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Separability Problems and Finitely Strictly Singular Operators Between James Spaces

مسائل قابلية الإنفصال والمؤثرات الشاذة فعلياً و المنتهية بين فضاءات جيمس

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Dedication

To my parents, brothers, sisters and husband.

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Abstract

We give characterizations of isometric shift operators and Backward shifts on Banach spaces with linear isometries between subspaces of continuous functions. We show the inverse spectral theory for the Ward equation and for the 2+1. Chiral model, we also consider the isometric shifts and metric spaces. We also study the Cauchy problem of the Ward equation. We discuss the relative Position of four subspaces in of Hilbert space, with an indecomposable representations ofQuivers on infinite-dimensional Hilbert spaces. We give the structure of type 1 shifts with the separability problem for isometric shifts on the space of continuous functions. Strictly Singular operators and the invariant subspace problems are shown. We establish the finitely Strictly Singular operators between James spaces

الخلاصة

أعطينا تشخيصات مؤثرات الإزاحة متساوية القياس والإزاحات إلي الوراء علي فضاءات باناخ مع تساوى القياسات الخطية بين الفضاءات الجزئية للدوال المستمرة. تم توضيح نظرية الطيف العكسية لمعادلة وارد لنموزج شيرال 1+2. أيضا أعتبرنا الإزاحات متساوية القياس والفضاءات المترية. أيضا درسنا مسألة كوشى لمعادلة وارد. تمت مناقشة الوضع النسبي للفضاءات الجزئية الأربعة لفضاء هلبرت مع التمثيلات التي لا تنحل لعناصر أساسية للإرتجاف على فضاءات هلبرت لانهائية البعد. أعطينا بناء إزاحات النوع 1 مع مسألة الفصوصية للإزاحات متساوية القياس على فضاء الدوال المستمرة. أوضحنا المؤثرات الشاذة التامة ومسألة الفضاء الجزئي اللامتغير. تم تاسيس المؤثرات الشاذة والمنتهية بين فضاءات جيمس.

Introduction

We obtain many significiant results concerning shift operators on Banach spaces.

Using a result of Holsztynski we classify isometric shift operators on $C(X)$ for any compact Hausdorff space X into two (not necessarily disjoint) classes. If there exists an isometric shift operator $T: C(X) \to C(X)$ of type II, they show that X is necessarily separable. In case T is of type I, we exhibit a *paticular* infinite countable set $D = \{p, \psi^{-1}(p), \psi^{-2}(p), \psi^{-3}(p), \ldots\}$ of isolated points in X . Under the additional assumption that the linear functional Γ carrying $f \in C(X)$ to $Tf(p) \in \mathbb{C}$ is identically zero, we show that D is dense in X. We show that the Banach space $C(X)$. of real valued continuous functions does not admit a backward shift, if X is a compact Hausdorff space with an infinite connected component. We say that a linear subspace A of $C_0(X)$ is strongly separating if given any pair of distinct points x_1, x_2 of the locally compact space X, then there exists $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$. We show that a linear isometry T of A onto such a subspace B of $C_0(Y)$ induces a homeomorphism h between two certain singular subspaces of the Shilov boundaries of B and A, sending the Choquet boundary of B onto the Choquet boundary of A. We also provide an example which shows that the above result is no longer true if we do not assume A to be strongly separating. We solve the Cauchy problem of the Ward model in light-cone coordinates using the inverse spectral (scattering) method. Let M be a complete metric space. If $C^*(M)$ admits an isometric shift, then M is separable. We generalize the results of study the inverse scattering problem of the Ward equation with non-small data and solve the Cauchy problem of the Ward equation with a non-small purely continuous scattering data. We study the relative position of several subspaces in a separable infinite-dimensional Hilbert space. In finite-dimensional case, Gelfand and Ponomarev gave a complete classification of indecomposable systems of four subspaces. We construct exotic examples of indecomposable systems of four subspaces in infinite-dimensional Hilbert spaces. We study indecomposable representations of quivers on separable infinite-dimensional Hilbert spaces by bounded operators. We exhibit several concrete examples and investigate duality theorem between reflection functors. We provide some "structure" theorems for analyzing type 1 isometric shifts by characterizing the functions in the range of T^n. We provide examples of nonseparable spaces X for which $C(X)$ admits an isometric shift, which solves in the negative a problem proposed by Gutek et al. Properties of strictly singular operators have recently become of topical interest because the work of Gowers and Maurey gives Banach spaces on which every continuous operator is of form $\lambda I + S$, where S is strictly singular. So if strictly singular operators had invariant subspaces, such spaces would have the property that all operators on them had invariant subspaces. An operator $T : X \to Y$ between Banach spaces is said to be finitely strictly singular if for every $\varepsilon > 0$ there exists n such that every subspace $E \subseteq X$ with $\dim E \geq n$ contains a vector x such that $\|Tx\| < \varepsilon \|x\|$. We show that, for $1 \leq p < q < \infty$, the formal inclusion operator from J_p to J_q is finitely strictly singular.

Chapter 1

Backward and Isometric Shifts on Banach Spaces

We give a negative answer to this question. In fact, given any integer $l \geq 1$, we construct an example of an isometric shift operator $T: C(X) \to C(X)$ of type I with $X \setminus \overline{D}$ having exactly l elements, where $\overline{D}$ is the closure of D in X. We show that for arbitrary infinite compact Hausdorff spaces of J. R. Holub $X, C(X)$. does not admit a backward shift.

Section (1.1): Operators of Isometric Shift on Continuous Function Spaces.

R. M. Crownover [200] was the first person to give a basis free definition of a shift on a general Banach space. In [201] J. R. Holub studied isometric shift operators on $C_{\mathbb{R}}(X)$, where $C_{\mathbb{R}}(X)$ is the real Banach space of real valued continuous functions on the compact Hausdorff space X. One of the results proved by him asserts that if X has only finitely many components then $C_{\mathbb{R}}(X)$ does not admit an isometric shift operator. However his techniques do not carry over to the complex Banach space $C_{\mathbb{C}}(X)$. In [202] Gutek *et al* study simultaneously the real as well as the complex case.

The convention that maps between topological spaces are necessarily continuous. In the work of Gutek *et al* [202] a crucial role is played by a result of W. Holsztynski [203] which essentially describes the form of a linear isometry $T: C(X) \to C(Y)$ where X and Y are any two compact Hausdorff spaces. Here $C(X)$ denotes the complex Banach space of complex valued continuous functions on X. Using Holsztynski's result they classify isometric shift operators $T: C(X) \to C(X)$ into two (We denote the range of an operator T by $R(T)$. After proving Theorem (1.1.1) correctly remark that the only element $f \in R(T)$ vanishing on X_0 (using the notation in [202]) of [202] they further assert that when $X_0 \neq X$, the above observation gives the "uniqueness" of p where $X_0 = X \setminus \{p\}$. In [202] turns out to be an isometric shift operator expressible as a shift operator in *two different ways.* Also it turns out that any isometric shift operator $T: C(X) \to C(X)$ expressible as an operator of type I in two different ways is automatically. But the converse is not true. We will give a specific example of an isometric shift operator which is simultaneously. But is expressible as an operator in exactly one way.

Let $T: C(X) \to C(X)$ be an isometric shift operator of type I which is not of type II. Then our observation in the earlier paragraph yields a *unique* isolated point p in X, a homeomorphism $\psi: X_0 \to X$ where $X_0 = X \setminus \{p\}$ and a map $w: X_0 \to S'$ satisfying

T

The statement "the only element $f \in R(T)$ vanishing on X_0 is 0" is equivalent to asserting that the characteristic function χ_p of p is not in $R(T)$. A natural question is whether p is the only isolated point in X with $\chi_p \notin R(T)$. We will also see that the answer to this question is negative. In [202] satisfies the condition that none of χ_1, χ_2 and χ_3 is $R(T)$.

Let $T: C(X) \to C(Y)$ be any linear isometry. In [203] Holsztynski gives a *specific construction* yielding a well determined closed subset Y_0 of Y and well determined maps $\psi: Y_0 \to X, w: Y_0 \to S^1$ with ψ surjective and satisfying

One of our major results is a *"universal property"* possessed by Holsztynski's triple $\{Y_0, \psi, w\}$ (Theorem (1.1.1)). This result has some important consequences which will be discussed.

Given any integer $l \geq 1$ we construct an isometric shift operator $T: C(X) \to C(X)$ with $X \setminus \overline{D}$ having exactly l elements. One of the results proved in [208] asserts that if $X = S^n$ the n-sphere or I^n the n-cube then $C(X)$ does not admit an isometric shift operator. We will show that if M^n is any compact topological manifold with or without boundary then $C(M^n)$ does not admit an isometric shift operator. Actually it turns out that *some* of the results proved in [202] are valid for linear isometries $T: C(X) \to C(X)$ with codimension of $R(T)$ in $C(X)$ equal to 1. T need not be a shift operator; namely T need not satisfy the condition $\bigcap_{n \geq 1} R(T^n) = \{0\}$. Our exposition will take this fact into account and clearly point out results which are valid for codimension 1 linear isometries. Actually we show that $C(M^n)$ does not admit a codimension 1 linear isometry when M^n is a compact manifold.

For any compact Hausdorff space X let $C(X)$ denote the complex Banach space of complex valued continuous functions on X. Throughout this section X, Y will denote compact Hausdorff spaces and $T: C(X) \to C(Y)$ a linear isometry. In [209] Holsztynski describes a *specific construction* yielding a closed subset Y_0 of Y, well determined maps $\psi: Y_0 \to X, w: Y_0 \to S^1$ with ψ surjective and satisfying

We will refer to $\{Y_0, \psi, w\}$ obtained as above as Holsztynski's triple associated to the linear isometry $T: C(X) \to C(Y)$. We actually need this specific construction. Hence we briefly describe this construction.

For any $x \in X$ let $S_x = \{f \in C(X) \mid \|f\| = 1 = |f(x)|\}$ and $Q_x = \{y \in Y \mid T(S_x) \subset S_y\}$ (where of course $S_y = \{g \in C(Y) \mid \|g\| = 1 = |g(y)|\}$. Holsztynski shows that $Q_x \neq \emptyset$ for any $x \in X, Q_x \cap Q_{x'} = \emptyset$ if $x \neq x'$ in $X, Y_0 = \bigcup_{x \in X} Q_x$ is closed in Y and that $\psi: Y_0 \to X$ defined by $\psi(y) = x$ for any $y \in Q_x$ is continuous. Since $Q_x \neq \emptyset$ for each $x \in X$, it is clear that ψ is surjective. If $w(y) = T1(y)$ where $1 \in C(X)$ is the constant function assigning 1 to each $x \in X$ then it is shown in [203] that Y_0, ψ, w satisfy (3). Also in (3) if we substitute $f = 1 \in C(X)$ we get $w(y) = T1(y)$ for all $y \in Y_0$. This shows that w is unique. The following theorem shows that Holsztynski's triple $\{Y_0, \psi, w\}$ possesses a universal property.

Theorem (1.1.1)[199]: *Let A be any subspace (not necessarily closed) of Y and $\varphi: A \to X, u: A \to S^1$ maps satisfying*

T

Then $A \subseteq Y_0, \varphi = \psi|A$ and $u = w|A$.

Proof: Before taking up the proof observe that we do not assume that $\varphi: A \to X$ is surjective.

We first show that any $a \in A$ satisfies $a \in Q_{\varphi(a)}$. Let $f \in S_{\varphi(a)}$. This means $\|f\| = 1 = \left|f\big(\varphi(a)\big)\right|$. From equation (4) we get $|Tf(a)| = \left|f\big(\varphi(a)\big)\right| = 1$. Since T is an isometry, we get $\|Tf\| = 1$. Thus $\|f\| = 1 = |Tf(a)|$, showing that $Tf \in S_a$. Hence $f \in S_{\varphi(a)} \Rightarrow Tf \in S_a$. This yields $a \in Q_{\varphi(a)}$. Since $Y_0 = \bigcup_{x \in X} Q_x$ we see that $A \subseteq Y_0$. From equation (4) we see that $u(a) = T1(a) = w(a)$ for all $a \in A$, yielding

$$u = w|A.$$

Since $Tf(y) = w(y)f(\psi(y))$ for all $y \in Y_0$ and $A \subseteq Y_0$, we get $Tf(a) = w(a)f(\psi(a))$. Again equation (4) yields $Tf(a) = w(a)f(\varphi(a)) = w(a)f(\varphi(a))$ since $w = w|A$. From $|w(a)| = 1$, we get $f(\psi(a)) = f(\varphi(a))$. This is valid for all $f \in C(X)$. Since functions in $C(X)$ separate points of X we get $\psi(a) = \varphi(a)$. This shows that

$$\varphi = \psi|A.$$

Corollary (1.1.2)[199]: *Let A, B be subspaces of $Y, \varphi: A \to X, \theta: B \to X, w: A \to S^1$, $v: B \to S^1$ be maps satisfying equation (4) and equation (5) below:*

Then $\varphi|A \cap B = \theta|A \cap B$ and $u|A \cap B = v|A \cap B$. Moreover $\gamma: A \cup B \to X, t: A \cup B \to S^1$ defined by $\gamma|A = \varphi, \gamma|B = \theta$; $t|A = u, t|B = v$ are continuous and

T

Proof. From Theorem (1.1.1), $\varphi = \psi|A, \theta = \psi|B$; $u = w|A$ and $v = w|B$. The first part is immediate now. Also we get $\gamma = \psi|A \cup B, t = w|A \cup B$ from which we get the second part.

Theorem (1.1.1) can be strengthened as follows:

Theorem (1.1.3)[199]: *Let A be a subspace of $Y, \varphi: A \to X$ and $v: A \to \mathbb{C}$ be maps satisfying*

Then $A \subseteq Y_0$ if and only if $v(A) \subset S^1$. Moreover when this condition is satisfied we have $\varphi = \psi|A$ and $v = w|A$.

Proof. In view of Theorem (1.1.1) we have only to show that $A \subseteq Y_0 \Rightarrow v(A) \subset S^1$. Assume $A \subseteq Y_0$. Then for any $a \in A, \exists$ an $x \in A$ with $a \in Q_x$. This means $T(S_x) \subset S_a$.

Clearly $1 \in S_x$. Hence $T1 \in S_a$ yielding $1 = |T1(a)| = |v(a)|\,|1(\varphi(A))| = |v(a)|$. Hence $v(A) \subset S^1$.

Remarks (1.1.4)[199]: (a) A bounded linear operator T on a Banach space E is defined to be a shift by Crownover [200] if T is injective, the range $R(T)$ of T has codimension 1 in E and $\bigcap_{n\geq 1} R(T^n) = \{0\}$. We now observe that Theorem (1.1.1) of [202] is valid for any codimension 1 linear isometry $T: C(X) \to C(X)$. Hence we may introduce the concepts of type I and type II codimension 1 linear isometries.

(b) From Theorem (1.1.1) it follows that Y_0 is the largest subset of Y admitting maps $\psi: Y_0 \to X, w: Y_0 \to S^1$ satisfying equation (3). It turns out that Y_0 is closed and $\psi: Y_0 \to X$ is surjective. It can very well happen that there exists a closed set $Y_1 \subsetneq Y_0$ and $\psi: Y_1 \to X$ is surjective. This is what happens in the case of a codimension 1 linear isometry $T: C(X) \to C(X)$ which is simultaneously of types I and II.

An immediate consequence of Corollary (1.1.2) is the following:

Proposition (1.1.5)[199]: *Suppose $T: C(X) \to C(X)$ is a codimension 1 linear isometry and there are closed subspaces $X_0 \subsetneq X, X_1 \subsetneq X$ with $X_0 \neq X_1$ maps $w: X_0 \to S^1$, $w': X_1 \to S^1$ and surjective maps $\psi: X_0 \to X, \psi': X_1 \to X$ satisfying*

as well as

T

Then T is simultaneously of types I and II.

Definition (1.1.6)[199]: When the hypotheses of Proposition (1.1.5) are satisfied we say that T can be expressed as an operator of type I in two different ways.

As an immediate consequence of Proposition (1.1.5) we obtain the following:

Corollary (1.1.7)[199]: *Let $T: C(X) \to C(X)$ be a codimension 1 linear isometry of type I which is not of type II. Then there exists a unique isolated point p in X, a unique homeomorphism $\psi: X_0 \to X$ where $X_0 = X \setminus \{p\}$, a unique map $w: X_0 \to S^1$ satisfying (7).*

Using Corollary (1.1.2) we can find a necessary and sufficient condition for a given codimension 1 linear isometry $T: C(X) \to C(X)$ of type I to be also of type II.

Proposition (1.1.8)[199]: *Let $T: C(X) \to C(X)$ be a codimension 1 linear isometry of type I. Let p be an isolated point in $X, \psi: X_0 \to X, w: X_0 \to S^1$ maps with $X_0 = X \setminus \{p\}$ and ψ homeomorphic satisfying (7).*

Then T will be of type II if and only if there exist elements $c \in X$ and $\lambda \in S^1$ satisfying

Proof. If T is also of type II, ψ and w admit extensions, also denoted by the same letters $\psi: X \to X, w: X \to S^1$ satisfying (7) for all $y \in X$. Choose $c = \psi(p)$ and $\lambda = w(p)$. Then clearly (9) is satisfied.

Conversely, assume that there exist $c \in X$ and $\lambda \in S^1$ satisfying (9). Then $\theta: \{p\} \to X, v: \{p\} \to S^1$ defined by $\theta(p) = c, v(p) = \lambda$ are clearly continuous. Taking $A = X_0, \varphi = \psi, u = w;\ B = \{p\}$ from Corollary (1.1.2) we immediately conclude that T is of type II.

We will discuss methods of constructing codimension 1 linear self isometries of $C(X)$. Using those methods we will construct a codimension 1 linear isometry $T: C(K) \to C(K)$ of type II which is not of type I when K is the Cantor set. However our methods do not yield an *isometric shift* operator on $C(K)$. Since K has no isolated points, if there is an isometric shift operator on $C(K)$ it will be of type II which is not of type I.

We proving the following:

Proposition (1.1.9)[199]: *Let $T: C(X) \to C(X)$ be a codimension 1 linear isometry of type I; p, $X_0 = X \setminus \{p\}, \psi: X_0 \to X$ and $w: X_0 \to S^1$ have their usual meanings. Let $q \in X_0$ be any isolated point. Then $\chi_q \in R(T) \Leftrightarrow T\chi_{\psi(q)}(p) = 0$.*

Proof. Suppose $\chi_q \in R(T)$ say $\chi_q = Th$ with $h \in C(X)$. Using the equation $Th(y) = w(y)h(\psi(y))\ \forall y \in X_0$ we immediately see that $h|\big(X - \psi(g)\big) = 0$ and that

$h(\psi(g)) = \frac{1}{w(q)}$. Hence $\chi_q = Th \Rightarrow h = \frac{1}{w(q)}\chi_{\psi(q)}$ From $\chi_q(p) = 0$ we now get $\frac{1}{w(q)}T\chi_{\psi(q)}(p) = 0$ yielding $T\chi_{\psi(q)}(p) = 0$.

Conversely, if $T\chi_{\psi(q)}(p) = 0$, straight-forward checking shows that $\chi_q = Th$ where $h = \frac{1}{w(q)}\chi_{\psi(q)}$

We will make use of this proposition.

Throughout X will denote a compact Hausdorff space. Let $T: C(X) \to C(X)$ be a codimension 1 linear isometry of type I. Then as seen already, there exist an isolated point p in X, a homeomorphism $\psi: X_0 \to X$ where $X_0 = X \setminus \{p\}$ and a map $w: X_0 \to S^1$ satisfying

Denoting the continuous linear functional $f \mapsto Tf(p)$ on $C(X)$ by T we see that $|\Gamma f| \leq \|f\|$ for all $f \in C(X)$.We will presently see that the converse to this is true.

Proposition (1.1.10)[199]: *Let p be an isolated point and $\psi: X_0 \to X$ a homeomorphism. Let T be a continuous linear functional on $C(X)$ satisfying $|\Gamma f| \leq \|f\|$ for all $f \in C(X)$ and $w: X_0 \to S^1$ a map. Then $T: C(X) \to C(X)$ defined by $Tf(y) = w(y)f(\psi(y))$ for all $y \in X_0$ and $Tf(p) = \Gamma f$, for any $f \in C(X)$ is a codimension 1 linear isometry.*

Proof. The proof given in [202] for the fact that $\chi_P \notin R(T)$ is valid here also. Still we spell it out. If $\chi_p = Tf$, since $p \notin X_0$, we get $Tf(y) = 0$ for all $y \in X_0$. It follows from the equation $Tf(y) = w(y)f(\psi(y))$ that $f = 0$, since $\psi: X_0 \to X$ is surjective and $|w(y)| = 1$ for every y. This will mean $\chi_p = 0$, a contradiction. Let $\Delta_1: C(X) \to C(X)$ be defined by $\Delta_1 f(x) = f(\psi^{-1}(x))/w(\psi^{-1}(x))$. A straight-forward verification shows that $f = T\Delta_1 f + \{f(p) - T\Delta_1 f(p)\}\chi_P$. This proves that $C(X)/R(T)$ is of dimension 1, with the class $[\chi_P]$ of χ_P in $C(X)/R(T)$ forming a basis element. Using the facts $\sup_{y\in X_0}|Tf(y)| = \sup_{y\in X_0}|f(x)| = \|f\|$ and $|Tf(p)| = |\Gamma f| \leq \|f\|$ we immediately get $\|Tf\| = \|f\|$.

Suppose $T: C(X) \to C(X)$ is a codimension 1 linear isometry of type II. Then we get $\psi: X \to X, w: X \to S^1$ with ψ surjective and satisfying

Moreover there exist two unique elements $a \neq b$ in X with $\psi(a) = \psi(b)$ and $\psi|X - \{a, b\}: X - \{a, b\} \to X - \{c\}$ bijective. Here $\psi(a) = \psi(b) = c$. If W denotes the quotient space obtained from X by identifying a and b, ψ induces a map $\bar{\psi}: W \to X$. Then $\bar{\psi}: W \to X$ is a homeomorphism. The following proposition yields a converse to this.

Proposition (1.1.11)[199]: *Let $\psi: X \to X, w: X \to S^1$ be given with ψ surjective. Suppose there exist $a \neq b$ in X with $\psi(a) = \psi(b)$ and $\psi|X - \{a, b\}: X - \{a, b\} \to X - \{c\}$ bijective, where $c = \psi(a) = \psi(b)$.*

Then $T: C(X) \to C(X)$ defined by

is a codimension 1 linear isometry.

Example (1.1.12)[199]: Let K denote the Cantor set. Given $a \neq b$ in K it is shown in [208] that there exists a surjection $\psi: K \to K$ satisfying $\psi(a) = \psi(b) = a$ with the additional property that $\psi|K \setminus \{a\}: K \setminus \{a\} \to K$ is bijective. Proposition (1.1.11) yields a codimension 1 linear isometry $T: C(K) \to C(K)$. Since K has no isolated points, it follows that T cannot be of type I.

Remark (1.1.13)[199]: Given an isolated point p in X, a homeomorphism $\psi: X_0 \to X$ (where $X_0 = X \setminus \{p\}$), a map $w: X_0 \to S^1$ and a continuous linear functional $\Gamma: C(X) \to C$ satisfying $|\Gamma f| \leq \|f\|$, Proposition (1.1.10) shows that $T: C(X) \to C(X)$ defined by $Tf(y) = w(y) f(\psi(y)) \forall y \in X_0$ and $Tf(p) = \Gamma f$ is a codimension 1 linear isometry of type I. Let $\Delta_1: C(X) \to C(X)$ be defined as earlier, namely $\Delta_1 f(x) = f(\psi^{-1}(x)) / w(\psi^{-1}(x))$ for any $x \in X$. Then Δ_1 is a surjective complex linear map, $\|\Delta_1\| = \|f\|$ and $\operatorname{Ker} \Delta_1 = \mathbb{C}\chi_P$. For any integer $n \geq 1$, let $\Delta_n: C(X) \to C(X)$ be defined by $\Delta_n = (\Delta_1)^n$; let $\Delta_0 = Id_{C(X)}$. It is easy to see that $f \in R(T^n)$ if and only if $\Delta_j f(p) = \Gamma \Delta_{j+1} f$ for $0 \leq j \leq n-1$. If $\beta_j: C(X) \to \mathbb{C}$ denotes the continuous linear functional $\beta_j f =$

$\Delta_j f(p) - \Gamma\Delta_{j+1} f$ then $f \in R(T^n) \Leftrightarrow f \in \bigcap_{j=0}^{n-1} \operatorname{Ker} \beta_j$. Thus T will be an isometric shift $\Leftrightarrow \bigcap_{j\geq 0} \operatorname{Ker} \beta_j = \{0\}$.

We now give an example of a codimension 1 linear isometry $T: C(X) \to C(X)$ which is not a shift.

Examples (1.1.14)[199]: Let $A = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. As usual we identify $C(A)$ with the space of convergent complex sequences $\underline{c} = (c_1, c_2, c_3, \dots)$. Then $T: C(A) \to C(A)$ given by $T(c_1, c_2, C_3, \dots) = (c_1, 0, c_2, c_3, c_4, \dots)$ is a codimension 1 linear isometry which is not a shift.

Example (1.1.15)[199]: Consider Example in [202]. T in this example is expressible as an isometric shift operator of type I in two different ways. If $p = 1, X_o = X \setminus \{1\}$ and $\psi: X_0 \to X, w: X_0 \to S^1$ are given by $\psi(n+1) = n \;\forall n \in \mathbb{N}, \psi(\infty) = \infty$ and $w(y) = 1 \;\forall y \in X_0$ then we clearly have

T

Similarly setting $q = 2, X_0' = X \setminus \{2\}$ and defining $\psi' = X_0' \to X, w': X_0' \to S^1$ by $\psi'(1) = 1, \psi'(n+1) = n$ for $n > 2, \psi'(\infty) = \infty, w'(1) = -1$ and $w'(x) = 1$ for all $x \in X_0' \setminus \{1\}$ we see that

T

Thus T is expressible as an isometric shift operator in two different ways.

Propositions (1.1.5), (1.1.8) and Corollary (1.1.7) were proved for codimension 1 linear isometries. In particular they are valid for isometric shift operators.

Example (1.1.16)[199]: Consider Example in [202]. In this example T is an isometric shift operator of type I which is not of type II. Thus T is expressible as an isometric shift operator of type I in only one way. $p = 1, X_0 = X \setminus \{1\}$; $\psi: X_0 \to X, w: X_0 \to S^1$ with $\psi(n+1) = n \;\forall n \in \mathbb{N}, \psi(\infty) = \infty$ and $w(y) = 1 \;\forall g \in X_0$ satisfy $Tf(y) = w(y) f(\psi(y)) \forall y \in X_0$ and $f \in C(X)$. However, straight-forward checking shows that 2 and 3 are isolated points with χ_2 as wel as χ_3 not in $R(T)$. This means the only function vanishing on either $X \setminus \{1\}$ or $X \setminus \{2\}$ or $X \setminus \{3\}$ and lying in $R(T)$ is the constant function 0.

As an immediate consequence of Proposition (1.1.10) we get the following:

Proposition (1.1.17)[199]: *Let $T: C(X) \to C(X)$ be an isometric shift operator expressible as a shift operator of type I in a unique way. Let $p, X_0 = X \setminus \{p\}, \psi: X_0 \to X$ and $w: X_0 \to S^1$ have their usual meanings. Let q be any isolated point in X with $q \neq p$. Then the following are equivalent:*

(i) $f \in R(T), f|(X - \{q\}) = 0 \Rightarrow f = 0$

(ii) $\chi_q \notin R(T)$

(iii) $T\chi_{(q)}(p) \neq 0$.

Example (1.1.18)[199]: Let $X = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. We identify $C(X)$ with the space of convergent complex sequences $\underline{c} = (c_1, c_2, c_3, \dots)$ under $f \leftrightarrow \underline{c}$ where $c_n = f(n)$. Under this identification $f(\infty)$ will correspond to $\lim_{n\to\infty} c_n$. We write C_∞ for $\lim_{n\to\infty} c_n$. Consider $T: C(X) \to C(X)$ defined by

Let $\psi: X \to X$ and $w: X \to S^1$ be defined by $\psi(n+1) = n \forall n \in \mathbb{N}, \psi(1) = \psi(\infty) = \infty$; $w(1) = 1, w(2) = i, w(3) = -1, w(4) = -i, w(n) = 1$ for $n \geq 5$ and $w(\infty) = 1$. Clearly $\psi|X - \{1, \infty\}: X - \{1, \infty\} \to X - \{\infty\}$ is bijective. T is the codimension (i) linear isometry of type II obtained from ψ and w applying Proposition (1.1.11). Since 1 is isolated in X we see that T is also of type I. Since ∞ is not isolated in X from the same remark we see that T cannot be expressed as a type I operator in two different ways. We will show that T satisfies $\bigcap_{n\geq 1} R(T^n) = \{0\}$. Then it will follow that T is an isometric shift operator simultaneously of types I and II but expressible as a shift operator of type I in exactly one way.

For any $\underline{a} = (a_1, a_2, a_3, \dots) \in C(X)$ let us denote the conventional shift $\underline{a} \longmapsto (0, a_1, a_2, a_3, \dots)$ by S. Given $\underline{c} = (c_1, c_2, c_3, c_4, \dots) \in C(X)$ let us denote the element $(-c_1, ic_2, -ic_3, c_4, c_5, c_6, \dots)$ by $\gamma(\underline{c})$. An easy calculation shows that

$T^6\underline{c}$

Denote the element $\left(c_\infty, ic_\infty, -ic_\infty, \overbrace{-c_\infty, -c_\infty, -c_\infty, \dots, -c_\infty}^{3/\text{terms}}, 0,0,0,0,\dots\right)$ of $C(X)$ by $\mu_l(C_\infty)$. Then by induction on l we show that

Supposes $\underline{a} = (a_1, a_2, a_3, \dots)$ is in $\bigcap_{l\geq 1} R\left(T^{3(l+1)}\right)$. Then from (14) we see that there should exist an element $c_\infty \in \mathbb{C}$ with $a_1 = c_\infty, a_2 = ic_\infty, a_3 = -ic_\infty$ and $a_k = -c_\infty$ for all $k \geq 4$. Writing λ for c_∞ we should have

Also $\underline{a} = T\underline{b}$ for some $\underline{b} = (b_1, b_2, b_3, \dots) \in C(X)$. This means $\underline{a} = (b_\infty, ib_1, -b_2, -ib_3, b_4, b_5, \dots)$ yielding $\lambda = b_\infty$ and $b_n = -\lambda$ for $n \geq 4$. But $\underline{b}$ being a convergent sequence, we should have $b_\infty = \lim_{n-\infty} b_n = -\lambda$. Thus we get $\lambda = -\lambda$ or $\lambda = 0$. This yields $\underline{a} = 0 \in C(X)$ thereby showing that $\bigcap_{n\geq 1} R(T^n) = \{0\}$. This completes the proof that T is an isometric shift operator.

$\overline{D} \neq X$. In this section, given any integer $l \geq 1$ we construct an isometric shift operator of type I with $X \backslash \overline{D}$ having exactly l elements. Let $A = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. As usual $C(A)$ will be identified with the space of convergent complex sequences $\underline{c} = (c_1, c_2, c_3, \dots)$. Let $\{a_1, a_2, \dots, a_l\}$ be a discrete space with l elements and $X = A \cup \{a_1, a_2, \dots, a_l\}$ (disjoint union). Any element of $C(X)$ can be uniquely written as $\underline{c} \oplus \sum_{j=1}^{l} \lambda_j \chi_{a_j}$ with $\underline{c} \in C(A)$. Let $T: C(A) \to C(A)$ be the usual lateral shift, namely $T\underline{c} = (0, c_1, c_2, c_3, \dots)$. Let $S: C(X) \to C(X)$ be defined by

$$S\left(\underline{c} \oplus \sum_{j=1}^{l}\right.$$

We could rewrite the formula for S as

$$S\left(\underline{c} \oplus \sum_{j=1}^{l} \lambda\right.$$

Let $X_0 = X \setminus \{1\} = (A \setminus \{1\}) \cup \{a_1, \dots, a_l\}$. Define $\psi: X_0 \to X, w: X_0 \to S^1$ by

ψ

ν

Then it is clear that

We will check that S is an isometric shift operator. One can check that

$$S^{l}\left(\underline{c}\oplus\sum_{j}\right.$$

$$S^{2l}\left(\underline{c}\oplus\sum_{j=1}^{l}\lambda_{j}\right.$$

Let us denote $(\lambda_l, \lambda_{l-1}, \dots, \lambda_1, 0,0,0, \dots)$ by $\underline{u}$ then we have

$$S^{(2n+1)l}\left(\underline{c}\oplus\sum_{j}\right.$$

$$S^{2nl}\left(\underline{c}\oplus\right.$$

From (22) and (23) we see that if

then $\underline{x}$ will not be convergent unless $\underline{x} = 0$ in $C(A)$ and $\mu_1 = \dots = \mu_1 = 0$. This proves that $\bigcap_{n\geq 1} R(S^n) = \{0\}$. Thus S is an isometric shift operator of type I. In this example $D = \mathbb{N}, \overline{D} = A$ and $X \setminus \overline{D} = \{a_1, \dots, a_l\}$.

In this example it is easily seen that $I_D \cap R(S^l) = \{0\}$. Also $I_D \cap R(S^{l-1}) \neq \{0\}$ because if $I_D \cap R(S^{l-1}) = \{0\}$, we would have $|X \setminus \overline{D}| \leq l - 1$, which is not the case here.

We proved the following theorem:

Theorem (1.1.19)[199]: *Let M be any compact manifold with or without boundary. Then $C(M)$ does not admit a codimension 1 linear isometry. In particular $C(M)$ does not admit an isometric shift operator.*

Proof. Any compact manifold M has only finitely many connected components. Hence M cannot admit an infinite number of isolated points. Thus to prove Theorem (1.1.19) we have only to show that $C(M)$ does not admit a codimension 1 linear isometry of type II. As remarked earlier, if there existed a codimension 1 linear isometry $T: C(M) \to C(M), M$ would be homeomorphic to a quotient of M obtained by identifying exactly two points. Let $a \neq b$ *be* any two points of M. *If* M were of dimension $0, M$ would be a finite discrete space. Hence $C(M)$ can not admit any injective linear map which is not surjective. Thus we may assume that $\dim M = n \geq 1$.

Suppose $\delta M = 0$. Let X be the quotient space obtained from M by identifying a and b. Let $c \in X$ be the point represented by a or b. Let $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ and $B^n \vee B^n$ the wedge where $0 \in B^n$ is chosen as the base point. The element $c \in X$ will have a fundamental system of neighbourhoods homeomorphic to $B^n \vee B^n$ with c corresponding to the base point in $B^n \vee B^n$. But $B^n \vee B^n$ is not locally Euclidean around the base point. Hence X cannot be homeomorphic to M^n.

Suppose $\delta M \neq \emptyset$. If a and b are both in $\operatorname{Int} M^n, c \in X$ will have a fundamental system of neighbourhoods homeomorphic to $B^n \vee B^n$ with c corresonding to the base point of $B^n \vee B^n$. Let $B_+^n = \{x \in \mathbb{R}^n | x_1 \geq 0, \|x\| < 1\}$. If one of a, b is in $\operatorname{Int} M^n$ and the other is in δM then c will admit a fundamental system of neighbourhoods homeomorphic to $B^n \vee B_+^n$ with c corresponding to the base point. If both a and b are in $\delta M, c$ will admit a fundamental system of neighbourhoods homeomorphic to $B_+^n \vee B_+^n$. For $B^n \vee B^n$ and $B^n \vee B_+^n$ the manifold condition fails at the base point. Also when $n \geq 2$, the manifold condition fails at the base point for $B_+^n \vee B_+^n$.

When $n = 1, M$ will be a disjoint union of k copies of S^1 and l copies of $[0, 1]$ for some integers $k \geq 0, l \geq 0$ and $k + l \geq 1$. If two boundary points in M are identified, the quotient X will have strictly less than l copies of $[0,1]$, hence cannot be homeomorphic to M.

Section (1.2): Backward Shifts on Banach Space of Continuous Functions:

Unilateral shifts on infinite dimensional separable Hilbert spaces are well known and have been studied in functional analysis, see Kato [205] and Rudin [206]. If H is a infinite dimensional Hilbert space then an operator $T : H \to H$ is a right or simply a shift if there is a complete orthonormal set $\{\varphi_n\}_{n\geq 1}$ in H such that $T(\varphi_n) = \varphi_{n+1}$ for $n \geq 1$, and it is a left (backward) shift if $T(\varphi_1) = 0$, and $T(\varphi_n) = \varphi_{n-1}, n \geq 2$. The generalization of shift operators to Banach spaces has been given by Crownover [207], and has been the subject of investigaton recently by Holub [208]; Gutek, Hart, Jamison, and Rajagopalan [209]; and Farid and Varadarajan [210]. In [208] Holub introduced backward shifts on Banach spaces, thus generalizing the concept of backward shifts on Hilbert spaces or l_p-spaces, $1 \leq p < \infty$ [205]. Holub discusses in [208] the problem of existence of a backward shift on Banach spaces $C(X), X$ compact Hausdorff, and conjectured that $C(X)$ does not admit a backward shift if X has an infinite connected component. We resolve the conjecture completely by proving that the space $C(X)$ does not admit backward shifts, if X is an arbitrary infinite compact Hausdorff space.

Here, E is an arbitrary real Banach space unless otherwise specified. By an operator on E, we understand a linear transformation on E into E.

Definition (1.2.1)[204]: An operator $T : E \to E$ is a backward shift if:

(i) $\dim \operatorname{Ker} T = 1$,

(ii) the induced operator $\hat{T} : E \mid \operatorname{Ker} T \to E$ is a linear isometry,

and

(iii) $\bigcup_{n\geq 1} \operatorname{Ker} T^n$ is dense in E.

As noted in [208] it is verified that if E is a separable Hilbert space then an operator $T : E \to E$ is a backward shift, if and only if there is a complete orthonormal set $\{\varphi_n\}_{n\geq 1}$ in E such that $T(\varphi_1) = 0$, and $T(\varphi_n) = \varphi_{n-1}, n \geq 2$, thus justifying Definition (1.2.1). Further since condition (i) in Definition (1.2.1) implies that $\operatorname{Ker} T^n$ is n-dimensional, condition (iii) implies that E is separable.

In the next proposition we state a known property of a backward shift in [208].

Since a proof is lacking we sketch a proof of the same here.

Proposition (1.2.2)[204]: *If T is a backward shift on an infinite dimensional Banach space E, then the range of T is all of E.* In (201, 2018, 2019, 220, 221, 222, 223].

Proof. Let $T : E \to E$ be a backward shift with $\operatorname{Ker} T = [z]$, the linear span of some nonzero vector z in E. Since $\hat{T} : E \mid [z] \to E$ is defined by $\hat{T}(\bar{x}) = T(x)$, where $\bar{x}$ is the equivalence class of x modulo $[z]$, it follows from the condition (2) of Definition (1.2.1), that $T(E)$ is a closed subspace of E. We verify that $z \in T(E)$. If $z \notin T(E)$, then $[z] \cap T(E) = \{0\}$. Hence $T^{-1}(\operatorname{Ker} T) = \operatorname{Ker} T$ from which it follows that $T^{-n}(\operatorname{Ker} T) = \operatorname{Ker} T$ for all $n \geq 1$. Hence condition (iii) fails to hold for T. Thus $z \in T(E)$. A repetition of the argument and induction yields that $T^{-n}(\operatorname{Ker} T) \subset T(E)$. for all $n \geq 1$. Hence $\bigcup_{n \geq 0} T^{-n}(\operatorname{Ker} T) \subset T(E)$. Thus it follows from (iii) of Definition (1.2.1) that $T(E) = E$, as desired.

As usual we identify the dual of $C(X)$, X compact Hausdorff with the Banach space of regular Borel measures μ on X, with the norm $\|\mu\| =$ total variation of $\mu = |\mu|(X)$, where $|\mu|$ is the variation measure associated with μ. If E is a Banach space, $\operatorname{Ext}(E)$ is the set of extreme points of the closed unit ball of E. Thus if $E = C(X)$, $\operatorname{Ext} E^* = \{\pm e_t \mid t \in X\}$, where E^* is the dual of E, and e_t is the point mass μ supported by $\{t\}, t \in X$ and $\mu\{t\} = 1$. It is verified that the distance between any two points in $\operatorname{Ext} E^*, E$ as above in [224, 225, 226, 227].

If $M \subset E, M^{\perp} = \{f \mid f \in E^*, f(x) = 0, \text{for all } x \in M\}$. As usual we identify the dual of the quotient $E \,|M = (E \mid M)^*$ with $M^{\perp}$. If $x \in E$, the linear span of x is noted as $[x]$.

For $f \in C(X)$, let us denote the support of f by $S(f)$, i.e., $\{S(f) = t \,|f(t) \leq 0\}$.

If $f \in C(X)$, and $\{t_1, t_2\} \subset S(f)$ we associate with the ordered pair (t_1, t_2), the Borel measure $\mu = f_{(t_1,t_2)}$ on X defined by $\mu\{t_1\} = f(t_2)/(|f(t_1)| + |f(t_2)|)$, $\mu\{t_2\} = -f(t_1)/(|f(t_1)| + |f(t_2)|)$, and $\mu(B) = 0$ for all Borel sets $B \subset X' = X \sim \{t_1, t_2\}$. It is verified that $\|f_{(t_1,t_2)}\| = 1, f_{(t_2,t_1)} = -f_{(t_1,t_2)}$, and $\pm f_{(t_1,t_2)} \in [f]^{\perp}$, i.e., $\int f d\mu = 0$, if $\mu = \pm f_{(t_1,t_2)}$.

Lemma (1.2.3)[204]: *If $f \in C(X)$ and $\{t_1, t_2\} \subset S(f)$, then $f_{(t_1,t_2)}, f_{(t_2,t_1)}$ are extreme points of the unit ball of $[f]^{\perp}$.*

Proof. Let us denote the measure of $f_{(t_1,t_2)}$. by μ. Assume, if possible, there are Borel measures $\mu_i, \mu_i \in [f]^{\perp}, \|\mu_i\| = 1, i = 1, 2$ such that $\mu = (\mu_1 + \mu_2)/2, \mu_1 \neq \mu_2$. It is now verified that the support of $\mu_i = \{t_1, t_2\}$, for $i = 1, 2$. Since the support of $\mu = \{t_1, t_2\}$,

Thus it is verified that $|\mu_i\{t_1\}| + |\mu_i\{t_2\}| = 1$, for $i = 1, 2$. From the definition of the norm of a measure it follows that the support of the measures μ_i is $\{t_1, t_2\}$ for $i = 1, 2$. Since the support of $\mu = \{t_1, t_2\}, \mu = (\mu_1 + \mu_2)/2, \mu_1 \neq \mu_2$ it follows from the proceding observation that there are nonzero real numbers δ_1, δ_2 such that

μ

Now since $\mu, \mu_1, \mu_2 \in [f]^{\perp}$, i.e., $\int_X f d\mu = \int_X f d\mu_i = 0, i = 1, 2$, by evaluating these integrals it follows that $\delta_1 f(t_1) + \delta_2 f(t_2) = 0$. Thus if $\lambda = \delta_1/\delta_2$, then

From the properties of μ_1, noted above,

$\|\mu_1$

A similar computation yields

Since $\|\mu_1\| = \|\mu_2\|$, and $\mu\{t_2\} \neq 0$, it follows that $\delta_2 = 0$, a contradiction. Thus $f_{(t_1,t_2)} \in \text{Ext}\,[f]^{\perp}$, which in turn implies $f_{(t_2,t_1)} \in \text{Ext}\,[f]^{\perp}$.

Lemma (1.2.4)[204]: *If* $f \in C(X)$, *and* card $S(f) \geq 3$, *then* $C(X)\,|[f]$ *is not linearly isometric with* $C(X)$. *Thus in particular if* $T : C(X) \to C(X)$ *is a backward shift*, X *infinite compact*, *and* $\text{Ker}\,T = [f]$, *then* card $S(f) \leq 2$.

Proof. Since $(C(X) \mid [f])$ is linearly congruent with $[f]^{\perp}$, it is enough to verify that $[f]^{\perp}$ is not linearly isometric with $\left(C(X)\right)^*$ if card $S(f) \geq 3$. Further since linear isometries preserve extreme points and the distance between any pair of extreme points of $C(X)^*$ is 2, it is enough to exhibit two extreme points α, β, of $[f]^{\perp}$, such that $\|\alpha - \beta\| < 2$, to complete the desired verification.

Let $\{t_1, t_2, t_3\} \subset S(f)$. It is assumed that $f(t_i) > 0$ for $i = 1, 3$, if necessary by relabelling the t's, and passing to the function $-f$. With this set up consider $\alpha = f_{(t_1,t_2)}$, and $\beta = f_{(t_3,t_2)}$. From Lemma (1.2.3) it follows that α, β are in $\text{Ext}\,[f]^{\perp}$. Let us denote $|f(t_i)| + \left|f\left(t_j\right)\right|$ by $C_{i,j}$ if $\left\{t_i, t_j\right\} \subset X$. With this notation,

$\|\alpha - \beta\|$

Thus

or

In either case it follows that $\|\alpha - \beta\| < 2$, noting that $f(t_i) > 0$, for $i = 1, 3$ and recalling that $C_{i,j} = |f(t_i)| + \left|f\left(t_j\right)\right|$ if $\left\{t_i, t_j\right\} \subset X$.

We continue to assume that X is an infinite compact Hansdorff space [212, 213, 214, 215, 216]. We denote the Banach space $C(X)$ by E. If $f \in E, f \neq 0$, let $\varphi : E \to E \mid [f]$ be the canonical quotient map where $\varphi(g) = \hat{g}$, the equivalence class of g modulo $[f]$. If $\hat{T} : E \mid [f] \to E$ is a linear isometry onto E, then the conjugate map $\hat{T}^*$ is a surjective linear isometry on E^* onto $(E \mid [f])^*$. As usual we identify $(E \mid [f])^*$ with $[f]^\perp$ by the map $\sigma : (E \mid [f])^*$ onto $[f]^\perp$, where $\sigma(l)(g) = l(\hat{g})$ for all $l \in (E \mid [f])^*$ and $g \in E$.

Lemma (1.2.5)[204]: *Let $f \in E, 0 < \operatorname{card} S(f) \leq 2$, and $\hat{T} : E \mid [f] \to E$ be a surjective linear isometry. If $T : E \to E$ is the operator defined by $T(g) = \hat{T}(\hat{g})$, then the subspace $\bigcup_{n\geq 1} T^{-n}[f]$ is not dense in E.*

Proof. The proof is accomplished by showing that if $\theta \in C(X), S(\theta)$ containing an accumulation point of X, then $\inf\|\theta - g\| > 0$ where the infimum is taken over $g \in \bigcup_{n\geq 1} T^{-n}[f]$.

It is verified that if $T(g) = f, f$ as in the proposition, then $S(g)$ is finite as follows.

Let $P = \left\{s \mid \left(\sigma \circ \hat{T}^*\right)(e_t) \in \{\pm e_s\}\right\}$ with $t \in S(f)$ where e_x is evaluation at x. Since $S(f)$ is finite, P as well as $P \cup S(f)$ is finite. If $\xi \notin S(f)$, then since $e_\xi \in \operatorname{Ext}[f]^\perp$, and s $\sigma \circ \hat{T}^*$ is a linear isometry on E^* onto $[f]^\perp$, it follows that there is a $t_0 \in X$ such that $\left(\sigma \circ \hat{T}^*\right)\left(e_{t_0}\right) \in \{\pm e_\xi\}$. If further $\xi \notin P \cup S(f)$, it follows from the definition of P, that $\left(\sigma \circ \hat{T}^*\right)\left(e_{t_0}\right) \in \{\pm e_\xi\}$ for some $t_0 \notin S(f)$.

Now $\left(\sigma \circ \hat{T}^*\right)\left(e_{t_0}\right)(g) = \hat{T}^*\left(e_{t_0}\right)(\hat{g}) = e_{t_0}\left(\hat{T}(\hat{g})\right) = T(g)\left(e_{t_0}\right) = f(t_0)$. Thus $f(t_0) = g(\xi)$ or $-g(\xi)$. Since $t_0 \notin S(f), g(\xi) = 0$. Thus $S(g) \subset P \cup S(f)$. Hence $S(g)$ is a finite set.

More generally it follows by induction that if $T^n(g) = f$, for some $n \geq 2$ then $S(g)$ is finite. Let $T^m(g) = f$ for some $m \geq 1$ imply $S(g)$ is finite. Let now $g \in C(X)$ be such that $T^{m+1}(g) = f$. Since $T^{m+1}(g) = T^m\left(T(g)\right)$, it follows by the induction hypothesis that if $T(g) = h$, then $S(h)$ is finite. Let $P = \left\{s \mid \left(\sigma \circ \hat{T}^*\right)(e_t) \in \{\pm e_s\} \text{ with } t \in S(h)\right\}$. P thus defined is a finite set since $S(h)$ is finite. Then proceeding as in the preceding paragraph if ξ is not $\in P \cup S(h) \cup S(f)$, then there is a $t_0 \in X$, such that

Thus evaluating both sides at g, noting $T(g) = h$, it follows that $g(\xi) = h(t_0)$ or $-h(t_0) = 0$, since $t_0 \notin S(h)$. Thus $S(g) \subset P \cup S(h) \cup S(f)$. Hence $S(g)$ is finite since $P \cup S(h) \cup S(f)$ is finite.

Now to complete the proof let us note that since $S(g)$ is open if $g \in C(X), S(g)$ finite implies $S(g)$ consists of isolated points. Let t_0 be an accumulation point of X, and $u \in C(X)$ be such that $\theta(t_0) \neq 0$. Since if $g \in T^{-n}[f]$ implies $S(g)$ is finite as shown earlier in the proof, it follows that $g(t_0) = 0$. Thus $\|\theta - g\| \geq |\theta(t)| > 0$. Hence the subspace $\bigcup_{n\geq 1} T^{-n}[f]$ is not dense in $C(X)$ as claimed.

Now we deduce the main result, stated as a theorem, from the preceding lemmas.

Theorem (1.2.6)[204]: *If X is an infinite compact Hausdorff space, then $C(X)$ does not admit backward shifts.*

Proof. Let $T : C(X) \to C(X)$ be a backward shift if possible, and $[f] = \operatorname{Ker} T$. It follows from Lemma (1.2.4), that $0 < \operatorname{card} S(f) \leq 2$. Hence from Lemma (1.2.5), it follows that $\bigcup_{n\geq 1} \operatorname{Ker} T^n$ is not dense in $C(X)$, contradicting that T is a backward shift. This completes the proof of the theorem.

We conclude with a remark and a corollary of the preceding theorem. Let C be the Banach space of real convergent sequences, with the supremum norm, and C_0 be the subspace of null sequences. Let e_n be the unit sequence $\{a_i\}_{i\geq 1}$ such that $a_i = 0$, if $i \neq n$, and $a_n = 1$.

Chapter 2

Inverse Spectral Theory on Ward Equation and Isometric Shifts on Meteric Spaces.

We obtain the following multiplicative representation of $T: (Tf)(y) = a(y)f\big(h(y)\big)$ for all $y \in \partial B$ and all $f \in A$, where a is a unimodular scalar-valued continuous function on ∂B. These results contain and extend some others by Amir and Arbel, Holsztyński, Myers and Novinger. Some applications to isometries involving commutative Banach algebras without unit are announced.

In particular we show that the solution can be constructed by solving a 2×2 local matrix Riemann-Hilbert problem which is uniquely defined in terms of the initial data. These results are also directly applicable to the $2+1$ Chiral model.

Section (2.1): Linear Isometries Between Subspaces:

Let K denote the field of real or complex numbers. For a locally compact Hausdorff space X, we denote by $C_0(X)$ the Banach space of all continuous $\mathbb{K}$-valued functions defined on X which vanish at infinity, equipped with its usual supremum norm. If X is compact, we write $C(X)$ instead of $C_0(X)$. $X \cup \{\infty\}$ denotes the Alexandroff compactification of X.

Let A be a linear subspace of $C_0(X)$. We will denote by σA the set of all $x_0 \in X$ such that for each neighborhood U of x_0, there is a function f in A such that $|f(x)| < \|f\|$ for all $x \in X - U$. Let us define the set

$\sigma_0 A :$

If it exists, we will denote by ∂A the Shilov boundary of A, that is, the minimal closed subset of X with the property that each function in A assumes its maximum on ∂A. On the other hand, it is said that $x_0 \in X$ is a strong boundary point of A if for each neighborhood U of x_0, there is a function f in A such that $\|f(x_0)\| = \|f\|$ and $|f(x)| < |f|$ for all $x \in X - U$. We will denote by τA the set of all strong boundary points of A.

We will denote by $\mathrm{Ch}\,A$ the Choquet boundary of A. Let us recall that each extreme point of the unit ball V of the dual space of A has the form μe_x, where μ is a complex number of modulus 1 and e_x is the evaluation map at the point $x \in X, e_x(f) = f(x)\ \big(f \in C_0(X)\big)$. The *Choquet boundary* for A is defined as $\{x \in X : e_x$ is an extreme $V\}$. Recall that although the Choquet boundary is usually defined in the case point of when X is compact and A separates points and contains the constants, both definitions agree in this case.

We say that a linear subspace A of $C_0(X)$ is separating (resp. strongly separating) if given any pair of distinct points x_1, x_2 of X, then there exists $f \in A$ such that $f(x_1) \neq f(x_2)$ (resp. $|f(x_1)| \neq |f(x_2)|$). It is well-known that the Shilov boundary of a separating subalgebra of $C_0(X)$ always exists.

A separating (resp. strongly separating) linear subspace A of $C_0(X)$ is said to be a separating (resp. strongly separating) function subspace if for all $x \in X$, there exists $f \in A$ such that $f(x) \neq 0$.

The source of this article is the classical Banach-Stone theorem. In its present form it states as follows: if there exists a linear isometry T of $C_0(X)$ onto $C_0(Y)$, then there are a homeomorphism h of Y onto X and a continuous map $a : Y \to \mathbb{K}, |a| \equiv 1$, such that T can be written as a weighted composition map, that is,

$(T$

This well-known theorem has been generalized in several directions, for instance, by considering injective (not necessarily surjective) linear isometries. Perhaps the most important result of this type is due to Holsztyński [15]: if there exists a linear isometry T of $C(X)$ into $C(Y)$, then we can find a closed subset Y_0 of Y and a continuous map h of Y_0 onto X and a continuous map $a : Y_0 \to \mathbb{K}, |a| \equiv 1$, such that

$(T$

Some years before, Geba and Semadeni [230] had obtained an analogue of Holsztyński's theorem though for isotonic injective linear isometries. Also a number of applications of Holsztyński's theorem can be found. Recently, for instance, it has played a crucial role in the classification of isometric shift operators on $C(X)$ ([231] and [232]). Generalizations of a similar type are provided by replacing $C_0(X)$ by its subspaces or subalgebras. Indeed, in [233] proved that, if $\mathbb{K} = \mathbb{R}$, then a sufficient condition for X and Y to be homeomorphic is that a completely regular linear subspace of $C(X)$ and such a subspace of $C(Y)$ be isometrically isomorphic. Let us recall that a closed linear subspace A of $C_0(X)$ is said to be completely regular if every $x \in X$ is a strong boundary point of A, i.e., $\tau A = X$.

In [234] (see also [235]) is extended the Banach-Stone theorem for function algebras, that is, closed separating subalgebras with unit of $C(X)$-spaces. He proved that two function algebras are isomorphic as algebras if and only if they are isometric as Banach spaces.

In [236] went a step further and extended some of the above generalizations: if there exists a linear isometry T from a linear subspace A of $C(X)$ which is separating and contains the constants into $C_0(Y)$, then there are a continuous map h of the Choquet boundary of $T(A)$, $\mathrm{Ch}\, T(A)$, onto $\mathrm{Ch}\, A$ and a continuous map $a : \mathrm{Ch}\, T(A) \rightarrow \mathbb{K}$, $|a| \equiv 1$, such that

$(Tf$

Similar extensions of the Banach-Stone theorem have been given for subspaces of $C_0(X)$ equipped with different norms. Among these subspaces we point out the following: spaces of differentiable functions (in [237]); spaces of absolutely continuous functions ([238]); spaces of Lipschitz functions ([239]).

If we weaken the geometric bond between $C_0(X)$ and $C_0(Y)$, the homeomorphism between X and Y may wither: Milutin [240] proved that if X is *any* uncountable compact metric space (for instance, $X = [0,1] \cup \{2\}$), then $C(X)$ is linearly homeomorphic to $C([0,1])$. However, if the isometry is not weakened too much, good results can still be accomplished: Amir [241] and Cambern [242] proved that if $C_0(X)$ and $C_0(Y)$ are isomorphic under an isomorphim T satisfying $\|T\| \cdot \|T^{-1}\| < 2$, which is the best constant, then X and Y must also be homeomorphic. This theorem has been extended to cover various subspaces of $C_0(X)$-spaces, for instance, extremely regular subspaces ([243], [244], [245] or [246]) and function algebras ([247]).

The corresponding Banach-Stone theorem for E-valued continuous functions is not true even when the Banach space E is the two dimensional space $\mathbb{R}^2$ and X, Y are compact metric spaces (in[248]). Thus, the main concern in this line is to determine the geometric properties of E which allow analogues of the Banach-Stone theorem. A systematic account of many of the generalizations in this and the above directions can be found in [249] or [250].

We deal with some of these generalizations. Indeed we focus on Holsztyński and Novinger's directions. Namely we study linear isometries of a strongly separating linear subspace A of $C_0(X)$ into $C_0(Y)$ or onto such a subspace B of $C_0(Y)$. We show that such isometries can be written as weighted composition maps on some subspaces of Y ($\sigma_0 B$ for the onto case). Furthermore, under the onto assumption, we prove that $\sigma_0 A$ and $\sigma_0 B$ are homeomorphic. As straightforward consequences of this result we first show that the set of strong boundary points of A and B are homeomorphic. Also ∂A and ∂B are homeomorphic if A and B are assumed to be strongly separating function subspaces. We also provide an example which shows that this latter result may fail if the hypothesis "*A* is strongly separating" is replaced by the weaker one "*A* is separating".

Next we extend some results by Amir and Arbel [251], Holsztyńsky [252], Myers [233] and Novinger [236]. We also apply our main results to study the isometries between separating function subalgebras of $C_0(X)$ and $C_0(Y)$ or, more generally, between semisimple commutative Banach algebras without unit and their Shilov boundaries.

Finally, we would like to remark that our techniques are not based on the usual concepts, such as extreme points of the unit ball of the dual of $C_0(X), T$ -sets or M-ideals, used to prove the Banach-Stone theorem and their generalizations. We only use straightforward concepts instead.

In the sequel we will assume that every linear subspace A of $C_0(X)$ has nonvoid Shilov boundary. Anyway, let us note that the Shilov boundary of a strongly separating linear subspace of $C_0(X)$ is nonvoid and coincides with the closure of its Choquet boundary ([253]).

Lemma (2.1.1)[229]: *Let A be a linear subspace of $C_0(X)$. Then $\partial A = \sigma A$.*

Proof. Let $x_0 \in \partial A$. Given an open neighborhood U of x_0, the closed set $X - U$ cannot be a boundary for A since it does not contain ∂A. Consequently, there exists a function $f \in A$ which does not attain its maximum value on $X - U$, that is, $|f(x)| < \|f\|$ for all x outside U.

Conversely, let $x_0 \in \sigma A$. If $x_0 \notin \partial A$, then there exists an open neighborhood U of x_0 such that $\partial A \cap U = \emptyset$. Hence, there exists a function $f \in A$ not attaining its maximum value on ∂A, which contradicts the definition of boundary.

Lemma (2.1.2)[229]: *Let A be a linear subspace of $C_0(X)$. Let T be a linear isometry from A into $C_0(Y)$. Let $x \in X$ such that there exists $f \in A$ with $\|f\| = |f(x)|$. Let*

For any $f \in A$, let

and let $I_x := \bigcap_{f \in C_x} L(f)$. Then I_x is a nonempty subset of Y.

Proof. For any $f \in C_x$, we have

and M_f is compact because $Tf \in C_0(Y)$. Hence, we only need to prove that if $f_1, \dots, f_n$ belong to C_x, then $\bigcap_{i=1}^{n} L(f_i) \neq \emptyset$. We have that $1 = \|f_i\| = |f_i(x)|$ for all $i = 1, \dots, n$. Let $f \in A$ be defined as

Clearly $|f(x)| = n = \|f\|$. Since T is an isometry, $\|Tf\| = n$ and there is $y \in Y$ such that

As $\|Tf_i\| \leq 1$ for all $i = 1, \dots, n$, we deduce that $|(Tf_i)(y)| = 1$ for all $i = 1, \dots, n$, that is, $y \in \bigcap_{i=1}^{n} L(f_i)$.

Remark (2.1.3)[229]: Let A be a linear subspace of $C_0(X)$ and let $x_0 \in \partial A$. We then define the following subset of Y:

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Lemma (2.1.4)[229]: *Let A be a linear subspace of $C_0(X)$ and let $x_0 \in \sigma_0 A$. Then $V_{x_0} \neq \emptyset$.*

Proof. Let $f_0 \in A$ such that $|f_0(x_0)| = 1$. Given $\epsilon > 0$, let

Let U be an open neighborhood of x_0. We will assume that $U \subseteq U_{f_0,\epsilon}$. Since $x_0 \in \sigma_0 A$ and U is an open neighborhood of x_0, there exists a function $g_0 \in A$ such that $\|g_0\| = 1$ and $|g_0(x)| < 1$ for all x outside U. Since $(X \cup \{\infty\}) - U$ is compact, we can consider

Then there exists $M > 0$ such that $\|f_0\| + Ms < 1 + \epsilon + M$. Take $x \in U$. Then

If $x \notin U$, then

As a consequence, $\|f_0 + Mg_0\| < 1 + \epsilon + M$. Hence, $\|T(f_0 + Mg_0)\| < 1 + \epsilon + M$. Furthermore, since

we can choose M in such a way that $|f_0 + Mg_0|$ attains its maximum value inside U. Otherwise,

which is a contradiction. Thus, let $x_1 \in U$ such that $\|f_0 + Mg_0\| = |(f_0 + Mg_0)(x_1)|$.

Let $x_2 \in U$ such that $\|g_0\| = |g_0(x_2)| = 1$. It is clear that we can choose g_0 such that $|(f_0 + Mg_0)(x_2)| = |f_0(x_2)| + M|g_0(x_2)|$. Thus,

Consequently,

From the definition of I_{x_1} (Lemma (2.1.2)), we infer that

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for all $y_1 \in I_{x_1}$. Since $|(Tg_0)(y)| \leq 1$ for all $y \in Y$, we deduce that $|(Tf_0)(y_1)| \geq 1 - \epsilon$ for all $y_1 \in I_{x_1}$.

Next we shall show that $(Tf_0)(y_1) \leq 1 + \epsilon$ for all $y_1 \in I_{x_1}$. Let us define the function

Hence $|g_1(x_1)| = 1 = \|g_1\|$ and $|g_1| < 1$ outside U. Since $(X \cup \{\infty\}) - U$ is compact, we can consider

Arguing as above, we find a number $N \in \mathbb{K}$ such that $|f_0 + Ng_1|$ attains its maximum value inside U and $|f_0 + Ng_1| < 1 + \epsilon + |N|$. Furthermore, we can choose N in such a way that

$|($

As a consequence, since $|(Tg_1)(y_1)| = 1$, we have

$|N| + 1 + \epsilon$

that is, $(Tf_0)(y_1) \leq 1 + \epsilon$ for all $y_1 \in I_{x_1}$. Gathering up the information we have obtained so far, it is clear that we can find a net (x_α) in X converging to x_0 and a net (y_α) in Y such that $y_\alpha \in I_{x_\alpha}$ for all α and such that there exists a subnet (y_β) of (y_α) converging to some $y_0 \in Y \cup \{\infty\}$ with $|(Tf_0)(y_0)| = 1$. This latter fact shows that $y_0 \neq 1$. Furthermore, it is apparent, from the above arguments, that, for any open neighborhood V of x_0, there exist a term x_{β_0} of the net (x_β) and a function $g_{\beta_0} \in A$ such that $|g_{\beta_0}(x_{\beta_0})| = 1 = \|g_{\beta_0}\|$ and $|g_{\beta_0}| < 1$ outside V.

On the other hand, let $g \in A$ such that $g(x_0) = 0$. We shall show that $(Tg)(y_0) = 0$. Given $\epsilon > 0$, let

Let V be an open neighborhood of x_0 such that $V \subseteq U_{g,\epsilon}$. Hence, as mentioned above, there exist a term x_{β_0} of the net (x_β) and a function $g_{\beta_0} \in A$ such that $g_{\beta_0}(x_{\beta_0}) = 1 = \|g_{\beta_0}\|$ and $|g_{\beta_0}| < 1$ outside V. Arguing as above, we find a number $P \in \mathbb{K}$ such that the function $|g + Pg_{\beta_0}|$ attains its maximum value inside V and $\|g + Pg_{\beta_0}\| < \epsilon + |P|$. Furthermore, we can choose P in such a way that

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As a consequence, since $(Tg_{\beta_0})(y_{\beta_0}) = 1$, we have

$|P| + \epsilon \geq$

that is, $(Tg)(y_{\beta_0}) \leq \epsilon$. Therefore, since the net (y_β) converges to y_0, we infer that $(Tg)(y_0) = 0$.

Let us now consider $l \in A$ such that $|l(x_0)| = 1$. Let us define the function

and let $g \in A$ such that $l' = f_0 + g$. It is clear that $g(x_0) = 0$. Consequently, by the above paragraph, $(Tg)(y_0) = 0$ and, since $l'(x_0) = 1$,

and, thus, $|(Tl)(y_0)| = 1$.

Finally, if $f \in A$, then we define $f' = f/|f(x_0)|$. Hence $|f'(x_0)| = 1$. As a consequence, by the previous paragraph, we infer that $|(Tf')(y_0)| = 1$, i.e., $|f(x_0)| = |(Tf)(y_0)|$. The proof is complete.

Lemma (2.1.5)[229]: *Let A be a strongly separating linear subspace of $C_0(X)$ and let T be a linear isometry from A into $C_0(Y)$. If x_0 is a strong boundary point of A, then* $V_{x_0} = I_{x_0}$.

