Sudan University of Science and Technology

College of Graduate Studies

On-Central Manifold and Normal Forms Theorem in Local Bifurcations

حول متعددة الطيات المركزية ومبرهنة الصيغ الناظمة في التفرعات الموضعية

A Thesis Submitted in Fulfillment of the requirements for the degree philosophy in Mathematics

By

Bashir Suliman Othman

Supervisor

Associate Professor: Adam Abdulla Abkar

2014

Dedication

To my family and all that help me to do this study.

At last, but by no means the least, my heart-left appreciations go to my parents and my suns Youssif and Mohammed for their “time- invariant” support.

Acknowledgments

First and foremost, it is an immense pleasure to express my deep and sincere gratitude to my supervisor, Prof. Adam Abdulla Abakar for his guidance, assistance, encouragement, and hearty support in all the phases of my doctoral program at Sudan University of Science and Technology. I am greatly indebted to Mathematics Department of Sudan University of Science and Technology and King Abdul-Aziz University, and King Abdul-Aziz University Liberally for their stimulating discussions. Here especially I want to show my gratitude to my co-supervisor, Dr. Tarig Alzaki, for his always understanding and support when I took days off or put their assignments aside to work on my thesis.

Abstract

The aim of this study is to introduce tools from local bifurcation theory which will be necessary in the following sections for the study of neural field equations. In a first step, we deal with a basic manifold, elementary bifurcations in low dimensions such as saddle-node, trans critical, pitchfork and Hopf bifurcations. Bifurcation analysis for infinite dimensional systems is subtle and can lead to difficult problems. If it is possible, the idea is to locally reduce the problem to a finite dimensional one. This reduction is called the center manifold theory and it will be the main theoretical result of this study. The center manifold theory requires some functional analysis tools which will be recalled, especially the notions of linear operator, spectrum, resolve, projectors etc... We also present some extensions of the center manifold theorem for parameter-dependent and equivarient differential equations. Directly related to the center manifold theory is the normal form theory which is a canonical way to write differential equations. We conclude this study by some applications.

الخلاصة

الهدف من هذه الدراسة هو ايجاد طريقة أو وسيلة بنظرية أنظمة التفرعات الموضعية التي يكون لها أهمية في دراسة المجال الحيادي للمعادلات، تبدأ بالخطوة الأولى، المفاهيم الأساسية لمتعددات الطيات في أقل الإحداثيات مثل شكل سرج صهوة الفرس، عبر الحرجة المذراة و أنظمة التفرعات لهوبف (Hopf). تحليل أنظمة التفرعات الموضعية ذات الإحداثيات اللانهائية التي تؤدي الى صعوبة المسائل. إذا كان من الممكن، الفكرة هي تنزيل (تخفيض) رتبة النظم اللانهائية الى منتهية مع الحفاظ على دور تمثيل النظام الأساسي، هذا التخفيض يسمى متعددة الطيات المركزية في التفرعات الموضعية وهي النظرية الأساسية في هذه الدراسة. نظرية متعددة الطيات المركزية التي تتطلب بعض طرق التحليل الدالي التي تسترجع خاصة فكرة المؤثر الخطي، الطيف،...وأيضاً نوجد بعض التوسيعات لنظرية الطيات المركزية وثوابت المعادلات التفاضلية.مباشرةً الى مبرهنة الصيغ الناظمة التي تساعد في كتابة المعادلات التفاضلية في صورتها القانونية (الأساسية) ثم نختتم هذه الدراسية ببعض التطبيقات.

The Contents

Subject	Page No.
Dedication	I
Acknowledgements	II
Abstract	III
Abstract (Arabic)	IV
The Contents	V
Introduction	1
Chapter 1 **Manifold Theory**	
Section (1.1): Basic Concepts	4
Section (1.2): Chart and Atlas	11
Section (1.3): Differentiable Manifolds	15
Chapter 2 **Bifurcation Theory**	
Section (2.1): Elementary bifurcation	21
Section (2.2): Stability Test	22
Section (2.3): Simplest Bifurcation condition	30
Chapter 3 **Centre Manifold Theory**	
Section (3.1): Linear systems	37
Section (3.2): The strategy	41
Chapter 4 **Normal Forms Theorem**	
Section (4.1): The Normal Forms Simplification	53
Section (4.2): Setting	54
Chapter 5 **Normal Form theory in local bifurcation**	
Section (5.1): Introduction	64
Section (5.2): Quadratic Normal Form	67
Section (5.3): Cubic Normal Form	69
Section (5.4): Hopf Bifurcation	72
Chapter 6 **Applications**	
Section (6.1): Introduction	76
Section (6.2): Singularity theory	78
Section (6.3): Scalar problems	79
References	86

Introduction

In this study we focus on two specific methods that arise in the analysis of local bifurcations in infinite-dimensional systems, namely the center manifold reduction and the normal form theory. Center manifolds provide a powerful method of analysis of such systems, as they allow one to reduce, under certain conditions, the infinite-dimensional dynamics near a bifurcation point to a finite-dimensional dynamics, described by a system of ordinary differential equations. An efficient way of studying the resulting reduced systems is with the help of normal form theory, which consists in suitably transforming a nonlinear system, in order to keep only the relevant nonlinear terms and to allow easier recognition of its dynamics. The combination of these two methods led over the recent years to significant progress in the understanding of various problems arising in applied sciences, and in particular in the study of nonlinear waves. A common feature of many of these problems is the presence of symmetries, as for instance reversibility symmetries. It turns out that both the center manifold reduction and the normal form transformations preserve symmetries, allowing an efficient treatment of such problems. In addition, they provide a detailed comprehensive study near a singularity in the solution set of the system, which might also orient a numerical treatment of such problems. The study is organized as follows. We start in Chapter one with a manifolds, define basic concepts, chart and atlas and differentiable manifolds. Chapter two bifurcations for one- and two-dimensional ordinary differential equations, Elementary bifurcation and Stability Test, saddle-node, pitchfork, Hopf, and steady bifurcations in the presence of a simple symmetry group. The purpose of this particular choice is to some of the techniques and notations used in the next chapters. Chapter three is devoted to the center manifold theory. Linear systems. We present the strategy for center manifold reduction for infinite-dimensional systems, together with simple examples and exercises illustrating the variety of possible applications.

The aim is to declare that the subject to use this reduction method simply by checking some clear assumptions. Chapter four is concerned with the normal form

theory. In particular, we show how to systematically compute the normal forms in concrete situations. We illustrate the general theory on different bifurcation problems, for which we provide explicit formulas for the normal form, allowing one to obtain quantitative results for the resulting systems. In Chapter five the normal form theory is applied to the study of reversible bifurcations, which appear to be of particular importance in applications, as this is shown in Chapter six. We focus on bifurcations of co-dimension 1, i.e., bifurcations involving a single parameter, which arise generically for systems in dimensions 2, 3, and 4. In all cases, we give the normal forms and collect some known facts on their dynamics. Finally, in Chapter six we present some applications of the methods described.

Historical Background: Many authors refer to the work of C. G. J. Jacobi from 1834, on equilibria of self gravitating rotating ellipsoids, as a first reference in the field of bifurcation theory. However, it seems that the first serious works on bifurcation problems were by Archimedes and Apollonius over 200 years BCE. Archimedes studied the equilibria of a floating paraboloid of revolution. In today's terminology his results would correspond to a pitchfork bifurcation which breaks a flip symmetry, or to a steady bifurcation with O(2) symmetry, when taking into account the invariance under rotations about the paraboloid axis. Apollonius studied the extreme a of the length of segments joining a point of the plane to a given conic . The number of solutionschanges from one to three in crossing the envelope of the normal to the conic. Here again, due to the symmetry of the conic, we have an example of a pitchfork bifurcation. Finally, it seems that the French word "bifurcation" was introduced by Poincaré in 1885 Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of R^n. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, so that many concepts from R^n, such as differentiability, point-derivations, tangent spaces, and differential forms,carry over to a manifold. Like most fundamental mathematical concepts, the idea of a manifold did not originate with a singlepers on, but is rather the distillation of years of collective

activity. In his masterpiece Disquisitiones generales circa superficies curvas (“General Investigations of Curved Surfaces”) published in 1827, Carl Friedrich Gauss freely used local coordinates on a surface, and so he already had the idea of charts. Moreover, he appeared to be the first to consider a surface as an abstract space existing in its own right, independent of a particular embedding in a Euclidean space. Bernhard Riemann’s inaugurallecture ÜberdieHypothesen, welche der Geometriezu Grundeliegen (“On the hypotheses that underliegeometry”) in Göttingen in 1854 laid thefoundations of higher-dimensional differential geometry. Indeed, the word “manifold” is a direct translation of the German word “Mannigfaltigkeit,” which Riemann used to describe the objects of his inquiry. This was followed by the work of Henri Poincaré in the late nineteenth century on homology, in which locally Euclidean spaces figured prominently. The late nineteenth and early twentieth centuries were also a period of feverish development in point-set topology. It was not until 1931 that one finds the modern definition of a manifold based on point-set topology and a group of transition functions Normal form theory for differential equations can be traced back to the original work of one hundred years ago, and most credit should be given to Poincaré [1879].

Chapter 1

Manifold Theory

The notion of a manifold S defined in the following chapter assumes S to be a subset of a Euclidean space R^n. However, a more axiomatic and abstractapproach to differential geometry is possible, and in many ways preferable.

Of course, a manifold in R^n must satisfy the axioms that we set up for anabstract manifold. Our axioms will be based on properties of charts. From the point of view of differential geometry the most important property of a manifold is that it allows the concept of a smooth function. We will define this notion and the more general notion of a smooth map between abstract manifolds.

Section (1.1): Basic Concepts:

The study of curves and surfaces in Geometry was mainly through parameterizations and measures, important examples of curves and surfaces arise more naturally as level sets, for example the circle $\{(x,,y) \mid x^2 + y^2 = 1\}$ and the sphere $\{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$. In order to deal with such sets, we shall define a notion of manifolds, which applies to subsets in R^n without the specification of a particular parameterization. The new notion will take into account the possibility that the given subset of R^n is not covered by a single parameterization. It is easy to give examples of subsets of R^3 that we imagine as surfaces, but whose natural parameterizations do not cover the entire set (at least if we require the parameterizations to be regular) For example, we have seen that for the standard spherical coordinates on the sphere there are two singular points, the poles. In order to have a regular parameterization we must exclude these points. A variation of the standard spherical coordinates with interchanged roles of y and z will have singular poles in two other points. The entire sphere can thus be covered by spherical coordinates if weallow two parameterizations covering different, overlapping subsets of the sphere.

Definition (1.1.1). A parameterized manifold in R^n is a smooth map $f: U \to R^n$, where $U \subset R^m$ is a non-empty open set. It is called regular at $x \in U$ if the $m \times n$ Jacobi matrix $Df(x)$ has rank m (that is, it has linearly independent columns), and it is called regular if this is the case at all $x \in U$. An m-dimensional parameterized manifold is a parameterized manifold $f: U \to R^n$ with $U \subset R^m$, which is regular (that is, regularity is implied at all points when we speak of the dimension)

Clearly, a parameterized manifold with $m = 2 \; and \; n = 3$is the sameas a parameterized surface, and the notion of regularity is identical to theone introduced in Geometry. For $m = 1$ there is a slight difference withthe notion of parameterized curves, because in Geometry we have requireda curve $\gamma: I \to R^n$ to be defined on an interval, whereas herewe are justassuming U to be an open set in R. Of course there are open sets in R which are not intervals, for example the union of two disjoint open intervals. Notice however, that if $\gamma: U \to R^n$ is a parameterized manifold with $U \subset R$, then for each $t_0 \in U$ there exists an open interval I around t_0 in U, and there striction of γ to that interval is aparameterized curve in the old sense. Infuture, when we speak of a parameterized curve, we will just assume that itis defined on an open set in R.Perhaps the case $m = 0$ needs some explanation. By definition R^0 is thetrivial vector space {0}, and a map $f: R^0 \to R^n$ has just one$u$ value $p = f(x)$. By definition the map $0 \to p$ is smooth and regular, and thus a 0-dimensionalparameterized manifold in $R^n$is a point$p \in R^n$ [9]

Example (1.1.1). Let $\sigma(u, v) = (\cos u \, , \sin u \, , \cos v \, , \sin v \,) \in R^4$. Then

$$D\partial(u, v) = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & -\sin v \\ 0 & \cos v \end{pmatrix}$$

has rank 2, so that σ is a 2-dimensional manifold in R^4. [9]

Example (1.1.2). The graph of a smooth function $h: U \to R^{n-m}$, is manifold in R^n. Let$\sigma(x) = \big(x, h(x)\big) \in R^n$, then $D\sigma(x)$ is an $n \times m$ matrix, of which the first mrows comprise a unit matrix. It follows that $D\sigma(x)$ has rank m for all x, sothat σ is regular.

Many basic results about surfaces allow generalization, often with proof sanalogous to the 2-dimensional case. Below is an example. By definition, are-parameterization of a parameterized manifold $\sigma: U \to R^n$ is a parameterizedmanifold of the form $\tau = \sigma \circ \emptyset$ where $\emptyset: W \to U$ is a diffeomorphism of open sets.

Definition (1.1.2). Is a differentiable mapping that has a differentiable inverse. Two sets are diffeomorphically equivalent if there is a diffeomorphism of one onto the other. For example, the reals and the interval $(0, \infty)$ are diffeomorphically equivalent, since the diffeomorphism [9]

$$f : R \mapsto (0, \infty): f(x) = e^x$$

has an inverse$g : (0, \infty) \mapsto R : g(x) = logx$

Theorem (1.1.1). Let $\sigma: U \to R^n$ be a parameterized manifold with $U \subset R^m$, and assume it is regular at $p \in U$. Then there exists a neighborhood of p in *U*, such that the restriction of σ to that neighborhood allows are parameterization which is the graph of a smooth function, where *n-m* among the variables $x_1, \dots, x_n$ are considered as functions of the remaining *m* variables.

Definition (1.1.3). Let $A \subset R^m \; and \; B \subset R^n$. A map $f: A \to B$ which iscontinuous, bijective and has a continuous inverse is called a homeomorphism.

The sets *A* and *B* are metric spaces, with the same distance functions as the surrounding Euclidean spaces, and the continuity of f and f^{-1} isassumed to be with respect to these metrics. [9]

Definition (1.1.4). A regular parameterized manifold $\sigma: U \to R^n$ which is ahomeomorphism $U \to \sigma(U)$, is called an embedded parameterized manifold. We shall define a concept of manifoldswhich applies to subsets of R^n rather than to parameterizations. In orderto understand the definition properly, we begin by the case of curves in R^2.

The idea is that a subset of R^2 is a curve, if in a neighborhood of each of itspoints it is the image of an embedded parameterized curve

Example (1.1.4). The graph of a smooth function $h: U \to R^{n-m}$, where $U \subset R^m$ is open, is an embedded parameterized manifold in R^n. It is regular by Example 1.1.2, and it is

clearly injective. The inverse map $\sigma(x) \to x$ isthe restriction to $\sigma(U)$ of the projection $R^n \ni x \mid \to (x_1, \ldots, x_m) \in R^m$ on the first *m* coordinates. Hence this inverse map is continuous.

Example (1.1.5). Consider the parameterized curve $\gamma(t) = (\cos t\,, \cos t \sin t\,)\; in\; R^2$. It is easily seen to be regular, and it has a self-intersection in (0, 0), which equals $\gamma\left(\frac{k\pi}{2}\right)$ for all odd integers *k* (see the figure below).The interval $I = \left]\frac{-\pi}{2}\,, \frac{3\pi}{2}\right[$ contains only one of the values $\frac{k\pi}{2}$, and the restriction of $\gamma$to*I* is an injective regular curve. The image $\gamma(I)$ is the fullset C in the figure.1 below.[9]

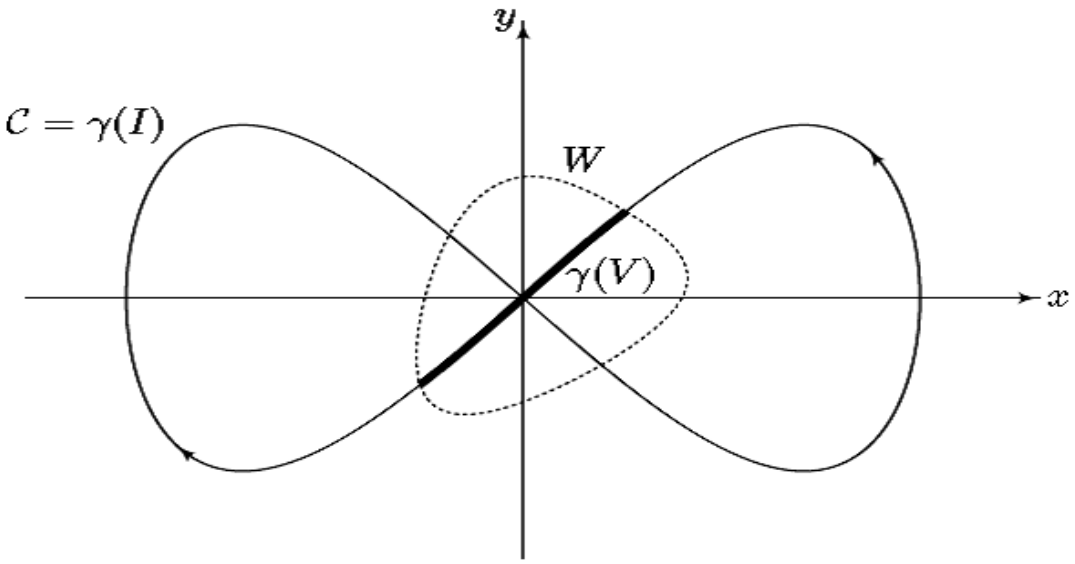


Figure 1.

The restriction $\gamma \mid$I is not a homeomorphism from *I* to *C*. The problem occurs in the point $(0,0) = \gamma\left(\frac{\pi}{2}\right)$. Consider an open interval $V = \left]\frac{\pi}{2} - \epsilon\,, \frac{\pi}{2} + \epsilon\right[$ where 0 < ϵ <π. The image $\gamma(V)$is shown in the figure, and it does not havethe form *C*∩*W* for any open set $W \subset R^2$, because *W* necessarily contains points from the other branch through (0, 0). Hence $\gamma \mid I$ is not an embeddedparameterized curve.

It is exactly the purpose of the homeomorphism requirement to exclude the possibility of a 'hidden' self-intersection, as in Example 1.1.4 Based on the example one can easily construct similar examples in higher dimension.

Definition (1.1.5). A curve in $R^2$is a non-empty set $C \subset R^2$ satisfying thefollowing for each$p \in C$. There exists an open neighborhood $W \subset R^2$ *of* p , an open set $I \subset R$, and an embedded parameterized curve $\gamma: I \to R^2$ withimage
$\gamma(I) = C \cap W$.

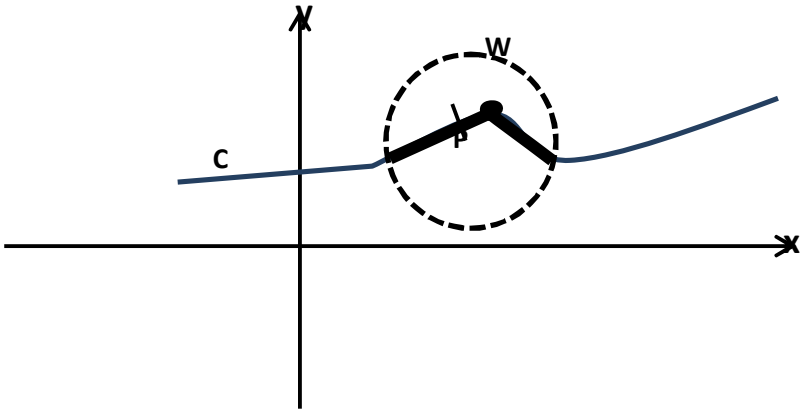


Figure 2.

Shows an embedded parameterized curve.

The definition of a curve allows the following useful reformulation [9].

Example (1.1.6). The image C = γ(I) of an embedded parameterized curveis a curve. In the condition above we can take $W = R^2$.

Example (1.1.7). The circle $C = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$is a curve. In order to verify the condition in Definition 1.1,5 let $p \in C$ be given. For simplicity we assume that $p = (x_0, y_0)$ with $x_0 > 0$.

Example (1.1.8). An 8-shaped set like the one in Example 1.1.5 is not a curve in R^2. In that example we showed that the parameterization by (cos t, cos t sin t) was not embedded, but of course this does not rule out that some other parameterization could satisfy the requirement in Definition 1.1.5 That this is not the case can be seen from Lemma 1.1.1 below.

It is of importance to exclude sets like this, because there is not a well defined tangent line in the point p of self-intersection. If a parameterization is given, we can distinguish the passages through p, and thus determine a tangent line for each branch.

However, without a chosen parameterization both branches have to be taken into account, and then there is not a unique tangent line in p. The definition of a curve allows the following useful reformulation.

Lemma (1.1.1). Let $C \subset R^2$ be non-empty. Then C is a curve if and only if it satisfies the following condition for each $p \in C$: There exists an open neighborhood $W \subset R^2$ of p, such that $C \cap W$ is thegraph of a smooth function h, where one of the variables x_1, x_2 is considered a function of the other variable.

Proof. Assume that C is a curve and let $p \in C$. Let $\gamma: I \to R^2$ be an embedded parameterized curve satisfying Definition (1.1.5) and with $\gamma(t_0) = p$. By the following Theorem 1.1.1 Let $\sigma: U \to R^2$ be a parameterized manifold d with $U \subset R^2$, and assume it is regular at $p \in U$. Then there exists a neighborhood of p in U, such that the restriction of σ to that neighborhood allows a reparameterization which is the graph of a smooth function, where (n – m) among the variables ($x_1, \ldots, x_n$) are considered as functions of the remaining m variables) in the special case m = 1, we find that there exists neighborhood V of $t_0$in I such that $\gamma|V$ allows a re-parameterization as a graph. It follows from Theorem (1.1.1) that there exists an open set W'⊂ R^2 such that $\gamma(V) = \gamma(I) \cap W' = C \cap W \cap W'$. The set $W \cap W'$ has all the properties desired ofW in the lemma. Conversely, assume that the condition in the lemmaholds, for a givenpoint p say with$C \cap W = \{(t, h(t)) \mid t \in I\}$, where $I \subset R$ is open and $h: I \to R$ is smooth. The curve $t \to (t, h(t))$ has the image $C \cap W$, and according to parameterized manifold in R^n it is an embedded parameterizedcurve. Hence the condition in definition 1.1.1 holds, and C is a curve. [9]

Theorem (1.1.2). Let $f: \Omega \to R$ be a smooth function, where $\Omega \subset R^2$ is open, and let $c \in R$. If it is not empty, the set $C = \{p \in \Omega \mid f(p) = c,\ p\ is\ not\ critical\}$ is a curve in R^2

Proof. By continuity of the partial derivatives, the set of non-critical points in Ω is an open subset. If we replace Ω by this set, the set C can be expressed asa level curve $\{p \in \Omega \mid f(p) = c\}$, to which we can apply the implicit functiontheorem. It then follows from Lemma 1.1.1that C is a curve

Example (1.3.9). The set $C = \{(x,y) \mid x^2 + y^2 = c\}$ is a curve in R^2 for each $c > 0$, since it contains no critical points for $f(x,y) = x^2 + y^2$ [9]

Definition (1.1.6). A surface in R^3 is a non-empty set $S \subset R^3$ satisfying thefollowing property for each point p $\in$ S. There exists an open neighborhood$W \subset R^3$ of p and an embedded parameterized surface σ: $U \to R^3$ with image$\sigma(U) = S \cap W$

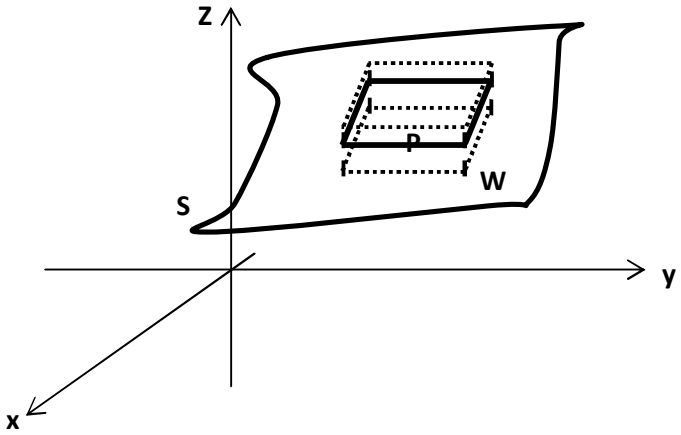


Fig 3

Shows for a given point p with S ∩W the surface has image according to Lemma 1.1.1 is non-empty

Example (1.1.10). The image $S = \sigma(U)$ of an embedded parameterized surfaceis a surface in R^3. In the condition above we can take $W = R^3$

Theorem (1.1.3). Let $f: \Omega \to R$ be a smooth function, where $\Omega \subset R^3$ is open,and let $c \in R$. If it is not empty, the set

$$S = \{p \in \Omega \mid f(p) = c, p \text{ is not critical}\}$$ is a surface in R^3.

Proof. The proof, which combines geometry,with Lemma 1.1.2 below, is entirely similar to that of Theorem 1.1.2

Example 1.1.11 Let us verify for the sphere that it contains no criticalpoints for the function $f(x,y,z) = x^2 + y^2 + z^2$. The partial derivativesare $f'x = 2x, f'y = 2y, f'z = 2z$, and they vanish simultaneously only at $(x,y,z) = (0,0,0)$. This point does not belong to the sphere, hence it is asurface. The verification for the cylinder is similar

Lemma 1.1.2. Let $S \subset R^3$ be non-empty. Then S is a surface if and only ifit satisfies the following condition for each $p \in S$: There exist an open neighborhood $W \subset R^3$ of p, such that $S \cap W$ isthe graph of a smooth function h, where one of the variables x_1, x_2, x_3 is considered a function of the other two variables [9].

Section (1.2): Chart and atlas:

As mentioned in the introduction there exist surfaces, for example the sphere, which we have seen that for the standard spherical coordinates on the sphere there are two singular points, the poles. In order to have a regular parameterization we must exclude these points. A variation of the standard spherical coordinates with interchanged roles of y and z will have singular poles in two other points. The entire sphere can thus be covered by spherical coordinates if we allow two parameterizations covering different, overlapping subsets of the sphere. Note that in contrast, the standard parameterization of the circle by trigonometric coordinates is everywhere regular, in general overlapping, parameterizations. This makes the following concepts relevant [9]

Definition of chart 1.2.1 another word for graph (Differential geometry) also called (local) coordinate system. a neighborhood of a point in a manifold together with its mapping into Euclidean n-space; formally, a pair $(\mathrm{U_l}, \mathrm{f_l})$where $\mathrm{U_l}$ is an element of a cover of the manifold and , $\mathrm{f_l}$is a homeomorphism that maps it to an open subset of $\mathrm{R^n}$. A collection of charts that cover the manifold is called an atlas.

