the space G/H of left cosets is a smooth manifold We define a left action of G on G/H by

$$g_1 . (g_2H) = (g_1g_2)H$$

This action is obviously transitive, and this implies that it is smooth.

In most examples, the group action preserves some property of the manifold (such as distances in some metric, or a class of curves such as straight lines in the plane); then the fact that the action is transitive means that the manifold looks the same" everywhere from the point of view of this property. Often, homogeneous spaces are models for various kinds of geometric structures, and as such they play a central role in many areas of differential geometry.

3.5.1 Invariants Under Lie Group Action on a Manifold

The concept of an invariant is quite complex. Throughout these years, the word has taken many different meanings and has been applied to many different objects. From our point of view, invariants are objects that remain in a way unchanged under some transformation. However, in general, the concept of transformation is not always present in the definition of invariants. For example, the dimension of a variety is sometimes called an invariant of the variety. While it is clear that this quantity does reveal something essential about the variety, no transformation is involved in that case.

A more correct way to define invariants substitutes the concept of transformation by the more general concept of equivalence relation. In this broader context, an invariant is something that is common to all the objects belonging to the same equivalence class.

In general, an invariant is defined as real valued function that is unaffected by a group transformations. The determination of a complete set of invariants of a given group action is a problem of supreme importance for the study of equivalence and canonical forms, for a group action are completely characterized by its invariants.

Proposition

Let $I : X \longrightarrow R$. The following conditions are equivalent

(a) I is a G-invariant function.

(b) I is constant on the orbits of G.

(c) All level sets $\{ I(x) = c \}$ are G-invariants subsets of X.

In particular, constant functions are trivially G-invariant. If G acts transitively on X ,then these are the only invariants.

Example

Consider the representation of the symmetric group S^n on R^n given by the permutation matrices, a function

$$I : R^n \longrightarrow R^n$$

is invariant under S^n if and only if it satisfies :

$I (x_{\pi(1)} \dots x_{\pi(n)}) = I (x_1 \dots x_n)$ for every $\pi \in S^n$.

The invariants of the permutation group S^n are known as symmetric functions. In our case, we shall be concerned strictly with equivalence classes defined by the orbits of a Lie group action on a smooth manifold. We will focus mostly on the concepts of differential invariant and joint invariant, which we shall define in precise terms shortly. We will investigate their existence and relationship.

In many cases, the existence of invariants is guaranteed by Frobenius' theorem which equates the number of functionally independent invariants to the difference between the dimension of the space acted on and the orbit dimension.

Let *G* be a Lie group acting smoothly on a *m*-dimensional smooth manifold *M*. For our purpose, invariants are defined as follows:

3.5.2 Definition

We say that a real valued function $I: U \subseteq M \longrightarrow R$ is an *invariant* if $I(g . x) = I(x)$, for all $g \in G$ and all $x \in U$. We say that a real valued function

$I: U \subseteq M \longrightarrow R$ is a *local invariant* if there exists a neighborhood *N* of the identity $e \in G$ such that $I(g .x) = I(x)$, for all $g \in N$ and all $x \in U$.

For instance, if $G = SO(2)$ acts on R^2, then the distance from a point to the origin is a global invariant of the group action.

In other words, invariants are quantities that remain constant on the orbits of a group action. Therefore the concept of orbit is closely related to the concept of invariant. We are interested in group actions on manifolds

3.5.3 Differential Invariant

A differential invariant is an invariant for the action of a Lie group on a space that involves the derivatives of graphs of functions in the space. Differential invariants are fundamental in projective differential geometry, and the curvature is often studied from this point of view. Differential invariants were introduced in special cases by Sophus Lie in the early 1880s and studied by Georges Henri Halphen at the same time. Lie (1884) was the first general work on differential invariants, and established the relationship between differential invariants, invariant differential equations, and invariant differential operators. The simplest case is for differential invariants for one independent variable *x* and one dependent variable *y*. Let *G* be a Lie group acting on R^2. Then *G* also acts, locally, on the space of all graphs of the form $y = f(x)$. Roughly speaking, a *k*-th order differential invariant is a function

$$I\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^k y}{dx^k}\right)$$

depending on *y* and its first *k* derivatives with respect to *x*, that is invariant under the action of the group.

The group can act on the higher-order derivatives in a nontrivial manner that requires computing the *prolongation* of the group action. The action of *G* on the first derivative, for instance, is such that the chain rule continues to hold: if

$$(\overline{x}, \overline{y}) = g \cdot (x, y),$$

then

$$g \cdot \left(x, y, \frac{dy}{dx}\right) \stackrel{\text{def}}{=} \left(\overline{x}, \overline{y}, \frac{d\overline{y}}{d\overline{x}}\right).$$

Similar considerations apply for the computation of higher prolongations. This method of computing the prolongation is impractical, however, and it is much simpler to work infinitesimally at the level of Lie algebras and the Lie derivative along the *G* action.

More generally, differential invariants can be considered for mappings from any smooth manifold *X* into another smooth manifold *Y* for a Lie group acting on the Cartesian product $X\times Y$. The graph of a mapping $X \rightarrow Y$ is a sub manifold of $X\times Y$ that is everywhere transverse to the fibers over *X*. The group *G* acts, locally, on the space of such graphs, and induces an action on the *k*-th prolongation $Y^{(k)}$ consisting of graphs passing through each point modulo the relation of *k*-th order contact. A differential invariant is a function on $Y^{(k)}$ that is Differential invariants can be applied to the invariant under the prolongation of the group action.

Applications

study of systems of partial differential equations: seeking similarity solutions that are invariant under the action of a particular group can reduce the dimension of the problem (i.e. yield a "reduced system").

3.5.4 Joint Invariants

Joint invariants appear when a transformation group acts simultaneously on several different spaces or, more typically , on a multiple copies of the same space. More specifically ,suppose G is a fixed group which acts on the spaces

$X_1, \ldots , X_m$. Then there is a naturally induced action of G on the Cartesian product space $X_1 \times \ldots \times X_m$ given by

$g. (x_1 \ldots,x_m) = (g.x_1 , \ldots ,g.x_m)$ for $x_1 \in X_i$, $g \in G$.

Definition :

A joint invariant is merely an invariant function

$$J : X_1\times \ldots\times X_m \longrightarrow R$$

for a Cartesian product action of a group G.

In other words

$J (g. x_1 , \dots ,g.x_m) = J (x_1 , \dots , x_m)$ for all $g \in G . x_i \in X_I$

We said before that "invariants are objects that remain in a way unchanged under some transformation". That means not all transformations as an example of transformations that keeps some objects unchanged there are :

1 rotation or "turn"

2 reflection or "flip "

3 translation or " slide "

(definitions was given in chapter 2)

After any of these transformations (turn, flip or slide), the shape still has the same size, area, angles and line lengths, in other words we say that size ,area , angles and line lengths are invariant under this action (transformation).

Rotation, reflection and translation are called " isometry " which is another word for rigid transformations.

In physics A symmetry of a physical system is a physical or mathematical feature of the system (observed or intrinsic) that is "preserved" under some change.

In real-world observations. For example, temperature may be constant throughout a room. Since the temperature is independent of position within the room, the temperature is *invariant* under a shift in the measurer's position. Similarly, a uniform sphere rotated about its center will appear exactly as it did before the rotation. The sphere is said to exhibit spherical symmetry. A rotation about any axis of the sphere will preserve how the sphere "looks.

3.6.1 Symmetric Spaces

The theory of symmetric spaces was initiated by E.CARTAN in 1926 and was vigorously developed by him in the late 1920"s

Symmetric spaces can be considered from many different points of view. They can be viewed as Riemannian manifolds with point reflections or with parallel curvature tensor,

or with special holonomy,

or as a homogeneous space with a special isotropy,

or special Killing vector fields,

or as Lie triple systems,

or as a Lie group with a certain involution.

And before we go through the definitions of the above I think that it is necessary to remember the following :

3.6.2 Geodesics:

The term "geodesic" comes from *geodesy*, the science of measuring the size and shape of Earth; in the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely, a segment of a great circle. The term has been generalized to include measurements in much more general mathematical spaces; A geodesic is a locally length-minimizing curve. Equivalently, it is a path that a particle which is not accelerating would follow. In the plane, the geodesics are straight lines. On the sphere, the geodesics are great circles (like the equator). The geodesics in a space depend on the Riemannian metric, which affects the notions of distance and acceleration.

Geodesics preserve a direction on a surface, and have many other interesting properties. The normal vector to any point of a geodesic arc lies along the normal to a surface at that point).,

3.6.3 Riemannian Manifold

(M,g) is a A Riemannian manifold Riemannian space or real differentiable manifold M in which each tangent space is equipped with an inner product g, a Riemannian metric, in a manner which varies smoothly from point to point. The metric g is a positive definite symmetric tensor: a metric tensor. In other words, a Riemannian manifold is a differentiable manifold in which the tangent space at each point is a finite-dimensional Euclidean space.

This allows one to define various geometric notions on a Riemannian manifold such as angles, lengths of curves, areas (or volumes), curvature, gradients of functions and divergence of vector fields.

Riemannian manifolds should not be confused with Riemann surfaces, manifolds that locally appear like patches of the complex plane.

The terms are named after German mathematician Bernhard Riemann.

Riemannian Metrics:

Let M be a differentiable manifold of dimension n. A Riemannian metric on M is a family of (positive definite) inner products

$$g_p\colon T_pM \times T_pM \longrightarrow \mathbf{R}, \qquad p \in M$$

such that, for all differentiable vector fields X,Y on M,

$$p \mapsto g_p(X(p), Y(p))$$

defines a smooth function $M \to \mathrm{R}$.

In other words, a Riemannian metric g is a symmetric (0,2)-tensor that is positive definite (i.e. $g(X, X) > 0$ for all tangent vectors $X \neq 0$).

3.6.4 Definition and Examples

A (Riemannian) symmetric space is a Riemannian manifold S with the property that the geodesic reflection at any point is an isometry of S. In other words, for any $x \in S$ there is some $s_x \in G = I(S)$ (the isometry group of S) with the properties

$$s_x(x) = x, \quad (ds_x)x = -I.$$

This isometry s_x is called symmetry at x.

S is said to be locally Riemannian symmetric if its geodesic symmetries are in fact isometric, and (globally) Riemannian symmetric if in addition its geodesic symmetries are defined on all of S.

As a first consequence of this definition, S is geodesically complete: If a geodesic γ is defined on [0, s), we may reflect it by $s_{\gamma(t)}$ for some $t \in (s/2, s)$, hence we may extend it beyond s. Moreover, S is homogeneous, i.e. for any two points $p, q \in M$ there is an isometry which maps p onto q.

In fact, if we connect p and q by a geodesic segment γ (which is possible since S is complete) and let $m \in \gamma$ be its midpoint, then $s_m(p) = q$. Thus G acts transitively. Let us fix a base point $p \in S$. The closed subgroup $G_p = \{g \in G;\ g(p) = p\}$ is called the isotropy group and will be denoted by K. The differential at p of any $k \in K$ is an orthogonal transformation of TpS. Recall that the isometric k is determined by its differential dk_p; thus we may view K also as a closed subgroup of $O(T_pS)$ (the orthogonal group on T_pS) using this embedding $k \to dk_p$ which is called isotropy representation. In particular, K is compact. Vice versa, if S is any homogeneous space, i.e. its isometric group G acts transitively, then S is symmetric if and only if there exists a symmetry s_p for some $p \in S$. Namely, the symmetry at any other point $q = g_p$ is just the conjugate $s_q = gs_pg^{-1}$.

. Thus we have seen:

Theorem 3.5.3

A symmetric space S is precisely a homogeneous space with a symmetry s_p at some point $p \in S$.

As usual, we may identify the homogeneous space S with the cosset space G/K using the G-equivariant diffeomorphism.

$gK \to gp$. In particular, $\dim S = \dim G - \dim K$.

Example 1: Euclidean Space. Let $S = R^n$ with the Euclidean metric. The symmetry at any point $x \in R^n$ is the point reflection $s_x(x + v) = x - v$. The isometric group is the Euclidean group E(n) generated by translations and orthogonal linear maps; the isotropy group of the origin O is the orthogonal group O(n). Note that the symmetries do not generate the full isometry group E(n) but only a subgroup which is an order-two extension of the translation group.

Example 2: The Sphere. Let $S = S^n$ be the unit sphere in R^{n+1} with the standard scalar product. The symmetry at any $x \in S^n$ is the reflection at the line R_x in R_{n+1}, i.e. $s_x(y) = -y + 2(y, x)x$ (the component of y in x-direction, (y, x)x, is preserved while the orthogonal complement y –(y, x)x changes sign). In this case, the symmetries generate the full isometry group which is the orthogonal group O(n + 1). The isotropy group of the last standard unit vector $e_{n+1} = (0, ..., 0, 1)^T$ is $O(n) \subset O(n + 1)$.

3.6.5 Definition:

A Riemannian manifold (*M*; *g*) is said to be a *symmetric space* if for every point *p* ϵ *M* there exists a isometry σ_ρ of (*M*; *g*) such that

(1) $\sigma_p(p) = p$, and

(2) $d\sigma_p = idTpM$:

Such an isometry is called an *involution* at *p* ϵ *M*.

Lemma 3.5.5. [6] *Let* (*M, g*) *be a symmetric space and let* $\sigma_p : (M, g) \to (M, g)$ *be an involution at p* ϵ *M. Then* σ_p *reverses the geodesics through p, i.e.*

$\sigma\, p(\gamma(t)) = \gamma(-t)$ *for all geodesics* $\gamma \in M$ such that $\gamma(0) = p$.

proof : A geodesic γ : I →M is uniquely determined by the initial data γ (0)and $\dot{\gamma}(0)$. Both the geodesic $\tau \to \sigma_\rho(\gamma(\tau))$and $t \to \gamma(-t)$ take the value γ(0)and have the tangent $-\dot{\gamma}(0)$ for t= 0.

The following lemma entails the core features of a symmetric space.

Lemma Let (M, g) be a symmetric space. If γ: I →M is a geodesic with γ(0) =p and γ(x) = q, then $\sigma_{\rho O \sigma \rho}\gamma(t)= \gamma(t+2x)$

.For $v \in T_{\gamma(t)}M$, $d\sigma_\rho(d\sigma_\rho(v)) \epsilon\, T_{\gamma(t=2x)}M$ is the vector at γ(t+2x) obtained by parallel transport of v along γ.

PROOF :Let $\bar{\gamma}(t) = \gamma(t+x)$then $\bar{\gamma}$is a geodesic with $\bar{\gamma}(0)$ =q, so by Lemma2.2 it follows that

$$\begin{aligned} \sigma_\rho(\sigma_\rho(\gamma(t))) &= \sigma_\rho(\gamma(-t)) \\ &= \sigma_\rho(\bar{\gamma}(-t-x)) \\ &= \bar{\gamma}(t+x) \\ &= \gamma(t+2x) \end{aligned}$$

If v $\in T_\rho M$ and V is a parallel vector field along γ with $V(\rho) = v$, then $d\sigma_\rho(V)$ is parallel, since σ_ρ is an isometry. Also

$$d\sigma_\rho O d\sigma_\rho(V(\gamma(t)))=V(\gamma(T+2X))$$

by the above and since $d\sigma$ applied twice cancels directions reversals.

3.6.6 Definition :

Asymmetric space is a Riemannian manifold whose curvature tensor is invariant under all parallel translations.

So by their definitions, symmetric spaces form a special topic in Riemannian geometry. Their theory ,however ,has emerged with the theory of semi simple Lie group.

3.6.7 Symmetric Pairs

The description of a symmetric space, in terms of a Lie group *G*, a closed subgroup *K* and an involution σ , leads to the concept of a symmetric pair which is defined as:

Definition:

A pair (*G*; *K*) is said to be a Riemannian symmetric pair if *G* is a Lie group, *K* a closed subgroup of *G*, and s_{p0} an involutive automorphism on *G* such that

(1) $(G_{sp0})_0 \subseteq K \subseteq G_{sp0}\sigma$

(2) $Ad(K)$ is a compact subset of $GL(\mathrm{g})$.

Here G_{sp0} are the elements of *G* that are left invariant by s_{p0} , i.e.

$$G_{sp0} = \{g \in G : s_{p0}g = g\}$$

The involutive automorphism s_{p0} is often uniquely defined, so it is common to only write (*G*; *K*) for a symmetric pair.

3.6.8 Basic Properties of Symmetric Spaces

The Cartan–Ambrose–Hicks theorem implies that *M* is locally Riemannian symmetric if and only if its curvature tensor is covariantly constant, and furthermore that any simply connected, complete locally Riemannian symmetric space is actually Riemannian symmetric.

Any Riemannian symmetric space *M* is complete and Riemannian homogeneous (meaning that the isometry group of *M* acts transitively on *M*). In fact, already the identity component of the isometry group acts transitively on *M* (because *M* is connected).

Locally Riemannian symmetric spaces that are not Riemannian symmetric may be constructed as quotients of Riemannian symmetric spaces by discrete groups of isometries with no fixed points, and as open subsets of (locally) Riemannian symmetric spaces.

3.6.9 Geometric Properties of Symmetric Spaces

1 – on a symmetric space every tensor field of odd degree which is invariant under S_x (the symmetry at x)vanishes at x

2 – If (M , g) is asymmetric space ,then ∇ R = 0 where ∇ is the covariant derivative and R is its curvature tensor

3– every symmetric space is locally symmetric.

4– every symmetric space is geodesically complete.

5 – Every symmetric space is a pseudo-Riemannian homogenous.

Examples

Basic examples of Riemannian symmetric spaces are Euclidean space, spheres, projective spaces, and hyperbolic spaces, each with their standard Riemannian metrics. More examples are provided by compact, semi-simple Lie groups equipped with a bi-invariant Riemannian metric. An example of a non-Riemannian symmetric space is anti-de Sitter space.

Any compact Riemann surface of genus greater than 1 (with its usual metric of constant curvature −1) is a locally symmetric space but not a symmetric space

.

3.6.10 Classification of Riemannian Symmetric Spaces

The algebraic description of Riemannian symmetric spaces enabled Élie Cartan to obtain a complete classification of them in 1926.

For a given Riemannian symmetric space *M* let (G,K,σ,g) be the algebraic data associated to it. To classify possibly isometry classes of *M*, first note that the universal cover of a Riemannian symmetric space is again Riemannian symmetric, and the covering map is described by dividing the connected isometry group *G* of the covering by a subgroup of its center. Therefore we may suppose without loss of generality that *M* is simply connected. (This implies *K* is connected by the long exact sequence of a vibration, because *G* is connected by assumption.)

A simply connected Riemannian symmetric space is said to be irreducible if it is not the product of two or more Riemannian symmetric spaces. It can then be shown that any simply connected Riemannian symmetric space is a Riemannian product of irreducible ones. Therefore we may further restrict ourselves to classifying the irreducible, simply connected Riemannian symmetric spaces.

The next step is to show that any irreducible, simply connected Riemannian symmetric space *M* is of one of the following three types:

1. Euclidean type: *M* has vanishing curvature, and is therefore isometric to a Euclidean space.
2. Compact type: *M* has nonnegative (but not identically zero) sectional curvature.

3-Non-compact type: *M* has nonpositive (but not identically zero) sectional curvature.

A more refined invariant is the rank, which is the maximum dimension of a subspace of the tangent space (to any point) on which the curvature is identically zero. The rank is always at least one, with equality if the sectional curvature is positive or negative. If the curvature is positive, the space is of compact type, and if negative, it is of noncompact type. The spaces of Euclidean type have rank equal to their dimension and are isometric to a Euclidean space of that dimension. Therefore it remains to classify the irreducible, simply connected Riemannian symmetric spaces of compact and non-compact type. In both cases there are two classes.

A. *G* is a (real) simple Lie group;

B. *G* is either the product of a compact simple Lie group with itself (compact type), or a complexification of such a Lie group (non-compact type).

The examples in class B are completely described by the classification of simple Lie groups. For compact type, *M* is a compact simply connected simple Lie group, *G* is *M*×*M* and *K* is the diagonal subgroup. For non-compact type, *G* is a simply connected complex simple Lie group and *K* is its maximal compact subgroup. In both cases, the rank is the rank of *G*.

The compact simply connected Lie groups are the universal covers of the classical Lie groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$and the five exceptional Lie groups E_6, E_7, E_8, F_4, G_2.

The examples of class A are completely described by the classification of noncompact simply connected real simple Lie groups. For non-compact type, *G* is such a group and *K* is its maximal compact subgroup. Each such example has a corresponding example of compact type, by considering a maximal compact subgroup of the complexification of *G* which contains *K*. More directly, the examples of compact type are classified by involutive automorphisms of compact simply connected simple Lie groups *G* (up to conjugation). Such involutions extend to involutions of the complexification of *G*, and these in turn classify non-compact real forms of *G*.

3.6.11 Duality

In both class A and class B there is thus a correspondence between symmetric spaces of compact type and non-compact type. This is known as duality for Riemannian symmetric spaces. : From any Lie algebra g with Cartan decomposition g = k + p we get another Lie algebra g' which is just g as a vector space, with the following Lie bracket $[\ ,\ ]'$: for any x, y ∈ p and a, b ∈ k we have $[a, b]' = [a, b]$, $[a, x]' = [a, x]$, $[x, y]' = -[x, y]$.

