Sudan University of Science and Technology

College of Science

Department of Mathematics

Introduction to Tensors Calculus with some applications

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Submitted By :

Tayseer Mirghani Abd alla

Azza Mohamed El-fatih

Ahmed Mohamed Ahmed

Supervisor :

Dr.Emad El din Abdel Rahim

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{بسم الله الرحمن الرحيم

صدق الله العظيم

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Dedication

To the fountain of patience and optimism and hope

To each of the following in the presence of God

And is messenger , our mothers dear

To those who have demonstrated to us what is

the most beautiful of our brothers life

to the big heart our fathers

To the people who paved our way of science and

knowledge

All our teachers Distinguished

Especially Dr.Emad El din Abdel Rahim

To the taste of the most beautiful moments with

Our friends.

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First of all we thank allah too much

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to complete the stages of our project

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colleagues and relatives who

assisted us in one way or another.

Abstract

The fundamental idea in Tensors calculus is the transformation of variables,In this research we studied some of this concepts. This work is divided in three chapters as follows :-

Firstly, we illustrate the coordinates,

Tensors and curvilinear coordinates via discussing of Cartesian Tensors, Tensors Derivatives and Tensor invariance, Also we presented some physical components of Tensors with some example.

In chapter Two, we presented some special Tensors, and we express their structures in curvilinear, cylindrical, Cartesian and Spherical Coordinates with some applications and examples.

Finally, we discuss the physical covariant and contravariant components of various coordinates System. Also we Studies the Linear, Bilinear and Multilinear forms with some examples.

الخلاصة

الفكرة الأساسية في حسبان الممتدات هي تحويل المتغيرات. في هذا البحث قمنا بدراسة بعض هذه المفاهيم. تم تقسيم هذا العمل إلى ثلاث أبواب على النحو التالي:-

أولاً ,قمنا بتوضيح الإحداثيات,الممتدات والإحداثيات الإنحنائية من خلال مناقشة الممتدات الكارتيزيه ,مشتقات الممتدات والممتدات اللامتغيره,أيضا قدمنا بعض مركبات الممتدات الفيزيائيه مع بعض الأمثلة.

في الباب الثاني ,قدمنا بعض الممتدات الخاصة ,ووضحنا تركيبها في الإحداثيات الإنحنائية ,الإحداثيات الاسطوانية,الإحداثيات الكارتيزيه والإحداثيات الكروية, مع بعض التطبيقات والأمثلة.

أخيراً ,ناقشنا المركبات المتغيرة المصاحبه واللامتغيره الفيزيائيه بالنسبه لمتجه وقمنا بإعطاء بعض الأمثله لتمثيلاتها في أنظمه إحداثيات متنوعه.أيضاً درسنا الصيغ الخطيه,الصيغ ثنائيه الخطيه والصيغ متتعده الخطيه مع بعض الامثله.

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Chapter One

Section (1-1): Coordinates and Tensors

Consider a space of real numbers of dimension R^n,,and a single real time, *t*. Continuum properties in this space can be described by arrays of different dimensions, m, such as scalars (m = 0), vectors (m = 1), matrices(m =2),and general multi-dimensional arrays. In this space we shall introduce acoordinate system,$x^i, i = 1, \dots . n$, as a way of assigning n real numbers for every point of space There can be a variety of possible coordinate systems. A general transformation rule between the coordinate systems is

$$\tilde{x}^i = \tilde{x}^i(x^1 \dots . x^n) \qquad (1.1)$$

Consider a small displacement dx^i . Then it can be transformed from coordinate system x^i to a new coordinate system $\tilde{x}^i$ using the partial differentiation rules applied to (1.1):

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j \qquad (1.2)$$

This transformation rule can be generalized to a set of vectors that we shall call contra variant vectors:

$$\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^j} A^j \qquad (1.3)$$

That is, a contravariant vector is defined as a vector which transforms to a new coordinate system according to (1.3). We can also introduce the transformation matrix as:

$$a^i_j \equiv \frac{\partial \tilde{x}^i}{\partial x^j} \qquad (1.4)$$

Reference:

[1] A.V Smirnov, 2004, Introduction to Ten for Calculus, © Draft, Internet.

[2] a. La Speedy Conzales, 2009, Ten for Calculus, Internet.

[3] JesperFerkingh off Borg, 2007, Introduction to vector and Ten for analysis Internet.

[4] Joseph C.Koleclci, 2002, An Introduction to Tensors for Student of Physics and Engineering, Internet.

Chapter One

Section (1-1): Coordinates and Tensors

Consider a space of real numbers of dimension R^n,,and a single real time, *t*. Continuum properties in this space can be described by arrays of different dimensions, m, such as scalars (m = 0), vectors (m = 1), matrices(m =2),and general multi-dimensional arrays. In this space we shall introduce acoordinate system,$x^i, i = 1, \dots . n$, asawayofassigning*n*realnumbersforeverypointofspace Therecan beavarietyofpossiblecoordinatesystems.A generaltransformation rulebetweenthecoordinatesystemsis

$$\tilde{x}^i = \tilde{x}^i(x^1 \dots . x^n) \qquad (1.1)$$

Considerasmalldisplacementdx^i.Thenitcanbetransformedfromcoordinates ystemx^itoanewcoordinatesystem$\tilde{x}^i$usingthepartialdifferentiationrules appliedto(1.1):

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j \qquad (1.2)$$

Thistransformationrulecan begeneralizedtoaset ofvectorsthatweshallcall contravariant vectors:

$$\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^j} A^j \qquad (1.3)$$

Thatis,acontravariantvectorisdefinedasavectorwhichtransformstoanew coordinatesystemaccording to(1.3). Wecanalsointroducethetransformation matrixas:

$$a^i_j \equiv \frac{\partial \tilde{x}^i}{\partial x^j} \qquad (1.4)$$

Withwhich(1.3)canberewrittenas

$$A^i = a^i_j A^j \qquad (1.5)$$

Transformationrule(1.3)willnotapplytoallthevectorsinourspace. For example,apartialderivative$\partial/\partial x_i$willtransformas:

$$\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial}{\partial \tilde{x}^i}\frac{\partial x^j}{\partial x^j} = \frac{\partial x^j}{\partial \tilde{x}^i} \qquad (1.6)$$

thatis,thetransformationcoefficientsaretheotherwayupcomparedto(1.2).Now wecangeneralizethistransformation rule,sothateachvectorthattransforms accordingto(1.6)willbecalledaCovariantvector

$$\tilde{A}_i = \frac{\partial x^j}{\partial \tilde{x}^i} A_j \qquad (1.7)$$

Thisprovides thereasonforusinglowerandupperindexesinageneral tensornotation.

Definition (1.1. 1) : Tensor

Tensoroforderm isasetofn^mnumbersidentifiedbym integerindexes.For example,a3rd ordertensor$A$can bedenotedasA_{ijk}andanmordertensorcan bedenotedas$A_{i_1,\ldots,i_m}$.Eachindexofatensorchangesbetween1andn.Forexample,ina3-dimensionalspace$(n = 3)$asecondordertensorwillberepresented by$3^2 = 9$components.

Eachindexofatensorshouldcomplytooneofthetwotransformationrules: (1.3)or (1.7).An index thatcompliestotherule(1.7)iscalleda**covariantindex**andis denotedasasub-index, andanindex complyingtothetransformationrule(1.3)is calleda**contravariantindex** andisdenotedasasuper-index.

Eachindexofatensorcanbecovariantoracontravariant,thustensorA^kisa2-covariant,1-contravarianttensorofthirdorder.

Tensorsareusuallyfunctionsofspaceandtime:

$$A_{i_1 \ldots\ldots i_m} = A_{i_1 \ldots\ldots i_m}(x^1 \ldots . x^n, t)$$

whichdefinesatensorfield,i.e.foreverypointx^iandtime$t$thereareasetofm^nnumbers$A_{i_1 \ldots i_m}$.

Remark (1.1.2):Tensorcharacterofcoordinatevectors

Note,thatthecoordinatesx^iarenottensors,sincegenerally,theyarenot transformedas(1.5).Transformationlawforthecoordinatesisactuallygivenby (1.1). Nevertheless,weshallusetheupper(contravariant)indexes forthecoordinates.

Definition (1.1.3):Kroneckerdeltatensor

Secondorderdeltatensor,δ_{ij}isdefinedas

$$i = j \Rightarrow \delta_{ij} = 1, \;\; i \neq j \Rightarrow \delta_{ij} = 0 \qquad (1.8)$$

Fromthisdefinitionandsincecoordinatesx^iareindependentofeachother it followsthat:

$$\frac{\partial x^i}{\partial x^j} = \delta_{ij} \qquad (1.9)$$

Corollary (1.1.4)Deltaproduct

Fromthedefinition(1.3)andthesummationconvention(21),followsthat

$$\delta_{ij} A_j = A_i \qquad (1.10)$$

Assumethatthereexiststhetransformationinverseto(1.5),whichwecallb^i:

$$x^i = b^i_j d\tilde{x}^j \qquad (1.11)$$

Thenbyanalogyto(1.4) canbedefinedas:

$$b^i_{j=} \frac{\partial x^i}{\partial \tilde{x}^j} \qquad (1.12)$$

From this relation and the independence of coordinates (1.9)

It follows that $a^i_j b^j_k = b^i_j a^j_k = \delta_{ik}$, namely:

$$a^k_i a^k_j = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^j}{\partial \tilde{x}^k} = \frac{\partial x^j}{\partial x^j}\frac{\partial \tilde{x}^i}{\partial \tilde{x}^k} = \frac{\partial \tilde{x}^i}{\partial \tilde{x}^k} = \delta_{ik} \qquad (1.13)$$

Definition (1.1.5):CartesianTensors

Cartesiantensorsareasubsetofgeneraltensorsforwhichthetransformation matrix(1.4)satisfiesthefollowingrelation:

$$a^k_i a^k_j = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^k}{\partial x^j} = \delta_{ij} \qquad (1.14)$$

ForCartesiantensorswehave:

$$\frac{\partial \tilde{x}^i}{\partial x^k} = \frac{\partial x^k}{\partial \tilde{x}^i} \, (1.15)$$

whichmeansthatboth(1.5)and(1.6)aretransformed withthe samematrixa^i.Thisinturnmeansthatthedifferencebetweenthecovariantand contravariantindexesvanishes fortheCartesiantensors. Consideringthiswe shallonlyusethesub-indexeswheneverwedealwithCartesiantensors.

Definition(1.1.6):TensorNotation

Tensornotationsimplifies writingcomplexequationsinvolvingmultidimensional objects.Thisnotationisbasedonasetoftensorrules.Therulesintroducedin thissectionrepresentacomplete setofrulesforCartesian tensors.Theimportanceoftensorrules isgivenbythefollowinggeneralremark:

Remark (1.1.7):Tensorrule

Tensorrulesguaranteethatifanexpressionfollows theserulesitrepresentsatensoraccordingtoDefinition(1.1).Thus,followingte nsorrules,onecanbuildtensorexpressionsthatwillpreservetensorpropertieso fcoordinatetransformations(Definition(1.1))andcoordinateinvariance. Tensorrulesarebasedonthefollowingdefinitionsandpropositions.

Definition(1.1 . 8) Tensorterms

A tensortermisaproductoftensors for example:

$$A_{ijk}B_{jk}C_{pq}E_qF_P (1.16)$$

Definition(1.1.9) Tensorexpression

Tensorexpressionisasumoftensorterms.Forexample:

$$A_{ijk}B_{jk} + C_iD_{pq}E_qF_p \qquad (1.17)$$

Generallythetermsintheexpressionmaycomewithplusorminussign.

Proposition (1.1.10): Allowedoperations

Theonlyallowedalgebraicoperationsintensorexpressions aretheaddition,subtractionandmultiplication.Divisionsareonlyallowed forconstants,like $1/_{c}$.Ifatensorindexappearsinadenominator,suchtermshouldberedefined,

soasnottohavetensorindexesinadenominator.Forexample, $1/_{A_i}$ shouldbe

redefinedas: $B_i \equiv 1/_{A_i}$

Definition(1.1.11): Tensorequality

Tensorequalityisanequalityoftwotensorexpressions.

Forexample:-(1.1.12)

$$A_{ij}B_j = C_{ikp}D_kE_p + E_jC_{jki}B_k \qquad (1.18)$$

Definition(1.1.13):Freeindexes

Afree indexisanyindexthatoccursonlyonce in atensorterm.For example,index$i$isafreeindexintheterm(1.16).

Proposition(1.1.14):Freeindexrestriction

Everyterminatensorequalityshouldhavethesamesetoffreeindexes.Forexample,ifindex$i$isafreeindexinanytermoftensorequality,suchas(1.18),itshouldbethefreeindexinallotherterms.

Forexample

$$A_{ij}B_j = C_jD_j$$

isnotavalid tensorequalitysince indexiisafreeindexinthetermontheRHSbutnotintheLHS.

Definition(1.1.15):Rankofaterm

A rankofatensortermisequaltothe numberofitsfreeindexes.Forexample,the rankofthetherm$A_{ijk}B_jC_k$isequalto1.

Itfollowsfrom(1.1.14)thatranksofallthetermsinavalid tensorexpression shouldbethesame.Note, thatthedifferencebetweentheorder andtherankis thattheorderisequaltothenumberofindexesofatensor,andthe rankisequal tothe numberoffreeindexesinatensorterm.

Proposition(1.1.16):Renamingoffreeindexes

Anyfreeindexin atensorexpressioncanbenamedbyanysymbolaslong asthissymboldoesnotalreadyoccurinthetensorexpression.

Forexample,theequality

$$A_{ij}B_j = C_iD_jE_j \qquad (1.19)$$

isequivalentto

$$A_{kj}B_j = C_kD_jE_j \qquad (1.20)$$

Herewereplacedthefreeindex$i$with$k$.

Definition(1.1.17):Dummyindexes

A dummyindexisanyindexthatoccurstwiceinatensorterm.

Forexample,indexes j, k, p, qin(1.16)aredummyindexes.

Proposition (1.1.18): Summationrule

Anydummyindeximpliessummation,i.e.

$$A_iB_i = \sum_{i}^{n} A_iB_i \qquad (1.21)$$

Proposition(1.1.19):Summationruleexception

Ifthereshouldbenosummation overtherepeatedindices,it canbeindicatedbyenclosingsuchindicesinparentheses.

Forexample,expression:

$$C_{(i)}A_{(i)}B_{(j)} = D_{ij}$$

doesnotimplysummationover$i$.