Definition1.2.2 Let S be a surface inR^3. A chart on S is an injective regular parameterized surface $\sigma: \mathrm{U} \to \mathrm{R}^3$ with image $\sigma(\mathrm{U}) \subset$ S. A collectionof charts $\sigma_i: \mathrm{U_i} \to \mathrm{R}^3$ on S is said to cover S if $\mathrm{S} = \mathrm{U_i}\sigma_i(\mathrm{U_i})$. In that casethe collection is called an atlas of S.

Example1.2.1. The image $\mathrm{S} = \sigma(\mathrm{U})$ of an embedded parameterized surfaceas in Example 1.1.7 has an atlas consisting just of the chart itself.

Example 1.2.2. The map$\sigma(\mathrm{u}, \mathrm{v}) = (\cos \mathrm{v}, \sin \mathrm{v}, \mathrm{u}), \mathrm{u}, \mathrm{v} \in \mathrm{R}$ is regular and covers the cylinder $\mathrm{S} = \{(\mathrm{x}, \mathrm{y}, \mathrm{z}) \mid \mathrm{x}^2 + \mathrm{y}^2 = 1\}$, but it is notinjective. Let $\mathrm{U_i} = \{(\mathrm{u}, \mathrm{v}) \in \mathrm{R}^2 \mid -\pi < v < \pi\}$,$\mathrm{U}_2 = \{(\mathrm{u}, \mathrm{v}) \in \mathrm{R}^2 \mid 0 < v < 2\pi\}$,and let σ_i denote the restriction of σ to U_i for

i $= 1, 2$. Then σ_1 and σ_2 areboth injective,σ_1 covers S with the exception of a vertical line on the backwhere $x = -1$, andσ_2 covers with the exception of a vertical line on the frontwhere $x = 1$. Together they cover the entire set and thus they constitute anatlas

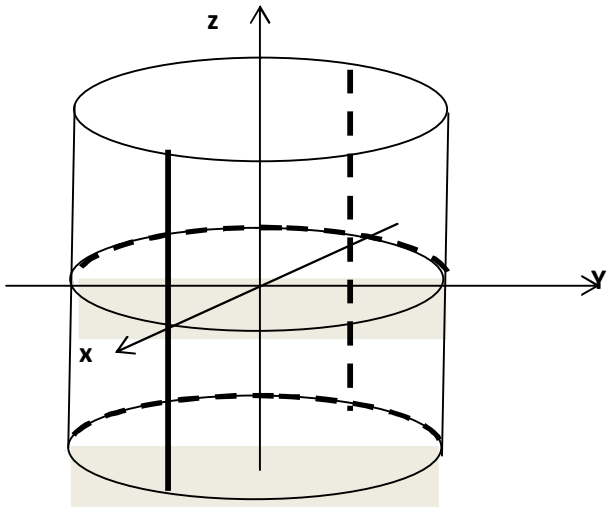


Figure 4.

describes cylinder S copse of two charts σ_1 and σ_2 as described in Example 1.2.2

Example 1.2.3. The spherical coordinate map[9]

$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$,

$\frac{-\pi}{2} < u < \frac{\pi}{2}$, $\frac{-\pi}{2} < v < \frac{\pi}{2}$,and its variation

$\tilde{\sigma}(u, v) = (\cos u \cos v, \sin u, \cos u \sin v)$,

$\frac{-\pi}{2} < u < \frac{\pi}{2} 0 < v < 2\pi$,

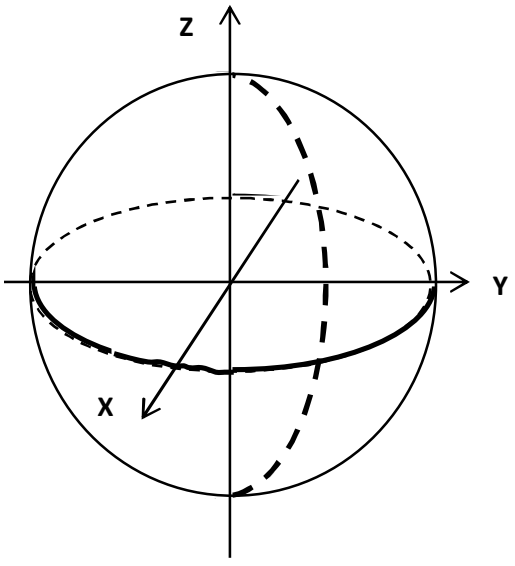


Figure 5.

are charts on the unit sphere. The restrictions on u and v ensure that they are regular and injective. The chart σ covers the sphere except a half circle (a meridian) in the xz-plane, on the back where $x \leq 0$, and the chart $\tilde{\sigma}$similarly covers with the exception of a half circle in the xy-plane, on the front where $x \geq 0$ (half of the 'equator'). As seen in the figure5 theexcepted half-circles are disjoints. Hence the two charts together cover thefull sphere and they constitute an atlas

Theorem 1.2.1 Let S be a surface. There exists an atlas of it.

Proof: For each $p \in S$ we choose an embedded parameterized surface σ as in [9]

Definition 1.2.3 Since a homeomorphism is injective, this parameterization isa chart on S. The collection of all these charts is an atlas

Definition 1.2.4 An m-dimensional manifold in R^n is a non-empty set$S \subset \mathrm{R}^{\mathrm{n}}$ satisfying the following property for each point$p \in S$. There existsan open neighborhood$W \subset \mathrm{R}^{\mathrm{n}}$ of p and an m-dimensional embedded follows from Definition1.2.2parameterized manifold $\sigma: \mathrm{U} \to \mathrm{R}^{\mathrm{n}}$ with image $\sigma(\mathrm{U}) = \mathrm{S} \cap \mathrm{W}$.

The surrounding space R^n is said to be the ambient space of the manifold.

Example 1.2.4. The case $m = 0$. It was explained in Section 1.1 that a0-dimensional parameterized manifold is a map $R^0 = \{0\} \to R^n$, whose imageconsists of a single point p. An element p in a set $S \subset R^n$is called isolatedif it is the only point from S in some neighborhood of p, and the set S iscalled discrete if all its points are isolated. By going over Definition 1.2.1 forthe case $m = 0$ it is seen that a 0-dimensional manifold in $R^n$is the same asa discrete subset.

Example 1.2.5 If we identify R^m with the set $\{(x_1, \ldots, x_m, 0 \ldots, 0)\} \subset R^m$,it is an m-dimensional manifold in R^n.

Example 1.2.6 An open set $\Omega \subset R^n$is an n-dimensionalmanifold in R^n. Indeed, we can take $W = \Omega$ and $\sigma =$ the identitymap in Definition 1.2.1.

Example 1.2.7 Let $\overline{S} \subset S$ be a relatively open subset of an m-dimensionalmanifold in R^n. Then S' is an m-dimensional manifold in R^n.

The following lemma generalizes Lemmas 1.1.2 and 1.2.1

Lemma 1.2.1 Let $S \subset R^n$ be non-empty. Then S is an m-dimensionalmanifold if and only if it satisfies the following condition for each $p \in S$: There exist an open neighborhood [9]. $W \subset R^n$ of p, such that $S \cap W$ is thegraph of a smooth function h, where $n - m$ of the variables $x_1, \ldots, x_n$ are considered as functions of the remaining m variables.

Proof. The proof is entirely similar to that of Lemma 1.2.1

Theorem 1.2.2 Let f: $\Omega \to R^k$ be a smooth function, where k ≤ n and where $\Omega \subset \mathrm{R^n}$ is open, and let $\mathrm{c} \in \mathrm{R^k}$. If it is not empty, the set $\mathrm{S} = \{\mathrm{p} \in \Omega \mid \mathrm{f(p)} = \mathrm{c}, \operatorname{rank} \mathrm{Df(p)} = \mathrm{k}\}$ is an $(n-k)$ −dimensional manifold in R^n.[9]

Proof. Similar to that of Theorem 1.2.1 for curves, by means of the implicit function theorem and Lemma (1.1.1) [8].

Intuitively, a smooth manifold is a space that, when examined closely enough, looks like Euclidean space. In this regard, manifolds provide a natural setting for defining many of the usual notions of calculus, including differentiation, tangent spaces, vector fields, differential forms, and integration. To begin our discussion, we need the definitions of a diffeomorphism and a k-dimensional manifold in $\mathrm{R^n}$:

Definition 1.2.5 If U and V are open sets in R^n, a diffeomorphism is a smooth(i.e., infinitely differentiable) function $\mathrm{h}: \mathrm{U} \to \mathrm{V}$ with a smooth inverse $h^{-1}: V \to U$.

Definition 1.2.6 A subset M of $\mathrm{R^n}$ is called a $\mathrm{k-dimensional}$ manifold (in $\mathrm{R^n}$)if for every point $\mathrm{x} \in \mathrm{M}$, there is an open set U containing x, an open set $\mathrm{V} \subset \mathrm{R^n}$ and a diffeomorphism $\mathrm{h}: \mathrm{U} \to \mathrm{V}$ such that

$$\mathrm{h(U \cap M)} = \mathrm{V} \cap (\mathrm{R^k} \times \{0\}) = \{(\mathrm{y_1}, \ldots, \mathrm{y_n}) \in \mathrm{V}: \mathrm{y_{k+1}} = \cdots = \mathrm{y_n} = 0\}.$$

In other words,$\mathrm{U \cap M}$ is equivalent to $\mathrm{R^k}$, 'up todiffeomorphism.We will use this definition of a manifold as we formally build up the machinery required to integrate on a manifold. However, it should also be noted that our definition need not rely (depend) on $\mathrm{R^n}$ as an ambient (surround)space. In fact, it is possible to define diffeomorphism and manifolds in this abstract sense:

Definition 1.2.7A function f is a diffeomorphism if it is bijective and smooth and if its inverse is also smooth.

Section (1.3): Differentiable manifolds:

Definition 1.3.1 An abstract manifold of dimension k is a second countableHausdorff space M, together with an open cover (U_i) of M, and homeomorphisms $\varphi_i : U_i \to R^k$ such that each φ_j and φ_i^{-1}isadiffeomorphismfrom$\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

Now, for a subset in R^n, the Euclidean definition of a manifold and the abstract definition of a manifold (Definition 1.3.1) are equivalent. Onedirection of this fact is proven below:

Theorem 1.3.1. Suppose M is a set in R^n. If M satisfies the Euclidean definitionof a manifold, then M satisfies the abstract definition of a manifold as well.

Proof. We know that for each point $x \in M$, there is a diffeomorphism $h : U \to V$ between sets U, $V \in R^n$as in Definition 1.3.1 (Note that here we are using"diffeomorphism" as we have defined it in Definition 1.3.1). Take $(u_i)_i \in J$, which isan open cover of M. For each U_ii, we define $\varphi_i = h|U_i \cap M : U_i \cap M \to V_i \cap (R^k \times \{0\})$. Clearly is a bijective, continuous function whose inverse is also continuous. Thus$\varphi_i \in J$ is a collection of homeomorphisms. For alli, $j \in J$, we have the $map\varphi_J \& \varphi_j^{-1}$defined on an open subset of$V_j \cap (R^k \times \{0\})$. We can extend this map's domain to anopen subset of V_i with the function$h$ and h^{-1}. Since this function is a diffeomorphism,one can show that its restriction $^{\varphi}{}_j \& ^{\varphi}{}_i^{-1}$is also diffeomorphism.

The other direction of the proof – that Definition 1.5 implies Definition 1.2– makes use of the Whitney embedding theorem, which states that any manifold can be smoothly embedded in R^n [8]

Definition 1.3.3. A coordinate chart on a set X is a subset $U \subseteq X$ together with abijection $\varphi: U \to \varphi(u) \subseteq R^n$ on to an open set $\emptyset(u)$ in R^n we now consider the situation where X is covered by such a chart and satisfies some consistency(strong) condition we have

Definition 1.3.4. An n-dimensional atlas on X is a collection of coordinate charts$\{ U_\infty, \varphi_\alpha \} \in I$ such that

1- X is covered by the$\{ U_\alpha \} \alpha \in I$

2- For each$\alpha, \beta \in I$, $\varphi_\alpha(u_\alpha) \cap u_\alpha)$ is open in R^n

3- The map $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(u_\alpha \cap \varphi_\alpha) \to \varphi_\beta\,(u_\alpha \cap \varphi_\beta)$ is C^∞ with C^∞ inverse

Definition 1.3.5. two atlases $\{(u_\alpha, \varphi_\alpha)\ , \{V_i\ , \psi_i\}$ are compatible if their union is on atlases.

It means that the extra maps , $\psi_i\varphi_\alpha^{-1}$ must be smooth compatibility is clearly an equivalence relation and we have the following

Definition 1.3.6. A differentiable structure on X is an equivalence class of atlas. Finally we come to the definition of a manifold:

Definition 1.3.7. An n-dimensional differentiable manifold is a space X with differentiable structure.

To prove something is a manifold, all you need is to find one atlasthe definition of manifold take in to account the existence of more atlases.

Proposition 1.3.8. with the topology above $\varphi_\alpha : u_\alpha \to \varphi(u_\alpha)$ is homeomorphism [1] .

Proof. if $V \subseteq U_\propto$ is open then $\varphi_\propto(V) = \varphi_\propto(v \cap U_\alpha)$ is open by the definition of the topology (the intersection of any two open set is open) so φ_α^{-1} is certainly continuous .

Now we let $w \subset \varphi_\alpha(U_\alpha)$ be an open set, then $\varphi_\alpha^{-1}(w) \subseteq U_\alpha$ and $U\alpha$ is open in M so we need to prove that the $\varphi_\alpha^{-1}(w)$ is open in M. But

$$\varphi_\beta \cap (\varphi_\alpha^{-1}(w) \cap U_\beta) = \varphi_\beta\varphi_\alpha^{-1}(w \cap \varphi_\alpha(U_\alpha \cap U_\beta)) \qquad (1)$$

From definition $\varphi_\alpha \cap (U_\propto \cap U_\beta)$ is open and hence its intersection with the open set w is open .Now $\varphi_\beta\varphi_\alpha^{-1}$ is C^∞ inverse and so certainly a homeomorphism , and it follows the right hand side of equation (1) is open thus the left hand side $\varphi_\beta \cap (\varphi_\alpha^{-1}(w) \cap U_\beta)$ is open and by definition of topology ,this means that $\varphi_\alpha^{-1}(w)$ is open hence φ_α is continuous. To make any reasonable further progress, we have to make two assumptions about this topology which will hold for the rest of these notes the manifold topology is Hausdorff (any two separated sets there exist $x \in V\ , y \in U.\ (V \cap U = \emptyset)$ in this topology we have countable basis of an open sets . Without these assumptions manifold are not even metric spaces, and there is not much analysis that can reasonably be done on them. [1]

Basic Definitions (1.3.8): An- manifold M, is a topological space with a maximal atlas or a maximal smooth structure.

There are two virtually identical definitions. The standard definition is as follows: There is an atlas A consisting of maps $F_{\propto} : U_{\propto} \rightarrow R^{n}{}_{\alpha}$ such that

(1) $U_{\propto}$is an open covering of M.

(2) $F_{\propto}$ is a homeomorphism on to its image.=$F_{\propto}(U_{\propto})$

(3) The transition functions

$F_{\propto} O F_{\propto}^{-1}$: $F_{\beta}\ (U_{\alpha} \cap U_{\beta}) \rightarrow F_{\alpha}(U_{\alpha} \cap U_{\beta})$ arediffeomorphism.

In condition 3 it suffices to show that the transition functions are smooth as they are already forced to be homeomorphisms.

A smooth structure is a collection D consisting of continuous functions whose domains are open subsets of **M** with the property that, For each $p \in M$, there is an open neighborhood $p \in U$ and functions$X_i \in D, i = 1, \ldots, n$ such that

(1) The domains of X_i contain U.

(2) The map $F = (x_1, \ldots, x_n) : U \rightarrow R^n$ is a homeomorphism or each $f : O \rightarrow R$ in A there is a smooth function $h : V \cap F\ (O) \rightarrow R$ such that $f = h\ (x_1, \ldots, x_n)$ on $U \cap O$.

Note that $h = f \circ F^{-1}$ in condition 3, but it is usually possible to find$h$ withouthaving to invert F. The map in 2 in both definitions is called a chart or coordinatesystem on U. The topology of M is recovered by these maps.

Note that it is very easy to see that these two definitions are virtually identical.

Definition 1.3.9 (i) A topological manifold of dimension **m** is a topological space with the property that every point has a neighborhood homeomorphic to an open set in R^m. It is usual to insist also that M be Hausdorff and second-countable (i.e. having a countable dense subset), and we will impose this requirement. We will also impose the requirement that M be para compact: every open cover has a locally finite refinement (A refinement of an open cover $\{U\}$ is an open cover {V} such that each V lies entirely in some U). Every subset of $R^N$is Hausdorff, second countable and Para compact, so these requirements hold automatically for the manifolds contained in R^N. [1]

(ii) A homeomorphism $\varphi : U \rightarrow V$, where U is open in M and V is open in R^m, is called a chart, and a collection $A = \{\varphi_\alpha : U_\alpha \rightarrow V_\alpha\}$ of charts such that M = U_α is called an atlas for M.

(iii) An atlas $\{\phi_\alpha: U_\alpha \rightarrow V_\alpha\}$ is smooth if whenever $U_\alpha \cap U_\beta \neq \emptyset$, the crossover homeomorphism $\phi_\beta \circ \varphi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is smooth.

(iv) If M is a manifold equipped with a smooth atlas A, then a map f : M → R^k is smooth with respect to A at a point $x \in M$ if $f \circ \varphi_\alpha^{-1}$is smooth, where$\varphi_\alpha \in A$ is a chart defined onsome neighborhood of x. And a map f : R^k → M is smooth with respect to A at $y \in R^k$if($\phi_\alpha \circ$ f)is smooth at y, for some chart ϕ_α whose domain contains f(y).Because of the smoothness of the crossover maps of charts in A, the criteria for smoothness described in (iv) are independent of the choice of chart $\varphi_\alpha \in A$ used to verify them.When $M \subset R^N$is a manifold, any two of its charts (whichwe required to be smooth (new)) automatically enjoyed the crossover property described in (i) and so an atlas of such charts was automatically smooth in the sense of (iii). Here, in our new more abstract situation, it makes no sense to speak of smooth (new or old) maps from M to R^k without reference to the charts of a smooth atlas, because M is not embedded in any bigger space already equipped with a notion of differentiability.

Examples of manifolds: We need better ways of recognizing manifolds than struggling to find explicit coordinate charts.

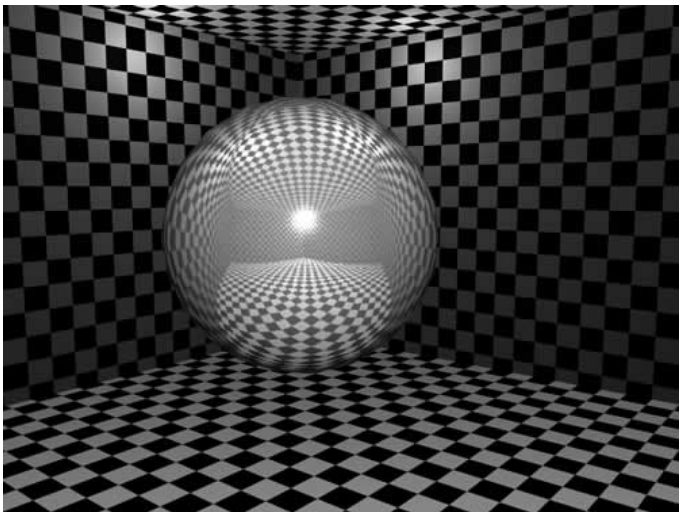
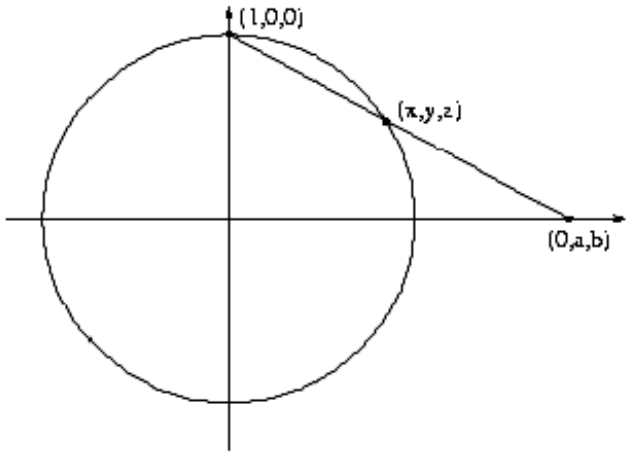


Figure 6

Figure 6 shows how we can use stereographic projection to get an atlas.

For example, the sphere is amanifoldand although we can use stereographic projection to get an atlas, there are other ways. Here is one. [1]

$$\begin{bmatrix} X-1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ a \\ b \end{bmatrix}$$



Theorem 1.3.2 Let $F : U \to R^m$ be a C^∞function on an open set $U \subseteq R^{m+n}$ andtake $c \in R^m$. Assume that for $each\ a \in F^{-1}(c)$, the derivative$DF_a : R^{m+n} \to R^m$is surjective. Then $F^{-1}(c)$ has the structure of an n-dimensional manifold which is Hausdorff and has a countable basis of open sets.

Proof: Recall that the derivative of F at a is the linear map $DF_a : R^{n+m} \to R^m$such that $F(a+h) = F(a) + DFa(h) + R(a,h)$ where

$$R(a,h)/||h|| \to 0 \ as\ h \to 0.$$

If we write $F(x_1,\ldots,x_{n+m}) = (F_1,\ldots,F_m)$ the derivative is the Jacobian matrix $\frac{\partial F_i}{\partial x_j}(a)$1≤i ≤ m, 1≤j ≤ n+m that this is surjective, so the matrix has rank m. Therefore byreordering the coordinates $x_1,\ldots,x_{m+n}$we may assume that the square matrix $\frac{\partial F_i}{\partial x_j}(a)$ 1≤i ≤ m, 1≤j ≤ mis invertible.

Now define

$$G : U \to F^{n+m} \text{ by } G(x_1,\ldots x_{n+m}) = (F_1,\ldots,F_m,x_{m+1},\ldots,x_{m+n}). \qquad (2)$$

Then DGa is invertible. We now apply the inverse function to G, a proof of which is given, tells us that there is a neighborhood V of$a$, and W of $G(a)$ such that $G : V \to W$ is invertible with smooth inverse. Moreover, the formula (2) showsthat G maps

$V \setminus F^{-1}(c)$ to the intersection of W with the copy of R^n given by $\{x \in R^{n+m}: x_i = c_i, 1 \leq i \leq m\}$. This is therefore a coordinate chart φ If we take two such charts $\varphi_{\alpha,}\varphi_{\beta}$ then$\varphi^{-1}$is a map from an open set in $\{x \in R^{n+m}: x_i = c_i, 1 \leq i \leq m\}$ to another one which is the restriction of the map $G_{\propto}G_{\beta}^{-1}$ of (an open set in) $R^{n+m}$to itself. but this is an invertible C^1map and so we have the requisite conditions for an atlas.

Finally, in the induced topology from$R^{m+n}, G\alpha$ is a homeomorphism, so open setsin the manifold topology are the same as open sets in the induced topology. Since$R^{m+n}$is Hausdorff with a countable basis of open sets, so is $F^{-1}(c)$ has structure of m-dimensional manifold,We can now give further examples of manifolds [1]

Examples 1.3.1: Lets$^n = \{x \in R^{n+1} : \sum_1^{n+1} x_i^2 = 1\}$ be the unit n-sphere. Define $F : R^{n+1} \rightarrow R\ by\ F(x) = \sum_1^{n+1} x_i^2$.

This is aC^{∞}map and$DF_a(h) = 2\sum_i a_i h_i$is non-zero (and hence surjective in the 1-dimensional case) so long as a is not identically zero. If $F(a) = 1$, then$\sum_1^{n+1} x_i^2 = 1 \neq 0$ so $a \neq 0$ and we can apply Theorem 1.3.2 and deduce that the sphere is a manifold.

Example 1.3.2: Let $O(n)$ be the space of n × n orthogonal matrices: $AA^T = 1$. Take the vectorspace M^n of dimension n^2 of all real n × n matrices and define the function $F(A) = AA^T$to the vector space of symmetric n × n matrices. This has dimension

n(n + 1)/2. Then $O(n) = F^{-1}(I)$.

Differentiating F we have $\mathrm{DF}_A(\mathrm{H}) = \mathrm{HA}^T + \mathrm{AH}^T$ and putting $\mathrm{H} = \mathrm{KA}$ this is$\mathrm{KAA}^T + \mathrm{AA}^T\mathrm{K}^T = \mathrm{K} + \mathrm{K}^T$ if $\mathrm{AA}^T = \mathrm{I}$, i.e. if $A \in F^{-1}(I)$. But given any symmetric matrix S, taking $\mathrm{K} = \mathrm{S}/2$shows that DF_I is surjective and so, applying Theorem 2.2 we find that O(n) is amanifold. Its dimension is$n^2 - n(n + 1)/2 = n(n - 1)/2$.[1]

Chapter 2

Bifurcation Theory

The goal of this chapter is to study the bifurcation relating with manifold, which allows us to introduce centre manifold (CM) theory and its importance in normal form theory.

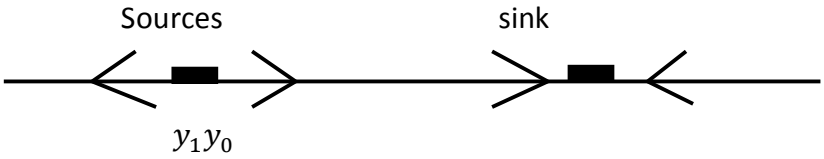
Section (2:1) Elementary bifurcation:

Definition 2.1.1: In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative" or topological change in its behavior. Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other invariant sets changes [5].
The change in the qualitative character of a solution as a control parameter is varied is known as a bifurcation. This occurs where a linear stability analysis yields an instability (characterized by a growth rate σ of a perturbation of the base solution with $\mathrm{Re}\,\sigma = 0$). The connection is through the implicit function theorem, the solution can be continued smoothly except where the Jacobean is singular. Typically a new solution develops at this point.

In practical applications that involve differential equations is very often happens that the differential equation contains parameters and the value of these parameters are often only known approximately. In particular they are generally determined by measurements which are not exact. For that reason it is important to study the behavior of solutions and examine their dependence on the parameters. This study leads to the area referred to as bifurcation theory. It can happen that a slight variation in a parameter can have significant impact on the solution. Bifurcation theory is a very deep and complicated area involving lots of current research. A complete examination of the field would be impossible. A fixed point (or equilibrium point) of a differential equation $y' = f(y)$ is a root of the equation $f(y) = 0$. As we have already seenfor autonomous problems fixed points can bevery useful in determining the long time behavior of solutions.Qualitative information about the equilibrium points of the

differential equation $y_0 = f(y)$can be obtained from special diagrams called phase diagrams.

line segment with labels for so-called sinks, sources or nodes , one for each root $f(y) = 0$ i.e. each equilibrium.