It is easy to check that this is again a Lie algebra with the same Cartan decomposition, but the sign of the curvature tensor of the corresponding symmetric space S is reversed.) Starting from a symmetric space of compact type we obtain one of noncompact type, called the dual symmetric space, and vice versa. E.g. the spheres and projective spaces are dual to the corresponding hyperbolic spaces, and the positive definite symmetric matrices GL(n,R)/O(n) are dual to the space U(n)/O(n) of real structures in C_n. The dual of a compact Lie group S = G = (G×G)/G is G^c/G where Gc corresponds to the complexified Lie algebra g^c = gC

3.6.12 Classification

Specializing to the Riemannian symmetric spaces of class A and compact type, Cartan found that there are the following seven infinite series and twelve exceptional Riemannian symmetric spaces *G*/*K*. They are here given in terms of *G* and *K*, together with a geometric interpretation, if readily available. The labeling of these spaces is the one given by Cartan.

Label	*G*	*K*	Dimension	Rank	Geometric interpretation
AI	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$(n-1)(n+2)/2$	$n-1$	Space of real structures on $\mathbb{C}^n$which leave the complex determinant invariant
AII	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	$(n-1)(2n+1)$	$n-1$	Space of

					quaternionic structures on $\mathbb{C}^{2n}$ compatible with the Hermitian metric
AIII	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$2pq$	min(*p*,*q*)	Grassmannian of complex *p*-dimensional subspaces of $\mathbb{C}^{p+q}$
BDI	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p)\times\mathrm{SO}(q)$	pq	min(*p*,*q*)	Grassmannian of oriented real *p*-dimensional subspaces of $\mathbb{R}^{p+q}$
DIII	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$n(n-1)$	[*n*/2]	Space of orthogonal complex structures on $\mathbb{R}^{2n}$
CI	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	$n(n+1)$	*N*	Space of complex structures on $\mathbb{H}^n$ compatible with the inner product
CII	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$4pq$	min(*p*,*q*)	Grassmannian

					of quaternionic *p*-dimensional subspaces of $\mathbb{H}^{p+q}$
EI	E_6	$\mathrm{Sp}(4)/\{\pm I\}$	42	6	
EII	E_6	$\mathrm{SU}(6)\cdot\mathrm{SU}(2)$	40	4	Space of symmetric subspaces of $(\mathbb{C}\otimes\mathbb{O})P^2$ isometric to $(\mathbb{C}\otimes\mathbb{H})P^2$
EIII	E_6	$\mathrm{SO}(10)\cdot\mathrm{SO}(2)$	32	2	Complexified Cayley projective plane $(\mathbb{C}\otimes\mathbb{O})P^2$
EIV	E_6	F_4	26	2	Space of symmetric subspaces of $(\mathbb{C}\otimes\mathbb{O})P^2$ isometric to $\mathbb{O}\mathbb{P}^2$
EV	E_7	$\mathrm{SU}(8)/\{\pm I\}$	70	7	
EVI	E_7	$\mathrm{SO}(12)\cdot\mathrm{SU}(2)$	64	4	Rosenfeld projective plane $(\mathbb{H}\otimes\mathbb{O})P^2$ over $\mathbb{H}\otimes\mathbb{O}$

EVII	E_7	$E_6 \cdot \mathrm{SO}(2)$	54	3	Space of symmetric subspaces of $(\mathbb{H} \otimes \mathbb{O})P^2$ isomorphic to $(\mathbb{C} \otimes \mathbb{O})P^2$
EVIII	E_8	$\mathrm{Spin}(16)/\{\pm vol\}$	128	8	Rosenfeld projective plane $(\mathbb{O} \otimes \mathbb{O})P^2$
EIX	E_8	$E_7 \cdot \mathrm{SU}(2)$	112	4	Space of symmetric subspaces of $(\mathbb{O} \otimes \mathbb{O})P^2$ isomorphic to $(\mathbb{H} \otimes \mathbb{O})P^2$
FI	F_4	$\mathrm{Sp}(3) \cdot \mathrm{SU}(2)$	28	4	Space of symmetric subspaces of $\mathbb{O}P^2$ isomorphic to $\mathbb{H}P^2$
FII	F_4	$\mathrm{Spin}(9)$	16	1	Cayley projective plane $\mathbb{O}P^2$
G	G_2	$\mathrm{SO}(4)$	8	2	Space of sub algebras of the

					octonion algebra 𝕆 which are isomorphic to the quaternion algebra

3.6.13 Weakly Symmetric Riemannian Spaces

In the 1950s Atle Selberg extended Cartan's definition of symmetric space to that of weakly symmetric Riemannian space, or in current terminology weakly symmetric space. These are defined as Riemannian manifolds *M* with a transitive connected Lie group of isometries *G* and an isometry σ normalizing *G* such that given *x*, *y* in *M* there is an isometry *s* in *G* such that $sx = \sigma y$ and $sy = \sigma x$. (Selberg's assumption that s^2 should be an element of *G* was later shown to be unnecessary by Ernest Vinberg.) Selberg proved that weakly symmetric spaces give rise to Gelfand pairs, so that in particular the unitary representation of *G* on $L^2(M)$ is multiplicity free.

Selberg's definition can also be phrased equivalently in terms of a generalization of geodesic symmetry. It is required that for every point *x* in *M* and tangent vector *X* at *x*, there is an isometry *s* of *M*, depending on *x* and *X*, such that

- *s* fixes *x*;
- the derivative of *s* at *x* sends *X* to –*X*.

When *s* is independent of *X*, *M* is a symmetric space.

3.6.14 Special Cases

Symmetric Spaces and Holonomy:

If the identity component of the holonomy group of a Riemannian manifold at a point acts irreducibly on the tangent space, then either the manifold is a locally Riemannian symmetric space, or it is in one of 7 families.

Hermitian Symmetric Spaces

A Riemannian symmetric space which is additionally equipped with a parallel complex structure compatible with the Riemannian metric is called a Hermitian symmetric space. Some examples are complex vector spaces and complex projective spaces, both with their usual Riemannian metric, and the complex unit balls with suitable metrics so that they become complete and Riemannian symmetric.

An irreducible symmetric space G/K is Hermitian if and only if K contains a central circle. A quarter turn by this circle acts as multiplication by i on the tangent space at the identity coset. Thus the Hermitian symmetric spaces are easily read off of the classification. In both the compact and the non-compact cases it turns out that there are four infinite series, namely AIII, BDI with $p=2$, DIII and CI, and two exceptional spaces, namely EIII and EVII. The non-compact Hermitian symmetric spaces can be realized as bounded symmetric domains in complex vector spaces.

Quaternion-Kähler Symmetric Spaces

A Riemannian symmetric space which is additionally equipped with a parallel sub bundle of End(TM) isomorphic to the imaginary quaternions at each point, and compatible with the Riemannian metric, is called Quaternion-Kähler symmetric space.

An irreducible symmetric space G/K is quaternion-Kähler if and only if isotropy representation of K contains an Sp(1) summand acting like the unit quaternions on a quaternionic vector space. Thus the quaternion-Kähler symmetric spaces are easily read off from the classification. In both the compact and the non-compact cases it turns out that there is exactly one for each complex simple Lie group, namely AI with $p = 2$ or $q = 2$ (these are isomorphic), BDI with $p = 4$ or $q = 4$, CII with $p = 1$ or $q = 1$, EII, EVI, EIX, FI and G.

Chapter 4

Representation Theory of Lie Groups

This chapter is about representation theory of Lie groups which describes abstract groups in terms of linear transformation of vector spaces and before we go into the details we have to give some related definitions.

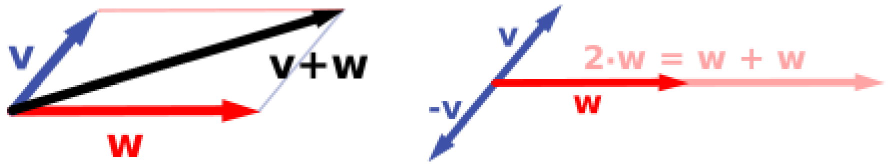
4.1.1 Definition : Vector Space

A vector space is a mathematical structure formed by a collection of elements called vectors, which may be added together and multiplied ("scaled") by numbers, called *scalars*. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called axioms. The concept of vector space will first be explained by describing two particular examples:

4.1.2 First Example: Arrows in the Plane

The first example of a vector space consists of arrows in a fixed plane, starting at one fixed point. This is used in physics to describe forces or velocities. Given any two such arrows, **v** and **w**, the parallelogram spanned by these two arrows contains one diagonal arrow that starts at the origin, too. This new arrow is called the *sum* of the two arrows and is denoted v + w. Another operation that can be done with arrows is scaling: given any positive real number a, the arrow that has the same direction as **v**, but is dilated or shrunk by multiplying its length by a, is called *multiplication* of **v** by a. It is denoted av. When a is negative, av is defined as the arrow pointing in the opposite direction, instead.

The following shows a few examples: if $a = 2$, the resulting vector aw has the same direction as **w**, but is stretched to the double length of **w** (right image below). Equivalently 2**w** is the sum w + w. Moreover, (−1)v = −v has the opposite direction and the same length as **v** (blue vector pointing down in the right image).



4.1.3 Second Example: Ordered Pairs of Numbers

A second key example of a vector space is provided by pairs of real numbers x and y. (The order of the components x and y is significant, so such a pair is also called an ordered pair.) Such a pair is written as (x, y). The sum of two such pairs and multiplication of a pair with a number is defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$a\,(x, y) = (ax, ay).$$

A vector space over a field F is a set V together with two operations that satisfy the eight axioms listed below. Elements of V are called *vectors*. Elements of F are called *scalars*. In the two examples above, our set consists of the planar arrows with fixed starting point and of pairs of real numbers, respectively, while our field is the real numbers. The first operation, *vector addition*, takes any two vectors **v** and **w** and assigns to them a third vector which is commonly written as **v** + **w**, and called the sum of these two vectors. The second operation takes any scalar a and any vector **v** and gives another vector a**v**. In view of the first example, where the multiplication is done by rescaling the vector **v** by a scalar a, the multiplication is called *scalar multiplication* of **v** by a.

To qualify as a vector space, the set V and the operations of addition and multiplication must adhere to a number of requirements called axioms. In the list below, let **u**, **v** and **w** be arbitrary vectors in V, and a and b scalars in F.

Axiom	**Meaning**
Associativity of addition	**u** + (**v** + **w**) = (**u** + **v**) + **w**
Commutatively of addition	**u** + **v** = **v** + **u**

Identity element of addition	There exists an element **0** ∈ V, called the *zero vector*, such that **v** + **0** = **v** for all **v** ∈ V.
Inverse elements of addition	For every **v** ∈ V, there exists an element −**v** ∈ V, called the *additive inverse* of **v**, such that **v** + (−**v**) = **0**.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$
Identity element of scalar multiplication	1**v** = **v**, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

These axioms generalize properties of the vectors introduced in the above examples. Indeed, the result of addition of two ordered pairs (as in the second example above) does not depend on the order of the summands:

$$(x_v, y_v) + (x_w, y_w) = (x_w, y_w) + (x_v, y_v).$$

Likewise, in the geometric example of vectors as arrows, **v** + **w** = **w** + **v**, since the parallelogram defining the sum of the vectors is independent of the order of the vectors. All other axioms can be checked in a similar manner in both examples. Thus, by disregarding the concrete nature of the particular type of vectors, the definition incorporates these two and many more examples in one notion of vector space.

Subtraction of two vectors and division by a (non-zero) scalar can be defined as

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}),$$

$$\mathbf{v}/a = (1/a)\mathbf{v}.$$

When the scalar field *F* is the real numbers **R**, the vector space is called a *real vector space*. When the scalar field is the complex numbers, it is called a *complex vector space*. These two cases are the ones used most often in engineering. The most general definition of a vector space allows scalars to be elements of any fixed field *F*. The notion is then known as an *F-vector spaces* or a *vector space over F*. A field is, essentially, a set of numbers possessing addition, subtraction, multiplication and division operations. For example, rational numbers also form a field.

In contrast to the intuition stemming from vectors in the plane and higher-dimensional cases, there is, in general vector spaces, no notion of nearness, angles or distances. To deal with such matters, particular types of vector spaces are introduced;.

4.2.1 Definition: Linear Transformation

A linear map (also called a linear mapping, linear transformation or, in some contexts, linear function) is a mapping $V \mapsto W$ between two modules (including vector spaces) that preserves (in the sense defined below) the operations of addition and scalar multiplication. An important special case is when $V = W$, in which case the map is called a **linear operator**, or an endomorphism of V. Sometimes the definition of a linear function coincides with that of a linear map, while in analytic geometry it does not.

A linear map always maps linear subspaces to linear subspaces (possibly of a lower dimension); for instance it maps a plane through the origin to a plane, straight line or point.

In the language of abstract algebra, a linear map is a homomorphism of modules. In the language of category theory it is a morphism in the category of modules over a given ring. Let *V* and *W* be vector spaces over the same field *K*. A function f: $V \to W$ is said to be a *linear map* if for any two vectors **x** and **y** in *V* and any scalar α in *K*, the following two conditions are satisfied:

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$ additivity

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$$ homogeneity of degree 1

This is equivalent to requiring the same for any linear combination of vectors, i.e. that for any vectors $x_1, ..., x_m \in V$ and scalars $a_1, ..., a_m \in K$, the following equality holds:

$$f(a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m) = a_1 f(\mathbf{x}_1) + \cdots + a_m f(\mathbf{x}_m).$$

Denoting the zero elements of the vector spaces V and W by $\mathbf{0}_V$ and $\mathbf{0}_W$ respectively, it follows that $f(\mathbf{0}_V) = \mathbf{0}_W$ because letting $\alpha = 0$ in the equation for homogeneity of degree 1,

$$f(\mathbf{0}_V) = f(0 \cdot \mathbf{0}_V) = 0 \cdot f(\mathbf{0}_V) = \mathbf{0}_W.$$

Occasionally, V and W can be considered to be vector spaces over different fields. It is then necessary to specify which of these ground fields is being used in the definition of "linear". If V and W are considered as spaces over the field K as above, we talk about K-linear maps. For example, the conjugation of complex numbers is an **R**-linear map **C** → **C**, but it is not **C**-linear.

A linear map from V to K (with K viewed as a vector space over itself) is called a linear functional.

These statements generalize to any left-module ${}_RM$ over a ring R without modification.

4.2.2 Examples

- The zero map is always linear.
- The identity map of any vector space is a linear operator.
- Any homothecy centered in the origin of a vector space, $v \mapsto cv$ where c is a scalar, is a linear operator.
- For real numbers, the map $x \mapsto x^2$ is not linear.
- For real numbers, the map $x \mapsto x+1$ is not linear (but is an affine transformation; $y = x+1$ is a linear equation, as used in analytic geometry.)

- If A is a real $m \times n$ matrix, then A defines a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$ by sending the column vector $\mathbf{x} \in \mathbf{R}^n$ to the column vector $A\mathbf{x} \in \mathbf{R}^m$. Conversely, any linear map between finite-dimensional vector spaces can be represented in this manner.
- Differentiation defines a linear map from the space of all differentiable functions to the space of all functions. It also defines a linear operator on the space of all smooth functions.
- The (definite) integral over some interval I is a linear map from the space of all real-valued integrable functions on I to **R.**
- The (indefinite) integral (or anti derivative) with a fixed starting point defines a linear map from the space of all real-valued integrable functions on **R** to the space of all real-valued functions on **R**. Without fixed starting point it does not define a mapping at all, as the presence of a constant of integration in the result means it produces an infinite number of outputs for a single input.
- If V and W are finite-dimensional vector spaces over a field F, then functions that send linear maps $f : V \rightarrow W$ to $\dim_F(W) \times \dim_F(V)$ matrices in the way described in the sequel are themselves linear maps (indeed linear isomorphisms).
- The expected value of a random variable (which is in fact a function, and as such member of a vector space) is linear, as for random variables X and Y we have $E[X + Y] = E[X] + E[Y]$ and $E[aX] = aE[X]$, but the variance of a random variable is not linear.

4.2.3 Matrices

If V and W are finite-dimensional vector spaces and a basis is defined for each vector space, then every linear map from V to W can be represented by a matrix. This is useful because it allows concrete calculations. Matrices yield examples of linear maps: if A is a real $m \times n$ matrix, then $f(\mathbf{x}) = A\mathbf{x}$ describes a linear map $\mathbf{R}^n \rightarrow \mathbf{R}^m$.

Let $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a basis for V. Then every vector $\mathbf{v}$ in V is uniquely determined by the coefficients $c_1, ..., c_n$ in the field $\mathbf{R}$:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

If $f : V \to W$ is a linear map,

$$f(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1 f(\mathbf{v}_1) + \cdots + c_n f(\mathbf{v}_n),$$

which implies that the function f is entirely determined by the vectors $f(\mathbf{v}_1), ..., f(\mathbf{v}_n)$. Now let $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be a basis for W. Then we can represent each vector $f(\mathbf{v}_j)$ as

$$f(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + \cdots + a_{mj}\mathbf{w}_m.$$

Thus, the function f is entirely determined by the values of a_{ij}. If we put these values into an $m \times n$ matrix M, then we can conveniently use it to compute the vector output of f for any vector in V. To get M, every column j of M is a vector whose coordinates are

$$(a_{1j}, ..., a_{mj})^T$$

corresponding to $f(\mathbf{v}_j)$ as defined above. To define it more clearly, for some column j that corresponds to the mapping $f(\mathbf{v}_j)$,

$$\mathbf{M} = \begin{pmatrix} & a_{1j} & \\ & . & \\ * & . & * \\ & . & \\ & a_{mj} & \end{pmatrix}$$

where $\mathbf{M}$ is the matrix of f. The symbol * denotes that there are other columns which together with column j make up a total of n columns of M. In other words, every column j = 1,...,n has a corresponding mapping $f(\mathbf{v}_j)$ whose coefficients $a_{1j},...,a_{mj}$ are the elements of column j, and $\mathbf{v}_j$ is a basis vector of the vector space V. It can be shown that M has full column rank. A single linear map may be represented by many matrices. This is because the values of the elements of a matrix depend on the bases chosen.

4.2.4 Examples of linear transformation matrices

In two-dimensional space $\mathbf{R}^2$ linear maps are described by 2×2 real matrices. Here are some examples:

- rotation by 90 degrees counterclockwise:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- rotation by angle θ counterclockwise:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- reflection against the x axis:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- reflection against the y axis:

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- scaling by 2 in all directions:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

- horizontal shear mapping:

$$\mathbf{A} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

- squeeze mapping:

$$\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$$

- projection onto the y axis:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

4.2.5 Forming New Linear Maps from Given Ones

The composition of linear maps is linear: if $f : V \rightarrow W$ and $g : W \rightarrow Z$ are linear, then so is their composition $g \circ f : V \rightarrow Z$. It follows from this that the class of all vector spaces over a given field K, together with K-linear maps as morphisms, forms a category.

The inverse of a linear map, when defined, is again a linear map.

If $f_1 : V \to W$ and $f_2 : V \to W$ are linear, then so is their sum $f_1 + f_2$ (which is defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$).

If $f : V \to W$ is linear and a is an element of the ground field K, then the map af, defined by $(af)(x) = a(f(x))$, is also linear.

Thus the set $L(V, W)$ of linear maps from V to W itself forms a vector space over K, sometimes denoted Hom(V, W). Furthermore, in the case that $V = W$, this vector space (denoted End(V)) is an associative algebra under composition of maps, since the composition of two linear maps is again a linear map, and the composition of maps is always associative. This case is discussed in more detail below.

Given again the finite-dimensional case, if bases have been chosen, then the composition of linear maps corresponds to the matrix multiplication, the addition of linear maps corresponds to the matrix addition, and the multiplication of linear maps with scalars corresponds to the multiplication of matrices with scalars.

4.2.6 Endomorphisms and Automorphisms

A linear transformation f: $V \to V$ is an endomorphism of V; the set of all such endomorphism End(V) together with addition, composition and scalar multiplication as defined above forms an associative algebra with identity element over the field K (and in particular a ring). The multiplicative identity element of this algebra is the identity map id: $V \to V$.

An endomorphism of V that is also an isomorphism is called an automorphism of V. The composition of two automorphisms is again an automorphism, and the set of all automorphisms of V forms a group, the automorphism group of V which is denoted by Aut(V) or GL(V). Since the automorphisms are precisely those endomorphisms which possess inverses under composition, Aut(*V*) is the group of units in the ring End(*V*).

If *V* has finite dimension *n*, then End(*V*) is isomorphic to the associative algebra of all $n \times n$ matrices with entries in *K*. The automorphism group of *V* is isomorphic to the general linear group GL(*n*, *K*) of all $n \times n$ invertible matrices with entries in *K*.

4.2.7 Kernel, Image and the Rank–Nullity Theorem

If $f : V \to W$ is linear, we define the kernel and the image or range of *f* by

$$\ker(f) = \{\, x \in V : f(x) = 0 \,\}$$

$$\operatorname{im}(f) = \{\, w \in W : w = f(x), x \in V \,\}$$

ker(*f*) is a subspace of *V* and im(*f*) is a subspace of *W*. The following dimension formula is known as the rank–nullity theorem:

$$\dim(\ker(f)) + \dim(\operatorname{im}(f)) = \dim(V).$$

The number dim(im(*f*)) is also called the *rank of f* and written as rank(*f*), or sometimes, ρ(*f*); the number dim(ker(*f*)) is called the *nullity of f* and written as null(*f*) or ν(*f*). If *V* and *W* are finite-dimensional, bases have been chosen and *f* is represented by the matrix *A*, then the rank and nullity of *f* are equal to the rank and nullity of the matrix *A*, respectively.