Corollary(1.1.20):Scalarproduct

A scalarproductnotationfrom vectoralgebra: $(A.B)$isexpressedin tensor notationasA_iB_i.

Thescalarproductoperationisalsocalleda*contractionofindexes*.

Proposition(1.1.21):Dummyindexrestriction

Noindexcanoccurmorethantwiceinanytensorterm.

Remark(1.1.22):Repeatedindexes

Incaseifanindexoccurs morethantwiceinatermthistermshould be redefinedsoasnottocontainmorethantwooccurrencesofthesameindex.For example,term$A_{ik}B_{jk}C_k$shouldberewrittenas$A_{ik}D_{jk}$,whereD_{jk}isdefinedas $D_{jk} \equiv B_{j(k)}C_{(k)}$withnosummationover$k$inthelastterm.

Proposition(1.1.23):Renamingofdummyindexes

Anydummyindexinatensortermcanberenamed toanysymbolaslong asthissymboldoesnotalreadyoccurinthisterm.

Forexample,term$A_iB_i$isequivalenttoA_jB_j,andsoareterms $A_{ijk}B_jC_k$ and$A_{ibq}B_pC_q$.

Remark(1.1.24):Renamingrules

Notethatwhilethedummyindexrenamingrule(2.16)isapplied toeach tensortermseparately,thefreeindexnamingrule(2.9)shouldapplytothewhole tensorexpression.Forexample,theequality(1.19)above$A_{ij}B_j = C_iD_jE_j$canalsoberewrittenas:

$$A_{kp}B_p = C_kD_jE_j (1.22)$$

withoutchangingitsmeaning.

Definition(1.1.25):Permutationtensor

Thecomponentsofathirdorderpermutationtensorε_{ijk}aredefinedtobe equalto0whenanyindexisequaltoanyotherindex;equalto1whenthesetof indexescanbeobtained bycyclicpermutationof123;and-1whentheindexes canbeobtainedbycyclicpermutationfrom132.Inamathematical languageit canbeexpressedas:

$$i = j \cup i = k \cup j = k \Rightarrow \varepsilon_{ijk} = 0$$

$$ijk \in PG(123) \Rightarrow \varepsilon_{ijk} = 1$$

$$ijk \in PG(132) \Rightarrow \varepsilon_{ijk} = -1 \qquad (1.23)$$

where$PG(abc)$isapermutationgroupofatripleofindexesabc,i.e. $PG(abc) = \{abc, bca, cab\}$.

Forexample,thepermutationgroupof123willconsistofthree combinations:123,231and312,andthepermutationgroupof123consistsof 132,321and213.

Corollary(1.1.26):Permutationofthepermutationtensorindexes

From thedefinitionofthepermutationtensoritfollowsthatthepermutation ofanyofitstwoindexeschangesitssign:

$$\varepsilon_{ijk} = -\varepsilon_{ikj} (1.24)$$

A tensorwiththispropertyiscalled*skew-symmetric*.

Corollary(1.1.27):Vectorproduct

Avectorproduct(cross-product)oftwovectorsinvectornotationisexpressedas

$$\overrightarrow{(A)} = \vec{B} \times \vec{C} \qquad (1.25)$$

whichintensornotationcanbeexpressedas

$$A_i = \varepsilon_{ijk} B_j C_k (1.26)$$

Remark (1.1.28):Crossproduct

Tensorexpression(1.26)ismoreaccuratethanitsvectorcounterpart(1.25), sinceit explicitlyshowshowtocomputeeachcomponentofavectorproduct.

Theorem(1.1.29): Symmetricidentity

IfA_{ij}isasymmetrictensor,thenthefollowingidentityistrue:

$$\varepsilon_{ijk} A_{jk} = 0 \qquad (1.27)$$

Proof:

FromthesymmetryofA_{ij}wehave:

$$\varepsilon_{ijk} A_{jk} = \varepsilon_{ijk} A_{kj} (1.28)$$

Let'srename index$j$into$k$and$k$into$j$intheRHSofthisexpression,according torule(1.1.23):

$$\varepsilon_{ijk} A_{kj} = \varepsilon_{ikj} A_{jk}$$

Using(1.24)wefinallyobtain:

$$\varepsilon_{ikj} A_{jk} = -\varepsilon_{ijk} A_{jk}$$

ComparingtheRHSofthisexpressiontotheLHSof(1.28)wehave:

$$\varepsilon_{ijk} A_{jk} = -\varepsilon_{ijk} A_{jk}$$

fromwhichweconcludethat(1.27)istrue.

Theorem(1.1.30):Tensoridentity

Thefollowingtensor identityistrue:

$$\varepsilon_{ijk} \varepsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp} (1.29)$$

Proof:

Thisidentitycanbeprovedbyexaminingthecomponentsofequality(1.29)com ponent-by-component.

Corollary(1.1.31):Vectoridentity

Usingthetensoridentity(1.29)itispossibletoprovethefollowingimportant vectoridentity:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})(1.30)$$

Definition (1.1.32): TensorDerivatives

ForCartesiantensorsderivativesintroducethefollowingnotation.

Definition(1.1.33):Timederivativeofatensor

A partialderivativeofatensorovertimeisdesignatedas

$$\dot{A} \equiv \frac{\partial A}{\partial t}$$

Definition(1.1.34):Spatialderivativeofatensor

ApartialderivativeofatensorAoveroneoritsspacialcomponentsisdenotedas A_i:

$$A_{,i} \equiv \frac{\partial A}{\partial x_i}(1.31)$$

thatis,theindexofthespatialcomponent thatthederivationisdoneoveris delimitedbyacomma(',')fromotherindexes.Forexample,A_{ijk}isaderivativeof asecondordertensorA_{ij}.

Definition(1.1.35): Nabla

NablaoperatoractingonatensorAisdefinedas

$$\nabla_i A \equiv A_{,i}(1.32)$$

Eventhoughthenotationin(1.31)issufficienttodefinethederivative,Insome instancesitisconvenienttointroducethenablaoperatorasdefinedabove.

Remark(1.1.36): Tensor derivative

InamoregeneralcontextofnonCartesiantensorsthecoordinateindependentderivativewillhaveadifferentformfrom (1.31).

Remark(1.1.37):Rankofatensorderivative

Thederivative ofazeroordertensor(scalar)asgivenby(1.31)formsafirst ordertensor(vector). Generally,thederivativeofanm-ordertensorformsanm+1 ordertensor.However,ifthederivationindex isadummyindex,thentherankof thederivativewillbelowerthanthatoftheoriginaltensor.Forexample,therank ofthederivative$A_{ij,j}$isone,sincethereisonlyonefreeindexinthisterm.

Remark(1.1.38): Gradient

Expression(1.31)representsagradient,whichinavectornotationis∇A:

$$\nabla A \to A_{,i}$$

Corollary(1.1.39): Derivativeofacoordinate

From(1.9)itfollowsthat:

$$x_{i,j} = \delta_{ij} (1.33)$$

Inparticular,thefollowingidentityistrue:

$$x_{i,j} = x_{1,1} + x_{2,2} + x_{3,3} = 1 + 1 + 1 = 3 \qquad (1.34)$$

Remark(1.1.40):Divergenceoperator

Adivergenceoperatorinavectornotationisrepresentedinatensornotation asA_{ii}:

$$(\nabla \cdot \vec{A}) \to A_{i,i}$$

Remark(1.1.41):Laplaceoperator

TheLaplaceoperatorinvectornotationisrepresentedintensornotationas $A_{,ii}$:

$$\Delta A \to A_{,ii}$$

Remark(1.1.42):Tensornotation

Examples(2.30),(2.32)and(2.33)clearlyshowthattensornotationismore concise andaccurate thanvectornotation,sinceitexplicitly showshoweach componentshouldbecomputed. Itisalsomoregeneralsinceitcoverscases thatdon'thaverepresentationin vectornotation,forexample:A_{ikkj}.

Section (1.2) :Curvilinearcoordinates

Inthissectionweintroducetheideaof*tensorinvariance*andintroducetherules forconstructing*invariantforms*

.**Definition(1.2.1): Tensorinvariance**

ThedistancebetweenthematerialpointsinaCartesiancoordinate systemis computedas $dl^2 = dx_i dx_i$.The*metrictensor*,g_{ij}isintroducedtogeneralizethe notionofdistance(1.39)tocurvilinearcoordinates

Definition(1.2.2):Metric tensor

Thedistanceelementincurvilinearcoordinatesystemiscomputedas:

$$dl^2 = g_{ij} dx^i dx^j (1.35)$$

whereg_{ij}iscalledthemetric tensor.

Thus,ifweknowthemetrictensorinagivencurvilinearcoordinatesystem then thedistance element iscomputed by(1.35). Themetrictensorisdefinedasa tensorsinceweneedtopreservetheinvarianceofdistanceindifferentcoordinate systems,thatis,thedistanceshouldbeindependent ofthecoordinate system, thus:

$$dl^2 = g_{ij} dx^i dx^j = \tilde{g}_{ij} d\tilde{x}^i \square \tilde{x}^j (1.36)$$

Themetrictensorissymmetric,which canbeshown byrewriting(1.35)as follows:

$$g_{ij} dx^i dx^j = g_{ij} dx^j dx^i = g_{ji} dx^i dx^j$$

wherewefirstswappedplacesofdx^i and, dx^jandthenrenamedindex i into j and j into i.Wecanrewritetheequalityaboveas:

$$g_{ij} dx^i dx^j - g_{ji} dx^i dx^j = (g_{ij} - g_{ji}) dx^i dx^j = 0$$

Sincetheequalityaboveshouldholdforany$dx^i dx^j$, weget:

$$g_{ij} = g_{ji} (1.37)$$

Themetrictensorisalsocalledthe*fundamentaltensor*.Theinverseofthe metrictensorisalsocalledthe*conjugatemetrictensor*,g^{ij},whichsatisfiesthe relation:

$$g^{ik} g_{kj} = \delta_{ij} (1.38)$$

Letx^ibeaCartesiancoordinatesystem,and$\tilde{x}^j$thenewcurvilinearcoordinatesy stem.Bothsystemsarerelatedbytransformationrules(1.5)and(1.11).Then from(1.36)weget:

$$dl^2 = dx^i dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j \frac{\partial x^i}{\partial \tilde{x}^k} \tilde{x}^k = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial x^i}{\partial \tilde{x}^k} d\tilde{x}^j d\tilde{x}^k (1.39)$$

WhenwetransformfromaCartesiantocurvilinearcoordinates themetric tensorincurvilinear coordinatesystem,$\tilde{g}_{ij}$canbedetermined bycomparingrelations(1.39)and(1.35):

$$\tilde{g}_{ij} = \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^k}{\partial \tilde{x}^j} (1.40)$$

Using(1.38)wecanalsofinditsinverseas:

$$\tilde{g}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^k} (1.41)$$

Usingtheseexpressiononecancomputeg_{ij}andg^{ij}in variouscurvilinearcoordinatesystems

Definition(1.2.3):Conjugatetensors

Foreachindexofatensorweintroduce theconjugate tensorwherethis indexistransferedtoitscounterpart(covariant/contravariant)usingtherelation s:

$$A^i = g^{ij} A_j (1.42)$$

$$A_i = g_{ij} A^j (1.43)$$

Conjugatetensorisalsocalledthe*associatetensor*.

Relations(1.42),(1.43)arealsocalledasoperationsof*raising/loweringofindex es*.

Remark(1.2.4):Tensorinvariance

Sincethetransformationrulesdefined by(1.11)haveasimplemultiplicative character,anytensorexpressionshouldretainit’soriginalformundertransform ationintoanewcoordinatesystem.Thusifanexpressionisgiveninatensorform it willbeinvariantundercoordinatetransformations.

Notalltheexpressionsconstructedfrom tensortermsincurvilinearcoordinateswillbetensorsthemselves.Forexample,ifvectors $A_i$and$B_i$aretensors, then$A_i B_i$isnotgenerallyatensor.However,ifweconsiderthesame operation onacontravarianttensorA^iandacovarianttensor$B_i$thentheproductwillforman invariant:

$$\bar{A}^i \bar{B}_i = A^i B_i (1.44)$$

Thusin curvilinearcoordinateswehavetorefinethedefinitionofthescalar product(Corollary1.1.27)ortheindexcontraction operationtomakeitinvariant.

Definition(1.2.5):InvariantScalarProduct

Theinvariantformofthescalarproduct betweentwocovariant vectorsA_i and B_i is$g^{ij} A_i B_j$.Similarly,theinvariantformofascalarproductbetweentwo contravariantvectorsA^i and B^i is$g_{ij} A^i B^j$,where$g_{ij}$is the metric tensor (1.40)and g^{ij} is its conjugate (1.38).

Corollary(1.2.6):Twoformsofascalarproduct

According to(1.42),(1.43)thescalarproductcanberepresentedbytwoinvariantforms : $A^i B_i$and $A_i B^i$.Itcanbeeasilyshownthatthesetwoformshavethe same values.

Corollary(1.2.7):Rulesofinvariantexpressions

Tobuildinvarianttensorexpressions weaddtwomorerulestoCartesian tensorrulesoutlinedin chapter (2):

1.Eachfreeindexshouldkeepitsvertical positionineveryterm,i.e. ifthe indexiscovariantinonetermitshouldbecovariantin everyotherterm,and viseversa.

2.Every pairofdummyindexesshouldbecomplementary,thatisoneshould becovariant,andanothercontravariant.

Forexample,aCartesianformulationofa*momentum equation*foranin-compressibleviscousfluidis

$$\dot{u}_i + u_k u_{i,k} = -\frac{P_{,i}}{\rho} + \nu \tau_{ik,k}$$

Theinvariantformofthisequationis:

$$\dot{u}_i + u^k u_{i,k} = -\frac{P_{,i}}{\rho} + \nu \tau^k_{i,k} (1.45)$$

wheretherisingofindexeswasdoneusingrelation(1.42):

$$u^k = g^{kj} u_j, and\ \tau^k = g^{kj} \tau_{ij}$$

definition (1.2.8): Covariantdifferentiation

Asimplescalarvalue,*S*,isinvariantundercoordinatetransformations.

Apartial derivativeofaninvariantisafirstordercovarianttensor(vector):

However,apartial derivativeofatensoroftheorder oneandgreaterisnot generallyaninvariantundercoordinatetransformationsoftype(1.7)and(1.3). Incurvilinearcoordinatesystemweshouldusemorecomplexdifferentiation rulestopreservetheinvarianceofthederivative.Theserulesarecalledtherules of*covariantdifferentiation*andtheyguaranteethatthederivativeitselfisatensor . According totheserulesthederivativesforcovariantandcontravariantindices willbeslightlydifferent.