Proof. It suffices to check that $I_{x_0} \subseteq V_{x_0}$ since the other inclusion is apparent. We will first show that, if $f \in A$ satisfies $f(x_0) = 0$, then $(Tf)(y) = 0$ for all $y \in I_{x_0}$. Let us suppose that there exists $y_0 \in I_{x_0}$ such that $(Tf)(y_0) \neq 0$ and $f(x_0) = 0$ for some $f \in A$. We will assume, without loss of generality, that $\|f\| = 1$ and $(Tf)(y_0) = \alpha > 0$. Let

Since x_0 is a strong boundary point of A and $y_0 \in I_{x_0}$, there exists $g \in A$ such that, multiplying by a constant if necessary, $|g(x_0)| = 1 = \|g\|, |g(x)| < 1$ for all $x \in U$ and $(Tg)(y_0) = 1$. Since, from the definition of $C_0(X)$, U is a compact set, we can consider

Thus there is a real number $M > 0$ such that $1 + Ms < \alpha + M$. We will distinguish two cases: If $x \in U$, then

If $x \notin U$, then

That is, we have that $\|f + Mg\| < \alpha + M$, but

which is absurd since T is an isometry.

Finally, let us suppose that there exists $y' \in I_{x_0}$ such that $|(Tf)(y')| \neq |f(x_0)|$ for some $f \in A$. Since x_0 is a strong boundary point of A, there will exist a function $k \in A$ such that $k(x_0) = 1 = \|k\|$. Hence it is straightforward to check that the function

belongs to A and, furthermore,

and

since $(Tk)(y') = 1$. This fact contradicts the paragraph above.

Theorem (2.1.6)[229]: *Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ into $C_0(Y)$. Then there are a subset Y_0 of Y, which is a boundary for $T(A)$, a continuous map h from Y_0 onto $\sigma_0 A$ and a continuous map $a : Y_0 \to \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in Y_0$, and*

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Furthermore, if $\sigma_0 A$ is compact, then Y_0 is closed.

Proof. Let Y_0 be the set $\bigcup_{x\in\sigma_0 A} V_x$. That Y_0 is nonvoid. In order to prove that Y_0 is a boundary for $T(A)$, let us suppose that there exists $f \in A$ such that

for all $y \in Y_0$. Then we can find $x_0 \in \sigma_0 A$ such that

Let $y_0 \in V_{x_0}$. Then $y_0 \in Y_0$ and

which contradicts the above assumptions.

Next, we define the map h of Y_0 onto $\sigma_0 A$ as $h(y) \coloneqq x$ if $y \in V_x$. Since A is strongly separating, given $x, x' \in \sigma_0 A$ with $x \neq x_0$, it is easy to check that $V_x \cap V_{x'} = \emptyset$.

Thus the map h is well-defined. Moreover, since $V_x \neq \emptyset$ for every $x \in \sigma_0 A, h$ is onto. In order to prove the continuity of h, suppose that $h(y_0) = x_0$ for some $y_0 \in Y_0$.

Let $f \in A$ such that $f(x_0) = 1$. Hence, $|(Tf)(y_0)| = 1$. Let (y_α) be a net in Y_0 converging to y_0 and let $h(y_\alpha) = x_\alpha$ for all α. Since $|(Tf)(y_0)| = 1$, we can assume, without loss of generality, that $\big||(Tf)(y_\alpha)| - 1\big| < 1/2$ for all α. Then, from the definition of $V_{x_\alpha}, |f(x_\alpha)| > 1/2$ for all α. Let (x_β) be a subnet of (x_α) converging to

$x_1 \in X \cup \{\infty\}$. Consequently, $|f(x_1)| \geq 1/2$. Hence, $x_1 \neq 1$. If $x_1 \neq x_0$, then we take $g \in A$ such that

Take a subnet (y) of (y_β) such that

Hence

and $(|g(x_\gamma)|)$ does not converge to $|g(x_1)|$, which is a contradiction. Hence every subnet of (x_α) has a subnet that converges to x_0 and then we have that (x_α) converges to x_0.

Now, let us define a map a of Y_0 into $\mathbb{K}$ as follows: given $y \in Y_0$, let f be any function in A such that $f(h(y)) = 1$. Hence, we define $a(y) \coloneqq (Tf)(y)$ for all $y \in Y_0$. This is a well-defined map because if we take another function g in A such that $g(h(y)) = 1$, then $(f-g)(h(y)) = 0$ and by the definition of h, $(Tf)(y) = (Tg)(y)$. On the other hand, it is clear that $|a(y)| = 1$ for all $y \in Y_0$.

Next we prove both that T can be written as a weighted composition map and, as a consequence, the continuity of a. We have already proved that if $f(h(y)) = 0$, then $(Tf)(y) = 0$ for all $y \in Y_0$ and all $f \in A$. If $f(h(y)) \neq 0$ for some $f \in A$ and some $y \in Y_0$, then let

k being any function in A such that $k(h(y)) = 1$. Clearly $g(h(y)) = 0$. Thus, $(Tg)(y) = 0$, that is,

In order to prove the continuity of a, we will show that for each $y \in Y_0$, there exists an open neighborhood of y where a is continuous. Let us consider any $f \in A$ such that $f(h(y)) \neq 0$ and let

It is clear that $h^{-1}(W)$ is an open neighborhood of y. Moreover the map $Tf/(f \circ h)$ is continuous on $h^{-1}(W)$, and *a* and $Tf/(f \circ h)$ coincide on $h^{-1}(W)$.

Finally, assume that $\sigma_0 A$ is compact and let $y_0 \in Y$ such that there exists a net (y_α) in Y_0 converging to y_0. For all α, y_α belongs to some V_{x_α} (see remark preceding Lemma (2.1.4)) with $h(y_\alpha) = x_\alpha \in \sigma_0 A$. Hence the net above has a subnet (x_β) which converges to some $x_0 \in \sigma_0 A$. Since

for all β and all $f \in A$, we deduce, by the continuity of f and Tf, that

for all $f \in A$, that is, $y_0 \in V_{x_0} \subset Y_0$.

Corollary (2.1.7)[229]: *In the same conditions as in Theorem (2.1.6), h sends* $\mathrm{Ch}T(A)$ *onto* $\mathrm{Ch}A$*.*

Proof. By [253], we know that the Choquet boundary of A is contained in ∂A. On the other hand, for every point x of $\mathrm{Ch}A$, there exists $f \in A$ such that $f(x) \neq 0$, so $\mathrm{Ch}A \subset \sigma_0 A$. Since $T^{-1} : T(A) \to A$ is a linear surjective isometry, we have that its adjoint $(T^{-1})' : A' \to \left(T(A)\right)'$ sends the extreme points of the unit ball of A' onto such points of $\left(T(A)\right)'$. So, if $x \in \mathrm{Ch}A, (T^{-1})' e_x = \mu e_y$, where $\mu \in \mathbb{K}, |\mu| = 1$ and $y \in Y$. If $f \in A$,

We conclude that $y \in V_x$ and, consequently, that $\mathrm{Ch}A \subseteq h\left(\mathrm{Ch}T(A)\right)$.

The other inclusion follows from the same arguments since $(T^{-1})'$ is bijective (see [242]).

Theorem (2.1.8)[229]: *Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ into $C_0(Y)$. Then $Tf = E(a \cdot f \circ h)$, where h is a continuous map defined from a subset Y_0 of Y onto $\sigma_0 A$, $a : Y_0 \to \mathbb{K}$ is a continuous map such that $|a| \equiv 1$ and $E : Z \to C_0(Y)$ is a norm-preserving linear extension with $Z = \{a \cdot f \circ h : f \in A\}$.*

Theorem (2.1.9)[229]: *Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ onto such a subspace B of $C_0(Y)$. Then there exist a homeomorphism h of $\sigma^0 B$ onto $\sigma^0 A$ and a continuous map $a : \sigma_0 B \to \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in \sigma_0 B$, and*

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Proof. Let h and Y_0 be as in Theorem (2.1.6). To prove the injectivity of h, we shall check that the sets V_x are singletons. Suppose that $y_0, y_1 \in V_x$ for some $x \in \sigma_0 A$ and $y_0 \neq y_1$. Consequently

for all $f \in A$, which contradicts the strongly separating property of B.

We now show that $Y_0 = \sigma_0 B$. Let $y_0 \in Y_0$. There exists $x_0 \in \sigma_0 A$ such that

Let U be any open neighborhood of y_0. It falls out of the way we obtain y_0 in Lemma (2.1.4) that there exists a set I_x, for some $x \in X$, contained in U. This means, according to the definition of I_x (Lemma (2.1.1)), that

that is,

Thus we have an intersection of closed subsets whose intersection with the compact set $(X \cup \{\infty\}) \backslash U$ is empty. Hence, there exist finitely many functions $\{f_1, \dots, f_n\} \subset C_x$ such that

We can assume, with no loss of generality, that

Then the function $f := \sum_{i=1}^{n} f_i$ satisfies $f(x) = n$. As a consequence,

for all $y \notin U$, which implies that $y_0 \in \sigma_0 B$.

Conversely, to prove that $\sigma_0 B \in Y_0$, take $y_0 \in \sigma_0 B$. We now consider the inverse of T, which is an isometry of B onto A. By Theorem (2.1.6), there exists a continuous map k from a subset X_0 of X, defined as $\bigcup_{y \in \sigma_0 B} V_y$, onto $\sigma_0 B$. As above, we can prove that $X_0 \subset \sigma_0 A$. Let us consider $x_0 \in X_0$ such that $k(x_0) = y_0$. It just remains to prove that $y_0 \in V_{x_0}$. We know that, for all $g \in B$,

that is, for all $f \in A$,

Consequently, $h(y_0) = x_0$. Hence, h is a homeomorphism of $\sigma_0 A$ onto $\sigma_0 B$. Finally, by Theorem (2.1.6), T is a weighted composition map.

Now the following corollary holds because of Theorem (2.1.11) and Corollary (2.1.8).

Corollary (2.1.10)[229]: *In the same conditions as in Theorem* (2.1.11)*, h is a homeomorphism of* $\mathrm{Ch}B$ *onto* $\mathrm{Ch}A$*.*

Remarks (2.1.11)[229]: (i) Let T be the isometric embedding of $C_0(\mathbb{N})$ into $C(\mathbb{N} \cup \{\infty\})$. This example shows both that, in Theorem (2.1.6), Y_0 may not be closed and that, in Theorem (2.1.8), the Shilov boundaries of A and B are not homeomorphic in general. However, if A and B are assumed to be strongly separating function subspaces, then it is straightforward, from the definition of $\sigma_0 A$ (resp. $\sigma_0 B$) and by Lemma (2.1.1), that their Shilov boundaries are homeomorphic.

(ii) The following example shows that this latter assertion may fail if we replace the hypothesis "A is strongly separating" by "A is separating": Let us define the compact set

and let A' be the set of all functions $f_n \in C(X), n = 2,3,4, \ldots$, defined as follows: if n is even,

If n is odd,

Given $x \in X$ (resp. $x \in Y$), we will denote by χ_x the characteristic function of the singleton $\{x\}$. Let us define the function $f \in C(X)$ in the following way:

Let A be the linear span of A' and f.

On the other hand, let us define the compact set

and let us define $g \in C(Y)$ as follows:

Let B be the linear span of the set $\left\{\chi_{\frac{1}{n}}: n \in \mathbb{N}\right\}$. and g. It is now a routine matter to verify that A is a (not strongly) separating linear subspace of $C(X)$. Also, A is linearly isometric to B. However,

is not homeomorphic to $\partial B = Y$.

(iii) The assertion of Theorem (2.1.9) cannot be strengthened to the effect "X homeomorphic to Y". A counterexample is obtained by taking the isometry T of $C_0(X) = C_0(0,1)$ into $C_0(Y) = C_0\big((0,1) \cup (1,2)\big)$ defined to be $(Tf)(x) = f(x)$ if $x \in (0,1)$, and $(Tf)(x) = f(x-1)/2$ if $x \in (1,2)$. Clearly X is not homeomorphic to Y because Y is not connected.

Corollary (2.1.12)[229]: *Let T, X, Y, A and B be as in Theorem (2.1.9). If, in addition, we assume that either τA or τB is a nonempty set, then τA and τB are homeomorphic.*

Proof. Let us define the set

By the definition of h (see the proof of Theorem (2.1.6)) and since it is injective (Theorem (2.1.9)), we infer that $h(Y_{00}) = \tau A$. Hence, by virtue of Theorem (2.1.9), it suffices to check that $Y_{00} = \tau B$. Let $y_0 \in Y_{00}$. There exists $x_0 \in \tau A$ such that $x_0 = h(y_0)$. Let U be any open neighborhood of y_0. If $y \notin U$, then $y \in V_{x_0}$ since, by Lemma (2.1.5), $V_{x_0} = I_{x_0}$ whenever x_0 is a strong boundary point. Thus, by Lemma (2.1.2), there is $f_y \in A$ such that

and

For each $y \in (Y \cup \{\infty\}) - U$, we take an open neighborhood U_y of y such that $\left|(Tf_y)(y')\right| < 1$ for all $y' \in U_y$. Since $(Y \cup \{\infty\}) - U$ is compact, we can find

$\{y_1, \dots, y_n\} \subset (Y \cup \{\infty\}) - U$ such that $(Y \cup \{\infty\}) - U \subset \bigcup_{i=1}^{n} U_{y_i}$. Now, let us define the map

It is clear that

and

Moreover, $|(Tg)(y)| < 1$ for all y outside U. Consequently, the elements of Y_{00} are strong boundary points for B.

Conversely, let $y_0 \in \tau B$. Arguing as in the preceding paragraph, we prove $X_{00} \subset \tau A$, where

Thus, there exists $x_0 \in \tau A$ such that $k(x_0) = y_0$, where k is the inverse of h (see the proof of Theorem (2.1.9)). That is, $y_0 \in V_{x_0} \subset Y_{00}$ and we are done.

The following corollary shows that Myers' theorem ([233]) is valid also for noncompact spaces and complex valued functions. Moreover we can write the isometry as a weighted composition map.

Corollary (2.1.13)[229]: *Let T be a linear isometry of a completely regular linear subspace A of $C_0(X)$ onto such a subspace B of $C_0(Y)$. Then there exist a homeomorphism h of Y onto X and a continuous map $a : Y \to \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in Y$, and*

(T

Corollary (2.1.14)[229]: *Let T be a linear isometry of a strongly separating subspace A of $C_0(X)$ into $C_0(Y)$. Then $\sigma_0 A$ is homeomorphic to a quotient of a subspace Y_0 of Y.*

Proof. With the same notation as in Theorem (2.1.6), let us define the following equivalence in Y_0: $y_0 \sim y_1$ if y_0, y_1 belong to the same V_x, for some $x \in \sigma_0 A$. If π denotes the natural quotient map of Y_0 onto $(Y_0/\sim)$, then the map $h^{\sim} = h \circ \pi^{-1}$ is a continuous bijection of $(Y_0/\sim)$ onto $\sigma_0 A$. To prove the continuity of $(h^{\sim})^{-1}$, take a net (x_α) in $\sigma_0 A$ converging to $x_0 \in \sigma_0 A$. For each α, take $y_\alpha \in V_{x_\alpha}$. Clearly there exists a subnet (y_β) of (y_α) converging to a point $y_0 \in Y \cup \{\infty\}$. Take any $f \in A$. Then $\left(|f(x_\beta)|\right)$ converges to $|f(x_0)|$ and $|(Tf)(y_\beta)|$ converges to $|(Tf)(y_0)|$. As

for all β, we have that

for all $f \in A$, that is, $y_0 \in V_{x_0}$. This implies that $\left((h^{\sim})^{-1}(x_\beta)\right)$ converges to $(h^{\sim})^{-1}(x_0)$. In this way, every subnet of $\left((h^{\sim})^{-1}(x_\alpha)\right)$ has a subnet converging to $(h^{\sim})^{-1}(x_0)$. So $\left((h^{\sim})^{-1}(x_\alpha)\right)$ converges to $(h^{\sim})^{-1}(x_0)$. Then $(h^{\sim})^{-1}$ is continuous.

Remark (2.1.15)[229]: Let us suppose that, in Corollary (2.1.14), X is compact and $1 \in A$. We now consider the quotient space Y/D, where D is a decomposition of Y which consists of the subsets $V_x, x \in \sigma_0 A$, and the singletons $\{y\}$ such that $y \in Y - Y_0$. Since now $a = T1$ then $\|a\| = 1$ and we can define an isometry $T^{\sim}$ of A into $C_0(Y/D)$ by the requirement that $(T^{\sim} f)(y^{\sim}) = (\bar{a} Tf)\left(\pi^{-1}(y^{\sim})\right)$ for all $f \in A$ and all $y^{\sim} \in Y/D$, where $\bar{a}$ denotes the complex conjugate of the map a and π the natural quotient map of Y onto Y/D. $T^{\sim}$ is well defined because, by Theorem (2.1.6), $\bar{a}Tf$ is constant on each $V_x, x \in \sigma_0 A$. As in Corollary (2.1.14), we prove that $\sigma_0 A$ is homeomorphic to a subspace $(Y/D)_0$ of Y/D, defined like $Y/0$ in Theorem (2.1.6). Moreover, with the hypothesis of Corollary (2.1.14), there exists a norm-preserving linear extension U from the subspace $\{A \circ h\}$ of $C(Y_0)$ into $C_0(Y)$ defined to be $U(g) := \bar{a}(Tf)$, where $g := f \circ h$ for some $f \in A$.

All these remarks show that Corollary (2.1.14) extends a result by D. Amir and B.

Arbel [251], if we assume that X is compact and $A = C(X)$. Furthermore, this assumption lets us claim that if $T(C(X))$ is a strongly separating linear subspace, then it is complemented in $C_0(Y)$. To prove this, we define a projection π from $C_0(Y)$ onto $T(C(X))$ as follows: given $f \in C_0(Y)$, let $\pi(f)$ be

where h^{-1} is the inverse of h defined as in Theorem (2.1.11). It is easy to check that $\pi \circ T = T$ and, consequently, $\pi^2 = \pi$.

Theorem (2.1.16)[229]: *Let T be a linear isometry of a separating function of subalgebra A of $C_0(X)$ onto such a subspace B of $C_0(Y)$. Then*

(i) *∂A is homeomorphic to ∂B.*

(ii) *There exists a continuous map $b : Y \to \mathbb{K}$ such that*

$T(fg)($

Proof. (i) It suffices to show that A is strongly separating. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. There exists $f \in A$ such that $f(x_1) = z_1$ and $f(x_2) = z_2$ with $z_1 \neq z_2$. If $|z_1| = |z_2|$, then we consider the function $g \coloneqq f + f^2 \in A$. Hence, $g(x_1) = z_1(1 + z_1)$ and $g(x_2) = z_2(1 + z_2)$. With no loss of generality, we can assume that $\mathrm{Re}\, z_1$ and $\mathrm{Re}\, z_2$ are different. Otherwise we multiply the function f by the complex number i and, since $z_1 \neq z_2$, we infer that $\mathrm{Re}\, iz_1 \neq \mathrm{Re}\, iz_2$. Then

and

Clearly, $|g(x_1)| \neq |g(x_2)|$. Summing up, A is a strongly separating linear subspace of $C_0(X)$ and the result follows from Theorem (2.1.11).

(ii) We know, by Theorem (2.1.9), that $a(y)T(fg)(y) = (Tf)(y)(Tg)(y)$ for all $f, g \in A$ and all $y \in \partial B$. Consequently, if b denotes the complex conjugate of a,

for all $y \in \partial B$ and all $f, g \in A$.

Let $y_0 \in Y - \partial B$ and let $f \in A$ such that $(Tf)(y_0) = 0$. We now show that $T(fg)(y_0) = 0$ for all $g \in A$. It is clear that the maps

and

coincide on ∂B, for $k \in A$ such that $(Tk)(y_0) \neq 0$. Thus they coincide on Y and consequently $T(fg)(y_0) = 0$.

To extend b from ∂B to Y, take, for each $y \in Y - \partial B, f, g$ in A such that $(Tf)(y) \neq 0$ and $(Tg)(y) \neq 0$. Then we define

This extension of the map b to the whole Y is well defined because if we consider $k, l \in A$ with $(Tk)(y) \neq 0$ and $(Tl)(y) \neq 0$, then

and

coincide on ∂B and, as a consequence, on Y. The continuity of b follows from the continuity of $T(fg), Tf$ and Tg in an open neighborhood of each y.

Theorem (2.1.17)[229]: *Let A, B be semisimple commutative Banach algebras (not necessarily with unit), such that $\|f\|^2 = \|f^2\|$ for all $f \in A$ (resp. $f \in B$). If T is a linear isometry of A into (resp. onto) B, then there exists a continuous map (resp. a homeomorphism) h of a subset Y_0 of the maximal ideal space Y of B (resp. the Shilov* *boundary ∂B) onto the Shilov boundary, ∂A, of A and a continuous map $a : Y_0 \to \mathbb{K}$ (resp. $a : \partial B \to \mathbb{K}$) with $|a(y)| = 1$ for all $y \in Y_0$ (resp. $y \in \partial B$) and*

$(Tf)(y$

Proof. Since $\|f\|^2 = \|f^2\|$ for all $f \in A$ (for all $f \in B$ respectively), the Gelfand transform is an isometry of A (resp. B) into $C_0(X)$ (resp. $C_0(Y)$), where X (resp. Y) is the maximal ideal space of A (resp. B). We can, therefore, regard A and B as separating function subalgebras of $C_0(X)$ and $C_0(Y)$ respectively, and the result follows from Theorem (2.1.6) and Theorem (2.1.11).

Corollary (2.1.18)[229]: (Nagasawa) *Two semisimple commutative Banach algebras with unit A and B such that $\|f\|^2 = \|f^2\|$ for all $f \in A$ (resp. $f \in B$) are isometric as Banach spaces if and only if they are isomorphic as algebras.*

Proof. Let us first regard A and B as in the proof of Theorem (2.1.17) and let T be a linear isometry of A onto B. It is clear, since both subalgebras have unit 1, that, in this context, the continuous function *a* which appears in Theorem (2.1.16) is $T1$. That is, $a \in B$. Furthermore, since T is onto, there exists $f \in A$ such that $Tf = 1$. Finally, from Theorem (2.1.6), we infer both that $b = a^{-1} \in B$ and that $b \cdot T$ is the desired algebra isomorphism.

The converse is clear (see [24]).

Section (2.2): Inverse Spectral Theory for the Ward Equation:

We study the Cauchy problem for the Ward model in light-cone coordinates:

where $[\,,]$ denotes the usual matrix commutator, $Q(x, y, t)$ is a traceless 2×2 anti-Hermitian matrix and $Q_0(x, y)$ is a 2×2 anti-Hermitian traceless matrix decaying sufficiently fast as $x^2 + y^2 \to \infty$.

We shall solve this problem using the so-called inverse spectral (scattering) method. This method is based on the fact that equation (1) is the compatibility condition of the following Lax pair,

where $\mu(x, y, t, k)$ is a 2×2 matrix. The transformation

maps equation (1) to the Ward model [255] in laboratory coordinates. The Cauchy problem in this model is defined by

Q_{tt}

This problem can be solved by using the fact that equation (6) possess the following Lax pair

2

The Cauchy problem (6), (7), was studied in [256] using the Lax pair (8), (9).

Here we study the Cauchy problem (1), (2), using the Lax pair (3), (4). We also make some remarks about the Cauchy problem (6), (7).

We note that the transformations

and

map equations (1) and (6) to equations

and

respectively. Thus, our results are directly applicable to the solutions of the Cauchy problem for equations (12) and (13). These equations are the $2+1$ integrable chiral equations in light-cone and laboratory coordinates, respectively.

In order to simplify the rigorous aspects of our formalism we first assume that $Q_0(x,y)$ is a Schwartz function which is small in the following sense

where $\hat{Q}_0$ is the Fourier transformation of Q_0 in the x variable. This assumption excludes soliton solutions. We then indicate how the formalism can be extended in the case that the above assumption is violated. In the case that Q_0 is sufficient small, the inverse spectral method yields a solution of the Ward model in light-cone coordinates through the following construction.

Theorem (2.2.1)[254]: *Let $Q_0(x,y), x,y \in R$ be a 2×2 anti-Hermitian traceless matrix which is a Schwartz function and which is small in the sense of equation (14).*

(i) Given $Q_0(x,y)$, define $\mu^+(x,y,k), k \in C^+ = \{k \in C : Imk \geq 0\}$ and $\mu^-(x,y,k), k \in C^- = \{k \in C : Imk \leq 0\}$ as the 2×2 matrix valued functions which are the unique solutions of the linear integral equations

$$\mu^+(x,y,k) = I + \cdots$$

and

$$\mu^-(x,y,k) = I + \cdots$$

where I denotes the 2×2 unit matrix.

(ii) Given $\mu^{\pm}$ define the 2×2 matrix $S(x + ky, k), x, y, k \in R$, by

$$I - S = \left(I - \frac{1}{4\pi} \right.$$

$$\left(I - \frac{1}{4\pi} \int_0 \right.$$

(iv) Given $S(x + ky, k)$ define the sectionally holomorphic function $M(x, y, t, k) = M^{+}(x, y, t, k)$ for $k \in C^{+}, M(x, y, t, k) = M^{-}(x, y, t, k)$ for $k \in C^{-}$ as the unique solution of the following 2×2 Riemann-Hilbert problem

M^{-}

$\det$

M

(v) Given $M(x, y, t, k)$ define Q as

Then Q solves equation (1) and $Q(x, y, 0) = Q_0(x, y)$.

We now make some remarks about related work. A method for solving the Cauchy problem for decaying initial data for integrable evolution equations in one spatial variable was discovered in [257]. This method which we refer to as the inverse spectral method, reduces the solution of the Cauchy problem to the solution of an inverse scattering problem for an associated linear eigenvalue equation (namely for the x-part of the associated Lax pair). Such an integrable evolution equation in one spatial dimension is the chiral equation; the associated x-part of the Lax pair is where the eigenfunction $\mu(x,t,k)$ is a 2×2 matrix, k is the spectral parameter and $Q(x,t)$ is a solution of the chiral equation.

Each integrable evolution equation in one spatial dimension has several two spatial dimensional integrable generalizations. An integrable generalization of the chiral equation is (1). A method for solving the Cauchy problem for decaying initial data for integrable evolution equations in two spatial variables appeared in ([258],[259]). For some equations such as the Kadomtsev-Petviashvili I equation, this method is based on a nonlocal Riemann-Hilbert problem, while for other equations such as the Kadomtsev-Petviashvili II equation, this method is based on a certain generalization of the Riemann- Hilbert problem called the $\bar{\partial}(\mathrm{DBAR})$ problem.

It is interesting that although equation (1) is an equation in two spatial variables, the Cauchy problem can be solved by a local Riemann-Hilbert problem. This is a consequence of the fact that the equation (3) is a first order ODE in the variable $x-ky$.

For integrable equations, there exist several different methods for constructing exact solutions. Such exact solutions for the Ward model in laboratory coordinates have been constructed in [255, 256]. In particular, Ward constructed soliton solutions using the so-called dressing method [260]. These solutions are obtained by assuming that $M(x,y,t,k)$ has simple poles. In this case the corresponding solitons interact trivially, that is they pass through each other without any phase-shift. Recently, new soliton [261, 262] and soliton-antisoliton solutions [262] were derived, by assuming that $M(x,y,t,k)$ has double or higher order poles. The corresponding lumps interact nontrivially, namely they exhibit π/N scattering between N initial solitons.

The formalism presented in this section can also be used to obtain exact soliton solutions. In particular, it is shown that if the assumption (14) is violated then $M(x,y,t,k)$ still satisfies the Riemann-Hilbert problem (18) but now it is generally a meromorphic as opposed to a holomorphic function of k. The solitonic part of the solution $Q(x,y,t)$ is generated by the poles of M. The main advantage of this approach is that it can be used to establish the generic role played by the soliton solutions. Namely, it is well known [263] that the long time behaviour of the solution of a local

Riemann-Hilbert problem of the type (18) where M is a meromorphic function of k, is dominated by the associated poles. Thus the long time behaviour of $Q(x,y,t)$ with arbitrary decaying initial data $Q_0(x,y)$ is given by the multisoliton solution.

In this section we prove Theorem (2.2.1).

We first consider the direct problem, i.e., we show that the spectral data $S(x+ky,k)$ are well defined in terms of the initial data $Q_0(x,y)$. Replacing $Q(x,y,t)$ by $Q_0(x,y)$ in equation (3) we find

Let $\hat{\mu}(p,y,k)$ denote the x-Fourier transform of $\mu(x,y,k)$. Then equation (23) gives

Equations (15) and (16) are integrable forms of equation (24) with different initial values. Under the small norm assumption (14), equations (15) and (16) are uniquely solvable in the space of bounded continuous functions $f(x,y)$ such that $f-I$ has a finite L_1 norm. Since the dependence on k of the kernel of the integral equations (15) and (16) is analytic, the functions $\mu^{\pm}(x,y,k)$ are analytic in k for $\pm Imk \geq 0$.

Equations (15) and (16) can also be written in the form

$\mu^{\pm}(x,y,k)$

where

We note that $G^{\pm}$ can be evaluated in closed form, where $\delta(y)$ and $\theta(x)$ denote the Dirac and the Heaviside functions, respectively.

Indeed, writing $1/k = (k_R - ik_I)/|k^2|$ and using

$$\int_R dx$$

we find

Recall that G^+ corresponds to $k_I \geq 0$; then in this case $(\mathrm{sgn}x)\theta(xlk_I) = \theta(x)\theta(l) - \theta(-x)\theta(-l)$, and the above equation becomes

$$G^+$$

Using

we find the expression for G^+ given by (27). Similarly for G^-.

Using equation (27) it is straightforward to compute the large k behaviour of $\mu^\pm$:

$$\mu^\pm(x, y, k)$$

$$= I \pm \frac{i}{2\pi k} \int_{R^2}$$

for $k \to \infty$. Thus

Taking the complex conjugate of equation (15), letting $p \to -p$ and using the fact

that

we find

$\overline{\mu_{11}^{+}}$

$\overline{\mu_{12}^{+}}$

Letting $\xi = x + ky, \eta = x - ky$, equation (23) becomes

Thus any two solutions of this equation are related by a matrix which is a function of $x + ky$ and of k. Hence

Equation (37) and the symmetry relations (35), imply that $I - S$ is a Hermitian matrix. In particular, the determinant of $I - S$ is real. The determinant of equation (37) yields

Taking the complex conjugate of this equation and using the symmetry relations (35), we find

Equations (38) and (39) imply $\det(I - S) = \pm 1$. However, equation (33) implies that

Thus equation (38) implies $\det(I - S) = 1 + O(1/k)$ as $k \to \infty$, and since $\det(I - S) = \pm 1$ it follows that

Equations (38) and (41) imply

Since $\mu^{\pm}$ are analytic in $C^{\pm}$, equations (40) and (42) define a local Riemann-Hilbert problem [264]. Its unique solution is

Evaluating equation (37) as $y \to -\infty$ (keeping $x + ky$ fixed) we find

which is equation (17).

We now consider the inverse problem, i.e., we show how to construct the solution of the Cauchy problem (1), (2), starting from $S(x+ky,k)$. Given $S(x+ky+k^2t,k)$, we define $M(x,y,t,k)$ as the solution of the Riemann-Hilbert problem (18). In general, if the L_2 norm with respect to k of S and of $\frac{\partial S}{\partial k}$ are sufficiently small, then the problem has a unique solution. However, in our particular case the solution exists without a small norm assumption. This is a consequence of the fact that $I - S$ is a Hermitian matrix. Using this fact it can be shown (in [265]) that the homogeneous problem, i.e., the problem

has only the zero solution.

Given M, we define $Q(x,y,t)$ by equation (21). A direct computation shows that if M^+ solves the Riemann-Hilbert problem (18), ie, if M^+ satisfies

$M^+(x,y,t,k) =$

and if $Q(x,y,t)$ is defined by equation (21) then M^+ satisfies equations (3) and (4). Hence Q satisfies equation (1). Furthermore the investigation of the Riemann-Hilbert problem (18) at $t = 0$, implies that $Q(x,y,0) = Q_0(x,y)$. Also since $I - S$ is Hermitian, M^+ and M^- have the proper symmetry properties (see equations (35)), which in turn imply that $Q(x,y,t)$ is a traceless anti-Hermitian matrix [272].

In this section we show how the formalism can be modified to include the soliton solutions.

Since the matrix $I - S$ is Hermitian of determinant one, it can be represented as where α is an arbitrary function of $(x + ky + k^2 t, k)$.

Then equation (18) becomes

where M_1^+ and M_2^- are 2-dimensional column vectors which are functions of (x, y, t, k).

In particular

Equations (15) and (16) are Fredholm integral equations of the second type; thus they may have homogeneous solutions. These homogeneous solutions which correspond to discrete eigenvalues are rather important since they give rise to solitons. We assume that there exists a finite number of discrete eigenvalues and that they are all simple. Then Fredholm theory implies that M_1^+ admits the representation

where $m_1^+(x, y, t, k)$ is analytic for $k \in C^+$ and the vectors $\phi_l(x, y, t), 1 \leq l \leq N$ are homogenous solutions of the first column vector of equation (15). Following the arguments of [267] it can be shown that

where c_l is a scalar function of the argument indicated. Hence equation (51) becomes

$M_1^+(x, y, t, k$

Substituting this equation into equation (50) solving the resulting Riemann-Hilbert problem we find

$M_1^-(x, y, t,$

Let

In what follows, for simplicity of notion we suppress the x, y, t dependence. Using the notation (55) together with the symmetry relation (35), equation (54) becomes

$$\Bigg(\begin{matrix} \overline{B(\bar{k})} \\ -\overline{A(\bar{k})} \end{matrix}$$

Equation (56) express the solution of the Riemann-Hilbert problem (18) in the case that solitons are included.

Soliton solutions correspond to $\alpha = 0$. In this case evaluating equation (56) at $k = \bar{k}_j$ we find

The complex conjugate of these equations yields

Equations (57) and (58) determine $A(k_l)$ and $B(k_l), l = 1, \ldots, N$. Equation (54) yields

Thus using equation (33) we find

$Q_{11} =$

In summary, the N-soliton solution is given by equations (60), where $c_l = c_l(x + k_l y + k_l^2 t)$ and $A(k_l), B(k_l)$ are the solutions of the equations (57) and (58). In the case of 1-soliton equations (57) and (58) yield

$A(k_1) =$

Thus

$Q_{11} = -Q_{22} =$

Figure 1 represent a snapshot of the solution of equation (1) by taking $c_1 = x + k_1 y + k_1^2 t$ for $k_1 = i$ at time $t = -3$.

Here we study the Cauchy problem (6), (7) using the Lax pair (8), (9). In the case that $Q_1 Q_2$ are sufficient small, the inverse spectral method yields a solution of the Ward model in laboratory coordinates through the following construction.

Theorem (2.2.2)[254]: *Let* $Q_1(x,y), Q_2(x,y), x, y \in R$ *be* 2×2 *anti-Hermitian traceless matrices which are Schwartz functions and which satisfy the small norm conditions*

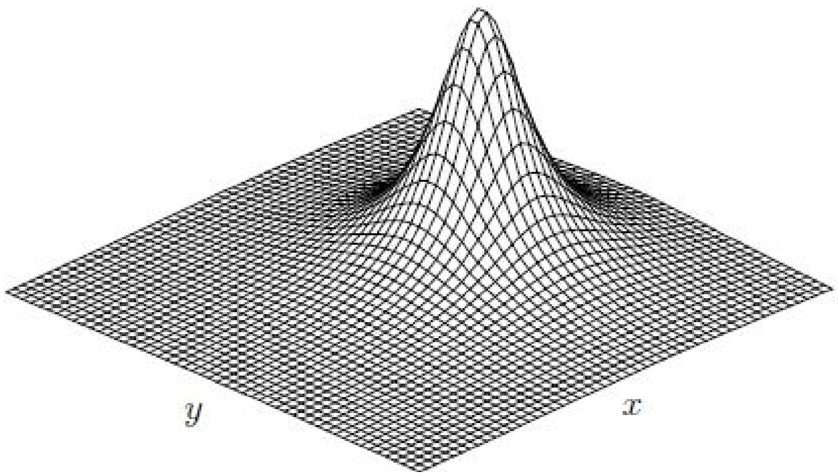


Figure 1: 1-soliton solution of (1) at $t = -3$.

where $\hat{Q}_j$ is the Fourier transformation of Q_j in the x variable.

(i) Given $Q_1(x, y), Q_2(x, y)$, define $\mu^{+}(x, y, k), k \in C^{+} = \{k \in C : Imk \geq 0\}$ and $\mu^{-}(x, y, k), k \in C^{-} = \{k \in C : Imk \leq 0\}$ as the 2×2 matrix valued functions which are the unique solutions of the linear integral equations

$$\mu^{+} = I + \frac{1}{4}$$

and

$$\mu^{-} = I + \frac{1}{4}$$

(ii) Given $\mu^{\pm}$ define the 2×2 matrix $S\left(x + \frac{k^2-1}{2k} y, k\right), x, y, k \in R$, by

$$I - S = \left(I - \frac{1}{4\pi} \right.$$

$$\left(I - \frac{1}{4\pi}\right.$$

(iv) Given $S\left(x+\frac{k^2-1}{2k}y,k\right)$ *define the sectionally holomorphic function* $M(x,y,t,k) = M^+(x,y,t,k)$ *for* $k \in C^+$, $M(x,y,t,k) = M^-(x,y,t,k)$ *for* $k \in C^-$ *as the unique solution of the following* 2×2 *Riemann-Hilbert problem*

$$M^-(x,y,t,k)$$

$$\det M$$

$$M$$

(v) Given $M(x,y,t,k)$ *define* Q *as*

Then Q solves equation (6) and $Q(x,y,0) = Q_1(x,y), Q_t(x,y,0) = Q_2(x,y)$.

The proof of Theorem (2.2.3).

Equations (64) and (65) can also be written in the form

$$\mu^{\pm}(x,y,k)$$

$$= I +$$

where

$$G^{\pm}($$

or

Substituting this equation into equation (71), it is straightforward to compute the large k behaviour of $\mu^{\pm}$,

$$\mu^{\pm} = I \pm \frac{i}{2\pi k} \int_{R^2}$$

for $k \to \infty$. Thus

The corresponding soliton solutions of equation (6) can be derived following the method of this section.

Section (2.3): Metric Spaces and Isometric Shifts:

Shift operators play an important role in many disciplines such as perturbation theory, engineering mathematics, scattering theory, stochastic processes, etc. (in [269]). Recently these operators have been applied in connection with wavelets and iteration attractors in complex analysis (in [270]). Crownover [271] was the first to extend the definition of shift operator from separable Hilbert spaces to arbitrary Banach spaces without using a basis. Namely, if $\mathscr{K}$ is a Banach space, then $T : \mathscr{K} \to \mathscr{K}$ is said to be an (isometric) shift operator if

(i) T is a linear isometry,

(ii) The codimension of $T(\mathscr{K})$ in $\mathscr{K}$ is 1,

(iii) $\bigcap_{1=n}^{\infty} T^n(\mathscr{K}) = \{0\}$.

If Condition (iii) is removed, then we have a codimension (i) linear isometry.

In [272], Gutek, Hart, Jamison and Rajagopalan extended many of the results obtained by Holub in [273] concerning isometric shift operators on the Banach space $C(X)$ (X compact Hausdorff). First, they classified codimension (i) linear isometries on $C(X)$ using the following result: let $T : C(X) \to C(X)$ be a codimension (i) linear isometry. Then there exists a closed subset X_0 of X such that either

(i) $X_0 = X \backslash \{p\}$

where p is an isolated point of X, or

(ii) $X_0 = X$

and such that there exists a continuous map h of X_0 onto X and a function $a \in C(X_0)$, $|a| \equiv 1$, such that

for all $x \in X_0$.

The proof of this result is based on a well known theorem of Holsztyński [274].

Those isometries that satisfy Condition (i). Those satisfying Condition (ii). These two classes are not disjoint. Farid and Varadarajan [275] devoted to clarify the above classification. Finally, [276] proposes an alternative (disjoint) classification based on the separation properties of the range of T. Thus, T is of type II if and only if $T\big(C(X)\big)$ separates all the points of X except two and is of type I which is not of type II if and only if $T\big(C(X)\big)$ separates all the points of X.

Codimension (i) linear isometries on arbitrary function algebras have also been studied and classified in [277] by using the results in [278]. Recently, Izuchi [279] has characterized Douglas algebras which admit codimension (i) linear isometries, thus solving the conjecture settled in [277].

Another question which has also been addressed in the context of isometric shifts is the characterization of those compact Hausdorff spaces X which admit such operators, that is, the existence of isometric shifts on $C(X)$. [272], proved that nonseparable spaces without isolated points do not admit isometric shifts and even that there is no nonseparable space which admits isometric shifts of type II. Haydon [280] proved the existence of isometric shifts of type II when X is either connected or the Cantor set. However it is still an open question whether there exists a nonseparable compact space X which admits an isometric shift. We show that no nonseparable metric (noncompact) space admits isometric shifts. We also provide an example of an isometric shift with several interesting features.

Let $\mathbb{K}$ denote the field of real or complex numbers. If X is a compact (respectively locally compact) Hausdorff space, then $C(X)$ (respectively $C_0(X)$) stands for the Banach space of all $\mathbb{K}$-valued continuous functions defined on X (respectively which vanish at infinity), equipped with its usual supremum norm. If M is a metric space, then we shall write $C^*(M)$ to denote the normed space of all bounded $\mathbb{K}$-valued continuous functions defined on M. As usual, βM stands for the Stone-Čech compactification of M. Given $f \in C(X)$, we shall consider that $c(f)$ is its cozero set. If U is a subset of X, then $\mathrm{cl}_X(U)$ and $\mathrm{int}_X(U)$ denote its closure and its interior in X, respectively.

Let M be a complete metric space and let $T : C^*(M) \to C^*(M)$ be an isometric shift. Then T induces an isometric shift (which we continue to denote by T) on $C(\beta M)$.

Theorem (2.3.1)[268]: *Let M be a complete metric space. If $C^*(M)$ admits an isometric shift T, then M is separable.*

Proof. Let us first assume T to be of type II which is not of type I. According to [272], the map $h : \beta M \to \beta M$ is a surjective continuous map such that there exists $x_0 \in \beta M$ in such a way that $h^{-1}(\{x\})$ consists of just one point for every $x \in \beta M \backslash \{x_0\}$ and $h^{-1}(\{x_0\})$ consists of two points of βM, say x_1, x_2. Also, since T is not of type I, then the points x_1 and x_2 are not isolated. Furthermore, in [272], it is proven that the set

is a countable dense subset in βM. We are going to see that this set is contained in M and in this way we give an explicit countable dense subset in M.

First we have, by [272], that if $\beta M/R$ is the quotient space for the equivalence relation defined as xRy whenever $h(x) = h(y)$, then the map $h^R : \beta M/R \to \beta M$ sending each class x^R into the image $h(x)$ of any $x \in x^R$ is a surjective homeomorphism. This implies in particular that the image of a G_δ-point in $\beta M/R$ is a G_δ-point in βM and vice versa. Let us recall that the only points in βM which are G_δ are those in M.

Let us check which of the G_δ-points in $\beta M/R$ are. Suppose that $x^R \in \beta M/R$ satisfies that there exists $x \in M$ with $x \in x^R$. Clearly, if x^R is the singleton $\{x\}$, then x^R is G_δ. Otherwise, as we remark above, x^R consists of two points, x_1, x_2, and is the only point in $\beta M/R$ which is not a singleton. Then it is apparent that x^R is G_δ if and only if both $x_1, x_2 \in M$.

Suppose next that $x^R = \{x_1, x_2\}$, and that $x_1 \in M$. Since T is not of type I, then the points x_1 and x_2 are clearly not isolated. Thus, there exists a sequence (y_n) in $M\backslash\{x_1, x_2\}$ converging to x_1. Also each y_n is a G_δ-point, and consequently so is $h^R(y_n) = h(y_n)$, that is, the sequence $\big(h(y_n)\big)$ is contained in M, and converges to x_0. But this implies in particular that $x_0 \in M$ [281]. Conversely, if we assume that $x_0 \in M$, then x_0 is a G_δ-point of βM and, consequently, so is $h^{-1}(\{x_0\}) = \{x_1, x_2\}$. This implies, as stated above, that both x_1 and x_2 belong to M. Summing up, we proved that $x_0 \in M$ if and only if $x_1 \in M$ or $x_2 \in M$, and that this fact yields $x_1, x_2 \in M$.

Let us now assume that $x_0 \notin M$, which is to say that $x_1, x_2 \notin M$. Then it is easy to check that the restriction of the map h to $M, h : M \to M$, is bijective and continuous, and its inverse $h^{-1} : M \to M$ is also continuous. Consequently, the map $T\colon C^*(M) \to C^*(M)$ sending each f into $a \cdot f \circ h, |a| \equiv 1$, is clearly a surjective linear isometry, that is, it is not a codimension (i) isometry, against our hypothesis. We deduce that x_0 must belong to M, and consequently x_1, x_2 belong to M. Hence, $h^{-1}(\{x_0\}) \subset M$.

A similar reasoning leads to the fact that $h^k(\{x_0\}) \subset M$ for every integer k. That is, D $\subset M$, as was to be proved.

Let us now assume that $T\colon C(\beta M) \to C(\beta M)$ is of type I. Thus, there exist an isolated point $p \in \beta M$ and a homeomorphism ([278]) h of $\beta M\backslash\{p\}$ onto βM and a function $a \in C(\beta M\backslash\{p\}), |a| \equiv 1$, such that

for all $x \in \beta M\backslash\{p\}$. Consider the set $A = \{p, h^{-1}(p), h^{-2}(p), \dots\}$. Then $Y \coloneqq \beta M\backslash \mathrm{cl}_{\beta M}(A)$ is a locally compact space and $h : Y \to Y$ is a surjective homeomorphism. Hence we have a surjective isometry $S : C_0(Y) \to C_0(Y)$ defined to be where $\hat{a}$ is the restriction to Y of a.

For any $f \in C_0(Y)$, we can define a function $\hat{f} \in C(\beta M)$ such that $\hat{f} = f$ on Y and 0 on $\beta M \backslash Y$. As a consequence, a linear continuous functional μ (indeed a regular complex measure) can be defined on $C_0(Y)$ to be $\mu(f) := (T\hat{f})(p)$.

Claim (2.3.2)[268]: *Assume that there is $f \in C_0(Y)$ such that $\mu(f) = 0$ and $(\mu \circ S^{-n})(f) = 0$ for all $n \in \mathbb{N}$. Then $f \equiv 0$.*

Let us suppose, contrary to what we claim, that there is $f \in C_0(Y)$, $f \neq 0$, such that $\mu(f) = 0$ and $(\mu \circ S^{-n})(f) = 0$ for all $n \in \mathbb{N}$. Let us check that $\hat{f} \in R(T^n)$ for all $n \in \mathbb{N}$.

Since $S : C_0(Y) \to C_0(Y)$ is a surjective isometry, there is $g \in C_0(Y)$ such that $S(g) = f$. If $x \in Y$, then

$(T\hat{g})(x) =$

That is, $T\hat{g} = \hat{f}$ on Y.

On the other hand, $(T\hat{g})(p) := \mu(g) = \mu(S^{-1}f) = (\mu \circ S^{-1})(f)$. By assumption, $(\mu \circ S^{-1})(f) = 0$. Hence, $(T\hat{g})(p) = 0 = \hat{f}(p)$.

Next, from the representation of the isometric shift T, we know that $(T\hat{g})\big(h^{-n}(p)\big) = a\big(h^{-n}(p)\big) \cdot \hat{g}\big(h^{-n+1}(p)\big)$, but $h^{-n+1}(p) \in \beta M \backslash Y$, which is to say that $\hat{g}\big(h^{-n+1}(p)\big) = 0$.

Finally, it is apparent, from the above two paragraphs and from density, that $T\hat{g} \equiv 0$ on $\beta M \backslash Y$. Hence, gathering the information above, we infer that $T\hat{g} = f$, i.e., $\hat{f} \in R(T)$.

Let us next check that $\hat{f} \in R(T^2)$. To see this, it suf®ces to prove that $\hat{g} \in R(T)$.
Since S is surjective, there is $g_1 \in C_0(Y)$ such that $S(g_1) = g$. Furthermore $(T\hat{g}_1)(p) := \mu(g_1) = \mu(S^{-2}f) = (\mu \circ S^{-2})(f) = 0 = \hat{g}(p)$. Hence, as above, we deduce that $T\hat{g}_1 = \hat{g}$. In a similar manner, we can obtain $g_2, g_3, \ldots, g_n, \ldots$ to show that $\hat{f} \in R(T^n)$ for all $n \in \mathbb{N}$. This fact contradicts the definition of isometric shift and the proof of Claim (2.3.2)[268] is complete.

It is well-known that every regular complex measure θ can be written as $\theta = (\theta_1 - \theta_2) + i(\theta_3 - \theta_4)$, where $\theta_i, i = 1, 2, 3, 4$, are regular positive measures. Hence each of the regular complex measures $\mu, \mu \circ S^{-1}, \mu \circ S^{-2}, \ldots, \mu \circ S^{-n}, \ldots$ can be divided into four regular positive measures. As a consequence, we get a new sequence of regular positive measures, which we shall denote by $\{\mu_n\}_{n\in\mathbb{N}}$. With no loss of generality, we can assume that all these measures are normalized.

Since the space of regular measures on a locally compact space is a Banach space, we can define a regular positive measure as follows:

Claim (2.3.3)[268]: *For every nonempty open subset U of $Y, \eta(U) > 0$.*

Let us suppose that there exists a nonempty open subset U of Y such that $\eta(U) = 0$. Hence we can find $f \in C_0(Y), f \neq 0$, such that $c(f) \subset U$. Consequently,

for all $n \in \mathbb{N}$. Finally, Claim (2.3.2) yields $f \equiv 0$, a contradiction.

Let us now define a (open) subset $N := M \backslash \mathrm{cl}_{\beta M}(\{p, h^{-1}(p), h^{-2}(p), \ldots\})$ of M.

Next we consider the family, say $\mathscr{F}_1$, of all subsets B of N which satisfy the following property: if $x, y \in B$, then $d(x, y) \geq 1$ or $d(x, y) = 0$, where d denotes the metric in N induced from M. Let us choose a chain $(A_\alpha)_\alpha$ of elements of $\mathscr{F}_1$ ordered by inclusion. Since

Zorn's lemma yields a maximal element, say M_1.

Claim (2.3.4)[268]: *M_1 is a countable set.*

Assume the contrary. Then there exists an uncountable family Δ of indices such that

Since N is an open subset of M, there is, for each $\alpha \in \Delta$, a constant $M_\alpha > 0$ such that the open ball $B(x_\alpha, M_\alpha) \subset N$.

Next, for each $\alpha \in \Delta$, take

and consider the open ball $B(x_\alpha, m_\alpha)$. It is clear, from the definition of $\mathscr{F}_1$, that if $\alpha, \beta \in \Delta, \alpha \neq \beta$, then

Now, for every $\alpha \in \Delta$, we can define the set

It is apparent that $V_\alpha \cap M = B(x_\alpha, m_\alpha)$ for each $\alpha \in \Delta$ and that $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$. Furthermore, each V_α is contained in Y since Y is open.

Summarizing, we have found an uncountable pairwise disjoint family of open subsets $\{V_\alpha : \alpha \in \Delta\}$ in Y.

We know, by Claim (2.3.3), that $\eta(V_\alpha) > 0$ for all $\alpha \in \Delta$. Hence, there is $n_0 \in \mathbb{N}$ such that the set

is not countable since neither is Δ. Let us choose a countable subset $\{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$ of indexes in γ Then,

This contradiction completes the proof of Claim (2.3.4).

As in the paragraph before Claim (2.3.4), we can define, for every $n \in \mathbb{N}$, the family $\mathscr{F}_n$ of all subsets B of N which satisfy the following property: if $x, y \in B$, then $d(x,y) \geq 1/n$ or $d(x,y) = 0$. In like manner, we obtain, for every $n \in \mathbb{N}$, a maximal element M_n of $\mathscr{F}_n$ which turns out to be countable.

Let us now see that the countable set

is dense in N. To this end, choose $x \in N\backslash \mathrm{D}$ and $\varepsilon > 0$. Then there exists $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \varepsilon$. Since $x \notin \mathrm{D}$, then $x \notin M_{m_0}$. This fact implies the existence of $y \in M_{m_0}$ such that $d(x, y) < 1/m_0$. That is, there is an element y of D in the open ball $B(x, \epsilon)$ and the density of D in N follows.

Finally, it is clear that the countable set

is dense in M and we are done.

In [280], Haydon presented a method to provide isometric shifts of type II. However, the scarcity of examples of isometric shifts of type I is remarkable. In this final section we provide an example of an isometric shift of type I, which is not of type II, with several additional features. Indeed, [272] raised the question whether, for an isometric shift of type I, the set $D \coloneqq \{p, h^{-1}(p), h^{-2}(p), \dots\}$ was always dense in X. The question was answered in the negative by Farid and Varadarajan [275] by providing an example of an isometric shift of type I such that $X\backslash \mathrm{cl}_X(D)$ was a (finite) nonempty subset. Our example shows somehow that D can be far from being dense in X in the sense that $X\backslash \mathrm{cl}_X(D)$ is uncountable. Our X also has, contrary to what Holub conjectured in [279], an infinite connected component (in [272]).

Example (2.3.5)[268]:. *Let ∂D denote the unit circle in $\mathbb{C}$, and let*

It is clear that X is a compact metric space. Let us show that X admits an isometric shift of type I by constructing it explicitely.

Let $T : C(X) \to C(X)$ be the following operator. Take any $f \in C(X)$ and define,

for each $e^{i\theta} \in \partial D$,

It is clear that, given any $e^{i\theta} \in \partial D$, the sequences $\left(e^{i(\theta+2n\sqrt{2})}\right)$ and $\left(e^{i(\theta+(2n-1)\sqrt{2})}\right)$ are dense in ∂D. Then we take in ∂D the point $1 = e^{i\theta}$.

Clearly the evaluation map δ_1 is continuous in $C(X)$ and its norm is equal to 1.

So, for $f \in C(X)$, we define

$$(Tf)($$

Next, for $n \geq 3$, we define

and

It is clear that $Tf \in C(X)$ and that T is an isometry. In fact T is a codimension 1 linear isometry of type I (being $p = 1/2$), which is not of type II since the range of T separates all the points of X (in [276]). Let us see that it is also a shift operator.

Suppose that $g \in C(X)$ satisfies $g \in \bigcap_{n=1}^{\infty} R(T^n)$. We have to prove that $g = 0$.

First we have that $\left(g(1/n)\right)$ must be a convergent sequence, and it converges to the value $g(0)$. Also, we have that $g(1/2) = -(T^{-1}g)(1)/2 - (T^{-1}g)\left(e^{-i\sqrt{2}}\right)/2$, by construction. In the same way $g(1/3) = -(T^{-1}g)(1/2) = (T^{-2}g)(1)/2 + (T^{2}g)\left(e^{-i\sqrt{2}}\right)/2 = (T^{-1}g)\left(e^{-i\sqrt{2}}\right)/2 + (T^{-1}g)\left(e^{-i2\sqrt{2}}\right)/2$ and, in general, for $n \geq 2, n \in \mathbb{N}$,

$$g(1/n) =$$

In particular, we have that the sequence

$$\Big($$

must converge to $g(0)$, because g is continuous.

On the other hand, by the density of points of the form $e^{i2n\sqrt{2}}, n \in \mathbb{N}$, we have that given any point $z_0 \in \partial D$, there exists a sequence (n_k) of even numbers such that $\left(e^{-in_k\sqrt{2}}\right)$ converges to z_0 as k tends to infinity. Also $\left(e^{-i(n_k-1)\sqrt{2}}\right)$ converges to $z_0 e^{-i\sqrt{2}}$. Since $T^{-1}g$ is continuous, this implies that $\left((T^{-1}g)\left(e^{-in_k\sqrt{2}}\right)\right)$ goes to $(T^{-1}g)(z_0)$, and that $(T^{-1}g)\left(e^{-i(n_k-1)\sqrt{2}}\right)$ goes to $(T^{-1}g)\left(z_0 e^{-i\sqrt{2}}\right)$. We deduce that

converges to

On the other hand, we know that the above sequence converges to $g(0)$. But a similar approach can be taken for a sequence of odd natural numbers (m_k) instead of (n_k). In this case we will obtain that

converges to

and on the other hand, it must converge to $-g(0)$. As a consequence, we deduce that $g(0). = -g(0). = 0$, and that, for every $z_0 \in \partial D$,

In particular, this implies that for every $z_0 \in \partial D, (T^{-1}g)\left(z_0 e^{-i\sqrt{2}}\right) = (T^{-1}g)\left(z_0 e^{i\sqrt{2}}\right)$. Consequently, the sequence is constant. By the density of points $e^{i2n\sqrt{2}}, n \in \mathbb{N}$, we conclude that $T^{-1}g$ is constant on ∂D. In particular, this implies that the sequence $(|g(1/n)|)$ is constant. Since it converges to $|g(0)| = 0$, we conclude that $g(1/n) = 0$ for every $n \in \mathbb{N}$. As a consequence it is easy to see that $T^{-1}g \equiv 0$ on ∂D. But this clearly implies that $g = 0$, as we wanted to prove (in [282, 283]).

Chapter 3

Cauchy Problem of the Ward Equation

We generalize the results of study the inverse scattering problem of the Ward equation with non-small data and solve the Cauchy problem of the Ward equation with a non-small purely continuous scattering data.

The Ward equation (or the modified $2+1$ chiral model)

$$\partial_t(J^{-1}$$

for $J : \mathbb{R}^{2,1} \to SU(n), \partial_w = \partial/\partial w$, is obtained from a dimension reduction and a gauge fixing of the self-dual Yang–Mills equation on $\mathbb{R}^{2,2}$ [32,33]. It is an integrable system which possesses the Lax pair [34,35,36]

with $\xi = \frac{t+y}{2}, \eta = \frac{t-y}{2}$. Note (2) implies that $J^{-1}\partial_\xi J = -\partial_x Q, J^{-1}\partial_x J = -\partial_\eta Q$. Then by a change of variables $(\eta, x, \xi) \to (x, y, t)$, (2) is equivalent to

(

(

see [37], and the Ward equation (1) turns into

The construction of solitons, the study of the scattering properties of solitons, and Darboux transformation of the Ward equation have been studied intensively by solving the degenerate Riemann–Hilbert problem and studying the limiting method [38,39,40,41,35,42,43]. In particular, Dai and Terng gave an explicit construction of all solitons of the Ward equation by establishing a theory of Backlund transformation [44, 182, 183, 184, 185, 186, 187, 188, 189].

For the investigation of the Cauchy problem of the Ward equation, Villarroel [50], Dai, Terng and Uhlenbeck [32] use Fourier analysis in the x, y-space to study the spectral theory of $\mathcal{L}_\lambda = \partial_y - \lambda\partial_x$ in (3), while Fokas and Ioannidou [37] invert $\mathcal{L}_\lambda$ by interpreting it as a 1-dimensional spectral operator with coefficients being the x-Fourier transform of functions [196, 190, 191, 192]. In both cases, small data conditions of Q are required to ensure the invertibility of $\mathcal{L}_\lambda$ and the solvability of the inverse problem. Under the small data condition, the eigenfunctions Ψ possesses continuous scattering data only and therefore the solutions for the Ward equation do not include the solitons in previous study.

Nontheless, the approach of Fokas and Ioannidou [37] shows that after taking the Fourier transform in the x-space, (3) looks similar to the spectral problem of the AKNS system

where J is a constant diagonal matrix with distinct eigenvalues. The solution of the forward and inverse scattering problem of the AKNS system is fairly complete, due to the work of Beals, Coifman, Deift, Tomei, Zhou [45,46,47]. In particular, the inverse scattering problem for the AKNS system and its associated nonlinear evolution equations is rigorously solved for generic $q \in L_1$ without small data condition [48].

The purpose is to remove the small data condition in solving the scattering and inverse scattering problem of (3) and the Cauchy problem of the Ward equation (5) with a purely continuous scattering data. We summarize principal results as follows.

Definition (3.1)[31]:

$$\mathbb{P}_{\infty,k_1,k_2} = \Big\{ qx(x$$

$$\sup_y \Big|$$

$$0 \le j$$

$$\mathbb{D}\mathbb{H}^k = \{f$$

To derive Theorem (3.21), we transform the existence problem of Ψ into a Riemann–Hilbert problem with a non-small continuous data by the translating invariant and the derivation properties of the spectral operator $\mathcal{L}_\lambda$, and an induction scheme. Hence the scheme of [45] can be adapted to solve the Riemann–Hilbert problem. That is, we first approximate the solution by a piecewise rational function. Then the correction is made by a solution of a Riemann–Hilbert problem with small data and a solution of a finite linear system. Since the eigenfunction obtained in each induction step consists the data of the Riemann–Hilbert problem in the next step, we need to obtain the H^2-estimate (8) of the eigenfunction. Besides, the boundary estimate (9) and the meromorphic property are derived in each step to assure the solvability of the linear system.

In general, the points in Z, i.e., poles of $\Psi(x, y, \lambda)$, will occur or accumulate on the real line, or the limit points will accumulate themselves. Assuming higher regularities on the potential Q and $Z = Z(\Psi) = \varphi$ (there are no poles of $\Psi(x, y, \lambda)$), we can extract the continuous scattering data:

Definition (3.2)[31]: Let $\mathfrak{S}_{c,k}, k \geq 7$, be the space consisting of continuous scattering data $v(x, y, \lambda), \lambda \in \mathbb{R}$, such that v satisfies the algebraic constraints:

and the analytic constraints: for $i + j \leq k - 4$,

$\mathcal{L}_\lambda v = 0,$ $\quad v$

$\partial_x^i \partial_y^j (v - 1)$ ar

$\partial_x^i \partial_y^j (v - 1) \to$

$\partial_\lambda v$ are in $L_2($

where $\mathcal{L}_\lambda = \partial_y - \lambda \partial_x$.

The characterization of the scattering data $v \in \mathfrak{S}_{c,k}$ is necessary. Since the Cauchy integral operator will play a key role in the inverse problem. The study of the asymptotic behavior of the scattering data v (hence the asymptotic behavior of the eigenfunctions Ψ) is important. Because the Cauchy operator is bounded in L_2 [49], in general, an L_2-estimate of Ψ and its derivatives will be good enough. However, a formal calculation will yield (112) if the inverse problem is solvable. Hence we provide the estimates (9)–(11).

The derivation of (9)–(11) basically relies on the L_2-boundedness of the Cauchy operator and the estimates obtained in the small-data problem. In particular, both of the 1-dimensional (Fokas and Ioannidou [37] or (16)) and the 2-dimensional formulation (Villarroel [50] or (28)) of the spectral problem are crucial in the derivation of the estimates with small data condition. That is, using (16), boundedness or integrability in x-variable of the eigenfunctions Ψ comes first from the differentiability and integrability of the potentials Q via the Fourier transform. Then, strong asymptote in x, y or λ-variable of the eigenfunctions Ψ can be obtained by (28) and previous estimates. We lose some regularities in deriving strong asymptote. See the proof of Theorem (3.12) for example.

For the inverse problem, the results are:

Definition (3.3)[31]:

$\mathbb{P}_1 =$

$\mathbb{X} =$

$\hat{\mathbb{X}} =$

where $\widehat{\ }$ is the Fourier transform with respect to the x-variable, $M_n(\mathbb{C})$ is the space of $n \times n$ matrices, and for $f \in M_n(\mathbb{C})$

Theorem (3.4)[31]: *Suppose* $Q \in \mathbb{P}_1$*. Then for all fixed* $\lambda \in \mathbb{C}^{\pm}$*, there is uniquely a solution* Ψ *of* (115) *and* (116) *such that* $\Psi - 1 \in \mathbb{X}$*. Moreover, for* $\lambda \in \mathbb{C}^{\pm}$*,*

Proof. Write $\Psi = 1 + W$. Then (20), (21) are transformed into

Taking the Fourier transform with respect to the x-variable (in distribution sense), we obtain

$\partial_y \widehat{W}$

Thus we are led to consider the following integral equations

$\widehat{W}(\xi, y, \lambda)$

where $*$ is the convolution operator with respect to the ξ -variable. Define

$\mathcal{K}_\lambda f(\xi, y, \lambda)$

Thus (13) turns into

$\widehat{W} =$

Where $\int_{-\infty}^{y} e^{i\lambda\xi(y-y')}\widehat{\partial_x Q}(\xi,y')dy', \int_{y}^{+\infty} e^{i\lambda\xi(y-y')}\widehat{\partial_x Q}(\xi,y')dy' \in \widehat{\mathbb{X}}$ by $Q \in \mathbb{P}_1$. Note that

$$|\mathcal{K}_\lambda f($$

Hence

So

$$\widehat{W} = \begin{cases} (\\ - \\ - \\ (\end{cases}$$

Hence (13) is solvable if $Q \in \mathbb{P}_1$. Furthermore, the eigenfunction of (115), (116) is given by

$$\Psi(x,y,\lambda) = \begin{cases} 1 \\ 1 \end{cases}$$

The uniqueness follows from (115), (116), (15), the definition of $\mathbb{X}$, and the contraction property of $\mathcal{K}_\lambda$.

The uniform boundedness of Ψ comes from Definition (3.3), (15) and $Q \in \mathbb{P}_1$. By (26), $\widehat{\partial_x Q} * \widehat{W}, \widehat{\partial_x Q} \in L_1(d\xi\, dy)$ and the Riemann–Lebesgue theorem, we obtain

$\Psi(\cdot,y,\lambda) \to 1$ as $|x| \to \infty$. On the other hand, (16), $\widehat{\partial_x Q} * \widehat{W}, \widehat{\partial_x Q} \in L_1(d\xi\, dy)$ and the Lebesgue convergence theorem imply that $\Psi(x,\cdot,\lambda) \to 1$ when $|y| \to \infty$.

Lemma (3.5)[31]: *Suppose Ψ satisfies* (115), (116). *Then for $\lambda \notin \mathbb{R}$,*

Proof. Let $e_1,\ldots,e_n$ denote the standard basis for $\mathbb{C}^n, \psi_k$ the kth column vector of the matrix Ψ. Let $\Lambda^k(\mathbb{C}^n)$ denote the space of alternating k forms on $\mathbb{C}^n$. Hence $\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n = (\det\Psi)(e_1 \wedge e_2 \wedge \ldots \wedge e_n)$. Taking derivatives of both sides, we derive

$\{$

So

by $\partial_x Q \in su(n)$. Moreover, for $\lambda \notin \mathbb{R}$, the equation turns into the debar equation

by the change of variables:

Therefore the Liouville's theorem and (12) imply that $\det\Psi \equiv 1$, for $\lambda \notin \mathbb{R}$.

Lemma (3.6)[31]: *Suppose that $Q \in \mathbb{P}_1$. Then the reality condition*

holds for the eigenfunction Ψ.

Proof. By Lemma (3.5), one derives

$(\partial_y - \lambda\partial_x)\Psi($

Besides, noting $\left|\widehat{f^n}\right|_{L_1(d\xi)} \leq \left|\hat{f}\right|^n_{L_1(d\xi)}$ and the boundary condition of Ψ, we obtain $\Psi^{-1} - 1 \in \mathbb{X}$. Hence the lemma follows from the uniqueness property in Theorem (3.4).

