Chapter 1

Approximation of H^p-function on compact Lie groups

In this chapter we will discuss the boundedness of the operators $f \to \delta_R^\delta(f)$ on $H^p(G)$, and we establish some approximation results.

Section (1.1): The Bochner-Riesz Means and H^P-boundedness

Let G be a compact connectedsemi simpleLie group (A Lie group is a group G, equipped with a manifold structure such that the group operations

$mult: G \times G \to G, (g_1, g_2) \to g_1 g_2$, $inv: G \times G \to G, g \to g^{-1}$are smooth) with dimension n. for any distribution $f \in s'(G)$ the Bochner - Riesz mean of order δ of f are given by:

$$\delta_R^\delta(f)(x) = \sum \left(1 - \frac{|\lambda + \beta|^2}{R^2}\right)_+^\delta d_\lambda Tr\left(\hat{f}(\lambda) U_\lambda(x)\right).$$

When $G = T^n$,the n-dimensional torus, it is well known that the operators $f \to s_R^\delta(f)$

Maps H^p boundedly into itself if $\delta > \delta_p = n/p - (n+1)/2 (0 < p \le 1)$, and $f \to \sup_{R>0} |s_R^\delta(f)|$ maps H^p to weak - L^p if $\delta = \delta_p (0 < p < 1)$. Here the order δ_p plays a key role and is called the critical index. In this setting of general compact Lie groups, weak-type estimates for the maximal operator $\sup_{R>0} |\delta_R^\delta(f)|$ at the critical index for H^p (G)-functions have been studied, we discuss the boundedness of the operator $f \to \delta_R^\delta(f)$ H^p (G)

Theorem (1.1.1) [1]:

Suppose $0 < p \le 1$ and $\delta > \delta_p = n/p - (n+1)/2$. Then $f \to S_R^\delta(f)$ is of type (H^p, H^p) and

$$\|S_R^\delta(f)\|_{H^p} \le C \|f\|_{H^p}$$

With constant C independent of f and R. Moreover for $f \in H^p(G)$, $\|S_R^\delta(f) - f\|_{H^p} \to 0$ as $R \to \infty$.

Theorem (1.1.1) cannot be extended to the case $\delta = \delta_p$. There is, however, a weak result:-

Proof:

Consider any $(p, 2, s)$-atom$a$, which we can assume to be regular with $\operatorname{supp} a \subset B(x_0, \rho)$ where $\rho < \varepsilon_0/8$. (Otherwise $S_R^\delta(a)$ differs from an exceptional 2-atom by a positive constant independent of R and$a$). Choose a positive integer $i_0 = i_0(\rho)$ such that$2^{i_0}\rho \leq \varepsilon_0/4 < 2^{i_0+1}\rho$, and put

$$U_0 = \{x \in G: d(x, e) < 2\rho\}$$

$$U_i = \left\{x \in G: 2^{i-1}\rho < d(x, e) < 2^{i+1}\rho\right\}$$

for $i = 1,2, \dots, i_0$ and

$$U = \left\{x \in G: d(x, e) > 2^{i_0}\rho\right\}.$$

Then $U_i \subset B(e, \varepsilon_0/2)$ for $i = 0,1, \dots, i_0$ and $U \subset G \backslash B(e, 2^{-3}\varepsilon_0)$. Writing $V_i = \exp^{-1} U_i$ we can choose a partition of unity $\{\psi_i\}_{i=0}^{i_0}$ subordinate to the open covering $\{V_i\}_{i=0}^{i_0}$ of $\bar{B}(0, \varepsilon_0/8) \subset \mathfrak{g}$, and satisfying

$$\left|\frac{\partial^{\boldsymbol{j}}}{\partial \boldsymbol{X}^{\boldsymbol{j}}} \psi_i(X)\right| \leq C_j \left(2^i \rho\right)^{-|\boldsymbol{j}|}$$

For each multi-index$\boldsymbol{j}$. For $0 \leq i \leq i_0$ define $\Phi_i(x) = \psi_i(X)$ for $x = \exp X \in B(e, \varepsilon_0)$, for $\Phi_i(x) = 0$ otherwise. Observing that $(B(e, \varepsilon_0), \exp^{-1} \quad)$ is chart of G we have for $x \in G$ and $i = 0,1, \dots, i_0$

$$\left|\boldsymbol{X}^{\boldsymbol{j}} \Phi_i(X)\right| \leq C_j \left(2^i \rho\right)^{-|\boldsymbol{j}|}.$$

Thus we can obtain a partition of unity $\{\phi_i\}_{i=0}^{i_0} \cup \{\phi\}$ subordinate to the open covering

$\{U_i\}_{i=0}^{i_0} \cup \{U\}$ of G, and satisfying

$$\left|\boldsymbol{X}^{\boldsymbol{j}} \phi(X)\right| \leq C_j (1)$$

for each $\boldsymbol{j} = (j_1, \dots, j_n)$ and $i = 0,1, \dots, i_0$.

Let $k_i(x) = k_i^{\delta,R}(x) = D_i^{\delta,R} * a(x)$ and $k(x) = k^{\delta,R} * a(x)$ where $D_i^{\delta,R}(x) = D_R^\delta(x)\phi_i(x)$ and $D^{\delta,R}(x) = D_R^\delta(x)\phi(x)$. It is obvious that

$$\operatorname{supp} k_i \subset B\left(x_0, 2^{i+2}\rho\right) \qquad (2)$$

and

$$\int_G k_i(x)P\big(\pi(x)\big)\,dx = 0 \qquad (3)$$

For any polynomial P on Ewithdeg$(P) \leq s$, and for all $i = 0,1,\dots,i_0$ by (2), (3) and theorem (1.1.1) it is easy to see that the proof of the first part of Theorem (1.1.1) can be reduced to showing the following estimates:

$$\|k_i\|_2 \leq C2^{-i\sigma}2^{in(1/2-1/p)}\|a\|_2, \;\; i = 0,1,\dots,i_0 \qquad (4)$$

$$\|k\|_2 \leq C \qquad (5)$$

With $C = C(\delta, s, p, n, \varepsilon_0)$ independent of R, i and a. Hence $\sigma = \min(\delta_0, s_0) > 0$,

$s_0 = s - n/p + n + 1$, and $\delta_0 = \delta - \delta_p$. Furthermore we can assume that $i \geq 3$. Then $k_i(x) = 0$ if $d(x, x_0) \leq 2\rho$.

The following estimates for the derivatives ofD_R^δ were established in.

$$|X^j D_R^\delta| \leq \begin{cases} CR^{k+n} & if\ d(x\,,e) < R^{-1} \\ CR^{k+\frac{1}{2}(n-1)-\delta}\,(d(x\,,e)^{\frac{-1}{2}+(n+1)-\delta} + \varDelta^R(x^{-1}) & other\ wise \end{cases} \qquad (6)$$

for any multi-index $\boldsymbol{j}$, $|\boldsymbol{j}| \leq k$, where $\Lambda^{(R)}$ is a central function on G given by

$$\varDelta^{(R)}(x) = \varDelta^{(R)}(\exp G) = \prod_{x\in\varDelta^+} \max(R^{-1}, \sin(ix\,(H)/2))$$

and $\exp H$ is the unique element in $\exp \bar{A}$ conjugate to $x \in G$. Let D be the Weyl denominator

$$D(x) = D(\exp H) = (-2i)^m \prod_{x\in\varDelta^+} \sin(ix\,(H)/2).$$

Then by Weyl's integral formula we have

$$\int_{U_i} \varDelta^{(R)}(x)^{-1}\,dx = C \int_{V_i\cap t} |D(\exp H)|^2 \varDelta^{(R)}(\exp H)^{-1}\,dH$$

$$\leq C\left(2^i\rho\right)^{m+t}.(7)$$

Using (6) and (7) we can deduce that (4) holds for $R > (2N\rho)^{-1}$ where $N = \sup_{\alpha\in\Delta}|x|$,

Let $T_s^x(\phi)$ be the Taylor polynomial of ϕ of order s at x. For $R < (2N\rho)^{-1}$ we use the cancellation property (iii)) of the atom a to obtain

$$|k_i(x)| = \int_{B(x_0,\rho)} a(y)\left(D_i^{\delta,R}(xy^{-1}) - T_s^{x,x_0^{-1}}\left(D_i^{\delta,R}(y)\right)\right)dy$$

$$\leq C\|a\|_2\rho^{s+1+n/2}\sup\left\{\left|\boldsymbol{X}^{\boldsymbol{j}}D_i^{\delta,R}(w)\right| : w \in B\left(xx_0^{-1},\rho\right), |\boldsymbol{j}|\leq s+1\right\}.$$

If$R > \left(2^{i-1}\rho\right)^{-1}$ then $d\geq R^{-1}$for $w \in U_i$. also for each $x \in G$we have $\Delta^{(R)}(w)^{-1} \lesssim \Delta^{(R)}(xx_0^{-1})^{-1}$ for all $w \in B(xx_0^{-1},\rho)$. Thus using (1) and (6) a straightforward calculation shows

$$\sup_{\substack{w\in B(xx_0^{-1},\rho)\\ |\boldsymbol{j}|\leq s+1}}\left|X^{\boldsymbol{j}}D_i^{\delta,R}(w)\right| \leq C_{\boldsymbol{j}}\boldsymbol{R}^{(n-1)/2-\delta+s+1}\left\{(2^i\rho)^{-\delta-(n+1)/2} + \Delta^{(R)}(xx_0^{-1})^{-1}\right\}.$$

If $\delta_0 \geq s_0$ then $\boldsymbol{R}^{s_0-\delta_0} \leq (2^i\rho)^{\delta_0-s_0}$ and

$$\|k_i\|_2 \leq C\|a\|_2\boldsymbol{R}^{s_0-\delta_0}\rho^{s+1+n/2}\left\{(2^i\rho)^{-\delta-n/2} + \left\|\Delta^{(R)-1}\right\|_2\right\}$$

$$\leq C\|a\|_2 2^{-i\sigma}2^{i(n/p-n/2)}$$

with $C = C(n,p,\delta,s,\varepsilon_0)$ independent of R, i and a. For $\delta_0 < s_0$ the same estimate for $\|k_i\|_2$ can be obtained if we use the inequality $\boldsymbol{R}^{s_0-\delta_0} \leq \rho^{\delta_0-s_0}$. Therefore (4) holds for $R > \left(2^{i-1}\rho\right)^{-1}$.

A similar argument shows that (4) holds for other values of R, and (5) can be proved by the same method.

For the second part of the theorem we observe that $\|P_t * f - f\|_{H^P} \to 0$ as $t \to 0$ for each $f \in H^P(G)$, and $P_t * f \in S(G)$ for all $t > 0$

$$\left\|S_R^\delta(P_t * f) - P_t * f\right\|_{H^P} \leq C\left\|S_R^\delta(P_t * f) - P_t * f\right\|_2 \to 0$$

as $R \to \infty$. Then we can easily deduce that $\left\|S_R^\delta(f) - P_t * f\right\|_2 \to 0$ as $R \to \infty$.

Theorem (1.1.2)[1]:

Suppose $0 < p < 1$ and $\delta = \delta_p$. There is, however, a weak boundedly and

$$\left\|S_R^\delta\right\|_{H(p,\infty)} \leq C\|f\|_{H^p}$$

with C independent of$f$ and R. Also if $f \in H^p(G)$ then $\left\|S_R^\delta(f) - f\right\|_{H(p,\infty)} \to 0$ as $R \to \infty$.

Let exp denote the exponential map of $\mathfrak{g}$ to G. We can choose a positive number ε_0 such that for each $x \in G$, $\exp^{-1} \circ L_{x-1}$ is an analytic diffeomorphism of $B(x, \varepsilon_0)$ onto $B(0, \varepsilon_0)$, an open ball centered at the origin of $\mathfrak{g}$, where L_x is the left translation by x given by $L_x(y) = xy$ for each $y \in G$.

Proof:

Following the idea in [4] we can prove that for any regular atom a with $a \subset B(x_0, \rho)$

$$\left(S_R^\delta(a)\right)_N^*(x) \leq C_{n,p}\{1 + (d(x, x_0) + \rho)^{-n/p} + M(|D|^{-1})(xx_0^{-1}) + M\big(M(|D|^{-1})\big)(xx_0^{-1})\} \quad (8)$$

where $N > n/p$, $C_{n,p}$ is a constant independent of$R$ and a, and

$$M(\phi)(x) = \sup_{0<r<\varepsilon_0/2} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\phi(y)|\, dy.$$

By (8) we have for $s > 1$

$$\lambda_{\left(S_R^\delta(a)\right)_N^*}(s) \leq C_{n,p} S^{-p}. (9)$$

It is easy to see that (9) follows for $s \leq 1$ owing to the compactness of G. This implies the first part of Theorem (1.1.2). We end the proof of the theorem by observing that if

$\phi \in L^p(G), 0 < p < \infty$ then$\phi \in L(p, \infty, G)$ and$\|\phi\|_{p,\infty} \leq \|\phi\|_p$.

Section (1.2):Approximation of H^p-function on compact Lie groups:

For $\sigma > 0$ we introduce two operators on $S'(G)$ as follows: The Riez differential operator $\Delta^{\sigma/2}$ (of order σ) is defined by

$$\left(\Delta^{\sigma/2}\mathrm{g}\right)^{\wedge}(\lambda) = \mu_\lambda^{\sigma/2}\hat{\mathrm{g}}(\lambda), \quad \lambda \in \hat{G}, \quad \mathrm{g} \in S'(G).$$

The fractional integral operator I^σ (of order σ) is given by

$$\left(I^{\sigma/2}\mathrm{g}\right)^{\wedge}(\lambda) = \mu_\lambda^{-\sigma/2}\hat{\mathrm{g}}(\lambda), \quad \lambda \in \hat{G}, \quad \mathrm{g} \in S'(G).$$

For $\sigma, p > 0$ we set

$$H^{p,\sigma}(G) = \left\{\mathrm{g} \in H^p(G) : \Delta^{\sigma/2}\mathrm{g} \in H^p(G)\right\}$$
$$H_2^{p,\sigma}(G) = \left\{\mathrm{g} \in H^p \cap L^2(G) : \Delta^{\sigma/2}\mathrm{g} \in H^p \cap L^2(G)\right\}.$$

Definition (1.2.1) [1]:

Let $0 < p, \sigma < \infty$ and $f \in H^p(G)$. The K-function of f (of order σ) is defined by

$$K_\sigma(f,t)_{H^p} = \inf\left\{\|f-h\|_{H^p} + t^\sigma\left\|\Delta^{\sigma/2}\right\|_{H^p} : h \in H^{p,\sigma}(G)\right\}.$$

As in the case when $G = T^n$ the K-function has the following properties:

Proposition (1.2.2) [1]:

For $0 < p, \sigma < \infty$ and $f, f_1 \in H^p(G)$

(i) $K_\sigma(f,t)_{H^p} \leq \|f\|_{H^p}$

(ii) $K_\sigma(f+\mathrm{g},t)_{H^p}^p \leq K_\sigma(f,t)_{H^p}^p + K_\sigma(\mathrm{g},t)_{H^p}^p$

(iii) $K_\sigma(f,t)_{H^p} = \inf\left\{\|f-h\|_{H^p} + \left\|\Delta^{\sigma/2}h\right\|_{H^p} : h \in H_2^{p,\sigma}(G)\right\}$

(iv) $K_\sigma(f,t)_{H^p} \to 0$ as $t \to 0$.

Property (iv) shows that it is appropriate to use the K-functionals as measures of approximation.

The proof of the proposition has been shown except for showing $\Delta^{\sigma/2}P_l * f \in H^p \cap L^2(G)$ which can be proved as follows. For $f \in H^p(G)$ we have

$$\left\|\hat{f}(\lambda)\right\| \leq C\|f\|_{H^p}|\lambda+\beta|^{n(1/p-1)}d_\lambda^{n+2}, \quad \lambda \in \hat{G} \qquad (10)$$

where $\|\cdot\|$ is the usual Hilbert-Schmidt norm. Observe that $d_\lambda \leq |\lambda+\beta|^m$ and $\mu_\lambda \sim |\lambda+\beta|^2$ where $\beta = \sum_{x\in\Delta} \alpha/2$. By Parseval's equality we know that $\Delta^{\sigma/2} P_l * f \in L^2(G)$. Furthermore $\Delta^{\sigma/2} P_l * f \in H^p(G)$ follows directly from the inclusion $L^2(G) \subset H^p(G)$ when $p \leq 2$ and $\Delta^{\sigma/2} P_l \in L^p(G)$.

We have the following results:-

Lemma (1.2.3) [1]:

Let a be any (p,∞,s)-atom with $s \geq [n(1/p-1)]$ and $0 < p \leqq 1$. Then

$$\left\|S_R^\delta(I^2 a) - I^2 a\right\|_2 \leq CR^{-2}\|a\|_2$$

With C independent of R and a.

Lemma (1.2.4) [1]:

Suppose that $\delta > \delta_p$ and $s = [n(1/p-1)]$ where $0 < p \leq 1$. For any (p,∞,s)-atom a we have

$$\left\|S_R^{\delta+k}(I^2 a) - I^2 a\right\|_p \leq CR^{-2}$$

with C independent of R and a, where $k \geq m+3$ is a positive integer.

Proof:

In view of Lemma (1.2.3) we can assume that a is regular and $\operatorname{supp} a \subset B(x_0,\rho)$ with $\rho < \varepsilon_0/8$. Write

$$\left\|S_R^{\delta+k}(I^2 a) - I^2 a\right\|_p^p$$

$$= \int_{d(x,e) \leq 6\rho} \left|S_R^{\delta+k}(I^2 a)(x) - I^2 a(x)\right|^p dx + \int_{d(x,e)>6\rho} \left|S_R^{\delta+k}(I^2 a)(x) - I^2 a(x)\right|^p dx.$$

Write this sum as $\sigma_R^{(1)} + \sigma_R^{(2)}$ where $\sigma_R^{(1)}$ represents the first integral and $\sigma_R^{(2)}$ the second. By Lemma (1.2.3) we have immediately $\sigma_R^{(1)} \leq CR^{-2p}$. The lemma follows once the same estimate for $\sigma_R^{(2)}$ is obtained.

Let W_t be the heat kernel on G given by

$$W_t(x) = \sum_{\lambda \in \widehat{G}} e^{-t\mu\lambda} d_\lambda \chi_\lambda(x)$$

and write

$$\psi(x) = \int_0^\infty (W_t(x) - 1)\, dt.$$

We can show that $\psi \in C^\infty(G\ \{e\})$ using the following estimates:

$$\left|X^j \chi_i(x)\right| \leq d_\lambda^n |\lambda + \beta|^s$$

And

$$\left|X^j W_i(x)\right| \leq C(1 + d(x,e)^{-n-s})$$

For each multi-index$\mathbf{j}, |\mathbf{j}| \leq s$, where the second inequality is assumed to hold for almost all$x$. Observe again that $(B(e,\varepsilon_0), \exp\quad^{-1})$ is a chart of G. Using the asymptotic expansion of the heat kernel

$$W_i(x) = t^{-n/2} \exp(-d^2\,(x,e)/4t)\, G_i(x)$$

WhereG_t satisfies $\|G_t\|_{C^k} \leq \quad c_k$ for every nonnegative integer k with c_k independent of $t > 0$ we obtain

$$\left|X^j \psi(x)\right| \leq C_j d(x,e)^{2-n-|\mathbf{j}|} \qquad (11)$$

for each multi-index $\mathbf{j}$and $x \in (e, \varepsilon_0)$. Now $I^2 a = a * \psi$. Let $F_i = \psi\phi_i$ and $F = \psi\phi$ where $\{\phi_i\}_{i=0}^{i_0} \cup \{\phi\}$ is the partition of unity subordinate to the open covering $\{U_i\}_{i=0}^{i_0} \cup \{U\}$ of . Then $F_i \in C^\infty(G)$ for $i \geq 1, F \in C^\infty(G)$, and by (11)

$$\left|X^j F_i(x)\right| \leq C_j \left(2^i \rho\right)^{2-n-|\mathbf{j}|}.$$
$$\left|X^j F_i(x)\right| \leq C_j \qquad (12)$$

Also

$$S_R^{\delta+k}(I^2a) - I^2a = \sum_{i=0}^{I_0-1}\left(S_R^{\delta+k}(a*F_i) - a*F_i\right) + S_R^{\delta+k}(a*F). \qquad (13)$$

We first consider the case $i=0$. Recalling that $\delta(x,e) > 6\rho$ we deduce that $F_0(xy^{-1}) = 0$ for $y \in B(e,\rho)$ since $\mathrm{supp}F_0 \subset U_0$. Using the cancellation property of the atom a we have

$$\left|S_R^{\delta+k}(a*F_0)(x) - a*F_0(x)\right| \le C\rho^{-n/p+n+s-1} \sup_{\substack{y\in B(x,p)\\ |j|=s-1}} \left|X^j\delta_R^{\delta+k}(F_0)(y)\right|.$$

Observe that $\frac{1}{2}d(x,e) \le d(yw^{-1},e) = d(y,w) \le \frac{3}{2}d(x,e)$ for $w \in B(e,\rho)$ and $y \in B(x,\rho)$. By (6) we have

$$\left|X^jS_R^{\delta+k}(F_0)(x)\right| \le \begin{cases} CR^{s-1+n}\rho^2 & \text{if } d(x,e) < R^{-1}. \\ CR^{s-1+(n-1)/2-\delta-k}\rho^2\left(d(x,e)^{-(n+1)/2-\delta-k} + R^m\right) & \text{otherwise.} \end{cases}$$

A direct calculation gives

$$\int_{d(x,e)>6\rho} \left|S_R^{\delta+k}(a*F_0)(x) - a*F_0(x)\right|^p dx \le CR^{-2p}. \qquad (14)$$

For $i \ge 1$ we observe

$$\int_{d(x,e)>6\rho} \left|S_R^{\delta+k}(a*F_i)(x) - a*F_i(x)\right|^p dx$$

$$\le C\rho^{-n+pn+ps+p}\left\{\int_{6\rho<d(x,e)\le 2^{i+2}\rho} + \int_{d(x,e)\ge 2^{i+2}\rho}\right\}\cdot\left(\sup_{\substack{y\in B(x,\rho)\\ |j|=s+1}}\left|X^jS_R^{\delta+k}(F_i)(y) - X^jF_i(y)\right|\right)^p dx.$$

Denote the last two integrals in ordered by I_1 and I_2. A similar argument as for the case when $i=0$ shows that

$$I_2 \le CR^{-2\rho}2^{-i(-n+pn+ps+p)}. \qquad (15)$$

We now estimate i_1. Put $f_i(y) = X^j F_i(y)$. By the Poisson summation formula for central functions on G we have

$$\left|X^j S_R^{\delta+k}(F_i)(y) - X^j F_i(y)\right| = \left|S_R^{\delta+k}(f_i)(y) - f_i(y)\right| \tag{16}$$

$$\leq CR^n \left| \int_Q \acute{f}_{i,y}(H) \prod_{x\in\Delta} (x,H)(R|H|)^{-\delta-k-n/2} J_{\delta+k+n/2}(R|H|) D(\exp H) sH \right|$$

$$+ CR^n \left| \int_Q \acute{f}_{i,y}(H) \sum_{\zeta\in t_e\{0\}} \prod_{\alpha\in\Delta} (\alpha, H+\zeta)\cdot(R|H+\zeta|)^{-\delta-n/2-k} J_{\delta+k+n/2}(R|H + \zeta|) D(\exp H)\, dH \right|$$

where$t_e = \{H \in t: \exp H = e\}$,$\acute{f}_{i,y}(H) = f_{i,y}(\exp H) + f_{i,y}(\exp(-H)) - 2f_i(y)$,$-f_{i,y}(\exp H) = \int_G f_i(yz(\exp H)z^{-1})dz$and $Q \subset t$ is the fundamental domain of T containing 0. Denote the first integral in (16) by I_3 and the second by I_4. For any $w \in G$ and $H \in Q$ we have $\mathrm{Ad}(w)H = \sum_{i=1}^n a_i(w,H)X_i$ and

$$\sum_{i=1}^n |a_i(W,H)|^2 = (\mathrm{Ad}(w)H, \mathrm{Ad}(w)H) = |H|^2$$

Observe that

$$\frac{d^k}{dt^k} f_i(y \exp(t\,\mathrm{Ad}(w)H)) = (\mathrm{Ad}(w)H)^k f_i(y \exp(t\,\mathrm{Ad}(w)H)).$$

Therefore by (12)

$$\left|\acute{f}_{i,y}(H)\right| \leq C|H|^2 \sum_{k,\ell=1}^n \|X_k X_\ell f_i\|_\infty \leq C|H|^2 \left(2^i\rho\right)^{-n-s-1}. \tag{17}$$

From (17) we can easily obtain

$$I_4 \leq CR^{-2}\left(2^i\rho\right)^{-n-s-1}$$

and

$$I_3 \leq CR^n \left| \int\limits_{\substack{H\in Q \\ R^{-1}<|H|<\varepsilon_0}} \grave{f}_{i,y}(H) \prod_{\alpha\in\Delta^+} (\alpha, H)\, (R|H|)^{-\delta-k-(n+1)/2} \cdot \cos\big((R|H|+\theta)D \exp H\big)\, dH \right|$$
$$+ CR^{-2}\big(2^i\rho\big)^{-n-s-1}$$

where $\theta = -\pi(\delta + k + (n+1)/2)/2$. Here we have used the asymptotic property of the Bessel function J_ν of order ν. Denote the integral in the above inequality by I' and put $\widetilde{D}(H) = D(\exp H) - \prod_{\alpha\in\Delta^+}(\alpha, H)$. Then we have $I' \leq C(I + J)$ where

$$I = R^{-\delta-k+(n-1)/2} \cdot \left| \int\limits_{R-1}^{\varepsilon_0} \left\{ \int\limits_{|\xi|=1} \grave{f}_{i,y}(t\xi) \prod_{\alpha\in\Delta^+} (\alpha,\xi)^2\, d\sigma_\xi \right\} t^{-\delta-k-1+(n-1)/2} \cos(Rt+\theta)\, dt \right|$$

$$J = R^{-\delta-k+(n-1)/2} \cdot \left| \int\limits_{R-1}^{\varepsilon_0} \left\{ \int\limits_{|\xi|=1} \grave{f}_{i,y}(t\xi) \prod_{\alpha\in\Delta^+} (\alpha,\xi)\, \widetilde{D}(t\xi) d\sigma_\xi \right\} t^{-\delta-k-1+(\ell-3)/2} \cos(Rt+\theta)\, dt \right|.$$

Observe that

$$1 \leqq CR^{\delta(k)} \left\{ \int\limits_{\alpha(R)}^{\beta(R)} \left[\int\limits_{|\xi|=1} \left|\grave{f}_{i,y}(t_R\xi) - \grave{f}_{i,y}(t\xi)\right| \prod_{\alpha\in\Delta^+} (\alpha,\xi)^2\, d\sigma_\xi \right] t^{\delta(k)-1} dt \right.$$
$$+ \int\limits_{\alpha(R)}^{\beta(R)} \left[\int\limits_{|\xi|=1} \left|\grave{f}_{i,y}(t_R\xi) - \grave{f}_{i,y}(t\xi)\right| \prod_{\alpha\in\Delta^+} (\alpha,\xi)^2\, d\sigma_\xi \right] \left| t_R^{\delta(k)-1} - t^{\delta(k)-1} \right| dt$$
$$+ \int\limits_{R-1}^{\alpha(R)} \left[\int\limits_{|\xi|=1} \left|\grave{f}_{i,y}(t_R\xi) - \grave{f}_{i,y}(t\xi)\right| \prod_{\alpha\in\Delta^+} (\alpha,\xi)^2\, d\sigma_\xi \right] t_R^{\delta(k)-1} dt$$
$$\left. + \int\limits_{\beta(R)}^{\varepsilon_0} \left[\int\limits_{|\xi|=1} \left|\grave{f}_{i,y}(t\xi)\right| \prod_{\alpha\in\Delta^+} (\alpha,\xi)^2\, d\sigma_\xi \right] t^{\delta(k)-1} \cdot dt \right\} \qquad (18)$$

where $\alpha(R) = (1+\pi)/R, \beta(R) = \varepsilon_0 - \pi/R, t_R = t + \pi/R$ and $\delta(k) = (n-1)/2 - \delta - k$. By (17) a straightforward calculation gives $I \leq \tilde{I} + CR^{-2}\big(2^i\rho\big)^{-n-s-1}$. Here $\tilde{I}$ is the first integral on the righthand side of (18). Put $u(t) = f_i(y \exp(t\, \mathrm{Ad}(w)\xi))$.