Theyareexpressedasfollows:

$$A_{i,j} \equiv \frac{\partial A_i}{\partial x^j} - \begin{Bmatrix} k \\ ij \end{Bmatrix} A_k \qquad (1.46)$$

$$A^i_{,j} \equiv \frac{\partial A^i}{\partial x^j} + \begin{Bmatrix} i \\ kj \end{Bmatrix} A^k \qquad (1.47)$$

wherethecontstruct$\left\{ {k \atop ij} \right\}$isdefinedas:

$$\left\{ {k \atop ij} \right\} = \frac{1}{2} g^{kj} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

andisalsoknownintensorcalculusas*Christoffel's symbol*ofthesecondkind. Tensorg^{ij}representstheinverseofthemetrictensorg_{ij}(1.38).Ascanbeseen differentiationofasinglecomponentofavectorwillinvolveallothercomponent s ofthisvector.

Indifferentiating higherordertensorseachindexshouldbetreatedindependently.Thusdifferenti ating asecondordertensor,A^{ij},shouldbeperformed as:

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \left\{ {m \atop ik} \right\} A_{mj} - \left\{ {m \atop jk} \right\} A_{im}$$

andascanbeseen alsoinvolvesallthecomponentsofthistensor.Likewisefor thecontravariantsecondordertensorA^{ij},wehave:

$$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \left\{ {i \atop mk} \right\} A^{mj} + \left\{ {j \atop mk} \right\} A^{im} (1.48)$$

Andforageneral$n$-covariant,m-contravarianttensorwehave:

$$A^{j_1 \ldots\ldots j_m}_{i_1 \ldots\ldots i_{n,P}} = \frac{\partial}{\partial x^p} A^{j_1 \ldots\ldots j_m}_{i_1 \ldots\ldots i_{n,K}}$$

$$+ \left\{ {j_1 \atop qp} \right\} A^{j_2 \ldots\ldots j_m}_{i_1 \ldots\ldots i_n} + \cdots + \left\{ {jm \atop qp} \right\} A^{j_1 \ldots\ldots j_{m-1q}}_{i_1 \ldots\ldots i_n}$$

$$+ \left\{ {q \atop i1p} \right\} A^{j_2 \ldots\ldots j_m}_{qi_1 \ldots\ldots i_n} + \cdots + \left\{ {q \atop i_{np}} \right\} A^{j_2 \ldots\ldots j_m}_{i_1 \ldots\ldots i_{n-1q}} (1.49)$$

Despitetheirseemingcomplexity,therelationsofcovariant differentiation canbeeasilyimplemented algorithmicallyandusedinnumericalsolutionson arbitrarycurvedcomputationalgrids.

Remark(1.2.9):Rulesofinvariantexpressions

Aswaspointedoutin Corollary(3.6),therulestobuild invariantexpressions involveraisingorloweringindexes(1.42),(1.43). However,sincewedidnotintro-ducethenotationforcontravariantderivative,theonlywaytoraisetheindex ofa covariantderivative,sayA_i,ittousetherelation(1.42)directly,thatis:$g^{ij}A_j$. Forexample,wecanre-formulatethemomentumequation(1.45)interms of contravariantfreeindex$i$as :

$$\dot{u}^i + u^k u^i_{,k} = -\frac{g^{ik}P_{,k}}{\rho} + \nu\tau^{ik}_{,k} (1.50)$$

wheretheindexofthepressureterm wasraisedbymeansof(1.42). Usingtheinvarianceofthescalar productonecanconstruct twoimportant differentialoperatorsin curvilinearcoordinates:*divergence*ofavector$divA \equiv A^i_i$ (1.51)and *Laplacian*,$\Delta A \equiv g^{ik}A_{ki}$ (1.55).

Definition(1.2.10):Divergence

DivergenceofavectorisdefinedasA^i_i:

$$divA \equiv A^i_{,i} (1.51)$$

Fromthisdefinitionandtheruleofcovariantdifferentiation(47)wehave:

$$A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \begin{Bmatrix} i \\ ki \end{Bmatrix} A^k (1.52)$$

thiscanbeshown[2]tobeequalto:

$$A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}\right)A^i = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right) (1.53)$$

where$g$isthedeterminantofthemetrictensorg_{ij}.

Thedivergenceofacovariantvector$A_i$isdefinedasadivergenceofits conjugate contravarianttensor(1.42):

$$A^{i}_{,i} = g^{ij} A_{j,i} (1.54)$$

Definition(1.2.11): Laplacian

A LaplaceoperatororaLaplacianofascalarAisdefinedas

$$\Delta A \equiv g^{ik} A_{,ki} (1.55)$$

The definitions(3.8),(3.9)ofdifferentialoperatorsare invariantundercoordinatetransformations.Theycanbeprogrammedusingasymbolicmanipulation packages andusedtoderiveexpressionsindifferentcurvilinearcoordinatesytems

Definition(1.2.12) Orthogonalcoordinates

Unitvectorsandstretchingfactors

Thecoordinatesystemis*orthogonal*ifthetangential vectorstocoordinatelines areorthogonalateverypoint.

Considerthreeunitvectors,a^i, b^i, c^i,eachdirectedalongoneofthecoordinateaxis(tangentialunitvectors),thatis:

$$a^i = \{a^1, 0, 0\} (1.56)$$

$$b^i = \{0, b^2, 0\} \quad (1.57)$$

$$c^i = \{0,0, c^3\} \qquad (1.58)$$

Theconditionoforthogonality meansthatthescalarproductbetweenany twooftheseunitvectorsshouldbezero.According tothedefinition ofascalar product.itshouldbewritteninform(1.44),thatis,ascalarproduct betweenvectors$a_i$and$b_i$canbewrittenas:$\left(a^i b_i \ or \ a_i b^i\right)$.Let'susethefirstform for definiteness.

Then,applyingtheoperationofrisingindexes(1.42),wecanexpress thescalarproductincontravariantcomponentsonly:

$$0 = a^i b_i = g_{ij} a^i b^j =$$

$$g_{11}a^1 0 + g_{12}a^1 b^2 + g_{13}00$$

$$g_{21}a^2 b^2 + g_{22}0b^2 + g_{23}00$$

$$g_{31}a^3 0 + g_{32}0b^2 + g_{33}00$$

$$= (g_{12} + g_{21})a^1 b^2 = 2g_{12}a^1 b^2 = 0 \qquad (1.59)$$

whereweusedthesymmetryofg_{ij},(1.37).Sincevectorsa^1andb^2werechosento benon-zero,wehave:$g_{12} = 0$.Applyingthesamereasoningforscalarproductsof other vectors,weconcludethatthemetrictensorhasonlydiagonalcomponents non-zero:

$$g_{ij} = \delta_{ij}g_{(ii)}(1.60)$$

Let'sintroducestretchingfactors,h_i,asthesquarerootsofthesediagonalcompononentsofg_{ij}:

$$h_1 \equiv (g_{11})^{1/2}\ ; h_2 \equiv (g_{22})^{1/2}; h_3 \equiv (g_{33})^{1/2}(1.61)$$

Now,consider thescalarproductofeachoftheunitvectors(1.56)-(1.58)with itself.Sinceallvectors areunit,thescalarproduct ofeachwithitselfshould be one:

$$a^i a_i = b^i b_i = c^i c_i = 1$$

Or,expressedincontravariantcomponentsonlytheconditionofunityis:

$$g_{ij}a^i a^j = g_{ij}b^i b^j = g_{ij}c^i c^j = 1$$

Now,considerthefirsttermaboveandsubstitutethecomponentsof$a$from(1.56). Theonlynon-zerotermwillbe:

$$g_{11}a^1 a^1 = (h_1)^2(a^1)^2 = 1$$

andconsequently:

$$a^1 = \pm\frac{1}{h_1} \quad (1.62)$$

wherethenegativesolutionidentifiesavectordirectedintotheoppositedirection, andwecanneglectitfordefiniteness.Applyingthesamereasoning foreachof thetreeunit vectorsa_i, b_i, c_i,wecanrewrite(1.56),(1.57)and(1.58)as:

$$a^i = \left\{\frac{1}{h_1}, 0,0\right\} \quad (1.63)$$

$$b^i = \left\{0, \frac{1}{h_2}, 0,\right\} \quad (1.64)$$

$$c^i = \left\{0,0, \frac{1}{h_3}\right\} \quad (1.65)$$

whichmeansthatthecomponents ofunitvectorsinacurvedspaceshouldbe scaledwithcoefficientsh_i.Itfollowsfromthisthattheexpressionfortheelement oflengthincurvilinearcoordinates,(1.35),canbewrittenas:

$$dl^2 = g_{ij}d\tilde{x}^i d\tilde{x}^j = h_i^2\left(d\tilde{x}^i\right)^2 \quad (1.66)$$

Similarly,weintroducetheh^icoefficientsfortheconjugatemetrictensor (1.38) :

$$g^{ij} = \delta_{ij}\left(h^{(i)}\right)^2 \quad (1.67)$$

Combiningthelatterwith(1.38),weobtain:$\delta_{ij}h_{(i)}h^i = \delta_{ij,}$fromwhichitfollowsthat

$$h_{(i)} = {}^{1}/_{h^{(i)}} \quad (1.68)$$

Definition(1.2.1 3) Physicalcomponentsoftensors

Consider adirectioninspacedeterminedbyaunitvectore_i. Thenthe*physical component*ofavector$A_i$inthedirection$e_i$isgiven byascalar productbetween $A_i$ande_i,namely:

$$A(e)=g^{ij}A_i \ e_j$$

AccordingtoCorollary(3.5)theabovecanalsoberewrittenas:

$$A(e) = A_i \ e^i \ = A^i e_i (1.69)$$

Supposetheunitvectorisdirectedalongoneoftheaxis:e^i={e^1, 0,0}.From (1.63)it follows that:

$$e^1 \ = {}^1/_{h_1}$$

Where$h_1$isdefinedby(1.61).

Thusaccordingto(1.69)thephysicalcomponentof vectorA_i indirection1inorthogonalcoordinatesystemisequalto:

$$A(1) = \ {}^A/_{h_1}$$

or,repeatingtheargumentforothercomponents,wehaveforthephysical componentsofacovariantvector:

$$ {}^{A_1}/_{h_1}, {}^{A_2}/_{h_2}, \ {}^{A_3}/_{h_3} (1.70)$$

Followingthesamereasoning,forthecontravariantvectorA_i ,wehave:

$$h_1 A^1, h_2 A^2 h_3 A^3$$

Generalrulesofcovariantdifferentiation introducedissimplify considerablyinorthogonalcoordinate systems.

Inparticular,wecandefinethe *nabla*operatorbythephysicalcomponents ofacovariantvectorcomposedof partialdifferentials:

$$\nabla_i = \frac{1}{h_{(i)}} \frac{\partial}{\partial x^i} (1.71)$$

wheretheparenthesesindicatethatthere'snosummationwithrespectto index$i$.

Inorthogonalcoordinatesystemthegeneralexpressionsfordivergence(1.53) andLaplacian(1.55)operatorscanbeexpressedintermsofstretchingfactorsonly:

$$A^i_{,i} = \frac{1}{H}\frac{\partial}{\partial x_i}\left(\frac{H}{h_{(i)}}A_i\right)$$

$$\Delta A = \frac{1}{H}\frac{\partial}{\partial x_i}\left(\frac{H}{h_{(i)}}\frac{\partial A}{\partial x_i}\right) \qquad (1.72)$$

$$H \equiv \prod_{i=1}^{n} h_i$$

Importantexamplesoforthogonalcoordinatesystemsaresphericalandcylindricalcoordinatesystems.Considerthe exampleofacylindricalcoordinatesystem: $X_i = \{x_1, x_2, x_3\}$ and $\tilde{x}_i = \{r, 0,1\}$

$$x_1 = r\cos\theta$$

$$x_2 = r\, sin\theta$$

$$x_3 = l$$

Accordingto(1.40)onlyfewcomponents ofthemetrictensorwillsurvive . Thenwecancomputenabla,divergence andLaplacian operatorsaccordingto(1.71),(1.52)and(1.55),or usingsimplifiedrelations(1.72)-(1.73):

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right)$$

$$divA = \frac{\partial A_1}{\partial\tilde{x}^1} + \frac{1}{\tilde{x}^1}\frac{\partial A_2}{\partial\tilde{x}^2} + \frac{\partial A_3}{\partial\tilde{x}^3} + \frac{1}{\tilde{x}^1}A_1$$

$$= \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} + \frac{1}{r}A_r$$

Note,thatinsteadofusingthecontravariantcomponentsasimpliedbythegeneral definitionofthedivergenceoperator(1.51)weare usingthecovariantcomponentsasdictatedbyrelation(1.70).Theexpressionofthe Laplacianbecomes:

$$\Delta \mathrm{A} = \frac{\partial^2 \mathrm{A}}{(\partial \tilde{\mathrm{x}}_1)^2} + \frac{1}{\tilde{x}_1^2}\frac{\partial^2 A}{(\tilde{x}_2)^2} + \frac{\partial^2 \mathrm{A}}{(\partial \tilde{\mathrm{x}}_3)^2} + \frac{1}{\tilde{x}_1}\frac{\partial A}{\partial \tilde{x}_1}$$

$$= \frac{\partial^2 \mathrm{A}}{(\partial \mathrm{r})^2} + \frac{1}{r^2}\frac{\partial^2 \mathrm{A}}{(\partial \theta)^2} + \frac{\partial^2 \mathrm{A}}{(\partial \mathrm{z})^2} + \frac{1}{r}\frac{\partial A}{\partial r}$$

Chapter Two

Special Tensors

Knowing howtensorsare defined and recognizing atensorwhen itpops up infrontofyou are two differentthings.Somequantities,whichare tensors, frequently ariseinapplied problems and youshould learn torecognizethesespecialtensorswhentheyoccur. Inthis section someimportanttensorquantities aredefined. Wealsoconsiderhowthesespecialtensors caninturnbeusedtodefineothertensors.