4.2.8 Cokernel

A subtler invariant of a linear transformation is the *co*kernel, which is defined as

$$\operatorname{coker} f := W/f(V) = W/\operatorname{im}(f).$$

This is the *dual* notion to the kernel: just as the kernel is a *sub*space of the *domain,* the co-kernel is a *quotient* space of the *target*. Formally, one has the exact sequence

$$0 \to \ker f \to V \to W \to \operatorname{coker} f \to 0.$$

These can be interpreted thus: given a linear equation $f(\mathbf{v}) = \mathbf{w}$ to solve,

- the kernel is the space of *solutions* to the *homogeneous* equation $f(\mathbf{v}) = 0$, and its dimension is the number of *degrees of freedom* in a solution, if it exists;
- the co-kernel is the space of *constraints* that must be satisfied if the equation is to have a solution, and its dimension is the number of constraints that must be satisfied for the equation to have a solution.

The dimension of the co-kernel and the dimension of the image (the rank) add up to the dimension of the target space. For finite dimensions, this means that the dimension of the quotient space $W/f(V)$ is the dimension of the target space minus the dimension of the image.

As a simple example, consider the map f: $\mathbf{R}^2 \to \mathbf{R}^2$, given by $f(x, y) = (0, y)$. Then for an equation $f(x, y) = (a, b)$ to have a solution, we must have $a = 0$ (one constraint), and in that case the solution space is (x, b) or equivalently stated, $(0, b) + (x, 0)$, (one degree of freedom). The kernel may be expressed as the subspace $(x, 0) < V$: the value of x is the freedom in a solution – while the cokernel may be expressed via the map $W \to \mathbf{R}$, $(a,b) \mapsto (a)$:given a vector (a, b), the value of a is the *obstruction* to there being a solution.

An example illustrating the infinite-dimensional case is afforded by the map f: $\mathbf{R}^\infty \to \mathbf{R}^\infty$, $\{a_n\} \mapsto \{b_n\}$with $b_1 = 0$ and $b_{n+1} = a_n$ for $n > 0$. Its image consists of all sequences with first element 0, and thus its cokernel consists of the classes of sequences with identical first element. Thus, whereas its kernel has dimension 0 (it maps only the zero sequence to the zero sequence), its co-kernel has dimension 1. Since the domain and the target space are the same, the rank and the dimension of the kernel add up to the same sum as the rank and the dimension of the co-kernel ($\aleph_0 + 0 = \aleph_0 + 1$), but in the infinite-dimensional case it cannot be inferred that the kernel and the co-kernel of an endomorphism have the same dimension ($0 \neq 1$). The reverse situation obtains for the map h: $\mathbf{R}^\infty \to \mathbf{R}^\infty$, $\{a_n\} \mapsto \{c_n\}$with $c_n = a_{n+1}$. Its image is the entire target space, and hence its co-kernel has dimension 0, but since it maps all sequences in which only the first element is non-zero to the zero sequence, its kernel has dimension1.

4.2.9 Index

For a linear operator with finite-dimensional kernel and co-kernel, one may define *index* as:

$$\operatorname{ind} f := \dim \ker f - \dim \operatorname{coker} f,$$

namely the degrees of freedom minus the number of constraints.

For a transformation between finite-dimensional vector spaces, this is just the difference dim(*V*) − dim(*W*), by rank–nullity. This gives an indication of how many solutions or how many constraints one has: if mapping from a larger space to a smaller one, the map may be onto, and thus will have degrees of freedom even without constraints. Conversely, if mapping from a smaller space to a larger one, the map cannot be onto, and thus one will have constraints even without degrees of freedom.

The index comes of its own in infinite dimensions: it is how homology is defined, which is a central theory in algebra and algebraic topology; the index of an operator is precisely the Euler characteristic of the 2-term complex 0 → *V* → *W* → 0. In operator theory, the index of Fredholm operators is an object of study, with a major result being the Atiyah–Singer index theorem

4.3.1 Group Representation

group representations describe abstract groups in terms of linear transformations of vector spaces; in particular, they can be used to represent group elements as matrices so that the group operation can be represented by matrix multiplication. The term *representation of a group* is also used in a more general sense as the study of the concrete ways in which abstract groups can be realized as groups of rigid transformations of R^n (or C^n).

The quintessential example might be the symmetry group of a square. A symmetry of the square is any rigid motion of Euclidean space which preserves the square. For example, we can rotate the square counterclockwise by a quarter turn (90 degrees); this is one symmetry of the square. Another is flipping the square over its horizontal axis. Of course, doing nothing" is the identity symmetry of the square Of course, any two symmetries can be composed to produce a third. For example, rotation though a quarter-turn followed by reflection over the horizontal axis has the same effect on the square as reflection over one of the diagonals. Also, every symmetry has an opposite" or inverse symmetry. For example, rotation counterclockwise by ninety degrees can be undone by rotation clockwise by ninety degrees so that these two rotations are inverse to each other. Each reflection is its own inverse.

The full set of symmetries of the square forms a group: a set with natural notion of composition of any pair of elements, such that every element has an inverse. This group is represented as a set of rigid transformations of the vector space R^2. It turns out that, despite the abstract definition, every group can be thought of concretely a group of symmetries of some set, usually in many different ways. The goal of representation theory is to understand the different ways in which abstract groups can be realized as transformation groups.

In practice, we are mainly interested in understanding how groups can be represented as groups of linear transformations of Euclidean space. More formally, a "representation" means a homomorphism from the group to the automorphism group of an object. If the object is a vector space we have a *linear representation*.

4.3.2 Branches of Group Representation Theory

The representation theory of groups divides into sub theories depending on the kind of group being represented. The various theories are quite different in detail, though some basic definitions and concepts are similar. The most important divisions are:

- *Finite groups* — Group representations are a very important tool in the study of finite groups. They also arise in the applications of finite group theory to crystallography and to geometry. If the field of scalars of the vector space has characteristic *p*, and if *p* divides the order of the group, then this is called *modular representation theory*; this special case has very different properties..

- *Compact groups* or *locally compact groups* — Many of the results of finite group representation theory are proved by averaging over the group. These proofs can be carried over to infinite groups by replacement of the average with an integral, provided that an acceptable notion of integral can be defined. This can be done for locally compact groups, using Haar measure. The resulting theory is a central part of harmonic analysis. The Pontryagin duality describes the theory for commutative groups, as a generalised Fourier transform..
- *Lie groups* — Many important Lie groups are compact, so the results of compact representation theory apply to them. Other techniques specific to Lie groups are used as well. Most of the groups important in physics and chemistry are Lie groups, and their representation theory is crucial to the application of group theory in those fields. *Linear algebraic groups* (or more generally *affine group schemes*) — These are the analogues of Lie groups, but over more general fields than just **R** or **C**. Although linear algebraic groups have a classification that is very similar to that of Lie groups, and give rise to the same families of Lie algebras, their representations are rather different (and much less well understood). The analytic techniques used for studying Lie groups must be replaced by techniques from algebraic geometry, where the relatively weak Zariski topology causes many technical complications.
- *Non-compact topological groups* — The class of non-compact groups is too broad to construct any general representation theory, but specific special cases have been studied, sometimes using ad hoc techniques. The *semi simple Lie groups* have a deep theory,

building on the compact case. The complementary *solvable* Lie groups cannot in the same way be classified. The general theory for Lie groups deals with semi direct products of the two types, by means of general results called *Mackey theory*, which is a generalization of Wigner's classification methods.

Representation theory also depends heavily on the type of vector space on which the group acts. One distinguishes between finite-dimensional representations and infinite-dimensional ones. In the infinite-dimensional case, additional structures are important (e.g. whether or not the space is a Hilbert space, Banach space, etc.).

One must also consider the type of field over which the vector space is defined. The most important case is the field of complex numbers. The other important cases are the field of real numbers, finite fields, and fields of p-adic numbers. In general, algebraically closed fields are easier to handle than non-algebraically closed ones. The characteristic of the field is also significant; many theorems for finite groups depend on the characteristic of the field not dividing the order of the group.

4.3.3 Definitions

A **representation** of a group G on a vector space V over a field K is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map

$$\rho\colon G \to GL(V)$$

such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \qquad \text{for all } g_1, g_2 \in G.$$

Here V is called the **representation space** and the dimension of V is called the **dimension** of the representation. It is common practice to refer to V itself as the representation when the homomorphism is clear from the context.

In the case where V is of finite dimension n it is common to choose a basis (a **basis** is a set of linearly independent vectors that, in a linear combination, can represent every vector in a given vector space or free module, or, more simply put, which define a "coordinate system" (as long as the basis is given a definite order). In more general terms, a basis is a linearly independent spanning set). for V and identify GL(V) with GL(n, K), the group of n-by-n invertible matrices on the field K.

- If G is a topological group and V is a topological vector space, a **continuous representation** of G on V is a representation ρ such that the application $\Phi : G \times V \to V$ defined by $\Phi(g, v) = \rho(g)(v)$ is continuous.
- The **kernel** of a representation ρ of a group G is defined as the normal subgroup of G whose image under ρ is the identity transformation:

$$\ker \rho = \{g \in G \mid \rho(g) = \mathrm{id}\}.$$

A faithful representation is one in which the homomorphism $G \to$ GL(V) is injective; in other words, one whose kernel is the trivial subgroup $\{e\}$ consisting of just the group's identity element.

- Given two K vector spaces V and W, two representations $\rho : G \to$ GL(V) and $\pi : G \to$ GL(W) are said to be **equivalent** or **isomorphic** if there exists a vector space isomorphism $\alpha : V \to W$ so that for all g in G

$$\alpha \circ \rho(g) \circ \alpha^{-1} = \pi(g).$$

4.3.4 Example

\` Consider the complex number $u = e^{2\pi i / 3}$ which has the property $u^3 = 1$. The cyclic group $C_3 = \{1, u, u^2\}$ has a representation ρ on $\mathbf{C}^2$ given by:

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \qquad \rho\left(u^2\right) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}.$$

This representation is faithful because ρ is a one-to-one map.

An isomorphic representation for C_3 is

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho\left(u^2\right) = \begin{bmatrix} u^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The group C_3 may also be faithfully represented on $\mathbf{R}^2$ by

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(u) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \rho\left(u^2\right) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Where

$$a = \mathrm{Re}(u) = -\tfrac{1}{2}, \qquad b = \mathrm{Im}(u) = \tfrac{\sqrt{3}}{2}.$$

4.3.5 Irreducible Representation of a Group

A subspace *W* of *V* that is invariant under the group action is called a *sub representation*. If *V* has exactly tow subrepresentations, namely the zero-dimensional subspace and *V* itself, then the representation is said to be *irreducible*; if it has a proper subrepresentation of nonzero dimension, the representation is said to be *reducible*. The representation of dimension zero is considered to be neither reducible nor irreducible, just like the number 1 is considered to be neither composite nor prime.

Under the assumption that the characteristic of the field *K* does not divide the size of the group, representations of finite groups can be decomposed into a direct sum of irreducible subrepresentations (Maschke's theorem). This holds in particular for any representation of a finite group over the complex numbers, since the characteristic of the complex numbers is zero, which never divides the size of a group.

In the example above, the first two representations given are both decomposable into two 1-dimensional sub representations (given by span{(1,0)} and span{(0,1)}), while the third representation is irreducible.

4.3.6 Induced Representation

The induced representation is one of the major general operations for passing from a representation of a subgroup *H* to a representation of the (whole) group *G* itself. It was initially defined as a construction by Frobenius, for linear representations of finite groups. It includes as special cases the action of *G* on the cosets G/H by permutation, which is the case of the induced representation starting with the trivial one-dimensional representation of H. If $H = \{e\}$ this becomes the regular representation of G. Therefore induced representations are rich objects, in the sense that they include or detect many interesting representations. The idea is by no means limited to the case of finite groups, but the theory in that case is particularly well-behaved

4.3.7 Examples

For any group, the induced representation of the trivial representation of the trivial subgroup is the right regular representation. More generally the induced representation of the trivial representation of any subgroup is the permutation representation on the cosets of that subgroup.

An induced representation of a one dimensional representation monomial matrices. Some groups have the property that all of their irreducible representations are monomial, the so-called monomial groupsis called a monomial representation.

4.4.1 Conjugacy Class

The elements of any group may be partitioned into **conjugacy classes**; members of the same conjugacy class share many properties, and study of conjugacy classes of non-abelian groups reveals many important features of their structure. In all abelian groups every conjugacy class is a set containing one element (singleton set).

Functions that are constant for members of the same conjugacy class are called class functions

4.4.2 Definition

Suppose G is a group. Two elements a and b of G are called **conjugate** if there exists an element g in G with

$$gag^{-1} = b.$$

(In linear algebra, this is referred to as matrix similarity.)

It can be readily shown that conjugacy is an equivalence relation and therefore partitions G into equivalence classes. (This means that every element of the group belongs to precisely one conjugacy class, and the classes Cl(a) and Cl(b) are equal if and only if a and b are conjugate, and disjoint otherwise.) The equivalence class that contains the element a in G is

$$\mathrm{Cl}(a) = \{ g \in G \text{:there exists } x \in G \text{ with } g=xax^{-1} \}$$

and is called the **conjugacy class** of a. The **class number** of G is the number of distinct (nonequivalent) conjugacy classes. All elements belonging to the same conjugacy class have the same order.

Conjugacy classes may be referred to by describing them, or more briefly by abbreviations such as "6A", meaning "a certain conjugacy class of order 6 elements", and "6B" would be a different conjugacy class of order 6 elements; the conjugacy class 1A is the conjugacy class of the identity. In some cases, conjugacy classes can be described in a uniform way – for example, in the symmetric group they can be described by cycle structure.

4.4.3 Examples

The symmetric group S_3, consisting of all 6 permutations of three elements, has three conjugacy classes:

- no change (abc → abc)
- interchanging two (abc → acb, abc → bac, abc → cba)
- a cyclic permutation of all three (abc → bca, abc → cab)

The symmetric group S_4, consisting of all 24 permutations of four elements, has five conjugacy classes, listed with their cycle structures and orders:

- $(1)_4$: no change (1 element: { {1, 2, 3, 4} })
- (2): interchanging two (6 elements: { {1, 2, 4, 3}, {1, 4, 3, 2}, {1, 3, 2, 4}, {4, 2, 3, 1}, {3, 2, 1, 4}, {2, 1, 3, 4} })
- (3): a cyclic permutation of three (8 elements: { {1, 3, 4, 2}, {1, 4, 2, 3}, {3, 2, 4, 1}, {4, 2, 1, 3}, {4, 1, 3, 2}, {2, 4, 3, 1}, {3, 1, 2, 4}, {2, 3, 1, 4} })
- (4): a cyclic permutation of all four (6 elements: { {2, 3, 4, 1}, {2, 4, 1, 3}, {3, 1, 4, 2}, {3, 4, 2, 1}, {4, 1, 2, 3}, {4, 3, 1, 2} })

- (2)(2): interchanging two, and also the other two (3 elements: { {2, 1, 4, 3}, {4, 3, 2, 1}, {3, 4, 1, 2} })

In general, the number of conjugacy classes in the symmetric group S_n is equal to the number of integer partitions of *n*. This is because each conjugacy class corresponds to exactly one partition of {1, 2, ..., *n*} into cycles, up to permutation of the elements of {1, 2, ..., *n*}..

4.4.4 Properties

- The identity element is always in its own class, that is Cl(*e*) = {*e*}
- If *G* is abelian, then $gag^{-1} = a$ for all *a* and *g* in *G*; so Cl(*a*) = {*a*} for all *a* in *G*.
- If two elements *a* and *b* of *G* belong to the same conjugacy class (i.e., if they are conjugate), then they have the same order. More generally, every statement about *a* can be translated into a statement about $b=gag^{-1}$, because the map $\varphi(x) = gxg^{-1}$ is an automorphism of *G*.
- An element *a* of *G* lies in the center Z(*G*) of *G* if and only if its conjugacy class has only one element, *a* itself. More generally, if $C_G(a)$ denotes the *centralizer* of *a* in *G*, i.e., the subgroup consisting of all elements *g* such that *ga* = *ag*, then the index $[G : C_G(a)]$ is equal to the number of elements in the conjugacy class of *a* (by the orbit-stabilizer theorem).
- If *a* and *b* are conjugate, then so are their powers a^k and b^k. (Proof: if $a = gbg^{-1}$, then $a^k = (gbg^{-1})(gbg^{-1})...(gbg^{-1}) = gb^kg^{-1}$.) Thus taking *k*th powers gives a map on conjugacy classes, and one may consider which conjugacy classes are in its preimage. For example, in the symmetric group, the square of an element of type (3)(2) (a 3-cycle and a 2-cycle) is an element of type (3), therefore one of the power-up classes of (3) is the class (3)(2); the class (6) is another.

4.4.5 Conjugacy as Group Action

If we define

$g . x = gxg^{-1}$

for any two elements g and x in G, then we have a group action of G on G. The orbits of this action are the conjugacy classes, and the stabilizer of a given element is the element's centralizer.

Similarly, we can define a group action of G on the set of all subsets of G, by writing

$g . S = gSg^{-1}$,

or on the set of the subgroups of G.

Theorem:4.1

The number of irreducible representation of the symmetric group s_n is the number of conjugacy classes (the number of partition of s .

4.5.1 Representation of a Lie Group

The idea of a **representation of a** Lie group plays an important role in the study of continuous symmetry. A great deal is known about such representations.

4.5.2 Representations on a Complex Finite-Dimensional Vector Space

Let us first discuss representations acting on finite-dimensional complex vector spaces. A representation of a Lie group G on a finite-dimensional complex vector space V is a smooth group homomorphism $\Psi:G\to\text{Aut}(V)$ from G to the automorphism group of V.

For n-dimensional V, the automorphism group of V is identified with a subset of the complex square matrices of order n. The automorphism group of V is given the structure of a smooth manifold using this identification. The condition that Ψ is smooth, in the definition above, means that Ψ is a smooth map from the smooth manifold G to the smooth manifold Aut(V).

If a basis for the complex vector space *V* is chosen, the representation can be expressed as a homomorphism into GL(*n*,C). This is known as a *matrix representation*.

4.5.3 Representations on a Finite-Dimensional Vector Space over an Arbitrary Field

A representation of a Lie group *G* on a vector space *V* (over a field *K*) is a smooth (i.e. respecting the differential structure) group homomorphism $G \to \mathrm{Aut}(V)$ from *G* to the automorphism group of *V*. If a basis for the vector space *V* is chosen, the representation can be expressed as a homomorphism into GL(*n*,*K*). This is known as a *matrix representation*. Two representations of *G* on vector spaces *V*, *W* are *equivalent* if they have the same matrix representations with respect to some choices of bases for *V* and *W*.

On the Lie algebra level, there is a corresponding linear mapping from the Lie algebra of G to End(*V*) preserving the Lie bracket [,].

If the homomorphism is in fact a monomorphism, the representation is said to be *faithful*.

A unitary representation is defined in the same way, except that G maps to unitary matrices; the Lie algebra will then map to skew-hermitian matrices.

If *G* is a compact Lie group, every finite-dimensional representation is equivalent to a unitary one.

4.5.4 Representations on Hilbert Spaces

A representation of a Lie group *G* on a complex Hilbert space *V* is a group homomorphism $\Psi: G \to B(V)$ from *G* to B(*V*), the group of bounded linear operators of *V* which have a bounded inverse, such that the map $G \times V \to V$ given by $(g,v) \to \Psi(g)v$ is continuous.

This definition can handle representations on **infinite-dimensional** Hilbert spaces. Such representations can be found in e.g. quantum mechanics, but also in Fourier analysis as shown in the following example.

Let G=**R**, and let the complex Hilbert space V be L^2(**R**). We define the representation Ψ:**R** → B(L^2(**R**)) by $\Psi(r)\{f(x)\} \rightarrow f(r^{-1}x)$.

4.5.5 Adjoint Representation

In mathematics, the **adjoint representation** (or **adjoint action**) of a Lie group G is a way of representing the elements of the group as linear transformations of the group's Lie algebra, considered as a vector space. For example, in the case where G is the Lie group of invertible matrices of size n, *GL(n)*, the Lie algebra is the vector space of all (not necessarily invertible) n-by-n matrices. So in this case the adjoint representation is the vector space of n-by-n matrices, and any element g in *GL(n)* acts as a linear transformation of this vector space given by conjugation: $x \mapsto gxg^{-1}$.

For any Lie group, this natural representation is obtained by linearizing (i.e. taking the differential of) the action of G on itself by conjugation. The adjoint representation can be defined for linear algebraic groups over arbitrary fields.

4.5.6 Adjoint Representation of a Lie Algebra

One may always pass from a representation of a Lie group G to a representation of its Lie algebra by taking the derivative at the identity.

