Chapter (1)

Basic concepts and definitions

Similar to other field theories such as fluid mechanics, heat conduction, and electromagnetics, the study and application of elasticity theory requires knowledge of several areas of applied mathematics. The theory is formulated in terms of a variety of variables including scalar, vector, and tensor fields, and this calls for the use of tensor notation along with tensor algebra and calculus. Through the use of particular principles from continuum mechanics, the theory is developed as a system of partial differential field equations that are to be solved in a region of space coinciding with the body under study. Solution techniques used on these field equations commonly employ Fourier methods, variational techniques, integral transforms, complex variables, potential theory, finite differences, and finite and boundary elements. Therefore, to develop proper formulation methods and solution techniques for elasticity problems, it is necessary to have an appropriate mathematical background. The purpose of this initial chapter is to provide a background primarily for the formulation part of our study. Additional review of other mathematical topics related to problem solution technique is provided in later chapters where they are to be applied.

Section (1.1):Scalar, Vector, Matrix, and Tensor Definitions:

Elasticity theory is formulated in terms of many different types of variables that are either specified or sought at spatial points in the body under study. Some of these variables are scalar quantities, representing a single magnitude at each point in space. Common examples include the material density ρ and material moduli such as Young's modulus E, Poisson's ratio ν, or the shear modulus μ. Other variables of interest are vector quantities that are expressible in terms of components in a two- or three-dimensional coordinate system. Examples of vector variables are the displacement and rotation of material points in the elastic continuum. Formulations within the theory also require the need for matrix variables, which commonly require more than three components to quantify. Examples of such variables include stress and strain. As shown in subsequent chapters, a three-dimensional formulation requires nine components (only six are independent) to quantify the stress or strain at a point. For this case, the variable is normally expressed in a matrix format with

three rows and three columns. To summarize this discussion, in a three-dimensional Cartesian coordinate system, scalar, vector, and matrix variables can thus be written as follows:

$$\text{mass density scalar } = \rho$$

$$\text{displacement vector } = ue_1 + ve_2 + we_3$$

$$\text{stress matrix } = [\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \sigma_{zy} & \sigma_z \end{bmatrix}$$

where e_1, e_2, e_3 are the usual unit basis vectors in the coordinate directions. Thus, scalars, vectors, and matrices are specified by one, three, and nine components, respectively.

The formulation of elasticity problems not only involves these types of variables, but also incorporates additional quantities that require even more components to characterize. Because of this, most field theories such as elasticity make use of a tensor formalism using index notation. This enables efficient representation of all variables and governing equations using a single standardized scheme. The tensor concept is defined more precisely in a later section, but for now we can simply say that scalars, vectors, matrices, and other higher-order variables can all be represented by tensors of various orders. We now proceed to a discussion on the notational rules of order for the tensor formalism. Additional information on tensors and index notation can be found in many texts such as Goodbody [3] or Chandrasekharaiah and Debnath [1]. Index notation is a shorthand scheme whereby a whole set of numbers (elements or components) is represented by a single symbol with subscripts. For example, the three numbers a_1, a_2, a_3 are denoted by the symbol a_i, where index i will normally have the range 1, 2, 3. In a similar fashion, a_{ij} represents the nine numbers $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, e_{31}, e_{32}, e_{33}$. Although these representations can be written in any manner, it is common to use a scheme related to vector and matrix formats such that

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1.1}$$

In the matrix format, a_{1j} represents the first row, while a_{i1} indicates the first column. Other columns and rows are indicated in similar fashion, and thus the first index represents the row, while the second index denotes the column.

In general a symbol $a_{ij\ldots k}$ with N distinct indices represents 3^N distinct numbers. It should be apparent that a_i and a_j represent the same three numbers, and likewise a_{ij} and a_{mn} signify the same matrix. Addition, subtraction, multiplication, and equality of index symbols are defined in the normal fashion. For example, addition and subtraction are given by

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, a_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{22} & a_2 \pm b_2 & a_{23} \pm b_{22} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix} \tag{1.2}$$

and scalar multiplication is specified as

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} \tag{1.3}$$

The multiplication of two symbols with different indices is called outer multiplication, and a simple example is given by

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{1.4}$$

The previous operations obey usual commutative, associative, and distributive laws, for example:

$$\begin{aligned} a_i + b_i &= b_i + a_i \\ a_{ij} b_k &= b_k a_{ij} \\ a_i + (b_i + c_i) &= (a_i + b_i) + c_i \\ a_i(b_{jk} c_l) &= (a_i b_{jk}) c_l \\ a_i(b_k + c_k) &= a_{ij} b_k + a_{ij} c_k \end{aligned} \tag{1.5}$$

Note that the simple relations $a_i = b_i$ and $a_{ij} = b_{ij}$ imply that $a_1 = b_1, a_2 = b_2, \ldots$ and $a_{11} = b_{11}, a_{12} = b_{12}, \ldots$. However, relations of the form $a_i = b_j$ or

$a_{ij} = b_{kl}$ have ambiguous meaning because the distinct indices on each term are not the same, and these types of expressions are to be avoided in this notational scheme. In general, the distinct subscripts on all individual terms in an equation should match.

It is convenient to adopt the convention that if a subscript appears twice in the same term, then summation over that subscript from one to three is implied; for example:

$$\begin{aligned} a_{ii} &= \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} \\ a_{ij}b_j &= \sum_{j=1}^{3} a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 \end{aligned} \tag{1.6}$$

It should be apparent that $a_{ii} = a_{jj} = a_{kk} = \cdots$ and therefore the repeated subscripts or indices are sometimes called dummy subscripts. Unspecified indices that are not repeated are called free or distinct subscripts. The summation convention may be suspended by underlining one of the repeated indices or by writing no sum. The use of three or more repeated indices in the same term (e.g., a_{iii} or $a_{iij}b_{ij}$) has ambiguous meaning and is to be avoided. On a given symbol, the process of setting two free indices equal is called contraction. For example, a_{ii} is obtained from a_{ij} by contraction on i and j. The operation of outer multiplication of two indexed symbols followed by contraction with respect to one index from each symbol generates an inner multiplication; for example, $a_{ij}b_{jk}$ is an inner product obtained from the outer product $a_{ij}b_{mk}$ by contraction on indices j and m.

A symbol $a_{ij\ldots m\ldots n\ldots k}$ is said to be symmetric with respect to index pair mn if

$$a_{ij\ldots m\ldots n\ldots k} = a_{ij\ldots n\ldots m\ldots k} \tag{1.7}$$

while it is anti symmetric or skew symmetric if

$$a_{ij\ldots m\ldots n\ldots k} = -a_{ij\ldots n\ldots m\ldots k} \tag{1.8}$$

Note that if $a_{ij\ldots m\ldots n\ldots k}$ is symmetric in mn while $b_{pq\ldots m\ldots n\ldots r}$ is antisymmetric in mn, then the product is zero:

$$a_{ij\ldots m\ldots n\ldots k}b_{pq\ldots n\ldots m\ldots r} = 0 \tag{1.9}$$

A useful identity may be written as

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) = a_{(ij)} + a_{[ji]} \tag{1.10}$$

The first term $a_{(ij)} = 1/2(a_{ij} + a_{ji})$ is symmetric, while the second term $a_{[ij]} = 1/2(a_{ij} - a_{ji})$ is antisymmetric, and thus an arbitrary symbol a_{ij} can be expressed as the sum of symmetric and antisymmetric pieces. Note that if a_{ij} is symmetric, it has only six independent components. On the other hand, if a_{ij} is antisymmetric, its diagonal terms a_{ii} (no sum on i) must be zero, and it has only three independent components. Note that since $a_{[ij]}$ has only three independent components, it can be related to a quantity with a single index, for example, a_i

Section (1.2):Kronecker Delta and Alternating Symbol:

A useful special symbol commonly used in index notational schemes is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \text{ (no sum)} \\ 0, & \text{if } i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.11}$$

Within usual matrix theory, it is observed that this symbol is simply the unit matrix. Note that the Kronecker delta is a symmetric symbol. Particular useful properties of the Kronecker delta include the following:

$$\begin{aligned} &\delta_{ij} = \delta_{ji} \\ &\delta_{ii} = 3, \delta_{i\underline{i}} = 1 \\ &\delta_{ij}a_j = a_i, \delta_{ji}a_i = a_j \\ &\delta_{ij}a_{jk} = a_{ik}, \delta_{jk}a_{ik} = a_{ij} \\ &\delta_{ij}a_{ij} = a_{ii}, \delta_{ij}\delta_{ij} = 3 \end{aligned} \tag{1.12}$$

Another useful special symbol is the alternating or permutation symbol defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{cases} \tag{1.13}$$

Consequently, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{321} = \varepsilon_{132} = \varepsilon_{231} = -1, \varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \cdots = 0$. Therefore, of the 27 possible terms for the alternating symbol, 3 are equal to 1, three are equal to -1, and all others are 0. The alternating symbol is antisymmetric with respect to any pair of its indices.

This particular symbol is useful in evaluating determinants and vector cross products, and the determinant of an array a_{ij} can be written in two equivalent forms:

$$\det[a_{ij}] = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3} \tag{1.14}$$

where the first index expression represents the row expansion, while the second form is the column expansion. Using the property

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{1.15}$$

another form of the determinant of a matrix can be written as

$$\det[a_{ij}] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr} a_{ip} a_{jq} a_{kr} \tag{1.16}$$

It is convenient and in fact necessary to express elasticity variables and field equations in several different coordinate systems.This situation requires the development of particular transformation rules for scalar, vector, matrix, and higher-order variables. This concept is fundamentally connected with the basic definitions of tensor variables and their related tensor transformation laws. We restrict our discussion to transformations only between Cartesian coordinate systems, and thus consider the two systems shown in Figure 1-1. The two Cartesian frames (x_1, x_2, x_3) and (x_1', x_2', x_3') differ only by orientation, and the unit basis vectors for each frame are $\{e_i\} = \{e_1, e_2, e_3\}$ and $\{e_i'\} = \{e_1', e_2', e_3'\}$.

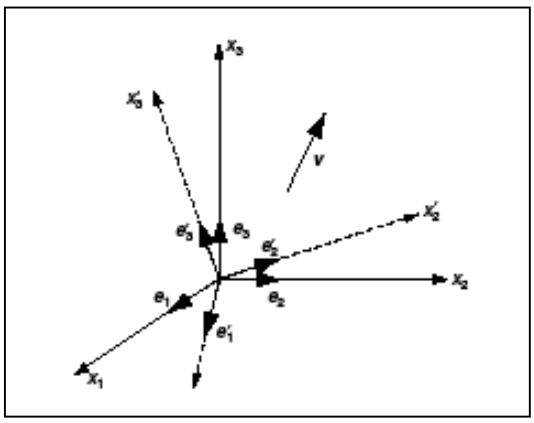


FIGURE 1-1 Change of Cartesian coordinate frames.

Let Q_{ij} denote the cosine of the angle between the x_i'-axis and the x_j-axis:

$$Q_{ij} = \cos(x_i', x_j) \tag{1.17}$$

Using this definition, the basis vectors in the primed coordinate frame can be easily expressed in terms of those in the unprimed frame by the relations

$$\begin{aligned} \boldsymbol{e}_1' &= \boldsymbol{Q}_{11}\boldsymbol{e}_1 + \boldsymbol{Q}_{12}\boldsymbol{e}_2 + \boldsymbol{Q}_{13}\boldsymbol{e}_3 \\ \boldsymbol{e}_2' &= \boldsymbol{Q}_{21}\boldsymbol{e}_1 + Q_{22}e_2 + \boldsymbol{Q}_{23}\boldsymbol{e}_3 \\ \boldsymbol{e}_3' &= \boldsymbol{Q}_{31}\boldsymbol{e}_1 + \boldsymbol{Q}_{32}\boldsymbol{e}_2 + \boldsymbol{Q}_{33}\boldsymbol{e}_3 \end{aligned} \tag{1.18}$$

or in index notation

$$\boldsymbol{e}_i' = \boldsymbol{Q}_{ij}\boldsymbol{e}_j \tag{1.19}$$

Likewise, the opposite transformation can be written using the same format as

$$\boldsymbol{e}_i = \boldsymbol{Q}_{ij}\boldsymbol{e}_j' \tag{1.20}$$

Now an arbitrary vector v can be written in either of the two coordinate systems as

$$\begin{aligned} v &= v_1e_1 + v_2e_2 + v_3e_3 \\ &= v_1'e_1' + v_2'e_2' + v_3'e_3' \end{aligned} \tag{1.21}$$

Substituting form (1.20) into $(1.21)_1$ gives

$$\boldsymbol{v} = v_i\boldsymbol{Q}_{ij}\boldsymbol{e}'_j$$

but from $(1.21)_2$, $\boldsymbol{v} = v_j' \boldsymbol{e}_j'$ and so we find that

$$\boldsymbol{v}_i' = \boldsymbol{Q}_{ij} \boldsymbol{v}_j \tag{1.22}$$

In similar fashion, using (1.19) in $(1.21)_2$ gives

$$v_i = Q_{ij} v_j' \tag{1.23}$$

Relations (1.22) and (1.23) constitute the transformation laws for the Cartesian components of a vector under a change of rectangular Cartesian coordinate frame. It should be understood that under such transformations, the vector is unaltered (retaining original length and orientation), and only its components are changed. Consequently, if we know the components of a vector in one frame, relation (1.22) and/or relation (1.23) can be used to calculate components in any other frame.

The fact that transformations are being made only between orthogonal coordinate systems places some particular restrictions on the transformation or direction cosine matrix Q_{ij}. These can be determined by using (1.22) and (1.23) together to get

$$v_i = Q_{ji} v_j' = Q_{ji} Q_{jk} v_k \tag{1.24}$$

From the properties of the Kronecker delta, this expression can be written as

$$\delta_{ik} v_k = Q_{ji} Q_{jk} v_k \quad \text{or} \quad \left(Q_{ji} Q_{jk} - \delta_{ik}\right) v_k = 0$$

and since this relation is true for all vectors v_k, the expression in parentheses must be zero, giving the result

$$Q_{ji} Q_{jk} = \delta_{ik} \tag{1.25}$$

In similar fashion, relations (1.22) and (1.23) can be used to eliminate v_i (instead of v_i' to get

$$Q_{ij} Q_{kj} = \delta_{ik} \tag{1.26}$$

Relations (1.25) and (1.26) comprise the orthogonality conditions that Q_{ij} must satisfy. Taking the determinant of either relation gives another related result:

$$\det\left[Q_{ij}\right] = \pm 1 \tag{1.27}$$

Matrices that satisfy these relations are called orthogonal, and the transformations given by (1.22 and (1.23) are therefore referred to as orthogonal transformations.

Scalars, vectors, matrices, and higher-order quantities can be represented by a general index notational scheme. Using this approach, all quantities may then be referred to as tensors of different orders. The previously presented transformation properties of a vector can be used to establish the general transformation properties of these tensors. Restricting the transformations to those only between Cartesian coordinate systems, the general set of transformation relations for various orders can be written as

$$\begin{aligned}
&a' = a, \text{zero order (scalar)}\\
&a'_i = Q_{ip}a_p, \text{first order (vector)}\\
&a'_{ij} = Q_{ip}Q_{jq}a_{pq}, \text{second order (matrix)}\\
&a'_{ijk} = Q_{ip}Q_{jq}Q_{kr}a_{pqr}, \text{third order}\\
&a'_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}a_{pqr}, \text{fourth order}\\
&\vdots\\
&a'_{ijk\ldots m} = Q_{ip}Q_{jq}Q_{kr}\ldots Q_{mt}Q_{pqr}a_{pqr\ldots t} \text{ general order}
\end{aligned} \tag{1.28}$$

Note that, according to these definitions, a scalar is a zero-order tensor, a vector is a tensor of order one, and a matrix is a tensor of order two. Relations (1.28) then specify the transformation rules for the components of Cartesian tensors of any order under the rotation Q_{ij}. This transformation theory proves to be very valuable in determining the displacement, stress, and strain in different coordinate directions. Some tensors are of a special form in which their components remain the same under all transformations, and these are referred to as isotropic tensors. It can be easily verified that the Kronecker delta δ_{ij} has such a property and is therefore a second-order isotropic tensor. The alternating symbol ε_{ijk} is found to be the third-order isotropic form. The fourthorder case can be expressed in terms of products of Kronecker deltas, and this has important applications in formulating isotropic elastic constitutive relations in Section 4.2.

The distinction between the components and the tensor should be understood. Recall that a vector v can be expressed as

$$
\begin{aligned}
v &= v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_i\boldsymbol{e}_i \\
&= v_1'\boldsymbol{e}_1' + v_2'\boldsymbol{e}_2' + v_3'e_3' = v_i'\boldsymbol{e}_i'
\end{aligned} \tag{1.29}
$$

In a similar fashion, a second-order tensor A can be written

$$
\begin{aligned}
A = {} & A_{11}\boldsymbol{e}_1\boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1\boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1\boldsymbol{e}_3 \\
& + A_{21}\boldsymbol{e}_2\boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\boldsymbol{e}_3 \\
& + A_{31}\boldsymbol{e}_3\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\boldsymbol{e}_3 \\
& = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A_{ij}'e_i'e_j'
\end{aligned} \tag{1.30}
$$

and similar schemes can be used to represent tensors of higher order. The representation used in equation (1.30) is commonly called dyadic notation, and some authors write the dyadic products $\boldsymbol{e}_i\boldsymbol{e}_j$ using a tensor product notation $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Additional information on dyadic notation can be found in Weatherburn [7] and Chou and Pagano [2].

Relations (1.29) and (1.30) indicate that any tensor can be expressed in terms of components in any coordinate system, and it is only the components that change under coordinate transformation. For example, the state of stress at a point in an elastic solid depends on the problem geometry and applied loadings. As is shown later, these stress components are those of a second-order tensor and therefore obey transformation law (1.28) . Although the components of the stress tensor change with the choice of coordinates, the stress tensor (representing the state of stress) does not.

An important property of a tensor is that if we know its components in one coordinate system, we can find them in any other coordinate frame by using the appropriate transformation law. Because the components of Cartesian tensors are representable by indexed symbols, the operations of equality, addition, subtraction, multiplication, and so forth are defined in a manner consistent with the indicial notation procedures previously discussed. The terminology tensor without the adjective Cartesian usually refers to a more general scheme in which the coordinates are not necessarily rectangular Cartesian and the transformations between coordinates are not always orthogonal. Such general tensor theory is not discussed or used in this text.

EXAMPLE 1-1: Transformation Examples:

The components of a first- and second-order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, a_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

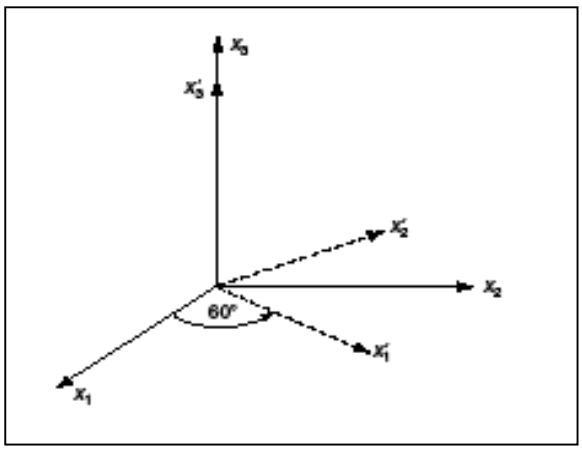


FIGURE 1-2 Coordinate transformation.

Determine the components of each tensor in a new coordinate system found through a rotation of 60°8 ($\pi/6$ radians) about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis (see Figure 1-2). The original and primed coordinate systems shown in Figure 1-2 establish the angles between the various axes. The solution starts by determining the rotation matrix for this case:

$$Q_{ij} = \begin{bmatrix} \cos 60° & \cos 30° & \cos 90° \\ \cos 150° & \cos 60° & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation for the vector quantity follows from equation $(1.5.1)_2$:

$$a_i' = Q_{ij} a_j = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 + \sqrt{3} \\ 2 - \sqrt{3}/2 \\ 2 \end{bmatrix}$$

and the second-order tensor (matrix) transforms according to (1.28):

$$\begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$
$$= \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2+\sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1-3\sqrt{3}/2 \\ 3/2+\sqrt{3} & 1-3\sqrt{3}/2 & 4 \end{bmatrix}$$

where $[\ ]^T$ indicates transpose . Although simple transformations can be worked out by hand, for more general cases it is more convenient to use a computational scheme to evaluate the necessary matrix multiplications required in the transformation laws (1.28). MATLAB software is ideally suited to carry out such calculations.

section (1.3):Principal Values and Directions for Symmetric Second-Order Tensors:

Considering the tensor transformation concept previously discussed, it should be apparent that there might exist particular coordinate systems in which the components of a tensor take on maximum or minimum values. This concept is easily visualized when we consider the components of a vector shown in Figure 1-1. If we choose a particular coordinate system that has been rotated so that the x_3-axis lies along the direction of the vector, then the vector will have components $\boldsymbol{\nu} = \{0, 0, |\boldsymbol{\nu}|\}$. For this case, two of the components have been reduced to zero, while the remaining component becomes the largest possible (the total magnitude).