Definition (2.1): Metric Tensor

Define$y^i, =$ $1, \dots, N$ asindependentcoordinatesinan$N$dimensionalorthogonal Cartesiancoordinate system. The distance squared between twopoints $y^i and\ y^i + dy^i, i = 1, \dots, N$isdefined bythe expression

$$ds^2 = dy^m dy^m = (dy^1)^2 + (dy^2)^2 + \cdots + (dy^N)^2 \qquad (2.1)$$

Assumethatthecoordinatesy^iarerelatedtoasetofindependentgeneraliz edcoordinates$x^i, i = 1, \dots, N$byasetoftransformationequations

$$y^i = y^i(x^1, x^2, \dots, x^N), i = 1, \dots., N \qquad (2.2)$$

Toemphasize thateachy^idepends upon the$x$coordinateswesometimes usethenotation$y^i = y^i(x)$,for$i = 1, \dots, N$. Thedifferential ofeachcoordinatecanbewrittenas

$$dy^m = \frac{dy^m}{dx^j} dx^j \ , m = 1, \dots \dots, n \qquad (2.3)$$

Andconsequentlyinthe$x$-generalized coordinatesthedistance squared, found fromthe equation(2.1),becomesaquadraticform. Substitutingequation (2.3)intoequation(2.1)wefind:

$$ds^2 = \frac{dy^m}{dx^i} \frac{dy^m}{dx^i} dx^i dx^j = g_{ij} dx^i dx^j \qquad (2.4)$$

where

$$g_{ij} = \frac{dy^m}{dx^i} \frac{dy^m}{dx^j} \ , i, j = 1, \dots, N \qquad (2.5)$$

arecalledthemetricesofthespacedefinedbythecoordinates$x^i, i = 1, \dots, N$.

Heretheg_{ij}arefunctionsof the $x$coordinatesand issometimes writtenas$g_{ij} = g_{ij}(x)$.Further,themetricesg_{ij}aresymmetric inthe indices$i$and $j$sothat$g_{ij} = g_{ji}$forallvalues of$i$and $j$overthe range ofthe indices. Ifwetransformto anothercoordinatesystem,say$x^i, i = 1, \dots, N$,thentheelementofarclengthsquaredisexpressedinterms ofthebarred coordinates and$ds^2 = g_{ij}dx^i dx^j, where\ g_{ij} = g_{ij}(x)$isafunction ofthebarred coordinates. Thefollowingexampledemonstratesthatthesemetricesaresecondordercovariant tensors.

Example (2.2)

Showthemetric components g_{ij}arecovarianttensorsofthesecondorder.

Solution:

Inacoordinatesystem $x^i, i = 1, \dots, N$theelementofarclengthsquared is$ds^2 = g_{ij}dx^i dx^j$ (2.6)

whileinacoordinatesystem$x^i, i = 1, \dots, N$theelementofarclengthsquared isrepresentedintheform

$$ds^2 = g_{mn}dx^m dx^n \qquad (2.7)$$

The element of arc length squared is to be an invariant and so we require that

$$g_{mn}dx^m dx^n = g_{ij}dx^i dx^j \qquad (2.8)$$

Hereit isassumed thatthereexists acoordinatetransformation, which relates thebarred and unbarred coordinates.

Ingeneral,if$x^i = x^i(x)$,then for $x^i, i = 1, \dots, N$wehave

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^m} d\bar{x}^m and\ dx^j = \frac{\partial x^j}{\partial \bar{x}^n} d\bar{x}^n \qquad (2.9)$$

Substitutingthesedifferentials inequation(2.8)givesustheresult

$$g_{mn}dx^m dx^n = g_{ij}\frac{\partial x^i}{\partial x^m}\frac{\partial x^j}{\partial x^n}dx^m dx^n \; or$$

$$\left(g_{mn} - g_{ij}\frac{\partial x^i}{\partial x^m}\frac{\partial x^j}{\partial x^n}\right)dx^m dx^n = 0$$

For arbitrary changes in $d\bar{x}^m$ this equation implies that $g_{mn} = g_{ij}\frac{\partial x^i}{\partial x^m}\frac{\partial x^j}{\partial x^n}$ and consequently g_{ij} transforms as a second order absolute covariant tensor

Example (2.3):Curvilinear coordinates

Consider asetofgeneraltransformationequations from rectangularcoordinates(x, y, z)tocurvilinear coordinates(u, v, w).Thesetransformationequations andthecorresponding inversetransformationsarerepresented

$x = x\ (u, v, w)$ $\qquad u = u(x, y, z)$

$y = y(u, v, w)$ $\qquad v = v(x, y, z)$

$z = z(u, v, w)$ $\qquad w = w(x, y, z)$ (2.10)

Here$y^1 = x, y^2 = y, y^3 = z \; and \; x^1 = u, x^2 = v, x^3 = w$arethe Cartesianand generalized coordinates and N=3.The intersectionofthecoordinatesurfaces$u = c_1, v = c_2, w = c_3$definecoordinatecurvesofthe curvilinear coordinatesystem. The substitutionofthe giventransformationequations(2.10)into the postion vector $\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$ produces the position vector which is a function of the generalized coordinates and

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\hat{e}_1 + y(u, v, w)\hat{e}_2 + z(u, v, w)\hat{e}_3$$

and consequently $d\vec{r} = \frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv + \frac{\partial \vec{r}}{\partial w}dw$ where

$$\vec{E}_1 = \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{e}_1 + \frac{\partial y}{\partial u}\hat{e}_2 + \frac{\partial z}{\partial u}\hat{e}_3$$

$$\vec{E}_2 = \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}\hat{e}_1 + \frac{\partial y}{\partial v}\hat{e}_2 + \frac{\partial z}{\partial v}\hat{e}_3$$

$$\vec{E}_3 = \frac{\partial \vec{r}}{\partial w} = \frac{\partial x}{\partial w}\hat{e}_1 + \frac{\partial y}{\partial w}\hat{e}_2 + \frac{\partial z}{\partial w}\hat{e}_3 \qquad (2.11)$$

Are tangent vectors to the coordinate curve. Theelementofarclength inthecurvilinear coordinatesis

$ds^2 = d\vec{r}.d\vec{r}$

$$= \frac{\partial \vec{r}}{\partial u}.\frac{\partial \vec{r}}{\partial u}dudu + \frac{\partial \vec{r}}{\partial u}.\frac{\partial \vec{r}}{\partial v}dudv + \frac{\partial \vec{r}}{\partial u}.\frac{\partial \vec{r}}{\partial w}dudw + \frac{\partial \vec{r}}{\partial v}.\frac{\partial \vec{r}}{\partial u}dvdu$$
$$+ \frac{\partial \vec{r}}{\partial v}.\frac{\partial \vec{r}}{\partial v}dvdv + \frac{\partial \vec{r}}{\partial v}.\frac{\partial \vec{r}}{\partial w}dvdw + \frac{\partial \vec{r}}{\partial w}.\frac{\partial \vec{r}}{\partial u}dwdu$$
$$+ \frac{\partial \vec{r}}{\partial w}.\frac{\partial \vec{r}}{\partial v}dwdv + \frac{\partial \vec{r}}{\partial w}.\frac{\partial \vec{r}}{\partial w}dwdw \qquad (2.12)$$

Utilizing the summationconvention, theabove can be expressed in theindex notation.Definethe quantities

$$g_{11} = \frac{\partial \vec{r}}{\partial u}.\frac{\partial \vec{r}}{\partial u} g_{12} = \frac{\partial \vec{r}}{\partial u}.\frac{\partial \vec{r}}{\partial v} g_{13} = \frac{\partial \vec{r}}{\partial u}.\frac{\partial \vec{r}}{\partial w}$$
$$g_{21} = \frac{\partial \vec{r}}{\partial v}.\frac{\partial \vec{r}}{\partial u}\ g_{22} = \frac{\partial \vec{r}}{\partial v}.\frac{\partial \vec{r}}{\partial v} g_{23} = \frac{\partial \vec{r}}{\partial v}.\frac{\partial \vec{r}}{\partial w}$$
$$g_{31} = \frac{\partial \vec{r}}{\partial w}.\frac{\partial \vec{r}}{\partial u} g_{32} = \frac{\partial \vec{r}}{\partial w}.\frac{\partial \vec{r}}{\partial v}\ \ g_{33} = \frac{\partial \vec{r}}{\partial w}.\frac{\partial \vec{r}}{\partial w}$$

And let $x^1 = u, x^2 = v, x^3 = w$. Then the above element of arc length can be expressed as

$$ds^2 = \vec{E}_i.\vec{E}_j dx^i dx^j = g_{ij} dx^i dx^j\ i,j = 1,2,3$$

Where

$g_{ij} = \vec{E}_i.\vec{E}_j = \frac{\partial \vec{r}}{\partial x^i}.\frac{\partial \vec{r}}{\partial x^i} = \frac{\partial y^m}{\partial x^i}.\frac{\partial y^m}{\partial x^i} i,j$ free indices (2.13)

are called the metric component of the curvilinear coordinate system.

The metric component may be thoughtof as the elements of symmetric matrix, since $g_{ij} = g_{ji}$. In the rectangular coordinate system$x, y, z,$ the element of arc length squared is $ds^2 = dx^2 + dy^2 + dz^2$.

In this space the metric components are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example (2.4): cylindrical coordinates (r,$\boldsymbol{\theta}$,z)

The transformation equations from rectangular coordinate to cylindrical coordinates can be expressed as $x = r\cos\theta\ , y = r\sin\theta\ ,\ z = z.$ Here $y^1 = x\ , y^2 = y\ , y^3 = z$ and $x^1 = r,\ x^2 = \theta, x^3 = z$ and the position vector can be expressed

$$\vec{r} = \vec{r}(r, \theta, z) = r\ \cos\theta\hat{e}_1 + r\ \sin\theta\hat{e}_2 + z\hat{e}_3$$

The derivatives of this postion vector are calculated and we find

$$\vec{E}_1 = \frac{\partial\vec{r}}{\partial r} = \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2\ , \vec{E}_2 = \frac{\partial\vec{r}}{\partial\theta} = -\sin\theta\hat{e}_1 + r\cos\theta\hat{e}_2\ , \vec{E}_3 = \frac{\partial\vec{r}}{\partial z} = \hat{e}_3$$

From theresultsinequation (2.13),themetriccomponentsofthisspaceare

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Wenotethatsince$g_{ij} = 0$when$i = j$,thecoordinate system isorthogonal.

Given aset oftransformationsofthe formfound inequation (2.10), onecanreadily determinethe metric components associatedwith the generalized coordinates.

For future referencewelistseveraldifferentcoordinatesystems togetherwith their metric components.Each ofthelisted coordinatesystems areorthogonal andso$g_{ij} = 0$ for$i = j$.Themetriccomponentsoftheseorthogonal systems havetheform

$$g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

andtheelementofarclengthsquared is

$$ds^2 = h_1^2(dx^1)^2 + h_2^2(dx^2)^2 + h_3^2(dx^3)^2$$

1. Cartesian coordinates (x,y,z)

$$x = x \qquad h_1 = 1$$

$$y = y \qquad h_2 = 1$$

$$z = z \qquad h_3 = 1$$

Thecoordinatecurvesareformedbytheintersectionofthecoordinatesurfaces *x*=Constant,*y*=Constantand*z*=Constant.

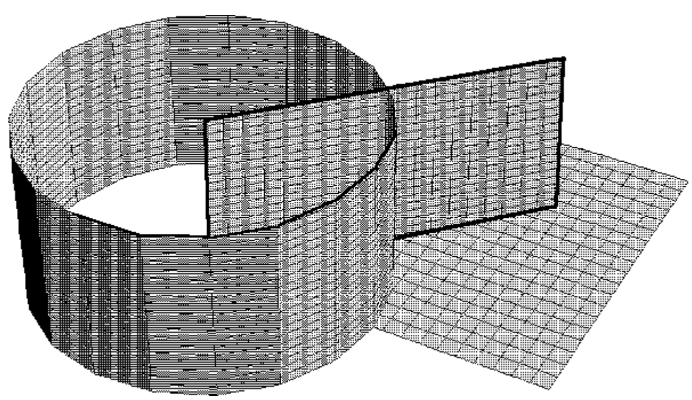

Figure (2.1)Cylindrical coordinates.

2. Cylindrical coordinates(*r*,*θ*,*z*)

$$x = r\cos\theta \qquad r \geq 0 \qquad h_1 = 1$$

$$y = r\sin\theta \qquad 0 \leq \theta \leq 2\pi \qquad h_2 = r$$

$$z = z \qquad -\infty < z < \infty \qquad h_3 = 1$$

The coordinatecurves, illustratedinthe figure(2.1),areformedbythe intersectionofthecoordinatesurfaces

$$x^2 + y^2 = r^2 \qquad cylinders$$

$$y/x = \tan\theta \qquad planes$$

$$z = constant \qquad planes$$

3. Spherical coordinates(ρ,θ,φ)

$$x = p\sin\theta\cos\varphi \qquad p \geq o \qquad h_1 = 1$$

$$y = p\sin\theta\cos\varphi \qquad 0 \leq \theta \leq \pi \qquad h_2 = p$$

$$z = p\cos\theta \qquad 0 \leq \varphi \leq 2\pi \qquad h_3 = p\sin\theta$$

The coordinatecurves, illustratedinthe figure(2.2),areformedbythe intersectionofthecoordinatesurfaces

$$x^2 + y^2 + z^2 = p^2 \qquad spheres$$

$$x^2 + y^2 = \tan^2\theta\, z^2 \qquad cones$$

$$y = x\tan\varphi \qquad plane$$

4. Paraboliccylindrical coordinates (ξ,η,z)

$$x = \xi\eta \qquad -\infty < \xi < \infty \qquad h_1 = \sqrt{\xi^2 + \eta^2}$$

$$y = {}^1/_2\,(\xi^2 - \eta^2) \qquad -\infty < z < \infty \quad h_2 = \sqrt{\xi^2 + \eta^2}$$

$$z = z \qquad \eta \geq 0 \qquad h_3 = 1$$

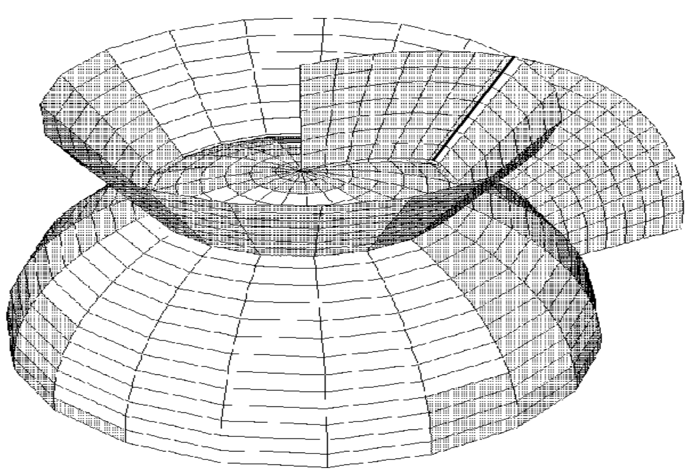

Figure (2.2)Spherical coordinates.