Taking the derivative of the adjoint map

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$$

gives the **adjoint representation** of the Lie algebra $\mathfrak{g}$:

$$d(\mathrm{Ad})_x : T_x(G) \to T_{Ad(x)}(\mathrm{Aut}(\mathfrak{g}))$$

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}).$$

Here $\mathrm{Der}(\mathfrak{g})$is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$which may be identified with the derivation algebra of $\mathfrak{g}$. The adjoint representation of a Lie algebra is related in a fundamental way to the structure of that algebra. In particular, one can show that

$$\mathrm{ad}_x(y) = [x, y]$$

for all

$$
\begin{aligned}
&x, y \in \mathfrak{g} \\
&\mathrm{ad}_x(y) = d(\mathrm{Ad}_x)_e(y) \\
&\qquad = \lim_{\varepsilon \to 0} \frac{(I+\varepsilon x)y(I+\varepsilon x)^{-1} - y}{\varepsilon} \\
&\qquad = \lim_{\varepsilon \to 0} \frac{(I+\varepsilon x)y(I-\varepsilon x+(\varepsilon x)^2+O(\varepsilon^3)) - y}{\varepsilon} \\
&\qquad = \lim_{\varepsilon \to 0} \frac{((I+\varepsilon x)yI-(I+\varepsilon x)y\varepsilon x+(I+\varepsilon x)y(\varepsilon x)^2+O(\varepsilon^3)) - y}{\varepsilon} \\
&\qquad = \lim_{\varepsilon \to 0} \frac{(IyI+\varepsilon xyI-Iy\varepsilon x-\varepsilon xy\varepsilon x+Iy(\varepsilon x)^2+\varepsilon xy(\varepsilon x)^2+O(\varepsilon^3)) - y}{\varepsilon} \\
&\qquad = \lim_{\varepsilon \to 0} \frac{y+xy\varepsilon-yx\varepsilon-xyx\varepsilon^2+yx^2\varepsilon^2+xyx^2\varepsilon^2+O(\varepsilon^3) - y}{\varepsilon} \\
&\qquad = \lim_{\varepsilon \to 0} xy-yx-xyx\varepsilon+yx^2\varepsilon+xyx^2\varepsilon+O(\varepsilon^2) \\
&\qquad = [x, y]
\end{aligned}
$$

4.5. 7 Examples

- If *G* is abelian of dimension *n*, the adjoint representation of *G* is the trivial *n*-dimensional representation.
- If *G* is a matrix Lie group (i.e. a closed subgroup of $GL(n,\mathbf{C})$), then its Lie algebra is an algebra of $n \times n$ matrices with the commutator for a Lie bracket (i.e. a sub algebra of $\mathfrak{gl}_n(\mathbf{C})$). In this case, the adjoint map is given by $\mathrm{Ad}_g(x) = gxg^{-1}$.
- If *G* is SL(2, R) (real 2×2 matrices with determinant 1), the Lie algebra of *G* consists of real 2×2 matrices with trace 0. The representation is equivalent to that given by the action of *G* by linear substitution on the space of binary (i.e., 2 variable) quadratic forms.

4.6.1 Roots of a Semi Simple Lie Group

If *G* is semi simple, the non-zero weights of the adjoint representation form a root system. To see how this works, consider the case $G = \mathrm{SL}(n, \mathbf{R})$. We can take the group of diagonal matrices $\mathrm{diag}(t_1, ..., t_n)$ as our maximal torus *T*. Conjugation by an element of *T* sends

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & t_1 t_2^{-1} a_{12} & \cdots & t_1 t_n^{-1} a_{1n} \\ t_2 t_1^{-1} a_{21} & a_{22} & \cdots & t_2 t_n^{-1} a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_n t_1^{-1} a_{n1} & t_n t_2^{-1} a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Thus, T acts trivially on the diagonal part of the Lie algebra of G and with eigenvectors $t_i t_j^{-1}$ on the various off-diagonal entries. The roots of G are the weights $\operatorname{diag}(t_1, ..., t_n) \to t_i t_j^{-1}$. This accounts for the standard description of the root system of $G = \mathrm{SL}_n(\mathbf{R})$ as the set of vectors of the form $e_i - e_j$.

4.6.2 Example SL(2, R)

Let us compute the root system for one of the simplest cases of Lie Groups. Let us consider the group SL(2, **R**) of two dimensional matrices with determinant 1. This consists of the set of matrices of the form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with a, b, c, d real and $ad - bc = 1$.

A maximal compact connected abelian Lie subgroup, or maximal torus T, is given by the subset of all matrices of the form

$$\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} = \begin{bmatrix} t_1 & 0 \\ 0 & 1/t_1 \end{bmatrix} = \begin{bmatrix} \exp(\theta) & 0 \\ 0 & \exp(-\theta) \end{bmatrix}$$

with $t_1 t_2 = 1$. The Lie algebra of the maximal torus is the Cartan sub algebra consisting of the matrices

$$\begin{bmatrix} \theta & 0 \\ 0 & -\theta \end{bmatrix} = \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \theta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \theta(e_1 - e_2).$$

If we conjugate an element of SL(2, R) by an element of the maximal torus we obtain

$$\begin{bmatrix} t_1 & 0 \\ 0 & 1/t_1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1/t_1 & 0 \\ 0 & t_1 \end{bmatrix} = \begin{bmatrix} at_1 & bt_1 \\ c/t_1 & d/t_1 \end{bmatrix} \begin{bmatrix} 1/t_1 & 0 \\ 0 & t_1 \end{bmatrix} = \begin{bmatrix} a & bt_1^2 \\ ct_1^{-2} & d \end{bmatrix}$$

The matrices

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

are then 'eigenvectors' of the conjugation operation with eigenvalues $1, 1, t_1^2, t_1^{-2}$. The function Λ which gives $t_1^2$is a multiplicative character, or homomorphism from the group's torus to the underlying field R. The function λ giving θ is a weight of the Lie Algebra with weight space given by the span of the matrices.

It is satisfying to show the multiplicativity of the character and the linearity of the weight. It can further be proved that the differential of Λ can be used to create a weight. It is also educational to consider the case of SL(3, R).

4.7.1 The Connection Between Lie Groups and Lie Algebras

Definitions of groups and subgroups were discussed in early chapters of this thesis, but we need to give a definition of one of the important class of subgroups which is normal subgroup

Definition (Normal subgroup). A subgroup H of a group G is normal, if $ghg^{-1} \in H$ for all g ∈ G and h ∈ H. Equivalently, H is normal, if gH = Hg for all g ∈ G.

Normal subgroups are very useful, because we can naturally induce a group structure on the set of cosets gH, and thus get the quotient group G/H. We now turn to continuous groups, which are equipped with a differentiable structure. These are so called Lie groups

4.7.2 Definition (Real Lie Group, Complex Lie Group).

Let G be a group.

1) If G is also a smooth (real) manifold, and the mappings (a, b) $\longrightarrow$ ab and a $\longrightarrow$ a^{-1} are smooth, G is a real Lie group.

2) If G is also a complex analytic manifold, and the mappings (a,b) $\longrightarrow$ ab and a $\longrightarrow$ a^{-1} are analytic, G is a complex Lie group.

The above definitions basically tell us that Lie groups can be locally parameterized by a number of either real or complex parameters. These parameters describe the elements of a Lie group in a continuous manner, which the multiplication and inversion maps must respect. It is now time to introduce the concept of an (abstract) Lie algebra. A Lie algebra is basically a vector space equipped αwith the "commutator".

4.7.3 Definition (Lie Algebra)

A real (or complex) vector space g is a real (or complex) Lie algebra, if it is equipped with an additional mapping [., .] : g x g $\longrightarrow$ g, which is called the Lie bracket and satisfies the following properties:

1. bilinearity: [αa + βb, c] = α [a, c] + β [b, c] and [c,α a +β b] = α [c, a] + β [c, b] for all a, b, c ∈ g and α, β ∈ F (where F = R or F = C),
2. [a, a] = 0 for all a ∈ g,
3. Jacobi identity: [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 for all a, b, c ∈ g.

Note that given the definition, the Lie bracket is automatically skew-symmetric: [a, b] = -[b, a]. We can naturally introduce the concept of a Lie subalgebra h, which is a vector subspace of g, and is also closed under the Lie bracket ([a, b] ∈ h for all a, b ∈ h).

Before advancing, let us introduce, in one big definition, all the relevant morphisms for our current structures. Morphisms, in the sense of category theory, are structure preserving mappings between objects. The most used morphisms, for example, are morphisms of vector spaces, which are the familiar linear maps. Another concept is that of an isomorphism -a bijective morphism; if two structures are isomorphic, they are, for the purposes of this structure, equivalent, so by knowing the structure related properties of a specific object, we automatically know the properties of all the objects isomorphic to it. We now turn to the definitions, where we just have to identify, which properties should these morphisms preserve in each case.

4.7.4Definition(Group/LieGroup/LieAlgebramorphism/Isomorphism)

1 _ Let $\varphi : G_1 \rightarrow G_2$ be a mapping between two groups. Then φ is a group homomorphism, if φ(ab) = φ(a)φ(b) for all a, b ϵ G_1. If φ is also bijective, it is called a group isomorphism.

2- Let $\varphi : G_1 \rightarrow G_2$ be a mapping between two Lie groups. If φ is smooth (or analytic in the case of a complex Lie group) and a group homomorphism, it is a Lie group homomorphism. If φ is also diffeomorphic (bijective and φ, φ^{-1} both smooth), it is called a Lie group isomorphism.

3- Let $\emptyset : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ be a mapping between two Lie algebras. If ∅ is linear and it preserves the Lie bracket, namely [∅(a), ∅(b)] = ∅([a, b]) for all a, b ∈ $\mathfrak{g}_1$, it is called a Lie algebra morphism. If it is also bijective, it is a Lie algebra isomorphism.

If we have a Lie group G, we automatically get it's algebra g, by taking the tangent space of the identity e (this tangent space is the set of equivalence classes of tangent curves $\gamma : R \rightarrow G$ with γ (0) = e).

If G is a n-dimensional manifold, then the tangent space at any point of the manifold has an induced structure of an n-dimensional vector space. Also, it is possible to construct a mapping $\mathfrak{g} \longrightarrow G$ from the Lie algebra to the group, called the exponential map, with some interesting properties (the construction is by one parameter Lie group morphisms $R \longrightarrow G$ generated by an element in the Lie algebra).

Proposition: If G is a connected Lie group, then the group law can be reconstructed from the Lie bracket in g.

It is important to note that this last result only allows to reconstruct multiplication, while it doesn't give the topology of the Lie group.

4.7.5 Further Connection Between the Lie Group and its Lie Algebra

In this section, we will state a few more results, which will eventually culminate in the fundamental theorems of Lie theory.

One thing to note, is the fact that Lie group morphisms induce Lie algebra morphisms. Namely, if $\emptyset : G1 \longrightarrow G2$, then this mapping induces a mapping $\emptyset_* : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ between tangent spaces, which is linear and respects the Lie bracket. Furthermore, the following two statements holds:

4.7.6 Proposition:

1. If $\emptyset : G_1 \longrightarrow G_2$is a Lie group morphism, then $\exp(\emptyset_*(a)) = \emptyset(\exp(a))$ for all a ϵ G_1.
2. If G1 is connected, then a Lie group morphism $\emptyset : G_1 \longrightarrow G_2$ is uniquely determined by the induced linear mapping $\emptyset_*$ between tangent spaces.

These two statements show that we can work on the level of the Lie algebras and use the linear mapping $\emptyset_*$ between them, instead of working directly with groups and the mapping $\emptyset$.

One thing we don't know, however, is whether we can work with any linear mapping between the Lie algebras (which preserves the Lie bracket), and expect that this represents some Lie group morphism on the group level. To answer this, we now move directly to the fundamental theorems of Lie theory and their consequences .

Theorem:4.2 (Fundamental Theorems of Lie Theory).

1. The subgroups H of a Lie group G and the sub algebras h of the corresponding Lie algebra $\mathfrak{g}$ are in bijection. In particular, we get the subalgebra by taking the tangent space at the identity in a subgroup.
2. If G_1 and G_2 are Lie groups, and G_1 is also connected and simply connected, then we have a bijective correspondence between Lie group morphisms $G_1 \longrightarrow G_2$ and Lie algebra morphisms $\mathfrak{g}1 \longrightarrow \mathfrak{g}2$.

3. Let a be an abstract finite-dimensional Lie algebra (real/complex). Then there exists a Lie group G (real/complex), so that its corresponding Lie algebra $\mathfrak{g}$ is isomorphic to a.

The first statement basically says that subalgebras of $\mathfrak{g}$ correspond to subgroups of G. The second statement answers our question: we can work with Lie algebra morphisms and expect them to also be morphisms on the level of groups, if the first group is connected and simply connected ("simply connected" is another term from topology, and it basically means that every loop in such a space can be contracted to a point; the plane with a point P removed is for example not simply connected, since one cannot contract the loop which goes around the point P). The third theorem is perhaps the most interesting: we can take any abstract Lie algebra (for example we take some real vector space V and define the commutator between (some) basis vectors), and with this automatically get some Lie group G, which locally has the structure of the given Lie algebra. But is this group unique? In other words, does a Lie algebra correspond to exactly one group? The following statement is just a corollary of the fundamental theorems, but it is important enough for us to promote it to a theorem:

Theorem4.3 (Connected Lie Groups of a Given Lie Algebra).

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there exists a unique (up to isomorphism) connected and simply connected Lie group G with $\mathfrak{g}$ as its Lie algebra. If $G^{/}$ is another connected Lie group with this Lie algebra, it is of the form G/Z, where Z is some discrete central subgroup of G.

We now finally have the whole picture between the connection between Lie groups and Lie algebras. Basically, we now know that we can work with Lie algebras, but thus losing (only) the topology of the group. By knowing possible Lie algebras, we know the possible Lie groups by the following line of thought: each Lie algebra g generates a unique connected and simply connected Lie group G. Then, we also have connected groups of the form G/Z, where Z is a discrete central subgroup (central means that it lies in the center of G, which is the set of all elements a, for which ab = ba for all b $\in$ G). We also have disconnected groups with algebra $\mathfrak{g}$, but they are just the previous groups G/Z overlayed with another (unrelated) discrete group structure

4.8.1 Matrix Algebras

We will consider Lie groups and Lie algebras of matrices.

We define the GL(n, F) as the group of all invertible n x n matrices, which have either real (F = R) of complex (F = C) entries. Multiplication in this group is defined by the usual multiplication of matrices (if A and B are invertible, then AB is also invertible, because $(AB)^{-1} = B^{-1}A^{-1}$ (and multiplication is therefore well defined). One can also verify other group properties. The manifold structure is automatic, since it is an open set of all n x n matrices (which form a n^2 dimensional vector space, which is isomorphic to F^{n2}).

We also define $\mathfrak{g}$l(n, F) as the set of ALL matrices of dimension n x n with entries in F. This set is of course a vector space under the usual addition of matrices and scalar multiplication. One can also define the commutator of two such matrices, as [A,B] = AB - BA, and this operation satisfies the requirements for the Lie bracket. The set gl(n, F) therefore has the structure of a Lie algebra.

The notation for $\mathfrak{g}$l(n, F) was suggestive. The matrices $\mathfrak{g}$l(n, F) are the Lie algebra of the Lie group GL(n, F), including the Lie bracket being the common commutator.

Now, by virtue of the first fundamental theorem, we can construct various matrix subgroups of GL(n, F) by taking Lie subalgebras of $\mathfrak{g}$l(n, F), namely subalgebras of n x n matrices. There are a number of important groups and algebras of this type, and they are called the classical groups. We will, for the purposes of future convenience and reference, list them in a large table ,together with the restrictions, by which they were obtained, as well as some other properties. These properties will be their null and first homotopic group, π^0 and π^{-1} (we will not go further into these concepts here, let us just mention that a trivial π^0 means G is connected, and a trivial π^{-1} means G is simply connected; furthermore, for G which are not connected, π^1 is specified for the connected component of the identity).

Another property, which we will also list, is whether a group is compact (as a topological space) and denote this by a C. Finally, dim will be the dimension of the group as a manifold, which is equal to the dimension of the Lie algebra as a vector space. It is easy to check the dimensionality in each case by noting that n x n matrices form a n^2 dimensional space by themselves, but then the dimensionality is gradually reduced by the constraints on its Lie algebra. However, the constraints on the Lie algebra are derived from the constraints on the group. In the orthogonal case for example, we have $e^A(e^A)^T = (e^A)^T e^A$, which implies $e^A e^{(AT\ )} = e^{(A+AT\ )} = I$, and consequently $A + A^T = 0$. The constraint $\det e^A = 1$ can be reduced to the constraint TrA = 0 (one can show this by putting A into its Jordan form).

Table 1: List of important real Lie subgroups and Lie subalgebras of the general linear group.

G		**G**		Dim	Π^0	Π^1	C
GL(n,R)	/	gl(n,)R	/	n^2	Z_2	$Z_2(n\geq3)$	
SL(n,R) ⊆ GL(n,R)	DetA=1	sl(n,R)	Tr A=0	n^2 -1	{1}	$Z_2(n\geq3)$	
SP(n,R) ⊆GL(2n,R)	$A^TJA=J$	sp(n,R)	$JA+A^TJ=0$	n (2 n +1)	{1}	Z	
O(n,R) ⊆GL(n,R)	$AA^T=1$	o(n;R)	$A+A^T=0$	n (n -1)/2	Z_2	$Z_2(n\geq3)$	C

SO(n,R) ⊆ GL(n,R)	DetA=1 , $AA^T=1$	so(n,R)	$A+A^T=0$	n (n -1)/2	{1}	$Z_2(n\geq3)$	C
U(n) ⊆ GL(n,C)	AAy = I	u(n)	A + Ay = 0	n^2	{1}	Z	C
SU(n) ⊆ GL(n,C)	detA = 1 ,AA^y = I	su(n)	TrA =0 ,A + A^y = 0	n^2 -1	{1}	{1}	C

Table 2: List of important complex Lie subgroups and Lie subalgebras of the general linear group.

G		**G**		Dim	Π^0	Π^1	C
GL(n,C)	/	gl(n,)C	/	n^2	{1}	Z	
SL(n,C) ⊆ GL(n,C)	DetA=1	sl(n,C)	Tr A=0	n^2 -1	{1}	{1}	
O(n,C) ⊆GL(n,C)	$AA^T=1$	o(n,C)	$A+A^T=0$	n (n -1)/2	Z_2	Z_2	
SO(n,C) ⊆ GL(n,C)	DetA=1 ,$AA^T=1$	so(n,C)	TrA=0, $A=A^T=0$	n (n -1)/2	{1}	Z_2	

The name of the groups in tables 1 and 2 are the following: GL is the general linear group, SL is the special linear group, Sp is the symplectic group, O is the orthogonal group, SO is the special orthogonal group, U is the unitary group and SU is the special unitary group. In the definition of the symplectic group, we have used the matrix J, which is, in the case of 2n x 2n matrices, written as

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

All this classification and hard work has not been in vain. We expect matrix groups to be very important, because when considering representations of Lie groups and Lie algebras on a vector space V , we will get linear operators, which can be presented as matrices. For the representations, matrices will thus present group and algebra elements. It turns out that this is not the only reason of their importance; as the last theorem of this section, we look at the following result :

Theorem 4.4 (Ado's Theorem).

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over F (either real or complex). Then $\mathfrak{g}$ is isomorphic to some subalgebra of $\mathfrak{g}$ l(n, F) for some n.

This theorem implies that not only do we know practically everything about Lie groups by considering abstract Lie algebras, but it is sufficient to consider just the Lie algebras of matrices, with the Lie bracket being the commutator and the exponential map into the group being the exponential of a matrix. This reduction to such a specific environment as matrices greatly simplify anything we want to do in a general Lie group.

Chapter 5

The Application of The Classification Theory

This chapter is devoted to classification theory and some applicants. Classification theory is one of major themes in model theory supplying uniform points of views for understanding algebraic or analytic mathematical structures in the same class. As model theory is a branch of mathematical logic, logical points of views to the structures are newly provided in addition to their own mathematical natures. The most intensively studied class of structures is stable structures such as modules, differential fields; and such study is called `stability theory'. As is well-known stability theory has strong application impacts towards geometry and number theory. And as usual we begin with some definitions.

5.1.1 Model Theory

model theory is the study of classes of mathematical structures (e.g. groups, fields, graphs, universes of set theory) from the perspective of mathematical logic. The objects of study are models of theories in a formal language. We call a **theory** a set of sentences in a formal language, and **model** of a theory a structure (e.g. an interpretation) that satisfies the sentences of that theory.

Model theory recognizes and is intimately concerned with a duality: It examines semantical elements (meaning and truth) by means of syntactical elements (formulas and proofs) of a corresponding language.

Model theory developed rapidly during the 1990s, and a more modern definition is provided by Wilfred Hodges (1997):

model theory = algebraic geometry − fields, (5.1)

although model theorists are also interested in the study of fields. Other nearby areas of mathematics include combinatorics, number theory, arithmetic dynamics, analytic functions, and non-standard analysis.

5.1.2 Branches of Model Theory

Informally, model theory can be divided into classical model theory, model theory applied to groups and fields, and geometric model theory. A missing subdivision is computable model theory, but this can arguably be viewed as an independent subfield of logic.