This situation is most useful for symmetric second-order tensors that eventually represent the stress and/or strain at a point in an elastic solid. The direction determined by the unit vector $\boldsymbol{n}$ is said to be a principal direction or eigenvector of the symmetric second-order tensor a_{ij} if there exists a parameter λ such that

$$a_{ij}n_j = \lambda n_i \tag{1.31}$$

where λ is called the principal value or eigenvalue of the tensor. Relation (1.31) can be rewritten as

$$(a_{ij} - \lambda\delta_{ij})n_j = 0$$

and this expression is simply a homogeneous system of three linear algebraic equations in the unknowns n_1, n_2, n_3. The system possesses a nontrivial solution if and only if the determinant of its coefficient matrix vanishes, that is:

$$\det[a_{ij} - \lambda\delta_i] = 0$$

Expanding the determinant produces a cubic equation in terms of λ:

$$\det[a_{ij} - \lambda\delta_{ij}] = -\lambda^3 + I_a\lambda^2 - II_a\lambda + III_a = 0 \qquad (1.32)$$

where

$$\begin{aligned} I_a &= a_{ii} = a_{11} + a_{22} + a_{33} \\ II_a &= \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ III_a &= \det[a_{ij}] \end{aligned} \qquad (1.33)$$

The scalars I_a, II_a, and III_a are called the fundamental invariants of the tensor a_{ij}, and relation (1.32) is known as the characteristic equation. As indicated by their name, the three invariants do not change value under coordinate transformation. The roots of the characteristic equation determine the allowable values for λ, and each of these may be back-substituted into relation (1.31) to solve for the associated principal direction $\boldsymbol{n}$.

Under the condition that the components a_{ij} are real, it can be shown that all three roots $\lambda_1, \lambda_2, \lambda_3$ of the cubic equation (1.32) must be real. Furthermore, if these roots are distinct, the principal directions associated with each principal value are orthogonal. Thus, we can conclude that every symmetric second-order tensor has at least three mutually perpendicular principal directions and at most three distinct principal values that are the roots of the characteristic equation. By denoting the principal directions $\boldsymbol{n}^{(1)}, \boldsymbol{n}^{(2)}, \boldsymbol{n}^{(3)}$ corresponding to the principal values $\lambda_1, \lambda_2, \lambda_3$, three possibilities arise:

1. All three principal values distinct; thus, the three corresponding principal directions are unique (except for sense).
2. Two principal values equal ($\lambda_1 \neq \lambda_2 \neq \lambda_3$); the principal direction $\boldsymbol{n}^{(1)}$ is unique (except for sense), and every direction perpendicular to $\boldsymbol{n}^{(1)}$ is a principal direction associated with λ_2, λ_3.

3. All three principal values equal; every direction is principal, and the tensor is isotropic, as per discussion in the previous section.

Therefore, according to what we have presented, it is always possible to identify a right-handed Cartesian coordinate system such that each axis lies along the principal directions of any given symmetric second-order tensor. Such axes are called the principal axes of the tensor. For this case, the basis vectors are actually the unit principal directions $\{\boldsymbol{n}^{(1)}, \boldsymbol{n}^{(2)}, \boldsymbol{n}^{(3)}\}$, and it can be shown that with respect to principal axes the tensor reduces to the diagonal form

$$a_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.34}$$

Note that the fundamental invariants defined by relations (1.33) can be expressed in terms of the principal values as

$$\begin{aligned} I_a &= \lambda_1 + \lambda_2 + \lambda_3 \\ II_a &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ III_a &= \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{1.35}$$

The eigenvalues have important extremal properties. If we arbitrarily rank the principal values such that $\lambda_1 > \lambda_2 > \lambda_3$, then λ_1 will be the largest of all possible diagonal elements, while λ_3 will be the smallest diagonal element possible. This theory is applied in elasticity as we seek the largest stress or strain components in an elastic solid.

EXAMPLE 1-2: Principal Value Problem:

Determine the invariants and principal values and directions of the following symmetric second-order tensor:

$$a_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

The invariants follow from relations (1.33)

$$I_1 = a_{ii} = 2 + 3 - 3 = 2$$
$$II_a = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = 6 - 25 - 6 = -25$$
$$III_a = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{vmatrix} = 2(-9 - 16) = -50$$

The characteristic equation then becomes

$$\det[a_{ij} - \lambda\delta_{ij}] = -\lambda^3 + 2\lambda^2 + 25\lambda - 50 = 0$$
$$\Rightarrow (\lambda - 2)(\lambda^2 - 25) = 0$$
$$\therefore \lambda_1 = 5, \lambda_2 = 2, \lambda_3 = -5$$

Thus, for this case all principal values are distinct.

For the $\lambda_1 = 3$ root, equation (1.31) gives the system

$$-3n_1^{(1)} = 0$$
$$-2n_2^{(1)} + 4n_3^{(1)} = 0$$
$$4n_2^{(1)} - 8n_3^{(1)} = 0$$

which gives a normalized solution $\boldsymbol{n}^{(1)} = \pm(2\boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{5}$. In similar fashion, the other two principal directions are found to be $\boldsymbol{n}^{(1)} = \pm\boldsymbol{e}_1, \boldsymbol{n}^{(1)} = \pm(\boldsymbol{e}_2 - 2\boldsymbol{e}_3)/\sqrt{5}$. It is easily verified that these directions are mutually orthogonal. Figure 1-3 illustrates their directions with respect to the given coordinate system, and this establishes the right-handed principal coordinate axes (x_1', x_2', x_3'). For this case, the transformation matrix Q_{ij} defined by (1.17) becomes

$$Q_{ij} = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

Notice the eigenvectors actually form the rows of the Q-matrix.

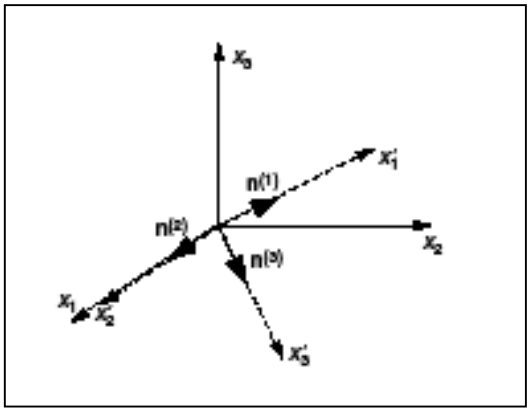


FIGURE 1-3 Principal axes for Example 1-2.

Using this in the transformation law (1.28), the components of the given second-order tensor become

$$a'_{ij} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

This result then validates the general theory given by relation (1.34) indicating that the tensor should take on diagonal form with the principal values as the elements.

Only simple second-order tensors lead to a characteristic equation that is factorable, thus allowing solution by hand calculation. Most other cases normally develop a general cubic equation and a more complicated system to solve for the principal directions.

Elasticity theory requires the use of many standard algebraic operations among vector, matrix, and tensor variables. These operations include dot and cross products of vectors and numerous matrix/tensor products. All of these operations can be expressed efficiently using compact tensor index notation. First, consider some particular vector products. Given two vectors a and b, with Cartesian components a_i and b_i, the scalar or dot product is defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i \tag{1.36}$$

Because all indices in this expression are repeated, the quantity must be a scalar, that is, a tensor of order zero. The magnitude of a vector can then be expressed as

$$|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2} = (a_i a_i)^{1/2} \tag{1.37}$$

The vector or cross product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} a_j b_k \boldsymbol{e}_i \tag{1.38}$$

where $\boldsymbol{e}_i$ are the unit basis vectors for the coordinate system. Note that the cross product gives a vector resultant whose components are $\varepsilon_{ijk} a_j b_k$. Another common vector product is the scalar triple product defined by

$$\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k \tag{1.39}$$

Next consider some common matrix products. Using the usual direct notation for matrices and vectors, common products between a matrix $\boldsymbol{A} = [\boldsymbol{A}]$ with a vector a can be written as

$$\begin{aligned} \boldsymbol{A}\boldsymbol{a} &= [\boldsymbol{A}]\{\boldsymbol{a}\} = A_{ij} a_j = a_j A_{ij} \\ \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{A} &= \{\boldsymbol{a}\}^T [\boldsymbol{A}] = a_i A_{ij} = A_{ij} a_i \end{aligned} \tag{1.40}$$

where $\boldsymbol{a}^{\boldsymbol{T}}$ denotes the transpose, and for a vector quantity this simply changes the column matrix (3×1) into a row matrix (1×3). Note that each of these products results in a vector resultant. These types of expressions generally involve various inner products within the index notational scheme, and as noted, once the summation index is properly specified, the order of listing the product terms does not change the result. We will encounter several different combinations of products between two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{B} &= [\boldsymbol{A}][\boldsymbol{B}] = A_{ij} B_{jk} \\ \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{T}} &= A_{ij} B_{kj} \\ \boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{B} &= A_{ji} B_{jk} \\ tr(\boldsymbol{A}\boldsymbol{B}) &= A_{ij} B_{ji} \\ tr(\boldsymbol{A}\boldsymbol{B}^{\boldsymbol{T}}) &= tr(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{B}) = A_{ij} B_{ij} \end{aligned} \tag{1.41}$$

where $\boldsymbol{A}^{\boldsymbol{T}}$ indicates the transpose and $tr\boldsymbol{A}$ is the trace of the matrix defined by

$$A_{ij}^T = A_{ji}$$
$$tr\,\boldsymbol{A} = A_{ii} = A_{11} + A_{22} + A_{33} \tag{1.42}$$

Similar to vector products, once the summation index is properly specified, the results in (1.41) do not depend on the order of listing the product terms. Note that this does not imply that $\boldsymbol{AB} = \boldsymbol{BA}$, which is certainly not true.

Calculus of Cartesian Tensors:

Most variables within elasticity theory are field variables, that is, functions depending on the spatial coordinates used to formulate the problem under study. For time-dependent problems, these variables could also have temporal variation. Thus, our scalar, vector, matrix, and general tensor variables are functions of the spatial coordinates (x_1, x_2, x_3). Because many elasticity equations involve differential and integral operations, it is necessary to have an understanding of the calculus of Cartesian tensor fields. Further information on vector differential and integral calculus can be found in Hildebrand [4] and Kreyszig [5].

The field concept for tensor components can be expressed as

$$\begin{aligned} a &= a(x_1, x_2, x_3) = a(x_i) = a(x) \\ a_i &= a_i(x_1, x_2, x_3) = a_i(x_i) = a(x) \\ a_{ij} &= a_{ij}(x_1, x_2, x_3) = a_{ij}(x_i) = a_{ij}(x) \\ &\vdots \end{aligned}$$

It is convenient to introduce the comma notation for partial differentiation:

$$a_{,i} = \frac{\partial}{\partial x_i} a, a_{i,j} = \frac{\partial}{\partial x_j} a_i, a_{ij,k} = \frac{\partial}{\partial x_k} a_{ij,} \ \ldots$$

It can be shown that if the differentiation index is distinct, the order of the tensor is increased by one. For example, the derivative operation on a vector $a_{i,j}$ produces a second-order tensor or matrix given by

$$a_{i,j} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}$$

Using Cartesian coordinates (x, y, z), consider the directional derivative of a scalar field function f with respect to a direction s:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

Note that the unit vector in the direction of s can be written as

$$\boldsymbol{n} = \frac{dx}{ds}\boldsymbol{e}_1 + \frac{dy}{ds}\boldsymbol{e}_2 + \frac{dz}{ds}\boldsymbol{e}_3$$

Therefore, the directional derivative can be expressed as the following scalar product:

$$\frac{df}{ds} = \boldsymbol{n} \cdot \boldsymbol{\nabla} f \tag{1.43}$$

where $\boldsymbol{\nabla} f$ is called the gradient of the scalar function f and is defined by

$$\boldsymbol{\nabla} f = \boldsymbol{grad}\ f = \boldsymbol{e}_1 \frac{\partial f}{\partial x} + \boldsymbol{e}_2 \frac{\partial f}{\partial y} + \boldsymbol{e}_3 \frac{\partial f}{\partial z} \tag{1.44}$$

and the symbolic vector operator $\boldsymbol{\nabla}$ is called the del operator

$$\boldsymbol{\nabla} = \boldsymbol{e}_1 \frac{\partial}{\partial x} + \boldsymbol{e}_2 \frac{\partial}{\partial y} + \boldsymbol{e}_3 \frac{\partial}{\partial z} \tag{1.45}$$

These and other useful operations can be expressed in Cartesian tensor notation. Given the scalar field ϕ and vector field $\boldsymbol{u}$, the following common differential operations can be written in index notation:

$$
\begin{aligned}
Gradient\ of\ a\ Scalar\ \boldsymbol{\nabla}\phi &= \phi_i \boldsymbol{e}_1 \\
Gradient\ of\ a\ Vector\ \boldsymbol{\nabla}\boldsymbol{u} &= u_{i,j}\boldsymbol{e}_i\boldsymbol{e}_j \\
Laplacian\ of\ a\ Scalar\ \boldsymbol{\nabla}^2\phi &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \phi_{,ii} \\
Divergence\ of\ a\ Vector\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= u_{i,i} \\
Curl\ of\ a\ Vector\ \boldsymbol{\nabla}\times\boldsymbol{u} &= \varepsilon_{ijk}u_k\boldsymbol{e}_i \\
Laplacian\ of\ a\ Vector\ \boldsymbol{\nabla}^2\boldsymbol{u} &= u_{i,kk}\boldsymbol{e}_i
\end{aligned}
\tag{1.46}
$$

If ϕ and ψ are scalar fields and $\boldsymbol{u}$ and $\boldsymbol{v}$ are vector fields, several useful identities exist:

$$
\begin{aligned}
&\boldsymbol{\nabla}(\phi\psi) = (\boldsymbol{\nabla}\phi)\psi + \phi(\boldsymbol{\nabla}\psi) \\
&\boldsymbol{\nabla}^2(\phi\psi) = (\boldsymbol{\nabla}^2\phi)\psi + \phi(\boldsymbol{\nabla}^2\psi) + 2\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi \\
&\boldsymbol{\nabla}\cdot(\phi\boldsymbol{u}) = \boldsymbol{\nabla}\phi\cdot\boldsymbol{u} + \phi(\boldsymbol{\nabla}\cdot\boldsymbol{u}) \\
&\boldsymbol{\nabla}\times(\phi\boldsymbol{u}) = \boldsymbol{\nabla}\phi\times\boldsymbol{u} + \phi(\boldsymbol{\nabla}\times\boldsymbol{u}) \\
&\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) \\
&\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = 0 \\
&\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}^2\phi \\
&\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times u = 0 \\
&\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}^2\boldsymbol{u} \\
&\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}
\end{aligned}
\tag{1.47}
$$

Each of these identities can be easily justified by using index notation from definition relations (1.46).

Next consider some results from vector/tensor integral calculus. We simply list some theorems that have later use in the development of elasticity theory.

Divergence or Gauss Theorem:

Let S be a piecewise continuous surface bounding the region of space V. If a vector field $\boldsymbol{u}$ is continuous and has continuous first derivatives in V, then

$$
\iint_S \boldsymbol{u}\cdot\boldsymbol{n}\, ds = \iiint_V \nabla\cdot\boldsymbol{u}\, dV \tag{1.48}
$$

where **n** is the outer unit normal vector to surface S. This result is also true for tensors of any order, that is:

$$\iint_S a_{ij\ldots k} n_k dS = \iiint_V a_{ij\ldots k,k}\, dV \tag{1.49}$$

Stokes Theorem:

Let S be an open two-sided surface bounded by a piecewise continuous simple closed curve C. If $\boldsymbol{u}$ is continuous and has continuous first derivatives on S, then

$$\oint_C \boldsymbol{u} \cdot \boldsymbol{dr} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n}\, dS \tag{1.50}$$

where the positive sense for the line integral is for the region S to lie to the left as one traverses curve C and n is the unit normal vector to S. Again, this result is also valid for tensors of arbitrary order, and so

$$\oint_C a_{ij\ldots k} dx_t = \iint_S \varepsilon_{rst} a_{ij\ldots k,s} n_r\, dS \tag{1.51}$$

It can be shown that both divergence and Stokes theorems can be generalized so that the dot product in (1.48) and/or (1.50) can be replaced with a cross product.

Green's Theorem in the Plane:

Applying Stokes theorem to a planar domain S with the vector field selected as $u = f\boldsymbol{e}_1 + \mathrm{g}\boldsymbol{e}_2$ gives the result

$$\iint_S \left(\frac{\partial \mathrm{g}}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \int_C (f dx + \mathrm{g} dy) \tag{1.52}$$

Further, special choices with either $f = 0$ or $\mathrm{g} = 0$ imply

$$\iint_S \frac{\partial \mathrm{g}}{\partial x} dxdy = \int_C \mathrm{g} n_x ds\,, \qquad \iint_S \frac{\partial f}{\partial y} dxdy = \int_C f n_y ds \tag{1.53}$$

Zero-Value Theorem:

Let $f_{ij\ldots k}$ be a continuous tensor field of any order defined in an arbitrary region V. If the integral of $f_{ij\ldots k}$...k over V vanishes, then $f_{ij\ldots k}$ must vanish in V, that is:

$$\iiint_V f_{ij\ldots k} dV = 0 \Rightarrow f_{ij\ldots k} = 0 \in V \tag{1.54}$$

Section (1.4):Orthogonal Curvilinear Coordinates:

Many applications in elasticity theory involve domains that have curved boundary surfaces, commonly including circular, cylindrical, and spherical surfaces. To formulate and develop solutions for such problems, it is necessary to use curvilinear coordinate systems. This requires redevelopment of some previous results in orthogonal curvilinear coordinates. Before pursuing these general steps, we review the two most common curvilinear systems, cylindrical and spherical coordinates. The cylindrical coordinate system shown in Figure 1-4 uses (r, θ, z)

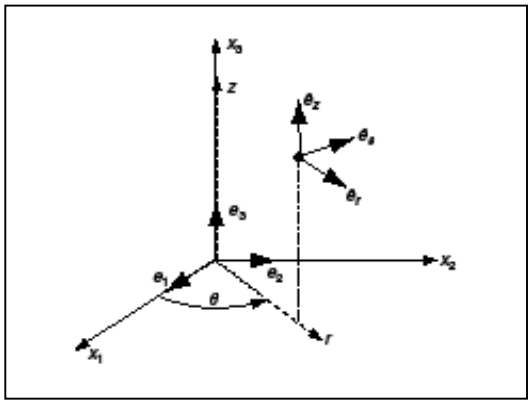


FIGURE 1-4 Cylindrical coordinate system.

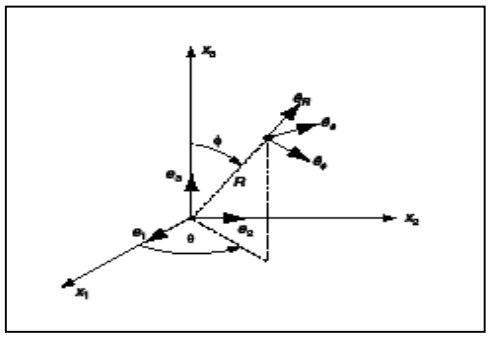


FIGURE 1-5 Spherical coordinate system.

coordinates to describe spatial geometry. Relations between the Cartesian and cylindrical systems are given by

$$x_1 = r\cos\theta\,, x_2 = \sin\theta, x_3 = z$$
$$r = \sqrt{x_1^2 + x_2^2}, \theta = \tan^{-1}\frac{x_2}{x_1}, z = x_3 \tag{1.55}$$

The spherical coordinate system is shown in Figure 1-5 and uses (R, ϕ, θ) coordinates to describe geometry. The relations between Cartesian and spherical coordinates are

$$x_1 = R\cos\theta\sin\phi, x_2 = \sin\phi, x_3 = R\cos\phi$$
$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \phi = \cos^{-1}\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \theta = \tan^{-1}\frac{x_2}{x_1} \tag{1.56}$$

The unit basis vectors for each of these curvilinear systems are illustrated in Figures 1-4 and 1-5. These represent unit tangent vectors along each of the three orthogonal coordinate curves.

Although primary use of curvilinear systems employs cylindrical and spherical coordinates, we briefly present a general discussion valid for arbitrary coordinate systems. Consider the general case in which three orthogonal curvilinear coordinates are denoted by ξ^1, ξ^2, ξ^3, while the Cartesian coordinates are defined by x^1, x^2, x^3 (see Figure 1-6). We assume there exist invertible coordinate transformations between these systems specified by

$$\xi^m = \xi^m(x^1, x^2, x^3), \;\; x^m = x^m(\xi^1, \xi^2, \xi^3) \tag{1.57}$$

In the curvilinear system, an arbitrary differential length in space can be expressed by

$$(ds)^2 = (h_1 d\xi^1)^2 + (h_2 d\xi^2)^2 + (h_3 d\xi^3)^2 \tag{1.58}$$

where h_1, h_2, h_3 are called scale factors that are in general nonnegative functions of position. Let $\boldsymbol{e}_k$ be the fixed Cartesian basis vectors and $\hat{\boldsymbol{e}}_k$ the curvilinear basis (see Figure 1-6). By

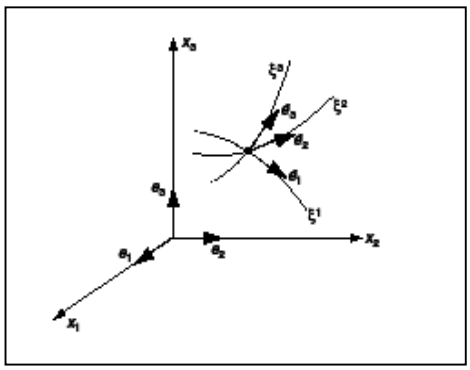


FIGURE 1-6 Curvilinear coordinates.

using similar concepts from the transformations discussed , the curvilinear basis can be expressed in terms of the Cartesian basis as

$$\begin{aligned}
\hat{\boldsymbol{e}}_1 &= \frac{dx^k}{ds_1}\boldsymbol{e}_k = \frac{1}{h_1}\frac{\partial x_k}{\partial \xi^1}\boldsymbol{e}_k \\
\hat{\boldsymbol{e}}_2 &= \frac{dx^k}{ds_2}\boldsymbol{e}_k = \frac{1}{h_2}\frac{\partial x_k}{\partial \xi^2}\boldsymbol{e}_k \\
\hat{\boldsymbol{e}}_3 &= \frac{dx^k}{ds_3}\boldsymbol{e}_k = \frac{1}{h_3}\frac{\partial x_k}{\partial \xi^3}\boldsymbol{e}_k
\end{aligned} \tag{1.59}$$

where we have used (1.58). By using the fact that $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$, relation (1.59) gives

$$\begin{aligned}
(h_1)^2 &= \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} \\
(h_2)^2 &= \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} \\
(h_3)^2 &= \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3}
\end{aligned} \tag{1.60}$$

It follows from (1.59) that the quantity

$$Q_r^k = \frac{1}{h_r}\frac{\partial x_k}{\partial \xi^r}, \quad \text{(no sum on } r\text{)} \tag{1.61}$$

represents the transformation tensor giving the curvilinear basis in terms of the Cartesian basis. This concept is similar to the transformation tensor Q_{ij} defined by (1.17) that is used between Cartesian systems.