The coordinatecurves, illustratedinthe figure(2.3),areformedbythe intersectionofthecoordinate surfaces

$$x^2 = -2\xi^2\left(y - \frac{\xi^2}{2}\right) \qquad parabolic\ cylinders$$

$$x^2 = 2n^2\left(y + \frac{n^2}{2}\right) \qquad parabolic\ cylinders$$

$$z = constant \qquad plane$$

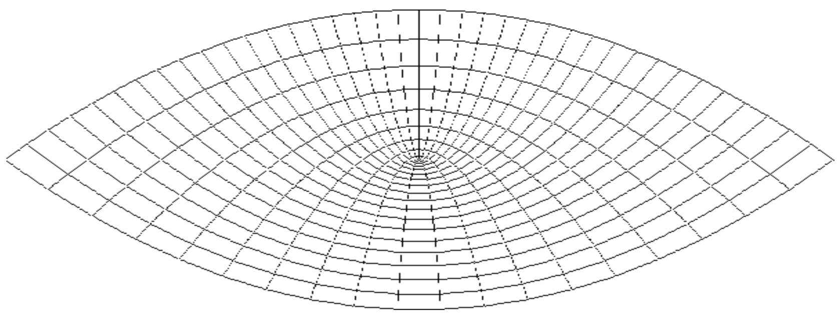

Figure (2.3)Paraboliccylindrical coordinates inplane$z$=0.

5. Paraboliccoordinates (*ξ*,*η*,*φ*)

$$x = \xi\eta\cos\varphi\ \xi \geq 0 \qquad h_1 = \sqrt{\xi^2 + \eta^2}$$

$$y = \xi\eta\sin\varphi \qquad \eta \geq 0 \qquad h_2 = \sqrt{\xi^2 + \eta^2}$$

$$z = {}^1/_2\,(\xi^2 - \eta^2) \quad 0 < \varphi < 2\pi h_3 = \xi\eta$$

The coordinatecurves, illustratedinthe figure(2.4),areformedbythe intersectionofthecoordinate surfaces

$$x^2 + y^2 = -2\xi^2\left(z - \frac{\xi^2}{2}\right) \qquad paraboloids$$

$$x^2 + y^2 = 2\eta^2\left(z + \frac{\xi^2}{2}\right) \qquad paraboloids$$

$$y = x\tan\varphi \qquad plane$$

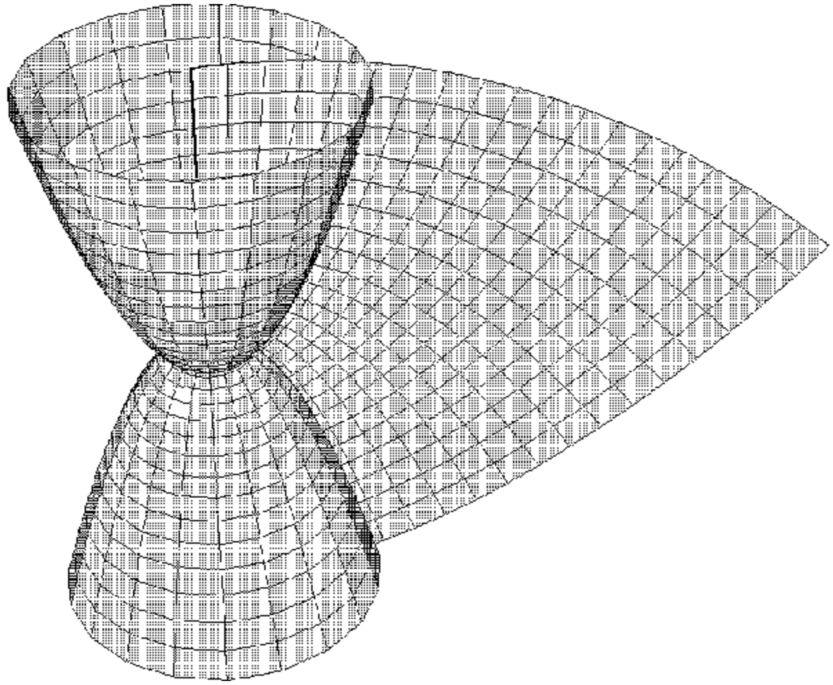

Figure (2.4)Paraboliccoordinates,$\varphi=\pi/4$.

6. Elliptic cylindrical coordinates(ξ,η,z)

$x = \cosh\xi\cos\eta\ \xi \geq 0$ $\qquad h_1 = \sqrt{\sinh^2\xi + \sin^2\eta}$

$y = \sinh\xi\sin\eta$ $\qquad 0 \leq \eta \leq 2\pi h_2 = \sqrt{\sinh^2\xi + \sin^2\eta}$

$z = z$ $\qquad -\infty < z < \infty \quad h_3 = 1$

The coordinatecurves, illustratedinthe figure(2.5),areformedbythe intersectionofthecoordinate surfaces

$$\frac{x^2}{\cosh^2\xi} + \frac{y^2}{\sinh^2\xi} = 1 \qquad Elliptical\ cylinders$$

$$\frac{x^2}{\cos^2\eta} - \frac{y^2}{\sin^2\eta} = 1 \qquad Hyperbolic\ cylinders$$

$$z = constant \qquad planes$$

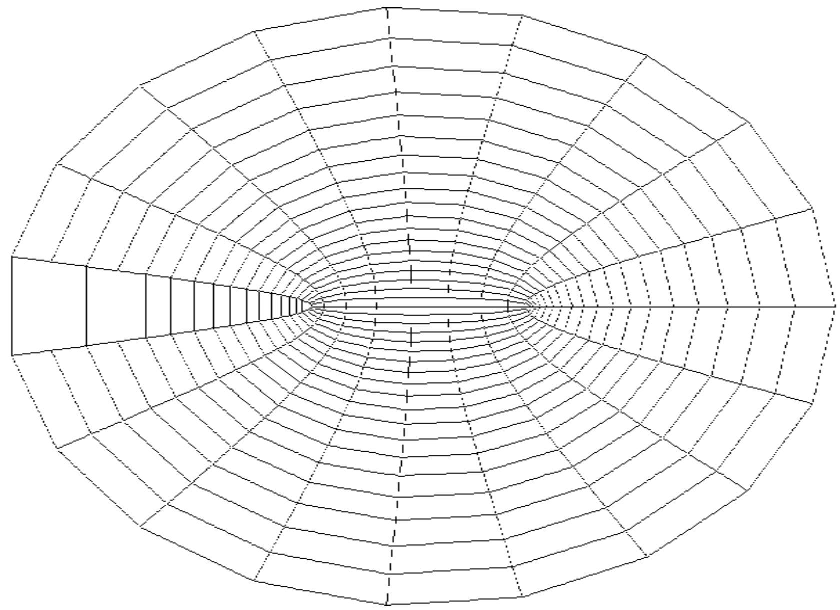

Figure (2.5)Elliptic cylindrical coordinatesintheplane$z$=0.

7. Elliptic coordinates(ξ,η,φ)

$x = \sqrt{(1-\eta^2)(\xi^2-1)}\cos\varphi \qquad 1 \le \xi \le \infty \qquad h_1 = \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}$

$y = \sqrt{(1-\eta^2)(\xi^2-1)}\sin\varphi \qquad -1 \le \eta \le 1 \qquad h_2 = \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}$

$z = \xi\,\eta \qquad 0 \le \varphi \le 2\pi \qquad h_3 = \sqrt{(1-\eta^2)(\xi^2-1)}$

The coordinatecurves, illustratedinthe figure (2.6),areformedbythe intersectionofthecoordinate surfaces

$$\frac{x^2}{\xi^2-1}+\frac{y^2}{\xi^2-1}+\frac{z^2}{\xi^2}=1 \qquad \textit{prolate ellipsoid}$$

$$\frac{z^2}{\eta^2}-\frac{x^2}{1-\eta^2}-\frac{y^2}{1-\eta^2}=1 \qquad \textit{two\_sheeted hyperboloid}$$

$$y = x\tan\varphi \qquad \textit{planes}$$

8. Bipolar coordinates(*u*,*v*,*z*)

$$x = \frac{a \sinh v}{\cosh v - \cos u} \quad ,0 \leq u < 2\pi \qquad h_1^2 = h_2^2$$

$$y = \frac{a \sin u}{\cosh v - \cos u} \quad ,-\infty < v < \infty \quad h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$z = z \qquad\qquad - \infty < v < \infty \quad h_3^2 = 1$$

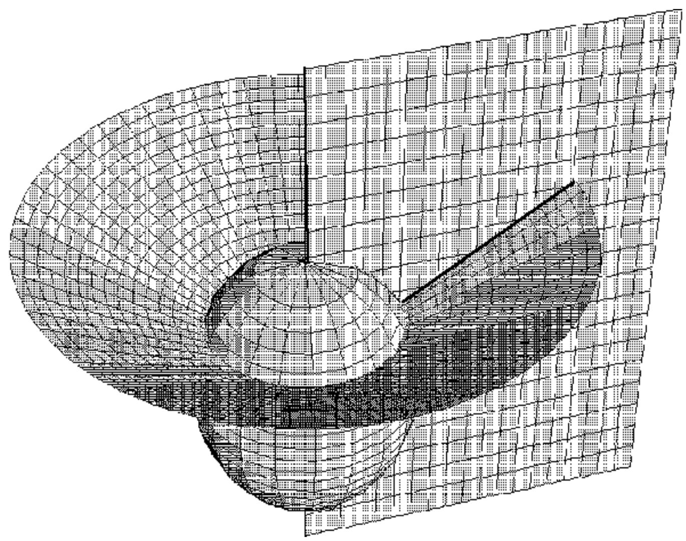

Figure (2.6)Elliptic coordinatesφ=π/4.

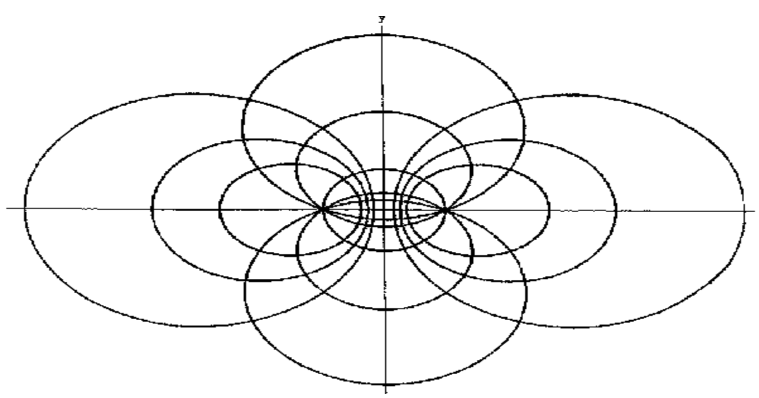

Figure (2.7)Bipolar coordinates.

The coordinatecurves, illustratedinthe figure(2.7),areformedbythe intersectionofthecoordinate surfaces

$$(x - a \coth v)^2 + y^2 = \frac{a^2}{\sinh^2 v} \qquad\qquad cylinders$$

$$x^2 + (y - a \cot u)^2 = \frac{a^2}{\sin^2 u} \qquad\qquad cylinders$$

$$z = constant \qquad\qquad planes$$

9. Conicalcoordinates(*u,v,w*)

$$x = \frac{uvw}{ab} \quad , \quad b^2 > u^2 > a^2 > w^2 \, , u \geq 0 \qquad h_1^2 = 1$$

$$y = \frac{u}{a}\sqrt{\frac{(v^2 - a^2)(w^2 - a^2)}{a^2 - b^2}} \, h_2^2 = \frac{u^2(v^2 - w^2)}{(v^2 - a^2)(b^2 - v^2)}$$

$$z = \frac{u}{b}\sqrt{\frac{(v^2 - b^2)(w^2 - b^2)}{b^2 - a^2}} \, h_3^2 = \frac{u^2(v^2 - w^2)}{(w^2 - a^2)(w^2 - b^2)}$$

The coordinatecurves, illustratedinthe figure(2.8),areformedbythe intersectionofthecoordinate surfaces

$$x^2 + y^2 + z^2 = u^2 \qquad spheres$$

$$\frac{x^2}{v^2} + \frac{y^2}{v^2 - a^2} + \frac{z^2}{v^2 - b^2} = 0 \quad , \qquad \text{cones}$$

$$\frac{x^2}{w^2} + \frac{y^2}{w^2 - a^2} + \frac{z^2}{w^2 - b^2} = 0 \quad , \qquad \text{cones}$$

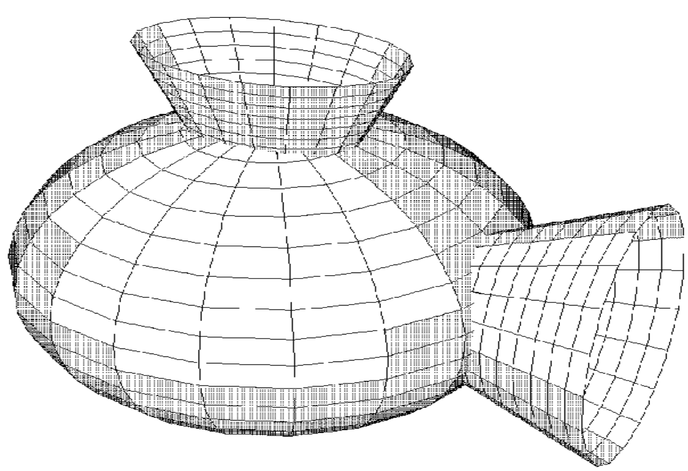

Figure (2.8)Conicalcoordinates.