Examples of early theorems from classical model theory include Gödel's completeness theorem, the upward and downward Löwenheim–Skolem theorems, Vaught's two-cardinal theorem, Scott's isomorphism theorem, the omitting types theorem, and the Ryll-Nardzewski theorem. Examples of early results from model theory applied to fields are Tarski's elimination of quantifiers for real closed fields, Ax's theorem on pseudo-finite fields, and Robinson's development of non-standard analysis. An important step in the evolution of classical model theory occurred with the birth of stability theory (through Morley's theorem on unaccountably categorical theories and Shelah's classification program), which developed a calculus of independence and rank based on syntactical conditions satisfied by theories.

During the last several decades applied model theory has repeatedly merged with the more pure stability theory. The result of this synthesis is called geometric model theory. An example of a theorem from geometric model theory is Hrushovski's proof of the Mordell–Lang conjecture for function fields. The ambition of geometric model theory is to provide a *geography of mathematics* by embarking on a detailed study of definable sets in various mathematical structures, aided by the substantial tools developed in the study of pure model theory.

5.1.3 Universal Algebra

Fundamental concepts in universal algebra are signatures σ and σ-algebras. Since these concepts are formally defined in the article on structures, the present article can content itself with an informal introduction which consists in examples of how these terms are used.

*The standard signature of rings is $\sigma_{ring} = \{\times,+,-,0,1\}$, where × and + are binary, − is unary, and 0 and 1 are nullary.

The standard signature of semirings is $\sigma_{smr} = \{\times,+,0,1\}$, where the arities are as above.

*The standard signature of groups (with multiplicative notation) is $\sigma_{grp} = \{\times,^{-1},1\}$, where × is binary, $^{-1}$ is unary and 1 is nullary.

*The standard signature of monoids is $\sigma_{mnd} = \{\times,1\}$.

A ring is a σ_{ring}-structure which satisfies the identities $u + (v + w) = (u + v) + w$, $u + v = v + u$, $u + 0 = u$, $u + (-u) = 0$, $u \times (v \times w) = (u \times v) \times w$, $u \times 1 = u$, $1 \times u = u$, $u \times (v + w) = (u \times v) + (u \times w)$ and $(v + w) \times u = (v \times u) + (w \times u)$.

*A group is a σ_{grp}-structure which satisfies the identities $u \times (v \times w) = (u \times v) \times w$, $u \times 1 = u$, $1 \times u = u$, $u \times u^{-1} = 1$ and $u^{-1} \times u = 1$.

*A monoid is a σ_{mnd}-structure which satisfies the identities $u \times (v \times w) = (u \times v) \times w$, $u \times 1 = u$ and $1 \times u = u$.

*A semi group is a $\{\times\}$-structure which satisfies the identity $u \times (v \times w) = (u \times v) \times w$.

*A magma is just a $\{\times\}$-structure.

This is a very efficient way to define most classes of algebraic structures, because there is also the concept of σ-homomorphism, which correctly specializes to the usual notions of homomorphism for groups, semi groups, magmas and rings. For this to work, the signature must be chosen well.

Terms such as the σ_{ring}-term $t(u,v,w)$ given by $(u + (v \times w)) + (-1)$ are used to define identities $t = t'$, but also to construct free algebras. An educational class is a class of structures which, like the examples above and many others, is defined as the class of all σ-structures which satisfy a certain set of identities.

Birkhoff's Theorem states:

A class of σ-structures is an equational class if and only if it is not empty and closed under sub algebras, homomorphic images, and direct products.

The first step, often trivial, for applying the methods of model theory to a class of mathematical objects such as groups, or trees in the sense of graph theory, is to choose a signature σ and represent the objects as σ-structures. The next step is to show that the class is an elementary class, i.e. axiomatizable in first-order logic (i.e. there is a theory *T* such that a σ-structure is in the class if and only if it satisfies *T*). E.g. this step fails for the trees, since connectedness cannot be expressed in first-order logic. Axiomatizability ensures that model theory can speak about the right objects. Quantifier elimination can be seen as a condition which ensures that model theory does not say too much about the objects.

5.1.4 Model Theory and Set Theory

Set theory (which is expressed in a countable language), if it is consistent, has a countable model; this is known as Skolem's paradox, since there are sentences in set theory which postulate the existence of uncountable sets and yet these sentences are true in our countable model. Particularly the proof of the independence of the continuum hypothesis requires considering sets in models which appear to be uncountable when viewed from *within* the model, but are countable to someone *outside* the model.

The model-theoretic viewpoint has been useful in set theory; for example in Kurt Gödel's work on the constructible universe, which, along with the method of forcing developed by Paul Cohen can be shown to prove the (again philosophically interesting) independence of the axiom of choice and the continuum hypothesis from the other axioms of set theory.

In the other direction, model theory itself can be formalized within ZFC set theory. The development of the fundamentals of model theory (such as the compactness theorem) rely on the axiom of choice, or more exactly the Boolean prime ideal theorem. Other results in model theory depend on set-theoretic axioms beyond the standard ZFC framework. For example, if the Continuum Hypothesis holds then every countable model has an ultra-power which is saturated (in its own cardinality). Similarly, if the Generalized Continuum Hypothesis holds then every model has a saturated elementary extension. Neither of these results are provable in ZFC alone. Finally, some questions arising from model theory (such as compactness for infinitary logics) have been shown to be equivalent to large cardinal axioms.

5.1.5 Stable Theory.

In model theory, a complete theory is called **stable** if it does not have too many types. One goal of **classification theory** is to divide all complete theories into those which models can be classified and those which models are too complicated to classify, and to classify all models in the cases where this can be done. Roughly speaking, if a theory is not stable then its models are too complicated and numerous to classify, while if a theory is stable there might be some hope of classifying its models, especially if the theory is super stable or totally transcendental.

Stability theory was started by Morley (1965), who introduced several of the fundamental concepts, such as totally transcendental theories and the Morley rank. Stable and super stable theories were first introduced by Shelah (1969), who is responsible for much of the development of stability theory. The definitive reference for stability theory is (Shelah 1990), though it is notoriously hard even for experts to read.

5.2.1 Classification of Lie Algebra

5.2.2 Semi Simple Lie Algebras and Root Systems

In previous sections, we have shown the relationships between the Lie groups and algebras and we have motivated, why we shall study only sub algebras of the matrix Lie algebra $\mathfrak{g}$l(n; F). We shall now proceed towards a classification of Lie algebras: we will be interested mainly in so called semi simple Lie algebras. We shall introduce the concept of semi simple Lie algebras and motivate, how they fit in the big picture of all Lie algebras.

First, we have to list a few definitions in order to be able to talk.

5.2.3 Definition (Ideal, Solvable, , Simple Lie Algebras).

Let $\mathfrak{g}$ be a Lie algebra.

1- A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, if $[h, g] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$.

2- $\mathfrak{g}$ is solvable, if $D^n \mathfrak{g} = 0$ for a large enough n, where $D^i\mathfrak{g}$ is defined as $D^0\mathfrak{g} = \mathfrak{g}$ and $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]]$ (where $[\mathfrak{a}, \mathfrak{b}] = \{[a, b], a \in \mathfrak{a} \wedge b \in \mathfrak{b}\}$).

3- $\mathfrak{g}$ is semi simple, if there are no nonzero solvable ideals in $\mathfrak{g}$.

4- $\mathfrak{g}$ is simple, if it is non-Abelian, and contains 0 and $\mathfrak{g}$ as the only ideals.

These definitions are presented here mainly for completeness and we will not be using them directly. One can imagine ideals as "hungry entities", such that a commutator with an element in the ideal gives you back an element of the ideal (they are called invariant sub algebras). Ideals can be used for making quotients of Lie algebras. Solvability is an important concept and is useful for example in representation theory. One can intuitively imagine solvable Lie algebras as "almost Abelian", while semi simple Lie algebras are "as far from Abelian as possible".

We now introduce the concept of a radical, so that we can write an important result, known as Levi decomposition :

Proposition 5.2.4 (Radicals).

Every Lie algebra $\mathfrak{g}$ contains a unique largest solvable ideal, This ideal rad($\mathfrak{g}$) is called the radical of $\mathfrak{g}$.

Theorem 5.1 (Levi Decomposition).

Any Lie algebra $\mathfrak{g}$ has a Levi decomposition

$$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss} \qquad (5.2)$$

where rad($\mathfrak{g}$) is the radical of $\mathfrak{g}$, and $\mathfrak{g}_{ss}$ is a semi simple sub algebra of $\mathfrak{g}$.

This result tells us that we can always decompose a Lie algebra into its "Abelian" and "Non-Abelian" parts, which are the maximum ideal (the radical), and some remainder, which doesn't contain any remaining ideals (a semi simple

Lie algebra). Thus, for any algebra, we will be interested in the "Non-Abelian" part, namely the semi simple Lie algebra $\mathfrak{g}_{ss}$.

We finish this subsection with an important result, which could alternatively be given for the definition of a semi simple Lie algebra:

Proposition. A Lie algebra g is semi simple if and only if $\mathfrak{g} = \oplus \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple Lie algebras.

We can therefore view a semi simple Lie algebra $\mathfrak{g}$ as a direct sum of simple Lie algebras $\mathfrak{g}_i$, which have only 0 and $\mathfrak{g}_i$ for their ideals. In particular, every simple Lie algebra is also semi simple.

The Root System of a Semi simple Lie Algebra

Semi simple Lie algebras have a very important property called the root decomposition, which will be the main concern in this subsection. But first, in order to be able to formulate this decomposition, we shall define Cartan sub algebras (not in general, but for semi simple Lie algebras) | another definition for the sake of completeness.

Definition5.2.5 (Toral, Cartan Sub Algebras).

1. A sub algebra h $\subseteq \mathfrak{g}$ is toral, if it is commutative and for all elements $h \in$ h, the operators $[h, .]$ are diagonalizable (as linear operators on the vector space $\mathfrak{g}$).
2. Let $\mathfrak{g}$ be a semi simple Lie algebra. Then h is a Cartan sub algebra, if it is a toral sub algebra, and C(h) = h (where C(h) = $\{x \in h, \forall h \in h : [x, h] \in h\}$ is the centralizer).

This gives the usual definition of the Cartan sub algebra we are familiar with. Namely, it turns out that h is a Cartan sub algebra, if it is a maximal toral sub algebra, and since $[h_1, h_2] = 0$ in this sub algebra, it is the maximal sub algebra of simultaneously diagonalizable elements (diagonalizable in the vector space g). We will not elaborate on the issue of existence of the Cartan sub algebra, we will just state that any complex semi simple Lie algebra indeed has a Cartan sub algebra.

We can now state the main result

Theorem5.2. (Root Decomposition). If $\mathfrak{g}$ is a semi simple Lie algebra, and h⊆ $\mathfrak{g}$ is the Cartan sub algebra, then

$$\mathfrak{g} = h \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad (5.3)$$

where sub algebras $\mathfrak{g}_\alpha$ are defined as $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}, \forall h \in h : [h, x] = \alpha(h)x\}$, where α are linear functional on the vector space h. The functional α go over $R = \{\alpha \in h^* \setminus \{0\}; \mathfrak{g}_\alpha \neq 0\}$.

Let's give some motivation for this result. Basically, the Cartan sub algebra consists of elements, which commute amongst themselves. Consequently, the linear operators of the type [*h*, .] (with $h \in$ h) commute amongst themselves, and can therefore be simultaneously diagonalized. That means we get a number of common Eigen spaces $\mathfrak{g}_i$ (and the total dimensionality of these Eigen spaces is equal to the dimensionality n of the vector space $\mathfrak{g}$). This means that all $x \in \mathfrak{g}_i$ are automatically characterized by specifying the eigenvalues for all the operators [*h*, .], where $h \in$ h. An Eigen space can therefore be characterized by a linear functionalα $\in h^*$, which sends a $h \in$ h to the eigenvalue of [*h*, .] in this Eigen space. Of course, since the dimension of $\mathfrak{g}$ is finite, there are a finite number of these Eigen spaces (consequently, R is finite). Also, the Cartan sub algebra h is an Eigen space under the operators [*h*, .] with the eigenvalue 0, so we get the zero functional 0 for the functional α on this Eigen space. The case α = 0 (which corresponds exactly to the Cartan sub algebra, since it is the maximal Lie sub algebra of commuting elements) is separated from the others, so we demand $\alpha \in h^* \setminus \{0\}$.

We have thus obtained a finite set R of linear functional. The functional α have many important properties. One thing to note is that it turns out all Eigen spaces $\mathfrak{g}_\alpha$ are one-dimensional, and that for α ϵ R we have α ϵ R. One other important property is that if we choose for some α ϵ R three elements, namely e

$\epsilon\ \mathfrak{g}_{\alpha}$, f $\epsilon\ \mathfrak{g}_{-\alpha}$ and *h* ϵ h as an appropriately normalized element corresponding to the root $\alpha \in R \subseteq h^*$ (with respect to a "a scalar product"), these three elements form a sub algebra isomorphic to **s**l(2,C) (or su(2), if we want a real Lie algebra), where *h* has the function of being diagonal, while e and f are "raising" and 'lowering" elements. Moreover, due to the representation theory of the **s**l(2,C) Lie algebra, it is possible to prove that the set R has the structure of a reduced root system, which is defined in the next section.

5.2.6 Abstract Root Systems

We already know of the root decomposition of a semi simple Lie algebra g into its Cartan sub algebra, and the corresponding root spaces. The root system R is a set of nonzero linear functional on the Cartan sub algebra, but which always has certain properties, which hold for the roots α.

. We will now define abstract root systems; as already stated, we claim (without proof) that the root system of a Lie algebra has the structure of an abstract root system.

5.2.7 Definition (Reduced Root System).

Let V be an Euclidean vector space (finite-dimensional real vector space with the canonical inner product (., .)). Then $R \subseteq V \setminus \{0\}$ is a reduced root system, if it has the following properties:

1. The set R is finite and it contains a basis of the vector space V.
2. For roots $\alpha, \beta \in R$, we demand $n_{\alpha\beta}$ to be

$$n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in z \qquad (5.4)$$

3. If $s_\alpha : V \longrightarrow V$, $s_\alpha(\Lambda) = \Lambda - \frac{2(\alpha, \Lambda))}{(\alpha,\alpha)}\,\alpha$ then $s_\alpha \in R$ for all α , β ϵ R
4. If α, cα ∈ R for some real c, then c = 1 or c = -1.

Although we have written these conditions in a formal manner, it is also instructive to try to visualize them (see Figure 1). So we start with a finite collection of points, which span the whole Euclidean space V .

The expression $\frac{(\alpha,\beta)}{(\beta,\beta)}\beta$

is the projection of α onto the direction of the root β. We can rewrite this as $proj_{\beta}\alpha = n_{\alpha\beta}\,\beta$ since $n_{\alpha\beta} \in Z$, then the projection 2 Of α is a half-integer of β. The third condition constructs a function s_{α} which subtracts from a vector ʌ the twofold projection of ʌ to the direction of α: a single subtraction would bring us to the orthogonal complement of the vector α (which is a hyper plane of codimension 1), but a twofold subtraction actually reverses the α -component of the vector ʌ: lambda is reflected over the hyper plane $L_{\alpha} = \{ \Lambda \in V ; (\Lambda, \alpha) = 0\} = \{R\alpha\}^{\perp}$. The third condition therefore states that for a root β ∈ R, the reduced root system R must also contain all the reflections of β over the hyper planes, constructed by the roots in R.

The fourth condition is the reason for calling this structure a reduced root system. Namely, say α ∈R. Then we want to know, which c α are also permissible to be in R given only the first three conditions. The second condition tells us that 2nc which implies that c must be half integer. The same holds for $2n_{c\alpha,\alpha} = 2/c \in Z$, If c and 1/c are half integer, then the only possibilities are C ∈ {±2 ,±1 ,±1/2}.Condition four puts a further restriction to this, so that we permit only 2 of the six possibilities. We will see that these conditions imply still further properties of roots. We know, for example, that in the Euclidean space V , the standard scalar product gives

$$(\alpha,\beta) = |\alpha|.|\beta|\cos\varphi \tag{5.5}$$

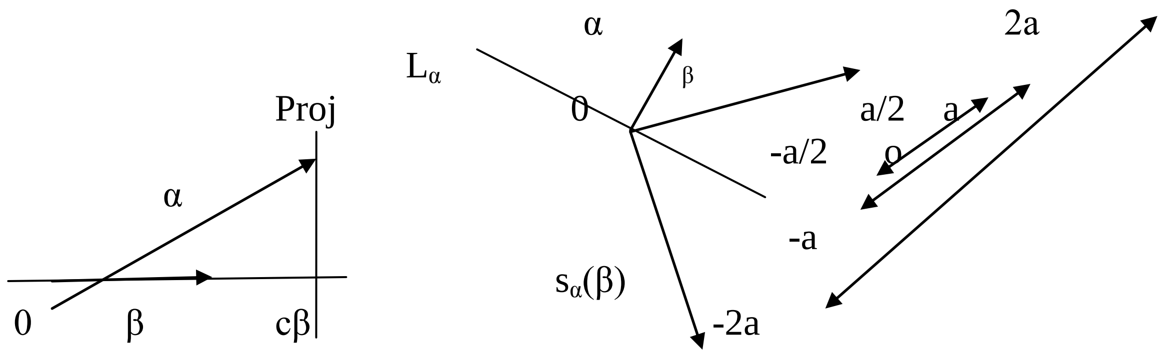
Where α, β∈ R are root vectors, while φ is the angle between them. With this, we can rewrite the numbers $n_{\alpha\beta}$ as

$$n_{\alpha\beta} = 2\frac{|\alpha|.|\beta|\cos\varphi}{|\beta|.|\beta|} = 2\frac{|\alpha|}{|\beta|}\cos\varphi \qquad (5.6)$$

$$\Rightarrow n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\varphi \qquad (5.7)$$

Because $n_{\alpha\beta}$, $n_{\beta\alpha}$ ∈ Z, and also $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|^2}{|\beta|^2}$, we get restrictions on the angle φ between two roots, as well as their relative length. We also have another restriction, due to the nonnegative right-hand side of equation (above): either both $n_{\alpha\beta}$ and $n_{\beta\alpha}$ are positive, both are negative, or both are zero. Analyzing these possibilities is straightforward, and we get a list as in table 3 this list is complete, and there are no other possibilities, since the product $n_{\alpha\beta}n_{\beta\alpha} < 4$ due to the cosine on the right-hand side of equation (above) (if the product equals 4, we have the trivial case α = β).

Condition2 : condition 3: condition4
:



2C is integer

Figure 1: A visual representation of the conditions which must hold for a reduced root system.

Table 3: Possible relative positions of two roots α and β, with the angle between them denoted as φ. We also, without loss of generality, suppose that α ≥ β and therefore $n_{\alpha\beta} \geq n_{\beta\alpha}$.

$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\|\alpha\| / \|\beta\|$	φ [rad] φ [°]
0	0	/	π/2 90
-1	-1	1	2π/3 120
1	1	1	π/3 60
-2	-1	√2	3 π/4 135
2	1	√2	π/4 45
-3	-1	√3	π/6 30
3	1	√3	5π/6 150

5.2.8 Definition (Root System Isomorphism).

Let $R_1 \subseteq V_1$ and $R_2 \subseteq V_2$ be two root systems.

Then $g: V_1 \longrightarrow V_2$ is a root system isomorphism, if it is a vector space isomorphism,

$$\varphi(R_1) = \varphi(R_2) \text{ and } n_{\alpha\beta} = n_{\varphi(\alpha)\propto(\beta)} \text{ for all } \alpha, \beta \in R_1.$$

We shall now look at rank 2 root systems (those, which are in a 2-dimensional Euclidean space) and try to classify them up to a root system isomorphism. In this 2-dimensional case, we must have at least two non-parallel roots α and β, so that they span the whole space.

Then the possible angles between them are listed in table 3. Because reflections in condition 3 for root systems demands that both β and $s_\alpha(\beta)$ are part of the root system, this angle between α and β can be chosen to be the greater angle amongst the two possibilities, thus $\varphi \geq 90$ and therefore we have four possibilities: 90°, 120°, 135° and 150°. We then choose a length of the root α, and according to the table of properties, we fix the length of β. Here, the orientation and length of the root α is not important, and also we can choose either α or β to be the longer root. These will be equivalent situations, since rotations and stretching a root system together with its ambient space are among root system isomorphisms. Then we proceed to draw all the reflections of α and β guaranteed to exist by condition 3. Also we have exactly two vectors of the type c α due to - α = s $_{\alpha}$ (α) and condition 4. We thus get 4 different rank 2 root systems, which are drawn in Figure 2, and we claim that every rank 2 root system is isomorphic to one of them. We cannot add any other roots because of the angle restrictions, and this can be checked for all of the 4 situations separately. Also, these are distinct root systems, since isomorphisms conserve the angles between two roots.

