بسم الله الرحمن الرحيم

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Department of mathematics

Integral Equation's

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In Mathematics

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بسم الله الرحمن الرحيم

قال تعالـــــــــــــــــــــــــــــــــــــــى :

(27) وَمِنَ النَّاسِ وَالدَّوَابِّ وَالأَنْعَامِ مُخْتَلِفٌ أَلْوَانُهُ

كَذَلِكَ إِنَّمَا يَخْشَى اللَّهَ مِنْ عِبَادِهِ الْعُلَمَاء

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صدق الله العظيم

سورة فاطـــــر

الأية (28)

الإهـــدآ :

إلي تلك الرائعة المتوسدة كل فؤادي ..... التـي أ رضعتني أول معاني الحياة ...... حباً ... ووداً دون كللٍ أ و ملل أو إنتظار مقابل إلا أ ن تراني أ ملاً مُفرحاً ... عِلماً وحُباً وصِدقاً

ترياق الحنايا فـي كل اللحظات

عُصارة قلبي ونبض حياتي

أ ميرتي الجميلة

...... أ مـي ......

إلي مُعلمي الأول...

فارسي المغوار ... الإنسان خُلقاً ... مُهدِ الليالي الحّيرني ... ثقةً وفخراً وعِذة

مُلبسي ثوب الكبرياء

أنيسُ روحي

... أبـــي ...

إلي تلك النُسيمات العزبة ...... ذات ذلك العقد الفريد ...

إخوتي ... أخواتي ...زُملائي ...

إلي مُنيري العِذ بقناديلهم ... شهدنا لهم منذ الحروف الأُول ...

أساتذتنا الأجلاّ

وكل من أعاننا ووقف معنا ...

دافعاً .. وصابراً .. ومشجعاً .. ولو بحرف ..

إلي أُ ســرة الكلية .....

نُهدي هذا البحث الذي نرجو أ ن يُفيد ....الباحثين.

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To those wonderful Proprietary all my heart.
That breastfed first the meaning of life
..... Love ... and friendlier...
Tirelessly or wait to see me versus only hope joyous note and love and honest
Nostalgia antidote in all moments
Squeezer my heart and pulse of my life
Beautiful princess
..... Mom .....
To the teachers of the first ...
Persian Commando ... human ethics ... threatened nights Puzzled me ...
confidence and pride and Honor
endue me dress Pride
Anis spiritual
... Dad...
To those Breezes Homestead ..... with this unique contract ...
My brothers ... sisters ... my colleagues
To honor the lights shining... we have seen them since the first letters
...
Our teachers Generous
anyone who helped us and stop us
Motive.And long-suffering. And encouraging

.. Even if the letter...
To the family of the college
we dedicate this research, which we hope will benefit ....
Researchers...

Dedication :

We would liketo express our sinceregratitude to thesupervisor ,and

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And our friend.

Abstract :

In this research, we studied a number of aspects of the integral equations. In chapter one we introduce the kernel an integral operator and symmetric integral transformations and also how to find eigenvalues in the integral operator.
In chapter tow we define differential operator and adjoint operator and their respective fields and also differential operator from second-order, and the symmetry faithfully to the ideals of some of the non-homogeneous problems and how to solve them. In chapter three we have some applications to eigenfunction and use a Green's function assigned to the processes to resolve its issues and also halt to clarify the representation of spectra and the Green's functions, and finally in chapter Four studied the classification and division of integral equations and the successive approximation methods for the solution processes and representation and equivalence with differential equations and we got a replacement Fredholm.

الخُلاصة :

في هذا البحث تطرقنا إلي دراسة عدد من الجوانب المتعلقة

بالمعادلات التكاملية.

ففي الباب الأول تعرفنا علي النواة كمؤثر تكاملي والتحويلات التكاملية المتماثلة وأيضاً كيفية إيجاد القيم الزاتية في المؤثرالتكاملي .

وفي الباب الثاني قمنا بتعريف المؤثر التفاضلي وتعرفنا علي المرافق لة ومجالِة وأيضاً المؤثرالتفاضلي من الدرجة الثانية ،والتماثل فية ومثلنا لبعض المسائل

غير المتجانسة وكيفية حلها .

وفي الباب الثالث قمنا ببعض التطبيقات علي الدوال الزاتية وإستخدام دالة قرين في عمليات حل مسائلها وأيضا عرجنا إلي توضيح تمثيل الأطياف

ودوال قرين ،

وأخيرا في الباب الرابع عمدنا إلي تصنيف وتقسيم المعادلات التكاملية والطرق التقريبية المتعاقبة لعمليات الحل وتماثلها وتكافؤها مع المعادلات التفاضلية وتعرفنا علي بديل فردهولم.

The contains :

Chapter (1)

Integral operators

In this chapter we shall be concerned with the study of linear transformations whose domain is an infinite-dimensional Euclidean vector space. In order to provide an introduction to the various concepts involved, we will restrict ourselves to one particular vector space and one particular linear transformation.

Section (1.1): The kernel of an integral operator

The vector space we shall consider will have as element function belonging to the set, R = R $(a,$ b),of all real functions which are Riemann integral on the finite interval $a \leq x \leq b$.

Definition (1.1.1):-

If f and g are in R then we define

$$(f + \mathrm{g})(x) = f(x) + \mathrm{g}(x)$$

$(c\, f)(x) = c\, f(x)$,real scalar c.

$$(f, \mathrm{g}) = \int_a^b f(x)\mathrm{g}(x)dx$$

Then it easily seen that R satisfies all the axioms of a Euclidean vector space , except possibly the positivity of the inner product. However , this difficulty can be overcome, as in the case of finite-dimensional spaces.

Definition (1.1.2):-

By introducing the relation of equivalence,≈,between vectors. We say that $f \approx \mathrm{g}$ if f and g differ only on a set of measure zero, or, which is equivalent:

$$f \approx \mathrm{g} \quad \text{if} \quad \int_a^b |f - \mathrm{g}|^2 dx = \mathbf{0}$$

Consequently, we have that $(f, f) = 0$ if and only if $f \approx 0$. In this way we obtain a Euclidean vector space with ≈ replacing =.

Definition (1.1.3):-

Let $K(x, y)$ be a real continuous function defined on the square $a \leq x \leq b, a \leq y \leq b$ .It induces a transformation K on R to R by means of the relation

$$Kf(x) = \mathrm{g}(x) = \int_a^b K(x, y) f(y) dy \quad (1.1)$$

The function $K(x, y)$*i*s called the kernel of integral transformation K.

Definition (1.1.4):-

The range of *K*is not all of R .In fact K carries every f into a continuous function. This can be seen as follow .Since $K(x, y)$ is continuous on bounded closed set, it is also uniformly continuous. Therefore, for any $\varepsilon > 0$ there is$\delta(x) > 0$ such that, for points x_1, x_2, y_1, y_2in $[a, b]$,

$$|K(x_1, y_1) - K(x_2, y_2)| < \varepsilon$$

When

$$|x_1 - x_2| < \delta(\varepsilon), |y_1 - y_2| < \delta(\varepsilon) \quad (1.2)$$

Hence,

$$|Kf(x) - Kf(t)| = \left| \int_a^b [K(x,y) - K(t,y)] f(y) dy \right|$$

$$\leq \int_a^b |K(x,y) - K(t,y)||f(y)| dy$$

$$\leq \varepsilon \int_a^b |f(y)| dy$$

Whenever $|x - t| < \delta(\varepsilon)$ and it follows that

$$\lim_{x \to t} Kf(x) = Kf(t)$$

From the continuity of Kf it follows that ifKf is equal to the zero vector then it is identically zero .Because, if $Kf(t) \neq 0$ then by continuity$Kf(x) \neq 0$ for all $x$in some interval containing t.Thus Kf is non-zero on a set which is not of measure zero.

Definition (1.1.5):-

Similarly we could consider the complex Euclidean vector space $\mathrm{R}^* = \mathrm{R}^*(a,b)$ of complex-valued integrable functions on $a \leq x \leq \mathrm{b}$ .The inner product in this case is given by

$$(f, \mathrm{g}) = \int_a^b f(x)\overline{\mathrm{g}(x)}\, dx. \quad (1.3)$$

Tow elements of R^* will be said to be equal if they differ at most on a set of measure zero.The complex-valued continuous function $K(x,y)$ defined on

$a \leq x \leq b$, $a \leq y \leq b$That induces by (1.1) .A linear transform k on R*to R*, the transformed function $Kf(x)$, being a complex continuous function.

Definition (1.1.6):-

In both R and R* we define the adjoint of k to be the integral transformation k*, given by

$$K^*f(x) = \int_a^b \overline{K(y,x)}\, f(y)dy \qquad (1.4)$$

From this definition it follows that

$$(Kf, \mathrm{g}) = \int_a^b \left\{ \int_a^b K(x,y)f(y) \right\} \overline{\mathrm{g}(x)}\, dx$$

$$= \int_a^b \left\{ \int_a^b K(x,y)\overline{\mathrm{g}(x)}dx \right\} f(y)d = (f, K^*\mathrm{g})$$

Hence we have that

$$(Kf, \mathrm{g}) = (f, K^*\mathrm{g}) \qquad (1.5)$$

The equation (1.5) determines the adjoint transformation K*uniquely, because if there is another transformation M such that $(\mathrm{K}f, \mathrm{g}) = (f, \mathrm{Mg})$ For all f and gin R*,then, if we subtract this result form (1.5), it follows that $(f, (\mathrm{R}^* - \mathrm{M})\mathrm{g}) = 0$ for every f and g in R.Therefore $\mathrm{R}^* = \mathrm{M}$ as required.

As may be anticipated, integral transformations, defined over an infinite-dimensional vector space, have many properties similar to those of the linear transformations defined on a finite dimensional vector space.

Theorem (1.1.7):-

$$\|K\| \leq \left(\int_a^b \int_a^b K|(x,y)|^2 dx\, dy \right)^{1/2}$$

Proof:-

If we choose f so that $\|f\| = 1$ Then

$$|Kf(x)|^2 = \left| \int_a^b K(x,y)\, f(y) dy \right|^2$$

$$\leq \int_a^b |K(x,y)|^2\, dy \int_a^b |f(y)|^2 dy = \int_a^b |K(x,y)|^2 dy$$

Therefore

$$\|Kf(x)\|^2 = \int_a^b |Kf(x)|^2 dx \leq \int_a^b \int_a^b |Kk(x,y)|^2\, dy dx$$

and we write

$$\|K\| = l.u.b.\{\|Kf(x)\| | \|f\| = 1\} \leq \left(\int_a^b \int_a^b K|(x,y)|^2 dx\, dy \right)^{1/2}.$$

The property of boundedness of the transformation is intimately connected with continuity. Recall for a moment some result for linear transformation.

Definition (1.1.8):-

The linear transformation T, form the vector space $V_1$to the vector space V_2 is called continuous at x in V_2if the sequence $\{Tx_n\}$ Converges to T$x$whenever the

sequence $\{x_n\}$ converges to x .It follows that an equivalent definition of continuity is given by the statement for every $\varepsilon > 0$,there is a$\delta(\varepsilon, x) > 0$ Such that

$\|Ty - Tx\| < \varepsilon$ Whenever $\|y - x\| < \delta(\varepsilon, x)$

The transformation T is bounded if and only if it is continuous for all x.Indeed, if T is bounded and $\{ x_n\}$ Converges to x we see that

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\|\|x_n - x\|$$

Since T is bounded .consequently, since $\{x_n\}$ Converges to $x$the right-hand side can be made arbitrarily small, and it follows that $\{Tx_n\}$Converges to Tx.

Conversely, if T is continuous at$x = 0$ then for $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $\|Tx\| < \varepsilon$whenever $\|x\| < \delta$, now if , $y \neq 0$ *i*s given element, write

$$z = \frac{\delta(1)}{2}\frac{y}{\|y\|^2}$$

and we have that

$$\|z\| = \frac{\delta(1)}{2} < \varepsilon(1)$$

and$\|Tz\| < 1$

Consequently, as we can write

$$\mathrm{y} = \frac{2}{\delta(1)}\|\mathrm{y}\|\mathrm{z},$$

It follow from the above inequality that

$$\|Ty\| \leq \frac{2}{\delta(1)}\|y\|$$

Which is true even when $y = 0$ .therefore T is bounded.

Definition (1.1.9):-

The integral transformations with which we are primarily concerned have an even stronger property than boundednes -that of complete countinuity

Definition (1.1.10):-

A linear transformation Ton V_1 toV_2 is completely continuous or compact if forevery sequence $\{Tx_n\}$.With uniformly bounded norms (that is $\{x_n\} < c$ for $c > 0$ and all$n$) there exists that $\{Tx_j\}$Converges to y .

Compactness implies boundedness for if T Were a compact but unbounded transformation there would exist a sequence $\{x_n\}$

Such that$\|Tx_n\| > n\|x_n\|$.Writing $y_n = x_n/\|x_n\|$we have$\|y_n\| \leq 1$ but$\|Ty_n\| > n$consequently no sequence $\{Ty_n\}$ would then converge.

Definition (1.1.11):-

Linear transformations T,whose domains are finite – dimensional are necessarily compact T is bound when $V_1 = V_2$and we see that if $\|x_n\| < c$

Then

$$\|Tx_n\| \leq \|T\|\|x_n\| \leq \|T\|c.$$

Consequently $\{Tx_n\}$ is uniformly bounded .It follows that a convergent subsequence exists for infinite dimensional space and integral transformations we can obtain the following result.

Theorem (1.1.12):-

A continuous kernel $K(x, y)$ defines a compact transformation

Proof:-

If $\{f_n\}$ is a sequence of function in r such that

$$\|f_n\| = \left(\int_a^b |f_n|^2\, dx\right)^{1/2} \leq c.$$

then

$$|Kf_n(x)| = \left|\int_a^b K(x,y) f_n(y) dy\right| \leq \left(\int_a^b K(x,y)^2 dx\right)^{1/2} \left(\int_a^b |f_n(y)|^2\, dy\right)^{1/2}$$

and it follows that the functions K$f_n(x)$ are uniformly bounded .Also see that for points x_1, x_2in $[a,b]$

$$|Kf_n(x_2) - Kf_n(x_1)| = \left|\int_a^b (K(x_2,y) - K(x_1,y)) f_n(y) dy\right|$$

$$\leq \left(\int_a^b |K(x_2,y) - K(x_1,y)|^2 dy\right)^{\frac{1}{2}} \left(\int_a^b |f_n(y)|^2 dy\right)^{\frac{1}{2}}$$

$\leq \varepsilon(b-a)^{1/2} c$ When $|x_2 - x_1| < \delta(\varepsilon)$

Where we have used equations (1.2) and (1.6).

Therefore the sequence $\{Kf_n(x)\}$ is equicontinuous and it follows that there is a sequence $\{Kf_n(x)\}$ which converges uniformly to a function $f(x)$,continuous on$[a,b]$.Because of the uniform convergence we can obtain the result that

$$\lim_{j\to\infty} \|Kf_{nj} - f\|^2 = \lim_{j\to\infty} \int_a^b |Kf_{nj} - f|^2\, dx$$

$$= \int_a^b \lim_{j\to\infty} |Kf_{nj} - f|^2\, dx$$

$$= 0$$

Which establishes the theorem.

Definition (1.1.13):-

The integral transformation k is said to be symmetric if for all elements f and g of R.

$(Kf, \mathrm{g}) = (f, K\mathrm{g})$

Then K is symmetric if and only if $K = K^*$

Definition (1.1.14):-

Areal kernel $K(x, y)$ is said to be symmetric if $K(x, y) = K(y, x)$for all x, y such that $a \leq x \leq b, a \leq y \leq b$.

Definition (1.1.15):-

A kernel $K(x, y)$defined on R* is called Hermitian if $K(x, y) = \overline{K(x, y)}$

Theorem (1.1.16):-

The integral transformation K is symmetric if and only if it has a symmetric kernel

Proof:-

If f and g are elements of R then

$$(Kf, \mathrm{g}) = \int_a^b \left\{ \int_a^b K(x, y) f(y) dy \right\} \mathrm{g}(x) dx = \int_a^b \int_a^b K(x, y) f(y) \mathrm{g}(x) dy dx$$

and

$$(f, K\mathrm{g}) = \int_a^b f(y) \left\{ \int_a^b K(y,x)\mathrm{g}(x)dx \right\} dy$$

$$= \int_a^b \int_a^b K(y,x) f(y)\mathrm{g}(x)dxdy$$

If $K(x,y)$is symmetric $K(x,y) = K(y,x)$and a change of order of integration establishes that $(Kf, \mathrm{g}) = (f, K\mathrm{g})$

Conversely, we assume that $K(x,y)$ is not symmetric, and we try to obtain a contradiction. If there are values$x_1$and x_2 in$[a, \mathrm{b}]$ for which $\mathrm{K}(x_1, x_2) < K(x_2, x_1)$ then from the continuity of $\mathrm{K}(x,y)$ there will exist closed intervals $I_1 : c_1 \leq d_1$ and $I_2 : c_2 \leq d_2$ in the interval [a, b] which are such that$x_1$is in $\mathrm{I}_1$and x_2 is in $\mathrm{I}_2$and

$$\mathrm{m} = \mathrm{l.u.b.}\{\mathrm{K}(x.y) \mid x \mathrm{in}\ \mathrm{I}_1, y\ \mathrm{in} \mathrm{I}_2\}, \mathrm{g.l.b.}\{K(x,y) \mid x in \mathrm{I}_2 y\ \mathrm{in}\ \mathrm{I}_1,\} = \mathrm{M}$$

Now define functions $\phi(\mathrm{I}_1, x)$ and $\phi(\mathrm{I}_2, x)$the *characteristic functions* of $\mathrm{I}_1$and I_2respectively to be such that

$$\phi(\mathrm{I}_j, x) = \begin{cases} 1, & \text{for } x in \mathrm{I}_j \\ 0, & \text{for } x notin \mathrm{I}_j \end{cases}$$

Where j=1, 2

Then if $f(x) = \phi(I_1, x)$ and$\mathrm{g}(x) = \phi(I_2, x)$we see that

$$(Kf, \mathrm{g}) = \int_a^b \left\{ \int_a^b K(x,y) f(y)dy \right\} \mathrm{g}(x)dx.$$

$$= \int_{c_1}^{d_1} \int_{c_2}^{d_2} K(x,y)dydx$$

$$\geq M(d_1 - c_1)(d_2 - c_2)$$

and

$$(Kf, \mathrm{g}) = \int_a^b \left\{ \int_a^b K(x,y) f(y) dy \right\} \mathrm{g}(x) dx.$$

$$= \int_{c_1}^{d_1} \int_{c_2}^{d_2} K(x,y) dy dx$$

$$\leq m(d_1 - c_1)(d_2 - c_2)$$

Consequently K could not be a symmetric transformation

We now establish a theorem relating to the eigenvalues of symmetric integral transformation

Theorem (1.1.17):-

The eigenvector corresponding to distinct eigenvalues of a symmetric integral transformation k are orthongonal

Proof:-

Let f and g be tow elements of R and λ,μ tow distinct eigenvalues of asymmetric integral transformations.

Then

$$Kf = \lambda f$$

$$K\mathrm{g} = \mu \mathrm{g}$$

And it follows that, since K is symmetric

$$\lambda(f,\mathrm{g}) = (\lambda f,\mathrm{g}) = (Kf,\mathrm{g}) = (f,K\mathrm{g}) = (f,\mu\mathrm{g}) = \mu(f,\mathrm{g})$$

Therefore

$$(\lambda - \mu)(f,\mathrm{g}) = 0$$

And since λ and μ are distinct, it follows that f and g must be orthogonal.

Definition (1.1.18):-

The simplest of symmetric kernel have the form

$$K(x,y) = \sum_{i,j=1}^{n} c_{ij} h_i(x) h_j(y) \qquad (1.7)$$

Where $\mathrm{c_{ij}} = \mathrm{c_{ji}}$ and $h_i(x)$ are continuous function defined on $[a,\mathrm{b}]$.

Definition (1.1.19):-

The kernel,$\mathrm{K}(x,y)$ which can be expressed as a finite linear combination of products of a function of x alone and a function of y alone is said to be a separable kernel, in the particular of (1.7) the kernel is said to be a symmetric separable kernel

Definition (1.1.20):-

Although we shall be discussing this kernel in detail, it is worth noting here that such a kernel induces an integral transformation which has only a finite number of non-zero eigenvalues, and a symmetric integral transformation which has a finite number of non-zero eigenvalues must have a separable kernel

Definition (1.1.21):-

Information concerning the eigenvalues and eigenvector of transformations with separable kernels can be obtained by using the corresponding results obtained when dealing with systems of finite order.

Theorem (1.1.22):-

Let K(x, y) be a symmetric separable kernel which is not identically zero, then

there exists an orthonormal set of vector $f_1(x), \ldots\ldots\ldots\ldots f_m(x)$ in R and non-zero real scalars $\lambda_1 \ldots\ldots \lambda_m$such the $Kf_i(x) = \lambda_i f_i(x)$ for $i = 1, 2, \ldots\ldots m$; also if g(x) is in R and orthogonal to $f_i(x)$ for all I then kg$(x) = 0$ .the scalars $\lambda_i$are the only non-zero eigenvalues of K ,and there is only a finite number of linear independent eigenvectors corresponding to each eigenvalues

Proof:-

Let V be the collection of all vector of the form

$$d_1 h_1(x) + \ldots\ldots + d_n h_n(x).$$

Such a collection as Vis in fact a vector space, and its dimension q, say is the number of linearly independent vector in the set

$$h_1(x) \ldots\ldots\ldots\ldots h_n(x)$$

Since the kernel $K(x, y)$ is also assumed to be defined in terms of $h_1(x) \ldots\ldots\ldots\ldots h_n(x)$as in (1.7), we find that for any arbitrary function p(y) in V

$$\int_a^b K(x, y)p(y)dy = \sum_{i,j=1}^{n} c_{ij} \int_a^b h_i(y)p(y)dy h_i(x) \qquad (1.8)$$

From (1.8) we readily see that k transformations V into itself. further since k(x, y) is symmetric ,as we have assumed dim V=q, that there exist q orthonormal vectors $f_i(x), i = 1,2, \ldots\ldots qinV$such that , $i = 1,2, \ldots\ldots q$ if $Kf_i(x) = \lambda_i f_i(x)$ for $i = 1,2, \ldots\ldots m$and$\lambda_i \neq 0$, i = m + 1, …. q,$\lambda_i = 0$we obtain the set of vector$f_i(x)$, i = 1,2, …… m required by the theorem however ,if $\lambda_i = 0\, i = 1,2, \ldots\ldots q$ then K$f_i(x) = 0$ and hence K$p(x) = 0$ for all function $p(x)$ in V since the function $f_i, i = 1,2 \ldots\ldots q$form a basis of V .Now for a fixed value of $x$the

kernel $K(x,y)$ given by (1.7) is a continues function of y in v and we can write in this case $K(x,y) = p(y)$hence

$$Kp = 0 = \int_a^b K(x,y)p(y)dy = \int_a^b |p(y)|^2\, dy$$

And we conclude that $p(x) = 0$ for all y since x can assume any arbitrary but fixed value, it follows that $p(x) = K(x,y) = 0$ for all $x$and y. This result contradicts the hypothesis of the theorem; therefore not all the λ_i are zero we thus assume $\lambda_i \neq 0$, $i = 1,2 \ldots\ldots m$ and $\lambda_i = 0$, $i = m + 1, \ldots\ldots q$,in R, let g be a function which is orthogonal to $f_i, i = 1,2, \ldots\ldots m$and define

$$p(x) = g(x) - \sum_{j=m+1}^{q} \Big(g(x), f_j(x)\Big) f_j(x) \qquad (1.9)$$

Then it is easily verified, by first forming the inner product (p, f_i) that p is orthogonal to $f_i, i = 1,2, \ldots\ldots q$ consequently ,since the functions $f_i, i = 1,2, \ldots\ldots q$form a basis for V the function $p(x)$must also be orthongonal to the functions $h_1(x) \ldots\ldots\ldots\ldots h_n(x)$, therefore , by (1.8) we see that $Kp(x) = 0$ .In addition we see that when $j = m + 1, \ldots\ldots q$, then $Kf_i = 0$. Combining these several results and using (1.5) we find that $Kg = 0$ as required .

To prove the remaining part of the theorem, we notice that if F is an arbitrary function in R then (1.8) implies that $KF(x)$ is in V .In particular,if$KF = \lambda F \neq 0$

Then $F(x) = (1/\lambda)KF(x)$ is in V and has the representation

$$F(x) = a_1 f_1(x) + \cdots + a_q f_q(x)$$

For scalars $a_i = (f, f_i)$, it follows from theorem (1.1.17) that $a_i = 0$unless $f_i$is associated with the eigenvector λ .This statement is sufficient to complete the proof of the theorem.