10. Prolatespheroidal coordinates (*u,v,φ*)

$$x = a \sinh u \sin v \cos \varphi \quad , u \geq o \qquad h_1^2 = h_2^2$$

$$y = a \sinh u \sin v \sin \varphi \quad , 0 \leq v \leq \pi \qquad h_2^2 = a^2(\sinh^2 u + \sin^2 v)$$

$$z = a \cosh u \cos v \qquad , 0 \leq \varphi < 2\pi \quad h_3^2 = a^2 \sinh^2 u \, \sin^2 v$$

The coordinatecurves, illustratedinthe figure(2.9),areformedbythe intersectionofthecoordinatesurfaces

$$\frac{x^2}{(a\sinh u)^2}+\frac{y^2}{(a\sinh u)^2}+\frac{z^2}{(a\cosh u)^2}=1\ , \qquad prolate\ ellipsoids$$

$$\frac{z^2}{(a\cos v)^2}-\frac{x^2}{(a\sin v)^2}-\frac{y^2}{(a\sin v)^2}=1\ , \quad Two\_sheeted\ hyperboloid$$

$$y=x\tan\varphi \qquad , \qquad planes$$

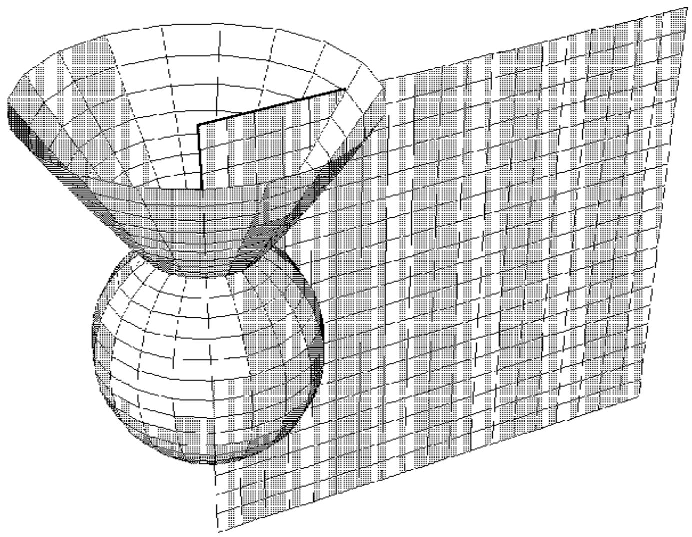

Figure (2.9)Prolatespheroidal coordinates

11. Oblate spheroidal coordinates(ξ,η,φ)

$x=a\cosh\xi\cos\eta\cos\varphi\ , \quad \xi\geq 0 \qquad h_1^2=h_2^2$

$y=a\cosh\xi\cos\eta\sin\varphi\ , \quad -\frac{\pi}{2}\leq\eta\leq\frac{\pi}{2} h_2^2=a^2(\sinh^2\xi+\sin^2\eta)$

$z=a\sinh\xi\sin\eta \qquad , \quad 0\leq\ \varphi\leq 2\pi \qquad h_3^2=a^2\cosh^2\xi\cos^2\eta$

Thecoordinatecurves,illustratedinthefigure(2.10),areformedbytheintersectionofthecoordinate surfaces

$$\frac{x^2}{(a\cosh\xi)^2}+\frac{y^2}{(a\cosh\xi)^2}+\frac{z^2}{(a\sinh\xi)^2}=1\ , \qquad oblate\ ellipsoids$$

$$\frac{x^2}{(a\cos\eta)^2}+\frac{y^2}{(a\cos\eta)^2}-\frac{z^2}{(a\sin\eta)^2}=1\ , \quad one\_sheet\ hyperboloids$$

$$y=x\tan\varphi \qquad , \qquad planes$$

12. Toroidalcoordinates(u,v,φ)

$$x = \frac{a \sinh v \cos\varphi}{\cosh \mathrm{v} - \cos \mathrm{u}} \quad , \qquad 0 \le u < 2\pi \qquad \mathrm{h}_1^2 = \mathrm{h}_2^2$$

$$y = \frac{a \sinh v \sin\varphi}{\cosh \mathrm{v} - \cos \mathrm{u}} \quad , \qquad -\infty \le v < \infty \qquad \mathrm{h}_2^2 = \frac{\mathrm{a}^2}{(\cosh \mathrm{v} - \cos \mathrm{u})^2}$$

$$z = \frac{a \sin u}{\cosh \mathrm{v} - \cos \mathrm{u}} \quad , \qquad 0 \le \varphi < 2\pi \qquad \mathrm{h}_3^2 = \frac{\mathrm{a}^2 \sinh^2 \mathrm{v}}{(\cosh \mathrm{v} - \cos \mathrm{u})^2}$$

Thecoordinatecurves,illustratedinthefigure(2.11),areformedbytheintersectionofthecoordinate surfaces

$$x^2 + y^2 + \left(z - \frac{a \cos u}{\sin u}\right)^2 = \frac{a^2}{\sin^2 \mathrm{u}} \qquad \textit{spheres}$$

$$\left(\sqrt{x^2 + y^2} - a\frac{\cosh v}{\sinh v}\right)^2 + z^2 = \frac{a^2}{\sinh^2 \mathrm{v}} \qquad \textit{Torus}$$

$$y = x \tan\varphi \qquad \textit{planes}$$

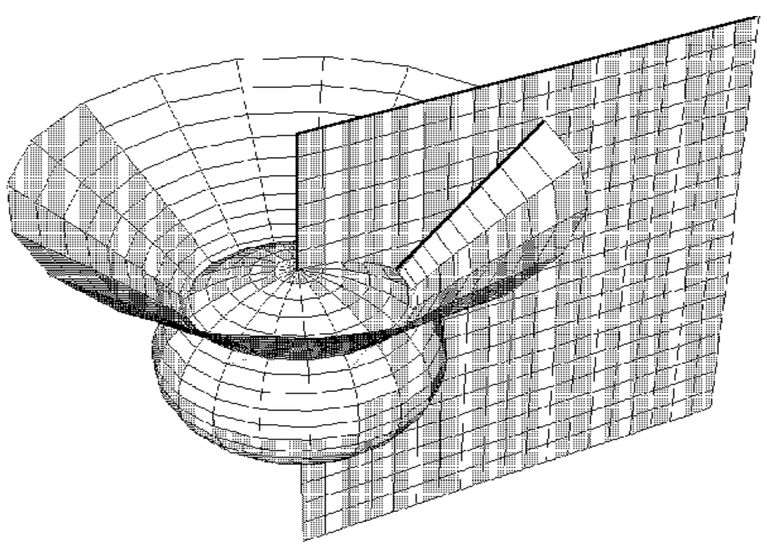

Figure (2.10)Oblate spheroidal coordinates

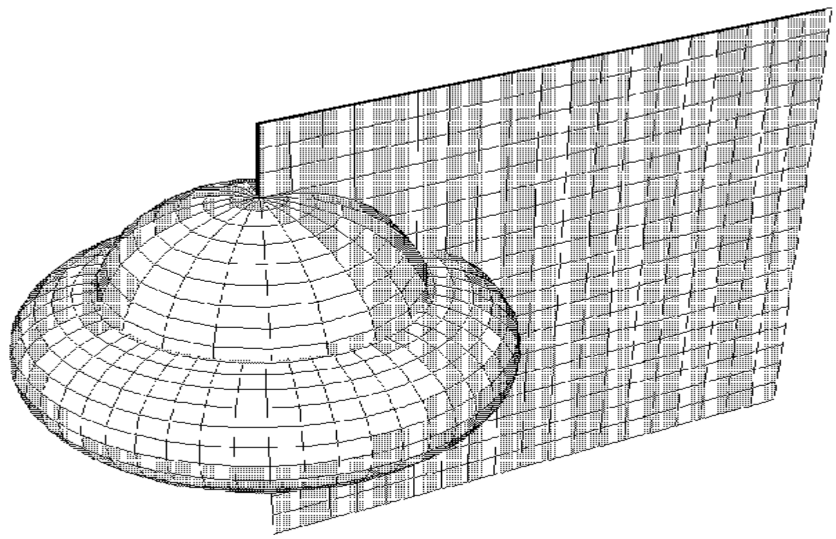

Figure (2.11)Toroidalcoordinates

Example (2.5):

ShowtheKroneckerdelta δ_j^iisamixedsecondordertensor.

Solution:

Assume wehaveacoordinatetransformation$x^i = x^i(\bar{x})$, $i = 1, \dots, n$oftheform(1.2.30)and

possessinganinversetransformationoftheform(1.2.32). Let $\bar{\delta}_j^i$and δ_j^idenote the Kroneckerdelta inthebarred andunbarredsystem ofcoordinates.Bydefinition theKroneckerdelta isdefined $\bar{\delta}_j^i = \delta_j^i = \begin{cases} 0, if\ i \neq j \\ 1, if\ j = i \end{cases}$

Employing the chain rule we write

$$\frac{\partial \overline{x}^m}{\partial \overline{x}^n} = \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^i}{\partial \overline{x}^n} = \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^n}\delta_k^i \qquad (2.14)$$

by hypothesis the $\overline{x}^i, i = 1, \dots, n$ are independent coordinates and there for we have

$\frac{\partial \overline{x}^m}{\partial \overline{x}^n} = \overline{\delta}_n^m$ and (2.14) simplifies to $\overline{\delta}_n^m = \delta_k^i \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^n}$ Therefore, theKroneckerdeltatransformsasamixedsecondordertensor.

Definition (2.6) : ConjugateMetricTensor

Let$g$denote thedeterminantofthematrixhavingthemetric tensor g_{ij} , $i, j = 1, \dots, n$asitselements. Inourstudyofcofactorelementsofamatrix wehaveshownthat

$cof(g_{1j})g_{1k} + cof(g_{2j})g_{2k} + \cdots + cof(g_{nj})g_{nk} = g\delta_k^j$ (2.15)

We can use this fact to find the elements in theinverse matrixassociatedwith the matrixhaving the components g_{ij} .Theelementsofthisinversematrix are

$$g^{ij} = \frac{1}{g} cof(g_{ij}) \qquad (2.16)$$

andarecalledtheconjugate metric components.Weexaminethesummation $g^{ij}g_{ik}$andfind:

$$g^{ij}g_{ik} = g^{1j}g_{1k} + g^{2j}g_{2k} + \cdots + g^{nj}g_{nk} = \frac{1}{g}[cof(g_{1j})g_{1k} + cof(g_{2j})g_{2k} + \cdots + cof(g_{nj})g_{nk}] = \frac{1}{g}[g\delta_k^j] = \delta_k^j$$

The equation:

$$g^{ij}g_{ik} = \delta_k^j \qquad (2.17)$$

isanexamplewherewecanusethequotientlawtoshowg^{ij}isasecondordercontr avarianttensor.Because ofthesymmetryofg^{ij}andg_{ij}theequation (2.17)canberepresentedinother forms.

Example (2.7)

Let$A_i$and A^idenoterespectively thecovariantand contravariantcomponentsofa vector$\tilde{A}$. Showthesecomponents arerelated bytheequations

$$A_i = g_{ij}A^j \qquad (2.18)$$

$$A^k = g^{jk}A_j \qquad (2.19)$$

Whereg_{ij}andg^{ij}arethemetric andconjugate metric componentsofthespace.

Solution:

Wemultiply the equation(2.18)byg^{im}(inner product)and useequation (2.17)tosimplify the results. This produces the equation$g^{im}A_i = g^{im}g_{ij}A^j = \delta^m A^j = A^m$.Changing indicesproducesthe result giveninequation(2.19). Conversely, ifwestartwith equation (2.19)andmultiply byg_{km}(innerproduct)weobtain $g_{km}A^k = g_{km}g^{jk}A_j = \delta^j A_j = A_m$whichisanotherformoftheequation

(2.18)withtheindiceschanged.

Noticetheconsequencesofwhattheequations

(2.18)and(2.19)implywhenweareinanorthogonalCartesiancoordinatesyste m where

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad and \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this special case, we have

$$A_1 = g_{11}A^1 + g_{12}A^2 + g_{13}A^3 = A^1$$
$$A_2 = g_{21}A^1 + g_{22}A^2 + g_{23}A^3 = A^2$$
$$A_3 = g_{31}A^1 + g_{32}A^2 + g_{33}A^3 = A^3$$

These equationstellusthatinaCartesiancoordinatesystem the contravariantand covariantcomponents areidenticallythesame.

Eample (2.8)

WehavepreviouslyshownthatifA$_i$isacovarianttensorofrank1itscomponentsi n abarred system ofcoordinates are

$$\bar{A}_i = A_j \frac{\partial x^j}{\partial x^i} \qquad (2.20)$$

Solveforthe$A_j$intermsofthe$\bar{A}_i$.(i.e. findtheinversetransformation).

Solution :

Multiply equation (2.20) by $\frac{\partial \bar{x}^i}{\partial x^m}$ (inner product) and obtain

$$\bar{A}_i \frac{\partial \bar{x}^i}{\partial x^m} = A_j \frac{\partial x^i}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^m} \qquad (2.21)$$

In the above product we have $\frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^m} = \frac{\partial x^j}{\partial x^m} = \delta_m^j$ since x^i and $x^m$are assumed to be independent coordinate. this reduces equation (2.21) to form

$$\bar{A}_i \frac{\partial \bar{x}^i}{\partial x^m} = A_j \delta_m^j = A_m \qquad (2.22)$$

whichisthedesiredinversetransformation.

Thisresultcanbeobtainedinanotherway. Examine thetransformationequation (2.20)andaskthe question, Whenwehavetwocoordinate systems,sayabarredandanunbarredsystem,doesitmatterwhich system wecallthebarred system. Withsomethoughtitshouldbeobviousthatitdoesn’tmatterwhich system youlabelasthebarred system. Therefore, wecaninterchangethebarred andunbarredsymbolsin equation (2.20) and obtain the result $A_i = \bar{A}_j \frac{\partial \bar{x}^j}{\partial x^i}$ which is the same form as equation (2.22), but with a different set of indices.

Definition (2.9) : Associated Tensors

Any tensor constructed by multiplying (inner product) a given tensor with meteric or conjugate metric tensor is called an associated tensor.

Associatedtensors are differentwaysofrepresentingatensor. The multiplicationofatensor bythe metric orconjugatemetric tensorhasthe effectofloweringorraising indices.

For example the covariant andcontravariantcomponentsof avectoraredifferentrepresentationsof thesamevectorindifferentforms.