The results and arguments will be applied or adapted. Denote

$(f *_\lambda$

$(f *$

By the change of variables (17), we then have

$(\partial_y -$

with $\lambda = \lambda_R + i\lambda_I$. Now let S be the set of Schwartz functions. If $Q \in \mathbb{P}_1 \cap S$, then the eigenfunction Ψ obtained by Theorem (3.4) satisfies

where

$G_\lambda f($

The following lemma is due to R. Beals.

Lemma (3.7)[31]: *Suppose $\varphi \in S$. For $|\lambda| \neq 0$ and $|\lambda_I| < 1$,*

$|G_\lambda \varphi$

where C is a constant.

Proof. Let $\frac{1}{s} = \frac{\lambda_R}{\lambda_R^2+\lambda_I^2}$. So

Write

$$G_\lambda \varphi = \frac{-1}{2\pi i\lambda}\Bigg($$

$$+\int$$

$$+\int$$

$$= I_1 + I_2$$

In view of (20), it is easy to see that

$$|I_1| \le$$

$$\le$$

$$|I_2| \le$$

Finally,

$$\Bigg|\operatorname{sgn}(\lambda_I)$$

This yields

Combining (21), (22), and (23), we prove the lemma.

Lemma (3.8)[31]: *Suppose that $Q \in \mathbb{P}_1 \cap S$. Then there exists a constant C_N such that*

where C_N is a constant depending on Q.

Proof. Since

it suffices to prove $\xi^i \widehat{W} \in \mathbb{X}$ for $0 \leq k \leq N$. This can be proved by induction on k and using the same argument as in the proof of Theorem (3.4) if $\left|\xi^N \widehat{\partial_x Q}\right|_{L_1(d\xi\, dy)} < \infty$.

Definition (3.9)[31]: Define

$$\mathbb{P}_{1,k} = \{\partial_x$$

$$|\xi^i \hat{q}$$

for

Note that $\mathbb{P}_1 \in \mathbb{P}_{1,k}$. For simplicity we abuse the notation $\partial_x^i \partial_y^j Q, \partial_x^i \partial_y^j \Psi$ by $Q \underbrace{x \cdots x}_{i} \underbrace{y \cdots y}_{j}$, and $\Psi \underbrace{x \cdots x}_{i} \underbrace{y \cdots y}_{j}$ in the remaining part of this section.

Lemma (3.10)[31]: *Suppose that $Q \in \mathbb{P}_{1,k}, k \leq 5$. Then*

Moreover, as $|\lambda| \to \infty$,

where C_N, C *is a constant depending on* Q.

Proof. The uniform boundedness of $\partial_x^N \Psi, 0 \leq N \leq 4$, in Lemma (3.8) will be used in the proof. A direct computation yields

So

$$\Psi_x + \frac{Q_x}{\lambda}$$

by (24). Therefore, inverting the operator $\partial_y - \lambda\partial_x$ in (25) and applying Lemmas (3.7), (3.8), we have

$$|I_1| =$$

$$\leq$$

$$\leq$$

$$\leq$$

$\leq$

$\leq$

as $|\lambda| \to \infty$. Taking the x-derivatives of both the sides of (25), we derive

$|I_2| =$

$\leq$

$\leq$

$\leq$

Here we have used (115) and Lemma (3.8).

By the same scheme as above and the following equalities:

$\Psi_{xx} + \frac{Q}{}$

$\Psi_{xxx} + \frac{Q}{}$

one derives

$|\Psi_{xx}| \leq \frac{C}{|\lambda|}\Big($

$$|\Psi_{xxx}| \leq \frac{C}{|\lambda|}\Big($$

Hence the estimates for Ψ_{xx} and Ψ_{xxx} follow.

Lemma (3.11)[31]: *Suppose that $Q \in \mathbb{P}_{1,k}, k \leq 5$. Then*

as $|\lambda| \to \infty$. Here C is a constant depending on Q.

Proof. Using the formula

$$\Psi_y + \frac{Q_y}{\lambda}$$

and Lemma (3.10), one can derive

$$|II_1| = \frac{1}{|\lambda|}\Big|$$

$$\leq \frac{C}{|\lambda|^2}$$

$$+\Big|$$

$$\leq \frac{C}{|\lambda|^2}$$

(by

$$\leq \frac{C}{|\lambda|^2}$$

$$|II_2| = \frac{1}{|\lambda|}\Big|$$

$$\leq \frac{C}{|\lambda|^2}$$

$$+\Big|$$

$$\leq \frac{C}{|\lambda|}$$

(by

$$\leq \frac{C}{|\lambda|}$$

where the estimate $\left|\Psi_{yy}\right| = \left|\lambda^2\Psi_{xx} + \lambda(Q_x\Psi)_x + (Q_x\Psi)_y\right|$ has been used. Thus (26) is proved. On the other hand, we write

$$\Psi_{xy} + \frac{Q}{}$$

Similarly, one can verify

$$|III_1| \leq \frac{C}{|\lambda|^2}$$

$$|III_2| \leq \frac{C}{|\lambda|^2}$$

$$|III_3| \leq \frac{C}{|\lambda|^3}$$

$$|III_4| \leq \frac{C}{|\lambda|}$$

by Lemma (3.10), (26).

Theorem (3.12)[31]: *If $Q \in \mathbb{P}_{1,k}$, $k \leq 5$, then as $|\lambda| \to \infty$,*

$$\left|\partial_x\right.$$

where C is a constant depending on Q.

Proof. Applying (25), Lemmas (3.10) and (3.11), we obtain

$$\Big|\Psi$$

as $|\lambda| \to \infty$. Therefore, (28) is proved.

To proved (29), we used the results of Lemmas (3.10) and (3.11) to improve the estimates of I_1, I_2, II_1, and II_2 in the proof of Lemmas (3.10), (3.11). More precisely,

$$|I_1| =$$

$$\leq$$

$$\leq$$

$$|I_2| =$$

$$\leq$$

$$\leq$$

$$|II_1| =$$

$$\leq$$

$$\leq$$

$$|II_2| =$$

$$\leq$$

$$\leq$$

Here $|\Psi_{yy}| = |\lambda\Psi_{xy} + Q_{xy}\Psi + Q_x\Psi_y|$ and (37) have been used in the estimation of II_2.

By induction, we can generalize the results of Lemmas (3.8)–(3.10)-(3.11) and Theorem (3.12) to

Corollary (3.13)[31]: *Suppose that* $Q \in \mathbb{P}_{1,k}$. *Then for* $i + h \leq \max\{k, 5\} - 4$ *and as* $|\lambda| \to \infty$,

Remark (3.14)[31]: In general, the scattering transformation is a generalized Fourier transform. That is, it maps smooth potentials to decaying scattering data, and decaying potentials to smooth scattering data. As is known, the asymptotic expansion of eigenfunctions is related to the decayness of the scattering data. However, in the case of Ward equation, even for the Schwartz potentials, the second order asymptotic expansion of Theorem (3.12) seems difficult to be improved. To see it, the second-order coefficient of the asymptotic expansion Ψ, and an analogue of (25) need to be introduced. That is

and

$($

where ϕ is a Schwartz function. Then $f(x,y), c(y)$ are Schwartz. It can be checked that Ψ_2 does not possess integrability in the x-variable. This causes troubles in estimating $\left|\Psi - \left(1 - \frac{Q}{\lambda} + \frac{\Psi_2}{\lambda^2}\right)\right|$ while inverting (30) to derive a higher order asymptotic expansion of Ψ.

First we introduce

Definition (3.15)[31]: The Cauchy operator C and its limits $C_\pm$ are defined as follows:

It is well known that $C_\pm$ are bounded operators on $L_p(\mathbb{R})$ for $1 < p < \infty$, and $C_\pm f(\lambda) = \lim_{\tilde{\lambda}\to\lambda} Cf(\tilde{\lambda})\,, \lambda \in \mathbb{R}, \tilde{\lambda} \in \mathbb{C}^\pm$ [49].

Definition (3.16)[31]: Suppose $v(\lambda)$ is defined on $\mathbb{R}$. A function $\Psi(\lambda)$ is called a solution of the Riemann–Hilbert problem $(\lambda \in \mathbb{R}, v)$ if

where $\Psi_{\pm}(\lambda) = \lim_{\tilde{\lambda}\to\lambda} \Psi(\tilde{\lambda})$, $\lambda \in \mathbb{R}, \tilde{\lambda} \in \mathbb{C}^{\pm}$. Moreover, the function $v(\lambda)$ is called the data of the Riemann–Hilbert problem $(\lambda \in \mathbb{R}, v)$.

Suppose the data $v(\lambda), \lambda \in \mathbb{R}$ satisfies $\partial_{\lambda}^{i}(\Psi - 1) \in L_2(\mathbb{R}, d\lambda)$, for $i = 0, 1, 2$. It can be seen that Ψ is a solution of the Riemann–Hilbert problem $(\lambda \in \mathbb{R}, v)$ if and only if

Lemma (3.17)[31]: *Suppose the data* $v(\lambda), \lambda \in \mathbb{R}$*, satisfies*:

Then the Riemann–Hilbert problem $(\lambda \in \mathbb{R}, v)$ *has a unique solution* Ψ *such that* $\Psi - 1 \in L_\infty(d\lambda) \cap L_2(d\lambda)$*. Moreover, if* $H^k = \{f \mid \partial_{\lambda}^{j} f \in L_2(d\lambda), 0 \le j \le k\}$ *and*

then

for some constant C in [45].

Lemma (3.18)[31]: *Suppose the data* $v(\lambda), \lambda \in \mathbb{R}$*, is a scalar function satisfying*:

(i) $v(\lambda) \neq 0, \forall\lambda$;

(ii) $\int_{-\infty}^{\infty} d \arg v(\lambda) = 0$;

(iii) $v - 1, \partial_{\lambda} v \in L_2(d\lambda)$.

Then the Riemann–Hilbert problem $(\lambda \in \mathbb{R}, v)$ *has a unique solution* Ψ*. Moreover, if* *then*

where $H^k(d\lambda) = \{f \mid \partial_\lambda^i f \in L_2(d\lambda), 0 \le i \le k\}$, *and* C *is a constant depending on* $|v|_{L_\infty}, |1/v|_{L_\infty}$.

Lemma (3.19)[31]: *Suppose* $Q \in \mathbb{P}_{\infty,2,0} \cap \mathbb{P}_1$. *Then the eigenfunction obtained in Theorem* (3.1.9) *satisfies*:

(a) $\partial_x^i(\Psi(\cdot, y, \lambda) - 1), i = 0, 1, 2$, *are uniformly bounded in* $L_2(dx)$;

(b) $\Psi(\cdot, y, \lambda) - 1, \partial_x \Psi(\cdot, y, \lambda) \to 0$ *uniformly in* $L_2(dx)$ *as* $\lambda \to \infty$.

Proof. By noting that the Fourier transform is an isometry on the L_2 spaces, to prove (a), it suffices to show that $\xi^i \widehat{W}, i = 0, 1, 2$, are uniformly bounded in $L_2(d\xi)$. We will only treat the case of $\lambda \in \mathbb{C}^+$ and $\xi \ge 0$ for simplicity. Other cases can be handled similarly. Note that

$|\mathcal{K}$

Denote $\widehat{\mathbb{X}}_2 = \{f(\xi, y, \lambda) : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \to M_n(\mathbb{C}) : \sup_{y,\lambda} |f(\xi, y, \lambda)|_{L_2(d\xi)} < \infty\}$. So

$\mathcal{K}_\lambda$

By the assumption $Q \in \mathbb{P}_{\infty,2,0}$, we have

Therefore the solution $\widehat{W}$ of (24) is in $\widehat{\mathbb{X}} \cap \widehat{\mathbb{X}}_2$. Moreover, one can derive

$$\xi \widehat{W} = \int_{-\infty}^{y} e^{i\lambda\xi(y-}$$

from (13). As a result, we have $\xi \widehat{W} \in \widehat{\mathbb{X}} \cap \widehat{\mathbb{X}}_2$, if $Q \in \mathbb{P}_{\infty,2,0} \cap \mathbb{P}_1$. The same argument can prove $\xi^2 \widehat{W} \in \widehat{\mathbb{X}} \cap \widehat{\mathbb{X}}_2$, if $Q \in \mathbb{P}_{\infty,2,0} \cap \mathbb{P}_1$. Hence (a) is justified.

To prove (b), by the definition of $\widehat{\mathbb{X}}$ and result of (a), the function $\widehat{W}(\xi, y, \lambda)$ can be approximated uniformly by g where

and g is a linear combination of step functions in ξ with uniformly bounded coefficients in y, λ. Hence

$$\left(\int_{-\infty}^{\infty} e^{i\xi x} \right.$$

where $\chi_{|x|>N}$ is the characteristic function of the set $\{|x| > N\}$. The above two inequalities imply that $(\Psi(x, y, \lambda) - 1)\chi_{|x|>N} \to 0$ uniformly in $L_2(dx)$ as $N \to \infty$. We can prove the case of $\left(\partial_x \Psi(x, y, \lambda)\right)\chi_{|x|>N}$ by the similar method. Combining with Theorem (3.12) and the Lebesgue convergence theorem, one can prove (b).

Lemma (3.20)[31]: *Let $x + \lambda y = z, \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$, and $f_{\pm,z}(x, \lambda) = \lim_{|y| \to 0^{\pm}} f(x, y, \lambda)$. If $f(x, y, \lambda)$ is the solution of the Riemann–Hilbert problem $\left(x \in \mathbb{R}, F(x, \lambda)\right)$ and*

$$F$$

then

Proof. For $y = 0$, the lemma follows from the Sobolev's theorem, Lemma (3.17) and the assumption on $f_{\pm,z}, F$.

For simplicity, we omit the words "for $|\lambda| \gg 1$" in the following proof.

Decompose $f(x, y, \lambda)$ into

$$f(x, y, \lambda) = 1$$

$$= 1 -$$

Note that $f\_(F - 1)(\cdot, \lambda)$ is uniformly Hölder continuous by the assumption on $F, f_{\pm}$ and the imbedding theorem of Morrey [51]. Hence one has $I(x, y, \lambda) \to I_{\pm,z}(x, \lambda)$

uniformly as $y \to 0^{\pm}$ [52]. The uniform convergence of $II(x,y,\lambda) \to II_{\pm,z}(x,\lambda)$ as $y \to 0^{\pm}$ can be justified by the Hölder inequality. Moreover, one can check that this convergence is independent of x. As a result, $f(x,y,\lambda) \to f_{\pm,z}(x,\lambda)$ uniformly as $y \to 0^{\pm}$.

Since the lemma holds on the x-axis the uniform convergence provided above implies: for any $\epsilon > 0$, one can find $N_{\epsilon_1}, \delta_\epsilon$ such that $|f(x,y,\lambda) - 1| < \epsilon$ for $\forall|\lambda| \geq N_{\epsilon_1}, \forall|y| \leq \delta_\epsilon$. Besides, by the Hölder inequality, we can find N_{ϵ_2} such that $|f(x,y,\lambda) - 1| < \epsilon$ for $\forall|\lambda| > N_{\epsilon_2}, |y| \geq \delta_\epsilon$. Hence for any $\epsilon > 0$, we obtain

Theorem (3.21)[31]: *Let $Q \in \mathbb{P}_{\infty,2,0}$. Then there is a bounded set $Z \subset \mathbb{C}$ such that*

(a) *$Z \cap (\mathbb{C} \setminus \mathbb{R})$ is discrete in $\mathbb{C} \setminus \mathbb{R}$;*
(b) *For $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup Z)$, the problem* (3) *has a unique solution Ψ and $\Psi - 1 \in \mathbb{DH}^2$;*
(c) *For $(x,y) \in \mathbb{R} \times \mathbb{R}$, the eigenfunction $\Psi(x,y,\cdot)$ is meromorphic in $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with poles precisely at the points of $Z \cap (\mathbb{C} \setminus \mathbb{R})$;*
(d) *$\Psi(x,y,\lambda)$ satisfies*:

$$\lim_{|x|\to\infty} \Psi(\cdot$$

(e) *$\Psi(x,0,\lambda)$ satisfies*:

$$\partial_x^i(\Psi$$

$$\lambda \in \mathbb{C}$$

and le

with u

where $\epsilon \geq \epsilon_j > 0$ are any given constants, $D_\epsilon(\lambda_j)$ denotes the disk of radius ϵ centered at λ_j.

Here the function spaces $\mathbb{P}_{\infty,2,0}$, and $\mathbb{DH}^2$ are defined as follows.

Proof. We will prove Theorem (3.1.21) by induction on the norm of

Step 1 *(The case of $n = 0$).* If $\left|\widehat{\partial_x Q}(\xi, y)\right|_{L_1(d\xi\, dy)} < \left(\frac{3}{2}\right)^0$, the existence and (31) are proved by Theorem (3.4). The conditions (32), (33) and (34) are shown by Theorem (3.12) and Lemma (3.19). The holomorphic property comes from (16).

Step 2 *(Transforming to a Riemann–Hilbert problem).* Suppose Theorem (3.21) holds for $\left|\widehat{\partial_x Q}(\xi, y)\right|_{L_1(d\xi\, dy)} < \left(\frac{3}{2}\right)^n$. Note the eigenfunction corresponding to a y-translate of Q is the y-translate of the eigenfunction. Thus after translation we may have

$$\int_R \int_{-\infty}^{0} \left|\widehat{\partial_x}\right.$$

for a potential $\partial_x Q(x, y)$ with $\left|\widehat{\partial_x Q}(\xi, y)\right|_{L^1(d\xi\, dy)} < \left(\frac{3}{2}\right)^{n+1}$. Let $\chi^{\pm} = \chi^{\pm}(y) \leq 1$ be smooth real-valued functions such that

$$\chi^- = \begin{cases} 1, & \text{for} \\ 0, & \text{for} \end{cases}$$

$$\chi^+ = \begin{cases} 1, & \text{for} \\ 0, & \text{for} \end{cases}$$

So $Q^{\pm} \in \mathbb{P}_{\infty,4,0}$ and $\left|\widehat{\partial_x Q^{\pm}}(\xi, y)\right|_{L^1(d\xi\, dy)} < \left(\frac{3}{2}\right)^n$. By the induction hypothesis there exist bounded sets $Z^{\pm}$ such that $Z^{\pm} \cap (\mathbb{C}\backslash\mathbb{R})$ are discrete in $\mathbb{C}\backslash\mathbb{R}$ and for all $\lambda \in \mathbb{C}\backslash Z^{\pm}$, $Q^{\pm}$ have eigenfunctions $\Psi^{\pm}$ which fulfill the statements of Theorem (3.21). Here we remark that the meaning of the notation Ψ^{+} is different from that of Ψ_{+}. The former is a function defined in the half plane $y \geq 0$, the latter means $\lim_{\lambda_I \to 0^+} \Psi(x, y, \lambda)$.

Hence any eigenfunction Ψ for Q, whenever it exists, must be of the form

where for $y \in \mathbb{R}^{\pm}$,

$$\begin{cases} a^{\pm}(x \\ a^{\pm}(\\ a^{\pm}_{\pm,z} \end{cases}$$

Conversely, if we can find $a^{\pm}$ such that $a^{\pm}$ satisfies (36) for $y \in \mathbb{R}^{\pm}$ and $a^{+}(a^{-})^{-1}(x,0,\lambda) = (\Psi^{+})^{-1}\Psi^{-}(x,0,\lambda)$ (the invertibility of $a^{\pm}, \Psi^{\pm}$ is implied by Lemma (3.1.5)). Then we can define $\Psi(x,y,\lambda)$ by (35) and prove Theorem (3.21) in case of $\left|\widehat{\partial_x Q}(\xi,y)\right|_{L_1(d\xi\ dy)} < \left(\frac{3}{2}\right)^{n+1}$. Therefore, we conclude this step by

Lemma (3.22)[31] *(Transforming into a Riemann–Hilbert problem). To prove Theorem* (3.21)*, it is equivalent to solving the problem*: *find a bounded set* $Z, f(\tilde{x},\tilde{y},\lambda)$*, and* $\tilde{f}(\tilde{x},\tilde{y},\lambda)$ *such that* $Z^{\pm} \subset Z$ *and*

a) $Z \cap (\mathbb{C}\backslash\mathbb{R})$ *is discrete in* $\mathbb{C}\backslash\mathbb{R}$;
b) *For* $\lambda \in \mathbb{C}^{+} \setminus (\mathbb{R} \cup Z), f$ *is the unique solution of the Riemann–Hilbert problem* $\left(\tilde{x} \in \mathbb{R}, F(\tilde{x},\lambda)\right)$;
c) *For* $\lambda \in \mathbb{C}^{-} \setminus (\mathbb{R} \cup Z), \tilde{f}$ *is the unique solution of the Riemann–Hilbert problem* $\left(\tilde{x} \in \mathbb{R}, F^{-1}(\tilde{x},\lambda)\right)$;
d) $f, \tilde{f}$ *are meromorphic in* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *with poles at the points of* $Z \cap (\mathbb{C} \setminus \mathbb{R})$;
e) $f_{\pm,z}, \tilde{f}_{\pm,z}$ *satisfy* (33)*,* (34)*,*

where

and

Proof. Note that if $f, \tilde{f}$ exist for Lemma (3.22), then by Lemma (3.20) $f, \tilde{f}$ satisfy (31), (88) as well. Therefore, the lemma can be proved by the change of variables (37) (or (88)) and setting

with $\tilde{x}, \tilde{y} \in \mathbb{R}$ where in the above discussion.

Step 3 *(Factorization: a diagonal problem, a Riemann–Hilbert problem with small data and a rational function).* For any square matrix A we let $d_k^+(A)$ denote the upper $(k \times k)$-principal minors. Also let $\beta_{ik}, i \leq k$ be the minor of A formed of the first i rows, the first $i-1$ columns, and the kth column, and γ_{ki} be the minor of A formed of the first i columns, the first $i-1$ rows, and the kth row. The following factorization theorem can be found in [53].

Lemma (3.23)[31]: *Suppose the principal minors $d_k^+(A) \neq 0$, for $1 \leq k \leq n$. Then the matrix A can be represented as*

where

$C =$

From now on, we only deal with the case of $\lambda \in \overline{\mathbb{C}^+}$ for simplicity. The other case can be proved in an analogous argument.

Lemma (3.24)[31]: *For $\lambda \in \overline{\mathbb{C}^+} \setminus [Z^+ \cup Z^-]$, we have a factorization* *where*

δ *is dia*

δ, g_u, g_l

$\partial_x^i(\delta -$

$\lambda \in \overline{\mathbb{C}^+}$

and lett

uniforn

$\delta - 1, g$

Proof. By the same technique of the proof of Lemma (3.5), one proves $\det \Psi^{\pm} = 1$ for $\lambda \notin \mathbb{R}$. So $\det F \equiv 1$. As a result, if $d_i^+(F)(\tilde{x}_0, \lambda_0) = 0$ for some $1 \le i < n$, then F must have a pole at $(\tilde{x}_0, \lambda_0)$. By $\det \Psi^{\pm} = 1$ and (44), we obtain $\lambda_0 \in [Z^+ \cup Z^-]$. Therefore for $\lambda \in \overline{\mathbb{C}^+} \backslash [Z^+ \cup Z^-]$, we obtain a factorization by Lemma (3.23). The properties (39)–(42) are implied by

$F(\tilde{x}, \lambda$

$F(\tilde{x}, \lambda$

which come from the induction hypothesis.

Lemma (3.25)[31]: *(A diagonal Riemann–Hilbert problem). For* $\lambda \in \overline{\mathbb{C}^+} \backslash [Z^+ \cup Z^-]$*, the Riemann–Hilbert problem* $(\tilde{x} \in \mathbb{R}, \delta(\tilde{x}, \lambda))$ *has a solution* $\Delta(z, \lambda)$*. Moreover,*

(i) Δ *is* λ*-meromorphic in* $\mathbb{C}^+$ *with poles at* $[Z^+ \cup Z^-] \cap \mathbb{C}^+$;

(ii) $\Delta_{\pm,z}$ *satisfies* (89), (112).

Proof. For $\lambda \in \overline{\mathbb{C}^+} \backslash [Z^+ \cup Z^-]$, the matrix δ is a diagonal matrix with nonvanishing entries. So the winding number of $\delta(\tilde{x}, \lambda)$ is well defined by $N(\lambda) = -\frac{1}{2\pi i} \int \frac{d}{dt} \arg \delta(t, \lambda) dt$. By (40) and (41), $N(\lambda)$ is a continuous integer-valued function for $x \in \overline{\mathbb{C}^+} \backslash [Z^+ \cup Z^-]$. Thus $N(\lambda) \equiv 0$ by (42).

Combining with (41), and (42), Lemma (3.23) implies the existence of Δ which satisfies the Riemann–Hilbert problem $(\tilde{x} \in \mathbb{R}, \delta(\tilde{x}, \lambda))$, (33), and (34). The meromorphic property of $\Psi(x, y, \cdot)$ is proved by [54]

Lemma (3.26)[31]: *For $\lambda \in \overline{\mathbb{C}^+} \setminus \cup_{\lambda_j \in [Z^+ \cup Z^-]} D_\epsilon(\lambda_j)$, there exists*

such that

$\left|\Delta_{-,z}\left(1 + (R\right.\right.$

$\left|\Delta_{-,z}\left(1 + (R\right.\right.$

$(R_\epsilon)_u (R_\epsilon)_l)$

R_ϵ *can be m*

$R_\epsilon \in H^2(\mathbb{R},$

(independe

matrix wit

tends to 0

Proof. By the condition (42), there exists δ_ϵ such that $\left|g_u \chi_{|\lambda| > \delta_\epsilon}\right|_{H^2(d\tilde{x})} < \epsilon$. Moreover, by (41), for each $\lambda_0 \in \mathbb{C}^+ \setminus \cup_{\lambda_j \in [Z^+ \cup Z^-]} D_\epsilon(\lambda_j)$, $|\lambda_0| \le \delta_\epsilon$, there exists $N = N(\epsilon, \lambda_0)$ such that

$|g_u -$

where

One can check that $p_{\epsilon,u} \in H^2(\mathbb{R}, d\tilde{x})$ satisfies (47), (48). Hence choosing a bigger N or δ_ϵ, there exists a z-rational function, denoted as $\tilde{p}_{\epsilon,u}$,

$|g_u$

and $\tilde{p}_{\epsilon,u}$ satisfies (47), (48).

Consequently, using (43), (44), Lemmas (3.24), (3.25), and the off-diagonal form of g_u, one can find a z-rational function $R_u(z,\lambda)$ which is an approximation of g_u on $z \in \mathbb{R}$ and satisfies (45)–(49).

The case of g_l can be done in analogy.

With Lemma (3.26), one can find a solution to the small-data Riemann–Hilbert problem $\left(\tilde{x} \in \mathbb{R}, \Delta_{-,z}(1 + (R_\epsilon)_{-,z}) F \left(1 + (R_\epsilon)_{+,z}\right)^{-1} \Delta_{+,z}^{-1}\right)$. However, it is difficult to analyze the meromorphic property of the solution in a neighborhood of points in $[Z^+ \cup Z^-]$. Hence we need to improve Lemma (3.26). First of all, let us denote $\mathbb{C}_\epsilon^+ = \{\lambda \in \mathbb{C}^+ |\ \lambda_I \geq \epsilon\}$, and $[Z^+ \cup Z^-]_\epsilon^+ = \{\lambda \in [Z^+ \cup Z^-] \mid \lambda_I \geq \epsilon\}$ for simplicity.

Lemma (3.27)[31]: *For $\lambda \in \mathbb{C}^+$, there exist*

such that

$\Big|\Delta_{-,z}\left(1 + (\tilde{R}_\epsilon\right.$

$\Big|\Delta_{-,z}\left(1 + (\tilde{R}_\epsilon\right.$

$(\tilde{R}_\epsilon)_u \left((\tilde{R}_\epsilon)_l\right)$

$\tilde{R}_\epsilon$ can be me

$\tilde{R}_\epsilon \in H^2(\mathbb{R}, d$

(independen

matrix with

tends to 0 as

Proof. One can multiply g_u (g_l respectively) by product

so that $\mathcal{G}_{\epsilon,u} = \mathcal{P}_{\epsilon,u} g_u$ is holomorphic in $\lambda \in \mathbb{C}_\epsilon^+$. Then using (41) and the same argument as the proof of Lemma (3.26), one can approximate $\mathcal{G}_{\epsilon,u}$ by a piecewise z-rational function $R'_{\epsilon,u}$. Let $\tilde{R}_{\epsilon,u} = \mathcal{P}_{\epsilon,u}^{-1} R'_{\epsilon,u}$.

Next, choose k_j sufficiently large in $\mathcal{U}_\epsilon(\lambda) = \prod_{\lambda_j \in [Z^+ \cup Z^-]_\epsilon^+} \left(\frac{\lambda - \lambda_j}{\lambda + i}\right)^{k_j}$ to make $\mathcal{U}_\epsilon \delta, \mathcal{U}_\epsilon \Delta$ holomorphic in $\lambda \in \mathbb{C}_\epsilon^+$. Hence the lemma can be proved by an adaptation of the proof of Lemma (3.26). (Note the factors $\mathcal{U}_\epsilon, \mathcal{P}_{\epsilon,u}, \mathcal{P}_{\epsilon,l}$ are cancelled out.)

Lemma (3.28)[31]: *(A Riemann–Hilbert problem with small data). The Riemann–Hilbert problem* $\left(\tilde{x} \in \mathbb{R}, \Delta_{-,z}\left(1 + \left(\tilde{R}_{\epsilon,u}\right)_{-,z}\right) F \left(1 + \left(\tilde{R}_{\epsilon,u}\right)_{+,z}\right)^{-1} \Delta_{+,z}^{-1}\right)$ *admits a solution* $f_{\epsilon,s}(z,\lambda)$ *for* $\lambda \in \mathbb{C}_\epsilon^+ \backslash [Z^+ \cup Z^-]_\epsilon^+$. *Moreover,*

(i) $f_{\epsilon,s}$ *is meromorphic in* $\lambda \in \mathbb{C}_\epsilon^+$ *with poles at* $[Z^+ \cup Z^-]_\epsilon^+$;

(ii) $\left(f_{\epsilon,s}\right)_{\pm,z}$ *satisfies* (33)*,* (34)*.*

Proof. By the assumption (50), (51), one can apply Lemma (3.17) to find $f_{\epsilon,s}$ which satisfies (33) and the Riemann–Hilbert problem $\left(\tilde{x} \in \mathbb{R}, \Delta_{-,z}\left(1 + \left(\tilde{R}_{\epsilon,u}\right)_{-,z}\right) F \left(1 + \left(\tilde{R}_{\epsilon,u}\right)_{+,z}\right)^{-1} \Delta_{+,z}^{-1}\right)$.

Moreover, $f_{\epsilon,s}$ satisfies (34) by Lemma (3.17), (44), Lemma (3.25), and (54). Finally, $f_{\epsilon,s}$ is meromorphic in $\lambda \in \mathbb{C}_\epsilon^+$ with poles at $[Z^+ \cup Z^-]_\epsilon^+$ by (43), Lemma (3.25), and (53).

We conclude this step by a characterization of Lemma (3.22).

Lemma (3.29)[31]: *(Factorization of the Riemann–Hilbert problem). Suppose* $f(z,\lambda)$ *fulfills the statement in Lemma* (3.22)*. Then there exist a unique function* $r_\epsilon(z,\lambda)$ *and a set* Z_ϵ *, such that* *for some integer $N_\epsilon, Z_\epsilon \subset Z$, and for $\lambda \in \mathbb{C}_\epsilon^+ \backslash Z$,*

Conversely, suppose there are uniformly bounded sets Z_ϵ, and functions $\{r_\epsilon\}$ which are λ-meromorphic in $\mathbb{C}_\epsilon^+$ with poles at Z_ϵ, satisfy (61)–(63)*, and*

for $\lambda \in \mathbb{C}_\epsilon^+ \backslash (Z_\epsilon \cup [Z^+ \cup Z^-]_\epsilon^+)$. Define $f_\epsilon = r_\epsilon f_{\epsilon,s} \Delta\big(1 + \tilde{R}_\epsilon\big)$ for $\lambda \in \mathbb{C}_\epsilon^+$. Then we have

f_ϵ *is r*

$f_{\epsilon_1} =$

Hence $f = f_\epsilon$ is well defined, and f satisfies the statements in Lemma (3.22) *with $Z = \bigcup_\epsilon Z(f_\epsilon) \cup \{\lambda_j \in \mathbb{R} \mid \limsup_{\epsilon\to 0} |f_\epsilon(D_{2\epsilon}(\lambda_j) \cap C_\epsilon^+)| = \infty\}$. Here $Z(f_\epsilon)$ denotes the poles of f_ϵ.*

Proof. First of all, by Lemma (3.5), $\det f_{\epsilon,s}(z,\lambda) = \det\left(1 + \tilde{R}_\epsilon(z,\lambda)\right) = \det \Delta(z,\lambda) = 1$. So they are invertible at regular λ. Besides, $f(z,\lambda)$ and $f_{\epsilon,s}(z,\lambda)\Delta(z,\lambda)\left(1 + \tilde{R}_\epsilon(z,\lambda)\right)$ are *z*meromorphic, possess the same jump singularity across $z \in \mathbb{R}$, and tend to 1 at infinity. Therefore

is z-rational and (55)–(57) are satisfied by Lemmas (3.25)–(3.28) and the assumption on f. For the converse part, (60) comes immediately from the definition of f_ϵ and the meromorphic properties of $r_\epsilon, \Delta, \tilde{R}_\epsilon, f_{\epsilon,s}$ implied by assumption and Lemmas (3.25)–(3.28).

Besides, by assumption, $f_{\epsilon_1}, f_{\epsilon_2}$ satisfy the same Riemann–Hilbert problem in Lemma (3.22) for $\lambda \in \mathbb{C}_{\epsilon_1}^+ \backslash Z_{\epsilon_1}$. Thus (58) follows from the Liouville's theorem and the meromorphic properties. As a result, the well-defined property follows from (60) and (61).

The conditions (33), (34) can be proved by Lemmas (3.25)–(3.28), and (55)–(57), $f = f_\epsilon$ (i.e., (58)), and $Z = \bigcup Z_\epsilon \cup \{\lambda_j \in \mathbb{R}\colon \limsup_{\epsilon\to 0}|f_\epsilon(D_{2\epsilon}(\lambda_j)\cap C_\epsilon^+)| = \infty\}$.

Step 4 *(Solving the Riemann–Hilbert problem).* We complete the proof of Theorem (3.21) by finding a rational function r_ϵ in Lemma (3.29).

Lemma (3.30)[31]: *(Existence of the rational function r_ϵ). There exist a function r_ϵ and a uniformly bounded set Z_ϵ such that r_ϵ is λ-meromorphic in $\mathbb{C}_\epsilon^+$ with poles at the points of Z_ϵ and satisfies* (55)–(57)*,* (59) *for* $\lambda \in \mathbb{C}_\epsilon^+\backslash(Z_\epsilon \cup [Z^+ \cup Z^-]_\epsilon^+)$.

Proof. For simplicity, we drop ϵ in the notation $r_\epsilon, f_\epsilon, s, R_\epsilon$, … in the following proof.

(a) A linear system for $r(z,\lambda)$**.** Let $\{z_k = \tilde{x}_k + i\tilde{y}_k\}, k = 1, \dots, N$ be the simple poles of R in $\mathbb{C}^\pm$ by (55). Denote

at z_j. Thus

$f_s\Delta(1 +$

Now let

Hence at z_j,

where

We then try to find c_j, such that $r(z,\lambda)f_s(z,\lambda)\Delta(z,\lambda)\big(1+R(z,\lambda)\big)$ is holomorphic at z_j. This yields the linear system for c_j:

The properties (47), (49) imply that n_j are invertible and $\left(d_j n_j^{-1}\right)^2=0$. Therefore, it can be justified that (66) are consequences of (67). Inserting (65) into (67), we obtain a system of Nn^2 linear equations in Nn^2 unknowns (the entries of c_k) with coefficients in entries of $d_j(\lambda), n_j(\lambda), \alpha_j(\lambda), \beta_j(\lambda)$. Observing that as $|\lambda|\to\infty$,

by Lemmas (3.25)–(3.28) we have (67) are solvable as $|\lambda|\to\infty$. Precisely, c_k can be written in rational forms of $d_j, n_j, \alpha_j, \beta_j$ which are all holomorphic in $\lambda\in\mathbb{C}_\epsilon^+\setminus[Z^+\cup Z^-]$. Therefore, (67) are solvable for $\in\mathbb{C}^+\setminus Z_\epsilon$ where Z_ϵ are uniformly bounded sets. Consequently, (55), (56), (57), and (59) are fulfilled.

By the same argument as the proof of Theorem (3.21), we have

Corollary (3.31)[31]: *Suppose that $Q\in\mathbb{P}_{\infty,k,0}, k\geq 2$, and $\Psi(x,y,\lambda)$ is the associated eigenfunction. Then*

$\Psi-1$ *is u*

In particular, if λ_0 is a removable singularity of $\Psi(x,y,\lambda)$, then

$\Psi-1$ *i*

By a similar argument as that in Lemmas (3.5) and (3.6) and using the uniqueness property in Theorem (3.21), we can derive the same algebraic characterization of the eigenfunctions:

Lemma (3.32)[31]: *Suppose that $Q\in\mathbb{P}_{\infty,k,0}, k\geq 2$. Then the eigenfunction Ψ satisfies* *for* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

We define the continuous scattering data and study its algebraic and analytic characteristics in this section. We first show that the existence of continuous scattering data for $Q \in \mathbb{P}_1$ is automatic.

Lemma (3.33)[31]: *If* $Q \in \mathbb{P}_1$, *then the eigenfunction* $\Psi(x, y, \cdot)$ *obtained by Theorem* (3.4) *has limits* $\Psi_{\pm}$ *on* $\mathbb{R}$.

Proof. Suppose $\{\lambda_k\} \subset \mathbb{C}^+$, and λ_k converge to a point of $\mathbb{R}$. Write $\widehat{W}_k$ instead of $\widehat{W}(\xi, y, \lambda_k)$ and

f_k

Then (13) and (14) imply

$\widehat{W}_k - \widehat{W}$

Now write

Note that (15) and $\sup_y |\widehat{W}_h|_{L_1(d\xi)} \leq \left(1 - |\widehat{\partial_x Q}(\xi, y)|_{L_1(d\xi\, dy)}\right)^{-1}$ imply

and

$|I_1''|_{L_1(d\xi)}$

On the other hand,

$|(K_{\lambda_k} - K_{\lambda_h})\widehat{W}$

by the Lebesgue convergence theorem and $Q \in \mathbb{P}_1$. So

$|I_1'|_{L_1}$

Hence $|I_1|_{L_1(d\xi)} \to 0$ as $k, h \to \infty$ by (71)–(73). A similar argument will induce $|I_2|_{L_1(d\xi)} = \left|(1 - K_{\lambda_k})^{-1}(f_k - f_h)\right|_{L_1(d\xi)} \to 0$ as well. Therefore, we have $|\widehat{W}_k - \widehat{W}_h|_{L_1(d\xi)} \to 0$ as $k, h \to \infty$ by (70). Taking the Fourier transform, we prove the lemma when $\lambda \in \mathbb{C}^+$.

The case of $\lambda \in \mathbb{C}^-$ can be proved by analogy.

Lemma (3.34)[31]: *Suppose that $Q \in \mathbb{P}_1$ and*

Then Ψ_+ and Ψ_- are continuously differentiable with respect to x and y.

Lemma (3.35)[31]: *For $Q \in \mathbb{P}_1$ and Q satisfies* (74), *the eigenfunction $\Psi(x, y, \cdot)$ is holomorphic in $\mathbb{C}^\pm$ and has limits $\Psi_\pm$ on* R.*Moreover, there exists a continuously differentiable function $v(x + \lambda y, \lambda)$ such that*

$\Psi_+(x,$

where $\mathcal{L}_\lambda = \partial_y - \lambda\partial_x$.

Proof. The holomorphicity has been proved in Theorem (3.21). By assumption, Lemmas (3.33) and (3.17), $\Psi_{\pm}$ is invertible. Hence Lemma (3.34) implies

$$(\partial_y - \lambda$$

We denote $Z = Z(\Psi) = \varphi$ if there are no poles of $\Psi(x, y, \lambda)$.

Lemma (3.36)[31]: *For $Q \in \mathbb{P}_{\infty,k,0}, k \geq 2$, if $Z = \varphi$, then there exists a continuously differentiable function $v(x + \lambda y, \lambda)$ such that*

$$\Psi_{+}(x,$$

Since we are going to solve the inverse problem by the Riemann–Hilbert problem $(\lambda \in \mathbb{R}, v)$. By the scheme, we need to investigate $L_2(\mathbb{R}, d\lambda)$ condition on v and $\partial_\lambda v$. Hence the λ-asymptote of v and $\partial_\lambda v$ will be investigated in the remaining part of this section.

We extend Theorem (3.21), and Corollary (3.12) as follows.

Lemma (3.37)[31]: *If $Q \in \mathbb{P}_{\infty,k,0}, k \geq 5$ and $Z = \varphi$, then for $i + j \leq k - 4$,*

as $|\lambda| \to \infty$. Where C is a constant depending on Q.

We improve the boundary properties (31), (32) of Theorem (3.21) as follows.

Lemma (3.38)[31]: *If $Q \in \mathbb{P}_{\infty,k,0}, k \geq 5$, and $Z = \varphi$, then for $i + j \leq k - 4$,*

$$\partial_x^i \partial$$

Proof. By the results of Lemma (3.37), it is sufficient to prove this lemma for $|\lambda| < c$ where c is any fixed constant. However, for $|\lambda| < c, i + j \leq k - 4$,

$$\partial_x^i \partial$$

follow from (115), Corollary (3.31), and the Sobolev's theorem. For $y \neq 0$, one can follow the argument of Lemma (3.20) to show the uniform convergence of $\partial_x^i \partial_y^j \Psi \to \partial_x^i \partial_y^j \Psi_{\pm,z}$. Then the lemma is proved by the uniform convergence and applying Hölder inequality to

Ψ

Lemma (3.39)[31]: *For* $Q \in \mathbb{P}_{\infty,k,1} \cap \mathbb{P}_1, k \geq 7$, *we have*

and C *depends continuously on* x, y.

Proof. By formula (16), we have

Write

Note that $\widehat{W} \in \widehat{\mathbb{X}}$ with $\widehat{\mathbb{X}}$ defined by Definition (3.3). Therefore Theorem (3.12) implies

Now we define

$\frac{B_1(\xi,}{\lambda}$

$\frac{B_2(\xi,}{\lambda}$

$\frac{B_3(\xi,}{\lambda}$

$$\frac{B_4(\xi, y}{\lambda}$$

By (14), (15), (76), (77), and Theorem (3.12), we obtain

Differentiating both the sides of (14), we obtain

$$(1 - \mathcal{K}_\lambda)\partial_\lambda \widehat{W} = i$$

$$-$$

$$= i$$

$$-$$

$$+$$

for $\lambda \in \mathbb{C}^+, \xi \geq 0$ (other cases can be done similarly). Define

Using the definition of $\mathbb{P}_{\infty,k,1}$, and following the way to prove (78), one can show that if $Q \in \mathbb{P}_{\infty,k,1}$ and $k \geq 6$. Combining (75), (76), (79), (80), and (15), we prove $\left|\partial_\lambda \Psi_\pm\right| < \frac{C}{|\lambda|}$ as $|\lambda| \to \infty$ and C depends continuously on x, y.

Since $\partial_x \Psi(x, y, \lambda) = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\xi\widehat{W}(\xi, y, \lambda)d\xi$, modifying the above argument and letting $k \geq 7$ in $\mathbb{P}_{\infty,k,1}$, one can obtain the estimate for $\left|\partial_\lambda\partial_x\Psi_\pm\right|$ as well.

Lemma (3.40)[31]: *If $Q \in \mathbb{P}_{\infty,k,1}, k \geq 7$, and $Z = \varphi$, then*

and C depends continuously on x, y.

Proof. Since the property we wish to justify is a local property, without loss of generality, we need only to show

where C depends continuously on x, y, and $\chi(x, y)$ is any fixed smooth function with compact support. Now by the induction scheme as the proof of Theorem (3.21), we have

and

By induction and applying Lemmas (3.38), and (3.39), it reduces to showing

where

By (82), one can derive the inhomogeneous Riemann–Hilbert problem

$(\chi\partial_\lambda$

with

Hence [55]

$$\chi\partial_{\lambda}$$

with $x + \lambda y = z$, and

Therefore by Lemma (3.39) and (83),

$$|\chi$$

as $|\lambda| \to \infty$. Furthermore, differentiating both the sides of (83) and using Corollary (3.31), Lemma (3.39), we obtain

$$|\partial_x(\chi\partial_\lambda a)|$$

Hence the lemma follows from (84), (85), and Sobolev's theorem.

Theorem (3.41)[31]: *For $Q \in \mathbb{P}_{\infty,k,1}, k \geq 7$, if $Z = \varphi$, then there exists uniquely a function $v(x,y,\lambda) \in \mathfrak{S}_{c,k}$ which satisfies*

Where the space $\mathfrak{S}_{c,k}$ is defined by

Proof. The condition (8) follows from Lemma (3.36). The identity (6) comes from (68) and Lemma (3.36). Besides, (69) and Lemma (3.36) imply that for $\lambda \in \mathbb{R}$

$$v(x + \lambda y, \lambda) =$$

Therefore (7) follows.

Next note that Lemma (3.37) implies that

$$\partial_x^i \partial_y^j (\Psi_\pm$$

So (9) follows. Combining Lemma (3.38), (86), one obtains $\partial_x^i \partial_y^j (v - 1) \to 0$ uniformly in L_∞. So condition (10) follows from (9), and the Lebesgue convergence theorem. Finally, condition (11) is derived by applying Lemma (3.40).

Definition (3.42)[31]: For $Q \in P_{\infty,k,1}, k \leq 7$, if the eigenfunction $\Psi(x,y,\cdot)$ has limits $\Psi_\pm$ on $\mathbb{R}$, then we define the continuous scattering data of Q to be $v \in Sc, k$ obtained by Theorem (3.41). Moreover, the continuous scattering transformation Sc on Q is defined by $Sc(Q) = v$.

Theorem (3.43)[31]: *Given $v(x,y,\lambda) \in \mathfrak{S}_{c,k}, k \geq 7$, there exists a unique solution $\Psi(x,y,\cdot)$ for the Riemann–Hilbert problem $\big(\lambda \in \mathbb{R}, v(x,y,\lambda)\big)$ such that*

$$\Psi$$

Moreover, for each fixed $\lambda \notin \mathbb{R}$, and $i + j \leq k - 4$,

Theorem (3.43) is proved by a Riemann–Hilbert problem with a non-small purely continuous scattering data. Without uniform boundedness of $\partial_\lambda v$, we need to handle separately the Riemann– Hilbert problem for $|\lambda| > M \gg 1$ and $|\lambda| \leq M$. For $|\lambda| > M \gg$

1, the Riemann–Hilbert problem is a small-data problem and hence can be solved. For $|\lambda| \leq M$, the Riemann–Hilbert problem is again factorized into a diagonal problem, a Riemann–Hilbert problem with small data, and a finite linear system. Note we obtain the globally solvability by applying the Fredholm property and the reality condition (7). Moreover, good estimates for Ψ can be derived only for $\lambda \notin \mathbb{R}$. However, it is enough to imply satisfactory analytical properties of the potentials.

Proof. First of all, (9), (10) and Lemma (3.12) imply that there exists a constant $M > 0$ such that, as $|x|$ or $|y| > M - 1$, the Riemann–Hilbert problem $\big(\lambda \in \mathbb{R}, v(x, y, \lambda)\big)$ can be solved and

for a constant C. Hence (87) holds as $|x|$ or $|y| > M - 1$. Applying Hölder inequality, (9), (10), and (90), we then derive:

Hence, to prove Theorem (3.43), it is sufficient to solve the Riemann–Hilbert problem $\big(\lambda \in \mathbb{R}, v(x, y, \lambda)\big)$ and establish (87), (88) for $\max(|x|, |y|) < M$. The scheme in particular Lemmas (3.24)–(3.30), can be adapted to the solving of this problem. More precisely,

Lemma (3.44)[31]: *For $\lambda, x, y \in \mathbb{R}$, we have a factorization*

and for $i + j \leq k - 4$,

χ *is dia*

$\partial_x^i \partial_y^j (\chi$

and the

$\chi - 1, h$

Lemma (3.45)[31]: *(A diagonal Riemann–Hilbert problem). For* $\max(|x|,|y|) < M$*, there exists a uniquely solution* $\Xi(x,y,\lambda)$ *to the Riemann–Hilbert problem* $(\lambda \in \mathbb{R}, \chi)$ *such that*

and for each fixed $\lambda \notin \mathbb{R}$*,*

Proof. Applying (92), and (93), one obtains that

$\partial_\lambda^i(\chi$

Hence the winding number $N(x,y) = -\frac{1}{2\pi i}\int \frac{d \arg \chi}{d\zeta}(x,y,\zeta)d\zeta$ is integer-valued. Moreover, the condition (93) implies that $N(x,y) \equiv 0$.

Thus for $\max(|x|,|y|) < M$, the existence of Ξ, and (94) can be implied by (92), the Sobolev's theorem, and Lemma (3.8). By (92), (94), and the formulas

$\Xi(x,y,\lambda)$

$\partial_x\Xi(x,y,\lambda)$

$\partial_y\Xi(x,y,\lambda)$

$\partial_x^2\Xi(x,y,\lambda)$

we derive (95). Finally, we obtain (96) by Hölder inequality.

Lemma (3.46)[31]: *For* $\max(|x|,|y|) < M$*, there exists a function* $H(x,y,\lambda)$ *satisfying*

and

(a) $H(x,y,\lambda) \in L_\infty \cap H^1(\mathbb{R}, d\lambda)$*, and* $\partial_x^i \partial_y^j H(x,y,\lambda) \in L_\infty \cap L_2(\mathbb{R}, d\lambda)$;

(b) $|\Xi_-(1+H_-)v(1+H_+)^{-1}\Xi_+^{-1}(x,y,\lambda) - 1|_{H^1(\mathbb{R},d\lambda)} < \infty$;

(c) $|\Xi_-(1+H_-)v(1+H_+)^{-1}\Xi_+^{-1}(x,y,\lambda) - 1|_{L_\infty}\|C_\pm\| < 1$;

(d) $H_u(H_l)$ *is strictly upper* (*lower*) *triangular*;

(e) H *is rational in* $\lambda \in \mathbb{C}^\pm$*, with only simple poles and each corresponding residue is off diagonal, with only one non-zero entry* κ *and* $\partial_x^i \partial_y^j \kappa \in L_\infty(dx\,dy)$.

Lemma (3.47)[31]: *(A Riemann–Hilbert problem with small data). For* $\max(|x|,|y|) < M$*, the Riemann–Hilbert problem* $(\lambda \in \mathbb{R}, \Xi_-(1+H_-)v(1+H_+)^{-1}\Xi_+^{-1})$ *admits a solution* $\varphi_s(x,y,\lambda)$*. Moreover,*

φ_s -

and for each fixed $\lambda \notin \mathbb{R}$*,*

Lemma (3.48)[31]: *(Factorization of the Riemann–Hilbert problem). Suppose* $\Psi(x,y,\lambda)$ *satisfies Theorem* (3.43)*. Then for* $\max(|x|,|y|) < M$*, there exists a unique function* u*,*

and

Conversely, if for $\max(|x|,|y|) < M, \exists u(x,y,\lambda)$ *satisfying* (97)*,* (98) *and*

Define $\Psi = u\varphi_s \Xi(1+H)$ *for* $\max(|x|,|y|) < M$*. Hence* Ψ *satisfies Theorem* (3.43).

We then use Lemma (3.48) to prove Theorem (3.43).

(a) A linear system for $u(x, y, \lambda)$. Let

Then at λ_j

with

Since λ_j is a simple pole of H and $\varphi_s \Xi$ is regular at λ_j, we can write

We then try to find a_k, such that $u(x, y, \lambda)\varphi_s(x, y, \lambda)\Xi(x, y, \lambda)\big(1 + H(x, y, \lambda)\big)$ is holomorphic at λ_j. This yields the linear system for a_k:

(b) Solving the linear system (106)–(107). Note by Lemma (3.46), one can conclude

Therefore, it can be justified that (106) is a consequence of (107). Note the off-diagonal form of h_l (h_u) in Lemma (3.44) is crucial here. Inserting (103) into (107), we obtain a system of pn^2 linear equations in pn^2 unknowns (the entries of a_k with coefficients in entries of $h_j(x, y), n_j(x, y), \alpha_j(x, y), \beta_j(x, y)$). Therefore, we conclude the existence problem of Ψ is Fredholm.

(c) Solving the Riemann–Hilbert problem. Using the Fredholm alternative, we need only to show that for any fixed x, y the homogeneous problem (with limit 0 rather than 1 as

$\lambda \to \infty$) has only the trivial solution. Suppose $f(x,y,\lambda)$ solves this homogeneous problem. Consider $g(x,y,\lambda) = f(x,y,\lambda)f(x,y,\lambda)^*$. Since $f(x,y,\cdot) \in L_2(\mathbb{R}, d\lambda)$, we have $g(\lambda) \in L_1(\mathbb{R}, d\lambda)$ and is holomorphic in $\mathbb{C}^{\pm}$. Thus the Cauchy's theorem implies

$$0 = \int_{\mathbb{R}}$$

Because of (7) we conclude $f_- \equiv 0$ on $\mathbb{R}$, so also $f_+ \equiv 0$ and $f \equiv 0$. Hence we prove the solvability of the Riemann–Hilbert problem in Theorem (3.43).

Lemma (3.49)[31]: *For the solution* Ψ *of the Riemann–Hilbert problem obtained in Theorem* (3.43)*, we have*

Proof. By (6), $\det \Psi(x,y,\cdot)$ has no jump across the real line. So applying the Liouville's theorem, (109) follows from the holomorphic property in $\mathbb{C}^{\pm}$ and $\Psi \to 1$ as $|\lambda| \to \infty$. Hence $\Psi(x,y,\lambda)$ is invertible for all $\lambda \in \mathbb{C}$, limits $\left(\Psi\left(x,y,z,\bar{\lambda}\right)^*\right)_{\pm}^{-1}$ for $\lambda \in \mathbb{R}$ exist, and $\left(\Psi\left(x,y,z,\bar{\lambda}\right)^*\right)^{-1}$ fulfills the boundary condition as $|\lambda| \to \infty$. Secondly, by (7) and $\Psi_+ = \Psi_- v$, we obtain

So

Therefore $\left(\Psi\left(x,y,\bar{\lambda}\right)^*\right)^{-1}$ satisfies the same Riemann–Hilbert problem in Theorem (3.43). Consequently $\Psi(x,y,\lambda) = \left(\Psi\left(x,y,\bar{\lambda}\right)^*\right)^{-1}$ by the uniqueness property of Theorem (3.43) (the Liouville's theorem) and (110) is established.

Theorem (3.50)[31]: *Given* $v(x,y,\lambda) \in \mathfrak{S}_{c,k}, k \geq 7$, *the eigenfunction* Ψ *obtained by Theorem* (3.43) *satisfies* (3) *with*

and $\Psi(x,\cdot,\lambda) \to 1$ *as* $y \to -\infty$, *where* $\partial_x Q(x,y) \in su(n)$, *and for* $i+j \leq k-4, i > 0, \partial_x^i \partial_y^j Q, \partial_y Q, Q \in L_\infty, \partial_x^i \partial_y^j Q, \partial_y Q, Q \to 0$ *as* x *or* $y \to \infty$.

Applying Theorems (3.41)–(3.50), we extend the results of [32,37,50] as follows.

Proof. By (89), the boundary condition (116) is satisfied. Besides, the Cauchy integral formula, and Theorem (3.43) imply

For fixed $x, y \in \mathbb{R}$, applying $\mathcal{L}_\lambda = \partial_y - \lambda \partial_x$ to (113) and using (87), (9), we obtain

With $Q(x,y)$ given by (112). Hence comparing (113) and (114) and using the uniqueness result of Theorem (3.43), we obtain (115).

Besides, (9), (87), (112), and Hölder inequality show that $Q, \partial_x Q$, and $\partial_y Q \in L_\infty$.

Furthermore, by (115), (88), (109), and the λ-independence of Q, we derive $\partial_x^i \partial_y^j Q \in L_\infty$ and $\partial_x^i \partial_y^j Q, \partial_y Q, Q \to 0$ as x or $y \to \infty$, for $i+j \leq k-4, i > 0$.

Finally, by (110) and (115), we have

(

Thus $\partial_x Q(x,y) \in su(n)$.

Definition (3.51)[31]: For a function $v \in \mathfrak{S}_c$, we define the inverse scattering transformation S_c^{-1} on v by $S_c^{-1}(v) = Q$, where Q is obtained by Theorems (3.43) and (3.50).

Theorem (3.52)[31]: *If* $Q_0 \in \mathbb{P}_{\infty,k,1}, k \geq 7$, *and there are no poles of the eigenfunction* Ψ_0 *of* Q_0, *then the Cauchy problem of the Ward equation* (5) *with initial condition* $Q(x,y,0) = Q_0(x,y)$ *admits a global solution satisfying*: *for* $i + j + \leq k - 4, i^2 + j^2 > 0$,

In this Section, we review an existence theorem of Fokas and Ioannidou [37] by

an analytical treatment. Under the small-data constraint, we analyze the asymptotic behavior of the eigenfunctions. We solve the direct problem by justifying Theorems (3.21) and (3.41). The inverse problem is complete in this Section by proving Theorems (3.43) and (3.50). Finally, Theorem (3.51) is proved.

Given a potential $\partial_x Q(x,y) : \mathbb{R} \times \mathbb{R} \to su(n)$, and a constant $\lambda \in \mathbb{C}$, we consider

the boundary value problem

∂

To investigate the problem, we denote throughout as follows.

Proof. We can apply Theorem (3.21) to find the eigenfunction $\Psi(x,y,0,\lambda)$. By assumption, and Theorem (3.41), $S_c(Q_0) \in \mathfrak{S}_{c,k}$.

Now let us define $\mathrm{v}(t)$ by

For each $t \in \mathbb{R}$, rewriting $x+\lambda y+\lambda^2 t = x+\lambda(y+\lambda t) = x+\lambda^2\left(t+\frac{1}{\lambda}y\right)$ and modifying the approach in proving lemmas in this one can justify that $\mathrm{v}(t) \in \mathfrak{S}_{c,k}$ (see Definition (3.2)). So v satisfies the algebraic constraints:

(i) $\det(v) \equiv 1$;

(ii) $v = v^* > 0$,

and the analytic constraints: for $i+j+h \le k-4$,

(a) $\mathcal{L}_\lambda v = 0, \mathcal{M}_\lambda v = 0$;

(b) $\partial_x^i \partial_y^j \partial_t^h (v-1)$ are uniformly bounded in $L_\infty \cap L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda)$;

(c) $\partial_x^i \partial_y^j \partial_t^h (v-1) \to 0$ uniformly in $L_\infty \cap L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda)$ as $|x|$ or $|y|$ or $t \to \infty$;

(d) $\partial_\lambda v \in L_2(\mathbb{R}, d\lambda)$ and the norms depend continuously on x, y,

where $\mathcal{L}_\lambda = \partial_y - \lambda\partial_x$, and $\mathcal{M}_\lambda = \partial_t - \lambda\partial_y$.

Now we apply Theorems (3.43) and (3.50) to show the existence of $\Psi(x,y,t,\lambda)$ and $Q(x,y,t)$ satisfying (115) and (116). More precisely,

$\Psi(x,y,t,$

$\Psi_\pm -$

and for each fixed $\lambda \notin \mathbb{R}, i+j+h \le k-4$,

In addition,

$Q(x,$

and for $i+j+h \le k-4, i^2+j^2 > 0$,

To prove (4), we note it is equivalent to prove

Applying $\mathcal{M}_\lambda$ to both sides of (118) and using similar approach as that in the proof of Theorem (3.50), we obtain

Comparing (118) and (123) and using the uniqueness result of Theorem (3.43), we obtain (122). The smooth and decay properties of Q can be derived by an argument similar to the proof of Theorem (3.50) and conditions (120)–(121).

Since we have obtain the differentiability of $\Psi(x, y, t, \lambda)$ and $Q(x, y, t)$. The compatibility condition of (115) and (122) yields (5).

We conclude this report by a brief remark on examples of $Q_0 \in \mathbb{P}_{\infty,k,1}, k \geq 7$, and the corresponding eigenfunction Ψ_0 has no poles. The first class of examples is $\mathbb{P}_1 \cap S$ (S is the set of Schwartz functions and $\mathbb{P}_1$ is defined by Definition (3.3)). To construct an example with large norm, we let $v(x, y, \lambda) = v(x + \lambda y, \lambda)$ satisfy

and for $\forall i, j, h \geq 0$,

∂

∂

We can solve the inverse problem and obtain $\Psi_0 \in S$ by the argument in proving Theorem (3.21). Note here we need to use the reality condition $v = v^* > 0$ to show the global solvability. Moreover, by using the fomula $Q_0(x, y) = \frac{1}{2\pi i} \int_{\mathbb{R}} \psi_{0,-}(v - 1) d\xi$, one obtains that Q_0 is Schwartz and possesses purely continuous scattering data.

Chapter 4

Indecomposable System of Four Subspaces and Representations of Quivers on Infinite-Dimensional Hilbert Spaces

We extend the Coxetor functors and defect using Fredholm index. The relative position of subspaces has close connections with strongly irreducible operators and transitive lattices. There exists a relation between the defect and the Jones index in a type II_1 factor setting. We also show a complement of Gabriel's theorem. Let Γ be a finite, connected quiver. If its underlying undirected graph contains one of extended Dynkin diagrams $\tilde{A}_n (n \geq 0), \widetilde{D}_n (n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$, then there exists an indecomposable representation of Γ on separable infinite-dimensional Hilbert spaces. We show a generalization of the system of four subspaces with certain considerations. We give a projection in a Hilbert space with respect to an invertible series of projections on a subspaces of the Hilbert space. We show a reflection function of abounded self-adjoint operator on an orthogonal complement projection.

Section (4.1): Exotic Indecomposable System of Four Subspaces and Coxetor Functions with a Factor Verison:

One of the main problem to attack is a classification of indecomposable systems $S = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a Hilbert space H. In the case when H is finite-dimensional, Gelfand and Ponomarev completely classified indecomposable systems and gave a complete list of them in [100]. The important numerical invariants are $dim\, H$ and the defect defined by

Theorem (4.1.1)[180]: *The set of possible values of the defect $\rho(S)$ for indecomposable systems S of four subspaces in a finite-dimensional space is exactly the set* $\{-2, -1, 0, 1, 2\}$.

The defect characterizes an essential feature of the system in the case of finite dimension as follows: If $\rho(S) = 0$, then S is isomorphic to a bounded operator system up to permutation of subspaces, that is, there exist a permutation σ on $\{1, 2, 3, 4\}$ and a pair of linear operators $A : E \to F$ and $B : F \to E$ such that $H = E \oplus F, E_{\sigma(1)} = E \oplus$

$0, E_{\sigma(2)} = 0 \oplus F, E_{\sigma(3)} = \{(x, Ax) \in H;\ x \in E\}$ and $E_{\sigma(4)} = \{(By, y) \in H;\ y \in F\}$. If $\rho(S) = \pm 1, S$ is represented up to permutation by $H = E \oplus F, E_1 = E \oplus 0, E_2 = 0 \oplus F, E_3$ and E_4 are subspaces of H that are not reduced to the graphs of the operators as in the case that $\rho(S) = 0$. A system with $\rho(S) = \pm 2$ cannot be described in the above forms.

Following [97, 98, 99, 100, 101, 102], we recall the canonical forms of indecomposable systems $S = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a finite-dimensional space H up to permutation in the following:

(A) The case when $dim\, H = 2k$ for some positive integer k.

There exist no indecomposable systems S with $\rho(S) = \pm 2$. Let H be a space with a basis $\{e_1, \dots, e_k, f_1, \dots, f_k\}$.

(a) $S_3(2k, -1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = -1$

(b) $S_3(2k, 1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 1$

(c) $S_{1,3}(2k, 0) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 0$

(d) $S(2k, 0;\ \lambda) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 0$

Every other system $S_i(2k,\rho), S_{i,j}(2k,0)$ can be obtained from the systems $S_3(2k,\rho), S_{i,3}(2k,0)$ by a suitable permutation of the subspaces. Let $\sigma_{i,j}$ be the transposition (i,j). We put $S_i(2k,\rho) = \sigma_{3,i}S_3(2k,\rho)$ for $\rho = -1, 1$. We also define

$S_{i,j}(2k,0) = \sigma_{1,i}\sigma_{3,j}S_{1,3}(2k,0)$ for $i,j \in \{1,2,3,4\}$.

(B) The case $dim\ H = 2k+1$ is odd for some integer $k \geq 0$. Let H be a space

with a basis $\{e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k\}$.

(e) $S_1(2k+1,-1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = -1$

(f) $S_2(2k+1,1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 1$

(g) $S_{1,3}(2k+1,0) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 0$

(h) $S(2k+1,-2) = (H; E_1, E_2, E_3, E_4)$ with $\rho(S) = -2$

(i) $S(2k+1,2) = (H; E_1, E_2, E_3, E_4))$ with $\rho(S) = 2$

We put $S_i(2k+1,-1) = \sigma_{1,i} S_1(2k+1,-1), S_i(2k+1,+1) = \sigma_{2,i} S_2(2k+1,+1), S_{i,j}(2k+1,0) = \sigma_{1,i}\sigma_{3,j} S_{1,3}(2k+1,0)$ for $i,j \in \{1,2,3,4\}$.

Theorem (4.1.2)[180]: *If a system S of four subspaces in a finite-dimensional space H is indecomposable, then S is isomorphic to one of the following systems*: $S_{i,j}(m,0)$, $(i < j, i,j \in \{1,2,3,4\}, m = 1,2,\ldots)$; $S(2k,0;\lambda)$, $(\lambda \in \mathbb{C}, \lambda \neq 0, \lambda \neq 1, k = 1,2,\ldots)$, $S_i(m,-1)$, $S_i(m,1)$, $(i \in \{1,2,3,4\}, m = 1,2,\ldots)$; $S(2k+1,-2)$, $S(2k+1,+2)$, $(k = 0,1,\ldots)$.

We shall construct uncountably many, exotic, indecomposable systems of four subspaces, that is, indecomposable systems which are not isomorphic to any closed operator system under any permutation of subspaces [103, 104, 105, 106, 107, 108, 109, 111].