It may be felt that more generality can be obtained by considering the kernel to the form

$$K(x,y)=\sum_{i=1}^{p} f_i(x)\mathrm{g}_i(y) \qquad (1.10)$$

This is not the case as such a kernel can always be reduced to the form (1.7) and when $\mathrm{K}(x,y)$ is symmetric we shall obtain coefficients $\mathrm{c_{ij}}=\mathrm{c_{ji}}$,all that is required is that we should express the 2p functions $f_j(x)$and $\mathrm{g_j}(x)$in terms of a set of q linearly independent functions $\mathrm{h_j}(x)$ provided of course that dimension of the space is q in this case (1.10) becomes

$$K(x,y)=\sum_{m=1}^{p}\sum_{i=1}^{q} a_{mi}h_i(x)\sum_{j=1}^{q} b_{mj}h_j(y)$$

$$=\sum_{i,j=1}^{q} c_{ij}h_i(x)h_j(y)$$

Where

$$c_{ij}=\sum_{m=1}^{p} a_{mi}b_{mj}$$

Similarly we can obtain

$$K(x,y)=\sum_{i,j=1}^{q} c_{ij}h_i(y)h_j(x)=\sum_{i,j=1}^{q} c_{ij}h_i(x)h_j(y)$$

When we have that $\mathrm{K}(x,y)=\mathrm{K}(y,x)$ the linear independence of $h_i(x)$assures us that for fixed x

$$\sum_{i=1}^{q} c_{ij} h_i(x) = \sum_{i=1}^{q} c_{ji} h_i(x)$$

and it follow that $c_{ij} = c_{ji}$

The practical evaluation of the non-zero eigenvalues $\lambda_i$is best obtained if we start with the kernel in the form (1.10).If $F(x)$ is in R and an eigenvector of k we may write

$$KF(x) = \int_a^b \sum_{m=1}^{p} f_m(x) g_m(x) F(y)\, dy = \lambda F(x)$$

Writing

$$a_m = \int_a^b g_m(y) F(y)\, dy \qquad (1.11)$$

We obtain

$$F(x) = \frac{1}{\lambda} \sum_{m=1}^{p} a_m f_m(x) \qquad (1.12)$$

Combining (1.12) and (1.11) we obtain

$$\lambda a_m = \sum_{j,m=1}^{p} a_j \int_a^b g_m(y) f_j(y)\, dy$$

This last equation is a representation of the set of linear algebraic equations for the unknown $a_m$and as such will only possess a non-trivial solution when the characteristic determinant vanishes that is when

$$det\left\{ \lambda \delta_{mj} - \int_a^b g_m(y) f_m(y)\, dy \right\} = 0 \qquad (1.14)$$

therefore once the eigenvalues have been determined from (1.14)

the corresponding eigenvectors follow from(1.12)

Example (1.1.23):-

Find the non-zero eigenvalues of the kernel

$$\mathrm{K}(x,y) = 1 + \cos(x-y), \qquad -\pi \le x \le \pi, -\pi \le y \le \pi$$

We must solve

$$KF(x) = \lambda f(x), \lambda \neq 0$$

that is

$$\int_{-\pi}^{\pi} (1 + \cos(x-y))F(y)dy = \lambda \mathrm{F}(x)\,, \quad \lambda \neq 0$$

Expanding the cosine term and using (1.11) we may write this equation in the form

$$a_1 + a_2 \cos x + a_3 \sin x = \lambda f(z)$$

Where

$$a_1 = \frac{1}{\lambda} \int_{-\pi}^{\pi} F(y)\, dy$$

$$a_2 = \frac{1}{\lambda} \int_{-\pi}^{\pi} \cos y\, F(y)\, dy$$

$$a_1 = \frac{1}{\lambda} \int_{-\pi}^{\pi} \sin y\, F(y)\, dy$$

From(1.13) we have the alternative forms

$$a_1 = \frac{1}{\lambda}\int_{-\pi}^{\pi}(a_1 + a_2\cos y + a_3\sin y)\,dy$$

$$a_2 = \frac{1}{\lambda}\int_{-\pi}^{\pi}\cos y\,(a_1 + a_2\cos y + a_3\sin y)\,dy$$

$$a_3 = \frac{1}{\lambda}\int_{-\pi}^{\pi}\sin y\,(a_1 + a_2\cos y + a_3\sin y)\,dy$$

That is on expanding the kernel K(x, y) we have written

$$f_1(x) = 1, \qquad f_2(x) = \ cosx\ ,\ \ f_3(x) = sinx$$

$$\mathrm{g}_1(y) = 1,\ \ \mathrm{g}_2(y) = \mathrm{cosy}, \mathrm{g}_3(y) = \mathrm{sin}y.$$

integratingtheseequationsweobtain

$$\lambda a_1 = 2\pi a_1$$

$$\lambda a_2 = \pi a_2$$

$$\lambda a_3 = \pi a_3$$

From which it follow that the possible values of λ for non-zero eigenvector

F$(x)$are$\lambda = 2\pi$ and $\lambda = \pi$.When $\lambda = \pi$ the equations show that $a_1 = 0$ whilst $a_2$and $a_3$can take arbitrary values . Consequently from (1.12) we obtain

$$F(x) = \frac{1}{\pi}\{a_2\cos x + a_3\sin y\}$$

Similarly when $\lambda = 2\pi$ it follows that $a_2 = a_3 = 0$ whilst $a_1$can be arbitrary and we obtain the corresponding eigenvector

$$F(x) = \frac{a_2}{2\pi}$$

Consequently it follow that there are three linearly independent solutions of the given equations (1.15), namely

$$\mathrm{F}_1(x) = cosx\ , F_2(x) = sinx\ , \mathrm{F}_3(x) = 1$$

As it happens these are orthogonal eigenvectors which when normalized may be written

$$G_1(x) = \frac{F_1(x)}{\|F_1(x)\|} = \frac{\cos x}{\sqrt{(\pi)}}$$

$$G_2(x) = \frac{F_2(x)}{\|F_2(x)\|} = \frac{\sin x}{\sqrt{(\pi)}}$$

$$G_3(x) = \frac{F_3(x)}{\|F_3(x)\|} = \frac{1}{\sqrt{(\pi)}}$$

Example (1.1.24):-

Repeat the above problem but with the kernel

$$K(x, y) = sin\, x\, cos\, y\ , -\pi \le x \le \pi\ , -\pi \le y \le \pi$$

We are now required to solve

$$KF(x) = \int_{-\pi}^{\pi} sin\, x\, cos\, y\, F(y)\, dy = \lambda F(x)$$

This we may write in the form

$$a \sin x = \lambda F(x)$$

Where

$$a = \int_{-\pi}^{\pi} cos\, y\, F(y)\, dy = \int_{-\pi}^{\pi} cos\, y \left(\frac{1}{\lambda} a \sin y\right) dy = 0$$

Therefore there are no non-zero eigenvalues of this kernel when the domain with which we are concerned is the square $-\pi \leq x \leq \pi\,, -\pi \leq y \leq \pi$ we remark that this kernel is not symmetric if we changed the domain so that $0 \leq x \leq \pi/2\ \,, 0 \leq y \leq \pi/2$ then we would obtain one non-zero eigenvalue $\lambda = \frac{1}{2}$

Section (1.2):Eigenvalues of symmetric integral operator

Let K be asymmetric integral transformation defined on the real Euclidean vector space R of integrable function on $a \leq x \leq b$.In this section we establish a number of important results for such operators which we shall find useful later.

Theorem (1.2.1):-

The number of distinct non-zero eigenvalues of a symmetric integral transformation K is at most afinite number of linearly independent eigenvector.

Proof:-

We have already seen that eigenvectors corresponding to distinct eigenvalues are orthogonal further since any finite set of linearly independent eigenvectors corresponding to the same eigenvalue may be replaced by orthonormal eigenvectors, we may assume that any finite set of linearly independent eigenvectors is in fact an orthonormal set.

Letf_i, $i = 1, 2, \ldots \ldots, m$ be a finite set of orthonormal eiginvectors corresponding to the eiginvalues$i$ which need not be distinct. Now for a fixed value of $x$the function $\mathrm{g}(y) = K(x, y)$ is a function in R and we have

$$\big(\mathrm{g}(y), f_i(y)\big) = \int_a^b K(x, y) f_i(y)\, dy = K f_i(x) = \lambda_i f_i(x)$$

Bessel's inequality enables us to write

$$\sum_{i=1}^{m} {\lambda_i}^2 |f_i(x)|^2 \leq \int_a^b |K(x, y)|^2 dy \qquad (1.17)$$

and on integrating this with respect to $x$we obtain

$$\sum_{i=1}^{m} \lambda^2 \leq \int_a^b \int_a^b |K(x,y)|^2 \, dydx \qquad (1.18)$$

The right-hand side is independent of m. This fact coupled with the result of theorem (1.1.22) indicates that there can be at most a finite number of linearly independent eigenvalues. This inequality also implies that there can be at most a finite number of linearly independent eigenvectors with eigednvalues greater in absolute value than unity or with eigenvalues whose absolute values lie in the range

$$\frac{1}{2n} < x < \frac{1}{n} \, , n = 1, 2, ...$$

The distinct eigenvalues can now be counted because we count first those with absolute value greater than unity then those in the range $½ < x < 1$ and so on thus, every non-zero eigenvalue can be counted.

It is customary to order the eigenvalues according to their absolute value and then to display them in the following form:

$$\lambda_{-1} \leq \lambda_{-2} \leq \ldots\ldots < 0 < \ldots\ldots \leq \lambda_1 \leq \lambda_2$$

where the final separation has been effected with respect to the signs of the eigenvalues . Eash eigenvalue is repeated in such adisplay a number of times equal to the number of linearly independent eigenvectors associated with it .such a display of the eigenvalues of the operator K is known as spectrum of the operator Acorollary theorem(1.2.1) having a direct bearing on the notion of the spectrum of an operator can be stated as follows.

Corollary(1.2.2):-

If K has at least one non-zero eigenvalue then there exist non-negative integers M and N not both zero but possibly in finite and an orthonormal set of vectors $f_i(x)$, $-\infty \leq -M \leq i \leq N \leq \infty, i \neq 0$

such that $Kf_i(x) = \lambda_i f_i(x)$ with

$$\lambda_{-1} \leq \lambda_{-2} \leq \ldots\ldots < 0 < \ldots\ldots \leq \lambda_2 \leq \lambda_1.$$

The set of numbers$\lambda_j$includes every non-zero eigenvalue of K, and every eigenvector of K corresponding to a non-zero eigenvalue λ is a finite linear combination of eigenvectors$f_i(x)$ associated with the eigenvalue λ. Finally we have that:

$$\sum_i \lambda_i^2 |f_i(x)|^2 \leq \int_a^b |K(x,y)|^2 dy \qquad (1.19)$$

$$\sum_i \lambda_i^2 \int_a^b \int_a^b |K(x,y)|^2 dy dx \qquad (1.20)$$

These last equations follow directly form (1.17) and (1.18). Also, from (1.20) it follows that if $N = \infty$ then $\lim_{i\to\infty} \lambda_i = 0$ and if $M = \infty$ then $\lim_{i\to\infty} \lambda_{-i} = 0$ The question of the existence of a non-zero eigenvalue can be answered if we first develop for integral transformations results an alogous to those already obtained for linear transformations we recall that if $T$is a linear transformations on afinite-dimensional Euclidean vector space the largest and smallest eigenvalues λ_1 and λ_n respectively are given by

$$\lambda_1 = l.u.b.\{(Tx,x)\ x = 1\}$$

$$\lambda_n = \text{g.}l.b.\{(Tx,x)\ x = 1\}.$$

Similar results can also be obtained for integral operators, as we now see.

Theorem (1.2.3):-

Let $p = l.u.b.\{(Kf, f)|\ f = 1\}\ and q = \mathrm{g}\,.l.b.\{(\mathrm{K}f, f)\ |\ f = 1\}$

then if $p > 0$ it is the largest non-zero eigenvalue and if $q < 0$ it is the smallest non-zero eigenvalue.

Proof:-

If

$$P` = l.u.b.\{(-\mathrm{K}f, f)|\ \|f\| = 1\} = l.u.b\ \{(\mathrm{K}f, f)\ |\|f\| = 1\ \}$$

is the largest eigenvalue of $-K$ then $-P`$ must be the smallest eigenvalue of $K$ and $-P` = q$ .Therefore , it will be sufficient for us to prove the theorem with regard to$p$,and the statement regarding q will follow if we consider the transformation $-K$

If

$Kf = \lambda f \text{for} f = 1$

Then it follows at once that

$$\lambda = \lambda(f, f) = (\mathrm{K}f, f) \leq p$$

consequently if $p$is an eigenvalue it must be the largest. Therefore to establish the theorem we must show that p is indeed an eigenvalue consider initially the case when the kernel of$K$ is separable. Then when $p > 0$ the kernel cannot be identically zero and theorem (1.1.22) implies the existence of m orthonormal vectors f_i, and non-zero eigenvalue λi having the$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ such that for any vector **g** orthogonal to f_i for all i, $K\mathrm{g}(x) = 0$ let Vbe the linear manifold spanned by f_i then if

$$h(x) = \sum_{i=1}^{m} a_i f_i(x)$$

is any function in Vit follows that $Kh(x)$ is also in Vsince $Kf_i(x) = \lambda_i f(x)$.Thus we have that

$$\lambda_{\mathrm{i}} = l.u.b.\{(\mathrm{K}h, h) | h = 1\ , h \text{ in V}\}$$

$$\leq l.u.b.\{(\mathrm{k}f, f)\ |\ \|f\| = 1\ , f in R\}\ = p \qquad (1.21)$$

If we can now show that the reverse inequality is also true then we can assert that since $p = \lambda_1$, p must be an eigenvalue of k . Now when $f$is in R and $f = 1$ we may write

$f = h + \mathrm{g}$,h in V and g in $\mathrm{V}_{\perp}$

Where $\mathrm{V}_{\perp}$ is the space of functions orthogonal to V therefore,
$K\mathrm{g}(x) = 0, \mathrm{K}f(x) = \mathrm{K}h(x)$ is in $V$and

$$(Kf, f) = (Kh, h + \mathrm{g}) = (Kh, h), \qquad (1.22)$$

also we have that

$$(f, f) = (h + \mathrm{g}, h + \mathrm{g}) = (h, h) + (\mathrm{g}, \mathrm{g}),$$

where

$$0 \leq h \leq f = 1$$

If we restrict our choice of f to ensure $(kf, f) > 0$ it follows from (1.22) that since $p > 0, (Kh, h) > 0, \|h\| \neq 0$ and

$$(Kf, f) = h^2(Kp, p) \leq (Kp, p),$$

where

$$p(x) = \frac{h(x)}{h(x)} \text{ is in } V$$

Hence we have that

$$\text{l. u. b.}\{(Kf,p)|f=1, finR\} \leq \text{l. u. b}\{(Kp,p)|p=1, pinV \qquad (1.23)$$

Which establishes that $p \leq \lambda_1$ combining (1.21) and (1.23) we see that $p = \lambda_1$, and the theorem follows for integral transformations with separable kernels.

We will now consider the case when $K(x,y)$is any continuous symmetric kernel. To establish the theorem in this case we shall need to use the following result

For every $\varepsilon > 0$ there is a symmetric separable kernel $H(x,y)$ such that

$$\int_a^b \int_a^b |K(x,y) - H(x,y)|^2 dxdy \leq \varepsilon$$

This means that we can find a sequence $k_n(x,y)$ of symmetric separable kernels such that

$$\|K - K_n\|^2 \leq \int_a^b \int_a^b |K(x,y) - K_n(x,y)|^2 \, dxdy$$

$$\leq \frac{1}{n^2} \, (say)$$

Where we have used theorem (1.1.7) we shall write:

$$p_n = \text{l. u. b.}\{(K_n f, f)|f = 1\},$$

and assert that

$$p = \lim_{n\to\infty} p_n$$

To prove this assertion we notice that if $\|f\| = 1$ then

$$|(Kf, f) - (K_n f, f)| = |(K - K_n)f, f|$$

$$\leq \|(K - K_n)f\|\|f\|$$

$$\leq \|K - K_n\|$$

$$\leq \frac{1}{n}$$

Therefore

$$(Kf, f) \leq (K_n f, f) + \frac{1}{n}$$

$$p \leq p_n + \frac{1}{n}$$

Similarly we can show that

$$p_n \leq p + \frac{1}{n}$$

Which finally establishes our assertion that $p = lim_{n\to\infty} p_n$ since$p > 0$ we may restrict any further discussion to sequences for which $p_n > 0$.We have proved the theorem for symmetric separable kernels so we can write $p_n$as eigenvalues of the operator K_n and denote by f_n the corresponding eigenvectors thus we have that

$$K_n f_n = p_n f_n \,, \|f_n\| = 1$$

We will now show that

$$\lim_{n\to\infty} \|Kf_n - pf_n\| = 0$$

Hence it follows that

$$\|Kf_n - pf_n\| \leq \|Kf_n - K_n f_n\| + \|p_n f_n - pf_n\|$$

$$\leq \|K - K_n\| + |p_n - p|$$

In view of the fact that K is a compact operator theorem (1.1.12) there is a g in $R$and a sub sequence $\{f_{nj}\}$ such that

$$\lim_{j\to\infty} \|Kf_{nj} - \mathrm{g}\| = 0 \qquad (1.24)$$

Then the inequality

$$\|\mathrm{g} - pf_{nj}\| \leq \|\mathrm{g} - k\mathrm{K}\| + \|Kf_{nj} - pf_{nj}\|$$

Together with (1.24) and (1.25) establishes that

$$\lim_{j\to\infty} \|\mathrm{g} - pf_{nj}\| = 0 \qquad (1.25)$$

Hence it follows that

$$\|\mathrm{g}\| = \lim_{j\to\infty} \|pf_{nj}\| = |p| \neq 0$$

Finally we must show that g is an eigenvector of K with eigenvalue P using (1.24) and (1.25) we see that since

$$\|K\mathrm{g} - p\mathrm{g}\| \leq \|K\mathrm{g} - Kpf_{nj}\| + \|pKf_{nj} - p\mathrm{g}\|$$

$$\leq \|\mathrm{K}\|\|\mathrm{g} - pf_{nj}\| + |p|\|Kf_{nj} - \mathrm{g}\|$$

The required result follows immediately this completes the proof of the theorem.

Notice that the result $p = lim_{n\to\infty} p_n$ enables quite readily to obtain approximate values of P.

This is due to the fact that the value of p_n is easily calculated by the method out lined in section (1.1) since

$$\|K\| = \text{l.u.b.}\{|(Kf,f)|\,|\,\|f\| = 1, finR\}$$

The theorem shows that whenever $\|k\| \neq 0$ the transformation K has a non-zero eigenvalue equal to $\pm\|K\|$. We will now show that a non-zero eigenvalue always exists unless the kernel is identically zero.

Theorem (1.2.4):-

If$K(x,y)$ is continuous and symmetric $(Kf,f) = 0$ for all f if and only if$K(x,y) = 0$.

Proof:-

$(Kf,f) = 0$for all$f$ implies that $(Kf,\mathrm{g}) = 0$ for all f and g. This follows from identity

$$(Kf,\mathrm{g}) = \frac{1}{2}\{(K(f+\mathrm{g}),f+\mathrm{g}) - (Kf,f) - (K\mathrm{g},\mathrm{g})\}$$

Now, taking $\mathrm{g} = Kf$ we have that

$$(Kf,\mathrm{g}) = (Kf,Kf) = 0$$

Hence using the fundamental property of scalar products that

$$Kf = \int_a^b K(x,y)f(y)dy = 0, a \leq x \leq b, finR.$$

For fixed x the kernel $K(x,y) = h(y)$is clearly in R so if we choose $f = h$ we obtain

$$\int_a^b |h|^2\,dy = 0$$

That is $K(x,y) = 0, a \le x \le b, a \le y \le b$.

Theorem (1.2.5):-

Let $Kf_j = \lambda_j f_j, j = 1,2, \dots \dots, m$.where the f_j are orthonormal eigenvectors of k then there is an eigenvector g of k with a corresponding non-zero eigenvalue λ which is orthogonal to $f_j, j = 1,2, \dots \dots, m$if and only if g is also an eigenvector with non-zero eigenvalue λ of operator H whose kernel is

$$H(x,y) = K(x,y) - \sum_{j=1}^{m} \lambda_j f_j(x) f_j(y) \qquad (1.26)$$

Proof:-

Suppose $H\mathrm{g} = \lambda \mathrm{g}, \quad \lambda \neq 0, \quad \mathrm{g} \neq 0$ then we have

$$Hf_j = Kf_j - \sum_{i=1}^{m} \lambda_i f_i(x)\big(f_i, f_j\big)$$

$$= Kf_j - \lambda_j f_j$$

$$= 0 f_j$$

Therefore by theorem (1.1.22) we must have that $g$is orthogonal to $f_j, j = 1,2, \dots \dots, m$ consequently

$$\lambda \mathrm{g} = H\mathrm{g} = K\mathrm{g} - \sum_{j=1}^{m} \lambda_j \, f_j(x)\big(\mathrm{g}, f_j\big) = K\mathrm{g}\,,$$

and it follows that g must be an eigenvector of K with eigenvalue λ.

Conversely if g is orthogonal to $f_j, j = 1,2, \dots \dots, m$ when $\mathrm{K} = \lambda_j f_j$ then $H\mathrm{g} = K\mathrm{g} = \lambda \mathrm{g}$.

As an immediate consequence of this theorem we can establish that a symmetric integral transformation with a finite number of non-zero eigenvalue must have a separable kernel .This result will be the converse of theorem (1.1.22).

Theorem (1.2.6):-

Let $f_j, j = 1,2, \dots \dots, m$be a basis of an orthonormal eigenvector of k corresponding non-zero eigenvalues $\lambda_j, j = 1,2, \dots \dots, m$ if $k$has no other non-zero eigenvalues then

$$K(x,y) = \sum_{j=1}^{m} \lambda_j f_j(x) f_j(y)$$

Proof:-

The operator Hcorresponding to the kernel $H(x,y)$ given by (1.26) has no non-zero eigenvalues in view of theorem (1.2.5) and hence by theorem (1.2.4) $H(x,y)$ must be identically zero the theorem follows immediately.

Finally in this section we use the notation introduced in theorem (1.2.1) to prove the following theorem.

Theorem (1.2.7):-

(a) If

$$i < N, \lambda_{i+1}$$

$$= \text{l. u. b.}\{(kf,f)\big|\|f\| = 1, f \text{ orthogonal to } f_j, j = 1, \dots, i\}$$

(b)If

$$-i > -M, \lambda_{-i-1}$$

$$= \text{g. l. b.}\{(Kf,f)\big|\|f\| = 1\ , f \text{ orthogonal to } \ f_j, j = -1, \dots, -i\}$$

Proof:-

When $N > 0$ theorem (1.2.3) allows us to write

$$l.u.b.\{(Kf,f)|\|f\| = 1\} = \lambda$$

If$N > i$, let

$$K_i(x,y) = K(x,y) - \sum_{j=1}^{i} \lambda_j f_j(x) f_j(y)$$

Then from the results of theorem (1.2.5) we have that the integral transformation $K_i$has exactly $\lambda_{i+1}, \lambda_{i+2}$, ... as positive eigenvalues with $f_{j+1}(x), f_{j+2}(x)$, ... as corresponding eigenvectors. It follows that

$$\lambda_{i+1} = \text{l. u. b.}\{(K_i f,f)|\|f\| = 1\}$$

However when f is orthogonal to $f_j, j = 1, \ldots\ldots, i$

$$K_j f = Kf$$

Therefore

$$\begin{aligned}\lambda_{i+1} &\geq \text{l. u. b.}\{(K_i f,f)\big|\|f\| = 1, f \text{ orthogonal to } f_j, j = 1,2, \ldots\ldots, i\} \\ &= \text{l. u. b.}\{(Kf,f)\big|\|f\| = 1, f \text{ orthogonal to } f_j, j = 1,2, \ldots\ldots, i\} \\ &\geq (Kf_{i+1}, f_{i+1}) = \lambda_{i+1}.\end{aligned}$$

The inequalities thus become equalities, and the proof of the first part of the theorem is complete the remaining result for negative eigenvalues can be obtained in a similar manner by considering the operator $-K$ with only a few slight modifications the corresponding theorems for integral transformations overR^*can be established.

Definition (1.2.8):-

When we were concerned with asymmetric linear transformation$T$ on n-dimensional Euclidean vector space E_n we saw that it was possible to find n orthononrmal vector $x_i, i = 1,2, \dots, n$ satifying

$$Tx_i = \lambda_i x_i$$

Such that if y is in E_n, then

$$y = \sum_{i=1}^{n} (y, x_i)x_i$$

$$Ty = \sum_{i=1}^{n} \lambda_i (y, x_i)x_i$$

That is, we obtained an expansion for any element y in E_n in terms of a system of orthonormal vectors $x_i,\ \ i = 1,2,3, \dots . n.$

We now enquire whether or not similar expansions are available in infinite dimensional spaces when linear transformations replaced by integral transformations.

We will start our investigation by considering again the kernel used in examples (1.1.23) .For the kernel:

$$K(x, y) = 1 + cos\, x\, cos\, y + sin\, x\, sin\, y,$$

$$-\pi \le x \le \pi\ ;\ -\pi \le y \le \pi \qquad (1.27)$$

We found only three orthogonal solutions of $Kf = \lambda f\ , \lambda \neq 0.$

These were 1, $cosx$and $sinx$.

Definition (1.2.9):-

Despite the fact that any linear combination of these three is a continuous function, this does not imply that they span R, since not every continuous function can be expressed as a linear combination of them. For example, suppose we assume that the following expansion is admissible:

$$cos2x = a + b\,cosx + c\,sinx, \quad -\pi \le x \le \pi \tag{1.28}$$

Integrating (1.28) over$-\pi \le x \le \pi$we find $a = 0$ similarly, by first multiplying by either $sinx$or$cosx$and then integrating over the same range, we find that $b = 0\ andc = 0$. Thus (1.28) is not an admissible expansion, as could have been anticipated from the results of elementary of view, since we have in fact discarded all those eigenvectors with zero eigenvalue.

Definition (1.2.10):-

In an attempt to include these additional eigenvectors, we first notice the following integral relationships:

$$\int_{-\pi}^{\pi} cos\,px\,cos\,qx\,dx = \int_{-\pi}^{\pi} sin\,px\,sin\,qx\,dx = \begin{cases} 0, p \neq q, p, q = 0,1, \dots \\ \pi, p = q, q, p = 1,2, \dots \end{cases}$$

$$\int_{-\pi}^{\pi} sin\,px\,cos\,qx\,dx = 0, p = 1,2, \dots ., q = 0,1,2, \dots .$$

Consequently it follows that:

$$1, \quad cons\,nx\,, \quad sin\,nx\,, \quad n = 1,2, \dots\,, -\pi \le x \le \pi\,,$$

are orthogonal. As a result,$cosnx, sinnx, n = 2,3,4, \dots .$, must be eigenvectors with zero eigenvalue for the integral transformation having the kernel (1.2.6).