Then$|u(t+\pi/R)+u(-t-\pi/R)-u(t)-u(-t)| \leq C\left(R^{-1}tu''(\theta_1)+R^{-2}\left(u''(\theta_2)u''(\theta_3)\right)\right)$ where $-t<\theta_1<t, t<\theta_2<t+\pi-R$ and $-t-\pi/R<\theta_3<-t$. Thus using the same method as in the proof of (17) we obtain

$$\left|\grave{f}_{i,y}(t_R\xi)-\grave{f}_{i,y}(t\xi)\right| \leq C(R^{-1}t+R^{-2})\left(2^i\rho\right)^{-n-s-1}.$$

This gives immediately

$$\tilde{I} \leq CR^{-2}\left(2^i\rho\right)^{-n-s-1}.$$

Observe that $\left|\widetilde{D}(t\xi)\right| \leq Ct^{m+2}$ and we obtain similarly $J \leq CR^{-2}\left(2^i\rho\right)^{-n-s-1}$. Therefore

$$I_1 \leq CR^{-2}\left(2^i\rho\right)^{-n+pn+ps+p}. \qquad (19)$$

Also it is easy to see that by (12)

$$\int_{d(x,e)>6\rho} \left|S_R^{\delta+k}(a*F)(x)-a*F(x)\right|^p dx \leq CR^{-2p}. \qquad (20)$$

Thus from (13)-(15) and (19)-(20) we have $\sigma_k^{(2)} \leq CR^{-2p}$ which completes the proof of the lemma.

Following the idea we have

Lemma (1.2.5) [1]:

Suppose $\sigma>0$ and $0<p\leq 1$. Then I^σ maps $H^p(G)$ into itself continuously.

For an integer $N\geq 0$ and a multi-index $\mathbf{j}=(j_1,\dots,j_N)\in\{0,1,\dots,n\}^N$ let $R_j(f)$ denote the generalized Riesz transform $R_j(f)=R_{j_1}\dots R_{j_m}f$ where R_jf is the jth Riesz transform of f if $j\neq 0$ and $R_0f=f$.

Lemma (1.2.6) [1]:

Let $p>(n-1)/(n-1+N)$ and choose $f\in H^p\cap L^2(G)$. Then there exist constants C_1, C_2 and C depending only on p, n and N such that

(i) $C_1 \sum_j \|R_j(f)\|_p \leq \|f\|_{H^p} \leq C_2 \sum_j \|R_j(f)\|_p$

(ii) $\|R_j(f)\|_{H^p} \leq C\|f\|_{H^p}$

where the sums in (i) are taken over all multi-indexs $j \in \{0,1,\dots,n\}^N$.

Proof:

For $x \in G, t > 0$ write

$$s(x,t) = \left(\sum_j |P_t * R_j(f)(x)|^2\right)^{q/2}.$$

Using a standard argument we can show that $s(x,t)$ is subharmonic for $q \geq (n-1)(n-1+N)$

$$\sum_{i=0}^{n} X_i^2 s(x,t) \geq 0$$

for $q \geq (n-1)/(n-1+N)$, where $X_0 = \frac{\partial}{\partial t}$. From this, arguing similarly as for the case $p = N = 1$, we obtain for $q = (n-1)/(n+k-1)$

$$\left(\sum_j |P_t * R_j(f)(x)|^2\right)^{1/2} \leq C\left(P_t * \left(\sum_j |P_j(f)(x)|^2\right)^{q/2}\right)^{1/q} \qquad (21)$$

for $r_0 > 0$ let

$$P_{r_0}^*(f)(x) = \sup_{\substack{d(x,y)<t \\ 0<t\leq r_0}} |P_t * f(y)|$$

$$P_{r_0}^+(f)(x) = \sup_{0<t\leq r_0} |P_t * f(x)|$$

$$M_{r_0}(f)(x) \sup_{0<h\leq r_0} |B(x,h)|^{-1} \int_{B(x,h)} |f(y)|\, dy.$$

We have

$$\|R_j(f)\|_{H^p} \leq C \left\| P^*_{\varepsilon_0}\left(R_j(f)\right)\right\|_P \quad (22)$$

$$\leq C \left\| M_{\varepsilon_0}\left(P^+_{2\varepsilon_0}\left(\sum_j |R_j(f)|^2\right)^{q/2}\right)\right\|^{1/q}_{p/q}$$

$$\leq C(n,N,p)\sum_j \|R_j(f)\|_p.$$

Again we can prove that for any (p,∞,s) atom a with $s = [n(1/p-1)]$

$$\|R_i a\|_p \leq C, \quad i = 0,1,2,\dots,n \qquad (23)$$

With$C$ independent of a. The lemma now follows directly from (22) and (23).

Consider the following kernel on G

$$B^\delta_R(x) = \sum_{\lambda\in\hat{G}} \phi(|\lambda+\beta|/R) d_\lambda \chi_\lambda(x), \ \ \delta > 0$$

where

$$\phi(t) = \begin{cases} t^2\,(1-t^2)^\delta/\left[1-(1-t^2)^\delta\right], & 0 \leq t < 1 \\ 0, & t \geq 1. \end{cases}$$

Proposition (1.2.7) [1]:

For $\delta > \delta_p$ and $0 \leq 1, f \to f * B^\delta_R$ maps $H^p(G)$ into itself boundedly, and $\|f * B^\delta_R\|_{H^p} \leq C\|f\|_{H^p}$ with C independent of R and f.

Proof:

Write

$$\sigma_{n,\delta}(H) = \int_0^\infty \phi(t) J_{n/2-1}(t|H|)(t|H|)^{1-n/2} t^{n-1} dt, \ \ H \in t.$$

Then using the Poisson summation formula for central functions on G

$$B_R^\delta(x) = B_R^\delta(\exp H) = CR^n D(\exp H)^{-1} \sum_{\xi\in\Gamma} \prod_{x\in\Delta^+} (\alpha, H+\xi)\, \sigma_{n,\delta}(R|H+\xi|)$$

where x is conjugate to $\exp \bar{A}$. Denote by k_p the largest integer less than or equal to

$n/p - (n+1)/2$, and choose a positive integer m_0 such that $m_0 > 2k_p + 4 - n$. Using the proposition of the Bessel function we can deduce that for $\delta > \delta_p$.

$$\sigma_{n,\delta}(H) = C \sum_{i=0}^{m_0} (-1)^i \binom{m_0}{i} \Gamma(\delta + i\delta + 1) J_{\delta+i\delta+n/2}(|H|)|H|^{-(\delta+i\delta+n/2)} + \gamma(H)$$

where Γ is the Euler gamma-function, and γ satisfies

$$\left| \left\{ \left(\frac{1}{r}\frac{d}{dr} \right)^k \gamma(r) \right\}_{r=|H|} \right| \leq C\left(1 + |H|^{-\delta-(n+1)/2-k} + |H|^{-k_p-1-(n+1)/2-k}\right)$$

for $k = 0,1,2,$ .... Thus arguing similarly we can obtain for any multi-index **j**

$|X^j B_R^\delta(x)| \leq$

$$\begin{cases} CR^{|j|+n} & \text{if } d(x,e) \leq R^{-1}, \\ CR^{|j|-\delta+(n-1)/2} \begin{pmatrix} d(x,e)^{-\delta-(n+1)/2} + \Delta^{(R)}(x)^{-1} + \\ CR^{|j|-k_p+(n-3)/2} d(x,e)^{-k_p-1-(n+1)/2} + \Delta^{(R)}(x)^{-1} \end{pmatrix} & \text{otherwise.} \end{cases}$$

Using these estimates the proof of the proposition follows in the same way at that of Theorem (1.1.1).

Theorem (1.2.8) [1]:

Suppose $f \in H^p(G)$ and write $\delta_p = n/p - (n+1)/2$.

(i) If $\delta > \delta_p$ and $0 < p \leq 1$ then

$$C_1(1 + |\beta| \|f\|_{H^p})^{-1} K_2(f, R^{-1})_{H^p} \leq \left\| S_R^\delta(f) - f \right\|_{H^p} \leq C_2 K_2(f, R^{-1})_{H^p}$$

(ii) If $\delta = \delta_n$ and $0 < p < 1$ then

$$\left\|S_R^{\delta}(f) - f\right\|_{H(p,\infty)} \leq CK_2(f, R^{-1})_{H^p}$$

where C_1, C_2 and C are positive constants independent of R and f, and $\beta = \sum_{x\in\Delta^+} \alpha/2$ which is zero for G commuatative.

To prove the theorem we need some preliminary results. We can easily obtain:

Proof:

By theorem (1.1.1) we have for any $\mathrm{g} \in H_2^{p,2}(G)$

$$\left\|S_R^{\delta}(f) - f\right\|_{H^p} \leq C_p\left\{\|f - \mathrm{g}\|_{H^p} + \sum_{i=1}^{k}\left\|S_R^{\delta+i-1}(\mathrm{g}) - S_R^{\delta+i}(\mathrm{g})\right\|_{H^p} + \left\|S_R^{\delta+k}(\mathrm{g}) - \mathrm{g}\right\|_{H^p}\right\}$$

where $k \geq m + 3$. Also Lemma (1.2.5) gives $\|\mathrm{g}\|_{H^p} \leq C\|\Delta\mathrm{g}\|_{H^p}$, for $\mathrm{g} \in H_2^{p,2}$, and hence

$$\left\|S_R^{\delta+i-1}(\mathrm{g}) - S_R^{\delta+i}(\mathrm{g})\right\|_{H^p} \leq C_p R^{-2}\|\Delta\mathrm{g}\|_{H^p}, \;\; i = 1,2,\dots,k.$$

Now we have the atomic decomposition $\Delta\mathrm{g} = \sum_i \lambda_i a_i$ where the a_i are (p,∞,s)-atoms and $s = [n(1/p - 1)]$. For any $0 < p \leq 1$ take $N > (1/p - 1)(n - 1)$. Then by Lemma (1.2.6)

$$\left\|S_R^{\delta+i-1}(\mathrm{g}) - \mathrm{g}\right\|_{H^p}^p \leq \sum_j |\lambda_i|^p \left\|S_R^{\delta+k}(I^2 a_i)\right\|_{H^p}^p$$

$$\leq C\sum_j |\lambda_i|^p \sum_j \left\|R_j\left(S_R^{\delta+k}\left(I^2 a_i\right) - I^2 a_i\right)\right\|_p^p.$$

Here $R_{\mathbf{j}}$ is the Riesz transform of order N. Obviously $R_{\mathbf{j}}(I^2 a_i) \in H_2^{p,2}(G)$. So $\Delta R_{\mathbf{j}}(I^2 a_i)$ has an decomposition $\Delta R_{\mathbf{j}}(I^2 a_i) = \sum_l \mu_l b_l$. Thus by Lemmas (1.2.4) and (1.2.6)

$$\left\|R_j\left(S_R^{\delta+k}(I^2 a_i) - I^2 a_i\right)\right\|_p^p \leq \sum_j |\mu_\ell|^p \left\|S_R^{\delta+k}(I^2 b_\ell) - I^2 b_\ell\right\|_p^p \leq CR^{-2}. \quad (24)$$

Therefore $\left\|S_R^{\delta+k}(\mathrm{g}) - \mathrm{g}\right\|_{H^p} \leq C_p R^{-2}\|\Delta\mathrm{g}\|_{H^p}$ for $\mathrm{g} \in H_2^{p,2}(G)$, and by proposition (1.2.2)

$$\left\|S_R^{\delta}(f) - f\right\|_{H^p} \leq C_p K_2(f, R^{-1})_{H^p}.$$

To complete the proof of (i) we observe that for $f \in H^p(G)$

$$\Delta S_R^\delta(f) = R^2\left(f - S_R^\delta(f)\right) * B_R^\delta - |\beta|^2 S_R^\delta(f).$$

Using Theorem (1.1.1) and Lemma (1.2.4)we obtain $\Delta_R^\delta \in H^p(G)$ and

$$R^{-2}\left\|\Delta S_R^\delta(f)\right\|_{H^p} \leq C_p\left(\left\|S_R^\delta(f) - f\right\|_{H^p} + |\beta|^2\|f\|_{H^p}R^{-2}\right).$$

By (10) we can deduce that the saturation degree of approximation in $H^p(G)$ by the Bochner-Riesz means is R^{-2} and if $f \in H^p(G)$ is non-constant there is a constant $C > 0$ such that

$$\left\|S_R^\delta(f) - f\right\|_{H^p} \geq CR^{-2}$$

for $\delta > \delta_p$ and R sufficiently large. Thus for $f \in H^p(G)$

$$K_2(f, R^{-1})_{H^p} \leq \left\|f - S_R^\delta(f)\right\|_{H^p} + R^{-2}\left\|\Delta S_R^\delta(f)\right\|_{H^p}$$

$$\leq C_p\left(1 + |\beta|^2\|f\|_{H^p}\right)\left\|S_R^\delta(f) - f\right\|_{H^p}$$

and this takes cares in (i) of Theorem (1.2.8).

For part (ii) of theorem observe that $H^p(G)$ is continuously embedded in $H(p,\infty)$, so that $\|f\|_{H(p,\infty)} \leq C\|f\|_{H^p}$ with C independent of $H^p(G)$. By Theorem (1.1.2), Lemma (1.2.5) and (24) we can deduce that for any $(p,\infty.s)$-atom a

$$\left\|S_R^\delta(I^2a) - I^2a\right\|_{H(p,\infty)} \leq \sum_{i=1}^{k}\left\|S_R^{\delta+i-1}(I^2a) - S_R^{\delta+i}(I^2a)\right\|_{H(p,\infty)}$$

$$+\left\|S_R^{\delta+k}(I^2a) - I^2a\right\|_{H(p,\infty)} \leq CR^{-2}$$

Where$\delta = \delta_p, k \geq m + 3$, and C is independent of $R$and a. Thus using the Stein-Taibleson-Weiss result on the weak-type estimate of operators an argument similar to the proof of (i) yields

$$\left\|S_R^\delta(\mathrm{g}) - \mathrm{g}\right\|_{H(p,\infty)} \leq C_pR^{-2}\|\Delta \mathrm{g}\|_{H^p}$$

for any $\mathrm{g} \in H_2^{p,2}(G)$ and $\delta = \delta_p$. We therefore obtain by Theorem (1.1.2)

$$\left\| S_R^{\delta}(f) - f \right\|_{H(p,\infty)} \leq C_p K_2(f, R^{-1})_{H^p}$$

And this completes the proof of the theorem.

Chapter 2

Compact Lie Group and Gårding Inequality

In this chapter weprove some of application of Hilbert space and Sobolev boundedness of pseudo-differential operators.

Section (2.1): Sharp Gårding Inequality

Let Gbe a compact Lie group of dimension $n$with the neutral element e.Its Liealgebra will be denoted by $\mathfrak{g}$. We now fix the necessary notation.Let $\hat{G}$denote theunitary dual of G, i.e. set of all equivalence classes of (continuous) irreducible unitaryrepresentations of $G$and let $\mathrm{Rep}(G)$ be the set of all such representations of G.

For $f \in \mathrm{C}^{\infty}(G)$ and $\xi \in \mathrm{Rep}(G)$, let

$$\hat{f}(\xi) = \int_G f(x)\xi(x)^* \, dx$$

be the (global) Fourier transform of f, where integration is with respect to the normalized Haar measure on G. For an irreducible unitaryrepresentation $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$we have the linear operator $\hat{f}(\xi): \mathcal{H}_\xi \to \mathcal{H}_\xi$. Denote by $\dim(\xi)$ the dimension of $\xi, \dim(\xi) = \dim \mathcal{H}_\xi$. If $\xi$is a matrix representation, we have $\hat{f}(\xi) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$. Since$G$iscompact, $\hat{G}$is discrete and all of its elements are finite dimensional. Consequently,by the Peter–Weyl theorem we have the Fourier inversion formula

$$f(x) = \sum_{|\xi|\in\hat{G}} \dim(\xi)\,\mathrm{Tr}\left(\xi(x)\hat{f}(\xi)\right).$$

The Perceval identity takes the form

$$\|f\|^2_{L^2(G)} = \sum_{|\xi|\in\hat{G}} \dim(\xi)\left\|\hat{f}(\xi)\right\|^2_{HS},$$

Where $\left\|\hat{f}(\xi)\right\|^2_{HS} = Tr\left(\hat{f}(\xi)\hat{f}(\xi)^*\right)$, which gives the norm on $\ell^2(\hat{G})$. For a linear continuousoperator from $\mathrm{C}^{\infty}(G)$ to $D'(G)$ we introduce its full matrix-valued symbol

$\sigma_A(x,\xi) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$by

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x).$$

Then it was shown that

$$Af(x) = \sum_{|\xi|\in\hat{G}} \dim(\xi)\mathrm{Tr}\left(\xi(x)\sigma_A(x,\xi)\hat{f}(\xi)\right) \qquad (1)$$

Holds in the sense of distributions, and the sum is independent of the choice of arepresentation $\xi$from each class $[\xi] \in \hat{G}$. Moreover, we have

$$\sigma_A(x,\xi) = \int_G R_A(x,y)\xi(y)^*\,dy$$

in the sense of distributions, where $R_A$is the right-convolution kernel of$A$:

$$Af(x) = \int_G K(x,y)f(y)\,dy = \int_G f(y)R_A(x,y^{-1}x)\,dy.$$

Symbols $\sigma_A$can be viewed as mappings on $G \times \hat{G}$: the symbol of acontinuous linearoperator $A : \mathrm{C}^\infty(G) \to \mathrm{C}^\infty(G)$ is a mapping

$$\sigma_A: G \times \mathrm{Rep}(G) \to \bigcup_{\xi\in\mathrm{Rep}(G)} \mathrm{End}(\mathcal{H}_\xi),$$

Where $\sigma_A(x,\xi) : \mathcal{H}_\xi \to \mathcal{H}_\xi$is linear for every $x \in G$and $\xi \in \mathrm{Rep}(G)$, and $\mathrm{End}(\mathcal{H}_\xi)$ is the space of all linear mappings from$\mathcal{H}_\xi$to $\mathcal{H}_\xi$. If$\eta \in [\xi]$, i.e. there is an intertwining isomorphism

$U: \mathcal{H}_\eta \to \mathcal{H}_\xi$such that $\eta(x) = U^{-1}\xi(x)U$, then $\sigma_A(x,\eta) = U^{-1}\sigma_A(x,\xi)U$. In thissense we may think that thesymbol $\sigma_A$is defined on $G \times \hat{G}$instead of$G \times \mathrm{Rep}(G)$.

A linear operator Pon a Hilbert space $\mathcal{H}$is calledpositive if$\langle Pv, v\rangle_{\mathcal{H}} \geq 0$ for every $v \in V$for a dense subset $V \subset \mathcal{H}$. A matrix $P \in \mathbb{C}^{n\times n}$is calledpositive if the natural corresponding linear operator $\mathbb{C}^n \to \mathbb{C}^n$is positive, where $\mathbb{C}^n$has the standard inner product.

A matrix pseudo-differential symbol σ_A is called positive if the matrix

$\sigma_A(x,\xi) \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)}$ is positive for every $x \in G$and $\xi \in \mathrm{Rep}(G)$. In this case we write$\sigma(x,\xi) \geq 0$. We note that for each $\xi \in \mathrm{Rep}(G)$, thecondition $\sigma_A(x,\xi) \geq 0$ implies

$\sigma_A(x,\eta) \geq 0$ for all $\eta \in [\xi]$. We can alsonote that this symbol positivity does notchange if we move from left symbols to right symbols:

$$\sigma_A(x,\xi) := \xi^*(x)(A\xi)(x) = \xi(x)^*\rho A(x,\xi)\xi(x),$$

$$\rho A(x,\xi) := (A\xi)(x)\xi^*(x) = \xi(x)\sigma_A(x,\xi)\xi(x)^*;$$

that is, $\sigma_A$is positive if and only if $\rho\ A$ is positive. Moreover, this positivity conceptis natural in the sense that a left- or right-invariant operator is positive if and only ifits symbol is positive, as it can be seen from the equalities

$$\langle a * f, f\rangle_{L^2(G)} = \sum_{|\xi|\in\hat{G}} \dim(\xi)\mathrm{Tr}\big(\hat{f}(\xi)\hat{a}(\xi)\hat{f}(\xi)^*\big), \qquad (2)$$

$$\langle f * a, f\rangle_{L^2(G)} = \sum_{|\xi|\in\hat{G}} \dim(\xi)\mathrm{Tr}\left(\hat{f}(\xi)^*\hat{a}(\xi)\hat{f}(\xi)\right), \qquad (3)$$

which can be shown by a simple calculation which we give in Proposition(2.1.9). At thesame time, the operator $M_f$of multiplication by a smooth function $f \in \mathrm{C}^\infty(G)$ ispositive if and only if the function satisfies$f(x) \geq 0$ for every $x \in G$. The symbolof such multiplication operator is$\sigma_{M_f}(x,\xi) = f(x)I_{\dim(\xi)}$, so that this means thepositivity of the matrix symbol again.

Now we can formulate the following:

Theorem (2.1.1) [2]:

Let Abe a linear continuous operator from $\mathrm{C}^\infty(G)$ to $D'(G)$, and let$m \in \mathbb{R}$. Then the following statements are equivalent:

(i) $A \in \Psi^m(G)$.

(ii) For every left-invariant differential operator $P_x \in \mathrm{Diff}^k(G)$ of order $k$and every difference operator $Q_\xi \in \mathrm{diff}^\ell(G)$ of order ℓthere is the symbol estimate

$$\left\|Q_\xi P_x \sigma_A(x,\xi)\right\|_{op} \leq \mathrm{C}_{Q_\xi P_x}\langle\xi\rangle^{m-\ell}.$$

(iii) For an admissible collection $\Delta_1,\ldots,\Delta_\mu \in \mathrm{diff}^1(\hat{G})$ we have

$$\left\|\Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right\|_{op} \leq \mathrm{C}_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}$$

for all multi-indices α,β. Moreover, sing supp $R_A(x,\cdot) \subseteq \{e\}$.

(iv) For a strongly admissible collection $\Delta_1,\ldots,\Delta_\mu \in \mathrm{diff}^1(\hat{G})$ we have

$$\left\|\Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right\|_{op} \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}$$

for all multi-indices α,β.

The set of symbols $\sigma_A$satisfying either of conditions (ii)–(iv) will be denoted by

$\mathscr{S}^{m}_{1,0}(G) = \mathscr{S}m\ \ (G)$. We note that if conditions (iii) or (iv) hold for one admissible(strongly admissible, resp.) collection of firstorder difference operators, theyautomatically hold for all admissible (strongly admissible, resp.) collections.

For the purposeswe will say that a matrix-valued symbol $\sigma_A(x,\xi)$ belongs to $\mathscr{S}^{m}_{\rho,\delta}(G)$if it is smooth in x and if for a strongly admissible collection $\Delta_1,\ldots,\Delta_\mu \in \mathrm{diff}^1(\hat{G})$we have

$$\left\|\Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right\|_{op} \leq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \qquad (4)$$

for all multi-indices α,β, uniformly in $x \in G$and $\xi \in \mathrm{Rep}(G)$.

Remark(2.1.2) [2]:

As it was pointed in Theorem (2.1.1) we still have theequivalenceof conditions (ii), (iii), (iv), also if we replace symbolicinequalities in Theorem (2.1.4) by inequalities of the form (4). Also inthis setting, if conditions (iii) or (iv)hold for one admissible collection of first order differenceoperators, theyautomaticallyhold for all admissible Collections.