Exotic examples: Let $L = \ell^2(\mathbb{N})$ with a standard basis $\{e_1, e_2, \ldots\}$. Put $K = L \oplus L$ and $H = K \oplus K = L \oplus L \oplus L \oplus L$. Consider a unilateral shift $S : L \to L$ by $Se_n = e_{n+1}$ for $n = 1, 2, \ldots$. For a fixed parameter $\gamma \in \mathbb{C}$ with $|\gamma| \geq 1$, we consider an operator

Let $E_1 = K \oplus 0, E_2 = 0 \oplus K$, $E_3 = \{(x, T_\gamma x) \in K \oplus K;\ x \in K\} + \mathbb{C}(0,0,0,e_1) = \text{graph}\, T_\gamma + \mathbb{C}(0,0,0,e_1)$, and $E_4 = \{(x,x) \in K \oplus K;\ x \in K\}$. Consider a system $S_\gamma = (H; E_1, E_2, E_3, E_4)$. We shall show that S_γ is indecomposable. If $|\gamma| > 1$, then S_γ is not isomorphic to any closed operator systems under any permutation. We could regard the system S_γ is a one-dimensional "deformation" of an operator system. First we start with an easy fact.

Lemma (4.1.3)[180]: *Assume that a bounded operator $A \in B(\ell^2(\mathbb{N}))$ is represented as an upper triangular matrix $A = (a_{ij})_{ij}$ by a standard basis $\{e_1, e_2, \ldots\}$. If the diagonal is constant λ, i.e., $a_{ii} = \lambda$ for $i = 1, \ldots$, and A is an idempotent, then $A = 0$ or $A = I$.*

Proof. Put $N = A - \lambda I$. Then N is an upper triangular matrix with zero diagonal. Comparing the diagonals for

we have $\lambda^2 = \lambda$. Hence $\lambda = 0$ or 1. If $\lambda = 0$, then $N^2 = N$. Since N is an idempotent and an upper triangular matrix with zero diagonal, $N = 0$, that is, $A = 0$. If $\lambda = 1$, then $(I - A)$ is an idempotent and an upper triangular matrix with zero diagonal, $I - A = 0$, that is, $A = I$.

Theorem (4.1.4)[180]: *If $|\gamma| \geq 1$, then the above system $S_\gamma = (H; E_1, E_2, E_3, E_4)$ is indecomposable.*

Proof. We shall show that $\{V \in End(S_\gamma);\ V^2 = V\} = \{0, I\}$. Let $V \in End(S_\gamma)$ satisfy $V^2 = V$. Since $V(E_i) \subset E_i$ for $i = 1, 2, 4$, we have

We write

for some $A = (a_{ij})_{ij}, B = (b_{ij})_{ij}, C = (c_{ij})_{ij}, D = (d_{ij})_{ij} \in B(L)$. We shall investigate the condition that $V(E_3) \subset E_3$. Since $E_3 = \text{graph } T_\gamma + \mathbb{C}(0, 0, 0, e_1)$, E_3 is spanned by

$$\left\{\begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\right.$$

We may write

$$E_3 = \left\{\left(\right.\right.$$

Since $(e_1, 0, 0, 0) \in E_3$, we have

$$\begin{pmatrix} A \\ C \\ 0 \\ 0 \end{pmatrix}$$

Then, for any $m = 1, 2, \ldots,$ we have $c_{m1} = \mu_m = 0$. Moreover $0 = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1}$. Hence $\lambda_{m+1} = 0$ because $\gamma \neq 0$. Therefore $a_{m+1,1} = \lambda_{m+1} = 0$. Thus the first column of C is zero and the first column of A is zero except a_{11}. We shall show that $C = 0$ and A is an upper triangular Toeplitz matrix by the induction of nth column. The case when $n = 1$ is already shown. Assume that the assertion holds for nth column. Since $(e_{n+1}, 0, \gamma e_n, 0) \in E_3$, we have

$$\begin{pmatrix} A & B \\ C & D \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then $c_{m,n+1} = \mu_m = \gamma c_{m+1,n} = 0$. And $\gamma a_{m,n} = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1}$. Since $\gamma \neq 0, a_{m,n} = \lambda_{m+1} = a_{m+1,n+1}$. Thus we have shown that $C = 0$ and A is an upper triangular Toeplitz matrix. Since V is an idempotent, so is

Hence A is also an idempotent. By Lemma (4.1.3), we have two cases $A = 0$ or $A = I$.

(i) The case $A = 0$: we shall show that $B = D = 0$. This immediately implies $U = 0$, so that $V = 0$.

(ii) The case $A = I$: Since $I - V \in End(S_\gamma)$ is also an idempotent and it can be reduced to the case (i), we have $V = I$.

Hence we may assume that $A = 0$. Since U is an idempotent, D is also an idempotent. Since $(0, 0, 0, e_1) \in E_3$, we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, for any $m = 1, 2, \ldots,$ we have $\lambda_m = \mu_m = 0$. Hence $b_{m_1} = \gamma\lambda_{m+1} + \mu_m = 0$ and $d_{m+1,1} = \mu_m = 0$. Thus the first column of B is zero and the first column of D is zero except d_{11}. We shall show that D is an upper triangular Toeplitz matrix by the induction of nth column. The case when $n = 1$ is already shown. Assume that the assertion holds for nth column. Since $(0, e_n, e_n, e_{n+1}) \in E_3$,

$$\begin{pmatrix} 0 & B \\ 0 & D \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We have $d_{m+1,n+1} = \mu_m = d_{mn}$. Hence D is an upper triangular Toeplitz matrix. Since D is also an idempotent, $D = O$ or $D = I$ by Lemma (4.1.3).

If $D = 0$, then $U = U^2 = 0$. Thus $B = 0$, and the assertion is verified. We shall show that the case when $D = I$ will not occur. On the contrary, suppose that $D = I$. We have

$$V \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \end{pmatrix} =$$

Then, for any $m = 1, 2, \ldots,$ we have $\mu_m = \lambda_m = 0$. Hence $b_{m1} = \gamma\lambda_{m+1} + \mu_m = 0$. Thus the first column of B is zero. We shall show that B should be the following form by the induction of nth column:

that is, $b_{ij} = \gamma^{k-1}$ if $j > i$ and $j - i = 2k - 1$, and $b_{ij} = 0$ if otherwise.

The case when $n = 1$ is already shown. Assume that the assertion holds for nth column. Since

$$\begin{pmatrix} 0 & B \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for any $m = 1, 2, \ldots$, we have $\mu_m = \delta_{m,n}$. And

that is,

$$((r$$

By the induction we have shown that B is the above form. But then

$$\|B$$

because $|\gamma| \geq 1$. This contradicts the fact that B is bounded. Therefore $D \neq I$. This finishes the proof.

Theorem (4.1.5)[180]: *If* $|\beta| \geq 1, |\gamma| \geq 1$ *and* $|\beta| \neq |\gamma|$*, then the above systems* $S_\beta = \left(H; E_1, E_2, E_3^\beta, E_4\right)$ *and* $S_\gamma = \left(H; E_1, E_2, E_3^\gamma, E_4\right)$ *are not isomorphic.*

Proof. On the contrary, suppose that there were an isomorphism $V : S_\beta \to S_\gamma$. We shall show a contradiction. We may and do assume that $|\beta| > |\gamma|$. Since $V(E_i) = E_i$ for $i = 1, 2, 4$, we have

$$V$$

We write

for some $A = \left(a_{ij}\right)_{ij}, B = \left(b_{ij}\right)_{ij}, C = \left(c_{ij}\right)_{ij}, D = \left(d_{ij}\right)_{ij} \in B(K)$. We shall investigate the condition that $V\left(E_3^\beta\right) = E_3^\gamma$. Since $E_3^\beta = \text{graph}\, T_\beta + \mathbb{C}(0, 0, 0, e_1), E_3^\beta$ is spanned by

$$\left\{\begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\right.$$

We also write

$$E_3^{\gamma} = \Bigg(\quad ($$

Since $(e_1, 0, 0, 0) \in E_3^{\beta}$, we have

$$0 \neq \begin{pmatrix} A \\ C \\ 0 \\ 0 \end{pmatrix}$$

Then, for any $m = 1, 2, \ldots,$ we have $c_{m1} = \mu_m = 0$. Moreover $0 = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1}$. Hence $\lambda_{m+1} = 0$ because $\gamma \neq 0$. Therefore $a_{m+1,1} = \lambda_{m+1} = 0$. Thus the first column of C is zero and the first column of A is zero except a_{11}. Since $Ae_1 \neq 0, a_{11} \neq 0$.

We shall show that $C = 0$ and A is an upper triangular matrix satisfying

and $a_{ij} = 0$ if $i > j$, by the induction of nth column. The case when $n = 1$ is already shown. Assume that the assertion holds for nth column. Since $(e_{n+1}, 0, \beta e_n, 0) \in E_3^{\beta}$, we have

$$\begin{pmatrix} A & B \\ C & D \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we have $c_{m,n+1} = \mu_m = \beta c_{m+1,n} = 0$. Moreover

Since $\gamma \neq 0, a_{m+1,n+1} = \frac{\beta}{\gamma} a_{m,n}$. This completes the induction. Then we have because $a_{11} \neq 0$ and $\left|\frac{\beta}{\gamma}\right| > 1$. But this contradicts the fact that the operator A is bounded. Therefore S_β and S_γ are not isomorphic.

Next we shall show that if $|\gamma| > 1$, then S_γ is not isomorphic to any closed operator system. We introduce a necessary criterion for the purpose.

Definition (4.1.6)[180]: Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of fours subspaces. The *intersection diagram* for a system S is an undirected graph $\Gamma S = (\Gamma_S^0, \Gamma_S^1)$ with the set of vertices Γ_S^0 and the set of edges Γ_S^1 defined by $\Gamma_S^0 = \{1, 2, 3, 4\}$ and for $i \neq j \in \{1, 2, 3, 4\}$

Lemma (4.1.7)[180]: *Let $S = S_{T,S} = (H; E_1, E_2, E_3, E_4)$ be a closed operator system. Then the intersection diagram ΓS for the system S contains*

that is, $E_4 \cap E_1 = 0$, $E_1 \cap E_2 = 0$ and $E_2 \cap E_3 = 0$. In particular, then the intersection diagram ΓS is a connected graph.

Proposition (4.1.8)[180]: *If $|\gamma| > 1$, then the system S_γ is not isomorphic to any closed operator system under any permutation of subspaces*.

Combining the preceding two propositions, we have the existence of uncountably many, exotic, indecomposable systems of four subspaces.

Theorem (4.1.9)[180]: *There exists uncountably many, indecomposable systems of four subspaces which are not isomorphic to any closed operator system under any permutation of subspaces*.

Proof. A family $\{S_\gamma; \gamma > 1, \gamma \in \mathbb{R}\}$ of indecomposable systems above is a desired one.

Gelfand and Ponomarev introduced an integer valued invariant $\rho(S)$, called *defect*, for a system $S = (H; E_1, E_2, E_3, E_4)$ of four subspaces by

They showed that if a system of four subspaces is indecomposable, then the possible value of the defect $\rho(S)$ is one of five values $\{-2, -1, 0, 1, 2\}$. We shall extend their notion of defect for a certain class of systems relating with Fredholm index.

Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. We first introduce elementary numerical invariants

m_i

Similarly put

$n_{ij} =$

If S is indecomposable and $dimH \geq 2$, then $m_{ijk} = 0$ and $n_{ijk} = 0$.

If H is finite-dimensional, then

$dimE_i +$

$= di$

$= di$

In order to make the numerical invariant unchanged under any permutation of subspaces, counting ${}_4C_2 = 6$ pairs of subspaces

(

we have the following expression of the defect:

ρ

Definition (4.1.10)[180]: Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. For any distinct $i, j = 1, 2, 3, 4$, define an adding operator

Then

and

We say $S = (H; E_1, E_2, E_3, E_4)$ is a *Fredholm* system if A_{ij} is a Fredholm operator for any $i, j = 1, 2, 3, 4$ with $i \neq j$. Then $\operatorname{Im} A_{ij} = E_i + E_j$ is closed and

$$\text{Index } A_{ij} = d$$

Kato called the number $dim(E_i \cap E_j) - dim\left(\left(E_i + E_j\right)^{\perp}\right)$ the index of the pair E_i, E_j in [114].

Definition (4.1.11)[180]: We say $S = (H; E_1, E_2, E_3, E_4)$ is a *quasi-Fredholm* system if $E_i \cap E_j$ and $\left(E_i + E_j\right)^{\perp}$ are finite-dimensional for any $i \neq j$. In the case we define the *defect* $\rho(S)$ of S by

$$\rho$$

which coincides with the Gelfand–Ponomarev original defect if H is finite-dimensional. Moreover, if S is a Fredholm system, then it is a quasi-Fredholm system and

Proposition (4.1.12)[180]: *Let $S_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. Then S_T is a Fredholm system if and only if*

T and $T-I$ are Fredholm operators. If the condition is satisfied, then the defect is given by

Similarly S_T is a quasi-Fredholm system if and only if $Ker\,T, Ker\,T^*, Ker\,(T-I)$ and $Ker\,(T-I)^*$ are finite-dimensional. If the condition is satisfied, then the defect is given by

$$\rho(S_T) = \frac{1}{3}(di$$

Proof. It is clear that $E_i \cap E_j = 0$ and $E_i + E_j = H$ for $(i,j) = (1,2),(1,4),(2,4),(2,3)$. Since $Ker A_{13} = E_1 \cap E_3 = Ker\,T \oplus 0$ and $(Im\,A_{13})^\perp = (E_1 + E_3)^\perp = (K \oplus \mathrm{Im}\,T)^\perp$, they are finite-dimensional if and only if $Ker\,T$ and $(Im\,T)^\perp = Ker\,T^*$ are finite-dimensional. And $\mathrm{Im}\,A_{13}$ is closed if and only if $\mathrm{Im}\,T$ is closed. We transform E_3 and E_4 by an invertible operator $R = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \in B(H) = B(K \oplus K)$, then $R(E_3) = \{(x,(T-I)x) \in K \oplus K;\ x \in K\}$ and $R(E_4) = K \oplus 0$. Hence $R(E_3 \cap E_4) = Ker\,(T-I) \oplus 0$ and $R(E_3 + E_4) = K \oplus \mathrm{Im}(T-I)$. Then

$$dim(($$

Thus $E_3 \cap E_4$ and $(E_3 + E_4)^\perp$ are finite-dimensional if and only if $Ker\,(T-I)$ and $\left(Im(T-I)\right)^\perp = Ker\,(T-I)^*$ are finite-dimensional. And $\mathrm{Im}\,A_{13} = E_3 + E_4$ is closed if and only if $\mathrm{Im}(T-I)$ is closed. It follows the desired conclusion.

We shall show that the defect could have a fractional value.

Example (4.1.13)[180]: Let S be a unilateral shift on $K = \ell^2(\mathbb{N})$. Then the operator system S_S is indecomposable. It is not a Fredholm system but a quasi-Fredholm system and $\rho(S_S) = -\frac{1}{3}$. The operator system $S_{S+\frac{1}{2}I}$ is a Fredholm system and $\rho\left(S_{S+\frac{1}{2}I}\right) = -\frac{2}{3}$. Moreover $(S_{T+\alpha I})_{\alpha\in\mathbb{C}}$ is uncountable family of indecomposable, quasi-Fredholm systems. Fredholm systems among them and their defect are given by

$$\rho(S_{S+\alpha I}) = \begin{cases} - \\ - \end{cases}$$

Corollary (4.1.14)[180]: *Let $S_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. If S_T is a Fredholm system, then S_{T^*} is a Fredholm system and $\rho(S_{T^*}) = -\rho(S_T)$. Similarly if S_T is a quasi-Fredholm system, then S_{T^*} is a quasi-Fredholm system and $\rho(S_{T^*}) = -\rho(S_T)$.*

Proposition (4.1.15)[180]: *Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. If S is a Fredholm system, then the orthogonal complement $S^\perp = (H; E_1^\perp, E_2^\perp, E_3^\perp, E_4^\perp)$ is a Fredholm system and $\rho(S^\perp) = -\rho(S)$. Similarly if S is a quasi-Fredholm system, then $S^\perp$ is a quasi-Fredholm system and $\rho(S^\perp) = -\rho(S)$.*

Example (4.1.16)[180]: For $\gamma \in \mathbb{C}$ with $|\gamma| \geq 1$, let $S_\gamma = (H; E_1, E_2, E_3, E_4)$ be an exotic system of four subspaces in Theorem (4.1.4). Then S_γ is a quasi-Fredholm system and

$$\rho(S_\gamma) =$$

In fact, $E_1 \cap E_3 = \mathbb{C}(e_1, 0, 0, 0), E_2 \cap E_3 = \mathbb{C}(0, 0, 0, e_1)$ and $E_4 \cap E_3 = \mathbb{C}(a, 0, a, 0)$, where $a = (\gamma^{n-1})_n \in L = \ell^2(\mathbb{N})$. All the other terms are zeros.

Definition (4.2.17)[180]: Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. We say that S is *non-degenerate* if $E_i + E_j = H$ and $E_i \cap E_j = 0$ for $i \neq j$. If S is non-degenerate, then S is clearly a Fredholm system with the defect $\rho(S) = 0$. But the converse is not true. We have the following example due to the referee: Let S be a unilateral shift. Consider $S_{S/2, S^*/2}$. Then $\rho(S_{S/2,S^*/2}) = 0$ by Proposition (4.1.19) below. Since $Ker\ S^* \neq 0$, it is seen that $E_2 \cap E_4 \neq 0$.

Proposition (4.1.18)[180]: *Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Then S is non-degenerate if and only if $S^\perp$ is non-degenerate.*

Proposition (4.1.19)[180]: *Let $S_{T,S}$ be a bounded operator system. Then $S_{T,S}$ is a Fredholm system if and only if S, T and $ST - I$ are Fredholm operators. And if the condition is satisfied, then*

Proof. It is clear that $E_i \cap E_j = 0$ and $E_i + E_j = H$ for $(i, j) = (1,2), (1,4), (2,3)$. Since $KerA_{13} = E_1 \cap E_3 = Ker\, T \oplus 0$ and $(Im\, A_{13})^{\perp} = (E_1 + E_3)^{\perp} = (K_1 \oplus \mathrm{Im}\, T)^{\perp}$, they are finite-dimensional if and only if $Ker\, T$ and $(Im\, T)^{\perp} = Ker\, T^*$ are finite-dimensional. And $\mathrm{Im}\, A_{13}$ is closed if and only if $\mathrm{Im}\, T$ is closed. Similarly $KerA_{24} = E_2 \cap E_4 = 0 \oplus Ker\, S$ and $(Im\, A_{24})^{\perp} = (E_2 + E_4)^{\perp} = (Im\, S \oplus K_2)^{\perp}$. Hence they are finite-dimensional if and only if $Ker\, S$ and $(Im\, S)^{\perp} = Ker\, S^*$ are finite-dimensional. And $\mathrm{Im}\, A_{24}$ is closed if and only if $\mathrm{Im}\, S$ is closed. Next,

$KerA$

$\mathrm{Im}\, A_{34} = \{$

Multiplying invertible operator matrices from both sides, we have

Hence $\mathrm{Im}\, A_{34}$ is closed if and only if $\mathrm{Im}(ST - I)$ is closed, and $(Im\, A_{34})^{\perp}$ is finite-dimensional if and only if $\left(Im(ST - I)\right)^{\perp}$ is finite-dimensional. Now it is easy to see the desired conclusions.

Let S and S' be two quasi-Fredholm systems of four subspaces. Then it is evident that $S \oplus S'$ is also a quasi-Fredholm system and

Therefore we should investigate the possible values of the defect for indecomposable systems.

Theorem (4.1.20)[180]: *The set of the possible values of the defect of indecomposable systems of four subspaces is exactly $\mathbb{Z}/3$.*

Proof. Let S be a unilateral shift on $L = \ell^2(\mathbb{N})$. Let $K = L \otimes \mathbb{C}^n$ and $H = K \oplus K$. For a positive integer n, put

V

Let $S_V = (H; E_1, E_2, E_3, E_4)$ be the operator system associated with the single operator V. We shall show that S_V is indecomposable. Let $T = \left(T_{ij}\right)_{ij} \in B(K)$ be an idempotent which commutes with V. It is enough to show that $T = 0$ or $T = I$.

Since $VT = TV$, we have

$ST_{11} =$

By the Kleinecke–Shirokov theorem, T_{1n} is a quasinilpotent. Since T_{1n} commutes with a unilateral shift S, T_{1n} is a Toeplitz operator. Then $\|T_{1n}\| = r(T_{1n}) = 0$. Thus $T_{1n} = 0$ by Halmos [115]. Inductively we can show that $T_{12} = T_{13} = \cdots = T_{1n} = 0$. Similar argument shows that T is a lower triangular operator matrix, i.e., $T_{ij} = 0$ for $i < j$. Since $T^2 = T$, we have $T_{ii}^2 = T_{ii}$ for $i = 1, \ldots, n$. The diagonal of $VT = TV$ shows that each T_{ii} commutes with a unilateral shift S. This implies that $T_{ii} = 0$ or I as in Lemma (4.1.3).

(i) The case that $T_{11} = 0$: The 2-1th component of $VT = TV$ shows that $T_{22} = ST_{21} - T_{21}S$. Hence T_{22} cannot be I. Thus $T_{22} = 0$. Similarly we can show that $T_{ii} = 0$ for $i = 1, \ldots, n$. Thus the diagonal of operator matrix T is zero. Furthermore T is a lower triangular operator matrix and idempotent. Hence $T = O$.

(ii) The case that $T_{11} = I$: Considering $I - T$ instead of T, we can use the case (i) and show that $T = I$. Therefore S_V is indecomposable.

The defect is given by

$\rho(S_V) =$

$=$

In fact,

$Ker\, V^* = \{$

is n-dimensional.

Similarly SV^* is an indecomposable system with $\rho(S_{V^*}) = \frac{n}{3}$.

For $n = 0$, consider an indecomposable system S_{S+3I} as in Example after Proposition (4.1.12). Then $\rho(S_{S+3I}) = 0$.

Therefore the defect for indecomposable systems of four subspaces can take any value in $\mathbb{Z}/3$.

Corollary (4.1.21)[180]: *For any $n \in \mathbb{Z}$ there exist uncountable family of indecomposable systems S of four subspaces with the same defect $\rho(S) = \frac{n}{3}$.*

Proof. For a positive integer n, consider a family $(S_{V+\alpha I})_{\alpha\in(0,1)}$ and $(S_{V^*+\alpha I})_{\alpha\in(0,1)}$ of bounded operator systems similarly as in the above theorem. Then any $S_{V+\alpha I}$ is also indecomposable and

If $\alpha \neq \beta$, then the spectrum $\sigma(V + \alpha I) \neq \sigma(V + \beta I)$. Since $V + \alpha I$ and $V + \beta I$ are not similar, $S_{V+\alpha I}$ and $S_{V+\beta I}$ are not isomorphic each other.

We also have $\rho(S_{V^*+\alpha I}) = \frac{n}{3}$. And they are not isomorphic each other.

For $n = 0$, consider a family $(S_{S+3I+\alpha I})_{\alpha\in[0,1]}$. They are indecomposable, not isomorphic each other and $\rho(S_{S+3I+\alpha I}) = 0$.

In [100] Gelfand and Ponomarev introduced two functors Φ^+ and Φ^- on the category of systems S of n subspaces in finite-dimensional vector spaces. They used the functors Φ^+ and Φ^- to give a complete classification of indecomposable systems of four subspaces with defect $\rho(S) \neq 0$ in finite-dimensional vector spaces. If the defect $\rho(S) < 0$, then there exists a positive integer ℓ such that $(\Phi^+)^{\ell-1}(S) \neq 0$ and $(\Phi^+)^{\ell}(S) = 0$. Combining the facts that indecomposable systems $\mathcal{T}$ with $\Phi^+(\mathcal{T}) = 0$ can be classified easily and that S is isomorphic to (and recovered as)

$(\Phi^{-})^{\ell-1}(\Phi^{+})^{\ell-1}(S)$, they provided a complete classification. A similar argument holds for systems S with defect $\rho(S) > 0$.

In their argument the finiteness of dimension is used crucially. In fact if an indecomposable system $S = (H; E_1, E_2, E_3, E_4)$ with $dim H > 1$ satisfies that the defect $\rho(S) < 0$, then $\Phi^{+}(S) = (H^{+}; E_1^{+}, E_2^{+}, E_3^{+}, E_4^{+})$ has the property that $dim H^{+} < dim H$. The property guarantees the existence of a positive integer ℓ such that $(\Phi^{+})^{\ell}(S) = 0$. Although we cannot expect such an argument anymore in the case of infinite-dimensional space, these functors Φ^{+} and Φ^{-} are interesting on their own right. Therefore we shall extend these functors Φ^{+} and Φ^{-} on infinite-dimensional Hilbert spaces and show that the Coxeter functors preserve the defect and indecomposability under certain conditions.

Definition (4.1.22)[180]: Let Sys^{n} be the category of the systems of n subspaces in Hilbert spaces and homomorphisms. Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $R := \bigoplus_{i=1}^{n} E_i$ and

Define $S^{+} = (H^{+}; E_1^{+}, \ldots, E_n^{+})$ by

H^{+}

Let $\mathcal{T} = (K;\ F_1, \ldots, F_n)$ be another system of n subspaces in a Hilbert space K and $\varphi : S \to \mathcal{T}$ be a homomorphism. Since $\varphi : H \to K$ is a bounded linear operator with $\varphi(E_i) \subset F_i$, we can define a bounded linear operator $\varphi^{+} : H^{+} \to K^{+}$ by $\varphi^{+}(x_1, \ldots, x_n) = \big(\varphi(x_1), \ldots, \varphi(x_n)\big)$. Since $\varphi^{+}(E_i^{+}) \subset F_i^{+}, \varphi^{+}$ defines a homomorphism $\varphi^{+} : S^{+} \to \mathcal{T}^{+}$. Thus we can introduce a covariant functor $\Phi^{+} : Sys^{n} \to Sys^{n}$ by

Example (4.1.23)[180]: If $S = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C})$, then $S^{+} \cong \big(\mathbb{C}^2; \mathbb{C}(1,0), \mathbb{C}(0,1), \mathbb{C}(1,1)\big)$.

Lemma (4.1.24)[180]: *Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces and consider $S^{+} = (H^{+}; E_1^{+}, E_2^{+}, E_3^{+}, E_4^{+})$. Then*

In particular, we have $dim\, E_1^+ \cap E_2^+ = dim\, E_3 \cap E_4$. *Same formulae hold under permutation of subspaces.*

Proof. Let $x = (x_1, x_2, x_3, x_4) \in E_1^+ \cap E_2^+$, then $x_1 = x_2 = 0$. Since $x \in H^+, \tau(x) = x_3 + x_4 = 0$. Thus $a \coloneqq x_3 = -x_4 \in E_3 \cap E_4$ and $x = (0, 0, a, -a)$. The converse inclusion is clear.

Lemma (4.1.25)[180]: *Let* $S = (H; E_1, E_2, E_3, E_4)$ *be a system of four subspaces and consider* $S^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$. *If* $E_3 \cap E_4 = 0$ *and* $E_3 + E_4 = H$, *then* $E_1^+ + E_2^+ = H^+$. *Same formulae hold under permutation of subspaces.*

Proof. Let $z = (z_1, z_2, z_3, z_4) \in H^+$. Put $y_1 \coloneqq z_1$ and $x_2 \coloneqq z_2$. Since $E_3 + E_4 = H$, there exist $y_3 \in E_3$ and $y_4 \in E_4$ such that $-y_1 = y_3 + y_4$. Since $y_1 + y_3 + y_4 = 0, y \coloneqq (y_1, 0, y_3, y_4) \in H^+$. Similarly there exist $x_3 \in E_3$ and $x_4 \in E_4$ such that $-x_2 = x_3 + x_4$, so that $x \coloneqq (0, x_2, x_3, x_4) \in H^+$.

Since $z \in H^+, z_1 + z_2 + z_3 + z_4 = 0$. Hence

$z_3 +$

Because $E_3 \cap E_4 = 0$, we have $z_3 = x_3 + y_3$ and $z_4 = x_4 + y_4$. Therefore $z = x + y \in E_1^+ + E_2^+$.

Example (4.1.26)[180]: Let $S_{S,T} = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system. Combining the preceding two Lemmas (4.1.24) and (4.1.25) with a characterization of bounded operator systems, we have that $S^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$ is a bounded operator system up to permutation of subspaces. More precisely, $(H^+; E_3^+, E_4^+, E_1^+, E_2^+)$ is a bounded operator system.

Let $0 \oplus E_i \oplus 0 \coloneqq 0 \oplus \cdots \oplus 0 \oplus E_i \oplus 0 \oplus \cdots \oplus 0 \subset R$ and $q_i \in B(R)$ be the projection onto $0 \oplus E_i \oplus 0$. Let $\iota_+ : H^+ \to R$ be a canonical embedding. Then we have an exact sequence:

Furthermore we have

$Ker\ \tau q_i$

These properties characterize $S^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$.

In general we have

Corollary (4.1.27)[284]: *Let $S = (H; E_n, E_{n+1}, E_{n+2}, E_{n+3})$ be a system of four subspaces and consider $S^+ = (H^+; E_n^+, E_{n+1}^+, E_{n+2}^+, E_{n+3}^+)$. If $E_{n+2} \cap E_{n+3} = 0$ and $E_{n+2} + E_{n+3} = H$, then $E_n^+ + E_{n+1}^+ = H^+$.* The *Same formular hold under permutation of subspaces*.

Proof. Let $z = (z_n, z_{n+1}, z_{n+2}, z_{n+3}) \in H^+$. Lut $y_n \coloneqq z_n$ and $x_{n+1} \coloneqq z_{n+1}$. Since $E_{n+2} + E_{n+3} = H$. Then $y_{n+2} \in E_{n+2}$ and $y_{n+3} \in E_{n+3}$ such that $-y_n = y_{n+2} + y_{n+3}$. Since $y_n + y_{n+2} + y_{n+3} = 0, y \coloneqq (y_n, 0, y_{n+2}, y_{n+3}) \in H^+$. Similarly there exist $x_{n+2} \in E_{n+2}$ and $x_{n+3} \in E_{n+3}$ such that $-x_{n+1} = x_{n+2} + x_{n+3}$, so that $x \coloneqq (0, x_{n+1}, x_{n+2}, x_{n+3}) \in H^+$.

Since $z \in H^+, z_n + z_{n+1} + z_{n+2} + z_{n+3} = 0$. Hence

$z_{n+2} + z_{n+3} = -z$

$= (y_{n+2} +$

Because $E_{n+2} \cap E_{n+3} = 0$, we have $z_{n+2} = x_{n+2} + y_{n+2}$ and $z_{n+3} = x_{n+3} + y_{n+3}$. Therefore $z = x + y \in E_n^+ + E_{n+1}^+$.

Proposition (4.1.28)[180]: *Let X, Y and Z be Hilbert spaces and $T : X \to Y$ and $S : Y \to Z$ be bounded linear maps. Suppose that a sequence*

is exact. Let $p_1, \dots, p_n \in B(Y)$ be projections with $\sum_i p_i = I$ and $p_i p_j = 0$ for $i \neq j$. Furthermore we assume that

Let $E_i \coloneqq \operatorname{Im} Sp_i \subset Z$ and $E_i' \coloneqq Ker\ p_i T \subset X$. Define $S = (Z; E_1, \dots, E_n)$ and $S' = (X; E_1, \dots, E_n)$. Then $S' \cong \Phi^+(S)$.

Definition (4.1.29)[180]: In [100] Gelfand and Ponomarev introduced a dual functor Φ^- using quotients of vector spaces. If H is a Hilbert space and K a subspace of H, then it is convenient to identify the quotient space H/K with the orthogonal complement $K^\perp$. Therefore we shall generalize their functor Φ^- in terms of orthogonal complements instead of quotients in our case of Hilbert spaces. Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $e_i^\perp \in B(H)$ be the projection onto $E_i^\perp \subset H$. Let $Q := \bigoplus_{i=1}^n E_i^\perp$ and

Then $\mu^* : Q \to H$ is given by $\mu^*(y_1, \ldots, y_n) = \sum_{i=1}^n y_i$. Define $H^- := Ker\, \mu^* \subset Q$. Let $\iota_- : H^- \to Q$ be a canonical embedding. Then $q_- := \iota_-^* : Q \to H^-$ is the projection. Let $0 \oplus E_i^\perp \oplus 0 := 0 \oplus \cdots \oplus 0 \oplus E_i^\perp \oplus 0 \cdots \oplus 0 \subset Q$ and $r_i \in B(Q)$ be the projection onto $0 \oplus E_i^\perp \oplus 0$. Define $S^- = (H^-; E_1^-, \ldots, E_n^-)$ by

We note that

We have an exact sequence

and a sequence

satisfying that $\overline{\operatorname{Im} \mu} = Ker\, q_-$ and q_- is onto. Thus it is easy to see that our definition of $S^- = (H^-; E_1^-, \ldots, E_n^-)$ coincides with the original one by Gelfand and Ponomarev up to isomorphism in the case of finite-dimensional spaces.

Define $\Phi^-(S) := S^- = (H^-; E_1^-, \ldots, E_n^-)$. Then there is a relation between S^+ and S^-. We recall some elementary facts first.

Lemma (4.1.30)[180]: *Let H and K be Hilbert spaces and M a closed subspace of H. Let $T : H \to K$ be a bounded operator. Consider $T^* : K \to H$. Then $\overline{T(M^\perp)} = \left((T^*)^{-1}(M)\right)^\perp \subset K$.*

Lemma (4.1.31)[180]: *Let L be a Hilbert space and M, K closed subspaces of L. Let $P_K \in B(L)$ be the projection onto K. Then $P_K(M^\perp) = K \cap (K \cap M)^\perp$.*

Proof. By the preceding lemma,

Decompose $x \in L$ such that $x = x_1 + x_2$ with $x_1 \in K, x_2 \in K^\perp$. Then $P_K x \in M$ if and only if $x_1 \in M$. Therefore $\left(\overline{P_K(M^\perp)}\right)^\perp = (K \cap M) + K^\perp$. Thus $\overline{P_K(M^\perp)} = K \cap (K \cap M)^\perp$.

Proposition (4.1.32)[180]: *Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Then we have*

Proof. Since $\Phi^\perp(S) = (H; E_1^\perp, \ldots, E_n^\perp)$, we have

where $H' = \{(y_1, \ldots, y_n) \in \bigoplus_{i=1}^n E_i^\perp;\ y_1 + \cdots + y_n = 0\}$. Therefore we have $H' = H^-$. Applying the preceding lemma by putting $L = \bigoplus_{i=1}^n E_i^\perp, M = \{(y_1, \ldots, y_n) \in L;\ y_k = 0\}$ and $K = H^- \subset L$, we have

$E_k^- = \overline{q_-((}$

Therefore $(E_k^-)^\perp = (E_k^\perp)^+$ in H^-. Hence $\Phi^\perp\Phi^-(S) = \Phi^+\Phi^\perp(S)$. This implies the conclusion.

Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H and $\mathcal{T} = (K;\ F_1, \ldots, F_n)$ be another system of n subspaces in a Hilbert space K. Let $\varphi : S \to \mathcal{T}$ be a homomorphism, i.e., $\varphi : H \to K$ is a bounded linear operator with $\varphi(E_i) \subset F_i$. Define $\varphi^- : \Phi^-(S) \to \Phi^-(\mathcal{T})$ by

Thus we can introduce a covariant functor $\Phi^- : Sys^n \to Sys^n$ by

Remark (4.1.33)[180]: Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $R := \bigoplus_{i=1}^{n} E_i$ and $\tau : R \to H$ is given by $\tau(x) = \sum_{i=1}^{n} x_i$. Let $H^0 := Ker\,\tau$ and $q_0 : R \to H^0$ be the canonical projection. Define $E_k^0 := \overline{q_0(0 \oplus E_k \oplus 0)}$. Let $S^0 := (H^0; E_1^0, \dots, E_n^0)$ and $\Phi^0(S) = S^0$. Then we have

Furthermore

$\Phi^-\Phi^+$

Suppose that H is finite-dimensional. Then

$dim\, H^0 =$

In particular, if $S = (H; E_1, E_2, E_3, E_4)$ is an indecomposable system of four subspaces with $dim\, H \geq 2$, then $dim\, H^0 = \sum_i dim\, E_i - dim\, H$ and the defect

We shall characterize $\Phi^-(S)$. The following fact is useful: Let H and K be Hilbert spaces and $T : H \to K$ be a bounded linear operator. Then $\mathrm{Im}\, T$ is closed in K if and only if $\mathrm{Im}\, T^*$ is closed in H.

Proposition (4.1.34)[180]: *Let U, V and W be Hilbert spaces and $A : U \to V$ and $B : V \to W$ be bounded linear operators. Suppose that a sequence*

is exact. Let $p_1, \dots, p_n \in B(V)$ be projections with $\sum_i p_i = I$ and $p_i p_j = 0$ for $i \neq j$. Furthermore we assume that

Let $L_i' := \overline{\mathrm{Im}\, B p_i} \subset W$ and $L_i := Ker\, p_i A \subset U$. Define $S = (U; L_1, \dots, L_n)$ and $S' = (W; L_1', \dots, L_n')$. Then $S' \cong \Phi^-(S)$.

Proof. Since $\mathrm{Im}\, B = W$ is closed, $\mathrm{Im}\, B^* \subset V$ is also closed. Then and $KerB^* = (Im\, B)^{\perp} = W^{\perp} = 0$. Hence the dual sequence

is exact. We shall apply Proposition (4.1.28) by putting $X = W, Y = V, Z = U, T = B^*$ and $S = A^*$. We can check the assumption of the proposition. In fact,

Ke

and $\mathrm{Im}\, Sp_i = \mathrm{Im}\, A^* p_i = \mathrm{Im}(p_i A)^*$ is closed, because $\mathrm{Im}\,(p_i A)$ is closed. Let

E

and

E_i'

Then $(X; E_1', \ldots, E_n') \cong \Phi^+(Z; E_1, \ldots, E_n)$, that is, we have

Thus $(S')^{\perp} \cong \Phi^+(S^{\perp})$. Hence

Proposition (4.1.35)[180]: *Let S and T be systems of n subspaces in a Hilbert space H. Then we have $\Phi^+(S \oplus \mathcal{T}) \cong \Phi^+(S) \oplus \Phi^+(\mathcal{T})$,*

$\Phi^-(S \oplus \mathcal{T})$

Definition (4.1.36)[180]: Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Then S is said to be *reduced from above* if for any $k = 1, \ldots, n$

In particular we have $E_k \subset \sum_{i \neq k} E_i$. Similarly S is said to be *reduced from below* if for any $k = 1, \ldots, n$

In particular we have $E_k^\perp \subset \sum_{i\neq k} E_i^\perp$ and $\bigcap_{i\neq k} E_i = 0$.

It is evident that $S \oplus \mathcal{T}$ is reduced from above if and only if both S and $\mathcal{T}$ are reduced from above. Similarly $S \oplus \mathcal{T}$ is reduced from below if and only if both S and $\mathcal{T}$ are reduced from below.

Example (4.1.37)[180]: (a) Any bounded operator system is reduced from above and reduced from below. In fact $E_1 + E_2 = H, E_1 + E_4 = H, E_2 + E_4 = H$ and $E_1^\perp + E_2^\perp = H, E_1^\perp + E_4^\perp = H, E_2^\perp + E_4^\perp = H$.

(b) The exotic examples are reduced from above and reduced from below.

We shall show a duality theorem between Coxeter functors Φ^+ and Φ^-.

Theorem (4.1.38)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that S is reduced from above. Then we have*

Proof. Let $R = \bigoplus_{i=1}^n E_i$. Consider a sequence

$$H^+ \overset{\iota_+}{\longrightarrow} R \overset{\tau}{\longrightarrow} H \longrightarrow 0.$$

Since S is reduced from above, $\operatorname{Im} \tau = \sum_{i=1}^n E_i = H$. Thus the above sequence is exact. Let $p_i \in B(R)$ be the projection onto $0 \oplus E_i \oplus 0$. We shall apply Proposition (4.1.34) by putting $U = H^+, V = R, W = H, A = \iota_+$ and $B = \tau$. We can check the assumption of the proposition. In fact, since S is reduced from above, for any $x_k \in E_k$, there exist $x_i \in E_i$ for $i \neq k$ such that $x_k = \sum_{i\neq k} -x_i$. Then $\sum_{i=1}^n x_i = 0$, that is, $x := (x_i)_i \in H^+$. Then

Thus $\operatorname{Im} p_k A = 0 \oplus E_k \oplus 0 = \operatorname{Im} p_k$ and $\operatorname{Im} p_k A$ is closed. Therefore $(W; L_1', \dots, L_n') \cong \Phi^-(U; L_1, \dots, L_n)$. Since

and

we have

$$S =$$

Similarly we have the following:

Theorem (4.1.39)[180]: *Let $S = H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that S is reduced from below. Then we have*

Proof. If S is reduced from below, then $S^\perp$ is reduced from above. Hence $\Phi^- \Phi^+ (S^\perp) \cong S^\perp$. Then

$$S \cong$$

Proposition (4.1.40)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\Phi^+(S) = 0$ if and only if for any $k = 1, \dots, n$*

Proof. It is easy to see that $\Phi^+(S) = 0$ if and only if for any $x_i \in E_i$ with $i = 1, \dots, n$ $\sum_i x_i = 0$ implies $x_1 = \cdots = x_n = 0$. The latter condition is equal to that $E_k \cap (\sum_{i \neq k} E_i) = 0$ for any $k = 1, \dots, n$.

The above condition that $E_k \cap (\sum_{i \neq k} E_i) = 0$ for any $k = 1, \dots, n$ is something like an opposite of that S is reduced from above.

Proposition (4.1.41)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\Phi^+(S) = 0$ and $\sum_{i=1}^n E_i$ is closed in H if and only if $(H; E_1, \dots, E_n, (\sum_{i=1}^n E_i)^\perp)$ is isomorphic to a system of direct sum decomposition, that is, there is an orthogonal direct sum decomposition $K = \bigoplus_{i=1}^{n+1} K_i$ of a Hilbert space K and $(H; E_1, \dots, E_n, (\sum_{i=1}^n E_i)^\perp)$ is isomorphic to a system $(K; K_1, \dots, K_{n+1})$, in particular S is isomorphic to a commutative system.*

Proof. Assume that $\Phi^+(S) = 0$ and $\sum_{i=1}^n E_i$ is closed in H. Let $E_{n+1} = (\sum_{i=1}^n E_i)^\perp$. Let $R := \bigoplus_{i=1}^{n+1} E_i$ and $K_i := 0 \oplus \cdots \oplus 0 \oplus E_i \oplus 0 \oplus \cdots \oplus 0 \subset R$. Define $\varphi : K \to H$ by $\varphi((x_i)_i) = \sum_i x_i$. Then the bounded operator φ is onto, because $\sum_{i=1}^n E_i$ is closed in H.

Since $\Phi^+(S) = 0, \varphi$ is one to one by the preceding proposition. It is clear that $\varphi(K_i) = E_i$. Hence $(H; E_1, \dots, E_{n+1})$ is isomorphic to $(K; K_1, \dots, K_{n+1})$. The converse and the rest are trivial.

Example (4.1.42)[180]: Let $T \in B(K)$ be a positive operator with dense range and $\operatorname{Im} T \neq K$. Let $H = K \oplus K, E_1 = K \oplus 0$ and $E_2 = \operatorname{graph} T$. Put $S = (H; E_1, E_2)$. Then $\Phi^+(S) = 0$ and $(E_1 + E_2)^\perp = 0$. But $(H; E_1, E_2, 0)$ is not isomorphic to a system of direct sum decomposition. In fact $E_1 + E_2 = K \oplus \operatorname{Im} T$ is not closed.

We also have the following:

Proposition (4.1.43)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\Phi^-(S) = 0$ if and only if for any $k = 1, \dots, n$*

Proposition (4.1.44)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. If S is reduced from above and $S \neq 0$, then $\Phi^+(S) \neq 0$. Similarly if S is reduced from below and $S \neq 0$, then $\Phi^-(S) \neq 0$.*

Proof. Suppose that $E_i = 0$ for any $i = 1, \dots, n$. Then $H = \sum_{i=1}^{n-1} E_i = 0$. This contradicts the assumption that $S \neq 0$. Therefore $E_k \neq 0$ for some k. Since $\sum_{i \neq k} E_i = H$, for a non-zero $x_k \in E_k$, there exist $x_i \in E_k$ for $i \neq k$ such that $-x_k = \sum_{i \neq 0} x_i$. Therefore $x := (x_1, \dots, x_n) \in H^+$ is non-zero, that is, $\Phi^+(S) \neq 0$. The other is similarly proved.

Remark (4.1.45)[180]: By Proposition (4.1.34), if a system of n subspaces $S = (H; E_1, \dots, E_n)$ is indecomposable and $dim\, H \geq 2$, then for any distinct $n - 1$ subspaces $E_{i_1}, \dots, E_{i_{n-1}}$, we have that

that is,

Unless H is finite-dimensional, these conditions seem to be weaker than that S is reduced from below and above.

Remark (4.1.46)[180]: Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H and consider $S^+ = (H^+; E_1^+, \dots, E_n^+)$. Then for any distinct $n-1$ subspaces $E_{i_1}^+, \dots, E_{i_{n-1}}^+$, we have that

In fact, for example, let $(x_1, \dots, x_n) \in \bigcap_{k=1}^{n-1} E_k^+$. Then $x_1 = x_2 = \cdots = x_{n-1} = 0$. Since $(x_1, \dots, x_n) \in H^+$, we have $\sum_{i=1}^{n} x_k = 0$. Hence $x_n = 0$. Thus $\bigcap_{k=1}^{n-1} E_k^+ = 0$.

On the other hand the above condition implies that

This condition is a little weaker than that S^+ is reduced from below unless H is finite-dimensional.

Consider $S^- = \Phi^\perp \Phi^+ \Phi^\perp(S)$ similarly. Then we have

The condition is a little weaker than that S^- is reduced from above unless H is finite-dimensional.

Theorem (4.1.47)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that S is reduced from above and $S^+ = \Phi^+(S)$ is reduced from below. If S is indecomposable, then $\Phi^+(S)$ is also indecomposable*.

Example (4.1.48)[180]: Let $S_\gamma = (H; E_1, E_2, E_3, E_4)$ be an exotic example. Since $E_i + E_j = H$ and $E_i \cap E_j = 0$ for distinct $i, j \in \{1, 2, 4\}$, we have $E_k^+ + E_m^+ = H$ and

$E_k^+ \cap E_m^+ = 0$ for distinct $k, m \in \{3, 4\}$ or $k, m \in \{1, 3\}$ or $k, m \in \{2, 3\}$ by Lemmas (4.1.24) and (4.1.25). Since $E_k^+ + E_m^+ = H$ is closed, $(E_k^+)^\perp + (E_m^+)^\perp$ is closed. Hence $(E_k^+)^\perp + (E_m^+)^\perp = H$. Therefore S_γ is reduced from above and $\Phi^+(S_\gamma)$ is reduced from below. Since S_γ is indecomposable, $\Phi^+(S_\gamma)$ is also indecomposable.

Similarly we have the following:

Theorem (4.1.49)[180]: *Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that S is reduced from below and $S^- = \Phi^-(S)$ is reduced from above. If S is indecomposable, then $\Phi^-(S)$ is also indecomposable.*

We shall show that the Coxeter functors Φ^+ and Φ^- preserve the defect under certain conditions.

Let $S = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Consider $S^+ = (H^+; E_1^+, \dots, E_n^+)$. Let $R = \bigoplus_{i=1}^n E_i$ and $p_0 \in B(R)$ be the projection of R onto H^+. Let $e_i \in B(H)$ be the projection of H onto E_i. Recall that $\tau : R \to H$ is given by $\tau(a) = \sum_{i=1}^n a_i$ for $a = (a_1, \dots, a_n) \in R$.

Lemma (4.1.50)[180]: Suppose that $\sum_{i=1}^n e_i$ *is invertible. Then for $a = (a_1, \dots, a_n) \in R$ we have*

Proof. Recall that $\tau^* : H \to R$ is given by $\tau^*(y) = (e_1 y, \dots, e_n y)$ for $y \in H$. Consider the orthogonal decomposition $R = H^+ \oplus (H^+)^\perp$. Since $H^+ = Ker\,\tau, (H^+)^\perp = \operatorname{Im}\tau^*$ in R. Define

Then

$$\tau(x) = \sum_{k=1}^{n} \Bigg(a_k$$

Therefore $x \in H^+$. Put $y := (\sum_{i=1}^{n} e_i)^{-1}(\tau(a)) \in H$. Then $\tau^*(y) = (e_1 y, \ldots, e_n y) \in (H^+)^\perp$. Since $a = x + \tau^*(y) \in H^+ \oplus (H^+)^\perp$, we have $p_0(a) = x$.

Corollary (4.1.51)[180]: *Suppose that $\sum_{i=1}^{n} e_i$ is invertible. Then $\operatorname{Im} \tau^*$ is closed and*

Lemma (4.1.52)[180]: *Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $e_i \in B(H)$ be the projection of H onto E_i. Then*

Moreover $\sum_{i=1}^{n} E_i$ is closed if and only if $\sum_{i=1}^{n} e_i$ has a closed range.

Proof. See [116] for several facts on operator ranges. Let $T = (T_{ij})_{ij} \in B(H^n)$ be an operator matrix defined by $T_{1j} = e_j$ and $T_{ij} = 0$ for $i \neq 1$. Recall that $\operatorname{Im} T = \operatorname{Im}((T\,T^*)^{1/2})$ for any operator T. Since $\operatorname{Im} T = (\sum_{i=1}^{n} E_i) \oplus 0 \oplus 0 \oplus 0$ and $\operatorname{Im}((T\,T^*)^{1/2}) = \left(\operatorname{Im}((\sum_{i=1}^{n} e_i)^{1/2})\right) \oplus 0 \oplus 0 \oplus 0$, we have $\sum_{i=1}^{n} E_i = \operatorname{Im}((\sum_{i=1}^{n} e_i)^{1/2})$. It is a known fact that $\operatorname{Im} A$ is closed if and only if $\operatorname{Im} A^{1/2}$ is closed for any positive operator $A \in B(H)$. This implies the rest.

Corollary (4.1.53)[180]: *Let $S = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. If S is reduced from above, then $f := \sum_{i=1}^{n} e_i$ is invertible.*

Proof. Let $x \in Ker\, f$. Then $(e_i x | x) = 0$ so that $e_i x = 0$. Since S is reduced from above, $x \in \bigcap_i E_i^\perp = 0$. Thus $Ker\, f = 0$. Then $\overline{\operatorname{Im} f} = (Ker\, f)^\perp = H$. Since S is reduced from above, $\sum_{i=1}^{n} E_i = H$ is clearly closed. By the preceding lemma, f has a closed range. Thus $\operatorname{Im} f = H$. Therefore f is invertible.

Lemma (4.1.54)[180]: *Suppose that S is reduced from above. Then for $k = 1, \ldots, n$*

$(E_k^+$

Proof. Since S is reduced from above, we have $\operatorname{Im} p_k p_0 = 0 \oplus E_k \oplus 0$. In fact, for any $a_k \in E_k$, there exist $a_i \in E_i, (i \neq k)$ such that $-a_k = \sum_{i \neq k} a_i$. Then $(a_1, \dots, a_n) \in H^+$ and

$$p_k$$

The converse inclusion is trivial. Since $\operatorname{Im} p_k p_0 = 0 \oplus E_k \oplus 0$ is closed, $(\operatorname{Im} p_k p_0)^* = \operatorname{Im} p_0 p_k$ is also closed. Hence

$$(E_k^+)$$

Therefore the conclusion follows from Lemma (4.1.49).

Proposition (4.1.55)[180]: *Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces and $S^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$. Suppose that S is reduced from above. Then $f := e_1 + e_2 + e_3 + e_4$ is invertible and*

$$(E_1^+)^\perp \cap$$
$$= \{(e$$

Moreover we have

The same formulae hold under permutation of subspaces.

Proof. Let $x = (x_1, x_2, x_3, x_4) \in (E_1^+)^\perp \cap (E_2^+)^\perp$. Then by the preceding lemma, there exist $a_1 \in E_1$ and $a_2 \in E_2$ such that

$$x$$

Put $u := f^{-1}(a_1 - a_2) \in H$. Then $a_1 = e_1 u, a_2 = -e_2 u, e_3 u = 0$ and $e_4 u = 0$. Therefore $u \in E_3^\perp \cap E_4^\perp$ and

$$x =$$

Conversely suppose that

$$x =$$

for some $u \in E_3^\perp \cap E_4^\perp$. Put $a_1 := e_1 u \in E_1$ and $a_2 := -e_2 u \in E_2$. Since $e_3 u = 0$ and $e_4 u = 0$, we have

$$a$$

Because f is invertible, $u = f^{-1}(a_1 - a_2)$. Therefore

$$x = (a$$

On the other hand, $a_1 = e_1 u = e_1 f^{-1}(a_1 - a_2)$. Hence

Since $a_2 = -e_2 u = -e_2 f^{-1}(a_1 - a_2)$, we have

Since $e_3 f^{-1}(a_1 - a_2) = e_3 u = 0$, we have $e_3 f^{-1} a_1 = e_3 f^{-1} a_2$. Similarly $e_4 f^{-1} a_1 = e_4 f^{-1} a_2$. Therefore

$$x = (-$$

Thus $x \in (E_1^+)^\perp \cap (E_2^+)^\perp$.

Moreover define $T : E_3^\perp \cap E_4^\perp \to (E_1^+)^\perp \cap (E_2^+)^\perp$ by

$$Tu =$$

for $u \in E_3^\perp \cap E_4^\perp$. Then T is a bounded, surjective operator. We shall show that T is one to one. Suppose that $Tu = 0$. Since $e_2 f^{-1} e_1 u = 0, f^{-1} e_1 u \in E_2^\perp$. Similarly $f^{-1} e_1 u \in E_3^\perp$ and $f^{-1} e_1 u \in E_4^\perp$. Since S is reduced from above,

$$f^-$$

Hence $e_1 u = 0$. Similarly we have $e_2 u = 0$. Therefore $fu = e_1 u + e_2 u + e_3 u + e_4 u = 0$. Since f is invertible, $u = 0$. Thus T is an invertible operator. Therefore $dim((E_1^+)^\perp \cap (E_2^+)^\perp) = dim(E_3^\perp \cap E_4^\perp)$.

Theorem (4.1.56)[180]: *Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Suppose that S is reduced from above. If S is a quasi-Fredholm system, then $\Phi^+(S)$ is also a quasi-Fredholm system and*

Theorem (4.1.57)[180]: *Let $S = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Suppose that S is reduced from below. If S is a quasi-Fredholm system, then $\Phi^-(S)$ is also a quasi-Fredholm system and*

Proof. Recall that S is reduced from below if and only if $\Phi^{\perp}(S)$ is reduced from above, and S is a quasi-Fredholm system if and only if $\Phi^{\perp}(S)$ is a quasi-Fredholm system. Applying the preceding theorem, $\Phi^{-}(S) = \Phi^{\perp}\Phi^{+}\Phi^{\perp}(S)$ is a quasi-Fredholm system and

ρ

Example (4.1.58)[180]: Let S be an operator system. Since $E_1 = K \oplus 0, E_2 = 0 \oplus K$, we have that $f = \sum_{i=1}^{4} e_i \geq I$ is invertible. Moreover if $S = S_T$ is associated with a single bounded operator T, then $E_4 = \{(x,x) \in H;\ x \in K\}$. Thus $E_i + E_j = H$ for $(i,j) = (1,2),(1,4),(2,4)$ and S is reduced from above. Therefore, if S_T is a quasi-Fredholm system, then $\Phi^{+}(S_T)$ is also a quasi-Fredholm system and $\rho\big(\Phi^{+}(S_T)\big) = \rho(S_T)$. Similarly, let S_γ be an exotic example. Then S_γ is reduced from above and f is invertible. Since S_γ is a quasi-Fredholm system, $\Phi^{+}(S_\gamma)$ is also a quasi-Fredholm system and

$$\rho\left(\Phi^{+}(S_\gamma)\right) = \rho(S_\gamma).$$

We consider the relative position of subspaces in a factor. There exists a relation between the defect and the Jones index [102] in a type II_1 factor setting.

Definition (4.1.59)[180]: Let M be a factor on a Hilbert space H. We say that $S = (M;\ e_1,\dots,e_n)$ is a system of n projections in M if $e_1,\dots,e_n$ are in fact n projections in M. If $M = B(H)$, then we can identify the system $S = (M;\ e_1,\dots,e_n)$ with the system $(H; E_1,\dots,E_n)$ of n subspaces in a Hilbert space H, where E_i is the range of e_i for $i = 1,\dots,n$. Two systems $S = (M;\ e_1,\dots,e_n)$ and $S' = (M;\ e_1',\dots,e_n')$ are isomorphic in M if there exists an invertible operator $t \in M$ such that

for $i = 1,\dots,n$.

Example (4.1.60)[180]: Let M be a factor of type II_1 and $N \subset M$ be a subfactor. Consider n intermediate subfactors $N \subset K_1,\dots,K_n \subset M$. Let e_{K_i} be the Jones projection of $L^2(M)$ onto the subspace $L^2(K_i)$ for $i = 1,\dots,n$. Since the Jones projections $e_{K_1},\dots,e_{K_n}$ are in the basic construction $< M, e_N >$, we have a system $S = \left(< M, e_N >;\ e_{K_1}, \dots, e_{K_n}\right)$ of n projections in $< M, e_N >$.

Definition (4.1.61)[180]: Let M be a factor of type II_1 with the normalized trace τ. Let $S = (M;\ e_1, e_2, e_3, e_4)$ be a system of four projections in M. We define the defect $\rho(S)$ of S (relative to M) by

In the setting above, we have a relation between the defect and the Jones index.

Proposition (4.1.62)[180]: *Let M be a factor of type II_1 and $N \subset M$ be a subfactor of finite index. Let $N \subset K_1, K_2, K_3, K_4 \subset M$ be intermediate subfactors. Consider the system $S = \left(< M, e_N >;\ e_{K_1}, e_{K_2}, e_{K_3}, e_{K_4}\right)$ of four projections in $< M, e_N >$. Then*

$$\rho(S$$

Corollary (4.1.63)[284]: Suppose that $\sum_{j=1}^{n} \sum_{i=1}^{n} e_i^j$ *is invertible. Then for $a = (a_1, \dots, a_n) \in R$, we have*

$$p_0^j$$

Proof. Let $\tau^* : H \to R$ is given by $\tau^*(y) = \left(e_1^j y, \dots, e_n^j y\right)$ for $y \in H$. For $R = H^+ \oplus (H^+)^\perp$. Since $H^+ = Ker\ \tau, (H^+)^\perp = \mathrm{Im}\ \tau^*$ in R. We can define

$$\tau(x) = \sum_{k=1}^{n}$$

Hence $x \in H^+$. If we let $y := \left(\sum_{j=1}^{n} \sum_{i=1}^{n} e_i^j\right)^{-1} \left(\tau(a)\right) \in H$. Then $\tau^*(y) = \left(e_1^j y, \dots, e_n^j y\right) \in (H^+)^\perp$. Now since $a = x + \tau^*(y) \in H^+ \oplus (H^+)^\perp$, we have $p_0^j(a) = x$.

Section (4.2): Indecomposable Representations of Quivers on Hilbert Spaces:

We studied the relative position of *several subspaces* in a separable infinite-dimensional Hilbert space in [2] after Gelfand and Ponomarev [3]. We extend it to the relative position of several subspaces along quivers. More generally we study representations of quivers on infinite-dimensional Hilbert spaces by bounded operators. We call them Hilbert representations for short.

Gabriel's theorem says that a finite, connected quiver has only finitely many indecomposable representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8 [4]. The theory of representations of quivers on finite-dimensional vector spaces has been developed by Bernstein, Gelfand and Ponomarev [5], Brenner [6], Donovan and Freislish [7], Dlab and Ringel [8], Gabriel and Roiter [9], Kac [10], Nazarova [11], ... Infinite-dimensional representatios of quivers have also been investigated in purely algebraic setting. See Krause and Ringel [12] and Reiten and Ringel [13].

Locally scalar representations of quivers in the category of Hilbert spaces were introduced by Kruglyak and Roiter [14]. They associate operators and their adjoint operators with arrows and classify them up to the unitary equivalence. They proved an analog of Gabriel's theorem. Their study is connected with representations of $*$-algebras generated by linearly related orthogonal projections, see for example, S. Kruglyak, V. Rabanovich and Y. Samoilenko [15].

We study duality theorem between reflection functors and the existence of indecomposable representations of quivers on infinite-dimensional Hilbert spaces. We associate bounded operators with arrows but we do not associate their adjoint operators simultaneously as in [14].

In particular if we consider a certain quiver Γ whose underlying undirected graph is the extended Dynkin diagram $\widetilde{D}_4$, then indecomposability of Hilbert representations of Γ is reduced to indecomposability of systems of four subspaces studied in [3] and [2]. We consider a complement of Gabriel's theorem for Hilbert representations and prove one direction: If the underlying undirected graph of a finite, connected quiver Γ contains one of extended Dynkin diagrams $\tilde{A}_n (n \geq 0), \widetilde{D}_n\ (n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$, then there exists an indecomposable representation of Γ on separable infinite-dimensional Hilbert spaces. The result does not depend on the choice of orientation. But we cannot prove the converse. In fact if the converse were true, then a long standing problem in [16] on transitive lattices of subspaces of Hilbert spaces would be settled.

Recall that we study relative position of n subspaces in a separable infinite-dimensional Hilbert space in [2]. See Y.P. Moskaleva and Y.S. Samoilenko [17] on a connection with $*$- algebras generated by projections. Let H be a Hilbert space and $E_1, \ldots E_n$ be n subspaces in H. Then we say that $S = (H; E_1, \ldots, E_n)$ is a system of n subspaces in H or an n-subspace system in H. A system S is called indecomposable if S cannot be decomposed into a non-trivial direct sum. For any bounded linear operator A on a Hilbert space K, we can associate a system S_A of four subspaces in $H = K \oplus K$ by

In particular on a finite-dimensional space, Jordan blocks correspond to indecomposable systems. Moreover on an infinite-dimensional Hilbert space, the above system S_A is indecomposable if and only if A is strongly irreducible, which is an infinite-dimensional analog of a Jordan block, see Jiang and Wang [18,19]. For example, a unilateral shift operator is a typical example of strongly irreducible operator. Such a system of four subspaces give an indecomposable Hilbert representation of a quiver with underlying undirected graph $\widetilde{D}_4$. We transform these representations and make up indecomposable Hilbert representations of other quivers. In purely algebraic case many such functors are introduced, in [7,9] and [20], for example. We follow some of their constructions. But we have not yet checked all such functors preserve indecomposability in infinite-dimensional Hilbert setting in general. We need to prove the indecomposability of the Hilbert representations in our concrete examples directly.

We have the following: Let Γ be a finite, connected quiver. If its underlying undirected graph contains one of extended Dynkin diagrams $\tilde{A}_n\ (n \geq 0), \widetilde{D}_n (n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$, then there exists an indecomposable representation of Γ on separable infinite-dimensional Hilbert spaces. There were two difficulties which did not appear in finite-dimensional case. Firstly we need to find indecomposable, infinite-dimensional representations of a certain class of Γ. We constructed them by studying the relative position of several subspaces along quivers, where vertices and arrows are represented by subspaces and natural inclusion maps. Secondly we need to change the orientation of the quiver preserving indecomposability. Here comes reflection functors. Being different from finite-dimensional case, we need to check the co-closedness condition at sources to show that indecomposability is preserved under reflection functors. We introduce a certain nice class, called positive-unitary diagonal Hilbert representations, such that co-closedness is easily checked and preserved under reflection functors at any source.

We believe that there exists an analogy between study of Hilbert representations of quivers and subfactor theory invented by V. Jones [21]. In fact Dynkin diagrams also appear in the classification of subfactors, see, for example, Goodman, de la Harpe and Jones [22], Evans and Kawahigashi [23]. But we have not yet understood the full relations between them.

There exists a close interplay between finite-dimensional representations of quivers and finite-dimensional representations of path algebras in purely algebraic sense. Any Hilbert representation of a quiver gives an operator algebra representation of the corresponding path algebra. Therefore we expect some relation between Hilbert representations of quivers and certain operator algebras associated with quivers. There exist some related works. See, for example, P. Muhly [24], D.W. Kribs and S.C. Power [25] and B. Solel [26]. But the relation is not so clear for us.

Throughout the paper a projection means an operator e with $e^2 = e = e^*$ and an idempotent means an operator p with $p^2 = p$. By a subspace we mean a closed subspace unless otherwise stated.

In purely algebraic setting, it is known that if a finite-dimensional algebra R is not of representation-finite type, then there exist indecomposable R-modules of infinite length as in M. Auslander [27]. Since we consider bounded operator representations on Hilbert spaces, the result in [27] cannot be applied directly.

A quiver $\Gamma = (V, E, s, r)$ is a quadruple consisting of the set V of vertices, the set E of arrows, and two maps $s, r : E \to V$, which associate with each arrow $\alpha \in E$ its support $s(\alpha)$ and range $r(\alpha)$. We sometimes denote by $\alpha : x \to y$ an arrow with $x = s(\alpha)$ and $y = r(\alpha)$. Thus a quiver is just a directed graph. We denote by $|\Gamma|$ the underlying undirected graph of a quiver Γ. A quiver Γ is said to be connected if $|\Gamma|$ is a connected graph. A quiver Γ is said to be finite if both V and E are finite sets.

Definition (4.2.1)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that (H, f) is a *Hilbert representation* of Γ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_\alpha)_{\alpha \in E}$ is a family of bounded linear operators $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$.

Definition (4.2.2)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (H, f) and (K, g) be Hilbert representations of Γ. A *homomorphism* $T : (H, f) \to (K, g)$ is a family $T = (T_v)_{v \in V}$ of bounded operators $T_v : H_v \to K_v$ satisfying, for any arrow $\alpha \in E$

The composition $T \circ S$ of homomorphisms T and S is defined by $(T \circ S)_v = T_v \circ S_v$ for $v \in V$. Thus we have obtained a category $H\ \mathrm{Rep}(\Gamma)$ of Hilbert representations of Γ. We denote by $Hom\big((H, f), (K, g)\big)$ the set of homomorphisms $T : (H, f) \to (K, g)$. We denote by $End(H, f) \coloneqq Hom\big((H, f), (H, f)\big)$ the set of endomorphisms. We denote by

the set of idempotents in $End(H, f)$. Let $0 = (0_v)_{v \in V}$ be the family of zero endomorphisms 0_v and $I = (I_v)_{v \in V}$ be the family of identity endomorphisms I_v. The both 0 and I are in $Idem(H, f)$.