Definition (1.2.11):-

Having gone this far we may as well try to find any remaining eigenvectors, g, which have zero eigenvalues. If there is such a function, g, then by Theorem (1.1.17) g must be orthogonal to $1, cosx,\ sinx$, the eigenvectors with non-zero eigenvalues. Further, g will also be orthogonal to the eigenvectors $cosnx, sinnx$, $n = 2,3,$ .... we shall see in the vectors $cosnx, sinnx$, $n = 0.1,$ ..., must necessarily be the zero vector.

Definition (1.2.12):-

Assuming this fact for the moment, we conclude that $1, cosnx, sinnx, n = 1,2$ ..., constitute a maximal set of eigenvectors of K.when normalized, the eigenvectors may be written.

$$f_0(x) = \frac{1}{\sqrt{(2\pi)}}, f_{2j-1}(x) = \frac{cos\, jx}{\sqrt{(\pi)}}, f_{2j}(x) = \frac{sin\, jx}{\sqrt{(\pi)}}, j = 1,2,..$$

Therefore, the required extension of the results for linear transformations on a finite- dimensional space to integral transformations will be achieved if we expand any element g of R in infinite series in the form:

$$\mathrm{g}(x) = \sum_{j=0}^{\infty} (\mathrm{g}, f_j) f_j(x) = \sum_{j=1}^{\infty} \left(\int_{-\pi}^{\pi} \mathrm{g}(y) f_j(y) dy \right) f_j(x).$$

This has the alternative form:

$$\mathrm{g}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{g}(y)\, cos\, ny\, dy, \quad n = 0,1,2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{g}(y)\, sin\, ny\, dy, \quad n = 1,2, \ldots$$

This last form of expansion is known as a Fourier series expansion. Whilst we shall not enter into a detailed study of such series, we will observe that, although not every continuous function can be expanded in a convergent Fourier series, an extension of the expansion theorem for linear operations to integral operators is still possible.

The extension of the result $Ty = \sum_{i=1}^{n} \lambda_i(y, x_i)x_i$to integral operators in more readily obtained, and is, in fact, given by the following theorem.

Theorem (1.2.13):-

Let $\{\lambda_i\}, -\infty \leq M \leq i \leq N \leq \infty, i \neq 0$ is the spectrum of the integral operator K associated with the eigenvectors$\{f_i\}$. If g is any element of$R$ then.

$$K\mathrm{g} = \sum_i \lambda_i(\mathrm{g}, f_i)f_i = \sum_i (K\mathrm{g}, f_i)f_i \qquad (1.29)$$

Andthe series is absolutely uniformly convergent an $a \leq x \leq b$

Proof:-

The series (1.29) converges absolutely uniformly on $a \leq x \leq b$ if for every$\varepsilon > 0$ there is an integer $N_1(\varepsilon) > 0$ such that:

$$\sum_{i=m}^{n} |\lambda_i a_i f_i(x)| < \varepsilon \; a_i = (g, f_i) \qquad (1.30)$$

Whenever $mandn$ have the same sign, with $|m| > N_1 and |n| > N_1$ to show that this is the case, let:

$$\mathrm{A} \geq \mathrm{l.u.b}\left\{\int_{\mathrm{a}}^{\mathrm{b}} |\mathrm{K}(x,\mathrm{y})|^2 \mathrm{dy} \mid \mathrm{a} \leq x \leq \mathrm{b}\right\}$$

Notice also that as a consequence of Bessel's inequality we can write:

$$\sum_i |a_i|^2 \leq \|\mathrm{g}\|$$

Where the prime denotes that $i = 0$ has been excluded from the summation. Then it follows that there exist numbers n, m having the same sign and satisfying $|n| > N_1, |m| > N_1$ for some integer N_1 such that:

$$\sum_{i=m}^{n} |\lambda_i a_i f_i| \leq \left(\sum_{i=m}^{n} |a_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=m}^{n} {\lambda_i}^2 |f_i|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\varepsilon^2}{A}\right)^{\frac{1}{2}} \left(\int_a^b |K(x,y)|^2 \, dy\right)^{\frac{1}{2}}$$

$$\leq \varepsilon.$$

Thus the inequality (1.30) is established. It remains to show that the series converges to Kg. It would perhaps be appropriate to digress for a moment and elaborate on this particular use of the Schwarz inequality.

Although it is a well-established result for finite series with a development which is independent of the notions of vector spaces and inner products, it in no way conflicts with our present requirements. In, fact, if we write $S = (\lambda_m f_m, \lambda_{2m+1} f_{m+1}, \dots, \lambda_n f_n)$ and$A = (a_m, a_{m+1}, \dots, a_n)$ then the left – hand side is clearly an inner product form defined in the usual manner for finite-dimensional spaces

by:

$$(S, A) = \sum_i \lambda_i f_i a_i$$

The application of Schwarz inequality is then clear. Returning to the proof of the theorem, we remark that if the number of eigenvalues is finite then by Theorem (1.2.6) the kernel $K(x, y)$ is separable, and we have:

$$K(x, y) = \sum_i \lambda_i f_i(x) f_i(y)$$

From this it follows that:

$$K\mathrm{g} = \sum_i \lambda_i (\mathrm{g}, f_i) f_i$$

As required.

Now consider the case when there is an infinite number of eigenvalues. We will assume $N = M = \infty$ the other cases being treated similarly. Then:

$$\lim_{i \to \infty} \lambda_i = 0 = \lim_{i \to \infty} \lambda_{-i}$$

If we denote by h any vector orthogonal to the vectors

$$f_1, f_2, \ldots, f_i, f_{-1}, \ldots, f_{-j} \qquad (1.31)$$

Then from Theorem (1.2.7) we have that:

$$\|h\|^2 \lambda_{-j-1} \leq (Kh, h) \leq \|h\|^2 \lambda_{j+1.}$$

Therefore, for any sequence of vectors (h_i) such that (h_i) is orthogonal to every member of the set (1.31) for all j, and having the property that,

$\|h_i\| \leq B$, we have:

$$\lim_{j\to\infty}\left|Kh_j, h_j\right| = 0 \qquad (1.32)$$

Let $\{H_j\}$ be any other sequence of vectors with $\|H_j\| \leq B$and H_i orthogonal to the set (1.31) for any j. Then vectors$(h_j \pm H_j)$are also orthogonal to set (1.31) for all j and$\|h_i \pm H_i\| \leq 2B$.If now we use the result (1.32) in conjunction with the identity:

$$(Kh, H) = \frac{1}{4}\{(K(h+H), h+H) - (K(h-H), h-H)\},$$

We see that:

$$\lim_{j\to\infty}\left|\left(Kh_j, H_j\right)\right| = 0. \qquad (1.33)$$

We establish the theorem by selecting particular sequences $\{h_j\}$and $\{H_j\}$as follows. Let:

$$h_j = \mathrm{g} - \sum_{i=-j}^{j} (\mathrm{g}, f_i) f_i$$

Then $h_i$iscertainly orthogonal to the set (1.31) for all j, and

$||h_i|| \leq ||\mathrm{g}||$.Corollary which states that:

$$\left\| \mathrm{g} - \sum_{i=-j} (\mathrm{g}, f_i) f_i \right\| \leq \left\| \mathrm{g} - \sum_{i=-j} c_i f_i \right\|$$

For arbitrary c_i, in particular$c_i = 0$

Similarly if we let:

$$H_j = Kh_j = K\mathrm{g} - \sum_{i=j}^{j} (\mathrm{g}, f_j)\lambda_j f_j$$

$$= K\mathrm{g} - \sum_{i=-j}^{j} (\mathrm{g}, Kf_j) f_j$$

$$= K\mathrm{g} - \sum_{i=-j}^{j} (K\mathrm{g}, f_j) f_j$$

Then we see that H_i is orthogonal to $f_1, f_2, \ldots., f_i, f_{-1}, \ldots, f_{-j}$ and

$$||H_j|| \leq ||K\mathrm{g}||.$$

With these choices of $h_i$and $H_i$equation (1.33) becomes:

$$\lim_{j=\infty} \left|(Kh_j, Kh_j)\right| = 0$$

That is:

$$\lim_{j\to\infty} \int_a^b \left| K\mathrm{g} - \sum_{i=-j}^{j} \lambda_i (\mathrm{g}, f_i) f_i \right|^2 dx = 0$$

In view of the uniform convergence, this may be written:

$$\int_a^b \left| K\mathrm{g} - \sum_{i} \lambda_i (\mathrm{g}, f_i) f_i \right|^2 dx = 0$$

Therefore, because of the continuity of the integral we have

$$K\mathrm{g} = \sum \lambda_i(\mathrm{g}, f_i) f_i$$

And the theorem is complete.

Definition (1.2.14):-

This theorem is an extension of one concerned with the diagonalization of linear transformations defined on an n-dimensional Euclidean vector space. The results of the theorem may be given an interesting, but at the moment an imprecise interpretation. If we can write the function g in the form$\sum_i(\mathrm{g}, f_i)\, f_i$ then, using the terminology of linear transformations, we can say that there is a basis (f_i) with respect to which the function g has the co-ordinate representation$[(\mathrm{g}, f_i)]$.It follows that Kg has the representation$[\lambda_i(\mathrm{g}, f_i)]$ with respect to the same basis.

That is, in some sense the operator K performs essentially an operation of multiplication. To make this interpretation more precise we need to introduce the concept of a iterated kernel, which will enable us to obtain an expansion for our kernel, similar to that in the case of only a finite number of eigenvalues. Such an expansion is not always possible, in general. However, we shall see that the operator $K^n, n > 1$ has such an expansion.

Definition (1.2.15):-

Let $P(x, y)\ and Q\ (x, y)$ be continuous on the square $a \le x \le b;\ a \le y \le b$ and define:

$$R(x, y) = \int_a^b P(x, t) Q(t, y)\, dt\ , a \le x \le b : a \le y \le b. \qquad (1.34)$$

We notice that $R(x, y)$ is continuous on the square $a \leq x \leq b, a \leq y \leq b$but need not be symmetric, even when $P(x, y)$ $and Q(x, y)$ are symmetric. However, if we also have:

$$\int_a^b P(x,t)Q(t,y)\,dt = \int_a^b Q(x,t)P(t,y)\,dt \qquad (1.35)$$

Then $R(x, y)$ is symmetric since:

$$R(x,y) = \int_a^b P(x,t)Q(t,y)\,dt = \int_a^b Q(x,t)P(t,y)\,dt$$

$$= \int_a^b P(y,t)Q(t,x)\,dt = R(y,x).$$

The kernel defined by (1.34) defines an operator R and we have:

$$Rf(x) = \int_a^b \int_a^b P(x,t)Q(t,y)\,dt\,f(y)dy = \int_a^b P(x,t)\int_a^b Q(t,y)f(y)\,dy\,dt$$

$$= \int_a^b P(x,t)\big(Qf(y)\big)(t)\,dt = P(Qf) = PQf$$

Where we have introduced two operators P and Q which have an obvious definition. Therefore we see that $R(x, y)$is in fact the kernel of the operator PQ and we have $R = PQ$. In the case when (1.35) holds we have $PQ = QP$ and it follows that:

$$PR = P(PQ) = P(QP) = (PQ)P = RP\,.$$

Hence, when (1.35) is valid, P and R commute. In particular if we define:

$$K^n(x,y) = \int_a^b K(x,t)K^{n-1}(t,y)dt\,, \qquad n = 1,2, \ldots\ldots$$

Where $K^1(x,y) = K(x,y)$ is symmetric then $K^1(x,y)$ is also symmetric and is the kernel of the integral transformation K^n. The kernel $K^n(x,y)$ is called the nth iterated kernelof $K(x,y)$. Such a kernel will be given a more natural development when we deal with the topic of integral equations.

Finally we notice that if $n = r + s$ then $K^n = K^r K^S$and:

$$K^n(x,y) = \int_a^b K^r(x,t)K^s(t,y)\,dt$$

Theorem (1.2.16):-

If $\{\lambda_i\}$ and $\{f_i\}$ are defined as in Theorem (1.2.13), then:

$$H^n(x,y) = \sum_i {\lambda_i}^n f_i(x)f_i(y), \qquad n = 2,3,\ldots. \qquad (1.36)$$

And the series converges absolutely uniformly on:

$$a \le x \le b : a \le y \le b\ .$$

Proof:-

For fixed y, define $g(x) = K^{n-1}(x,y)$, then:

$$Kg(x) = \int_a^b K(z,t)g(t)dt = \int_a^a K(z,t)K^{n-1}(t,y)dt = K^n(z,y)$$

Theorem (1.2.13) states that every function of the form $Kg(x)$ has an expansion in eigenvectors given by:

$$Kg(x) = \sum_i (Kg, f_i)f_i(x)$$

Where, in this case,

$$(Kg, f_i) = \int_a^b K^n(u,y) f_i(y) du = K^n f_i(y) = {\lambda_i}^n f_i(y)$$

This yields the expansion (1.36) which converges uniformly in *x* for each fixed y by Theorem (1.2.13). To complete the proof we must show that the series is absolutely uniformly convergent in both *x* and y. To this end, first notice that in the case when $n = 2$ we have:

$$K^2(x,x) = \sum_i {\lambda_i}^2 |f_i(x)|^2 \, , a \le x \le b \, .$$

This series converges uniformly. Therefore there exists an integer $N(\varepsilon)$ such that if$mandn$have the same sign and $|n| < N(\varepsilon)$ then:

$$\sum_{i=m}^{n} {\lambda_i}^2 |f_i(x)|^2 < \varepsilon \, , a \le x \le b.$$

Further we may choose $N(\varepsilon)$large enough to ensure that $|\lambda_i| < 1$ for $i > N(\varepsilon)$consequently, on applying Schwarz's inequality we obtain:

$$\left| \sum_{i=m}^{n} {\lambda_i}^2 f_i(x) f_i(y) \right|^2 \le \sum_{i=m}^{n} |\lambda_i|^n |f_i(x)|^2 \sum_{i=m}^{n} |\lambda_i|^n |f_i(y)|^2$$

$$\le \sum_{i=m}^{n} |\lambda_i|^2 |f_i(x)|^2 \sum_{i=m}^{n} |\lambda_i|^2 |f_i(y)|^2$$

$$\le \varepsilon^2 \, ,$$

$$a \le x \le b; a \le y \le b.$$

In view of the Cauchy criterion for convergence, the theorem follows.

Definition (1.2.17):-

We will now examine the eigenvalues of the operator K^n. We shall find that just as in the finite dimensional case, the eigenvalue are the nth powers of the eigenvalues of K.

Theorem (1.2.18):-

If $\{\lambda_i\}$ and $\{f_i\}$ are defined as in theorem (1.2.13), the non-zero eigenvalues of $K^n$are ${\lambda_i}^n$. Further, if $S(r)$ is the set of indices i such that ${\lambda_i}^n = r \neq 0$, then the number of linearly independent eigenvectors of K^n with eigenvalue $r$is the number of indices in $S(r)$.

Proof:-

Let $K^n \mathrm{g} = r\mathrm{g}, r \neq 0$. Then by means of the uniform convergence established in Theorem (1.2.16), we can write:

$$K^n \mathrm{g} = \sum_i {\lambda_i}^n (\mathrm{g}, f_i) f_i$$

Every eigenvector of K corresponding to an eigenvalue $\lambda_i$is also an eigenvector ofK^n but corresponding to an eigenvalue ${\lambda_i}^n$ since:

$$K^n f_i = K^{n-1} K f_i = \lambda_i K^{n-1} f_i.$$

Therefore by Theorem (1.2.13) we must have $(\mathrm{g}, f_i) = 0$ unless$i$is in $S(r)$ consequently we have

$$r\mathrm{g} = K^n \mathrm{g} = \sum_{i \in s(r)} {\lambda_i}^n (\mathrm{g}, f_i) f_i$$

This shows that g is a linear combination of eigenvectors of K^n corresponding to the eigenvalue ${\lambda_i}^n$.Hence$r = {\lambda_i}^n$and the number of linearly independent eigenvectors of K^n for eigenvalue$r$is exactly the number of indices in S(r) .

The expansions developed in Theorems (1.2.16) and (1.2.18) are not true, in general for n=1 as can be seen by examining once again theorem (1.2.13).However, such expansions are not the only ones possible as the next theorem shows.

Theorem (1.2.19):-

$$\lim_{n\to\infty} \int_a^b \left| K(x,y) - \sum_{i=-n}^{n} \lambda_i f_i(x) f_i(y) \right|^2 dx = 0$$

Uniformly on $a \le y \le b$

Proof:-

Writing

$$\int_a^b \left| K(x,y) - \sum_{i=-n}^{n} \lambda_i f_i(x) f_i(y) \right|^2 dx = \int_a^b |X - Y|^2 \, dx$$

We see that

$$\int_a^b |X - Y|^2 dx = (X - Y, X - Y) = (X,X) - (Y,Y) - (Y,X) + (Y,Y).$$

Then

$$(X,X) = \int_a^b K(x,y)K(x,y)dx = K^2(y,y).$$

$$(X,Y) = \int_a^b K(x,y) \sum_{i=-n}^{n} \lambda_i f_i(x) f_i(y)\, dx$$

$$= \sum_{i=-n}^{n} \lambda_i f_i(y) Kf(y)$$

$$= \sum_{i=-n}^{n} |f_i(y)|^2$$

$$(Y,Y)^2 = \int_a^b \sum_{i=-n}^{n} \lambda_i f_i(x) f_i(y) \sum_{i=-n}^{n} \lambda_i f_i(x) f_i(y)\, dy$$

$$= \sum {\lambda_i}^2 |f_i(y)|^2$$

This last result follows from the orthonormality of the vectorsf_i.

Therefore we have:

$$\int_a^b \left| K(x,y) - \sum_{i=-n}^{n} \lambda_i f_i(x) f_i(y) \right|^2 dx = K^2(y,y) - \sum_{i=-n}^{n} {\lambda_i}^2 |f_i(y)|^2$$

The theorem then follows as a result of Theorem (1.2.16).

Definition (1.2.20):-

The symmetric kernel $K(x,y)$ is said to be positive definite when it defines a positive definite transformation, that is, when $(Kf,f) > 0$ for all f in R and $f \neq 0$,the symmetric kernel is said to be positive semi-definite if $(Kf,f) \geq$

0,for all f in R It follows directly from Theorem (1.2.7) that the kernel is positive semi-definite if and only if all its eigenvalues are non-negative.

Theorem (1.2.21):-

If $K(x,y)$ is positive semi-definite then $K(x,x) \geq 0$for $a \leq x \leq b$.

Proof:-

Suppose $K(z,z) < 0$for some z, then there is an interval I containing z such that $K(x,y) < 0$for$x$in I and y in I. Now choose $f(x) = 1$ for x in J and zero elsewhere, and we find that:

$$(Kf,f) = \int_I \int_I k(x,y)dxdy < 0.$$

This contradiction consequently establishes the theorem.

Theorem (1.2.22):-

If $K(x,y)$ is positive semi-definite then:

$$K(x,y) = \sum_i \lambda_i f_i(x) f_i(y)$$

And the series converges absolutely uniformly for $a \leq x \leq b;\ a \leq y \leq b$.

Proof:-

The kernel

$$H(x,y) = K(x,y) - \sum_{i=1}^{n} \lambda_i f_i(x) f_i(y)$$

Generates a transformation which has non-zero eigenvalues

$$\lambda_{i+1}, \lambda_{i+2}, \ldots \ldots$$

Since they are non-zero the kernel $H(x, y)$must be positive semi definite and we have by Theorem (1.2.21) that

$$K(x,x) - \sum_{i=1}^{n} \lambda_i |f_i(x)|^2 \geq 0$$

In the limit as $n \to \infty$ we have

$$\sum_{i=1}^{n} \lambda_i |f_i(x)|^2 \leq K(x,x) \qquad (1.37)$$

Therefore for fixed x we can find an integer $N(\varepsilon)$, for all $\varepsilon > 0$, such that:

$$\sum_{i=1}^{n} \lambda_i |f_i(x)|^2 \leq \varepsilon^2$$

Whenever $n \geq m \geq N(\varepsilon)$

Now by Schwarz's inequality

$$\left| \sum_{i=m}^{n} \lambda_i f_i(x) f_i(y) \right|^2 \leq \sum_{i=m}^{n} \lambda_i |f_i(x)|^2 \sum_{i=m}^{n} \lambda_i |f_i(y)|^2 \leq \varepsilon^2 A^2$$

Where:

$$A = \text{l. u. b.} \{K(y,y) | a \leq y \leq b\} \geq \sum_{i} \lambda_i |f_i(y)|^2, by (1.37)$$

This inequality together with the Cauchy criterion establishes that our series converges uniformly in $y$for fixed x(Similarly with respect to $x$for fixed y). Now

$$\int_a^b \left| K(x,y) - \sum_i \lambda_i f_i(x) f_i(y) \right|^2 dy$$

$$\leq 2 \int_a^b \left| K(x,y) - \sum_{i=1}^{n} \lambda_i f_i(x) f_i(y) \right|^2 dy$$

$$+ 2 \int_a^b \left| K(x,y) - \sum_{i=1}^{n} \lambda_i f_i(x) f_i(y) \right|^2 dy,$$

And by theorem (1.2.19) and the uniform convergence just established we obtain

$$\int_b^a \left| K(x,y) - \sum \lambda_i f_i(x) f_i(y) \right|^2 dy = 0, \qquad a \leq x \leq b.$$

Finally, we must establish the uniform convergence in both x and y.

Putting $x = y$ in (1.38) we obtain

$$K(x,y) = \sum_i \lambda_i |f_i|^2$$

This series is uniformly convergent. Therefore, there exists $\mathrm{M}(\varepsilon)$ for $\varepsilon > 0$ such that

$$\sum_{i=m}^{n} \lambda_i |f_i(x)|^2 < \varepsilon \quad , n > m > M(\varepsilon) \qquad (1.39)$$

Hence, on using the result

$$\left(\sum_{\mathrm{i=m}}^{\mathrm{n}} \lambda_i f_i(x) f_i(y)\right)^2 \leq \sum_{\mathrm{i=m}}^{\mathrm{n}} \lambda_i |f_i(y)|^2 \sum_{i=m}^{n} \lambda_i |f_i(y)|^2$$

In conjunction with (1.39) we readily establish the absolute uniform convergence.

In this, as in previous cases, similar results can be developed for the space R^* provided the usual minor modifications are made.

Chapter (2)

Differential operators

Section (2.1):Inverse operators and the δ–function

Definition (2.1.1):-

Let L be a linear ordinary differential operators acting on the space of functions $u(x)$. The inverse operator to L is $L^{-1}$and it is such that $LL^{-1} = L^{-1}L$.We assume that L^{-1} is an integral operator with kernel $K(x,t)$ so that

$$L^{-1}u = \int K(x,t)u(t)dt$$

Then formally at least we may write

$$u(x) = Iu = LL^{-1}u = L\int K(x,t)u(t)dt$$

Since L is a differential operator with respect to the variable x , we see that, formally,

$$u(x) = \int LK(x,t)u(t)dt \qquad (1.2)$$

We may write the kernel of this integral operator in the form

$$LK(x,t) = \mathrm{g}(x,t)$$

and obtain

$$u(x) = \int \mathrm{g}(x,t)u(t)dt \qquad (2.2)$$

Now, if this result is to be true for all continuous $u(t)$ it follows that $\mathrm{g}(x,t)$ must be zero whenever $x \neq t$ and when $x = t$ the integral on the right must reduce identically

to $u(x)$.To ensure that this was always the case, Dirac introduced his celebrated δ-functions in place of $g(x\ ,t)$ and obtained

$$\int \delta(t-x)u(t)dt \qquad (2.3)$$

Where $\delta(x)=0$, if $x \neq 0$

Such a function is zero everywhere except at the origin, where it becomes infinite in such a way as to ensure

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

At first sight, such a functions ,would appear to be nothing short of sheer nonsense. We conclude therefore that at best ,$\delta(x)$ is not a function in the ordinary sense .That this is indeed the case has been elegantly demonstrated by Laurent Schwartz in this theory of distributions where he justifies not only the use of the $\delta-$function as defined above but also the use of all its derivatives the theory of distributions provides a powerful mathematical tool on two main counts first it allows us to interchange limiting operations where such an interchange is not valid for ordinary functions. Secondly it allows us to use series which under normal circumstances we would call divergent for example the fact that $\delta(x)$ is not a function in the ordinary sense indicates that (2.3) is not a valid consequence of (2.1) this in turn stems from the fact that in order to obtain (2.3) we had to interchange the operations of integration and differentiation and this was not justified however Schwartz showed that if the equations are understood in the sense of the theory of distributions such interchange are perfectly justified and (2.3) becomes a valid consequence of(2.1) we shall make extensive use of the result form this theory whilst such results will be stated before they are used no proof will be given and we referred to original sources

Definition (2.1.2):-

The most important property of the δ-function and that which makes it so useful is the following for ever continuous function $\phi(x)$

$$\int_{-\infty}^{+\infty} \delta(x)\phi(x)dx = \phi(0) \qquad (2.4)$$

That is the δ -function can be handled algebraically as if it were an ordinary function. Whilst this may be, so we, must always interpret any equation involving $\delta(x)$as follow. If the equation is multiplied throughout by an arbitrary continuous function $\phi(x)$ and integrated over the range $(-\infty, +\infty)$ and, if (2.4) is used to evaluate integrals involving $\delta(x)$, then the resulting equation is correct and involves only ordinary functions.