We will also write $a \in \mathscr{S}^{m}_{\rho,\delta\#}(G)$if for every multi-index β and for every $x_0 \in G$ wehave $\partial_x^\beta a(x_0,\cdot) \in \mathscr{S}^{m+\delta|\beta|}_{\rho\#}(G)$, where for a multiplier $b = b(\xi)$ we write $b \in \mathscr{S}^{\mu}_{\rho\#}(G)$ iffor every multi-index α there is a constant $C_\alpha$such that

$$\left\|\Delta_\xi^\alpha b(\xi)\right\|_{op} \leq C_\alpha\langle\xi\rangle^{\mu-\rho|\alpha|}$$

holds for all $[\xi] \in \hat{G}$. We record the following straightforward lemma that follows from the smothness of symbols in x and the compactness of$G$:

Lemma (2.1.3) [2]:

We have $a \in \mathscr{S}^{\ m}_{\rho,\delta}(G)$if and only if $a \in \mathscr{S}^{\ m}_{\rho,\delta\#}(G)$.

Another tool which will be required for the proof is the finite version of the Leibnizformula given a continuous unitary matrix representation$\xi^0 = \left[\xi^0_j\right]_{1\le i,j\le \ell}: G \to \mathbb{C}^{\ell\times\ell}, \ell = \dim(\xi^0)$, let$q(x) = \xi^0(x) - I$(i.e. $q_{ij} = \xi^0_{ij} - \delta_{ij}$with Kronecker's deltas δ_{ij}), and define

$$\mathbb{D}_{ij}\hat{f}(\xi) := \widehat{q_{ij}f}(\xi).$$

In the previous notation, we could also write $\mathbb{D}_{ij} = q_{ij}$. For a multi-index $\gamma \in \mathbb{N}_0^{\ell^2}$,we write $|\gamma| = \sum_{i,j=1}^{\ell}|\gamma_{ij}|$, and for higher order difference operators we write $\mathbb{D}^\gamma = \mathbb{D}_{11}^{\gamma_{11}}, \mathbb{D}_{12}^{\gamma_{12}} \dots \mathbb{D}_{\ell,\ell-1}^{\gamma_{\ell,\ell-1}}\mathbb{D}_{\ell\ell}^{\gamma_{\ell\ell}}$. In contrast to theasymptotic Leibniz rulefor arbitrary differenceoperators, operators $\mathbb{D}$ satisfy the finite Leibniz formula:

Proposition(2.1.4) [2]:

For all $\gamma \in \mathbb{N}_0^{\ell^2}$ we have

$$\mathbb{D}^\gamma(a,b) = \sum_{|\varepsilon|,|\delta|\le|\gamma|\le|\varepsilon|+|\delta|} \mathrm{C}_{\gamma\in\delta}(\mathbb{D}^\varepsilon a)\left(\mathbb{D}^\delta b\right),$$

with the summation taken over all $\varepsilon, \delta \in \mathbb{N}_0^{\ell^2}$satisfying $|\varepsilon|, |\delta| \le |\gamma| \le |\varepsilon| + |\delta|$. Inparticular, for $|\gamma| = 1$, we have

$$\mathbb{D}_{ij}(ab) = \left(\mathbb{D}_{ij}a\right)b + a\left(\mathbb{D}_{ij}b\right) + \sum_{k=1}^{\ell}(\mathbb{D}_{ik}a)\left(\mathbb{D}_{kj}b\right). \qquad (5)$$

Difference operators $\mathbb{D}$ lead to strongly admissible collections.

Lemma(2.1.5) [2]:

The family of difference operators associated to the family of functions

$\{q_{ij} = \xi_{ij} - \delta_{ij}\}[\xi] \in \hat{G}, 1 \le i,j \le \dim(\xi)$ is strongly admissible.Moreover, this family has

a finite subfamily associated to finitely many representations which is still stronglyadmissible.

We now give a simple proof of the equalities (2) and (3).

Proposition(2.1.6) [2]:

We have

$$\langle a * f, f\rangle_{L^2(G)} = \sum_{|\xi|\in\hat{G}} \dim(\xi)\mathrm{Tr}\big(\hat{f}(\xi)\hat{a}(\xi)\hat{f}(\xi)^*\big),$$

$$\langle f * a, f\rangle_{L^2(G)} = \sum_{|\xi|\in\hat{G}} \dim(\xi)\mathrm{Tr}\left(\hat{f}(\xi)^*\hat{a}(\xi)\hat{f}(\xi)\right),$$

Proof:

The second claimed equality follows from the following calculation:

$$\langle f * a, f\rangle_{L^2(G)} = \int_G (f * a)(x)\overline{f(x)}\, dx$$

$$= \int_G \sum_{|\xi|\in\hat{G}} \dim(\xi)\mathrm{Tr}\left(\xi(x)\hat{a}(\xi)\hat{f}(\xi)\right) \overline{\sum_{|\eta|\in\hat{G}} \dim\eta\, \mathrm{Tr}\left(\eta(x)\hat{f}(\eta)\right)}\, dx$$

$$= \int_G \sum_{|\xi|\in\hat{G}} \dim(\xi) \sum_{k,l,m=1}^{\dim(\xi)} \xi(x)_{kl}\hat{a}(\xi)_{lm}\hat{f}(\xi)_{mk} \overline{\sum_{|\eta|\in\hat{G}} \dim(\eta) \sum_{p,q=1}^{\dim(\eta)} \eta(x)_{pq}\hat{f}(\eta)_{qp}}$$

$$= \sum_{|\xi|\in\hat{G}} \dim(\xi) \sum_{k,l,m=1}^{\dim(\xi)} \hat{a}(\xi)_{lm}\hat{f}(\xi)_{mk}\overline{\hat{f}(\xi)_{lk}} = \sum_{|\xi|\in\hat{G}} \dim(\xi)\, Tr\big(\hat{a}(\xi)\hat{f}(\xi)\hat{f}(\xi)^*\big),$$

where we used the orthogonality of the matrix elements of the representations. Thefirst claimed equality can be proven in an analogous way.

We also record the Sobolev boundedness resut.

Theorem(2.1.7) [2]:

Let Gbe a compact Lie group. Let A be a continuous linear operatorfrom $\mathrm{C}^\infty(G)$ to $\mathrm{C}^\infty(G)$ and let $\sigma_A$be its symbol. Assume that there exist constants$m, \mathrm{C}_\alpha \in \mathbb{R}$ such that

$$\|\partial_x^\alpha \sigma_A(x,\xi)\|_{op} \leq \mathrm{C}_\alpha \langle \xi \rangle^m$$

holds for all $x \in G, \xi \in \mathrm{Rep}(G)$, and all multi-indices α. Then A extends to abounded operator from $H^s(G)$ to $H^{s-m}(G)$ for all $s \in \mathbb{R}$.

Let $0 \leq \delta, \rho \leq 1$. An amplitude $a \in \mathcal{A}^m_{\rho,\delta}(G)$ is a mappingdefined on$G \times G \times \mathrm{Rep}(G)$, smooth in $x$and y, such that for an irreducible unitaryrepresentation $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$ we have linear operators

$$a(x,y,\xi) : \mathcal{H}_\xi \to \mathcal{H}_\xi,$$

and for a strongly admissible collection of difference operators $\Delta_\xi^\alpha$the amplitudesatisfies the amplitude inequalities

$$\left\| \Delta_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi) \right\|_{op} \leq \mathrm{C}_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-\rho+|\alpha|+\delta|\beta+\gamma|},$$

for all multi-indices $\alpha, \beta, \gamma$and for all $(x, y, [\xi]) \in G \times G \times \hat{G}$. For an amplitude a,the amplitude operator $\mathrm{Op}(a) : \mathrm{C}^\infty(G) \to D'(G)$ is defined by

$$\mathrm{Op}(a)u(x) := \sum_{|\xi| \in \hat{G}} \dim(\eta) \mathrm{Tr}\left(\eta(x) \int_G a(x,y,\eta) u(y) \eta(y)^* \, dy \right). \quad (6)$$

Notice that if here $\mathrm{a}(x,y,\eta) = \sigma_A(x,\eta)$ then $\mathrm{Op}(a) = A$ as in (1). This definitioncan be justified as follows:

Proposition(2.1.8) [2]:

Let $0 \leq \delta < 1$ and $0 \leq \rho \leq 1$, and let $a \in \mathcal{A}^m_{\rho,\delta}(G)$. Then $\mathrm{Op}(a)$is a continuous linear operator from $\mathrm{C}^\infty(G)$ to $\mathrm{C}^\infty(G)$.

Proof:

By the definition of $\langle \eta \rangle$we have $(1 - \mathcal{L}_G)\eta(\eta) = \langle \eta \rangle^2(y)$. On the otherhand, theWeyl spectral asymptotics formula for the Laplace operator $\mathcal{L}_G$ implies that$\langle \eta \rangle^{-1} \leq \mathrm{C} \dim \langle \eta \rangle^{-2/\dim(G)}$. Consequently, integratingby parts in thedy-integral in (6) with operator$\langle \eta \rangle^{-2}(I - \mathcal{L}_G)$ arbitrarily many times,we see that the η-series in (6) converges, so that $\mathrm{Op}(a)u \in \mathrm{C}^\infty(G)$ provided that$u \in \mathrm{C}^\infty(G)$. The continuity of $\mathrm{Op}(a)$ on $\mathrm{C}^\infty(G)$ follows by a similar argument.

Proposition(2.1.9) [2]:

Let $0 \leq \delta < \rho \leq 1$ and let $a \in \mathcal{A}^{m}_{\rho,\delta}(G)$. Then $A = \mathrm{Op}(a)$ is a pseudo-differential operator on G with a matrix symbol$\sigma_A \in \mathscr{S}^{\ m}_{\rho,\delta}(G)$. Moreover, $\sigma_A$has the asymptotic expansion

$$\sigma_A(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_y^{(\alpha)} \Delta_\xi^{(\alpha)} a(x,y,\xi)\Big|_{y=x}.$$

Proof:

If $\sigma_A$is the matrix symbol of the continuous linear operator $A = \mathrm{Op}(a) : \mathrm{C}^\infty(G) \to \mathrm{C}^\infty(G)$, we can find it from the formula $\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x)$. Byfixing some basis in the representation spaces, we have

$$\sigma_A(x,\xi)_{mn} = \sum_{l=1}^{\dim(\xi)} \xi(x^{-1})_{ml}(A\xi_{ln})(x)$$

$$= \sum_{l=1}^{\dim(\xi)} \xi(x^{-1})_{ml} \int_G \sum_{|\eta| \in \hat{G}} \dim(\eta)\, \mathrm{Tr}(\eta(x) a(x,y,\eta) \xi(y)_{ln} \eta(y)^*)\, dy$$

$$= \int_G \xi(x^{-1}y)_{mn} \sum_{|\eta| \in \hat{G}} \dim(\eta)\, \mathrm{Tr}\big(\eta(y^{-1}x) a(x,y,\eta)\big) dy$$

$$= \int_G \xi(x^{-1}y)_{mn} \sum_{|\eta| \in \hat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\xi)} \eta(y^{-1}x)_{jk} a(x,y,\eta)_{kj}\, dy$$

$$= \int_G \xi(x^{-1}y)_{mn} \sum_{|\eta| \in \hat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\xi)} \eta(z)_{jk} a(x, xz^{-1}, \eta)_{kj}\, dz$$

$$\sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_u^{(\alpha)} \sum_{|\eta| \in \hat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\xi)} a(x,u,\eta)|_{u=x} \int_G \xi(z^{-1}y)_{mn} \eta(z)_{jk} q_\alpha(z)\, dz,$$

by the Taylor expansion . Using difference operators $\Delta_\xi^\alpha\, \hat{s}(\xi) := \widehat{q_\alpha s}(\xi)$, we find

$$\sum_{|\eta|\in\hat{G}} \dim(\eta) \sum_{j,k=1}^{\dim(\xi)} a(x,u,\eta)_{kj} = \int_G \xi(z^{-1})\eta(z)_{jk} q_\alpha(z)\, dz$$

$$= \int_G \xi(z)^* q_\alpha(z) \sum_{|\eta|\in\hat{G}} \dim(\eta)\, \mathrm{Tr}\big(\eta(z)a(x,u,\eta)\big)dz = \Delta_\xi^\alpha a(x,u,\xi).$$

Thus

$$\sigma_A(x,\xi) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!} \partial_u^{(\alpha)} \int_G \xi(z)^* q_\alpha(z) \sum_{|\eta|\in\hat{G}} \dim(\eta)\, \mathrm{Tr}\big(\eta(z)a(x,u,\eta)\big)dz\Big|_{u=x}$$

$$= \sum_{\alpha\geq 0} \frac{1}{\alpha!} \partial_u^{(\alpha)}\, \Delta_\xi^\alpha a(x,u,\xi)|_{u=x}.$$

The remainder in this asymptotic expansion can be dealt with in a way similar tothe argument for the composition formulae, so we omit the proof.

On a group G, function $f : G \to \mathbb{C}$ iscalled even if it is inversion-invariant, i.e. if

$f(x-1) = f(x)$ for every $x \in G$. Function$f : G \to \mathbb{C}$ is called odd if $f(x-1) = -f(x)$ for every $x \in G$. Recall that $f : G \to \mathbb{C}$ is central if $f(xy) = f(yx)$ for all $x, y \in G$. Linear combinations of characters $\chi_\xi = \big(x \mapsto \mathrm{Tr}\big(\xi(x)\big)\big)$ of irreducible unitary representations $\xi$of

a compact group G are central, and such linear combinations are dense among the central functions of$\mathrm{C}(G)$. When G is a compact Lie group, for $Y \in \mathrm{g}$ and $f \in \mathrm{C}^\infty(G)$ we define

$$\mathrm{L_Y f(x)} := \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{f(x\exp(tY))}|_{\mathrm{t=0}}, \quad \mathrm{R_Y f(x)} := \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{f(x\exp(tY)x)}|_{\mathrm{t=0}},$$

so that $L_Y, R_Y$are the first order differential operators, $L_Y$being left-invariant and $R_Y$right-invariant. For a central function $f$we have$L_Y f = R_Y f$, which would notbe true for an arbitrary smooth function f. Moreover, if f is even and central then

$$L_Y f(x^{-1}) = L_Y f(x),$$

i.e. $L_Y f$is odd in this case. Similarly $L_Y f$is even for odd central functions f, but$L_Y f$does not have to be central. More precisely, for central $f \in \mathrm{C}^\infty(G)$ we obtain

$$L_Y f(u^{-1}xu) = L_{uY_{u^{-1}}} f(x),$$

where $u \in G$. For higher order derivatives of even and odd functions, taking thedifferential of

$$f(x\exp(t_1X_1)\dots\exp(t_kX_k)) = \pm f\big(x^{-1}\exp\big((-t_k)X_k\big)\dots\exp\big((-t_1)X_1\big)\big)$$

at $t_1 = \cdots = t_k = 0$, we obtain:

Proposition(2.1.10) [2]:

Let $f \in \mathrm{C}^\infty(G)$ be even and central, and $X_1, \dots, X_k \in \mathfrak{g}$. Then

$$L_{X_1}L_{X_2}\dots L_{X_{k-1}}L_{X_k}f(x^{-1}) = (-1)^k L_{X_k}L_{X_{k-1}}\dots L_{X_2}L_{X_1}f(x).$$

Similarly, if$f \in \mathrm{C}^\infty(G)$ is an odd central function, then we have the equality

$$L_{X_1}L_{X_2}\dots L_{X_{k-1}}L_{X_k}f(x^{-1}) = (-1)^{k+1} L_{X_k}L_{X_{k-1}}\dots L_{X_2}L_{X_1}f(x).$$

We notice that if a linear operator $Q : H^{(m-1)/2}(G) \to H^{-(m-1)/2}(G)$ is bounded then

$$\begin{aligned} R_e(Qu,u)_{L^2} &\geq -|(Qu,u)_{L^2}| \\ &\geq -\|Qu\|_{H^{(m-1)/2}}\|u\|_{H^{(m-1)/2}} \\ &\geq -\|Q\|_{\mathcal{L}(H^{(m-1)/2},H^{(m-1)/2})}\|u\|^2_{H^{(m-1)/2}}. \end{aligned}$$

Hence Theorem (2.1.1) would follow if we could show that $A = P + Q$, where P is positive(on $\mathrm{C}^\infty(G) \subset L^2(G)$) and $Q : H^{(m-1)/2}(G) \to H^{-(m-1)/2}(G)$ is bounded. The proofof this decomposition will be done in several steps.

First, we construct an auxiliary function $w_\xi$which willplay a crucial role for our proof.

We can treat Gas a closed subgroup of $\mathrm{GL}(N,\mathbb{R}) \subset \mathbb{R}^{N\times N}$for some$N \in \mathbb{N}$. Thenits Lie algebra $\mathfrak{g} \subset RN \times N$is an n-dimensional vector subspace (hence identifiablewith $\mathbb{R}^n$) such that

$[A,B] := AB - BA \in \mathfrak{g}$ for every $A,B \in \mathfrak{g}$. Let $U \subset G$ be aneighbourhood of the neutral element $e \in G$, and let $V \subset \mathfrak{g}$ be aneighbourhoodof$0 \in \mathfrak{g} \cong \mathbb{R}^n$, such that the matrix exponential mapping is a diffeomorphismexp $: V \to U$.

For the construction and for the notation we definethe centralnorm $|\cdot|$on g as follows. Take the Euclidean norm $|\cdot|$ on $\mathfrak{g}$ and define

$$|X| = \int_G |uXu^{-1}|_0 \, du, \qquad (7)$$

where we may view the product under the integral as the product of matrices in $\mathbb{R}^{N\times N}$. Then by definition the norm (7) is invariant by the adjoint representation,and we have, in particular $|\exp^{-1}(xy)| = |\exp^{-1}(yx)|$, etc.

We may assume that $V$is the open ball $V = \mathbb{B}(0,r) = \{z \in \mathbb{R}^n : |z| < r\}$of radius$r > 0$. Let $\varphi : [0,r) \to [0,\infty)$ be a smooth function such that$\big(Z \mapsto \phi([Z])\big) : \mathfrak{g} \to \mathbb{R}$ is supported in $V$and $\phi(s) = 1$ for small $s > 0$. For every $\xi \in \mathrm{Rep}(G)$ we define

$$w_\xi(x) := \phi\big(|\exp^{-1}(x)|\langle\xi\rangle^{1/2}\big)\psi(\exp^{-1}(x))\langle\xi\rangle^{n/4}, \qquad (8)$$

where

$$\psi(Y) = \mathrm{C}_0|\det D\exp(Y)|^{-1/2} f(Y)^{-1/2},$$

D exp is the Jacobi matrix of exp, $f(Y)$ is the density with respect to the Lebesguemeasure of the Haar measure on Gpulled back to $\mathfrak{g} \cong \mathbb{R}^n$by the exponential mapping,and with constant

$\mathrm{C}_0 = \left(\int_{\mathbb{R}^n} \phi([Z])^2 \, dZ\right)^{-1/2}$. By $I_{\dim(\xi)}$ we denote the identitymapping on $\mathbb{C}^{\dim(\xi)}$.

For $x, y \in G$close to each other, $\mathrm{dist}(x,y)$ is the geodesicdistance between $x$and y.

Section (2.2): Proof of Sharp Gårding Inequaity

Lemma(2.2.1) [2]:

We have $w_\xi \in \mathrm{C}^\infty(G), w_\xi(e) = \mathrm{C}_0\langle\xi\rangle^{n/4}, w_\xi$is central and inversion-invariant, i.e.

$w_\xi(xy) = w_\xi(yx)$ and $w_\xi(x^{-1}) = w_\xi(x)$ for every$x, y \in G$.

Also,$\mathrm{dist}(x,e) \approx |\exp^{-1}(x)| \le r\langle\xi\rangle^{-1/2}$ on the support ofw_ξ. Moreover, $\|w_\xi\|_{L^2(G)} = 1$ forall $\xi \in \mathrm{Rep}(G)$. Finally, we have$\big((x,\xi) \mapsto w_\xi(x) I_{\dim(\xi)}\big) \in \mathscr{S}^{n/4}_{1,1/2}(G)$.

Proof:

It is easy to see that $w_\xi \in \mathrm{C}^\infty(G), w_\xi(e) = \mathrm{C}_0\langle\xi\rangle^{n/4}$, and that w_ξ is inversion-invariant.Clearly $\mathrm{dist}(x,e) \approx |\exp^{-1}(x)| \le r\langle\xi\rangle^{-1/2}$ on the support of$w_\xi$in viewof properties of the function ϕ. In

particular, (8) is well-defined and support$w_\xi \subset U$. From (7) it also follows that $w_\xi$is central since f is invariant under adjointrepresentation as a density of two bi-invariant measures.

Let us now show that $\|w_\xi\|_{L^2(G)} = 1$ for all $\xi \in \mathrm{Rep}(G)$. Indeed,

$$\int_G |w_\xi(x)|^2\, dx = \langle\xi\rangle^{n/2} \int_{\mathbb{R}^n} \phi\big(|Y|\langle\xi\rangle^{1/2}\big)^2\, |\psi(Y)|^2 |\det D\exp(Y)| f(Y) dY$$

$$= \mathrm{C}_0^2 \int_{\mathbb{R}^n} \phi(|Z|)^2\, dZ,$$

so that $\|w_\xi\|_{L^2(G)} = 1$ in view of the choice of the constant C_0. Thus, the main thingis to check that $w_\xi I_{\dim(\xi)} \in \mathscr{S}^{\,n/4}_{1,1/2}$. By Lemma (2.1.6), weneed to check that for everymulti-index β and every $x_0 \in G$we have $\partial_x^\beta w_\xi(x_0) \in \mathscr{S}^{\,n/4+|\beta|/2}_{1\#}(G)$. We observe thatthe x-derivatives of $w_\xi$are sums of terms of the form

$$\chi(\exp^{-1}(x))\tilde{\phi}\big(|\exp^{-1}(x)|\langle\xi\rangle^{1/2}\big)\langle\xi\rangle^{n/4+1/2} I_{\dim(\xi)}, \qquad (9)$$

where $\chi \in \mathrm{C}_0^\infty(V), \tilde{\phi} \in \mathrm{C}_0^\infty(R), \tilde{\phi}$is constant near the origin, and l is an integer suchthat

$0 \le l \le |\beta|$. We note that $\langle\xi\rangle^{n/4+1/2} I_{\dim(\xi)}$ is thesymbol of the pseudo-differentialoperator $\langle 1 - \mathcal{L}_G\rangle^{n/8+l/4}$, and hence$\langle\xi\rangle^{n/4+l/2} I_{\dim(\xi)} \in \mathscr{S}^{\,n/4+l/2}_{1\#} \subset \mathscr{S}^{\,n/4+|\beta|/2}_{1\#}$. Moreover,we can eliminateit from the formulae by the composition formulae for thematrix-valuedsymbols. Thus we have to check that for everyx$_0$ $\in$ G, the other terms in (9) fixed at x = x$_0$ are in $\mathscr{S}^{\,0}_{1\#}$(G), i.e. that

$$\tilde{\phi}\big(|\exp^{-1}(x_0)|\langle\xi\rangle^{1/2}\big) I_{\dim(\xi)} \in \mathscr{S}^{\,0}_{1\#}(G). \qquad (10)$$

If $\exp^{-1}(x_0) = 0$, then this symbol is a constant times the identity$I_{\dim(\xi)}$ and hence itis in $\mathscr{S}^{\,0}_{1\#}(G)$. On the other hand, if $\exp^{-1}(x_0) \neq 0$, then the symbol (10) is compactlysupported in ξ, and hence defines a smoothing operator. Indeed, in this case it hasdecay of any order in $\langle\xi\rangle$, together with all difference operators applied to it, withconstants depending on x_0, so it is smoothing by Theorem (2.1.4).

Let us also give an alternative argument relating this operator to acorrespondingoperators on $\mathfrak{g}$. Writing $\varphi_v(t) := \tilde{\phi}(|v|t)$ and using the characterisation of pseudo-differentialoperators in Theorem (2.1.4), we notice that (10) holds if for all $x_0 \in G$,the operators $\varphi_{\exp^{-1}(x_0)}\big((I - \mathcal{L}_G)^{1/4}\big)$

belong to$\Psi^0(G)$. Looking at these operatorslocally near every point $x \in G$and introducing $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\theta \circ \exp_x^{-1}$issupported in a small neighbourhood near x, with

$\exp x := (Z \mapsto x \exp(Z)) : \mathfrak{g} \to G$the exponential mapping centred at x, we have to show that

$$\theta(y)\varphi_v(B) \in \Psi^0(\mathbb{R}^n) \qquad (11)$$

holds locally on the support of θ, for all $v = \exp^{-1}(x_0)$, where operator$B$ is thepullback by expx of the operator $(I - \mathcal{L}_G)^{1/4}$ near x. Inparticular, we have$B \in \Psi_{1,0}^{1/2}(\mathbb{R}^n)$, B is elliptic on the support of θ, and its symbol is real-valued.

We now observe that if $v = 0$, then the operator in (11) is the multiplicationoperator by a smooth function, so that (11) is true in this case. If $v \neq 0$, we can show that the operator in (11) is actually a smoothing operator, so that (11) is also true.Here $\varphi_v \in C_0^\infty(R)$ since$v \neq 0$. We denote $D_t = \frac{1}{i2\pi}\partial_t$. Let $f \in L^2(\mathbb{R}^n)$ be compactlysupported, and let $u = u(x,t)$ be the solution to the Cauchy problem

$$D_t u = Bu, \;\; u(0,\cdot) = f$$

We can write $u(t,\cdot) = e^{i2\pi tB} f$and we have $u(t,\cdot) \in L^2(\mathbb{R}^n)$.Consequently,

$$\varphi_v(B)f = \int_{\mathbb{R}} \left(e^{i2\pi tB}\right)\widehat{\varphi_v}(t)\, dt = \int_{\mathbb{R}} B^{-k}u(t,\cdot)D_t^k\widehat{\varphi_v}(t)\, dt,$$

where we integrated by parts $k$times using the relation $u = B^{-1}D_t u$, and where wecan localise to a neighbourhood of a point $x$at each step. Consequently, we obtainthat $\varphi_v(B)f \in H_{\mathrm{loc}}^{k/2}(\mathbb{R}^n)$ for all $k \in \mathbb{Z}^+$, so that actually $\varphi_v(B)f \in C^\infty(\mathbb{R}^n)$. Thus,the operator $\varphi_v(B)$ is smoothing and (11) holds also for $v \neq 0$.