Let $\Gamma = (V, E, s, r)$ be a finite quiver and $(H, f), (K, g)$ be Hilbert representations of Γ. We say that (H, f) and (K, g) are *isomorphic*, denoted by $(H, f) \simeq (K, g)$, if there exists an isomorphism $\varphi : (H, f) \to (K, g)$, that is, there exists a family $\varphi = (\varphi_v)_{v\in V}$ of bounded invertible operators $\varphi_v \in B(H_v, K_v)$ such that, for any arrow $\alpha \in E$,

We say that (H, f) is a finite-dimensional representation if H_v is finite-dimensional for all $v \in V$. And (H, f) is an infinite-dimensional representation if H_v is infinite-dimensional for some $v \in V$.

In this section we shall introduce a notion of indecomposable representation, that is, a representation which cannot be decomposed into a direct sum of smaller representations anymore.

Definition (4.2.3)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (K, g) and (K', g') be Hilbert representations of Γ. Define the direct sum $(H, f) = (K, g) \oplus (K', g')$ by

$H_v = K$

We say that a Hilbert representation (H, f) is zero, denoted by $(H, f) = 0$, if $H_v = 0$ for any $v \in V$.

Definition (4.2.4)[1]: A Hilbert representation (H, f) of Γ is called *decomposable* if (H, f) is isomorphic to a direct sum of two non-zero Hilbert representations. A non-zero Hilbert representation (H, f) of Γ is said to be *indecomposable* if it is not decomposable, that is, if $(H, f) \cong (K, g) \oplus (K', g')$ then $(K, g) \cong 0$ or $(K', g') \cong 0$.

We start with an easy fact. Let H be a Hilbert space and K_1, K_2 be closed subspaces of H. Assume that $K_1 \cap K_2 = 0$ and $H = K_1 + K_2$. But we do not assume that K_1 and K_2 are orthogonal. Let $T : H \to H$ be a bounded operator with $TK_i \subset K_i$ for $i = 1, 2$. Define $S_i = T|_{K_i}: K_i \to K_i$. Consider the (orthogonal) direct sum $K_1 \oplus K_2$ and the bounded operator $S_1 \oplus S_2$ on $K_1 \oplus K_2$. Define a bounded invertible operator $\varphi : H \to K_1 \oplus K_2$ by $\varphi(h) = (h_1, h_2)$ for $h = h_1 + h_2$ with $h_i \in K_i$, as in the proof of [2] Then we have $T = \varphi^{-1} \circ (S_1 \oplus S_2) \circ \varphi$.

The following proposition is used frequently to show the indecomposability in concrete examples.

Proposition (4.2.5)[1]: *Let (H, f) be a Hilbert representation of a quiver Γ. Then the following conditions are equivalent*:

(a) (H, f) *is indecomposable.*

(b) $Idem(H, f) = \{0, I\}$.

Proof. $\neg(a) \Rightarrow \neg(b)$: Assume that (H, f) is not indecomposable. Then there exist non-zero representations (K, g) and (K', g') of Γ, such that $(H, f) \cong (K, g) \oplus (K', g')$. For any $x \in V$, define the projection $Q_x \in B(K_x \oplus K'_x)$ of $K_x \oplus K'_x$ onto K_x. Then $Q := (Q_x)_{x \in V}$ is in $End(K \oplus K', g \oplus g')$, because

for any $\alpha \in E$. Since there exist $v, w \in V$ such that $K_v \neq 0$ and $K'_w \neq 0$, we have $Q_v \neq 0$ and $Q_w \neq I$. Thus $Q \neq 0$ and $Q \neq I$. Let $\varphi = (\varphi_x)_{x \in V} : (H, f) \to (K, g) \oplus (K', g')$ be an isomorphism. Put $P_x = (\varphi_x)^{-1} Q_x \varphi_x$ for $x \in V$ and $P := (P_x)_{x \in V} \in Idem(H, f)$. Then $P \neq 0$ and $P \neq I$.

$\neg(b) \Rightarrow \neg(a)$: Assume that there exists $P \in Idem(H, f)$ with $P \neq 0$ and $P \neq I$.

Thus there exist $v \in V$ and $w \in V$ such that $P_v \neq 0_v, P_w \neq I_w$. For any $x \in V$, define closed subspaces

Then $K := (K_x)_x \neq 0, K' := (K'_x)_x \neq 0$ and $H \neq K \oplus K'$. For any $\alpha \in E$, let $x = s(\alpha)$ and $y = r(\alpha)$. Since $f_\alpha P_x = P_y f_\alpha$, we have $f_\alpha K_x \subset K_y$. Similarly, $f_\alpha(I - P_x) = (I - P_y) f_\alpha$ implies that $f_\alpha K'_x \subset K'_y$. We can define $g_\alpha = f_\alpha|_{K_x}: K_x \to K_y$ and $g'_\alpha = f_\alpha|_{K'_x}: K'_x \to K'_y$. Put $g = (g_\alpha)_\alpha$ and $g' = (g'_\alpha)_\alpha$. Then (K, g) and (K', g') are representations of Γ. Define $\varphi_x : H_x \to K_x \oplus K'_x$ by $\varphi_x(\xi) = (P_x \xi, (I - P_x)\xi)$ for $\xi \in H_x$. Then $\varphi := (\varphi_x)_{x \in V} : (H, f) \to (K, g) \oplus (K', g')$ is an isomorphism. Since $K := (K_x)_x \neq 0$ and $K' := (K'_x)_x \neq 0, (H, f)$ is decomposable.

Remark (4.2.6)[1]: (a) The proof of the above Proposition (4.2.5) shows that (H, f) is decomposable if and only if there exist non-zero families $K = (K_x)_{x \in V}$ and $K = (K'_x)_{x \in V}$ of closed subspaces K_x and K'_x of H_x with $K_x \cap K'_x = 0$ and $K_x + K'_x = H_x$ such that $f_\alpha K_x \subset K_y$ and $f_\alpha K'_x \subset K'_y$ for any arrow $\alpha : x \to y$.

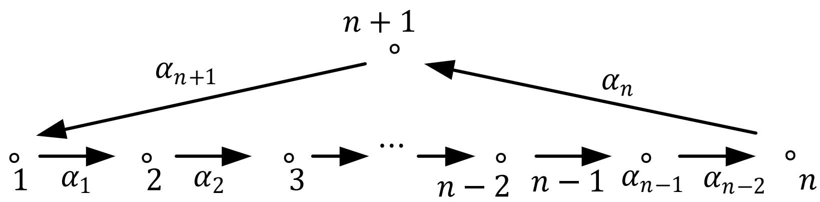
(b) In the statement of the above Proposition (4.2.5), we cannot replace the set $Idem(H, f)$ of idempotents of endomorphisms by the set of projections of endomorphisms. For example, let $H_0 = \mathbb{C}^2$. Fix an angle θ with $0 < \theta < \pi/2$. Put $H_1 = \mathbb{C}(1, 0)$ and $H_2 = \mathbb{C}(\cos\theta, \sin\theta)$. Then the system $(H_0; H_1, H_2)$ of two subspaces is isomorphic to

Hence $(H_0; H_1, H_2)$ is decomposable. Now consider the following quiver Γ:

Define a Hilbert representation (H, f) of Γ by $H = (H_i)_{i=0,1,2}$ and canonical inclusion maps $f_i = f_{\alpha_i} : H_i \to H_0$ for $i = 1, 2$. Then the Hilbert representation (H, f) is also decomposable, see Example (4.2.9) below in this Section. But for any $P = (P_i)_{i=0,1,2} \in End(H, f)$, if $P_i \in B(H_i)$ is a projection for $i = 0, 1, 2$, then $P = 0$ or $P = I$. In fact $P_0(H_i) \subset H_i$ for $i = 1, 2$. Let $e_1 \in B(H_0)$ and $e_2 \in B(H_0)$ be the projections of H_0 onto H_1 and H_2. Then the C^*-algebra $C^*(\{e_1, e_2\})$ generated by e_1 and e_2 is exactly $B(H_0) \cong M_2(\mathbb{C})$. Since P_0 commutes with e_1 and $e_2, P_0 = 0$ or $P_0 = I$. Because $P_i = P_0|_{H_i}, P_i = 0$ or $P_i = I$ simultaneously.

Example (4.2.7)[1]: Let Γ be a loop with one vertex 1 and one arrow $\alpha : 1 \to 1$, that is, the underlying undirected graph is an extended Dynkin diagram $\tilde{A}_0$. Let $H_1 = \ell^2(\mathbb{N})$ and $f_\alpha = S : H_1 \to H_1$ be a unilateral shift. Then the Hilbert representation (H, f) is infinite-dimensional and indecomposable. In fact, any $T \in Idem(H, f)$ can be identified with $T \in B(\ell^2(\mathbb{N}))$ with $T^2 = T$ and $TS = ST$. Since T commutes with a unilateral shift S, the operator T is a lower triangular Toeplitz matrix. Since T is an idempotent, $T = 0$ or $T = I$. Thus (H, f) is indecomposable. Replacing S by $S + \lambda I$ for $\lambda \in \mathbb{C}$, we obtain a family of infinite-dimensional, indecomposable Hilbert representations (H^λ, f^λ) of Γ. Since (H^λ, f^λ) and (H^μ, f^μ) are isomorphic if and only if $S + \lambda I$ and $S + \mu I$ is similar, we have uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ.

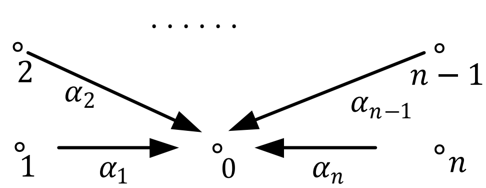
Example (4.2.8)[1]: Let $\Gamma = (V, E, s, r)$ be a quiver whose underlying undirected graph is an extended Dynkin diagram $\tilde{A}_n, (n \geq 1)$. Then there exist uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ. For example, consider



Define a Hilbert representation (H, f) of Γ by $H_1 = H_2 = \cdots = H_{n+1}\ell^2(\mathbb{N})$, $f_{\alpha_2} = f_{\alpha_3} = \cdots = f_{\alpha_{n+1}} = I$ and $f_{\alpha_1} = S$, the unilateral shift. Let $P = (P_k)_{k\in V} \in Idem(H, f)$. Then

Since P_1 is an idempotent and $SP_1 = P_1S$, we have $P_1 = 0$ or $P_1 = I$. This implies $P = 0$ or $P = I$. Therefore (H, f) is indecomposable. Replacing S by $S + \lambda I$ for $\lambda \in \mathbb{C}$, we obtain uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ.

Example (4.2.9)[1]: Let L be a Hilbert space and $E_1, \ldots E_n$ be n subspaces in L. Then we say that $S = (L; E_1, \ldots, E_n)$ is a system of n subspaces in L. A system S is called indecomposable if S cannot be decomposed into a non-trivial direct sum, see [2]. Consider the following quiver $\Gamma_n = (V, E, s, r)$



Define a Hilbert representation (H, f) of Γ_n by $H_k \coloneqq E_k (k = 1, \ldots, n)$, $H_0 \coloneqq L$ and $f_k = f_{\alpha_k} : H_k = E_k \to H_0 = L$ be the inclusion map. Then the system S of n subspaces is indecomposable if and only if the corresponding Hilbert representation (H, f) of Γ is indecomposable. In fact, assume that S is indecomposable. Let $P = (P_k)_{k\in V} \in Idem(H, f)$. Then $f_kP_k = P_0f_k$. This implies $P_0(H_k) \subset H_k$ for $k = 1, \ldots, n$. Since P_0 is an idempotent and S is indecomposable, $P_0 = 0$ or $P_0 = I$ by [2]. Since $f_kP_k = P_0f_k, P_k = 0$ or $P_k = I$ simultaneously. Thus $P = 0$ or $P = I$, that is, (H, f) is indecomposable.

Conversely assume that (H,f) is indecomposable. Let $R \in B(L)$ be an idempotent with $R(E_k) \subset E_k$ for $k = 1,\dots,n$. Define $P = (P_k)_{k\in V}$ by $P_0 = R$ and $P_k = P_0|_{H_k}$. Then $P \in Idem(H,f)$. Therefore $P = 0$ or $P = I$. Thus $R = O$ or $R = I$. Hence S is indecomposable.

We can also show that two systems S and S' of n subspaces are isomorphic if and only if the corresponding Hilbert representations (H,f) and (H',f') of Γ are isomorphic.

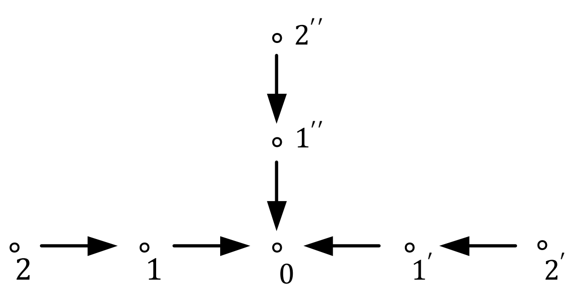
Since there exist uncountably many, indecomposable systems of fours subspaces in an infinite dimensional Hilbert space as in [2], there exist uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ_4 whose underlying undirected graph is the extended Dynkin diagram $\widetilde{D}_4$.

In particular, let $K = \ell^2(\mathbb{N})$ and $A \in B(K)$ be a strongly irreducible operator studied in [18,19] for example, a unilateral shift. Define

$H_3 = \{(x$

Let $f_k = f_{\alpha_k}: H_k \to H_0$ be the inclusion map for $k = 1,2,3,4$. Put $H^{(A)} = (H_v)_{v\in V}$ and $f^{(A)} = (f_\alpha)_{\alpha\in E}$. Then $\left(H^{(A)}, f^{(A)}\right)$ is an infinite-dimensional, indecomposable Hilbert representation of $\widetilde{D}_4$. Moreover let A and B be strongly irreducible operators on $\ell^2(\mathbb{N})$. Then two indecomposable Hilbert representations $\left(H^{(A)}, f^{(A)}\right)$ and $\left(H^{(B)}, f^{(B)}\right)$ of $\widetilde{D}_4$ are isomorphic if and only if two operators A and B are similar.

Example (4.2.10)[1]: Consider the following quiver $\Gamma = (V,E,s,r)$



Then underlying undirected graph is an extended Dynkin diagram $\tilde{E}_6$. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V},$ $(f_\alpha)_{\alpha \in E})$ of Γ as follows:

Put

$$H_0 = F$$

$$H_{1''} = \{(x, x, x)$$

Then $H_{1''}$ is a closed subspace of H_0. In fact, let

$$(x_n,$$

converges to $(a, b, c) \in H_0$. Then $x_n \to c, y_n \to a - c$ and $c + S(a - c) = b$. Define $x = c$ and $y = a - c$. Then $(a, b, c) = (x, x, x) + (y, Sy, 0) \in H_{1''}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. We shall show that the Hilbert representation (H, f) is indecomposable. Take $T = (T_v)_{v \in V} \in Idem(H, f)$. Since $T \in End(H, f)$, for any $v \in \{1, 2, 1', 2', 1'', 2''\}$ and any $x \in H_v$, we have $T_0 x = T_v x$. In particular, $T_0 H_v \subset H_v$. Since $H_1 \cap H_{1'} = K \oplus 0 \oplus 0, H_{2'} = 0 \oplus K \oplus 0$ and $H_2 = 0 \oplus 0 \oplus K$, T_0 preserves these subspaces. Hence T_0 is a block diagonal operator with

$$T_0 = P \oplus Q \oplus R \in B(K \oplus K \oplus K).$$

Since $T_0(H_{2''}) \subset H_{2''}$, for any $x \in K$,

for some $y \in K$. Therefore $P = Q = R$ and $T_0 = P \oplus P \oplus P$. Moreover P is an idempotent, because so is T_0. Since T_0 preserves $H_{1'} \cap H_{1''} = \{(y, Sy, 0) \in K^3 | \ y \in K\}$, for any $y \in K$, there exists $z \in K$ such that

Therefore $PSy = Sz = SPy$ for any $y \in K$, i.e., $PS = SP$. Since P is an idempotent, $P = 0$ or $P = I$. This means that $T_0 = 0$ or $T_0 = I$. Because $T_0 x = T_v x$ for any $x \in H_v$ for $v \in \{1, 2, 1', 2', 1'', 2''\}$, we have $T_v = 0$ or $T_v = I$ simultaneously. Thus $T = 0$ or $T = I$, that is $Idem(H, f) = \{0, I\}$. Therefore (H, f) is indecomposable.

Example (4.2.11)[1]: We have a different kind of infinite-dimensional, indecomposable Hilbert representation $(L, g) = ((L_v)_{v \in V}, (g_\alpha)_{\alpha \in E})$ of the same Γ in Example (4.2.10) as follows: Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. Define

$$L_0 = K \oplus K \oplus$$

$$L_{1'}$$

$$L_1$$

For any arrow $\alpha \in E$, let $g_\alpha : L_{s(\alpha)} \to L_{r(\alpha)}$ be the canonical inclusion map. We can similarly prove that the Hilbert representation (L, g) is indecomposable.

We shall show that two Hilbert representations in Examples (4.2.10) and (4.2.11) are not isomorphic. In fact, on the contrary, suppose that there were an isomorphism $\varphi = (\varphi_v)_{v \in V} : (H, f) \to (L, g)$. Since any arrow is represented by the canonical inclusion, $\varphi_0 : H_0 \to L_0$ satisfies that $\varphi_v = \varphi_0|_{H_v} : H_v \to L_v$. This implies that $\varphi_0(H_v) \subset L_v$ for any $v \in V$. Since $\varphi_0(H_{1'}) \subset L_{1'}$ and $\varphi_0(H_1) \subset L_1, \varphi_0$ has a form such that

Since $\varphi_0(H_2) \subset L_2$, for any $z \in K$ there exists $y \in K$ such that $(0, Dz, Ez) = (0, y, Sy)$. Hence $Ez = Sy = SDz$, so that $E = SD$. Then $\mathrm{Im}\varphi_0 \subset K \oplus K \oplus \mathrm{Im}\, S \neq L_0$. This contradicts the assumption that $\varphi_0 : H_0 \to L_0$ is onto. Therefore Hilbert representations (H, f) and (L, g) of Γ are not isomorphic.

Reflection functors are crucially used in the proof of the classification of finite-dimensional, indecomposable representations of tame quivers. In fact many indecomposable representations of tame quivers can be reconstructed by iterating reflection functors on simple indecomposable representations. We cannot expect such a best situation in infinite-dimensional Hilbert representations. But reflection functors are still useful to show that some property of representations of quivers on infinite-dimensional Hilbert spaces does not depend on the choice of orientations and does depend on the fact underlying undirected graphs are (extended) Dynkin diagrams or not.

Let $\Gamma = (V, E, s, r)$ be a finite quiver. A vertex $v \in V$ is called a *sink* if $v \neq s(\alpha)$ for any $\alpha \in E$. Put $E^v = \{\alpha \in E \mid r(\alpha) = v\}$ .We denote by $\bar{E}$ the set of all formally reversed new arrows $\bar{\alpha}$ for $\alpha \in E$. Thus if $\alpha : x \to y$ is an arrow, then $\bar{\alpha} : x \leftarrow y$.

Definition (4.2.12)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we construct a new quiver $\sigma_v^+(\Gamma) = (\sigma_v^+(V), \sigma_v^+(E), s, r)$ as follows: All the arrows of Γ having v as range are reversed and all the other arrows remain unchanged. More precisely,

where $\overline{E^v} = \{\alpha \mid \alpha \in E^v\}$.

Definition (4.2.13)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a sink $v \in V$, we define a *reflection functor* at v

between the categories of Hilbert representations of Γ and $\sigma_v^+(\Gamma)$ as follows: For a Hilbert representation (H, f) of Γ, we shall define a Hilbert representation $(K, g) = \Phi_v^+(H, f)$ of $\sigma_v^+(\Gamma)$. Let

be a bounded linear operator defined by

Define

K_v

Consider also the canonical inclusion map $i_v : K_v \to \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

For $\beta \in E^v$, let

be the canonical projection. Then define

g

that is $g_{\overline{\beta}}((x_\alpha)_{\alpha\in E^v}) = x_\beta$.

For $\beta \notin E^v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H, f) \to (H', f')$, we shall define a homomorphism

$S = (S_u)$

If $u = v$, a bounded operator $S_v : K_v \to K'_v$ is given by

It is easy to see that S_v is well-defined and we have the following commutative diagram:

For other $u \in V$ with $u \neq v$, we put

We shall consider a dual of the above construction. A vertex $v \in V$ is called a *source* if $v \neq r(\alpha)$ for any $\alpha \in E$. Put $E_v = \{\alpha \in E | s(\alpha) = v\}$.

Definition (4.2.14)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we construct a new quiver $\sigma_v^-(\Gamma) = (\sigma_v^-(V), \sigma_v^-(E), s, r)$ as follows: All the arrows of Γ having v as source are reversed and all the other arrows remain unchanged. More precisely,

where $\overline{E_v} = \{\bar{\alpha} \mid \alpha \in E_v\}$.

In order to define a reflection functor at a source, it is convenient to consider the orthogonal complement $M^\perp$ of a closed subspace M of a Hilbert space H instead of the quotient H/M. Define an isomorphism $f : M^\perp \to H/M$ by $f(y) = [y] = y + M$ for $y \in M^\perp \subset H$. Then the inverse $f^{-1} : H/M \to M^\perp$ is given by $f^{-1}([x]) = P_M^\perp(x)$ for

$x \in H$, where $P_M^{\perp}$ is the projection of H onto $M^{\perp}$. We shall use the following elementary fact frequently:

Lemma (4.2.15)[1]: *Let K and L be Hilbert spaces, $M \subset K$ and $N \subset L$ be closed subspaces. Let $A : K \to L$ be a bounded operator. Assume that $A(M) \subset N$. Let $\tilde{A} : K/M \to L/N$ be the induced map such that $\tilde{A}([x]) = [Ax]$ for $x \in K$. Identifying K/M and L/N with $M^{\perp}$ and $N^{\perp}$, $\tilde{A}$ is identified with the bounded operator $S : M^{\perp} \to N^{\perp}$ such that $S(x) = P_N^{\perp}(Ax)$. Then $S = (A^*|_{N^{\perp}})^*$.*

Proof. Consider $A^* : L \to K$. Since $A(M) \subset N$, we have $A^*(N^{\perp}) \subset M^{\perp}$. Hence the restriction $A^*|_{N^{\perp}} : N^{\perp} \to M^{\perp}$ has the adjoint

For any $m \in M^{\perp}$ and $n \in N^{\perp}$

$((A^*|_{N^{\perp}})^* m$

Corollary (4.2.16)[284]: *upon considering Lemma (4.2.15) and letting $A : K \to L$ be a bounded self-adjoint operator. We assume that $A^*(M) \subset N$. Let $\tilde{A}^* : K/M \to L/N$ be the induced map that $\tilde{A}^*([x]) = [A^* x]$ for $x \in K$. $\tilde{A}^*$ is identified with the bounded self-adjoint operator $S^* : M^{\perp} \to N^{\perp}$ such that $S^*(x) = P_N^{\perp}(A^* x)$. Then $S^* = (A^*|_{N^{\perp}})$.*

Proof. Consider $A : L \to K$. Since $A^*(M) \subset N$, then $A(N^{\perp}) \subset M^{\perp}$. Hence $A|_{N^{\perp}} : N^{\perp} \to M^{\perp}$. has the *self*-adjoint

For any $m \in M^{\perp}$ and $n \in N^{\perp}$

$((A^*|_{N^{\perp}}) m$

Definition (4.2.17)[1]: *(Reflection functor Φ_v^-).* Let $\Gamma = (V, E, s, r)$ be a finite quiver. For a source $v \in V$, we define a *reflection functor* at v

between the categories of Hilbert representations of Γ and $\sigma_v^-(\Gamma)$ as follows: For a Hilbert representation (H,f) of Γ, we shall define a Hilbert representation $(K,g) = \Phi_v^-(H,f)$ of $\sigma_v^-(\Gamma)$. Let

be a bounded linear operator defined by

Define

where $\hat{h}_v^* : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \to H_v$ is given $\hat{h}_v^*\big((x_\alpha)_{\alpha \in E_v}\big) = \sum f_\alpha^*(x_\alpha)$. For $u \in V$ with $u \neq v$, put $K_u = H_u$.

Let $Q_v : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \to K_v$ be the canonical projection. For $\beta \in E_v$, let

be the canonical inclusion. Define

g

For $\beta \notin E_v$, let $g_\beta = f_\beta$.

For a homomorphism $T : (H,f) \to (H',f')$, we shall define a homomorphism

$S = (S_u)$

recalling the above Lemma (4.2.14). For $u = v$, a bounded operator $S_v : K_v \to K_v'$ is given by

where Q_v': $\bigoplus_{\alpha \in E_v} H'_{r(\alpha)} \to K_v'$ be the canonical projection.

We have the following commutative diagram:

For other $u \in V$ with $u \neq v$, we put

We shall explain a relation between two (covariant) functors Φ_v^+ and Φ_v^-. We need to introduce another (contravariant) functor Φ^* in the first place. Let $\Gamma = (V, E, s, r)$ be a finite quiver. We define the opposite quiver $\Gamma = (V\ , E, s, r)$ by reversing all the arrows, that is,

Definition (4.2.18)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver and $\bar{\Gamma} = (\bar{V}, \bar{E}, s, r)$ its opposite quiver. We introduce a contravariant functor

between the categories of Hilbert representations of Γ and Γ as follows: For a Hilbert representation (H, f) of Γ, we shall define a Hilbert representation $(K, g) = \Phi^*(H, f)$ of Γ by

For a homomorphism $T : (H, f) \rightarrow (H', f')$, we shall define a homomorphism

$S = (S_u$

by bounded operators $S_u : K'_u = H'_u \rightarrow K_u = H_u$ given by $S_u = T_u^*$.

Proposition (4.2.19)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver. If $v \in V$ is a source of Γ, then v is a sink of $\bar{\Gamma}$, $\sigma_v^-(\Gamma) = \sigma_v^+(\Gamma)$ and we have the following*:

(i) *For a Hilbert representation (H, f) of Γ,*

(ii) *For a homomorphism $T : (H, f) \rightarrow (H', f')$,*

Proof. (i) It is enough to consider around a source v. For each $\alpha \in E_v$ with $\alpha : v \to u = r(\alpha)$, a bounded operator $f_\alpha : H_v \to H_u$ is assigned in (H, f). Taking Φ^*, we have $\Phi^*(H_u) = H_u$ and $\Phi^*(f_\alpha) = f_\alpha^* : H_u \to H_v$ in $\Phi^*(H, f)$. Let

be a bounded operator given by

Define

Then $\Phi_v^+\big(\Phi^*(H_v)\big) = W_v$ and $\Phi_v^+\big(\Phi^*(H_u)\big) = H_u$ in $\Phi^+\big(\Phi^*(H, f)\big)$. Consider the canonical inclusion map $i_v : W_v \to \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$. For $\beta \in E_v$, let

be the canonical projection. Then $\Phi_v^+\left(\Phi^*(f_\beta)\right) = P_\beta \circ i_v$. Finally take Φ^* again. Since $h_v^* : H_v \to \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by

we have

$\Phi^*\left(\Phi_v^+(\right.$

Moreover $i_v^* = Q_v : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \to W_v$ is the canonical projection. For $\beta \in E_v$, we have

Therefore

$\Phi^*\left(\Phi\right.$

(ii) If $u \neq v$, then

If $u = v$, then, apply Lemma (4.2.3) by putting that $K = \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$, $L = \bigoplus_{\alpha \in E_v} H'_{r(\alpha)}$, M is the closure of $\left\{ \left(f_\alpha(x) \right)_{\alpha \in E_v} \in K \mid x \in H_v \right\}$ in K, N is the closure of $\left\{ \left(f'_\alpha(x) \right)_{\alpha \in E_v} \in L \mid x \in H'_v \right\}$ in L and $A : K \to L$ with $A\left((y_\alpha)_{\alpha \in E_v} \right) = \left(T_{r(\alpha)} y_\alpha \right)_{\alpha \in E_v}$. Then

$$\left(\Phi^* \left(\Phi_v^+ \left(\Phi^*(T) \right) \right) \right)_v = \left(\Phi_v^-(T) \right)_v.$$

Proposition (4.2.20)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver. If $v \in V$ is a sink of Γ, then v is a source of $\bar{\Gamma}$, $\sigma_v^+(\Gamma) = \sigma_v^-(\bar{\Gamma})$ and we have the following*:

(a) *For a Hilbert representation (H, f) of Γ,*

(b) *For a homomorphism $T : (H, f) \to (H', f')$,*

$$\Phi_v^+(T) = \Phi^* \left(\Phi_v^- \left(\Phi^*(T) \right) \right).$$

We shall show a certain duality between reflection functors, which is analogous to Takesaki duality in operator algebras. Bernstein, Gelfand and Ponomarev [5] introduced reflection functors and Coxeter functors and clarify a relation with the Coxeter–Weyl group and Dynkin diagrams in the case of finite-dimensional representations of quivers. In the case of infinite-dimensional Hilbert representations, duality theorem between reflection functors does not hold as in the purely algebraic setting. We need to modify and assume a certain closedness condition at a sink or a source.

Definition (4.2.21)[1]: Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Recall that $E^v = \{\alpha \mid r(\alpha) = v\}$. We say that a Hilbert representation (H, f) of Γ is *closed* at v if $\sum_{\alpha \in E^v} \operatorname{Im} f_\alpha \subset H_v$ is a closed subspace. We say that (H, f) is *full* at v if $\sum_{\alpha \in E^v} \operatorname{Im} f_\alpha = H_v$.

Remark (4.2.22)[1]: Recall that a bounded operator $h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \to H_v$ is given by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$. Then a Hilbert representation (H, f) of Γ is *closed* at v if and only if $\mathrm{Im}h_v$ is closed. A Hilbert representation (H,f) is *full* at v if and only if h_v is onto.

Definition (4.2.23)[1]: Let $\Gamma=(V,E,s,r)$ be a finite quiver and $v\in V$ a source. Recall that $E_v=\{\alpha\mid s(\alpha)=v\}$. We _ say that a Hilbert representation (H,f) of Γ is *co-closed* at v if $\sum_{\alpha\in E_v}\mathrm{Im}\, f_\alpha^*\subset H_v$ is a closed subspace. We say that (H,f) is *co-full* at v if $\sum_{\alpha\in E_v}\mathrm{Im}\, f_\alpha^*=H_v$.

Remark (4.2.24)[1]: Recall that a bounded operator $\hat{h}_v: H_v\to\bigoplus_{\alpha\in E_v}H_{r(\alpha)}$ is given by $\hat{h}_v(x)=\left(f_\alpha(x)\right)_{\alpha\in E_v}$ for $x\in H_v$. Then a Hilbert representation (H,f) of Γ is co-closed at v if and only if $\mathrm{Im}\,\hat{h}_v^*$ is closed. A Hilbert representation (H,f) is co-full at v if and only if $\hat{h}_v^*$ is onto if and only if $\mathrm{Im}\,\hat{h}_v$ is closed and $\bigcap_{\alpha\in E_v}\mathrm{Ker}\, f_\alpha=0$. In fact the latter condition is equivalent to $\left(\mathrm{Im}\,\hat{h}_v^*\right)^\perp=\mathrm{Ker}\,\hat{h}_v=0$. We also see that (H,f) is co-closed at v if and only if $\Phi_v^*(H,f)$ is closed at v. And (H,f) is co-full at v if and only if $\Phi_v^*(H,f)$ is full at v.

In order to prove a duality theorem, we need to prepare a lemma.

Lemma (4.2.25)[1]: *Let H and K be Hilbert spaces and $T: H\to K$ be a bounded operator. Let $T=U|T|$ be its polar decomposition and U a partial isometry with supp $U=\mathrm{Im}|T|$ and $\mathrm{Im}U=\mathrm{Im}T$. Suppose that $\mathrm{Im}\,T$ is closed. Then we have the following*:

(i) $\mathrm{Im}|T|=\mathrm{Im}\,T^*$ *is a closed subspace of H.*

(ii) *Under the orthogonal decomposition*

the restriction $|T|\big|_{\mathrm{Im}\,|T|}:\mathrm{Im}\,|T|\to\mathrm{Im}|T|$ is a bounded invertible operator.

(iii) *Let $S=\left(|T|\big|_{\mathrm{Im}\,|T|}\right)^{-1}$ be its inverse. Define a bounded operator $B: K\to\mathrm{Im}T^*$ by $Bx=SU^*x$ for $x\in K$. Let $Q: H\to\mathrm{Im}T^*$ be the canonical projection. Then $BT=Q$. Moreover $B|_{\mathrm{Im}\,T}:\mathrm{Im}\,T\to\mathrm{Im}\,T^*$ is a bounded invertible operator.*

Proof. (i) Since $\mathrm{Im}\,T$ is closed, $\mathrm{Im}T^*$ is also closed. Since $U(|T|x)=Tx$ by definition of U and $\mathrm{Im}\,T$ is closed, $\mathrm{Im}\,|T|$ is closed.

(ii) Since Ker $|T|^{\perp} = \mathrm{Im}|T|, |T||_{\mathrm{Im}\,|T|}$ is one to one. Since $|T|(H) = |T|(\mathrm{Im}\,|T|)$ is closed, $|T||_{\mathrm{Im}\,|T|}$ is onto. Hence $|T||_{\mathrm{Im}\,|T|}$ is bounded invertible.

(iii) For any $x = x_1 + x_2 \in H$ with $x_1 \in \mathrm{Im}|T| = \mathrm{Im}\, T^*$ and $x_2 \in \mathrm{Ker}\,|T|$,

It is clear that $B|_{\mathrm{Im}\,T}$ is a bounded invertible operator.

Theorem (4.2.26)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Assume that a Hilbert representation (H, f) of Γ is closed at v. Let $h_v : \bigoplus_{\alpha\in E^v} H_{s(\alpha)} \to H_v$ be a bounded operator defined by $h_v((x_\alpha)_{\alpha\in E^v}) = \sum_{\alpha\in E^v} f_\alpha(x_\alpha)$. Define a Hilbert representation $(\widetilde{H}, \tilde{f})$ of Γ by $\widetilde{H}_v = (\mathrm{Im}\, h_v)^{\perp} \subset H_v, \widetilde{H}_u = 0$ for $u \neq v$ and $\tilde{f} = 0$. Then we have*

Proof. Let $(H^+, f^+) = \Phi_v^+(H, f)$ and $(H^{+-}, f^{+-}) = \Phi_v^-(\Phi_v^+(H, f))$. Then $H_v^+ = \mathrm{Ker}\, h_v = \{(x_\alpha)_{\alpha\in E^v} \in \bigoplus_{\alpha\in E^v} H_{s(\alpha)} \mid \sum_{\alpha\in E^v} f_\alpha(x_\alpha) = 0\}$, and $H_u^+ = H_u$ for $u \neq v$. We have $f_{\bar{\beta}}^+((x_\alpha)_{\alpha\in E^v}) = x_\beta$ for $\beta \in E^v$, and $f_\beta^+ = f_\beta$ for $\beta \notin E^v$.

Let $\hat{h}_v : H_v^+ \to \bigoplus_{\alpha\in E^v} H_{s(\alpha)}$ be a bounded operator given by

$\hat{h}_v((x$

Hence $\hat{h}_v$ is the canonical embedding. Since (H, f) is closed at v, $\mathrm{Im} h_v$ and $\mathrm{Im}\, h_v^*$ are closed subspaces. Therefore

For any other $u \in V$ with $u \neq v, H_u^{+-} = H_u$. Let $Q_v : \bigoplus_{\alpha\in E^v} H_{s(\alpha)} \to H_v^{+-}$ be the canonical projection. For $\beta \in E^v$, let

be the canonical inclusion. Then $f_\beta^{+-} : H_{s(\beta)} \to H_v^{+-}$ is given by $f_\beta^{+-} = Q_v \circ j_\beta$. For other $\beta \notin E^v$, we have $f_\beta^{+-} = f_\beta$.

We shall define an isomorphism

Apply Lemma (4.2.25) by putting $T = h_v, H = \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ and $K = H^v$. Consider the polar decomposition $h_v = U|h_v|$. Put $S = \left(|h_v|\big|_{\mathrm{Im}\,|h_v|}\right)^{-1}$. Define a bounded operator $B : H_v \to \mathrm{Im}\, h_v^*$ by $B = SU^*$. Then Bh_v is the canonical projection Q_v of H_v onto $\mathrm{Im} h_v^*$. We define

$\varphi_v : H$

by $\varphi_v(x, y) = \left(B|_{\mathrm{Im}\, h_v} x, y\right)$ for $x \in \mathrm{Im}\, h_v$ and $y \in (\mathrm{Im}\, h_v)^{\perp}$. By Lemma (4.2.23) (ii), φ_v is a bounded invertible operator. For $u \in V$ with $u \neq v$, put $\varphi_u : H_u \to H_u \oplus 0$ by $\varphi_u(x) = (x, 0)$ for $x \in H_u$. For any $\beta \in E^v$ and $x \in H_{s(\beta)}$,

$\varphi_v \circ f_\beta(x)$

On the other hand,

$(f_\beta^{+-} \oplus 0)$

For other $\beta \notin E^v$, we have

$\varphi_{r(\beta)} \circ$

Hence $\varphi : (H, f) \to \Phi_v^-\left(\Phi_v^+(H, f)\right) \oplus \left(\widetilde{H}, \tilde{f}\right)$ is an isomorphism.

If we do not assume that a Hilbert representation (H, f) of Γ is closed at v, then the above Theorem (4.2.26) does not hold in general. In fact, consider the following quiver $\Gamma = (V, E, s, r)$:

Let $K = \ell^2(\mathbb{N})$ with the canonical basis $(e_n)_{n \in \mathbb{N}}$. Define a Hilbert representation (H, f) of Γ by $H_0 = K \oplus K, H_1 = K \oplus 0$ and H_2 is the closed subspace of H_0 spanned by $\left\{\left(\cos\frac{\pi}{n+2} e_n, \sin\frac{\pi}{n+2} e_n\right) \in K \oplus K \mid n \in \mathbb{N}\right\}$. Then $H_1 \cap H_2 = 0$ and $H_1 + H_2$ is a dense subspace of H_0 but not closed in H_0. Let $f_k = f_{\alpha_k} : H_k \to H_0$ be the inclusion map for $k = 1, 2$. Then (H, f) is not closed at a sink $v = 0$. It is easy to see that $H_0^+ = \mathrm{Ker} h_0 = 0, f_1^+ = 0$ and $f_2^+ = 0$. Therefore $H_0^{+-} = H_1 \oplus H_2$ and $H_1^{+-} = H_1, H_2^{+-} = H_2$. We have

$f_k^{+-}: H_k \to H_1 \oplus H_2$ is a canonical inclusion for $k = 1, 2$. Since $\widetilde{H}_0 = (\operatorname{Im} h_0)^{\perp} = 0$, we have $(\widetilde{H}, \tilde{f}) = (0, 0)$. Therefore

Φ_0^-

is closed at a sink $v = 0$. But (H, f) is not closed at a sink $v = 0$. Therefore there exists no isomorphism between (H, f) and $\Phi_0^-\left(\Phi_0^+(H, f)\right) \oplus (\widetilde{H}, \tilde{f})$. Note that (H, f) is not full at a sink $v = 0$ and $\Phi_0^-\left(\Phi_0^+(H, f)\right)$ is full at a sink $v = 0$. Therefore this example also shows that, if we do not assume that a Hilbert representation (H, f) of Γ is full at v, then the following duality theorem (Corollary (4.2.27)) does not hold in general.

Corollary (4.2.27)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. If a Hilbert representation (H, f) of Γ is full at v, then*

We have a dual version.

Theorem (4.2.28)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Assume that a Hilbert representation (H, f) of Γ is co-closed at v. Let $\hat{h}_v : H_v \to \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ is a bounded operator defined by $\hat{h}_v(x) = \left(f_\alpha(x)\right)_{\alpha \in E_v}$ for $x \in H_v$. Define a Hilbert representation $(\check{H}, \check{f})$ of Γ by*

$\check{H}_u = 0$ for $u \neq v$ and $\check{f} = 0$. Then

Proof. We see that v is a sink in $\bar{\Gamma}$, because v is a source in Γ. Since a Hilbert representation (H, f) of Γ is co-closed at v, a Hilbert representation $\Phi^*(H, f)$ is closed at v. By Theorem (4.2.26), there exists a Hilbert representation $(\widetilde{H}, \tilde{f})$ of $\bar{\Gamma}$ such that

Put $(\breve{H},\check{f}) = \Phi^*(\widetilde{H},\tilde{f})$. Then

$(H,$

Moreover it is easy to see that

If we do not assume that a Hilbert representation (H,f) of Γ is co-closed at the source v, then the above Theorem (3.2.28) does not hold in general. In fact, consider the following quiver $\Gamma = (V,E,s,r)$:

Let $K = \ell^2(\mathbb{N})$ with the canonical basis $(e_n)_{n\in\mathbb{N}}$. Define a Hilbert representation (H,f) of Γ by $H_0 = K \oplus K, H_1 = K \oplus 0$ and H_2 is the closed subspace H_0 spanned by $\left\{\left(\cos\frac{\pi}{n+2}e_n, \sin\frac{\pi}{n+2}e_n\right) \in K \oplus K \mid n \in \mathbb{N}\right\}$. Let $f_k = f_{\alpha_k}: H_0 \to H_k$ be the canonical projection for $k = 1,2$. Then (H,f) is not co-closed at a source $v = 0$. It is easy to see that $H_0^- = \left(\mathrm{Im}\,\hat{h}_0\right)^\perp = 0, f_1^- = 0$ and $f_2^- = 0$. Therefore $H_0^{-+} = H_1 \oplus H_2$ and $H_1^{-+} = H_1, H_2^{-+} = H_2$. We have that $f_k^{-+}: H_1 \oplus H_2 \to H_k$ is the canonical projection for $k = 1,2$. Since $\breve{H}_0 = Ker\,\hat{h}_0 = 0$, we have $(\breve{H},\check{f}) = (0,0)$. Therefore

Φ_0^+

is co-closed at a source $v = 0$. But (H,f) is not co-closed at a source $v = 0$. Therefore there exists no isomorphism between (H,f) and $\Phi_0^+\left(\Phi_0^-(H,f)\right) \oplus (\breve{H},\check{f})$. Note that (H,f) is not co-full at a source $v = 0$ and $\Phi_0^+\left(\Phi_v^-(H,f)\right)$ is co-full at a source $v = 0$. Therefore this example also shows that, if we do not assume that a Hilbert representation (H,f) of Γ is co-full at v, then the following duality theorem (Corollary (3.2.29)) does not hold in general.

Corollary (4.2.29)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. If a Hilbert representation (H, f) of Γ is co-full at v, then*

Lemma (4.2.30)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Then for any Hilbert representation (H, f) of Γ , $\Phi_v^+(H, f)$ is co-full at v.*

Proof. Put $(H^+, f^+) = \Phi_v^+(H, f)$. Recall that $h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \to H_v$ is given by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$, and $H_v^+ = \mathrm{Ker} h_v$. And for $\beta \in E^v$, let $i_v : H_v^+ \to \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ be the canonical inclusion and $P_\beta : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \to H_{s(\beta)}$ the canonical projection. We define

f

Therefore $f_{\overline{\beta}}^{+*} : H_{s(\beta)} \to H_v^+$ is given by $f_{\overline{\beta}}^{+*} = i_v^* \circ P_\beta^*$. Since $P_\beta^*: H_{s(\beta)} \to \bigoplus_{\alpha \in E^v} H_{s(\alpha)}$ is the canonical inclusion and $i_v^*: \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \to H_v^+$ is the canonical projection, we have

Therefore (H^+, f^+) is co-full at v.

Proposition (4.2.31)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. If (H, f) is a Hilbert representation of Γ, then*

Lemma (4.2.32)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Then for any Hilbert representation (H, f) of Γ, $\Phi_v^-(H, f)$ is full at v.*

Proof. Put $(H^-, f^-) = \Phi_v^-(H, f)$. Recall that $\hat{h}_v : H_v \to \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by $\hat{h}_v(x) = \left(f_\alpha(x)\right)_{\alpha \in E_v}$ for $x \in H_v$ and $H_v^- = \left(\mathrm{Im}\, \hat{h}_v\right)^\perp \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$. Let $Q_v : \bigoplus_{\alpha \in E_v} H_{r(\alpha)} \to H_v^-$ be the canonical projection. For $\beta \in E_v$, let $j_\beta : H_{r(\beta)} \to \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ be the canonical inclusion. Then

f_β

Therefore

Thus (H^{-}, f^{-}) is full at v.

Proposition (4.2.33)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. If (H, f) is a Hilbert representation of Γ, then*

We examine on which representation a reflection functor vanishes.

Lemma (4.2.34)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Then, for any Hilbert representation (H, f) of Γ, the following are equivalent*:

(1) $\Phi_v^{+}(H, f) \cong (0, 0)$,

(2) $H_u = 0$ *for any* $u \in V$ *with* $u \neq v$.

Furthermore if the above conditions are satisfied and (H, f) is indecomposable, then $H_v \cong \mathbb{C}$. If the above conditions are satisfied and (H, f) is full at the sink v, then $(H, f) \cong (0, 0)$.

Proof. Put $(H^{+}, f^{+}) = \Phi_v^{+}(H, f)$. Recall that $h_v : \bigoplus_{\alpha \in E^v} H_{s(\alpha)} \to H_v$ is given by $h_v((x_\alpha)_{\alpha \in E^v}) = \sum_{\alpha \in E^v} f_\alpha(x_\alpha)$, and $H_v^{+} = \mathrm{Ker} h_v$. For other $u \in V$ with $u \neq v$, $H_u^{+} = H_u$.
(a) ⇒ (b): Assume that $\Phi_v^{+}(H, f) = 0$. Then, for any $u \in V$ with $u \neq v$ we have

$$H_u = H_u^{+} = 0.$$

(b) ⇒ (a): Assume that $H_u = 0$ for any $u \in V$ with $u \neq v$. Then $H_v^{+} = 0$, because $H_v^{+} = \mathrm{Ker} h_v \subset \bigoplus_{\alpha \in E^v} H_{s(\alpha)} = 0$. For other $u \in V$ with $u \neq v$, $H_u^{+} = H_u = 0$.
Furthermore assume that the above conditions are satisfied and (H, f) is indecomposable. Then $f = 0$. Suppose that $\dim H_v \geq 2$. Then a non-trivial decomposition $H_v = K \oplus L$ gives a non-trivial decomposition of (H, f). This contradicts that (H, f) is indecomposable. Hence $H_v \cong \mathbb{C}$. Assume that the above conditions are satisfied and (H, f) is full at v. Then $f = 0$, so that $H_v = \sum_{\alpha \in E^v} \mathrm{Im}\, f_\alpha = 0$. Hence $(H, f) \cong (0, 0)$.

Lemma (4.2.35)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a source. Then, for any Hilbert representation (H, f) of Γ, the following condition are equivalent*:

(1) $\Phi_v^-(H, f) \cong (0,0)$,

(2) $H_u = 0$ *for any* $u \in V$ *with* $u \neq v$.

Furthermore if the above conditions are satisfied and (H, f) is indecomposable, then $H_v \cong \mathbb{C}$. If the above conditions are satisfied and (H, f) is co-full at the source v, then $(H, f) \cong (0,0)$.

Proof. Put $(H^-, f^-) = \Phi_v^-(H, f)$. Recall that $\hat{h}_v : H_v \to \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$ is given by $\hat{h}_v(x) = \left(f_\alpha(x)\right)_{\alpha \in E_v}$ for $x \in H_v$, and $H_v^- = \left(\operatorname{Im} \hat{h}_v\right)^\perp \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)}$. For other $u \in V$ with $u \neq v, H_u^- = H_u$.

(a) ⇒ (b): Assume that $\Phi_v^-(H, f) = 0$. Then, for any $u \in V$ with $u \neq v$ we have

$$H_u = H_u^- = 0.$$

(b) ⇒ (a): Assume that $H_u = 0$ for any $u \in V$ with $u \neq v$. Then $H_v^- = 0$, because $H_v^- = \left(\operatorname{Im} \hat{h}_v\right)^\perp \subset \bigoplus_{\alpha \in E_v} H_{r(\alpha)} = 0$. For other $u \in V$ with $u \neq v, H_u^- = H_u = 0$.

Assume that the above conditions are satisfied and (H, f) is co-full at v. Since $f_\alpha^* = 0$ for any $\alpha \in E, H_v = \sum_{\alpha \in E_v} \operatorname{Im} f_\alpha^* = 0$. Hence $(H, f) \cong (0,0)$. The rest is clear.

We shall show that a reflection functor preserves indecomposability of a Hilbert representation unless vanishing on it, under the assumption that the Hilbert representation is closed (resp. coclosed) at a sink (resp. source).

Theorem (4.2.36)[1]: *Let $\Gamma = (V, E, s, r)$ be a finite quiver and $v \in V$ a sink. Suppose that a Hilbert representation (H, f) of Γ is indecomposable and closed at v. Then we have the following*:

(1) *If* $\Phi_v^+(H, f) = 0$, *then* $H_v = \mathbb{C}, H_u = 0$ *for any* $u \in V$ *with* $u \neq v$ *and* $f_\alpha = 0$ *for any* $\alpha \in E$.

(2) *If* $\Phi_v^+(H, f) \neq 0$, *then* $\Phi_v^+(H, f)$ *is also indecomposable and* $(H, f) \cong \Phi_v^-\left(\Phi_v^+(H, f)\right)$.

Proof. Recall an operator $h_v : \bigoplus_{\alpha\in E^v} H_{s(\alpha)} \to H_v$ defined by $h_v((x_\alpha)_{\alpha\in E^v}) = \sum_{\alpha\in E^v} f_\alpha(x_\alpha)$. Since (H,f) is closed at a sink v, we have a decomposition such that

by Theorem (4.2.24), where $\widetilde{H}_v = (\operatorname{Im} h_v)^\perp \subset H_v, \widetilde{H}_u = 0$ for $u \neq v$ and $\tilde{f} = 0$.
Since (H,f) is indecomposable, $\Phi_v^-(\Phi_v^+(H,f)) \cong (0,0)$ or $(\widetilde{H},\tilde{f}) \cong (0,0)$.

Case 1. Suppose that $\Phi_v^-(\Phi_v^+(H,f)) \cong (0,0)$. Then $(H,f) \cong (\widetilde{H},\tilde{f})$. Hence $H_u \cong \widetilde{H}_u = 0$ for $u \neq v$. This implies that $\Phi_v^+(H,f) \cong (0,0)$ by Lemma (4.2.34). Since (H,f) is indecomposable, $H_v \cong \mathbb{C}$.

Case 2. Suppose that $(\widetilde{H},\tilde{f}) \cong (0,0)$. Then $(H,f) \cong \Phi_v^-(\Phi_v^+(H,f))$. Since (H,f) is nonzero, $\Phi_v^+(H,f)$ is non-zero. We shall show that $\Phi_v^+(H,f)$ is indecomposable. Assume that $\Phi_v^+(H,f) \cong (K,g) \oplus (K',g')$. Then

Since (H,f) is indecomposable, $\Phi_v^-(K,g) \cong (0,0)$ or $\Phi_v^-(K',g') \cong (0,0)$. By Lemma (4.2.32), $\Phi_v^+(H,f)$ is co-full at v, so are its direct summands (K,g) and (K',g'). Then $(K,g) \cong (0,0)$ or $(K',g') \cong (0,0)$ by Lemma (4.2.35). Thus $\Phi_v^+(H,f)$ is indecomposable.

Since Cases 1 and 2 are mutually exclusive and either of them occurs, we get the conclusion.

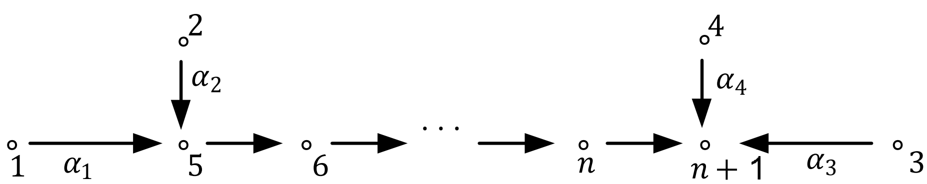
We have a dual version.

Theorem (4.2.37)[1]: *Let $\Gamma = (V,E,s,r)$ be a finite quiver and $v \in V$ a source. Suppose that a Hilbert representation (H,f) of Γ is indecomposable and co-closed at v. Then we have the following*:

1. *If $\Phi_v^-(H,f) = 0$, then $H_v = \mathbb{C}, H_u = 0$ for any $u \in V$ with $u \neq v$ and $f_\alpha = 0$ for any $\alpha \in E$.*
2. *If $\Phi_v^-(H,f) \neq 0$, then $\Phi_v^-(H,f)$ is also indecomposable and $(H,f) \cong \Phi_v^+\Phi_v^-(H,f))$.*

Gabriel's theorem says that a finite, connected quiver has only finitely many indecomposable representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8. We consider a complement of Gabriel's theorem for Hilbert representations. We need to construct some examples of indecomposable, infinite-dimensional representations of quivers with the underlying undirected graphs extended Dynkin diagrams $\widetilde{D}_n (n \geq 4), \tilde{E}_7$ and $\tilde{E}_8$. We consider the relative position of several subspaces along the quivers, where vertices are represented by a family of subspaces and arrows are represented by natural inclusion maps.

Lemma (4.2.38)[1]: *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\widetilde{D}_n$ for $n \geq 4$:*



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ.

Proof. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:

Define

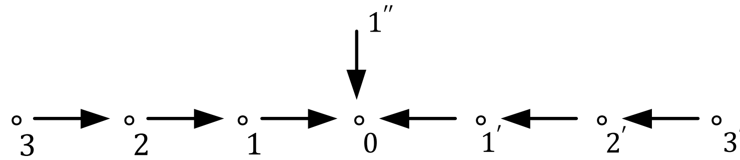
$H_1 = K$

$H_4 = \{(x$

Let $f_{\alpha_k}: H_{s(\alpha_k)} \to H_{r(\alpha_k)}$ be the inclusion map for any $\alpha_k \in E$ for $k = 1, 2, 3, 4,$ and $f_\beta = id$ for other arrows $\beta \in E$. Then we can show that (H, f) is indecomposable.

Let $\Gamma = (V, E, s, r)$ be the quiver of Example (4.2.10) as in Example (4.2.9) in this Section with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_6$. We have already shown that there exists an infinitedimensional, indecomposable Hilbert representation (H, f) of Γ.

9)[1]: *Let* $\Gamma = (V, E, s, r)$ *be the following quiver with the underlying* **3Lemma (4.2.** *undirected graph an extended Dynkin diagram* $\tilde{E}_7$:



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ.

Proof. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:

Let

$H_2 =$

$H_{1'} = 0$

$H_{3'} = 0$

For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. We shall show that the Hilbert representation (H, f) is indecomposable. Take $T = (T_v)_{v \in V} \in Idem(H, f)$. Since $T \in End(H, f)$ and any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in \{1, 2, 3, 1', 2', 3', 1''\}$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Since T_0 preserves $H_3 = K \oplus 0 \oplus 0 \oplus 0, H_{3'} = 0 \oplus K \oplus 0 \oplus 0$, and $H_{1'} \cap H_1 = 0 \oplus 0 \oplus K \oplus K, T_0$ is written

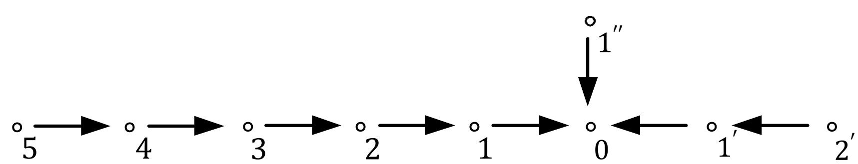
for some $A, B, X, Y, Z, W \in B(K)$.

Because $H_{1''} = \{(x, y, x, y) \in K^4 \mid x, y \in K\}$ is also invariant under T_0, for any $x, y \in K$, there exist $x', y' \in K$ such that

Putting $y = 0$, we have $Ax = Xx$ and $0 = Zx$ for any $x \in K$. Hence $A = X$ and $Z = 0$. Similarly, letting $x = 0$, we have $Y = 0$ and $W = B$. Therefore T_0 has a block diagonal form such that

Furthermore, as T_0 preserves $H_{1'} \cap H_2 = \{(0,0,x,x) \in K^4 \mid x \in K\}$, for any $x \in K$ there exists $y \in K$ such that $(0,0,Ax,Bx) = (0,0,y,y)$. Hence $A = B$. Therefore $T_0 = A \oplus A \oplus A \oplus A$. Moreover $H_1 \cap H_{2'} = \{(0,0,x,Sx) \in K^4 \mid x \in K\}$ is also invariant under T_0. Hence for any $x \in K$, there exists $y \in K$ such that $(0,0,Ax,ASx) = (0,0,y,Sy)$. Thus $AS = SA$. Since $T \in Idem(H,f), T_0$ is an idempotent, so that A is also an idempotent. Because $AS = SA$ and $A^2 = A$, we have $A = 0$ or $A = I$. Thus $T_0 = 0$ or $T_0 = I$. Since for any $v \in V$ and any $x \in H_v T_0 x = T_v x$, we have $T_v = 0$ or $T_v = I$ simultaneously. Thus $T = (T_v)_{v \in V} = 0$ or $T = I$, that is, $Idem(H,f) = \{0, I\}$. Therefore (H,f) is indecomposable.

Lemma (4.2.40)[1]: *Let $\Gamma = (V,E,s,r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_8$:*



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H,f) of Γ.

Proof. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H,f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:

Let

$H_2 = 0 \oplus$

$H_4 = 0 \oplus 0 \oplus 0$

$H_{1'} = K \oplus K \oplus$

For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. We shall show that the Hilbert representation (H, f) is indecomposable. Take $T = (T_v)_{v \in V} \in Idem(H, f)$. Since $T \in End(H, f)$ and any arrow is represented by the inclusion map, we have $T_0 x = T_v x$ for any $v \in V$ and any $x \in H_v$. In particular, $T_0 H_v \subset H_v$. Since T_0 preserves subspaces $H_{2'} = K \oplus K \oplus 0 \oplus 0 \oplus 0 \oplus 0, H_2 = 0 \oplus 0 \oplus K \oplus K \oplus K \oplus K$, T_0 has a form such that

for some $A \in B(K \oplus K)$ and $B \in B(K \oplus K \oplus K \oplus K)$.

Moreover $H_{1''} \cap H_2 = 0 \oplus 0 \oplus K \oplus 0 \oplus 0 \oplus 0$ and $H_3 = 0 \oplus 0 \oplus 0 \oplus K \oplus K \oplus K$ are invariant under T_0. Furthermore $H_5 = 0 \oplus 0 \oplus 0 \oplus K \oplus 0 \oplus 0$ and $T_0(H_5) \subset H_5$. Therefore T_0 is written as

for some $a, b, c, d, e, f, g, h, i, j, k, l \in B(K)$.

Since $H_{1'} \cap H_3 = 0 \oplus 0 \oplus 0 \oplus \{(y, 0, y) \in K^3 \mid y \in K\}$ is invariant under T_0,

for any $y \in K$, there exists $y' \in K$ such that

$B\Big($

Therefore $f+h=l$ and $j=0$. Next consider $H_{1'}\cap H_2 = 0\oplus 0\oplus\{(x,y,x,y);\ x,y\in K\}$. Since $H_{1'}\cap H_2$ is invariant under T_0, for any $x,y\in K$ there exist $x',y'\in K$ such that

$$B\begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix}$$

Putting $y=0$, we have

Hence $e=i$ and $g=k$.

Letting $x=0$, we have $fy+hy=y'=ly$ for any $y\in K$. Hence $f+h=l.$

Since T_0 preserves $H_{2'}\cap H_1=\{(x,x)\in K^2 \mid x\in K\}\oplus 0\oplus 0\oplus 0\oplus 0,$ for any $x\in K$, there exists $x'\in K$ such that

Hence $ax+bx=cx+dx,$ for any $x\in K$, so that $a+b=c+d.$

Furthermore $H_{1''}=\{(y,z,x,0,y,z)\in K^6 \mid x,y,z\in K\}$ is invariant under T_0. Therefore for any $x,y,z\in K$ there exist $x',y',z'\in K$ satisfying

Put $x=z=0$. Then for any $y\in K$, we have $ay=y'=ey, cy=z'=gy$ and $gy=0$. Hence we have $a=e$ and $c=g=0$.

Letting $x=y=0$, for any $z\in K$ we have $bz=y'=0, dz=z'=lz$ and $hz=0$. Therefore $b=0, d=l$ and $h=0$. Combining these with $f+h=l$ and $a+b=c+d$, we have $a=d$ and $f=l=d$. Thus T_0 is a block diagonal such that

Since T_0 is an idempotent, a is also an idempotent.

Finally consider that $H_4 = 0 \oplus 0 \oplus 0 \oplus K \oplus \{(y, Sy) \in K^2 \mid y \in K\}$ is invariant under T_0. Then for any $x, y \in K$, there exist $x', y' \in K$ such that

$$T_0(0,0$$

Hence $aSy = Sy' = Say$, so that $aS = Sa$. Since S is a unilateral shift and a is an idempotent, we have $a = 0$ or $a = I$. This implies that $T_0 = 0$ or $T_0 = I$. Since for any $v \in V$ and any $x \in H_v T_0 x = T_v x$, we have $T_v = 0$ or $T_v = I$ simultaneously. Thus $T = (T_v)_{v \in V} = 0$ or $T = I$, that is, $Idem(H, f) = \{0, I\}$. Therefore (H, f) is indecomposable.

We shall show that the existence of indecomposable, infinite-dimensional representations does not depend on the choice of the orientation of quivers. Suppose that two finite, connected quivers Γ and Γ' have the same underlying undirected graph and one of them, say Γ, has an infinite dimensional, indecomposable, Hilbert representation. We need to prove that another quiver Γ' also has an infinite-dimensional, indecomposable, Hilbert representation. Reflection functors are useful to show it. But we need to check the co-closedness at a source. We introduce a certain nice class of Hilbert representations such that co-closedness is easily checked and preserved under reflection functors at any source.

Definition (4.2.41)[1]: Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n. We count the arrows from the left as $\alpha_k : s(\alpha_k) \to r(\alpha_k) (k = 1, \ldots, n-1)$. Let (H, f) be a Hilbert representation of Γ. We denote f_{α_k} by f_k for short. For example,

We say that (H, f) is *positive-unitary diagonal* if there exist $m \in \mathbb{N}$ and orthogonal decompositions (admitting zero components) of Hilbert spaces and decompositions of operators

f_k

such that each $f_{k,i} : H_{s(\alpha_k),i} \to H_{r(\alpha_k),i}$ is written as $f_{k,i} = 0$ or $f_{k,i} = \lambda_{k,i} u_{k,i}$ for some positive scalar $\lambda_{k,i}$ and onto unitary $u_{k,i} \in B\left(H_{s(\alpha_k),i}, H_{r(\alpha_k),i}\right)$.

It is easy to see that if (H, f) is positive-unitary diagonal, then $\Phi^*(H, f)$ is also positiveunitary diagonal.

Example (4.2.42)[1]: Consider the following quiver Γ:

Let H_3 be a Hilbert space and $H_1 \subset H_2 \subset H_3$ inclusions of subspaces. Define a Hilbert representation (H, f) of Γ by $H = (H_i)_{i=1,2,3}$ and canonical inclusion maps $f_i = f_{\alpha_i}: H_i \to H_{i+1}$ for $i = 1, 2$. Then (H, f) is positive-unitary diagonal. In fact, define

Consider orthogonal decompositions $H_k = \bigoplus_{i=1}^{3} H_{k,i}$ $(k = 1, 2, 3)$ by

$H_1 = K_1$

Then $f_1 = I \oplus 0 \oplus 0$ and $f_2 = I \oplus I \oplus 0$. Hence (H, f) is positive-unitary diagonal. It is trivial that the example can be extended to the case of inclusion of n subspaces.

Lemma (4.2.43)[1]: *Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n and (H, f) be a Hilbert representation of Γ. Assume that (H, f) is positive-unitary diagonal. Then (H, f) is closed at any sink of Γ and co-closed at any source of Γ.*

Proposition (4.2.44)[1]:*Let Γ be a quiver whose underlying undirected graph is Dynkin diagram A_n and (H, f) be a Hilbert representation of Γ. Let v be a source of Γ. Assume that (H, f) is positive-unitary diagonal. Then $\Phi_v^-(H, f)$ is also positive-unitary diagonal.*

Proof. If $(H, f) \cong (H', f') \oplus (H'', f'')$, then $\Phi_v^-(H, f) \cong \Phi_v^-(H', f') \oplus \Phi_v^-(H'', f'')$. Therefore $H_k^- = \bigoplus_{i=1}^{m} H_{k,i}^-$. Hence it is enough to consider orthogonal components. We may and do examine locally the following cases:

Case 1. A Hilbert representation (H, f) is given by

with $T_1 = \lambda_1 U_1$ and $T_2 = \lambda_2 U_2$ for some positive scalars λ_1, λ_2 and onto unitaries U_1, U_2. Put $(H^-, f^-) = \Phi_0^-(H, f)$:

Then $(a, b) \in H_1 \oplus H_2$ is in $H_0^- = \left(Im\, \hat{h}_0\right)^{\perp}$ if and only if $\left((a, b)\middle|(T_1 z, T_2 z)\right) = 0$ for any $z \in H_0$, so that $T_1^* a + T_2^* b = 0$. Hence

Solving

$(x, 0)$

we have

$$T_1^- x = \left(\frac{\lambda_2^2}{\lambda_1^2+\lambda_2^2} x, -\frac{\lambda_1 \lambda_2}{\lambda_1^2+\lambda_2^2} U_2 U_1^* x\right) \quad \text{for } x \in H_1.$$

Similarly we have

Let

$\lambda_1^- :=$

Then U_1^- is an onto unitary and $T_1^- = \lambda_1^- U_1^-$. Similarly T_2^- is a positive scalar times unitary.

Case 2. A Hilbert representation (H, f) is given by

with $T_1 = 0$ and $T_2 = 0$.

Then it is easy to see that $H_0^- = H_1 \oplus H_2, T_1^-$ and T_2^- are canonical inclusions: $T_1^- x = (x,0) \in H_1 \oplus H_2$ for $x \in H_1$ and $T_2^- y = (0,y) \in H_1 \oplus H_2$ for $y \in H_2$. We may write that $T_1^- = I \oplus 0 : H_1 \oplus 0 \to H_1 \oplus H_2$ and $T_2^- = 0 \oplus I : 0 \oplus H_2 \to H_1 \oplus H_2$. Hence (H^-, f^-) is positive-unitary diagonal.

Case 3. A Hilbert representation (H, f) is given by

with $T_1 = \lambda_1 U_1$ and $T_2 = 0$ for some positive scalar λ_1 and onto unitary U_1. Then we see that $H_0^- = 0 \oplus H_2, T_1^- = 0$ and $T_2^- y = (0,y) \in 0 \oplus H_2$ for $y \in H_2$.