The physical components of a vector or tensor are simply the components in a local set of Cartesian axes tangent to the curvilinear coordinate curves at any point in space. Thus, by using transformation relation (1.61), the physical components of a tensor $\boldsymbol{a}$ in a general curvilinear system are given by

$$a_{\langle ij\ldots k\rangle} = Q_i^p Q_j^q \ldots Q_k^s a_{pq\ldots s} \tag{1.62}$$

where $a_{pq\ldots s}$ are the components in a fixed Cartesian frame. Note that the tensor can be expressed in either system as

$$\begin{aligned} a &= a_{ij\ldots k}\boldsymbol{e_i}\boldsymbol{e_j}\ldots\boldsymbol{e_k} \\ &= a_{\langle ij\ldots k\rangle}\hat{\boldsymbol{e}}_{\boldsymbol{i}}\hat{\boldsymbol{e}}_{\boldsymbol{j}}\ldots\hat{\boldsymbol{e}}_{\boldsymbol{k}} \end{aligned} \tag{1.63}$$

Because many applications involve differentiation of tensors, we must consider the differentiation of the curvilinear basis vectors. The Cartesian basis system $\boldsymbol{e}_k$ is fixed in orientation and therefore $\partial\boldsymbol{e_k}/\partial x^j = \partial\boldsymbol{e_k}/\partial\xi^j = 0$. However, derivatives of the curvilinear basis do not in general vanish, and differentiation of relations (1.59) gives the following results:

$$\begin{aligned} \frac{\partial\hat{\boldsymbol{e}}_m}{\partial\xi^m} &= -\frac{1}{h_n}\frac{\partial h_m}{\partial\xi^n}\hat{\boldsymbol{e}}_n - \frac{1}{h_r}\frac{\partial h_m}{\partial\xi^r}\hat{\boldsymbol{e}}_r; \quad m \neq n \neq r \\ \frac{\partial\hat{\boldsymbol{e}}_m}{\partial\xi^n} &= -\frac{1}{h_m}\frac{\partial h_n}{\partial\xi^m}\hat{\boldsymbol{e}}_n; \quad m \neq n, \ \text{no sum on repeated indices} \end{aligned} \tag{1.64}$$

Using these results, the derivative of any tensor can be evaluated. Consider the first derivative of a vector $\boldsymbol{u}$:

$$\frac{\partial}{\partial\xi^n}\boldsymbol{u} = \frac{\partial}{\partial\xi^n}\left(u_{\langle m\rangle}\hat{\boldsymbol{e}}_{\boldsymbol{m}}\right) = \frac{\partial u_{\langle m\rangle}}{\partial\xi^n}\hat{\boldsymbol{e}}_{\boldsymbol{m}} + u_{\langle m\rangle}\frac{\partial\hat{\boldsymbol{e}}_{\boldsymbol{m}}}{\partial\xi^n} \tag{1.65}$$

The last term can be evaluated using (1.64), and thus the derivative of u can be expressed in terms of curvilinear components. Similar patterns follow for derivatives of higher-order tensors.

All vector differential operators of gradient, divergence, curl, and so forth can be expressed in any general curvilinear system by using these techniques. For example, the vector differential operator previously defined in Cartesian coordinates in (1.45) is given by

$$\nabla = \hat{\boldsymbol{e}}_{\mathbf{1}} \frac{1}{h_1} \frac{\partial}{\partial \xi^1} + \hat{\boldsymbol{e}}_{\mathbf{2}} \frac{1}{h_2} \frac{\partial}{\partial \xi^2} + \hat{\boldsymbol{e}}_{\mathbf{3}} \frac{1}{h_3} \frac{\partial}{\partial \xi^3} = \sum_i \hat{\boldsymbol{e}}_{\boldsymbol{i}} \frac{1}{h_i} \frac{\partial}{\partial \xi^i} \tag{1.66}$$

and this leads to the construction of the other common forms:

$$\begin{aligned} Gradient\ of\ a\ Scalar\ \ \nabla f &= \hat{\boldsymbol{e}}_{\mathbf{1}} \frac{1}{h_1} \frac{\partial f}{\partial \xi^1} + \hat{\boldsymbol{e}}_{\mathbf{2}} \frac{1}{h_2} \frac{\partial f}{\partial \xi^2} + \hat{\boldsymbol{e}}_{\mathbf{3}} \frac{1}{h_3} \frac{\partial f}{\partial \xi^3} \\ &= \sum_i \hat{\boldsymbol{e}}_{\boldsymbol{i}} \frac{1}{h_i} \frac{\partial f}{\partial \xi^i} \end{aligned} \tag{1.67}$$

$$Divergence\ of\ a\ Vector\ \boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial \xi^i} \left(\frac{h_1 h_2 h_3}{h_i} u_{\langle i \rangle} \right) \tag{1.68}$$

$$Laplacian\ of\ a\ Scalar\ \nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial \xi^i} \left(\frac{h_1 h_2 h_3}{(h_i)^2} \frac{\partial \phi}{\partial \xi^i} \right) \tag{1.69}$$

$$Curl\ of\ a\ Vector\ \nabla \times \boldsymbol{u} = \sum_i \sum_j \sum_k \frac{\varepsilon_{ijk}}{h_j h_k} \frac{\partial}{\partial \xi^i} \left(u_{\langle k \rangle} h_k \right) \hat{\boldsymbol{e}}_{\boldsymbol{i}} \tag{1.70}$$

$$Gradient\ of\ a\ Vector\ \nabla \boldsymbol{u} = \sum_i \sum_j \frac{\hat{\boldsymbol{e}}_{\boldsymbol{i}}}{h_i} \left(\frac{\partial u_{\langle j \rangle}}{\partial \xi^i} \hat{\boldsymbol{e}}_{\boldsymbol{j}} + u_{\langle j \rangle} \frac{\partial \hat{\boldsymbol{e}}_{\boldsymbol{j}}}{\partial \xi^i} \right) \tag{1.71}$$

$$\begin{aligned} &Laplacian\ of\ a\ Vector\ \nabla^2 \boldsymbol{u} \\ &= \left(\sum_i \frac{\hat{\boldsymbol{e}}_{\boldsymbol{i}}}{h_i} \frac{\partial}{\partial \xi^i} \right) \cdot \left(\sum_j \sum_k \frac{\hat{\boldsymbol{e}}_{\boldsymbol{k}}}{h_k} \left[\frac{\partial u_{\langle j \rangle}}{\partial \xi^k} \hat{\boldsymbol{e}}_{\boldsymbol{j}} + u_{\langle j \rangle} \frac{\partial \hat{\boldsymbol{e}}_{\boldsymbol{j}}}{\partial \xi^k} \right] \right) \end{aligned} \tag{1.72}$$

It should be noted that these forms are significantly different from those previously given in relations (1.46) for Cartesian coordinates. Curvilinear systems add additional terms not found in rectangular coordinates. Other operations on higher-order tensors can be developed in a similar fashion (see Malvern [6], app. II).

EXAMPLE 1-3: Polar Coordinates:

Consider the two-dimensional case of a polar coordinate system as shown in Figure 1-7. The differential length relation (1.58) for this case can be written as

$$(ds)^2 = (dr)^2 + (rd\theta)^2$$

and thus $h_1 = 1$ and $h_2 = r$. By using relations (1.59) or simply by using the geometry shown in Figure 1-7,

$$\begin{aligned} \hat{\boldsymbol{e}}_{\boldsymbol{r}} &= \cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2 \\ \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} &= -\sin\theta \boldsymbol{e}_1 + \cos\theta \boldsymbol{e}_2 \end{aligned} \tag{1.73}$$

and so

$$\frac{\partial \hat{\boldsymbol{e}}_{\boldsymbol{r}}}{\partial \theta} = \hat{\boldsymbol{e}}_{\boldsymbol{\theta}}, \frac{\partial \hat{\boldsymbol{e}}_{\boldsymbol{m}}}{\partial \theta} = -\hat{\boldsymbol{e}}_{\boldsymbol{r}}, \qquad \frac{\partial \hat{\boldsymbol{e}}_{\boldsymbol{r}}}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_{\boldsymbol{\theta}}}{\partial r} = 0 \tag{1.74}$$

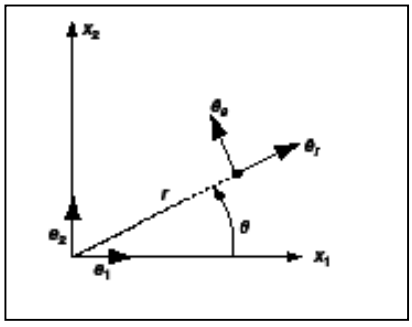


FIGURE 1-7 Polar coordinate system.

The basic vector differential operations then follow to be

$$
\begin{aligned}
\nabla &= \hat{\boldsymbol{e}}_{\boldsymbol{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_{\boldsymbol{r}} \frac{1}{r} \frac{\partial}{\partial \theta} \\
\nabla \phi &= \hat{\boldsymbol{e}}_{\boldsymbol{r}} \frac{\partial \phi}{\partial r} + \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial \phi}{\partial \theta} \\
\nabla \cdot u &= \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\
\nabla^2 \phi &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\
\nabla \times \boldsymbol{u} &= \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
\nabla \boldsymbol{u} &= \frac{\partial u_r}{\partial r} \hat{\boldsymbol{e}}_{\boldsymbol{r}} \hat{\boldsymbol{e}}_{\boldsymbol{r}} + \frac{\partial u_\theta}{\partial r} \hat{\boldsymbol{e}}_{\boldsymbol{r}} \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \hat{\boldsymbol{e}}_{\boldsymbol{r}} + \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} - u_r \right) \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \\
\nabla^2 \boldsymbol{u} &= \left(\nabla^2 u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right) \hat{\boldsymbol{e}}_{\boldsymbol{r}} + \left(\nabla^2 u_\theta - \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \hat{\boldsymbol{e}}_{\boldsymbol{\theta}}
\end{aligned}
\tag{1.75}
$$

where $u = u_r \hat{\boldsymbol{e}}_{\boldsymbol{r}} + u_\theta \hat{\boldsymbol{e}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{e}}_{\boldsymbol{z}} = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$. Notice that the Laplacian of a vector does not simply pass through and operate on each of the individual components as in the Cartesian case. Additional terms are generated because of the curvature of the particular coordinate system. Similar relations can be developed for cylindrical and spherical coordinate systems .

Chapter (2)

Deformation: Displacements and Strains

We begin development of the basic field equations of elasticity theory by first investigating the kinematics of material deformation. As a result of applied loadings, elastic solids will change shape or deform, and these deformations can be quantified by knowing the displacements of material points in the body. The continuum hypothesis establishes a displacement field at all points within the elastic solid. Using appropriate geometry, particular measures of deformation can be constructed leading to the development of the strain tensor. As expected, the strain components are related to the displacement field. The purpose of this chapter is to introduce the basic definitions of displacement and strain, establish relations between these two field quantities, and finally investigate requirements to ensure single-valued, continuous displacement fields. As appropriate for linear elasticity, these kinematical results are developed under the conditions of small deformation theory. Developments in this chapter lead to two fundamental sets of field equations: the strain-displacement relations and the compatibility equations. Further field equation development, including internal force and stress distribution, equilibrium and elastic constitutive behavior, occurs in subsequent chapters.

Section (2.1):General Deformations:

Under the application of external loading, elastic solids deform. A simple two-dimensional cantilever beam example is shown in Figure (2-1). The undeformed configuration is taken with the rectangular beam in the vertical position, and the end loading displaces material points to the deformed shape as shown. As is typical in most problems, the deformation varies from point to point and is thus said to be nonhomogenous. A superimposed square mesh is shown in the two configurations, and this indicates how elements within the material deform locally. It is apparent that elements within the mesh undergo extensional and shearing deformation. An elastic solid is said to be deformed or strained when the relative displacements between points in the body are changed. This is in contrast to rigid-body motion where the distance between points remains the same.

In order to quantify deformation, consider the general example shown in Figure

(2-2). In the undeformed configuration, we identify two neighboring material points P_o and P connected with the relative position vector $\boldsymbol{r}$ as shown. Through a general deformation, these points are mapped to locations P_o' and P' in the deformed configuration. For finite or large deformation theory, the

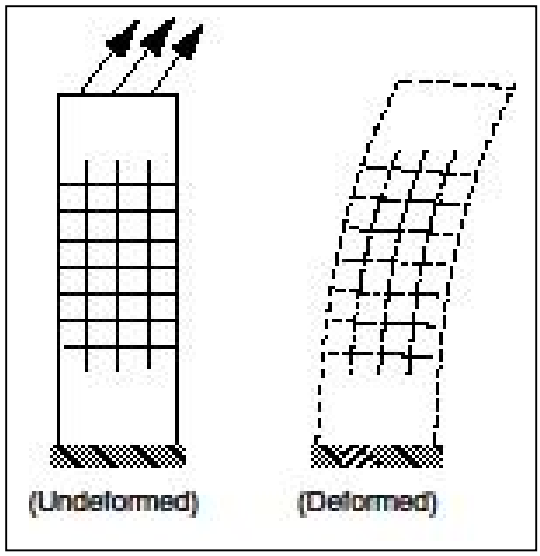


Figure (2-1) Two-dimensional deformation example.

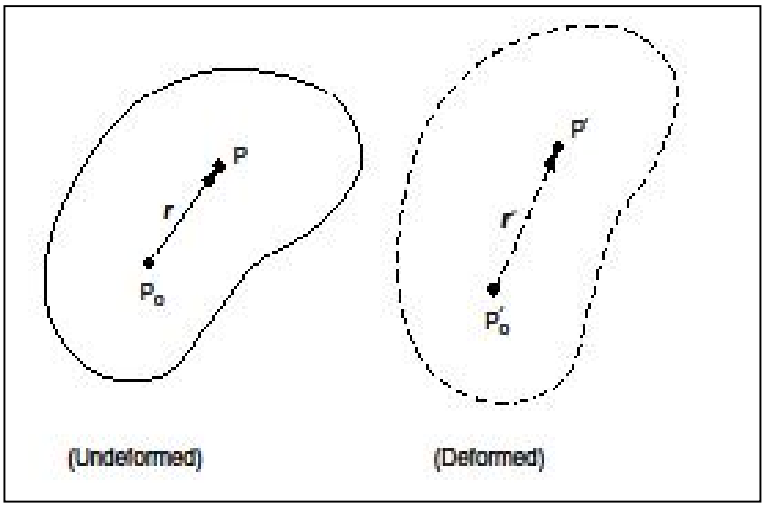


Figure (2-2) General deformation between two neighboring points.

undeformed and deformed configurations can be significantly different, and a distinction between these two configurations must be maintained leading to Lagrangian and Eulerian descriptions; see, for example, Malvern [6] or Chandrasekharaiah and Debnath [1]. However, since we are developing linear

elasticity, which uses only small deformation theory, the distinction between undeformed and deformed configurations can be dropped.

Using Cartesian coordinates, define the displacement vectors of points P_o and P to be $\boldsymbol{u^o}$ and u, respectively. Since P and P_o are neighboring points, we can use a Taylor series expansion around point P_o to express the components of $\boldsymbol{u}$ as

$$\begin{aligned} \boldsymbol{u} &= u^o + \frac{\partial \boldsymbol{u}}{\partial x}\boldsymbol{r}_x + \frac{\partial \boldsymbol{u}}{\partial y}\boldsymbol{r}_y + \frac{\partial \boldsymbol{u}}{\partial z}\boldsymbol{r}_z \\ \boldsymbol{v} &= v^o + \frac{\partial \boldsymbol{v}}{\partial x}\boldsymbol{r}_x + \frac{\partial \boldsymbol{v}}{\partial y}\boldsymbol{r}_y + \frac{\partial \boldsymbol{v}}{\partial z}\boldsymbol{r}_z \\ \boldsymbol{w} &= w^o + \frac{\partial \boldsymbol{w}}{\partial x}\boldsymbol{r}_x + \frac{\partial \boldsymbol{w}}{\partial y}\boldsymbol{r}_y + \frac{\partial \boldsymbol{w}}{\partial z}\boldsymbol{r}_z \end{aligned} \tag{2.1}$$

Note that the higher-order terms of the expansion have been dropped since the components of $\boldsymbol{r}$ are small. The change in the relative position vector $\boldsymbol{r}$ can be written as

$$\Delta\boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{u} - \boldsymbol{u}^o \tag{2.2}$$

and using (2.1) gives

$$\begin{aligned} \Delta\boldsymbol{r}_x &= \frac{\partial \boldsymbol{u}}{\partial x}\boldsymbol{r}_x + \frac{\partial \boldsymbol{u}}{\partial y}\boldsymbol{r}_y + \frac{\partial \boldsymbol{u}}{\partial z}\boldsymbol{r}_z \\ \Delta\boldsymbol{r}_y &= \frac{\partial \boldsymbol{v}}{\partial x}\boldsymbol{r}_x + \frac{\partial \boldsymbol{v}}{\partial y}\boldsymbol{r}_y + \frac{\partial \boldsymbol{v}}{\partial z}\boldsymbol{r}_z \\ \Delta\boldsymbol{r}_z &= \frac{\partial \boldsymbol{w}}{\partial x}\boldsymbol{r}_x + \frac{\partial \boldsymbol{w}}{\partial y}\boldsymbol{r}_y + \frac{\partial \boldsymbol{w}}{\partial z}\boldsymbol{r}_z \end{aligned} \tag{2.3}$$

or in index notation

$$\Delta\boldsymbol{r}_i = \boldsymbol{u}_{i,j}\boldsymbol{r}_j \tag{2.4}$$

The tensor $u_{i,j}$ is called the displacement gradient tensor, and may be written out as

$$u_{i,j} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{2.5}$$

From relation (1.10), this tensor can be decomposed into symmetric and antisymmetric parts as

$$u_{i,j} = e_{ij} + \omega_{ij} \tag{2.6}$$

where

$$\begin{aligned} e_{ij} &= \frac{1}{2}\left(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}\right) \\ \omega_{ij} &= \frac{1}{2}\left(\boldsymbol{u}_{i,j} - \boldsymbol{u}_{j,i}\right) \end{aligned} \tag{2.7}$$

The tensor e_{ij} is called the strain tensor, while ω_{ij} is referred to as the rotation tensor. Relations (2.4) and (2.6) thus imply that for small deformation theory, the change in the relative position vector between neighboring points can be expressed in terms of a sum of strain and rotation components. Combining relations (2.2), (2.4), and (2.6), and choosing $r_i = dx_i$, we can also write the general result in the form

$$\boldsymbol{u}_i = \boldsymbol{u}_i^o + \boldsymbol{e}_{ij}\boldsymbol{dx}_j + \omega_{ij}\boldsymbol{dx}_j \tag{2.8}$$

Because we are considering a general displacement field, these results include both strain deformation and rigid-body motion. A dual vector ω_i can be associated with the rotation tensor such that $\omega_i = -1/2\, \varepsilon_{ijk}\omega_{jk}$. Using this definition, it is found that

$$
\begin{aligned}
\omega_1 = \omega_{32} &= \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_3}{\partial x_2} - \frac{\partial \boldsymbol{u}_2}{\partial x_3}\right) \\
\omega_2 = \omega_{13} &= \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_1}{\partial x_3} - \frac{\partial \boldsymbol{u}_3}{\partial x_1}\right) \\
\omega_3 = \omega_{21} &= \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_2}{\partial x_1} - \frac{\partial \boldsymbol{u}_1}{\partial x_2}\right)
\end{aligned}
\tag{2.9}
$$

which can be expressed collectively in vector format as $\boldsymbol{\omega} = (1/2)(\nabla \times \boldsymbol{u})$. As is shown in the next section, these components represent rigid-body rotation of material elements about the coordinate axes. These general results indicate that the strain deformation is related to the strain tensor e_{ij}, which in turn is a related to the displacement gradients. We next pursue a more geometric approach and determine specific connections between the strain tensor components and geometric deformation of material elements.

Although the previous section developed general relations for small deformation theory, we now wish to establish a more geometrical interpretation of these results. Typically, elasticity variables and equations are field quantities defined at each point in the material continuum. However, particular field equations are often developed by first investigating the behavior of infinitesimal elements (with coordinate boundaries), and then a limiting process is invoked that allows the element to shrink to a point. Thus, consider the common deformational behavior of a rectangular element as shown in Figure (2-3). The usual types of motion include rigid-body rotation and extensional and shearing deformations as illustrated. Rigid-body motion does not contribute to the strain field, and thus also does not affect the stresses. We therefore focus our study primarily on the extensional and shearing deformation.