The names are borrowed from the theory of fluids and they are defined as follows

1. Sink an equilibrium y_0 which attracts nearby solutions at $t = \infty$ i.e., there exists$M > 0$ so that if $|y(0) - y_0| < M$, then $|y(x) - y_0| \to 0 \quad when\, t \to \infty$
2. Source an equilibrium y_1 which repels nearby solutions at $t = \infty$i.e., here exists $M > 0$ so that if $|y(0) - y_1| < M$, then $|y(x) - y_1|$ increases as $t \to \infty$.
3. Node An equilibrium y_2 which is neither a sink or a source. In fluids, sink means fluid is lost and source means fluid is created

Section (2.2): Stability Test:

The term stable means that solutions that start near the equilibrium will stay nearby as $t \to \infty$. The term unstable means not stable. Therefore, a sink is stable and a sourceis unstable. Precisely, an equilibrium y_0 is stable provided for given $\epsilon > 0$ thereexists some $\delta > 0$ such that $|y(0) - y_0| < \delta$ implies $y(t)$ exists for $t \geq 0 and\, |y(t) - y_0| < \epsilon$.

Lemma 2.2.1 If a system is structurally stable in a region D_0 with the boundary B_0 and all its orbits point strictly inside B_0, then it is strictly structurally stable in $\mathrm{U} = \mathrm{D}_0$.

(Proof of Lemma 2.2.1): (a) Prove that a smooth planar system $x^{\cdot} = f(x)$, $x \in R^2$,is topologically equivalent (in fact, diffeomorphic) in a region U, that is, free of equilibria and periodic orbits and is bounded by two orbits and two smooth curves transversal to orbits to the system

$$y_1^{\cdot} = 1,$$
$$y_2^{\cdot} = 0,$$

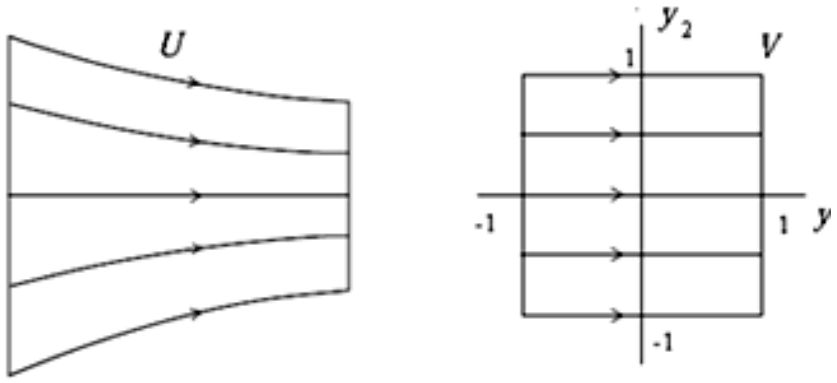


Fig 2.1: Phase portraits in U and V are equivalent.

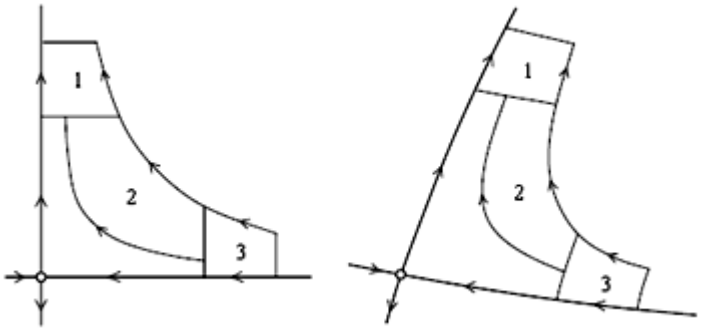


Fig. 2.2 Saddles are topologically equivelent.

(a) in the unit square $V = \{(y_1, y_2) : |y_1| \leq 1, |y_2| \leq 1\}$ (Figure 2.1)

(b) Generalize this result to n-dimensional systems and prove Lemma 2.2.1.

(c) Prove, using part (a),that two hyperbolic saddle points on the plane have locally topologically equivalent phase portraits. Figure 8Where is the differentiability lost?

Lemma 2.2.2 The system

$$x^{\cdot} = \alpha + x^2 + O(x^3) \quad (1)$$

is locally topologically equivalent near the origin to the system

$$x^{\cdot} = \alpha + x^2. \quad (2)$$

Proof: The proof goes through two steps. It is based on the fact that for scalar systems a homeomorphism mapping equilibria into equilibria will also map their connecting orbits.

Step 1 (Analysis of equilibria). Introduce a scalar variable y and write the first system as

$$y^{\cdot} = F(y,\alpha) = \alpha + y^2 + \psi(y,\alpha), \qquad (3)$$

where ψ = O(y^3) is a smooth functions of (y,α) near (0, 0). Consider the equilibrium manifold of (2.2.3) near the origin (0, 0) of the (y,α)-plane,$M = \{(y,\alpha) : F(y,\alpha) = \alpha + y^2 + \psi(y,\alpha) = 0\}$.The curve M passes through the origin $F(0,0) = 0$. By the Implicit Function Theorem (since $F\alpha(0,0) = 1$),it can be locally parameterized by y:

$$M = \{(y,\alpha), \alpha = g(y)\},$$

where g is smooth and defined for small $|y|$. Moreover,

$$g(y) = -y^2 + O(y^3)$$

Thus,for any sufficiently small $\alpha < 0$,there are two equilibria of (3) near the origin in (3), $y_1(\alpha)$ and y_1(α),which are close to the equilibria of (1),i.e., x_1(α) = $+\sqrt{-\alpha}$and x_2(α) = $-\sqrt{-\alpha}$,for the same parameter value . fig 9

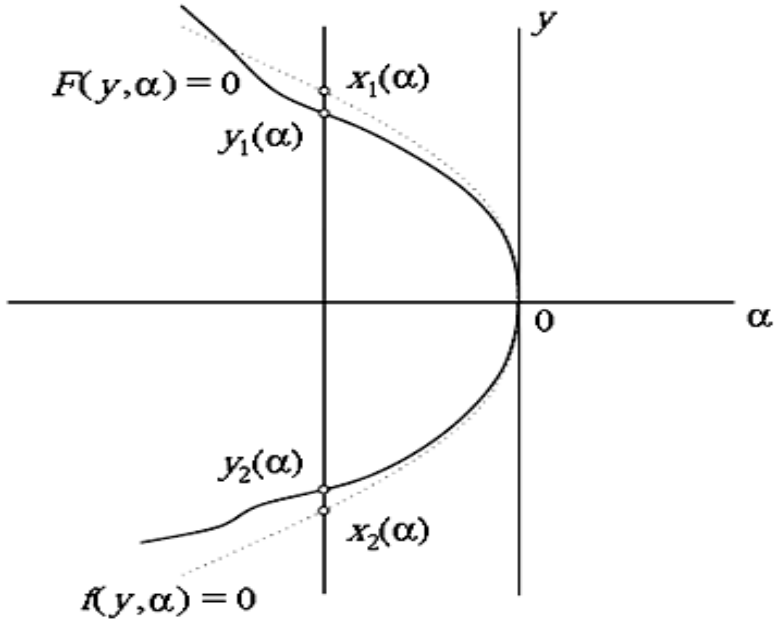


Fig 2.3 Fold bifurcation for the perturbed system.

Step 2 (Homeomorphism construction). For small $|\alpha|$,construct a parameter- dependent map y = hα(x) as following. For $\alpha \geq 0$ take the identity map $h_\alpha(x) = x$. For $\alpha < 0$

take a linear transformation $h_\alpha(x) = a(\alpha) + b(\alpha)x$, where the coefficients a, b are uniquely determined by the conditions

$$h_\alpha(x_j(\alpha)) = y_j(\alpha),\ j = 1, 2, \quad (4)$$

The constructed map $h_\alpha : R^1 \to R^1$ is a homeomorphism mapping orbits of $\dot{x} = \alpha + x^2 \equiv f(x, \alpha)$. (3) near the origin into the corresponding orbits of (3), preserving the direction of time. Although it is not required in the study for the homeomorphism h_α to depend continuously on α,this property holds here, since hα tends to the identity map as negative

α→ 0.

Theorem 2.2.1 (Stability Conditions). Let f and f' be continuous. The equation $y' = f(y)$has a sink at $y = y_0$ provided $f(y_0) = 0$ and $f'(y_0) < 0$. An equilibrium $y = y_1$is a sourceprovided $f(y_1) = 0$ and $f'(y_1) > 0$. There is no test when f' is zero at an equilibrium.

Equilibria.[5]

$$let \quad \dot{x} = f(x) \ldots\ldots \to \quad (5)$$

where x is the variable subject to change. Note, that when x depends on time t, we call Equation (1) an autonomous system If we know the value of x at time t = 0, we have an initial value problem

$$\dot{x} = f(x), x(0) = x_0 \ldots \to \quad (6)$$

where x_0 is the known value. When we plot the change in x during time we have an orbit .All orbits together with the direction of arrows gives a phase portrait. It is possible that for $t \to \infty$ all orbits converge to a single value of x, denoted as x^*. We call this value a stable equilibrium of Eqn. (5). Equilibria can also be unstable; orbits then go into the direction away from x^*. Of course, when time is reversed the equilibrium can be perceived as stable again. To find an equilibrium all one has to do is set Eqn. (5) equal to zero and solve the resulting equation. To investigate the stability of the found equilibria linearization is needed, i.e. we need to differentiate the function. Now, if $f'(x) < 0$, the equilibrium is stable. Vice versa, if $f'(x) > 0$ the equilibrium is unstable. If $f'(x) = 0$ we cannot sayanything about the equilibrium at this point.[5]

Now consider a dynamical system that depends on parameters. In thecontinuous-time case we will write it as

$$x^{\cdot} = f(x, \alpha), \tag{6}$$

while in the discrete-time case it is written as

$$\mathrm{x} \rightarrow \mathrm{f}(\mathrm{x}, \alpha), \tag{7}$$

where $x \in R^n$ and $\alpha \in R^m$ represent phase variables and parameters, respectively. Consider the phase portrait of the system. (consider the phase portrait in a parameter-dependent region $U_{\propto} \subset R^n$.) As the parametersvary,the phase portrait also varies. There are two possibilities: either thesystem remains topologically equivalent to the original one,or its topology changes.

Definition 2.2.1. The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.

Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

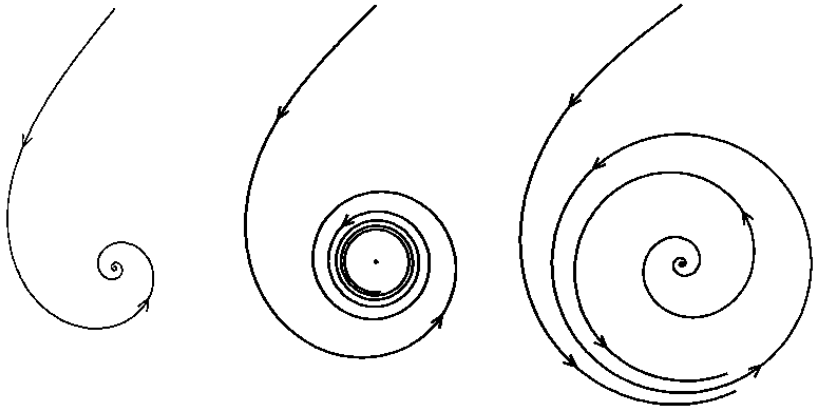
Example 2.2.1. Consider the following planar system that depends on one parameter:

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_1^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2) \end{aligned} \tag{8}$$

Equivalence and Bifurcations: in polar coordinates (ρ, θ) it takes the form

$$\rho^{\cdot} = \rho(\alpha - \rho^2), \theta^{\cdot} = 1, \tag{9}$$

and can be integrated explicitly , Since the equations for



α< 0α = 0α> 0

FIGURE 2.4. Hopf bifurcation.

ρ and θ are independent in (2.2.9),we can easily draw phase portraits of the system in a fixed neighborhood of the origin, which is obviously the only equilibrium point (see Figure 2.4). For α≤ 0,the equilibrium is a stable focus, since $\rho^{\cdot} < 0$ and $\rho(t) \to 0$,if we start from any initial point. On the other hand, if α> 0,we have $\rho^{\cdot} > 0$ for small ρ> 0 (the equilibrium becomes an unstable focus),and $\rho^{\cdot} < 0$ for sufficiently large ρ. It is easy to see from (2.2.9) that the system has a periodic orbit for any $\alpha > 0$ of radius $\rho_0 = \sqrt{\alpha}$ (at $\rho = \rho_0$we have $\rho^{\cdot} = 0$). Moreover,this periodic orbit is stable,since $\rho^{\cdot} > 0$ inside and $\rho^{\cdot} < 0$ outside the cycle. Therefore, $\alpha = 0$ is a bifurcation parameter value. Indeed,a phase portraitwith a limit cycle cannot be deformed by a one-to-one transformationinto a phase portrait with only an equilibrium. The presence of a limit cycleis said to be a topological invariant. As αincreases and crosses zero, wehave a bifurcation in system (2.2.8) called the Andronov-Hopf bifurcation.[12]

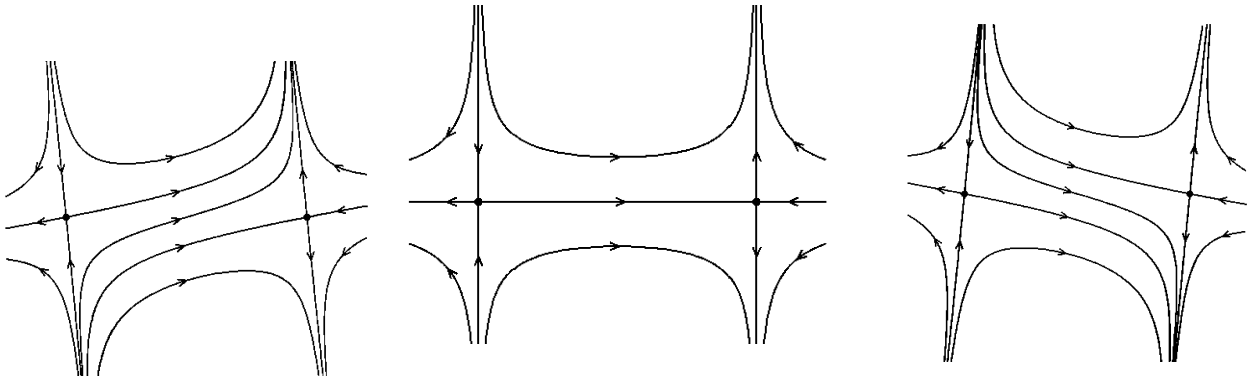
It leads to the appearance, from the equilibrium state, of small-amplitude periodic oscillations. As should be clear, an Andronov-Hopf bifurcation can be detected if we fix any small neighborhood of the equilibrium. Such bifurcations are called local. One can also define local bifurcations in discrete-time systemsas those detectable in any small neighborhood of a fixed point. We will often refer to local bifurcations as bifurcations of equilibria or fixed points, although we will analyze not just these points but the whole phase portraits near the equilibria. Those bifurcations of limit cycles which correspond to local bifurcations of associated Poincare maps are called local bifurcations of cycles.

There are also bifurcations that cannot be detected by looking at small neighborhoods of equilibrium (fixed) points or cycles. Such bifurcations are called global.

Example 2.2.2. (Heteroclinic bifurcation) Consider the following planar system that depends on one parameter:

$$\begin{aligned} \dot{x}_1 &= 1 - x_1^2 - \alpha x_1 x_2, \\ \dot{x}_2 &= x_1 x_2 + \alpha(1 - x_1^2). \end{aligned} \tag{10}$$

The system has two saddle equilibria $x(1) = (-1, 0), x(2) = (1, 0)$,for all values of α (see Figure 2.2). At α = 0 the



α > 0α = 0α < 0

FIGURE 2.5. Heteroclinic bifurcation [12]

horizontal axis is invariant and, therefore, the saddles are connected by an orbit that is asymptotic tone of them for t → +∞and to the other for t→−∞. Such orbits are called heteroclinic. Similarly,an orbit that is asymptotic to the same equilibriumas t → +∞ and t→−∞ is called homoclinic. For α = 0, the x_1-axis is nolonger invariant,and the connection disappears. This is obviously a globalbifurcation. To detect this bifurcation we must fix a region U covering bothsaddles.

There are global bifurcations in which certain local bifurcations are involved .In such cases, looking at the local bifurcation provides only partial information on the behavior of the system. The following example illustrates this possibility.

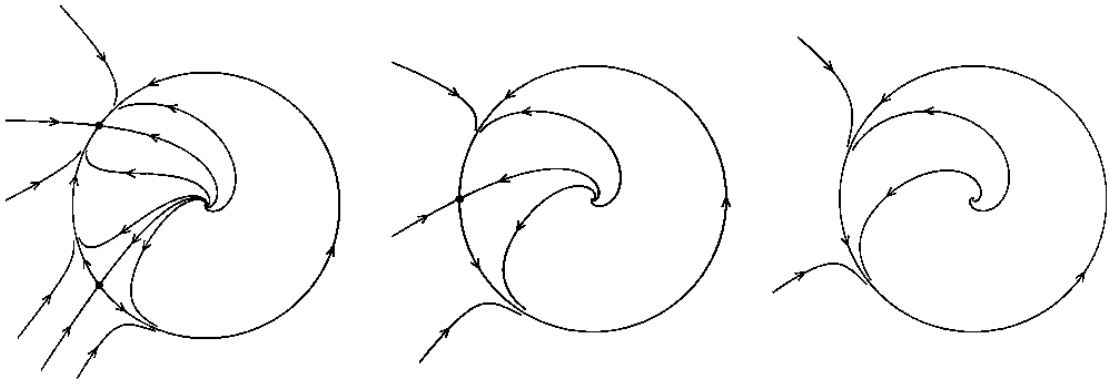
Example 2.1.3. (Saddle-node homoclinic bifurcation) Let us analyze the following system on the plane:[12]

$$\dot{x}_1 = x_1(1 - x_1^2 - x_1^2) - x_2(1 + \alpha + x_1),$$
$$\dot{x}_2 = x_1(1 + \alpha + x_1) + x_2(1 - x_1^2 - x_2^2), \qquad (11)$$

where α is a parameter. In polar coordinates (ρ, θ) system (9) takes the form $\dot{\rho} = \rho(1 - \rho^2)$,

$$\dot{\theta} = 1 + \alpha + \rho \cos\theta. \qquad (12)$$

Fix a thin annulus U around the unit circle $\{(\rho,\theta) : \rho = 1\}$. At $\alpha = 0$, there is a no hyperbolic equilibrium point of system (2.2.10) in the annulus: $x_0 = (\rho_0,\theta_0) = (1,\pi)$(see Figure 2.3). It has eigenvalues $\lambda_1 = 0, \lambda_2 = -2$ For small positive values of α the equilibrium disappears, while for small negative α it splits into a saddle and a node (this bifurcation is called a saddlenodeor fold bifurcation[12]



α < 0 α = 0 α > 0

FIGURE 2.6. Saddle-node homoclinic bifurcation

Local Bifurcation Theorem: In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoretical ecology, there is an application of what is referred to as bifurcation analysis; the analysis of a system of ordinary differential equations (ODE's) under parameter variation. Performing a local bifurcation analysis is often a powerful way to analysis the properties of such systems, since it predicts what kind of behavior (system is in equilibrium, or there is cycling) occurs where in parameter space. With local bifurcations linearization in state space at the critical point of the parameter space provides sufficient information..[7]

In this chapter we formulate conditions defining the simplest bifurcationsof equilibrium in n-dimensional continuous-time systems: the fold and the Hopf bifurcations. Then we study these bifurcations in the lowest possible dimensions: the fold bifurcation for scalar systems and the Hopf bifurcation for planar systems.[6]

Section (2.3): Simplest bifurcation conditions:

Consider a continuous-time system depending on a parameter

$$x' = f(x, \alpha), x \in R^n, \alpha \in R^1,$$

where f is smooth with respect to both x and α. $Let\, x = x_0$ be a hyperbolicequilibrium in the system for $\alpha = \alpha_0$. under a small parameter variation the equilibrium moves slightly but remains hyperbolic. Therefore, we can vary the parameter further and monitor the equilibrium. It is clear that there are, generically only two ways in which the hyperbolicity condition can be violated. Either a simple realeigenvalue approaches zero and we have$\lambda_1 = 0$ (see Figure 2.4(a)),or apair of simple complex eigenvalues reaches the imaginary axis and we have $\lambda_{1,2} = i\omega_0$, $\omega_0 > 0$ (see Figure 2.4(b)) for some value of the parameter. It is obvious (and can be rigorously formalized) that we need more parameters to allocate extra eigenvalues on the imaginary axis. Notice that this mightnot be true if the system has some special properties, such as a symmetry

One-Parameter Bifurcations of Equilibrium

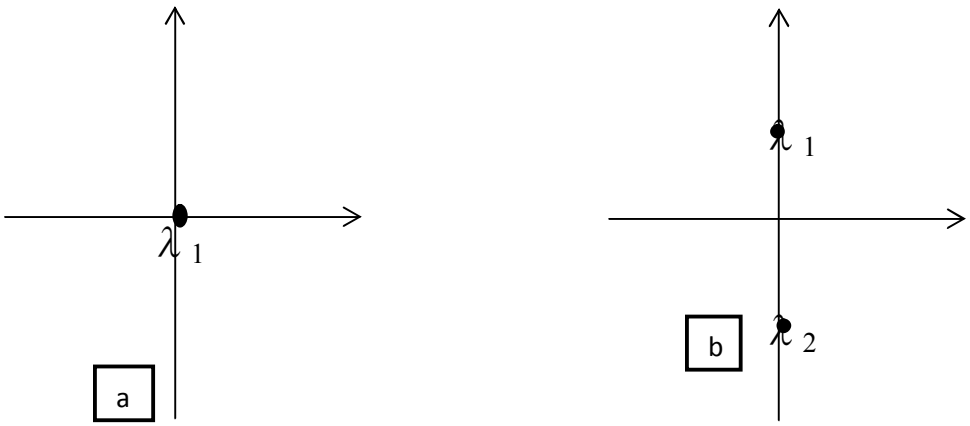


FIGURE 2.7 Codim 1 critical cases.

Definition 2.3.1 The bifurcation associated with the appearance of $\lambda_1 = 0$is called a fold (or tangent) bifurcation.

Remark: This bifurcation has a lot of other names, including limit point, saddle nodebifurcation, and turning point.

Definition 2.3.2 The bifurcation corresponding to the presence of $\lambda_{1,2} = \pm\, i\omega_0,\ \omega_0 > 0$, is called a Hopf (or Andronov-Hopf) bifurcation.

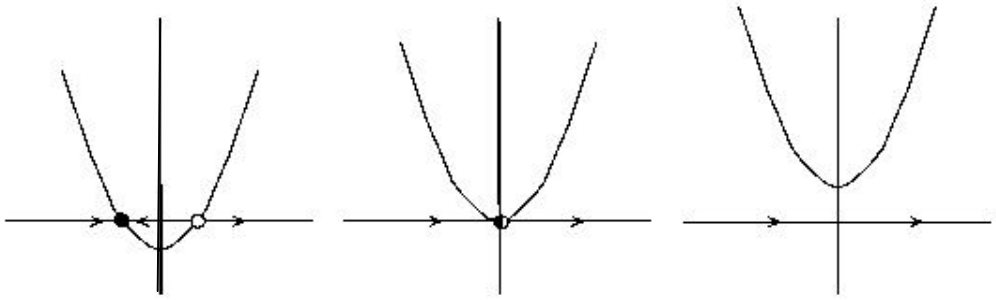
Notice that the tangent bifurcation is possible if n ≥ 1,but for the Hop bifurcation we need n ≥ 2.[6]

Definition 2.3.3. The appearance of a topologically nonequivalent phaseportrait under variation of parameters is called a bifurcation.

Thus, a bifurcation is a change of the topological type of the system as itsparameters pass through a bifurcation (critical) value. [6].

Saddle Node Bifurcation: We begin with the Saddle Node bifurcation (also called the blue sky bifurcation) corresponding to the creation and destruction of fixed points. The normal form for this type of bifurcation is given by the example$x' = r + x^2$

The three cases of $r < 0$, $r = 0$ and $r > 0$ give very different structure for the solutions.



$r < 0 r = 0 r > 0$

Figure 2.8

We observe that there is a bifurcation at $r = 0$. For $r < 0$ there are two fixed points given $by\ x = \pm\sqrt{-r}$. The equilibrium $x = -\sqrt{-r}$ is stable, i.e., solutions beginning near this equilibrium converge to it as time increases. Further, initial conditions near $\sqrt{-r}$ divergefrom it.At $r = 0$ there is a single fixed point at $x = 0$ and initial conditions less than zero givesolutions that converge to zero while positive initial conditions give solutions that increase without bound.

Finally if $r > 0$ there are no fixed points at all. For any initial condition solutions increasewithout bound.[6]

There are several ways we depict this type of bifurcation one of which is the so called bifurcation diagram

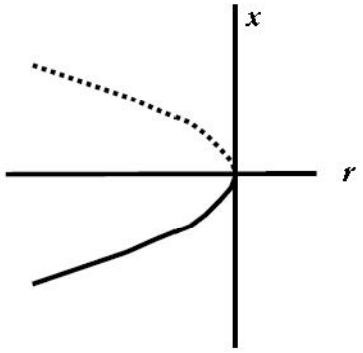


Figure 2.9

Note that if instead we consider$x_0 = r - x^2$ the the so-called phase line can be drawn as

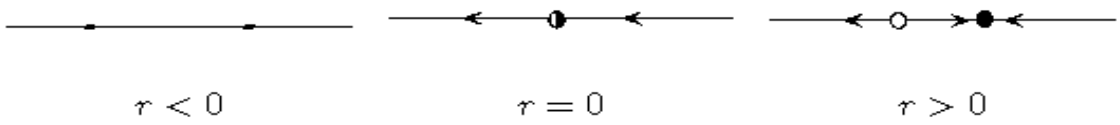


Figure 2.10

We return now to a general discussion of bifurcations in a parameter-dependent system (1) (or (2)). Take some value $\alpha = \alpha_0$ and considera maximal connected parameter set (called a stratum) containing x_0 andcomposed by those points for which the system has a phase portrait that is topologically equivalent to that at α_0. Taking all such strata in the parameter space R^m, we obtain the parametric portrait of the system. For example, system (5) exhibiting the Andronov-Hopf bifurcation has a parametric portrait with two strata: $\{\alpha \leq 0\}$ and {α> 0}. In system (7) there are three strata: $\{\alpha < 0\}$, $\{\alpha = 0\}$,and $\{\alpha > 0\}$. Notice, however, that the phase portrait of figure (2.1) for $\alpha < 0$ is topologically equivalent to that for $\alpha > 0$. The parametric portrait together with its characteristic phase portraits constitute a bifurcation diagram.