We now introduce a positive operator Pwhich will be important for theproof of the sharp Gårding inequality. This operator$P$will give a positive approximation to our operator A.

Proposition (2.2.2) [2]:

Let $\sigma_A \in S_{1,0}^{m}(G)$. Let us define an amplitude $p$by

$$P(x,y,\xi) := \int_G w_\xi(xz^{-1})w_\xi(yz^{-1})\sigma_A(z,\xi)\, dz, \qquad (12)$$

Where$w_\xi \in \mathrm{C}^\infty(G)$ is as in (8). Let the amplitude operator $P = \mathrm{Op}(p)$ be given by

$$Pu(x) = \int_G \sum_{|\xi|\in\hat{G}} \dim(\xi)\,\mathrm{Tr}\big(\xi(y^{-1}x)p(x,y,\xi)\big)\,u(y)\,dy.$$

Then $p \in \mathcal{A}^m_{1,1/2}(G)$ and the operator P is positive.

Proof:

We observe that

$$\|p(x,y,\xi)\|_{op} \le \int_G \left|w_\xi(xz^{-1})w_\xi(yz^{-1})\right| dz \left(\sup_{z\in G}\|\sigma_A(z,\xi)\|_{op}\right) \le \mathrm{C}\langle\xi\rangle^{\mathrm{m}}$$

because $\|w_\xi\|_{L^2(G)} = 1$ by Lemma (2.2.1). Then $p \in \mathcal{A}^m_{1,1/2}(G)$ followsfrom Lemma (2.2.1) and the Leibniz formula in Proposition (2.2.3) by anargument similar to the one whichwill be given in the proof of Lemma (2.2.4), so we omit it. Let $(e_k)_{k=1}^{\ell}$be an orthonormalbasis for $\mathbb{C}^\ell$. For matrices $M, Q \in \mathbb{C}^{\ell\times\ell}$, where Q is positive, we have

$$\mathrm{Tr}(M^*QM) = \sum_{k=1}^{\ell}\langle M^*QMe_k, e_k\rangle_{\mathbb{C}^\ell} = \sum_{k=1}^{\ell}\langle QMe_k, Me_k\rangle_{\mathbb{C}^\ell} \ge 0. \qquad (13)$$

Let us denote

$$M(z,\xi) := \int_G w_\xi(yz^{-1})\xi(yz^{-1})^* u(y)\,dy.$$

We can now show that the operator P is positive:

$$\langle Pu, u\rangle_{L^2(G)} = \int_G Pu(x)\overline{u(x)}\,dx$$

$$= \int_G\int_G \sum_{|\xi|\in\hat{G}} \dim(\xi)\,\mathrm{Tr}(\xi(x)p(x,y,\xi)u(y)\xi(y)^*)\,dy\overline{u(x)}dx$$

$$= \int_G \sum_{|\xi|\in\hat{G}} \dim(\xi)\int_G \mathrm{Tr}\left(\xi(x)\int_G w_\xi(xz^{-1})w_\xi(yz^{-1})\sigma_A(z,\xi)\,dzu(y)\xi(y)^*dy\right)\overline{u(x)}dx$$

$$= \int_G \sum_{|\xi| \in \hat{G}} \dim(\xi)\, \mathrm{Tr}\big(M(z,\xi)^* \sigma_A(z,\xi) M(z,\xi)\big)\, dz,$$

which is non-negative because of (13).

In the earlier notation, we show here that$p(x,x,\xi) - \sigma_A(x,\xi)$ is a symbol of a bounded operator from $H^s(G)$ to $H^{s-(m-1)}$.

Lemma (2.2.3) [2]:

Let $s \in \mathbb{R}$. Then the pseudo-differential operator with the symbol$p(x,x,\xi) - \sigma_A(x,\xi)$ is bounded from $H^s(G)$ to $H^{s-(m-1)}$.

Proof:

By Theorem (2.1.10) it is enough to show that

$$\left\| \partial_x^\beta \big(p(x,x,\xi) - \sigma_A(x,\xi)\big) \right\|_{op} \leq \mathrm{C}_\beta \langle \xi \rangle^{m-1}$$

holds for every multi-index β. By Lemma (2.2.1) we have

$$\partial_x^\beta \big(p(x,x,\xi) - \sigma_A(x,\xi)\big) = \int_G w_\xi(z)^2 \left(\partial_x^\beta \sigma_A(xz^{-1},\xi) - \partial_x^\beta \sigma_A(x,\xi) \right) dz.$$

We notice that $\mathrm{dist}(z,e) \leq \mathrm{C}\langle \xi \rangle^{-1/2}$ on the support of w_ξ, and we can use the Taylorexpansion of $\partial_x^\beta \sigma_A(xz^{-1},\xi)$ at $x$to get

$$\partial_x^\beta \sigma_A(xz^{-1},\xi) = \partial_x^\beta \sigma_A(x,\xi) + \sum_{|\gamma|=1} \partial_x^{(\gamma)} \partial_x^\beta (x,\xi) q_\gamma(z) + O(\mathrm{dist}(x,e)^2). \qquad (14)$$

Taking the Taylor polynomials $q_\gamma$to be odd, $q_\gamma(z) = -q_\gamma(z^{-1})$, and using the evennessof ξ from Lemma (2.2.1), we can conclude that$\int_G w_\xi(z)^2 q_\gamma(z)\, dz = 0$. Since for all $\beta$and γ we have$\left\| \partial_x^{(\gamma)} \partial_x^\beta \sigma_A(x,\xi) \right\|_{op} \leq \mathrm{C}\langle \xi \rangle^m$, we can estimate

$$\left\|\partial_x^\beta\left(p(x,x,\xi)-\sigma_A(x,\xi)\right)\right\|_{op} \leq C\langle\xi\rangle^m \sum_{|\gamma|=2}\int_G w_\xi(z)^2\left|q_\gamma(z)\right| dz \leq C\langle\xi\rangle^{m-1}$$

because $\left|q_\gamma(z)\right| \leq C\langle\xi\rangle^m$ on the support of w_ξ, for $|\gamma| = 2$.

Let $\sigma_P$be the matrix symbol of the operator$P$from Proposition (2.2.2).

Lemma(2.2.4) [2]:

Let $s \in \mathbb{R}$. Then the pseudo-differential operator with the symbol$\sigma_P(x,\xi) - \sigma_P(x,x,\xi)$ is bounded from $H^s(G)$ to $H^{s-(m-1)}$.

Proof:

Observe that for a fixed $s \in \mathbb{R}$, it is enough to take sufficiently many derivatives(and not infinitely many) for the Sobolev boundedness in Theorem (2.1.10). Thus it isenough to prove that for sufficiently many$\beta \in \mathbb{N}_0^n$it holds that

$$\left\|\partial_x^\beta\left(\sigma_P(x,\xi)-p(x,x,\xi)\right)\right\|_{op} \leq C_\beta\langle\xi\rangle^{m-1}.$$

By an argument in the proof of Proposition (2.1.13) we have theexpansion

$$\sigma_P(x,\xi) \sim \sum_{\alpha\geq 0}\frac{1}{\alpha!}\Delta_\xi^\alpha\partial_y^{(\alpha)}P(x,y,\xi)|_{y=x},$$

whose asymptotic properties we will discuss below. Instead of studying the terms$\partial_x^\beta\left(\Delta_\xi^\alpha\partial_y^{(\alpha)}P(x,x,\xi)|_{y=x}\right)$, we may study$\partial_x^\beta\left(\Delta_\xi^\alpha\partial_y^\alpha P(x,x,\xi)|_{y=x}\right)$as well. Moreover,abusing the notation slightly, without loss of generality we can look only at the right-invariantderivatives $\partial_y^\alpha$and left-invariant derivatives ∂_x^β. Recalling that

$$P(x,y,\xi) = \int_G w_\xi(xz^{-1})w_\xi(yz^{-1})\sigma_A(z,\xi)\,dz,$$

we notice that

$$\partial_x^\beta\left(\Delta_\xi^\alpha\partial_y^\alpha P(x,y,\xi)|_{y=x}\right) = \Delta_\xi^\alpha\int_G w_\xi(z)\left(\partial_z^\alpha w_\xi\right)(z)\partial_x^\beta\sigma_A(z^{-1}x,\xi)\,dx. \qquad (15)$$

We notice also that by Remark (2.1.5) we can replace differences $\Delta_\xi$by$\mathbb{D}_\xi$with a suitablecorrection for multi-indices. The application of $\mathbb{D}^\alpha$here introduces (due to the Leibnizformula in Proposition (2.1.7)) a finite sum of terms of the type

$$\int_G \left(\mathbb{D}_\xi^\kappa w_\xi(z)\right)\left(\mathbb{D}_\xi^\lambda \partial_z^\alpha w_\xi(z)\right)\left(\mathbb{D}_\xi^\mu \sigma_A(z^{-1}x,\xi)\right) dz, \qquad (16)$$

where $|\kappa+\lambda+\mu| \geq |\alpha|$. Recalling that $w_\xi \in \mathscr{S}_{1,1/2}^{n/4}$ by Lemma (2.2.1),we get that

$$\left|\left(\mathbb{D}_\xi^\kappa w_\xi(z)\right)\left(\mathbb{D}_\xi^\lambda \partial_z^\alpha w_\xi(z)\right)\left(\mathbb{D}_\xi^\mu \sigma_A(z^{-1}x,\xi)\right)\right| \leq \mathrm{C}\langle\xi\rangle^{m+n/2-|\alpha|/2}.$$

Taking into account that the support of $z \longmapsto w_\xi(z)$ is contained in the set of measure$\mathrm{C}\langle\xi\rangle^{-n/2}$ by Lemma (2.2.1), and that taking differences in ξdoes not increase the supportin z, we get that the integral in (16) can be estimated by $\mathrm{C}\langle\xi\rangle^{m-|\alpha|/2}$. Thus, we get

$$\left\|\partial_x^\beta\left(\Delta_\xi^\alpha \partial_y^\alpha p(x,x,\xi)|_{y=x}\right)\right\|_{op} \leq \mathrm{C}\langle\xi\rangle^{m-|\alpha|/2}. \qquad (17)$$

For $|\alpha| \geq 2$ this implies the desired bound by $\mathrm{C}\langle\xi\rangle^{m-1}$ for the Sobolev boundedness ofthe corresponding operator. Now, assume that $|\alpha| = 1$. Taking the Taylor expansionof $\sigma_A(z^{-1}x,\xi)$ at $x$similar to the one in (14) we see that the first termvanishes:

$$\int_G w_\xi(z)\left(\partial_z^\alpha w_\xi\right)(z)\, dz = 0$$

for $|\alpha| = 1$ because functions $w_\xi$and $\partial_z^\alpha w_\xi$are even and odd, respectively, see Proposition (2.2.14). Consequently, for $|\gamma| \geq 1$, we can estimate

$$\left|w_\xi(z)\left(\partial_z^\alpha w_\xi\right)(z) q_\gamma(z)\right| \leq \mathrm{C}\langle\xi\rangle^{n/2+|\alpha|/2-|\gamma|/2},$$

which together with (15) gives

$$\left\|\partial_x^\beta\left(\Delta_\xi^\alpha \partial_y^\alpha p(x,x,\xi)|_{y=x}\right)\right\|_{op} \leq \mathrm{C}\langle\xi\rangle^{m-|\alpha|/2-|\gamma|/2} \leq \mathrm{C}\langle\xi\rangle^{m-1}$$

because $|\alpha| = 1$ and $|\gamma| \geq 1$. Finally, let us look at the remainder

$$\sigma_{R_N}(x,\xi) = \sigma_P(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \Delta_\xi^\alpha \partial_y^\alpha p(x,y,\xi)|_{y=x}.$$

By the arguments similar to the above we can see that

$$\left\| \partial_x^\beta \sigma_{R_N}(x,\xi) \right\|_{op} \leq C_\beta \langle \xi \rangle^{m+n/2-|\beta|/2-N/2},$$

so that for every $s, t \in \mathbb{R}$ there exists a sufficiently large N_{st}such that $R_N$is boundedfrom $H^s(G)$ to $H^t(G)$ whenever $N \geq N_{st}$. This concludes the proof.

Theorem (2.2.5) [2]:

Let $A \in \Psi^m(G)$ be such that its full matrix symbol $\sigma_A$satisfies$\sigma_A(x,\xi) \geq 0$for all

$(x,[\xi]) \in G \times \hat{G}$. Then there exists $C < \infty$such that

$$\mathrm{Re}(Au,u)_{L^2(G)} \geq -C\|u\|^2_{H^{(m-1)/2}(G)}$$

for every $u \in C^\infty(G)$.

As a corollary of Theorem (2.1.1) we obtain the following statement oncompact Liegroups, analogous to the corresponding result on $\mathbb{R}^n$, which is often necessary in theproofs of pseudo-differential inequalities.

Proof

Let $Q = A - P$ with operator Pas in Proposition (2.2.2).Let $u \in C^\infty(G)$. Then $A = P + Q$and the positivity of Pimplies

$$\mathrm{Re}(Au,u)_{L^2(G)} = \mathrm{Re}(Pu,u)_{L^2(G)} + \mathrm{Re}(Qu,u)_{L^2(G)} \geq \mathrm{Re}(Qu,u)_{L^2(G)}.$$

Let now $P_0 = \mathrm{Op}\big(p(x,x,\xi)\big)$. Writing $Q = (A - P_0) + (P_0 - P)$, we have

$$\sigma_{A-P_0}(x,\xi)\sigma_A(x,\xi) - P(x,x,\xi)\text{and}\sigma_{P_0-P}(x,\xi) = P(x,x,\xi) - \sigma_P(x,\xi).$$

Consequently, $A - P_0$ and $P_0 - P$are bounded from $H^{(m-1)/2}(G)$ to$H^{-(m-1)/2}(G)$ byLemma (2.2.3) and Lemma (2.2.4), respectively. Hence$Q$ is bounded from $H^{(m-1)/2}(G)$ to$H^{-(m-1)/2}(G)$, so that

$$\left|\mathrm{Re}(Qu,u)_{L^2(G)}\right| \leq \|Qu\|_{H^{-(m-1)/2}(G)}\|u\|_{H^{(m-1)/2}(G)} \leq \mathrm{C}\|u\|^2_{H^{(m-1)/2}(G)},$$

completing the proof of Theorem (2.1.1).

Corollary (2.2.6) [2]:

Let $A \in \Psi^1(G)$ be such that its matrix symbol $\sigma_A$satisfies

$$\|\sigma_A(x,\xi)\|_{op} \leq \mathrm{C}$$

for all $(x,[\xi]) \in G \times \hat{G}$. Then$A$ is bounded from $L^2(G)$ to $L^2(G)$.

Here $\|\cdot\|_{op}$denotes the $\ell^2 \to \ell^2$ operator norm of the linear finiteDimensionalmapping (matrix multiplication by) $\sigma_A(x,\xi)$, i.e.

$$\|\sigma_A(x,\xi)\|_{op} = \sup\{\|\sigma_A(x,\xi)v\|_{\ell^2} : v \in \mathbb{C}^{\dim(\xi)}, \|v\|_{\ell^2} \leq 1\}.$$

The weights for measuring the orders of symbols are expressed in termsof theeigenvalues of the bi-invariant Laplacian $\mathcal{L}_G$. Matrix elements of every representationclass $[\xi] \in G$span an eigenspace of the bi-invariant Laplace–Beltrami operator $\mathcal{L}_G$on $G$with the corresponding eigenvalue $-\lambda_\xi^2$. Based on these eigenvalues we define

$$\langle\xi\rangle = \left(1+\lambda_\xi^2\right)^{1/2}.$$

In particular,for the usual Sobolev spaces, we have $f \in H^s(G)$ if and only if

$\langle\xi\rangle^s \hat{f}(\xi) \in \ell^2(\hat{G})$. Tofix the norm on $H^s(G)$ for the followingstatement, we can then set

$$\|\mathrm{f}\|_{\mathrm{H}^s(\mathrm{G})} := \left(\sum_{|\xi|\in\hat{\mathrm{G}}} \dim(\xi)\langle\xi\rangle^{2s}\mathrm{Tr}\left(\hat{\mathrm{f}}(\xi)^*\hat{\mathrm{f}}(\xi)\right)\right)^{1/2},$$

and we can write this also as $\left\|\langle\xi\rangle^s\hat{f}(\xi)\right\|_{\ell^2(\hat{G})}$. Also, we note that if$A \in \Psi^m(G)$, then thereis a constant $0 < M < \infty$ suchthat $\|\sigma_A(x,\xi)\|_{op} \leq M\langle\xi\rangle^m$holds for all$x \in G$and $[\xi] \in \hat{G}$.

As another corollary of Theorem (2.1.1) we can get a norm-estimate forpseudodifferentialoperators on compact Lie groups:

Proof

We note that the assumption $\|\sigma_A(x,\xi)\|_{op} \leq \mathrm{C}$ impliesthat for any $\theta \in \mathbb{R}$ be have the inequality $\mathrm{Re}\left(\mathrm{C} - \mathrm{e}^{\mathrm{i}\theta}\sigma_A(x,\xi)\right) \geq 0$. Consequently,the sharp Gårding inequality in implies that we have

$$\mathrm{Re}\left(\left(\mathrm{C} - \mathrm{e}^{\mathrm{i}\theta}A\right)u,u\right)_{L^2(G)} \geq -\mathrm{C}'\|u\|^2_{L^2(G)}$$

For all $u \in L^2(G)$. From this it follows that $\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}(Au,u)_{L^2(G)}\right) \leq \mathrm{C}'''\|u\|^2_{L^2(G)}$, so that $\left|(Au,u)_{L^2(G)}\right|\mathrm{C}''\|u\|^2_{L^2(G)}$.

Corollary(2.2.7) [2]:

Let $A \in \Psi^m(G)$ and let

$$M = \sup_{(x,[\xi])\in G\times\hat{G}} \left(\langle\xi\rangle^{-m}\|\sigma_A(x,\xi)\|_{op}\right).$$

Then for every $s \in \mathbb{R}$ there exists a constant $\mathrm{C} > 0$ such that

$$\|Au\|^2_{H^s(G)} \leq M^2\|u\|^2_{H^{s+m-1/2}(G)}$$

for all $u \in \mathrm{C}^\infty(G)$.

Proof

Let us define

$$B(x,\xi) = M^2\langle\xi\rangle^{2m+2s}I_{\dim\xi} - \sigma_A(x,\xi)^*\sigma_A(x,\xi)\langle\xi\rangle^{2s}.$$

By the Leibniz formula, $B \in \mathscr{S}^{2m+2s}(G)$, and $B(x,\xi) \geq 0$ due to the definition of$M$. Consequently, we have

$$\mathrm{Re}(\mathrm{Op}(B)u,u)_{L^2(G)} \geq -\mathrm{C}\|u\|_{H^{m+s-1/2}(G)}.$$

Recall that for the bi-invariant Laplace-Beltrami operator $\mathcal{L}_G$on G, the symbol of$I - \mathcal{L}_G$is $\langle\xi\rangle^2$, so that $\|Au\|^2_{H^2(G)} = (A^*(I-\mathcal{L}_G)^sAu,u)_{L^2(G)}$. On the other hand,

$$\mathrm{Op}(B) + A^*(I-\mathcal{L}_G)^sA - M^2(I-\mathcal{L}_G)^{m+s} \in \Psi^{2\mathrm{m}+2\mathrm{s}-1}(G)$$

because its symbol is in $\mathscr{S}^{2m+2s-1}(G)$ by the composition formula for pseudo-differentialoperators combined with the formula for the adjoint operators combined with the formula for the adjoint operator. Combining thesefacts we obtain the statement of Corollary.

Chapter 3

Compact Lie groups and Central oscillating multipliers

In this chapter weshow results for a compact Lie group.

Section (3.1): Estimates on kernels

Let G be a connected, compact semisimple Lie group (a compact Lie group is a compact differentiable manifold together with smooth multiplication map $(G\times G)\to G$ which gives G the structure of a group) of dimension $n$and rank .anyc^{∞}function f on G Ghas Fourier expansion $f(x)=\sum d_{\lambda}x_{\lambda}*f(x)$,and we will deal oscillating multiplier operator

$$T_{\Omega,\alpha,\beta}(f)(x)=k_{\Omega,\alpha,\beta}*f(x),\alpha,\beta\geq 0.$$

Here $k_{\Omega,\alpha,\beta}$ is a central kernel defined by

$$k_{\Omega,\alpha,\beta}(y)=\sum_{\lambda+\delta\in\Lambda\setminus\{0\}}\frac{e^{i\|\lambda+\delta\|^{\beta}\,\Omega\left(\frac{\lambda+\delta}{\|\lambda+\delta\|^{\alpha}}\right)}}{\|\lambda+\delta\|^{\alpha}}d_{\lambda}\chi_{\lambda}(\xi),$$

Where δ is the sum of all positive roots , y the conjugate to the element exp ξ in a fixed maximal tours of G and Ω is asufficiently smooth function that is symmetric under the dual weyl group .thus $T_{\Omega,\alpha,\beta}$ is a convolution operator ,and

$T_{\Omega,\alpha,\beta}$(f) has Fourier expansion$T_{\Omega,\alpha,\beta}(f)(x)=\sum\frac{e^{i\|\lambda+\delta\|^{R}\,\Omega(\frac{(\lambda+\delta)}{\|\lambda+\delta\|})}}{\|\lambda+\delta\|^{\alpha}}d_{\lambda}X_{\lambda}*f(x)$

For any $f\in C^{\infty}(G)$.

Lemma (3.1.1) [3]:

Let $A\subseteq\mathbb{R}^{m}$denote an open set and$\Phi\in C_0^{\infty}(A)$. If $\Psi\in C^{\infty}(A)$satisfies

$$|det(\partial^2/\partial x_i\partial x_j(x))|\geq C>0$$

for all $x\in supp(\Phi)$then for large $|\lambda|$,

$$\left|\int_A e^{i(\lambda\Psi(x)+\langle\xi,x\rangle)}\Phi(x)dx\right| \lesssim |\lambda|^{-m/2}\|\Phi\|_{c^{2m}}.$$

Let

$$D(\xi) = \sum_{w\in W} e^{i}\langle w\delta,\xi\rangle$$

be theWeyl denominator. By the definition, and the fact that Ω is symmetric under the actionof the dual Weyl group, it is easy to see

$$K_{\Omega,\alpha,\beta}(y) = \frac{\sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{e^{i\|\lambda+\delta\|^{\beta}}}{\|\lambda+\delta\|^{\alpha}}\Omega\left(\frac{\lambda+\delta}{\|\lambda+\delta\|}\right)\left(\prod_{a\in A}\langle\delta+\lambda,a\rangle\right)\sum_{w\in W}\epsilon(w)\, e^{i\,\langle w(\lambda+\delta),\xi\rangle}}{D(\xi\,)\prod_{\alpha\in A}\langle\alpha,\delta\rangle}$$

$$= \frac{\sum_{\lambda\in P\{0\}} \frac{e^{i\|\lambda\|^{\beta}}}{\|\lambda\|^{\alpha}}\Omega\left(\frac{\lambda}{\|\lambda\|}\right)\left(\prod_{a\in A}\langle\lambda,a\rangle\right)e^{i\,\langle\lambda,\xi\rangle}}{D(\xi\,)\prod_{a\in A}\langle a,\delta\rangle},$$

where y is conjugate to exp $\xi \in T$. Choose a C^{∞}function $\Phi(t)$on the real line with supportin the interval $\left[\frac{1}{2},2\right]$and satisfying

$$\sum_{k=-\infty}^{\infty}\Phi_k(t) = 1$$

for all $t \in (0,\infty)$, where $\Phi_k(t) = \Phi(2^{-k}\,t)$. Noting that all $\lambda \in P\setminus\{0\}$can be representedas a countable set

$$\{\lambda_j : j \in N \;\text{ and }\; 0 < \|\lambda_1\| \le \|\lambda_2\| \le, \ldots\},$$

without loss of generality, we may write

$$\sum_{\lambda\in P\setminus\{0\}} \frac{e^{i\|\lambda\|^{\beta}}\Omega\left(\frac{\lambda}{\|\lambda\|}\right)}{\|\lambda\|^{\alpha}}\left(\prod_{a\in A}\langle\lambda,a\rangle\right)e^{i\,\langle\lambda,\xi\rangle}$$

$$= \sum_{k=0}^{\infty} \sum_{\lambda \in P} \Phi_k(\|\lambda\|) \frac{e^{i\|\lambda\|^\beta}}{\|\lambda\|^\alpha} \Omega\left(\frac{\lambda}{\|\lambda\|}\right) \left(\prod_{a \in A} \langle \lambda, a \rangle \right) e^{i\,\langle \lambda, \xi \rangle}.$$

Now we have

$$T_{\Omega,\alpha,\beta}(f)(x) = K_{\Omega,\alpha,\beta} * f(x) = \sum_{k=0}^{\infty} H_k * f(x)$$

with

$$H_k(y) = \frac{\sum_{\lambda \in P} \Phi_k(\|\lambda\|) \frac{e^{i\|\lambda\|^\beta}}{\|\lambda\|^\alpha} \Omega\left(\frac{\lambda}{\|\lambda\|}\right) \left(\prod_{a \in A} \langle \lambda, a \rangle\right) e^{i\,\langle \lambda, \xi \rangle}}{D(\xi\,) \prod_{a \in A} \langle a, \delta\,\rangle}.$$

By the Plancherel Theorem and the choice of Φ, it is easy to see that we have

$$\|H_k * f\|_{L^2(G)} \simeq \|f\|_{L^2(G)} 2^{-k\alpha}, \qquad k = 0,1,\ldots$$

By the Poisson summation formula, we may write

$$H_k(y) \simeq \frac{\sum_{\nu \in \mathbb{N}} G_k(\xi + \nu)}{D(\xi) \prod_{a \in A} \langle a, \delta\,\rangle},$$

where

$$G_k(\xi) = \int_{\mathbb{R}^m} \frac{e^{i\|y\|^\beta}}{\|y\|^\alpha} \Phi_k(\|y\|) \Omega\left(\frac{\lambda}{\|\lambda\|}\right) \left(\prod_{a \in A} \langle y, a \rangle \right) e^{-i\,\langle \xi, y \rangle} dy.$$

Without loss of generality, we may use the Euclidean norm $\|\cdot\|$instead of $|\cdot|$. Rescaling,we have

$$G_k(\xi) \simeq 2^{-k\left(\alpha - \frac{n+m}{2}\right)} \int_{\mathbb{R}^m} \frac{e^{i\left(2^{\beta k}|y|^{\beta - 2k} < \xi, y >\right)} \Omega(y')}{|y|^{\alpha - (n-m)/2}} \Phi(|y|) \left(\prod_{a \in A} \langle y', a \rangle \right) dy$$

where $y' = \frac{y}{|y|}$for $y \neq 0$. Denote

$$\Psi(|y|) = \frac{\Phi(|y|)}{|y|^{\alpha-\frac{n-m}{2}}}, \qquad \text{and}\Theta(y') = \Omega(y')\prod_{a\in A}\langle y', a\rangle.$$

Then Ψ is a C^∞ function with support in $\{y: \frac{1}{2} \le |y| \le 2\}$, and $\Theta(y)$is a sufficientlysmooth function defined on the unit sphere. Furthermore, we have

$$G_k(\xi) \simeq 2^{-k\left(\alpha-\frac{n+m}{2}\right)} \int_{\mathbb{R}^m} e^{i\,(2^{\beta k}|y|^\beta - 2^k<\xi,y>)}\,\Psi(|\,y\,|)\Theta(y')dy.$$

We will now carefully estimate $G_k(\xi)$as a function of β.