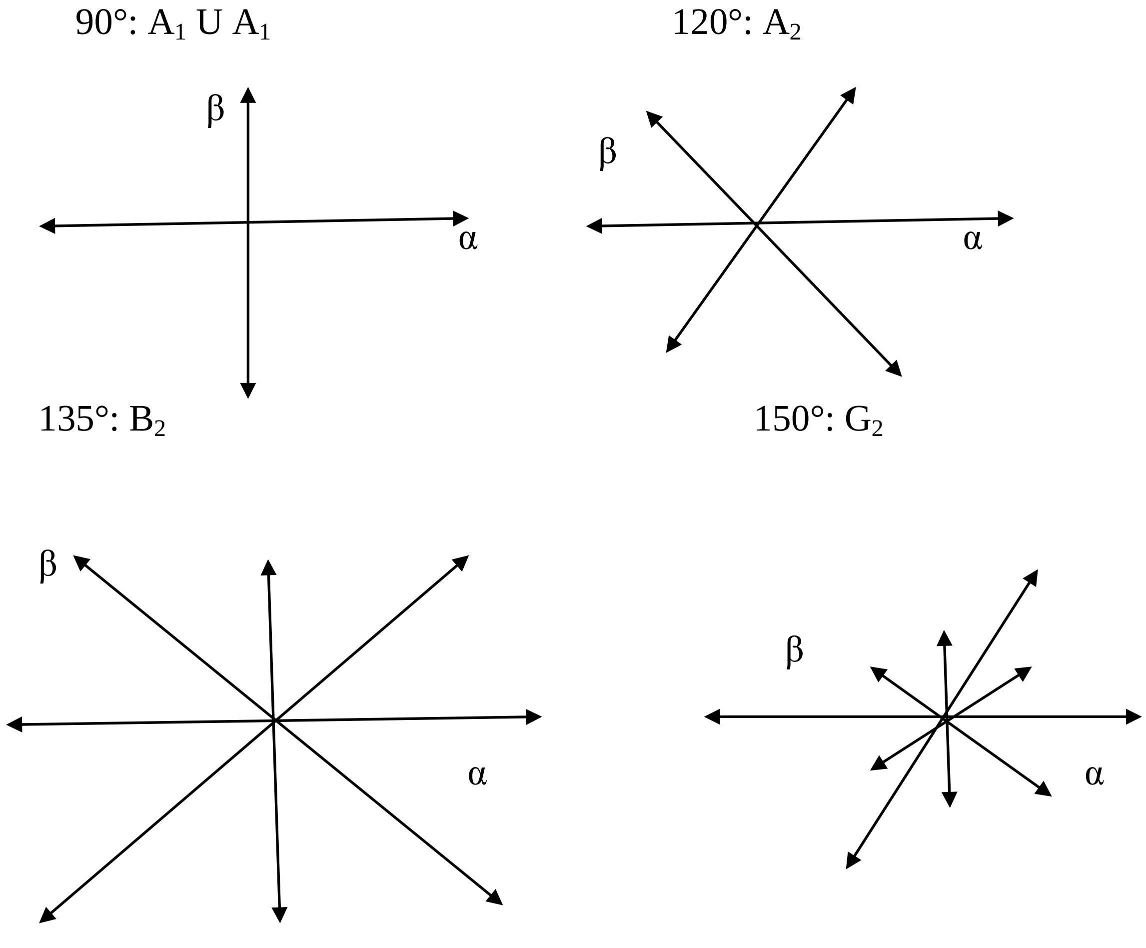


Figure 2: All possible (up to root system isomorphism) reduced root systems of rank 2 along with their traditional names.

Simple Roots

We have defined a root system as a finite collection of vectors in an Euclidean vector space, which satisfy certain properties. Condition 1 states that the root system must span the whole space. If the space is n-dimensional, we only need n linearly independent vectors.

Since R spans V , we know that R contains a basis for V . The only remaining question is, whether we can choose these n vectors among the many root vectors in such a way, to be able to reconstruct the whole root system out of them. This is our motivation for introducing simple roots.

First we notice that a root system is symmetric with respect to the zero vector; namely, we have $-\alpha \in R$ if $\alpha \in R$. Therefore we get the idea that we separate a root system into two parts, which we will call positive and negative roots. We will do this by choosing a polarization vector $t \in V$, which is not located on any of the orthogonal hyper planes (which go through the zero vector) of the roots in R, so that all roots point in one of the two half spaces divided by the orthogonal hyper plane of t. Since α and $-\alpha$ are in different subspaces, we have thus separated the root system into two parts. We then look at only "positive"
roots and define the concept of a simple root.

5.2.9 Definition (Polarization, Simple Roots).

1- Let $t \in V$ be such that $(\alpha , t) \neq 0$ for all $\alpha \in R$. Then the polarization of R is the decomposition $R = R_+ \cup R_-$ ($\cup$ denotes the disjoint union), where $R_+ = \{ \alpha \in R; (\alpha, t) > 0\}$ and $R_- = \{\alpha \in R; (\alpha, t) < 0\}$. The elements of R_+ are called positive roots and the elements of R_- are called negative roots.

2- A positive root $\alpha \in R_+$ is simple, if it cannot be written as $\alpha_1 + \alpha_2$, where $\alpha_1, \alpha_2 \in R_+$.

The simple roots are a very useful concept. Every positive root can be written as a finite sum of simple roots, since it is either a simple root, or if it is not, it can be written as a sum of two other positive roots. Since the root system R is finite, this has to stop after a finite number of steps. Also, every negative root can be written as $-\alpha$ for some positive root α.

Together that means that for any root $\beta \in R$, we can write it as a linear combination of simple roots with integer coefficients. Let us denote the set of simple roots as S. Therefore, every $\beta \in R$ is a linear combination of vectors in S.

Because S spans R and R spans V , simple roots S span the whole space V . Also, it can be proven that simple roots are linearly independent.

We will need another property of simple roots in the future: the scalar product $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ simple roots. We see this in two steps:

1. First off, note that if α, β ɡ R are two roots, such that $(\alpha, \beta) \leq 0$ and $\alpha \neq c\,\beta$, then $\alpha + \beta \in R$. This can easily be seen by introducing a rank 2 root subsystem, which contains both α and β (we choose these two roots, which have to satisfy Properties from table 3 and consequently generate − by reflections $s_{\wedge}$ − one of four rank 2 root systems). This means that it is sufficient to check that $\alpha + \beta \in R$ for the particular cases in Figure 2.

2. We claim that for simple roots $\alpha, \beta \in S$, we have $(\alpha , \beta) \leq 0$, if $\alpha \neq \beta$. If that were not true, we would have $(\alpha , \beta) > 0$. Then, we have $(-\alpha, \beta) < 0$, and also $\beta \neq c\,\alpha$ (c can only be 1 or −1, but only the case $c = 1$ gives that α and β are positive, and that is forbidden by $\alpha \neq \beta$). By the first step we then have $(-\alpha) + \beta \in R$ so either $\beta - \alpha \in R_+$ or $\beta - \alpha \in R_-$. In the first case, we would have $\beta = \alpha + (\beta - \alpha)$ − a contradiction, since β is simple (and therefore cannot be written as a sum of other positive roots), and in the second case, we would have a contradiction in $\alpha = \beta +(\alpha - \beta)$ being simple (now $\beta - \alpha \in R_-$ and thus $\alpha - \beta \in R+$). We have thus proved that $(\alpha , \beta) > 0$ is impossible for two simple roots.

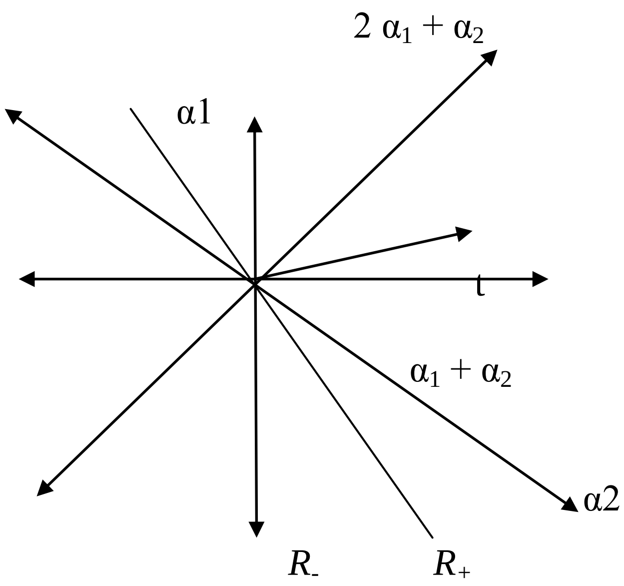


Figure 3: A construction of a simple root system in the case of B_2 of rank 2. Choosing the polarization t we make the decomposition $R = R_+ \cup R_-$, and identify the two simple roots α_1 and α_2.

We now sum up the results we obtained about simple roots:

Proposition. Let $\{ \alpha_i \}_{i\epsilon I} = S \subseteq R \subseteq V$ be the set of simple roots due to a polarization t. Then S is a basis of V and every $\alpha \in R$ can be written as

$$\alpha = \sum_{i\epsilon I} n_i \alpha_i, \text{ where } n_i \in Z,$$

and all n_i are non-negative if $\alpha \epsilon R_+$ and non-positive if $\alpha \in R_-$. Thus, when choosing a polarization, one immediately gets a "basis" S for the root system R. We would like to reduce the question of classification of reduced root systems R to simple root systems S. For that to work, two important issues have to be resolved; it turns out that for any choice of polarization t we get equivalent simple root systems (which are related by an orthogonal transformation), and that the root system R can be uniquely reconstructed from its simple root system S.

First, let's consider the equivalence of simple root systems under polarizations. Given a root system R, we can divide the space V by drawing all the hyper planes L_α corresponding to the roots α, where α goes over all R. With this, the space V is divided into so called Weyl chambers. Every polarization t, since it is not on any hyper plane L_α is in some Weyl chamber. If two polarizations t_1 and t_2 belong to the same Weyl chamber, then the scalar product $sgn(\alpha, t_1) = sgn(\alpha, t_2)$ for all $\alpha \in R$, and we get the same decomposition $R = R_+ \cup R_-$, with both polarizations, and thus also the same simple root system S.

Therefore, we have a bijection between Weyl chambers and different S polarizations. Every Weyl chamber can be transformed to an adjacent Weyl chamber, which is separated from the original by L_α via the mapping s_α. For two polarizations t_1 and t_2, we can therefore construct a composite mapping of s_α so that the Weyl chamber associated with t_1 is transformed to the chamber associated with t_2; this mapping also transforms S_1 to S_2 (where S_i is the simple root system associated with t_i), and since it is orthogonal (and thus preserves angles), the two root systems S_1 and S_2 are equivalent.

We now turn to the reconstruction of R from its simple root system S.. Here, we again use the reflections s_α the group of transformations, which is generated by such reflections, is called the Weyl group. It turns out it suffices to generate this group with just simple reflections (mappings s_α where $\alpha \in S$, so α is just a simple root). Furthermore, it turns out the set R can be written precisely as the set of all the elements $w(\alpha_i)$, where w goes over all the elements in the Weyl group and α_i goes over all the simple roots. We therefore start with S and repeatedly apply the associated simple reflections, and we end up with exactly the whole root system R.

With this, we can give the following result:

5.2.10 Proposition

(Correspondence between a reduced and simple root system).

There is a bijection between reduced root systems and simple root systems. Every reduced root system R has a unique (up to an orthogonal transformation) simple root system S, and conversely, we can uniquely reconstruct R from a simple root system S.

It is thus sufficient to classify all possible simple root systems instead of all root systems.

Furthermore, we know that for a root system in a n-dimensional space, the simple root system has exactly n linearly independent elements.

5.2.11 Classification of Simple Root Systems

Irreducible Root Systems

Armed with the knowledge obtained in previous sections, we know the whole chain of structures, associated with a Lie group. Every Lie group G has a Lie algebra g, which in turn, if it is semi simple, has a reduced root system R, which in turn has a simple root system S. We shall proceed with the classification of simple root systems.

We know that the root system R is by definition a finite set of vectors. Therefore, $S \subseteq R$ is also a finite set. Since S is a basis for the vector space V, an n-dimensional Lie algebra has a simple root system with n elements.

An important concept is that of the root system, which is associated with the decomposition into systems in distinct orthogonal subspaces. Namely, if one can make a decomposition $R = R_1 \cup R_2$ into two subsystems, so that $R_1 \subseteq V_1$ and $R_2 \subseteq V_1^{\perp}$?for some linear subspace $V_1 \subseteq V$, we say that R is reducible. We will not prove this explicitly, but it is intuitively expected that reducible root systems always break up into irreducible ones, which cannot be broken down further, and that the same thing happens with the associated simple root system S. Let us now state, what we have described, more formally

5.2.12 Definition (Reducible, Irreducible Root System).

Let R be a (reduced) root system. Then R is reducible, if $R = R_1 \cup R_2$, with $R_1 \perp R_2$. R is irreducible, if it is not reducible.

5.2.13 Proposition.

1- Every reducible root system R can be written as a finite disjoint union $R = \cup R_I$, where R_i are irreducible root systems and $R_i \perp R_j$ if $i \neq j$.

2- If R is a reducible root system with the decomposition $R == \cup R_I$, we have $S = \cup S_i$, where S is the simple root system of R (under some polarization), and $S_i = R_i \cap S$ is the simple root system of R_i ("under the same polarization").

3- If S_i are simple root systems, $S_i \perp S_j$ for $i \neq j$, $S = \cup S_i$, and R, R_i are the root systems generated by the simple systems S, S_i, we have $R = \cup R_i$.

It thus suffices to do a classification of irreducible simple root systems, since reducible ones are built from a finite number of irreducible ones.

5.2.14 The Cartan Matrix and Dynkin Diagrams

Suppose that we have a simple root system S. We can ask ourselves, what is the relevant information contained in such a system. Certainly, it is not the absolute position of the roots, or their individual length, since we can take an orthogonal transformation and still obtain an equivalent root system. The important properties are their relative length to each other and the angle between them. Since we have for simple roots $\alpha, \beta \in S$ the inequality $(\alpha, \beta) \leq 0$, the angle between simple roots is $\geq 90^{\circ}$, and with the help of table 3, we have the four familiar possibilities. Of course, the angle between them also dictates their relative length, so the only relevant information are the angles between the roots (and which root is longer).

We can present this information economically as a list of numbers. Instead of angles, we specify the numbers

$$n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\beta,\beta)} \qquad (5.8)$$

which are conserved via root system isomorphisms. We call this list the Cartan matrix

5.2.15 Definition (Cartan Matrix).

Let $S \subseteq R$ be a simple root system in n dimensional space, and let us choose an order of labeling for the elements $\alpha_i \in S$, where $i \in \{1 . . . , n\}$. The Cartan matrix a is then a n x n matrix, which has the following entries component wise: $a_{ij} = n_{\alpha_i\alpha_j}$.

Due to the definition of $n_{\alpha\beta}$, we clearly have $a_{ii} = 2$ for all $i \in \{1 . .. , n\}$. Also, since the scalar product of simple roots $(\alpha_i,\alpha_j) \leq 0$ for $i \neq j$, the non-diagonal entries in the Cartan matrix are not positive: $a_{ij} \leq 0$ for $i \neq j$.

It is also possible to present the information in the Cartan matrix in a graphical way via Dynkin diagrams. We will now define these diagrams by telling the recipe, of how such a diagram is drawn.

5.2.16 Definition (Dynkin Diagram)

Suppose $S \subseteq R$ is a simple root system. The Dynkin diagram of S is a graph constructed by the following prescription:

1. For each $\alpha_i \in S$ we construct a vertex (visually, we draw a circle).

2. For each pair of roots α_i; α_j , we draw a connection, depending on the angle φ between them.

* If $\varphi = 90^\circ$, the vertices are not connected (we draw no line).
* If $\varphi = 120$, the vertices have a single edge (we draw a single line).
* If $\varphi = 135^\circ$, the vertices have a double edge (we draw two connecting lines).
* If $\varphi = 150^\circ$, the vertices have a triple edge (we draw three connecting lines).

3. For double and triple edges connecting two roots, we direct them towards the shorter root (we draw an arrow pointing to the shorter root).

There is no need to direct single edges, since they represent $\varphi = 120^\circ$, which lead to $|\alpha_i| = |\alpha_j|$ while there are no edges in the orthogonal case, when there is no restriction to the relative length of the pair of roots. Also, the choices for the number of edges in the recipe for the Dynkin diagram is no coincidence: no edge between a pair of roots means that they are orthogonal. It is then clear from the definition of reducible roots that a Dynkin diagram is connected if and only if the simple root system S is irreducible. Moreover, each connected component of the Dynkin diagram corresponds to a irreducible simple root system S_i in the decomposition S =∪ Si.

Also, it comes as no surprise that the information in the Cartan matrix can be reconstructed from the Dynkin diagram, since the entries $a_{ij} = n_{\alpha i \alpha j}$ can be reconstructed from the number of edges and their direction. For example, if $\varphi = 150^\circ$, drawn by a directed triple line, we know from table 3 that $a_{ij} = -3$ and $a_{ji} = -1$ (where $\alpha_i > \alpha_j$). This full reconstruction of the information of a simple root system S from a Dynkin diagram can be stated more formally.

5.2.17 Proposition

Let R and $R^/$ be two (reduced) root systems, constructed from the same Dynkin diagram. Then R and $R^/$ are isomorphic.

5.2.18 Classification of Connected Dynkin Diagrams

Dynkin diagrams are a very effective tool for classifying simple root systems S, and consequently the reduced root systems R. Since reducible root systems are a disjoint union of mutually orthogonal sub root systems, the Dynkin diagram is just drawn out of many connected graphs. It is thus sufficient to classify connected Dynkin diagrams. We will state the result of this classification and will sketch a simplified proof.

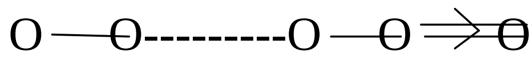
Theorem 5.7 (Classification of Dynkin Diagrams).

Let R be a reduced irreducible root system. Then its Dynkin diagram is isomorphic to a diagram from the list in Figure 4, which is also equipped with labels of the diagrams. The index in the label is always equal to the number of simple roots, and each of the diagrams is realized for some reduced irreducible root system R.

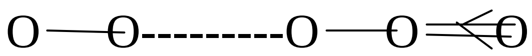
The 4 families:

An $(n \geq 1)$

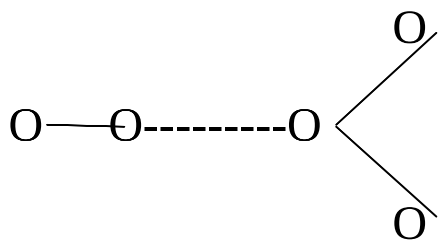
Bn $(n \geq 2)$:



Cn $(n \geq 2)$:

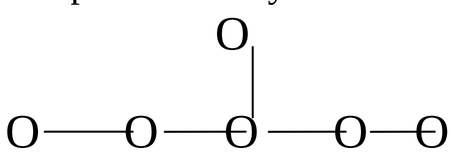


Dn $(n \geq 4)$:

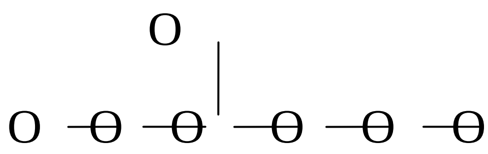


The 5 exceptional root systems:

$E6$:



$E7$:



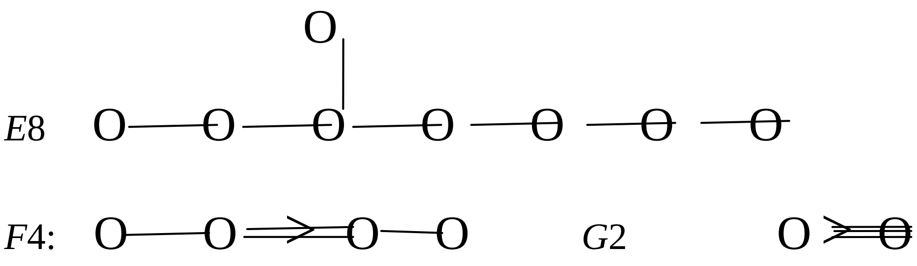


Figure 4: Dynkin diagrams of all irreducible root systems R.

We now mention some important rules as a motivation for the classification theorem

1. If I is a connected Dynkin diagram with 3 vertices, then the only two possibilities are shown in Figure 5. We shall now derive this result. Consider a Dynkin diagram with 3 vertices: ignoring the relative lengths of the simple roots, the diagram is specified by the 3 angles between the roots. At most one of these angles is equal to 90°, because I is connected. Furthermore, the sum of the angles between 3 vectors is 360° in a plane, and less than 360°, if they are linearly independent. This excludes all possible diagrams with 3 vertices, as shown in Figure 5, except the two from the statement.

Also, if there is no 90° angle, we have a loop: it suffices to check a loop of 3 vertices with just single connections, since double or triple connections would increase the sum of the angles even further.

Allowed Dynkin diagrams

Figure 5: Possible diagrams with 3 vertices; if the sum of the angles between vertices $\sum \geq 360^\circ$, the diagram is forbidden.

2. If I is a Dynkin diagram and a simple graph, then I contains no cycles (sub graphs with vertices connected in a loop). If that were not the case, there would exist a sub graph J with $k \geq 3$ vertices, which would be a loop (with only single connections). In this loop, the neighboring vertices would give -1 to the Cartan

matrix, the diagonal elements would give 2, while all others would be 0. We reliable the indices, so that i runs from 1 to k, and we label $\alpha_{k+1} = \alpha_1$ and

$\alpha_0 = \alpha_k$. For $x = \sum_{j \in J} \alpha_j$

, with the normalization of roots $(\alpha_i, \alpha_i) = 2$, we then have (^ax, x)= 0which is a contradiction for the positive definite Cartan matrix of a sub graph of a Dynkin diagram (we found x ∈ Ker ^a).