Figure (2-4) illustrates the two-dimensional deformation of a rectangular element with original dimensions dx by dy. After deformation, the element takes a rhombus form as shown in the dotted outline. The displacements of various corner reference points are indicated

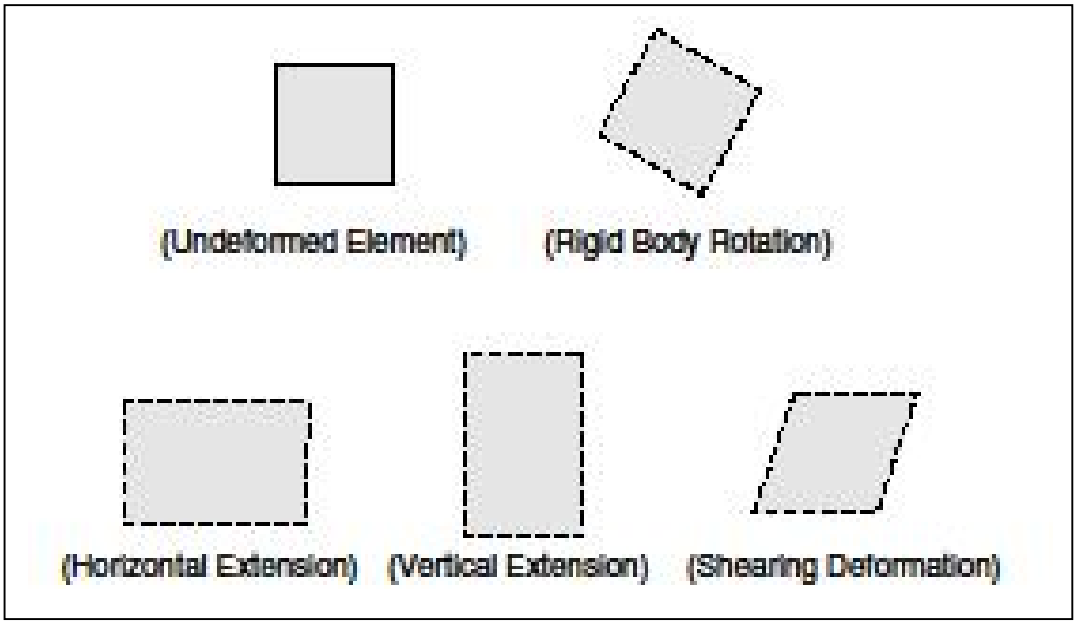


Figure (2-3) Typical deformations of a rectangular element.

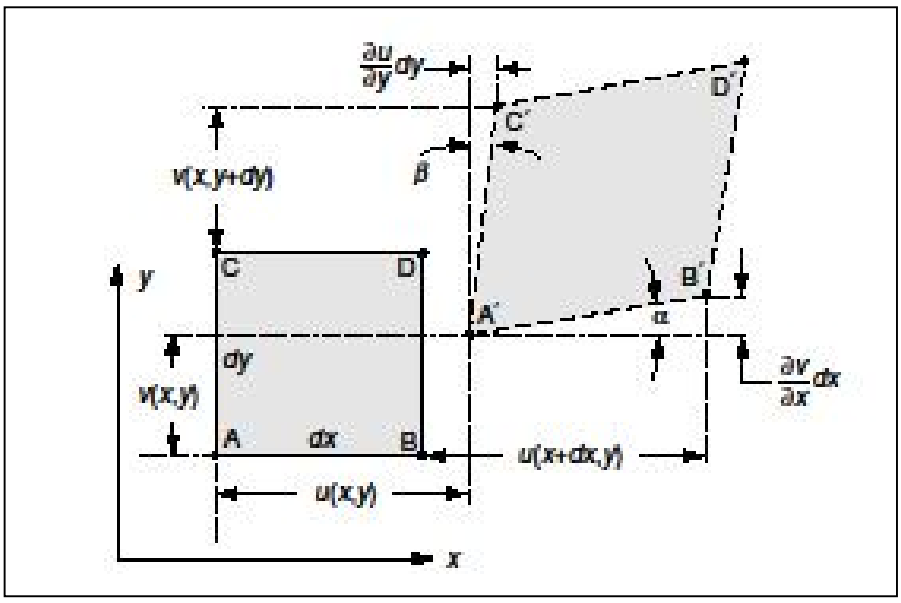


Figure (2-4) Two-dimensional geometric strain deformation.

in the figure. Reference point A is taken at location (x, y), and the displacement components of this point are thus $u(x, y)$ and $v(x, y)$. The corresponding displacements of point B are $u(x + dx, y)$ and $v(x + dx, y)$, and the displacements of the other corner points are defined in an analogous manner.

According to small deformation theory, $u(x+dx,y) \approx u(x,y) + (\partial u/\partial x)dx$, with similar expansions for all other terms.

The normal or extensional strain component in a direction n is defined as the change in length per unit length of fibers oriented in the n-direction. Normal strain is positive if fibers increase in length and negative if the fiber is shortened. In Figure (2-4), the normal strain in the x direction can thus be defined by

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

From the geometry in Figure (2-4),

$$A'B' = \sqrt{\left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\, dx$$
$$\approx \left(1 + \frac{\partial u}{\partial x}\right) dx$$

where, consistent with small deformation theory, we have dropped the higher-order terms. Using these results and the fact that $AB = dx$, the normal strain in the x-direction reduces to

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{2.10}$$

In similar fashion, the normal strain in the y-direction becomes

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{2.11}$$

A second type of strain is shearing deformation, which involves angles changes (see Figure (2-3)). Shear strain is defined as the change in angle between two originally orthogonal directions in the continuum material. This definition is actually referred to as the engineering shear strain. Theory of elasticity applications generally use a tensor formalism that requires a shear strain definition corresponding to one-half the angle change between orthogonal axes; see previous relation (2:1:7)1. Measured in radians, shear strain is positive if the right angle between the positive directions of the two axes decreases. Thus, the sign of the shear strain depends on the coordinate

system. In Figure (2-4), the engineering shear strain with respect to the x- and y-directions can be defined as

$$\gamma_{xy} = \frac{\pi}{2} - \angle C'A'B' = \alpha + \beta$$

For small deformations, $\alpha \approx \tan\alpha$ tan a and $\beta \approx \tan\beta$, and the shear strain can then be expressed as

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} + \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.12}$$

where we have again neglected higher-order terms in the displacement gradients. Note that each derivative term is positive if lines AB and AC rotate inward as shown in the figure. By simple interchange of x and y and u and v, it is apparent that $\gamma_{xy} = \gamma_{yx}$.

By considering similar behaviors in the y-z and x-z planes, these results can be easily extended to the general three-dimensional case, giving the results:

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned} \tag{2.13}$$

Thus, we define three normal and three shearing strain components leading to a total of six independent components that completely describe small deformation theory. This set of equations is normally referred to as the strain-displacement relations. However, these results are written in terms of the engineering strain components, and tensorial elasticity theory prefers to use the strain tensor e_{ij} defined by (2:1:7)1. This represents only a minor change because the normal strains are identical and shearing strains differ by a factor of one-half; for example, $e_{11} = e_x = \varepsilon_x$, and $e_{12} = e_{xy} = 1/2\,\gamma_{xy}$ and so forth.

Therefore, using the strain tensor e_{ij}, the strain-displacement relations can be expressed in component form as

$$e_x = \frac{\partial u}{\partial x}, \qquad e_y = \frac{\partial v}{\partial y}, \qquad e_z = \frac{\partial w}{\partial z}$$
$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}\right) \tag{2.14}$$

Using the more compact tensor notation, these relations are written as

$$e_{ij} = \frac{1}{2}\left(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}\right) \tag{2.15}$$

while in direct vector/matrix notation as the form reads:

$$\boldsymbol{e} = \frac{1}{2}[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T] \tag{2.16}$$

where $\boldsymbol{e}$ is the strain matrix and $\nabla\boldsymbol{u}$ is the displacement gradient matrix and $(\nabla\boldsymbol{u})^T$ is its transpose.

The strain is a symmetric second-order tensor $\left(e_{ij} = e_{ji}\right)$ and is commonly written in matrix format:

$$\boldsymbol{e} = [\boldsymbol{e}] = \begin{bmatrix} e_x & e_{xy} & e_{xz} \\ e_{xy} & e_y & e_{yz} \\ e_{xz} & e_{yz} & e_z \end{bmatrix} \tag{2.17}$$

Before we conclude this geometric presentation, consider the rigid-body rotation of our twodimensional element in the x-y plane, as shown in Figure (2-5). If the element is rotated through a small rigid-body angular displacement about the z-axis, using the bottom element edge, the rotation angle is determined as $\partial v/\partial x$, while using the left edge, the angle is given by $-\partial u/\partial y$. These two expressions are of course the same; that is, $\partial v/\partial x = -\partial u/\partial y$ and note that this would imply $e_{xy} = 0$. The rotation can then be expressed as $\omega_z = [(\partial v/\partial x) - (\partial u/\partial y)]/2$, which matches with the expression given earlier in (2:1:9)3. The other components of rotation follow in an analogous manner.

Relations for the constant rotation ω_z can be integrated to give the result:

$$\begin{aligned} \boldsymbol{u}^* &= u_0 - \omega_z y \\ \boldsymbol{v}^* &= v_0 + \omega_z x \end{aligned} \tag{2.18}$$

where u_0 and v_0 are arbitrary constant translations in the x- and y-directions. This result then specifies the general form of the displacement field for two-dimensional rigid-body motion. We can easily verify that the displacement field given by (2.18) yields zero strain.

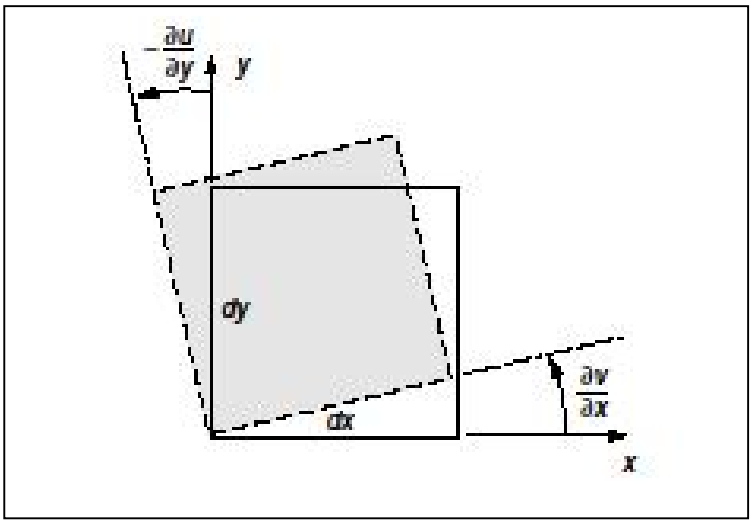


Figure (2-5) Two-dimensional rigid-body rotation.

For the three-dimensional case, the most general form of rigid-body displacement can be expressed as

$$\begin{aligned} \boldsymbol{u}^* &= u_0 - \omega_z y + \omega_y z \\ \boldsymbol{v}^* &= v_0 - \omega_x z + \omega_z x \\ \boldsymbol{w}^* &= w_0 - \omega_y x + \omega_x y \end{aligned} \tag{2.19}$$

As shown later, integrating the strain-displacement relations to determine the displacement field produces arbitrary constants and functions of integration, which are equivalent to rigidbody motion terms of the form given by (2.18) or (2.19). Thus, it is important to recognize such terms because we normally want to drop them from the analysis since they do not contribute to the strain or stress fields.

Section(2.2):Strain Transformation:

Because the strains are components of a second-order tensor . Transformation relation (1:5:1)3 is applicable for second-order tensors, and applying this to the strain gives

$$e'_{ij} = Q_{ip} Q_{jq} e_{pq} \tag{2.20}$$

where the rotation matrix $Q_{ij} = \cos(x_i', x_j)$. Thus, given the strain in one coordinate system, we can determine the new components in any other rotated system. For the general three-dimensional case, define the rotation matrix as

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{2.21}$$

Using this notational scheme, the specific transformation relations from equation (2.21) become

$$\begin{aligned}
e_x' &= e_x l_1^2 + e_y m_1^2 + e_z n_1^2 + 2(e_{xy} l_1 m_1 + e_{yz} m_1 n_1 + e_{zx} n_1 l_1) \\
e_y' &= e_x l_2^2 + e_y m_2^2 + e_z n_2^2 + 2(e_{xy} l_2 m_2 + e_{yz} m_2 n_2 + e_{zx} n_2 l_2) \\
e_z' &= e l_3^2 + e_y m_3^2 + e_z n_3^2 + 2(e_{xy} l_3 m_3 + e_{yz} m_3 n_3 + e_{zx} n_3 l_3) \\
e_{xy}' &= e_x l_1 l_2 + e_y m_1 m_2 + e_z n_1 n_2 + e_{xy}(l_1 m_2 + m_2 l_2) + e_{yz}(m_1 n_2 + n_1 m_2) + e_{zx}(n_1 l_2 + l_1 n_2) \\
e_{yz}' &= e_x l_2 l_3 + e_y m_2 m_3 + e_z n_2 n_3 + e_{xy}(l_2 m_3 + m_2 l_3) + e_{yz}(m_2 n_3 + n_2 m_3) + e_{zx}(n_2 l_3 + l_2 n_3) \\
e_{zx}' &= e_x l_3 l_1 + e_y m_3 m_1 + e_z n_3 n_1 + e_{xy}(l_3 m_1 + m_3 l_1) + e_{yz}(m_3 n_1 + n_3 m_1) + e_{zx}(n_3 l_1 + l_3 n_1)
\end{aligned} \tag{2.22}$$

For the two-dimensional case shown in Figure (2-6), the transformation matrix can be expressed as

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.23}$$

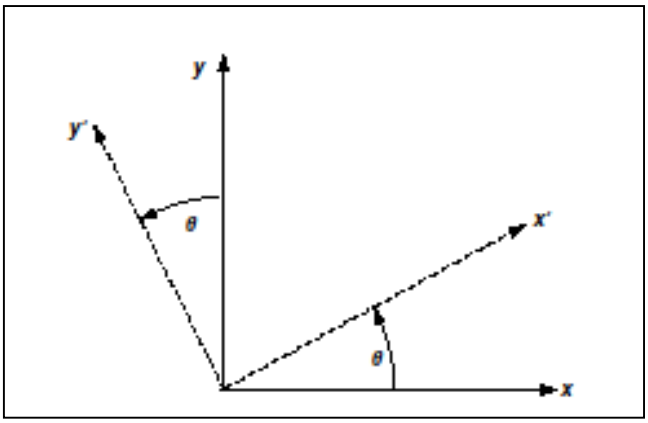


Figure (2-6) Two-dimensional rotational transformation.

Under this transformation, the in-plane strain components transform according to

$$
\begin{aligned}
e'_x &= e_x \cos^2\theta + e_y \sin^2\theta + 2e_{xy}\sin\theta\cos\theta \\
e'_y &= e_x \sin^2\theta + e_y \cos^2\theta + 2e_{xy}\sin\theta\cos\theta \\
e'_{xy} &= -e_x \sin\theta\cos\theta + e_y \sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned}
\tag{2.24}
$$

which is commonly rewritten in terms of the double angle:

$$
\begin{aligned}
e'_x &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta \\
e'_y &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \\
e'_{xy} &= \frac{e_y + e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta
\end{aligned}
\tag{2.25}
$$

Transformation relations (2.25) can be directly applied to establish transformations between Cartesian and polar coordinate systems . Additional applications of these results can be found when dealing with experimental strain gage measurement systems. For example, standard experimental methods using a rosette strain gage allow the determination of extensional strains in three different directions on the surface of a structure. Using this type of data, relation (2:3:6)1 can be repeatedly used to establish three independent equations that can be solved for the state of strain (e_x, e_y, e_{xy}) at the surface point under study . it follows that because the strain is a symmetric second-order tensor, we can identify and determine its principal axes and values. According to this theory, for any given strain tensor we can establish the principal value problem and solve the characteristic equation to explicitly determine the principal values and directions. The general characteristic equation for the strain tensor can be written as

$$
\det[e_{ij} - e\delta_{ij}] = -e^3 + \vartheta_1 e^2 - \vartheta_2 e + \vartheta_3 = 0 \tag{2.26}
$$

where e is the principal strain and the fundamental invariants of the strain tensor can be expressed in terms of the three principal strains e_1, e_2, e_3 as

$$
\begin{aligned}
\vartheta_1 &= e_1 + e_2 + e_3 \\
\vartheta_2 &= e_1 e_2 + e_2 e_3 + e_3 e_1 \\
\vartheta_3 &= e_1 e_2 e_3
\end{aligned}
\tag{2.27}
$$

The first invariant $\vartheta_1 = \vartheta$ is normally called the cubical dilatation, because it is related to the change in volume of material elements.

The strain matrix in the principal coordinate system takes the special diagonal form

$$e_{ij} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \tag{2.28}$$

Notice that for this principal coordinate system, the deformation does not produce any shearing and thus is only extensional. Therefore, a rectangular element oriented along principal axes of strain will retain its orthogonal shape and undergo only extensional deformation of its sides.

In particular applications it is convenient to decompose the strain tensor into two parts called spherical and deviatoric strain tensors. The spherical strain is defined by

$$\tilde{e}_{ij} = \frac{1}{3} e_{kk} \delta_{ij} = \frac{1}{3} \vartheta \delta_{ij} \tag{2.29}$$

while the deviatoric strain is specified as

$$\tilde{e}_{ij} = e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \tag{2.30}$$

Note that the total strain is then simply the sum

$$e_{ij} = \tilde{e}_{ij} + \hat{e}_{ij} \tag{2.31}$$

The spherical strain represents only volumetric deformation and is an isotropic tensor, being the same in all coordinate systems . The deviatoric strain tensor then accounts for changes in shape of material elements. It can be shown that the principal directions of the deviatoric strain are the same as those of the strain tensor. We now investigate in more detail the nature of the strain-displacement relations (2.14), and this will lead to the development of some additional relations necessary to ensure continuous, single-valued displacement field solutions. Relations (2.14), or the index notation form (2.15), represent six equations for the six strain components in terms of three displacements. If we specify continuous, single-valued displacements u, v, w, then through differentiation the resulting strain field will be equally well behaved. However, the converse is not necessarily true; that is, given the six strain components, integration of the strain-displacement relations (2.14)

does not necessarily produce continuous, single-valued displacements. This should not be totally surprising since we are trying to solve six equations for only three unknown displacement components. In order to ensure continuous, single-valued displacements, the strains must satisfy additional relations called integrability or compatibility equations. Before we proceed with the mathematics to develop these equations, it is instructive to consider a geometric interpretation of this concept. A two-dimensional example is shown in Figure (2-7) whereby an elastic solid is first divided into a series of elements in case (a). For simple visualization, consider only four such elements. In the undeformed configuration shown in case (b), these elements of course fit together perfectly. Next, let us arbitrarily specify the strain of each of the four elements and attempt to reconstruct the solid. For case (c), the elements have been carefully strained, taking into consideration neighboring elements so that the system fits together thus yielding continuous, single-valued displacements. However, for case (d), the elements have been individually deformed without any concern for neighboring deformations. It is observed for this case that the system will not fit together without voids and gaps, and this situation produces a discontinuous displacement field. So, we again conclude that the strain components must be somehow related to yield continuous, single-valued displacements. We now pursue these particular relations.

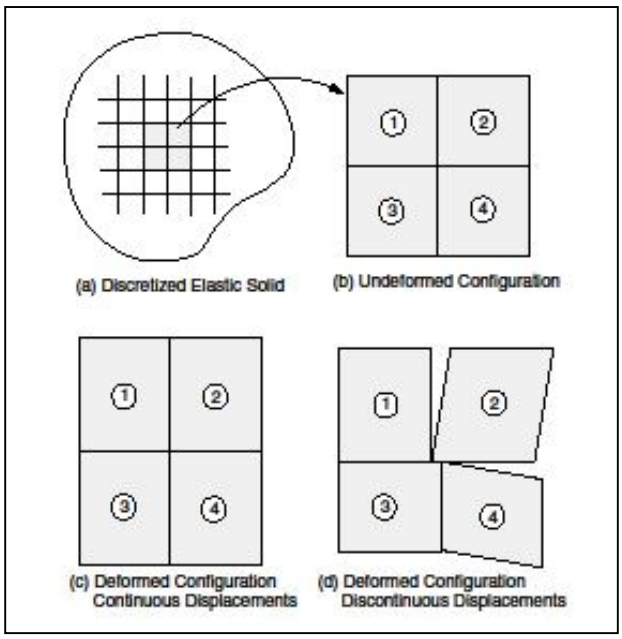


Figure (2-7) Physical interpretation of strain compatibility.

The process to develop these equations is based on eliminating the displacements from the strain-displacement relations. Working in index notation, we start by differentiating (2.15) twice with respect to
x_k and x_l:

$$e_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right)$$

Through simple interchange of subscripts, we can generate the following additional relations:

$$e_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right)$$
$$e_{jl,ik} = \frac{1}{2}\left(u_{j,lik} + u_{l,jik}\right)$$
$$e_{ik,jl} = \frac{1}{2}\left(u_{i,kjl} + u_{k,ijl}\right)$$

Working under the assumption of continuous displacements, we can interchange the order of differentiation on u, and the displacements can be eliminated from the preceding set to get

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \qquad (2.32)$$

These are called the Saint Venant compatibility equations. Although the system would lead to 81 individual equations, most are either simple identities or repetitions, and only 6 are meaningful. These six relations may be determined by letting $k = l$, and in scalar notation, they become

$$
\begin{aligned}
&\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}\\
&\frac{\partial^2 e_y}{\partial z^2}+\frac{\partial^2 e_z}{\partial y^2}=2\frac{\partial^2 e_{yz}}{\partial y\partial z}\\
&\frac{\partial^2 e_z}{\partial x^2}+\frac{\partial^2 e_x}{\partial z^2}=2\frac{\partial^2 e_{zx}}{\partial z\partial x}\\
&\frac{\partial^2 e_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right)\\
&\frac{\partial^2 e_y}{\partial z\partial x}=\frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}\right)\\
&\frac{\partial^2 e_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}\right)
\end{aligned}
\tag{2.33}
$$

It can be shown that these six equations are equivalent to three independent fourth-order relations . However, it is usually more convenient to use the six second order equations given by (2.33).