For example

$$x\delta(x) = 0 \qquad (2.5)$$

Because for any arbitrary continuous function $\phi(x)$ we have on writing $x\phi(x) = \psi(x)$

$$\int_{-\infty}^{+\infty} \delta(x)x\phi(x)dx = \int_{-\infty}^{+\infty} \omega(x)\delta(x)dx = \psi(0) = 0$$

Finally in this introduction to the use of the δ-function we remark that the familiar techniques of integration such as integration by part and substitution can be shown to apply to integrals involving δ-functions as an example, consider the integral

$$I = \int_{-\infty}^{+\infty} \delta\big(f(x)\big)\emptyset(x)dx$$

Where $\phi(x)$is an arbitrary continuous function and $f(x)$ is a monotonic function of x which vanishes when $x = x_0$write $y = f(x)$ and it follows that d$y = f'(x)dx$.

The integral then becomes

$$I = \int_{-\infty}^{+\infty} \delta(y)\psi(y)dy$$

Where$\psi(y) = (x)/|f'(x)|$, and the modulus sign is to ensure that the integration is always from $+\infty$ to$-\infty$ hence

$$I = \psi(0) = \frac{\phi(x_0)}{|f'(x_0)|}$$

Consequently if by $\delta(x - x_0)$ we understand

$$\phi \int_{-\infty}^{+\infty} \delta(x - x_0)\phi(x)dx = \emptyset(x_0)$$

It follows that

$$\delta\big(f(x)\big) = \frac{\delta(x - x_0)}{|f'(x_0)|} \qquad (2.6)$$

As two special cases of (2.6) we have

$$\int_{-\infty}^{+\infty} \delta(ax - b)\phi(x)dx = |a|^{-1}\phi(ba^{-1})$$

and

$$\delta(x) = \delta(-x)$$

The arbitrary continuous function ϕ we have used to test the validity of (2.5) and (2.6) we shall in future to as a testing function for our study of differential equation it will be convenient to restrict the term testing function to mean those function ϕ which are continuous have continuous derivatives of all orders and vanish outside a certain finite interval since ϕ has continuous derivatives of all orders we say that it belongs to the space of C^{∞}functions. in addition as ϕ vanishes outside a finite interval we say that it has compact support where by the support of ϕ we mean the closure of the set of points where ϕ is non-vanishing these two properties of the

testing functions which we shall use we can sum up in notational form by saying ϕ is in $C_0^\infty$the space of functions with continuous derivatives of all orders and having compact support we notice that the restriction of testing function to C_0^∞ functions in no way invalidates (2.5) or (2.6)

Definition (2.1.3):-

The set of all such testing functions as we have described can easily be seen to form a linear vector space we find however that it is more convenient to work with the basic notions of convergence than to introduce an inner product into this space. We say that a sequence of testing functions $\{\phi\}$ converges to zero if the functions ϕ_n and all their derivatives converge uniformly to zero and if all the function $\phi_n$vanish identically outside the same finite interval.

We define a linear functional $F(\phi)$ on the space of testing functions ϕ as follows :$F(\phi)$ is a linear functional on the space of testing functions if to every testing function$\emptyset$in this space a real or complex number $F(\emptyset)$ is assigned such that

$$F(\phi_1 + \phi_2) = F(\phi_1) + F(\phi_2)$$

$$F(R\phi) = RF(\phi)$$

For any scalar quantity R such a functional is said to be continuous if the sequence of numbers $F(\phi)$ converges to zero whenever the sequence of testing function $\{\phi_n(x)\}$ converges to zero in the sense described above. Schwartz refers to any continuous linear functional on the space of testing functions as a distribution.

Typical examples of continuous linear functional are

$$F_1(\phi) = \phi'(0)$$

$$F_2(\phi) = \int_0^1 \phi(x)dx$$

Definition (2.1.4):-

Any continuous linear functional on space S for which there is defined an inner product can be expressed as an inner product of element of S it therefore seems strange that we have deliberately avoided introducing an inner product into our space of testing function it would seem natural to introduce the inner product

$$(\phi, \psi) = \int_{-\infty}^{+\infty} \phi(x)\, \psi(x) dx$$

However if this is done we find that the space of testing functions so restricted is not complete, the following example illustrates this point

Example (2.1.5):-

Consider the linear vector space of functions $f(t)$ continuous on [0,1] we define the scalar product by

$$(f, \mathrm{g}) = \int_0^1 f(x)\mathrm{g}(x) dx$$

Now consider the sequences of continuous functions

$$f_n(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2} - \frac{1}{2n} \\ n\left(t - \frac{1}{2}\right) + \frac{1}{2}, & \frac{1}{2} - \frac{1}{2n} \le t \le \frac{1}{2} + \frac{1}{2n} \\ 1, & \frac{1}{2} + \frac{1}{2n} \le t \le 1 \end{cases}$$

For $n = 1,2,3$ ... ... it is easy to show that these functions converge in the Cauchy sense that is given $\varepsilon > 0$

$$\|f_n - f_m\|^2 = \int_0^1 (f_n - f_m)^2 dt < \varepsilon$$

Whenever n and m are greater than some suitably chosen $N(\varepsilon)$.However the limit of this sequence is the function

$$f(x) = \begin{cases} 0; & 0 \le x \le \frac{1}{2} \\ 1; & \frac{1}{2} \le x \le 1 \end{cases}$$

Which is discontinuous and therefore does not belong to the space under consideration.

Nevertheless, we would still like to express a continuous linear functional $F(\phi)$ as an integral preferably in the form

$$F(\phi) = \int_{-\infty}^{+\infty} s(x)\phi(x)dx$$

Sometimes this is possible as for example in the case of the functional $F_2(\phi)$

$$F_2(\phi) = \int_0^1 \phi(x)\, dx$$

Where in this case

$$s(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Such a representation is not so immediate in the case of $F_1(\phi) = \phi'(0)$ or.
$F_3(\phi) = \phi'(0)$

However we have seen that

$$\phi(0) = \int_{-\infty}^{+\infty} \delta(x)\phi(x)dx$$

and consequently

$$F_3(\phi) = \phi(0) = \int_{-\infty}^{+\infty} \delta(x)\phi(x)dx$$

Where we have chosen $s(x) = \delta(x)$ the use of the δ-function in this context is an example of a symbolic function by the term symbolic function we mean a symbol $s(x)$ which enables us to write any continuous linear functional, $F(\phi)$ on the space of testing function in the form

$$F(\phi) = \int_{-\infty}^{+\infty} s(x)\phi(x)dx$$

These symbolic functions need not necessarily have any numerical values. They only have values when multiplied by a testing function and integrated. We notice that if $f(x)$ is an integral function, then

$$\int_{-\infty}^{+\infty} f(x)\phi(x)dx$$

Is a continuous linear functional on the space of testing functions.

consequently every integrable function is a symbolic function but there are many symbolic functions as for example the function which are neither integrable nor functions in the ordinary sense we summarize what is proved in the theory of distributions by saying that symbolic functions may be manipulated as if they were ordinary functions however any equation involving symbolic functions to be equation involving symbolic function is to be understood in the sense that if that equation is first multiplied throughout by an arbitrary testing function and then integrated over$(-\infty, +\infty)$,the result is a correct equation involving only ordinary functions.

Definition (2.1.6):-

The symbolic function is defined by the functional it produces and it would seem natural to try to define the derivative of a symbolic function in terms of the derivative of an ordinary function this we can do if first we notice that for any integrable function f possessing a continuous first derivative,

$$\int_{-\infty}^{+\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} f(x)\phi'(x)dx$$

This follow as a result of one integration by parts and noticing that the testing function has compact support we use the result to define the derivative of a symbolic function as follow we say that $s'(x)$ is the derivative of $s(x)$ if

$$\int_{-\infty}^{+\infty} s'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} s(x)\phi'(x)dx \quad (2.7)$$

For every testing function $\phi(x)$.

For example the symbolic function $\delta'(x)$is defined by

$$\int_{-\infty}^{+\infty} \delta'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} \delta(x)\phi'(x)dx = -\phi'(0)$$

and we see that $\delta'(x)$is produces that functional which assigns the value$-\phi'(x)$ to the testing function $\phi(x)$.Similarly we define by $\delta''(x)$is

$$\int_{-\infty}^{+\infty} \delta"(x)\phi(x)dx = -\int_{-\infty}^{+\infty} \delta'(x)\phi'(x)dx = -\phi"(0)$$

With this definition of a derivative a available we can show that the function has a representation in terms of the derivative of the Haeaviside unit function $H(x)$ defined as

$$H(x) = 1, \quad x > 0$$

$$= 0, \quad x < 0$$

To see this we use (2.7) to obtain

$$\int_{-\infty}^{+\infty} H'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} H(x)\phi'(x)dx$$

$$= -\int_{0}^{\infty} \phi'(x)dx$$

$$= \phi(0)$$

Since the testing function has compact support hence by comparison with (2.4) we see that

$$H'(x) = \delta(x)$$

It can be shown that the definition of derivative as afforded by (2.7) enables us to use all the usual rules of differentiation which are employed when dealing with ordinary functions.

These notions of symbolic functions and symbolic derivative enable us to attach a meaning to the derivative of a function that has a jump discontinuity at $x = x_0$ of magnitudea_1 but that everywhere else has a piecewise continuous derivative then the derivative of $f(x)$ is $f'(x)$ for $x < x_1$and $x > x_1$but is undefined for

$x = x_1$however we can define a symbolic derivative $f'_s(x)$ of $f(x)$

by means of (2.7) to this end let

$$g(x) = f(x) - a_1 H(x - x_1)$$

Where $H(x - x_1) = 1; \quad x > x_1$

$= 0, \quad x < x_1$

Then for any testing function $\phi(x)$,

$$\int_{-\infty}^{+\infty} f(x)\phi'(x)dx = \int_{-\infty}^{+\infty} g(x)\phi'(x)dx + a_1 \int_{-\infty}^{+\infty} H(x - x_1)\phi'(x)dx$$

$$= \int_{-\infty}^{\infty} g'(x)\phi(x)dx - a_1\phi(x_1)$$

Using (2.7) to define $f'_s(x)$we also have

$$\int_{-\infty}^{\infty} f_s'\phi(x)dx = \int_{-\infty}^{\infty} \{g'(x) + a_1\delta(x - x_1)\}\phi'(x)dx$$

and it follows that we can write

$$f_s' = g'(x) + a_1\delta(x - x_1)$$

Since $g'(x) = f'(x) + a_1\delta(x - x_1)$ It follow that $f' = g'$ except at $x = x_1$

This is easily generalized to functions $f(x)$ having jumps magnitude $a_1, \ldots\ldots, a_m$at the points $x_1, \ldots\ldots, x_m$ we then obtain

$$f_s' = f' + a_1\delta(x - x_1) + a_2\delta(x - x_2) + \cdots . + a_m\delta(x - x_m)$$

For the symbolic derivative of such a piecewise differentiable function

We shall be using the symbolic derivative very frequently and in the light of what we have just said the omission of the subscript s will cause no ambiguity.

Example (2.1.7):-

Consider $f(x) = |x|$.We will determine the symbolic derivative of such a function $\phi(x)$if is any testing function we have from (2.7)

$$\int_{-\infty}^{+\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} f(x)\phi'(x)dx$$

$$= -\int_{-\infty}^{0} (-x)\,\phi'(x)dx - \int_{0}^{\infty} x\phi'(x)dx$$

$$= -\int_{-\infty}^{0} \phi(x)dx + \int_{0}^{\infty} \phi(x)dx$$

$$= \int_{-\infty}^{\infty} (sgnx)\phi(x)dx$$

Therefore

$$f'(x) = |x|' = \text{sgn}(x)$$

Where sgn x is the function signum$x$ and is defined to be $(-1)$for x negative and $(+1)$ for x positive

Example (2.1.8):-

The function

$$\text{sgn}\, x = 1, \qquad x > 0$$

$$= -1, \qquad x < 0$$

has a jump magnitude 2 at $x = 0$ consequently the symbolic derivative of such a function is given by (2.8) as

$$\frac{d}{dx}(sgnx) = 2\delta(x)$$

These functions and their associated function are useful in many other fielder of analysis. For instance we can evaluate

$$I = \int_{-1}^{+1} |x| \quad \psi"(x)dx$$

By means of repeated integration by parts as follow

$$I = \{|x|\psi'(x)\}\Big|_{-1}^{+1} - \int_{-1}^{+1} |x|'\psi'(x)dx$$

$$= \psi'(1) - \psi'(-1) - \{x'\psi(x)\}\Big|_{-1}^{+1} + \int_{-1}^{+1} |x|"\psi(x)dx$$

$$= \psi'(1) - \psi'(-1) - \{sgn(x)\psi(x)\}\Big|_{-1}^{+1} + 2\int_{-1}^{+1} \delta(x)\psi(x)dx$$

$$= \psi'(1) - \psi'(-1) - \psi(1) - \psi(-1) + 2\psi(0)$$

This process should be compared with the method of evaluation which we begin by writing

$$I = -\int_{-1}^{0} x\psi"(x)dx + \int_{0}^{1} x\psi"(x)dx$$

We now return to the problem of inverting the differential operator *L* Let $\phi(x)$and $\psi(x)$be two testing function and consider the equation

$$L\psi = \phi$$

We will assume that when the inverse operator exists it does so in the form of an integral operator with a kernel $K(x,t)$, such that

$$L^{-1}\phi(x) = \int K(x,t)\phi(t)dt$$

But now we allow $K(x,t)$to be a symbolic function in the sense defined above applying the operator L to both sides of this equation we obtain:

$$LL^{-1}\phi(x) = \phi(x) = \int LK(x,t)\phi(t)dx$$

This equation will be satisfied if we can find $K(x,t)$ such that

$$LK(x,t) = \delta(x-t)$$

Where all differentiation is to be understood as being symbolic differentiation

Example (2.1.9):-

Invert the operator $L \equiv d^2/dx^2$in this case equation (2.9) educes to

$$\frac{d^2}{dx^2}K(x,t) = \delta(x-t) \qquad (2.10)$$

Since $H'(x) = \delta(x)$ One integration yields

$$\frac{dK}{dx}(x,t) = H(x-t)\alpha(t)$$

Where $\alpha(t)$ is some arbitrary function.

Integrating, we get

$$K(x,t) = \int H(x-t)dx + x\alpha(t) + \beta(t)$$

$$= (x-t)H(x-t) + x\,\alpha(t) + \beta(t) \qquad (2.11)$$

Where $\beta(t)$ is an arbitrary function .It can be shown (Schwartz) that any symbolic function which is a solution of (2.10) can be written in the form (2.11) we see that $K(x,t)$ in (2.11) is a continuous piecewise differentiable function, further we notice that if $f(x)$ is any integrable function with compact support the function

$$u(x) = \int K(x,t)f(t)dt$$

Satisfies the equation

$$\frac{d^2u}{dx^2} = f(x) \qquad (2.12)$$

The final expression (2.11) for $K(x,t)$ contains two arbitrary functions and so in general we would expect to be able to satisfy two boundary conditions for the equation (2.12)

Example (2.1.10):-

Find the function $u(x)$ which satisfies

$$\frac{d^2u}{dx^2} = f(x) \;\; ; u(0) = u(1) = 0 \qquad (2.13)$$

This equation has as we have seen a solution in the form

$$u(x) = \int K(x,t)f(t)dt$$

Where $k(x,t)$ is given by (2.11) consequently

$$u(x) = \int_{-\infty}^{x} (x-t)f(t)dt + x\int_{-\infty}^{+\infty} \alpha(t)f(t)dt + \int_{-\infty}^{+\infty} \beta(t)f(t)ddt$$

Substituting $x = 0$ and $x = 1$ into this equation and using the boundary values imposed on $u(x)$ we obtain

$$0 = -\int_{-\infty}^{0} tf(t)dt + \int_{-\infty}^{\infty} \beta(t)f(t)dt$$

$$0 = \int_{-\infty}^{1} (1-t)f(t)dt + \int_{-\infty}^{+\infty} \alpha(t)f(t)dt + \int_{-\infty}^{\infty} \beta(t)f(t)dt$$

From the first of these equations we see that

$$\beta(t) = tH(-t)$$

Whilst from the second

$$\alpha(t) = -1 + tH(t), -\infty \leq t \leq 1$$

$$= 0, \qquad t > 1$$

Inserting these results in the form of the solution, we obtain

$$u(x) = \int_{0}^{x} (x-t)f(t)dt - x\int_{0}^{1} (1-t)f(t)dt \qquad (2.14)$$

In this case we see that the kernel has the particular form

$$K(x,t) = (x-t)H(x-t) - x(1-t) \qquad (2.15)$$

For $0 \leq x \leq 1$ and$t \leq 1$

We readily see that the kernel $K(x,y)$ satisfies the same boundary conditions as $u(x)$that is

$$K(0,t) = K(1,t) = 0$$

We shall find that even for more general differential operators .The kernel of the inverse operators considered as a function of x satisfies the same boundary conditions as does the solution of the differential equation

Definition (2.1.11):-

Before we can continue with the problem of inverting a differential operator we must be more precise in our definition of the operator itself. since in this chapter we shall be primarily concerned with linear, ordinary second-order differential operators we will write

$$L \equiv a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$

Where for the time being we will assume that $a(x)$,$b(x)$ and $c(x)$ are continuous functions of x having defined the form of the operator, it now remains to specify the linear vector space S of function on which the operator L acts we shall be concerned primarily with differential equations defined over a finite interval and this we may conveniently take to be the interval $[0,1]$ consequently we will take S to be the space of all real-valued functions which are Lebesgus square integrable over $[0,1]$ that is S contains all real functions $u(x)$ defined for $0 \leq x \leq 1$ and such that

$$\int_0^1 u(x)^2 dx < \infty$$

For physical applications the distinction between Riemann and Lebesgue integrals is unimportant mathematically we require the Lebesgue integral formulation to ensure that the space S is complete if$u$ and v are functions belonging to S we introduce an inner product into S by the definition

$$(u, v) = \int_0^1 u(x)v(x)dx$$

Definition (2.1.12):-

The operator L is a differential operator and consequently cannot be applied to all elements of S since there is no reason to suppose that every function in S is in fact differentiable .Also even when a certain function u is in S the result Lu may not be for example $u(x) = x\ sin\ (x-1)$ is in S and is differentiable but its derivative

$$u'(x) = sin(x^{-1}) - x^{-1}\ cos(x^{-1})$$

is not in S consequently we will consider L to act only on those functions u in S which ensure that $Lu(x)$ is also in S

Finally from our previous discussion of differential operators we appreciate that in order to obtain a unique solution of the differential equation

$$Lu(x) = f(x) \qquad (2.16)$$

It is not sufficient simply to specify L we also require conditions on $u(x)$ itself for convenience we take these conditions to be

$$B_1(u) = \alpha_{10}u(0) + \alpha_{11}u'(0) + \beta_{10}u(1) + \beta_{11}u'(1) = 0$$

$$B_2(u) = \alpha_{20}u(0) + \alpha_{21}u'(0) + \beta_{20}u(1) + \beta_{21}u'(1) = 0 \qquad (2.17)$$

Where the α_{ij} and $\beta_{ij}$are known constants strictly, for each different set of conditions (2,17) we should use a different symbol for the operator L although it is only the conditions which change however we shall not do this as the resulting complication in notation would serve no useful practical purpose

Definition (2.1.13):-

The domain of the operator L as follow: it is the set of all functions u in S which have piecewise continuous second derivatives satisfy (2.17) and ensure that Lu is also in S

The domain of L which is clearly a linear manifold of S need not necessarily be a subspace (i.e. a complete linear manifold) of S that is there could exist a sequence of functions $u_n(x)$ in the domain which converge to limit $u(x)$in S, although this limit, $u(x)$ is not in the domain of L

Definition (2.1.14):-

In order we may apply the theory of linear operator to the differential operator L we need to be able to define the adjoin operator L^*previously we defined the adjoin operator $L^*$by the equation

$(v\ , L\ u) = (L^* v, u)$

When dealing with differential operators we may employ a similar method to fi x our ideas, we will consider a particular example

Example (2.1.15):-

Let $L\ = \frac{d}{dx}$ on a manifold ,M of S defined by the condition $v(0)\ =\ 2v(1)$. Then ,

$$(u, Lv) = \int_0^1 u \frac{dy}{dx} dx$$

$$= [uv]_0^1 - \int_0^1 v \frac{du}{dx} dx$$

$$= v(1)[u(1) - 2u(0)] - \int_0^1 v \frac{du}{dx} dx$$

$$= (L^* u, v)$$

We see that L^* consists of two parts ; a differential operator $-d/dx$, and some boundary terms .In this example the differential operator $-d/dx$ is called the formal adjoint of the differential operator d/dx. The adjoint to L on the manifold M will de the operator $-d/dx$ on the manifold defined by $u(1) = 2u(0)$. Therefore we have

$$(u, Lv) = (L^* u, v)$$

Where $L^* u = -\frac{du}{dx}$

And $u(x)$ satisfies $u(1) = 2u(0)$

We notice that in this example L acts on the manifold of square integrable functions $u(x)$ which satisfy $(0) = 2v(1)$,but L^*acts on the manifold of square integrable functions $u(x)$ which are such that $u(0) = u(1)/2$. In general the manifold on which L^*acts is different form that on which L acts .One manifold is said to be the dual of the other , if $L = L^*$the differential operator is said to be formally self-adjoint . If, in addition the boundary conditions for L and L^* are equivalent in the sense that they define the same manifold then the differential operator is said to be self-adjoint.

Example (2.1.16):-

Let

$$L \equiv e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx}$$

On the manifold defined by

$$u'(0) = 0; u(1) = 0$$

Then

$$(v, Lu) = \int_0^1 v\left\{e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx}\right\} u dx$$

$$= \int_0^1 v\{e^x u'\}' dx$$

$$= [ve^x u']_0^1 - [v'e^x u]_0^1 + \int_0^1 u(e^x v')' dx$$

$$= u'(1)v(1)e^1 + v'(0)u(0) + \int_0^1 u\{e^x v" + e^x v'\} dx$$

Thus we see that the differential operator in the adjoin operator has the form:

$$e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx}$$

and L is consequently formally self-adjoin

In order that

$$(v\,, L\,u) = (L^* v, u)$$

For all u the boundary conditions satisfied by v must be

$$V'(0) = 0; v(1) = 0$$

Therefore, since $u(x)$ and $v(x)$ now satisfy the same boundary conditions it follows that L is self-adjoint

Notice, that the general second-order operator

$$Lu = a(x)\frac{d^2 u}{dx^2}$$

has a formal adjoins defined by

$$L^* v = \frac{d^2}{dx^2}(a, v) - \frac{d}{dx}(bv) + cv$$

It then follows that

$$\int_{\alpha}^{\beta}[vLu - uL^{*}v]dx = [J(v,u)]_{\alpha}^{\beta} \qquad (2.18)$$

Where

$$J(v,u) = avu' - u(av)' + buv$$

$J(v,u)$is call the conjunct of the functions v and u.

Section(2.2):

Green's functions and second-order differential operators

A large number of problems in mathematical physics reduce to the study of second-order differential equations. Consequently in this section we consider second-order differential operators in some detail unless otherwise stated e will take as our second-order operators the general form

$$Lu = -\frac{1}{\omega}(pu')' + qu \qquad (2.19)$$

and this is self-ad joint, provided the scalar product is chosen to be

$$(u\,,v) = \int_{0}^{1} u(x)v(x)\omega\,(x)dx \qquad (2.20)$$

The minus sign in the definition of L is to ensure that the operator is positive definite. That this is the case can be seen as follows:

$$(u, Lu) = \int_{0}^{1} u\left(-\frac{1}{\omega}(pu')' + qu\right)\omega dx$$

$$= \int_0^1 (pu'^2 + uq\omega u^2)dx - upuu' \Big|_0^1$$

Consequently if, $p > 0, q > 0, \omega > 0$, and the boundary conditions are such that the integrated terms vanish we have that

$(u, Lu) > 0$

as required

A part from this one mention of boundary conditions all we have achieved so far is the requirement for L to be formal self-adjoint we now wish to find the conditions which will ensure that L is self-adjoint to this end if we examine the difference $(v\ , Lu) - (Lv, u)$, using (2.18)we see that

$$(v, Lu) - (Lv, u) = J(v\ , u)|_0^1 = -p(x)(vu' - uv')|_0^1$$

Therefore, J will be self-adjoint if $J(v, u)$ vanishes identically when u and v are in the same manifold. Tow particular special cases arise which will have important applications later.

(1) Unmixed boundary conditions

Boundary conditions are said to be unmixed if they involve the function and its derivative at either $x = 0\ or$ or $x = 1$ but not at both. A typical example of an unmixed boundary condition is

$$au(0) + bu'(0) = 0$$

It is easy to show that if u satisfies an unmixed condition at $x = 0$ and an unmixed condition at $x = 1$ then L as defined by (2.19) is self-adjoint.

(2) Periodic boundary conditions

Bounded conditions are said to be periodic if they have the form:

$$u(0) = u(1);\ \ u'(0) = u'(1)$$

Again it is easy to show that, for such conditions,L is self-adjoint. We now prove the following theorem.

Theorem (2.2.1):-

If u is any solution of $Lu = 0$ and if v is any solution of $L^*v = 0$, the conjunct of u and v is a constant whose value depends on u and v.

Proof

L and$L^*$are formal self-adjoint in the space S thus from the definition of the conjunct we have that

$$J(u,v)|_{\alpha}^{\beta} = \int_{\alpha}^{\beta} (vLu - uL^*v)dx = 0$$

Therefore the value of $J(v,u)$at $x = \alpha$ and $x = \beta$ must be the same for arbitrary α and β hence $J\ (v,u)$ must be a constant

Corollary (2.2.2):-

If L is a formally self-adjoint operator and $u_1$and u_2 are two solutions of $Lu = 0$ then the conjunct of $u_1$and u_2 is a constant whose value depends on $u_1$and u_2

Corollary (2.2.3): -

If L is self-ad joint and $u_1$and u_2 are two solutions of $Lu = 0$ and if $J(u_1,u_2)$ vanishes for some value of x for which $p(x) \neq 0$ then $u_1$and u_2 are linearly dependent.