Hence (H^-, f^-) is positive-unitary diagonal.

Case 4. A Hilbert representation (H, f) is given by

with $T_1 = \lambda_1 U_1$ for some positive scalar λ_1 and onto unitary U_1. Put $(H^-, f^-) = \Phi_0^-(H, f)$:

Then we see that $H_0^- = 0$ and $T_1^- = 0$.

Case 5. A Hilbert representation (H, f) is given by

with $T_1 = 0$.

Then we have that $H_0^- = H_1$ and $T_1^- = I : H_1 \to H_1 = H_0^-$.

We shall show that we can change the orientation of Dynkin diagram A_n using only the iteration of σ_v^- at sources v except the right end.

Lemma (4.2.45)[1]: *Let Γ_0 and Γ be quivers whose underlying undirected graphs are the same Dynkin diagram A_n for $n \geq 2$. We assume that Γ_0 is the following*:

Then there exists a sequence $v_1, \dots, v_m$ of vertices in Γ_0 such that

(1) *for each* $k = 1, \dots, m, v_k$ *is a source in* $\sigma^-_{v_{k-1}} \dots \sigma^-_{v_2}\sigma^-_{v_1}(\Gamma_0)$,

(2) $\sigma^-_{v_m} \dots \sigma^-_{v_2}\sigma^-_{v_1}(\Gamma_0) = \Gamma$,

(3) *for each* $k = 1, \dots, m, v_k \neq n$.

Proof. The proof is by induction on the number n of vertices. Let $n = 2$. Since $\sigma^-_1(\circ_1 \rightarrow \circ_2) = \circ_1 \leftarrow \circ_2$, the statement holds. Assume that the statement holds for $n - 1$. If Γ has an arrow $\circ_{n-1} \rightarrow \circ_n$, then we can directly apply the assumption of the induction. If Γ has an arrow $\circ_{n-2} \rightarrow \circ_{n-1} \leftarrow \circ_n$, replace only this part by $\circ_{n-2} \leftarrow \circ_{n-1} \rightarrow \circ_n$ to get Γ'. Then $n - 1$ is a source of Γ', and $\sigma^-_{n-1}(\Gamma') = \Gamma$. Applying the induction assumption for Γ', we can construct the desired iteration. Consider the case that Γ has an arrow $\circ_{n-2} \leftarrow \circ_{n-1} \leftarrow \circ_n$. If there exists a vertex u such that $\circ_{u-1} \rightarrow \circ_u$ and $\circ_k \leftarrow \circ_{k+1}$ for $k = u, \dots, n-1$, then define a new quiver Γ'' by putting $\circ_{u-1} \leftarrow \circ_u, \circ_{n-1} \rightarrow \circ_n$ and other arrows unchanged with Γ. By the induction assumption, there exists a sequence $v_1, \dots, v_m$ of vertices in Γ_0 such that $\sigma^-_{v_m} \dots \sigma^-_{v_2}\sigma^-_{v_1}(\Gamma_0) = \Gamma''$ and, for each $k = 1, \dots, m, v_k \neq n$ and $v_k \neq n - 1$. Then

If all the arrows between 1 and n are of the form $\circ_k \leftarrow \circ_{k+1}$ for $k = 1, \dots, n-1$, then $\sigma^-_{n-1} \dots \sigma^-_2\sigma^-_1(\Gamma_0) = \Gamma$.

Lemma (4.2.46)[1]: *Let* $\Gamma = (V, E, s, r)$ *and* $\Gamma' = (V', E', s', r')$ *be finite, connected quivers and* Γ' *contains* Γ *as a subgraph, that is,* $V \subset V', E \subset E', s = s'|E$ *and* $r = r'|E$. *If there exists an infinite-dimensional, indecomposable, Hilbert representation of* Γ, *then there exists an infinite dimensional, indecomposable, Hilbert representation of* Γ'.

We prove the following theorem.

Theorem (4.2.47)[1]: *Let* Γ *be a finite, connected quiver. If the underlying undirected graph* $|\Gamma|$ *contains one of the extended Dynkin diagrams* $\tilde{A}_n (n \geq 0), \tilde{D}_n (n \geq 4), \tilde{E}_6, \tilde{E}_7$ *and* $\tilde{E}_8$, *then there exists an infinite dimensional, indecomposable, Hilbert representation of* Γ.

Proof. By Lemma (4.2.46), we may assume that the underlying undirected graph $|\Gamma|$ is exactly one of the extended Dynkin diagrams $\tilde{A}_n (n \geq 0), \tilde{D}_n (n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$.

The case of extended Dynkin diagrams $\tilde{A}_n (n \geq 0)$ was already verified in Examples (4.2.7) and (4.2.8).

Next suppose that $|\Gamma|$ is $\tilde{E}_6$. Let Γ_0 be the quiver of Example (4.2.10) and we denote here by $(H^{(0)}, f^{(0)})$ the Hilbert representation constructed there. Then $|\Gamma_0| = |\Gamma| = \tilde{E}_6$, but their orientations are different in general. Three "wings" of $|\Gamma_0|$ $2-1-0, 2'-1'-0, 2''-1''-0$ can be regarded as Dynkin diagrams A_3. Applying Lemma (4.3.45) for these wings locally, we can find a sequence $v_1, \dots, v_m$ of vertices in Γ_0 such that

(1) for each $k = 1, \dots, m, v_k$ is a source in $\sigma_{v_{k-1}}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0)$,

(2) $\sigma_{v_m}^- \dots \sigma_{v_2}^- \sigma_{v_1}^- (\Gamma_0) = \Gamma$,

(3) for each $k = 1, \dots, m, v_k \neq 0$.

We note that co-closedness of Hilbert representations at a source can be checked locally around the source. Since the restriction of the representation $(H^{(0)}, f^{(0)})$ to each "wing" is positive unitary diagonal and the iteration of reflection functors does not move the vertex 0, we can apply Lemma (4.2.43) and Proposition (4.3.44) locally that $\Phi_{v_{k-1}}^- \dots \Phi_{v_2}^- \Phi_{v_1}^- (H^{(0)}, f^{(0)})$ is co-closed at v_k for $k = 1, \dots, m$. Therefore Theorem (4.2.37) implies that $(H, f) := \Phi_{v_m}^- \dots \Phi_{v_2}^- \Phi_{v_1}^- (H^{(0)}, f^{(0)})$ is the desired indecomposable, Hilbert representation of Γ. Since the particular Hilbert space $H_0^{(0)}$ associated with the vertex 0 is infinite-dimensional and remains unchanged under the iteration of the reflection functors above, (H, f) is infinite-dimensional.

The case that the $|\Gamma|$ is $\tilde{E}_7$ or $\tilde{E}_8$ is shown similarly if we apply iteration of reflection functors on the representations in Lemma (4.2.39) or Lemma (4.2.41).

Finally consider the case that the $|\Gamma|$ is $\tilde{D}_n$. Let Γ_0 be the quiver of Lemma (4.2.38) and $(H^{(0)}, f^{(0)})$ the Hilbert representation constructed there. Then $|\Gamma_0| = |\Gamma| = \tilde{D}_n$, but their orientations are different in general. Let Γ_1 be a quiver such that $|\Gamma_1| = \tilde{D}_n$ and the orientation is as same as Γ on the path between 5 and $n+1$ and as same as Γ_0 on the rest four "wings." Define a Hilbert representation $(H^{(1)}, f^{(1)})$ of Γ_1 similarly as $(H^{(0)}, f^{(0)})$. For any arrow β in the path between 5 and $n+1, f_\beta^{(1)} = I$.

Hence the same proof as for $(H^{(0)}, f^{(0)})$ shows that $(H^{(1)}, f^{(1)})$ is indecomposable. By a certain iteration of reflection functors at a source 1, 2, 3 or 4 on $(H^{(1)}, f^{(1)})$ yields an infinite-dimensional, indecomposable, Hilbert representation of Γ. Here the co-closedness at a source 1, 2, 3 or 4 on $(H^{(1)}, f^{(1)})$ is easily checked, because the map is the canonical inclusion. Thus we can apply Theorem (4.2.39) in this case too.

Corollary (4.2.48)[1]: *Let Γ be a finite, connected quiver. If there exists no infinite-dimensional, indecomposable, Hilbert representation of Γ, then the underlying undirected graph $|\Gamma|$ is one of the Dynkin diagrams $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7$ and E_8 (see [16, 28, 29, 30]).*

We have a partial evidence for a certain quiver whose underlying undirected graph is A_n. We prepare an elementary lemma. Let H be a Hilbert space. For $a, b \in H$ we denote by $\theta_{a,b}$ a rank one operator on H such that $\theta_{a,b}(x) = (x \mid b)a$ for $x \in H$. Then $\theta_{a,b}^2 = \theta_{a,b}$ if and only if $(a \mid b) = 1$ or $a = 0$ or $b = 0$. Moreover if $\dim H \geq 2$ and $(a \mid b) = 1$, then $\theta_{a,b}$ is an idempotent such that $\theta_{a,b} \neq 0$ and $\theta_{a,b} \neq I$.

Lemma (4.2.49)[1]: *Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ a bounded operator. Take $a, b \in H_1$ and $c, d \in H_2$. Suppose that there exists a scalar λ such that $Ta = \lambda c$ and $T^* d = \bar{\lambda} b$. Then $T\, \theta_{a,b} = \theta_{c,d} T$.*

Proposition (4.2.50)[1]: *Let Γ be the following quiver whose underlying undirected graph is A_n for $n \geq 1$:*

Then there exists no infinite-dimensional, indecomposable, Hilbert representation of Γ.

Proof. The case $n = 1$ is clear by a non-trivial decomposition $H_1 = L_1 \oplus K_1$. We may assume that $n \geq 2$. Suppose that there were an infinite-dimensional, indecomposable, Hilbert representation (H, f) of Γ. Put $T_k = f_{\alpha_k}: H_k \to H_{k+1}$ for $k = 1, \ldots, n-1$.

Case 1. Suppose that $T_{n-1}T_{n-2} \ldots T_1 \neq 0$. Then there exists $a_1 \in H_1$ such that $T_{n-1}T_{n-2} \ldots T_1 a_1 \neq 0$. Consider non-zero vectors $a_k = T_{k-1}T_{k-2} \ldots T_1 a_1 \in H_k$ for

$k = 1, \ldots, n$. Put $b_n = \|a_n\|^{-2} a_n \in H_n$. Define $b_i = T_i^* T_{i+1}^* \cdots T_{n-1}^* b_n \in H_i$ for $i = 1, 2, \ldots, n-1$. Then

$$(a_i \mid b_i) = (a$$

Since $T_k a_k = a_{k+1}$ and $T_k^* b_{k+1} = b_k$, the above Lemma (4.2.49) implies that $T_k \theta_{a_k, b_k} = \theta_{a_{k+1}, b_{k+1}} T_k$ for $k = 1, \ldots, n-1$. Define the non-zero idempotents $P_k = \theta_{a_k, b_k}$. Since (H, f) is infinite-dimensional, there exists some vertex m such that H_m is infinite-dimensional. Then $P_m \neq I$. Define $P = (P_k)_k$, then $P \in Idem(H, f)$ and $P \neq O$ and $P \neq I$. This contradicts the assumption that (H, f) is indecomposable.

Case 2. Suppose that there exists r such that $T_{r-1} T_{r-2} \ldots T_1 \neq 0$ and $T_r T_{r-1} \ldots T_1 = 0$ for some $r = 1, \ldots, n-1$ and $\dim H_m \geq 2$ for some $m = 1, \ldots, r$. Then there exists $a_1 \in H_1$ such that $T_{r-1} T_{r-2} \ldots T_1 a_1 \neq 0$. Consider non-zero vectors $a_k = T_{k-1} T_{k-2} \ldots T_1 a_1 \in H_k$ for $k = 1, \ldots, r$. Put $b_r = \|a_r\|^{-2} a_r \in H_r$. Define $b_i = T_i^* T_{i+1}^* \cdots T_{r-1}^* b_r \in H_i$ for $i = 1, 2, \ldots, r-1$. Then we have $T_k \theta_{a_k, b_k} = \theta_{a_{k+1}, b_{k+1}} T_k$ for $k = 1, \ldots, r-1$ as Case 1. Define non-zero idempotents $P_k = \theta_{a_k, b_k}$ for $k = 1, \ldots, r$. Put $P_k = 0$ for $k = r+1, \ldots, n$. Then $T_r \theta_{a_r, b_r} = \theta_{T_r a_r, b_r} = \theta_{0, b_r} = 0$ and $T_k P_k = P_{k+1} T_k = 0$ for $k = r, \ldots, n-1$. Since $\dim H_m \geq 2$, the non-zero idempotent $P_m \neq I$. Define $P = (P_k)_k$, then $P \in Idem(H, f)$ and $P \neq O$ and $P \neq I$. This is a contradiction.

Case 3. Suppose that there exists r such that $T_{r-1} T_{r-2} \ldots T_1 \neq 0$ and $T_r T_{r-1} \ldots T_1 = 0$ for some $r = 1, \ldots, n$ and $\dim H_k = 1$ for $k = 1, \ldots, r$. Therefore $T_r = 0$. We may put $P_k = 0$ for $k = 1, \ldots, r$. Then for any $a, b \in H_{r+1}$ and $P_{r+1} = \theta_{a,b}$, we have $T_k P_k = P_{k+1} T_k = 0$ for $k = 1, \ldots, r$. Hence we may choose freely P_k for $k = r+1, \ldots, n$. Starting form H_{r+1}, we can repeat the argument from the beginning. After finite steps, we can reduce to the situation of Case 1 or Case 2. And finally we obtain a contradiction.

Chapter 5

Problems for Isometric Shift of Continuous Functions on Compact Spaces

One immediate consequence is that a space which admits such a shift must be ccc (countable chain condition). We then construct several new examples of type 1 shifts. We provide examples of nonseparable spaces X for which $C(X)$ admits an isometric shift, which solves in the negative a problem proposed by Gutek et al. We show that the operator has a shift if the sequence of the functions of all ranges of the operators is equal to zero.

Section (5.1): Type One Shifts on Continuous Spaces:

This section is concerned with shifts on Banach spaces of the form $C(X)$ (i.e., the space of continuous, real or complex valued functions defined on a compact Hausdorff space X). For motivation, consider the following simple example. Let $X = \omega + 1$ (the one point compactification of the integers ω) and identify $C(X)$ with the space of convergent sequences of numbers. Shift each member of $C(X)$ one place to the right: let $T(\langle x_1, x_2, \dots \rangle) = \langle 0, x_1, x_2, \dots \rangle$. Note that T has the following properties:

(i) T is a linear isometric operator ($\|T(f)\| = \|f\| \text{ for all } f$);
(ii) T is co-dimension 1 (the quotient space $C(X)/\operatorname{ran}(T)$ is one-dimensional); and
(iii) $\bigcap_{n=1}^{\infty} \operatorname{ran}(T^n) = \{0\}$.

Roughly, these three conditions say that T is rigid, T shifts by just one coordinate, and that all of $C(X)$ is shifted. Define an *isometric shift* [82] on $C(X)$ to be any $T : C(X) \to C(X)$ which satisfies these three conditions.

It is unknown whether there is a non-separable compact X for which $C(X)$ admits an isometric shift. We will study the structure of isometric shifts in [92, 93, 95]. We will also give examples of shifts which, while still occurring over separable spaces, have more complex behaviors than previously known examples.

In [82], a representation theorem of Holsztyński [83] is used to divide isometric shifts into two classes. Holsztyński's theorem applies to arbitrary linear isometric maps between function spaces. For a mapping from $C(X)$ to itself, it asserts the existence of a closed subset X_0 of X, a continuous map ψ from X_0 onto X, and a continuous (real or complex valued) function w such that $(Tf)(x) = w(x)f(\psi(x))$ for all $x \in X_0$.

Furthermore, w has the property that $\|w(x)\| = 1$ for all x. In [82] it is shown that the assumption that T is co-dimension 1 places severe restrictions on X_0 and ψ. Either $X\backslash X_0$ is just a single point and ψ is 1:1, or $X_0 = X$ and there is exactly one point whose inverse image under ψ has more than one point (and this inverse image consists of exactly two points). They labeled these cases "type 1" and "type 2" shifts, respectively. For the type 2 case, it is shown in [82] that the union of the iterated inverse images of the special point where ψ is not 1:1 forms a dense subset of X. Thus, for the question of the existence of a shift on $C(X)$ where X is non-separable, one need only consider the type 1 case. (see [84,85,86,87]).

We will consider only the type 1 shifts. It is convenient to rephrase Holsztyński's theorem as follows: There is an isolated point p_1 of X, a homeomorphism ψ of $X\backslash\{p_1\}$ onto X, a continuous map $w: X\backslash\{p_1\} \to S^1$, and a measure μ on X with $|\mu| \leq 1$ such that, for all $f \in C(X)$

The measure μ is either a signed or a complex Borel measure, and $|\mu|$ is its total variation of [88]. The existence of μ follows from the Rietz Theorem, and it is easily checked that $|\mu| \leq 1$ iff the resulting T is isometric.

As noted in [85], any mapping T defined as in (1) will be a co-dimension 1 linear isometric operator (assuming ψ, w, and μ satisfy the conditions above). We will refer to a T defined in this way as the *type* 1 *pre-shift* generated by ψ, *w*, and μ. So, a pre-shift T will be a shift iff $\bigcap_{n=1}^{\infty} \operatorname{ran}(T^n) = \{0\}$. Also, if p_1 is the (unique) isolated point of X which is not in the domain of ψ, thenwe let p_n denote $\psi^{-1}(p_{n-1})$ for each integer $n \geq 2$, and we let $D_\psi = \{p_n: n = 1, 2 \ldots\}$.We refer to this particular ordering of the points of D_ψ as the *standard listing* of D_ψ.

We begin by giving a characterization of the functions which are in $\operatorname{ran}(T^n)$, where T is any type 1 pre-shift. This result (Theorem (5.1.1)) will be the basis for many of our later arguments. Some easy corollaries of this theorem are that X must be ccc (countable chain condition) if $C(X)$ admits a shift, and that if one can find any ψ (as above) which makes D_ψ dense in X, then $C(X)$ does admit a shift. This second fact was also established in [82]. All the type 1 shifts produced have did have D_ψ dense, we will refer to shifts produced in this way as "primitive" shifts. We give some very general techniques for constructing these sorts of type 1 shifts. One can think of the non-separability question as asking how non-primitive a shift can be—i.e., how big can X minus the closure of D_ψ be? The existence of a non-primitive shift was established in [85]. They showed that for any finite n one can produce a type 1 shift on $C(\omega + 1)$ for which D_ψ misses n of the isolated points—these points are rotated in a simple cycle by ψ. Another consequence of Theorem (5.1.1) is that, for any type 1 shift, it must be that every isolated point of $X \backslash D_\psi$ has finite order under ψ. Despite this fact, we produce examples of shifts for which $X \backslash D_\psi$ has infinitely many isolated points. These have a somewhat complex structure, since ψ must divide the isolated points of $X \backslash D_\psi$ into infinitely many finite cycles.

As usual, the term *compact space* means a space which is both compact and Hausdorff. We use the standard sup norm on $C(X)$. When we use the symbol $C(X)$, we are *simultaneously* considering the spaces of real valued and complex valued functions on X. When we need to distinguish between these function spaces, we use $C_{\mathbb{R}}(X)$ and $C_{\mathbb{C}}(X)$. We will denote the unit circle in $\mathbb{C}$ by S^1. When we are simultaneously considering the real and complex cases, we will abuse notation somewhat and also let S^1 represent the "unit circle" $\{-1, 1\}$ of $\mathbb{R}$, even though S^0 would be a more proper notation. One should also note that the "co-dimension 1" condition in the definition of shift means, in the complex case, that $C(X)/\operatorname{ran}(T) \cong \mathbb{C}$.

Theorem (5.1.1)[81]: *Let X be a compact space, let T be the type 1 pre-shift on $C(X)$ generated by $\psi, w,$ and μ, and let $\{p_1, p_2, \ldots\}$ be the standard listing of D_ψ. Define constants $\alpha_1, \alpha_2, \ldots$ by letting $\alpha_1 = 1$ and $\alpha_n = \alpha_{n-1} w(p_n)$ for $n \geq 2$. Define functions $w_1, w_2, \ldots$ by letting $w_1 = w \circ \psi^{-1}$ and $w_n = (w_{n-1})(w \circ \psi^{-n})$ for $n \geq 2$. Then, for any $f \in C(X), f \in \operatorname{ran}(T^n)$ iff*

Proof. The proof is by induction on n. First, consider the case $n = 1$. Suppose that $f \in \mathrm{ran}(T)$ so that $f = Tg$ for some $g \in C(X)$. Then $f(x) = w(x)g(\psi(x))$ for $x \neq p_1$, and thus $g = \frac{f \circ \psi^{-1}}{w \circ \psi^{-1}}$ (this holds for all $x \in X$ since $\psi^{-1} : X \to X \setminus \{p_1\}$). Thus

Now suppose that $f \in C(X)$ and that $f(p_1) = \int_X \frac{f \circ \psi^{-1}}{w_1} d\mu$. Let $g = \frac{f \circ \psi^{-1}}{w \circ \psi^{-1}}$. It is easily checked that $f = Tg$, and thus $f \in \mathrm{ran}(T)$.

Note that in the previous paragraph we have actually proven the following fact: for any $f, g \in C(X)$,

f

This result is essentially the same of [85].

Now, fix $n \geq 2$ and suppose that the theorem has been proven for $1, 2, \ldots, n-1$.

Let $f \in \mathrm{ran}(T^n)$. Since $f \in \mathrm{ran}(T^i)$ for $i = 1, 2, \ldots, n-1$, we know that (2) holds for each $i < n$, and so we just need to show that $f(p_n) = \alpha_n \int_X \frac{f \circ \psi^{-n}}{w_n} d\mu$. Since $f \in \mathrm{ran}(T^n), f = Tg$ where $g \in \mathrm{ran}(T^{n-1})$. Thus,

$f($

Finally, suppose that $f \in C(X)$ and that f satisfies (2) for $i = 1, \ldots, n$. Let $g = \frac{f \circ \psi^{-1}}{w \circ \psi^{-1}}$. Then $f = Tg$, so to prove that $f \in \operatorname{ran}(T^n)$, we need to show that $g \in \operatorname{ran}(T^{n-1})$. By induction, it is sufficient to show that $g(p_i) = \alpha_i \int_X \frac{g \circ \psi^{-i}}{w_i} d\mu$ for $i = 1, \ldots, n-1$. Fix such an i, then

g

(Note that $\psi^{-(i+1)}(x)$ cannot equal p_1, which justifies replacing Tg with $(w)(g \circ \psi)$ in the fourth step.)

Note that $\psi^{-n} : X \backslash \{p_1, \ldots, p_n\} \to X$. Thus, in condition (2) of Theorem (5.2.1)[81], the value of each $f(p_i)$ depends only on $f \restriction_{X \backslash \{p_1, \ldots, p_n\}}$. This gives a clear picture of each $\operatorname{ran}(T^n)$. Each $f \in C(X \backslash \{p_1, \ldots, p_n\})$ extends uniquely to a function $f \in C(X)$ which is $\operatorname{ran}(T^n)$, with the values of $f(p_n), f(p_{n-1}), \ldots, f(p_1)$ being determined (in this order) by the integrals (2) from Theorem (5.1.1). Unfortunately, the "picture" for $\bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$ is not as clear. But we can say that

$f \in$

The following important theorem from [82] follows easily from Theorem (5.1.1).

Corollary (5.1.2)[81]: *If X is a compact space which admits a ψ for which D_ψ is dense, then there exists a type 1 shift on $C(X)$. (More precisely, our assumption is that there exists a homeomorphism $\psi : X\backslash\{p_1\} \to X$ for which $\{\psi^{-n}(p_1): n \in \omega\}$ is dense.)*

Proof. Let ψ be such that D_ψ is dense, let $w \equiv 1$, let $\mu \equiv 0$, and let T be the type 1 preshift generated by ψ, w, and μ. Suppose $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$. By Theorem (5.2.1)[81], $f(p_n) = 0$ for each n, and since D_ψ is dense, $f \equiv 0$.

When a shift T is generated as in Corollary (5.1.2), we will refer to T as a *primitive type* 1 *shift*. Obviously, a primitive shift can only occur for a separable space. The existence of non-primitive shifts was first shown in [85]. In the remainder of this section, we will use Theorem (5.1.1) to prove further theorems about the "structure" of non-primitive shifts.

We first derive some relatively easy consequences of Theorem (5.1.1).

Theorem (5.1.3)[81]: *Let X be a compact space, and suppose that ψ, w, and μ generate a type 1 shift T on $C(X)$. Then there does not exist a non-empty open subset U of $X\backslash D_\psi$ such that $|\mu|(U) = 0$ and $\psi(U) \subset U$.*

Proof. Suppose such an open set U exists. Let f be a non-zero function whose support is contained in U. Then the support of each $f \circ \psi^{-n}$ is contained in $\psi^n(U) \subset U$, so $\int_X \frac{f \circ \psi^{-n}}{w_m} d\mu = 0$. But $f(p_n) = 0$ for all n, so $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$.

Theorem (5.1.4)[81]: *If a compact space X admits a type 1 shift, then X has the countable chain condition (ccc).*

Proof. Suppose that ψ, w, and μ generate a type 1 shift T on $C(X)$. Let C be an uncountable pairwise-disjoint collection of open subsets of X. Since D_ψ is countable, we can assume that no member of C intersects D_ψ. For each integer $i \geq 0$ let $C_i = \left\{U \in C: |\mu|\left(\psi^i(U)\right) > 0\right\}$ (as usual, we take ψ^0 to be the identity function). Since each

C_i is countable, choose a set $U \in C \setminus \bigcup_{i=0}^{\infty} C_i$. Then the open set $\bigcup_{i=0}^{\infty} \psi^i(U)$ contradicts the conclusion of Theorem (5.1.3).

Theorem (5.1.5)[81]: *Let X be a compact space which admits a type 1 shift generated by ψ, w and μ. If μ has separable support, then X is separable.*

Proof. Suppose that S is separable subspace of X which contains the support of μ. If $\bigcup_{n=0}^{\infty} \psi^n(S)$ is not dense in X, then the complement of the closure of this set would be an open set which violates the conclusion of Theorem (5.1.3).

We next show that it was not really necessary to let $\mu \equiv 0$ in Corollary (5.1.2). In fact, all that is needed is that $|\mu| < 1$.

Theorem (5.1.6)[81]: *Let X be a compact space and let T be a type 1 pre-shift generated by ψ, w, and μ. If D_ψ is dense in X and $|\mu| < 1$, then T is a shift.*

Proof. Assume that D_ψ is dense and that $|\mu| = r < 1$. Suppose that $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$, and let $M = \sup\{|f(x)|: x \in X\}$. Let $\{p_1, p_2, \dots\}$ be the standard listing of D_ψ. By Theorem (5.1.2), each $|f(p_n)| \le \int_X |f \circ \psi^{-n}|\, d\mu \le rM$. Since D_ψ is dense, $|f(x)| < rM$ for all $x \in X$, so $M \le rM$. Thus $M = 0$.

We next show that in order to prove that T is a shift, it is enough to show that each $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$ is zero on $X \setminus D_\psi$.

Theorem (5.1.7)[81]: *Let X be a compact space and let T be a type 1 pre-shift generated by ψ, w, and μ. Let $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$. Then if $f(x) = 0$ for all $x \in X \setminus D_\psi$, then $f \equiv 0$.*

Proof. Let $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$ be such that $f(x) = 0$ for all $x \in X \setminus D_\psi$. Thus $\lim_{n\to\infty} f(p_n) = 0$. Fix $\varepsilon > 0$. Choose N such that $|f(p_n)| < \varepsilon$ for all $n > N$. Then

Thus, $|f(p_n)| < \varepsilon$ for all $n \geq N$. Thus (by induction) $|f(p_n)| < \varepsilon$ for all n. Hence, $f \equiv 0$.

In the example from [85], D_ψ contains all but finitely many of the isolated points of X, and the remaining isolated points are rotated in a cycle by ψ. The next theorem shows that this sort of behavior must happen.

Theorem (5.1.8)[81]: *Let X be a compact space, and suppose that ψ, w, and μ generate a type 1 shift T on $C(X)$. Then each isolated point of X which is not in D_ψ has finite order under ψ (i.e., for each such x there is an n such that $\psi^n(x) = x$).*

Proof. Let $\{p_1, p_2, \dots\}$ be the standard listing of D_ψ, and suppose that s_0 is an isolated point of X with $s_0 \notin D_\psi$ and $\{\psi^n(s_0): n = 1, 2, \dots\}$ infinite. For each integer $n \geq 1$, let $f_n \in C(X)$ be the (unique) function in $\operatorname{ran}(T_n)$ such that $f_n(s_0) = 1$ and $f_n(x) = 0$ for $x \notin \{s_0, p_1, \dots, p_n\}$ (see the remark following Theorem (5.1.1)). We will establish that the functions f_n converge to a function $f \in C(X)$. Since each $\operatorname{ran}(T^n)$ is closed (this is easy to see because $C(X)$ is complete and T^n is a isometric), it follows that $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$, which contradicts the fact that T is a shift.

We will establish the convergence of the functions f_n by showing that they form a Cauchy sequence in $C(X)$. Let $s_n = \psi^n(s_0)$, and let $S = \{s_0, s_1, \dots\}$. Since $s_i \neq s_j$ if $i \neq j$, we have that $\sum_{n=0}^{\infty} |\mu(\{s_n\})| \leq 1$. Let n and k be integers with $k \leq n$. Then by Theorem (5.1.1),

$f_n($

In order to simplify the notation, let

$$a(k)$$

So $f_n(p_k) = a(k) + \sum_{i=1}^{\infty} b(k,i) f_n(p_{k+i})$ for $k \leq n$, and $f_n(p_k) = 0$ for $k > n$. We may have noted that the above summations are actually finite, since $f_n(p_{k+i}) = 0$ for $i > n - k$. Note also that $f_n(p_n) = a(n)$. Our reason for leaving the summations infinite is to simplify the induction argument below. Since $\sum_{n=1}^{\infty} |a(n)|$ converges, the proof will be complete if we establish that $\|f_n - f_{n-1}\| \leq |a(n)|$. Fix $n \geq 2$. We show that $|f_n(p_k) - f_{n-1}(p_k)| < |a(n)|$ for all k by inducting "backwards" on k. If $k > n$, then $f_n(p_k) = f_{n-1}(p_k) = 0$, and if $k = n$ then $f_n(p_k) = a(n)$ and $f_{n-1}(p_k) = 0$. So let $k < n$ and suppose that $\left|f_n(p_j) - f_{n-1}(p_j)\right| < |a(n)|$ for all $j > k$. Then

$$|f_n(p_k)$$

$$= \Big| a$$

$$\leq \sum_{i=}^{\infty}$$

$$\leq \sum_{i=}$$

$$= |a$$

The next theorem uses a counting argument to put further restrictions on spaces which can admit shifts. It eliminates, for example, $C(X)$ where X consists of a convergent sequence adjoined to a non-separable Cantor cube.

Recall that for a locally compact space X, a function $f \in C(X)$ is said to *vanish at infinity* if for every $\varepsilon > 0$ there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$. The set of all such functions is denoted by $C_0(X)$.

Theorem (5.1.9)[81]: *Let X be a non-separable compact space which satisfies the following condition: for every countable set D of isolated points of X, $|C_0(X \backslash \overline{D})| > 2^{\aleph_0}$. Then $C(X)$ does not admit an isometric shift.*

Proof. Suppose $C(X)$ admits a shift T. Since X non-separable, T must be type 1, so it must be generated by some ψ, w, and μ. Identify each $f \in \left|C_0\left(X \backslash D_\psi\right)\right|$ with the natural extension of f to X by letting $f(x) = 0$ for all $x \in D_\psi$. Now, for each such f, let s_f be the infinite sequence whose nth term is $\int_X \frac{f \circ \psi^{-n}}{w_n} d\mu$. By our cardinality assumption, there are distinct functions f and g such that $s_f = s_g$. But then $f - g \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$, a contradiction.

First, consider the example of [85]. Here, X is once again $\omega + 1$. The map ψ is a simple cycle on a set of the form $\{0, \ldots, k\}$, ψ sends n to $n - 1$ for $n > k + 1$, and ψ fixes ω. The measure μ is concentrated at $0, \mu(\{0\}) = 1, w(0) = -1$, and $w(x) = +1$ otherwise.

Another interesting example is found in [87]. Start with the Cantor set C, and construct a homeomorphism $\psi : C \to C$ such that ψ has a fixed point x and such that there is also a point y whose forward orbit under ψ is dense in C. Now, form the space X by adding to C a sequence $\{p_1, p_2, \ldots\}$ of isolated points which converges to x, and extend ψ by letting $\psi(p_i) = p_{i-1}$. Let $w \equiv 1$, and concentrate the measure at y, but let $\mu(\{y\}) = -1$.

Recall that a *primitive type* 1 *shift* is one of the form given by Corollary (5.1.2).

Since the w and μ are, in a sense, irrelevant in this case, we will refer simply to a homeomorphism $\psi : X \backslash \{\rho_1\} \to X$ for which D_ψ is dense as a *primitive shift on* X. We give some new constructions for building primitive shifts on compact metric spaces. We will make use in this section of the notion of a *weak chain*, which is a finite sequence

$C = (U_1, U_2, \dots, U_n)$ of open sets such that the intersection $U_i \cap U_j$ is nonempty whenever $|i - j| \leq 1$.

Theorem (5.1.10)[81]: *Let X be an infinite compact metric space and let D be a dense set of isolated points. If $X \setminus D$ is connected, then there exists a primitive shift on X.*

Proof. The set $X \setminus D$ is compact. Let $\mathcal{U}_1$ be a minimal cover of $X \setminus D$ consisting of balls of radius 1 centered at points of $X \setminus D$. Suppose we have defined minimal covers $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n$ of $X \setminus D$. We define $\mathcal{U}_{n+1}$ to be a minimal cover of $X \setminus D$ consisting of balls of radius r_{n+1} which are centered at points of $X \setminus D$ such that $r_{n+1} < 1/(n+1)$ and each ball in $\mathcal{U}_{n+1}$ is a subset of some ball in $\mathcal{U}_n$. (in [89, 85, 90, 88].) Define functions V_n from finite sets of integers $\{1, 2, \dots, k_n\}$ onto $\mathcal{U}_n$ in such a way that the intersection $V_n(i) \cap V_n(j)$ is not empty whenever $|i - j| \leq 1$ (i.e., the functions V_n define weak chains) and every $V_n(1) = V_n(k_n)$. Note that the set D is countable. Order points of D in a list as follows. If there are points in $D \setminus \bigcup \mathcal{U}_1$ then place them at the start of the list in any order (note that there are at most finitely many such points). Now add the points of $D \cap (V_1(1) \setminus \bigcup \mathcal{U}_2)$ to the list, unless there are no such points, in which case just add any single point from $D \cap V_1(1)$. Next, add the points of D which are in $V_1(2) \setminus \bigcup \mathcal{U}_2$ and which have not been picked before, or if no such points exist, add any single point from $D \cap V_1(2)$. Continue this procedure for $V_1(j) \setminus \bigcup \mathcal{U}_2$, where $j = 3, \dots, k_1$. Then repeat the above process beginning with $V_2(1) \cup \mathcal{U}_3$, and so on. Label the resulting listing of D as $\{p_1, p_2, \dots\}$. Define ψ from $X \setminus \{p_1\}$ onto X by

It is easy to see that ψ is continuous. Therefore ψ is a primitive shift on X.

Theorem (5.1.11)[81]: *Let X be an infinite compact metric space and let D be a dense set of isolated points. If there exists a homeomorphism f from $X \setminus D$ onto $X \setminus D$ such that the set $\{f^n(x): n = 1, 2, \dots\}$ is dense in $X \setminus D$ for some x in $X \setminus D$, then there is a primitive shift on X.*

Proof. Let x_0 be a point in $X\backslash D$ such that the set $\{f^n(x): n = 1, 2, \dots\}$ is dense in $X\backslash D$. Note that $X \setminus D$ is dense in itself. As in Theorem (5.1.10), define minimal covers $\mathcal{U}_n$ of $X \setminus D$ consisting of balls of radius r_n which are centered at points of $X \setminus D$ such that $r_n < 1/n$ and each ball in $\mathcal{U}_{n+1}$ is a subset of some ball in $\mathcal{U}_n$. Note again that D is countable. Order points of D as follows. If there are points in $D \setminus \bigcup \mathcal{U}_1$, then order them arbitrarily as $p_1, \dots, p_{s_1}$. If there are no points in $D \setminus \bigcup \mathcal{U}_1$, then pick any point in D and call it p_1 (in that case $p_1 = p_{s_1}$). Let p_{s_1+1} be a point in $D \setminus \bigcup \mathcal{U}_2$ that has not been picked before and that is in the same ball from $\mathcal{U}_1$ to which $f(x_0)$ belongs, if any. If none, take any point in D not picked before that is in this ball. If we may use more than one ball, use the one that has more points from $D \setminus \bigcup \mathcal{U}_2$ that have not been picked before. Let p_{s_1+2} be a point in $D \setminus \bigcup \mathcal{U}_2$ not picked before that is in the ball from $\mathcal{U}_1$ that contains $f^2(x_0)$, if any, If none, pick any point from D in this ball not picked before. Continue on, choosing points from $D \setminus \bigcup \mathcal{U}_2$ in the ball from $\mathcal{U}_1$ to which $f^3(x_0)$, then $f^4(x_0)$, and so on, belong, until there are no more points in $D \setminus \bigcup \mathcal{U}_2$ left. If the last point picked was p_{s_2} and it corresponds to the point $f^k(x_0)$, we let p_{s_2+1} to be a point of $D \setminus \bigcup \mathcal{U}_3$ not picked before that belongs to a ball from $\mathcal{U}_2$ to which $f^{k+1}(x_0)$ belongs, if any. If none, we pick any point of D in this ball. Continue this process.

Define ψ from $X \setminus \{p_1\}$ onto X by

It is easy to see that ψ is continuous. Therefore ψ is a primitive shift on X.

Theorem (5.1.12)[81]: *Let X be an infinite compact metric space and let D be a dense set of isolated points in X. If there exists a primitive shift on X then there is a primitive shift on a disjoint union of finitely many copies of X.*

Proof. Let φ be a primitive shift on X, and let $\{p_1, p_2, \dots\}$ be the standard listing of D_φ. Let Y be a disjoint union of finitely many copies of X, say $Y = X \times \{1, 2, \dots, n\}$. Define ψ from $Y \setminus \{(p_0, n)\}$ onto Y by

It is easy to see that ψ is a primitive shift on Y.

The following example generalizes a construction from [87].

Theorem (5.1.13)[81]: *Let X be an infinite compact metric space and let D be a dense set of isolated points. If there is a primitive shift on X then there is a primitive shift on the following compactification of $X \times \mathbb{Z}$. The space is $Y = (X \times \mathbb{Z}) \cup \big((X \setminus D) \times \{-\infty, \infty\}\big)$. The topology on Y is defined by a basis of open sets $U \times \{n\}, \big((U \setminus D) \times \{-\infty\}\big) \cup (U \times \{j \in \mathbb{Z}: j < k\}), \big((U \setminus D) \times \{\infty\}\big) \cup (U \times \{j \in \mathbb{Z}: j > k\})$, where U is an open subset of X, and n and k are integers.*

Proof. Let φ be a primitive shift on X, and let $\{p_0, p_1, p_2, \dots\}$ be the standard listing of D_φ (for this construction, it is more convenient to start the indexing at zero). Define ψ from $Y \setminus \{(p_0, 0)\}$ onto Y by

$\psi(x$
$\psi(x$
$\psi(p$
$\psi(x$
$\psi(x$

It is easy to see that ψ is a primitive shift on Y.

Theorem (5.1.14)[81]: *Let X be an infinite compact metric space and let D be a dense set of isolated points in X. If there exists a primitive shift on $X \setminus D$ then there is a primitive shift on X.*

Proof. Let φ be a primitive shift on $X \setminus D$, and let $\{p_0, p_1, p_2, \dots\}$ be the standard listing of D_φ. Note that D_φ is dense in $X \setminus D$.

First, we will define a homeomorphism h on $X \setminus D$ such that $D_\varphi = \{h^n(p_0): n \in \mathbb{Z}\}$.

Let $0 < n_1 < n_2 < \cdots$ be such that the points $p_0, p_{n_1}, p_{n_2}, \dots$ form a sequence converging to a point x_0 of $(X \setminus D) \setminus D_\varphi$. Define h on $X \setminus D$ as follows:

$h(p_{n_{2k}}) = p_{n_{2k-1}-}$

$h(p_0) = p_{n_2-1},$

$h(p_{n_{2k-3}}) = p_{n_{2k}-}$

$h(x) = \varphi($

The function h is one-to-one and onto $X \setminus D$. It is also continuous. Therefore it is a homeomorphism.

Let $\mathcal{U}_1$ be a minimal cover of $X \setminus D$ consisting of balls of radius 1 centered at points of $X \setminus D$ and such that a ball centered at x_0 is one of them. Suppose we have defined minimal covers $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n$ of $X \setminus D$. We define $\mathcal{U}_{n+1}$ to be a minimal cover of $X \setminus D$ consisting of balls of radius r_{n+1} which are centered at points of $X \setminus D$ such that $r_{n+1} < 1/(n+1)$, each ball in $\mathcal{U}_{n+1}$ is a subset of some ball in $\mathcal{U}_n$, and such that a ball centered at x_0 is one of them.

Denote a ball in $\mathcal{U}_n$ centered at x_0 by $U_n(x_0)$. Let $k_1, k_2, \dots$ be a decreasing sequence of negative integers and let $m_1, m_2, \dots$ be an increasing sequence of positive integers such that both $h^{k_n}(p_0)$ and $h^{m_n}(p_0)$ are elements of $U_n(x_0)$, and such that for any ball V in U_n there is j between k_n and m_n such that $h^j(p_0)$ is an element of V. Denote that ball by $V_n(j)$. Assume in addition that $\mathcal{U}_n = \{V_n(j) : k_n \le j \le m_n\}$, where $V_n(k_n) = V_n(m_n) = U_n(x_0)$ is a ball centered at x_0.

Let q_0 be any point of D. Define ψ from $D \setminus \{q_0\}$ onto D by induction as follows.

Step 1. Let $\psi^{-1}(q_0)$ be any point of $D \setminus \{q_0\}$ in $V_1(m_1 - 1) \setminus \cup \mathcal{U}_2$, if any. If none, put $\psi^{-1}(q_0)$ to be any point of $D \setminus \{q_0\}$ in $V_1(m_1 - 1)$. Denote this point by q_1. Let $\psi^{-2}(q_0)$ be any point of $D \setminus \{q_0, q_1\}$ in $V_1(m_1 - 2) \setminus \cup \mathcal{U}_2$, if any. If none, put $\psi^{-2}(q_0)$ to be any point of $D \setminus \{q_0, q_1\}$ in $V_1(m_1 - 2)$. Denote this point by q_2. Continue in this way, choosing q_3, q_4, etc. Finally, let $\psi^{-m_1+k_1}(q_0)$ be any point of $D \setminus \{q_0, q_1, \dots, q_{m_1-k_1-1}\}$ in $V_1(k_1) \setminus \cup \mathcal{U}_2$, if any. If none, put $\psi^{-m_1+k_1}(q_0)$ to be any point of $D \setminus \{q_0, q_1, \dots, q_{m_1-k_1-1}\}$ in $V_1(k_1)$. Denote this point by $q_{m_1-k_1}$. Also, note that $V_1(k_1) = V_1(m_1) = U_1(x_0)$.

If there are any points of $D \setminus \{q_0, q_1, \dots, q_{m_1-k_1}\}$ left in $\cup \mathcal{U}_1 \setminus \cup \mathcal{U}_2$, we repeat the whole procedure again. That means we put $\psi^{-m_1+k_1-1}(q_0)$ to be any point of $D \setminus \{q_0, q_1, \dots, q_{m_1-k_1}\}$ in $V_1(m_1 - 1) \setminus \cup \mathcal{U}_2$, if any. If none, put $\psi^{-m_1+k_1-1}(q_0)$ to be any point of $D \setminus \{q_0, q_1, \dots, q_{m_1-k_1}\}$ in $V_1(m_1 - 1)$. Denote this point by $q_{m_1-k_1+1}$, and so on.

Step 2. Once we exhausted points of D in $\bigcup \mathcal{U}_1 \setminus \bigcup \mathcal{U}_2$, we repeat the procedure for m_2, k_2, and the remaining points of D in $\bigcup \mathcal{U}_2 \setminus \bigcup \mathcal{U}_3$, and so on. Extend ψ to $X \setminus \{q_0\}$ by putting $\psi|_{X\setminus D} = h$. It is fairly easy to see that ψ is continuous. It is one-to-one and onto X. So it is a primitive shift.

Theorem (5.1.8) raises the question: can there actually be infinitely many isolated points not in D_ψ? We give two examples showing that the answer is yes. The space for the first example is just simply $\omega + 1$. For this example, however, there is a sharp distinction between the space of real valued functions and the space of complex valued functions. Only the latter space admits a shift of this kind.

Theorem (5.1.15)[81]: *If* $T : C_{\mathbb{R}}(\omega + 1) \to C_{\mathbb{R}}(\omega + 1)$ *is a type* 1 *shift generated by* ψ, w*, and* μ*, then* D_ψ *contains all but finitely many points of* $\omega + 1$*.*

Proof. Suppose ψ, w, and μ generate a type 1 shift on $\omega + 1$ such that D_ψ is co-infinite. By Theorem (5.1.8), the isolated points of $\omega + 1$ which are not in D_ψ can be written as the disjoint union of infinitely many finite sets such that the restriction of ψ to each one is a simple cycle. Since the function $w: \omega + 1 \to \{+1, -1\}$ is continuous, it must be eventually constant. Hence, we can find disjoint finite subsets A and B of ω, each invariant under ψ, such that $w(x) = w(y)$ for all $x, y \in A \cup B$. Note that $\mu(A)$ and $\mu(B)$ must be non-zero (otherwise, the characteristic function of one of these sets would be in $\bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$ by the same argument as in Theorem (5.1.3)). So choose non-zero $a, b \in \mathbb{R}$ such that $a\,\mu(A) + b\,\mu(B) = 0$. Now let $f \in C(\omega + 1)$ be defined by $f(x) = a$ for $x \in A, f(x) = b$ for $x \in B$, and $f(x) = 0$ otherwise. For each n,

where W equals the common value of the function w on $A \cup B$. Thus $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$.

In order to construct the desired shift on $C_{\mathbb{C}}(\omega + 1)$, we need the following lemma. This generalizes the fact that if z is point on the unit circle whose angle is irrational, then $\{z^n : n \in \omega\}$ is dense in the circle.

Lemma (5.1.16)[81]: *There exists an infinite sequence $\langle z_1, z_2, \dots \rangle$ of points in S^1 (the unit circle in $\mathbb{C}$) such that $\lim_{n\to\infty} z_n = 1$ and, for each k, the set $\{\langle z_1^n, z_2^n, \dots, z_k^n\rangle : n \in \omega\}$ is dense in $(S^1)^k$.*

Theorem (5.1.17)[81]: *There is a type 1 shift T on $C_{\mathbb{C}}(\omega + 1)$ generated by ψ, w, and μ such that D_ψ is co-infinite.*

Proof. Partition ω into two infinite sets: $\{p_1, p_2, \dots\}$ and $\{q_1, q_2, \dots\}$. We use ∞ instead of ω to denote the non-isolated point of $\omega + 1$. Define $\psi, w,$ and μ as follows. Let $\psi(p_n) = p_{n-1}$ for $n > 1$ (as usual), and let each $w(p_n) = 1$ and $\mu(\{p_n\}) = 0$. Also, let $w(\infty) = 1$ and $\mu(\{\infty\}) = 0$. Let each $\psi(q_n) = q_n$ and $\mu(\{q_n\}) = 1/2^n$. Fix a sequence $\langle z_1, z_2, \dots \rangle$ which satisfies Lemma (5.1.16), and let each $w(q_n) = 1/z_n$. Let T be the type 1 pre-shift generated by ψ, w, and μ. We will prove that T is a shift.

Let $f \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$. We make the convenient definition $a_i = f(q_i)/2^i$ in order to simplify the notation. By Theorem (5.1.1),

for each $n \geq 1$. Let $L = \sum_{n=1}^{\infty} |a_n|$ (this sum converges since f is bounded). Note that in order to prove that $f \equiv 0$, it is sufficient to show that $L = 0$, since this will imply that each $f(p_i) = 0$ and that each $f(q_i) = 0$.

For each k, choose (by Lemma (5.1.16)) an integer n_k such that $\sum_{i=1}^{k} \left| z_i^{n_k} a_i - |a_i| \right| < 1/k$ (choose n_k so that $z_i^{n_k}$ is close to $|a_i|/a_i$ for each $i \in \{1, \dots, k\}$ such that $a_i \neq 0$). Then each $\left| f(p_{n_k}) - L \right| \leq 1/k + \sum_{i=k+1}^{\infty} \left| z_i^{n_k} a_i - |a_i| \right| \leq 1/k + 2\sum_{i=k+1}^{\infty} |a_i|$. Thus, $f(\infty) = \lim_{n\to\infty} f(p_n) = L$. But, if we repeat the above argument by choosing n_k so that $z_i^{n_k}$ is close to $-|a_i|/a_i$, we get that $f(\infty) = -L$. Thus, $L = 0$.

In fact, the construction in Theorem (5.1.17) did not require that the space was $\omega+1$. All that was needed was that the sequence $\langle p_1, p_2, \ldots\rangle$ was convergent. Thus, we have actually proven the following.

Theorem (5.1.18)[81]: *If X is any compactification of ω which contains a convergent infinite sequence of isolated points, then $C_{\mathbb{C}}(X)$ admits a type 1 shift.*

Finally, we present our most interesting example. Here, we return to considering $C_{\mathbb{C}}(X)$ and $C_{\mathbb{R}}(X)$ simultaneously.

Theorem (5.1.19)[81]: *There exists a compact space X and a type 1 shift on $C(X)$ generated by ψ, w, and μ such that infinitely many isolated points of X are in $X \setminus D_\psi$.*

Proof. Let X be the discrete sum of $\omega+1$ and $\beta\omega$ (the Stone–Čech compactification of ω). Since we have two copies of ω in X, we need some notation to distinguish them. List (all of) the isolated points of $\omega+1$ as $\{p_1, p_2, \ldots\}$, and again denote the non-isolated point of $\omega+1$ as ∞. Partition (all of) the isolated points of $\beta\omega$ into countably many finite sets $A_0, A_1, \ldots$ where each A_i has exactly 2^i points.

As before, let $\psi(p_n)=p_{n-1}$ and $w(p_n)=1$ for each $n>1$, and let $\psi(\infty)=\infty$ and $w(\infty)=1$. For each $i\geq 0$, choose a cyclic permutation of A_i of order 2^i, and let $\psi\restriction_{A_i}$ be this permutation. Then extend ψ to a homeomorphism of $\beta\omega$. Now, for each $i\geq 0$ choose a point $a_i\in A_i$—this choice of points will remain fixed for the rest of the construction. Let each $w(a_i)=-1$, and let $w(x)=1$ for each isolated point x of $\beta\omega$ which is not in $\{a_1, a_2, \ldots\}$. Then extend w continuously from the remainder of $\beta\omega$ into $\{-1,1\}$. Finally, let each $\mu(\{a_i\})=1/4^{i+1}$, and let $\mu(S)=0$ for any $S\subset X$ which does not intersect $\{a_1, a_2, \ldots\}$.

To show that T is shift, let $f\in\bigcap_{n=1}^{\infty}\operatorname{ran}(T^n)$. We will establish that $f\equiv 0$ in several steps. Without loss of generality, assume that $\|f\|\leq 1$. For each $n\geq 1$, we have

To simplify the notation, let

so that $f(p_n) = \sum_{i=0}^{\infty} g(i,n)$. Since $\|f\| \leq 1$, each $|g(i,n)| \leq 1/4^{i+1}$. Note that it is sufficient to prove that each $g(i,n) = 0$. For, if we can show this, then each $f(p_n) = 0$ and $f\left(\psi^{-n}(a_i)\right) = 0$ for all i, n. The latter equation implies that $f(x) = 0$ for every isolated point x of $\beta\omega$.

Now, fix some i and consider $g(i,n)$ for various values of n. Since $\psi \restriction_{A_i}$ is a cyclic permutation of order 2^i, we have that $\psi^{-n}(a_i) = \psi^{-n+2^i}(a_i)$. Also, exactly one of the values $w\left(\psi^{-(n+1)}(a_i)\right), w\left(\psi^{-(n+2)}(a_i)\right), \ldots, w\left(\psi^{-(n+2^i)}(a_i)\right)$ is -1. It follows that $g\left(i, n+2^i\right) = -g(i,n)$. Thus, for any k,

This formula will be the main tool for the rest of the proof.

The next step is to establish that $\lim_{n\to\infty} f(p_n) = 0$. Fix i, and consider the sum of $g(i,n)$ for 2^{i+1} consecutive second indices:

Thus, $\sum_{n=1}^{M} g(i,n) = 0$ for any even multiple M of 2^i. So for any k,

$$\sum_{n=1}^{2^k} f(p$$

Thus,

So, $\sum_{n=1}^{2^k} f(p_n) \to 0$ as $k \to \infty$. But this limit would be $+\infty$ if $\lim_{n\to\infty} f(p_n) > 0$. Similarly, $f(p_n)$ cannot converge to a negative number. Since the limit must converge, we have established that $\lim_{n\to\infty} f(p_n) = 0$.

In order to complete the proof, we use induction to establish the following somewhat cryptic claim:

Claim (4.1.20)[81]. *Suppose that $\varepsilon > 0$ and $N \in \omega$ are such that $\left|\sum_{i=0}^{N} g(i,n)\right| < \varepsilon$ holds for every $n \geq 1$. Then $|g(i,n)| < \varepsilon$ for each $i \in \{0, \dots, N\}$ and every $n \geq 1$.*

Note that the claim is trivial when $N = 0$. Suppose the claim holds for $N - 1$ and that $\left|\sum_{i=0}^{N} g(i,n)\right| < \varepsilon$ for all n. Now,

$$\sum_{i=0}^{N}$$

Thus

so $|g(N,n)| < \varepsilon$ for all n. On the other hand,

$$\sum_{i=0}^{N}[$$

Thus, $2\left|\sum_{i=0}^{N-1} g(i,n)\right| < 2\varepsilon$, so by induction, $|g(i,n)| < \varepsilon$ for all n and all $i \leq N - 1$. This establishes the claim.

Now, finally, fix some $\varepsilon > 0$. Choose an integer N such that $|f(p_n)| < \varepsilon/2$ for all $n \geq N$ and such that $\sum_{i=N+1}^{\infty} 1/4^{i+1} < \varepsilon/2$. Then for any $n \geq N$,

and

Thus, $\left|\sum_{i=1}^{N} g(i,n)\right| < \varepsilon$ for all $n \geq N$. But $g(i,n)$ is periodic in n, so $\left|\sum_{i=1}^{N} g(i,n)\right| < \varepsilon$ for all n, and thus by our claim $|g(i,n)| < \varepsilon$ for all $i \in \{0,\ldots,N\}$ and every n. Since ε is arbitrary and N can be chosen arbitrarily large, it follows that every $g(i,n) = 0$.

Obviously, the example of Theorem (5.1.19) is separable, but it does have some properties not found in previous examples. Not only does D_ψ fail to be dense, but no finite $F \subset X$ has the property that $\bigcup_{n=-\infty}^{\infty} \psi^n(F)$ dense. Also, even though the isolated points are dense, the space does not admit any ψ for which D_ψ would be dense.

Note also that it was not really necessary to use $\beta\omega$ in Theorem (5.1.19), although it allowed us to easily extend both ψ and w to the whole compactification. In fact, we really showed that the discrete sum of $\omega + 1$ and Y will admit a shift provided that Y is a compactification of ω whose isolated points can be partitioned into sets A_i of size 2^i such that there is a homeomorphism of Y which is cyclic on each A_i and such that there is also a continuous $w: Y \to \{-1, 1\}$ such that $w(x) = -1$ for exactly one x in each A_i. For our last example, we show that a metrizable such space exists.

Theorem (5.1.21)[81]: *There exists a compactmetric space X and a type 1 shift on $C(X)$ generated by ψ, w, and μ such that infinitely many isolated points of X are in $X \setminus D_\psi$.*

Proof. We will define as subset Y of the plane which admits a ψ and w as mentioned above. Then the discrete sum of $\omega + 1$ and Y will then be the desired space.

Let B denote of the subset of the x-axis consisting of those points whose x-coordinates are either 0 or $\pm 1/n$ for some $n \in \mathbb{N}$ (i.e., the union of two simple sequences converging to the origin). The set B will be the non-isolated points of Y. Each set A_i will be a subset of $B \times \{1/2^i\}$. The set A_0 consists of just the point $\langle 1, -1\rangle$. For

$i > 0$, A_i consists of the points whose x-coordinates are $\pm 1/n$ where $n \in \{1, \ldots, 2^{i-1}\}$.

Now let $Y = B \cup_{i \in \omega} A_i$.

Let $w(p) = -1$ for each point p of Y whose x-coordinate is -1, and let $w(p) = +1$ otherwise. To define ψ on B, fix the origin, send $\langle -1, \rangle 0$ to $\langle +1, 0 \rangle$, and move each other, point to the nearest point of B to its left. On A_0, ψ is the identity. For $i > 0$, define ψ by sending $\langle -1, 1/2^i \rangle$ to $\langle +1, 1/2^i \rangle$, and also move each other point of A_i to the nearest point of A_i to its left. Note that $\langle +1/2^{i-1}, 1/2^i \rangle$ goes to $\langle -1/2^{i-1}, 1/2^i \rangle$.

It is easily checked that w is continuous and ψ is a homeomorphism.

Corollary (5.1.22)[284]: *Let T_j be a type 1 pre-shift generated by ψ, w, and μ. Let $f_j \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T_j^n)$. Then if $f_j(x) = 0$ for all $j \geq 1$ and $x \in X \setminus D_\psi$, then $f_j \equiv 0$.*

Proof. For $f_j \in \bigcap_{n=1}^{\infty} \operatorname{ran}(T_j^n)$ and $f_j(x) = 0$ for all $x \in X \setminus D_\psi$. Thus $\lim_{n \to \infty} f_j(p_n) = 0$. For $\varepsilon > 0$ fixed, given N such that $|f_j(p_n)| < \varepsilon$ for all $n > N$.

Therefore

Hence $|f_j(p_n)| < \varepsilon$ for all $n \geq N$. By induction we get that $|f_j(p_n)| < \varepsilon$ for all $n \geq N$.

Hence, $f_j \equiv 0$.

Section (5.2): Problems for Isometric Shifts and Continuous Spaces:

The usual concept of shift operator in the Hilbert space ℓ^2 has been introduced in the Banach spaces in the following way in [57, 58]: Given a Banach space E over $\mathbb{K}$ (the field of real or complex numbers), a linear operator $T : E \to E$ is said to be an *isometric shift* if

(a) T is an isometry,

(b) The codimension of $T(E)$ in E is 1,

(c) $\bigcap_{n=1}^{\infty} T^n(E) = \{0\}$.

One of the main settings where isometric shifts have been studied is $E = C(X)$, that is, the Banach space of all $\mathbb{K}$-valued continuous functions defined on a compact and Hausdorff space X, equipped with its usual supremum norm. In this setting, major breakthroughs were made in [59] and [60]. On the one hand, in [59], Gutek, Hart, Jamison, and Rajagopalan studied in depth these operators. In particular, using the well-known Holsztyński's Theorem ([61]), they classified them into two types, called type I and type II. On the other hand, in [60], Haydon showed a general method for providing isometric shifts of type II, as well as concrete examples.

However, a very basic question has remained open since the publication in 1991 of [59]: If $C(X)$ admits an isometric shift, must X be separable? This question is only meaningful for type I isometric shifts since it was already proved in [59] isometric shifts yield the separability of X. Let us recall the definitions. If $T : C(X) \to C(X)$ is an isometric shift, then there exist a closed subset $Y \subset X$, a continuous and surjective map $\phi : Y \to X$, and a function $a \in C(Y), |a| \equiv 1$, such that $(Tf)(x) = a(x) \cdot f(\phi(x))$ for all $x \in Y$ and all $f \in C(X)$. T is said to be of type I if Y can be taken to be equal to $X\setminus\{p\}$, where $p \in X$ is an isolated point, and is said to be of type II if Y can be taken equal to X. Moreover, if T is of type I, then the map $\phi : X\setminus\{p\} \to X$ is indeed a homeomorphism.

Not much is known about the possibility of finding a nonseparable space X such that $C(X)$ admits an isometric shift since the problem was proposed. Interesting results in this direction say that such an X must have the countable chain condition (in [63] or

[62]). In [63], it is even proved that $C_0(X \setminus \mathrm{cl}_X\{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\})$ must have cardinality at most equal to c, that is, the cardinality of $\mathbb{R}$ (where, as usual, $\mathrm{cl}_X A$ denotes the closure of A in X and $C_0(Z)$ is the space of $\mathbb{K}$-valued continuous functions on Z vanishing at infinity).

From this fact, we can easily deduce that if $C(X)$ admits an isometric shift, then there exists a set S of cardinality at most c that is dense in X. To see it, we write $C_0(X \setminus \mathrm{cl}_X\{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\}) = \{f_\alpha : \alpha \in I \subset \mathbb{R}\}$. For each $\alpha \in I$ such that $f_\alpha \neq 0$, we pick a point $x_\alpha \in X$ such that $f_\alpha(x_\alpha) \neq 0$. Obviously, given any (nonempty) open set $U \subset X \setminus \mathrm{cl}_X\{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\}$, there exists $f_\alpha \neq 0$ whose support is contained in U. This implies that the set S consisting of the union of all points x_α and $\{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\}$ is dense in X.

We will give an answer in the negative to the separability question: There are indeed examples of isometric shifts on $C(X)$, with X not separable, and even having 2^c infinite components. The latter example can be connected with the question addressed in [58], where it was conjectured that the space X cannot have an infinite connected component (the only examples which appeared so far in the literature for type I isometric shifts, both of spaces containing exactly one infinite component, can be found in [59] and [64]; for the case of type II isometric shifts in the complex setting, see ([60], [61]). Related to this, one of the main results in [59] states that $C(X)$ does not admit any isometric shifts, whenever X has a countably infinite number of components, all of whom are infinite.

Some other papers have recently studied questions related to isometric shifts (also defined on other spaces of functions). Among them, we will mention for instance (see [65, 66, 67, 68, 69, 70, 71, 62, 72, 73]).

The unit circle in $\mathbb{C}$ will be denoted by $\mathbb{T}$. $L^\infty(\mathbb{T})$ will be the space of all Lebesgue-measurable essentially bounded complex-valued functions on $\mathbb{T}$, and $\mathfrak{M}$ will be its maximal ideal space. m will denote the Lebesgue measure on $\mathbb{T}$.

It is well known that if ρ is an irrational number, then the rotation map $[\rho] : \mathbb{T} \to \mathbb{T}$ sending each $z \in \mathbb{T}$ to $ze^{2\pi\rho i}$ satisfies that $\{[\rho]^n(z) : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$

for every $z \in \mathbb{T}$ (in [74]). Indeed, it is easy to see that this fact can be generalized to separable powers of $\mathbb{T}$, that is, those of the form $\mathbb{T}^\kappa$ for $\kappa \leq c$ (similarly as it is mentioned for finite powers in [73, 65, 6]): Let $\Lambda := \{\rho_\alpha : \alpha \in \mathbb{R}\}$ be a set of irrational numbers linearly independent over $\mathbb{Q}$; if $\mathbb{P}$ is any nonempty subset of $\mathbb{R}$ and $[\rho_\alpha]_{\alpha\in\mathbb{P}} : \mathbb{T}^\mathbb{P} \to \mathbb{T}^\mathbb{P}$ is defined as $[\rho_\alpha]_{\alpha\in\mathbb{P}} \ ((z_\alpha)_{\alpha\in\mathbb{P}}) := \left(z_\alpha e^{2\pi\rho_\alpha i}\right)_{\alpha\in\mathbb{P}}$, then the set $\{[\rho_\alpha]_{\alpha\in\mathbb{P}} \ ((z_\alpha)_{\alpha\in\mathbb{P}}) : n \in \mathbb{N}\}$ is dense in $\mathbb{T}^\mathbb{P}$ for every $(z_\alpha)_{\alpha\in\mathbb{P}} \in \mathbb{T}^\mathbb{P}$.

Given two topological spaces Z and W, we denote by $Z + F$ their topological sum, that is, the union $Z \cup W$ endowed with the topology consisting of unions of open subsets of these spaces (on [75]).

$\mathcal{N} := \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}, \ldots\}$ will be a discrete infinite countable space, and $\mathcal{N} \cup \{\infty\}$ will denote its one-point compactification. In our examples, the point **1** will play the same role as ρ in the definition of isometric shift of type I.

Throughout "homeomorphism" will be synonymous with "*surjective* homeomorphism".

We will usually write $T = T[a, \phi, \Delta]$ to describe a codimension 1 linear isometry $T : C(X) \to C(X)$, where X is compact and contains $\mathcal{N}$. It means that $\phi : X\backslash\{\mathbf{1}\} \to X$ is a homeomorphism, satisfying in particular $\phi(\boldsymbol{n} + \mathbf{1}) = \boldsymbol{n}$ for all $n \in \mathbb{N}$. It also means that $a \in C(X\backslash\{\mathbf{1}\}), |a| \equiv 1$, and that Δ is a continuous linear functional on $C(X)$ with $\|\Delta\| \leq 1$. Finally, the description of T we have is $(Tf)(x) = a(x)f(\phi(x))$, when $x \neq \mathbf{1}$, and $(Tf)(\mathbf{1}) = \Delta(f)$, for every $f \in C(X)$.

In general, given a continuous map f defined on a space X, we also denote by f its restrictions to subspaces of X and its extensions to other spaces containing X.

All results will be valid in the real and complex settings, unless otherwise stated.

The only exceptions are the following: Results exclusively given for $\mathbb{K} = \mathbb{C}$. The only result valid just for the case $\mathbb{K} = \mathbb{R}$ is given in Example (5.1.10). $C_\mathbb{C}(X)$ and $C_\mathbb{R}(X)$ will denote the Banach spaces of continuous functions on X taking complex and real values, respectively.