Theseformsareassociated withoneanotherbywayofthemetricandconjugate metric tensorand

$$g^{ij}A_i = A^j g_{ij}A^j = A_i$$

Example(2.10)

Thefollowingaresomeexamplesofassociatedtensors

$$A^j = g^{ij}A_iA_j = g_{ij}A^i$$

$$A^m_{.jk} = g^{mi}A_{ijk}A^{i.k}_{m} = g_{mj}A^{ijk}$$

$$A^{nm}_{i..} = g^{mk}g^{nj}A_{ijk}A_{mjk} = g_{im}A^i_{.jk}$$

Sometimes'dots'areusedasindicesinordertorepresentthelocationoftheindext hatwasraisedorlowered. Ifatensorissymmetric, theposition oftheindexisimmaterialandsoadotisnotneeded. Forexample, ifA_{mn}isasymmetric tensor, then itiseasytoshowthat $A^n_{.m}$and $A^{.n}_{m}$areequalandtherefore canbewritten as A^n_m

Higher order tensors are similarly related. For example, if we find a fourth order covariant tensor T_{ijkm} we can then construct the fourth order contravariant tensor T^{pqrs} from the relation

$$T^{pqrs} = g^{pi}g^{qj}g^{rk}g^{sm}T_{ijkm}$$

This fourth order tensor can also be expressed as a mixed tensor. Some mixed tensors associated with the given fourth order covariant tensor are

$$T^p_{.jkm} = g^{pi}T_{ijkm}, \quad T^{pq}_{..km} = g^{qi}T^p_{.jkm}.$$

Definition (2.11): Riemann Space V_N

ARiemannianspace$V_N$issaidtoexistiftheelementofarclength squared hastheform

$$ds^2 = g_{ij}dx^i dx^j \qquad (2.23)$$

wherethemetrices$g_{ij} = g_{ij}(x^1, x^2, \ldots, x^N)$are continuousfunctions ofthecoordinatesand are different fromconstants.Inthe specialcase$g_{ij} = \delta_{ij}$the Riemannianspace$V_N$reduces toaEuclidean spaceE_N. The elementofarc length squared defined byequation(2.23) iscalled theRiemannianmetric andany geometry whichresults byusingthis metric iscalledaRiemanniangeometry. Aspace$V_N$iscalledflatifitispossibletofindacoordinatetransformationwheretheelementofarclengthsquared is$ds^2 = i\left(dx^i\right)^2$whereeach$\boldsymbol{\iota}$iseither +1 or−1. Aspacewhichisnotflatiscalledcurved.

Definition (2.12): Geometry in V_N

Giventwovectors$\vec{A} = A^i\vec{E}_I \text{ and } \vec{E} = B^j\vec{E}_jA$ thentheirdotproductcanberepresented

$$\vec{A}.\vec{B} = A^iB^j\vec{E}_I.\vec{E}_j = g_{ij}A^iB^j = A_JB^j = g^{ij}A_jB_i = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta \qquad (2.24)$$

Consequently,inan$N$dimensional Riemannianspace$V_N$thedotorinner productoftwovectors $\vec{A}$and$\vec{B}$ isdefined:

$$g_{ij}A^iB^j = A_jB^j = A^iB_i = g^{ij}A_jB_i = AB\cos\theta. \qquad (2.25)$$

Inthisdefinition AisthemagnitudeofthevectorA^i,thequantity$B$isthemagnitudeofthevector$B_i$and θistheanglebetweenthevectorswhentheir originsaremadetocoincide.

Inthespecialcasethatθ=90° wehave$g_{ij}A^iB^j = 0$ asthe condition thatmust besatisfiedinorderthatthe givenvectors A^iandB^iareorthogonal tooneanother.Consider alsothespecialcaseofequation(2.25)when$A^i = B^i$andθ=0.Inthiscasetheequations(2.25)informusthat

$$g^{in}A_nA_i = A^iA_i = g_{in}A^iA^n = (A)^2 \qquad (2.26)$$

From this equationonecandeterminethemagnitudeofthevector A^i.The magnitudes$A$and Bcan be written$A = \left(g_{in}A^iA^n\right)^{1/2}$ and $B = \left(g_{pq}B^pB^q\right)^{1/2}$andsowecanexpressequation (2.24)intheform

$$\cos\theta = \frac{g_{ij}A^iB^j}{(g_{mn}A^mA^n)^{1/2}\left(g_{pq}B^pB^q\right)^{1/2}} \qquad (2.27)$$

Animportapplicationoftheaboveconcepts arisesinthedynamics ofrigidbody motion.Note thatifa victor A^i has constant magnitude and the magnitude of $\frac{dA^i}{dt}$ is different from zero , then the vectors A^i and $\frac{dA^i}{dt}$ must be orthogonal to one another due to the fact that $g_{ij}A^i\frac{dA^j}{dt} = 0$. As an example consider the unit vectors $\hat{e}_1, \hat{e}_2, and\ \hat{e}_3$ on the rotating system of Cartesian axes. We have for constants C_i , $i = 1, \dots, 6$ that

$$\frac{d\hat{e}_1}{dt} = C_1\hat{e}_2 + C_2\hat{e}_3\ , \frac{d\hat{e}_2}{dt} = C_3\hat{e}_3 + C_4\hat{e}_1\ , \frac{d\hat{e}_3}{dt} = C_5\hat{e}_1 + C_6\hat{e}_2$$

Because the derivative of any $\hat{e}_1$ (i fixed) constant victors $\hat{e}_i$and $\hat{e}_x(j \neq i, k \neq i, and\ j \neq k)$, since any vector in this planr must be perpendicular to $\hat{e}_i$. the above definition of a dot product inV_N can be used to define unit vectors in V_N.

Definition (2.13) : Unit vector

When ever the magnitude of a vector A^i is unity, the vector is called a unit vector. In this case we have

$$g_{ij}A^iA^j = 1 \qquad (2.28)$$

Example (2.14): Unit Vector

In $V_N$theelementofarclengthsquared isexpressed $ds^2 = g_{ij}dx^idx^j$whichcanbeexpressed inthe form $1 = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$. this equation states that the vector $\frac{dx^i}{ds}$ $, i = 1, \dots, N$ is a unit vector. One application of this equation is to consider a partical moving a long a curve in V_N which is described by the parametric equation $x^i = x^i(t)$ for$i = 1, \dots, N$. the vector $V^i = \frac{dx^i}{dt}, i = 1, \dots., N$ represents a velocity vector of the particle. By chain rule differentiation we have

$$V^i = \frac{dx^i}{dt} = \frac{dx^i}{ds}\frac{ds}{dt} = V\frac{dx^i}{ds} \qquad (2.29)$$

Where $v = \frac{ds}{dt}$ is the scaler speed of the particle and $\frac{dx^i}{ds}$ is unit tangent vector to the curve. The equation (2.29) shows that the velocity is directed a long the tangent to the curve and has a magnitude V. that is

$$\left(\frac{ds}{dt}\right)^2 = (V)^2 = g_{ij}V^iV^j$$

Example (2.15): Curvilinear Coordinates

Find an expression for thecosine ofthe angles between the coordinatecurves associatedwith the transformationequations

$$x = x(u, v, w), \quad y = y(u, v, w,), \quad z = z(u, v, w).$$

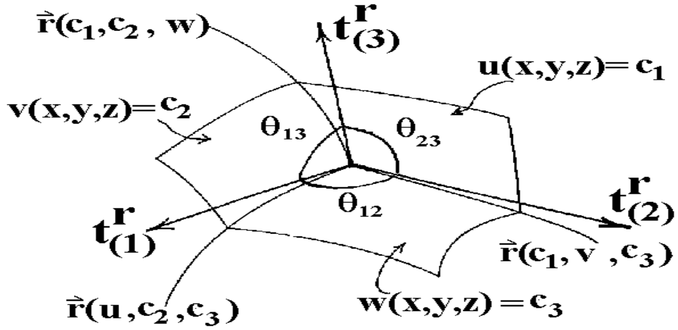

Figure (2.12)Anglesbetweencurvilinear coordinates.

Solution :

Let$y^1 = x,\ y^2 = y, y^3 = z$denote the Cartesianand curvilinear coordinatesrespectively. With referencetothefigure(2.12)wecaninterprettheintersectionofthesurfaces $v = c_2 and\ w = c_3$as the curve $\vec{r} = \vec{r}(u, c_2, c_3)$ which is a function of parametric U. by moving only a long this curve we have $d\vec{r} = \frac{d\vec{r}}{du}$ and consequently

$$ds^2 = \overrightarrow{dr}.\overrightarrow{dr} = \frac{\overrightarrow{\partial r}}{\partial u}.\frac{\overrightarrow{\partial r}}{\partial u}dudu = g_{11}(dx^1)^2\ , or\ 1 = \frac{\overrightarrow{dr}}{ds}.\frac{\overrightarrow{dr}}{ds}g_{11}\frac{(dx^1)^2}{ds}$$

This equation shows that the vector $\frac{dx^1}{ds} = \frac{1}{\sqrt{g_{11}}}$ is a limit vector a long this curve. this tangent vector can be represented by $t^r_{(1)} = \frac{1}{\sqrt{g_{11}}}S^r_1$.

The curve which is defined by the intersection of the surfaces $u = c_1$and$w = c_3$has the limit tangent vector $t^r_{(2)} = \frac{1}{\sqrt{g_{22}}}S^r_2$ .similarly, the curve which is defined as the intersection of the surfaces $u = c_1$and$v = c_2$has the limit tangent vector $t^r_{(3)} = \frac{1}{\sqrt{g_{33}}}S^r_3$ .the cosine of the angle between the unit vectors $t^r_{(1)}$ and $t^r_{(2)}$ is obtained from the result of the equation (2.25). we find

$$\cos\theta_{12} = g_{pq}t^p_{(1)}t^q_2 = g_{pq}\frac{1}{\sqrt{g_{11}}} = \delta^p_1\frac{1}{\sqrt{g_{22}}}\delta^q_2 = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

For $\theta_{13}$the angle between the direction$t^i_{(1)}$ and $t^i_{(3)}$ we find $\cos\theta_{13} = \frac{g_{13}}{\sqrt{g_{11}}\sqrt{g_{33}}}$. finally for θ_{23} the angle between the direction $t^i_{(2)}$ and $t^i_{(3)}$ we find $\cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}}\sqrt{g_{33}}}$

When $\theta_{13}=\theta_{12}=\theta_{23}=90°$,wehave$g_{12} = g_{13} = g_{23} = 0$and thecoordinatecurveswhichmakeupthecurvilinear coordinate system areorthogonal tooneanother.

Inanorthogonal coordinate system weadopt thenotation

$$g_{11} = (h_1)^2, \; g_{22} = (h_2)^2, \; g_{33} = (h_3)^2 \quad and \; g_{ij} = 0, i \neq j$$

Epsilon permutation symbol:

Associated with the epermutation symbol there are the epsilon permutation symbols defined by the reaction

$$\epsilon_{ijk} = \sqrt{g e_{ijk}} \; and \; \epsilon^{ijk} = \frac{1}{\sqrt{g}} e^{ijk} \qquad (2.30)$$

Where g is determinant of the metricsg_{ij}.

Itcanbedemonstratedthatthee_{ijk}permutationsymbolisarelativetensor ofweight-1whereasthe ϵ_{ijk}permutationsymbolisanabsolute tensor.Similarly, thee^{ijk}permutationsymbolisarelativetensorof weight+1 andthecorrespondingϵ^{ijk}permutationsymbolisanabsolute tensor.

Example (2.16): ϵ permutation symbol

Showthat e_{ijk}isarelative tensorofweight$-1$and thecorrespondingϵ_{ijk}permutationsymbolisan absolutetensor.

Solution :

examine the Jacobian

$$J\left(\frac{x}{\bar{x}}\right)=\begin{vmatrix}\frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^1}{\partial \bar{x}^3}\\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^3}\\ \frac{\partial x^3}{\partial \bar{x}^1} & \frac{\partial x^3}{\partial \bar{x}^2} & \frac{\partial x^3}{\partial \bar{x}^3}\end{vmatrix}$$

And make the substitution $\propto_j^i=\frac{\partial x^i}{\partial \bar{x}^j}\quad i,j=1,2,3$. from this definition of a determinant we may write

$$e_{ijk}\propto_m^i\propto_n^j\propto_p^k=J\left(\frac{X}{\bar{X}}\right)e_{mnp}\qquad (2.31)$$

By definition , $\bar{e}_{mnp}=e_{mnp}$ in all coordinate systems and hence equation (2.31) can be expressed in the form

$$\left[J\left(\frac{x}{\bar{x}}\right)\right]^{-1}e_{ijk}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^n}\frac{\partial x^k}{\partial \bar{x}^p}=\bar{e}_{mnp}\qquad (2.32)$$

whichdemonstratesthate_{ijk}transformsasarelativetensorofweight−1.

We have previously shown themetric tensorg_{ij}isa second order covarianttensorand transforms according to the rule$\bar{g}_{ij}=g_{mn}\frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j}$. taking the determinant of this result we find

$$\bar{g}=|\bar{g}_{ij}|=|g_{mn}|\left|\frac{\partial x^m}{\square\bar{x}^i}\right|^2=g\left[J\left(\frac{x}{\bar{x}}\right)\right]^2\qquad (2.33)$$

where$g$isthe determinantof$(g_{ij})$and $\bar{g}$isthe determinantof$(\bar{g}_{ij})$.This result demonstratesthat$g$isa scalarinvariantofweight+2.Taking thesquarerootofthisresultwefindthat

$$\sqrt{\bar{g}}=\sqrt{g}J\left(\frac{x}{\bar{x}}\right)\ (2.34)$$

_

Consequently,wecall$\sqrt{g}$ascalar invariantofweight+1.Nowmultiply both sidesofequation(2.32)by$\sqrt{\bar{g}}$anduse(2.34)toverifytherelation

$$\sqrt{g}e_{ijk}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^n}\frac{\partial x^k}{\partial \bar{x}^p} = \sqrt{\bar{g}}\bar{e}_{mnp} \qquad (2.35)$$

Thisequation demonstratesthatthequantity$\epsilon_{ijk} = \sqrt{g}e_{ijk}$transformslikeanabsolutetensor.

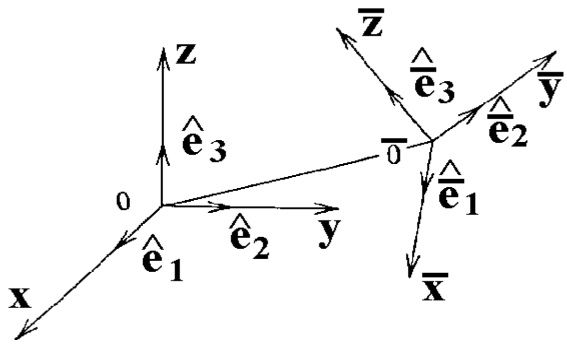

Figure (2.14)Translationfollowedbyrotationofaxes

Inasimilarmanner onecanshowee^{ijk}isarelative tensorofweight+1and $\epsilon^{ijk} = \frac{1}{\sqrt{g}}e^{ijk}$isanabsolute tensor. Thisisleftasanexercise.