Definition 2.3.5. A bifurcation diagram of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.

It is desirable to obtain the bifurcation diagram as a result of the qualitative analysis of a given dynamical system. It classifies in a very condensed way all possible modes of behavior of the system and transitions between them (bifurcations) under parameter variations. Note that the bifurcation diagram depends, in general, on the region of phase space considered.

Remark: If a dynamical system has a one- or two-dimensional phase space and depends on only one parameter, its bifurcation diagram can be visualized in the direct product of the phase and parameter spaces, $R^{1,2} \times R^1$ with the phase portraits represented by one- or two-dimensional slices α = const.Consider,for example,a scalar system [6]

$$\dot{x} = \alpha x - x^3, x \in R^1, \alpha \in R^1$$

$\dot{x}$ =0, simply $x_{1,2} = \mp\sqrt{\alpha}$ This system has an equilibrium $x_0 = 0$ for all α. This equilibrium is stablefor $\alpha < 0$ and unstable for $\alpha > 0$ (α is the eigenvalue of this equilibrium). For $\alpha > 0$,there are two extra equilibria branching from the origin (namely, $x_{1,2} = \mp\sqrt{\alpha}$) which are stable. This bifurcation is often called a pitchforkbifurcation,the reason for which becomes immediately clear if one has a look at the bifurcation diagram of the system presented in (x, α)-space(see Figure 2.11). Notice that the system demonstrating the pitchfork bifurcation is invariant under the transformation

$x \to -x$.

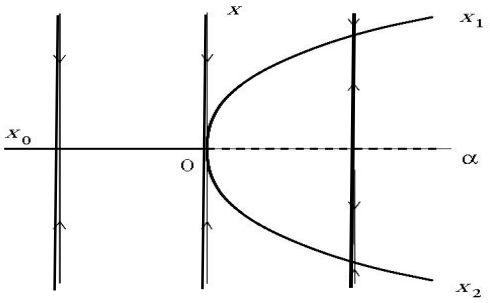


Figure 2.11

In the simplest cases, the parametric portrait is composed by a finite number of regions in R^m. Inside each region the phase portrait is topologicallyequivalent. These regions are separated by bifurcation boundaries, which are smooth submanifolds in R^m(i.e., curves,surfaces). The boundaries can intersect, or meet. These intersections subdivide the boundaries in to subregions, and so forth. A bifurcation boundary is defined by specifying phase object (equilibrium, cycle, etc.) and some bifurcation conditions determining the type of its bifurcation (Hopf, fold, etc.). For example, the Andronov-Hopf bifurcation of an equilibrium is characterized by one bifurcation condition namely ,the presence of a purely imaginary pair of eigenvalues of the Jacobian matrix evaluated at this equilibrium $Re\ \lambda_{1,2} = 0$.When a boundary is crossed, the bifurcation occurs.

Example 2.3.1 (Saddle fixed points $in\ R^2$) Suppose in $x_0 = 0$ is a fixed point of a two-dimensional discrete-time system (now $n = 2$). Assume that$n_- = n_+ = 1$,so that there is one (real) multiplier $\mu_1$outside the unit circle ($|\mu_1| > 1$) and one (real) multiplierμ_2 inside the unit circle($|\mu_2| < 1$). In our case,there are two invariant manifolds passing throughthe fixed point,namely the one-dimensional manifold$\mathrm{W^s(x_0)}$ formed byorbits converging to x_0 under iterations of f,and the one-dimensional manifold$W^u(x_0)$ formed by orbits tending to x_0 underiterations of $\mathrm{f^{-1}}$. Recallthat the orbits of a discrete-time system are sequences of points. All orbitsnot belonging to the aforementioned manifolds pass near the fixed pointand eventually leave its neighborhood in both “time” directions. Figure 2.9 shows two types of saddles in R^2. In the case (a) of positivemultipliers,$0 < \mu_2 < 1 < \mu_1$,an orbit starting at a point on $W^s(x_0)$converges to x_0 monotonously. Thus,the stable manifold $W^s(x_0)$ is formedby two invariant branches, ${W^s}_{1,2}(x_0)$,separated by x_0. The same can besaid about the unstable manifold $W^u(x_0)$ upon replacing f by its inverse.

The restriction of the map onto both manifolds preserves orientation. If the multipliers are negative (case (b)figure 4), $\mu_1 < -1 < \mu_2 < 0$,the orbits on the manifolds “jump” between the two components ${W^{s,u}}_{1,2}$separated by x_0. The map reverses orientation in both manifolds. The branches ${W^{s,u}}_{1,2}$are invariant with respect to the second iteratef^2 of the map.[6]

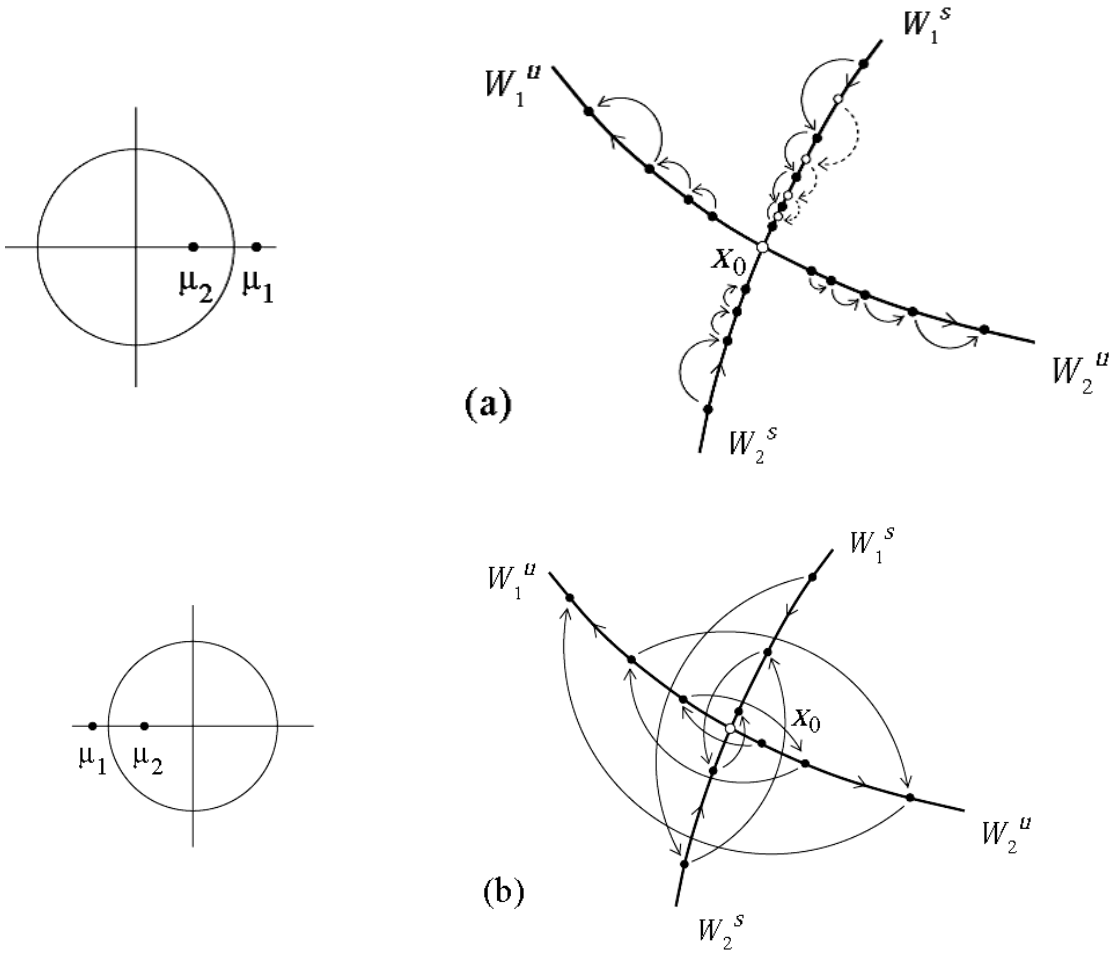


FIGURE 2.12. Shows Invariant manifolds of saddle fixed points on the plane: (a) positive multipliers; (b) negative multipliers

Definition 2.3.6. The codimension of a bifurcation in system (2.2.3) or (2.2.5) is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary. Equivalently, the codimension (codim for short) is the number of independent conditions determining the bifurcation. This is the most practical definition of the codimension. It makes it clear that the codimension of a certain bifurcation is the same in all generic systems depending on a sufficient number of parameters.

Remark: The bifurcation diagram of even a simple continuous-time system in a bounded region on the plane can be composed by an infinite number of strata. The situation

becomes more involved for multidimensional continuous-time systems (with $n > 3$). In such systems the bifurcation values can be dense in some parameter regions and the parametric portrait can have a Cantor (fractal) structure with certain patterns repeated on smaller and smaller scales to infinity. Clearly, the task of fully investigating such a bifurcation diagram is practically impossible. Nevertheless, even partial knowledge of the bifurcation diagram provides important information about the behavior of the system being studied.[6]

Chapter 3

Centre Manifold

Center manifolds theory is of fundamental importance in the study of nonlinear dynamical systems when analyzing bifurcations of a given type. In fact, this theory allows us to reduce the study of a differential equation with delay near a non-hyperbolic equilibrium point to that of an ordinary differential equation on a finite-dimensional invariant manifold.

One of the most useful non-linear methods to reduce systems at near-equilibrium point is the centre manifold approach [10]. This approach assumes that then on-linear dynamical system at near-equilibrium point is governed by the dynamics on the centre manifold when certain eigenvalues have zero real parts (and all other eigenvalues have negative real parts). It may be noted that the centre manifold appears to be an extremely powerful method due to the fact that if m eigenvalues of the n eigenvalues of the non-linear system have zero real parts, then the number of equations for the non-linear system is reduced to(n − m)by applying the centre manifold approach. However, this method can only be used for model reduction when the system has an eigenvalue with zero real parts at an equilibrium point. Usually, the centre manifold has complicated non-linear terms. In this case, the non-linear system can be simplified by using further non-linear coordinate transformations [10]. The normal form theory is often applied after the centre manifold approach. The main objective of the method of normal forms is to obtain the simplest possible non-linear system by the use of successive non-linear co-ordinate transformations [10]. At the end of these non-linear transformations, only the resonant terms are retained: they cannot be eliminated and are essential to the non-linear system dynamics.(we will discus Normal forms in chapter four).

Section (3.1): Linear systems [4]:

Consider the following system of linear differential equations:

$$\dot{x} = Ax, \quad x \in R^n. \tag{1}$$

where $A \in R^{n\times n}$. The global behavior of equation (1) and the stability ofthe zero solution are completely determined by the eigenvalues of A, let $\sigma(A)$denote the spectrum of A(the value of all eigenvalues of λ). Then we have

$\sigma(\mathrm{A}) = \sigma_s(\mathrm{A}) \cup \sigma_u(\mathrm{A}) \cup \sigma_c(\mathrm{A})$, where

$$\sigma_s(\mathrm{A}) = \{\lambda \in \sigma(\mathrm{A}) \mid \mathrm{Re}\,\lambda < 0\}$$
$$\sigma_u(A) = \{\lambda \in \sigma(A) \mid \mathrm{Re}\,\lambda > 0\}$$
$$\sigma_c(\mathrm{A}) = \{\lambda \in \sigma\mathrm{A}) \mid \mathrm{Re}\,\lambda = 0\}$$

let the corresponding (generalized) eigenspaces be denotednow suppose that we add nonlinear terms to Eq (1)

$$x^{\cdot} = A(\lambda)x + f(x,\lambda), x \in R^u, \ \lambda \in R^k \qquad (2)$$

where $f(x,\lambda) \in C^k, k \geq 2$, and $f(0,0) = Df(0,0) = 0$. Then 0 is still a solutionto equation (2). We now must ask how the behavior of the nonlinear system (2) is related to that of the linear system (1). The answer is easy if the system is hyperbolic $\sigma_c(A) = \emptyset$;), the phase portraits are topologically equivalent by theGr¨obman-Hartman theorem. The answer is not so simple if $\sigma_c A \neq \emptyset$, This isthe topic of the center manifold (CM) theory.[4]

In the non-hyperbolic case there exists invariant manifolds M_s, M_u, and $M_c$analogous to the generalized eigenspaces.Since we are usually interested in the stability of the zero solution, it willbe assumed that $\sigma_u A = \emptyset$;. While this assumption is notnecessary for theformulation of the center manifold theorem, $\sigma_u A \neq \emptyset$; guarantees that the zerosolution is unstable.It can be useful to include this case since an unstablesolution can undergo a secondary bifurcation and become stable.

We write $R^n = E^c \oplus E^u$ and rewrite equation (2) as

$$x^{\cdot} = Ax + f(x,y)$$
$$y^{\cdot} = By + g(x,y) \qquad (3)$$

where $\sigma A = \sigma_c A$ and $\sigma B = \sigma_s B$. Note that we still have $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$. The re-use of the matrix A is clearlyan abuse of notation, but its use should be clear in context. We assume $A \in R^{c\times c}$and $B \in R^{s\times s}$where $n = c + s$. We can now state the center manifold theorem [4].

Theorem 3.1.1: (Center Manifold Theorem). Given system (3) with the zero solution and $\sigma A = \sigma_c A$ and $\sigma B = \sigma_s B$, then there exists (locally) aninvariant center manifold $M^c(0)$ that can be represented as

$$M^c(0) = \{(x, y) \mid y = h(x), h(0) = Dh(0) = 0, |x| < \sigma\}$$

for some sufficiently small σ. Moreover, $M^c(0) \in C^k$, same as f, g.The notation $M^c(0)$ is used to emphasize the local nature of the center manifold.From now on it will simply be referred to as M^c.A few notes on the CM theorem:[4]

1. With $y = h(x)$, we can reduction of the dynamics of (2) to the CM isgiven by:

$$u^{\cdot} = Au + f(u, h(u)).$$

The use of the variable u is to emphasize the fact that the CM is in generalnot a linear subspace. Use of the variable x would not make this specific.

2. The condition that $h(0) = Dh(0) = 0$ implies that the CM is tangent toEc at the origin.

3. The CM is C^k whenever $f, g \in C^k$with the exception of some cases when$k = \infty$,this is due to the local nature of the CM. as$k \neq \infty$, theneighborhood on which the CM is defined can shrink such that a C^1manifold does not exist.

4. The center manifold is not unique! However, in practice this non-uniquenessdoesn't really pose a problem.

5. The CM has similar properties to that of E^c: M^c must contain all solutionscontained in a small neighborhood of zero, including fixed points,small periodic solutions, homo- and hetero-clinic orbits.let $x_0$be a point of the intersection. By Definition,it belongs to both invariant manifolds. Therefore, the orbit starting at this point converges to the saddle point $x_0$under repeated iteration of either f orf^{-1} : $f^k(x^0) \to x_0$ as $k \to \mp\infty$. Each point of this orbit is a point of intersection of$W^s(x_0)$ and $W^u(x_0)$. This infinite number of intersections forces the manifolds to "oscillate" in a complex manner Near x_0,as sketched in Figure 3.1(b). The resulting "web" is called the Poincare homoclinic structure.

The orbit starting at$x_0$is said to be homoclinic to x_0. It is the presence of the homoclinic structure that can make the intersection of W^s,$u(x_0)$ with any neighborhood of the saddle $x_0$highly nontrivial.[4]

Y=h(x) is (CM)

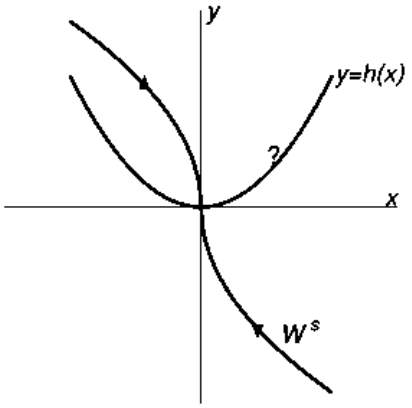


Figure 3.2 [4]

Theorem (Local Center Manifold Theorem) 3.1.2: Let $f \in C^r(E)$, where E is an open subset of R^n containingthe origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has c eigenvalues with zero real part, and$s = n - c$ eigenvalues with negative real part. The system (1) then can be written in diagonal form

$$x' = Cx + F(x, y)$$
$$y' = Py + G(x, y) \quad (4)$$

where $(x, y) \in R^c \times R^s, C$ is a square matrix with c eigenvalues with zero real parts, P is a square matrix with s eigenvalues with negative real parts, and $F(0) = G(0) = 0, DF(0) = DG(0) = 0$; furthermore, there exists $\sigma > 0$ and a function $h \in C^r(N\sigma(0))$, $h(0)\ 0, Dh(0) = 0$ defines the local center manifold $W_c(0) := \{(x; y) \in R^c \times R^s | y = h(x) \text{ for} |x| < \sigma\}$and satisfies

$$Dh(x)\ [Cx\ F(x, h(x))] = P.h(x) + G(x, h(x)) \quad (5)$$

for $|x| < \sigma$; and the flow on the center manifold $W^c(0)$ is defined by the system of differential equations

$$x' = Cx + F(x, h(x)) \quad (6)$$

for all $x \in R^c$ with $|x| < \sigma$.

This theorem can be used to determine the flow near nonhyperbolic equilibrium points.

Section (3.2): The strategy:

To do this:

(1) Convert the equation (1) in diagonal form equation (4)

(2) Use a series expansion for the components of $\mathrm{h(x)}$ (up to the degree $c$of accuracy we need, providedthat r is sufficiently large)

(3) Determine the components of the expansion of $\mathrm{h(x)}$(without constants a linear terms in order to satisfy) using (5)

(4) Substitute this approximate expression of $\mathrm{h(x)}$ into (6) to determine the flow.[3]

To solve equation (5), equate the coefficient of different term in the polynomials on both sides and we obtain a system of algebraic equations for the coefficients of the polynomial. By solving these equations, we obtainan approximation to the centre manifold $y = h(x)$. After h is identified, the reduced order structural dynamicmodel, which is only a function of x' , is given by$\mathrm{x' = Cx + F(x, h(x))}$[10]

Example 3.1.1 [11]: Consider the system$\dot{\mathrm{x}} = \mathrm{ax^2 + xb^T y + cx^3 + \ldots, x \in R^1}$, $\dot{\mathrm{y}} = \mathrm{By + dx^2 + \ldots, y \in R^m}$. Where B is a matrix without imaginary eigenvalues. The Taylor expansion for center manifold should have form

$$\mathrm{y = h(x) = \sigma x^2 + O(x^3)}$$

with yet unknown vector coefficient σ.$\dot{\mathrm{y}} = \mathrm{2\sigma x\dot{x} + O(x^4)}$

Substitute this $\dot{y}$ into the second equation of the system, we get

$\mathrm{2\sigma x(ax^2 + xb^T y + cx^3 + \ldots) + \ldots = B(\sigma x^2 + O(x^3)) + dx^2 + ..}$**).**

Equating terms of order x^2 we get $B\sigma + d = 0$. Thus $\sigma = -B^{-1}d$.

Reduced onto the center manifold equation is

$$\dot{x} = ax^2 + (c - bTBd)x^3 + O(x^4).$$

Example 3.1.2: Consider an ODE$\dot{x} = Ax + O(|x|^2), x \in R^n$ with the right hand side of smoothness C^r , $r < 1$. Assume that the matrix A hasn_s ,n_u andn_c eigenvalues in the left complex half-plane, right complex half-plane and on imaginary axis respectively, $n_s + n_u + n_c = n$. Denote T^s ,$T^u$and T^c the corresponding invariant planes of A . (Note: “s” is for “stable”, “u” is for “unstable”, “c” is for “center ”).

Local Center Manifolds:

Hypotheses:

Let X , Z , Y be (real or complex) Banach spaces such that

$$Z \rightarrow Y \rightarrow X ,$$

with continuous embeddings. We consider a differential equation in X of the form

$$\frac{du}{dt} = Lu + R(u), \tag{7}$$

in which we assume that the linear part L and the nonlinear part $R$are such that the following holds.

Hypothesis 3.1 We assume that L and R in (7) have the following properties:

(i) $\mathrm{L} \in \mathrm{L}(\mathrm{Z}, \mathrm{X})$

(ii) for some k ≥ 2, there exists a neighborhood $\mathrm{V} \subset \mathrm{Z}$ of 0 such that $\mathrm{R} \in \mathrm{C}^{\mathrm{k}}(\mathrm{V}, \mathrm{Y})$and

$$\mathrm{R}(0) = 0, \mathrm{DR}(0) = 0.$$

Hypothesis 3.2 (Linear equation) For any $\eta \in [0, \gamma]$ and any

$\mathrm{f} \in \mathrm{L}_{\eta}(\mathrm{R}, \mathrm{Y}_{\mathrm{h}})$the linear problem

$$\frac{\mathrm{du_h}}{\mathrm{dt}} = \mathrm{L_h u_h} + \mathrm{f}(\mathrm{t}), \tag{8}$$

has a unique solution $\mathrm{u_h} = \mathrm{K_h f} \in \mathrm{L}_{\eta}(\mathrm{R}, \mathrm{L_h})$. Furthermore, the linear map Khbelongs to $\mathrm{L}(\mathrm{L}_{\eta}(\mathrm{R}, \mathrm{Y_h}), \mathrm{L}_{\eta}(\mathrm{R}, \mathrm{Z_h}))$, and there exists a continuous map $\mathrm{C} : [0, \gamma] \rightarrow \mathrm{R}$ such that

$$\left|\left|\mathrm{K_h}\right|\right|\mathrm{L}(\mathrm{L}_{\eta}(\mathrm{R}, \mathrm{Y_h}), \mathrm{L}_{\eta}(\mathrm{R}, \mathrm{Z_h})) \leq \mathrm{L}_{\eta}.$$

Theorem (3.1.3) (Center manifold theorem) Assume that Hypotheses 3.1, 3.2, and spectrum of A hold. Then there exists a $\mathrm{map}\Psi \in \mathrm{L}^{\mathrm{k}}(\mathrm{E_0}, \mathrm{Z_h})$, with

$$\Psi(0) = 0, \mathrm{D}\Psi(0) = 0, \tag{9}$$

and a neighborhood O of 0 in Z such that the manifold

$$\mathrm{M_0} = \{\mathrm{u_0} + \Psi(\mathrm{u_0}), \mathrm{u_0} \in \mathrm{E_0}\} \subset \mathrm{Z} \tag{10}$$

has the following properties:

(i) $\mathrm{M_0}$ is locally invariant, i.e., if u is a solution of (7) satisfying $\mathrm{u}(0) \in \mathrm{M_0} \cap \mathrm{O}$

and u(t) ∈ O for all t ∈ [0,T], then u(t) ∈M0 for all t ∈ [0,T].

(ii)$\mathrm{M_0}$ contains the set of bounded solutions of (7) staying in O for all t ∈ R, i.e., if u is a solution of (3.1.7) satisfying u(t) ∈ O for all t ∈ R, then u(0) ∈$\mathrm{M_0}$.

Proof of theorem (3.1.3) (Center Manifolds)

Consider system (7), and assume that Hypotheses 3.1, 3.2, and spectrum of A hold. For any u ∈ Z we set

$$u = u_0 + uh \in Z, u_0 = P_0 u \in E_0, u_h = P_h u \in Z_h,$$

and rewrite the system (7) as

$$\frac{du_0}{dt} - L_0 u_0 = P_0 R(u)$$

$$\frac{du_h}{dt} - L_h u_h = P_h R(u). \text{ (B.1)}$$

Modified System

We take a cut-off function $\chi : E_0 \to R$ of class C such that

$$\chi(u0) = \begin{cases} 1 \; for ||u_0|| \leq 1 \\ 0 \text{ for } ||u_0|| \geq 2 \end{cases}$$

$$\chi(u_0) \in [0,1] \text{ for all } u_0 \in E_0.$$

Since E_0 is finite-dimensional such a function always exists. We use this function to modify the nonlinear terms R(u) outside a neighborhood of the origin, in order to be able to control the norm of the u0-component of the system (B.1) in the space of exponentially growing functions L_η (R,E_0). We set

$$R^\varepsilon(u) = \chi(\tfrac{u_0}{\varepsilon}) R(u) \text{ for all } \varepsilon \in (0, \varepsilon_0),$$

where ε_0 is chosen such that

$$\{u = u_0 + u_h, ||u_0|| E_0 \leq 2\varepsilon_0, ||u_h|| Z_h \leq \varepsilon_0\} \subset V,$$

with V the neighborhood of the origin in Hypothesis 3.1. Then R^ε is well definedin the closed set

$$O_\varepsilon = E_0 \times B_\varepsilon(Z_h), B_\varepsilon(Z_h) = \{u_h \in Z_h, \; ||u_h|| \leq \varepsilon\},$$

and satisfies

$$R_\varepsilon(u) = R(u) \text{for all } u \in O_\varepsilon, ||u_0|| \leq \varepsilon.$$

Consider the modified system

$$\frac{du_0}{dt} - L_0 u_0 = P_0 R^\varepsilon(u)$$

$$\frac{du_h}{dt} - L_h u_h = P_h R^\varepsilon(u). \text{ (B.2)}$$

The nonlinear terms in this system now satisfy

$$\delta_0(\varepsilon) \stackrel{\text{def}}{=} sup_{u\in O_\varepsilon}(\left|\left|P_0R^\varepsilon(u)\right|\right|)E_0\,, ||P_hR^\varepsilon(u)||Y_h) = O(\varepsilon^2)$$

$$\delta_1(\varepsilon)\, sup_{u\in O_\varepsilon}(||DuP_0R^\varepsilon(u)||L(Z, E_0),$$

$$||DuP_hR^\varepsilon(u)||L(Z, Yh)|| = O(\varepsilon). \text{ (B.3)}$$

We prove below the existence of a "global" center manifold for this system which,due to the fact that R^ε and R coincide for $||u_0||E_0 \leq \varepsilon$, will give the local centermanifold for the system (7) in the Theorem).