Choose C^∞nonnegative functions$\Gamma_{j,k}, j = 1,2,3$ such that for all $k = 1,2,\ldots,$

$$\sum_{j=1}^{3} \Gamma_{j,k}\,\Gamma(\xi\,) \;\equiv\; 1,$$

and

$$\text{supp}(\Gamma_{1,k}) \subset \{|\,\xi\,| \le\; c_1 2^{k(\beta-1)}\}, \text{and}\Gamma_{1,k}(\xi\,) \;\equiv\; 1 \text{ if}|\xi| \le c_1 2^{k(\beta-1)-1};$$

$$\text{supp}(\Gamma_{2,k}) \subset \{|\,\xi\,| \ge\; c_2 2^{k(\beta-1)}\}, \text{and}\Gamma_{2,k}(\xi\,) \;\equiv\; 1 \text{ if}|\xi| \ge c_2 2^{k(\beta-1)+1};$$

$$\text{supp}\left(\Gamma_{3,k}\right) \subset \left\{\frac{4c_2}{3} 2^{k(\beta-1)} > |\,\xi\,| > \frac{3c_1}{4} 2^{k(\beta-1)}\right\},$$

where c_2 is a sufficiently large constant, c_1 is a sufficiently small constant, and both constantsmay depend on β, n and$m$, but are independent of k and ξ. Clearly, we may assume that$\Gamma_{j,k}(\xi)$satisfies

$$|\nabla\Gamma_{j,k}(\xi)| \le 2^{-k(\beta-1)}.$$

Now we have

$$G_k(\xi) = \sum_{j=1}^{3} \Gamma_{j,k}\,(\xi\,)G_k(\xi\,).$$

We give the following estimates on $\Gamma_{j,k}(\xi)G_k(\xi), j = 1,2,3.$

Lemma (3.1.2) [3]:

For any non-negative integer N, we have

$$|\Gamma_{1,k}(\xi)G_k(\xi)| \leq \Gamma_{1,k}(\xi\,)2^{-k(\alpha-\frac{n+m}{2}+N\beta)};$$

$$|\Gamma_{2,k}(\xi\,)G_k(\xi\,)| \leq \Gamma_{2,k}(\xi\,)2^{-k(\alpha-\frac{n+m}{2}+N)}|\xi|^{-N};$$

$$|\Gamma_{3,k}(\xi)G_k(\xi)| \leq \Gamma_{3,k}(\xi)2^{-k(\alpha-\frac{n+m}{2})}|\xi|^{-\frac{m}{2}}2^{\frac{-km}{2}}.$$

Proof:

Using polar coordinates, we have

$$\left|\Gamma_{1,k}(\xi)G_k(\xi\,)\right| \simeq \Gamma_{1,k}(\xi\,)\left|2^{-k(\alpha-m+n/2)}\int_{S^{m-1}}\Theta(y')\int_0^\infty e^{i\theta}(t)\Psi(t)dt\,d\sigma(y')\right|,$$

where $d\sigma(y')$is the induced Lebesgue measure on the unit sphere$S^{m-1}$and

$$\theta(t) = 2^{\beta k}t^{\beta} - 2^k t\langle\xi, y'\rangle.$$

Note that when

$$t \in \operatorname{supp}(\Psi) = \left[\frac{1}{2,2}\right], \qquad c_1 > \frac{2^\beta}{\beta}100, \qquad \text{and} \quad \xi \in \operatorname{supp}(\Gamma_{1,k})$$

we have

$$\left|\frac{d\theta(t)}{dt}\right| \geq |\beta 2^{k\beta}t^{\beta-1}| - 2^k|\xi| \;\geq 2^{k\beta}.$$

Thus, an easy integration by parts in t-variable yields that

$$|\Gamma_{1,k}(\xi)G_k(\xi)| \leq \Gamma_{1,k}(\xi)2^{-k(\alpha-\frac{n+m}{2}+N\beta)}\text{for any } N > 0.$$

If $\xi \in \operatorname{supp}\Gamma_{2,k}$, without loss of generality, we assume $|\xi_1| \geq \frac{|\xi|}{m}$. It is easy to see that for $1/2 < |y| \leq 2$ we have

$$\left|\frac{\partial}{\partial y_1}(2^{k\beta}|y|^{\beta} - 2^k\langle\xi, y\rangle)\right| \geq 2^k|\xi|.$$

Using integration by parts again we obtain

$$|\Gamma_{2,k}(\xi)G_k(\xi)| \leq \Gamma_{2,k}(\xi)2^{-k(\alpha-\frac{n+m}{2}+N)}|\xi\,|^{-N}\text{for any } N > 0.$$

By Lemma (3.1.1) and noting$2^{k\beta} \simeq (|\xi|2k)$in the support of $\Gamma_{3,k}$, we have

$$|\Gamma_{3,k}(\xi)G_k(\xi)|\Gamma_{3,k}(\xi)2^{-k(\alpha-(n+m)/2)}|\xi\,|^{-\frac{m}{2}}2^{\frac{-km}{2}}.$$

This completes the proof of the lemma.

By Lemma (3.1.2), we have

Proposition (3.1.3) [3]:

Let G_k be defined as in Lemma (3.1.2). Then there is an $M > 0$ such that

$$\sum_{k=0}^{\infty}|G_k(\xi)| \simeq O\left(|\xi\,|^{-\left(\frac{\alpha-\frac{n}{2}}{\beta-1}\right)-\frac{m}{2}}\right), \qquad \text{if}|\xi| \leq M$$

and

$$\sum_{k=0}^{\infty}|G_k(\xi)| = O(|\,\xi\,|^{-N})\text{ for any integer } N, \qquad \text{if}|\xi| \geq M.$$

Proof:

We know

$$\sum_{k=0}^{\infty}|G_k(\xi)| \simeq \sum_{j=1}^{3}\sum_{k=0}^{\infty}\Gamma_{j,k}(\xi)|G_k(\xi)|\,.$$

Clearly, if$M$ is sufficiently large, $\Gamma_{1,k}(\xi)$and $\Gamma_{3,k}(\xi)$vanish for all$k = 0,1,..$ByLemma (3.1.2),

$$\sum_{k=0}^{\infty}\Gamma_{2,k}(\xi)|G_k(\xi)| \leq \sum_{k=0}^{\infty}2^{-k(\alpha-(n+m)/2)+N)}\,|\xi|^{-N} \leq |\,\xi\,|^{-N}.$$

If$|\xi| \leq M$, then Lemma (3.1.2)implies

$$\sum_{k=0}^{\infty} \Gamma_{1,k}(\xi)|G_k(\xi)| \leq \sum_{2^{-k} \leq \left(\frac{|\xi|}{c_1}\right)^{\frac{1}{1-\beta}}} \Gamma_{1,k}(\xi) 2^{-k\left(\alpha - \frac{(n+m)}{2} + N\beta\right)} \leq |\xi|^{\frac{\left(\alpha - \frac{(n+m)}{2} + N\beta\right)}{\beta-1}} \text{for any } N.$$

Again, by Lemma (3.1.2)and choosing$N = m/2$, we have

$$\sum_{k=0}^{\infty} \Gamma_{2,k}(\xi)|G_k(\xi)| \leq \sum_{2^k \leq (|\xi|/c_2)^{\frac{1}{\beta-1}}} \Gamma_{2,k}(\xi) 2^{-k\left(\alpha - \frac{n}{2}\right)} |\xi|^{-\frac{m}{2}} \leq |\xi|^{-\left(\frac{\alpha - \frac{n}{2}}{\beta-1}\right) - \frac{m}{2}}$$

Also, by Lemma (3.1.1),

$$\sum_{k=0}^{\infty} |\Gamma_{3,k}(\xi) G_k(\xi)| \leq \sum_{k=0}^{\infty} \Gamma_{3,k}(\xi) 2^{-k(\alpha - n/2)} |\xi|^{-\frac{m}{2}} \leq \sum_{k=0}^{\infty} \Gamma_{3,k}(\xi) |\xi|^{\left(\frac{\alpha - \frac{n}{2}}{1-\beta}\right) - \frac{m}{2}} \leq |\xi|^{-\left(\frac{\alpha - \frac{n}{2}}{\beta-1}\right) - \frac{m}{2}}.$$

This completes the proof.

By the proofs of Proposition (3.1.3)and Lemma (3.1.1), we easily obtain

Proposition (3.1.4) [3]:

Let $\Gamma_{j,k}(\xi), j = 1,2,3$ and $G_k(\xi)$be the same as in Proposition (3.1.3). Then

$$\left| \nabla \left(\sum_{k=0}^{\infty} \Gamma_{3,k}(\xi) G_k(\xi) \right) \right| \leq |\xi|^{-\left(\frac{\alpha - \frac{n}{2}}{\beta-1}\right) - \frac{m}{2}} \left\{ |\xi|^{\frac{1}{\beta-1}} + |\xi|^{-1} \right\};$$

$$\nabla \left| \sum_{k=0}^{\infty} \Gamma_{1,k}(\xi) G_k(\xi) \right| \leq 1 \text{ if} |\xi| \leq 100;$$

$$\left| \nabla \left(\sum_{\infty}^{k=0} \Gamma_{2,k}(\xi) G_k(\xi) \right) \right| \leq |\xi|^{-\left(\frac{\alpha - \frac{n}{2}}{\beta-1}\right) - \frac{m}{2}} \left\{ |\xi|^{\frac{1}{\beta-1}} + |\xi|^{-1} \right\} \text{if} |\xi| \leq 100;$$

and

$$\left|\nabla\left(\Gamma_{2,k}(\xi)G_k(\xi)\right)\right| \leq \Gamma_{2,k}(\xi)2^{-k\left(\alpha-\frac{n+m}{2}+N-1\right)}|\xi|^{-N}$$
$$+\gamma_{2,k}(\xi)2^{-k\left(\alpha-\frac{n+m}{2}+N\right)}2^{k(1-\beta)}|\xi|^{-N}\text{for any } N>0;$$

$$\left|\nabla\left(\Gamma_{1,k}(\xi)G_k(\xi)\right)\right| \leq \Gamma_{1,k}(\xi)2^{-k\left(\alpha-\frac{n+m}{2}+N\beta-1\right)}+\gamma_{1,k}(\xi)2^{-k\left(\alpha-\frac{n+m}{2}+N\beta\right)}2^{k(1-\beta)}$$

for any$N>0$ where $\gamma_{2,k}$, and $\gamma_{1,k}$, are C^∞functions supported in a set

$$\left\{\xi\in\mathbb{R}^m: |\xi|\simeq 2^{k(\beta-1)}\right\}.$$

Proof:

The proofs for $j=1,2,3$ are similar. Thus, we only prove the case $j=3$. Note thatfor each ξ_0, there is a small neighborhood of ξ_0 such that in the neighborhood the sum

$$\sum_{k=0}^{\infty}\Gamma_{3,k}(\xi)G_k(\xi)$$

has only finitely many nonzero terms. We have

$$\nabla\left(\sum_{k=0}^{\infty}\Gamma_{3,k}(\xi)G_k(\xi)\right)=\sum_{k=0}^{\infty}\nabla\Gamma_{3,k}(\xi)G_k(\xi)+\sum_{k=0}^{\infty}\Gamma_{3,k}(\xi)\nabla G_k(\xi)\,.$$

By the choice of $\Gamma_{3,k}(\xi)$and Proposition (3.1.3), it suffices to prove the proposition for

$$\sum_{k=0}^{\infty}\Gamma_{3,k}(\xi)\nabla G_k(\xi).$$

By the formulation of G_k, without loss of generality, we may write

$$\nabla G_k(\xi)=O\left(2^{-k\left(\alpha-\frac{n+m}{2}-1\right)}\int_{\mathbb{R}^m}e^{i\left(2^{\beta k}|y|^{\beta-2k<\xi,y>}\right)}(\Psi_1(|y|)\Theta(y')\,dy\right),$$

where Ψ_1 is a C^∞function supported in $\{1/2<|y|\leq 2\}$. Thus, the lemma follows triviallyby the estimate of

$$\sum_{k=0}^{\infty} |\, \Gamma_{3,k}(\xi) G_k(\xi)|$$

in Proposition (3.1.3).

Proposition (3.1.5) [3]:

Let $0 < \beta < 1$. Then there is a constant C independent of k such that

$$\|H_k * f\|_{L^p}(G) \leq C 2^{-k\left(\alpha - \frac{\beta n}{2}\right)} \|f\|_{L^p}(G) \text{for } 1 \leq p \leq \infty.$$

Proof:

By Young's inequality, it suffices to show that

$$\|H_k\|_{L^1(G)} \leq 2^{-k\left(\alpha - \frac{\beta n}{2}\right)}.$$

Recall the Weyl integration formula

$$\int_G f(x)dx \simeq \int_Q f(exp\, \xi)|D(\xi)|^2 d\xi$$

for any central function $f(x) = f(exp\xi)$. Applying this formula on the central function

$$H_k(y) \simeq \frac{\sum_{\nu \in \mathbb{N}} G_k(\xi + \nu)}{D(\xi) \prod_{a \in A} \langle a, \delta \rangle},$$

we have

$$\|H_k\|_{L^1(G)} = \int_G |\, H_k(x)|dx \leq \sum_{j=1}^{3} \sum_{\nu \in \mathbb{N}} \int_Q |\Gamma_j(\xi + \nu) G_k(\xi + \nu)) D(\xi)| d\xi\,.$$

Therefore, by Lemma (3.1.2), and noting

$$|D(\xi)| \leq \min\left\{1, |\xi|^{\frac{n-m}{2}}\right\},$$

we have

$$\|H_k\|_{L^1(G)} \leq \sum_{j=1}^{3} \int_{\Gamma_m} |\Gamma_j(\xi)G_k(\xi))D(\xi)|d\xi \leq 2^{-k(\alpha-(n+m)/2+N\beta)} \int_{|\xi|\leq c_1 2^{k(\beta-1)}} |D(\xi)\,|d\xi$$

$$+2^{-k(\alpha-(n+m)/2+N)} \int_{|\xi|\geq c_2 2^{k(\beta-1)}} |\xi|^{-N}|D(\xi)|d\xi$$

$$+\,2^{-k(\alpha-(n+m)/2)}2^{\frac{-km}{2}}. \int_{|\xi|\simeq 2^{k(\beta-1)}} |\xi|^{-m/2}|D(\xi)|d\xi \leq 2^{-k\left(\alpha-\frac{\beta n}{2}\right)}.$$

This completes the proof.

In this case, we write $K_{\alpha,1}(y) = K_\alpha(y)$. Choose a $C^\infty(\mathbb{R}^m)$function Ψ such that $\Psi(y) = 0$ if $|y| < r$, and $\Psi(y) = 1$ if $|y| > 2r$. Clearly, if we fix a suitably small $r > 0$, thenfollowing the same argument as before, we have

$$K_\alpha(y) = K_\alpha(\exp\xi) \simeq \frac{\left(\prod_{a\in A}\frac{\partial}{\partial a}\right)\sum_{\lambda\in P} e^{\frac{i\|\lambda\|}{\|\lambda\|^\alpha}}\Psi(\|\lambda\|)e^{i\,\langle\lambda,\xi\rangle}}{D(\xi)\prod_{a\in A}\langle a,\delta\,\rangle}.$$

By the Poisson summation formula,

$$K_\alpha(y) = \frac{\left(\prod_{a\in A}\frac{\partial}{\partial a}\right)\sum_{\nu\in\mathbb{N}} G(\xi+\nu)}{D(\xi)\prod_{a\in A}\langle a,\delta\,\rangle}.$$

where

$$G(\xi) = \int_{\mathbb{R}^m} \frac{e^{i|u|}}{|u|^\alpha}\Psi(|u|)e^{i\langle u,\xi\rangle}du.$$

hence

$$G(\xi) \simeq \int_0^\infty \frac{e^{it}}{t^\alpha}\Psi(t)V_{\frac{m-2}{2}}(t\,|\xi|)t^{m-1}dt\,,$$

where

$$V_\nu(t) = \frac{J_\nu(t)}{t^\nu},$$

And$J_\nu\ (t)$is the Bessel function of order ν. We have,

$$\left(\prod_{a\in A}\frac{\partial}{\partial a}\right)V_{\frac{m-2}{2}}(t\ |\xi|) \simeq \left(\prod_{a\in A}\langle a,\xi\rangle\right)V_{\frac{n-2}{2}}(t\ |\xi|)t^{n-m}.$$

Thus

$$K_\alpha(x) \simeq \frac{\sum_{a\in A}\left(\prod_{\nu\in\mathbb{N}}\langle a,\xi+\nu\rangle\right)Y(\xi+\nu)}{D(\xi)\prod_{a\in A}\langle a,\delta\ \rangle}, \qquad (1)$$

where

$$Y(\xi) = \int_0^\infty \frac{e^{it}}{t^\alpha}\Psi(t)V_{\frac{m-2}{2}}(t\ |\xi|)t^{n-1}dt = |\xi|^{\frac{2-n}{2}}\int_0^\infty e^{it}\Psi(t)J_{\frac{n-2}{2}}(t|\xi|)t^{\frac{n}{2}-\alpha}dt.$$

Choose the function Φ as before, we have

$$Y(\xi) = \sum_k Y_k(\xi)\,,$$

where

$$Y_k(\xi) = |\xi|^{\frac{2-n}{2}}2^{-k\left(\alpha-\frac{n}{2}-1\right)}\int_0^\infty e^{i2^k t}\Phi(t)(2^k t)J_{\frac{n-2}{2}}(2^k\ t\ |\xi|t^{\frac{n}{2}-\alpha}dt.$$

when

$$2^k|\ \xi\ | \leq\ 10,$$

using the well-known estimate

$$J_{\frac{n-2}{2}}(t) = O\left(t^{\frac{n-2}{2}}\right), \qquad \text{as } t\to 0,$$

it is easy to see

$$|Y_k(\xi)| \leq |\xi|^{\frac{2-n}{2}} 2^{-k\left(\alpha-\frac{n}{2}-1\right)} (2^k|\xi|)^{\frac{n-2}{2}} = 2^{-k(\alpha-n)}. \qquad (2)$$

when

$$2^k \mid \xi \mid > 10,$$

we use the asymptotic expansion of$J_{\frac{n-2}{2}}(t \mid \xi \mid)$:

$$J_{\frac{n-2}{2}}(t|\xi|) \simeq e^{\pm i\, t|\xi|} \left(\sum_{j=0}^{N} \frac{c_j}{(t|\xi|)^{j+1/2}} \right) + O(t\,|\xi|)^{-(N+1)-1/2},$$

where $c_1, c_2, \ldots, c_N$are constants.

$$Y_k(\xi) \simeq \sum_{j=0}^{N} c_j\, |\xi|^{\frac{1-n}{2}-j} 2^{-k\left(\alpha-\frac{n}{2}-\frac{1}{2}+j\right)} \int_0^\infty e^{i2^k t(1\pm|\xi|)}\, \Phi(t)\Psi(2^k t) t^{\frac{n}{2}-\alpha-j} dt$$
$$+ O\left(|\xi|^{\frac{-n+1}{2}-N-1} 2^{-k\left(\alpha-\frac{n}{2}+\frac{1}{2}+N\right)} \right). \qquad (3)$$

Using integration by parts and noting $\Psi(2^k t) \equiv 1$ if $k > 2$ and $t \in \mathrm{supp}(\Phi)$, it is easy to seethat for $k = 3,4,\ldots$one has

$$|Y_k(\xi)| \leq |\xi|^{\frac{1-n}{2}} 2^{-k\left(\alpha-\frac{n}{2}-\frac{1}{2}\right)} \min\left\{1, 2^{-Nk}\left|1-|\xi|\right|^{-N}\right\}$$
$$+ O\left(|\xi|^{\frac{-n-1}{2}} 2^{-k\left(\alpha-\frac{n}{2}+\frac{1}{2}\right)} \right). \qquad (4)$$

We now obtain

Proposition (3.1.6) [3]:

If $\alpha > \frac{n-1}{2}$, then

$$\|K_\alpha\|_{L^1}(G) \leq 1.$$

Proof:

As in the proof for the case $0 < \beta < 1$, by (2) we have

$$\|K_a\|_{L^1(G)} \leq \sum_{k=0}^{\infty} \int_{\Gamma_m} \left| \prod_{a \in A} \langle a, \xi \rangle \, Y_k(\xi) || D(\xi) d\xi \right|$$

$$\leq \sum_{k=5}^{\infty} 2^{-k(\alpha-n)} \int_{|\xi| \leq \frac{10}{2^k}} |\xi|^{n-m} d\xi + \sum_{k=0}^{\infty} \int_{\frac{10}{2^k} < |\xi| \leq \frac{1}{2}} |\xi|^{n-m} |Y_k(\xi)| d\xi$$

$$+ \sum_{k=0}^{\infty} \int_{|\xi| > 100} |\xi|^{\frac{n-m}{2}} |Y_k(\xi)| d\xi + \sum_{k=0}^{\infty} \int_{\frac{1}{2} < |\xi| \leq 100} |Y_k(\xi)| d\xi$$

$$= I_1 + I_2 + I_3 + I_4.$$

Clearly,

$$I_1 \leq \sum_{k=5}^{\infty} 2^{-k\alpha} \leq 1.$$

Noting that $\alpha - \frac{n-1}{2} > 0$, by (4) we have

$$I_2 \leq \sum_{k=0}^{\infty} 2^{-k(\alpha - \frac{n}{2} - 1/2)} \int_{10/2^k < |\xi| \leq 1/2} 2^{-Nk} |1 - |\xi||^{-N} d\xi$$

$$+ \left(O \sum_{k=0}^{\infty} \int_{10/2^k < |\xi| \leq 1/2} |\xi|^{n-m} |\xi|^{\frac{-n-1}{2}} 2^{-k(\alpha - \frac{n}{2} + 1/2)} d\xi \right) \leq 1.$$

Similarly, using (4) we have

$$I_3 \leq 1.$$

Let

$$E = \{\xi : 1/2 < |\xi| \leq 100\},$$

$$E_k = \{\xi : |1 - |\xi|| < 2^{-k}\},$$

We write

$$I_4 \leq \sum_{k=0}^{\infty} 2^{-k(\alpha-\frac{n}{2}-1/2)-Nk} \int_{E\setminus E_k} (1-|\xi|)^{-N} d\xi + \sum_{k=0}^{\infty} 2^{-k(\alpha-\frac{n}{2}-1/2)} \int_{E_k} d\xi \leq \sum_{k=0}^{\infty} 2^{-k(\alpha-\frac{n}{2}+1/2)} \leq 1.$$

This proves the proposition.

Section (3.2): Theorems by complex interpolation

Now we can complete the proof by a standard complex interpolation. In fact, define ananalytic family of operators $T_z$by

$$T_z(f)(x) = K_z * f(x) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{e^{i\|\lambda+\delta\|}}{\|\lambda+\delta\|^z} d_\lambda \chi_\lambda * f(x), \qquad z \in \mathbb{C}.$$

By the Plancherel formula, if$Rez = 0$, we have

$$\|T_z(f)\|_{L^2(G)} \leq \|f\|_{L^2(G)}.$$

It is easy to see from the proof of Proposition (3.1.6)that if $Rez > \frac{n-1}{2}$, then

$$\|T_z(f)\|_{L^1(G)} \leq \|K_z\|_{L^1(G)} \|f\|_{L^1(G)} \leq \|f\|_{L^1(G)}.$$

Thus, for $1 \leq p \leq 2$,easily follows by a complex interpolation onthese two inequalities. For $p > 2$. the theorem follows by a trivial dual argument.