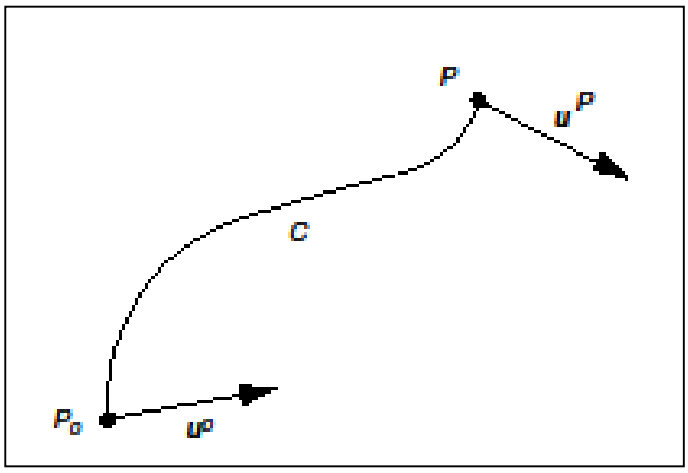


Figure (2-8) Continuity of displacements.

In the development of the compatibility relations, we assumed that the displacements were continuous, and thus the resulting equations (2.33) are actually only a necessary condition. In order to show that they are also sufficient, consider two arbitrary points P and P_oo in an elastic solid, as shown in Figure (2-8). Without

loss in generality, the origin may be placed at point P_o.

The displacements of points P and P_o are denoted by u_i^P and u_i^o, and the displacement of point P can be expressed as

$$u_i^P = u_i^o + \int_C du_i = u_i^o + \int_C \frac{\partial u_i}{\partial x_j} dx_j \tag{2.34}$$

where C is any continuous curve connecting points P_o and P. Using relation (2.6) for the displacement gradient, (2.34) becomes

$$u_i^P = u_i^o + \int_C \left(e_{ij} + \omega_{ij}\right) dx_j \tag{2.35}$$

Integrating the last term by parts gives

$$\int_C \omega_{ij}\, dx_j = \omega_{ij}^P x_j^P - \int_C x_j \omega_{ij,k}\, dx_k \tag{2.36}$$

where ω_{ij}^P is the rotation tensor at point P. Using relation (2:1:7),

$$\begin{aligned} \omega_{ij,k} &= \frac{1}{2}\left(u_{i,jk} - u_{j,ik}\right) = \frac{1}{2}\left(u_{i,jk} - u_{j,ik}\right) + \frac{1}{2}\left(u_{k,ji} - u_{k,ji}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial x_j}\left(u_{i,k} - u_{k,i}\right) - \frac{1}{2}\frac{\partial}{\partial x_i}\left(u_{j,k} - u_{k,j}\right) = e_{ik,j} - e_{jk,i} \end{aligned} \tag{2.37}$$

Substituting results (2.36) and (2.37) into (2.35) yields

$$u_i^P = u_i^o + \omega_{ij}^P x_j^P + \int_C U_{ik}\, dx_k \tag{2.38}$$

where $U_{ik} = e_{ik} - x_j\left(e_{ik,j} - e_{jk,i}\right)$.

Now if the displacements are to be continuous, single-valued functions, the line integral appearing in (2.38) must be the same for any curve C; that is, the integral must be independent of the path of integration. This implies that the integrand must be an exact differential, so that the value of the integral depends only on the end points. Invoking Stokes theorem, we can show that if the region is simply connected (definition of the term simply connected is postponed for the moment), a necessary and sufficient condition for the integral to be path independent is for $U_{ik,l} = U_{il,k}$.

Using this result yields

$$e_{ik,l} - \delta_{jl}\left(e_{ik,j} - e_{jk,i}\right) - x_j\left(e_{ik,jl} - e_{jk,il}\right) \\ = e_{il,k} - \delta_{jk}\left(e_{il,j} - e_{jl,i}\right) - x_j\left(e_{il,jk} - e_{jl,ik}\right)$$

which reduces to

$$x_j\left(e_{ik,jl} - e_{jk,il} - e_{il,jk} + e_{jl,ik}\right) = 0$$

Because this equation must be true for all values of x_j, the terms in parentheses must vanish, and after some index renaming this gives the identical result previously stated by the compatibility relations (2.32):

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

Thus, relations (2.32) or (2.33) are the necessary and sufficient conditions for continuous, single-valued displacements in simply connected regions.

Now let us get back to the term simply connected. This concept is related to the topology or geometry of the region under study. There are several places in elasticity theory where the connectivity of the region fundamentally affects the formulation and solution method. The term simply connected refers to regions of space for which all simple closed curves drawn in the region can be continuously shrunk to a point without going outside the region. Domains not having this property are called multiply connected. Several examples of such regions are illustrated in Figure (2-9). A general simply connected two-dimensional region is shown in case (a), and clearly this case allows any contour within the region to be shrunk to a point without going out of the domain. However, if we create a hole in the region as shown in case (b), a closed contour surrounding the hole cannot be shrunk to a point without going into the hole and thus outside of the region. Thus, for two-dimensional regions, the presence of one or more holes makes the region multiply connected. Note that by introducing a cut between the outer and inner boundaries in case (b), a new region is created that is now simply connected. Thus, multiply connected regions can be made simply connected by introducing one or more cuts between appropriate boundaries. Case (c) illustrates a simply connected three-dimensional example of a solid circular cylinder. If a spherical cavity is placed inside this cylinder as shown in case (d), the region is still simply connected because any closed contour can still be shrunk to a

point by sliding around the interior cavity. However, if the cylinder has a through hole as shown in case (e), then an interior contour encircling the axial through hole cannot be reduced to a point without going into the hole and outside the body. Thus, case (e) is an example of the multiply connected three-dimensional region.

It was found that the compatibility equations are necessary and sufficient conditions for continuous, single-valued displacements only for simply connected regions. However, for multiply connected domains, relations (2.32) or (2.33) provide only necessary butnot sufficient conditions. For this case, further relations can be developed and imposed on the problem, and these are found through the introduction of cuts within the region to make it simply connected as per our early discussion. These developments have been given by Fung (1965) or the reissue Fung and Tong (2001) and are not developed here.

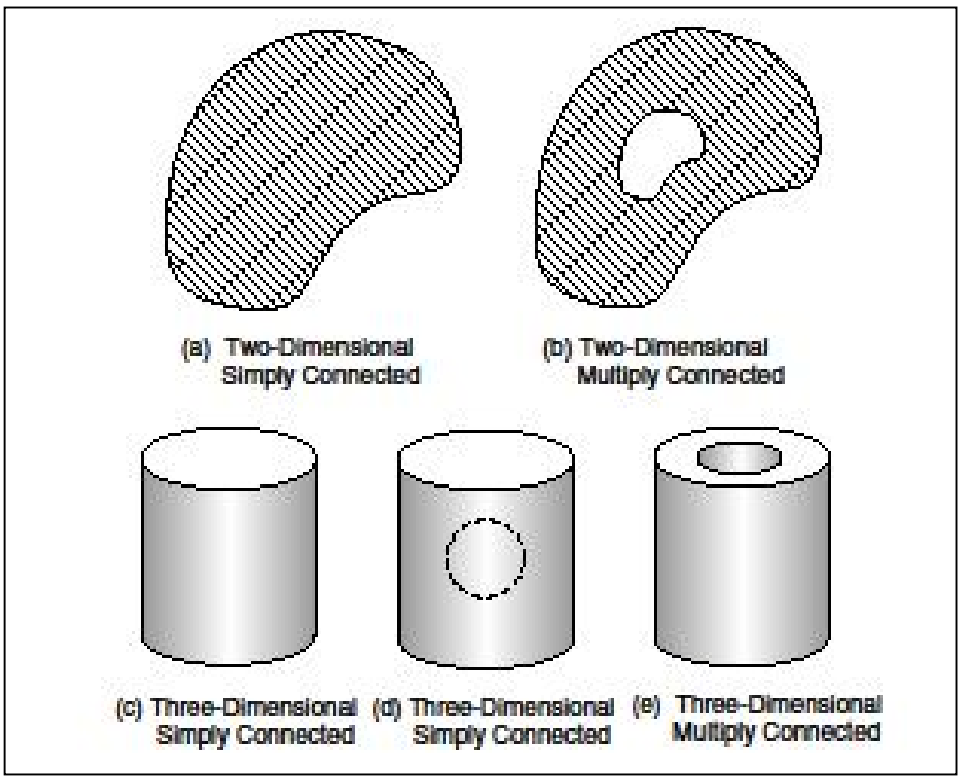


Figure (2-9) Examples of domain connectivity.

Although the compatibility relations guarantee (under appropriate conditions) continuous displacements, they do not ensure uniqueness of the displacement field. At the end of Section 2.2 we mentioned that, through integration of the strain displacement relations, the displacements can be determined only up to an arbitrary rigid-body motion. In some elasticity problems (e.g., thermal stress, crack problems, and dislocation modeling), it is necessary to use multivalued displacement fields to properly model the problem .

The solution to many problems in elasticity requires the use of curvilinear cylindrical and spherical coordinates. It is therefore necessary to have the field equations expressed in terms of such coordinate systems. We now pursue the development of the strain-displacement relations in cylindrical and spherical coordinates. Starting with form (2.16)

$$\boldsymbol{e} = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$$

the desired curvilinear relations can be determined using the appropriate forms for the displacement gradient term $\nabla \boldsymbol{u}$.

The cylindrical coordinate system previously defined in Figure (1-4) establishes new components for the displacement vector and strain tensor

$$\boldsymbol{u} = \boldsymbol{u}_r \boldsymbol{e}_r + \boldsymbol{u}_\theta \boldsymbol{e}_\theta + \boldsymbol{u}_z \boldsymbol{e}_z$$

$$\boldsymbol{e} = \begin{bmatrix} e_r & e_{r\theta} & e_{rz} \\ e_{r\theta} & e_\theta & e_{\theta z} \\ e_{rz} & e_{\theta z} & e_z \end{bmatrix} \tag{2.39}$$

Notice that the symmetry of the strain tensor is preserved in this orthogonal curvilinear system. Using results (1.71) and (1.64), the derivative operation in cylindrical coordinates can be expressed by

$$\begin{aligned} \nabla \boldsymbol{u} = {} & \frac{\partial \boldsymbol{u}_r}{\partial r} \boldsymbol{e}_r \boldsymbol{e}_r + \frac{\partial \boldsymbol{u}_\theta}{\partial r} \boldsymbol{e}_r \boldsymbol{e}_\theta + \frac{\partial \boldsymbol{u}_z}{\partial r} + \frac{1}{r}\left(\frac{\partial \boldsymbol{u}_r}{\partial \theta} - \boldsymbol{u}_\theta\right) \boldsymbol{e}_\theta \boldsymbol{e}_r \\ & + \frac{1}{r}\left(\boldsymbol{u}_r + \frac{\partial \boldsymbol{u}_\theta}{\partial \theta}\right) \boldsymbol{e}_\theta \boldsymbol{e}_\theta + \frac{1}{r}\frac{\partial \boldsymbol{u}_z}{\partial \theta} \boldsymbol{e}_\theta \boldsymbol{e}_z + \frac{\partial \boldsymbol{u}_r}{\partial z} \boldsymbol{e}_z \boldsymbol{e}_r + \frac{\partial \boldsymbol{u}_\theta}{\partial z} \boldsymbol{e}_z \boldsymbol{e}_\theta \\ & + \frac{\partial \boldsymbol{u}_z}{\partial z} \boldsymbol{e}_z \boldsymbol{e}_z \end{aligned} \tag{2.40}$$

Placing this result into the strain-displacement form (2.16) gives the desired relations in cylindrical coordinates. The individual scalar equations are given by

$$
\begin{aligned}
e_r &= \frac{\partial u_r}{\partial r}, \quad e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad e_z = \frac{\partial u_z}{\partial z} \\
e_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
e_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
e_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right)
\end{aligned}
\tag{2.41}
$$

For spherical coordinates defined by Figure (1-5), the displacement vector and strain tensor can be written as

$$
\boldsymbol{u} = u_R \boldsymbol{e}_R + u_\phi \boldsymbol{e}_\phi + u_\theta \boldsymbol{e}_\theta
$$

$$
\boldsymbol{e} = \begin{bmatrix} e_R & e_{R\phi} & e_{R\theta} \\ e_{R\phi} & e_\phi & e_{\phi\theta} \\ e_{R\theta} & e_{\phi\theta} & e_\theta \end{bmatrix}
\tag{2.42}
$$

Following identical procedures as used for the cylindrical equation development, the straindisplacement relations for spherical coordinates become

$$
\begin{aligned}
e_R &= \frac{\partial u_R}{\partial R}, \quad e_\phi = \frac{1}{R}\left(u_R + \frac{\partial u_\phi}{\partial \phi}\right) \\
e_\theta &= \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + \sin\phi\, u_R + \cos\phi\, u_\phi\right) \\
e_{R\phi} &= \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right) \\
e_{\phi\theta} &= \frac{1}{2R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - \cos\phi\, u_\theta\right) \\
e_{\theta R} &= \frac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right)
\end{aligned}
\tag{2.43}
$$

We can observe that these relations in curvilinear systems contain additional terms that do not include derivatives of individual displacement components. For example, in spherical coordinates a simple uniform radial displacement

u_R gives rise to transverse extensional strains
$e_\phi = e_\theta = u_R/R$. This deformation can be simulated by blowing up a spherical balloon and observing the separation of points on the balloon's surface. Such terms were not found in the Cartesian forms given by (2.14), and their appearance is thus related to the curvature of the spatial coordinate system . Clearly, the curvilinear forms (2.41) and (2.43) appear more complicated than the corresponding Cartesian relations. However, for particular problems, the curvilinear relations, when combined with other field equations, allow analytical solutions to be developed that could not be found using a Cartesian formulation.

Chapter (3)

Stress and Equilibrium

The previous chapter investigated the kinematics of deformation without regard to the force or stress distribution within the elastic solid. We now wish to examine these issues and explore the transmission of forces through deformable materials. Our study leads to the definition and use of the traction vector and stress tensor. Each provides a quantitative method to describe both boundary and internal force distributions within a continuum solid. Because it is commonly accepted that maximum stresses are a major contributing factor to material failure, primary application of elasticity theory is used to determine the distribution of stress within a given structure. Related to these force distribution issues is the concept of equilibrium. Within a deformable solid, the force distribution at each point must be balanced. For the static case, the summation of forces on an infinitesimal element is required to be zero, while for a dynamic problem the resultant force must equal the mass times the element's acceleration. In this chapter, we establish the definitions and properties of the traction vector and stress tensor and develop the equilibrium equations, which become another set of field equations necessary in the overall formulation of elasticity theory. It should be noted that the developments in this chapter do not require that the material be elastic, and thus in principle these results apply to a broader class of material behavior.

Section (3.1):Body and Surface Forces:

When a structure is subjected to applied external loadings, internal forces are induced inside the body. Following the philosophy of continuum mechanics, these internal forces are distributed continuously within the solid. In order to study such forces, it is convenient to categorize them into two major groups, commonly referred to as body forces and surface forces.

Body forces are proportional to the body's mass and are reacted with an agent outside of the body. Examples of these include gravitational- weight forces, magnetic forces, and inertial forces. Figure (3-1) (a) shows an example body force of an object's self-weight. By using continuum mechanics principles, a body force density (force per unit volume) $\boldsymbol{F}(x)$ can be defined such that the total resultant

body force of an entire solid can be written as a volume integral over the body

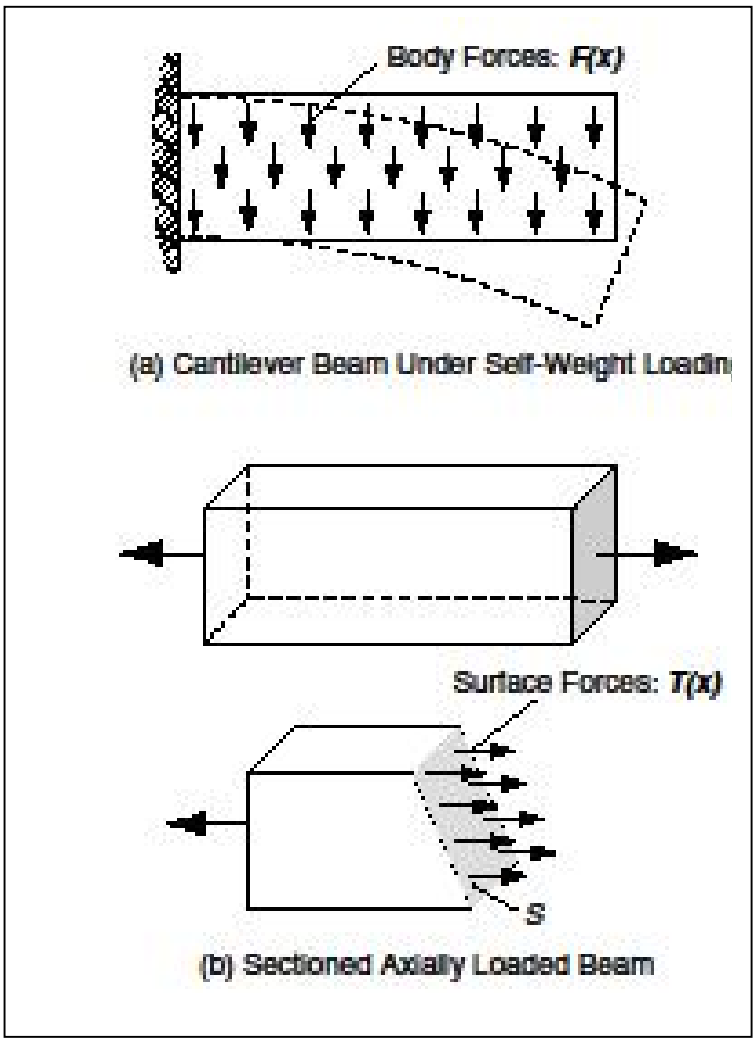


Figure (3-1) Examples of body and surfaces.

$$F_R = \iiint_V \boldsymbol{F}(\boldsymbol{x})\, dV \tag{3.1}$$

Surface forces always act on a surface and result from physical contact with another body. Figure (3-1) (b) illustrates surface forces existing in a beam section that has been created by sectioning the body into two pieces. For this particular case, the surface S is a virtual one in the sense that it was artificially created to investigate the nature of the internal forces at this location in the body. Again the resultant surface force over the entire surface S can be expressed as the integral of a surface force density function $\boldsymbol{T^n}(\boldsymbol{x})$

$$F_S = \iint_S \boldsymbol{T^n}(\boldsymbol{x})\, dS \tag{3.2}$$

The surface force density is normally referred to as the traction vector and is discussed in more detail in the next section. In the development of classical elasticity, distributions of body or surface couples are normally not included. Theories that consider such force distributions have been constructed in an effort to extend classical elasticity for applications in micromechanical modeling. Such approaches are normally called micropolar or couplestress theory (see Eringen 1968) .

In order to quantify the nature of the internal distribution of forces within a continuum solid, consider a general body subject to arbitrary (concentrated and distributed) external loadings, as shown in Figure (3-2). To investigate the internal forces, a section is made through the body as shown. On this section consider a small area ΔA with unit normal vector $\boldsymbol{n}$. The resultant surface force acting on ΔA is defined by $\Delta \boldsymbol{F}$. Consistent with our earlier discussion, no resultant surface couple is included. The stress or traction vector is defined by

$$\boldsymbol{T^n}(\boldsymbol{x}, \boldsymbol{n}) = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta \boldsymbol{A}} \tag{3.3}$$

Notice that the traction vector depends on both the spatial location and the unit normal vector to the surface under study. Thus, even though we may be investigating the same point, the traction vector still varies as a function of the orientation of the surface normal. Because the traction is defined as force per unit area, the total surface force is determined through integration as per relation (3.2). Note, also, the simple action-reaction principle (Newton's third law)

$$\boldsymbol{T^n}(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{T^n}(\boldsymbol{x}, -\boldsymbol{n})$$

Consider now the special case in which $\Delta \boldsymbol{A}$ coincides with each of the three coordinate planes with the unit normal vectors pointing along the positive coordinate axes. This concept is shown in Figure (3-3), where the three coordinate surfaces for $\Delta \boldsymbol{A}$ partition off a cube of material. For this case, the traction vector on each face can be written as

$$
\begin{aligned}
\boldsymbol{T^n}(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_1) &= \sigma_x \boldsymbol{e}_1 + \tau_{xy} \boldsymbol{e}_2 + \tau_{xz} \boldsymbol{e}_3 \\
\boldsymbol{T^n}(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_2) &= \tau_{yx} \boldsymbol{e}_1 + \sigma_y \boldsymbol{e}_2 + \tau_{yz} \boldsymbol{e}_3 \qquad (3.4) \\
\boldsymbol{T^n}(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_3) &= \tau_{zx} \boldsymbol{e}_1 + \tau_{zy} \boldsymbol{e}_2 + \sigma_z \boldsymbol{e}_3
\end{aligned}
$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the unit vectors along each coordinate direction, and the nine quantities $\{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{xz}\}$ are the components of the traction vector on each of three coordinate planes as illustrated. These nine components are called the stress components,

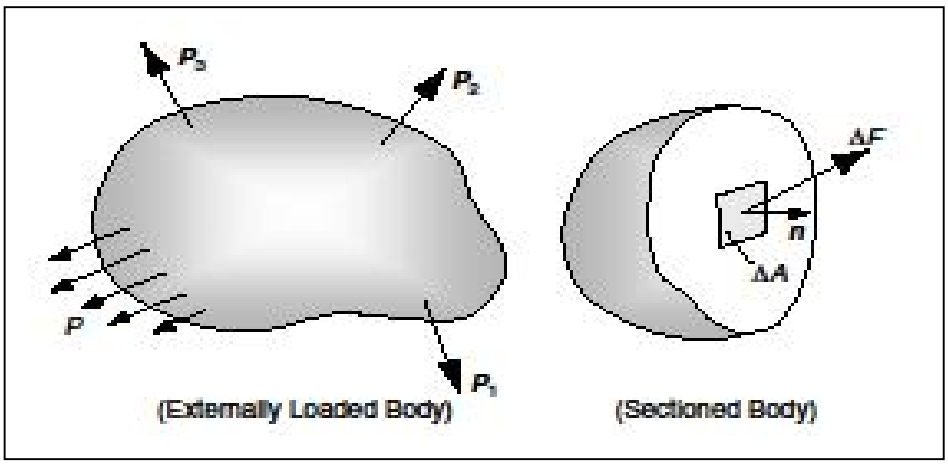


Figure (3-2) Sectioned solid under external loading.