Integral Formulation:

We replace system (B.2) by the integral formulation

$$u_0(t) = S_{0,\varepsilon}(u, t, u_0(0)) \stackrel{\text{def}}{=} e^{L_0t}u_0(0) + \int_0^t e^{L_0(t-s)}P_0R^\varepsilon(u(s))ds$$

$$u_h = S_{h,\varepsilon}(u \stackrel{\text{def}}{=} K_hP_0R^\varepsilon(u). \text{ (B.4)}$$

The first equation in this system is obtained by the variation of constant formula from the first equation in (B.1). Here $u_0(0) \in E_0$is arbitrary, and the exponential e^{L_0t} exists since E_0 is finite-dimensional. The second equation in (B.4) is obtained from Hypothesis 3.2, used with $f \in L_\eta(R, y_h)$.. It is now straightforward to checkthat this integral system is equivalent to (B.2) for

$$u = (u_0, u_h) \in N_{\eta,\varepsilon} \stackrel{\text{def}}{=} L_\eta(R, E_0) \times L_\eta(R, B_\varepsilon(Z_h)),$$

with $0 < \eta \leq \gamma$ and $\varepsilon \in (0, \varepsilon_0)$. Notice that Nη,ε is a closed subspace of$L_\eta(R, Z)$ so that it is complete when equipped with the norm of$L_\eta(R, Z)$.

Fixed Point Argument:

Our aim now is to show that (B.4) has a unique solution u = $(u_0, u_h) \in N_{\eta,\varepsilon}$,for any $u_0(0) \in E_0$. For this we use a fixed point argument for the map

$$S_{\varepsilon(u,u_0(0))} \stackrel{\text{def}}{=} (S_{0,\varepsilon(u,u_0(0))}S_{h,\varepsilon(u))}\,, S_{\varepsilon(u_0(0))} : N_{\eta,\varepsilon} \to N_{\eta,\varepsilon}$$

We show that $S_{\varepsilon(u_0(0))}$ is well defined and that it is a contraction with respect to the norm of$L_\eta(R, Z)$. for $\eta \in (0, \gamma]$, with γ the constant in Hypothesis 3.2, and εsufficiently small.

First, Hypothesis 2.4 implies that for any $\delta > 0$ there is a constant $c\delta > 0$ such that

$$||e^{L_0 t}||L(E_0) \leq c\delta\, e^{\delta|t|} \text{ for all } t \in R. \text{ (B.5)}$$

Using this equality with δ =η, we find

$$\sup_{t\in R}(e^{-\eta|t|}\big||e^{L_0 t}u_0(0)|\big|E_0) \leq L_\eta||u_0(0)||E_0\,,$$

which shows that the first term in $S_{0,\varepsilon\,(u,u_0(0))}$belongs to $L_\eta\,(R,E_0)$, for any $\eta > 0$. Next, for any $u \in N_{\eta,\varepsilon}$, we have the estimates

$$\big||P_0 R^\varepsilon(u)|\big|E_0 \leq \delta_0(\varepsilon), \quad \big||P_h R^\varepsilon(u(t))\big||Y_h \leq \delta_0(\varepsilon),$$

which together with (B.5) for $\delta = \eta/2$, and Hypothesis 3.2 imply

$$\sup_{t\in R}(e^{-\eta|t|}||\int_0^t e^{L_0(t-s)}P_0 R^\varepsilon(u(s))ds\ ||_{E_0}) \leq c_\delta\delta_0(\varepsilon)\sup_{t\in R}(e^{-\eta|t|}\int_0^t e^{\delta|t-s|}ds$$
$$\leq \frac{2c\eta/2\delta 0(\varepsilon)}{\eta},$$

and

$$||K_h P_h R^\varepsilon(u)||L_\eta\,(R,Z_h) \leq C(0)\delta_0(\varepsilon).$$

This shows that $S_{\varepsilon\,(u_0(0))} \in N_{\eta,\varepsilon}$, provided$C(0)\delta_0(\varepsilon)) \leq \varepsilon$, which holds for εsufficiently small since $\delta_0(\varepsilon) = O(\varepsilon^2)$.

Now we show that the map $S_{\varepsilon\,(u_0(0))}$ is a contraction with respect to the norm of$L_\eta\,(R,Z)$ for $\eta \in (0,\gamma]$ and sufficiently small ε . From equality (B.3) we find that

$$\big||R^\varepsilon(u_1)-R^\varepsilon(u_2)|\big|L_\eta(R,y) = \sup_{t\in R}(e^{-\eta|t|}\Big|\big|R^\varepsilon(u_1(t))-R^\varepsilon(u_2(t))\big|\Big|Y)$$
$$\leq \delta_1(\varepsilon)\sup_{t\in R}(e^{-\eta|t|}\big||u_1(t)-u_2(t)|\big|Z)$$
$$\leq \delta_1(\varepsilon)||u_1-u_2||L_\eta\,(R,Z)$$

for any $u_1,u_2 \in N_{\eta,\varepsilon}$. Now, using (B.5) with δ =η/2 we obtain

$$||S_{0,\varepsilon\,(u_1,u_0(0))}-S_{0,\varepsilon\,(u_2,u_0(0))}||L_\eta\,(R,E_0)$$
$$\leq c_\delta\,\delta_1(\varepsilon)\sup_{t\in R}(e^{-\eta|t|}(|\int_0^t e^{\eta|s|+\delta|t-s|}ds|)||u_{1-}u_2||L_\eta\,(R,Z))$$
$$\leq \frac{2c\eta/2\delta 0(\varepsilon)}{\eta}\,||u_1-u_2||L_\eta\,(R,Z),$$

and using the estimate in Hypothesis 3.2 we find

$$||S_{0,\varepsilon\,(u_1)}-S_{0,\varepsilon\,(u_2)}||L_\eta\,(R,Z_h)$$
$$\leq C(\eta)\,\delta_1(\varepsilon)\,||u_1-u_2||L_\eta\,(R,Z),$$

Since $\delta_1(\varepsilon) = O(\varepsilon)$ for any $\eta \in (0,\gamma]$, we can choose ε small enough such that

$$||S_{0,\varepsilon\,(u_1,u_0(0))} - S_{0,\varepsilon\,(u_2,u_0(0))}||L_\eta\,(R,Z) \le \frac{1}{2}||u_1 - u_2||L_\eta\,(R,Z).$$

Consequently, the map $S_\varepsilon u_0(0))$ is a contraction in the complete metric space $N_{\eta,\varepsilon}$. Applying the fixed point theorem we now have the existence of a unique solution of (B.4),

$$u \overset{def}{=} \Phi(u_0(0)) \in N_{\eta,\varepsilon}$$

for any $u_0(0) \in E_0$, for any $\eta \in (0,\gamma]$, and ε sufficiently small. Clearly, this is also a solution of (B.2).

Properties of Φ: Recall that ε is chosen such that

$$C(0)\delta_0(\varepsilon)) \le \varepsilon, \qquad \frac{2c\eta/2\delta_1(\varepsilon)}{\eta} \le \frac{1}{2}, C(\eta)\delta_1(\varepsilon) \le \frac{1}{2}$$

Then the continuity on [0,γ] of the map η →C(η) in Hypothesis 3.2 implies that for any$\hat{\eta} \in (0,\gamma)$, we can choose ε > 0 such that these inequalities hold for all $\eta \in [\hat{\eta},\gamma]$.Consequently, for any$\hat{\eta} \in (0,\gamma)$, there exists ε > 0 such that the uniquefixed point $\Phi(u_0(0))$ belongs to $N_{\eta,\varepsilon}$for any $\eta \in [\hat{\eta},\gamma]$. This property is used later when showing that the center manifold is of class C^k.

Next, notice that the map $u_0(0) \to S_{0,\varepsilon}(u, u_0(0))$ is Lipchitz from E_0into $L_\eta\,(R,E_0)$, so that the map$u_0(0) \to S_{0,\varepsilon}(u, u_0(0))$ is also Lipchitz. Consequently, Φ is a Lipchitz map. In addition, the uniqueness of the fixed point implies that

$$\Phi(0) = 0.$$

Construction of Ψ: We define now the map$\Psi: E_0 \to Z_h$in the Theorem, through

$$(u_0\,(0), \Psi(u_0) \overset{def}{=} \Phi(u_0)(0)\ for\ all\ u_0 \in E_0,$$

i.e., by taking the component inZ_h of the fixed point $\Phi(u_0)(0))$ at t = 0. Since Φ is a Lipchitz map, we have thatΨ is also a Lipchitz map, and since $\Phi(0) = 0$, we have

$$\Psi(0) = 0.$$

We prove now thatΨ has the properties (i) and (ii) in the theorem.

First, we show that the manifold

$$M_{\eta,\varepsilon} = \{(u_0\,(0), \Psi(u_0)\,, u_0 \in E_0\}$$

is a global invariant manifold for the flow defined by (B.2). We define the shiftoperator $\Gamma_s$through

$$(\Gamma\, su)(t) = u(t + s) \text{ for all } t, s \in R.$$

Since system (B.2) is autonomous, it is equivarient under the actionΓ s for any s ∈R, so that if u is a solution of (B.2), then $\Gamma\, su$ is also a solution of (B.2). Moreover,$\Gamma\, su \in N_{\eta,\varepsilon}$ when

$$u \in N\eta, \varepsilon .$$

Consider a solution u of (B.2) with u(0) = $\{(u_0(0), \Psi(u_0)\ (0))$ for some$u_0(0) \in E_0$. Then u = Φ($u_0(0)$) ∈ $N\eta, \varepsilon$, and since $\Gamma\, su \in N\eta, \varepsilon$ is also a solution, from the uniqueness of the fixed point we conclude that

$$\Gamma\, su = \Phi(u_0(s)) \text{ for all } s \in R.$$

Consequently,

$$u(s) = (u_0\ (s), \Psi(u_0)\ (s)) \text{ for all } s \in R,$$

which shows that$M_{\eta,\varepsilon}$ is globally invariant under the flow defined by (B.2). Sincethe system (B.1) coincides with (B.2) in $O_\varepsilon = B_\varepsilon(E_0) \times B_\varepsilon\ (Z_h)$,this proves part **(i)** of the Theorem with $M_0 = M_{\eta,\varepsilon}$and $O = O_\varepsilon$. Indeed, assume that u is a solution of (B.1) such that $u(0) \in M_0 \cap O$ and $u(t) \in O$ for all $t \in [0, T]$.

Then u satisfies (B.2) for all t ∈[0,T], and since u(0) ∈$M_{\eta,\varepsilon}$and $M_{\eta,\varepsilon}$ is an invariant manifold, we have $u(t) \in M_{\eta,\varepsilon} = M_0$for all t ∈[0,T].

Consider now a solution u of (B.1) which belongs to $O = O_\varepsilon$ for all t ∈R. Then u $N_{\eta,\varepsilon}$and it is also a solution of (B.2). Consequently, $u = \Phi(u_0(0))$, so that $u(0) \in M_{\eta,\varepsilon} = M_0$ which proves part (ii) of the Theorem

Regularity of Ψ: We have proved so far thatΨ is a Lipchitz map. Notice that for this proof we have only used the fact that R is of class C^1. It remains to show thatΨ is of class $C^k$when R is of class C^k. For this, it is enough to prove that Φ is of class C^k.

The major difficulty in proving this property comes from the fact that the Nemitsky operator

$$R^\varepsilon : L_\eta\ (R, Z) \to L_\eta\ (R, y)$$

is not continuously differentiable, due to the growth of

$$u \in L_\eta\ (R, Z)\ as\ t \to \pm\infty.$$

(i) $R^{\varepsilon} : L_{\eta}(R, Z) \to L_{\xi}$ (R, y is continuous for any $\eta \geq 0$ and $\zeta > 0$;

(ii) $R^{\varepsilon} : L_{\eta}(R, Z) \to L_{\xi}$ (R, y is of class C k for any $0 \leq \eta < \zeta / k$ and ζ > 0.

We point out that the kth order derivative exists for $\eta = \zeta /k$, but this derivative is continuous only if $\eta < \zeta / k$.

the integral system (B.4) is written as

$$u = S_{u_0}(0) + KR^{\varepsilon}(u), \quad (B.6)$$

with S and K linear maps defined by

$$(S_{u_0}(0)(t) = e^{L_0 t} u_0(0),$$

and

$$(Kv)(t) = \int_0^t e^{L_0(t-s)} P_0(v(s))ds + (K_h P_h(v))(t).$$

We already showed that

$S \in L\left(E_0, L_{\hat{\eta}}(R, E_0)\right)$, $\left|\left|S_{u_0}(0)\right|\right|_{L_{\eta}(R, E_0)} \leq C_{\eta/2} ||u_0(0)||_{E_0}$, and that $KR^{\varepsilon} : N_{\eta,\varepsilon} \to N_{\eta,\varepsilon}$ is a contraction for any $\eta \in [\hat{\eta}, \gamma]$, when $\hat{\eta} \in (0, \gamma)$ and ε is sufficiently small.

The idea is to consider the fixed point $u = \Phi(u_0(0) \in N_{\eta,\varepsilon} \subset L_{\eta}(R, Z)$ of (B.4) found for $\eta \in [\hat{\eta}, \gamma]$, with $\hat{\eta}$ taken such that $0 < \hat{\eta} < \gamma / k$, and to show that the map $\Phi : E_0 \to L_{\eta}(R, Z)$ is of class C^k for all $\eta \in (k\hat{\eta}, \gamma]$, with

$$D^p \Phi(u_0(0)) \in L^p(E_0, L_{k\hat{\eta}}(R, Z)).$$

Here $L^p(E_0, L_{k\hat{\eta}}(R, Z))$ denotes the Banach space of p-linear continuous maps from E_0 into $L_{k\hat{\eta}}(R, Z)$. Several proofs of this result are available in the literature, all being quite long and technical. While we refer to these works for further details, we only point out that the derivative $\Phi(u_0(0))$ is the fixed point in $L(E_0, L_{k\hat{\eta}}(R, Z))$ of the linear equation

$$D\Phi(u_0(0)) = S + KD_u R^{\varepsilon}(\Phi(u_0(0))D\Phi(u_0(0)),$$

which may be differentiated up to order k. In particular, this implies that $DP_h \Phi(0) = 0$ and $D\Psi(0) = 0$, and ends the proof of Theorem 3.1.2.[30]

Theorem 3.1.3 (The center manifold theorem: Pliss-Kelley-Hirsch-Pugh-Shub) In some neighborhood U of the origin this ODE has C^r-smooth invariant manifolds W^s , W^u and

C^{r-1}-smooth invariant manifold W^c, which are tangent at the origin to the planes T^s , T^u andT^crespectively. Trajectories in the manifolds W^s and W^u exponentially fast tend to the origin as $t \to +\infty$ and $t \to -\infty$ respectively. Trajectories which remain in U for all $t \geq 0\ (t \leq 0)$ tend to W^c as $t \to +\infty\ (t \to -\infty)$. W^s , W^u and$W^c$are called the stable, the unstable and a center manifolds of the equilibrium zero respectively.[11]

Remarks: Behavior of trajectories on W^c is determined by nonlinear terms

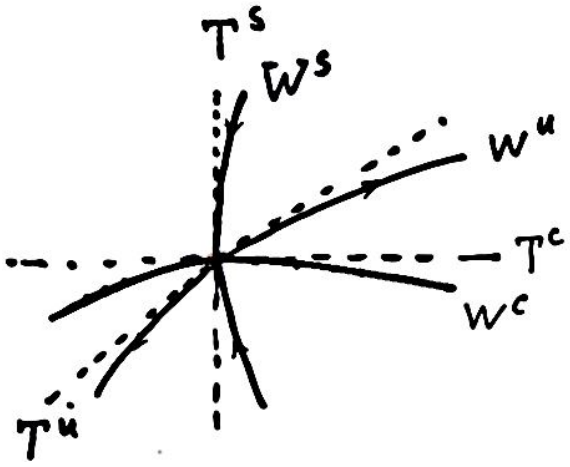


Figure 3.3

1- If the original equation has smoothnessC^∞ or C^ω, then $W^s$and W^u also have smoothness C^∞ or C^ω. However W^c in general has only a finite smoothness.

2- If $n_s = 0$ or $n_u = 0$ and the original equation has smoothness $C^r, r < \infty$, then W^c has smoothness C^r .[11]

Theorem 3.1.4 The smooth system

$$y^{\cdot} = \propto + y^2 + O(y^3) \tag{6}$$

is locally topologically equivalent near the origin to the system

$$x^{\cdot} = \propto + x^2. \tag{7}$$

Proof: The proof involves two steps. It is based on the fact that for scalar systems a homeomorphism mapping equilibria onto equilibria will automatically map their connecting orbits onto each other

Step 1 (Analysis of equilibria). Write system (6) as

$$y^{\cdot} = F(y, \propto) = \propto + y^2 + y^3 \Psi(y, \propto), \tag{7}$$

where is a smooth function of $(y, \propto)$ near (0, 0). Consider the equilibrium manifold of (7) near the origin (0, 0) of the (y, _)-plane:

$$M = \{(y, \propto) : F(y, \propto) = \propto + y^2 + y^3 \Psi(y, \propto) = 0\}.$$

The curve M passes through the origin (F(0, 0) = 0). By the Implicit Function Theorem, it can be locally parameterized by y (since $F_{\propto}(0, 0) = 1 \neq 0$)

$$M = \{(y, \propto) : \propto = g(y)\},$$

where g is smooth and defined for small |y|. Moreover,

$$g(y) = -y^2 + O(y^3).$$

Thus, for any sufficiently small $\propto < 0$, there are two equilibria of (3.1.7) near the origin, $y_1(\propto)$ and $y_2(\propto)$, which are close to the equilibria of (3.1.6), i.e., $x_1(\propto) = -\sqrt{-\propto}$ and $x_2(\propto) = \sqrt{-\propto}$, for the same parameter value.

Step 2 (Homeomorphism construction). For small $|\propto|$, construct a parameter-dependent map $y = h_{\propto}(x)$ as follows. For $\propto \geq 0$ take the identity map $h_{\propto}(x) = x$.

For $\propto < 0$ take a linear transformation

$$h_{\propto}(x) = A(\propto) + B(\propto)x,$$

where the coefficients A,B are uniquely determined by the conditions $h_{\propto}(x_j(\propto)) = y_j(\propto)$, j = 1, 2.

Namely,

$$A(\propto) = \frac{y_2(\propto) + y_1(\propto)}{2}$$

$$B(\propto) = \frac{y_2(\propto) - y_1(\propto)}{2}$$

Notice that $A(\propto) \to 0$ and $B(\propto) \to 1$ as $\propto \uparrow 0$. The map $h_{(\propto)} : R \to R$thus, constructed is a homeomorphism mapping orbits of (7) near the origin into the corresponding orbits of (3.1.8), preserving the direction of time.

Remarks:

(1) Although we do not require for the homeomorphism$h_{(\propto)}$to depend continuously on $\propto$, this property holds here. In particular, $h_{(\propto)}$ tends to the identity map $as \propto \uparrow 0$.

(2) The equivalence between $\dot{y} = \propto -y^2 + O(y^3)$ and $\dot{x} = \propto -x^2$ can be established by similar arguments.

Theorem 3.1.5 Suppose that a one-dimensional system

$$x^{\cdot} = f(x, \propto), x \in R, \propto \in R, \tag{8}$$

with smooth f, has at $\propto = 0$ the equilibrium $x = 0$, and is such that $\lambda = f_x(0,0) = 0$. Assume that the following two conditions are satisfied:

$$\text{(A.1) } f_{xx}(0,0) \neq 0;$$

$$\text{(A.2) } f_{\propto}(0,0) \neq 0.$$

Then there are invertible smooth coordinate and parameter changes transforming the system into

$$y^{\cdot} = \beta \pm y^2 + O(y^3). \tag{9}$$

Remark: One can reformulate the statement of the theorem by saying that near the origin (8) is locally smoothly conjugate to (9).

Proof of Theorem 3.1.5:

Step 0 (Taylor expansion). Write the right-hand side of (8) as $f(x, \propto) = f_0(\propto) + f_1(\propto)x + f_2(\propto)x^2 + O(x^3)$, where

$$f_0(0) = f(0,0) = 0, f_1(0) = f_x(0,0) = 0,$$

$$f_2(0,0) = \frac{1}{2} f_{xx}(0,0).$$

Step 1 (Shift of the coordinate). Perform a linear coordinate shift by introducing a new variable ξ:

$$x = \xi + \delta, \tag{10}$$

Where $\delta = \delta(\alpha)$ is an a priori unknown function that will be defined later. We will see that $\delta = O(\alpha)$ as $\propto \to 0$. The inverse coordinate transformation is $\xi = x - \delta$ Substituting (10) into (8) we find

$$\dot{\xi} = x^{\cdot} = f_0(\propto) + f_1(\propto)(\xi + \delta) + f_2(\propto)(\xi + \delta)^2 + \cdots$$

Therefore,

$$\dot{\xi} = [f_0(\propto) + f_1(\propto)\delta + f_2(\alpha)\delta^2 + O(\delta^3)] + [f_1(\alpha) + 2f_2(\alpha) + O(\delta^2)]\xi + [f_2(\alpha) + O(\delta)]\xi^2 + O(\xi^3).$$

Assumption (A.1) implies that

$$f_2(0) = \frac{1}{2} f_{xx}(0,0) \neq 0.$$

Then, according to the Implicit Function Theorem, there is a smooth function $\delta(\alpha)$that annihilates theξ_-term in the equation above for all sufficiently small | ∝ |. Indeed, the condition for the _-term to vanish can be written as

$$F(\propto, \delta) \equiv f_1(\alpha) + 2f_2(\alpha)\delta + \delta^2 \Psi(\propto, \delta) = 0$$

with some smooth function . We have

$$F(0,0) = 0, \mathrm{F}_\delta(0,0) = 2\mathrm{f}_2(0) \neq 0, \mathrm{F}_\propto(0,0) = \mathrm{f}_1'(0)$$

which implies (local) existence and uniqueness of a smooth function $\delta = \delta(\alpha)$such that $\delta(0) = 0$ and $F(\alpha, \delta(\alpha)) \equiv 0$. It follows that $\delta(\alpha) = \frac{-f_1'(0)}{2f_2(0)}\alpha + O(\alpha^2)$ (cf. (5.7)). The equation for ξ now no longer contains any ξ-term. Since $\mathrm{f}_1(\_) = \mathrm{f}_1'(0) + \mathrm{O}(\propto^2)$, we can write

$$\dot{\xi} = [\mathrm{f}_0'(0) + \mathrm{O}(\propto^2)] + [\mathrm{f}_2(0) + \mathrm{O}(\propto)]\xi^2 + \mathrm{O}(\xi^3). \quad (11)$$

Step 2 (Introduce a new parameter). Consider as a new parameter $\mu = \mu(\propto)$ the constant (-independent) term of (11), that we can write in the form

$$\mu = f_0'(0) \propto + \propto^2 \emptyset(\propto),$$

for some smooth function ∅. We have:

(a) $\mu(0) = 0$;

(b) $\dot{\mu}(0) = f_0'(0) = \mathrm{f}_\propto(0,0)$.

Since $f_\propto(0,0) \neq 0$ due to (A.2), the Inverse Function Theorem implies local existence and uniqueness of a smooth inverse function $\propto = \propto(\mu)$ with $\propto(0) = 0$. Therefore, equation (11) now reads

$$\dot{\xi} = \mu + a(\mu)\xi^2 + O(\xi^3),$$

where $a(\mu)$ is a smooth function with due to the first assumption $a(0) = f2(0) \neq 0$(A.1).

Step 3 (Final scaling). Let $y = |a(\mu)|\xi$ and $\beta = |a(\mu)|\mu$. Then we get

$$y^{\cdot} = \beta + sy^2 + O(y^3),$$

where $s = sign\, a(0) = \pm 1$. This is equation (11).

Chapter 4

Normal Forms Theorem

Section (4.1): The Normal forms simplification:

As explained in chapter 3, the centre manifold reduces the non-linear system on the centre variables. However, the associated centre manifold equations can have complicated on-linear terms due to the approximation of the stable variables in a power series in centre variables and their substitutions in the centre manifold equations. For such systems, non-linear successive coordinate transformations can be used to reduce this non-linear system to its simplest form, called the normal form. The idea of the normal transformation comes from Poincare in year ([145] and [146]). The main idea of the normal form approach is to eliminate as many non-linear terms as possible through a non-linear change of variables [10].

This normal forms theory is a very powerful tool for the analysis of the local dynamical behavior near a singularity.

The general idea of normal form is to put a complicated system into a form as simple as possible by means of change of coordinates. depending on the purpose of simplification and this concept may vary greatly. It depend on the change of coordinates that are tolerated(linear, polynomial, formal series, smooth , analytic) and on the possible structures that preserved (i.e. simplistic, volume-preserving, symmetric, reversible)[2]

Definition 4.1.1. A cycle is a periodic orbit, namely a non -equilibrium orbit L_0, such that each point $x_0 \in L_0$ satisfies $\phi^{t+T_0}x_0 = \phi^t x_0$ withsome$T_0 > 0$, for all $t \in T$ [12].

Let us restrict on local normal forms, i.e. in the vicinity of a stationary point of a vector field or a diffeomorphism (the latter can be applied to the Poincare 1879 [17] map of a periodic orbit) .We concentrate on the simplification of the Taylor series. The general idea is to apply consecutive polynomial changes of variables; at each step we simplify terms of a degree higher than in the step before. The ideal simplification would be to put all higher order terms to zero, which would (at least at the level of formal

series) linearize the system. But as soon as there are resonances, this is impossible the planar system $2x\partial/\partial x + (y + x^2)\partial/\partial y$ cannot be formally linearized.

Section (4.2) Setting:

Let X be a C^{r+1}vector field defined on a neighborhood of$0 \in R^n$, and denote $A = dX(0)$ (its linear approximation at $zero$). The Taylorexpansion of X at 0 takes the form

$$X(x) = A.x + \sum_{k=2}^{r} X_k(x) + O(|x|^{r+1}) \quad (1)$$

Where $X_k \in H^k$, the space of vector fields whose components are homogeneous polynomials of degree k. The classical formal normal form theorem is as follows.We define the operator$L_A$on$H^k$by putting $L_A h(x) = dh(x).A.x - A.h(x)$, onecalls L_A the homological operator. One checks that $L_A(H^k) \subset H^k$. One alsodenotes this by add $A(h)(x)$. Let R^k be the range of L_A, i.e. $R^k = L_k(H^k)$. Let G^k denote any complementary subspace toR^kin H^k. The formal normal form theorem states, under the above settings.[2]

Main Theorem: We consider a differential equation in R^n of the form

$$\frac{du}{dt} = Lu + R(u), \quad (2)$$

in which L and R represent the linear and nonlinear terms, respectively. More precisely, we assume that the following holds.

Hypothesis 4.1 Assume that L and R in (2) have the following properties:

(i) L is a linear map in R^n

(ii) for some $k \geq 2$, there exists a neighborhood $V \subset R^n$ of 0 such that $R \in C^k(V, R^n)$and $R(0) = 0,\ DR(0) = 0$.

Our purpose is to transform this system, in a neighborhood of the origin, in such a way that the Taylor expansion of the transformed nonlinear vector field contains a minimal number of terms at every order. The following result shows the existence of a polynomial change of variables leading to a transformed vector field, has this property.