It is well known that $\mathfrak{M}$ is extremally disconnected, that is, the closure of each open subset is also open. In fact, each measurable subset A of $\mathbb{T}$ determines via the

Gelfand transform an open and closed subset $\mathbf{G}(A)$ of $\mathfrak{M}$, and the sets obtained in this way form a basis for its topology (see [76]). Now it is straightforward to see that $\mathfrak{M}$ is not separable: Let (x_n) be a sequence in $\mathfrak{M}$, and consider a partition $(a.e)$ of $\mathbb{T}$ by k arcs of equal length, $k \geq 3$. This determines a partition of $\mathfrak{M}$ into k closed and open subsets of $\mathbb{T}$. Select the arc A_1 such that $\mathbf{G}(A_1)$ contains x_1. Next do the same process with k^2 arcs of equal length, and pick A_2 with $x_2 \in \mathbf{G}(A_2)$. Repeat the process infinitely many times, in such a way that each time we take A_n of length $1/k^n$ such that $x_n \in \mathbf{G}(A_n)$. It is clear that if $A := \bigcup_{n=1}^{\infty} A_n$, then $m(A) < 2\pi$, so $\mathbf{G}(\mathbb{T}\backslash A)$ is a nonempty closed and open subset of $\mathfrak{M}$ containing no point x_n.

Notice that, since $\mathfrak{M}$ is not separable, every isometric shift on $C(\mathfrak{M})$ must be of type I. But there are none because $\mathfrak{M}$ has no isolated points. Even more, in [59], it is proved that no space $L^\infty(Z, \sum, \mu)$ admits an isometric shift if μ is non-atomic.

As usual, we consider $\mathbb{T}$ oriented counterclockwise, and denote by $A(\alpha, \beta)$ the (open) arc of $\mathbb{T}$ beginning at $e^{i\alpha}$ and ending at $e^{i\beta}$.

Theorem (5.2.1)[56]: $C(\mathfrak{M} + \mathcal{N} \cup \{\infty\})$ *admits an isometric shift*.

Once we have a first example, we can get more. For instance, the next result is essentially different in that it provides examples with 2^c infinite connected components.

Proof. We start by defining a linear and surjective isometry on $L^\infty(\mathbb{T})$. We first consider the rotation $\psi(z) := ze^i$ for every $z \in \mathbb{T}$, and then define the isometry $S : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ as $Sf := f \circ \psi$ for every $f \in L^\infty(\mathbb{T})$. On the other hand, using the Gelfand transform we have that the Banach algebra $L^\infty(\mathbb{T})$ is isometrically isomorphic to $C(\mathfrak{M})$, so S determines a linear and surjective isometry $T_S : C(\mathfrak{M}) \to C(\mathfrak{M})$. Also, by the Banach-Stone theorem, there exists a homeomorphism $\phi : \mathfrak{M} \to \mathfrak{M}$ such that $T_S f = -f \circ \phi$ for every $f \in C(\mathfrak{M})$. Notice that this is valid both in the real and complex cases (see for instance [77]).

Let $X := \mathfrak{M} + \mathcal{N} \cup \{\infty\}$. The definition of T_S can be extended to a new isometry $T : C(X) \to C(X)$ in three steps. First, for each $f \in C(X)$, we put $(Tf)(x) := (T_S f)(x)$ if $x \in \mathfrak{M}$. In the same way $(Tf)(\mathbf{n}) := (f \circ \phi)(\mathbf{n})$ if $\mathbf{n} \in \mathcal{N} \cup \{\infty\} \backslash \{1\}$ (where $\phi :$

$\mathcal{N}\backslash\{\mathbf{1}\} \to \mathcal{N}$ is the canonical map sending each $\mathbf{n}$ into $\mathbf{n}-\mathbf{1}$, which obviously can be extended as $\phi(\infty) := \infty$). Finally, we put

where $\Phi := (\sqrt{5}-1)/2$ is the golden ratio conjugate. It is easy to verify that T is a codimension one linear isometry, so we just need to prove that $\bigcap_{i=1}^{\infty} T^i(C(X)) = \{\mathbf{0}\}$. Suppose then that $f \in \bigcap_{i=1}^{\infty} T^i(C(X))$. It is easy to check that

On the other hand, if we fix any $\alpha \in \mathbb{T}$, then there exist two increasing sequences (n_k) and (m_k) in $2\mathbb{N}$ and $2\mathbb{N}+1$, respectively, converging to α mod 2π. An easy application of the Dominated Convergence Theorem proves that $\int_{A(\alpha,\alpha+2\pi\Phi)} f dm = 2\pi \lim f(\mathbf{n}_k)$, and $\int_{A(\alpha,\alpha+2\pi\Phi)} f dm = -2\pi \lim f(\mathbf{m}_k)$. By continuity, we deduce that

Obviously, this implies that $\int_{A(\alpha,\alpha+2\pi\Phi)} f dm = 0$ for every $\alpha \in \mathbb{T}$, and $f(\infty) = 0$. In particular this proves that $f(\mathbf{n}) = 0$ for every $\mathbf{n} \in \mathcal{N}$. As a consequence we can identify $f \in \bigcap_{n=1}^{\infty} T^n(C(X))$ with an element $f \in L^\infty(\mathbb{T})$ satisfying $\int_{A(\alpha,\alpha+2\pi\Phi)} f dm = 0$ for every $\alpha \in \mathbb{T}$. On the other hand, it is clear that we may assume that f takes values just in $\mathbb{R}$.

Claim (5.2.2)[56]: *$\int_{A(\alpha,\alpha+2\pi\Phi^n)} f dm = (-1)^n F(n-1) \int_{\mathbb{T}} f dm$ for every $\alpha \in \mathbb{T}$ and $n \in \mathbb{N}$, where $F(n)$ denotes the nth Fibonacci number.*

Let us prove the claim inductively on n. We know that it holds for $n = 1$. Also notice that $\Phi + \Phi^2 = 1$, so $\Phi^n + \Phi^{n+1} = \Phi^{n-1}$ for every $n \in \mathbb{N}$. The case $n = 2$ is immediate because, since

$\mathbb{T}$

a. e., then we have $\int_{\mathbb{T}} \; fdm = \int_{A(\alpha,\alpha+2\pi\Phi^2)} fdm$ for every $\alpha \in \mathbb{T}$.

Now assume that, given $k \geq 2$, the claim is true for every $n \leq k$. Then we see that, for any $\alpha \in \mathbb{T}$,

$A(\alpha, \alpha + 2\pi\Phi^{k-}$

a. e., so

$(-1)^{k-1}F($

and the conclusion proves the claim (5.2.2).

The claim, combined with the fact that f is essentially bounded, implies that $\int_{\mathbb{T}} \; fdm = 0$, and consequently $\int_{A(\alpha,\alpha+2\pi\Phi^n)} fdm = 0$ for every $\alpha \in \mathbb{T}$ and every $n \in \mathbb{N}$. Now, it is easy to see that if U is an open subset of $\mathbb{T}$, then U is the union of countably many pairwise disjoint arcs whose lengths belong to the set $\{2\pi\Phi^n : n \in \mathbb{N}\}$. Now, applying again the Dominated Convergence Theorem, we see that $\int_U \; fdm = 0$. Obviously, this implies that $\int_K \; fdm = 0$ whenever $K \subset \mathbb{T}$ is compact.

Finally take $C^+ := \{z \in \mathbb{T} : \; f(z) > 0\}$. We know that there exists a sequence of compact subsets K_n of C^+, with $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$, and such that $\lim_{n\to\infty} m(C^+ \backslash K_n) = 0$. Clearly, the above fact and the Monotone Convergence Theorem imply that $\int_{C^+} fdm = 0$, and then $m(C^+) = 0$. Now we can easily conclude that $f \equiv 0$ a. e., and consequently T is a shift.

Next we prove Theorem (5.2.3). It provides nonseparable examples with 2^c infinite connected components, each homeomorphic to a (finite or infinite dimensional) torus: It follows from the fact that $\mathfrak{M}$ is homeomorphic to an infinite closed subset of $\beta\mathbb{N}\backslash\mathbb{N}$ that its cardinality must be 2^c (on [78] and [79]).

Theorem (5.2.3)[56]: *Let κ be any cardinal such that $1 \leq \kappa \leq c$. Then $C(\mathfrak{M} \times \mathbb{T}^\kappa + \mathcal{N} \cup \{\infty\})$ admits an isometric shift.*

Finally, we can also give examples with just one infinite component.

Proof. Write the isometric shift $T : C(\mathfrak{M} + \mathcal{N} \cup \{\infty\}) \to C(\mathfrak{M} + \mathcal{N} \cup \{\infty\})$ given in the proof of Theorem (5.2.1) as $T = T[a, \phi, \Delta]$. Obviously, $\Delta \equiv 0$ on $C(\mathcal{N} \cup \{\infty\})$, and it can be considered as an element of $C(\mathfrak{M})'$.

Consider a subset $\mathbb{P}$ of $\mathbb{R}$ with cardinal equal to κ, and suppose that $\{1/2\pi\} \cup \{\rho_\alpha : \alpha \in \mathbb{P}\}$ is a family of real numbers linearly independent over $\mathbb{Q}$. Then put $\rho_\kappa \coloneqq [\rho_\alpha]_{\alpha \in \mathbb{P}}$.

Define $\phi_\kappa : \mathfrak{M} \times \mathbb{T}^\kappa \to \mathfrak{M} \times \mathbb{T}^\kappa$ as $\phi_\kappa(x, \boldsymbol{z}) \coloneqq (\phi(x), \rho_\kappa(\boldsymbol{z}))$ for every $x \in \mathfrak{M}$, and $\boldsymbol{z} \in \mathbb{T}^\kappa$. Select now a point $\mathbf{v}_\kappa$ in $\mathbb{T}^{\mathbb{P}} = \mathbb{T}^\kappa$, and consider the evaluation map $\Gamma_{\mathrm{v}_\kappa} \in C(\mathbb{T}^\kappa)'$. Both Δ and $\Gamma_{\mathrm{v}_\kappa}$ are positive linear functionals, and so is the product $\Delta \times \Gamma_{\mathrm{v}_\kappa} \in C(\mathfrak{M} \times \mathbb{T}^\kappa)'$, which also satisfies $\|\Delta \times \Gamma_{\mathrm{v}_\kappa}\| \leq 1$ (see [80]).

Given $f \in C(\mathfrak{M} \times \mathbb{T}^\kappa)$ and $\boldsymbol{z} \in \mathbb{T}^\kappa$, we write $f_{\boldsymbol{z}} : \mathfrak{M} \to \mathbb{K}$ meaning $f_{\boldsymbol{z}}(x) \coloneqq f(x, \boldsymbol{z})$ for every $x \in \mathfrak{M}$. Obviusly $f_{\boldsymbol{z}}$ belongs to $C(\mathfrak{M})$, and $(\Delta \times \Gamma_{\mathrm{v}_\kappa})(f) = \Gamma_{\mathrm{v}_\kappa}\left(\Delta(f_{\mathrm{v}_\kappa})\right) = \Delta(f_{\mathrm{v}_\kappa})$.

Now, for $X_\kappa \coloneqq \mathfrak{M} \times \mathbb{T}^\kappa + \mathcal{N} \cup \{\infty\}$, define $a_\kappa \in C(X_\kappa \backslash \{\mathbf{1}\})$ as $a_\kappa \equiv -1$ on $\mathfrak{M} \times \mathbb{T}^\kappa$, and $a_\kappa \equiv 1$ everywhere else, and put $T_\kappa \coloneqq T[a_\kappa, \phi_\kappa, \Delta \times \Gamma_{\mathrm{v}_\kappa}]$.

Let $\psi : \mathbb{T} \to \mathbb{T}$ and Φ be as in the proof of Theorem (5.2.1). Given $f \in \bigcap_{i=1}^\infty T_\kappa^i(X_\kappa)$, we have that for every $k \in \mathbb{N}$, To continue with the proof, we need an elementary result:

Claim (5.2.4)[56]. *Suppose that* $(\boldsymbol{z}_\lambda)_{\lambda\in D}$ *is a net in* $\mathbb{T}^\kappa$ *converging to* $\boldsymbol{z}_0$. *Then* $\lim_\lambda \left\| f_{\boldsymbol{z}_\lambda} - f_{\boldsymbol{z}_0} \right\| = 0.$

Let us prove the claim (5.2.4). If it is not true, then there is an $\epsilon > 0$ such that, for every $\lambda \in D$, there exists $\nu \in D, \nu \geq \lambda$, such that $\left\| f_{\boldsymbol{z}_\nu} - f_{\boldsymbol{z}_0} \right\| \geq \epsilon$. It is easy to see that the set E of all $\nu \in D$ satisfying the above inequality is a directed set, and that $(\boldsymbol{z}_\nu)_{\nu\in E}$ is a subnet of $(\boldsymbol{z}_\lambda)_{\lambda\in D}$. Moreover there is a net $(x_\nu)_{\nu\in E}$ in $\mathfrak{M}$ such that $|f(x_\nu, \boldsymbol{z}_\nu) - f(x_\nu, \boldsymbol{z}_0)| \geq \epsilon$ for every $\nu \in E$. Since $\mathfrak{M} \times \mathbb{T}^\kappa$ is compact, there exist a point $(x_0, \boldsymbol{z}'_0) \in \mathfrak{M} \times \mathbb{T}^\kappa$ and a subnet $\left(x_\eta, \boldsymbol{z}_\eta\right)_{\eta\in F}$ of $(x_\nu, \boldsymbol{z}_\nu)_{\eta\in E}$ converging to $(x_0, \boldsymbol{z}'_0)$. Obviously $\left(\boldsymbol{z}_\eta\right)_{\eta\in F}$ is a subnet of $(\boldsymbol{z}_\nu)_{\eta\in E}$, so $\boldsymbol{z}_0 = \boldsymbol{z}'_0$. Consequently both $\left(x_\eta, \boldsymbol{z}_\eta\right)_{\eta\in F}$ and $\left(x_\eta, \boldsymbol{z}_0\right)_{\eta\in F}$ converge to $(x_0, \boldsymbol{z}_0)$. Taking limits, this implies $|f(x_0, \boldsymbol{z}_0) - f(x_0, \boldsymbol{z}_0)| \geq \epsilon$, which is absurd.

Now, fix $(\alpha, \boldsymbol{w}) \in \mathbb{T} \times \mathbb{T}^\kappa$ and $\epsilon > 0$. We know that $(\alpha, \boldsymbol{w})$ belongs to the closure of both

$j = 0, 1$. We first consider the case $j = 0$, and take a net $(y_\lambda)_{\lambda\in D} = \left(e^{in_\lambda}, \rho_\kappa^{-n_\lambda}(\mathbf{v}_\kappa)\right)_{\lambda\in D}$ in $\mathbf{N}_0$ converging to $(\alpha, \boldsymbol{w})$. Since $\left(e^{in_\lambda}\right)_{\lambda\in D}$ converges to α, there exists $\lambda_1 \in D$ such that

for every $\lambda \geq \lambda_1$.

On the other hand, by the claim, there exists $\lambda_2 \in D$ such that, if $\lambda \geq \lambda_2$, then $\left\| f_{\mathrm{w}} - f_{\rho_\kappa^{-n_\lambda}(\mathrm{v}_\kappa)} \right\| < \epsilon/4\pi$, so for every $\nu \in D$. We easily deduce that

and consequently $2\pi f(\infty) = \int_{A(\alpha,\alpha+2\pi\Phi)} f_{\mathrm{w}} dm$. In a similar way, working with $\mathbf{N}_1$, we see that $2\pi f(\infty) = -\int_{A(\alpha,\alpha+2\pi\Phi)} f_{\mathrm{w}} dm$. With the same arguments as in the proof of Theorem (5.2.1), we conclude that $f_{\mathrm{w}} \equiv 0$, and finally $f \equiv 0$, as we wanted to prove.

Theorem (5.2.5)[56]: *Let κ be any cardinal such that $1 \leq \kappa \leq c$. Then $C(\mathfrak{M} \times \mathbb{T}^\kappa + \mathcal{N} \cup \{\infty\})$ admits an isometric shift.*

Proof. Notice first that $L^\infty(\mathbb{T})$ is isometrically isomorphic to $L^\infty(\mathbb{T}_1 \cup \mathbb{T}_2)$, where $\mathbb{T}_i, i = 1, 2$, are disjoint copies of $\mathbb{T}$ endowed with the Lebesgue measure. It is not hard to see that this implies that $C(\mathfrak{M})$ and $C(\mathfrak{M} + \mathfrak{M})$ are isometrically isomorphic, so $\mathfrak{M}$ and $\mathfrak{M} + \mathfrak{M}$ are homeomorphic. Assume that $T = T[a, \phi, \Delta]$ is the isometric shift given in the proof of Theorem (5.2.1). We first define a homeomorphism $\chi : \mathfrak{M} \times \{0, 1\} \to \mathfrak{M} \times \{0, 1\}$ as $\chi(x, i) = (\phi(x), i + 1 \bmod 2)$ for every (x, i). For $i = 0, 1$, and $f \in C(\mathfrak{M} \times \{0, 1\})$, denote by $f \times \{i\}$ its restriction to $\mathfrak{M} \times \{i\}$, and put $\Delta_i(f) := \Delta(f \times \{i\})$.

Let $\rho_\kappa : \mathbb{T}^\kappa \to \mathbb{T}^\kappa, \mathbf{v}_\kappa$, and ⍰$\Gamma_{\mathbf{v}_\kappa}$ be as in the proof of Theorem (5.2.3).

Finally consider $X_\kappa := \mathfrak{M} \times \{0, 1\} + \mathbb{T}^\kappa + \mathcal{N} \cup \{\infty\}$, and define $T_\kappa : C(X_\kappa) \to C(X_\kappa)$ to be $T_\kappa := T[a_\kappa, \phi_\kappa, \Delta_\kappa]$, where

(i) $a_\kappa \equiv -1$ on $\mathfrak{M} \times \{0\} \cup \mathbb{T}^\kappa$, and $a_\kappa \equiv 1$ everywhere else.

(ii) $\phi_\kappa = \chi$ on $\mathfrak{M} \times \{0, 1\}$, and $\phi_\kappa = \rho_\kappa$ on $\mathbb{T}^\kappa$.

(iii) $\Delta_\kappa := (\Delta_0 + \Delta_1 +$ ⍰$\Gamma_{\mathbf{v}_\kappa})/3$.

As above, if $f \in \bigcap_{n=1}^\infty \mathbb{T}_\kappa^n(C(X_\kappa)), k \in \mathbb{N}$, and then

$3f(\mathbf{k})$

$=$

$=$

Next fix $\alpha \in \mathbb{T}, \mathbf{w} \in \mathbb{T}^{\kappa}$, and for $j = 0, 1, 2, 3$, take increasing sequences $\left(n_k^j\right)$ in $4\mathbb{N} + j$ such that $\lim_{k\to\infty} n_k^j = \alpha \bmod 2\pi$, and $\lim_{k\to\infty} f\left(\rho_{\kappa}^{-n_k^j}(\mathbf{v}_{\kappa})\right) = f(\mathbf{w})$. Now put

for $i = 0, 1$. Taking into account that $\tau\left(n_k^j\right)$ is constant for each j, and that $\tau(2) = 1 = \tau(3)$, and $\tau(1) = 0 = \tau(4)$, we have that the following equalities hold:

3

We deduce that $\mathbf{X}_i^\alpha = 0$ for every $\alpha \in \mathbb{T}$ and $i = 0, 1,$ and that $f \equiv 0$ on $\mathbb{T}^\kappa$. As in the proof of Theorem (5.2.1), we easily conclude that $f \equiv 0$.

We show that in the complex setting, it is possible to obtain nonseparable examples with arbitrary (finitely many) infinite connected components. For the different behavior in the real setting, see Example (5.1.10).

The first result is indeed given for separable examples. The idea of the proof is used in Theorem (5.2.9) to obtain nonseparable examples. In both cases $\mathbb{T}^0$ denotes the set $\{0\}$.

Definition (5.2.6)[56]: Let X be compact and Hausdorff, and suppose that $T = T[a, \phi, \Delta] : C(X) \to C(X)$ is an isometric shift of type I. For $n \in \mathbb{N}$, we say that T is n-*generated* if n is the least number with the following property: There exist n points $x_1, \dots, x_n \in X \setminus \mathrm{cl}_X\mathcal{N}$ such that the set

is dense in $X \setminus \mathrm{cl}_X\mathcal{N}$.

Notice that isometries simultaneously of types I and II are always 1-generated (in [59]), so the next theorem provides a way for constructing isometries that are not of type II.

Theorem (5.2.7)[56]: *Let $\mathbb{K} = \mathbb{C}$. Suppose that $n \in \mathbb{N}$, and that $(\kappa_j)_{j=1}^n$ is a finite sequence of cardinals satisfying $0 \le \kappa_j \le c$ for every j. Then there exists an n-generated isometric shift $T_n : C_\mathbb{C}(X_n) \to C_\mathbb{C}(X_n)$, where $X_n = \mathbb{T}^{\kappa_1} + \cdots + \mathbb{T}^{\kappa_n} + \mathcal{N} \cup \{\infty\}$.*

Proof. Let $\mathbb{P}_1, \dots, \mathbb{P}_n$ be any pairwise disjoint subsets of $\mathbb{R}$ of cardinalities $\kappa_1, \dots, \kappa_n$, respectively. Consider any family $\Lambda := \{\rho_\alpha : \alpha \in \mathbb{R}\}$ of real numbers linearly independent over $\mathbb{Q}$, and put $\sigma_j := [\rho_\alpha]_{\alpha \in \mathbb{P}_j}$ for each $j \le n$ (in the case when $\kappa_j = 0$, that is, $\mathbb{P}_j = \emptyset, \sigma_j$ is the identity). Also let $\mathbf{v}_j$ be a point in $\mathbb{T}^{\mathbb{P}_j}$.

Next write $X_n := \mathbb{T}^{\mathbb{P}_n} + \cdots + \mathbb{T}^{\mathbb{P}_1} + \mathcal{N} \cup \{\infty\}$, and define $\phi_n : X_n \to X_n$ as σ_j on each $\mathbb{T}^{\mathbb{P}_j}$. For $j \leq n$, let $z_j \in C\backslash\{0\}$, with $|z_j| \leq 1/2^j$, and $\zeta_j := e^{i\pi/2^{j-1}}$. Define a codimension 1 linear isometry T_n on $C_{\mathbb{C}}(X_n)$ as $T_n := T[a_n, \phi_n, \Delta_n]$, where $a_n \equiv \zeta_j$ on $\mathbb{T}^{\mathbb{P}_j}$ for each $j \leq n$, and $a_n \equiv 1$ on $\mathcal{N} \cup \{\infty\}$, and where $\Delta_n(f) := \sum_{i=1}^{n} z_j f(\mathbf{v}_j)$ for every f.

Of course, the construction of T_n depends on our choice of the sets $\mathbb{P}_j$ and Λ, the points $\mathbf{v}_j$, and the numbers z_j. We will prove that for any choices, the operator T_n satisfies the theorem.

We will do it inductively on n. We start at $n = 1$. It is easy to see that $T_1 : C_{\mathbb{C}}(X_1) \to C_{\mathbb{C}}(X_1)$ is an isometric shift (both of type I and type II). Now let us show that if T_n is an n-generated isometric shift for $n = l \in \mathbb{N}$, then T_{l+1} is an $(l+1)$-generated isometric shift.

Suppose that $f \in \bigcap_{m=1}^{\infty} T_{l+1}^{m}\left(C_{\mathbb{C}}(X_{l+1})\right)$. It is easy to check that

whenever $k \in \mathbb{N}$.

Fix $x_1 \in \mathbb{T}^{\mathbb{P}_1}, \ldots, x_{l+1} \in \mathbb{T}^{\mathbb{P}_{l+1}}$. For $j = 0, 1$, we can take increasing sequences $\left(n_k^j\right)$ in $2^{l+1}\mathbb{N}$ and $2^{l+1}\mathbb{N} + 2^l$, respectively, such that the sequences

converge to $\left(f(x_1), \ldots, f(x_{l+1})\right) \in \mathbb{C}^{l+1}$ for $j = 0, 1$.

This means, on the one hand, that

And, on the other hand,

We deduce that $f(x_{l+1}) = 0$, that is, $f \equiv 0$ on $\mathbb{T}^{\mathbb{P}_{l+1}}$, and consequently $f \in \bigcap_{m=1}^{\infty} T_l^m(C_{\mathbb{C}}(X_l))$. Since T_l is a shift, we conclude that $f \equiv 0$ on X_{l+1}. It is also easy to see that T_{l+1} is $(l+1)$-generated.

Theorem (5.2.8)[56]: Let $\mathbb{K} = \mathbb{C}$. Suppose that $n \in \mathbb{N}$, and that $(\kappa_j)_{j=1}^{n}$ is a finite sequence of cardinals satisfying $0 \leq \kappa_j \leq c$ for every j. Then there exists an isometric shift $T_n^{\mathfrak{M}}: C_{\mathbb{C}}(X_n^{\mathfrak{M}}) \to C_{\mathbb{C}}(X_n^{\mathfrak{M}})$, where $X_n^{\mathfrak{M}} = \mathfrak{M} + \mathbb{T}^{\kappa_1} + \cdots + \mathbb{T}^{\kappa_n} + \mathcal{N} \cup \{\infty\}$.

Proof. The proof is similar to that of Theorem (5.2.7). We consider the homeomorphism ϕ on $\mathfrak{M}$ coming from the rotation $\psi : \mathbb{T} \to \mathbb{T}$ given in the proof of Theorem (5.2.1). Fix $n \in \mathbb{N}$, and assume that X_n and $T_n = T[a_n, \phi_n, \Delta_n]$ are as in the proof of Theorem (5.2.7). Take $z_{n+1} \in \mathbb{C} \setminus \{0\}$ such that $|z_{n+1}| \leq 1/2^{n+1}$, and put $\zeta_{n+1} := e^{i\pi/2^n}$. We are going to define an isometric shift on $X_n^{\mathfrak{M}}$. First put

Obviously we are assuming that $1/2\pi$ does not belong to the linear span (over $\mathbb{Q}$) of $\{\rho_\alpha : \alpha \in \mathbb{P}_1 \cup \cdots \cup \mathbb{P}_n\}$. Let $a_n^{\mathfrak{M}} \in C_{\mathbb{C}}(X_n^{\mathfrak{M}})$ be equal to ζ_{n+1} on $\mathfrak{M}$, and equal to an on X_n, and let $\phi_n^{\mathfrak{M}}: X_n^{\mathfrak{M}} \to X_n^{\mathfrak{M}}$ be defined as ϕ_n on X_n, and as ϕ on $\mathfrak{M}$. We consider $T_n^{\mathfrak{M}} := T[a_n^{\mathfrak{M}}, \phi_n^{\mathfrak{M}}, \Delta_n^{\mathfrak{M}}]$. Following the same process as in the proof of Theorem (5.2.8), we easily obtain that $0 = z_{n+1} \int_{A(\alpha, 2\pi\Phi+\alpha)} f dm$ for every $\alpha \in \mathbb{T}$. As in the proof of Theorem (5.2.1), we see that $f \equiv 0$ on $\mathbb{T}$, which is to say that $f \equiv 0$ on $\mathfrak{M}$. We deduce that $f \in \bigcap_{m=1}^{\infty} T_n^m(C_{\mathbb{C}}(X_n))$, and consequently $f \equiv 0$.

The next example shows in fact that the procedure followed above is no longer valid when dealing with $\mathbb{K} = \mathbb{R}$.

Example (5.2.9)[56]: Let $\mathbb{K} = \mathbb{R}$. Suppose that $X = Y + X_1 + X_2 + X_3$ is compact, where each X_j is connected and nonempty, and $\mathcal{N} \subset Y$. Let $T = T[a, \phi, \Delta]$ be a codimension 1 linear isometry on $C_{\mathbb{R}}(X)$, and assume that $\phi(X_j) = X_j, j = 1, 2, 3$. Let us see that T is not a shift. First, there are $j, k, j \neq k$, with $a(X_j) = a(X_k) \in \{-1, 1\}$. There are also $\alpha_j, \alpha_k \in \mathbb{R}$ such that $|\alpha_j| + |\alpha_k| > 0$ and $\Delta(\alpha_j \xi X_j + \alpha_k \xi X_k) = 0$, where ξ_A denotes the characteristic function on A. It is easy to check that $\alpha_j \xi X_j + \alpha_k \xi X_k$ belongs to $T^n(C_{\mathbb{R}}(X))$ for every $n \in \mathbb{N}$, and consequently T is not a shift.

In particular, we see that neither $C_{\mathbb{R}}(\mathbb{T} + \mathbb{T}^2 + \mathbb{T}^3 + \mathcal{N} \cup \{\infty\})$ nor $C_{\mathbb{R}}(\mathfrak{M} + \mathbb{T} + \mathbb{T}^2 + \mathbb{T}^3 + \mathcal{N} \cup \{\infty\})$ admit an isometric shift.

Chapter 6

Finite Strictly Singular Operators on James Spaces

However, we exhibit examples of strictly singular operators without nontrivial closed invariant subspaces. So, though it may be true that operators on the spaces of Gowers and Maurey have invariant subspaces, yet this cannot be because of a general result about strictly singular operators. The general assertion about strictly singular operators is false. As a consequence, we obtain that the strictly singular operator with no invariant subspaces constructed by C. Read is actually finitely strictly singular. These results are deduced from the following fact: if $k \leq n$ then every k-dimensional subspace of $\mathbb{R}^n$ contains a vector x with $\|x\|_{\ell_\infty} = 1$ such that $x_{m_i} = (-1)^i$ for some $m_1 < \cdots < m_k$. In addition we deduce different examples of Strictly Singular operators of Cauchy sequences. Without nontrivial closed invariant subspaces. We show that the formal inclusion operator, in James space, is finitely Strictly Singular.

Section (6.1): Invariant subspace Problems and Strictly Singular Operators:

Operators without invariant subspaces were first found independently by P.Enflo by c_0 and l_1 on an unknown Banach space. They were found on and ([164], [165]), ([166], [167]) and various extensions of the method were found ([168], [169], [170], [171], [172]), of which the nearest to the present construction is the construction of a in [172]. A general account of l_1 quasinilpotent operator without invariant subspaces on ,the theory of invariant subspaces, written before all these counterexamples were found will J will be found in [173]. A short account of the basic properties of the James space be found in Singer, [174]. The original article is [175].

are normed spaces, is *norm* F and E where $T : E \to F$, A continuous linear map

$\varepsilon > 0$ such that if there is an *increasing*

.$x \in E$ for all

are Banach spaces, is said to F and E where A continuous linear map $T : E \to F$,

is $T|_W$ $W \subset E$ such that if there is no infinite dimensional subspace *strictly singular* be

.norm increasing

$(a_i)_{i=1}^{\infty} \in c_0$ such is the set of all sequences $J_p\big(1 < p < 1\big)$ space-p The James that

$$\|a\| = \Bigg\{$$

It is a fact that J_p is nonreflexive, $\dim\big(J_p^{**}/J_p\big) = 1$, but that every infinite dimensional subspace of J_p contains a subspace isomorphic to l_p.

It is well known that on $l_p(1 \le p < \infty)$ or c_0, any strictly singular operator is compact. On the other hand the inclusion map $l_p \hookrightarrow J_q(1 < p < q < \infty)$ is strictly singular but not compact [135, 136, 139, 140, 141]. For our purposes we want something like the inclusion map $l_p \hookrightarrow l_q$, but which happens between non-reflexive Banach spaces; (and which happens, let it be said, in a manner which has respect for the nonreflexivity, in the sense that there is a sequence of unit vectors in the domain space with no weak -* convergent subsequence, which is mapped to a sequence in the image space which also has no weak-* convergent subsequence).

For such a map we look to the James p-spaces J_p.

Lemma (6.1.1)[163]: *The natural inclusion $i : J_p \hookrightarrow J_q(1 < p < q < \infty)$ is strictly singular.*

Proof. If not, there is an infinite dimensional subspace $E \subset J_p$ on which the norms $\|\cdot\|_{J_p}$ and $\|\cdot\|_{J_q}$ are equivalent. Taking a subspace of E as necessary, this tells us that $\big(E, \|\cdot\|_{J_q}\big)$ is isomorphic to l_p (for every infinite dimensional subspace of J_p contains a subspace isomorphic to l_p (0.5)). Taking a further subspace, we find l_q embedded up to isomorphism in l_p; which is nonsense.

Definition (6.1.2)[163]: *Let us choose, once and for all, a strictly increasing sequence $(p_i)_{i=1}^{\infty}$ of real numbers strictly greater than 2. The Banach space X is defined as the l_2-direct sum*

It is on this Banach space X that we will construct a strictly singular operator without invariant subspaces.

We will write $(f_{ij})_{j=0}^{\infty}$ for the unit vector basis of J_{p_i}, and $(f_{0j})_{j=0}^{\infty}$ for the unit vector basis of the space l_2.

An element $\mathbf{x} \in X$ can be regarded as a sequence $(\boldsymbol{x}_i)_{i=0}^{\infty}$ with $\mathbf{x}_0 \in l_2, \mathbf{x}_i \in J_{p_i} (i > 0)$. It can be shown that if $(\delta_i)_{i=0}^{\infty}$ is a sequence of scalars tending to zero, then the "weighted shift" operator

is strictly singular. We will construct an operator on X without invariant subspaces, which has a good deal in common with a weighted shift W.

The next few definitions follow [166].

Definition (6.1.3)[163]: *Our construction will be built around a strictly increasing sequence $\mathrm{d} = (d_i)_{i=1}^{\infty}$ of positive integers.*

This sequence will be required to "increase sufficiently rapidly" in the sense of [165]. We will write $a_i = d_{2i-1} (i = 1, 2, \dots)$ and $b_i = d_{2i}$. Thus, $a_1 < b_1 < a_2 < b_2 < \cdots$. We define $a_0 = 1, v_0 = 0, v_n = n(a_n + b_n)\ (n > 0)$. We will use the symbol $p\mathbf{d}$ to mean, "provided $\mathbf{d}$ increases sufficiently rapidly", as we did in [R4]. We define $w_n = 1 + \sum_{r=0}^{n-1}(1 + v_r), w_0 = 1$.

Definition (6.1.4)[163]: *F will denote the dense subspace of X spanned by the unit vectors $\{f_{ij}, i \geq 0, j \geq 0\}$. If $S \subset \mathbb{Z}^+ \times \mathbb{Z}^+, F_S \subset F$ will denote the linear span of the set $\{f_{ij} : (i,j) \in S\}$. π_S will denote the projection $F \to F_S$ such that $\pi_S(f_{ij}) = f_{ij}((i,j) \in S)$ or $0\ ((i,j) \notin S)$. π_S is continuous only for certain choices of S; we shall not be using any S for which it is discontinuous, however.*

f_{ij}^* will denote the norm-1 linear functional on F such that $f_{ij}^*(f_{kl}) = \delta_{ik}\delta_{jl}$.

Definition (6.1.5)[163]: *Let $|p|$ denote the sum of the absolute values of the coefficients of the polynomial p. For a finite set S, let $|S|$ denote the number of elements of S.*

We will now define, in terms of the sequence $\mathbf{d}$ as in Definition (6.1.3), a sequence $(e_i)_{i=0}^{\infty}$ whose linear span is the dense subspace F of X.

We shall begin by rearranging the fundamental set $\left(f_{j,k}\right)_{j,k}^{\infty} = 0$ into a fundamental sequence $(f_i)_{i=0}^{\infty}$. Each $f_{j,k}$ is equal to $f_{\boldsymbol{I}(j,k)}$, where $\boldsymbol{I} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ is a suitable bijection (see Definition (6.1.9)). We will write F_n for the linear span $\operatorname{lin}\{f_0, \ldots, f_n\}$ - a special case of the subspaces F_S as in Definition (6.1.4). This particular choice of S will be called S_n, the unique subset $\boldsymbol{I}^{-1}([0,n])$ of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that

We then define linear relationships of general form $f_i = \sum_{j=0}^{i} \lambda_{i,j} e_j$, with $\lambda_{i,i} \neq 0$, for each $i \in \mathbb{Z}^+$ (this is done in Definition (6.1.10)). Because the matrix with entries $\lambda_{i,j}$ is lower triangular with nonzero diagonal entries, this linear map can be inverted; giving us an alternative vectorspace basis $(e_i)_{i=0}^{\infty}$ of F, given uniquely as linear combinations of general form $e_i = \sum_{j=0}^{i} \rho_{i,j} f_j$.

So there is a unique linear map $T : F \to F$ that acts as a right shift operator sending each e_i to e_{i+1}. It turns out that p$\mathbf{d}$, T extends to a continuous operator $X \to X$ that is strictly singular [148, 149, 150, 151, 152], and has no nontrivial closed invariant subspaces,

Definition (6.1.6)[163]: *Let the sequence $\mathbf{d}$ be given. Let $\Omega \subset \mathbb{Z}^+$ be the set*

$$\bigcup_{n=1}^{\infty} \bigcup_{r=}$$

Provided $\mathbf{d}$ increases sufficiently rapidly, the union (4) is disjoint, and both Ω and $\mathbb{Z}^+ \backslash \Omega$ are infinite sets. If $\mathbf{d}$ does indeed increase sufficiently fast for this to happen, we make the following definitions:

Definition (6.1.7)[163]: *Let γ be the unique increasing bijection $\mathbb{Z}^+ \backslash \Omega \to \mathbb{Z}^+$.*

Definition (6.1.8)[163]: *(a) For each* $s \geq 0$*, let* σ_s *be the natural bijection from the Set* $\bigcup_{n=s+1}^{\infty}[(n-s)a_n, (n-s)a_n + v_s] \subset \mathbb{Z}^+$ *to the set* $[0, v_s] \times \mathbb{Z}^+ \subset \mathbb{Z}^+ \times \mathbb{Z}^+$*, that sends the integer* $(n-s)a_n + i(0 \leq i \leq v_s)$ *to the pair* $(i, n-s-1)$*.*

(b) Define maps χ_s*, each with the same domain as* σ_s*, by* $\chi_s(j) = \sigma_s(j) + (w_s, 0)$*, so that the image of* χ_s *is equal to* $[w_s, w_s + v_s] \times \mathbb{Z}^+ = [w_s, w_{s+1}) \times \mathbb{Z}^+$ *(for* $w_{s+1} = w_s + v_s + 1$*, by Definition (6.1.3)).*

(c) Let the map $\chi : \Omega \to \mathbb{Z}^+ \times \mathbb{Z}^+$ *be the unique map whose restriction to each subset* $\bigcup_{n=s+1}^{\infty}[(n-s)a_n, (n-s)a_n + v_s]$ *of* Ω *is equal to* χ_s*.*

Now the map χ is a bijection from Ω onto $[w_0, \infty) \times \mathbb{Z}^+$, that is, onto $\mathbb{N} \times \mathbb{Z}^+$.

We may obtain a bijection $\mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ by making sure that $\mathbb{Z}^+ \backslash \Omega$ gets mapped onto $\{0\} \times \mathbb{Z}^+$, thus:

Definition (6.1.9)[163]: *Let us extend* χ *to a map* $\mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ *by defining* $\chi(i) = \left(0, \gamma^{-1}(i)\right)$ *for each* $i \notin \Omega$*. Since* χ *is always a bijection, we may also define the map* $\boldsymbol{I} = \chi^{-1} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$*.*

Definition (6.1.10)[163]: *Let the sequence* $\mathbf{d}$ *be given, and let it increase sufficiently fast that the maps* χ *and* $\boldsymbol{I}$ *are defined. For each* i *then, we define* $f_i = f_{j,k}$*, where* $(j,k) = \chi(i)$*. We shall show that,* $p\mathbf{d}$*, there is a unique sequence* $(e_i)_{i=0}^{\infty}$ *in* F*, with the following properties. Firstly,*

Secondly, if integers r, n, i *satisfy* $0 < r \geq n, i \in [0, v_{n-r}] + ra_n$*, then*

$f_i = (n -$

Thirdly, if $0 < r < n, i \in (ra_n + v_{n-r}, (r+1)a_n)$ *(respectively, if* $1 \leq n, i \in (v_{n-1}, a_n)$*), then*

where $h = \left(r + \frac{1}{2}\right)a_n$ *(respectively,* $h = \frac{1}{2}a_n$*). If integers* r, n, i *satisfy* $0 < r \leq n, i \in [r(a_n + b_n), na_n + rb_n]$ *then*

If integers r, n, i satisfy $0 \leq r < n, i \in \left(na_n + rb_n, (r+1)(a_n + b_n)\right)$, then

where $h = \left(r + \frac{1}{2}\right) b_n$.

Lemma (6.1.11)[163]: *$p\mathbf{d}$, the sequence $(e_i)_{i=0}^{\infty}$ satisfying ((5) - (9)) does indeed exist, is unique, and is a vector space basis of F. There is a unique linear map $T : F \to F$ such that $Te_i = e_{i+1}$ for each i.*

Proof. Each definition is of form $f_i = \sum_{l=0}^{i} \lambda_{il} e_l$, with $\lambda_{ii} \neq 0$. The values taken by the index i in formulae (5)-(7) include zero, $[0, v_{n-r}] + ra_n (0 < r \leq n)$; $(ra_n + v_{n-r}, (r+1)a_n)$ $(0 < r < n)$; and (v_{n-1}, a_n) $(1 \leq n)$. $P\mathbf{d}$, this means each value $i = 0$ or $i \in (v_{n-1}, na_n]$ $(n \geq 1)$ is mentioned once and only once.

The remaining values of i are taken care of by (8),(9). These cases cover intervals $[r(a_n + b_n), na_n + rb_n]$ $(0 < r \leq n)$ and $\left(na_n + rb_n, (r+1)(a_n + b_n)\right)$ $(0 \leq r < n)$, whose union is $(na_n, n(a_n + b_n)] = (na_n, v_n]$. As the index n varies, we catch the rest of $\mathbb{Z}^+$.

$P\mathbf{d}$, then, each $f_i (i \geq 0)$ is defined once and only once, and has the general form $\sum_{l=0}^{i} \lambda_{il} e_l$.

Because $\lambda_{ii} \neq 0$ the linear relationship between the e_i and the f_i is invertible (we have a lower triangular matrix with nonzero entries in the diagonal) so the e_i do exist, are unique, and span F. Note by the way that if $i = \boldsymbol{I}(j, k)$ then

since $f_{jk}^*(f_i) = 1$, and obviously $f_{jk}^*(e_m) = 0$ for $m < i$.

It is then also true that for each n,

lin

say where $S_n = \chi\{0, 1, 2 \ldots n\}, |S_n| = n + 1$. As we remarked at the beginning, we will abbreviate F_{S_n} to F_n. $(e_i)_0^{\infty}$ is an alternative vector space basis for F, so of course there is a unique map T such that $Te_i = e_{i+1}$ for all i - as yet we say nothing about continuity!

From now on, we will always assume that the given sequence $\mathbf{d}$ increases sufficiently rapidly that Lemma (6.1.11) holds.

Obviously we must now prove that $(p\mathbf{d}), T$ is continuous and strictly singular.

The method of achieving this result is to approximate T by an appropriate "weighted shift" operator W, and then estimate the norm of the "error term" $T - W$ by ad hoc methods. This also gives us a natural direction to take when proving that T is strictly singular.

Definition (6.1.12)[163]: *Let $W_0 : l_2 \to l_2$ be a weighted shift operator with $W_0 f_{0j} = \alpha_j f_{0,j+1}$; we define the weights α_j as follows. Writing $i = \gamma^{-1}(j)$, we know that either i is zero, or it lies in one of the intervals $(v_{n-1}, a_n), (ra_n + v_{n-r}, (r+1)a_n), \big(na_n + rb_n, (r+1)(a_n + b_n)\big)$ or $[r(a_n + b_n), na_n + rb_n]$ that feature in parts (7), (8) and (9) of Definition (6.1.2). With an eye on that definition, we define:*

$$\alpha_j = \begin{cases} \frac{2^{1/}}{1 +} \\ \frac{1}{1 + r} \\ \frac{1}{1 + r} \\ 0, \end{cases}$$

It is easily checked from Definition (6.1.10) that in cases when $\alpha_j \neq 0, W_0 f_{0j} = T f_{0j}$. For example, if $j = \gamma(i)$ with $i \in \big((na_n + rb_n), (r+1)(a_n + b_n) - 1\big)$, then both i and $i + 1$ lie in the interval $\big((na_n + rb_n), (r+1)(a_n + b_n)\big)$, hence for suitable h, we have

and

Hence,

W_0 is a weighted shift operator on l_2, obviously of norm $\frac{1}{2} \cdot 2^{1/\sqrt{b_1}}$ (if we assume the interval $(a_1, a_1 + b_1 - 1)$ is nonempty, a rather mild condition of "rapid increase" on the sequence $\mathbf{d}$). Note it is also compact, for the weights tend to zero.

Definition (6.1.13)[163]: *Let* $W_1 : \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{l_2} \to \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{l_2}$ *be the map such that the sequence* $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots)$ *with* $\mathbf{x}_i \in J_{p_i}$, *is sent to the sequence* $(0, \beta_1 \mathbf{x}_1, \beta_2 \mathbf{x}_2, \beta_3 \mathbf{x}_3, \dots)$ *where the coefficients* β_i *are as follows:*

Once again, in cases when $\beta_j \neq 0$, the action of T on J_{p_j} and of W_1 are identical. For if $j \in [w_s, w_s + v_s)$ and $k \geq 0$, let us write $n = k + 1 + s$ and $i = j - w_s$. Then Definition (6.1.8) gives us $\chi\big((n-s)a_n + i\big) = (w_s + i, n - s - 1) = (j, k)$; and likewise $\chi\big((n-s)a_n + i + 1\big) = (j + 1, k)$. Therefore, Definition (6.1.4) gives us $f_{j,k} = f_{(n-s)a_n+i} =$

$(s+1)^i \cdot$

and $f_{j+1,k} = f_{(n-s)a_n+i+1} =$

$(s+1)^{i+1} \cdot a$

Hence

for all $k \geq 0$; this of course agrees with $W_1 f_{j,k}$.

Since the embedding $J_{p_i} \hookrightarrow J_{p_{i+1}}$ is strictly singular [137, 154, 155, 156, 157], we conclude that $\|W_1\| = 1/2$. Writing W for the unique continuous linear map $X \to X$ which agrees with W_0 on l_2 and W_1 on $\bigoplus_1^{\infty} J_{p_i}$, we have $\|W\| = \frac{1}{2} \cdot 2^{1/\sqrt{b_1}}$. The "error term" $T - W$ acts as follows. By (5), (6) By (7),if $i = (r+1)a_n - 1$ and $j = \gamma(i)$ $(0 \leq r < n)$ then

$(T -$

By (6), if $j = w_m + v_m$ and $k \geq 0$ then writing $r = k+1, n = m+k+1$ and $i = ra_n + j - w_m = ra_n + v_{n-r}$, we have

$(T - W)f_{j,k} =$

$(n - r +$

By (8), if $j = \gamma(i), i = na_n + rb_n, 0 < r \leq n$, then

$(T -$

$= (1 + n$

Lastly by (9), if $j = \gamma(i), i = (r+1)(a_n + b_n) - 1, 0 \leq r < n$, then

$(T - W)f_{($

In all other cases, $(T - W)f_{i,j} = 0$.

Lemma (6.1.14)[163]: *For every $\eta > 0$ the following is true. $P\mathbf{d}$, $T - W$ is a nuclear operator of nuclear norm at most η.*

Proof. It is necessary to estimate the sum of the norms of all the vectors in (13)-(16), add up the estimates and check that ($p\mathbf{d}$) the sum is less than η. These sort of details will be very familiar (see [165]-[172]).

Obviously (13) contributes $2^{-1+\left(1-\frac{1}{2}a_1\right)/\sqrt{a_1}}$ to our sum (which is less than $\eta/5$ $\mathrm{p}\mathbf{d}$, let us say). Now (6) gives us (for $0 < r \leq n$)

$(1 + n)^r$

Now the J_p spaces have the special property - closely related to their nonreflexivity- that for all r, j we have $\left\|\sum_{s=0}^{r-1} f_{j,s}\right\| = 1$. Hence,

and

Hence, (14) contributes to our sum at most

p**d**. In view of (7), if $0 < r < n$ then

If $r = n > 0$, (9) gives

$\|e_{1+$

If $r = 0 < n$, (7) gives

$\|e_1$

Hence the contribution made by (15) to our sum is at most

$$\sum_{n=1}^{\infty}\sum_{r=1}^{n}(n - r +$$

$$= \sum_{n=1}^{\infty}$$

$$\sum_{n=1}^{\infty}$$

p**d** (the first two terms in the middle of (25) are summing appropriate multiples of the norms of vectors $e_{1+ra_n+v_{n-r}}$ on the left hand side; the last two terms do the same for vectors $e_{1+(r-1)a_{n-1}+v_{n-r}}$).

Then again, (9) gives us

when $0 \leq r < n$; if $r = n$ we are looking at $\|e_{1+v_n}\|$ which is given by (24). Hence the contribution to our sum made by (16) is at most

$$\sum_{n=2}^{\infty} \sum_{r=1}^{n-1} (1 +$$

p**d** (here the first two terms in the equation sum the norms of vectors $e_{1+na_n+rb_n}$ appearing in (16), with appropriate weights; and the last term does the same for vectors $e_{1+na_n+(r-1)b_n}$).

Lastly, (8) gives us (for each $0 < r \leq s \leq n$)

$$\|(1+n)$$

hence for $0 < r \leq n$

$$\|(1$$

(by (20))

Therefore the contribution to our sum from (17) is at most

p**d**. Adding up our estimates ((25),(21),(27),(29) and our remark about Tf_{00}) we find that

p**d**,

$\sum_{i,j}$

which gives the result.

Corollary (6.1.15)[163]: $P\mathbf{d}, \|T\| < 1.$

Corollary (6.1.16)[163]: $P\mathbf{d}, T$ *is strictly singular.*

Proof. Strict singularity is not affected if an operator is perturbed by an operator in the norm closure of the finite rank operators. Since $T - W$ is nuclear p$\mathbf{d}$, it is enough to show that W is strictly singular. Now with slight abuse of notation, we have

where W_0 is a compact operator on l_2. So it is enough to show that W_1 is strictly singular. Now W_1 is the map $\left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{l_2} \to \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{l_2}$ which sends the sequence $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots)$ to $(0, \beta_1\mathbf{x}_1, \beta_2\mathbf{x}_2, \dots)$. Furthermore $\beta_j \to 0$ as $j \to 1$ (see (12)). All we need then for our Corollary is the easy lemma:

Lemma (6.1.17)[163]: If $W_1 : \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{l_2} \to \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{l_2}$ *is the map sending*

$\left(\mathbf{x}_i \in J_{p_i}\right)$*, then* W_1 *is strictly singular provided* $\beta_i \to 0$ *as* $i \to \infty$.

Proof. If not, write $X_1 = \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)$ and let $E \subset X_1$ be an infinite-dimensional subspace, and $\varepsilon > 0$, such that for all $\mathbf{x} \in E$,

Let P_N denote the natural projection onto $\bigoplus_1^N J_{p_i}$ sending $(\mathbf{x}_1, \mathbf{x}_2, \dots)$ to $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, 0, 0, 0, \dots)$ Let $P_0 = 0$. Now $\|W_1 P_N - W_1\| \to 0$ as $N \to \infty$ because $\beta_i \to 0$. Therefore there is an N such that for all $\mathbf{x} \in E$,

So, W_1 is norm increasing on an infinite dimensional subspace of $P_N X_1$ (namely, $P_N E$). Let $S \subset \mathbb{Z}^+ \times \mathbb{Z}^+ = \{(M, N) : M < N$, and there is an infinite dimensional subspace of $(P_N - P_M)X_1$ on which W_1 is norm increasing $\}$. We have shown $(O, N) \in S$ for large N.

Let $(M,N)\in S$ be such that $N-M$ is minimal, and let E be a subspace of $(P_N-P_M)X_1$ on which W_1 is norm increasing, spanned by vectors $\mathrm{x}^{(i)}=\left(0,\dots,0,\mathrm{x}_{M+1}^{(i)},\mathrm{x}_{M+2}^{(i)},\dots\mathrm{x}_N^{(i)},0,\dots,0\right)$ $(i=1,\dots,\infty)$. If $N-M=1$ we find the inclusion map $J_{p_N}\to J_{p_{N+1}}$ is norm increasing on E, contradicting Lemma (6.1.1). If not, then taking a subspace of E and perturbing slightly as necessary, we can assume that for each j the $\mathrm{x}_j^{(i)}$ are a block basis in J_{p_i} (here we allow a "block basis" to perhaps include some zero vectors). Now the subspace of $J_{p_{M+1}}$ spanned by the $\mathrm{x}_{M+1}^{(i)}$, must be infinite dimensional, or we can remove the $\mathrm{x}_{M+1}^{(i)}$ (and reduce $N-M$) by passing to a subspace. So taking a subsequence as necessary we may assume the $\mathrm{x}_{M+1}^{(i)}$ independent, and likewise we may assume the $\mathrm{x}_N^{(i)}$ are independent. Consider the two norms

and

on the finitely nonzero sequences $\lambda\in c_{00}$. If on any infinite dimensional subspace of c_{00} they are equivalent, we have a subspace of $J_{p_{M+1}}$ isomorphic to one of J_{p_N}; this leads to $l_{p_{M+1}}\subset J_{p_N}$, a contradiction as in Lemma (6.1.1). If not, there is a normalized block basis $\mathrm{y}^{(i)}$ of the $\mathrm{x}^{(j)}$ such that writing $\mathrm{y}^{(i)}=\left(0,\dots,0,\mathrm{y}_{M+1}^{(i)},\dots\mathrm{y}_N^{(i)},0,\dots,0\right)$ we either have $\left\|\mathrm{y}_{M+1}^{(i)}\right\|\to 0$ or $\left\|\mathrm{y}_N^{(i)}\right\|\to 0$. Perturbing a subsequence of this block basis very slightly, we can obtain an increasing sequence (n_i), and vectors $\mathbf{z}^{(i)}$ close to $\mathbf{y}^{(n_i)}$, spanning a norm increasing subspace of X_1, for which $\mathbf{z}_N^{(i)}=0$ (or $\mathbf{z}_{M+1}^{(i)}=0$). Hence, either $(M,N-1)\in S$ or $(M+1,N)\in S$; so $N-M$ was not minimal. This contradiction shows that W_1 is strictly singular.

$\|T^{a_n+b_n}\|$ this of course is essentially a repeat of arguments given in [165]-[172].

Lemma (6.1.18)[163]: *Recall that $F_n = \mathrm{lin}(e_o, \dots, e_n)$. On F_n, one may consider the two norms*

and

We claim that for suitable functions $N_1 : \mathbb{N}^2 \to \mathbb{N}$ and $N_2 : \mathbb{N}^2 \to \mathbb{N}$, we have

for all $\mathrm{x} \in F_{na_n}$; and

for all $\mathrm{x} \in Fvn$, provided $\mathbf{d}$ increases sufficiently rapidly that Definition (6.1.10) is meaningful.

Proof. For F_{na_n} is $F_{S_{na_n}} = \mathrm{lin}\{f_{ij} : (i,j) \in S_{na_n}\}$ where S_n is the set mentioned in (3). Furthermore, the matrix of the map on F_{na_n} sending e_i to f_{j_k} (where $i = \boldsymbol{I}(j,k)$, is determined by the values $a_1, b_1, \dots a_n$ as used in Definition (6.1.10). If we take the releveant f_{j_k} as our basis for F_{na_n} then of course the norm is between the c_0 norm and the l_1 norm. If we make the change of basis to e_i $(i = 1, \dots, na_n)$ then for a suitable function $M_1(n, a_1, b_1, \dots a_n)$ the inequality (31) must hold. Because $\mathbf{d}$ is an increasing sequence, we can write $M_1(n, a_1, b_1, \dots a_n) \le N_1(n, a_n)$ for a suitable function N_1. Similarly, a suitable function N_2 exists such that (32) holds.

Definition (6.1.19)[163]: *Recall that we write f_i for the vector $f_{j,k}$, where $i = \boldsymbol{I}(j,k)$. Let $Q_m^0 : F \to F_{ma_m}$ be the projection such that*

$Q_m^0(f_j) =$

We shall establish the following lemma (in [167]).

Lemma (6.1.20)[163]: Pd*, for all n we have*

Later on, we will establish that, for an arbitrary norm-1 vector $x \in l_1$, and for any $\varepsilon > 0$, there is a polynomial q and an integer n such that

and

This will show that $e_0 \in \lim\{T_n^r \mathbf{x} : r \geq 0\}$ and hence that $\mathbf{x}$ is cyclic, since e_0 obviously is. So T is strictly singular, and has no invariant subspaces.

Proof. As with proving T continuous, we split the operator involved (in this case, $T^{a_n+b_n} \circ (I - Q_n^0)$) into a part that looks like a weighted shift operator, and a nuclear operator. In certain cases, we now find that $T^{a_n+b_n} \circ (I - Q_n^0) f_i$ is of form $\varepsilon_i \cdot f_{i+a_n+b_n}$. These cases are as follows:

Case (A1). *If $i \in [0, na_n]$ then $Q_n^0 f_i = f_i$ so of course*

Case (A2). *If $i \in [r(a_n + b_n), (n-1)a_n + rb_n]$ with $0 < r < n$, then by two applications of (2.2.3) we find that*

T^{a}

Case (A3). *If $i \in \big(na_n + rb_n, (r+1)(a_n + b_n)\big)$ with $0 \leq r < n-1$ then two applications of (2.2.4) likewise give us*

T^{a_n+b}

Case (A4). *If $i \in [0, v_{m-r} - a_n - b_n] + ra_m$ with $0 < r \leq m-n$ then (2.2.1) gives*

T^{a_n+}

Case (A5). *If* $i \in (ra_m + v_{m-r}, (r+1)a_m - a_n - b_n)$ *with* $0 < r < m, m > n$, *or if* $i \in (v_{m-1}, a_m - a_n - b_n), m > n$ *then by (6),*

$T^{a_n+b_n} \circ$

Case (A6). *If* $i \in e[r(a_m + b_m), ma_m + rb_m - a_n - b_n]$ *with* $0 < r \le m, m > n$*, then (2.2.3) gives*

T^{a}

Case (A7). *Finally if* $i \in (ma_m + rb_m, (r+1)(a_m + b_m) - a_n - b_n))$ *with* $0 \le r < m, m > n$ *then (2.2.4) gives*

T

Lemma (6.1.21)[163]: $P\mathbf{d}$ *the following is true. The operator* $W = W^{(n)} : F \to F$ *such that* Wf_i *is as in ((33)-(39)) if the integer* i *is mentioned in these cases, and* $Wf_i = 0$ *otherwise, has norm* $(1+n)^{-a_n-b_n} \cdot 2^{a_n/\sqrt{b_n}} < \frac{3}{2} \cdot (1+n)^{-a_n-b_n}$.

Proof. As earlier, we split up W into an operator W_0 acting on l_2, and W_1 acting on $\bigoplus_1^\infty J_{p_i}$. The operator W_0 covers all cases except Case (A4), and it acts as

with $\varepsilon_i \le (1+n)^{-a_n-b_n} \cdot 2^{a_n/\sqrt{b_n}}\, p\mathbf{d}$, and equality achieved for certain values i such that $\gamma^{-1}(i)$ is covered by Case (A3). The operator W_1 deals with Case (A4). It sends (for $i \in [0, v_{m-r} - a_n - b_n] + ra_m, 0 < r \le m - n)$ f_i to $(m - r + 1)^{-a_n-b_n} f_{i+a_n+b_n}$; that is, writing $j = i - ra_m \in [0, v_{m-r} - a_n - b_n]$,

j

Writing $m - r = k \ge n$ we find that W_1 sends $f_{j+w_k,s}$ to

for all $s \ge 0$ and $j \in [0, v_k - a_n - b_n]$.

Otherwise, $W_1 f_{l,s} = 0$. So W_1 acts on each J_{p_i} space as a multiple $\beta_i \times \varepsilon_i^{(n)}$, where $\beta_i \times \varepsilon_i^{(n)}$ is the inclusion $J_{p_i} \to J_{p_i+a_n+b_n}$, and β_i does not exceed $(n+1)^{-a_n-b_n}$. So, $\|W_1\| = (n+1)^{-a_n-b_n}$. Hence,

|

Lemma (6.1.22)[163]: *For each $\eta > 0$, the following is true. Pd, for every $n > 0$ the operator $T^{a_n+b_n} \circ (I - Q_n^0) - W^{(n)}$ is nuclear, of nuclear norm at most*

Proof. We must consider the error terms $T^{a_n+b_n} \circ (I - Q_n^0) f_i - W^{(n)} f_i$, sum all their norms, and obtain at most $\eta \cdot (n+1)^{-a_n-b_n}$. This is not in fact difficult to do, there are roughly six cases, corresponding to values i which were "missed out" of Cases (A2)- (A7) above.

Case (A8). *If $i \in \left((n-1)a_n + rb_n, na_n + rb_n\right]$ with $0 < r < n$; (these are some of the "missing values" from Case (A2) above).*

Then, (8) gives us

and hence, writing $j = i + a_n + b_n - na_n - (r+1)b_n - 1 \geq 0$, we have

T^j

Now (26) and (24) together give us, for all $0 \leq r \leq n$,

$\|$

(for pd, (24) is less than (27) for the same value of n). Applying this twice, we find that (40) is at most

$(1 +$

pd.

Case (A9). *If $i = v_n$ (the final value "missed out" of Case (A2) above). Then (8) and (24), together with the ever-useful fact that $\|T\| < 1$, give us*

$\leq (1 +$

$\leq ((1 +$

$\leq ((1 +$

for all n, p**d**.

Case (A10). *If $i \in \big(na_n + (n-1)b_n, n(a_n + b_n)\big)$ (not covered by Case (A3) above).*

Then by (9), f_i is a multiple $\lambda_i e_i$ with λ_i (crudely) at most $(1+n)^{v_n - 1}$. Therefore, $T^{a_n+b_n} f_i$ is $T^k\big(\lambda_i e_{1+v_n}\big)$ for some $k \geq 0$. Since $\|T\| \leq 1$,

$\|T^{a_n+b_n} f_i\|$

by (24);

p**d**.

Case (A11). *If $i \in (v_{m-r} - a_n - b_n, v_{m-r}] + ra_m$ with $0 < r \leq m - n$ (not covered by Case (A4) above), or if $i \in [0, v_{m-r}] + ra_m$ with $0 < r \leq m > n, m - r < n$.*

Then we see by (6) that

$f_i = ((1 +$

and in either case, $i + a_n + b_n > ra_m + v_{m-r}$. (In the second case, $i + a_n + b_n \geq ra_m + a_n + b_n > ra_m + v_{m-r}$ p**d**, since $m - r < n$). Writing

we have

$|T^{a_n+b_n} f$

$$\leq$$

Now (22),(23) tell us that $\mathrm{p}\mathbf{d}$, for all $0 \leq r \leq n$ we have

So, (49) is at most

$$(r$$

(the worst case in this estimate is when $r = 1$). Now if we have $m > n + 1$, then $(I - Q_n^0)f_i$ is just f_i (6.1.22), and (53) above is the upper bound we need; thus,

If on the other hand $m = n + 1$, $(I - Q_n^0)f_i$ is in fact not f_i but just $(1 + m)^{ra_m}(m - r + 1)^{i - ra_m}e_i$ (see Definition (6.1.19)). Our argument then gives the better estimate

$$\|T^{a_n+b_n}$$

$\mathrm{p}\mathbf{d}$.

Case (A12). *If $i \in [(r + 1)a_m - a_n - b_n, (r + 1)a_m)$, with $0 \leq r < m, m > n$, (these are the "missing values" from Case (A5) above).*

Then (7) gives us

$$T^{a_n+b_n} \circ ($$

for some $j \geq 0$. As we have remarked (20), $\left\|(1 + n)^{ra_n}e_{ra_n}\right\| = \sqrt{1 + a_{n-r}^{-2}}$; so since $\|T\| \leq 1$ this is at most (since $i \geq (r+1)a_m - a_n - b_n$)

for all $0 \leq r < m > n$ p$\mathbf{d}$.

Case (A13). *If $i \in (ma_m + rb_m - a_n - b_n, ma_m + rb_m]$ with $0 < r \leq m > n$; then we have one of the "missing values" from Case (A6) above.*

(8) gives us

and hence, writing $j = i + a_n + b_n - ma_m - rb_m - 1 \geq 0$, we have

$T^{a_n+b_n} f_i$

Now (26) and (24) together give us, for all $0 \leq r \leq n$,

$\|e$

(for p$\mathbf{d}$, (24) is less than (27) for the same value of n). Applying this twice, we find that (58) is at most

$(1 +$

p$\mathbf{d}$.

Case (A14). *Finally if $i \in \big[(r+1)(a_m + b_m) - a_n - b_n, (r+1)(a_m + b_m)\big)$ $(0 \leq r < m > n)$ then we are among the "missing values" from Case (A7).*

Then, (9) gives us

and hence, for $j = i + a_n + b_n - (r+1)(a_m + b_m) \geq 0$,

$T^{a_n+b_n} f_i =$

As in (28),

so

$$\|T^{a_n+b}$$

in all cases, $p\mathbf{d}$. We now add up all our estimates (42), (44), (46), (54), (56), (19), (60), and (63), counting according to the multiplicity of values i that are involved. We obtain this estimate of the nuclear norm.

$$\sum_i \|(T^{a_n}$$

$$b_n \cdot 2^{-\frac{1}{3}\sqrt{}}$$

$$(a_n + b_n$$

$$\sum_{\substack{m>n \\ 0\le r<m}} (a_n$$

$$\sum_{\substack{m>n \\ 0\le r<m}} (a_n$$

We will observe that as a function of n and d, $p\mathbf{d}$ this sum is at most $\eta \cdot (n+1)^{-a_n-b_n}$.

Definition (6.1.23)[163]: *Let $Q_m (m \ge 1)$ be the projection $F \to F_{ma_m}$, such that*

Note that in terms of what happens to the e_j, this amounts to much the same as of [172], though it does not look the same.

Lemma (6.1.24)[163]: $\|Q_m\| = 1$ *for all m.*

Proof. For we claim that for each i, the vectors $f_{ik}(k=0,1,2,\ldots)$ appear as a subsequence $\left(f_{jk}\right)_{k=0}^{\infty}$ of the f_j in their proper order $(j_0 < j_1 < j_2 < \ldots)$. This is true because is an increasing function and $f_{0j} = f_{\gamma(j)}$; and because for $i > 0$, say $i \in [w_m, w_m + v_m]$, we have $f_{ik} = f_{(k+1)a_{m+k+1}+i-w_m}$ (2.3). Hence, for each i there is a k such that

The norm of the projection that thus "truncates" a sequence is 1 on l_2 (of course), and also on any J_{p_i}. Hence, $\|Q_m\| = 1$.

Definition (6.1.25)[163]: *Let* $P_{n,m}(m > n \geq 1)$ *be the operator* $\tau_{nm} \circ Q_m : F_{ma_m} \to F_{ma_m}$, *where*

Lemma (6.1.26)[163]: $\|Q_m^0\| \leq a_m$ *for all* m, p**d**.