We write

$$T_{\Omega,\alpha,\beta}(f)(x) = J_\beta * I_\alpha f(x) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{e^{i\|\lambda+\delta\|^\beta} \Omega\left(\frac{\lambda+\delta}{\|\lambda+\delta\|}\right)}{\|\lambda+\delta\|^\alpha} d_\lambda \chi_\lambda * f(x),$$

where

$$J_\beta * f(x) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} e^{i\|\lambda+\delta\|^\beta} \Omega\left(\frac{\lambda+\delta}{\|\lambda+\delta\|}\right) d_\lambda \chi_\lambda * f\,(x),$$

and

$$I_\alpha f(x) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{1}{\|\lambda+\delta\|^\alpha} d_\lambda \chi_\lambda * f(x).$$

For any $1 < p \leq 2$, Hölder's inequality and the Plancherel theorem imply that

$$\left\|T_{\Omega,\alpha,\beta}(f)\right\|_{L^p} \leq \left\|J_\beta * I_\alpha f\right\|_{L^2} \simeq \|I_\alpha f\|_{L^2}.$$

Now, by the Sobolev imbedding theorem we have

$$\left\|T_{\Omega,\alpha,\beta}(f)\right\|_{L^{p_0}} \leq \|I_\alpha f\|_{L^2} \leq \|f\|_{L^{p_0}}, \text{with } 1/p_0 - 1/2 = \alpha/n.$$

Interpolating this inequality with the obvious L^2 boundedness

$$\left\|T_{\Omega,\alpha,\beta}(f)\right\|_{L^2} \leq \|f\|_{L^2},$$

we obtain for $2 \geq p > 1$. By a dual argument, we provefor all $p > 1$. To prove the theorem for any $p < 1$ with$\frac{1}{p} - \frac{1}{2} \leq \frac{\alpha}{n}$, it suffices to show that forany p-atom a,

$$\left\|T_{\Omega,\alpha,\beta}(a)\right\|_{L^p} \leq 1.$$

Using the same argument as in $p > 1$, by the Sobolev imbedding theorem we have

$$\left\|T_{\Omega,\alpha,\beta}(a)\right\|_{L^p} \leq \|I_\alpha(a)\|_{L^2} \leq \|a\|_{L^p} \leq 1.$$

This proves.

Recall that the kernel has an expression of the form

$$K_{\Omega,\alpha,\beta}(y) = \frac{\sum_{k=0}^{\infty}\sum_{\nu\in\mathbb{N}} G_k(\xi+\nu)}{D(\xi)a\prod_{a\in A}\langle a,\delta\rangle},$$

where

$$D(\xi) = \prod_{a\in A} \sin\left(\frac{\langle\alpha,\xi\rangle}{2}\right)$$

First, we need an estimate on $\frac{1}{D(\xi)}$.

ma (3.2.1)[3]:

Let x be conjugate to $\exp\theta \in T$ and $x\,y^{-1}$ be conjugate to $\exp\zeta \in T$. There is a small $r_0 > 0$, and a large $M > 0$ such that for $r < r_0$, if$d(y, I) < r$ and$|\sin\frac{\langle\alpha,\theta\rangle}{2}| \geq Mr$ for all $\alpha \in A$, then

$$|D(\zeta)^{-1} - D(\theta)^{-1} \leq \left(\frac{\sum_{j=1}^{\frac{n-m}{2}} \left|D_{\alpha_j}(\theta)\right|}{|D(\theta)|^2}\right) d(y, I),$$

where

$$D_{\alpha_j}(\theta) = \prod_{a\in A\setminus\{\alpha_j\}} \sin\left(\frac{\langle\alpha,\theta\rangle}{2}\right)$$

Proof:

Let $\tilde{I}$be the line segment between $\zeta$and θ. By themean value theorem and the productrule,

$$|D(\zeta)^{-1} - D(\theta)^{-1}| \leq \sup_{\sigma\in\tilde{I}} \left(\frac{\sum_{j=1}^{\frac{n-m}{2}} \left|\prod_{a\in A\setminus\{\alpha_j\}} \sin\left(\frac{\langle\alpha,\sigma\rangle}{2}\right)\right|}{\prod_{a\in A} \sin^2\left(\frac{\langle\alpha,\sigma\rangle}{2}\right)}\right) |\zeta - \theta|.$$

By continuity, if$d(y, I) \to 0$, we have

$$\left(\frac{\sum_{j=1}^{\frac{n-m}{2}} \left|\prod_{a\in A\setminus\{\alpha_j\}} \sin\left(\frac{\langle\alpha,\sigma\rangle}{2}\right)\right|}{\prod_{a\in A} \sin^2\left(\frac{\langle\alpha,\sigma\rangle}{2}\right)}\right) \to \left(\frac{\sum_{j=1}^{\frac{n-m}{2}} \left|\prod_{a\in A\setminus\{\alpha_j\}} \sin\left(\frac{\langle\alpha,\theta\rangle}{2}\right)\right|}{\prod_{a\in A} \sin^2\left(\frac{\langle\alpha,\theta\rangle}{2}\right)}\right).$$

Thus, we may choose an$r_0 > 0$, such that if$d(y, I) < r_0$, then

$$\sup_{\sigma\in\tilde{I}} \left(\frac{\sum_{j=1}^{\frac{n-m}{2}} \left|\prod_{a\in A\setminus\{\alpha_j\}} \sin\left(\frac{\langle\alpha,\sigma\rangle}{2}\right)\right|}{\prod_{a\in A} \sin^2\left(\frac{\langle\alpha,\sigma\rangle}{2}\right)}\right) \leq \left(\frac{100\sum_{j=1}^{\frac{n-m}{2}} \left|\prod_{a\in A\setminus\{\alpha_j\}} \sin\left(\frac{\langle\alpha,\theta\rangle}{2}\right)\right|}{\prod_{a\in A} \sin^2\left(\frac{\langle\alpha,\theta\rangle}{2}\right)}\right).$$

By compactness, this r_0 can be chosen to be independent of $x \in G$. Since

$$|\zeta - \theta| \leq d(xy^{-1}, x) = d(y, I)$$

when $d(y, I)$is small, we have

$$|D(\zeta)^{-1} - D(\theta)^{-1}| \leq \left(\frac{\sum_{j=1}^{\frac{n-m}{2}} \left| \prod_{a \in A \setminus \{\alpha_j\}} \sin\left(\frac{\langle \alpha, \theta \rangle}{2}\right) \right|}{\prod_{a \in A} \sin^2\left(\frac{\langle \alpha, \theta \rangle}{2}\right)} \right) d(y, I).$$

Now we are in a position to complete the proof. It is known that $T_{\Omega,\alpha,\beta}$is bounded on L^2, and exceptional atoms are uniformly bounded by 1. By discussions in, it suffices to show that for any regular$p$-atom a, one has

$$\left\| T_{\Omega,\alpha,\beta}(f) \right\|_{L^p}^p \leq 1.$$

The operato $T_{\Omega,\alpha,\beta}$is translation invariant. Without loss of generality, we may assume that $a(x)$is supported in a neighborhood $B(I, r)$of the identity I, where $r$is sufficiently small.We may also assume $r < r0$, where $r0$ is the number chosen in Lemma (3.2.3). Let$\varsigma = 1 - \beta$.Then we have

$$\left\| T_{\Omega,\alpha,\beta}(a) \right\|_{L^p}^p \leq \int_{d(x,I) \leq 100 r^{\varsigma}} \left| T_{\Omega,\alpha,\beta}(a)(x) \right|^p dx + \int_{d(x,I) > 100 r^{\varsigma}} \left| T_{\Omega,\alpha,\beta}(a)(x) \right|^p dx = I_1 + I_2.$$

By Hölder's inequality and the Sobolev imbedding theorem, we have

$$I_1 \leq \left\| T_{\Omega,\alpha,\beta}(a) \right\|_{L^2(G)}^p r^{\frac{2-p}{2}} \leq \|a\|_{L^2_{-a}(G)}^p r^{\frac{2-p}{2}} \leq \|a\|_{L^q(G)}^p r^{\left(1-\frac{p}{2}\right) n\varsigma},$$

where $1/2 = 1/q - \alpha/n$. Note that$\frac{1}{p} - \frac{1}{2} \leq \frac{\alpha}{n\beta}and\varsigma = 1 - \beta$. An easy computation shows

$$I_1 \leq r^{p\alpha - n}(1 - p/2) r^{\frac{2-p}{2}\varsigma n} \leq 1.$$

We now estimate I_2. As before, let xy^{-1} be conjugate to $\exp \zeta$, $x$conjugate to $\exp \theta$and$y$conjugate to $\exp \xi$, where $\exp \zeta$, $\exp \theta$and $\exp \xi$are elements in T. For $a \in A$, wedefine sets

$$B = \{x : d(x, I) > 100 r^{\varsigma}\},$$

$$E_a = E_{a,M} = \left\{ x \sim \exp \theta : \left| \sin\left(\frac{\langle a, \theta \rangle}{2}\right) \right| < Mr \right\},$$

and

$$E = \cap \{B - E_{a,M}, a \in A\}.$$

By the discussion inand continuity, we have

$$\int_{E_a\cap B} \left|T_{\Omega,\alpha,\beta}(a)(x)\right|^p \leq \left(\int_{E_a\cap B} \left|T_{\Omega,\alpha,\beta}(a)(x)\right|dx\right)^p r^{1-p}$$

$$\leq \|a\|^p_{L^1(G)} r^{1-p}\left(\int_{\{|\theta|10r^{\varsigma}\}\cap E_a,100M} \left|K_{\Omega,\alpha,\beta}(\exp\theta)\right| |D(\theta)|^2 d\theta\right)^p.$$

By Proposition (3.1.3), for each $a \in A$ we have

$$\int_{E_a} |T_{\Omega,\alpha,\beta}(a)(x)|^p dx \leq r^{-n+np+1-p}\left(\int_{\{|\theta|>10r^{\varsigma}\}\cap E_a,100M} |\theta|^{\frac{\alpha-\frac{n}{2}}{1-\beta}-\frac{m}{2}}|D(\theta)|d\theta\right)^p$$

$$\leq r^{-n+np+1-p}\int_{\{|\theta|>10r^{\varsigma}\}\cap E_a,100M} |\theta|^{\frac{\alpha-\frac{n}{2}}{1-\beta}-\frac{m}{2}+\frac{n-m}{2}-1}\left|\sin\frac{\langle a,\theta\rangle}{2}\right|(d\theta)^p$$

$$\leq r^{-n+np+1} r^{\zeta\, p\left(\frac{\alpha-\frac{n}{2}}{1-\beta}+\frac{n}{2}-1\right)} \leq 1.$$

Finally, we estimate

$$\int_E |T_{\Omega,\alpha,\beta}(a)(x)|^p\, dx.$$

As discussed in, we may write

$$\int_E |T_{\Omega,\alpha,\beta}(a)(x)|^p\, dx \leq \sum_{j=1}^{3}\int_{\{|\theta|>10r^{\varsigma}\}\cap E}\left|\int_{B(I,r)}\sum_{k=0}^{\infty}\Gamma_{j,k}(\zeta)G_k(\zeta)a(y)D(\zeta)^{-1}dy|\quad {}^p|D(\theta)\right|^2 d\theta$$

$$= \sum_{j=1}^{3} £_j\,,$$

where

$$G_k(\zeta) = \int_{\mathbb{R}^m} e^{\frac{i|y|^\beta}{|y|^\alpha}} \Phi_k(|y|)\Omega\left(\frac{y}{|y|}\right)\left(\prod_{a\in A}\langle y, a\rangle\right) e^{-i\langle\zeta,y\rangle}\, dy.$$

The estimates for $£_j, j = 1,2,3,$ can be achieved using Proposition (3.1.4). The crucial term is $£_3$,and the estimates of the other two terms are similar or easier.. Thus we only estimate $£_3$. By the cancellationcondition of the atom, we have

$$\int_{\{|\theta|>10r^\varsigma\}\cap E} \left| \int_{B(I,r)} \sum_{k=0}^{\infty} \Gamma_{3,k}(\zeta) G_k(\zeta) a(y) D(\zeta)^{-1} dy \right|^p |D(\theta)|^2 d\theta$$

$$\leq \int_{\{|\theta|>10r^\varsigma\}\cap E} \left| \int_{B(I,r)} \frac{\sum_{k=0}^{\infty}\Gamma_{3,k}(\zeta)G_k(\zeta) - \sum_{k=0}^{\infty}\Gamma_{3,k}(\theta)G_k(\theta)}{D(\zeta)} a(y)dy| \quad {}^p|D(\theta)\right|^2 d\theta$$

$$+ \sum_{k=0}^{\infty} \int_{\{|\theta|>10r^\varsigma\}\cap E} \left\{ \sum_{k=0}^{\infty} \left| |\Gamma_{3,k}(\theta)G_k(\theta)| \int_{B(I,r)} a(y)\left\{\frac{1}{D(\zeta)} - \frac{1}{D(\theta)}\right\} dy \right| \right\}^p |D(\theta)|^2 d\theta$$

$$= \mathfrak{I}_1 + \mathfrak{I}_2$$

By Lemma (3.2.3)and Proposition (3.1.3), the second term $\mathfrak{I}_2$ is dominated by

$$\int_{\{|\theta|>10r^\varsigma\}\cap E} |\theta|^{-p\left(\frac{\alpha-\frac{n}{2}}{\beta-1}\right)-\frac{m}{2}p} \left| \int_{B(I,r)} a(y)\left\{\frac{1}{D(\zeta)} - \frac{1}{D(\theta)}\right\} dy \right|^2 p|D(\theta)|^2 d\theta$$

$$\leq r^{-n+pn+p} \int_{\{|\theta|>10r^\varsigma\}\cap E} |\theta|^{-p\left(\frac{\alpha-\frac{n}{2}}{\beta-1}\right)-\frac{n}{2}p-m-p+n} d\theta$$

$$\leq r^{-n+pn+p} r^{p\left(\alpha-\frac{n}{2}\right)+(1-\beta)\left(-\frac{n}{2}p-p+n\right)} \leq r^{p\alpha+\frac{\beta np}{2}+\beta p-n\beta} \leq r^{\frac{\beta n(2\beta+\alpha)}{\beta n+\alpha}} \leq 1.$$

Using the mean value theorem and Proposition (3.1.4), and since r is small, we have

$$\mathfrak{I}_1 = \int_{\{|\theta|>10r^\varsigma\}\cap E} \left| \sum_{k=0}^{\infty} \int_{B(I,r)} \{\Gamma_{j,k}(\zeta)G_k(\zeta) - \Gamma_{j,k}(\theta)G_k(\theta)\} a(y) D(\zeta)^{-1} dy \right|^p |D(\theta)|^2 d\theta$$

$$\leq r^{-n+pn+p} \int_{|\theta|>10r^{\varsigma}} |\theta|^{-p\left(\frac{\alpha-\frac{n}{2}}{\beta-1}\right)-p\frac{m}{2}} |\theta|^{\frac{p}{\beta-1}} |D(\theta)|^{2-p} 2d\theta \leq r^{-n+pn+p} r^{p\left(a-\frac{n}{2}\right)+(1-\beta)\left(-\frac{n}{2}p-\frac{p}{1-\beta}+n\right)}$$
$$\leq r^{p\alpha-\beta n+\frac{np\beta}{2}}.$$

The last one is bounded by 1 if and only if$\frac{1}{p}-\frac{1}{2}\leq\frac{\alpha}{n\beta}$. This completes the proof.

The sufficient part of (1) can be proved by following the method of complexinterpolation from the proof of , but using the inequality at the end. We turn now to the proof of necessity by considering $T_{\alpha,\beta}(\Omega\equiv 1)$.

Fix an orthonormal basis $X_1, X_2, \ldots, X_n$of the Lie algebra $\mathbf{g}$ such that $X_1, X_2, \ldots, X_m$form an orthonormal basis of $\boldsymbol{\tau}$. Let $r>0$ be a fixed small positive number and$\widetilde{\Phi}$bethe exponential mapping from $g$to G.$\widetilde{\Phi}$ is a C^{∞}-diffeomorphism from the neighborhood

$V_I(r)=\{y\in G: d(y,I)<r\}$to a neighborhood$\widetilde{N}$ of the origin in $\mathbb{R}^n$, which satisfies$d\ (u,v)\approx|\Phi(u)-\Phi(v)|$, for all $u,v\in V_I(r)$, where$\left|\widetilde{\Phi}(u)-\widetilde{\Phi}(v)\right|$ is the Euclideandistance between$\widetilde{\Phi}(u)$ and $\widetilde{\Phi}(v)$.Recall that $d(u,v)\approx\left|\widetilde{\Phi}(u)-\widetilde{\Phi}(v)\right|$ means that thereare positive constants c_1 and c_2 such that

$$c_1\left|\widetilde{\Phi}(u)-\widetilde{\Phi}(u)\right|\leq d(u,v)\leq c_2\left|\widetilde{\Phi}(u)-\widetilde{\Phi}(u)\right|$$

for all $u,v\in V_I(r)$. Without loss of generality, we may assume $c_1=1/2$ and $c_2=2$.Let $U_r=\{\xi\in\boldsymbol{\tau}:\exp\xi\in V_I(r)\}$. $U_r$is a neighborhood of 0 in $\boldsymbol{\tau}$, and we can choose $r$sufficiently small such that all translations of $U_r$by elements of $\mathbb{N}$ are disjoint. Let $\eta(\xi)$be aC^{∞}function supported in $U_{r/10}$ and identically one on $U_{r/20}$.We also choose $0\leq\eta(\xi)\leq 1$.Let f_{ε}be a central function defined on $V_I(r)$by $f_{\varepsilon}(\exp\xi)=\sum_{n\in\mathbb{N}}\eta(\frac{\xi+n}{(0.001\varepsilon)^{\gamma+1}})$, where$\gamma=\frac{\beta}{1-\beta}$. It is easy to check that for all εin the interval $(0,r)$, and for any $1\leq p\leq\infty$,we have

$$\int_G |f_{\varepsilon}(x)|^p dx\approx\int_Q\left|\sum_{n\in\mathbb{N}}\eta\left(\frac{\xi+n}{(0.001\varepsilon)^{\gamma+1}}\right)\right|^p|D(\xi)|^2 d\xi=\int_Q\left|\eta\left(\frac{\xi}{(0.001\varepsilon)^{\gamma+1}}\right)\right|^p|D(\xi)|^2 d\xi$$
$$\simeq\varepsilon^{n(\gamma+1)}.$$

On the other hand, let $\exp\xi\in T$be the conjugate element of$y$. Using the same argument we have

$$K_{\alpha,\beta} * f(x) = H_\infty * f(x) + H_0 * f(x),$$

where

$$H_0(y) = \prod_{a\in A}\langle\alpha,\xi\rangle \frac{|\xi|^{\frac{2-n}{2}}\int_0^\infty e^{it^\beta}\Psi(t)J_{\frac{n-2}{2}}(t|\xi|)t^{\frac{n}{2}-\alpha}dt}{D(\xi)\prod_{a\in A}\langle\,\alpha,\delta\rangle},$$

$$H_\infty(y) = \frac{\sum_{\nu\in\mathbb{N}\setminus\{0\}}\prod_{a\in A}\langle\alpha,\xi+\nu\rangle)Y(\xi+\nu)}{D(\xi)\prod_{a\in A}\langle\alpha,\delta\rangle}$$

and

$$Y(\xi) = |\xi|^{\frac{2-n}{2}}\int_0^\infty e^{it^\beta}\Psi(t)J_{\frac{n-2}{2}}(t|\xi|)t^{\frac{n}{2}-\alpha}dt\,.$$

As in the proof 0f Proposition (3.1.6), using the asymptotic expansion of $\Psi(t)J_{\frac{n-2}{2}}(t|\xi|)$andintegration by parts, we have

$$\|H_\infty\|_{L^1(G)} \le 1.$$

Thus, for any $1 \le p \le \infty$,

$$\|H_\infty * f\|_{L^p(G)} \le \|f\|_{L^p(G)}.$$

By an estimate we have

$$|\xi|^{\frac{2-n}{2}}\int_0^\infty e^{it^\beta}(t)\Psi(t)J_{\frac{n-2}{2}}(t|\xi|)t^{\frac{n}{2}-\alpha}dt$$

$$= |\xi|^{-(n-\alpha-\frac{\beta n}{2})/(1-\beta)}e^{ic|\xi|-\beta/(1-\beta)}\sum_{m=0}^{M}a_m|\xi|^{m\beta/(1-\beta)} + O(|\xi|^{(M+1)\beta/(1-\beta)})$$

where $a_0 \neq 0$ and M is any fixed positive integer. Noting that

$$D(\xi) = \prod_{a\in A}\sin\left(\frac{\langle\alpha,\xi\rangle}{2}\right),$$

when $d(y, I)$is small, we have

$$H_0(y) \simeq h_1(y) + h_2(y),$$

with

$$h_2(y) = O\left(|\xi|^{-(n-\alpha-\frac{\beta n}{2}-\beta)/(1-\beta)}\right),$$

$$h_1(y) = |\xi|^{-(n-\alpha-\frac{\beta n}{2})/(1-\beta)} e^{ic|\xi|^{-\beta/(1-\beta)}}.$$

Let $\{J_k\}$be a sequence of sets defined by

$$J_k = \left\{\theta \in \Gamma_m : (2\pi k + \frac{\pi}{4})^{-\frac{1}{\gamma}} \leq |\theta| \leq \left(2\pi k - \frac{\pi}{4}\right)^{-\frac{1}{\gamma}}\right\}.$$

Without loss of generality, we may assume

$$\operatorname{diam}(J_k) = 10k^{-1-\frac{1}{\gamma}}.$$

Fix the integer k_0 for which

$$k_0^{-\frac{1}{\gamma}} \leq r < (k_0 - 1)^{-\frac{1}{\gamma}}.$$

Now, for all $k_0 \leq k \leq \varepsilon^{-\gamma}$, we define another sequence of sets $I_k$by

$$I_k = \left\{\theta \in \Gamma_m : \left(2\pi k + \frac{\pi}{4}\right)^{-\frac{1}{\gamma}} + 2\varepsilon^{\gamma+1} \leq |\theta| \leq \left(2\pi k - \frac{\pi}{4}\right)^{-\frac{1}{\gamma}} - 2\varepsilon^{\gamma+1}\right\}.$$

Using the above two sequences of sets, we may further construct two sequences of sets on G by

$$A_k = \{x \in G : x \text{ is conjugate to } \exp\theta \in T \text{ with} \theta \in I_k\},$$

$$B_k = \{x \in G : x \text{ is conjugate to } \exp\theta \in T \text{ with} \theta \in J_k\}.$$

All sets $A_k$are mutually disjoint and

$$Vol(A_k) \simeq k^{-1-\frac{n}{y}}.$$

Thus,

$$\left\|T_{\alpha,\beta}(f_\varepsilon)\right\|_{L^p} \geq -\|f_\varepsilon\|_{L^p}$$

$$+\left\{\sum_{k=k_0}^{\epsilon^{-\gamma}} \int_{A_k} \left(\left|Re \int_G h_1(y) f_\varepsilon(xy^{-1})dy\right| - \left|\int_G \left| \quad h_2(y)\, f_\varepsilon(xy^{-1})dy\right|\right)^p dx\right\}^{1/p}.$$

When $x \in A_k, k \geq k_0$, x is conjugate to the element $\exp\theta \in T$with $\theta \in I_k$. Let $\exp\zeta$ and$\exp\xi$ be elements in Tthat are conjugate to xy^{-1} and y, respectively. If

$f_\varepsilon(xy^{-1}) = 0$,then by the choice of $f d(x,y) \simeq |\zeta|$is less than εr. Thus $y$is in$U_{r/5}$, and we have

$$|\xi - \theta| \leq d(x,y) \leq |\zeta|.$$

Noting also that

$$|\theta| - |\xi - \theta| \leq |\xi| \leq |\theta| + |\xi - \theta|,$$

we see that if $x \in A_k$and $f_\varepsilon(xy-1) = 0$, then $\xi \in J_k$. Thus, for each $x \in I_k$,

$$\left|Re \int_G h_1(y) f_\varepsilon(xy^{-1})dy\right| dy - \left|\int_G h_2(y) f_\varepsilon(xy^{-1})dy\right|$$

$$\geq |(2\pi k)^{(n-\alpha-\frac{\beta n}{2})/\beta} - O(k^{(n-\alpha-\frac{\beta n}{2}-\beta)/\beta})| \int_G f_\varepsilon(xy^{-1})dy$$

$$\geq k^{(n-\alpha-\frac{\beta n}{2})/\beta} \int_G f_\varepsilon(xy^{-1})dy\,.$$

Therefore, for any $1 < p \leq 2$, we have

$$\left(\int_G \left|\int_G K_{\alpha,\beta}(y) f_\varepsilon(xy^{-1})dy\right|^p dx\right)^{1/p}$$

$$\geq \left\{\sum_{k=k_0}^{\varepsilon^{-\gamma}} k^{p(n-\alpha-\frac{\beta n}{2})/\beta - \frac{n}{\gamma} - 1} \left|\int_G f_\varepsilon(xy^{-1})dy\right|^p\right\}^{1/p} - \|f_\varepsilon\|_{L^p}$$

$$\simeq \varepsilon^{\frac{1}{(1-\beta)}(\alpha+\frac{\beta n}{2})+n/p} - \|f_\varepsilon\|_{L^p}.$$

Thus, if the operator is bounded on L^p for $1 < p \leq 2$, namely,

$$\|f_\varepsilon\|_{L^p} \geq \left\|T_{\alpha,\beta}(f_\varepsilon)\right\|_{L^p},$$

then for any small $\varepsilon > 0$ wemusthave

$$\varepsilon^{n(\gamma+1)} \geq \varepsilon^{\frac{p}{(1-\beta)}(\alpha+\frac{\beta n}{2})+n}.$$

This implies

$$1/p - 1/2 \leq \frac{\alpha}{n\beta}.$$

By duality, we conclude that $T_{\alpha,\beta}$is bounded on L^p for $1 < p < \infty$only if

$$|1/p - 1/2| \leq \frac{\alpha}{n\beta}.$$

This proves necessity in (1).