Proof

From corollary (2.2.3) it follows that the conjunct must vanish for all x consequently

$$u_1 u_2' - u_2' u_1 = 0$$

This implies that

$$\frac{u_1 u_2' - u_2 u_1'}{u_2^2}$$

One integration establishes that (u_1/u_2) is a constant and hence u_1 and u_2 are linearly dependent.

Definition (2.2.4):-

So far we have only examined homogeneous boundary value problems however we shall often encounter the following type of non-homogeneous problem :to find $u(x)$ such that

$$u''(x) = f(x); \quad u(0) = a, \quad u(1) = b$$

A possible approach to the solution of such a problem is to reduce the given problem to the consideration of homogeneous boundary values problems by writing

$$u = u_1 + u_2$$

Where

$$u_1'' = f(x);\ u_1(0) = 0;\ u_1(1) = 0$$

and

$$u_2'' = 0;\ \ u_2(0) = a;\ u_2(1) = b$$

A more convenient method however is to extend the ideas we presently have of an operator in much the same way as we extended the notion of a derivative by defining

symbolic differentiation. when we extended the processes of different ion by defining symbolic differentiation we considered a space of testing functions and then defined the symbolic derivative of a function by integrating by parts the product of the function with the ordinary derivative of a testing function we will do just this to extend the definition of a differential operator.

Definition (2.2.5):-

First we consider an example

$$K = -\frac{d^2}{dx^2} \qquad (2.21)$$

With associated boundary conditions,

$$u(0) = u(1) = 0$$

Then it is easily seen that K is self-adjoint and that M the domain of k is the set of functions u in S such that u" exists and belongs to S and such that $u(0) = u(1) = 0$

If now v belongs to M and if W belongs to S we will write

$$(Kv, w) = (v, Kw)$$

And use the left –hand side to define the symbolic function kw we must consider kw as a symbolic function because w may not belong to the domain of k even though it is in S by hypothesis. Therefore we have

$$(Kv, w) = -\int_0^1 v"wdx = -v'(1)w(1) + v'(0)w(0) - \int_0^1 vw"dx$$

Since v belongs to M the right-hand side of this equation is clearly a linear function for functions in M and consequently it may be used to define a symbolic function kw.

We see then that

$$\int_0^1 vkwdx = -\int_0^1 vw"dx - v'(1)w(1) + v'(0)w(0)$$

$$= \int_0^1 v\{-w" + w(1)\delta'(x-1) - w(0)\delta'(x)\}\,dx$$

Therefore

$$Kw = -w'' + w(1)\delta'(x-1) - w(0)\delta'(x) \qquad (2.22)$$

We should note at this point that in our previous discussions of theδ-function, we obtained the results

$$\int \phi(x)\delta(x-a)dx = \phi(a)$$

$$\int \phi(x)\delta'(x-a)dx = -\phi'(a) \qquad (2.23)$$

Provided $x = a$ was an interior point in the rang of integration we extend the definition of the δ-function by assuming that (2.23) holds, even when $x = a$ is an end point of the range we could equally well define

$$\int \phi(x)\delta(x-a)dx = \frac{1}{2}\phi(a)$$

$$\int \phi(x)\delta'(x-a)dx = -\frac{1}{2}\phi'(a)$$

When $x = a$both these extensions of the definition of the δfunction can be justified (Schwartz) and one or the other can be used provided it is used consistently for our purposes (2.23) will be quite adequate.

In the event that W belongs to M then (2.22) reduces to

$$Kw = -w"$$

as it should however, (2.22) is also applicable to function not in M for example if $w(x) = c$ a constant in [0,1] then

$$Kw = c\delta'(x-1) - c\delta'(x)$$

And we say that Kw, as defined by (2.22) is the result of applying the symbolic operator K to w

Example (2.2.6):-

To find $u(x)$ such that

$$u"(x) = f(x); \quad u(0) = a; \quad u(1) = b \qquad (2.24)$$

Using the ideas of symbolic operators introduced above and in particular (2.21) and (2.22) we can restate the problem as follows: to find a function $u(x)$ in S, such that

$$Ku = -f(x) + b\delta'(x-1) - a\delta'(x)$$

Now, we have seen in example (2.1.9) that the function $\mathrm{g}(\mathrm{x},\mathrm{t})$ satisfying

$$K\mathrm{g} = -\delta(x-1) \qquad (2.25)$$

Is

$$\mathrm{g}(x,\mathrm{t}) = (x-\mathrm{t})\mathrm{H}(x-\mathrm{t}) - x(1-\mathrm{t}), \qquad 0 \le x : t \le 1 \qquad (2.26)$$

If we multiply (2.26) by $f(t)$ and integrate with respect to t from 0 to 1 we obtain a function $u_1(x)$ given by

$$xu_1(x) = \int_0^1 f(t)\mathrm{g}(x,\mathrm{t})\mathrm{dt} = \int_0^x (x-\mathrm{t})f(t)dt - x\int_0^1 (1-t)\,dt.$$

This is a solution of $Ku_1 = -f$ as can be seen by noticing that

$$\mathrm{Ku}_1 = \mathrm{K}\int_0^1 f(t)\mathrm{g}(x,t)dt = \int_0^1 f(t)k\mathrm{g}(x,t)dt$$

$$= -\int_0^1 f(t)\delta(x-t)dt$$

$$= -f(x)$$

To complete the solution we require a function u_2 such that

$$\mathrm{K}u_2 = \mathrm{b}\delta'(-1) - \mathrm{a}\delta'(x) \qquad (2.27)$$

Thes we can obtain by differentiation of (2.25) and (2.26) with respect to t.

$$\mathrm{K}\frac{\partial \mathrm{g}}{\partial \mathrm{t}} = \delta'(x-1) \qquad (2.28)$$

$$\frac{\partial \mathrm{g}}{\partial \mathrm{t}} = -H(x-t) + x \qquad (2.29)$$

From these last two results we see that if we write

$$\mathrm{u}_2 = \mathrm{b}\frac{\partial \mathrm{g}}{\partial \mathrm{t}}\Big|_{\mathrm{t}=1} - \mathrm{a}\frac{\partial \mathrm{g}}{\partial \mathrm{t}}\Big|_{\mathrm{t}=0}$$

Then u_2 satisfies (2.27)

Since

$$\frac{\partial \mathrm{g}}{\partial \mathrm{t}}\Big|_{\mathrm{t}=1} = -H(x-1) + x = x, \quad \text{for all } x \text{ in } [0,1]$$

$$\frac{\partial \mathrm{g}}{\partial \mathrm{t}}\Big|_{t=0} = -H(x) + x = x - 1, \quad \text{for all } x \text{ in } [0,1]$$

It follows that

$$u_2 = bx - a(x-1)$$

Consequently the solution to the given problem is

$$u = u_1 + u_2$$

$$u = \int_o^x (x-t)f(t) - x\int_0^1 (1-t)f(t) + bx - a(x-1)$$

These several results which we have obtained for a particular differential operator may be extended to arbitrary differential operators.

Definition (2.2.7):-

let L be an arbitrary differential operator with domain M and denote by $L^*$the adjoint of which is assumed to have a domain M^*we say that the functions belonging to M* are testing functions for L.

Consider a function w belonging to S but not to M we wish to define Lw but cannot do so immediately because w is not in M however if v is a testing function for L that is v is contained in M* then L*v is defined and the scalar product (L*v,w) has a meaning to define the symbolic function L w we combine the properties of symbolic operators and ad joint operators by writing

$$(L^*v, w) = (v, Lw) \qquad (2.30)$$

The left-hand side of (2.30) is used to define the symbolic function Lw this extended definition of an operator as provided by (2.30) enables us to restrict ourselves to the consideration of operator with homogeneous boundary conditions because as we saw in the case of the operator k any problem with non-homogeneous boundary conditions may be changed into a non-homogeneousproblem involving an operator

having homogeneous boundary conditions consequently we will only consider homogeneous boundary conditions in future.

Definition (2.2.8):-

We are now in a position to consider more general forms of second –order differentional operators and their inverses we have already seen that the differentional operators $K = d^2/dx^2$with the associated boundary conditions has an inverse in the form of an integral operator whose kernel $\mathrm{g}(x,t)$ satisfies

$$\frac{d^2}{dx^2}\mathrm{g}(x,t) = \delta(x-t)$$

We saw in this case

$$\mathrm{g}(x,t) = (x-t)H(x-t) - x(x-t), 0 \le x,\ \ t \le 1$$

Now, suppose that L is a general second-order differential operators in x with domain M. We wish to find a function $\mathrm{g}(x,t)$ such that

$$L\mathrm{g}(x,t) = \delta(x-t) \qquad (2.31)$$

We notice that here L must be considered as a symbolic operator when applied to $\mathrm{g}(x,t)$ though of as a function of x since $\mathrm{g}(x,t)$ is not necessarily in the domain of L. Also we must use the extended definition of the operator; consequently the boundary conditions are automatically satisfied.

Definition (2.2.9):-

In conformity with our previous, rather particular discussion, we will call $\mathrm{g}(x,t)$ the Green's function of the operator L.It is readily seen that $\mathrm{g}(x,t)$ is the kernel of an integral operator which inverts$L$, for, if we write

$$u(x) = \int \mathrm{g}(x,t) f(t) dt$$

then

$$Lu = \int L\mathrm{g} f dt = \int \delta(x-t) f(t) dt = f(t)$$

Which implies that

$$u(x) = L^{-1} f(x) = \int \mathrm{g}(x,t) f(t) dt$$

So that we may proceed with the solution of (2.31) we will assume that *L* has the self-adjoint form:

$$L \equiv -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \qquad (2.32)$$

together with some homogeneous boundary conditions. We assume that $p(x)$ and $q(x)$ are continuous functions in $[0, 1]$, and further that $p(x)$ is non-vanishing in this interval.

Consider initially the case when $q(x) = 0$ .The solution of (2.31) in this case we will denote by $\mathrm{g}_0(x, \mathrm{t})$. Combining (2.31) and (2.32) we have in this case

$$-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)\mathrm{g}_0(x,t) = \delta(x-t) \qquad (2.33)$$

This we can integrate to obtain

$$\mathrm{g}_o(x,t) = -H(x,t)\int_t^x \frac{du}{p(u)} + \alpha(t)\int_o^x \frac{du}{p(u)} + \beta(t) \qquad (2.34)$$

Where β(t) is a constant of integration.

From equations (2.33) and (2.34) we notice that $g(x,t)$ is a continuous function of x and also its derivative is a continuous function of x except at the point $x = t$, where it has a jump of magnitude $(-1/p(t))$.

In order to show that $\mathrm{g}(x,t)$ has properties similar to those of $\mathrm{g}_0(x,t)$ we shall use the following theorem quoted without proof , from the theory of differential equations which concerns functions which are differentiable in the ordinary sense

Theorem (2.2.10):-

Let $p(x)$, $q(x)$ and $f(x)$ be piecewise continuous functions of x in $[0,1]$ and assume that $p(x)$ is positive in this interval. Then there exists a continuous function u such that $p(x)u'(x)$ exists and is continuous for all x , and which satisfies $u(0) = u'(0) = 0$ and $(pu')' - qu = f$, for all values of x for which both sides are continuous functions of x .

We can now show that $\mathrm{g}(x,t)$ has properties similar to those of $\mathrm{g}_0(x,t)$.If we write:

$$\mathrm{g}(x,t) = \mathrm{g}_0(x,t) + h(x,t)$$

and substitute this into equation (2.31) ,we find that

$$Lh(x,t) = -q(x)\mathrm{g}_0(x,t) \qquad (2.35)$$

Theorem (2.2.11):-

When L is given by (2.32) the Green's function satisfies the homogeneous equation. When L has an extended definition according to (2.30) the Green's function is everywhere continuous, but its derivative has a jump discontinuity, at $x = t$, of magnitude $(-1\backslash p(t))$.

Since we have seen that $L\mathrm{g} = 0$ everywhere save at $x = t, \mathrm{g}(x,t)$ is not a symbolic function, it is an ordinary function. Further, since we are using the extended

definition of the operator L it follows that $\mathrm{g}(x,\mathrm{t})$ considered as a function of x, mast satisfy the boundary conditions, If this were not the case then $\mathrm{g}(x,\mathrm{t})$would necessarily involve certain symbolic functions such as $\delta(x-t)$ and $\delta'(x-t)$ .Therefore, we shall always consider$\mathrm{g}(x,\mathrm{t})$ as an ordinary function of x , satisfying the same boundary conditions which help to specify the operator.

Once the Green's function has been determined the non-homogeneous equation

$$Lu = f \qquad (2.36)$$

With assigned boundary conditions can be solved immediately. The solution is

$$u(x) = \int_0^1 f(x)\mathrm{g}(x,t)dt \qquad (2.37)$$

That such a function as is defined by (2.37) does indeed satisfies (2.37) can be as follows:

$$Lu = \int_0^1 f(t)L\mathrm{g}(x,\mathrm{t})dt = \int_0^1 f(t)\delta(x-t)dt = f(x)$$

And $u(x)$ must satisfy the boundary conditions since $\mathrm{g}(x,\mathrm{t})$ as a function of x satisfies them.

Example (2.2.12):-

Find the Green's function for the operator

$$L \equiv -d^2/dx^2\ ;\ u(0) = u'(0) = 0$$

The required Green's function,$\mathrm{g}(x,\mathrm{t})$ is geven as the solution of the equation

$$\frac{d^2\mathrm{g}}{dx^2} = -\delta(x-t) \qquad (2.38)$$

With the conditions

$$\mathrm{g}(0,\mathrm{t}) = \mathrm{g}_x(0,\mathrm{t}) = 0. \qquad (2.39)$$

We have already seen that $\mathrm{g}(x,\mathrm{t})$satisfies

$$\frac{d^2\mathrm{g}}{dx^2} = 0 \qquad (2.40)$$

Everywhere except at $x = t$ where it is continuous but its derivative has a jump discontinuity which for this particular operator is equal to (-1) .

An arbitrary solution of (2.40) is

$$\mathrm{g}(x,\mathrm{t}) = \alpha(t)x + \beta(t). \qquad (2.41)$$

When $x < t$ the application of (2.39) to (2.40) shows that

$$\alpha(t) = \beta(t) = 0.$$

Therefore

$$\mathrm{g}(x,\mathrm{t}) = 0\ , x < t\ . \qquad (2.42)$$

For the case $x \geq t$ we use (2.41) together with the particular properties of Green's functions, namely the continuity of the solutions at $x = t$ and the discontinuity condition on the derivative at$x = t$.

From (2.41) we have

$$\frac{\partial \mathrm{g}}{\partial x} = \alpha(t)$$

And it follows from (2.42) and the fact that the magnitude of the jump in the derivative at $x = t$ is (-1) that

$$\alpha = -1.$$

Finally, since $\mathrm{g}(x,\mathrm{t})$is continuous at $x = t$ we must have

$$\lim_{x \to t^-} \mathrm{g}(x,\mathrm{t}) = \lim_{x \to t^+} \mathrm{g}(x,\mathrm{t}),$$

From which it follows that

$$\beta = t$$

Since $\mathrm{g}(x,\mathrm{t}) = 0,\ x < t.$

Therefore, we have as a solution to (2.38)

$$\mathrm{g}(x,\mathrm{t}) = 0\,, \qquad x < t$$

$$= t - x, \qquad x > t$$

or

$$\mathrm{g}(x,\mathrm{t}) = -(x-t)H(x-t) \qquad (2.43)$$

For all x.to see that this value of $\mathrm{g}(x,\mathrm{t})$satisfies (2.38) and (2.39) we differentiate (2.43) twice with respect to x to obtain

$$\mathrm{g}''(x,\mathrm{t}) = -2H'(x-t) - (x-t)H''(x-t)$$

$$= -\delta(x-t) - (x-t)\delta'(x-t).$$

If $\phi(x)$ is a testing function, multiply this last result throughout by $\phi(x)$ and integrate over the range of x we obtain

$$\int_0^1 \phi(x)\mathrm{g}''(x,\mathrm{t})\,dx = -2\int_0^1 \delta(x-t)\phi(x)\,dx - \int_0^1 \phi(x)(x-t)\delta'(x-t)\,dx$$

$$= -2\phi(t) + \{(x-t)\phi(x)\}' \Big|_x$$

$$= -\phi(t).$$

This is precisely the result we would obtain by multiplying (2.38) by $\phi(x)$ and integrating over the range of x.

Example (2.2.13):-

Solve the non-homogeneous equation

$$-\frac{d^2}{dx^2}u(x) = f(x)$$

Subject to the conditions

$$u(0) = u'(0) = 0.$$

Usung (2.37) and the the particular form for $\mathrm{g}(x, \mathrm{t})$ given by (2.43) we see that the solution is given by

$$u(x) = \int_0^1 f(t)(x-t)\,H(x-t)dt$$

$$= \int_0^x f(t)(x-t)\,dt.$$

This is easily seen, by direct substitution, to be a solution of the given boundary value problem.

Example (2.2.14):-

To solve

$$-\frac{d^2}{dx^2}u(x) = f(x)$$

Subject to the conditions:

$$u(0) = b \ ; \ u'(0) = a$$

We use the extended definition of . If S the space of square integrable functions defined over [0,1], is the space of functions with which we are dealing, then

$$L^* = -\frac{d^2}{dx^2}$$

With boundary conditions

$$v(1) = v'(1) = 0$$

Let $v(x)$ be a testing function for L, that is let v be a function in M^* the domain of L^*. The extended definition of L acting on u is given by

$$(Lu, v) = (u, L^*v) = -\int_0^1 uv''\,dx$$

$$= -\int_0^1 vu''\,dx - (uv' - u'v)\Big|_0^1$$

$$= -\int_0^1 vu''\,dx + bv'(0) - av(0).$$

Consequently, we see that

$$Lu = f(x) - a\delta(x) - b\delta'(x) \qquad (2.44)$$

As the extended form of L in this case. As might be expected, when $a = b = 0$ we see that L is actually, rather than formal, self-adjoint.

Writing

$$u_1 = \int_0^x f(t)(x-t)\,dt$$

We know that

$$Lu_1 = f(x).$$

Therefore, it remains to find a function u_2 such that

$$Lu_2 = -a\delta(x) - b\delta'(x)$$

When this is done then the solution of (2.44) will be

$$u = u_1 + u_2$$

Now since

$$L\mathrm{g}(x,t) = \delta(x-t)$$

We see that

$$L\left\{-a\mathrm{g}(x,0) + b\frac{\partial \mathrm{g}}{\partial x}(x,0)\right\} - a\delta(x) - b\delta'(x).$$

But, from (2.43), we see that

$$\mathrm{g}(x,0) = -x$$

$$\frac{\partial \mathrm{g}}{\partial x}(x,0) = 1$$

Therefore

$$u_2 = ax + b$$

And the solution of our given problem is

$$u = \int_0^x f(t)(x-t)\,dt + ax + b.$$

Admittedly this solution could have been obtained much more directly however the example was chosen primarily to illustrate, as simply as possible, a general method.

Example (2.2.15):-

We now change the boundary value problems so that both end-points of range are involved. We wish to solve

$$Lu = \frac{d^2u}{dx^2} = f(x)$$

Subject to the conditions

$$u(0) = 0\,; \quad u(1) = 0$$

The Green's function,$\mathrm{g}(x,\mathrm{t})$, in this case is given by the solution of

$$-\frac{d^2}{dx^2}\mathrm{g}(x,\mathrm{t}) = \delta(x-t)$$

On the manifold defined by the conditions

$$\mathrm{g}(0,\mathrm{t}) = \mathrm{g}(1,\mathrm{t}) = 0$$

The Green' s function is given by a solution of the homogeneous equation at all values of x save $x = t$ .For $x < t$ the solution of the homogeneous equation which satisfies $\mathrm{g}(0,\mathrm{t}) = 0$ must be proportional to x , and similarly for $x > t$ that solution which satisfies$\mathrm{g}(1,\mathrm{t}) = 0$ must be proportional to $(1-x)$.

Therefore as a trial form for $\mathrm{g}(x,\mathrm{t})$we could write

$$g(x,t) = x, \qquad x < t$$

$$= 1 - x, \quad x > t$$

This certainly satisfies the homogeneous equation everywhere saves at $= t$, and also ensures that the boundary conditions are satisfied.

However, such a solution does not satisfy the requirements of a Green's function, if only because it is not continuous at $x = t$.

That is

$$\lim_{x \to t^-} g(x,t) = t$$

$$\lim_{x \to t^+} g(x,t) = 1 - t\,.$$

To overcome this and , as it turns out , also meet the requirements of the jump discontinuity in the first derivative , multiply the value of $g(x,t)$for $x > t$ by the value of $g(x,t)$ for $x > t$ evaluated at $x = t$, and multiply the value of $g(x,t)$ for $x > t$ by the value of $g(x,t)$ for $x > t$ evaluated at $x = t$. This gives:

$$g(x,t) = x(1-t), \qquad x > t \qquad (2.45)$$

$$= (1-x)t, \qquad x > t$$

We readily see that this form for $g(x,t)$,apart from satisfying the required equation and boundary conditions, is continuous at $x = t$ and has there a jump discontinuity in its first derivative of magnitude.

$$-t - (1-t) \; = \; -1$$

As required.

Definition (2.2.16):-

The Green's function $\mathrm{g}(x,\mathrm{t})$for the operator $L \equiv -d^2/dx^2$ in the manifold defined by $u(0) = u(1) = 0$ is given by (7.45) and may be written in the more convenient form:

$$\mathrm{g}(x,t) \;=\; x(1-t)H(t-x) + t(1-x)H(x-t)$$

$$= \frac{1}{2}(x+t) - \frac{1}{2}|(x-t)| - xt \qquad (2.46)$$

We notice that $\mathrm{g}(x,\mathrm{t}) = \mathrm{g}(\mathrm{t},x)$ in this case. This symmetry of the Green's function is generally true for self-adjoint operators. We Check that (2.46) is in fact a solution by differentiation as follows:

$$\frac{d\mathrm{g}}{dx} = \frac{1}{2} - t - \frac{1}{2}sgn(x-t)$$

$$\frac{d^2\mathrm{g}}{dx^2} = -\delta(x-t).$$

Example (2.2.17):-

We wish to solve the following non-homogeneous boundary value problem

$$u'' + k^2u = -f(x) \qquad (2.47)$$

$$u(0) = a \;; u'(1) = b \qquad (2.48)$$

Our first task is to determine the Green's function $\mathrm{g}(x,\mathrm{t})$for the operator $L \equiv -(d^2/d\,x^2 + k^2)$subject to the homogeneous condition $u(0) = u'(1) = 0$once this is achieved we extend the definition of $L$to cater for the actual boundary value (2.48).

To determine the Green's function $\mathrm{g}(x,\mathrm{t})$we use the same technique here as in Example (2.2.16) since $\mathrm{g}(\mathrm{x},\mathrm{t})$is a solution of

$$\frac{\mathrm{d}^2\mathrm{g}}{\mathrm{d}x^2} + \mathrm{k}^2\mathrm{g} = -\delta(x - \mathrm{t})$$

$$\mathrm{g}(0, \mathrm{t}) = \mathrm{g}_x(1, \mathrm{t}) = 0,$$

We first consider solutions of the homogeneous equation for $\mathrm{g}(\mathrm{x}, \mathrm{t})$and choose them so that the boundary conditions at $x = 0$ and $x = 1$are satisfied.

Doing this we obtain

$$\mathrm{g}(x, \mathrm{t}) = sin\, Kx\ , \qquad x < t$$

$$= \cos K(1 - x)\,, x > t.$$

Since this form of $\mathrm{g}(x, \mathrm{t})$is not continuous at $x = t$ we multiply the first expression by the value of the second at $x = t$ and vice-versa, to obtain a function which is continuous at $x = t$,namely

$$\mathrm{g}(x, \mathrm{t}) = \sin Kx \cos K(1 - t)\,, \quad x < t \qquad (2.49)$$

$$= \cos K(1 - x) \sin Kt\,, \quad x > t.$$

We now examine the derivative of this function in the neighborhood of $x = t$ the jump in the magnitude of the derivative at $x = t$ is given by

$$K \sin K(1 - t) \sin Kt - K \cos Kt \cos K(1 - t) = -K \cos K.$$

It follows from the form of L ,that this jump should be (-1) to correct the value of $\mathrm{g}(x, \mathrm{t})$we must divide the value obtained in (2.49) by $K \cos K$ to obtain the find from:

$$\mathrm{g}(x, \mathrm{t}) = \sin Kx\, cosK(1 - t)/K \cos K\ , \ \ x < t$$

$$= \cos K(1 - x) \sin Kt/K \cos K\,, \ \ x > t$$

Or (2.50)

$$g(x,t) = \frac{\sin Kx \cos K(1-t)}{K\cos K} H(t-x) + \frac{\cos K(1-x)\sin Kt}{K\cos K} H(x-t)$$

For all x.that this is in fact a solution of the equation defining the Green's function for the problem in hand is readily established by direct differentiation.We notice that once again the Green's function is symmetric in $x$and t. Returning to the original problem we first consider the extended definition of . Evidently L is formally self-adjoint and so $L = L^*$.let v be a testing function of L (in this case any function in the domain of L will do) then extending the definition of L we obtain:

$$(Lu, v) = (u, Lv) = \int_0^1 u(v'' + k^2 v)\, dx$$

$$= \int_0^1 v(u'' + k^2 u)\, dx - (uv' - u'v)\Big|_0^1$$

$$= \int_0^1 vf(x)\, dx + av'(0) + bv(1).$$

Consequently

$$Lu = f(x) + b\delta(x-1) - a\delta'(x).$$

Proceeding as before we first write

$$Lu_1 = f$$

We now need to find a function u_2 such that

$$Lu_2 = b\delta(x-1) - a\delta'(x) \qquad (2.51)$$

And then $u = u_1 + u_2$ will be the required solution.