Proof. We know $\|Q_m\| = 1$; and Definition (6.1.12) tells us that $(Q_m - Q_m^0)f_j$ is zero unless $j \in [0, v_{m+1-r}] + ra_{m+1}, 1 < r \leq m+1$. In this case, it is

e_{j-ra_m}

Hence, crudely,

$\|Q_m -$

(by (20))

for all m, p**d**.

Lemma (6.1.27)[163]: $\|P_{nm}\| \leq a_{n+1}$ *for all* $n < m$ p**d**.

Proof. $\|Q_m\| = 1$ so $\|P_{nm}\| = \|\tau_{nm}\|$. Examining Definitions (6.1.25) and (6.1.10) we find

$$\tau_{nm} f_i =$$

Now the projection τ' such that

has norm 1, for the same reasons as in Definition (6.1.10). Therefore

$$1 + \sum_{\substack{i \in [0, v_m \\ m-n}}$$

$$\leq 1 + \sum_{r=m-n}^{m}$$

since $\|T\| < 1$. Recall from (20) that $\left\|(1+n)^{ra_n} e_{ra_n}\right\| = \sqrt{1 + a_{n-r}^{-2}}$. Substituting into (70) we have

$$\|\tau_{nm}\| \leq$$

$$=$$

$$<$$

for all n, p**d**. Thus Lemma (6.1.27) is proved.

Definition (6.1.28)[163]: *For each $1 < n \leq m$, let $K_{n,m} \subset F_{ma_m}$ be the set of vectors such that $\|\mathbf{x}\| \leq a_m$ and $\|\tau_{n,m}\mathbf{x}\| \geq 1/am$.*

Let $T_m : F_{ma_m} \to F_{ma_m}$ be the "truncated" version of $T, T_m(e_i) = e_{i+1}$ $(i < ma_m)$ or zero $(i = ma_m)$.

Lemma (6.1.29)[163]: *There is a function $N_3 : N^2 \to N$ with the following property: P$\mathbf{d}$, for all $1 < n < m$ and $\mathbf{x} \in K_{n,m}$, there is a polynomial p such that $|p| < N_3(m, a_m), p(t)$ is of form $\sum_{i=a_m}^{ma_m} \lambda_i t^i$, and*

Proof. For any y $\in K_{n,m}$ we can write $y = \sum_{i=1}^{\alpha} \lambda_i e_i$ where $\lambda_\alpha \neq 0$. Then,

$\mathrm{lin}\{T$

Since $\tau_{nm}\mathbf{y} \neq 0$ we know $\alpha < (m-n)a_m$ so certainly $e_{(m-n+1)a_m} \in \mathrm{lin}\{T_m^r e_i : a_m \leq r \leq ma_m\}$. Since K_{nm} is compact, there are a finite number of polynomials $p_1, \ldots, p_k$ of form $p_j(t) = \sum_{i=a_m}^{ma_m} \lambda_{ji} t^i$, such that for all $\mathbf{x} \in K_{nm}$ there is a j such that

Writing $N = \max_j |p_j|$, note that N depends only on elements of the underlying sequence $\mathbf{d}$ up to and including am; so $N < N_3(m, a_m)$ for a suitable function $N_3 : N^2 \to N$. Since in view of (19) we have

this concludes the proof.

We now extend the previous lemma as follows.

Lemma (6.1.30)[163]: *With is notation of Definition (6.1.12), the polynomial $q(t) = t^{b_m}(m+1)^{b_m}/b_m \cdot p(t)$ satisfies $t^{a_m+b_m} | q(t), \mathrm{degq} \leq b_m + ma_m, |q| \leq N_3(m, a_m)(m+1)^{b_m}/b_m$ and*

Proof. Given $\mathbf{x} \in K_{n,m}$ let p be the polynomial as in Definition (6.1.12), and write $q(t) = t^{b_m}(m+1)^{b_m}/b_m p(t)$. Let us consider the vector $q(T)\mathbf{x}$. For all $i \in [a_m + b_m, ma_m + b_m]$ we have $f_i = (m+1)^i e_i - b_m(m+1)^{i-b_m} e_{i-b_m}$; so if we write $p(T_m)\mathbf{x} = \sum_{i=a_m}^{ma_m} \lambda_i e_i$ then

$$\left\| \frac{(m+1)^{b_m}}{b_m} \right.$$

(since $\mathbf{x} \in K_{nm}$ so $\|\mathbf{x}\| \leq a_m$)

$\mathrm{p}\mathbf{d}$. Furthermore,

T

Since $\|T\| \leq 1$ $\mathrm{p}\mathbf{d}$, we deduce that

$$\frac{(m+1)^{i}}{b_m}$$

as above;

$$\leq \frac{(m+1)^{b_m}}{b_m}$$

by (27);

for all m, p$\mathbf{d}$. Adding up (83) and (77) we have

Using Definition (6.1.12) we have our result.

We now have the following very convenient lemma (see [165]):

Lemma (6.1.31)[163]: *For all $j \in [0, v_{k-r}], 1 \leq r < k-n$ and $s \geq r$, we have*

f_j

Proof. The vector $f^*_{j+w_{k-r},s-1} = f_{j+sa_{k+s-r}}$ Definition (6.1.7) is in the image of the projection $\tau_{n,k+s-r} \circ Q^0_{+s-r}k$ and is fixed by it. The vector $f_{j+w_{k-r},s} = f_{j+(s+1)a_{k+s-r+1}}$ is mapped (by Definitions (6.1.12),(6.1.13)) to

u

which by (10), (6), satisfies $f^*_{j+w_{k-r},s-1}(u) = -1$. It is easily seen that for all other vectors f_m , $\tau_{n,k+s-r} \circ Q^0_{k+s-r}(f_m)$ is either f_m, or zero, or another vector similar to u above, being therefore of form $f_{j',s'} + h$ with the pair (j', s') not equal to $(j + w_{k-r,s})$, and with $h \in F_{v_{k+s-r-1}}$. In all such cases $f^*_{j+w_{k-r},s}(\tau_{n,k+s-r}Q^0_{k+s-r}f_m) = 0$, hence the result.

Theorem (6.1.32)[163]: $P\mathbf{d}, T$ *has no invariant subspace.*

Proof. Let $\mathbf{x} \in X$, $\|\mathbf{x}\| = 1$ and $n > 0$. Since e_0 is cyclic for T, it is enough to show that for all such $\mathbf{x}$ and n there is a polynomial q such that $\|q(T)\mathbf{x} - e_0\| \leq 2/a_{n-1}$. We claim there is an $m > n$ such that

Now $\|P_{nk}\| \leq a_{n+1}$ for all k, and certainly for all $x \in F$ we have $P_{nk}\mathbf{x} = \mathbf{x}$ for all but finitely many k. Therefore, $P_{nk}\mathbf{x} \to \mathbf{x}$ as $k \to \infty$ for any vector $\mathbf{x} \in X$. Choose, then, a k so large that $\|P_{nk}\mathbf{x}\| = \|\tau_{nk} \circ Q_k\mathbf{x}\| > 1/2$. If $\|\tau_{nk} \circ Q^0_k\mathbf{x}\| >$ 1/4 our assertion is proved; if not then

For all $j > 0$ we either have

$(Q_k$

if $j \in [0, v_{k-r}] + ra_{k+1}, 1 < r \leq n$; or else, $(Q_k - Q_k^0)f_j = 0$. Hence, $\tau_{nk} \circ (Q_k - Q_k^0)f_j$ is either

if $j \in [0, v_{k-r}] + ra_{k+1}, 1 < r \leq k - n$; or else it is zero. Thus, $\tau_{nk} \circ (Q_k - Q_k^0) = \tau_{nk} \circ (Q_k - Q_k^0) \circ \pi_S$ where S is the finite set

say. Crudely, then, we may say that there is an $i \in S_{nk}$ such that

Now if $i = j + (r+1)a_{k+1}, j \in [0, v_{k-r}], 1 \leq r < k - n$ we know by (6) that $f_i = f_{j+w_{k-r},r}$. Because any $\mathbf{x} \in J_{p_{j+w_{k-r}}}$ is necessarily in c_0, we know that as $s \to \infty$, $\left|f_{j+w_{k-r},s}^*(\mathbf{x})\right| \to 0$. Therefore there is an $s > r$ such that

$|$

If (93) holds then we may deduce from (.1.14)[163] that

$\|\tau_{n,k+s}$

(for $\tau_{nk} \circ (Q_k - Q_k^0) = \tau_{nk} \circ Q_k \circ (Q_k - Q_k^0) = P_{nk} \circ (Q_k - Q_k^0)$; and $|S_{nk}| = w_{k-1}$).

$\mathrm{p}\mathbf{d}$. This proves our assertion that there is indeed an $m > n$ such that (87) holds. Pick such an m, and write $\mathbf{y} = Q_m^0\mathbf{x}$. We know that $\|\mathbf{y}\| \leq \|Q_m^0\| \leq a_m\,\mathrm{p}\mathbf{d}$. But $\|\tau_{nm}\mathbf{y}\| \geq$

$1/a_m$ so $\mathbf{y} \in K_{nm}$. Therefore by (6.3) there is a polynomial q such that $\|q(T)\mathbf{y} - e_0\| \leq 1/a_{n-1} + 3/a_m, t^{a_m+b_m}|q(t)$, deg $q \leq b_m + ma_m$, and $|q| \leq (m+1)^{b_m}/b_m$. Using our estimate on $\|T^{a_m+b_m} \circ (I - Q_m^0)\|$ and the fact that $\|T\| < 1$, we find that

Therefore

p$\mathbf{d}$. This inequality (which can be repeated with different values of n by choosing suitable alternative q) shows that in fact $\mathbf{x}$ is cyclic; and so we conclude the proof.

Corollary (6.1.33)[284]: p$\mathbf{d}$*, the Cauchy sequence* $(e_i)_{i=0}^{\infty}$ *satisfying ((5) - (9)) does indeed exist, is unique, and is a vector space basis of* F*. There is a unique linear map* $T : F \to F$ *such that* $Te_i = e_{i+1}$ *for each* i*.*

Proof. Each definition is of form $f_i = \sum_{l=0}^{i} \lambda_{il} e_l$, with $\lambda_{ii} \neq 0$. The values taken by the index i in formulae (5)-(7) include zero, $[0, v_\beta] + (n-\beta)a_n (\beta \geq 0)$; $\big((n-\beta)a_n + v_\beta, (n-\beta+1)a_n\big)$ $(\beta > 0)$; and (v_{n-1}, a_n) $(1 \leq n)$. $P\mathbf{d}$, this means each value $i = 0$ or $i \in (v_{n-1}, na_n]$ $(n \geq 1)$ is mentioned once and only once.

The remaining values of i are taken care of by (8),(9). These cases cover intervals $[(2a_n + r_n)(n-\beta), na_n + (n-\beta)(a_n + r_n)]$ and $\big(na_n + (n-\beta)(a_n + r_n), (2a_n + r_n)(n-\beta+1)\big)$ $(\beta \geq 0)$, whose union is $(na_n, (2a_n + r_n)] = (na_n, v_n]$ which implies $(2a_n + r_n) = v_n$. As the index n varies, we catch the rest of $\mathbb{Z}^+$.

$P\mathbf{d}$, then, each $f_i (i \geq 0)$ is defined once and only once, and has the general form $\sum_{l=0}^{i} \lambda_{il} e_l$.

Because $\lambda_{ii} \neq 0$ the linear relationship between the e_i and the f_i is invertible (see [163]) so the e_i do exist, are unique, and span F. Note by the way that if $i = \boldsymbol{I}(j, k)$ then since $f_{jk}^*(f_i) = 1$, and obviously $f_{jk}^*(e_m) = 0$ for $m < i$.

It is then also true that for each n,

lin

say where $S_n = \chi\{0, 1, 2 \ldots n\}, |S_n| = n + 1$. As we remarked before, we will abbreviate F_{S_n} to F_n. $(e_i)_0^\infty$ is an alternative sequence if vector space basis for F, so of course there is a unique map T such that $Te_i = e_{i+1}$ for all i - as yet we say nothing about continuity! We will always assume that the given Cauchy sequence $\mathbf{d}$ increases sufficiently rapidly that Lemma (6.1.33) holds (See [163]).

[163] proves that $(p\mathbf{d}), T$ is continuous and strictly singular.

Corollary (6.1.34)[284]: *For every $\eta > 0$ the following is true. $P\mathbf{d}$, $T - W$ is a nuclear operator of nuclear norm at most η.*

Proof. It is necessary to estimate the sum of the norms of all the vectors in (13)-(17), add up the estimates and check that ($p\mathbf{d}$) the sum is less than η (see [7]-[14]). Obviously (13) contributes $2^{-1+\left(1-\frac{1}{2}a_1\right)/\sqrt{a_1}}$ to our sum (which is less than $\eta/5$ $\mathrm{p}\mathbf{d}$, let us say). Now (6) gives us (for $\beta > 0$)

$$(1+n)^{(n-\beta)a_n} e_{(n}$$

Now the J_p spaces have the special property - closely related to their nonreflexivity- that for all $n - \beta, j$ we have $\left\|\sum_{n=\alpha}^{n-\beta-1} f_{j,n-\alpha}\right\| \le \sum_{n=\alpha}^{n-\beta-1}\left\|f_{j,n-\alpha}\right\| \le 1$. Hence,

and

Hence, (14) contributes to our sum at most

pd. In view of (17), if $\beta > 0$ then

$$\left\| e_{1+(n-\beta)} \right.$$

If $r_n = n > 0$, (9) gives

$$\left\| e_{1+(n-\beta)a_n+v_\beta} \right.$$

If $n - \beta = 0 < n$, (7) gives

$$\left\| e_{1+(n-\beta} \right.$$

Hence the contribution made by (15) to our sum is at most

$$\sum_{n=1}^{\infty}\sum_{\beta=1}^{\infty}(\beta+1)^{v_\beta}$$

$$=$$

pd (the first two terms in the middle of (108) are summing appropriate multiples of the norms of sequence of vectors $e_{1+(n-\beta)a_n+v_\beta}$ on the left hand side; the last two terms do the same for sequence of vectors $e_{1+(n-\beta-1)a_{n-1}+v_\beta}$).

Then again, (9) gives us

$$\left\|e_{1+na_n+(n-\beta)(a_n-}\right.$$

when $\beta > 0$; if $\beta = 0$ we are looking at $\left\|e_{1+v_n}\right\|$ which is given by (107). Hence the contribution to our sum made by (16) is at most

$$\sum_{n=2}^{\infty}(1+n)^{-1}2$$

p**d** (here the first two terms in the equation sum the norms of sequence of vectors $e_{1+na_n+(n-\beta)a_n}$ appearing in (16), with appropriate weights; and the last term does the same for sequence of vectors $e_{1+(2n-\beta-1)a_n}$).

Lastly, (8) gives us (for each $\beta \geq \alpha, \alpha \geq 0$)

$$\left\|(1+n)^{(n-\alpha)a_n+(}\right.$$

$$\leq 1$$

hence for $\beta \geq 0$

$$\left\|(1\right.$$

(by (103))

Therefore the contribution to our sum from (17) is at most

$$\sum_{\beta\geq 0}$$

p**d**. Adding up our estimates ((108),(104),(110),(112) and our remark about Tf_{00}) we find that p**d**,

$\sum_{i,j}$

which gives the result.

Corollary (6.1.35)[284]: Pd*, for all n we have*

Later on, we will establish that, for an arbitrary norm-1 vector $\mathrm{x} \in l_1$, and for any $\varepsilon > 0$, there is a polynomial q and an integer n such that

and

This will show that $e_0 \in \lim\{T_n^r \mathbf{x} : r \geq 0\}$ and hence that $\mathbf{x}$ is cyclic, since e_0 obviously is. So T is strictly singular, and has no invariant subspaces.

Proof. As with proving T continuous, we split the operator involved (in this case, $T^{2a_n+r_n} \circ (I - Q_n^0)$) into a part that looks like a weighted shift operator, and a nuclear operator. In certain cases, we now find that $T^{2a_n+r_n} \circ (I - Q_n^0) f_i$ is of form $\varepsilon_i \cdot f_{i+(2a_n+r_n)}$. These cases are as follows:

Case A1. *If $i \in [0, na_n]$ then $Q_n^0 f_i = f_i$ so of course*

Case A2. *If $i \in [(n-\beta)(2a_n+r_n), (n-1)a_n + (n-\beta)(a_n+r_n)]$ with $\beta > 0$, then by two applications of [7] we find that*

$T^{(2a_n+r_n}$

Case A3. *If $i \in \big(na_n + (n-\beta)(a_n+r_n), (n-\beta+1)(2a_n+r_n)\big)$ with $\beta > 0$ then two applications of [7] likewise give us*

$T^{(2a_n+r_n)} \circ (I -$

Case A4. *If $i \in [0, v_n - (2a_n+r_n)] + (n-\beta)a_{(2n-\beta)}$ with $n > 0$ then [7] gives*

$T^{(2a_n+r_n)} \circ$

Case A5. *If* $i \in \left((n-\beta)a_{(2n-\beta)} + v_n, \left((n-\beta)+1\right)a_{(2n-\beta)} - (2a_n + r_n)\right)$ *with* $n > 0$*, or if* $i \in \left(v_{(2n-\beta)-1}, a_{(2n-\beta)} - (2a_n + r_n)\right), n > \beta$ *then by (6),*

$T^{(2a_n+r_n)} \circ (I - Q$

Case A6. *If* $i \in e\left[(n-\beta)\left(a_{(2n-\beta)} + a_{(2n-\beta)+\epsilon}\right), (2n-\beta)a_{(2n-\beta)} + (n-\beta)a_{(2n-\beta)+\epsilon} - (2a_n + r_n)\right]$ *with* $n > 0$*, then [7] gives*

$T^{(2a_n+r_n)} \circ (I$

Case A7. *Finally if* $i \in \left((2n-\beta)a_{(2n-\beta)} + (n-\beta)a_{(2n-\beta)+\epsilon}, \left((n-\beta)+1\right)\left(a_{(2n-\beta)} + a_{(2n-\beta)+\epsilon}\right) - (2a_n + r_n)\right)$ *with* $n > 0$ *then [7] gives*

$T^{(2a_n+r_n)} \circ (I - Q$

Corollary (6.1.36)[284]: $\|Q_{(2n-\beta)}\| = 1$ *for all* $(2n-\beta)$*.*

Proof. For we claim that for each i, the sequence of vectors $f_{ik}(k = 0, 1, 2, \dots)$ appear as a subsequence $\left(f_{jk}\right)_{k=0}^{\infty}$ of the f_j in their proper order $(j_0 < j_1 < j_2 < \dots)$. This is true because is an increasing function and $f_{0j} = f_{\gamma(j)}$; and because for $i > 0$, say $i \in \left[w_{(2n-\beta)}, w_{(2n-\beta)} + v_{(2n-\beta)}\right]$, we have $f_{ik} = f_{(k+1)a_{(2n-\beta)+k+1}+i-w_{(2n-\beta)}}$ (Definition (6.1.8)). Hence, for each i there is a k such that

The norm of the projection that thus "truncates" a Cauchy sequence is 1 on l_2, and also on any J_{p_i}. Hence, $\|Q_{(2n-\beta)}\| = 1$.

Corollary (6.1.37)[284]: $\|Q^0_{(2n-\beta)}\| \leq a_{(2n-\beta)}$ *for all* $(2n-\beta), p$d.

Proof. We know $\|Q_{(2n-\beta)}\| = 1$; and (Definition (6.1.19)) tells us that $\left(Q_{(2n-\beta)} - Q^0_{(2n-\beta)}\right)f_j$ is zero unless $j \in [0, v_{n+1}] + (n-\beta), n > 0$. In this case, it is

$e_{j-(n-\beta)a_{(2n-\beta)+1}+($

Hence, crudely,

$$\left\|Q_{(2n-\beta)} - Q^{0}_{(2n-\ldots}\right.$$

$$\leq \sum_{n=2+\beta}^{(2n-\beta)+\ldots}$$

$$\max\left\{\left\|e_{j\ldots}\right.\right.$$

$$\leq \sum_{n=2+\beta}^{(2n-\beta)+\ldots}$$

$$\leq \sum_{n=2+\beta}^{(2n-\beta)+\ldots}$$

(by (103))

for all $(2n-\beta)$, $\mathrm{p}\mathbf{d}$.

Corollary (6.1.38)[284]: $\left\|P_{n(2n-\beta)}\right\| \leq a_{n+1}$ *for all* $n-\beta > 0$ $p\mathbf{d}$.

Proof. $\left\|Q_{(2n-\beta)}\right\| = 1$ so $\left\|P_{n(2n-\beta)}\right\| = \left\|\tau_{n(2n-\beta)}\right\|$. Examining (Definition (6.1.25)) and (Definition (6.1.10)) we find

$$\tau_{n(2n-\beta)} f_i =$$

$$\begin{cases} f_i, 0 \leq i < (n-\beta \ldots \\ -e_{i-(n-\beta)a_{(2n-\beta)}+\ldots} \\ \quad i \in [0, v_n] + (\ldots \\ 0, otherwise. \end{cases}$$

Now the projection τ' such that

has norm 1, for the same reasons as in (Definition (6.1.24)). Therefore

$$\left\|\tau_{n(2n-\beta)}\right\| \leq 1 + \ldots$$

$\leq 1 +$

since $\|T\| < 1$. Recall from (103) that $\left\|(1+n)^{(n-\beta)a_n} e_{(n-\beta)a_n}\right\| = \sqrt{1 + a_\beta^{-2}}$.

Substituting into (124) we have

$\|\tau_{\gamma}$

for all n, $\mathrm{p}\mathbf{d}$. Thus (Lemma (6.1.27)) is proved.

Corollary (6.1.39)[284]: Show that

(i) $\|a_\beta\| \leq \frac{2\|}{\|e_0\|}$

(ii) $\|e_0\| = \left(\frac{1}{a_\beta}\right.$

(iii) $\|q(T)\| < \frac{\epsilon}{3}$

Proof: (i) Equation (102) implies that

Substituting (103) in (126) we get

and

Hence

(ii) Since $\|e_0\|^2 - \frac{2}{a_\beta}\|e_0\| - 1 = 0$ we can get

(iii) From Lemma (6.1.20)

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because since $\|T\| < 1$ then $\|T^n\| < 1$ and $\|x\| = 1$.

If $Q_n^0 f_i = f_i$ then

hence

Corollary (6.1.40)[284]: *There is a function* $N_3 : N^2 \to N$ *with the following property:* $P\mathbf{d}$*, for all* $n > 0$ *and* $\mathbf{x} \in K_{n,(2n-\beta)}$*, there is a polynomial* p *such that* $|p| < N_3\left((2n-\beta), a_{(2n-\beta)}\right)$*,* $p(t)$ *is of form* $\sum_{i=a_{(2n-\beta)}}^{(2n-\beta)a_{(2n-\beta)}} \lambda_i t^i$*, and*

Proof. For any y $\in K_{n,(2n-\beta)}$ we can write $y = \sum_{i=1}^{\alpha} \lambda_i e_i$ where $\lambda_\alpha \neq 0$. Then,

$\mathrm{lin}\{T$

Since $\tau_{n(2n-\beta)}\mathbf{y} \neq 0$ we know $\alpha < (n-\beta)a_{(2n-\beta)}$ so certainly $e_{((n-\beta)+1)a_{(2n-\beta)}} \in \mathrm{lin}\left\{T_{(2n-\beta)}^{(n-\beta)} e_i : a_{(2n-\beta)} \leq (n-\beta) \leq (2n-\beta)a_{(2n-\beta)}\right\}$. Since $K_{n(2n-\beta)}$ is compact, there are a finite number of polynomials $p_1, \ldots, p_k$ of form $p_j(t) = \sum_{i=a_{(2n-\beta)}}^{(2n-\beta)a_{(2n-\beta)}} \lambda_{ji} t^i$, such that for all $\mathbf{x} \in K_{n(2n-\beta)}$ there is a j such that

$$\left\| p_j\left(T_{(2n-\beta)}\right)\mathbf{x} -$$

Writing $N = \max_j |p_j|$, note that N depends only on elements of the underlying Cauchy sequence $\mathbf{d}$ up to and including $a_{(2n-\beta)}$; so $N < N_3\left((2n-\beta), a_{(2n-\beta)}\right)$ for a suitable function $N_3 : N^2 \to N$. Since in view of (102) we have

$$\left\| \left((2n -\right.\right.$$

this concludes the proof.

We now extend the previous lemma as follows.

Corollary (6.1.41)[284]: *For all* $j \in [0, v_n]$*, we have*

$$f^*_{j+w_n, n-\beta+\epsilon_3-1}$$

Proof. The vector $f^*_{j+w_n, n-\beta+\epsilon_3-1} = f_{j+n-\beta+\epsilon_3 a_{k+\epsilon_3}}$ (Definition (6.1.8)) is in the image of the projection $\tau_{n,2n-\beta+\epsilon_3} \circ Q^0_{\epsilon_3} k$ and is fixed by it. The vector $f_{j+w_n, n-\beta+\epsilon_3} = f_{j+(n-\beta+\epsilon_3+1)a_{n+n-\beta+\epsilon_3}}$ is mapped (by (Definition (6.1.19)), (Definition (6.1.25))) to

$$u = -e_{j+n-\beta}$$

which by (100), (6), satisfies $f^*_{j+w_n, n-\beta+\epsilon_3-1}(u) = -1$. It is easily seen that for all other vectors $f_{(2n-\beta)}$, $\tau_{n,2n-\beta+\epsilon_3} \circ Q^0_{2n-\beta+\epsilon_3}\left(f_{(2n-\beta)}\right)$ is either $f_{(2n-\beta)}$, or zero, or another vector similar to u above, being therefore of form $f_{j',s'} + h$ with the pair (j', s') not equal to $\left(j + w_{n,n-\beta+\epsilon_3}\right)$, and with $h \in F_{v_{2n-\beta+\epsilon_3-1}}$. In all such cases $f^*_{j+w_n, 2n-\beta+\epsilon_3}$ $\left(\tau_{n,2n-\beta+\epsilon_3} Q^0_{2n-\beta+\epsilon_3} f_{(2n-\beta)}\right) = 0$, hence the result.

Corollary (6.1.42)[284]: Show that

(i) $\left\| Q^0_{(2n-\beta)} \right\| \le$

when $n = \beta$

(ii) $\left\| P_{n(2n-\beta)} \right\| \le$

when $n = \beta - 1$

Proof. (i) From Lemma (6.1.26) and Corollary (6.1.39) when butting $n=\beta$ give the result.

(ii) Similarly we can find the result by setting $n=\beta-1$ we can deduce that (i) and (ii) are equal.

between James Spaces: Section (6.2): Strictly Singular Operators

Recall that an operator $T:X\to Y$ between Banach spaces is said to be *strictly singular* if for every $\varepsilon>0$ and every infinite-dimensional subspace $E\subseteq X$ there is a vector x in the unit sphere of E such that $\|T\,x\|<\varepsilon$. Furthermore, T is said to be *finitely strictly singular* if for every $\varepsilon>0$ there exists $n\in\mathbb{N}$ such that for every subspace $E\subseteq X$ with $\dim E\geq n$ there exists a vector x in the unit sphere of E such that $\|T\,x\|<\varepsilon$. Finitely strictly singular operators are also known in literature as *superstrictly singular*. Note that

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and that each of these three properties defines a closed subspace in $L(X,Y)$. Actually, each property defines an operator ideal. We refer to [119, 120, 121, 122, 123, 124] for more information about strictly and finitely strictly singular operators.

We say that a subspace $E\subseteq X$ is invariant under an operator $T:X\to X$ if $\{0\}\neq E\neq X$ and $T(E)\subseteq E$. Every compact operator has invariant subspaces by [125]. On the other hand, Read constructed in [126] an example of a strictly singular operator [138, 143, 158, 159, 160]. Without nontrivial closed invariant subspaces (this answered a question of Pełczy´nski). Read's operator acts on an infinite direct sum which involves James spaces. Recall that James' p-space J_p is a sequence space consisting of all sequences $x=(x_n)_{n=1}^{\infty}$ in c_0 satisfying $\|x\|_{J_p}<\infty$ where

$\|x\|_{J_p}$

is the norm in J_p. For more information on James' spaces we refer to [127, 128, 120, 129, 130].

It was an open question whether every finitely strictly singular operator has invariant subspaces. Some partial results in this direction were obtained in [119, 123]. We answer this question in the negative by showing that the operator in [126] is, in fact, finitely strictly singular. As an intermediate result, we prove that the formal inclusion operator from J_p to J_q with $1 \le p < q < \infty$ is finitely strictly singular. The latter statement in a certain sense refines the result of Milman [121] that the formal inclusion operator from ℓ_p to ℓ_q with $1 \le p < q < \infty$ is finitely strictly singular. Milman's proof is based on the fact that every k-dimensional subspace E of $\mathbb{R}^n$ contains a vector "with a flat," namely, a vector x with sup-norm one with (at least) k coordinates equal in modulus to 1. For such a vector, one has $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$. The proofs of the results are based on the following refinement of this observation. We will show that x can be chosen so that these k coordinates have alternating signs. For such a "highly oscillating" vector x one has $\|x\|_{J_q} \ll \|x\|_{J_p}$. More precisely, a finite or infinite sequence of real numbers in $[-1, 1]$ will be called a *zigzag* of order k if it has a subsequence of the form $(-1, 1, -1, 1, \ldots)$ of length k. Our results will be based on the following theorem.

Corollary (6.2.1)[118]: *Let $k \in \mathbb{N}$; then every k-dimensional subspace of c_0 contains a zigzag of order k.*

Proof. Let F be a subspace of c_0 with $\dim F = k$. For every $n \in \mathbb{N}$, define $P_n : c_0 \to \mathbb{R}^n$ via $P_n : (x_i)_{i=1}^{\infty} \longmapsto (x_i)_{i=1}^{n}$. Let n_1 be such that $\dim P_{n_1}(F) = k$. There exists n_2 such that every vector in F attains its norm on the first n_2 coordinates. Indeed, define $g : F\backslash\{0\} \to \mathbb{N}$ via $g(x) = \max\{i: |x_i| = \|x\|_\infty\}$. Then g is upper semi-continuous, hence bounded on the unit sphere of F, so that we put $n_2 = \max\{g(x): x \in F, \|x\| = 1\}$. Put $n = \max\{n_1, n_2\}$. Since $P_n(F)$ is a k-dimensional subspace of $\mathbb{R}^n$, by Theorem (6.2.1) there exists $x \in F$ such that $P_n x$ is a zigzag of order k. It follows from our definition of n that x is a zigzag of order k in F.

Suppose that $1 \leq p < q$. Since $\|x\|_{J_p}$ is defined as the supremum of ℓ_p-norms of certain sequences, $\|\cdot\|_{\ell_q} \leq \|\cdot\|_{\ell_p}$ implies $\|\cdot\|_{J_q} \leq \|\cdot\|_{J_p}$. It follows that $J_p \subseteq J_q$ and the formal inclusion operator $i_{p,q} : J_p \to J_q$ has norm 1. We show next that it is finitely strictly singular. The main difference, though, is that we use Corollary (6.2.1) instead of the simpler lemma from [121,124].

Theorem (6.2.2)[118]:. *If* $1 \leq p < q < \infty$ *then the formal inclusion operator* $i_{p,q} : J_p \to J_q$ *is finitely strictly singular.*

Proof. Given any $x \in J_p$, then $\left|x_i - x_j\right|^q \leq (2\|x\|_\infty)^{q-p}\left|x_i - x_j\right|^p$ for every $i, j \in \mathbb{N}$, so that $\|x\|_{J_q} \leq (2\|x\|_\infty)^{1-\frac{p}{q}} \|x\|_{J_p}^{\frac{p}{q}}$. Fix an arbitrary $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $(k-1)^{\frac{1}{p}-\frac{1}{q}} > \frac{1}{\varepsilon}$. Suppose that E is a subspace of J_p with $\dim E = k$. By Corollary (6.2.1), there is a zigzag $z \in E$ of order k. By the definition of norm in J_p, we have $\|z\|_{J_p} \geq 2(k-1)^{\frac{1}{p}}$.

Put $y = \frac{z}{\|z\|_{J_p}}$. Then $y \in E$ with $\|y\|_{J_p} = 1$. Obviously, $\|y\|_\infty \leq \frac{1}{2}(k-1)^{-\frac{1}{p}}$, so that

Hence, $i_{p,q}$ is finitely strictly singular.

We will now use Theorem (6.2.2) to show that the strictly singular operator T constructed by Read in [126] is finitely strictly singular. Let us briefly outline those properties of T that will be relevant for our investigation. The underlying space X for this operator is defined as the ℓ_2-direct sum of ℓ_2 and Y, $X = (\ell_2 \oplus Y)_{\ell_2}$, where Y itself is the ℓ_2-direct sum of an infinite sequence of J_p-spaces $Y = \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{\ell_2}$, with (p_i) a certain strictly increasing sequence in $(2, +\infty)$. The operator T is a compact perturbation of $0 \oplus W_1$, where $W_1 : Y \to Y$ acts as a weighted right shift, that is,

$W_1($

with $\beta_i \to 0$. Note that one should rather write $\beta_i i_{p_i,p_i+1} x_i$ instead of $\beta_i x_i$. Clearly, it suffices to show that W_1 is finitely strictly singular.

For $n \in \mathbb{N}$, define $V_n : Y \to Y$ via

$V_n(x$

It follows from $\beta_i \to 0$ that $\|V_n - W_1\| \to 0$. Since finitely strictly singular operators from Y to Y form a closed subspace of $L(Y)$, it suffices to show that V_n is finitely strictly singular for every n. Given $n \in \mathbb{N}$, one can write

where $P_i : Y \to J_{p_i}$ is the canonical projection and $j_i : J_{p_i} \to Y$ is the canonical inclusion. Thus, V_n is finitely strictly singular because finitely strictly singular operators form an operator ideal. This yields the following result.

Corollary (6.2.3)[284]: *If $1 \le p + \epsilon = q < \infty$ then the formal inclusion operator $i_{p,p+\epsilon} : J_p \to J_{p+\epsilon}$ is strictly singular.*

Proof. For $x \in J_p$, then $|x_i - x_j| \le (2\|x\|_\infty)$, $i, j \in \mathbb{N}$, and $\|x\|_{J_{p+\epsilon}} \le (2\|x\|_\infty)^{\frac{\epsilon}{p+\epsilon}} \|x\|_{J_p}^{\frac{p}{p+\epsilon}}$. Given $\varepsilon > 0$ fixed. Let $(k-1)^{\frac{\epsilon}{p(p+\epsilon)}} > \frac{1}{\epsilon_1}$, $k \in \mathbb{N}$. Suppose $E \subseteq J_p$ with $\dim E = k < \infty$. By Corollary (6.2.1), we have a zigzag $z \in E$ of order k, then $\|z\|_{J_p} \ge 2(k-1)^{\frac{1}{p}}$. Set $y = \frac{z}{\|z\|_{J_p}}$ and $\|y\|_{J_p} = 1$ where $y \in E$. Moreover, $\|y\|_\infty \le \frac{1}{2}(k-1)^{-\frac{1}{p}}$ and have

Hence, $i_{p,p+\epsilon}$ is finitely strictly singular.

Theorem (6.2.4)[118]: *Read's operator T is finitely strictly singular.*

In the remaining two sections, we present two different proofs of Theorem (6.2.1), one based on combinatorial properties of polytopes and the other based on the geometry of the set of all zigzags and algebraic topology.

By a *polytope* in $\mathbb{R}^k$ we mean a convex set which is the convex hull of a finite set.

A set is a polytope iff it is bounded and can be constructed as the intersection of finitely many closed half-spaces. A *facet* of P is a face of (affine) dimension $k-1$. We refer to [131,132] for more details on properties of polytopes.

A polytope P is centrally symmetric iff it can be represented as the absolutely convex hull of its vertices, that is, $P = \mathrm{conv}\{\pm\bar{u}_1, \dots, \pm\bar{u}_n\}$ where $\pm\bar{u}_1, \dots, \pm\bar{u}_n$ are the vertices of P. Clearly, P is centrally symmetric iff it can be represented as the intersection of finitely many centrally symmetric "bands." More precisely, there are vectors $\bar{a}_1, \dots, \bar{a}_m \in \mathbb{R}^k$ such that $\bar{u} \in P$ iff $-1 \leq \langle \bar{u}, \bar{a}_i \rangle \leq 1$ for all $i = 1, \dots, m$, and the facets of P are described by $\{u \in P \colon \langle \bar{u}, \bar{a}_i \rangle = 1\}$ or $\{u \in P \colon \langle \bar{u}, -\bar{a}_i \rangle = 1\}$ as $i = 1, \dots, m$.

A *simplex* in $\mathbb{R}^k$ is the convex hull of $k+1$ points with non-empty interior. A polytope P in $\mathbb{R}^k$ is *simplicial* if all its faces are simplexes (equivalently, if all the facets of P are simplexes). Every polytope can be perturbed into a simplicial polytope by an iterated "pulling" procedure, in [5] for details. We will outline a slight modification of the procedure such that it preserves the property of being centrally symmetric. Suppose that P is a centrally symmetric polytope with vertices, say $\pm\bar{u}_1, \dots, \pm\bar{u}_n$ (see Fig. 1). Pull $\bar{u}_1$ "away from" the origin, but not too far, so that it does not reach any affine hyperplane spanned by the facets of P not containing $\bar{u}_1$; denote the resulting point $\bar{u}_1$. Let $Q = \mathrm{conv}\{\bar{u}_1, -\bar{u}_1, \pm\bar{u}_2, \dots, \pm\bar{u}_n\}$. By [131, 125] this procedure does not affect the facets of P not containing $\bar{u}_1$, while all the facets of Q containing $\bar{u}_1'$ become pyramids having apex at $\bar{u}_1'$. Note that no facet of P contains both $\bar{u}_1$ and $-\bar{u}_1$. Hence, if we put $R = \mathrm{conv}\{\pm\bar{u}_1', \pm\bar{u}_2, \dots, \pm\bar{u}_n\}$, then, by symmetry, all the facets of R containing $-\bar{u}_1'$ become pyramids with apex at $-\bar{u}_1'$, while the rest of the facets (in particular, the facets containing $\bar{u}_1'$) are not affected.

Now iterate this procedure with every other pair of opposite vertices. Let P' be the resulting polytope, $P' = \mathrm{conv}\{\pm\bar{u}_1', \dots, \pm\bar{u}_n'\}$. Clearly, P' is centrally symmetric and simplicial as in [131]. It also follows from the construction that if F is a facet of P' then all the vertices of P corresponding to the vertices of F belong to the same facet of P.

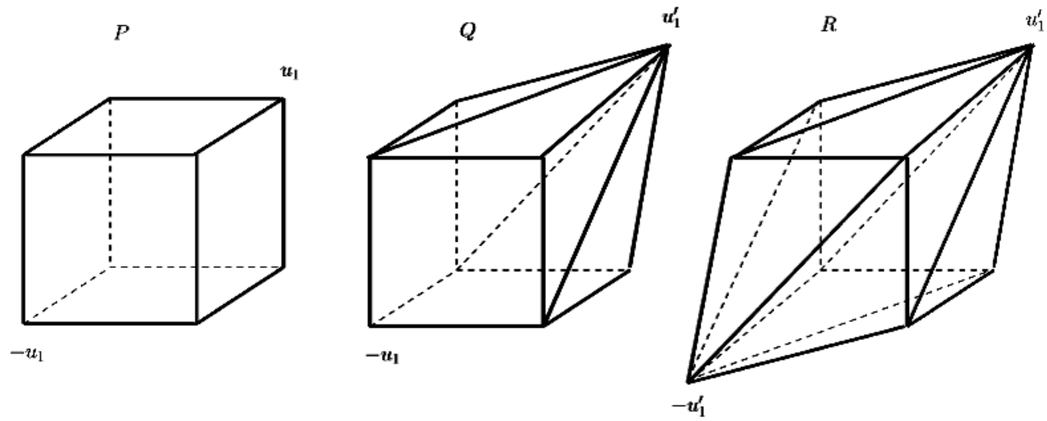

Fig. (1). Pulling out the first pair of vertices.

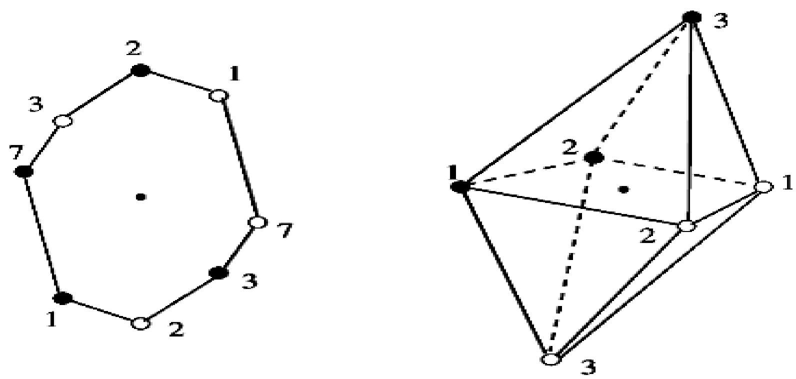

Fig. (2). Examples of marked polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We will call a polytope P *marked* if the following assumptions are satisfied:

(i) P is simplicial, centrally symmetric, and has a non-empty interior.

(ii) Every vertex is assigned a natural number, called its *index*, such that two vertices have the same index iff they are opposite to each other.

(iii) All the vertices of P are painted in two colors, say, black and white, so that opposite vertices have opposite colors.

See Fig. (2) for examples of marked polytopes. A face of a marked polytope is said to be *happy* if, when one lists its vertices in the order of increasing indices, the colors of the vertices alternate. For example, the front top facet of the marked polytope in the right-hand side of Fig. (2) is happy. See Fig. (3) for more examples of happy faces. We will reduce Theorem 1 to the claim that every marked polytope has a happy facet, which we will prove afterwards. Suppose that $k \leq n$ and E is a subspace of $\mathbb{R}^n$

with $\dim E = k$. Let $\{\bar{b}_1, \ldots, \bar{b}_k\}$ be a basis of E. We need to find a linear combination of these vectors $\bar{x} \coloneqq a_1\bar{b}_1 + \cdots + a_k\bar{b}_k$ such that $\bar{x}$ is a zigzag. Let B be the $n \times k$ matrix with columns $\bar{b}_1, \ldots, \bar{b}_k$, and

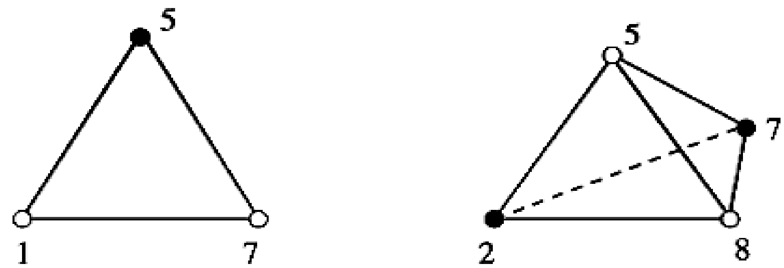

Fig. (3). Examples of happy simplexes in R2 and R3.

let $\bar{u}_1, \ldots, \bar{u}_n$ be the rows of B. If $\bar{a} = (a_1, \ldots, a_k)$, then $x_i = \langle \bar{u}_i, \bar{a} \rangle$ as $i = 1, \ldots, n$.

Thus, it suffices to find $\bar{a} \in \mathbb{R}^k$ such that the vector $(\langle \bar{u}_i, \bar{a} \rangle)_{i=1}^n$ is a zigzag of order k. Let P be the centrally symmetric convex polytope spanned by $\bar{u}_1, \ldots, \bar{u}_n$, i.e., $P = \operatorname{conv}\{\pm\bar{u}_1, \ldots, \pm\bar{u}_n\}$. Then some of the $\pm\bar{u}_i$'s will be the vertices of P, while the others might end up inside P. Suppose that $\pm\bar{u}_{m_1}, \ldots, \pm\bar{u}_{m_r}$ are the vertices of P, so that $P = \operatorname{conv}\{\pm\bar{u}_{m_1}, \ldots, \pm\bar{u}_{m_r}\}$. Following the "pulling" procedure that was described before, construct a simplicial centrally symmetric polytope $P' = \operatorname{conv}\{\pm\bar{u}'_{m_1}, \ldots, \pm\bar{u}'_{m_r}\}$. Every vertex of P' is either $\bar{u}'_{m_i}$ or $-\bar{u}'_{m_i}$ for some i. Paint the vertex white in the former case and black in the latter case; assign index i to this vertex. This way we make P' into a marked polytope.

We claim that happy facets of P' correspond to zigzags. Indeed, suppose that P' has a happy facet. Then this facet (or the facet opposite to it) is spanned by some $-\bar{u}'_{m_{i_1}}, \bar{u}'_{m_{i_2}}, -\bar{u}'_{m_{i_3}}, \bar{u}'_{m_{i_4}}$, etc., for some $1 \le i_1 < \cdots < i_k \le r$. It follows that $-\bar{u}_{m_{i_1}}, \bar{u}_{m_{i_2}}, -\bar{u}_{m_{i_3}}, \bar{u}_{m_{i_4}}$, etc., are all contained in the same facet of P. Hence, they are contained in an affine hyperplane, say L, such that P "sits" between L and $-L$. Let $\bar{a}$ be the vector defining L, that is, $L = \{\bar{u} : \langle \bar{u}, \bar{a} \rangle = 1\}$. Since P is between L and $-L$, we have $-1 \le \langle \bar{u}, \bar{a} \rangle \le 1$ for every $\bar{u}$ in P. In particular, $-1 \le x_i = \langle \bar{u}_i, \bar{a} \rangle \le 1$ for $i = 1, \ldots, n$. On the other hand, it follows from $-\bar{u}_{m_{i_1}}, \bar{u}_{m_{i_2}}, -\bar{u}_{m_{i_3}}, \bar{u}_{m_{i_4}}, \ldots \in L$ that $x_{m_{i_1}} = -1, x_{m_{i_2}} = 1, x_{m_{i_3}} = -1, x_{m_{i_4}} = 1$, etc. Hence, $\bar{x}$ is a zigzag of order k.

Thus, to complete the proof, it suffices to show that *every marked polytope has a happy facet*. Throughout the rest of this section, P will be a marked polytope in $\mathbb{R}^k$; $\mathcal{F}_j$ stands for the set of all j-dimensional faces of P for $j = 0, \dots, k-1$. In particular, $\mathcal{F}_{k-1}$ is the set of all facets of P, while $\mathcal{F}_0$ is the set of all vertices of P. By [131], every $(k-2)$-dimensional face E of P is contained in exactly two facets, say F and G; in this case $E = F \cap G$. Suppose that $R \subseteq \mathcal{F}_{k-1}$. For $E \in \mathcal{F}_{k-2}$, we say that E is a boundary face of R if $E = F \cap G$ for some facets F and G such that $F \in R$ and $G \notin R$. The set of all boundary faces of R will be referred to as the *face boundary* of R and denoted $\tilde{\partial} R$. Clearly, $\tilde{\partial} R \subset \mathcal{F}_{k-2}$. If F is a single facet, we put $\tilde{\partial} F = \tilde{\partial}\{F\}$. Clearly, $\tilde{\partial} F$ is the set of all the facets of F.

For a face F of P we define its *color code* to be the list of the colors of its vertices in the order of increasing indices. For example, the color codes of the simplexes in Fig. (3) are (wbw) and $(bwbw)$. Here b and w correspond to "black" and "white" respectively. A face in P will be said to be a *b-face* if its color code starts with b and a *w-face* otherwise.

Lemma (6.2.5)[118]: *Suppose that F is a facet of P. The following are equivalent*:

(i) *F is happy*;

(ii) *$\tilde{\partial} F$ contains exactly one happy b-face*;

(iii) *$\tilde{\partial} F$ has an odd number of happy b-faces.*

Proof. Note that since F is a simplex, every face of F can be obtained by dropping one vertex of F and taking the convex hull of the remaining vertices. Hence, the color code of the face is obtained by dropping one symbol from the color code of F.

(i)⇒(ii). Suppose that F is happy, then its color code is either $(bwbw\dots)$ or $(wbwb\dots)$. In the former case, the only happy b-face of F is obtained by dropping the last vertex, while in the latter case the only happy b-face of F is obtained by dropping the first vertex.

(ii)⇒(iii). Trivial.

(iii) $\Rightarrow$ (i). Suppose that $\tilde{\partial} F$ has an odd number of happy b-faces. Let E be a happy b-face in $\tilde{\partial} F$. Then the color code of E is the sequence $(bwbw\ldots)$ of length $k-1$. The color code of F is obtained by inserting one extra symbol into this sequence. Note that inserting the extra symbol should not result in two consecutive b's or w's, as in this case F would have exactly two happy b-faces (corresponding to removing each of the two consecutive symbols), which would contradict the assumption. Hence, the color code of F should be an alternating sequence, so that F is happy.

Lemma (6.2.6)[118]: *For every $R \subseteq \mathcal{F}_{k-1}$, the number of happy facets in R and the number of happy b-faces in $\tilde{\partial} R$ have the same parity.*

Proof. For $R \subseteq \mathcal{F}_{k-1}$, define the *parity* of R to be the parity of the number of happy b-faces in $\tilde{\partial} R$. Observe that if R and S are two disjoints subsets of $\mathcal{F}_{k-1}$, then the parity of $R \cup S$ is the sum of the parities of R and S (mod 2). It follows that the parity of R is the sum of the parities of all of the facets that make up R (mod 2). But this is exactly the parity of the number of happy facets in R by Lemma (6.2.5).

For every face F of P we write $-F$ for the opposite face. If R is a set of facets, we write $-R = \{-F: F \in R\}$. Also, we write R for the set theoretic union of all the facets in R.

Theorem (6.2.7)[118]: *Every marked polytope has a happy facet.*

Proof. We will prove a stronger statement: *every marked polytope in $\mathbb{R}^k$ has an odd number of happy b-facets*. The proof is by induction on k. For $k=1$, the statement is trivial. Let $k>1$ and let P be a marked polytope in $\mathbb{R}^k$.

For every facet F, let $\bar{n}_F$ be the normal vector of F, directed outwards of P. Fix a vector $\tilde{v}$ of length one such that $\tilde{v}$ is not parallel to any of the facets of P (equivalently, not orthogonal to $\tilde{n}_F$ for any facet F); it is easy to see that such a vector exists. By rotating P we may assume without loss of generality that $\tilde{v} = (0,\ldots,0,1)$. Let T be the projection from $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$ such that $T:(x_1,\ldots,x_{k-1},x_k) \mapsto (x_1,\ldots,x_{k-1})$. We can think of T as the orthogonal projection onto the "horizontal" hyperplane $\{\tilde{x} \in \mathbb{R}^k: x_k =$

$0\}$ in $\mathbb{R}^k$. Let $Q = T(P)$. Since T is linear and surjective, Q is again a centrally symmetric convex polytope in $\mathbb{R}^{k-1}$ with a non-empty interior (see Fig. 4).

It follows from our choice of $\tilde{v}$ that the kth coordinate of $\tilde{n}_F$ is non-zero for every facet F. Let R be the set of all the facets of P that "face upward," that is,

R

Clearly, a facet F is in $-R$ iff the kth coordinate of $\tilde{n}_F$ is negative. Hence, $-R \cap R = \emptyset$ and $-R \cup R = \mathcal{F}_{k-1}$. Observe that $\tilde{\partial} R = \tilde{\partial}(-R)$; hence $\tilde{\partial} R$ is centrally symmetric. Clearly, every vertical line (i.e., a line parallel to $\bar{v}$) that intersects the interior of P meets the boundary of P at

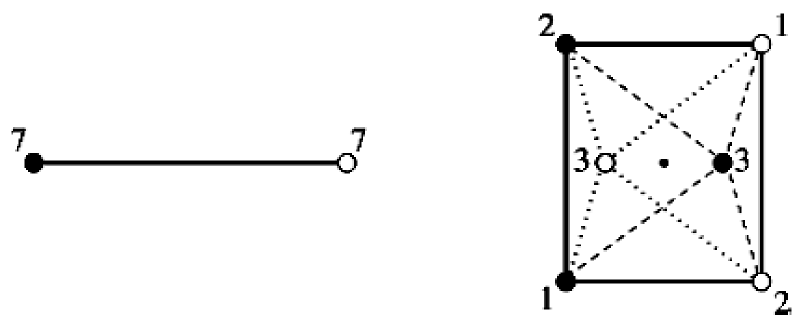

Fig. (4). The images $T(P)$ of the polytopes in Fig. 2.

exactly two points and meets the interior of Q at exactly one point. It follows that the restriction of T to $\bigcup R$ is a bijection between $\bigcup R$ and Q. The same is also true for $-R$. Therefore, the restriction of T to $\bigcup \tilde{\partial} R$ is a face-preserving bijection between $\bigcup \tilde{\partial} R$ and the boundary of Q. Under this bijection, the faces in $\tilde{\partial} R$ correspond to the facets of Q. Hence, this bijection induces a structure of a marked polytope on the boundary of Q, making Q into a marked polytope. It follows, by the induction hypothesis, that the boundary of Q has an odd number of happy b-facets. Hence, $\tilde{\partial} R$ has an odd number of happy b-faces. It follows from Lemma (6.2.6) that R has an odd number of happy facets. Let m and ℓ be the numbers of all happy b-facets and w-facets in R, respectively.

Then $m + \ell$ is odd. Observe that F is a happy b-facet iff $-F$ is a happy w-facet. It follows that $-R$ contains ℓ happy b-facets and m happy w-facets. Thus, the total number of happy b-facets of P is $m + \ell$, which we proved to be odd.

Fix a natural number n and let B_∞^n and S_∞^{n-1} be, respectively, the unit ball and the unit sphere of ℓ_∞^n, i.e., $B_\infty^n = \{x \in \mathbb{R}^n : \max|x_i| \le 1\}$ and $S_\infty^{n-1} = \{x \in \mathbb{R}^n : \max|x_i| = 1\}$. For $k \ge 1$ we define

$\Gamma_k = \{x$

$A_k^+ = \{x$

$A_k^- = -A$

Note that A_k^- is exactly the set of all zigzags of order k in $\mathbb{R}^n$. Put also $A_0^+ = A_0^- = \Gamma_0 = B_\infty^n$. For $k \ge 1, \Gamma_k, A_k^\pm \subset S_\infty^{n-1}$ and we have

Note that the first relation above is true also for $k = 0$.

We start with a simple lemma.

Lemma (6.2.8)[118]: *Suppose* p *is a real polynomial of degree* m*, and there are* $m+2$ *real numbers* $t_1 < t_2 < \cdots < t_{m+2}$*, such that* $p(t_i) \ge 0$ *for* i *odd and* $p(t_i) \le 0$ *for* i *even. Then* $p \equiv 0$*.*

Lemma (6.2.9)[118]: *There exists a sequence of subspaces* $\pi_k \subset \mathbb{R}^n, \pi_k \supset \pi_{k+1}$, $\dim \pi_k = n-k$*, such that, if* P_k *is the orthogonal projection onto* π_k*, then* $P_k|A_k^+$ *is injective.*

Proof. For $1 \le j \le n$ we define the vectors $\zeta^j \in \mathbb{R}^n$ by the formula $\zeta_i^j = i^{j-1}$. One checks easily that the ζ^j's are linearly independent. Define $\pi_0 = \mathbb{R}^n$, and, for $k \ge 1$, $\pi_k = (\operatorname{span}\{\zeta^1, \dots, \zeta^k\})^\perp$.

Suppose that $x, y \in A_k^+$, and $P_k x = P_k y$. There exist scalars $\alpha_1, \dots, \alpha_k$, such that $x - y = \sum_{j=1}^k \alpha_j \zeta^j$. We have indices $1 \le r_1 < \cdots < r_k \le n$ and $1 \le s_1 < \cdots < s_k \le n$, such that $x_{r_l} = y_{s_l} = (-1)^{l-1}$. It follows that $x_{r_l} - y_{r_l} \ge 0$ for l odd and ≤ 0 for l even, while $x_{s_l} - y_{s_l} \le 0$ for l odd and ≥ 0 for l even.

Let the polynomial p of degree $k-1$ be given by $p(t) = \sum_{j=1}^k \alpha_j t^{j-1}$. If $r_l = s_l$ for all l, we obtain for all $l = 1, \ldots, k$. Thus p has k distinct zeros; it must be identically 0, whence $x = y$. Suppose now that we have $r_l \neq s_l$ for at least one index l. We claim then that among the union of the indices r_l and s_l we can find $\iota_1 < \iota_2 < \cdots < \iota_{k+1}$, such that $x_{\iota_l} - y_{\iota_l}$ have alternating signs. This can be achieved by induction with respect to k. For $k = 1$ we must have $r_1 \neq s_1$, so we may take $\iota_1 = \min\{r_1, s_1\}, \iota_2 = \max\{r_1, s_1\}$. For $k > 1$, there are two cases. If $r_1 = s_1$, we take $\iota_1 = r_1 = s_1$ and apply the induction hypothesis to obtain the rest. If $r_1 \neq s_1$, we take ι_1 as the lesser of the two and ι_2 as the other one, and then we continue "accordingly" to ι_2 (that is, taking as ι's the rest of r's if $\iota_2 = r_1$ and the rest of s's if $\iota_2 = s_1$).

Now, the way ι_l have been chosen implies that $p(t)$ defined above satisfies the hypotheses of Lemma (6.2.8): it has degree $k - 1$ and the values it takes in $\iota_1, \ldots, \iota_{k+1}$ have alternating signs. It must then be identically 0, which implies $x = y$.

Since $A_k^- = -A_k^+$, it follows that $P_k | A_k^-$ is also injective.

Lemma (6.2.10)[118]: *If π_k, P_k are obtained in Lemma* (6.2.9)*, then*

is a balanced, convex subset of π_k, with 0 *as an interior point (in π_k). Moreover, $\Delta k = P_k(A_k^-) = P_k(A_k^+)$ and $\partial \Delta_k = P_k(\Gamma_{k+1})$ (the boundary in the relative topology of π_k).*

Proof. We will use induction with respect to k. The statement is immediately checked for $k = 0$ (note that $P_0 = I_{\mathbb{R}^n}$ and $\partial \Delta_0 = S_\infty^{n-1} = \Gamma_1$). Assume the statement true for k; we will prove its validity for $k + 1$. By the induction hypothesis, we have

and is therefore a balanced, convex subset of π_{k+1}, with 0 as an interior point. Take then $y \in \mathring{\Delta}_{k+1}$. Suppose $P_{k+1}^{-1}(y) \cap \partial \Delta_k$ contains a single point. Then $P_{k+1}^{-1}(y) \cap \Delta_k$ also contains a single point, and therefore $P_{k+1}^{-1}(y) \cap \pi_k$ is a support line for the convex set Δ_k. This line is contained in a support hyperplane (in π_k); but then the whole of Δ_k projects onto π_{k+1} on one side of this hyperplane, and thus y belongs to the boundary of this projection. Therefore y cannot be in $\mathring{\Delta}_{k+1}$.

The contradiction obtained shows that $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains at least two points. But

whence

Since P_{k+1} restricted to each of the two terms in the right-hand side is injective by Lemma (6.2.10), there exists a unique $z_+ \in A_{k+1}^+$ such that $y = P_{k+1}z_+$ and a unique $z_- \in A_{k+1}^-$ such that $y = P_{k+1}z_-$.

Take $x \in P_{k+1}^{-1}(y) \cap \partial\Delta_k$. Then either $x \in P_k(A_{k+1}^+)$ or $x \in P_k(A_{k+1}^-)$. If $x \in P_k(A_{k+1}^+)$ then $x = P_k z$ for some $z \in A_{k+1}^+$, so that $y = P_{k+1}x = P_{k+1}z$, which yields $z = z_+$; hence $x = P_k z_+$. Similarly, if $x \in P_k(A_{k+1}^-)$ then $x = P_k z_-$. It follows that $P_{k+1}^{-1}(y) \cap \partial\Delta_k \subseteq \{P_k z_+, P_k z_-\}$. Since $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains at least two points, we conclude that $P_{k+1}^{-1}(y) \cap \partial\Delta_k = \{P_k z_+, P_k z_-\}$ and $P_k z_+ \neq P_k z_-$. It follows from $y = P_{k+1}z_\pm$ that $\mathring{\Delta}_{k+1} \subset P_{k+1}\left(A_{k+1}^\pm\right)$. But, Δ_{k+1} being a closed convex set with a nonempty interior, it is the closure of its interior $\mathring{\Delta}_{k+1}$; since the two sets on the right are closed, we have actually $\Delta_{k+1} = P_{k+1}\left(A_{k+1}^\pm\right)$.

We want to show now that $\partial\Delta_{k+1} = P_{k+1}(\Gamma_{k+2})$. Suppose first that $y \in P_{k+1}(\Gamma_{k+2}) = P_{k+1}(A_{k+1}^+ \cap A_{k+1}^-)$; that is, $y = P_{k+1}z$ with $z \in A_{k+1}^+ \cap A_{k+1}^-$. Clearly, $y \in \Delta_{k+1}$. If $y \in \mathring{\Delta}_{k+1}$, then, defining z_+ and z_- as before, the injectivity of P_{k+1} on $A_{k+1}^\pm$ implies $z = z_- = z_+$. This contradicts $P_k z_+ = P_k z_-$; consequently, $y \in \partial\Delta_{k+1}$.

Conversely, take $y \in \partial\Delta_{k+1} = \partial\left(P_{k+1}(\Delta_k)\right)$. Again, take $z_+ \in A_{k+1}^+, z_- \in A_{k+1}^-$, such that $P_{k+1}z_+ = P_{k+1}z_- = y$. We have then $P_k z_+ \in \partial\Delta_k$ (if $P_k z_+ \in \mathring{\Delta}_k$, then $P_{k+1}z_+ = P_{k+1}P_k z_+$ must be in the interior of $P_{k+1}\Delta_k$, which is $\mathring{\Delta}_{k+1}$). Similarly, $P_k z_- \in \partial\Delta_k$.

If $P_k z_+ = P_k z_-$, then P_{k+1} applied to the whole segment $[P_k z_+, P_k z_-]$ is equal to y. Therefore the segment belongs to $\partial\Delta_k$. Since $\partial\Delta_k = P_k(A_{k+1}^+ \cup A_{k+1}^-)$, there exist two values x_1, x_2 either both in A_{k+1}^+ or both in A_{k+1}^-, such that $P_k x_1, P_k x_2 \in$

$[P_k z_+, P_k z_-]$, and thus $P_{k+1}x_1 = P_{k+1}x_2 = y$. This contradicts the injectivity of P_{k+1} on $A^{\pm}_{k+1}$.

Therefore $P_k z_+ = P_k z_-$. But z_+ and z_- both belong to A^+_k, on which P_k is injective. It follows that $z_+ = z_- \in A^+_{k+1} \cap A^-_{k+1} = \Gamma_{k+2}$, and $P_{k+1}z_+ = y$. This ends the proof.

The main consequence of Lemma (6.2.10), in combination with Lemma (6.2.9), is the fact that the linear map P_{k-1} maps homeomorphically Γ_k into $\partial\Delta_{k-1}$, which is the boundary of a convex, balanced set, containing 0 in its interior.

Theorem (6.2.11)[118]: *For every $k \le n$, every k-dimensional subspace of $\mathbb{R}^n$ contains a zigzag of order k.*

Proof. As noted above, P_{k-1} maps homeomorphically Γ_k onto the boundary of a convex, balanced set, containing 0 in its interior. Composing it with the map $x \mapsto \frac{x}{\|x\|}$, we obtain a homeomorphic map ϕ from Γ_k to S^{n-k}, which satisfies the relation $\phi(-x) = -\phi(x)$. Suppose that E is a k-dimensional subspace of $\mathbb{R}^n$ with no zigzags. Then $E \cap \Gamma_k = \emptyset$, so that the projection of Γ_k onto $E^\perp$ does not contain 0. Composing this projection with the map $x \mapsto \frac{x}{\|x\|}$, we obtain a continuous map from $\psi : \Gamma_k \to S^{n-k-1}$, that satisfies $\psi(-x) = -\psi(x)$. Then the map $\Phi := \psi \circ \phi^{-1} : S^{n-k} \to S^{n-k-1}$ is continuous and satisfies $\Phi(-x) = -\Phi(x)$. This is however impossible: it is known that such a map does not exist (see, for instance, [133]).

Remark (6.2.12)[284]: *For every $k = n - \epsilon$, every $(n-\epsilon)$-dimensional subspace of $\mathbb{R}^n$ contains a zigzag of order $(n-\epsilon)$.*

Proof. We have $P_{(n-\epsilon-1)}$ maps homeomorphically $\Gamma_{(n-\epsilon)}$ onto the boundary of a convex, balanced set, containing zero. We obtain, upon composing $x \mapsto \frac{x}{\|x\|}$, homeomorphic map $\varphi: \Gamma_{(n-\epsilon)} \to S^\epsilon$, satisfying that $\varphi(-x) = -\varphi(x)$. If $E \subseteq \mathbb{R}^n$ of $(n-\epsilon)$-dimensional with no zigzag, then $E \cap \Gamma_{(n-\epsilon)} = \phi$, so that $P\left(\Gamma_{(n-\epsilon)}\right)$ onto the orthogorolity of E, such that $O \notin P\left(\Gamma_{(n-\epsilon)}\right)$. By composing with the map $x \mapsto \frac{x}{\|x\|}$.

We get that $\varphi: \Gamma_{(n-\epsilon)} \to S^{\epsilon-1}$. Which satisfy $\psi(-x) = -\psi(x)$. Then $\Phi \coloneqq \psi \circ \varphi^{-1} : S^{\epsilon} \to S^{\epsilon-1}$ is bounded and hence $\varphi(-x) = -\varphi(x)$ does not exist (see Theorem (6.2.11)).

List of Symbols

Symbol	Page
$C_R(x)$: Real Branch Space	3
sup : Supremum	9
Ker : Kernel	10
$\bigoplus$: Direct Sum	13
dim : Dimension	16
Ext : Extreme	17
inf : infimum	20
Card : Cardinality	21
Ch : Choquet	23
Re : Real	40
sgn : Sgnature	48
det : determinant	49
Im : Imajinary	54
int : Interior	62
C : Clusure	63
max : maximum	68
H^2 : Hilbert space	69
L^p : lebesgue space	69
L : Hilbert space	69
$\mathbb{C}$: unitary space	73
L : lebesgue space	83
arg : argument	84
H : Hardy space	107
L : Hilbert space of sequences	118
End : Endomorphism	119
codim : codimension	126
$\otimes$: Tensor product	130
Hom : homomorphism	152
Idem : Idempotent	152
H : Hilbert Representation	160
ran : range	189
ccc : countable chain condition	194
mod : modular	218
J_1 : James p-space	225
pd : provided dincreases sufficiently rapidly	226
deg : degree	251
ℓ : Hilbert space	262
conv : convex	265

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