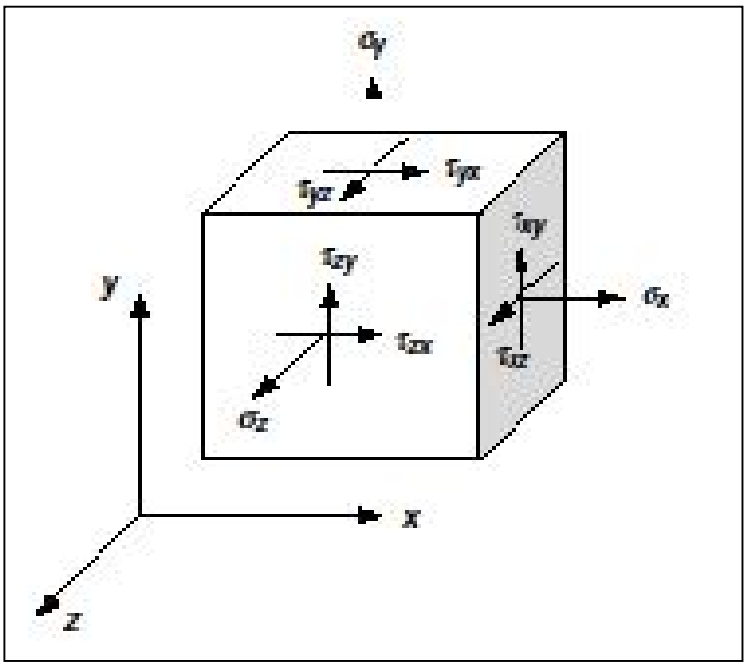


Figure (3-3) Components of the stress.

with $\sigma_x, \sigma_y, \sigma_z$ referred to as normal stresses and $\tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{zx}, \tau_{xz}$ called the shearing stresses. The components of stress σ_{ij} are commonly written in matrix format

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{3.5}$$

and it can be formally shown that the stress is a second-order tensor that obeys the appropriate transformation law (1:5:3)3.

The positive directions of each stress component are illustrated in Figure (3-3). Regardless of the coordinate system, positive normal stress always acts in tension out of the face, and only one subscript is necessary because it always acts normal to the surface. The shear stress, however, requires two subscripts, the first representing the plane of action and the second designating the direction of the stress. Similar to shear strain, the sign of the shear stress depends on coordinate system orientation. For example, on a plane with a normal in the positive x direction, positive τ_{xy} acts in the positive y direction. Similar definitions follow for the other shear stress components. In subsequent chapters, proper formulation of elasticity problems requires knowledge of these basic definitions, directions, and sign conventions for particular stress components.

Consider next the traction vector on an oblique plane with arbitrary orientation, as shown in Figure (3-4). The unit normal to the surface can be expressed by

$$\boldsymbol{n} = \boldsymbol{n}_x \boldsymbol{e}_1 + \boldsymbol{n}_y \boldsymbol{e}_2 + n_z \boldsymbol{e}_3 \tag{3.6}$$

where n_x, n_y, n_z are the direction cosines of the unit vector n relative to the given coordinate system. We now consider the equilibrium of the pyramidal element interior to the oblique and coordinate planes. Invoking the force balance between tractions on the oblique and coordinate faces gives

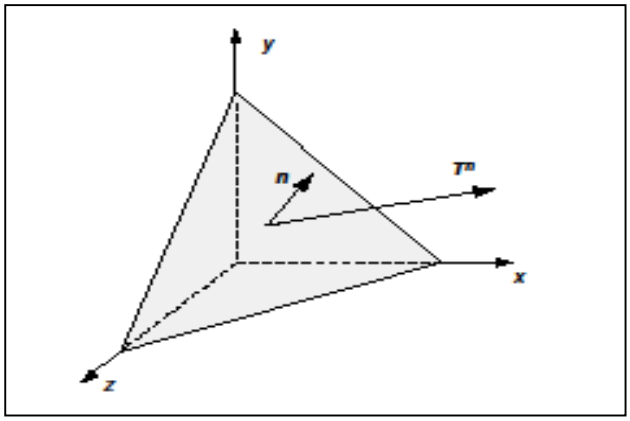


Figure (3-4) Traction on an oblique plane.

$$\boldsymbol{T^n} = n_x \boldsymbol{T^n}(\boldsymbol{n} = \boldsymbol{e}_1) + n_y \boldsymbol{T^n}(\boldsymbol{n} = \boldsymbol{e}_2) + n_z \boldsymbol{T^n}(\boldsymbol{n} = \boldsymbol{e}_3)$$

and by using relations (3.4), this can be written as

$$\boldsymbol{T^n} = (\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z)\boldsymbol{e}_1 + (\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z)\boldsymbol{e}_2 + (\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z)\boldsymbol{e}_3 \quad (3.7)$$

or in index notation

$$\boldsymbol{T}_i^n = \sigma_{ji} = n_j \quad (3.8)$$

Relation (3.7) or (3.8) provides a simple and direct method to calculate the forces on oblique planes and surfaces. This technique proves to be very useful to specify general boundary conditions during the formulation and solution of elasticity problems.

Following the principles of small deformation theory, the previous definitions for the stress tensor and traction vector do not make a distinction between the deformed and undeformed configurations of the body. As mentioned in the previous chapter, such a distinction only leads to small modifications that are considered higher-order effects and are normally neglected. However, for large deformation theory, sizeable differences exist between these configurations, and the undeformed configuration (commonly called the reference configuration) is often used in problem formulation. This gives rise to the definition of an additional stress called the Piola-Kirchhoff stress tensor that represents the force per unit area in the reference configuration (see Chandrasekharaiah and Debnath [1]. In the more general scheme, the stress s_{ij} is referred to as the Cauchy stress tensor. Throughout the text only small deformation

theory is considered, and thus the distinction between these two definitions of stress disappears, thereby eliminating any need for this additional terminology.

Section(3.2):Stress Transformation:

Analogous to our previous discussion with the strain tensor, the stress components must also follow the standard transformation rules for second-order tensors established . Applying transformation relation (1.5) for the stress gives

$$\sigma'_{ij} = Q_{ip}Q_{jq}\sigma_{pq} \tag{3.9}$$

where the rotation matrix $Q_{ij} = \cos(x'_i, x_j)$. Therefore, given the stress in one coordinate system, we can determine the new components in any other rotated system. For the general three-dimensional case, the rotation matrix may be chosen in the form

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{3.10}$$

Using this notational scheme, the specific transformation relations for the stress then become

$$\begin{aligned}
\sigma'_x &= \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1) \\
\sigma'_y &= \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2) \\
\sigma'_z &= \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3) \\
\tau'_{xy} &= \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_2 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2) \\
\tau'_{yz} &= \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3) \\
\tau'_{zx} &= \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_3 n_1)
\end{aligned} \tag{3.11}$$

For the two-dimensional case originally shown in Figure (2-6), the transformation matrix was given by relation (3.12). Under this transformation, the in-plane stress components transform according to

$$\begin{aligned}
\sigma'_x &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta \\
\sigma'_y &= \sigma_x \cos^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta \\
\tau'_{xy} &= -\sigma_x \sin\theta \cos\theta + \sigma_y \sin\theta \cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned} \tag{3.12}$$

which is commonly rewritten in terms of the double angle

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (3.13)$$
$$\tau_{xy}' = \frac{\sigma_y + \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Similar to our discussion on strain in the previous chapter, relations (3.13) can be directly applied to establish stress transformations between Cartesian and polar coordinate systems . Both two- and three-dimensional stress transformation equations can be easily incorporated in MATLAB to provide numerical solution to problems of interest . It is shown later in the chapter that the stress is a symmetric tensor. Using this fact, appropriate theory has been developed to identify and determine principal axes and values for the stress. For any given stress tensor we can establish the principal value problem and solve the characteristic equation to explicitly determine the principal values and directions. The general characteristic equation for the stress tensor becomes

$$\det[\sigma_{ij} - \sigma\delta_{ij}] = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \qquad (3.14)$$

where s are the principal stresses and the fundamental invariants of the stress tensor can be expressed in terms of the three principal stresses s_1, s_2, s_3 as

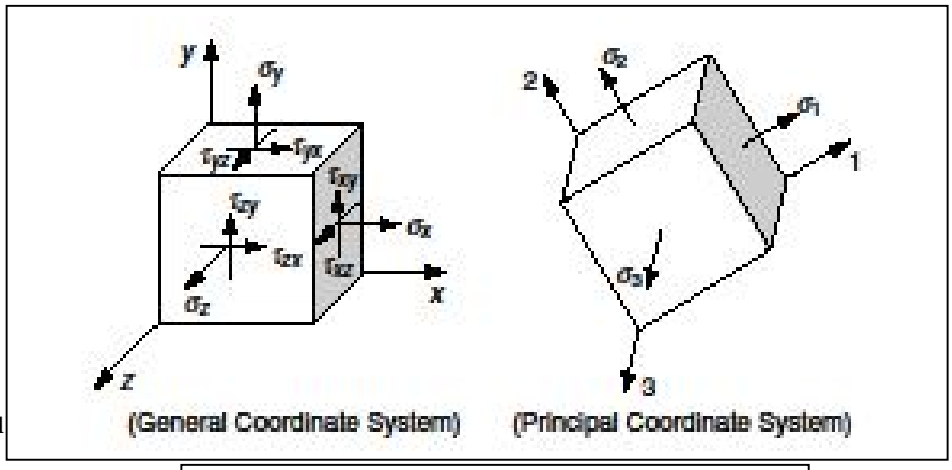
$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \qquad (3.15)$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

In the principal coordinate system, the stress matrix takes the special diagonal form

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \qquad (3.16)$$

A comparison of the general and principal stress states is shown in Figure (3-5). Notice that for the principal coordinate system, all shearing stresses vanish and thus the state includes only normal stresses. These issues should be compared to the equivalent comments made for the strain tensor at the end of Section 2.4.

Figu



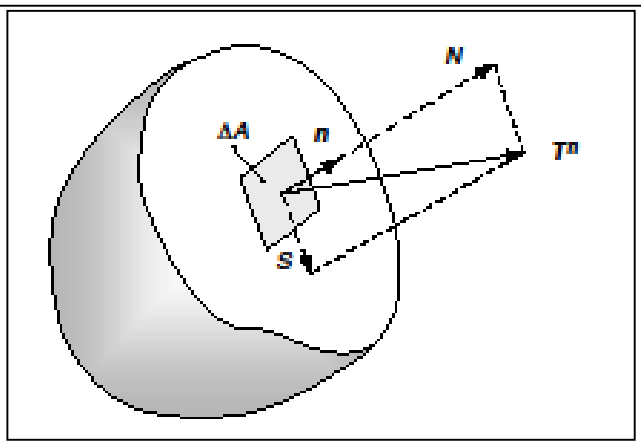


Figure (3-6) Traction vector decomposition.

We now wish to go back to investigate another issue related to stress and traction transformation that makes use of principal stresses. Consider the general traction vector $\boldsymbol{T^n}$ that acts on an arbitrary surface as shown in Figure (3-6). The issue of interest is to determine the traction vector's normal and shear components N and S. The normal component is simply the traction's projection in the direction of the unit normal vector n, while the shear component is found by Pythagorean theorem,

$$N = \boldsymbol{T^n} \cdot \boldsymbol{n}$$
$$S = (|\boldsymbol{T^n}|^2 - N^2)^{1/2} \qquad (3.17)$$

Using the relationship for the traction vector (3.7) into (3.4) gives

$$
\begin{aligned}
N = T^n \cdot n = T_i^n n_i = \sigma_{ji} n_j n_i \\
= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2
\end{aligned}
\tag{3.18}
$$

where, in order to simplify the expressions, we have used the principal axes for the stress tensor. In a similar manner,

$$
\begin{aligned}
|\boldsymbol{T^n}|^2 = \boldsymbol{T^n} \cdot \boldsymbol{T^n} = T_i^n T_i^n = \sigma_{ji} n_j \sigma_{ki} n_k \\
= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2
\end{aligned}
\tag{3.19}
$$

Using these results back into relation (3.17) yields

$$
\begin{aligned}
N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \\
S^2 + N^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2
\end{aligned}
\tag{3.20}
$$

In addition, we also add the condition that the vector $\boldsymbol{n}$ has unit magnitude

$$
1 = n_1^2 + n_2^2 + n_3^2 \tag{3.21}
$$

Relations (3.20) and (3.21) can be viewed as three linear algebraic equations for the unknowns $n_1^2 + n_2^2 + n_3^2$. Solving this system gives the following result:

$$
\begin{aligned}
n_1^2 &= \frac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\
n_2^2 &= \frac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\
n_3^2 &= \frac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}
\end{aligned}
\tag{3.22}
$$

Without loss in generality, we can rank the principal stresses as $\sigma_1 > \sigma_2 > \sigma_3$. Noting that the expressions given by (3.22) must be greater than or equal to zero, we can conclude the following

$$
\begin{aligned}
S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0 \\
S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0 \\
S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0
\end{aligned}
\tag{3.23}
$$

For the equality case, equations (3.23) represent three circles in an $S - N$ coordinate system, and Figure (3-7) illustrates the location of each circle. These results were originally generated by Otto Mohr over a century ago, and the circles

are commonly called Mohr's circles of stress. The three inequalities given in (3.23) imply that all admissible values of N and S lie in the shaded regions bounded by the three circles. Note that, for the ranked principal stresses, the largest shear component is easily determined as $S_{max} = 1/2\,|\sigma_1 - \sigma_3|$. Although these circles can be effectively used for two-dimensional stress transformation, the general tensorial-based equations (3.11) are normally used for general transformation computations.

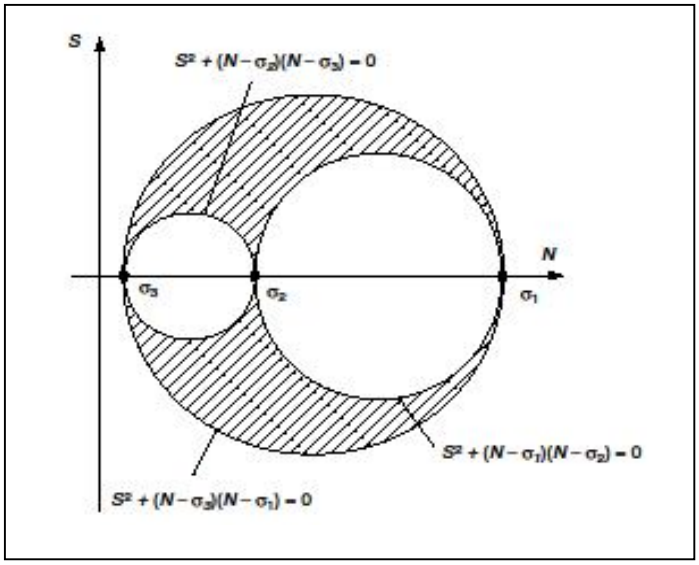


Figure (3-7): Mohr's circles of stress.

Example (3.1): Stress Transformation

For the following state of stress, determine the principal stresses and directions and find the traction vector on a plane with unit normal $\boldsymbol{n} = (0,1,1)/\sqrt{2}$

$$\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

The principal stress problem is started by calculating the three invariants, giving the result $I_1 = 3, I_2 = -6, I_3 = -8$. This yields the following characteristic equation:

$$\sigma^3 + 3\sigma^2 + 6\sigma - 8 = 0$$

The roots of this equation are found to be $\sigma = 4, 1, -2$. Back-substituting the first root into the fundamental system gives

$$-n_1^{(1)} + n_2^{(1)} + n_3^{(1)} = 0$$

$$n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} = 0$$

$$n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} = 0$$

Solving this system, the normalized principal direction is found to be $\boldsymbol{n}^{(1)} = (2,1,1)/\sqrt{6}$. In similar fashion the other two principal directions are $\boldsymbol{n}^{(2)} = (-1,1,1)/\sqrt{3}$, $\boldsymbol{n}^{(3)} = (0,-1,1)/\sqrt{2}$.

The traction vector on the specified plane is calculated by using the relation

$$T_i^n = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

Section(3.3):Spherical and Deviatoric Stresses:

As mentioned in our previous discussion on strain, it is often convenient to decompose the stress into two parts called the spherical and deviatoric stress tensors. Analogous to relations (2.29) and (2.30), the spherical stress is defined by

$$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3.24}$$

while the deviatoric stress becomes

$$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3.25}$$

Note that the total stress is then simply the sum

$$\sigma_{ij} = \tilde{\sigma}_{ij} + \hat{\sigma}_{ij} \tag{3.26}$$

The spherical stress is an isotropic tensor, being the same in all coordinate systems It can be shown that the principal directions of the deviatoric stress are the same as those of the stress tensor .

The stress field in an elastic solid is continuously distributed within the body and uniquely determined from the applied loadings. Because we are dealing primarily

with bodies in equilibrium, the applied loadings satisfy the equations of static equilibrium; that is, the summation of forces and moments is zero. If the entire body is in equilibrium, then all parts must also be in equilibrium. Thus, we can partition any solid into an appropriate subdomain and apply the equilibrium principle to that region. Following this approach, equilibrium equations can be developed that express the vanishing of the resultant force and moment at a continuum point in the material. These equations can be developed by using either an arbitrary finite subdomain or a special differential region with boundaries coinciding with coordinate surfaces.

Consider a closed subdomain with volume V and surface S within a body in equilibrium. The region has a general distribution of surface tractions $\boldsymbol{T^n}$ body forces F as shown in Figure (3-8). For static equilibrium, conservation of linear momentum implies that the forces acting on this region are balanced and thus the resultant force must vanish. This concept can be easily written in index notation as

$$\iint_S T_i^n \, dS + \iiint_V F_i \, dV = 0 \qquad (3.27)$$

Using relation (3.7) for the traction vector, we can express the equilibrium statement in terms of stress:

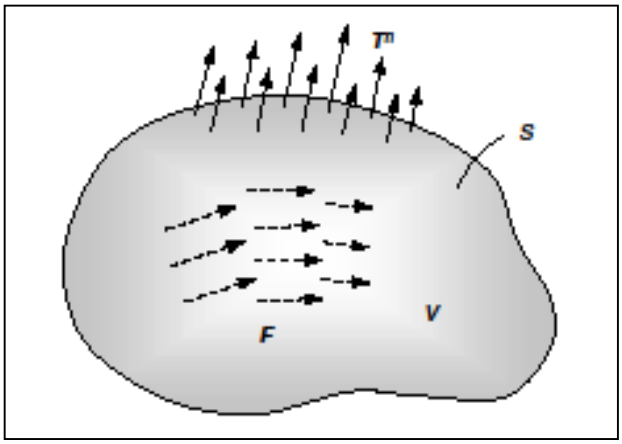


Figure (3-8) Body and surface forces acting on arbitrary portion of a continuum.

$$\iint_S \sigma_{ji} n_j \, dS + \iiint_V F_i \, dV = 0 \qquad (3.28)$$

Applying the divergence theorem (1.49) to the surface integral allows the

conversion to a volume integral, and relation (3.28) can then be expressed as

$$\iiint_V \left(\sigma_{ji,j} + F_i\right) dV = 0 \tag{3.29}$$

Because the region V is arbitrary (any part of the medium can be chosen) and the integrand in (3.29) is continuous, then by the zero-value theorem (1.54), the integrand must vanish:

$$\sigma_{ji,j} + \boldsymbol{F}_i = 0 \tag{3.30}$$

This result represents three scalar relations called the equilibrium equations. Written in scalar notation they are

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \boldsymbol{F}_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \boldsymbol{F}_y &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \boldsymbol{F}_z &= 0 \end{aligned} \tag{3.31}$$

Thus, all elasticity stress fields must satisfy these relations in order to be in static equilibrium.