Theorem 4.2.1(Normal form theorem) Consider the system (2) and assume that

Hypothesis 4.1 holds. Then for any positive $integer\, p,\ 2 \leq p \leq k$, there exists a polynomial $\Phi : R^n \to R^n$ of degree p, with $\Phi(0) = 0, D\Phi(0) = 0,$ and such that the change of variable

$$u = v + \Phi(v) \tag{3}$$

defined in a neighborhood of the origin in R^n transforms the equation (2) into the "normal form"

$$\frac{dv}{dt} = Lv + N(v) + \rho(v), \tag{4}$$

with the following properties:

(i) $N : R^n \to R^n$ is a polynomial of degree p, satisfying

$$N(0) = 0, DN(0) = 0.$$

(ii) The equality

$$N(e^{tL^*}v) = e^{tL^*}N(v), \tag{5}$$

holds for all $(t, v) \in R \times R^n$, where L∗ represents the adjoint of L.

(iii) ρ is a map of class C^k in a neighborhood of 0, such that

$$\rho(v) = o(|v|^p).$$

Proof of Theorem 4.2.1: Consider the Taylor expansion of R,

$$R(u) = \sum_{2 \leq q \leq p} R_q(u^{(q)}) + o(||u||^p)$$

for a given p, $2 \leq p \leq k$, where $u^{(q)} = (u, \ldots, u) \in (R^n)^q$, with $u \in R^n$ repeated q times, and R_q is the q-linear symmetric map on $(R^n)^q$ given through

$$R_q(u^{(q)}) = \frac{1}{q!} D^{(q)} R(0)(u^{(q)}).$$

Similarly, we write the polynomials Φ and N in the form

$$\Phi(v) = \sum_{2 \leq q \leq p} \Phi_q(v^{(q)}), N(v) = \sum_{2 \leq q \leq p} N_q(v^{(q)}),$$

with Φ_q and N_q q-linear symmetric maps on $(R^n)^q$ Differentiating (3) with respect to t and replacing du/dt and dv/dt from (2) and (4), respectively, leads to the identity

$$(I + D\Phi(v))\,(Lv + N(v) + \rho(v)) = L(v + \Phi(v)) + R(v + \Phi(v)), \tag{6}$$

which should be valid for all v in a neighborhood of 0. Our purpose it to determine Φ and N from this equality. By identifying the Taylor expansions on both sides, we obtain at order 2

$$\mathrm{D}\Phi_2\left(\mathrm{v}^{(2)}\right)\mathrm{Lv} - \mathrm{L}\Phi_2\left(\mathrm{v}^{(2)}\right) = \mathrm{R}_2\left(\mathrm{v}^{(2)}\right) - \mathrm{N}_2\left(\mathrm{v}^{(2)}\right), \tag{7}$$

and then at any order q, 3 ≤ q ≤ p, we have

$$\mathrm{D}\Phi_q\left(\mathrm{v}^{(\mathrm{q})}\right)\mathrm{Lv} - \mathrm{L}\Phi_q\left(\mathrm{v}^{(\mathrm{q})}\right) = \mathrm{R}_q\left(\mathrm{v}^{(\mathrm{q})}\right) - \mathrm{N}_q\left(\mathrm{v}^{(\mathrm{q})}\right), \tag{8}$$

with

$$Q_q(v^{(q)}) = -\sum_{2\le \mathrm{r}\le \mathrm{q}-1} \mathrm{D}\Phi_r(\mathrm{v}^{(\mathrm{r})})\mathrm{N}_{\mathrm{q}-\mathrm{r}+1}(\mathrm{v}^{(\mathrm{q}-r+1)}) +$$

$$+\sum_{r_1+\cdots+r_l=q} \mathrm{R}_l\Phi_{\mathrm{r}_1}\left(\mathrm{v}^{(\mathrm{r}_1)}\right), \Phi_{\mathrm{r}_2}\left(\mathrm{v}^{(\mathrm{r}_2)}\right) \ , \dots, \Phi_{\mathrm{r}_l}\left(\mathrm{v}^{(\mathrm{r}_l)}\right.$$

$\mathrm{r}_j \ge 1, \dots$

where we have set$\Phi_1(v) = v$. Notice that if Φ_l and N_l are known for any l, $2 \le l \le q-1$, then Q_q is known. Therefore, we can determine Φ and N by successively finding (Φ_2, N_2), (Φ_3, N_3), and so on, from (7) and (8).

The equations (7) and (8) have the same structure, more precisely, they are both of the form

$$\mathrm{A}_\mathrm{L}\,\Phi_\mathrm{q} = \mathrm{Q}_q - \mathrm{N}_\mathrm{q}, \tag{9}$$

in which A_L is a linear map (also called "homological operator") acting on the space of polynomials $\Phi : \mathrm{R}^\mathrm{n} \to \mathrm{R}^\mathrm{n}$ through

$$(\mathrm{A}_\mathrm{L}\,\Phi)(\mathrm{v}) = \mathrm{D}\Phi(\mathrm{v})\mathrm{Lv} - \mathrm{L}\Phi(\mathrm{v}). \tag{10}$$

A key property of A_L is that it leaves invariant the subspaceH_q of homogeneous polynomials of degree q, for any positive integer q. In the equality (9), Q_q is known, and we have to determine Φ_q and N_q. It is clear that if $\mathrm{A}_\mathrm{L}\,|H_q$ is invertible, then we can take N_q = 0, which gives the simplest solution here. However, this isnot always the case, and the condition for solving (9) is that$\mathrm{Q}_q - \mathrm{N}_\mathrm{q}$ lies in therange of the operator A_L. We claim that this condition is achieved when (6), or equivalently (5), is satisfied by N_q. Indeed, we define below a scalar product in the space H of polynomials of degree p, such that the adjoint operator $(\mathrm{A}_\mathrm{L})*$ of A_L with respect to this scalar product is $\mathrm{A}_{\mathrm{L}^*}$,

where L^* is the adjoint of L with respect to the canonical Euclidean scalar product in R^{n}. Then $\mathrm{Q}_q - \mathrm{N_q}$belongs to the range of $\mathrm{A_L}$ if $\mathrm{Q}_q - \mathrm{N_q} \in ker_{\mathrm{A_{L^*}}}$ ($\perp = im_{(\mathrm{A_L})}$ or, equivalently,$p_{(\ker(\mathrm{AL}*)}(\mathrm{Q}_q - \mathrm{N_q}) = 0$,where $p_{(\ker(\mathrm{AL}*)}$ is the orthogonal projection on $(\ker(\mathrm{AL}*)$ in the space H of polynomials of degree p. It is then natural to choose

$$\mathrm{N_q} = P_{ker\,(AL*)} Q_q.$$

Ofcourse, this choice is not unique, since we can add to $\mathrm{N_q}$any term in the range of $\mathrm{A_L}$. leaves invariant the subspace H_q, so that $\mathrm{N_q} \in ker_{\mathrm{AL}*} | H_q$. In particular, this shows that (5) holds for $\mathrm{N_q}$. With this choice for$\mathrm{N_q}$, we can now solve (9) and obtain a solution Φ_q, which is determined up to an arbitrary element in the kernel of AL. A possible, but not unique, choice is to choose the unique solution Φ_q,orthogonal to ker_{AL} inH_q. Summarizing, this shows that (9) possesses a solution ($\Phi_q, \mathrm{N_q}$) with $\mathrm{N_q}$ satisfying (5). Solving successively for q = 2, . . . , p, we obtain the polynomials Φ and N in the Theorem, with N satisfying (5). To finish the proof, it remains to define the scalar product in the space H such that (AL)∗ = AL∗ , (11) and to check that the orthogonal projection $\mathrm{P}_{\ker(\mathrm{AL}*)}$ on $\ker(\mathrm{AL}*$leaves invariantthe subspace H_q. For a pair of scalar polynomials P,$\mathrm{P}' : \mathrm{R}^n \to \mathrm{R}$ we define

$$def\langle p|p'\rangle = P(\partial u)P'(u)|u = 0, \qquad (12)$$

where $u = (u_1, \ldots, u_n) \in R^n$and $\partial u = (\partial/\partial u_1, \ldots, \partial/\partial u_1)$. The equality (12) defines a scalar product in the linear space of scalar polynomials $\mathrm{P} : \mathrm{R}^n \to \mathrm{R}$. consisting of monomials $u_1^{\propto} \ldots u_n^{\propto}$ .n , and

$$\left\langle u_1^{\propto} \ldots . u_n^{\propto} \middle| u_1^{\beta} \ldots . . u_n^{\beta} \right\rangle = \alpha_1! \ldots . \alpha_n! \, \delta_{\alpha_1\beta_1} \ldots \delta_{\alpha_n\beta_n}$$

where $\delta_{\alpha_j\beta_j} = 1$ if $\alpha_j = \beta_j$, and $\delta_{\propto_j\beta_j} = 0$ otherwise. (Notice that this scalar product can be extended to complex-valued polynomials $\mathrm{P} : \mathrm{C}^{\mathrm{n}} \to \mathrm{C}$ by taking

$$\mathrm{def}\,\langle \mathrm{P}|\mathrm{p}'\rangle = \mathrm{P}(\partial \mathrm{u})\mathrm{P}'(\mathrm{u})|\mathrm{u} = 0,$$

Finally, the identity above also holds in the subspaces$H_q$of homogeneous polynomials of degree q, which are all invariant under the actions of both AL and AL∗. Consequently,

$$\ker_{(\mathrm{AL}*|\mathrm{Hp})} = \ker_{\mathrm{AL}*} \cap \mathrm{H_p},$$

and since monomials with different degrees are orthogonal to each other, this implies the invariance of Hp under the orthogonal projection $\mathrm{P}_{\mathrm{kerAL}*}$This ends the proof of Theorem (1) [30]

Theorem 4.2.2[13,14] There exists a composition of near identity changes of variables of the form

$$x = y + \xi^k(y) \tag{13}$$

where the components of ξ^{k} are homogeneous polynomials of degree k, such that the vector field X is transformed into[2]

$$\mathrm{Y}(\mathrm{y}) = \mathrm{A}.\mathrm{y} + \sum_{k=2}^{r} \mathrm{g}^k(\mathrm{y}) + \mathrm{O}(|\mathrm{y}|^{r+1}) \text{ where } \mathrm{g}^k \in \mathrm{G}^k,$$

$\mathrm{k} = 2, \ldots, \mathrm{r}.$

in the latter case one can let r→∞ and obtaina normal form on the level of formal Taylor series (also called ∞-jets). we can assume that A is an upper triangular matrix. [2] Let the eigenvaluesbe $\lambda_1, \ldots, \lambda_n$. It can be calculated that the eigenvalues of L_A, as an operator$H^k \to H^k$,are then the numbers $< \lambda, \alpha > -\lambda_{\mathrm{j}}$where $\alpha \in N^n, \sum_{j=1}^{n}$ α_{j}=k, and $1 < j < n$. Hence, if these would all be nonzero then $B^k = H^k$ and then we have an ideal simplification i.e. allg^kequal to zero. However, if such a number is zero, that is,

$$< \lambda, \alpha > - \lambda_j = 0 \tag{14}$$

this is called a resonance between the eigenvalues. In such a case we have tochoose a complementary space G^{k}. From linear algebra it follows that one canalways choose [2]

$$G^k = \ker(L_{A^*}) \tag{15}$$

where $A^*$is the adjoint operator. But this choice of equation (3) is not unique and isfrom the computational point of view not always optimal, especially if thereare nilpotent blocks. This fact has been exploited by many authorsYu, 1999, Yu &Yuan, 2001, Yuan & Yu, 2001[17]. A typicalexample for this is the case where $\mathrm{A} = \mathrm{y}\,\partial/\,\partial\mathrm{x}$. On the other hand if A issemi-simple we can choose the complementary space to be $\ker(\mathrm{LA})$, so $L_A g_k$ = 0we can assume it to be the (complex) diagonal[λ_1, . . . , λ_{n}]. In that case we

canbe more explicit as follows. Let $e_j = \partial/\partial x_j$ denote the standard basis on C^n. For a monomial one can calculate that

$$L_A(x^\alpha e_j) = (<\lambda, \alpha_i> - \lambda_j)x^\alpha e_j \quad (16)$$

If the latter is zero, then the monomial is called resonant. This implies that the normal form can be chosen so that it only contains resonant monomials. Putting a system into normal form not only simplifies the original system, it also gives more geometric insight on the Taylor series. To be more precise, suppose (for simplicity: this can be generalized that A is semi simple. One can calculate that the condition [2] $L_A g_K = 0$ implies: $exp(-At)g_K(exp(At)x) = g_K(x)$ for all $t \in R$. This means that g_k is invariant for the 1-parameter group exp(At). A typical example in the plane is: A has eigenvalues $i\lambda, -i\lambda$. Note that the (only) resonances are $< (i\lambda, -i\lambda), (p+1, p) > -i\lambda = 0$ and $< (i\lambda, -i\lambda), (p, p+1) + i\lambda = 0$for all $p \in N$. We suppose that the original system was real i.e. on R^2;we can choose linear coordinates such that for $z = x + iy$, $\bar{z} = x - iy$ the linearpart is A =diagonal[iλ,−iλ]. Applying the remarks above we conclude that the normal form only contains the monomials $(z\bar{z})pz\partial/\partial z \; and \; (z\bar{z})p\bar{z}\partial/\partial\bar{z}$. Thegeometric interpretation here is that these monomials are invariant for rotationsaround (0, 0). This can also be seen on the real variant of this: the Taylorseries of the (real) normalized system has the form $(\lambda + f(x^2 + y^2))(x\,\partial/\partial y - y\,\partial/\partial x) + g(x^2\,y^2)(x\,\partial/\partial x + y\,\partial/\partial y)$ and is invariant for rotations. Warning the dynamic behavior of a formal normal form in the central manifold can be very different from that of the original vector field, since we are only looking at the formal level. A trivial example is (take $f = g = 0$ in the foregoing example)

$$X(x, y) = \lambda(x\,\partial y - y\,\partial x) - \exp(-1/(x^2))\,\partial/\partial x,$$

where orbits near (0, 0) spiral to(0, 0), whereas the normal form is just a linear rotation. This difference is due to the so called flat terms , i.e. the difference between the transformed vector field and a C^∞ realization of its normalized Taylor series (or polynomial).One could roughly say that, in the central manifold, the normal form has too much symmetries and is too poor to model more complicated dynamics of the system, which can be 'hidden in the flat terms'. To quote In the theory of normal forms

of analytic differential equations, divergence is the rule and convergence the exception, to use changes of variables like where ξ^k is then generated and this will guarantee that all changes of variables are compatible' with the extra structure The question of an analytic normal form, also in the hyperbolic case, leads to convergence questions and calls upon so called small divisor problems. The classical results are due to Poincace and Siegel; let us summarize them; they are formulated in the complex analytic setting:[2]

Theorem 4.2.2 (a) If the convex hull of the spectrum of A does not contain $0 \in C$ then X can locally be put into normal form by an analytic change of variables. [2]

Moreover this normal form is polynomial.

(b) If the spectrum $\{\lambda_1, \ldots, \lambda_n\}$ of A satisfies the condition that there existsC > 0 and μ> 0 such that for any m $\in N^n$ with

$$\sum_j m_j \geq 2| < h(\lambda_1, \ldots, \lambda_n), m > -\lambda_j | \geq \frac{c}{|m|^\mu} \qquad (17)$$

for $1 \leq j \leq n$, then X can be locally linearized by an analytic change of variables.

Note that case (a) contains the case where 0 is a hyperbolic source or sink. This case (a) in Theorem 4.2.2 can be extended if there are parameters: if X depends analytically on a parameter $\varepsilon \in C^p$ near $\varepsilon = 0$ then the change of variables isalso analytic in ε moreover the normal form is then a polynomial in the space variables whose coefficients are analytically dependent on the parameter ε. For case (b) this is surely not the case, since the condition (4.2.6) is fragile, a small distortion of the parameter generically causes resonances, be it of a high order. To fix the ideas: consider n = 2 and suppose $\lambda_1 < 0 < \lambda_2$. By a genericbut arbitrary small perturbation we can have that the ratio of these eigenvalues becomes a negative rational number - p/q, which gives a s of the form (2)$< \lambda, \tau > - \lambda_j$ with j = 1 and $\alpha = (q+1, p) \leq 0$, so (4.2.6) is violated (succeed) .[2]Soanalytic linearization, or even a polynomial analytic normal form, is un-generic for families of such hyperbolic stationary points. The search for analytic normal forms, i.e. simplified models, for families is still under investigation. A first simplification is obtained via the stable and unstable manifold from theorem 4.13 below , that is: the graphs of $\phi_{ss}$and ϕ_{uu}. When $X$is analytic near 0 then these manifolds are also analytic. So up to an analytic change of variables we can assume that E_s and E_u

are invariant, which gives a simplification of theexpression of X. Moreover there is analytic dependence on parameters. For local diffeomorphisms there are completely similar theorems about all the above.

Theorem 4.2.3 Let the vector field (X-invariant manifolds in the neighborhood of 0) be of class $\mathrm{C}^{\mathrm{r}}(1 \leq \mathrm{r} < 1)$. There exist map germs $\emptyset_{ss} : (Es, 0) \to Ec \oplus Eu$,$\emptyset_{sc} : (Es \oplus Ec, 0) \to Eu$, $\emptyset_{uu} : (Eu, 0) \to Es \oplus Ec$, $\emptyset_{cu} : (Ec \oplus Eu, 0) \to Es$ and $\emptyset c : (Ec, 0) \to Es \oplus Eu$of class$C^r$ such that the graphs of these maps are invariant for the flow of X. Moreover these maps are of classC^r, and their linear approximation at 0 is zero, i.e. their graphs are tangent to respectively$Es, Es \oplus Ec, Eu, Ec \oplus Eu$ and Ec. If X is of classC^1then$\emptyset_{ss}$ and $\emptyset_{\mathrm{uu}}$ are also of classC^1. If X is analytic then$\emptyset_{\mathrm{ss}}$and$\emptyset_{\mathrm{uu}}$ are also analytic.So analytic linearization, or even a polynomial analytic normal form, is un-generic for families of such hyperbolic stationary points.

The basic idea of normal form theory is to find a near-identity nonlinear transformation [17]

$$x = y + h(y) \equiv y + h_2(y) + h_3(y) + \cdots + h_k(y) + \tag{2}$$

such that the resulting system

$$\dot{y} = Ly + g(y) \equiv Ly + g_2(y) + g_3(y) + \cdots . + gk(y) + \tag{3}$$

becomes as simple as possible. Here $h_k(y)$ and $g_k(y)$denote the kth order vector homogeneous polynomials of y.

To apply normal form theory, first define an operator as follows:

$$L_k : H_k \to H_k ,$$
$$U_k \in H_k \to L_k(U_k) = [U_k, v_1] \in H_k, \tag{4}$$

where H_n denotes a linear vector space consistingof the kth-order vector homogeneous polynomialsThe operator $[U_k, v_1]$ is called Lie bracket, definedas

$$[U_k, v_1] = Dv_1.Uk - DU_k.v_1, \tag{5}$$

Next, define the space R_k as the range of L_k, and the complementary space of $\mathrm{R_k}$ as $\mathrm{K_k}$. Thus,

$$H_k = R_k \oplus K_k, \tag{6}$$

and we can then choose the basis for R_k and K_k. Consequently, a vector homogeneous polynomial $f_k \in H_k$ can be split into two parts: one is spannedon the basis of R_k and

the other on that ofK_k. Normal form theory shows that the part of $f_k$belonging to R_k can be eliminated while the part belonging to K_k must be retained, which is called normal form. [17]

Example 4.2.1

Consider the following general system with randomly chosen coefficients up to seventhorder

$$\dot{x}_1 = x2 + x_1^2 + \frac{1}{2}x_1x_2 + 2x_2^2 + 2x_1^3 + \frac{1}{7}x_1^2x_2 + \frac{5}{3}x_1x_2^2 + \frac{1}{2}x_2^3 + 5x_1^4 + \frac{1}{3}x_1^3x_2 - 15x_1^2x_2^2 + \frac{7}{3}x_1x_2^3 + 2x_2^4 - 2x_2^5 + 5x_1^4x_2 + \frac{1}{4}x_1^3x_2^2 + x_1^2x_2^3 + \frac{7}{4}x_1x_2^4 + 20x_2^5 + \frac{1}{2}x_1^6 + \frac{2}{5}x_1^5x_2 - x_1^4x_2^2 + \frac{1}{3}x_1^3x_2^3 + 2x_1^2x_2^4 + \frac{7}{5}x_1x_2^5 - 2x_2^6 + 2x_1^7 + x_1^6x_2 - 5x_1^5x_2^2 + \frac{1}{10}x_1^4x_2^3 + 3x_1^3x_2^4 + \frac{7}{2}x_1^2x_2^5 + 5x_1x_2^6 + x_2^7;$$

$$\dot{x}_2 = -x_1 + 3x_1^2 + \frac{1}{4}x_1x_2 + 5x_2^2 + \frac{2}{5}x_1^3 + 3x_1^2x_2 + 10x_1x_2^2 + \frac{4}{7}x_2^3 + \frac{5}{3}x_1^4 - \frac{2}{3}x_1^3x_2 + 10x_1^2x_2^2 + 3x_1x_2^3 + x_2^4 + 7x_1^5 - \frac{3}{5}x_1^4x_2 + 7x_1^2x_2^3 + \frac{3}{4}x_1x_2^4 + \frac{1}{8}x_2^5 - 2x_1^6 + 5x_1^5x_2 + \frac{1}{3}x_1^4x_2^2 + 7x_1^3x_2^3 + 4x_1^2x_2^4 - \frac{1}{5}x_1x_2^5 + 3x_2^6 + x_1^7 + 5x_1^6x_2 + \frac{5}{3}x_1^5x_2^2 + \frac{1}{2}x_1^4x_2^3 - 3x_1^3x_2^4 + 7x_1^2x_2^5 + \frac{5}{8}x_1x_2^6 + 3x_2^7$$

The linearized system of example(4.1)above about the equilibrium $x_1 = x_2 = 0$ has a pair of purely imaginary eigenvalues$\pm$I ,The Maple program is executed on a PC machineto obtain the followingsimplest normal form(SNF) in the complex form

$$\dot{u} = iu - \left[\frac{47}{336} + i\frac{233651}{13440}\right]u^2\bar{u} - \frac{25933399}{1354752}u^3\bar{u}^2$$

which can be directly transformed to the polar form using the real and imaginary parts:

$$\dot{R} = -\frac{47}{336}R^3 - \frac{25933390}{1354752}R^5 ; \dot{\theta} = 1 - \frac{233651}{13440}R^2$$

Since the coefficient of $R^3$is nonzero, this is Hopfbifurcation.

Example 4.2.2: The following system, described by a CNF (given in Cartesian coordinates), was obtained from a five dimensional system

$$\dot{x}_1 = x_2 + \frac{3}{40}(x_1^2 + x_2^2)x_1 - \frac{14867}{68000}(x_1^2 + x_2^2)^2 x_1 - \frac{26912070343}{103873536000}(x_1^2 + x_2^2)^3 x_1$$
$$- \frac{8114283157415584063}{19110912837120000000}(x_1^2 + x_2^2)^4 x_1 + \cdots$$
$$- \frac{7}{12}(x_1^2 + x_2^2)x_2 + \frac{8093503}{14688000}(x_1^2 + x_2^2)^2 x_2$$
$$- \frac{1887495055097}{3895257600000}(x_1^2 + x_1^2)^3 x_2$$
$$- \frac{765399918373406351207}{1375985724272640000000}(x_1^2 + x_2^2)^4 x_2 + \ldots$$
$$\dot{x}_2 = -x_2 + \frac{3}{40}(x_1^2 + x_2^2)x_2 - \frac{14867}{68000}(x_1^2 + x_2^2)x_2$$
$$- \frac{26912070343}{1038736000}(x_1^2 + x_2^2)^3 x_2$$
$$- \frac{811428315584063}{1911091283720000000}(x_1^2 + x_2^2)^4 x_2 + \cdots$$
$$\dot{R} = \frac{3}{40}R^3 - \frac{14867}{68000}R^5 ; \dot{\theta} = 1 - \frac{7}{12}R^2:$$

Since the coefficient of $R^3$is nonzero, this is Hopfbifurcation.

Chapter 5

Normal Form theory in local bifurcation

Section (5.1): Introduction:

The theory of normal forms and bifurcations of nonlinear difference equations is well known [17], [21], [25], [29], it is as follows. Consider two smooth (C^4) n-dimensional difference equations with equilibrium points

$$x^+ = f(x) \tag{1}$$
$$0 = f(0)$$

and

$$z^+ = g(z) \tag{2}$$

$0 = g(0)$ where $x^+(t) = x(t+1)$. These are locally diffeomorphic if there exists a local diffeomorphism

$$z = \emptyset(x) \tag{3}$$
$$0 = \emptyset(0)$$

which carries (1) to (2),

$$g(\emptyset(x)) = \emptyset(f(x)).$$

Such a local diffeomorphism carries trajectories x(t) in its domain onto trajectories z(t) in its range,

$$\mathrm{z(t)} = \emptyset(\mathrm{x(t)});$$

hence the two dynamics are locally smoothly equivalent. There is a weaker notion of equivalence; (1) is locally topologically conjugate to (2) if there is a local homeomorphism (3) which carries trajectories x(s) in its domain onto trajectories z(t) in its range while preserving the orientation of time, but not the exact time. The linear approximation of (1) around the fixed point x = 0 is

$$\delta x^+ = \frac{\partial f}{\partial x}(0)\delta x \tag{4}$$

and this is a hyperbolic fixed point if $\frac{\delta f}{\delta x}(0)$ has no eigenvalues on the unit circle. Thediscrete time Grobman–Hartman theorem states that if the equilibrium x = 0 of (1) is hyperbolic, then it is locally topologically conjugate to its linear approximation

(4). A related theorem is that two hyperbolic equilibria are locally topologically conjugate if their linear approximations have the same number of eigenvalues strictly inside the unit circle, the signs of their products are the same, and the same number of eigenvalues strictly outside and the signs of their products are the same [20]. A parameterized system

$$x^+ = f(x, \mu) \tag{5}$$

can have a locus of equilibria

$$x_e = f(x_e, \mu_e).$$

It undergoes a local bifurcation at an equilibrium xe, μ_e that is not locally topologically conjugate to every nearby equilibrium. In light of the above, such a bifurcation can happen only if one or more eigenvalues of the linearized system cross the unit circle, or the sign of the product of the strictly stable eigenvalues changes, or the sign of the product of the strictly unstable eigenvalues changes. A standard approach to analyzing the behavior of the parameterized system (5) around a bifurcation point is to add the parameter as an additional state with trivial dynamics [25].

$$\mu^+ = \mu \tag{6}$$

The next step is to compute the Poincar´e normal form of the center manifold dynamics. This is a normal form under smooth changes of coordinates

$$z = \emptyset x) = Tx - \emptyset^{[2]}](x) - \emptyset^{[3]}(x) - \cdots, \tag{7}$$

where $\emptyset^{[d]}(x)$ denotes a vector field that is a homogeneous polynomial of degree $d$in x. The linear part of the change of coordinates T puts the linear part of the center manifold dynamics in Jordan form. The quadratic, cubic, and higher parts of the change of coordinates $\emptyset^{[2]}$and $\emptyset^{[3]}$ simplify the quadratic, cubic, and higher parts of the center manifold dynamics by putting them in Poincare normal form. From its normal form the bifurcation is recognized and understood. Examples are the fold (or saddle-node), the flip, and the Neimark–Sacker bifurcations. The first depends on thenormal form of degree two, and the last two depend on the normal form of degree three. These are the only ones that are generic and of codimension 1, i.e., depend on a single parameter, so these are the most important.