The proof of sufficiency is exactly the same as the proofby Proposition (3.1.5),

$$\left\|T_{\alpha,\beta}(f_\varepsilon)\right\|_{L^1} \leq \sum_{k=0}^{\infty} \|H_k * f\|_{L^1} \leq \sum_{k=0}^{\infty} \|H_k\|_{L^1} \|L^1\| \|f\|_{L^1} \leq \|f\|_{L^1}, \qquad \text{if} \quad \alpha > \frac{n\beta}{2}.$$

Let $\Omega \equiv 1$. By the proof of necessity, it is easy to see that $T_{\Omega,\alpha,\beta}$ is notbounded on $L^1(G)$if $< \frac{n\beta}{2}$. For $\alpha = \frac{n\beta}{2}$, the inequality

$$\|f_\varepsilon\|_{L^p} \geq \left(\int_G \left| \int_G K_{\alpha,\beta}\,(y) f_\varepsilon(xy^{-1}) dy \right|^p dx \right)^{1/p}$$

is

$$\|f_\varepsilon\|_{L^1} \geq \left\{ \sum_{k=k_0}^{\varepsilon^{-\gamma}} k^{(n-n\beta)/\beta - \frac{n}{\gamma} - 1} \left| \int_G f_\varepsilon(xy^{-1}) dy \right| \right\} - \|f_\varepsilon\|_{L^1} \simeq \|f_\varepsilon\|_{L^1} \log \epsilon - \|f_\varepsilon\|_{L^1}$$

which is clearly not possible.

Chapter 4

Operators on Compact Lie group and Global Functional Calculus

In chapter we show a constructive symbolic proof of *Ga°rding* inequalityfor operators in (ρ,δ)-classes in the setting of compact Lie groups**.**

Section (4.1): Matrix symbolic Calculus and Ellipticity

Let $\mathcal{G}$ be a compact Lie group and let $\hat{\mathcal{G}}$denote the set of equivalenceclasses of irreducibleunitary representations $\xi : \mathcal{G} \to \mathrm{U}(d_\xi)$ of $\mathcal{G}$. It hasbeen shown in, thatthe Hörmander class $\Psi^m_{1,0}(\mathcal{G})$ of pseudo-differential operators on $\mathcal{G}$is characterised in termsof its global symbols

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x)$$

by symbolic estimates

$$\left\| \partial_x^\alpha \mathbb{D}_\xi^\beta \sigma_A(x,\xi) \right\|_{op} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \qquad (1)$$

in terms of (usual) derivatives ∂_xacting on points of $\mathcal{G}$ and difference operators $\mathbb{D}_\xi$actingon the representation lattice. This symbolic estimate resembles symbolic properties oflocal Hörmander symbols defined in terms of local coordinates, however, it applies to aglobal object definedon $\mathcal{G} \times \widehat{\mathcal{G}}$. The set of all symbols satisfying (1) will be denoted as$S^m_{1,0}(\mathcal{G} \times \hat{\mathcal{G}})$, we postpone a formal definition to the end of. First we will recallpreliminaries concerning Fourier series, globaloperator quantization and the associatedglobal calculus.

The Fourier transform on $\mathcal{G}$ isdefined in terms of the Peter–Weyldecomposition

$$L^2(\mathcal{G}) = \bigoplus_{|\xi| \in \hat{\mathcal{G}}} \mathcal{H}_\xi, \;\; \mathcal{H}_\xi = \left\{ x \mapsto \mathrm{trace}(\xi(x)E) : E \in \mathbb{C}^{d_\xi \times d_\xi} \right\}$$

of $L^2(G)$ into spaces of matrix coefficients. Orthogonal projections onto the subspaces $\mathcal{H}_\xi$are given in terms of convolutions with the characters$\chi_\xi(x) = \mathrm{trace}\ \xi(x)$ multiplied by thedimension d_ξ. We identify thespaces $\mathcal{H}_\xi$with $\mathbb{C}^{d_\xi \times d_\xi}$in terms of the matrices E in theabove formula and write in short

$$\widehat{f}(\xi) = \int_{\mathcal{G}} f(x)\xi(x)^* \, dx$$

for the Fourier coefficient of a function $f \in L^2(\mathcal{G})$ at the repesentation ξ. The abovedecomposition implies that

$$f(x) = \sum_{|\xi| \in \hat{\mathcal{G}}} d_\xi \operatorname{trace}\left(\xi(x)\widehat{f}(\xi)\right)$$

as orthogonal direct sum and, in particular, it follows that the Parseval identity

$$\|f\|^2_{L^2(G)} = \sum_{|\xi| \in \hat{\mathcal{G}}} d_\xi \left\|\widehat{f}(\xi)\right\|^2_{\mathrm{HS}} \quad (2)$$

holds true. Note, that the matrix norm involved in this expression is the Hilbert–Schmidtnorm induced by the trace inner product,$\left\|\widehat{f}(\xi)\right\|^2_{\mathrm{HS}} = \operatorname{trace}\left(\widehat{f}(\xi)\widehat{f}(\xi)^*\right)$.

Function spaces on $\mathcal{G}$can be characterised via Fourier transform in terms of sequencespaces on $\hat{\mathcal{G}}$. To simplify the notation, let

$$\sum(\hat{\mathcal{G}}) = \left\{\sigma : \hat{\mathcal{G}} \ni \xi \mapsto \sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}\right\}$$

denote the set of all sequences of matrices of appropriate dimensions. Then we define

$$\ell^p(\widehat{\mathcal{G}}) = \left\{\sigma \in \sum(\hat{\mathcal{G}}) : \sum_{|\xi| \in \hat{\mathcal{G}}} d_\xi \|\sigma(\xi)\|^p_{S_p} < \infty\right\}$$

in terms of Schatten p-norms of matrices. The Parseval identity implies that the Fouriertransform F is unitary between $L^2(G)$ and $\ell^2(\hat{\mathcal{G}})$ and bystraightforward estimates $\mathcal{F}$maps$L^1(G)$ to $\ell^2(\hat{\mathcal{G}})$ with inverse mapping$\ell^1(\hat{\mathcal{G}}) \to L^\infty(\mathcal{G})$. With this notation, $\ell^\infty(\hat{\mathcal{G}})$ is themultiplier algebra for $\ell^2(\hat{\mathcal{G}})$. This explains in particular the appearance of the operator-normin the symbol estimate (1) and the Hilbert–Schmidt norm in (2).

We note that there exists another version of ℓ^p-spaces on $\hat{\mathcal{G}}$based onfixing the Hilbert-Schmidt norm of $\sigma$and varying the powers of d_ξ. For a discussion on the relation between the spaces $\ell^p(\hat{\mathcal{G}})$ and Schatten classes ofoperators on $L^2(G)$.

As $\mathcal{H}_\xi$are minimal bi-invariant subspaces of $L^2(G)$, they are eigenspaces of all bi-invariantoperators and, in particular, of the (negative) Laplacian (Casimir element) $\mathcal{L}_\mathcal{G}$on $\mathcal{G}$. In the following we denote by $\langle\xi\rangle$the corresponding eigenvalues for the operator$\left(1-\mathcal{L}_\mathcal{G}\right)^{1/2}$, i.e., we have$f-\mathcal{L}_\mathcal{G}f\ =\langle\xi\rangle^2 f$for all $f\in\mathcal{H}_\xi$. Then the standard L^2-basedSobolevspaces, spaces of smooth functions and distributions are characterised as

$$f\in H^s(\mathcal{G})\quad\Leftrightarrow\quad\langle\xi\rangle^s\widehat{f}\,(\xi)\in\ell^2\left(\hat{\mathcal{G}}\right),$$

$$f\in C^\infty(\mathcal{G})\quad\Leftrightarrow\quad\forall N:\langle\xi\rangle^N\widehat{f}\,(\xi)\in\ell^2\left(\hat{\mathcal{G}}\right),$$

$$f\in\mathcal{D}'(\mathcal{G})\quad\Leftrightarrow\quad\exists N:\langle\xi\rangle^{-N}\widehat{f}\,(\xi)\in\ell^2\left(\hat{\mathcal{G}}\right).$$

In the last two lines we can replace $\ell^2\left(\hat{\mathcal{G}}\right)$ by any other $\ell^p\left(\hat{\mathcal{G}}\right)$. Wedenote the correspondingsequences as fast decaying (giving the spaces$\mathfrak{s}\left(\hat{\mathcal{G}}\right)$) and moderately growing (giving its duals$\mathfrak{s}'\left(\hat{\mathcal{G}}\right)$), respectively.

Difference operators acting on matrix-sequencesare defined in terms ofsmooth functions $q\in C^\infty(\mathcal{G})$ vanishing in the identity elementof the group. If $\sigma(\xi)$ is a moderately growing sequence of matrices, then$\mathcal{F}^{-1}[\sigma]\in\mathcal{D}'(\mathcal{G})$is a distribution and it makes sense to define thedifference operator $\Delta_q$acting on σ as

$$\Delta_\mathrm{q}\sigma:\mathcal{F}[\mathrm{q(x)}\mathcal{F}^{-1}\sigma].$$

Difference operators useful to us here are best defined in terms of matrixcoefficients ofrepresentations. For a fixed representation ξ we denote by $\Delta_{ij}$the difference operatorassociated to the function$(\xi(g)-\mathrm{I})_{ij}$. Thena simple calculation showsthat the finite Leibniz rule

$$\Delta_{ij}(\sigma\tau)=\left(\Delta_{ij}\sigma\right)\tau+\sigma\left(\Delta_{ij}\tau\right)+\sum_{k=1}^{d_\xi}\left(\Delta_{ik}\sigma\right)\left(\Delta_{kj}\tau\right)\qquad(3)$$

holds true. As inwe fix a finite selection of representations ${}_\kappa\xi$such that $({}_\kappa\xi(x)-\mathrm{I})_{ij}$vanishes to first order in the identity element $e\in\mathcal{G}$and that e is the only commonzero of all these,

$$\{e\}=\bigcap_k\{x:\quad{}_\kappa\xi(x)=\mathrm{I}\}.$$

We collect all these difference operators into a vector $\mathbb{D}$and use multi-index notation $\mathbb{D}_{\xi}^{\alpha}$for them. Similarly we write $q^{\alpha}(x)$ for the corresponding function so that $\mathbb{D}_{\xi}^{\alpha} = \Delta_{q^{\alpha}}$.Associated to the difference operators we find invariant differential operators $\partial_x^{(\alpha)}$suchthat theTaylor expansion

$$f(x) \sim \sum_{\alpha} \frac{\partial_x^{(\alpha)} f(e)}{\alpha!} q^{\alpha}(x^{-1})$$

holds for all smooth functions on $\mathcal{G}$ and in the vicinity of the identity element $e \in \mathcal{G}$. Thesedifferential operators are not powers of first order operators.

Combining all these ingredients we can characterise Hörmander classes $\Psi_{1,0}^{m}(\mathcal{G})$ of pseudo-diffferentialoperators, defined in local coordinates, in terms of global symbols. The resultsofimply that $A \in \Psi_{1,0}^{m}(\mathcal{G})$if and only if

$$Af(x) = \sum_{|\xi| \in \hat{\mathcal{G}}} d_{\xi} \operatorname{trace}\left(\xi(x)\sigma_A(x,\xi)\hat{f}(\xi)\right) \quad (4)$$

with a symbol $\sigma_A$satisfying (1). The constants in the symbolic estimates (1) definesemi-norms on $S_{1,0}^{m}(\mathcal{G} \times \hat{\mathcal{G}})$and the operator quantization

$$\mathrm{Op}: S_{1,0}^{m}(\mathcal{G} \times \hat{\mathcal{G}}) \ni \sigma_A \mapsto A \in \mathcal{L}\left(H^{s}(\mathcal{G}), H^{s-m}(\mathcal{G})\right) \quad (5)$$

is continuous. For later use we define slightly more general classes of symbols.

Definition(4.1.1) [4]:

Let $0 \leq \rho, \delta \leq 1$ and $m \in \mathbb{R}$. Then we denote by $\mathscr{S}_{\rho,\delta}^{m}(\mathcal{G} \times \hat{\mathcal{G}})$the setofall symbols $\sigma = \sigma(x,\xi) \in C^{\infty}(\mathcal{G}; \mathfrak{s}')(\hat{\mathcal{G}}) = C^{\infty}(\mathcal{G}) \hat{\otimes}_{\pi} \mathfrak{s}'(\hat{\mathcal{G}})$ satisfying the symbolic estimates

$$\left\| \partial_x^{\alpha} \mathbb{D}_{\xi}^{\beta} \sigma_A(x,\xi) \right\|_{op} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|} \quad (6)$$

for all multi-indices $\alpha$and β. The set of all operators associated to such symbolsvia formula (4) will be denoted as $\mathrm{Op}S_{\rho,\delta}^{m}(\mathcal{G} \times \hat{\mathcal{G}})$.

In fact, it can be shown that for $\rho > \delta$ the definition above is independent of thechoiceof difference operators as long as they form a so-called strongly admissiblecollection, see. If $\rho \leq \delta$, we consider the class $S_{\rho,\delta}^{m}(\mathcal{G} \times \hat{\mathcal{G}})$of symbols satisfying (6) for a fixedcollection of (strongly admissible) difference operators, as chosen above.

We will refer to operators associated to symbols from $S^m_{\rho,\delta}(\mathcal{G}\times\hat{\mathcal{G}})$as pseudo-differentialoperators of type (ρ,δ) in the sequel. If A and $B$are such operators oftype $(\rho,\delta), \rho > \delta$,with symbols $\sigma_A$and σ_B, then their composition $AB$is again of the same type with symbolgiven in terms of an asymptotic expansion

$$\sigma_{AB}(x,\xi) = \sigma_A(x,\xi)\#\sigma_B(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\left(\mathbb{D}^{\alpha}_{\xi}\sigma_A(x,\xi)\right)\left(\partial_x^{(\alpha)}\sigma_B(x,\xi)\right) \qquad (7)$$

modulo smoothing operators. Similarly, one obtains for the (formal) adjoint $A^*$of anoperator A

$$\sigma_{A^*}(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\partial_x^{(\alpha)}\mathbb{D}^{\alpha}_{\xi}(\sigma_A(x,\xi)^*). \qquad (8)$$

The class of left-invariant operatorsOp$S^m_{\rho,\delta}(\hat{\mathcal{G}})$corresponds to symbols from $S^m_{\rho,\delta}(\mathcal{G}\times\hat{\mathcal{G}})$, which are independent ofthe variable x. Similarly to the localcalculus of pseudo-differential operators, such left-invariantoperators can beobtained by freezing coefficients. Thus if

$A \in \mathrm{Op}\, S^m_{\rho,\delta}(\mathcal{G}\times\hat{\mathcal{G}})$with symbol $\sigma_A(x,\xi)$ and if $x_0 \in \mathcal{G}$is fixed, then A_{x_0} denotes the left-invariant operator defined by (4) in terms of $\sigma_A(x_0,\xi)$. The operator A_{x_0} can be described withoutresortingto symbols. Indeed, let $\phi_\epsilon \in C^\infty(\mathcal{G})$ be a family of smooth functions with shrinkingsupports in balls with radius of size $\epsilon > 0$ around the identity element$e \in \mathcal{G}$and with

$$\int_{\mathcal{G}} \phi_\epsilon(x)\,\mathrm{d}x = 1, \qquad \lim_{\epsilon\to 0}\int_{\mathcal{G}} f(x)\phi_\epsilon(x)\,\mathrm{d}x = f(e)$$

for any $f \in C(\mathcal{G})$. This implies that $\phi_\epsilon(x)$ gives an approximate convolutionidentity.

Proposition(4.1.2) [4]:

Let $0 \le \delta, \rho \le 1$. Let $\mathrm{Op}\, S^m_{\rho,\delta}(\mathcal{G}\times\hat{\mathcal{G}})$ and let $x_0 \in \mathcal{G}$be fixed. Thenthe associated left-invariant operator $A_{x_0} \in \mathrm{Op}S^m_{\rho,\delta}(\hat{\mathcal{G}})$is given by

$$A_{x_0}f(x) = \lim_{\epsilon\to 0}\int_{\mathcal{G}} \phi_\epsilon(x_0^{-1}x')[f(x(x')^{-1}\,\cdot)](x')\,\mathrm{d}x'.$$

Proof:

On the level of symbols the definition of $\phi_\epsilon$implies

$$\lim_{\epsilon\to 0}\int_{\mathcal{G}} \phi_\epsilon(x_0^{-1}x')\sigma_A(x',\xi)\,\mathrm{d}x' = \sigma_A(x_0,\xi) = \sigma_{A_{x_0}}(\xi).$$

and the statement follows by rewriting this in terms of associated operators (and denoting $x' = yx$)

$$A_{x_0}f(x) = \lim_{\epsilon\to 0}\sum_{|\xi|\in\hat{\mathcal{G}}} d\xi \int_{\mathcal{G}} \phi_\epsilon(x_0^{-1}yx)\,\mathrm{trace}\left(\xi(yx)\sigma_A(yx,\xi)\widehat{f}\,(\xi)\xi(y^{-1})\right)\mathrm{d}y$$

$$= \lim_{\epsilon\to 0}\int_{\mathcal{G}} \phi_\epsilon(x_0^{-1}yx)A[f(y^{-1}\,\cdot)](yx)\,\mathrm{d}y,$$

since

$$f\widehat{(y^{-1}\,\cdot)}(\xi) = \int_{\mathcal{G}} f(y^{-1}z)\xi(z)^*\,\mathrm{d}z = \int_{\mathcal{G}} f(w)\xi(yw)^*\,\mathrm{d}w$$

$$= \int_{\mathcal{G}} f(w)\xi(w)^*\,\mathrm{d}w\xi(y)^* = \widehat{f}\,(\xi)\xi(y^{-1}),$$

completing the proof.

Theorem (4.1.3) [4]:

Assume $\sigma_A \in S^m_{\rho,\delta}(\mathcal{G}\times\hat{\mathcal{G}})$ is parameter-elliptic with respect to Λ. Then

1. For any $0 \le \delta \le 1$ and $0 < \rho \le 1$, the resolvent of the symbol satisfies

$$\left\|\partial_x^k \mathbb{D}_x^\alpha \mathbb{D}_\xi^\beta (\sigma_A(x,\xi) - \lambda \mathrm{I})^{-1}\right\|_{op} \le C_{\kappa,\alpha,\beta}\left(|\lambda|^{1/m} + \langle\xi\rangle\right)^{-m(k+1)}\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \quad (9)$$

uniformly in $x$and $\lambda \in \Lambda$ and for co-finitely many $[\xi]$, and

2. For any $0 \le \delta < \rho \le 1$, there exists a parameter-dependent parametrix to $A - \lambda I$with symbol $\sigma_A^\sharp(x,\xi,\lambda)$ satisfying

$$\left\|\partial_x^k \mathbb{D}_x^\alpha \mathbb{D}_\xi^\beta \sigma_A^\#(x,\xi,\lambda)\right\|_{op} \leq C_{\kappa,\alpha,\beta}\left(|\lambda|^{1/m} + \langle\xi\rangle\right)^{-m(k+1)} \langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \tag{10}$$

uniformly in $x$and $\lambda \in \Lambda$ and for co-finitely many $[\xi]$.

Proof:

The key point is to prove the first statement, the second follows by the calculus.We split the proof of (9) into two parts and prove it first for symbols of left-invariantoperators. The third part will show (10).

Part (1):

Let $\sigma_A(\xi)$ be the symbol of a left-invariant operator of order $m > 0$ and assumewithout loss of generality that spec $\sigma_A(\xi) \cap \Lambda = \emptyset$. Then $\sigma_A^\#(\xi,\lambda) = (\sigma_A(\xi) - \lambda \mathrm{I})^{-1}$is justthe resolvent of the symbol and, since σ_A is parameter-elliptic withrespect to Λ, we havethat

$$\left\|\sigma_A^\#(\xi,\lambda)\right\|_{op} \leq C\left(|\lambda|^{1/m} + \langle\xi\rangle\right)^{-m} \tag{11}$$

holds true for all ξ and all $\lambda \in \Lambda$. Our aim is to prove the corresponding estimatesforhigher order differences and for derivatives with respect to λ.

Step (1.1):

We consider one of the first order differences $\Delta_{ij}$from. The finiteLeibniz rule (3) applied to $I = (\sigma_A - \lambda \mathrm{I})^{-1}(\sigma_A - \lambda \mathrm{I})$ implies that

$$= 0\left(\Delta_{ij}\, \sigma_A^\#(\xi,\lambda)\right)(\sigma_A(\xi) - \lambda \mathrm{I}) + \sigma_A^\#(\xi,\lambda)\left(\Delta_{ij}\, \sigma_A(\xi)\right) + \sum_k \left(\Delta_{ik}\, \sigma_A^\#(\xi,\lambda)\right)\left(\Delta_{kj}\, \sigma_A(\xi)\right)$$

(using that constants are annihilated by taking differences). We treat

$$\Delta_{ij}\, \sigma_A^\#(\xi,\lambda) + \sum_k \left(\Delta_{ik}\, \sigma_A^\#(\xi,\lambda)\right)\left(\Delta_{kj}\, \sigma_A(\xi)\right)\sigma_A^\#(\xi,\lambda) = -\sigma_A^\#(\xi,\lambda)\, \Delta_{ij}\, \sigma_A^\#(\xi,\lambda)$$

as a linear equation for $\Delta_{\mathrm{ij}}\, \sigma_{\mathrm{A}}^\#(\xi,\lambda)$. Indeed, regarding these equations as equationsforthe whole family $\left\{\Delta_{\mathrm{ij}}\, \sigma_{\mathrm{A}}^\#(\xi,\lambda)\right\}_{i,j}$at once allows us to treat the family of secondterms(i.e. the sums) on the left hand side as the perturbation of the family of thefirst terms.More precisely, by assumption (11), the 'big' coefficient block-matrixof the left-hand sideis a perturbation of the identity of size

$\langle\xi\rangle^{m-\rho}\left(|\lambda|^{1/m}+\langle\xi\rangle\right)^{-m} \lesssim \langle\xi\rangle^{-\rho}$and thus uniformlyinvertible for large ξ. Therefore, $\Delta_{ij}\,\sigma_A^{\#}(\xi,\lambda)$is of the size of the right-hand side, i.e., weobtain

$$\left\|\Delta_{ij}\,\sigma_A^{\#}(\xi,\lambda)\right\|_{op} \leq C\langle\xi\rangle^{m-\rho}\left(|\lambda|^{1/m}+\langle\xi\rangle\right)^{-2m} \leq C'\left(|\lambda|^{1/m}+\langle\xi\rangle\right)^{-m}\langle\xi\rangle^{-\rho}. \quad (12)$$

Step (1.2):

We proceed by induction over the order of differences and corresponding higherorder Leibniz rules. They yield linear equations for $\mathbb{D}_\xi^\beta\sigma_A^{\#}(\xi,\lambda)$in terms ofalready estimatedquantities $\mathbb{D}_\xi^\gamma\sigma_A^{\#}(\xi,\lambda), \gamma<\beta$, and known estimates for $\sigma_A(x,\xi)$. The coefficient matrixof this linear equation is the same, the right hand sidecontains only more terms. Termswith the worst estimates contain all differences applied to $\sigma_A$and no differences appliedto $\sigma_A^{\#}$. Hence, they can be estimated by$\left(|\lambda|^{1/m}+\langle\xi\rangle\right)^{-2m}\langle\xi\rangle^{m-\rho|\beta|}$and thus similarly toabove we obtain

$$\left\|\mathbb{D}_\xi^\beta\sigma_A^{\#}(\xi,\lambda)\right\|_{op} \leq C\left(|\lambda|^{1/m}+\langle\xi\rangle\right)^{-m}\langle\xi\rangle^{m-\rho|\beta|}.$$

Step (1.3):

Derivatives with respect to λ are treated by means of the resolvent identity

$$\partial_\lambda\sigma_A^{\#}(\xi,\lambda) = \partial_\lambda(\sigma_A(\xi)-\lambda\mathrm{I})^{-1} = (\sigma_A(\xi)-\lambda\mathrm{I})^{-2} = \sigma_A^{\#}(\xi,\lambda)^2.$$

This implies, after combination with the Leibniz rule (3),

$$\begin{aligned}\partial_\lambda\mathbb{D}_\xi^\beta\sigma_A^{\#}(\xi,\lambda) &= \mathbb{D}_\xi^\beta\partial_\lambda\sigma_A^{\#}(\xi,\lambda) = \left(\mathbb{D}_\xi^\beta\sigma_A^{\#}(\xi,\lambda)\sigma_A^{\#}(\xi,\lambda)\right)\\ &= \left(\mathbb{D}_\xi^\beta\sigma_A^{\#}(\xi,\lambda)\right)\sigma_A^{\#}(\xi,\lambda) + \sigma_A^{\#}(\xi,\lambda)\left(\mathbb{D}_\xi^\beta\sigma_A^{\#}(\xi,\lambda)\right) + \mathrm{l.o.t.}\\ &\lesssim \left(|\lambda|^{1/m}+\langle\xi\rangle\right)^{-2m}\langle\xi\rangle^{-\rho|\beta|},\end{aligned}$$

and the desired statement follows again by induction over the order of derivatives. Hence, we obtain (9) and (10) in the invariant case.

Part (2):

Now we consider the general case of a symbol $\sigma_A(x,\xi)$ depending on both variables.In order to show (9) we observe that above considerations are uniform in$x$so that thestatement for $|\alpha| = 0$ is already proven. Applying the Leibniz rule to $I = (\sigma_A - \lambda \mathrm{I})^{-1}(\sigma_A - \lambda \mathrm{I})$ yields for first order

x-derivatives,

$$\partial_j(\sigma_A(x,\xi) - \lambda \mathrm{I})^{-1} = -(\sigma_A(x,\xi) - \lambda \mathrm{I})^{-1}\big(\partial_j \sigma_A(x,\xi) - \lambda \mathrm{I}\big)^{-1},$$

and similarly also expressions for higher order derivatives of the resolvent in terms ofalready known terms. Applying difference operators to these combined with thefiniteLeibniz rule for differences yields (recursively) the estimates. This proves

$$\left\| \partial_x^\alpha \mathbb{D}_\xi^\beta (\sigma_A(x,\xi) - \lambda \mathrm{I})^{-1} \right\|_{\mathrm{op}} \leq C_\alpha \big(|\lambda|^{1/m} + \langle \xi \rangle\big)^{-m} \langle \xi \rangle^{-\rho|\beta| + \delta|\alpha|}$$

for all multi-indices $\alpha$and β. Together with the idea of Step (1.3) this proves (9).