Next consider the angular momentum principle that states that the moment of all forces acting on any portion of the body must vanish. Note that the point about which the moment is calculated can be chosen arbitrarily. Applying this principle to the region shown in Figure (3-8) results in a statement of the vanishing of the moments resulting from surface and body forces:

$$\iint_S \varepsilon_{ijk} x_j T_k^n \, dS + \iiint_V \varepsilon_{ijk} F_k \, dV = 0 \tag{3.32}$$

Again using relation (3.7) for the traction, (3.32) can be written as

$$\iint_S \varepsilon_{ijk} x_j \sigma_{lk} n_l \, dS + \iiint_V \varepsilon_{ijk} x_j F_k \, dV = 0$$

and application of the divergence theorem gives

$$\iiint_V \left[\left(\varepsilon_{ijk} x_j \sigma_{lk}\right)_{,l} + \varepsilon_{ijk} x_j F_k\right] dV = 0$$

This integral can be expanded and simplified as

$$\iiint_V \left[\varepsilon_{ijk} x_{j,l} \sigma_{lk} + \varepsilon_{ijk} x_j \sigma_{lk} + \varepsilon_{ijk} x_j F_k\right] dV =$$

$$\iiint_V \left[\varepsilon_{ijk} \delta_{jl} \sigma_{lk} + \varepsilon_{ijk} x_j \sigma_{lk,l} + \varepsilon_{ijk} x_j F_k\right] dV =$$

$$\iiint_V \left[\varepsilon_{ijk} \sigma_{jk} - \varepsilon_{ijk} x_j F_k + \varepsilon_{ijk} x_j F_k\right] dV = \iiint_V \varepsilon_{ijk} \sigma_{jk} \, dV$$

where we have used the equilibrium equations (3.30) to simplify the final result. Thus, (3.32) now gives

$$\iiint_V \varepsilon_{ijk} \sigma_{jk} \, dV = 0$$

As per our earlier arguments, because the region V is arbitrary, the integrand must vanish, giving $\varepsilon_{ijk} \sigma_{jk} = 0$. However, because the alternating symbol is antisymmetric in indices jk, the other product term σ_{jk} must be symmetric, thus implying

$$\sigma_{ij} = \sigma_{ji} \Rightarrow \begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{yz} &= \tau_{zy} \\ \tau_{zx} &= \tau_{xz} \end{aligned} \tag{3.33}$$

We thus find that, similar to the strain, the stress tensor is also symmetric and therefore has only six independent components in three dimensions. Under these conditions, the equilibrium equations can then be written as

$$\sigma_{ij,j} + F_i = 0 \tag{3.34}$$

Chapter (4)

Potentials and Stress Functions

We now wish to investigate the method of potentials to generate solutions to elasticity problems. Several different potential techniques have been developed in order to solve problems within both displacement and stress formulations. Methods related to the displacement formulation include the scalar and vector potentials from the Helmholtz decomposition, Galerkin vector, and Papkovich-Neuber functions. These schemes provide general solution forms for Navier's equations. Potentials used in the stress formulation are those related to the Maxwell and Morera stress functions, and these lead to Airy and other common stress functions that we have already used for the solution of particular elasticity problems. As previously observed, these stress functions normally satisfy the equilibrium equations identically and when combined with the compatibility relations they yield a simpler and more tractable system of equations.

For either displacement or stress formulations, these solution schemes bring up the question—are all solutions of elasticity expressible by the particular potential representation? This issue is normally referred to as the completeness of the representations, and over the past several decades these theoretical questions have generally been answered in the affirmative. For many cases these approaches are useful to solve particular three-dimensional elasticity problems, and we will investigate several such solutions. Some potential methods are also particularly useful in formulating and solving dynamic elasticity problems involving wave propagation (see Fung 1965 or Graff 1991).

Section(4.1):Helmholtz Displacement Vector Representation:

A useful relation called the Helmholtz theorem states that any sufficiently continuous vector field can be represented as the sum of the gradient of a scalar potential plus the curl of a vector potential. Using this representation for the displacement field, we can write

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{4.1}$$

where ϕ is the scalar potential and φ is the vector potential. The gradient term in

the decomposition has a zero curl and is referred to as the lamellar or irrotational part, while the $curl$ term in (4.1) has no divergence and is called solenoidal. Note that this representation specifies three displacement components in terms of four potential components, and furthermore the divergence of $\boldsymbol{\varphi}$ is arbitrary. In order to address these problems, it is common to choose w with zero divergence; that is,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\varphi} = 0 \tag{4.2}$$

It can be easily shown that the volume dilatation ϑ and the rotation vector $\boldsymbol{\omega}$ are related to these potentials by

$$\vartheta = e_{kk} = \phi_{,kk}, \omega_i = -\frac{1}{2}\varphi_{i,kk} \tag{4.3}$$

General solutions of these relations can be determined (see Fung 1965), and thus the scalar and vector potentials can be expressed in terms of the displacement field.

Using representation (4.1) in the general three-dimensional Navier equations we find

$$(\lambda + 2\mu)\nabla(\nabla^2\phi) + \mu\nabla \times (\nabla^2\boldsymbol{\varphi}) + F = 0 \tag{4.4}$$

Notice that if the divergence and curl is taken of the previous equation with zero body forces, the following relations are generated

$$\nabla^2\nabla^2\phi = \nabla^4\phi = 0, \nabla^2\nabla^2\boldsymbol{\varphi} = \nabla^4 = 0 \tag{4.5}$$

and thus we find that both potential functions are biharmonic functions. Further reduction of (4.4) will now be made for specific applications.

It is noted that for the case of zero body forces, special solutions of (4.4) occur with $\nabla^2\phi =$ constant and $\nabla^2\boldsymbol{\varphi} =$ constant. We consider the special case with

$$\nabla^2\phi = \text{constant}, \quad \varphi = 0 \tag{4.6}$$

Because our goal is to determine simply a particular solution, we can choose the constant to be zero, and thus the potential ϕ will be a harmonic function. For this case, the displacement representation is commonly written as

$$2\mu\mu_i = \phi_{,i} \tag{4.7}$$

and the function f is called Lamé's strain potential. Using this form, the strains and stresses are given by the simple relations

$$e_{ij} = \frac{1}{2\mu}\phi_{,ij}$$
$$e_{ij} = \phi_{,ij} \tag{4.8}$$

In Cartesian coordinates, these expressions would give

$$u = \frac{1}{2\mu}\frac{\partial\phi}{\partial x}, v = \frac{1}{2\mu}\frac{\partial\phi}{\partial y}, w = \frac{1}{2\mu}\frac{\partial\phi}{\partial z}$$
$$e_x = \frac{1}{2\mu}\frac{\partial^2\phi}{\partial x^2}, e_y = \frac{1}{2\mu}\frac{\partial^2\phi}{\partial y^2}, \ldots \tag{4.9}$$
$$\sigma_x = \frac{\partial^2\phi}{\partial x^2}, \sigma_y = \frac{\partial^2\phi}{\partial y^2}, \tau_{xy} = \frac{\partial^2\phi}{\partial x\partial y}$$

Thus, for this case any harmonic function can be used for Lamé's potential. Typical forms of harmonic functions are easily determined, and some examples include

$$x^2 - y^2, xy, r^n \cos n\theta, \log r, \frac{1}{R}, \log(R + z)$$
$$\text{with } r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}, R = \sqrt{x^2 + y^2 + z^2} \tag{4.10}$$

In the previous sections, the displacement vector was represented in terms of first derivatives of the potential functions ϕ and $\boldsymbol{\varphi}$. Galerkin (1930) showed that it is also useful to represent the displacement in terms of second derivatives of a single vector function. The proposed representation is given by

$$2\mu\boldsymbol{u} = 2(1 - \nu)\nabla^2\boldsymbol{V} - \nabla(\nabla \cdot \boldsymbol{V}) \tag{4.11}$$

where the potential function V is called the Galerkin vector. Substituting this form into Navier's equation gives the result

$$\nabla^4 V = -\frac{F}{1 - \nu} \tag{4.12}$$

Note that for the case of zero body forces, the Galerkin vector is biharmonic. Thus, we have reduced Navier's equation to a simpler fourth-order vector equation.

By comparing the representations given by (4.1) with that of (4.11), the Helmholtz potentials can be related to the Galerkin vector by

$$\begin{aligned} \phi &= -\frac{1}{2\mu}\boldsymbol{\nabla}\cdot\mathbf{V} \\ \nabla\times\varphi &= \frac{2(1-\nu)}{2\mu}\nabla^2\mathrm{V} \end{aligned} \tag{4.13}$$

Notice that if V is taken to be harmonic, then the curl of $\boldsymbol{\varphi}$ will vanish and the scalar potential ϕ will also be harmonic. This case then reduces to Lamé's strain potential presented in the previous section. With zero body forces, the stresses corresponding to the Galerkin representation are given by

$$\begin{aligned} \sigma_x &= 2(1-\nu)\frac{\partial}{\partial x}\nabla^2 V_x + \left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)\nabla\cdot\mathrm{V} \\ \sigma_y &= 2(1-\nu)\frac{\partial}{\partial y}\nabla^2 V_y + \left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)\nabla\cdot\mathrm{V} \\ \sigma_z &= 2(1-\nu)\frac{\partial}{\partial z}\nabla^2 V_z + \left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\nabla\cdot\mathrm{V} \\ \tau_{xy} &= (1-\nu)\left(\frac{\partial}{\partial y}\nabla^2 V_x + \frac{\partial}{\partial x}\nabla^2 V_y\right) - \frac{\partial^2}{\partial x\partial y}\nabla\cdot\mathrm{V} \\ \tau_{yz} &= (1-\nu)\left(\frac{\partial}{\partial z}\nabla^2 V_y + \frac{\partial}{\partial y}\nabla^2 V_z\right) - \frac{\partial^2}{\partial y\partial z}\nabla\cdot\mathrm{V} \\ \tau_{zx} &= (1-\nu)\left(\frac{\partial}{\partial y}\nabla^2 V_z + \frac{\partial}{\partial z}\nabla^2 V_x\right) - \frac{\partial^2}{\partial z\partial x}\nabla\cdot\mathrm{V} \end{aligned} \tag{4.14}$$

As previously mentioned, for no body forces the Galerkin vector must be biharmonic. In Cartesian coordinates, the general biharmonic vector equation would decouple, and thus each component of the Galerkin vector would satisfy the scalar biharmonic equation. However, in curvilinear coordinate systems (such as cylindrical or spherical), the unit vectors are functions of the coordinates, and this will not in general allow such a simple decoupling. Equation (1.72) provides the general form for the Laplacian of a vector, and the expression for polar coordinates is given in Example 1-3 by relation (1.75). Therefore, in curvilinear coordinates the individual components of the Galerkin vector do not necessarily satisfy the biharmonic equation. For cylindrical coordinates, only the z component of the

Galerkin vector satisfies the biharmonic equation, while the other components satisfy a more complicated fourth-order partial differential equation.

Before moving on to specific applications, we investigate a few useful relationships dealing with harmonic and biharmonic functions. Consider the following identity:

$$\nabla^2(xf) = x\nabla^2 f + 2\frac{\partial f}{\partial x}$$

Taking the Laplacian of this expression gives

$$\nabla^2(xf) = \nabla^2(x\nabla^2 f) + 2\frac{\partial}{\partial x}(\nabla^2 f)$$

and thus if f is harmonic, the product xf is biharmonic. Obviously, for this result the coordinate x could be replaced by y or z. Likewise we can also show by standard differentiation that the product $R^2 f$ will be biharmonic if f is harmonic, where $R^2 = x^2 + y^2 + z^2$. Using these results, we can write the following generalized representation for a biharmonic function g as

$$\mathrm{g} = f_0 + xf_1 + yf_2 + zf_3 + \frac{1}{2}R^2 f_4 \tag{4.15}$$

where f_i are arbitrary harmonic functions. It should be pointed out that not all of the last four terms of (4.15) are independent.

Consider now the special Galerkin vector representation where only the z component of $\boldsymbol{V}$ is nonvanishing; that is, $V = V_z \boldsymbol{e}_z$. For this case, the displacements are given by

$$2\mu u = 2(1-\nu)\nabla^2 V_z \boldsymbol{e}_z - \nabla\left(\frac{\partial \mathrm{V_z}}{\partial \mathrm{z}}\right) \tag{4.16}$$

With zero body forces, V_z will be biharmonic, and this case is commonly referred to as Love's strain potential. A special case of this form was introduced by Love (1944) in studying solids of revolution under axisymmetric loading.

For this case the displacements and stresses in Cartesian coordinates become

$$
\begin{aligned}
&2\mu u = \frac{\partial^2 V_z}{\partial x \partial z}, 2\mu v = -\frac{\partial^2 V_z}{\partial y \partial z}, 2\mu w = 2(1-\nu)\nabla^2 V_z - \frac{\partial^2 V_z}{\partial z^2} \\
&\sigma_x = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)V_z, \tau_{xy} = -\frac{\partial^3 V_z}{\partial x \partial y \partial z} \\
&\sigma_y = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)V_z, \tau_{yz} = \frac{\partial}{\partial y}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z \\
&\sigma_z = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z, \tau_{zx} = \frac{\partial}{\partial x}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z
\end{aligned}
\tag{4.17}
$$

The corresponding relations in cylindrical coordinates are given by

$$
\begin{aligned}
&2\mu u_r = \frac{\partial^2 V_z}{\partial r \partial z}, 2\mu u_\theta = -\frac{1}{r}\frac{\partial^2 V_z}{\partial \theta \partial z}, 2\mu u_z = 2(1-\nu)\nabla^2 V_z - \frac{\partial^2 V_z}{\partial z^2} \\
&\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)V_z, \tau_{r\theta} = -\frac{\partial^3}{\partial r \partial \theta \partial z}\left(\frac{V_z}{r}\right) \\
&\sigma_\theta = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)V_z, \tau_{\theta z} = \frac{1}{r}\frac{\partial}{\partial \theta}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z \\
&\sigma_z = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z, \tau_{zr} = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)V_z
\end{aligned}
\tag{4.18}
$$

We now consider some example applications for axisymmetric problems where the field variables are independent of θ.

EXAMPLE 4-1: Kelvin's Problem: Concentrated Force Acting in the Interior of an Infinite Solid:

Consider the problem (commonly referred to as Kelvin's problem) of a single concentrated force acting at a point in the interior of an elastic solid. For convenience we choose a coordinate system such that the force is applied at the origin and acts in the z direction (see Figure 4-1). The general boundary conditions on this problem would require that the stress field vanish at infinity, be singular at the origin, and give the resultant force system $P\boldsymbol{e}_z$ on any surface enclosing the origin.

The symmetry of the problem suggests that we can choose the Love/Galerkin

potential as an axisymmetric form $V_z(r,z)$. In the absence of body forces, this function is biharmonic, and using the last term in representation (4.15) with $f_4 = 1/R$ gives the trial potential

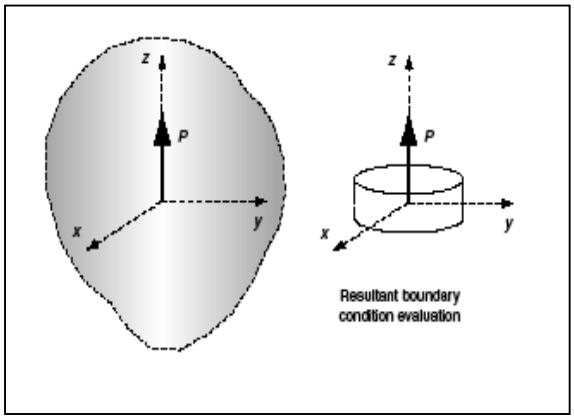


FIGURE 4-1 Kelvin's problem: concentrated force in an infinite medium.

$$V_z = AR = A\sqrt{r^2 + y^2} \tag{4.19}$$

where A is an arbitrary constant to be determined. We shall now show that this potential produces the correct stress field for the concentrated force problem under study.

The displacement and stress fields corresponding to the proposed potential follow from relations (4.18)

$$\begin{aligned}
&2\mu u_r = \frac{Arz}{R^3}, 2\mu u_\theta = 0, 2\mu u_z = A\left(\frac{2(1-2\nu)}{R} + \frac{1}{R} + \frac{z^2}{R^3}\right)\\
&\sigma_r = A\left(\frac{2(1-2)z}{R^3} - \frac{3r^2 z}{R^5}\right), \tau_{r\theta} = 0\\
&\sigma_\theta = A\frac{(1-2)z}{R^3}, \tau_{\theta z} = 0\\
&\sigma_z = -A\left(\frac{(1-2)z}{R^3} + \frac{3z^3}{R^5}\right), \tau_{zr} = -A\left(\frac{(1-2)z}{R^3} + \frac{3rz^2}{R^5}\right)
\end{aligned} \tag{4.20}$$

Clearly, these stresses (and displacements) are singular at the origin and vanish at infinity. To analyze the resultant force condition, consider an arbitrary cylindrical surface enclosing the origin as shown in Figure 4-1. For convenience, we choose the cylinder to be bounded at $z = \pm a$ and will let the radius tend to infinity. Invoking

vertical equilibrium, we can write

$$\int_0^\infty 2\pi r\sigma_z(r,a)dr - \int_0^\infty 2\pi r\sigma_z(r,-a)dr + \int_{-a}^{a} 2\pi r\tau_{rz}(r,z)dz + P = 0 \qquad (4.21)$$

The first two terms in (4.21) can be combined, and in the limit as $r \to \infty$ the third integral is found to vanish, thus giving

$$\begin{aligned} P &= -2\int_a^\infty 2\pi r\sigma_z(r,a)dR \\ &= 4\pi A\left[(1-2\nu)a\int_a^\infty \frac{RdR}{R^3}3a^3 + \int_a^\infty \frac{RdR}{R^5}\right] \\ &= 8\pi(1-\nu)A \end{aligned} \qquad (4.22)$$

The constant is now determined and the problem is solved. Of course, the stress field is linearly related to the applied loading, and typically for such three-dimensional problems the field also depends on Poisson's ratio.

EXAMPLE 4-2: Boussinesq's Problem: Concentrated Force Acting Normal to the Free Surface of a Semi-Infinite Solid:

Several other related concentrated force problems can be solved by this method. For example, consider Boussinesq's problem of a concentrated force acting normal to the free surface of a semi-infinite solid, as shown in Figure 4-2. Recall that the corresponding two-dimensional problem was solved in (Flamant's problem) .

This problem can be solved by combining a Galerkin vector and Lamé's strain potential of the forms

$$\begin{aligned} &V_x = V_y = 0, V_z = AR \\ &\phi = B\log(R+z) \end{aligned} \qquad (4.23)$$

Using similar methods as in the previous example, it is found that the arbitrary constants become

$$A = \frac{P}{2\pi}, B = -\frac{(1-2\nu)P}{2\pi} \qquad (4.24$$

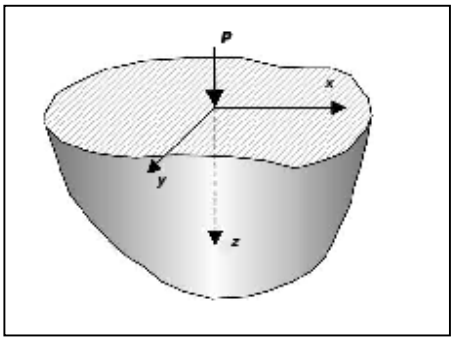


FIGURE 4-2 Boussinesq's problem: normal force on the surface of a half space.

EXAMPLE 4-3: Cerruti's Problem: Concentrated Force Acting Parallel to the Free Surface of a Semi-Infinite Solid:

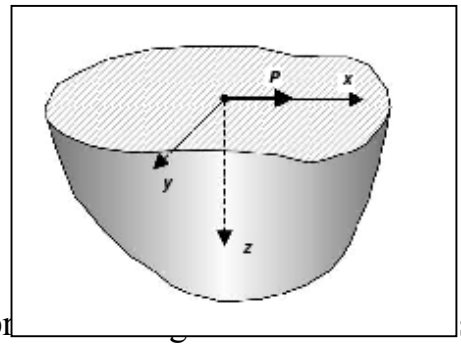
Another related example is Cerruti's problem of a concentrated force acting parallel to the free surface of an elastic half space (see Figure 4-3). For convenience, the force is chosen to be directed along the x-axis as shown. Although this problem is not axisymmetric, it can be solved by combining a particular Galerkin vector and Lamé's strain potential of the following forms:

$$V_x = AR, V_y = 0, V_z = Bx\log(R+z)$$
$$\phi = \frac{Cx}{R+z} \qquad (4.25)$$

Again, using methods from the previous examples, the constants are found to be

$$A = \frac{P}{4\pi(1-\nu)}, B = \frac{(1-2\nu)P}{4\pi(1-\nu)}, C = \frac{(1-2\nu)P}{2\pi} \qquad (4.26)$$

The displacements and stresses follow from relations (4.9) and (4.17) .

FIGURE 4-3 Cerruti's pr 

surface of a half space.

Section(4.2):Papkovich-Neuber Representation:

Using scalar and vector potential functions, another general solution to Navier's equations was developed by Papkovich [8] and later independently by Neuber (1934). The completeness of this representation was shown by Eubanks and Sternberg [10], and thus all elasticity solutions are representable by this scheme. We outline the development of this solution by first writing Navier's equation in the form

$$\nabla^2 u + \frac{1}{1-2\nu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot u) = -\frac{\boldsymbol{F}}{\mu} \tag{4.27}$$

Using the Helmholtz representation (4.1) and relation (4.3), this previous equation can be written as

$$\nabla^2\left[u + \frac{1}{(1-2\nu)}\boldsymbol{\nabla}\phi\right] = -\frac{\boldsymbol{F}}{\mu} \tag{4.28}$$

Define the vector term in the brackets as

$$\boldsymbol{h} = \boldsymbol{u} + \frac{1}{(1-2\nu)}\boldsymbol{\nabla}\phi \tag{4.29}$$

We note that

$$\nabla^2 h = -\boldsymbol{F}/\mu, \boldsymbol{\nabla}\cdot\boldsymbol{h} = \frac{2(1-\nu)}{1-2\nu}\nabla^2\phi \tag{4.30}$$

Using the identity $\nabla^2(\boldsymbol{R}\cdot\boldsymbol{h}) = \boldsymbol{R}\cdot\nabla^2\boldsymbol{h} + 2(\boldsymbol{\nabla}\cdot\boldsymbol{h})$, it can be shown that

$$\boldsymbol{\nabla}\cdot\boldsymbol{h} = \frac{1}{2}\left(\nabla^2(\boldsymbol{R}\cdot\boldsymbol{h}) + \frac{\boldsymbol{R}\cdot\boldsymbol{F}}{\mu}\right) \tag{4.31}$$

Combining results (4.4.5) with (4.4.4) gives

$$\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\phi-\frac{1}{2}\boldsymbol{R}\cdot\boldsymbol{h}\right]=\frac{\boldsymbol{R}\cdot\boldsymbol{F}}{\mu} \tag{4.32}$$

Defining the term in brackets by scalar h, we get

$$\nabla^2=\frac{\boldsymbol{R}\cdot\boldsymbol{F}}{\mu} \tag{4.33}$$

Using the definition of h, we can eliminate ϕ from relation (4.4.3) and obtain an expression for the displacement vector.