Since

$$\mathrm{Lg}(x,\mathrm{t}) = -\delta(x-\mathrm{t})$$

We see that

$$\mathrm{L}\left\{\mathrm{bg}(x,1) + \mathrm{a}\frac{\partial \mathrm{g}(x,0)}{\partial \mathrm{t}}\right\} = b\delta(x-1) - a\delta'(x). \qquad (2.52)$$

From (2.50) we see that

$$\mathrm{g}(x,1) = \frac{\sin Kx}{K\cos K}$$

$$\frac{\partial \mathrm{g}(x,0)}{\partial \mathrm{t}} = \frac{\cos K(1-x)}{\cos K}.$$

Consequently on comparing (2.51) and (2.52) we obtain

$$u_2(x) = \frac{b\sin Kx}{K\cos K} + \frac{a\cos K(1-x)}{\cos K}.$$

The final solution to our problem is given by

$$u = \int_0^1 f(t)\,\mathrm{g}(x,\mathrm{t})\mathrm{dt} + \frac{b\sin Kx}{K\cos K} + \frac{a\cos K(1-x)}{\cos K}, \qquad (2.53)$$

Where $\mathrm{g}(x,\mathrm{t})$ is given by (2.51).

The method used to determine the Green's function $\mathrm{g}(x,t)$in the last example is very useful in practice as we now demonstrate by considering the following unmixed boundary value problem : to find the Green's function for the general second-order ordinary differential operator.

$$L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

When the domain is defined by the general unmixed conditions (2.17)

$$B_1(u) = B_2(u) = 0$$

It is assumed that $B_1(u)$ only involves values at $x = 0$ whilst $B_2(u)$ only involves values at $x = 1$ further we will assume $p(x) \neq 0$ in the range $0 \leq x \leq 1$.

The Green's function $\mathrm{g}(x, \mathrm{t})$is that solution of the equation

$$\mathrm{Lg} = -\delta(x - t)$$

That is

$$(p\mathrm{g}_x)_x - q\mathrm{g} = -\delta(x - \mathrm{t}), \qquad (2.54)$$

Which satisfies the boundary conditions?

$$B_1(\mathrm{g}) = B_2(\mathrm{g}) = 0 \qquad (2.55)$$

In order to construct the Green's function we first of all examine the homogeneous form of equation (2.54) and consider solutions independent of any boundary conditions. Formally, such solutions exist and we will suppose that $v_1(x)$ and $v_2(x)$ are any two linearly independent solutions of the homogenous equations. Now let$\mathrm{w}_1(x)$ be a linear combination of $\mathrm{v}_1(x)$ and $v_2(x)$ which satisfies $B_1(\mathrm{w}_1) = 0$ and let $w_2(x)$ be a linear combination of $v_1(x)$ and $v_2(x)$ which ensures $B_2(w_2) = 0$ then as a first possibility for the form of the required Green's function we write

$$\mathrm{g}(x, \mathrm{t}) = \mathrm{w}_1(x), \qquad x < t$$

$$= \mathrm{w}_2(x), \qquad x > t$$

This function $\mathrm{g}(x, \mathrm{t})$ clearly satisfies (2.54) for $x \neq t$ and also the boundary conditions (2.55). However it does not necessarily satisfies the required properties of a Green's function namely continuity at $x = t$ and a jump discontinuity in derivative of magnitude$(-1/p(x))$. To make $\mathrm{g}(x, \mathrm{t})$ continuous at $x = t$ we proceed as before to multiply the first expression by the value of the second at $x = t$ and multiply the second expression by the value of the first at$x = t$.

We thus obtain

$$\mathrm{g}(x,\mathrm{t}) = \mathrm{w}_1(x)\mathrm{w}_2(t), \quad x < t \qquad (2.56)$$

$$= \mathrm{w}_2(x)\mathrm{w}_1(t), \quad x > t.$$

The jump in the derivative with respect to x evaluated $x = t$ is

$$\mathrm{w'}_2(t)\mathrm{w}_1(t) - \mathrm{w'}_1(t)\mathrm{w}_2(t) = \frac{-\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)}{p(t)}$$

Where $\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)$ is the conjunct of the solutions $\mathrm{w}_1(x)$ and$\mathrm{w}_2(x)$. To give this jump discontinuity the correct magnitude we divide the right-hand side of (2.56) by $\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)$ to obtain

$$\mathrm{g}(x,t) = w_1(x)w_2(t)/J(w_2,w_1)\,, \qquad x < t \qquad (2.57)$$

$$= w_2(x)w_1(t)/J(w_2,w_1)\,, \qquad x > t.$$

Dividing throughout by $\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)$ does not alter the fact that $\mathrm{g}(x,\mathrm{t})$ is a solution of (2.54) since $\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)$ is in general only a function of t. In addition in this case since L is self-adjoint we have the further assurance that our solution is not disturbed since by theorem (2.2.2). $\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)$most be a true constant

The fact that $\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)$ is a constant is of great practical significance since it means that we need not evaluate it at $x = t$ but may do so at any convenient value of x for instance in Example (2.2.18) the conjunct of $\cos K(1-x)$ and $\sin Kx$ has to be determined.

We obtain

$$\mathrm{J}(\cos K(1-x)\,,\sin Kx) = K\cos K(1-x)\cos Kx - K\sin K(1-x)\sin Kx$$

and this must be a constant consequently its value at $x = 0$ must be the same as at any other point thus on setting $x = 0$ in the above we obtain

$$\mathrm{J}(\cos K(1-x), \sin Kx) = K\cos K.$$

The final form for the Green's function is

$$\mathrm{g}(x,\mathrm{t}) = \frac{1}{\mathrm{J}(\mathrm{w}_2, \mathrm{w}_1)}\{\mathrm{w}_1(x)\mathrm{w}_2(t)H(t-x) + \mathrm{w}_2(x)\mathrm{w}_1(t)H(x-t)\}. \qquad (2.58)$$

This formula breaks down whenever$\mathrm{J}(\mathrm{w}_2, \mathrm{w}_1) = 0$. By theorem (2.2.2) and its corollaries we see that a vanishing conjunct implies that one solution $\mathrm{w}_2(x)$ is a multiple of the other solution $\mathrm{w}_1(x)$. This is contrary to our hypothesis that $\mathrm{w}_1(x)$and $\mathrm{w}_2(x)$ are linearly independent. However we will examine the consequences of a vanishing conjunct. If $\mathrm{w}_2(x)$ is a multiple of $\mathrm{w}_1(x)$ then since $\mathrm{B}_1(\mathrm{w}_1) = 0$ and $\mathrm{B}_2(\mathrm{w}_2) = 0$ it follows that$\mathrm{B}_1(\mathrm{w}_1) = \mathrm{B}_2(\mathrm{w}_2) = 0$. Therefore we see that (w_2) is a non-trivial solution of the equation

$$(pu')' - qu = 0$$

With boundary conditions

$$\mathrm{B}_1(\mathrm{u}) = \mathrm{B}_2(\mathrm{u}) = 0$$

Consequently $\mathrm{w}_2(x)$ must be an eigenfunction of the operator L with eigenvalue $= 0$.

We now turn our attention to the solution of a non-homogeneous equation with non-homogeneous boundary condition we proceed as befor to extend the definition of the following example.

Example (2.2.18):-

To find a solution $Lu = f$ such that $\mathrm{B}_1(\mathrm{u}) = a$, and $\mathrm{B}_2(\mathrm{u}) = \mathrm{b}$ the operator L is as before the formally self-ad joint operator

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

And the boundary conditions are the general unmixed conditions in (2.17). We know that the function

$$\mathrm{u}_1(x) = \int_0^1 f(x)\,\mathrm{g}(x,\mathrm{t})\mathrm{dt}$$

Where $\mathrm{g}(x,\mathrm{t})$ is the Green's function for L and homogeneous boundary conditions is solution of

$$\mathrm{Lu}_1 = f$$

Such that

$$\mathrm{B}_1(\mathrm{u}) = \mathrm{B}_2(\mathrm{u}) = 0$$

To solve our original problem we require finding a further function $\mathrm{u}_2(\mathrm{x})$ which is a solution of the equation

$$L\mathrm{u}_2 = 0; x \neq 1; x \neq 0$$

But satisfying

$$\mathrm{B}_1(\mathrm{u}_2) = \mathrm{a}\ ;\ \mathrm{B}_2(\mathrm{u}_2) = b$$

For definiteness we will assume that the boundary conditions have that precise form

$$\mathrm{B}_1(\mathrm{u}) = \mathrm{u}(0)\cos\alpha + \mathrm{u}'(0)\sin\alpha$$

$$B_2(u) = u(1)\cos\beta + u'(1)\sin\beta$$

Where α and β are constants.

Since L is formula self-adjoint any function v in the domain of L is a testing function for L and we obtain the extended definition for L acting on u by considering the inner product (u, Lv) and(Lu, v).

$$(Lu, v) = (v, Lu)$$

$$= \int_0^1 u[-(pv')' + qv]\,dx$$

$$= \int_0^1 v[-(pu')' + qu]\,dx - p(uv' - vu')\Big|_0^1. \qquad (2.59)$$

From the specific form of the boundary conditions we have adapted, we obtain

$$p(uv' - vu')\Big|_0^1 = -p(1)b[v(1)\sin\beta - v'(1)\cos\beta]$$

$$+ p(0)a[v(0)\sin\alpha - v'(0)\cos\alpha]. \qquad (2.60)$$

Consequently from (2.59) and (2.60) we obtain as the extended definition of Lu the expression

$$Lu = f(x) + bp(1)[\delta(x-1)\sin\beta - \delta'(x-1)\cos\beta]$$
$$- ap(0)[\delta(x)\sin\alpha - \delta'(x)\cos\alpha].$$

Therefore to complete our solution we require a function $u_2(x)$ such that

$$Lu_2 = bp(1)[\delta(x-1)\sin\beta - \delta'(x-1)\cos\beta]$$
$$- ap(0)[\delta(x)\sin\alpha - \delta'(x)\cos\alpha]. \qquad (2.61)$$

Now as

$$L\mathrm{g}(x,\mathrm{t}) = -\delta(x,t)$$

We see that

$$\mathrm{u}_2 = bp(1)\left\{\mathrm{g}(x,\mathrm{t})\sin\beta - \frac{\partial \mathrm{g}(x,\mathrm{t})}{\partial t}\cos\beta\right\}$$

$$-ap(0)\left\{\mathrm{g}(x,\mathrm{t})\sin\alpha - \frac{\partial \mathrm{g}(x,0)}{\partial t}\cos\alpha\right\}$$

Is a solution of (2.61).

We remark here that $\partial \mathrm{g}/\partial t$ appears in the above expression rather than $\partial \mathrm{g}/\partial x$ because $\partial/\partial x$ and L do not commute but $\partial/\partial t$ and L do, thus enabling us to obtain an alternative expression for $\delta'(x-t)$ from the equation defining $\mathrm{g}(x,t)$.

To obtain the precise form of $\mathrm{u}_2(x)$ we use (2.58) to eliminate $\mathrm{g}(\mathrm{x},\mathrm{t})$ from the above expression and obtain

$$\mathrm{u}_2(x) = \frac{bp(1)\mathrm{w}_1(\mathrm{x})}{\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)}\{\mathrm{w}_2(1)\sin\beta - \mathrm{w'}_2(1)\cos\beta\}$$

$$-\frac{ap(1)\mathrm{w}_2(\mathrm{x})}{\mathrm{J}(\mathrm{w}_2,\mathrm{w}_1)}\{\mathrm{w}_1(0)\sin\alpha - \mathrm{w'}_1(0)\cos\alpha\} \qquad (2.62)$$

Similarly we obtain

$$u_1(x) = \frac{w_2(x)}{J(w_2, w_1)} \int_0^x w_1(t) f(t)\, dt + \frac{w_1(x)}{J(w_2, w_1)} \int_x^1 w_2(t) f(t)\, dt \qquad (2.63)$$

The required solution is then

$$u = u_1 + u_2$$

Chapter (3)

The Problem of Eigenfunctions

Section (3.1):Eigenfunctions

The Green's function technique so far as we have presently developed it, requires that we find a function $g(x,t)$defined as that solution of the differential equation $Lg(x,t) = -\delta(x-t)$ which satisfies the same boundary conditions as the unknown function in the original problem. We have seen that in certain cases when the boundary conditions are homogeneous the method breaks down (see Example(2.2.18)). This we noticed, was due essentially to the fact that the equation $Lu = 0$, with associated homogeneous boundary condition possesses a non-trivial solution, that is to say $u(x)$is an eigenfunction of the operator L, and corresponds to the eigenvalue zero. The main purpose of this section is to examine this situation in more detail and the principal result which we shall obtain is contained in the following theorem.

Theorem (3.1.1):-

Let $w(x)$ be a unique solution a part possibly form an additive constant of the self-adjoint homogeneous differential equation:

$$Lu \equiv -(pw')' + qw = 0,$$

which satisfies the conditions

$$B_1(w) = B_2(w) = 0.$$

Then the non-homogeneous equation

$$Lu = f \qquad (3.1)$$

With boundary conditions

$$B_1(u) = B_2(w) = 0$$

has a solution if and only if

$$(w, f) = \int_0^1 f(x)w(x)\,dx = 0. \qquad (3.2)$$

Proof:-

We will assume that the boundary conditions are unmixed. The theorem however is also valid for arbitrary boundary conditions.

Let u be a solution of (3.1) then since self-adjoint we have

$$0 = (u, Lu) - (w, Lu) = -(w, f)$$

and we see that (3.2) is satisfied.

Conversely, suppose (3.2) is true. Let $v(x)$be a solution of

$$Lu = 0$$

Which is independent of $w(x)$. Then by using the Green's function technique as illustrated in Examples(2.2.12) and (2.2.13) we can find a solution of (3.1) satisfying the boundary condition $u(0) = u'(0) = 0$, rather than $B_1(u) = B_2(u) = 0$ as follows. The required Green's function g(x, t)can be assumed to have the form

$$\mathrm{g}(x, t) = \alpha(t)v(x) + \beta(t)w(x).$$

In the range $x < t$ the requirement $v(0) = v'(0) = 0$ yields,

$$\mathrm{g}(x, t) = \alpha(t)v(x), \qquad x < t.$$

For $x > t$ we are at liberty to choose

$$\mathrm{g}(x,t) = \beta(t)w(x), \quad x > t.$$

Then adjusting $\alpha(t)$and $\beta(t)$to ensure the continuity of $\mathrm{g}(x,t)$at $x = t$ we obtain

$$\mathrm{g}(x,t) = v(x)w(t), \quad x < t$$

$$= w(x)v(t), \qquad x > t.$$

Finally to ensure the correct magnitude of the jump in the discontinuity of the derivative we must divide by $J(w,v)$. Therefore the required Green's function in this case is:

$$\mathrm{g}(x,t) = \frac{w(t)v(x)H(t-x)}{J(v,w)} + \frac{v(t)w(x)H(x-t)}{J(v,w)} \qquad (3.3)$$

The solution (3.1) which satisfies the conditions $u(0) = u'(0) = 0$ is then given by:

$$u(x) = \frac{v(x)}{J(w,v)}\int_x^1 \big(v(t) + w(t)\big)f(t)\,dt + \frac{(v(x) + w(x))}{J(w,v)}\int_0^x v(t)f(t)\,dt$$

Re-arranging and using (3.2) we finally obtain

$$u(x) = \frac{v(x)}{J(w,v)}\int_0^x f(t)v(t)\,dt - \frac{v(x)}{J(w,v)}\int_0^x f(t)w(t)\,dt. \qquad (3.4)$$

We remark that since L is formally self-adjoint $J(w,v)$is aconstant for all values of x it now remains to show that the function $u(x)$ defined by (3.4) is a solution to the original problem. That such a function satisfies the equation $Lu = f$ is clear enough and we need only investigate the boundary conditions since the boundary conditions are linear homogeneous and unmixed we may write them in the form

$$B_1(u) = \alpha_1 u(0) + \beta_1 u'(0)$$

$$B_2(u) = \alpha_2 u(1) + \beta_2 u'(1)$$

Where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constant.

From the fact that $u(x)$in (3.4) satisfies $u(0) = u'(0) = 0$ it is clear that $B_1 = 0$ is satisfied. To establish the section that having used (3.2) we have form (3.4):

$$u(1) = \frac{w(x)}{J(w,v)} \int_0^1 f(t)v(t)\, dt.$$

Also on differentiating $u(x)$ in (3.4) with respect to x and setting $x = 1$ we have

$$u'(1) = \frac{w'(x)}{J(w,v)} \int_0^1 f(t)v(t)\, dt.$$

Combining these last two results we see that

$$B_2(u) = \frac{B_1(w)}{J(w,v)} \int_0^1 f(t)v(t)\, dt = 0$$

Because of the assumption regarding the nature of w. This completes the proof of the theorem.

In more general cases when the boundary conditions are mixed, the Green's function can be found in a straightforward manner as is illustrated by the following example.

Example (3.1.3):-

The Green's function $\mathrm{g}(x,t)$ is defined to be that solution of the equation

$$(p\mathrm{g}_x)_x - qp = -\delta(x-t)$$

Which satisfies the conditions

$$B_1(\mathrm{g}) = B_2(\mathrm{g}) = 0.$$

Let $v(x)$and $w(x)$be two independent solutions of the equation

$$(pu_x)_x - qu = 0.$$

Then

$$\mathrm{g}(x,t) = \alpha v(x) + \beta w(x) + \frac{1}{J(w,v)}\{v(x)w(t)H(t-x) + w(x)v(t)H(x-t)\} \quad (3.5)$$

Where α and $\beta$are two constant which must be chosen to ensure that $\mathrm{g}(x,t)$satisfies the conditions at $x = t$ and $B_1(\mathrm{g}) = B_2(\mathrm{g}) = 0$. as usual $J(w,v)$ is the conjunct of the solutions $w(x)$and $v(x)$evaluated at $x = t$ and since $w(x)$and $v(x)$are independent the conjunct is non-vanishing of $\mathrm{g}(x,t)$ defined in (3.5) is obvious. The magnitude in the derivative of $\mathrm{g}(x,t)$ at $x = t$is

$$\frac{w'(t)v(t) - w(x)v'(x)}{J(w,v)} = -\frac{1}{p}$$

Therefore $\mathrm{g}(x,t)$ defined in (3.5) satisfies the requirements of a Green's functions at $x = t$ we now examine the boundary conditions since they are linear we have

$$B_1(\mathrm{g}) = \alpha B_1(v) + \beta B_1(w) + B_1(r) = 0$$
$$B_2(\mathrm{g}) = \alpha B_2(v) + \beta B_2(w) + B_2(r) = 0$$

Where

$$r = \frac{1}{J(w,v)}\{v(x)w(t)H(t-x) - v(t)w(x)H(x-t)\}.$$

From these two algebraic equations we can obtain non-trivial solutions for α and β provided

$$\begin{vmatrix} B_1(v) & B_1(w) \\ B_2(v) & B_2(w) \end{vmatrix} \neq 0.$$

If this determinant vanishes then either $B_1(v) + B_2(v) = 0$ which implies that $v(x)$is an eigenfunction of the given operator, corresponding to the eigenvalue zero, or there is a constant R such that

$$B_1(v) + KB_2(u) = 0$$
$$B_1(v) + KB_2(w) = 0$$

Which imply that $B_1(v + Kw) = B_2(v + Kw) = 0$ and that $(v + kw)$ is an eigenvector of the given operator again corresponding to the eigenvalue zero.

Therefore provided there is no eigenfunction of the given operator corresponding to the eigenvalue zero then the function defined in (3.5) is a Green's function for the related problem.

Section (3.2):Green's functions and the adjoint operator

In this section we establish a theorem the result of which whilst certainly of importance in the theory of ordinary differential operators has perhaps greater significance in the study of partial differential equations.

Theorem (3.2.1):-

Let $\mathrm{g}(x,t)$be the Green's function for an operator Lon a manifold defined by certain boundary conditions and let $h(x,t)$be the Green's function for the adjoint operator L^*on the manifold defined by adjoint boundary conditions.

Then

$$\mathrm{g}(x,t) = h(t,x)$$

In the case when L is self-adjoint $\mathrm{g}(x,t)$is a symmetric function of $x$and t.

Proof:-

By hypothesis we have

$$\begin{aligned} Lg(x,t) &= -\delta(x-t) \\ L^*h(x,t) &= -\delta(x-t) \end{aligned} \qquad (3.6)$$

From the defining property of an adjoint operator we have

$$\int h(x,t)L\mathrm{g}(x,s)\,dx = \int L^*h\big(x,t)\mathrm{g}(x,s)\big)\,dx.$$

This can be reduced by using (3.5) to yield

$$\int h(x,t)\delta(x-s)\,dx = \int \delta(x-t)\mathrm{g}(x,s)\,dx.$$

Therefore

$$h(s,t) = \mathrm{g}(t,s)$$

If $L = L^*$ then $\mathrm{g} = h$ and $\mathrm{g}(s,t) = \mathrm{g}(t,s)$ which with an insignificant change in notation provides us with the statement of the theorem

Section (3.3):Spectral representation and Green's functions

Finally we will discuss in a purely formal manner on alternative method for solving boundary value problems, which is based on the theory of eigenfunctions of linear operators, and in so doing we will indicate its connection with our own approach.

In our previous discussion of abstract linear operators we saw that there exist two fundamental methods for solving the equation

$$Lu = f.$$

Where $L$is a linear operator, $f$agiven vector and u an unknown vector

One method is to construct the inverse operator $L^{-1}$and so obtain

$$u = L^{-1}f.$$

When L is a differential operator, this method leads as we have seen to the consideration of integral operators which have as kernels the Green's function of the given differential operator. The other method is to use the spectral representation of the operator L. That is to use the say we assume that the eigenvectors of L span the space and that

$$f = \sum a_n v_n$$

Where v_n is an eigenvector of L corresponding to the eigenvalue λ_n.

Then we write

$$u = \sum b_n v_n. \qquad (3.7)$$

Where b_n the coefficients have to be determined this is accomplished by using the orthonormality of the eigenvectors as follows

$$Lu = \sum \lambda_n b_n v_n = \sum a_n v_n \qquad (3.8)$$

Consequently

$$b_n = \frac{a_n}{\lambda_n}$$

and

$$u = \sum \left(\frac{a_n v_n}{\lambda_n}\right)$$

The representation of u and Lu in (3.7) and (3.8) is called the spectral representation of the operator L.

These ideas are particularly simple to apply when L is an operator in a finite-dimensional space. However, when L acts in an infinite-dimensional space there are difficult questions to answer regarding the convergence of (3.7) and (3.8) and whether or not the eigenvectors of L span the space. For the time being however we will tacitly assume that such aspectral.

Representation of L is available and further for our immediate convenience that L is self-adjoint. Then by successive application of the above reasoning we obtain for any polynomial $p(\lambda)$ inλ the result

$$p(L)u = \sum a_n p(\lambda_n) v_n \qquad (3.9)$$

This may be further generalized to embrace continuous function of λ by means of the following definition

$$f(L)u = \sum a_n f(\lambda_n) v_n \qquad (3.10)$$

When rigorously established this last result has many important applications, for instance the inverse operator $(L-\lambda)^{-1}$ is given by

$$(L-\lambda)^{-1}u = \sum (\lambda_n - \lambda)^{-1} a_n v_n.$$

Of course if λ is an eigenvalue of the operator L the result is completely meaning unless the appropriate a_n is zero.

The spectral representation of a_n operator L depends upon the study of the inverse of the operator $(L-\lambda)$ for all complex values of λ.

We say that the operator $(L-\lambda)$ has an inverse for any vector f which lies in the range of $(L-\lambda)$ if there exist a unique vector u in the domain of $(L-\lambda)$ such that

$$(L-\lambda)u = f.$$

We have seen that a necessary and sufficient condition for the existence of an inverse is that the homogeneous equation

$$(L-\lambda)u = 0.$$

Possesses only the trivial solution $u \equiv 0$. If this homogeneous equation has a non-trivial solutionλ is an eigenvalue of the operator L and u the corresponding eigenvector. In this case λ is said to belong to the point spectrum or discrete spectrum of L.

Suppose that the homogeneous equation has only the trivial solution $u = 0$ then as we know $(L-\lambda)$ may be the whole space S and the inverse is a bounded operator. This means that for every vector $f\ in\ S$ there exist a unique vector u which satisfies $(L-\lambda)u = f$. and such that $\frac{(u,u)}{(f,f)}$ is bounded.

In this case λ is said to belong to the resolvent set of the operator L. Secondly, the range or the closer of the range of $(L-\lambda)$ may be the whole space but the inverse operator$l$.In this case λ is said belong to the continuous spectrum of the operator L.

Finally the range or the closer of the range $(L-\lambda)$ may be a proper sub set of S.In this case λ belongs residual spectrum of L.

Definition (3.3.1):-

The spectrum of the operator$L$ consists of all values of λ which belong to the discrete the continuous or the residual spectrum.

The application of this theory to differential operators define over space with infinite dimension has many difficulties which are not in evidence when dealing with linear operators defined over finite-dimensional space. One such difficulty we have already mentioned the uncertainty concerning whether or not the eigenvectors of the operators span the space that is the domain of Lin the case of ordinary linear self-adjoint differential operators this question can be answered the eigenvectors of the operators in this case do indeed span the domain of the operator. Whilst this knowledge is comforting in it self we recall that there are many differential operator which are not self-adjoint.