3- If I is a Dynkin diagram, and α, β are two roots connected with a single connection, as shown in Figure 7, the two roots can be substituted by a single root, and we obtain a new Dynkin diagram. We will not prove this statement, but it can be shown by constructing a new root system, with the same roots as previously, but taking the root α + β instead of roots α and β. One can easily check (via scalar products) that the angles between the new root α+β and other roots are consistent with the contraction of the two vertices.

As a consequence, it is possible to eliminate some further diagrams by contracting vertices. The reasons why there can be at most one branching point (vertex with 3 connections), and why there cannot be 2 double connections, are illustrated in Figure 7.

Substitution ……. O ——O …..—> … O...

Forbidden diagrams as a consequence:

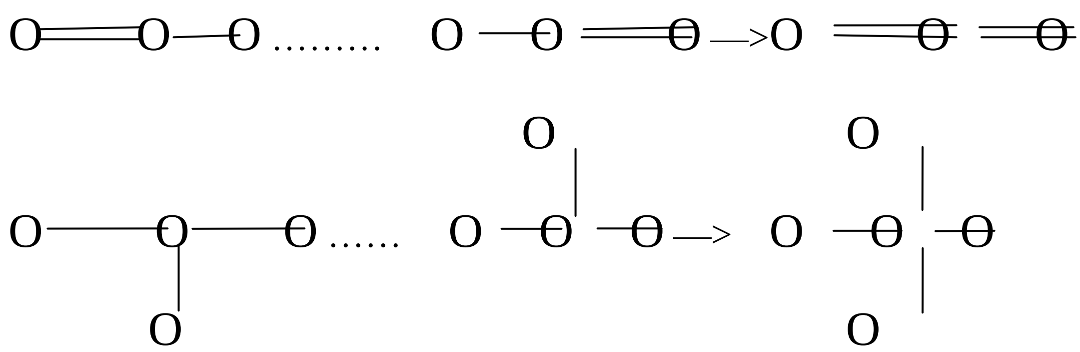


Figure 7: Contractions of two vertices with a single connection in valid Dynkin diagrams give valid Dynkin diagrams. As a consequence, we conclude the diagrams left of the arrow are not valid, since they give invalid contractions.

Serre relations and the Classification of semi simple Lie algebras

Now we will turn to the classification of semi simple Lie algebras, and explain how that is related to the classification of irreducible simple root systems.

One thing to note is that the decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ of a semi simple Lie algebra into simple Lie algebras is related to the decomposition of the root system $R = \cup R_i$ in particular, $\mathfrak{g}$ is simple if and only if its root system R is irreducible. That means that we will be classifying simple Lie algebras by considering only connected Dynkin diagrams. We will now describe an important result, which will eventually enable us to backtrack from root systems to Lie algebras.

Theorem5.8. (Sierre Relations).

Let $\mathfrak{g}$ be a complex semi simple Lie algebra with Cartan sub algebra h and its root system $R \subseteq h^*$, and choosing a polarization we have S as its simple root system. Let (., .) be a scalar product (a non-degenerate symmetric bilinear form) on $\mathfrak{g}$

1- We have the decomposition $\mathfrak{g} = h \oplus n+ \oplus n-$, where $n_\pm = \oplus_{\alpha\epsilon R\pm g\alpha}$.

2- Let H_α g h be the element, which corresponds to α g h g, and $h_i = h_\alpha i = 2H_{\alpha i}/(\alpha_i, \alpha_i)$. If we choose $e_i \epsilon g_{\alpha i}$, $f_i \in g_{-\alpha i}$ and $h_i = h_{\alpha i}$, with the constraint $(e_i, f_i) = 2/(\alpha_i; \alpha_i)$, then e_i generate n_+, f_i generate n_- and h_i form a basis for h (where in all cases $i \epsilon \{1, \ldots, r\}$), and thus $\{e_i, f_i, h_i\}_{i\epsilon\{1,\ldots,r\}}$ generates $\mathfrak{g}$.

3--The elements e_i, fi; $\mathfrak{g}_I$ satisfy the Serre relations (where a_{ij} are the elements of the Cartan matrix):

$$[h_i,h_j] = 0 \qquad [h_i,e_j] = a_{ij}e_j,$$

$$[h_i,f_i] = -a_{ij}f_i, \qquad [e_i,f_j] = \delta_{ij}h_j,$$

$$([e_i,.])^{1-aij}e_j = 0 \qquad ([f_i,.])^{1-aij}f_j = 0$$

Knowing the Serre relations, , we can turn the construction around:

Theorem (Construction of a Lie Algebra from the Root System). Let $R = R_{+} \cup R_{-}$ be a reduced irreducible root system with a chosen polarization, and $S = \{\alpha_1,\ldots, \alpha_r\}$ the corresponding simple root system. Let $\mathfrak{g}$ (R) be the complex Lie algebra generated by e_i,f_i, h_i, with the Serre relations giving the commutators. Then $\mathfrak{g}$. (R) is a semi simple Lie algebra with its root system R, and if g is another Lie algebra with its root system R, then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$ (R).

Theorem 5.9. (Classification of Semi sSimple Lie Algebras).

A simple complex finite dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a Lie algebra, constructed from one of the Dynkin diagrams in Figure 4. Semi simple Lie algebras are all possible finite direct sums of simple Lie algebras.

With this, we have classified semi simple Lie algebras. Here, it is important to note that Dynkin diagrams classify complex semi simple Lie algebras and not real ones. A classification of real semi simple Lie algebras is done with something called Satake diagrams ,. We can still identify real forms of the classified complex semi simple Lie algebras; a real form of a complex Lie algebra g is a real Lie algebra $\mathfrak{g}R$, which has the original algebra as its complexification: $\mathfrak{g} = \mathfrak{g}\, R \oplus i\mathfrak{g}\, R$ with the obvious commutator. The real form is not necessarily unique. For example, both su(n) and (n,R) have the complexification (n,C). It is possible to identify the Dynkin diagrams with known Lie algebras and their real forms (see table 4), but some Lie algebras (the exceptional ones) are new, and cannot be found among the classical matrix algebras.

Table 4: A table of Dynkin diagrams, corresponding complex semi simple Lie algebras, and their real forms.

Dynkin	Complex $\mathfrak{g}$	real form $\mathfrak{g}$ R
A_n(n≥1)	SL(n+1,C)	SU(n+1)
B_n(n≥1)	SO(2n+1,C)	SO(2n+1,R)
C_n(n≥1)	SP(n,C)	SP(n,R)
D_n(n≥2)	SO(2n,C)	SO(2n,R)
E_6	complex e_6	real e_6
E_7	complex e_7	real e_7
E_8	complex e_8	real e_8
F_4	complex f_4	real f_4
G_2	**complex $\mathfrak{g}_2$**	real $\mathfrak{g}_2$

It is noteworthy that the restrictions on n in Figure 4 are due to either small diagrams not existing, or they are the same as a previous one. For example, we would have $A_1 = B_1 = C_1$, which would correspond with SL(2,C)≅' SO (3,C) ≅ SP (1,C) on the Lie algebra level. We have thus readjusted the possible values n for the purposes of table 4.

One could ask, where in this classification are the familiar Lie algebras o(n, F), u(n) and $\mathfrak{g}$ l(n, F). We already have o(n, F), since o(n, F) = so(n, F). The others are not semi simple, since we have the Levi decompositions

$\mathfrak{g}$ l(n, F)= SL(n; F) $\oplus$ U(1) and U(n) = SU(n) $\oplus$ U(1), where u(1) is not simple, since it is Abelian.

5.3 APPLICATION

5.3.1 How to Use Invariants in Solving Variants

One of the key ideas in mathematical problem solving is exploiting *invariance*. Let us start with a very simple example: Say we are going to try to prove some theorem of Euclidean geometry involving a circle by using algebra and introducing Cartesian coordinates. We say "Without loss of generality, we may suppose the circle is centered at the origin." That is nice, instead of center (a,b) we have (0,0), simplifying some calculations. By what right do we make such an assumption? The answer is "invariance" - we are studying properties of figures which remain unchanged under rigid motions, so we lose nothing if we suppose the figure translated so as to place the center of our circle at the origin. Possibly we can go on to say "Without loss of generality "the radius of the circle may be taken equal to 1 "... this will be justified if the features of the problem under study are invariant w.r.t. scale change (dilation, or affine transformation); and so on - the more invariance, the greater the simplifying assumptions.

In some problems, already this first step *solves the whole problem*! Here is an example:

example: Prove that the inequality $(x + y)^p < x^p + y^p$ holds whenever x, y are positive numbers and $0 < p < 1$.

Solution: we may assume $x + y = 1$ (because the inequality in question remains invariant under the transformation of scaling $x \to tx$, $y \to ty$ with t positive). But, if $x+y = 1$, the inequality is obvious since then $x < 1$, $y < 1$ so $x < x^p$ and $y < y^p$, etc.

In the cases just discussed the transformations needed to "trivialize" the problem were simple and fairly obvious, but in other cases this is not so. Consider for example:

Steiner's porism Let A and B denote circles, one wholly enclosed by the other. A finite collection of circles $C_1, C_2, \ldots, C_n$ is called a Steiner chain (w.r.t. A and B) if a) Each of the circles C_i is tangent to both A and B, and b) Each of the pairs $(C_1, C_2), (C_2, C_3), \ldots, (C_n, C_1)$ exhibit external tangency.

It is clear that, given A and B, and the integer n, we cannot in general expect that there will exist a Steiner chain with n circles. However, Jacob Steiner made the beautiful discovery that: If there exists a Steiner chain with n circles, then there exists such a chain regardless of the choice of C_1, provided only it is tangent to A and B.

Now, observe: the theorem is obvious if A and B are concentric. So, how nice it would be to be able to say " A and B are concentric."? But, is this true? It would be if there were a one-to-one transformation of the plane P of the figure onto another plane P' such that: circles are carried to circles, and moreover the pair A, B is carried to a pair of *concentric* circles. Well, this is almost true: if we replace the Euclidean plane by a projective one it is, and that is good enough to do the business.

5.3.2 Another Aspect of Invariance is *Transformations*

An important theme in studying mathematical objects is *transformations*. They are ubiquitous, part of the very soul of mathematics, so much so that often we use them without being aware of them. For instance, one sometimes speaks of "the circle $x^2 + y^2 = 1$" ... but this is not a circle, it is an equation, and we have here to do with the wondrous phenomenon that geometric objects have algebraic counterparts and *vice versa*.. A common pattern in problem-solving is "Transform. Solve. Invert." This was developed and richly illustrated in an article of M. Klamkin and D. J. Newman. A very simple and amusing illustration is the problem to do arithmetic in the Roman system of numeration, e.g. compute the product VIII times V. The solution consists of

1. Transform (to decimal notation): The transformed problem is to compute 8 x 5.
2. Solve: 8 x 5 = 40 .
3. Invert: 40 = XL (Conclusion: VIII x V = XL.)

Indeed, when your pocket calculator computes 8 x 5, it uses this exact same approach, transforming from decimal to binary notation, solving and transforming back. From a logical point of view this is the same schema we employ when we solve a differential or difference equation by using Laplace transforms, or a geometric problem by using "analytic geometry". And before we go on to more examples I think that we should state the following important theorem.

Theorem 5.10 Noether's Theorem

Noether's Theorem states that if the Lagrangian function for a physical system is not affected by changes in the coordinate system used to describe them, then there will be a corresponding conservation law. For example, if the Lagrangian is independent of the location of the origin then the system will preserve (or conserve) linear momentum.

Application of Noether's theorem allows physicists to gain powerful insights into any general theory in physics, by just analyzing the various transformations that would make the form of the laws involved invariant. For example:

1-the invariance of physical systems with respect to spatial translation (in other words, that the laws of physics do not vary with locations in space) gives the law of conservation of linear momentum;

2-invariance with respect to rotation gives the law of conservation of angular momentum;

3-invariance with respect to time translation gives the well-known law of conservation of energy

In quantum field theory, the analog to Noether's theorem, the Ward–Takahashi identity, yields further conservation laws, such as the conservation of electric charge from the invariance with respect to a change in the phase factor of the complex field of the charged particle and the associated gauge of the electric potential and vector potential.

The Noether charge is also used in calculating the entropy of stationary black holes.

We now state Noether theorem:

Suppose the coordinates {q_i} are a function of a continuous parameter s. if $\frac{dl}{ds}$ =o then $\frac{dH}{dt}=0$ where H=$\sum H_i$ and $H_i = p_i(\partial q_i/\partial s)$

Proof:

Consider a quantity $(\partial q_i/\partial s)$ and its product with the corresponding momentum p_i. Call this product H_i; i.e. $H_i = p_i(\partial q_i/\partial s)$

Now consider the time derivative of H_i:

$$\frac{dH_i}{dt} = (\frac{dp_i}{dt})(\frac{\partial q_i}{\partial s}) + p_i \frac{d(\frac{\partial q_i}{\partial s})}{dt} \qquad (5.9)$$

The order of the differentiations (by t and by s) in the second term on the right may be interchanged to give

$$\frac{d(\frac{\partial q_i}{\partial s})}{dt} = \frac{\partial(\frac{dq_i}{dt})}{\partial s} = \frac{\partial v_i}{\partial s} \qquad (5.10)$$

Since $\frac{dp_i}{dt} = \frac{\partial L}{\partial v_i}$ and $p_i = \frac{\partial L}{\partial q_i}$ the time derivative of H_i reduces to

$$\frac{dH_i}{dt} = (\frac{\partial L}{\partial q_i})(\frac{\partial q_i}{\partial s}) + (\frac{\partial L}{\partial v_i})(\frac{\partial v_i}{\partial s}) = \frac{\partial L}{\partial s} \qquad (5.11)$$

But the right-hand-side of this equation is merely the rate of change of L having to do with the effect of a change in s. It is the change in L which occurs as a result of the effect of the change in s on q_i and v_i. A change in s may affect all of the coordinates and not just one. If everything is summed over the n coordinates the result is the total derivative of L with respect to s; i.e., The left-hand-side is just $\frac{dH}{dt}$ where $H = \sum H_i$.

Thus if L is independent of s; i.e., $\frac{dL}{ds} = 0$, then $\frac{dH}{dt} = 0.$ and thus H is constant over time; i.e., H is conserved.

Now we continue with a more sophisticated example:

Example: Let E be a measurable set of real numbers of positive measure. Prove that the set D := {x - y: x and y in E} contains an interval.

Solution: We'll transform the given information about *sets* to corresponding information about *functions*, by associating to E its "characteristic function" f, defined by

f(x) = 1 if x is in E, 0 otherwise. (5.12)

Now, for fixed t, the function

g(x) = f(x+t)f(x) (5.13)

equals 1 if both x and x+t are in E, and 0 otherwise. Thus, if this function g(x) is not identically 0, there is an x for which x and x+t are in E, and hence t is in D. From here on assume that E is bounded. Certainly g(x) is not identically 0 if its integral w.r.t. x is positive. But

h(t):= Int[f(x+t)f(x) dx] (5.14)

is a continuous function of t and, since h(0) equals the measure of E, which is positive, g is positive for all t with |t| sufficiently small; so those t belong to D, and the proof is finished**.**

5.3.3 Scaling is Another Aspect of Invariance

Consider for example the famous "Liouville theorem": A bounded entire function is constant.

Proof: Suppose f is entire and $|f(z)| \leq M$ for all complex z. We'll assume as known the maximum modulus theorem, and its immediate consequence, *Schwarz' lemma*: If g is holomorphic in the unit disk D and |g| is bounded by 1, and g(0)= 0, then $|g(z)| \leq |z|$. (we grasp at once that the Schwarz lemma is the "high ground" for Liouville's theorem (and many others!).

So, define g(z) = (f(Rz) - f(0))/2M, where R is an arbitrary positive number. Since Schwarz' lemma is applicable to g we get

$|f(Rz) - f(0)| \leq 2M |z|$.

Hence, for any complex number w, setting z = w/R in the last inequality, we obtain

$$|f(w) - f(0)| \leq 2M\,|w|/R\,. \qquad (5.15)$$

Since this is valid for all positive R, we conclude that f(w) - f(0) vanishes identically**.**

5.3.4 Another Aspect of Invariance: Invariants

There are still many other facets of the invariance concept, One such involves *conserved quantities* or "invariants" (here the "quantity" need not be a number; it might be a parity, or an algebraic entity.)

We look first at a deceptively simple example: Can a 5 x 13 rectangle be decomposed into finitely many pieces which may be reassembled into an 8 x 8 square?

"Obviously, no. Because the areas of the two figures are unequal." is the immediate reaction. But, suppose the concept of area were not known to us. It would be very difficult to demonstrate the impossibility. To appreciate this, consider the three dimensional analog: It is known that there are two convex polyhedral of equal volume such that neither can be decomposed into finitely many pieces which may be reassembled to give the other (Hilbert's third problem, solved by Max Dehn). Here it turns out that there are other "invariants" besides the volume that must match before such construction is possible. (This is in contrast to the planar situation: for two polygons of equal area one can always dissect and reassemble the one to obtain the other.

A substantial portion of higher mathematics is concerned with identifying, for a given class of objects and a given family of transformations of these objects, a "complete set of invariant quantities" the matching of which, for two objects in the class is necessary, or sufficient (or, preferably both) for them to be mutually transformable one to the other. (Example: The objects are n x n complex matrices with n distinct eigenvalues. Two matrices in this family are similar if and only if the two sets of eigenvalues coincide; if we allow multiple eigenvalues, this criterion is necessary but no longer sufficient for similarity.

5.3.5 Symmetry

In attacking mathematical problems, one should always look for symmetries that can be exploited to simplify matters. A typical exploitation of symmetry, which occurs in studying a boundary value problem, is to observe that the problem considered has rotational invariance and therefore use polar coordinates. Other kinds of symmetry are more subtle, such as the *duality* (or auto orphism) inherent in projective plane geometry, interchanging the roles of points and lines and thus yielding for free a new theorem "dual" to any given one. In the case of differential equations, the discovery of a symmetry may enable one to reduce the number of variables e.g. transform from a partial to an ordinary differential equation, or from an ordinary differential equation to one of lower order.(These observations were the point of departure for the researches of Sophus Lie.)

Example: The one dimensional heat equation $u_t = u_{xx}$ has the symmetry

$$v = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}. \qquad (5.16)$$

We find invariants of the group generated by v by solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2 + 2t)u} \qquad (5.17)$$

To demonstrate the method ,we will solve this in steps. First, solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2} \qquad (5.18)$$

Gives $ln\, x = ln\, t + c$. Therefore $c = ln\,(x / t)$.

We could take $\eta = ln\,(x / t)$, but since we may actually take any function of $ln\,(x / t)$ for η, it makes sense just to write $\eta = e^{\ln(x/t)} = x/t$.

Notice that we then have $x = \eta t$. Returning to the previous equation we have to solve $\dfrac{dt}{4t^2} = -\dfrac{du}{(\eta^2 t^2 + 2t)u}$

Integration leads to

$$\ln u = -\frac{1}{4}\eta^2 t - \frac{1}{2}\ln t + D.$$

Where D is the result of combining the constants of integration from both sides of the equation. Since $\eta = x/t$ we have

$$D = \ln(\sqrt{t}u) + \frac{x^2}{4t}.$$

This gives us our second invariant . In fact, any function of D will be a second invariant.

Let us take $v = e^D$ as our second invariant. That is, we set

$$v = \sqrt{t}e^{-x^2/4t}u$$

To be the second invariant. Notice that the two sets of invariants we have obtained are functionally independent. Let us find the group invariant solutions for the heat equation $u_t = u_{xx}$ where the group action is generated by the vector field $v = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}.$

We found two invariants, namely $\eta = x/t$ and $v = \sqrt{t}e^{-x^2/4t}u$, let our change of variables be $v = x/t$, and $v = \sqrt{t}e^{-x^2/4t}u$.

Applying the chain rule we have

$$u_t = \left(\frac{(-2t + x^2)v(y) - 4xv'(y)}{4t^{\frac{5}{2}}}\right)e^{-x^2/4t} \qquad (5.19)$$

Turning to the x derivatives gives

$$u_{xx} = \left(\frac{(-2t + x^2)v(y) - 4xv'(y) + 4xv''(y)}{4t^{\frac{5}{2}}}\right)e^{-x^2/4t} \qquad (5.20)$$

So using our expression for u_t and u_{xx}, the heat equation becomes $v''(y) = 0$. The general solution is just v(y) = $Ay + B$. Hence the group invariant solutions are of the form

$$u(x,t) = \frac{1}{\sqrt{t}}e^{-x^2/4t}\left(A\frac{x}{t} + B\right) \qquad (5.21)$$

Taking A=o, $B = \frac{1}{\sqrt{4\pi}}$ will give the fundamental solution of the heat equation.