Redefining new scalar and vector potentials in terms of h and $\boldsymbol{h}$, we can write

$$2\mu u=A-\boldsymbol{\nabla}\left[B+\frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)}\right] \tag{4.34}$$

where

$$\nabla^2 A=-2F,\nabla^2 B=\frac{\boldsymbol{R}\cdot\boldsymbol{F}}{(1-\nu)} \tag{4.35}$$

This general displacement representation is the Papkovich-Neuber solution of Navier's equations. For the case with zero body forces, the two potential functions $\boldsymbol{A}$ and B are harmonic. The four individual functions A_x, A_y, A_z, and B, however, are not all independent, and it can be shown that for arbitrary three-dimensional convex regions, only three of these functions are independent. Note that a convex region is one in which any two points in the domain may be connected by a line that remains totally within the region.

Comparing the Galerkin vector representation (4.11) with the Papkovich solution (4.34) it is expected that a relationship between the two solution types should exist, and it can be easily shown that

$$\boldsymbol{A}=2(1-\nu)\nabla^2\boldsymbol{V},B=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}-\frac{\boldsymbol{A}\cdot\boldsymbol{R}}{4(1-\nu)} \tag{4.36}$$

As with the Galerkin vector solution, it is convenient to consider the special case of axisymmetry where

$$A_r = A_\theta = 0, A_z = A_z(r,z), B = B(r,z)$$
$$\text{with } \nabla^2 B = 0 \quad \text{and} \quad \nabla^2 A_z = 0 \tag{4.37}$$

For this axisymmetric case, B and A_z are commonly called the Boussinesq potentials, and as before with zero body forces they are harmonic functions.

EXAMPLE 4-4: Boussinesq's Problem Revisited:

We consider again the problem shown previously in Figure 4-2 of a concentrated force acting normal to the stress-free surface of a semi-infinite solid. Because the problem is axisymmetric, we use the Boussinesq potentials defined by (4.37) These potentials must be harmonic functions of r and z, and using (4.10), we try the forms

$$A_z = \frac{C_1}{R}, B = C_2 \log(R + z) \tag{4.38}$$

where C_1 and C_2 are constants to be determined.

The boundary conditions on the free surface require that $\sigma_z = \tau_{rz} = 0$ everywhere except at the origin, and that the summation of the total vertical force be equal to P. Calculation of these stresses follows using the displacements from (4.34) in Hooke's law, and the result is

$$\sigma_z = -\frac{3C_1 z^3}{4(1-\nu)R^5}$$
$$\tau_{rz} = \frac{r}{R^3}\left(C_2 - \frac{(1-2\nu)}{4(1-\nu)}C_1 - \frac{3C_1 z^2}{4(1-\nu)R^2}\right) \tag{4.39}$$

Note that the expression for σ_z vanishes on $z = 0$, but is indeterminate at the origin, and thus this relation will not directly provide a means to determine the constant C_1. Rather than trying to evaluate this singularity at the origin, we pursue the integrated condition on any typical plane $z =$ constant

$$P = -\int_0^\infty \sigma_z(r,z) 2\pi r dr \tag{4.40}$$

Invoking these boundary conditions determines the two constants

$$C_1 = \frac{2(1-\nu)}{\pi}P, C_2 = \frac{(1-2\nu)}{2\pi}P \tag{4.41}$$

The results for the displacements and stresses are given by

$$\begin{aligned} u_r &= \frac{P}{4\pi\mu R}\left[\frac{rz}{R^2} - \frac{(1-2\nu)r}{R+z}\right] \\ u_z &= \frac{P}{4\pi\mu R}\left[2(1-\nu)\frac{z^2}{R^2}\right] \\ u_\theta &= 0 \end{aligned} \tag{4.42}$$

$$\begin{aligned} \sigma_r &= \frac{P}{2\pi R^2}\left[-\frac{3r^2 z}{R^3} + \frac{(1-2\nu)R}{R+z}\right] \\ \sigma_\theta &= \frac{(1-2\nu)P}{2\pi R^2}\left[\frac{z}{R} - \frac{R}{R+z}\right] \\ \sigma_z &= -\frac{3Pz^3}{2\pi R^5}, \tau_{rz} = -\frac{3Prz^2}{2\pi R^5} \end{aligned} \tag{4.43}$$

Many additional problems can be solved using the Papkovich method, and some of these are given in the exercises. This technique also is used in the next chapter to generate solutions for many singular stress states employed in micromechanics modeling.

An interesting connection can be made for the two-dimensional case between the Papkovich-Neuber scheme and the complex variable method discussed . For the case of plane deformation in the x, y-plane, we choose

$$A_x = A_x(x,y), A_y = A_y(x,y), A_z = 0, B = B(x,y) \tag{4.44}$$

Using the general representation (4.38), it can be shown that for the plane strain case

$$2\mu(u+iv) = (3-4\nu)\gamma(z) - \overline{z\gamma'(z)} - \overline{\psi(z)} \tag{4.45}$$

with appropriate selection of $\gamma(z)$ and $\psi(z)$ in terms of A_x, A_y, and B. It is noted that this form is identical to (10.2.9) found using the complex variable formulation.

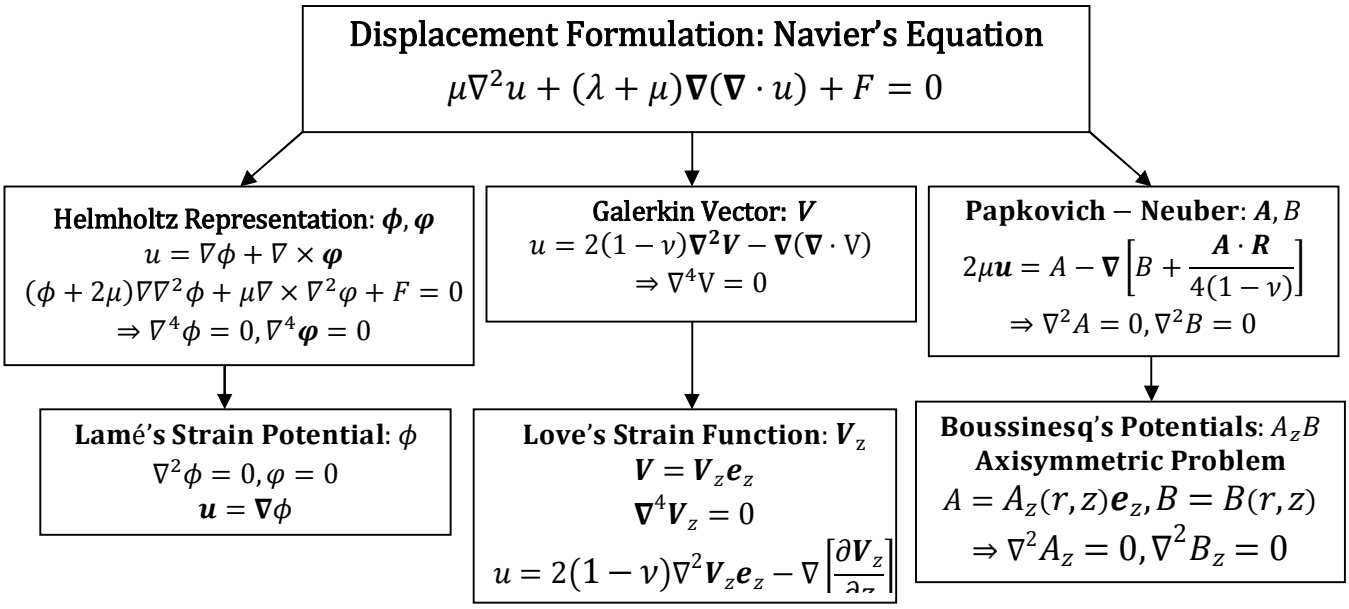


FIGURE 4-4 Displacement potential solutions.

A convenient summary flow chart of the various displacement functions discussed in this chapter is shown in Figure 4-4. The governing equations in terms of the particular potential functions are for the zero body force case. Chou and Pagano [2] provide additional tables for displacement potentials and stress functions.

Spherical Coordinate Formulations:

The previous solution examples employing displacement potentials simply used preselected forms of harmonic and biharmonic potentials. We now investigate a more general scheme to determine appropriate potentials for axisymmetric problems described in spherical coordinates. Referring to Figures 1-4 and 1-5, cylindrical coordinates (r, θ, z) are related to spherical coordinates (R, ϕ, θ) through relations

Restricting attention to axisymmetric problems, all quantities are independent of y, and thus we choose the axisymmetric Galerkin vector representation. Recall that this lead to Love's strain potential V_z, and the displacements and stresses were given by relations (4.16) to (4.18). Because this potential function was biharmonic, consider

first solutions to Laplace's equation. In spherical coordinates the Laplacian operator becomes

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\cot\phi\frac{\partial}{\partial\phi} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} \tag{4.46}$$

We first look for separable solutions of the form $R^n\Phi_n(\phi)$, and substituting this into Laplace's equation gives

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi_n}{d\phi}\right) + n(n+1)\Phi_n = 0 \tag{4.47}$$

Next, making the change of variable $x = \cos\phi$, relation (4.47) becomes

$$(1-x^2)\frac{d^2\Phi_n}{dx^2} - 2x\frac{d\Phi_n}{dx} + n(n+1)\Phi_n = 0 \tag{4.48}$$

and this is the well-known Legendre differential equation. The two fundamental solutions are the Legendre functions $P_n(x)$ and $Q_n(x)$ of the first and second kinds. However, only $P_n(x)$ is continuous for $|x| \leq 1, (0 \leq \phi \leq n)$, and so we drop the solution $Q_n(x)$. Considering only the case of integer values of parameter n, the solution reduces to the Legendre polynomials given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)}{dx^n} \tag{4.49}$$

where $P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2-1), \ldots$. Putting these results together gives the following harmonic solution set:

$$\{R^n\Phi_n\} = \left\{1, z, z^2 - \frac{1}{3}(r^2+z^2), z^3 - \frac{3}{5}z(r^2+z^2), \ldots\right\} \tag{4.50}$$

These terms are commonly referred to as spherical harmonics.

Our goal, however, is to determine the elasticity solution that requires biharmonic functions for the Love/Galerkin potential. In order to construct a set of biharmonic functions,weemploy the last term in relation (4.15) and thus argue that if $R^n\Phi_n$ are harmonic, $R^{n+2}\Phi_n$ will be biharmonic. Thus, a representation for the Love strain potential may be written as the linear combination

$$V_z = B_0(r^2+z^2) + B_1 z(r^2+z^2) + B_2(z^2-r^2)(r^2+z^2) + A_0 + A_1 z + A_2\left[z^2 - \frac{1}{3}(r^2+z^2)\right] + \cdots \tag{4.51}$$

It can be shown that this solution form is useful for general problems with finite domains. However, for the case involving infinite regions, this form will result in unbounded displacements and stresses at infinity. Therefore, (4.51) must be modified for use in regions that extend to infinity. This modification is easily developed by noting that the coefficient $n(n+1)$ in governing equation (4.47) will be the same if we were to replace n by $(-n-1)$. This then implies that solution forms $R^{-n-1}\Phi_{-n-1} = R^{-n-1}\Phi_n$ will also be harmonic functions. Following our previous construction scheme, another set of biharmonic functions for the potential function can be expressed as

$$V_z = B_0(r^2+z^2)^{1/2} + B_1 z(r^2+z^2)^{-1/2} + \cdots + A_0(r^2+z^2)^{-1/2} + A_1 z(r^2+z^2)^{-3/2} + \cdots \tag{4.52}$$

and this form will be useful for infinite domain problems. For example, the solution to the Kelvin problem in Example 4-1 can be found by choosing only the first term in relation (4.52). This scheme can also be employed to construct a set of harmonic functions for the Papkovich potentials; see Little (1973).

EXAMPLE 4-5: Spherical Cavity in an Infinite Medium Subjected to Uniform Far-Field Tension:

Consider the problem of a stress-free spherical cavity in an infinite elastic solid that is subjected to a uniform tensile stress at infinity. The problem is shown in Figure 4-5, and for convenience we have oriented the z-axes along the direction of the uniform far-field stress S.

We first investigate the nature of the stress distribution on the spherical cavity caused solely by the far-field stress. For the axisymmetric problem, the spherical stresses are related to the cylindrical components by the equations

$$\begin{aligned} \sigma_R &= \sigma_r \sin^2\phi + \sigma_z \cos^2\phi + 2\tau_{rz}\sin\phi\cos\phi \\ \sigma_\phi &= \sigma_z \sin^2\phi + \sigma_r \cos^2\phi - 2\tau_{rz}\sin\phi\cos\phi \\ \tau_{R\phi} &= (\sigma_r - \sigma_z)\sin\phi\cos\phi - \tau_{rz}(\sin^2\phi - \cos^2\phi) \end{aligned} \tag{4.53}$$

Therefore, the far-field stress $\sigma_z^\infty = S$ produces normal and shearing stresses on the spherical cavity of the form

$$\sigma_R = S\cos^2\phi\,, \tau_{R\phi} = -S\sin\phi\cos\phi \tag{4.54}$$

Using particular forms from our general solution (4.52), we wish to superimpose additional stress fields that will eliminate these stresses and vanish at infinity.

It is found that the superposition of the following three fields satisfies the problem requirements:

1. Force doublet in z direction: This state corresponds to a pair of equal and opposite forces in the z direction acting at the origin. The solution is formally determined from the combination of two equal but opposite Kelvin solutions from Example 4-1. The two forces are separated by a distance d, and the limit is taken as $d \to 0$. This summation and limiting process yields a state that is actually the derivative $(\partial/\partial z)$ of the original Kelvin field with a new coefficient of $-Ad$. This coefficient is denoted as K_1.

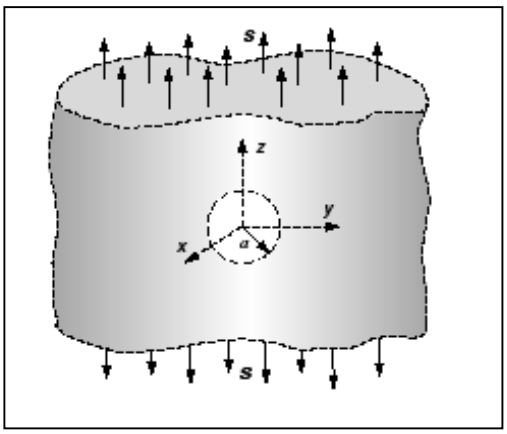


FIGURE 4-5 Spherical cavity in an infinite medium under tension.

2. Center of dilatation: This field is the result of three mutually orthogonal doubleforce pairs from the previous state (1) . The coefficient of this state is denoted by K_2.
3. Particular biharmonic term: A state corresponding to the A_1 term from equation (4.5).

Combining these three terms with the uniform far-field stress and using the

condition of zero stress on the spherical cavity provide sufficient equations to determine the three unknown constants. Details of this process can be found in Timoshenko and Goodier [9], and the results determine the coefficients of the three superimposed fields

$$\begin{aligned} K_1 &= -\frac{5Sa^3}{2(7-5\nu)} \\ K_2 &= -\frac{S(1-5\nu)a^3}{(7-5\nu)} \\ A_1 &= -\frac{Sa^3}{2(7-5\nu)} \end{aligned} \tag{4.55}$$

Using these constants, the stress and displacement fields can be determined.

The normal stress on the x, y-plane $(z = 0)$ is given by

$$\sigma_z(r, 0) = S\left(1 + \frac{4 - 5\nu}{2(7 - 5\nu)r^3} + \frac{9}{2(7 - 5\nu)}\frac{a^5}{r^5}\right) \tag{4.56}$$

At $r = a$, this result produces the maximum stress

$$\sigma_z(a, 0) = (\sigma_z)_{\max} = \frac{27 - 15\nu}{2(7 - 5\nu)}S \tag{4.57}$$

Typically, for many metals, $\nu = 0.3$, and this would give a stress concentration factor of

$$\frac{(\sigma_z)_{\max}}{S} = \frac{45}{22} = 2.04 \tag{4.58}$$

It should be noted that in three dimensions the stress concentration factor is generally a function of Poisson's ratio. A plot of this general behavior given by equation (4.57) is shown in Figure 4-6. It can be observed that the value of Poisson's ratio produces only small variation on the stress concentration.

Note that the corresponding two-dimensional case was previously developed in Example 8-7 and produced a stress concentration factor of 3. Plots of the corresponding two- and three-dimensional stress distributions are shown in Figure 4-7. For each case the normal stress component in the direction of loading is plotted versus radial distance away from the hole. It is seen that the three-dimensional

stresses are always less than two-dimensional predictions. This is to be expected because the three-dimensional field has an additional dimension to decrease the concentration caused by the cavity. Both stress concentrations rapidly decay away from the hole and essentially vanish at $r > 5a$. Additional information on this problem is given by Timoshenko and Goodier [9].

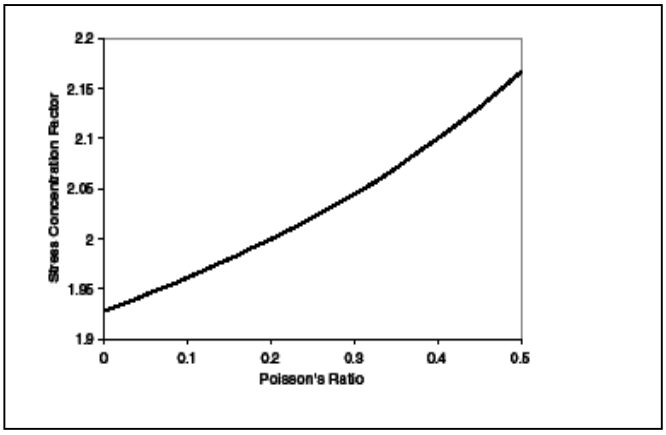


FIGURE 4-6 Stress concentration factor behavior for the spherical cavity problem.

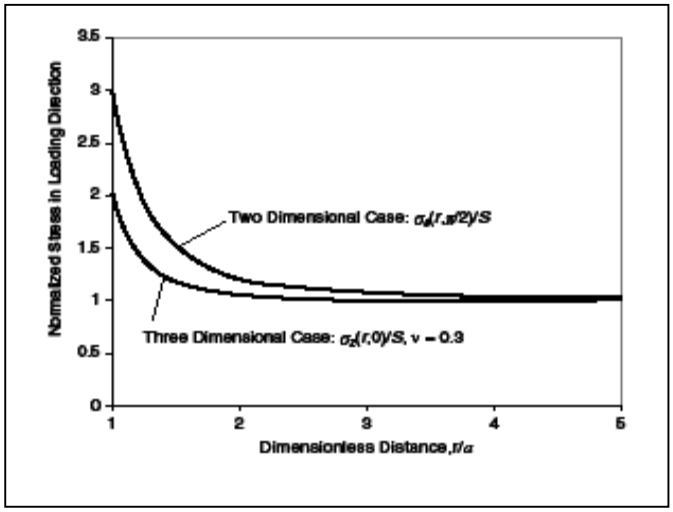


FIGURE 4-7 Comparison of two- and three-dimensional stress concentrations around .In the absence of body forces, the stress formulation of elasticity theory includes the equilibrium and Beltrami-Michell equations:

$$\sigma_{ij,j} = 0 \tag{4.59}$$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,il} = 0 \tag{4.60}$$

In order to develop a general solution to this system, stress functions are commonly used. Of course, we have already seen the use of several special stress functions earlier in the text, including Airy's form for the plane problem and Prandtl's function for the torsion example. Here, we investigate the general three-dimensional case and later specialize these results to some of the particular cases just mentioned. The concept of developing a stress function involves the search for a representation of the form

$$\sigma_{ij} = F_{ij}\{\Phi\} \tag{4.61}$$

where F_{ij} is some differential operator and F is a tensor-valued variable. Normally, the search looks for forms that automatically satisfy the equilibrium equations (4.59), and these are called self-equilibrated forms.

It is apparent that the equilibrium equations will be satisfied if σ_{ij} is expressed as the curl of some vector function, because the divergence of a curl vanishes identically. It can be shown that one such equilibrated form that provides a complete solution to the elasticity problem is given by

$$\sigma_{ij} = \varepsilon_{imp}\varepsilon_{jkl}\Phi_{mk,pl} \tag{4.62}$$

where Φ is a symmetric second-order tensor. Relation (4.62) is sometimes referred to as the Beltrami representation, and Φ is called the Beltrami stress function. It was shown by Carlson (1966) that all elasticity solutions admit this representation. It is easily demonstrated that (4.62) is an equilibrated form, since

$$\sigma_{ij,j} = \left(\varepsilon_{imp}\varepsilon_{jkl}\Phi_{mk,pl}\right)_{,j} = \varepsilon_{imp}\varepsilon_{jkl}\Phi_{mk,plj} = 0$$

because of the product of symmetric and antisymmetric forms in indices jl.

Property (1.15) allows expansion of the alternating symbol product, and thus relation (4.62) can be expressed as

$$\sigma_{ij} = \delta_{ij}\Phi_{kk,ll} - \delta_{ij}\Phi_{kl,kl} - \Phi_{ij,kk} + \Phi_{li,lj} + \Phi_{lj,li} - \Phi_{kk,ij} \tag{4.63}$$

or

$$
\begin{aligned}
\sigma_{11} &= \Phi_{33,22} + \Phi_{22,33} - 2\Phi_{23,23} \\
\sigma_{22} &= \Phi_{11,33} + \Phi_{33,11} - 2\Phi_{31,31} \\
\sigma_{33} &= \Phi_{22,11} + \Phi_{11,22} - 2\Phi_{12,12} \\
\sigma_{12} &= -\Phi_{12,33} - \Phi_{33,12} + \Phi_{23,13} + \Phi_{31,23} \\
\sigma_{23} &= -\Phi_{23,11} - \Phi_{11,23} + \Phi_{31,21} + \Phi_{12,31} \\
\sigma_{31} &= -\Phi_{31,22} - \Phi_{22,31} + \Phi_{12,32} + \Phi_{23,12}
\end{aligned}
\tag{4.64}
$$

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