Another and even more profound difficulty arises whenever we consider the possibility of λbelonging to the spectrum of L. It can be shown that an ordinary differential operator has no residual spectrum so we need only consider when λ is in either the discrete spectrum or the continuous spectrum. When λ is in the discrete spectrum no inverse of$(L-\lambda)$ exists and the problems almost resolve themselves. The nature of the continuous spectrum is the main difficulty in the theory of differential operators. It is important to be able decide when an operator has a continuousspectrumand when it dose how this continuous spectrum can be used to give a spectral representation of the operator. These several questions can be answered by a consideration of the Green's function approach would appear to be more fundamental, to illustrate the connection between the two approaches, let L be an ordinary self-adjoint differential operator with

eigenfunction $u_1, u_2, \ldots\ldots$ and associated eigenvalues $\lambda_1, \lambda_2, \ldots\ldots$. We well assume that the eigenfunctions span the domain of the given operator and that in consequence any square integrable function u can be expanded in the form:

$$u(x) = \sum \alpha_k u_k(x)$$

where$\alpha_k = (u_k, u)$

now it follows that:

$$Lu(x) = \sum \alpha_k L u_k(x) = \sum \alpha_k \lambda_k u_k$$

And if $f(t) = (\lambda - t)^{-1}$ we obtain

$$\frac{1}{(\lambda - t)} u(x) = \sum \frac{\alpha_k u_k(x)}{(\lambda - \lambda_k)}. \qquad (3.11)$$

The left-hand side of these equation can be expressed in terms of the Green's function for the differential operator $(L - \lambda)$. To see this let $w(x) = (L - \lambda)u(x)$;then if $G(x, y; \lambda)$ is the Green's function for the operator$(L - \lambda)$. We have shown that

$$w(x) = -\int G(x, \xi; \lambda)\, u(\xi) d\xi.$$

If now, we integrate (3.11) over the large circle of radius R in the complex λ plane, we obtain

$$\frac{1}{2\pi i} \int \frac{u(x)}{L - \lambda} d\lambda = \sum \frac{1}{2\pi i} \int \frac{\alpha_k u_k(x)}{(\lambda - \lambda_k)} d\lambda$$

When the radius of the circle increases without bound, the right-hand side includes more and more residues, and we obtain:

$$\lim_{R \to \infty} \frac{1}{2\pi i} \int \frac{u(x)}{L - \lambda} d\lambda = -\sum \alpha_k u_k(x) = -u(x). \qquad (3.12)$$

This result, which connects the Green's function with the eigenfunctions, was obtained, as already stressed, on the basis of a number of assumptions regarding the existence and completeness of the eigenfunctions. In practice, to overcome the difficulties, the above reasoning is usually followed through in the reverse direction. Starting with a knowledge of the Green's function, $G(x,\xi;\lambda)$,for the operator $(L-\lambda)$ we consider the following integral in the complex λ-plane;

$$\frac{1}{2\pi i}\int \frac{u(x)}{L-\lambda}d\lambda = \frac{1}{2\pi i}\int d\lambda \int G(x,\xi;\lambda)u(\xi)d\xi,$$

And by evaluating this in terms of residues we hope to obtain (3.12),that in expansion of $u(x)$ in terms of the eigenfunctions of L.

Definition (3.3.2):-

The spectral representation of operators despite the difficulties mentioned has many useful applications. For example a number of partial differential equations can be solved if we can assign a meaning to functions of certain ordinary differential operators.

Although such functions can be made meaningful in terms of the spectrum of the ordinary differential operators, in practice as we have seen we also really require a knowledge of the associated Green's functions of the differential operators. Therefore a part from this one brief and purely formal mention of spectral representations of operators, we shall confine ourselves to determining the Green's functions of ordinary and partial differential operators. Once such function are determined then if it is felt necessary, a spectral representation approach can be followed with greater degree of confidence.

Chapter (4)

Integral Equation's

Section (4.1):Classification of integral equations

Definition (4.1.1):-

Equations in which the unknown appears under an integral sign are known as integral equations.

Definition (4.1.2):-

If the range of integration is assumed to be fixed the equations are known as Fredholm equations.

Definition (4.1.3):-

If the range of integration is not fixed the equations are known as Volterra equations.
To be precise the equation

$$f(x) = \int_a^x K(x,z)y(z)\,dz. \qquad (4.1)$$

Where y the unknown is function and f is a known function, is a Volterra equation of the first kind with kernel $K(x,z)$.
A Fredholm equation of the first kind with kernel $K(x,z)$ is

$$f(x) = \int_a^b K(x,z)y(z)\,dz. \qquad (4.2)$$

Corresponding integral equations of the second kind involve an unknown function which appears not only under the integral sign, but outside as well for instance

$$y(x) + \int_a^x K(x,z)y(z)\,dz = f(x) \qquad (4.3)$$

And

$$y(x) + \int_a^b K(x,z)y(z)\,dz = \mathrm{g}(x) \qquad (4.4)$$

are respectively non-homogeneous Volterra and Fredholm equations of the kind with kernel $K(x,z)$ when $f(x)$ and $\mathrm{g}(x)$ are zero the equations become homogeneous.

We shall see that boundary value problems for elliptic partial differential equations lead to Fredholm integral equations where the region of integral has a fixed size. Initial value problems for hyperbolic partial differential equations, on the other hand lead to Volterra integral equations where the region of integration is no longer fixed.

Definition (4.1.4):-

In practice integral equations of the second kind are much easier to handle, than those of the first kind, and it is a very fortunate fact that the majority of problems of mathematical physics happen to reduce to the former rather than the latter. For this reason and also because Volterra equations can be thought of as special cases of the Fredholm equations for which the kernel $K(x,t)$, vanishes if t lies outside a region depending on x we shall only consider Fredholm equations of the second kind.

That is given

$$\phi(x) - \lambda \int K(x,t)\phi(t)\,dt = f(x), \qquad (4.5)$$

where the kernel $K(x,t)$ and scalar λ are assumed known, we require to determine the unknown function ϕ for all functions f of some suitable class. The integral in (4.5) is understood to extend over a fixed domain, which also constitutes the range of x.

Definition (4.1.5):-

In keeping our previous discussion of integral operator, we introduce the notation

$$K\phi = \int K(x,t)\phi(t)\,dt. \qquad (4.6)$$

The integral equation (4.5) then assumes the form

$$\phi - \lambda K\phi = f \qquad (4.7)$$

Probably the first method which comes to mind for solving (4.7) is that of successive approximation.

To this end we will rewrite (4.7) in the form

$$\phi = f + \lambda K\phi, \qquad (4.8)$$

as a zero-order approximation we take

$$\phi_0(x) = f(x),$$

we now substitute this zero-order approximation into the right-hand side of (4.8) to obtain the first order approximation $\phi_1(x)$

$$\phi_1(x) = f + \lambda Kf.$$

Similarly we substitute this first-order approximation into the right-hand side of (4.8) to obtain the second-order approximation and so on if the nth-order approximation $\phi_n(x)$ has been obtained in this way the $(n+1)$Theapproximation,$\phi_{n+1}(x)$ is taken to be the result of substituting $\phi_n(x)$ into the right-hand side of (4.8). In this manner successive approximations are obtain from the recurrence relation.

$$\phi_{n+1} = f + \lambda K\phi_n. \qquad (4.9)$$

If these successive approximations tend uniformly to a limit than as we will presently show, this limit must certainly be a solution of (4.8). If the limit does not exist then clearly, the method of successive approximations has now meaning.

We will now examine the detailed structure of the successive that the approximations. For definiteness, in this examination we will assume that the range of integration is $a \le t \le b$.Consequently

$$\phi_1(x) = f(x) + \lambda Kf(t) = f(x) + \lambda \int_a^b K(x,t)f(t)\,dt.$$

And

$$\phi_2(x) = f(x) + \lambda K\phi_1(t)$$

$$= f(x) + \lambda \int_a^b K(x,t)\phi_1(t)\,dt$$

$$= f(x) + \lambda \int_a^b K(x,t)f(t)t\,dt + \lambda^2 \int_a^b K(x,t)\,dt \int_a^b K(x,s)f(s)\,ds.$$

In the double integral appearing in the expression for $\phi_2(x)$, let us change the order of integration and write for brevity.

$$K_2(x,s) = \int_a^b K(x,t)K(x,s)\,dt.$$

Then with this notation, we obtain

$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,s)f(s)\,ds + \lambda^2 \int_a^b K_2(x,s)f(s)\,ds.$$

Similarly we fined

$$\phi_3(x) = f(x) + \lambda \int_a^b K(x,s)f(s)\,ds + \lambda^2 \int_a^b K_2(x,s)f(s)\,ds$$

$$+\lambda^3 \int_a^b K_2(x,s)f(s)\,ds.$$

Where

$$K_3(x,s) = \int_a^b K(x,t)K(x,s)\,dt.$$

Continuing in this manner we obtained the general from

$$\phi_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,s)f(s)\,ds. \qquad (4.10)$$

Where $K_m(x,s)$ is determined by the recurrence relations

$$K_1(x,s) = K(x,s);\ K_m(x,s) = \int_a^b K(x,t)K_{m-1}(t,s)\,dt. \qquad (4.11)$$

The function $K_m(x,s)$ is called the mth iterated kernel in relation to the given kernel $K(x,s)$. It can readily be shown that $K_m(x,s)$ also satisfies the more general recurrence

$$K_m(x,s) = \int_a^b K_r(x,t)K_{m-r}(t,s)\,dt, \qquad r \le m\,. \tag{4.12}$$

In keeping with the notation introduced in (4.6) it is evident that

$$K^2\phi = K(K\phi) = \int_a^b K(x,t)\int_a^b K(t,s)\phi(s)\,ds\,dt$$

And hence

$$K^2\phi = \int_a^b K_2(x,s)\phi(s)\,ds$$

Continuing in this manner we find that.

$$K^n\phi = \int_a^b K_n(x,s)\phi(s)\,ds \tag{4.13}$$

Where $K_n(x,s)$ is the nTh iterated kernel obtained from $K(x,s)$. Thus (4.10) may be written in the form

$$\phi_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m K^m f\,. \tag{4.14}$$

Once we established the uniform convergence of the series in (4.14) it will be seen that this interchange of integration and summation sign is also permissible as n tends to infinity.

Definition (4.1.6):-

In the limit as n tends to words infinity we are led to a consideration of the series

$$f(x) + \sum_{m=1}^{\infty} \lambda^m K^m f\,. \tag{4.15}$$

If this series converges uniformly to a function $\phi(x)$ then this function will satisfy the equation (4.7).

This readily seen since the assumed uniform convergence of the series (4.15) allows term-by-term integration, and have that.

$$\lambda K\phi = \lambda Kf + \sum_{m=1}^{\infty} \lambda^{m+1} K^{m+1} f$$
$$= \sum_{m=1}^{\infty} \lambda^m K^m f$$
$$= \phi - f$$

As required therefore the solution of the integral equation

$$\phi - \lambda K\phi = f$$

Is given by

$$\phi(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m K^m f \qquad (4.16)$$

Whenever the series is uniformly convergent. To examine the natural of the convergence of the series in (4.16) we first notice that

$$(Kf)^2 = \left\{ \int_a^b K(x,t) f(t)\, dt \right\}^2$$

And by means of the Schwarz inequality we obtain the inequality:

$$(Kf)^2 \leq \|f\|^2 \int_a^b K(x,t)^2\, dt.$$

To be precise we have define in our solution space the familiar inner product

$$(\phi, \psi) = \int \phi(x)\psi(x)\, dx$$

And associated norm

$$\|\phi\| = (\phi, \phi)^{1/2}$$

If we integrate each side of the above inequality with respect to x we obtain:

$$\|Kf^2\| \leq \|f\|^2 \int_a^b \int_a^b K(x,t)^2\, dt\ \ dx.$$

If we assume that the kernel $K(x,t)$ is square integrable for all x and t in $[a,b]$ then we may set

$$B^2 = \int_a^b \int_a^b K(x,t)^2\, dt\, dx < \infty$$

And obtain finally

$$\|Kf\| \leq \|f\|B.$$

Similarly we see that

$$(K^2 f)^2 = [K(Kf)]^2 \leq \|Kf^2\| \int_a^b K(x,t)^2\, dt,$$

And on integrating each side with respect to x we obtain

$$\|K^2 f\| \leq \|f\|B^2.$$

Proceeding inductively we find that

$$\|K^n f\| \leq \|f\|B^n \qquad (4.17)$$

Consequently the series in (4.16) is majorized by the series

$$\|f\| \sum_{m=0}^{\infty} |\lambda|^m B^m$$

This is a geometric series which is convergent provided

$$|\lambda|B < 1. \qquad (4.18)$$

Therefore by the Weierstrass$M$-test, the series in (4.16) is uniformly convergent whenever (4.18) is satisfied. Under these conditions it is evident that (x) , the required solution, is unique. Suppose the contrary and let ϕ_1 and $\phi_2$be two solutions of (4.7) then

$$\phi_1 - \lambda K\phi_1 = f$$
$$\phi_2 - \lambda K\phi_2 = f$$

Subtracting these tow equations and setting $w(x) = \phi_1(x) - \phi_2(x)$ we obtain

$$w(x) - \lambda K w(x) = 0.$$

From the Schwarz inequality it follows that

$$w^2(x) \leq |\lambda|^2 \int_a^b K(x,t)\, dt \int_a^b w^2(t)\, dt$$

And on integrating with respect to x we obtain

$$\|w(x)\|^2 \leq \lambda^2 B^2 \|w(x)\|^2.$$

Therefore

$$(1 - \lambda^2 B^2)\|w(x)\|^2 \leq 0$$

Now the first factor on the left is strictly positive, hence from the definition of norm, it follows that $w(x) \equiv 0$ and the uniqueness is established.

These results we can summarize in the form of a theorem

Theorem (4.1.7):-

The Fredholm equation of the second kind

$$\phi - \lambda k\phi = f \qquad (4.17)$$

With kernel $K(x,t)$ has a unique solution given by the uniformly convergent Neumann series

$$\phi(x) = f(x) + \sum_{m=0}^{\infty} \lambda^m K^m f \qquad (4.16)$$

Provided

(1) f Is square integrable (4.19)

$$(2) \int_a^b \int_a^b K(x,t)^2 dx\, dt = B^2 < \infty \qquad (4.20)$$

$$(3) |\lambda| B < 1. \qquad (4.21)$$

An alternative representation of the solution may be obtained as follows. We can rewrite the equation (4.7) in the operator form

$$(1 - \lambda K)\phi = f \qquad (4.22)$$

Where I is identity operator consequently, whenever the inverse operator $(I - \lambda K)^{-1}$ exist we may write

$$\phi = (I - \lambda K)^{-1} f$$

A formal expansion by the binomial theorem yields

$$\phi = f + \lambda K f + \lambda^2 K^2 f + \cdots$$

By definition the norm of the operator K is given by

$$\|K\| = \underset{f}{l.u.b} \cdot \frac{\|Kf\|}{\|f\|} = \underset{\|f\| = 1}{l\,.u\,.b} \cdot \|Kf\|$$

Consequently

$$\|Kf\| \leq \|K\| \|f\|$$

Similarly by first writing $Kf = \mathrm{g}$ say and applying Schwarz's inequality we find that

$$\|K^2 f\| = \|K(Kf)\| = \|K\mathrm{g}\| \leq \|K\| \|\mathrm{g}\| = \|Kf\|$$

Therefore

$$\|K^2 f\| \leq \|K\|^2 \|f\|$$

Thus the solution to integral equation in the form

$$\phi = (1 + \lambda K + \lambda^2 K^2 + \cdots) f$$

Is majorized by the series

$$(1 + |\lambda| \|K\| + |\lambda|^2 \|K\|^2 + \cdots) \|f\|$$

Which is uniformly convergent provided that f is bounded in norm and

$$|\lambda|\|K\| < 1$$

Under these condition it is evident that the solution ϕ is unique.

Introducing the reciprocal or resolvent operator R by

$$R = K + \lambda K^2 + \lambda^2 K^3 + \cdots \qquad (4.23)$$

We may represent the solution ϕ as

$$\phi = f + \lambda R f = (1 + \lambda R) f \qquad (4.24)$$

So for the only restriction that we have placed on f is that it should be square integrable and we have not really enquired whether or not this condition is both necessary and sufficient for solution ϕ of (4.7) to exist

Definition (4.1.8):-

In order to establish a necessary condition for the existence of a solution to (4.7) we first introduce the transposed integral equation to (4.7)

Equation (4.7) is

$$\phi - \lambda K \phi = f$$

Where the Fredholm operator K has kernel $K(x,t)$ and is given by

$$K\phi = \int K(x,t)\phi(t)\, dt$$

The transposed integral equation is

$$\psi - \lambda K^* \psi = \mathrm{g} \qquad (4.25)$$

Where that conjugate Fredholm operator $K^*\psi$ has a kernel $K(x,t)$ and is given by

$$K^*\psi = \int K(x,t)\psi(t)\, dt \qquad (4.26)$$

Where $K(x,t)$ is complex, the transposed kernel is given by the complex conjugate of (x,t) , the arguments transposed.

We will not introduce any special notation to emphasize this point but will simply assume that whenever it is appropriate to do so the complex conjugate is taken.

Conjugate operators are connected by the important relationship

$$(K\phi, \psi) = (\phi, K^*\psi) \qquad (4.27)$$

Definition (4.1.9):-

Let ψ be an eigenfunction of the transposed equation that is, let ψ is satisfy:

$$\psi - \lambda K^* \psi = 0 \qquad (4.28)$$

Now let us examine the inner product(ψ, f).

$$(\psi, f) = (\psi, [\phi - \lambda K\phi]) = (\psi, \phi) - \lambda(\psi, K\phi)$$
$$= (\psi, \phi) - \lambda(K^*\psi, \phi)$$
$$= ([\psi - \lambda K^*\phi], \phi).$$

Therefore from (4.28) and the defining properties of the inner product it follows that

$$(\psi, f) = 0 \qquad (4.29)$$

Definition (4.1.10):-

With the notation introduced in the previous section we will examine in detail the following four integral equations

$$\phi - \lambda K\phi = f \qquad (4.30)$$
$$\phi - \lambda K\phi = 0 \qquad (4.31)$$
$$\psi - \lambda K^*\psi = \mathrm{g} \qquad (4.32)$$
$$\psi - \lambda K^*\psi = 0 \qquad (4.33)$$

Theorem (4.1.11):(The Fredholm Alternative)

If the homogeneous equation (4.31) and (4.33) possess only the trivial solution $\phi = \psi = 0$ than the non-homogenous equation (4.30) and (4.32) have unique solutions ϕ and ψ respectively for all square integrable functions f and g.

If the homogenous equations (4.31) and (4.33) have non-trivial solution (eigenfunction) $\phi_1, \phi_2, \dots\dots, \phi_K$ and $\psi_1, \psi_2, \dots\dots, \psi_K$ then the number of these solutions which are linearly independent is the same for each equation.

In such case the non-homogenous equation (4.30) has a solution if and only if f is orthogonal to the m eigenfunction ψ_i of (4.33) that is

$$(\psi_i, f) = 0 \quad , \quad i = 1,2, \dots\dots, m \ . \qquad (4.34)$$

Similarly (4.32) will have a solution if and only if

$$(\phi_i, \mathrm{g}) = 0 \quad , \quad i = 1,2, \dots\dots, k \ . \qquad (4.35)$$

The proof of this theorem will be our main concern for the remainder of this section.

Definition (4.1.12):-

We will first demonstrate the theorem for integral equations with a particular type of kernel, namely, a degenerate kernel. We have already encountered such kernels under the heading of separable kernels (section (1.1)) which is their more usual title in the theory of abstract linear operator .For the purposes of this section, in order to conform with the language of the theory of integral equations, we will use the synonymous title of degenerate kernel, and write such kernels in the form

$$K_n(x,y) = \sum_{j=i}^{n} \alpha_j(x)\beta_j(t) \qquad (4.36)$$

The more general problem will be treated afterwards by approximating the arbitrary kernel by a degenerate kernel.

With no loss of generality we can assume that $\lambda = 1$and examine the Fredholm equation:

$$\phi - k_n\phi = f \qquad (4.37)$$

Which may also be written

$$\phi(x) - \sum_{j=1}^{n} \alpha_j(x)\left(\beta_j(t), \phi(t)\right) = f(x) \qquad (4.38)$$

The homogeneous transposed equations

$$\psi(x) - k^*\psi = 0$$

may be written in a like manner as

$$\psi(x) - \sum_{j=1}^{n} \left(\psi(t), \alpha_j(t)\right)\beta_j(x) = 0 \qquad (4.39)$$

Which shows that the solutions $\psi(x)$ are linear combinations of $\beta_1, \dots\dots, \beta_n$ which are themselves a linearly indendent and even an orthonormal set of functions.

We therefore decompose given function$f$ into

$$f = f_1 + f_2$$

Where $f_1$is a linear combination of the functions $\beta_1, \ldots\ldots, \beta_n$ and f_2 is orthogonal to function$\beta_i, i = 1,2, \ldots\ldots, n$, that is

$$(f_2, \beta_i) = 0, \quad i = 1,2, \ldots\ldots n$$

Thus, it follows that

$$K_n f_2 = \sum_{j=1}^{n} (\alpha_{j(x)}(\beta(t), f_2(t)) = 0,$$

and the substitution

$$\phi = \phi_1 + f_2,$$

reduces the equation (4.38) to

$$\phi_1 = \sum_{j=1}^{n} \alpha_j(x)\big(\beta_j(t), \phi_1(t)\big) = f_1 \qquad (4.40)$$

It is therefore sufficient to discuss the special case when $f_2 = 0$which will imply that $f = f_1$and $\phi = \phi_1$.

To solve the integral equation (4.40) we reduce it to an equivalent system of algebraic equations. To this end we write

$$c_i = (\beta_i, \phi_1), \quad i = 1,2, \ldots\ldots n \qquad (4.41)$$

and$y_i = (\beta_i, f), \quad i = 1,2, \ldots\ldots n \qquad (4.42)$

Then (4.40) takes on the form:

$$c_k - \sum_{j=1}^{n} a_{jk} c_j = y_k \qquad (4.43)$$

Where

$$a_{jk} = \big(\alpha_j, \beta_k\big) (4.44)$$

Notice that throughout we have assumed that the α_i(x) and $\beta_i(x)$ are linearly independent. If this were not case, the number of terms in (4.36) would simply be reduced. The assigning of the function$f$ in the original

integral equation (4.37) is thus equivalent to assigning the numbers $y_k$in the algebraic set (4.43). It follows that whenever a solution vector $\{c_1, c_2, \dots \dots, c_n\}$ of (4.43) exists we can construct a function ϕ in the form

$$\phi(x) = f(x) + \sum_{j=1}^{n} c_j \alpha_j(x) \qquad (4.45)$$

and clearly such a function satisfies (4.41) and the integral equation (4.80). Thus we see that the given integral equation can be solved if and only if the system of simultaneous algebraic equation (4.43) can be solved.

In order that (4.43) should possess a unique solution for $C_j, j = 1,2, \dots, n$, it is necessary and sufficient that the determinant of the coefficients appearing on the left-hand side should be non-zero. It follows that, when this the case, the integral equation (4.37) also has a unique solution. In particular we notice that if $\phi = 0$is the only solution of (4.37) for $f = 0$, then the above reasoning indicates that the determinant of the coefficients in (4.43) is again non-zero. Therefore (4.37) is soluble for ever choice of$f$.

Definition (4.1.13):-

Turning for a moment to the homogeneous transposed equation (4.39) ,we obtain , on forming the inner product with α_j, the corresponding system of algebraic equations to (4.43) , namely :

$$b_k - \sum_{j=1}^{n} a_{jk} b_k = 0 \qquad (4.46)$$

Where $$b_k = (\psi, \alpha_k) \qquad (4.47)$$

Form this system, and in particular (4.47), we see that any solution ψ of (4.39) which does not vanish identically cannot be orthogonal to all the functions α_k. Therefore such solutions must generate a non-trivial

solution vector $\{b_1, \ldots . b_n\}$of the system (4.46). Conversely, any non-trivial solution of (4.46) defines a function

$$\psi = \sum_{i=1}^{n} b_i \beta_i \qquad (4.48)$$

Which is always different from zero somewhere, and satisfies both (4.47) and (4.39). Furthermore, given such an eigenfunction as (4.48) of (4.39) we can multiply (4.43) by $b_i$and sum over all$i$to obtain:

$$\sum_{i=1}^{n} b_i y_i = \sum_{i=1}^{n} b_i c_i - \sum_{i,j=1}^{n} b_i a_{ij} c_j$$

$$= \sum_{j=1}^{n} \left[b_j - \sum_{j=1}^{n} b_j a_{ij} \right] c_j$$

$$= 0 \qquad (4.49)$$

Finally, form (4.42) we have

$$\sum_{i=1}^{n} b_i y_i = ((\sum b_i \beta_i), f) = (\varphi, f) = 0 \qquad (4.50)$$

So that (4.49) is simply a re-statement of the orthogonality condition (4.29).Since the transpose of a determinant has the same numerical value as determinant itself, it follows that the determinants of the coefficients of the systems (4.43) and (4.46) have the same value. Thus, in the case under discussion, when the determinant of coefficients in non-vanishing it follows that non-trivial solutions of (4.46) , and hence (4.39), do not exist. Consequently the orthogonality condition (4.29) imposes no restrictions at all on the function f.

When the determinant of the coefficients of (4.43) vanishes , the corresponding matrix of the coefficients has a rank $r < n$,and the vectors $(y_1, \ldots \ldots y_n)$ for which (4.43) possesses a solution only span subspace of the n-dimensional Euclidean space ; the subspace has the smaller dimension r. since the matrix of coefficients of the transposed system (4.46) of the homogeneous algebraic equations has the same rank r , the

eigenvectors $(b_1, b_2, \ldots, b_n)$ of (4.46) generate another subspace of dimension $m = n - r$, now according to (4.49) the m-dimensional and r-dimensional manifolds must be orthogonal , and consequently their union must be the whole n-dimensional Euclidean space. It therefore follows that the vectors $(y_1, \ldots, y_n)$ for which (4.45) is soluble are precisely those which fulfill the orthogonality condition (4.49) for every solution $(b_1, \ldots, b_n)$ of (4.46).