Kang and Krener [22] developed a quadratic normal form for continuous time nonlinear systems whose linear part is controllable. This was extended to discrete time systems by Barbot, Monaco, and Normand-Cyrot[18]. These authors considered a larger group of transformations to bring the system to normal form, including invertible state feedback as well as change of state coordinates. Kang [23], [24] also developed a quadratic normal form for continuous time nonlinear systems whose linear part may have uncontrollable modes. Krener, Kang, and Chang [26], [20] described the quadratic and cubic normal forms of continuous time nonlinear control systems and also their bifurcations. we will develop quadratic and cubic normal forms for discrete time nonlinear control systems of the form

$$x^+ = f(x,u) = Ax + Bu + f^{[2]}(x,u) + f^{[3]}(x,u) + O(x,u)^4, \tag{8}$$

where x, u are of dimensions n, 1 and $f^{[d]}(x,u)$ denotes a vector field that is a homogeneous polynomial of degree d in x, u. We do not assume that the linear part of the system is controllable. Moreover, our linear and quadratic normal forms differ from that of [18] for linearly controllable systems.

We also describe some of the simplest bifurcations of discrete time nonlinear control systems. A control system does not need a parameter to bifurcate; the control can play the same role. The equilibria of a controlled difference equation,

$$x^+ = f(x,u), \tag{9}$$

are those values of x_e, u_e such that $f(x_e, u_e) = x_e$. The equilibria are conveniently parameterized by u or one of the state variables. Two key facts differentiate bifurcationsof a control system (5.1.8) from that of a parameterized system (5.1.5). The first is thatfor the latter the structural stability of the equilibria is the crucial issue, but for the former the stabilizability by state feedback is the crucial issue. A control system (5.1.8) is linearly controllable (linearly stabilizable) at x_e, u_e if the local linear approximation

$$\delta x^+ = \frac{\partial f}{\partial x}(x_e, u_e)\delta x$$

is controllable (stabilizable). If the linear approximation is stabilizable, then the nonlinear system is locally stabilizable. If the linear approximation is not stabilizable, then the nonlinear system may or may not be locally stabilizable, depending on higher degree terms. A control bifurcation of (8) takes place at an equilibrium where the linear approximation loses stabilizability. Notice that this is different from the bifurcation of a parameterized system (5), which takes place at an equilibrium where thereis a loss of structural stability with respect to parameter variations. To emphasize this distinction, we shall refer to the latter as a classical bifurcation.

The other difference between control and classical bifurcations is that when bringing the control system into normal form, a different group of transformations is used.

For classical bifurcations, we use parameter dependent change of state coordinates and change of parameter coordinates, but for control bifurcations we use change of state coordinates and state dependent change of control coordinates (invertible state feedback) to simplify the dynamics.

Section (5.2): Quadratic normal form:

Consider a smooth (C^3) system of the form (5.8)under the action of linear and quadratic change of state coordinates and state feedback

$$z = \emptyset(x) = Tx - \emptyset^{[2]}(x), \tag{5.2.1}$$

$$v = \propto (x,u) = Kx + Lu - \propto^{[2]} (x,u), \tag{5.2.2}$$

where T, L are invertible. It is well known that there exist a linear change of coordinates T and a linear feedback K, L that transform the system into the linear normal form

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} o \\ B_2 \end{bmatrix} u + \begin{bmatrix} f_1^{[2]}(x_1,x_2,u) \\ f_2^2(x_1,x_2,u) \end{bmatrix} + O(x_1,x_2,u)^3 \tag{5.2.3}$$

where x_1, x_1 are n_1, n_2dimensional, $n_1 + n_2 = n$, A_1 is in Jordan form, and $A_2, B_2$are in controller (Brunovsky) form:

$$A_2 = \begin{bmatrix} 0\,1 & \ddots & 0 \\ \ddots\ \ddots & \ddots & \ddots \\ 0\,0 & \ddots & 1 \end{bmatrix} B_2 = \begin{bmatrix} 0 \\ \ddots \\ 1 \end{bmatrix}$$

The following result generalizes [27].

Theorem 5.2.1 Consider the system (5.2.3), where $A_1$is diagonal and $A_2, B_2$are in Brunovsky form. There exist a quadratic change of coordinates and a quadratic

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1^{[2]}(x_1, x_2) \\ x_2^{[2]}(x_1, x_2) \end{bmatrix}$$

$$v = u - \propto^2 (x_1, x_2, u)$$

which transform the system (5.2.3) into the quadratic normal form

$$\begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v$$

$$+ \begin{bmatrix} \tilde{f}_1^{[2,0]}(z_1, z_2, v) + \tilde{f}_1^{[1,1]}(z_1, z_2, v) + \tilde{f}_1^{[0,2]}(z_1, z_2, v) \\ 0 \qquad\qquad + 0 \qquad\qquad + \tilde{f}_2^{[0,2]}(z_1, z_2, v) \end{bmatrix} + O(z_1, z_2, v)^3 \quad (5.2.4)$$

Where $\tilde{f}_i^{[d_1,d_2]}(z_1, z_2, v)$ is a polynomial vector field homogeneous of degree d_1 inz_1 and homogeneous of degreed_2 inz_2, v. For notational convenience, we define $z_2, n_2 + 1 = v$.

The vector field $\tilde{f}_1^{[2,0]}$is in the quadratic normal form of Poincare,

$$\tilde{f}_1^{[2,0]} = \sum_{\lambda_i = \lambda_j \lambda_k} \beta_i^{jk} e_1^i z_1, j z_2, k \quad (5.2.5)$$

where $\lambda_1, \ldots, \lambda_n 1$ are the eigenvalues ofA_1, e_r^i

is the ith unit vector in$z_r$space, and $z_{r,i}$is the ith component of z_r. The other vector fields are as follows:

$$\tilde{f}_1^{[1,1]} = \sum_{\lambda_i = 0} \sum_{\lambda_j} \sum_{k=1}^{n_2 + 1} \gamma_i^{jk} e_1^i z_{1,j} z_{2,k}$$

$$+ \sum_{\lambda_{i \neq 0}} \sum_{\lambda_j \neq 0} \gamma_i^{j1} e_1^i z_1 z_{2,1} \quad (5.2.6)$$

$$\tilde{f}_1^{[0,2]} = \sum_{\lambda_{i \neq 0}} \sum_{\lambda_{j \neq 0}}^{n_2 + 1} \delta_i^{1k} e_1^i z_{2,1} z_{2,k} \quad (5.2.7)$$

$$\tilde{f}_2^{[0,2]} = \sum_{i=1}^{n_2-1} \sum_{k=i+2}^{n_2+1} \in_i^{1k} e_2^i z_{2,1} z_{2,k} \qquad (5.2.8)$$

The normal form is unique; that is, each system (5.2.4) can be transformed into only one such normal form (5.2.5) to (5.2.8) by a quadratic change of coordinates (5.2.2) and quadratic feedback (5.2.3). This follows from the fact that the numbers in the above, β_i^{jk} , λ_i^{jk} , δ_i^{1k} , ϵ_i^{1k} for the indicated indices, are moduli, i.e., continuous invariants ofthe system (5.2.4) under a quadratic change of coordinates and quadratic feedback.

Section (5.3): Cubic normal form:

We present the cubic normal form of a system that is already in linear and quadratic normal form.

Theorem 5.3.1 Consider a smooth (C^4) system (5.2.5)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u$$

$$+ \begin{bmatrix} f_1^{[2,0]}(x_1, x_2, u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1,1]}(x_1, x_2, u) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[0,2]}(x_1, x_2, u) \\ f_2^{[0,2]}(x_1, x_2, u) \end{bmatrix}$$

$$+ \begin{bmatrix} f_1^{[3]}(x_1, x_2, u) \\ f_2^{[3]}(x_1, x_2, u) \end{bmatrix} + O(x_1, x_2, u)^4$$

where A_1 is diagonal,A_2,B_2 are in Brunovsky form, and the quadratic terms are in the normal form of Theorem 5.2.1 There exist a cubic change of coordinates and a cubic feedback

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \emptyset_1^{[3]}(x_1, x_2) \\ \emptyset_2^{[3]}(x_1, x_2) \end{bmatrix}$$

$$v = u - \propto^{[3]} (x_1 , x_2 , u)$$

which transform the system (5.2.5) into the cubic normal form

$$\begin{bmatrix} Z_1^+ \\ Z_2^+ \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} v$$

$$+ \begin{bmatrix} f_1^{[2,0]}(z_1 , z_2 , v) \\ 0 \end{bmatrix} + \begin{bmatrix} f_1^{[1,1]}(z_1 , z_2 , v) \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} f_1^{[0,1]}(z_1, z_2, v) \\ f_2^{[0,2]}(z_1, z_2, v) \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[3.0]}(z_1, z_2, v) \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} \tilde{f}_1^{[2,1]}(z_1, z_2, v) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1^{[1,2]}(z_1, z_2, v) \\ \tilde{f}_2^{[1,2]}(z_1, z_2, v) \end{bmatrix}$$

$$+\begin{bmatrix} \tilde{f}_1^{[0,3]}(z_1, z_2, v) \\ \tilde{f}_2^{[0,3]}(z_1, z_2, v) \end{bmatrix} + O(z_2, z_2\, v)^4$$

The vector field $\check{f}_1^{[3,0]}$is in the cubic normal form of Poincar´e,(5.2.6)

$$\tilde{f}_1^{[3;0]} = \sum_{\lambda_i = \lambda_j \lambda_k \lambda_l} \beta_i^{jkl}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}\ z_{1,l},$$

and the other vector fields are as follows

$$\tilde{f}_1^{[2;1]} = \sum_{\lambda_i = 0}\ \sum_{\lambda_j \lambda_k = 0}\ \sum_{l=1}^{n_2+1} \gamma_i^{jkl}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}\ z_{2,l}$$

$$+ \sum_{\lambda_i \neq 0}\ \sum_{\lambda_j \lambda_k \neq 0} \gamma_i^{jk1}\ \mathbf{e}_1^i\ z_{1,j}\ z_{1,k}\ z_{2,1},$$

$$\tilde{f}_1^{[1;2]} = \sum_{\lambda_i = 0}\ \sum_{\lambda_j = 0}\ \sum_{k=1}^{n_2+1}\ \sum_{l=k}^{n_2+1} \delta_i^{jkl}\ \mathbf{e}_1^i\ z_{1,j}\ z_{2,k}\ z_{2,l}$$

$$+ \sum_{\lambda_i \neq 0}\ \sum_{\lambda_j \neq 0}\ \sum_{l=1}^{n_2+1} \delta_i^{j1l}\ \mathbf{e}_1^i\ z_{1,j}\ z_{2,1}\ z_{2,l},$$

$$\tilde{f}_1^{[0;3]} = \sum_{\lambda_i \neq 0}\ \sum_{k=1}^{n_2+1}\ \sum_{l=k}^{n_2+1} \epsilon_i^{1kl}\ \mathbf{e}_1^i\ z_{2,1}\ z_{2,k}\ z_{2,l},$$

The normal form is unique, that is, each system (5.3.1) can be transformed in to only one such normal form (5.3.2) to(5.3.8). This follows from the fact that the numbers in the aboveβ_i^{jkl} , λ_i^{jkl} , δ_i^{1kl} , $\in_i^{1kl}$, ξ_i^{j1l} , $\eta_i^{1kl}$for the indicated indices, are moduli of thesystem (5.2.3) under a cubic change of coordinates and cubic feedback. Let $j = 6$ if $j = k = l$ and $\sigma_{jkl} = \sigma_{jk}\sigma_{kl}\sigma_{jl}$ otherwise. These moduli are defined as follows:

$$\beta_i^{jkl} = \frac{1}{\sigma_{jkl}} \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}\partial x_{1,l}}(0,0,0)$$
$$for\ 1 \le i,\ j,k,l \le n_1,\ and\ \lambda_i = \lambda_j\lambda_k\lambda_l,$$
$$\gamma_i^{jkl} = \frac{1}{\sigma_{jk}} \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}\partial x_{2,l}}(0,0,0)$$

$$\gamma_i^{jk1} = \frac{1}{\sigma_{jk}} \sum_{r=0}^{n_2-k+1} \left(\frac{\lambda_i}{\lambda_j\lambda_k}\right)^r \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{1,k}\partial x_{2,r+1}}(0,0,0)$$
$$for\ 1 \le i \le n_1,\ 1 \le j \le k \le n_1,\ and\ \lambda_i\lambda_j\lambda_k \ne 0,$$

$$\delta_i^{jkl} = \frac{1}{\sigma_{kl}} \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{2,k}\partial x_{2,l}}(0,0,0)$$
$$for\ 1 \le i,\ j \le n_1,\ 1 \le k \le l \le n_2+1,\ and\ \lambda_i = \lambda_j = 0,$$
$$\delta_i^{j1l} = \frac{1}{\sigma_{1l}} \sum_{r=0}^{n_2-l+1} \left(\frac{\lambda_i}{\lambda_j}\right)^l \frac{\partial^3 f_{1,i}}{\partial x_{1,j}\partial x_{2,1+r}\partial x_{2,l+r}}(0,0,0)$$
$$for\ 1 \le i,\ j \le n_1,\ 1 \le k \le n_2+1,\ and\ \lambda_i\lambda_j\lambda_k \ne 0,$$
$$\epsilon_i^{1kl} = \frac{1}{\sigma_{1kl}} \sum_{r=0}^{n_2-l+1} \lambda_i^r \frac{\partial^3 f_{1,i}}{\partial x_{2,1+r}\partial x_{2,k+r}\partial x_{2,l+r}}(0,0,0)$$

$$for\ 1 \le i \le n_1,\ 1 \le k \le l,\ i+2 \le l \le n_2+1,\ and\ \lambda_i \ne 0,$$
$$\zeta_i^{j1l} = \frac{1}{\sigma_{1l}} \sum_{r=0}^{n_2-l+1} \lambda_j^{-r} \frac{\partial^3 f_{2,i+r}}{\partial x_{1,j}\partial x_{2,1+r}\partial x_{2,l+r}}(0,0,0)$$
$$for\ 1 \le i \le n_2-1,\ i+2 \le l \le n_2+1,\ and\ \lambda_j \ne 0,$$
$$\eta_i^{1kl} = \frac{1}{\sigma_{1kl}} \sum_{r=0}^{n_2-l+1} \frac{\partial^3 f_{2,i+r}}{\partial x_{2,1+r}\partial x_{2,k+r}\partial x_{2,l+r}}(0,0,0)$$
$$for\ 1 \le i \le n_2-1,\ 1 \le k \le l,\ and\ i+2 \le l \le n_2+1.$$

Remarks. Once again, if some of the eigenvalues of $A_1$are complex, then a linear complex change of coordinates is required to bring it to Jordan form. In this case, some of the coordinates of z_1 are complex conjugate pairs, and some of the coefficients in the normal form are complex. These complex coefficients occur in conjugate pairs so that the real dimension of the coefficient space of the normal form is unchanged. In the normal form of Poincar´e (5.2.6), the eigenvalues satisfying $\lambda_i = \lambda_j\lambda_k\lambda_l$are said to be in cubic resonance.

The basic idea of normal form theory consists of employing successive, near identity nonlinear transformations to eliminate the so-called non-resonant nonlinear terms, and the terms called resonant which cannot be eliminated are remained in normal forms.

For computing the normal form of systems with bifurcation (perturbation)parameters, one usually takes two steps. First, at a critical point (at which the dynamic system has a singularity) one sets the parameters to zero to obtain a so called "reduced" (or "simplified") system and then normal form theory is applied to this system to obtain the normal form. Having found the normal form of the reduced system, one adds "unfolding" terms to get a parametric normal form for bifurcation analysis. However, this way one usually does not know the relationship between the original system parameters and the unfolding.

Section (5.4): Hopf Bifurcation: [16]

Now suppose that system (5.1.1), which is rewritten below for convenience,

$$\dot{x} = f(x,\mu), x \in R^n, \mu \in R, f : R^{n+1} \to R^n,, \qquad (5.1.1)$$

has an equilibrium, given by $x = p(\mu)$. Suppose the Jacobian, $Df(\mu_0)$, of the system evaluated on the equilibrium at a critical point $\mu_0$has a simple pair of purely imaginary eigenvalues, $\pm i\omega(\omega > 0)$, and no other eigenvalues with zero real part. The implicit function theorem guarantees (since $Df(\mu_0)$is invertible) that for each μ near μ_0 there will be an equilibrium $p(\mu)$ near $p(\mu_0)$which varies smoothly with μ. Nonetheless, the dimensions of stable and unstable manifolds of $p(\mu)$ do change if the eigenvalues of $Df\ (p(\mu))$ cross the imaginary axis at μ_0.

This qualitative change in the local flow near $p(\mu)$ must be marked by some other local changes in the phase portraits not involving fixed points.A clue to what happens in the generic bifurcation problem involving an equilibriumwith purely imaginary eigenvalues can be gained from examining linear systems in which there is a change of this type. For example, consider the system $\dot{x} = \mu x - \omega y$,

$$\dot{y} = \omega x + \mu y, \tag{5.1.2}$$

whose solutions have the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\mu t} \begin{bmatrix} cos\ \omega t & -sin\ \omega t \\ sin\ \omega t & cos\ \omega t \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{5.1.3}$$

When $\mu < 0$, solutions spiral into the origin, and when $\mu > 0$, solutions spiral away from the origin. When $\mu = 0$, all solutions are periodic. Even in a one-parameter family of equations, it is highly special to find a parameter value at which there is a whole family of periodic orbits, but there is still a surface of periodic orbits which appears in the general problem.

The normal form theorem gives us the required information about how the generic problem differs from system (5.1.2). By smooth changes of coordinates, the Taylor series of degree 3 for the general problem can be brought to the following form

$$\begin{aligned} \dot{x} &= [\, d\mu + a(x^2 + y^2)]x - [\omega + c\mu + b(x^2 + y^2)]y, \\ \dot{y} &= [\, \omega + c\mu + b(x^2 + y^2)]x + [d\mu + a(x^2 + y^2)]y,) \end{aligned} \tag{5.1.4}$$

which is expressed in polar coordinates as

$$\begin{aligned} \dot{r} &= (d\mu + ar^2)r, \\ \dot{\theta} &= \omega + c\mu + br^2. \end{aligned} \tag{5.1.5}$$

Since the $\dot{r}$ equation in (5.5) separates from θ , we see that there are periodic orbits of (5.1.4) which are circles $r = const.$, obtained from the nonzero solutions of $\dot{r} = 0$in (5.1.5). If $a \neq 0$ and $d \neq 0$ these solutions lie along the parabola $\mu = -(\frac{a}{d})r^2$. This implies that the surface of periodic orbits has a quadratic tangency with its tangent plane $\mu = 0$in$R^2 \times R$.

In the following, we first introduce the Hopf bifurcation theorem and then discuss in detail the computation of normal forms associated with various singularities (including Hopf bifurcation)..[16]

Theorem 5.2.1 Suppose that the system $\dot{x} = f(x, \mu)$,

$x \in R^n$, $\mu \in R$, has an equilibrium (x_0, μ_0) at which the following properties are satisfied.

(H1) $D_x f(x_0, \mu_0)$ has a simple pair of purely imaginary eigenvalues and no other eigenvalues with zero real parts.

Then (H1) implies that there is a smooth curve of equilibria (x(μ), μ) with $x(\mu_0) = x_0$. The eigenvalues $\lambda(\mu)$, $\bar{\lambda}(\mu)$ of

$Dxf(x(\mu), \mu_0)$, which are imaginary at $\mu = \mu_0$, vary smoothly with μ. If, moreover,

(H2) $$\frac{d}{d\mu}(\mathrm{Re}\,\lambda(\mu))|_{\mu=\mu_0} = d \neq 0,$$

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) in $R^n \times R$ and a smooth system of coordinates (preserving the planes $\mu = \text{const.}$)

for which the Taylor expansion of degree 3 on the center manifold is given by (5.1.4). If $a \neq 0$, there is a surface of periodic solutions in the center manifold which has

quadratic tangency with the eigenspaces of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid

$$\mu = -\frac{a}{d}(x^2 + y^2). \tag{5.1.6}$$

If $a < 0$, then these periodic solutions are stable limit cycles, while if $a > 0$, the periodic solutions are repelling.[16]

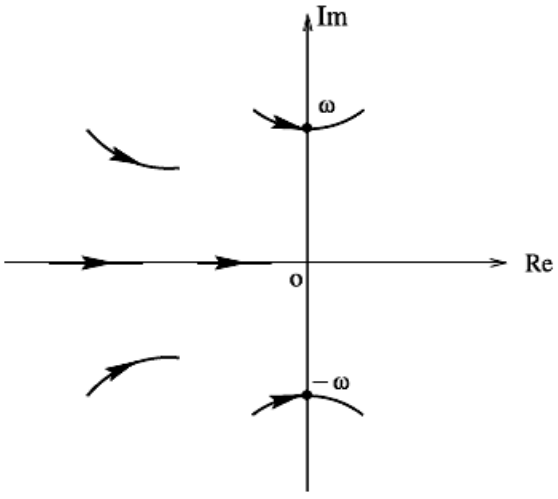


Fig 5.1 Transversality of Hopfbifurcation

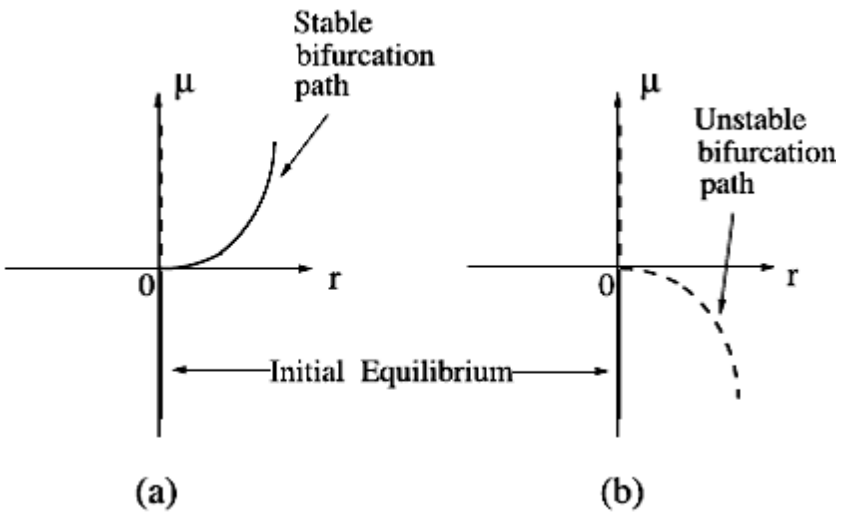


Fig 5.2 Post-critcal Bifurcation Path for A Hopfbifurcation

(a) Stable

(b) unstable

The transversality conditions given in (H1) and (H2) are illustratedin Fig. 5.1. The parameter–amplitude relation (5.1.6) is shown in Fig.5.2, where $r = \sqrt{x^2 + y^2}$, and the bifurcating periodic solutions depicted in three dimensional space are given [16] in Fig. 5.3.

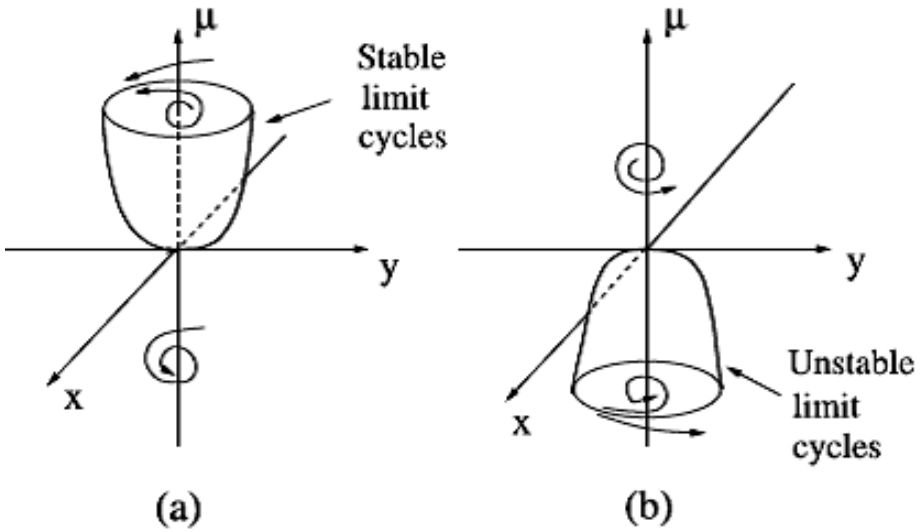


Fig 5.3 BirfurcatingPeriodic solutions (limit cycles)(a)stable(b)unstable

Chapter 6

Applications

Section (6.1): Introduction:

The numerical analysis of bifurcation problems is concerned with the stable, reliable and efficient computation of solutions to multi-parameter nonlinear problems. We shall consider numerical methods for solving nonlinear equations of the form

$$F(x, \lambda) = 0, \tag{6.1}$$

where F is a smooth operator in an appropriate Banach space setting, x is a state variable and λ represents one or more parameters. In applications the main interest is often the determination of qualitative changes in x as λ varies. Problems like (6.1) arise in the consideration of steady states of the dynamical system

$$\frac{dx}{dt} + F(x, \lambda) = 0, \tag{6.2}$$

and indeed the study of the solution set of (6.1) is usually the first step in an analysis of the behavior of solutions to (6.2).