5.3.6 Another aspect of invariance: impossibility proofs

We are familiar with "geometric constructions" using straightedge and compass, subject to certain strict ground rules as to how these may be used. And we know e.g. that certain constructions are impossible, such as inscribing a regular heptagon in a circle because it entails, algebraically, constructing the root of an irreducible cubic equation with integer coefficients, from a given segment of length one. (Galois theory teaches us that a necessary condition for constructability of a segment of length x is that f(x) vanish for some nontrivial polynomial f with integer coefficients, irreducible over the rational field, and having a degree which is a power of 2.) Less familiar to most students, and very elegant and instructive, are problems of construction with more restricted means, such as allowing use only of a straightedge (ruler). It is remarkable that certain *a priori* very unlikely looking configurations can be constructed using a straightedge. Given a circle in the plane and a point P outside, it is required to construct a line through P tangent to the circle. (Caution! It is not permitted to place the ruler so that it touches P, and rotate it about P until it also touches the circle.)

Amazingly, this problem has a solution (nontrivial and very instructive. But now consider the problem: Can one bisect a given segment using only a straightedge?

The answer is **NO**. To see why, consider a line L in the plane P, and two marked points A, B on it. It is desired to construct the midpoint M of the segment AB using the straightedge. Suppose we have found a procedure which works. Now, suppose we have a one-to-one mapping of plane P onto another plane P' which carries lines to lines, but which does not preserve the relation "M is the midpoint of the segment AB", in other words A, M, B are carried to points A', M', B' with A'M' unequal to B'M'. Then, this leads to a contradiction, because the construction of the midpoint in the plane P induces a construction in P' which also would have to lead to the midpoint of A'B'. Well, can the mapping of P onto P' be supplied? Again, the answer is "almost" - we again have to work with a projective plane, but with care concerning "improper points" it work.

The following problem illustrates the role of *invariants* (that is, conserved quantities) in impossibility proofs. We introduce it by recalling an old favorite: From an 8 x 8 chessboard two diagonally opposite squares are removed. Can the resulting board be tiled using 31 dominoes (each domino being a 1 x 2 rectangle congruent to a couple of adjacent squares of the chessboard)?

The answer is "No", the reason being that the mutilated chessboard has unequally many black and white squares (since the two removed squares have the same color), whereas a domino always covers one white and one black square.

This solution is very elegant, especially if one considers that in posing the problem there is no need to speak of a chessboard with colored squares, it could as well have been a white board ruled by lines into an 8 x 8 grid of squares. Then the cleverness of the solution is to realize that one should color the squares in an alternating pattern. This suggests a strategy for similar but more intricate tiling problems.

5.4 Group Invariants Solutions of Partial Differential Equations

5.4,1 Methods for Constructing Invariants

In this section we show how one finds the invariants of a given group action. First suppose G is a one-parameter group of transformation acting on M, with infinitesimal generator

$$\upsilon = \xi^1(x)\frac{\partial}{\partial x^1} + \dots + \xi^m(\chi)\frac{\partial}{\partial x^m} \quad (5.22)$$

expressed in some given local coordinates. A local invariant $\xi(\chi)$ of G is a solution of the linear ,homogenous first order partial differential equation

$$\upsilon(\xi)=\xi^1(x)\frac{\partial\xi}{\partial x^1}+\cdots+\xi^m(x)\frac{\partial\xi}{\partial x^m} \qquad (5,23)$$

By a theorem if $\nu\big|_\chi \neq 0$, then there exists m−1 functionally independent invariants, hence m −1 functionally independent solutions of the partial differential equation (5,1)in a neighborhood of χ0.

The classical theory of such equations shows that the general solution of (5,1) can be found by integrating the corresponding characteristic system of ordinary differential equations, which is :

$$\frac{d\chi^1}{\xi^1(\chi)}=\frac{d\chi^2}{\xi^2(\chi)}=\ldots=\frac{d\chi^m}{\xi^m(\chi)} \qquad (5,24)$$

The general solution of (5,2) can be written in the form :

$$\xi^1(\chi^1,\ldots,\chi^m)=c_1,\ldots,\xi^{m-1}(\chi^1,\ldots,\chi^m)=c_{m-1} \quad (5.25)$$

In which $c_1,\ldots,c_{m-1}$ re the constants of integration, and $\xi^i(\chi)$ are functions independent of the c_j,s. It is then easily seen that the functions $\xi^1,\ldots,\xi^{m-1}$ are the required functionally independent solutions to (5,1).

Any other invariant, I,e any other solution of(5,1), will necessarily be a function of $\xi^1,\ldots,\xi^{m-1}$. We illustrate this technique with a couple of examples

Example : (a) Consider the Rotation group so(2) , which has infinitesimal generator

$$v=-y\,\partial_\chi+x\,\partial_y \qquad (5.26)$$

The corresponding characteristic system is

$$\frac{d\chi}{-y}=\frac{dy}{\chi} \qquad (5.27)$$

This first order ordinary equation is easily solved ,the solutions are x2+ y2 = c for c an arbitrary constant . Thus $\xi(\chi, y) = \chi^2 + y^2$, or any function ,thereof,is the single independent invariant of the rotation group ,

(b) Consider the vector field

$$\upsilon = -y\frac{\partial}{\partial x} + \chi\frac{\partial}{\partial y} + \left(1+z^2\right)\frac{\partial}{\partial z} \qquad (5.28)$$

Defined onR3 , Note that v never vanishes, so we can find two independent invariants of the one- parameter group generated by v , in a neighborhood of any point in R3. The characteristic system in this case is:

$$\frac{d\chi}{-y} = \frac{dy}{\chi} = \frac{dz}{1+z^2} \qquad (5.29)$$

The first of these tow equations was solved in part (a) , so one of the invariants is the radius $\sqrt{\left(\chi^2 + y^2\right)}$.

. To find the other invariant ,note tat r is a constant for all solutions of the characiristic system , so we can replace x by $\sqrt{\left(r^2 - y^2\right)}$., before integrating .This lead to the equation:

$$\frac{dy}{\sqrt{\left(r^2 - y^2\right)}} = \frac{dz}{1+z^2} \qquad (5.30)$$

Which has the solution

$$\arcsin\frac{y}{r} = \arctan z + k \qquad (5.31)$$

For k an arbitrary constant , thus

$$\arctan z - \arcsin\frac{y}{r} = \arctan z - \arctan\frac{y}{\chi}$$

Is a second independent invariant for v.A slightly simple expression comes by taking the tangent of this invariant , which is (xz+y)/(yz+x) ,so

$$r=\sqrt{\chi^2+y^2} \text{ and } \xi=\frac{xz-y}{yz+x}$$

Provide a complete set of functionally independent invariants (provided yz≠ −x)

As usual any function of r and ξ is also an invariant, so, for instance

$$\bar{\xi}=\frac{r}{\sqrt{1+\xi^2}}=\frac{x+yz}{\sqrt{1+z^2}}$$

Is also an invariant , which in conjunction with r forms yet another pair of independent invariants.

EXAMPLE : Consider the vector fields

$$v=-y\frac{\partial}{\partial\chi}+\chi\frac{\partial}{\partial y},w=2xz\frac{\partial}{\partial\chi}+2yz\frac{\partial}{\partial y}+\left(z^2+1-\chi^2-y^2\right)\frac{\partial}{\partial z}$$

On R^3.An invariant ξ(x,y,z) is a solution to the pair of equations v(ξ)=0=w(ξ)

First note that independent invariant of v are just r= $\sqrt{x^2}+y^2$ and z . We now re-express w in terms of r and z

$$w=2rz\frac{\partial}{\partial r}+\left(z^2+1-r^2\right)\frac{\partial}{\partial z} \tag{5.32}$$

Since ξ must be a function of the invariants r , z, of v , it must be a solution to the differential equation

$$w(\xi)=2rz\frac{\partial\xi}{\partial r}+\left(z^2=1-r^2\right)\frac{\partial\xi}{\partial z} \tag{5.33}$$

The Characteristic system here is

$$\frac{dr}{2rz}=z^2+1-r^2$$

Solving this ordinary deferential equation, we find that

$$\xi = \frac{z^2 + r^1 + 1}{r} = \frac{\chi^2 + y^2 + z^2 + 1}{\sqrt{\chi^2 + y^2}} \qquad (5.34)$$

Is the single independent invariant of this group.

5.4.2 Reducing The Order of the ODE,s

Knowledge of a group of symmetries of a system of first order ordinary differential equation has much the same consequences as a knowledge of a similar group of symmetries of a single higher order equation .If we know a one-parameter symmetry group , then we can find the solution by quadrature from the solution to a first order system with one fewer equation in it. Similarly ,knowledge of an r- parameter solvable group of symmetries allows us to reduce the number of equations by r. These results clearly extend to higher order system s as well, so that invariants of an nth order system under a one –parameter group , say , allows us to reduce the order of one of the equations in the system by one .However, a higher order system can always be reduced by an equivalent first order system , so we are justified in restricting our attention to the latter case.

Theorem : Let

$$\frac{du^2}{dx} = F_\upsilon(\chi, u), \upsilon = 1, \ldots, q \qquad (5,35)$$

Be a first order system of q ordinary differential equations .Suppose G is a one-parameter group of symmetries of the system .Then there is a change of variables (y,w)=ψ(x,u) under which the system takes the form

$$\frac{dw^\upsilon}{dy} = \mathrm{H}_\upsilon\left(y, w^1, \ldots, w^{q-1}\right), \upsilon = 1, \ldots, q \qquad (5,36)$$

Thus the system reduces to a system of q-1 ordinary differential equations for $w^1, \ldots, w^{q-1}$ together with the quadrature

$$w^q = \int \mathrm{H}_q\left(y, w^1(y), \ldots, w^{q-1}(y)\right)dy + c. \qquad (5,37)$$

Proof: Let v be the infinitesimal generator of G .Assuming $v\big|_{\chi} \neq 0$, we can locally find new coordinates

$$y = \eta(x,u), w^{\nu} = \xi^{\nu}(x,u), \nu = 1,\dots,q,$$

Such that $v = \frac{\partial}{\partial w^{q}}$ in these coordinates .In fact

$$\eta(x,u), \xi^{1}(x,u), \dots, \xi^{q-1}(x,u)$$

Will be a complete set of functionally independent invariants of G , so

$$v(\eta) = v(\xi^{\nu}) = 0, \nu = 1,\dots,q-1 \qquad (5,38)$$

while $\xi^{q}(x,u)$ satisfies

$$v(\xi^{q}) = 1 \qquad (5.39)$$

i.e. it is of the form

It is then a simple matter to check that the equivalent first order system for w1 ,…,wq is invariant under the translation group generated by $\upsilon = \frac{\partial}{\partial w^{q}}$ if and only if the right hand –sides are all independent of wq i.e. it is of the form (5,6).

5.4.2 Example: Consider an autonomous system of tow equations

$$\frac{du}{dx} = F(u,v), \frac{dv}{dx} = \mathrm{H}(u,v).$$

Clearly $\upsilon = \frac{\partial}{\partial x}$ generates a one-parameter symmetry group ,so we can reduce this to a single first order equation plus a quadrature. The new coordinates are y=u, w=v and z=x in which we are viewing w and z as functions of y. Then ,

$$\frac{du}{dx} = \frac{1}{dz/\partial y}, \frac{d\upsilon}{dx} = \frac{dw/dy}{dz/dy}.$$

So we have the equivalent system

$$\frac{dw}{dx} = \frac{\mathrm{H}(y,w)}{F(Y,W)} = \frac{1}{F(Y,W)}.$$

We thus are left with a single first order equation for w=w(y),the corresponding value of z = z(y) is determined by a quadrature

$$z = \int \frac{dy}{F(Y, W)} + C.$$

If we revert to our original variables x,u,v, we see that we just have the equation

$$\frac{d\upsilon}{dx} = \frac{\mathrm{H}(u, w)}{F(U, \upsilon)}.$$

For the phase plane trajectories of the system ,the precise motion along these trajectories being then determined by quadrature:

$$\chi = \int \frac{du}{F(u, \upsilon(u))} + c.$$

5.4.3 Theorem: Suppose $du/dx = F(x, u)$ is a system of first order ordinary differential equations, and suppose G is an r-parameter solvable group of symmetries, acting regularly with r-dimensional orbits. Then the solution u=f(x) can be found quadrature from the solutions of a reduced system $dW/dy = \mathrm{H}(Y, W)$ q-r first order equations. In particular, if the original system is invariant under a q-parameter solvable group ,its general solution can be found by quadrature alone,

5.4.4 Example : Any linear two-dimensional system

$$u_t = \alpha(t)u + \beta(t)v,$$

$$v_t = \gamma(t)u + \delta(t)v.$$

Is invariant under the one-parameter group of scale transformation(t,u,v) $\rightarrow$ $(t, \lambda u, \lambda v)$ with infinitesimal generator $v = u\partial_u + v\partial_v$, and hence can be reduced to a single first order equation. We set $w = \log u, z = v/u$, which straightens out $v = \partial_w$. These new variables satisfy the transformed system

$$w_t = \alpha(t)u + \beta(t)z ,$$
$$z_t = \gamma(t)u + \left(\delta(t) - \alpha(t)\right)z - \beta(t)z^2$$

So if we can solve the Riccati equation for z , we can find w (and hence u and v) by quadrature.

However , if the original system possesses some additional symmetry property ,it may be unwise to carry out this preliminary reduction, as the resulting Riccati equation may no longer be invariant under some "reduced " system group.

For example , the system

$$u_1 = -u + (t-1)v$$
$$v_t = u - tv$$

Has an additional one-parameter symmetry group with generator $w = t\partial_w + \partial_{\upsilon}$, but the associated Riccati equation

$$z_1 = 1 + (1-t)z - (1-t)z^2$$

Has no obvious symmetry property. The problem is that the vector fields v and w generates a solvable two-dimensional Lie group, but have the commutation relation $[v, w] = -w$, so we should be reducing first with respect to w .

We need to first straighten out $w = \partial_w$ by choosing coordinates

$$\breve{w} = v , \breve{z} = u - tv .$$

The scaling group still has generator $v = \breve{w}\partial_{\breve{w}} + \breve{z}\partial_{\breve{z}}$ in these variables .To straighten its $\breve{z}$ -component we further set $\breve{z} = \log \breve{z} = \log(u - tv)$, in terms of which

$$w = \partial_{\breve{w}} , v = \breve{w}\partial_{\breve{w}} + \partial_{\breve{z}}$$

The system now takes the form

$$\frac{d\breve{w}}{dt} = e^{\breve{z}} , \frac{d\breve{z}}{dt} = -t - 1$$

Which can be integrated by quadrature's , we find

$$\breve{z}(t) = -\frac{1}{2}(t+1)^2 + \breve{c}$$

$$\breve{w}(t) = c\,erf\left[(t+1)/\sqrt{2}\right] + k ,$$

Where $\breve{c} = \log\left(c\sqrt{2/\pi}\right)$ and erf(y)= $\frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}dx$

Is the standard error function. Thus the general solution to the original system is

$$u(t) = \sqrt{\frac{2}{\pi}}ce^{-(t+1)^2/2} + ct\,erf\left(\frac{t+1}{\sqrt{2}}\right) + kt , v(t) = c\,erf\left(\frac{t+1}{\sqrt{2}}\right) + k ,$$

Where c and k are arbitrary constants.

Summary

So as we have seen in this thesis that , group theory is the most major theory of the 20th century, and it was an abstraction of ideas that were common to a number of major areas which were being studied essentially simultaneously ,and that groups have arisen in almost every branch of mathematics.

One of the most important groups is the transformation group which I have defined as ((a collection of transformations which forms a group with composition as operation)).

I have shown that not every collection of transformations will form a group, so I have stated the properties that any collection of transformations should have to be a group.

In chapter 1 there are basic concepts of groups with definitions and examples.

Definition and properties of transformation groups were stated in chapter 2 with isometry as an example of transformation groups.

Lie groups and symmetric spaces were studied in chapter 3, In chapter 4 a study of representation of Lie groups were given

In chapter 5 classification of semi simple Lie groups were given as an application and other applications such as: "how to use invariants in solving variants "which is the main problem discussed by this thesis.

A list of references was given at the end and I hope that I have done it well.

I hope that I have done this thesis well and of course I enjoyed doing this and I hope that I will write more about transformation groups in the future.

REFERENCES

1. Adams, John Frank (1969), *Lectures on Lie Groups*, Chicago Lectures in Mathematics, Chicago: Univ. of Chicago Press, ISBN 0-226-00527-5.

2. Baldwin, J. T. (2001), "Stability theory (in logic)", in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

3. Baldwin, J. T. (2001), "Stability theory (in logic)", in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

4. Borel, Armand (2001), *Essays in the History of Lie Groups and Algebraic Groups*, Providence, R.I.: American Mathematical Society, ISBN 978-0-8218-0288-5.

5. Borel, Armand (2001), *Essays in the history of Lie groups and algebraic groups*, History of Mathematics **21**, Providence, R.I.: American Mathematical Society, ISBN 978-0-8218-

6. Buechler, Steven (1996), *Essential stability theory*, Perspectives in Mathematical Logic, Berlin: Springer-Verlag, pp. xiv+355, ISBN 3-540-61011-1, MR 1416106

7. Buechler, Steven (1996), *Essential stability theory*, Perspectives in Mathematical Logic, Berlin: Springer-Verlag, pp. xiv+355, ISBN 3-540-61011-1, MR 1416106

8. D. Lascar, "Stability in model theory", Wiley (1987)

9. Edwin Thompson Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, 2003. ISBN 0-521-59271-2.

10. Erdmann, Karin; Wildon, Mark (2006), *Introduction to Lie Algebras* (1st ed.), Springer, ISBN 1-84628-040-0.

11. Fulton, William; Harris, Joe (1991), Representation theory. A first course, Graduate Texts in Mathematics, Readings in Mathematics 129, New York: Springer-Verlag, ISBN 978-0-387-97495-8, MR 1153249, ISBN 978-0-387-97527-6.

12. Hall,BrianC.(2003),Lie groups,Lie algebras,and representations:An elementary introduction,Springer

13. Helgason, Sigurdur (1978). *Differential Geometry, Lie Groups, and Symmetric Spaces*. New York: Academic Press..ISBN 0-12-338460-5.

14. Hodges, Wilfrid (1993), *Model theory*, Cambridge University Press, ISBN 978-0-521-30442-9

15. Hodges, Wilfrid (1993), *Model theory*, Cambridge University Press, ISBN 978-0-521-30442-9

16. Humphreys, James E. (1972), *Introduction to Lie Algebras and Representation Theory*, Berlin, New York: Springer-Verlag, ISBN 978-0-387-90053-7.

17. J.T. Baldwin, "Fundamentals of stability theory", Springer (1988)

18. J.T. Baldwin, "Fundamentals of stability theory", Springer (1988)

19. Kleiner, Israel (1986), "The evolution of group theory: a brief survey", *Mathematics Magazine***59** (4): 195–215, doi:10.2307/2690312, MR 863090.

20. Knapp, Anthony W. *Representation theory of semisimple groups. An overview based on examples*. Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, New Jersey, 2001. xx+773 pp. ISBN 0-691-09089-0

21. Kurzweil, Hans; Stellmacher, Bernd (2004), *The theory of finite groups*, Universitext, Berlin, New York: Springer-Verlag, ISBN 978-0-387-40510-0, MR 2014408.

22. Lay, David (2003), *Linear Algebra and Its Applications*, Addison-Wesley, ISBN 978-0-201-70970-4

23. Ledermann, Walter (1953), *Introduction to the theory of finite groups*, Oliver and Boyd, Edinburgh and London, MR 0054593.

24. Ledermann, Walter (1973), *Introduction to group theory*, New York: Barnes and Noble, OCLC 795613.

25. LEE,JOHN,M (2002)Introduction to smooth manifolds,Springer.

26. Marker, David (2002), *Model Theory: An Introduction*, Graduate Texts in Mathematics, Berlin, New York: Springer-Verlag, ISBN 978-0-387-98760-6

27. Marker, David (2002), *Model Theory: An Introduction*, Graduate Texts in Mathematics, Berlin, New York: Springer-Verlag, ISBN 978-0-387-98760-6

28. Miller, Willard Jr. (1972). *Symmetry Groups and Their Applications*. New York: Academic Press. OCLC 589081. Retrieved 2009-09-28

29. Munkres, James; *Topology*, Prentice Hall; 2nd edition (December 28, 1999). ISBN 0-13-181629-2.

30. O'Connor, J.J; Robertson, E.F. (1996), *The development of group theory*

31. Robinson, Derek John Scott (1996), *A course in the theory of groups*, Berlin, New York: Springer-Verlag, ISBN 978-0-387-94461-6.

32. Schubert, Horst (1968), *Topology*, Allyn and Bacon

33. Scientific Puplishing

34. Serre, Jean-Pierre (1977), *Linear representations of finite groups*, Berlin, New York: Springer-Verlag, ISBN 978-0-387-90190-9, MR 0450380

35. Serre, Jean-Pierre (1977). *Linear Representations of Finite Groups*. Springer-Verlag. ISBN 0-387-90190-6.

36. Steeb,Willi-HANS (2007), continuous symmetries , World

37. Varadarajan, V. S. (2004), *Lie Groups, Lie Algebras, and Their Representations* (1st ed.), Springer, ISBN 0-387-90969-9.

38. Warner, Frank (1983), *Foundations of Differentiable Manifolds and Lie Groups*, Berlin, New York: Springer-Verlag, ISBN 978-0-387-90894-6.

39. Weyl, Hermann (1952), *Symmetry*, Princeton University Press, ISBN 978-0-691-02374-8

40. Willard, Stephen (2004). *General Topology*. Dover Publications. ISBN 0-486-43479-6.