Definition (4.1.14):-

The established equivalence of integral and algebraic equation in both the homogeneous and non-homogeneous cases implies, in con-junction with the relation (4.50), that a solution of the fredholm integral equation of the second kind (4.37) exists if and only if the right-hand side, f, is orthogonal to every solution ψ of the homogeneous transposed equation (4.39). Since the eigenvectors $(b_1, b_2, \ldots, b_n)$ span a subspace of dimension m , this amounts to the m condition :

$$(\psi_i, f) = 0, i = 1,2, \ldots . m,$$

Where the ψ_i are m linearly independent eigenfunctions of the homogeneous transposed equation (4.39). This completes the prove of the Fredholm alternative in the case of degenerate kernels. It remains to show that theorem is also true for more general kernels.

Given an arbitrary, square integrable kernel $k$we will use the Neumann series representation (4.16) to reduce the given problem to one associated with a degenerate kernel which we now know how to solve.

In our previous discussions of integral operators we saw that it was always possible to approximate a given kernel by a degenerate kernel in such a manner that the difference between the two kernels was as small as we wished. In this particular case we will choose the approximating degenerate kernel,$K_n(x.t)$to be such that the difference

$$\widehat{K}(x,t) = K(x,t) - K_n(x,t) \qquad (4.51)$$

Satisfies the inequality

$$\iint \widetilde{K}(x,t)^2\, dtds < 1 \qquad (4.52)$$

Imposing this restriction on the kernel $\widetilde{K}(x,t)$ensures that we can use the results of theorem (4.1.7).that is, the Fredholm equation

$$\tilde{\phi} - \widetilde{K}\tilde{\phi} = \tilde{f} \qquad (4.53)$$

Where $\widetilde{K}$ is an integral operator with kernel $\widetilde{K}(x,t)$,has a solution $\tilde{\phi}$ which may be represented as

$$\tilde{\phi} = \tilde{f} + \tilde{R}\tilde{f} \qquad (4.54)$$

Where $\tilde{R}$ is the resolvent operator associated with the operator$\widetilde{K}$. We will use this representation of the solution of an integral equation having a kernel similar to (4.53), to reduce the unrestricted fredholm equation:

$$\phi - K\,\phi = f \qquad (4.55)$$

To a degenerate integral equation. This we accomplish as follows. Emphasizing the fact that we will denote operator s by the same letter as used to denote their kernel , but without any arguments attached (that is , K denotes the operator generated by the kernel $K(x,t)$) we see that in terms of the approximation(4.51) our given equation (4.55) may be written:

$$\phi - \widetilde{K}\phi = f + K_n\phi \qquad (4.56)$$

This equation has the same structure as (4.54); in particular it has the same kernel, thus its solution has the representation

$$\phi = (f + K_n\phi) + \tilde{R}(f + K_n\phi) \qquad (4.57)$$

Or, which is the same thing:

$$\phi - \left(K_n + \tilde{R}K_n\right)\phi = f + \tilde{R}f \qquad (4.58)$$

Clearly, this is a Fredholm equation of the second kind, and it must have a solution ϕ which is identical to the equation (4.51). At first sight it would appear that no great progress has been made. However, this is not the case since we can show that (4.58) has a degenerate kernel, and we have already established the Fredholm Alternative for such equations. That K_n is degenerate follow by hypothesis. To show that $\tilde{R}K_n$ has a

degenerate kernel we first observe that, since we have introduced the inner product

$$(f, \mathrm{g}) = \int f(x)\mathrm{g}(x)dx$$

Into our space, we may write $K_n\phi$ in the form

$$K_n\phi = \int K_n(x,t)\phi(t)dt = (K_n(x,t), \phi(t))$$

Similarly for all other operators appearing in our work ; in particular

$$\begin{aligned}\check{R}\phi &= \left[\widetilde{K} + \widetilde{K}^2 + \widetilde{K}^3 + \cdots\right]\phi \\ &= \left(\widetilde{K}(s,x), \phi(x)\right) + \left(\widetilde{K}^2(s,x), \phi(x)\right) + \cdots \\ &= \left(\tilde{R}(s,x), \phi(x)\right)\end{aligned}$$

It then follows immediately that

$$\tilde{R}K_n\phi = \sum_{i=1}^{n}\big(R(s,x), \alpha_i(x)\big)(\beta_i(t), \phi(t))$$

And clearly $\tilde{R}K_n$ has a degenerate kernel

$$\sum_{i=1}^{n}\big(R(s,x), \alpha_i(x)\big)\beta_i(t)$$

Therefore , the kernel of equation (4.58) is degenerate . since we have already established the Fredholm Alternative for degenerate Fredholm equations , it follows that (4.58), and consequently(4.55), is soluble if and only if the orthogonality condition

$$\left(\tilde{\psi}, [f + \tilde{R}f]\right) = 0 \qquad (4.59)$$

Is satisfied for every eigenfunction$\tilde{\psi}$, of the homogeneous transposed equation

$$\tilde{\psi} - \left(K_n + \tilde{R}K_n\right)^*\tilde{\psi} \qquad (4.60)$$

If we are to prove the theorm for such kernels as we presently consideried , we must show that the condition (4.60) is equivalent to

$$(\psi, f) = 0 \qquad (4.61)$$

Where ψ is any eigenfunction of homogeneous transposed equation

$$\psi - K^*\psi = 0 \qquad (4.62)$$

Admittedly we have anticipated (4.61), as such a result would prove the theorem.

That it is indeed true we can establish fairly easily , as follows. let us write:

$$\psi = \tilde{\psi} + \widetilde{R^*}\tilde{\psi} = \left(I + \widetilde{R^*}\right)\tilde{\psi} \qquad (4.63)$$

Where $\tilde{R}^*$ is the resolvent operator associated with $\widetilde{K}^*$ and

$$(1 - \widetilde{K}^*)^{-1} = \left(I + \tilde{R}^*\right)$$

Consequently, from (4.63) we obtain

$$\tilde{\psi} = \left(I - \widetilde{K}^*\right)\psi \qquad (4.64)$$

Which gives $\tilde{\psi}$ in terms of, since K and K^* are related by (4.27) it can easily be shown that the operator in (4.60) can be re-written

$$(K_n + \tilde{R}K_n)^* = {K_n}^* + {K_n}^*\tilde{R}^* \qquad (4.65)$$

Thus on substituting (4.65) and (4.60) can be obtain

$$\left(I - \widetilde{K}^*\right)\psi - \left({K_n}^* + {K_n}^*\tilde{R}^*\right)\left(I - \widetilde{K}^*\right)\psi$$

$$= \psi - \left(\widetilde{K}^* + \widetilde{K_n}^*\right)\psi + {K_n}^*\left(\tilde{R}^* + \tilde{R}^*\widetilde{K}^* - \tilde{R}^*\right)\psi = 0 \qquad (4.66)$$

From the definition of $\tilde{R}^*$ we see that

$$\tilde{R}^* = \widetilde{K}^* + \widetilde{K}^{2*} + \widetilde{K}^{3*} + \cdots$$

$$= \widetilde{K}^*(I - \widetilde{K}^*)^{-1}$$

$$= K^*\left(I + \tilde{R}^*\right)$$

Consequently , the last term in (4.66) vanishes, and we have

$$\psi - \left(\tilde{k}^* + k_n^*\right)\psi = \psi - k^*\psi = 0$$

Thus (4.60) and (4.62) are synonymous, and the eigenfunctions$\tilde{\psi}$, of (4.60), transform into the eigenfunctionsψ ,of (4.62). consequently (4.59)and(4.61) must have the same significance ,and the demonstration of Fredholm's alternative for square integrable kernels is complete .

In many applications of the Fredholm theory to boundary value problems, the kernel involved in the integral equations are not square integrable. However, under certain suitable conditions, the Fregholm theory is still applicable in such instances, as we will now show.

Definition (4.1.15):-

In the foregoing demonstration of the Fredholm alternative we laid great emphasis on the requirement that the associated kernel should be square integrable. This ensured that the resulting Neumann series was convergent. However ,the square integrability of the kernel was only essential if we were working in an inner product space (usually a Hillbert space); all that really is required is that

$$|\lambda|\|k\| < 1$$

Where $\|k\|$ is the norm of the integral operator involved. In fact the Fredholm alternative can be shown to be valid in general Banach space. It follows, then ,from the above discussion, that the reduction of an integral equation with arbitrary kernel $K(x,t)$to degenerate integral equation may always be carried out provided $K(x,t)$ can be approximated arbitrarily well , in whatever norm is used , by a degenerate kernel. That is, given $\varepsilon > 0$, on matter how small, we can write K in the form

$$K = K_n + \tilde{K}\left\|\tilde{K}\right\| < \varepsilon \qquad (4.67)$$

where$K_n$is integral operator with degenerate kernel. Such kernels are called completely continuous. A restatement of this property, which is more in keeping with our previous discussions linear operators is that K transforms any bounded set into a compact set ;that is ,for any sequence of functions $\{\phi_n\}$ which are such that $\|\phi_n\| < K$ the sequence $\{\psi_n\}$ where $\psi_n = K\phi_n$ contains a convergent subsequence.

Therefore the Fredholm alternative is valid for any integral equation

$$\phi - \lambda K\phi = f$$

Which has an operator with a completely continuous kernel.

With this slight preamble we are now in a position to extend the Fredholm alternative to certain kernel which are not square integreble

Definition (4.1.16):-

The kernel in question arise frequently in the study of boundary value problems , and are known as weakly singular kernels. They have the form

$$K(x,t) = \frac{k(x,t)}{r^{\lambda}} \qquad 0 \le \lambda < n \qquad (4.68)$$

Where $k(x,t)$is bounded function of $x$and t , r is the distance between x and t , and n is the dimension of the region of interest , D.Unlike the case of square integrable kernels, we shall find it necessary in this case to assume that the domain D is bounded . we will first show that the kernel (4.68) are completely continuous in the sense of the definition (4.67)

Writing

$$K'(x,t) = \begin{cases} K(x,t), & r \ge \delta \\ 0, & r < \delta \end{cases}$$

$$K''(x,t) = \begin{cases} 0, & r \ge \delta \\ K(x,t), & r < \delta \end{cases}$$

For some > 0 , we effectively truncate the singularity $r^{-\lambda}$ and obtain the decomposition

$$K(x,t) = K'(x,t) + K''(x,t).$$

Because of our assumption regarding the boundedness of D, the kernel $K'(x,t)$ is clearly square integrable and may , therefore , be approximated arbitrarily well in norm by a degenerate kernel. In fact if $K_n(x,t)$ is a degenerate kernel , Schwarz's inequality shows that

$$\|K'(x,t) - K_n(x,t)\|^2 \le \int\int [K'(x,t) - K_n(x,t)]^2 \, dx \, dt$$

Consequently we must now show that $K''(x,t)$can be made arbitrarily small in norm, to this end we notice that if $M$is a bound for $k(x,t)$

$$(K''\phi)^2 = (K''(x,t),\phi(t))^2 \leq M^2\left[\int_{r<\delta}\frac{|\phi|}{r^\lambda}dt\right]^2.$$

Applying Schwarz's inequality we obtain

$$\left[\int_{r<\delta}\frac{|\phi|}{r^\lambda}dt\right]^2 = \left[\int_{r<\delta}\frac{|\phi|}{r^{\frac{\lambda}{2}}}\frac{dt}{r^{\frac{\lambda}{2}}}\right]^2$$

$$\leq \left[\int_{r<\delta}\frac{|\phi|^2}{r^\lambda}dt\right]\left[\int_{r<\delta}\frac{dt}{r^2}\right]$$

As we have stated, r is the distance between the points $x$and t and it should be emphasized that, when we are working in more than one dimension,x represents the point with co-ordinates $(x_1, x_2, \ldots . x_n)$

And similarly for t. therefore the retraction $r > \delta$ confines our attention to the interior of a small 'circle' centred at $x$and radius δ. Consequently ,since$\lambda < n$,

$$\int_{r<\delta}\frac{dt}{r^\lambda} \leq D(\delta) < \infty$$

Where the number $D(\delta)$, as indicated, depends on the value of δ ;for example, in two dimensions we get:

$$\int\limits_{r<\delta}\frac{dt}{r^\lambda} \leq \frac{2\pi}{2-\lambda}(\delta)^{2-\lambda}$$

Clearly as a result of the restriction on λ, $D(\delta)$will tend to zero with δ with these several results available we can now write

$$\|K"\phi\|^2 = \int\limits_{D_z}(K\phi)^2 dx$$

$$\leq M^2 D(\delta)\int\limits_{D_z}\int\limits_{D_t}\frac{|\phi|^2}{r^\lambda}dtdx$$

Where D_z and D_t indicate the domain of integration together with the variable in the integration process. This can be further simplified by Fubini'stheorem , as follows

$$\int_{D_x}\int_{D_t}\frac{|\phi|^2}{r^\lambda}dtdx = \int_{D_t}|\phi|^2\left\{\int_{D_x}\frac{dx}{r^\lambda}\right\}dt$$

$$\leq \left[\int_{D_t}\frac{|\phi|^2}{r^\lambda}dt\right]\left[\int_{r<d}\frac{dx}{r^\lambda}\right]$$

Where d is the diameter D, arguing as before it is apparent that since D is bounded the integral

$$\int_{r<d}\frac{dx}{r^\lambda}$$

Must also be bounded and equal to B, say.

Therefore combining these result we have

$$\|K"\phi\|^2 \leq M^2 D(\delta)\|\phi\|^2 B.$$

Consequently since M and B are by definition finite, and since $D(\delta)$ can be made arbitrarily small by choice of δ sufficiently small it follows that

$$\|K''\| = l.u.b.\frac{\|K"\phi\|}{\|\phi\|}$$

Can be made arbitrarily small. Therefore it follows that kernels such as $K(x,t)$, which have a weak singularity may be approximated arbitrarily well in norm by degenerate kernels which establishes the validity of the Fredholm alternative for such kernels.

Section (4.2):Symmetric integral equations

Definition (4.2.1):-

The theory of integral equations can be developed in much greater detail when the kernel $K(x,t)$happens to be symmetric. Such a development will be a special case of the Fredholm theory, which we have already briefly examined, and is usually referred to as the Hilbert-Schmidt theory of integral equations. We have already studied in some detail in chapter1 the properties of integral operators which symmetric kernel, and the results we obtained there may be summarized as follows.

Theorem (4.2.2):-

Every continuous symmetric kernel which does not vanish identically possesses eigenvalues and eigenfunction; their number is denumerable infinite if and only if the kernel is non-degenerate. All the eigenvalues of a real symmetric kernel are real. In the application of the theory of integral equations, in particular when the kernel is symmetric we often find it necessary to consider the possibility of expanding a given function in$f$ a series of the form

$$f(x) = \sum (f, \phi_i)\phi_i(x)$$

Where ϕ_i is anorthonormal sequence of functions. Consequently we must examine in what sense such a formal expansion might represent the function $f$and under what circumstances the sequence of functions ϕ_i is complete. The next two theorems will answer these questions.

Theorem (4.2.3):-

Any square integrable function f is orthogonal to all the eigenfunctions ϕ_i of the symmetric kernel $K(x,y)$ if and only if $(K(x,t), f(t)) \equiv 0$ for the sequence of eigenfunctions $\{\phi_i\}$to be complete, it is necessary and sufficient that $(K(x,t), f(t)) \not\equiv 0$ for every $f \not\equiv 0$

Prove:-

The eigenfunctions$\emptyset_i$are the non-trivial solutions of

$$\lambda_i K\phi_i = \lambda_i\big(K(x,t),\phi_i(t)\big) = \phi_i(x) \qquad (4.69)$$

Consequently if the square integrable function f is orthogonal to the kernel $K(x,t)$ we have

$$\big(K(x,t),f(x)\big) = Kf = 0 \qquad (4.70)$$

Forming the inner product of (4.69)with$f$ we have

$$(\phi_i, f) = \lambda_i(K\phi_i, f) = \lambda_i(\phi_i, Kf)$$

The right-hand side vanishes by virtue of (4.70), and so $f$must be orthogonal to ϕ_i for all i.

If now we assume that f is orthogonal to all the eigenfunctions ϕ_i of $K(x,t)$ we must show that $\|Kf\| = 0$.to this end, introduce thekernel

$$K_n(x,t) = K(x,t) - \sum_{i=1}^{n} \frac{1}{\lambda_i}\{\phi_i(x)\phi_i(t)\} \qquad (4.71)$$

Then because f is orthogonal to all the ϕ_i

$$Kf = \left(\left\{K(x,t) - \sum \frac{\phi_i(x)\phi_i(t)}{\lambda_i}\right\}, f(t)\right) = K_n f \qquad (4.72)$$

Where K_n is the integral operator with kernel (4.71)

Therefore

$$\|Kf\| = \|K_n f\| \le \|K_n\|\|f\| \qquad (4.73)$$

Since the eigenvalue λ_i tend to infinity enables us to assert that

$$\|K_n\| = \frac{1}{\lambda_{n+1}} \to 0$$

As $n \to \infty$ thus the system of eigenfunctions spans the space and the theorem is established.

It would perhaps be instructive to establish that $\|k_n\| \to 0$ independently of $n$and reference to the aforementioned theorem. To this end, we notice that for an integral operator K with symmetric kernel $K(x,t)$ and eigenvalues λ_i with associated eigenvector ϕ_i

$$(K\phi_i, \phi_i) = \left(\frac{1}{\lambda_i}\phi_i, \phi_i\right) \leq \frac{\|\phi_i\|^2}{|\lambda_i|}$$

And

$$(K\phi_i, \phi_i) \leq \|K\phi_i\|\|\phi_i\| \leq \|K\|\|\phi_i\|^2$$

If we combine these results, it follows from the definition of $\|K\|$ that

$$\|K\| \geq \frac{1}{\lambda_i}$$

To prove that the reverse inequality is also true, we write

$$\psi_i = K\phi_i$$

Then

$$\|K\phi_i\|^2 = (K\phi_i, K\phi_i) = (K\phi_i, \psi_i) = (\phi_i, K\psi_i) = \left(\phi_i, \frac{K\phi_i}{\lambda_i}\right)$$

Form which it follows that

$$\|K\phi_i\|^2 \leq \|\phi_i\| \frac{\|K\phi_i\|}{|\lambda_i|}$$

And the reverse inequality is established which allows us to write

$$\|K\| = \frac{1}{|\lambda_i|}$$

Using this result conjunction with the symmetric kernel $K_n(x,s)$ it is easily established that

$$\|K_n\| = \frac{1}{\lambda_{n+1}}$$

In the special case when the kernel is degenerate its system of eigenfunction cannot be complete since they span only a finite-

dimensional space. When the kernel is not degenerate its system of eigenfunctions span an infinite-dimensional space. However this infinite-dimensional space may, or may not be the whole Hilbert space in which are situated our solutions and therefore it still remains to examine the sense in which a formal eigenfunction expansion of an arbitrary function f is to be interpreted.

This point can be resolved by means of the celebrated Hilbert-Schmidt theorem.

Theorem (4.2.4): (Hilbert-Schmidt)

Any function f which can be expressed in the form

$$f = K\mathrm{g} = \big(K(x,t), \mathrm{g}(x)\big) \qquad (4.74)$$

Where gis some square integrable function and K a linear integral operator with symmetric kernel $K(x,t)$ has an absolutely and uniformly convergent representation

$$f(x) = \sum_{i=1}^{\infty} (\phi_i, f)\phi_i(x) = \sum_{i=1}^{\infty} \frac{(\phi_i, \mathrm{g})}{\lambda_i}\phi_i(x) \qquad (4.75)$$

In terms of the eigenfunctionsϕ_i of K

Proof:-

All question regarding the completeness of the system of eigenfunctions are answered by the previous theorem. Consequently it only remains to examine the uniform convergence.

Since

$$(\phi_i, f) = (\phi_i, K\mathrm{g}) = (K\phi_i, \mathrm{g}) = \frac{1}{\lambda_i}(\phi_i, \mathrm{g})$$

An application of Schwarz's inequality enables us to write

$$\left(\sum_{i=m}^{n}(\phi_i, f)\phi_i\right)^2 = \left(\sum_{i=m}^{n}\frac{(\phi_i, \mathrm{g})}{\lambda_i}\phi_i\right)^2$$

$$\leq \left(\sum_{i=m}^{n}(\phi_i, \mathrm{g})^2\right)\left(\sum_{i=m}^{n}\frac{{\phi_i}^2}{\lambda_i^2}\right).$$

Bessel's inequality

$$\sum_{i=1}^{\infty}(\phi_i, \mathrm{g})^2 \leq \|\mathrm{g}\|^2 < \infty$$

Shows that the first factor on the right can be made arbitrarily small by taking m and n large enough (Cauchy criterion of convergence).

Further from the orthonormality of the ϕ_i it follows that

$$\sum_{i=1}^{n}\frac{\phi_i^2}{\lambda_i} \leq \int K(x,t)^2\, dt < \infty$$

Since the kernel $K(x,t)$ is bonded. Thus it follows that the infinite series in (4.75) converges absolutely and uniformly.

Tow show that the series actually converges to f it will be sufficient to show that the series converges to f in the mean since f and ϕ_i are uniformly continuous. This follows at once since

$$\left\| f - \sum_{i=1}^{n}(\phi_i, f)\phi_i \right\| = \left\| K\mathrm{g} - \sum_{i=1}^{n}(\phi_i, \mathrm{g})\phi_i / \lambda_i \right\|$$

$$= \|K_n \mathrm{g}\|$$

Where K_n is given by (4.71)

Applying Schwarz's inequality we obtain

$$\|K_n \mathrm{g}\| \leq \|K_n\|\|\mathrm{g}\| = \frac{\|\mathrm{g}\|}{\lambda_{n+1}}$$

And the right-hand side clearly tends to zero as n tends to infinity thus the series converges to f in norm, and the proof of the theorem is complete.

Definition (4.2.5):-

Consider, initially, the differential equation

$$Lu = -f \qquad (4.76)$$

Where L is the self-adjoint, second –order ordinary differential operator introduced in equation (2.32) and f is a piecewise continuous function defined over $[0.1]$ .in section (2.2)we were concerned with the boundary value problem; to find a solution ,u, of (4.76) which satisfied given homogeneous boundary conditions on the boundary of $[0,1]$.we saw that the required solution could be expressed as

$$u(x) = \int_0^1 G(x,t)f(t)dt \qquad (4.77)$$

Where $G(x,t)$was that Green's function associated with the operator L over $[0.1]$ which satisfied the same boundary conditions as the unknown function u.

We now propose to examine the more general boundary value problem associated with the linear family of differential equations

$$Lu + \lambda pu = -f \qquad ,p > 0 \qquad (2.78)$$

Which depend upon a parameterλ. As before, the operator L is defined by (2.32),f is assumed to be piecewise continuous over $[0,1]$ and homogeneous boundary conditions are imposed. If the Green's functions,$G(x,t)$, exists for the operator Lunder the given boundary conditions, then the solution,u, to (4.78) can be obtained in the form (4.77) if we first write

$$\emptyset(x) = \lambda pu - f(x)$$

It then follows that

$$u(x) = \lambda \int_0^1 G(x,t)p(t)u(t)dt + \mathrm{g}(x) \qquad (4.79)$$

Where

$$\mathrm{g}(x) = -\int_0^1 G(x,t)f(t)dt \qquad (4.80)$$

Is a known continuous function of x.therefore (4.79) is equivalent to (4.78). Hence finding a solution $u(x)$,of (4.78), subject to the prescribed boundary conditions is equivalent to solving the integral equation (4.79).also the homogeneous equation

$$Lu + \lambda pu = 0$$

Corresponds to the homogeneous integral equation

$$u(x) = \lambda \int_0^1 G(x,t)p(t)u(t)dt \qquad (4.82)$$

This last equation can be cast into a particularly convenient form if we introduce

$$z(x) = \lambda \int_0^1 k(x,t)z(t)dt \qquad (4.83)$$

The kernel of (4.88) is symmetric since L is self-adjoint. Therefore the theory of symmetric integral equations is applicable. By combining the several results which we obtained in the previous sections we readily see that the follow alternatives hold for the relation between the boundary values problems for non-homogeneous and homogeneous differential equations.

Either, for fixed λ, every solution of the homogeneous differential equation (4.81) vanishes identically (λ is not an eigenvalue of (4.81)); then the non-homogeneous differential equation (4.78) has a unique solution for an arbitrary choice of$f$, or for some value $\lambda = \lambda_i$,the

homogeneous equation (4.81) has a non-trivial solution u_i($\lambda_i$is an eigenvalue of (4.81) with corresponding eigenfunction u_i); the solution of the non-homogeneous equation (4.78) for $\lambda = \lambda_i$ exists if and only if

$$\int_0^1 pu_i f dx = 0$$

Is true for every eigenfunctionu_i associated with λ_i.

In addition, there exists a sequence of eigenvalues$\{\lambda_i\}, i = 1,2, \ldots.$,with associated eigenfunctions u_i which form an infinite set of orthogonal functions satisfying

$$\int_0^1 pu_i u_k dx = 0, \qquad i \neq k$$

$$\int_0^1 pu_i^2 dx = 1$$

If, with the appropriate Green's function as kernel, we can represent a function $\mathrm{g}(x)$ by a piecewise continuous $\phi(x)$in the form

$$\mathrm{g}(x) = \int_0^1 G(x,t)\phi(t)dt$$

Then $\mathrm{g}(x)$ can be expanded in terms of eigenfunction into a series

$$\mathrm{g}(x) = \sum_{n=1}^{\infty} c_n u_n(x)$$

$$c_n = \int_0^1 \mathrm{g}pudx$$

Which is absolutely and uniformly convergent.

List symbols:-

symbols	The meaning	No.page
$\int$	Integration	10
$\| \ \|$	Determinate	11
$\lim$	Limit	12
$\| \ \|$	Norm	13
l. u. b	Least upper bounded	19
$\sum$	Summation	21
$\det\{ \}$	Determinant	25
$\sqrt{\ }$	Square root	28
g. l. b	Great lower bounder	32
$\frac{d}{dx}$	Differentiation	74
$\frac{\partial}{\partial x}$	Partial	89
$\iint$	Double integral	131

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