The material in this review is applicable to a wide range of problems although we shall concentrate on problems arising in fluid dynamics, and so for us (6.2) represents the dynamical Navier–Stokes equations. The nonlinear character of the Navier–Stokes equations gives rise to multiple solutions and possibly complicated dynamics and this nonlinear behavior is central to problems in fluid dynamics, where the idea of dynamical similarity introduces various non-dimensional groups. In fluid mechanics we are therefore confronted with nonlinear partial differential equations that depend on a number of parameters. This is precisely the domain of bifurcation [15] theory. The overall goal, when studying a fluid mechanics problem, is to understand the complete behavior of the system as a 'function' of the parameters. Relevant questions are. How many steady states are there? Are they stable or unstable? (It is important to have theability to compute unstable steady states as well as stable ones, since solutions arising from bifurcations along unstable branches often interact with stable solutions producing otherwise inexplicable phenomena.) How does the structure of the steady

state solution set change as the parameters are varied? Do solutions always respect the symmetry of the domain or is there symmetry breaking? How do time-dependent solutions arise? We shall address some aspects of these questions in this study. Other very important questions about which we have nothing to say here include: How do the initial conditions affect the evolution of the system? What types of long-term dynamical behavior are possible? How does fluid turbulence arise? In fluid mechanics the nonlinearity of the governing equations combined with the nontrivial geometry of the domain means that there are many problems where limited progress can be made with analytical techniques and one needs to use numerical methods. There are two main numerical approaches to help answer some of the above questions for the Navier–Stokes equations. Either the time-dependent problem is discretized in space and the resulting system of ordinary equations is evolved forwards in time for various fixed values of the parameters. This approach is called 'simulation', and is the main technique used in the computational fluids community. The alternative approach is to discretize the steady problem to obtain a system of nonlinear equations, and then use methods from nonlinear analysis (e.g., the implicit function theorem, singularity theory) to compute paths of steady solutions and provide stability assignment using numerical continuation methods and eigenvalue information. We shall concentrate on the latter approach here. The numerical analysis of continuation methods was developed though many of the key ideas appear earlier in applications .Several codes were then [15] developed for numerical continuation and bifurcation analysis, the earliest being once reliable algorithms for numerical path following and simple bifurcation phenomena were devised then attention naturally shifted to multi-parameter problems and the construction of numerical approaches based on the use of singularity theory.

At the same time the convergence theory for discretization methods was concerned with the obvious questions: If a continuous problem has a particular singularity, under what conditions can it be guaranteed that the discretized problem has a singularity of the same type? Does the numerical method converge with the same rate

of convergence as at nonsingular points? Do we observe superconvergence when using projection methods [15]

One of the successes of numerical bifurcation techniques has been the ability to reproduce and help understand experimental results of the Taylor–Couette flow of a fluid confined between two concentric cylinders. Because this flow may be controlled quite precisely in the laboratory it provides anopportunity for rigorous experimental and numerical comparison. Of course the numerical techniques have been applied in a wide variety of other problems in fluid mechanics and have contributed significantly to the theoretical understanding of confined flows. The detailed plan of this study is as follows. some of the main ideas in singularity theory are outlined first for scalar equations, thenfor multi-parameter problems and problems with a simple reflection symmetry.

Section (6.2): Singularity theory: [15]

Gives a comprehensive account of numerical methods for bifurcation problems using singularity theory and minimally extended systems with bordered systems playing a key role in the linear algebra. There are many different aspects to singularity theory for bifurcation problems and we cannot hope to cover them all in this study, rather we concentrate on a few ideas to help motivate the material in later sections. However, we believe that a good understanding of the concepts and techniques is essential in order to develop reliable numerical techniques for multipara meter nonlinear problems.

The Lyapunov–Schmidt reduction procedure is a process by which information about solutions near a singular point of a nonlinear problem defined on a Banach space may be obtained by studying an equivalent reduced problem on a space of, typically, very small dimension. In fact,if the singularity is such that the linearization of the problem evaluated at the singularity has a one-dimensional kernel, then the reduced problem isone-dimensional. Thus, it is appropriate to study nonlinear scalar problemsof the form

$$f(x, \lambda, \propto) = 0, f : R \times R \times R^p \to R, \qquad (6.2.1)$$

where x is a scalar state variable, λ a distinguished parameter, and $\alpha \in \mathrm{R}^{\mathrm{p}}$a vector of control parameters. It is important to note that the view taken in the singularity theory

is that in applications one will wish to plot the state variable x against the special parameter λ for several fixed values of α. Thus we do not interchange λ with one of the $\propto_s$ and λ plays a different role than the other 'control' parameters. This approach leads to a different classification of singularities than that obtained from standard singularity theory.

Section (6.3): Scalar problems: [15]

In this section we consider the numerical calculation of singular points of the scalar problem

$$f(x,\lambda) = 0, x \in R, \lambda \in R, \qquad (6.3.1)$$

where $f(x,\lambda)$ is sufficiently smooth. Analysis of this very simple case introduces some important ideas and provides considerable insight into the behavior of more complicated equations. First, note that it is convenient to write f^0 for $f(x_0,\lambda_0), f_\lambda^0 \; for \; f_\lambda(x_0,\lambda_0)$, etc. Now, if $f^0 = 0$and$f_x^0 \neq 0$existence of a smooth path, $x(\lambda)$, near (x_0,λ_0)satisfying $f(x(\lambda),\lambda) = 0$. In this case we call (x_0,λ_0) a regular point. Of more interest are singular points where $f_x^0 = 0$. Consider the calculation of a singular point of (6.2.2). It is natural to form the system

$$F(y) = \begin{bmatrix} f(x,\lambda) \\ fx(x,\lambda) \end{bmatrix} = 0 \in R^2, y = \begin{pmatrix} x \\ \lambda \end{pmatrix} \qquad (6.3.2)$$

and seek a zero of $F(y)$. A solution y_0 is regular if $Fy(y_0)$ is nonsingular, which, as is easily checked, holds provided $f_\lambda^0 f_{xx}^0 \neq 0$, or, equivalently,$f_\lambda^0 \neq 0$ and $f_{xx}^0 \neq 0. (6.2.4)$

If (6.2.3) and (6.2.4) hold then (x_0,λ_0) is a quadratic fold point. The reasonfor the name is clear when one sketches the solution curve near (x_0,λ_0), noting that near (x_0,λ_0), $\lambda = \lambda(x)$ with $\lambda(x_0) = \lambda_0$, and $\frac{dy}{dx}(x_0) = 0 \quad \frac{dy^2}{dx^2}(x_0) = -\frac{f_{xx}^0}{f_\lambda^0}$(6.3.4)

We call (6.2.3) an extended system, and (6.2.4) provides two side constraints. Together, (6.3.2) and (6.3.3) provide the defining conditions for a quadratic fold point.

Quadratic fold points have several nice properties. First, Newton's method applied to (6.2.3) will converge quadratic ally for a sufficiently accurate initial guess. Second, a sensitivity analysis shows they are stable under perturbation. Assume $f(x,\lambda)$

is perturbed to $\hat{f}(x,\lambda,\varepsilon) = f(x,\lambda) + \varepsilon p(x,\lambda)$ and consider $\hat{F}(y,\varepsilon) = (f + \varepsilon p, fx + \varepsilon px) = 0$. Now $\hat{F}(y_0,0) = 0$ and $\hat{F}_y(y_0,0)$ is nonsingular and so it shows that $y = y(\varepsilon)$ near $\varepsilon = 0$, with $y(\varepsilon) = y_0 + O(\varepsilon)$, and $\hat{F}y(y(\varepsilon),\varepsilon$ nonsingular. Hence the perturbed problem $\hat{f}(x,\lambda,\varepsilon) = 0$ has a quadratic fold point $(x(\varepsilon),\lambda(\varepsilon))$ satisfying

$$x(\varepsilon) = x_0 + O(\varepsilon), \quad \lambda(\varepsilon) = \lambda_0 + O(\varepsilon).$$

This type of sensitivity analysis is common in structural mechanics where the various physical imperfections in a system are 'lumped together' as asingle artificial parameter. One might also consider $\varepsilon = hm$ where $\hat{f}$ is a discretization of f, h is a stepsize and m is the order of consistency. Clearly quadratic folds in f are preserved in $\hat{f}$ and it is not surprising that a similarresult holds for more general problems under certain assumptions.

Multi-parameter problems 6.2.2 [15]: Let us change perspective now, and think of ε in the previous section as acontrol parameter to be varied rather than merely a perturbation parameter. The above analysis still applies, and provided $\hat{f}_\lambda(x(\varepsilon),\lambda(\varepsilon),\varepsilon) \neq 0$ and $\hat{f}_{xx}(x(\varepsilon),\lambda(\varepsilon), \varepsilon \neq 0$ there is no requirement that ε remain small. Thus, we change notation by setting $\varepsilon = \alpha$, and dropping the '^' symbol over the f, and consider the two-parameter problem

$$f(x,\lambda,\alpha) = 0, \qquad x,\lambda,\alpha \in R. \tag{6.2.6}$$

Provided the side constraints $f_\lambda \neq 0$ and $f_{xx} \neq 0$ continue to hold, then a path of quadratic fold points can be computed using Newton's method applied to

$$F(y,\alpha) = \begin{bmatrix} f(x,\lambda,\alpha) \\ f_x(x,\lambda,\alpha) \end{bmatrix} = 0, \qquad y = \begin{pmatrix} x \\ \lambda \end{pmatrix}. \tag{6.2.7}$$

Since the side constraints appear in F_y, they can be easily monitored. If a zero occurs in a side constraint then a higher-order singularity has beendetected.

Possible types of behavior of solutions of (6.2.1) near a singular point are classified according to contact equivalence, namely, equivalence up to a smooth change of coordinates. This classification associates a number, the codimension, with each singularity, and if the codimension is finite then the singularity is equivalent to a polynomial canonical form. For example, the simplest singularity is the quadratic fold

point, which has canonical form $f(x,\lambda) = x^2 - \lambda$and has codimension zero. Clearly at $y_0 = (x_0,\lambda_0)^T = (0,0)^T$ then (6.2.3) and (6.2.4) are satisfied , conversely any f satisfying (6.2.3) and (6.2.4) is contact equivalent to $x^2 - \lambda$. In Jepson the singularities of codimension less than 4 are arranged in a hierarchy and this was used to provide an algorithm to obtain suitable extended systems and side constraints for the calculation of the singularities. For example, there are two codimension 1 singularities: a transcritical bifurcation ($\alpha = 0$in Figure 6.1below) that arises in a path of fold points when $f_\lambda = 0$; and a hysteresis bifurcation ($\alpha = 0$ in Figure 6. 2) below, that arises in a path of fold points when $f_{xx} = 0$. To compute a transcritical bifurcation in a stable manner we need 2 parameters, namely λ and α, and the extended system is $F(y) = (f, f_x, f_\lambda)^T = 0$, $y = (x,\lambda,\alpha)^T$. A transcritical bifurcation point, $y_0 = (x_0,\lambda_0,\alpha_0)^T$say, will be a regular solution if (a) $f_\alpha^0 \neq 0$, and (b)the side constraints $f_{xx}^0 \neq 0$ and $(f_{xy}^0)^2 - f_{xx}^0 f_{\lambda\lambda}^0 \neq 0$ hold. The canonicalform is $f(x,\lambda) = x^2 - \lambda^2$. The condition $f_\alpha^0 \neq 0$ is a universal unfoldingcondition that, roughly speaking, ensures that the control parameter α entersin f in such a way as to provide all qualitatively distinct solutions of$f(x,\lambda,\alpha) = 0$ as α varies near α_0. The transcritical bifurcation has codimension1, since 1 control parameter is needed in the universal unfolding$f(x,\lambda,\alpha) = 0$. Figure 6.1 shows the unfoldingof a trans critical bifurcation, and Figure 6.2 shows the unfolding of a hysteresis point (also of codimension1) which has extended system $F(y) = (f, f_x, f_{xx})^T = 0$ and side constraints $f_\lambda \neq 0, f_{xxx} \neq 0$. It is important to note that one would not expect to see the codimension 1 singularities, that is, trans-critical or hysteresis bifurcation points, in a oneparameter physical problem. Rather, two parameters are needed to observe them and to locate them numerically. Also, as we see in Figures 6. 1 and 6. 2, they are destroyed by perturbations. It is not surprising, then, that the convergence theory of discretizations near bifurcation points in one-parameter ,problems proves very technical and is perhaps of limited usefulness. [15]

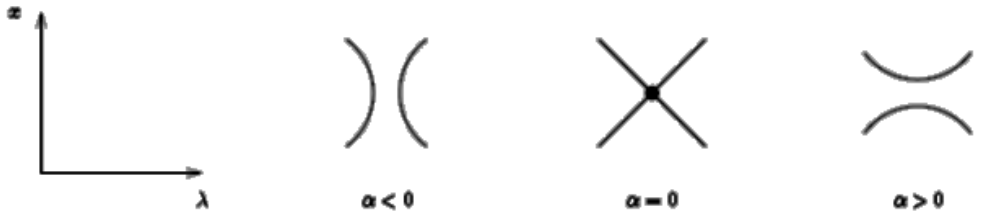


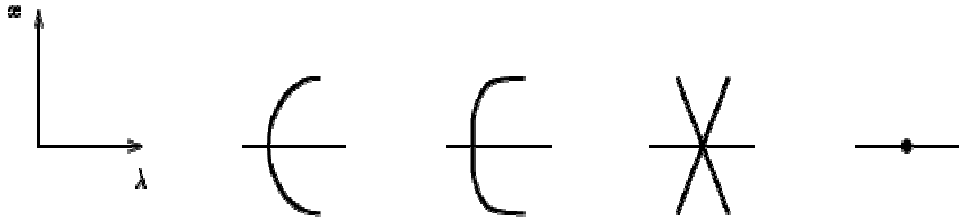
Figure 6.1 Solution diagrams for $\mathbf{f(x,\lambda,\alpha) = x^2 - \lambda^2 + \alpha = 0}$. The transcritical bifurcation point is destroyed for $\alpha \neq 0$

Fig.6. 2. Solution diagrams for$f(x,\lambda,\alpha) = x^3 + \alpha x - \lambda = 0$. The hysteresis point isdestroyed for $\propto = 0$ and there are no singular points for $\alpha > 0$

Problems with reflection symmetry (6.2.3) [15]: A classification of singularities satisfying various symmetries can also be given. We content ourselves here with a few remarks about the simpleZ_2 (i.e., reflection) symmetry. If$f(x,\lambda)$ satisfies the equivariance (symmetry) condition

$$f(-x,\lambda) = -f(x,\lambda), \qquad (6.2.8)$$

then a classification of singularities arises that reflects the symmetry in the problem and is different from that for problems with no symmetry. First note that if (6.2.8) is satisfied then $f(x,\lambda)$is odd in x and so we may write $f(x,\lambda) = xa(x^2,\lambda)$ for some functions$a(x^2,\lambda)$. Also, if (x,λ) satisfies$f(x,\lambda) = 0$ then so does $(-x,\lambda)$. Thus the solution diagrams are symmetric about the λaxis: see Figure 6.3. The simplest singularity (i.e., codimension 0) has the canonical form $f(x,\lambda) = x^3 - \lambda x = x(x^2 - \lambda)$ and gives rise to the common symmetric pitchfork bifurcation diagram (see Figure 6.3(a)). The singularities given in Figure 6.3(b), (c) and (d) have codimension 1 and typically will only be observed in a two-parameter setting .



(a) (b) (c) (d)

Fig. 6.3. Canonical solution diagrams for Z-symmetric singularities of$\text{codimension} \leq 1$: (a)$\text{f}(\text{x},\lambda) = \text{x}^3 - \lambda\text{x} = 0$, (b) $\text{f}(\text{x},\lambda) = \text{x}^5 - \lambda\text{x} = 0$,a quadratic symmetry breaking bifurcation, (c) $\text{f}(\text{x},\lambda) = \text{x}^3 - \lambda^2\text{x} = 0$,a C–coalescence point, and (d) $\text{F}(\text{x},\lambda) = \text{x}^3 + \lambda^2\text{x} = 0$, a $\text{C}+$ coalescencepoint. Unfolding's of (b), (c) and (d) are given on Golubitsky and Schaeffer (1985) theory for Hopf bifurcation is intimately connected to that for Z_2-symmetric problems. In particular, small amplitude periodic orbits of an autonomous system of ODEs are in one-to-one correspondence with zeros of a nonlinear problem that satisfies the Z-equivariance condition (6.2.8). The simplest Hopf bifurcation corresponds to a codimension0 singularity and hence is likely to be observed in one-parameter problems.

Non-linear model[10]

In the field of mechanical engineering, the non-linear dynamical system defined in Figure (6.4) is a classic example of friction-induced vibrations in a brake system. It presents the grabbing vibration in heavy trucks that results from coupling between the normal mode (k_1, m_1) of the brake control and the torsion mode of the front axle (k_2, m_2). In order to simulate a braking system placed crosswise due to overhanging caused by a static force effect, we consider the moving belt slopes with an angle θ This slope couples the normal and tangential degree-of-freedom induced only by the friction coefficient μ that is assumed to be constant. The braking force Fbrake transits through the braking command, that has non-linear behavior. Therefore, we consider the possibility of a non-linear contribution. This non-linearity is applied in order to indicate the influence and the importance of non-linear terms in under-standing the dynamic behavior of systems with non-linear phenomena, the prediction of dangerous or

favorable conditions, and the exploitation of the full capability of structures by using systems in the non-linear range. In this study, the nonlinear behavior dynamic of the brake command of the system (k_1, m_1), and the non-linear behavior dynamic of the front axle assembly and the suspension (k_2, m_2) are concerned, respectively. These non-linearities are defined as non-linear stiffness's. The non-linear behavior is then expressed as a quadratic and cubic polynomial in the relative

displacement:

$$k_1 = k_{11} + k_{12}(Y - y) + k_{13}(Y - y)^2$$
$$k_2 = k_{21} + k_{22}X + k_{23}X^2 \qquad (6.2.9)$$

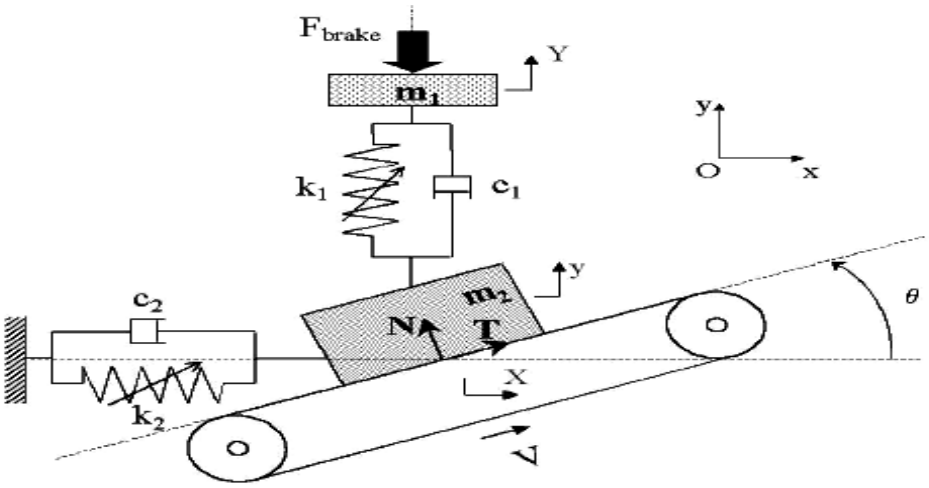


Figure 6. 4: Non-linear model of the braking system

With reference to Figure 4, and considering the non-linear expression of the stiffnesses defined in equations (6.2.9), the three equations of motion can be expressed as

$$\begin{cases} m_1\ddot{Y} + c_1(\dot{Y} - \dot{y}) + k_{11}(Y - y) + k_{12}(Y - y)^2 + k_{13}(Y - y)^3 = -Fbrake \\ m_2\ddot{X} + c_2\dot{X} + k_{21}X + k_{22}X^2 + k_{23}X^3 = -N\sin\theta + T\cos\theta \\ m_2\ddot{y} + c_1(\dot{y} - \dot{Y}) + k_{11}(y - Y) + k_{12}(y - Y)^2 + k_{13}(y - Y)^3 \\ = N\cos\theta + T\sin\theta \end{cases} \qquad (6.2.10)$$

Considering Coulomb's friction law $T = \mu N$, and the transformations $y = X\tan\theta$ and $x = \{X \;\; Y\}^T$, the nonlinear 2-degrees-of-freedom system is given by

$$M\ddot{x} + C\dot{x} + Kx = F + FNL(x) \; (\qquad 6.2.11)$$

where $\ddot{x}$,$\dot{x}$ and x define the acceleration, velocity, and displacement response 2-dimensional vectors of the degreesof- freedom, respectively. M, C and K are the mass, damping and stiffness matrices of the mechanical system. Fdefines the vector due to the brake force, and $FNL(x)$ is the vector containing all the non-linear terms of the system (6.2.10). By considering the equations of (6.2.10) the expressions of all the matrices and vectors are

$$M = \begin{bmatrix} m_2(tan^2\theta + 1) & 0 \\ 0 & m_1 \end{bmatrix} \quad (6.2.12)$$

$$\mathrm{C} = \begin{bmatrix} \mathrm{c}_1(\tan^2\theta - \mu\tan) + \mathrm{c}_2(1 + \mu\tan\theta) & \mathrm{c}_1(-\tan\_ + \mu) \\ -\mathrm{c}_1\tan\theta & \mathrm{c}1 \end{bmatrix} \quad (6.2.13)$$

$$K = \begin{bmatrix} k_{21}(1 + \mu\tan\theta) + k_{11}(tan^2\theta - \mu tan\theta) & k_{11}(-tan\theta + \mu) \\ -k_{11}\,tan\,\theta & k_{11} \end{bmatrix} \quad (6.2.14)$$

$$\mathrm{F}_{NL} = \begin{Bmatrix} (-\tan\theta + \mu)(\mathrm{k}_{12}(\mathrm{X}\tan\theta - \mathrm{Y})^2 + \mathrm{k}_{13}(\mathrm{X}\tan\theta - \mathrm{Y})^3) \\ +\mathrm{k}_{22}(1 + \mu\tan\theta)\mathrm{X}^2 + \mathrm{k}_{23}(1 + \mu\tan\theta)\mathrm{X}^3 \\ -\mathrm{k}_{12}(\mathrm{Y} - \mathrm{X}\,\mathrm{ta}\theta)^2 - \mathrm{k}_{13}(\mathrm{Y} - \mathrm{X}\tan\theta)^3 \end{Bmatrix} \quad (6.2.15)$$

$$\mathrm{F} = \begin{Bmatrix} 0 \\ -\mathrm{Fbrake} \end{Bmatrix} \quad (6.2.16)$$

The general form of the equation of motion for the non-linear system is given in the following way:

$$\mathrm{M}\ddot{\mathrm{x}} + \mathrm{C}\dot{\mathrm{x}} + \mathrm{Kx} = \mathrm{F} + \sum_{i=1}^{2}\sum_{j=1}^{2} f_2^{ij}\,\mathrm{x}_i\mathrm{x}_j + \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f_{(3)}^{ijk}\,\mathrm{x}_i\mathrm{x}_j\mathrm{X}_k \quad (6.2.16)$$

where $\mathrm{f}_{(2)}^{ij}$and $\mathrm{f}_{(3)}^{ijk}$) are the vectors of quadratic and cubic non-linear terms, respectively [10]

References

[1] Nigel Hitchin, Differentiable Geometry Course (hitchin@maths.ox.ac.uk) C3,2b 2010

[2] Patrick Bonckaet , Hasselt University , Agoralaan,gebouww D Central manifolds, normal forms B-3590 Diebenbeek, Belgium . September 5,2005

[3] H. K. Khalil. Nonlinear Systems. Prentice Hall, 3rd edition, 2002.

[4] Christopher ScheperCenter for Applied Mathematics Cornell University, May 15, 2007

[5] Gre'gory Faye11NeuroMathComp Laboratory, INRIA, Sophia Antipolis, CNRS, An introduction to bifurcation theory ENS Paris, FranceOctober 6, 2011

[6] G.A.K. van Voorn Elements of AppliedBifurcation Theory, Second Edition

[7] George.van.voorn introduction to bifurcation analysis Vrije Universite itde Boelelaan 1087, 1081 HV Amsterdam, the Nether land sgeorge. van. voorn@ falw. vu.nl June 19,20,21,22,26, 2006 PhD mini course:

[8] SHIV SUBRAMANIAM Date: August 21, 2009 INTEGRATION ON MANIFOLDS

[9] Henrik Schlichtkrull, Differentiable Manifolds for Geometry , University of Copenhagen

[10] ARNOLD V.I. Mathematical Methods of Classical Mechanics. Springer-Verlag, Berlin (1978). jean-jacques.sinou@ec-lyon.fr

[11] J.D Crawford.Rev.Mod.phys. Bifurcation theory 2001

[12] Nigel Hitchin, DIFFERENTIABLE MANIFOLDS Course C3.2b 2010

[13] S.-N. Chow, C. Li, and D. Wang. Normal forms and bifurcations of planar vector fields. Cambridge University Press, 1994

[14] F. Dumortier. Local study of planar vector fields: singularities and their unfoldings. In Structures in Dynamics, Mathematical Physics, Elsevier, 1991.

[15] K. A. Cliffe,S. J. Tavener , The Numerical analysis of bifurcation problems with application to fluid mechanicsActaNumerica (2008), pp. 39–131 c Cambridge University Press, 2000 E-mail: andrew.cliffe@aeat.co.uk

[16] M. Han, P. Yu ,Normal Forms, Melnikov Functions and Bifurcations of Limit Cycles, Applied Mathematical Sciences 181, DOI 10.1007/978-1-4471-2918-9_2, © Springer-Verlag London Limited 2012

[17] P. YU* and Y. YUAN, AN EFFICIENT METHOD FOR COMPUTING THE SIMPLEST NORMAL FORMS OF VECTOR FIELDS pyu@pyu1.apmaths.uwo.caReceived February 1, 2001; Revised November 14, 2001

[18] V. I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations, Springer-Verlag, New York, 1983

[19] J.-P. Barbot, S. Monaco, and D. Normand-Cyrot, Quadratic forms and feedback linearization in discrete time, Internat. J. Control, 67 (1997), pp. 567–586..

[20] J. Carr, Applications of the Centre Manifold Theory, Springer-Verlag, New York, 1981.

[21] D.-E. Chang, W. Kang, and A. J. Krener, Normal forms and bifurcations of control systems, Sydney, Australia, 2000, pp. 1602–1607.

[22] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.

[23] W. Kang and A. J. Krener, Extended Guadratic controller normal form and dynamic state ,linearization of nonlinear systems, SIAM J. Control Optim., 30 (1992), pp. 1319–1337.

[24] W. Kang, Bifurcation and normal form of nonlinear control systems, Part I, SIAM J. Control Optim., 36 (1998), pp. 193–212..

[25] Y. A. Kuznetsov, Elements of Applied Bifurcation Theory, Springer-Verlag, New York, 1998.

[26] A. J. Krener, W. Kang, and D.-E. Chang, Control bifurcations, IEEE Trans. Automat. Control, to appear.

[27] L. Li, Normal Forms of Controllable and Uncontrollable Discrete Applied Mathematics, University of California, Davis,CA, 1999.

[28] L. Li and A. J. Krener, Quadratic and cubic normal forms of discrete time nonlinear control systems, Brazil, 2000, pp. 19–25.

[29] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 1990.

[30] Mariana Haragus Gerald Looss, Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems ,Springer