Part (3):

In order to prove (10) we follow the usual parametrix construction in terms ofasymptotic series. We restrict the detailed consideration to left parametrices and constructa sequence $\sigma_j(x,\xi,\lambda)$ starting with

$$\sigma_0(x,\xi,\lambda) = (\sigma_A(x,\xi) - \lambda \mathrm{I})^{-1},$$

which satisfies the estimates (9), and define

$$\sigma_{j+1}(x,\xi,\lambda) = -\sum_{\substack{|\alpha|+k=j+1 \\ 0 \leq k \leq j}} \frac{1}{\alpha!} \Big(\mathbb{D}_\xi^\alpha \sigma_k(x,\xi,\lambda)\Big)\Big(\partial_x^\alpha \sigma_A(x,\xi)\Big)\sigma_0(x,\xi,\lambda). \quad (13)$$

Based on the Leibniz rule we can prove symbolic estimates. Plugging (9) into (13)yields

$$\left\| \partial_\lambda^k \partial_x^\alpha \mathbb{D}_\xi^\beta \sigma_j(x,\xi,\lambda) \right\|_{\mathrm{op}} \leq C_{k,\alpha,\beta,j} \big(|\lambda|^{1/m} + \langle \xi \rangle\big)^{-m(k+1)} \langle \xi \rangle^{-\rho|\beta| + \delta|\alpha| - j}.$$

The left parametrix $\sigma_A^\#(x,\xi,\lambda)$is obtained by forming an asymptotic sum

$$\sigma_A^\#(x,\xi,\lambda) \sim \sum_j \sigma_j(x,\xi,\lambda).$$

Since we assumed $\rho > \delta$the estimates are improving in the hierarchy and the desiredestimate for the parametrix follows and the remark follows by construction we have $\sigma_A^{\#}\#\sigma_A - \mathrm{I} \in \mathrm{S}^{-\infty}(\mathcal{G} \times \hat{\mathcal{G}})$.

An analogous construction changing (13) in the obvious way constructs a right parametrix$\sigma_A^{\flat}(x, \xi, \lambda)$with$\sigma_A\#\sigma_A^{\flat} - I \in \mathrm{S}^{-\infty}(\mathcal{G} \times \hat{\mathcal{G}})$. Then again by calculus

$$\sigma_A^{\#} - \sigma_A^{\flat} = \sigma_A^{\#} - \sigma_A^{\#}\#\sigma_A\sigma_A^{\flat} + \sigma_A^{\#}\#\sigma_A^{\flat} - \sigma_A^{\flat}$$

$$= \sigma_A^{\#}\#(\mathrm{I} - \sigma_A\#\sigma_A^{\flat}) + (\sigma_A^{\#}\#\sigma_A - \mathrm{I})\#\sigma_A^{\flat} \in \mathrm{S}^{-\infty}(\mathcal{G} \times \hat{\mathcal{G}}). \qquad (14)$$

Remark (4.1.4) [4]:

For $m = 0$ a similar statement is valid. Let $\sigma_A(x, \xi) \in S^0_{\rho,\delta}(\mathcal{G} \times \hat{\mathcal{G}})$with $0 \leq \delta \leq 1$ and $0 < \rho \leq 1$. Let further $M := \sup_{x,\xi} \|\sigma_A(x, \xi)\|_{op}$, then the set of λ with$|\lambda| \geq M$ belongs to the resolvent set of all matrices $\sigma_A(x, \xi)$. We replace theparameter-ellipticityby a similar condition for small λ from the sector Λ andassume

$$\|(\sigma_A(x, \xi) - \lambda\mathrm{I})^{-1}\|_{\mathrm{op}} \leq C(1 + |\lambda|)^{-1-k}, \ \lambda \in \Lambda \cup \{|\lambda| \geq 2M\}. \qquad (15)$$

This assumption is only non-trivial for $|\lambda| \leq 2M$on Λ, the estimate for large $|\lambda|$followsfrom the Neumann series representation of the resolvent.

Then for all multi-indices α, β, and all $k$we have

$$\left\|\partial_\lambda^k \partial_x^\alpha \mathbb{D}_\xi^\beta (\sigma_A(x, \xi) - \lambda\mathrm{I})^{-1}\right\|_{\mathrm{op}} \leq C(1 + |\lambda|)^{-1-k}\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \qquad (16)$$

uniformly in $\lambda \in \Lambda \cup \{|\lambda| \geq 2M\}$and if $\delta < \rho$there exists a parameter-dependentparametrix satisfying the estimate

$$\left\|\partial_\lambda^k \partial_x^\alpha \mathbb{D}_\xi^\beta \sigma_A^{\#}(x, \xi, \lambda)\right\|_{\mathrm{op}} \leq C(1 + |\lambda|)^{-1-k}\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \qquad (17)$$

uniformly in $\lambda \in \Lambda \cup \{|\lambda| \geq 2M\}$. The proof goes in analogy.

Remark (4.1.5) [4]:

We can use the ellipticity of the symbol to extend the estimates from Λ to

$\Lambda_\epsilon = \Lambda \cup \{\lambda: |\lambda| < \epsilon\}$for small ϵ. This can be done similar to the previous remark basedon the resolvent bound in $\lambda = 0$ and the Neumann series representation of resolvents.

The parametrix constructed in Theorem (4.1.3) represents the resolvent of theoperator ifit exists, in particular, resolvents are pseudo-differential operators in oursetting.

Corollary (4.1.6) [4]:

Let $0 \leq \delta < \rho \leq 1$ and assume that $\sigma_A \in S^m_{\rho,\delta}(\mathcal{G} \times \hat{\mathcal{G}})$is parameter-ellipticwith respect to some sector Λ. If $\lambda \in \Lambda$ belongs to the resolvent set of the operator A(i.e.,there exist no nonzero smooth functions $f \in C^\infty(\mathcal{G})$ with $Af = \lambda f$ or$A^* f = \bar{\lambda} f$), then$A - \lambda I$ is invertible on $\mathcal{D}'(\mathcal{G})$ and the resolvent $(A - \lambda I)^{-1}$ belongs to$\Psi^{-m}_{\rho,\delta}(\mathcal{G})$.

Proof:

As the parametrix is a regulariser of the operator, it follows by Atkinsons' theoremthat $A - \lambda \mathrm{I}$ is Fredholm between Sobolev spaces$H^{s+m}(\mathcal{G})$ and $H^s(\mathcal{G})$ for all $s \in \mathbb{R}$. The nullspace of $A - \lambda \mathrm{I}$ is independent of $s$and consists of functions from$C^\infty(\mathcal{G}) = \bigcap_{s\in\mathbb{R}} H^s(\mathcal{G})$. The same is true for the null space of $(A - \lambda \mathrm{I})^*$. If both are trivial, the operator $A - \lambda \mathrm{I}$ isinvertible on all $H^s(\mathcal{G})$ and thus on $\mathcal{D}'(\mathcal{G}) = \bigcup_{s\in\mathbb{R}} H^s(\mathcal{G})$. Denote $R_\lambda = (A - \lambda \mathrm{I})^{-1}$ and by$(A - \lambda \mathrm{I})^\sharp$the parametrix. Then

$$R_\lambda - (A - \lambda \mathrm{I})^\sharp = R_\lambda - R_\lambda \circ (A - \lambda \mathrm{I}) \circ (A - \lambda \mathrm{I})^\sharp$$

$$= R_\lambda \circ \left(\mathrm{I} - (A - \lambda \mathrm{I}) \circ (A - \lambda \mathrm{I})^\sharp\right) : \mathcal{D}'(\mathcal{G}) \to C^\infty(\mathcal{G}) \qquad (18)$$

is smoothing, has a smooth Schwartz kernel and is thus element of $\mathrm{Op}S^\infty(\mathcal{G} \times \hat{\mathcal{G}})$.

Section (4.2): Sobolev boundedness of operators with symbols and Gårding inequality

In this section we discuss functions of symbols and of operators.

Functions of left-invariant operators are naturallydefined in terms of the spectral calculus. Later we will show that under naturalassumptions on the function the resulting operators are again pseudo-differential.

To be precise, let $A \in OpS^m_{\rho,\delta}(\hat{\mathcal{G}})$ be left-invariant and parameter-elliptic with respectto some sector $\Lambda \subset C$. We denote $\{\Lambda_\epsilon := \Lambda \cup U_\epsilon, U_\epsilon = \{\lambda : |\lambda| \leq \epsilon\}$, and assume that thefunction F is holomorphic in $\mathbb{C}\backslash\Lambda_\epsilon$and continuous on its closure. In particular it mayhave branch points in 0 and ∞. Let $\Gamma = \partial\Lambda_\epsilon$be the (oriented) contour encircling the sectorΛ_ϵ. We assume decay of F along Γ. Then similar to the treatment of Seeley for complexpowers of classical parameter-elliptic operators, we define the function $F(A)$ interms of contour integrals but for our global symbols on $\mathcal{G} \times \widehat{\mathcal{G}}$ instead of homogeneouslocal symbolic components. Thus, we define $B = F(A)$ in terms of the symbol $\sigma_B(\xi)$ givenby

$$\sigma_B(\xi) = \frac{1}{2\pi i} \oint_\Gamma F(\lambda)\sigma_A^\#(\xi,\lambda)d\lambda. \qquad (19)$$

Theorem (4.2.1) will show that $\sigma_B \in S^{ms}_{\rho,\delta}(\hat{\mathcal{G}})$ provided Fdecays on Γ like $|\lambda|^s$for some $s < 0$.

Note, that for each fixed ξ the contour can be altered and replaced by a closed contouraround the spectrum of the matrix $\sigma_A(\xi)$ and thus $\sigma_B(\xi) = F(\sigma_A(\xi))$ in the sense of usualmatrix spectral calculus. As $\mathcal{H}_\xi$are finite dimensional invariant subspaces of A with A acting as multiplication by $\sigma_A(\xi)$ on $\mathcal{H}_\xi$(identified with the space of matrix coefficients),it is thus clear that the operator $F(A)$ coincides with the operator defined by the spectralcalculus and based on the same cut in the complex plane.

Of particular importance are complex powers $F(\lambda) = \lambda^s, s \in \mathbb{C}$. If $\sigma_A$is parameterellipticwith respect to Λ and $Re\ s < 0$, combining the decays in λ, the above integralconverges by Theorem (4.1.3) and defines a family of operators $B_s$which is holomorphic in $s$and satisfies the group property $B_{s+t} = B_s B_t$. Among other things, we will show that allthese $B_s$are pseudo-differential of the same type as A and of appropriate orders, makingthem in particular all elliptic.

For calculus reasons it is important to show that certainfunctions of symbols define again symbols. As an example, we will treat positive squareroots of positive symbols and use it to show a version of the Gårding inequality.

Remark (4.2.1) [4]:

Given Theorem (4.2.1), by the calculus (7) we can also define functions ofsymbols for functions which are growing at most polynomially and show that they satisfysymbolic estimates again.

Indeed, let $\sigma_A$and Fbe as in Theorem (4.2.1) but assume now that (20) holds with

$s \geq 0$.Writing$F(\lambda) = F(\lambda)\lambda^{-[s]-1}\lambda^{[s]+1}$, we can define

$$F(\sigma_A) \coloneqq \tilde{F}(\sigma_A)\sigma_A^{[s]+1},$$

where $\tilde{F}(\lambda) := F(\lambda)\lambda^{-[s]-1}$ satisfies (20) with $\tilde{s} = s - [s] - 1 < 0$. Consequently, byTheorem (4.2.1) and the calculus (7) we conclude that $F(\sigma_A) \in S_{\rho,\delta}^{ms}(\mathcal{G} \times \hat{\mathcal{G}})$.

The particular choice of factorising Fdoes not influence the resulting symbol. If wewrite $F(\lambda) = F_1(\lambda)\lambda^{k_1} = F_2(\lambda)\lambda^{k_2}$ for different integers κ_1 and κ_2 larger than $[s]$ anddefine $F(\sigma_A)$ through both formulae, then $F(\sigma_A)$ is still uniquely defined. If$\kappa_1 > \kappa_2$, then$F_1(\sigma_A(x,\xi))\sigma_A(x,\xi)^{k_1-k_2} = F_2(\sigma_A(x,\xi))$ for each x and ξdue to Cauchy integral theoremand the functional calculus of matrices.

Remark (4.2.2) [4]:

Arbitrary complex powers of parameter-elliptic symbols are again elliptic.

If$\sigma_A \in S_{\rho,\delta}^{m}(\mathcal{G} \times \hat{\mathcal{G}})$is parameter-elliptic with respect to some sector, then by the previousremark $\sigma_A^s \in S_{\rho,\delta}^{m\,Re\,s}(\mathcal{G} \times \hat{\mathcal{G}})$as well as $(\sigma_A^s)^{-1} = \sigma_A^{-s} \in S_{\rho,\delta}^{-m\,Re\,s}(\mathcal{G} \times \hat{\mathcal{G}})$.

Theorem (4.2.3) [4]:

Let $0 \leq \delta \leq 1$ and $0 < \rho \leq 1$. Let $\sigma_A \in S^m_{\rho,\delta}(\mathcal{G} \times \hat{\mathcal{G}}), m \geq 0$, beparameter-elliptic with respect to a sector Λ and assume that $F$is analytic in $\mathbb{C}\backslash\Lambda_\epsilon$andsatisfies

$$|F(\lambda)| \leq C|\lambda|^s (20)$$

for some $s < 0$. Then

$$\sigma_B(x,\xi) = \frac{1}{2\pi i}\oint_\Gamma F(\lambda)(\sigma_A(x,\xi) - \lambda I)^{-1} d\lambda,$$

defines a symbol $\sigma_B \in S^{ms}_{\rho,\delta}(\mathcal{G} \times \hat{\mathcal{G}})$.

We can write $F(\sigma_A) := \sigma_B$, so that $F(\sigma_A) \in S^{ms}_{\rho,\delta}(\mathcal{G} \times \hat{\mathcal{G}})$.

Proof

As $\|(\sigma_A(x,\xi) - \lambda I)^{-1}\|_{\mathrm{op}} \leq (1 + |\lambda|^{1/m} + \langle\xi\rangle)^{-m} \lesssim \lambda^{-1}$ the contourintegral converges for $s < 0$ and $\sigma_B(x,\xi)$ is well-defined. We now show the symbolicestimates.

We restrict considerations for $m > 0$. For $m = 0$ the proof is analogous and uses theestimate (16) instead. We split the proof into two parts and consider first the particularcase $s \in (-1,0)$. Then the elementary estimate

$$\int_0^\infty (1 + r^{1/m} + \langle\xi\rangle)^{-m}(1 + r)^s dr \leq \langle\xi\rangle^{ms} \int_0^\infty (1 + \tilde{r}^{1/m})^{-m}\tilde{r}^s d\tilde{r} \leq C\langle\xi\rangle^{ms}$$

is based on the substitution $r = (1 + \langle\xi\rangle)^m\tilde{r}$and the integrability of the remaining integralis ensured for large $\tilde{r}$by $s < 0$ and for small $\tilde{r}$by $s > -1$. This estimate can be directlyapplied to the contour integral. Indeed, for all fixed $x \in \mathcal{G}$and $[\xi] \in \hat{\mathcal{G}}$the condition ofparameter-ellipticity yields

$$\|\sigma_B(x,\xi)\|_{\mathrm{op}} \leq \frac{1}{2\pi}\oint_\Gamma |F(\lambda)|\|(\sigma_A(x,\xi) - \lambda I)^{-1}\|_{\mathrm{op}}|d\lambda|$$

$$\leq C\oint_\Gamma |\lambda|^s \left(|\lambda|^{\frac{1}{m}} + \langle\xi\rangle\right)^{-m} |d\lambda| \leq C'\langle\xi\rangle^{ms}$$

with all appearing constants independent of $x$and ξ. Furthermore, for all multi-indices $\alpha$and βwe obtain (by linearity and from the definition of improper Riemann integrals)

$$\partial_x^\alpha \mathbb{D}_\xi^\beta \sigma_B(x,\xi) = \frac{1}{2\pi i}\oint_\Gamma F(\lambda)\partial_x^\alpha \mathbb{D}_\xi^\beta(\sigma_A(x,\xi) - \lambda I)^{-1} d\lambda\,,$$

so that the first statement of Theorem (4.1.3) and the above argumentation imply the desiredestimates.

It remains to treat the general case. Let now $s < 0$ be arbitrary. Then we find afunction Ganalytic in $\mathbb{C}\backslash\Lambda_\epsilon$such that $F(\lambda) = G(\lambda)^{[-s]+1}$, $[-s]$ the integer part of $-s$.

Then $|G(\lambda)| \leq |\lambda|^{(s/(1+[-s])}$ has an exponent in the range $(-1,0)$ and

$$\sigma_C(x,\xi) = \frac{1}{2\pi i}\oint_\Gamma G(\lambda)(\sigma_A(x,\xi) - \lambda I)^{-1} d\lambda\,,$$

defines a symbol from $S_{\rho,\delta}^{ms/(1+[-s])}(\mathcal{G}\times\hat{\mathcal{G}})$ by the previous case. Furthermore, by the spectralcalculus of matrices, $\sigma_B(x,\xi) = (\sigma_C(x,\xi))^{1+[-s]}$ and hence$\sigma_B \in S_{\rho,\delta}^{ms}(\mathcal{G}\times\hat{\mathcal{G}})$.

In the following we use the standard cut $\mathbb{R}_- = \{\zeta \in \mathbb{R}: \zeta \leq 0\}$for defining thecomplex logarithm log on $C\backslash\mathbb{R}_-$. In particular, we can define for any matrix $\sigma_A(x,\xi)$with eigenvalues separated from $\mathbb{R}_-$the matrix function $\log\sigma_A\,(x,\xi)$ by the usual spectralcalculus. We will use this in particular for positive matrices.

Theorem (4.2.4) [4]:

Let $0 \leq \delta \leq 1$ and $0 < \rho \leq 1$. Assume $\sigma_A \in S_{\rho,\delta}^m(\mathcal{G}\times\hat{\mathcal{G}}), m \geq 0$, is positivedefinite, invertible, and satisfies

$$\|\sigma_A(x,\xi)^{-1}\|_{\mathrm{op}} \leq C\langle\xi\rangle^{-m} (21)$$

for all x and for all but finitely many ξ. Then $\sigma_A(x,\xi)$ is parameter-elliptic with respectto $\mathbb{R}_-$. Furthermore, for any number $s \in \mathbb{C}$,

$$\sigma_B(x,\xi) := \sigma_A(x,\xi)^s = \exp(s\log\sigma_A\,(x,\xi))$$

defines a symbol $\sigma_B \in S_{\rho,\delta}^{m'}(\mathcal{G}\times\hat{\mathcal{G}}), m' = \mathrm{Re}(ms)$.

Proof:

First we show parameter-ellipticity. From ellipticity we know that the spectrumof the positive matrix $\sigma_A(x,\xi)$ satisfies

$$\langle\xi\rangle^{-m}\mathrm{spec}\sigma_A(x,\xi) \subset [c,C] \subset \mathbb{R}_+$$

for some (fixed) constants $c$and C independent of $x$and ξ. Since the matrix $\sigma_A(x,\xi)$ isnormal, the operator norm of the resolvent $(\sigma_A(x,\xi)-\lambda I)^{-1}$ is given by the distance of λ to the spectrum and hence

$$\|(\sigma_A(x,\xi)-\lambda I)^{-1}\|_{\mathrm{op}} \leq (c\langle\xi\rangle^m + |\lambda|)^{-1} \leq C'(|\lambda|^{1/m} + \langle\xi\rangle)^{-1/m}$$

for any $\lambda \in \mathbb{R}_-$.

It remains to prove the symbol property. For $\mathrm{Re}s < 0$ we can directly apply theTheorem (4.2.1), while for $\mathrm{Re}s \geq 0$ we use the semigroup property combined with Leibnizrule. Indeed,

if $\mathrm{Re}s > 0$ we find an integer k such that $\mathrm{Re}s - k < 0$ and therefore

$(\sigma_A(x,\xi))^{s-k} \in S_{\rho,\delta}^{m(\mathrm{Re}s-k)}(\mathcal{G}\times\hat{\mathcal{G}})$ by Theorem (4.2.1). But from that we conclude

$\sigma_A(x,\xi)^s = \sigma_A(x,\xi)^{s-k}\sigma_A(x,\xi)^k \in S_{\rho,\delta}^{m'}(\mathcal{G}\times\hat{\mathcal{G}})$ for $m' = m\,\mathrm{Re}\,s$. The semigroup property holds due tothe positivity of the matrices.

In the sequel, we will use the particular case of Theorem (4.2.2) applied to square roots ofsymbols. Here we note that condition (21) for $\delta < \rho$is equivalent to the ellipticity of theoperator A in view.

Corollary (4.2.5) [4]:

Let $0 \leq \delta < \rho \leq 1$ and let $m \geq 0$. Assume $\sigma_A \in S_{\rho,\delta}^{2m}(\mathcal{G}\times\hat{\mathcal{G}})$ is elliptic(i.e. (21) holds) and satisfies $\sigma_A(x,\xi) > 0$ for all $x$and ξ. Then the square root

$$\sigma_B(x,\xi) = \sqrt{\sigma_A(x,\xi)}$$

in the sense of positive matrices is a symbol satisfying$\sigma_B \in S_{\rho,\delta}^{m}(\mathcal{G}\times\hat{\mathcal{G}})$.

So far we considered functions of symbols, which for thecase of invariant operators correspond to symbols of functions of operators defined interms of the spectral calculus. We can also

reformulate Theorem (4.2.1) for approximatefunctions of operators defined in terms of the parameter-dependent parametrix $\sigma_A^{\#}(x,\xi,\lambda)$of a parameter-elliptic symbol $\sigma_A = \sigma_A(x,\xi) \in S_{\rho,\delta}^m(\mathcal{G} \times \hat{\mathcal{G}})$ with respect to some sector Λ.Assume that $F$is analytic in $\mathbb{C}\backslash\Lambda_\epsilon$and satisfies (20). Then

$$\sigma_{F(A)}(x,\xi) := \frac{1}{2\pi i}\oint_\Gamma F(\lambda)\sigma_A^{\#}(x,\xi,\lambda)d\lambda \quad (22)$$

converges and defines a matrix-valued function $\sigma_{F(A)}(x,\xi)$ on $\mathcal{G} \times \hat{\mathcal{G}}$. Under the strongerassumption that the exponent $s \in (-1,0)$ we can apply the proof of the first part ofTheorem (4.2.1) and obtain

Corollary (4.2.6) [4]:

Let $0 \le \delta < \rho \le 1$. Let $\sigma_A \in S_{\rho,\delta}^m(\mathcal{G} \times \hat{\mathcal{G}})$ be parameter-elliptic with respectto Λ and let $\sigma_A^{\#}(x,\xi,\lambda)$ be the corresponding parameter-dependent parametrix. Let Fbeanalytic in

$\mathbb{C}\backslash\Lambda_\epsilon$satisfying (20) for some $s \in (-1,0)$ and define the operator $F(A)$ bythe symbol (22). Then $\sigma_{F(A)} \in S_{\rho,\delta}^{ms}(\mathcal{G} \times \hat{\mathcal{G}})$.

Again we want to extend this to all negative s. For this we establish an approximatefunctional calculus for the operators $F(A)$. First, we denote the set of all parameterdependentsymbols $\sigma(x,\xi,\lambda)$ satisfying

$$\left\|\partial_t^k \partial_x^\alpha \mathbb{D}_\xi^\beta \sigma(x,\xi,\lambda)\right\|_{\mathrm{op}} \le C_{j,k,\alpha,\beta}(|\lambda|^{1/m} + \langle\xi\rangle)^{-m(k+1)}\langle\xi\rangle^{-j}, \qquad \lambda \in \Lambda, \quad (23)$$

for some $m > 0$ and all $j \in \mathbb{N}$as $S^{-\infty}(\Lambda)$. They are smoothing parameter-dependentsymbols in our sense and by the same integration argument as in Theorem (4.2.1) we obtain

Corollary (4.2.7) [4]:

Assume that the parameter-dependent symbol $\sigma \in S^{-\infty}(\Lambda)$ is smoothing,that F is analytic in $\mathbb{C}\backslash\Lambda_\epsilon$and satisfies (20). Then the symbol

$$\tilde{\sigma}(x,\xi) := \frac{1}{2\pi i}\oint_\Gamma F(\lambda)\sigma(x,\xi,\lambda)d\lambda$$

is smoothing, $\tilde{\sigma} \in S^{-\infty}(\mathcal{G} \times \hat{\mathcal{G}})$.

List of symbols

Symbol	page
H^P: Hardy space	1
Mult: multi	1
Sup: Supperum	1
Supp: support	2
deg: degree	3
max: maximum	3
inf: infimum	6
L^2: Hilbert space	6
L^P: lebesgue space	7
Rep: representation	19
dim: dimension	19
l^2: Hilbert space	19
End: Endomorphism	20
diff: differential	21
op: operator	25
Tr: Trace	27
Re:Real	28
det: determinant	29

dirt: dietant 29

loc: local 31

L^1: Lebesgue of the real line 48

min: minimum 48

diam: diameter 61

vol: volume 61

Spec: spectrum 80

Res: Residue 80

Refrences

[1] Whalter Bloom andXuzengfu :Approximation of H^P-Function by Bochner-Riesz means on Compact Lie Group , Mathematical Lische Zritschrift , Springer-Verlag 1994

[2] Micharl Ruzhansky and Ville-Turcinen : Sharp Garding Inequality on Compact Lie Groups , Journal of Functional Analysis 206 (2011) 288-901.

[3] Jiechcng Chen and Dashan Fan : Central Oscillating Multipliers on Compact Lie Groups . Math . Z . (2011) 267 : 235-259 .

[4] Michael Ruzhansky and Jens Wirth : Global Functional Calculus for Operators on Compact Lie Groups , Journal of Functional Analysis 267(2014) 144-172 .

[5] A.Sagle and Walde : Introduction to Lie Groups and Lie Algebras , Pure and Applied Mathematics , Vol .51 , Academic Press , New York , 1973 .

[6] W.G.Dwyer and C.W.Wilkerson : The Center of Compact Group , Contemporary Mathematics 181 , Proceedings of the 1993 Cech Conferences (Northeastern Univ) , Amer . Math . Soc . Providence , 1995 , 199-157 .