Anotherexercisefoundatthe endofthis section istoshowthatageneralizationofthe$e-\delta$identity istheepsilonidentity

$$g^{ij}\epsilon_{ipt}\epsilon_{jrs} = g_{pr}g_{ts} - g_{ps}g_{tr} \qquad (2.36)$$

Definition (2.17) : Cartesian Tensors

Consider themotion ofarigidrodintwodimensions. Nomatterhowcomplicatedthe movementof the rodiswecandescribe themotion asatranslationfollowedbyarotation. Consider therigidrod$\overline{AB}$illustratedinthefigure(2.13).

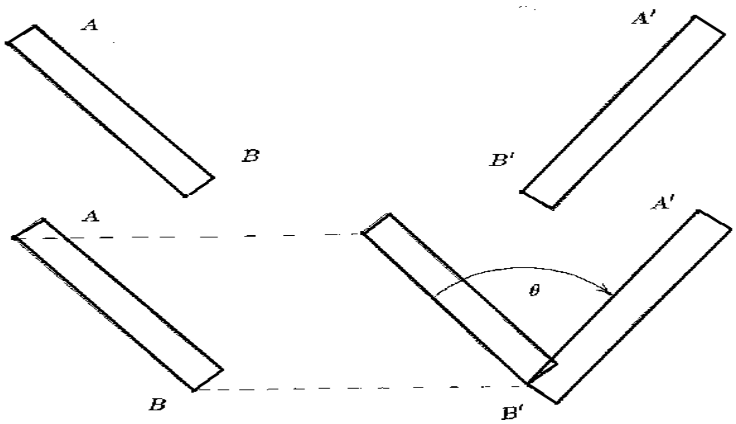

Figure (2.13)Motionofrigidrod

Inthisfigurethereisabeforeandafter picture oftherod’sposition. Bymovingthepoint$B$toB'we haveatranslation.Thisisthen followedbyarotationholding$B$fixed.

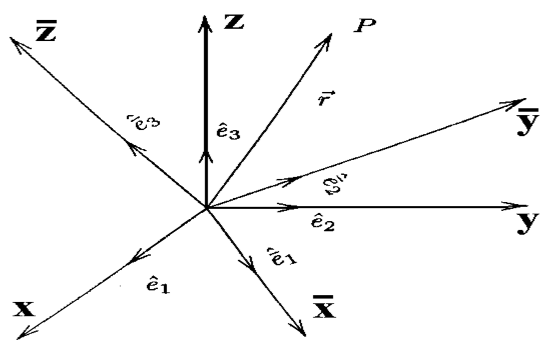

Figure (2.15)Rotationofaxes

Asimilar situationexistsinthreedimensions. Consider twosetsofCartesianaxes,sayabarred and unbarredsystem asillustratedinthefigure(2.14). Letustranslatetheorigin0to$\overline{0}$and thenrotatethe (x, y, z)axesuntilthey coincide with the $(\bar{x}, \bar{y}, \bar{z})$axes.
We consider firsttherotationofaxes when the origins0and $\overline{0}$coincideasthe translationaldistance canberepresentedbyavector $b^k, k = 1, 2, 3.$

When theorigin0istranslatedto$\bar{0}$wehavethesituationillustratedinthefigure (2.15),wherethebarred axes canbethoughtofasatransformationduetorotation.

Let :

$$\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 \qquad (2.37)$$

denotethepositionvectorofavariablepoint$P$withcoordinates(x, y, z)withrespecttotheorigin0andthe unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$. this same point, when referenced with respect to the origin$\bar{0}$ and the unit vectors $\hat{\bar{e}}_1, \hat{\bar{e}}_2, \hat{\bar{e}}_3$has the representation

$$\vec{r} = x\hat{\bar{e}}_1 + y\hat{\bar{e}}_2 + z\hat{\bar{e}}_3 \qquad (2.38)$$

Byconsideringtheprojectionsof$\vec{r}$uponthebarred andunbarredaxeswecanconstructthetransformation equationsrelating thebarred andunbarredaxes.

Wecalculatetheprojectionsof$\vec{r}$ontothe$x, y and z$axesandfind:

$$\vec{r}.\hat{e}_1 = x = \bar{x}(\hat{\bar{e}}_1.\hat{e}_1) + \bar{y}(\hat{\bar{e}}_2.\hat{e}_1) + \bar{z}(\hat{\bar{e}}_3.\hat{e}_1)$$

$$\vec{r}.\hat{e}_2 = y = \bar{x}(\hat{\bar{e}}_1.\hat{e}_2) + \bar{y}(\hat{\bar{e}}_2.\hat{e}_2) + \bar{z}(\hat{\bar{e}}_3.\hat{e}_2)$$

$$\vec{r}.\hat{e}_3 = z = \bar{x}(\hat{\bar{e}}_1.\hat{e}_3) + \bar{y}(\hat{\bar{e}}_2.\hat{e}_3) + \bar{z}(\hat{\bar{e}}_3.\hat{e}_3) \qquad (2.39)$$

We also calculate the projection of$\vec{r}$on to the $\bar{x}, \bar{y}, \bar{z}$ axes and find :

$$\vec{r}.\hat{\bar{e}}_1 = \bar{x} = x(\hat{e}_1.\hat{\bar{e}}_1) + y(\hat{e}_2.\hat{\bar{e}}_1) + z(\hat{e}_3.\hat{\bar{e}}_1)$$

$$\vec{r}.\hat{\bar{e}}_2 = \bar{y} = x(\hat{e}_1.\hat{\bar{e}}_2) + y(\hat{e}_2.\hat{\bar{e}}_2) + z(\hat{e}_3.\hat{\bar{e}}_2)$$

$$\vec{r}.\hat{\bar{e}}_3 = \bar{z} = x(\hat{e}_1.\hat{\bar{e}}_3) + y(\hat{e}_2.\hat{\bar{e}}_3) + z(\hat{e}_3.\hat{\bar{e}}_3) \qquad (2.40)$$

Byintroducingthe notation$(y_1, y_2, y_3) = (x, y, z)$

$(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (\bar{x}, \bar{y}, \bar{z})$and defining θ_{ij} as the angle between the unit vectors $\hat{e}_i$ and $\hat{\bar{e}}_j$ we can represent the above transformation equations in a more concise form. We observe that the direction cosine can be written as :

$$L_{11} = \hat{e}_1.\hat{\bar{e}}_1 = \cos\theta_{11}, L_{12} = \hat{e}_1.\hat{\bar{e}}_2 = \cos\theta_{12}, L_{13} = \hat{e}_1.\hat{\bar{e}}_3 = \cos\theta_{13}$$

$$L_{21} = \hat{e}_2.\hat{\bar{e}}_1 = \cos\theta_{21}, L_{22} = \hat{e}_2.\hat{\bar{e}}_2 = \cos\theta_{22}, L_{23} = \hat{e}_2.\hat{\bar{e}}_3 = \cos\theta_{23}$$

$$L_{31} = \hat{e}_3.\hat{\bar{e}}_1 = \cos\theta_{31}, L_{32} = \hat{e}_3.\hat{\bar{e}}_2 = \cos\theta_{32}, L_{33} = \hat{e}_3.\hat{\bar{e}}_3 = \cos\theta_{33}$$

(2.41)

Which enable us to write the equations (2.39) and (2.40) in the form

$$y_i = L_{ij}\bar{y}_j \quad and \quad \bar{y}_i = L_{ji}y_j \qquad (2.42)$$

Using the index notation we represent the unit vectors as :

$$\hat{\bar{e}}_r = L_{pr}\hat{e}_p \quad or \ \hat{e}_p = L_{pr}\hat{\bar{e}}_r \qquad (2.43)$$

Where L_{pr}arethedirection cosines.Inboththebarred andunbarredsystemtheunit vectorsareorthogonalandconsequently wemust havethedotproducts

$$\hat{\bar{e}}_r.\hat{\bar{e}}_p = \delta_{rp} \quad and \quad \hat{e}_m.\hat{e}_n = \delta_{mn} \qquad (2.44)$$

Where δ_{ij} istheKronecker delta. Substitutingequation (2.43)intoequation (2.44)wefindthedirection cosines L_{ij}must satisfytherelations:

$$\hat{\bar{e}}_r.\hat{\bar{e}}_s = L_{pr}\hat{e}_p.L_{ms}\hat{e}_m = L_{pr}L_{ms}\hat{e}_p.\hat{e}_m = L_{pr}L_{ms}\delta_{pm} = L_{mr}L_{ms} = \delta_{rs}$$

and

$$\hat{e}_r.\hat{e}_s = L_{mn}\hat{\bar{e}}_m.L_{sn}\hat{\bar{e}}_n = L_{mn}L_{sn}\hat{\bar{e}}_m.\hat{\bar{e}}_n = L_{rm}L_{sn}\delta_{mn} = L_{rm}L_{sm} = \delta_{rs}$$

The relations

$$L_{mr}L_{ms} = \delta_{rs} \quad and \quad L_{rm}L_{sm} = \delta_{rs} \qquad (2.45)$$

withsummationindex$m$,areimportantrelationswhicharesatisfiedbythedirection cosinesassociatedwith arotationofaxes.

Combining therotationandtranslationequationswefind

$$y_i = \underbrace{L_{ij}\bar{y}_j}_{rotation} + \underbrace{b_i}_{translation} \qquad (2.46)$$

Wemultiply thisequation by L_{ik}andmakeuseoftherelations (2.45)tofindtheinversetransformation

$$\bar{y}_k = L_{ik}(y_i - b_i) \qquad (2.47)$$

Thesetransformationsarecalledlinearoraffinetransformations.

Consider the $\bar{x}_i$ axes are fixed. While the x_i axes are rotating with respect to the $\bar{x}_i$ axes where both sets of axes have a common origin. Let $\vec{A} = A^i\hat{e}_i$ denote a vector fixed in and rotating with $\bar{x}_i$ axes. We denote by $\frac{d\vec{A}}{dt}|f$ and $\frac{d\vec{A}}{dt}|r$ the derivatives of $\vec{A}$ with respect to the fixed (f) and rotating (r) axes. We can write , with respect to the fixed axes that $\frac{d\vec{A}}{dt}|f = \frac{dA^i}{dt}\hat{e}_i + A^i\frac{d\hat{e}_i}{dt}$. Note that $\frac{d\hat{e}_i}{dt}$ is the derivative of vector with constant magnitude.

Therefore there exists constant w_i , $i = 1, \ldots\ldots, 6$ such that

$$\frac{d\hat{e}_1}{dt} = w_3\hat{e}_2 - w_2\hat{e}_3 , \frac{d\hat{e}_2}{dt} = w_1\hat{e}_3 - w_4\hat{e}_1 , \frac{d\hat{e}_3}{dt} = w_5\hat{e}_1 - w_6\hat{e}_2$$

From the dot product $\hat{e}_1 . \hat{e}_2 = 0$ we obtain by differentiation$\hat{e}_1 . \frac{d\hat{e}_2}{dt} + \frac{d\hat{e}_1}{dt} . \hat{e}_2 = 0$ which implies $w_4 = w_3$. Similarly from dot products $\hat{e}_1 . \hat{e}_3 \ and\ \hat{e}_2 . \hat{e}_3$ we obtain by differentiation the additional relations $w_5 = w_2 \ and\ w_6 = w_1$.

The derivative of $\vec{A}$ with respect to the fixed axes can now be represented

$$\frac{d\vec{A}}{dt}|f = \frac{dA^i}{dt}\hat{e}_i + (w_2A_3 - w_3A_2)\hat{e}_1 + (w_3A_1 - w_1A_3)\hat{e}_2$$

$$+ (w_1A_2 - w_2A_1)\hat{e}_1 = \frac{d\vec{A}}{dt}|r + \vec{w} \times \vec{A}$$

Where $\vec{w} = w_i e_i^{\wedge}$ is called an angular velocity vector of the rotating system. The term $\vec{w} \times \vec{A}$ represents the velocity of the rotating system relative to the fixed system and $\frac{d\vec{A}}{dt}|r = \frac{dA^i}{dt}e_i^{\wedge}$ represents the derivative with respect to the rotating system.

Employing thespecialtransformationequations(2.46)letusexaminehowtensorquantitiestransform whensubjected toatranslationandrotationofaxes. Theseareourspecialtransformationlawsfor Cartesian tensors.Weexamineonlythetransformation lawsforfirstandsecondorderCartesiantensorashigherorder transformationlawsareeasilydiscerned. Wehavepreviouslyshownthatingeneralthefirstandsecondordertensorquantitiessatisfythetransformationlaws:

$$\vec{A}_i = A_j \frac{\partial y_j}{\partial \bar{y}_i} \quad (2.48)$$

$$\vec{A}^i = A^j \frac{\partial \bar{y}_i}{\partial y_j} \quad (2.49)$$

$$\bar{A}^{mn} = A^{ij} \frac{\partial \bar{y}_m}{\partial y_i} \frac{\partial \bar{y}_n}{\partial y_j} \quad (2.50)$$

$$\bar{A}_{mn} = A_{ij} \frac{\partial y_i}{\partial \bar{y}_m} \frac{\partial y_j}{\partial \bar{y}_n} \quad (2.51)$$

$$\bar{A}_n^m = A_j^i \frac{\partial \bar{y}_m}{\partial y_i} \frac{\partial y_j}{\partial \bar{y}_n} \quad (2.52)$$

ForthespecialcaseofCartesiantensorsweassumethat$y_i$and $\bar{y}_i, i = 1,2,3$arelinearly independent.We differentiatetheequations(2.46)and(2.47)andfind

$$\frac{\partial y_i}{\partial \bar{y}_k} = L_{ij} \frac{\partial \bar{y}_j}{\partial \bar{y}_k} = L_{ij}\delta_{jk} = L_{ik} \quad and \quad \frac{\partial \bar{y}_n}{\partial y_m} = L_{ik} \frac{\partial y_i}{\partial y_m} = L_{ik}L_{im}$$
$$= L_{ik} = L_{mk}$$

Substituting these derivatives into the transformation equations (2.48) through (2.52) we produce the transformation equations

$$\vec{A}_i = A_j L_{ji}$$

$$\vec{A}^i = A^j L_{ji}$$

$$\bar{A}^{mn} = A^{ij} L_{im} L_{jn}$$

$$\bar{A}_{mn} = A_{ij} L_{im} L_{jn}$$

$$\bar{A}^m_n = A^i_j L_{im} L_{jn}$$

Chapter Three

Wehave previously shownan arbitraryvector $\tilde{A}$can berepresentedinmany formsdepending upon thecoordinatesystem andbasisvectorsselected. Forexample,considerthe figure(2.16) which illustatesacartesian coordinate system and curvilinear coordinatesystem.

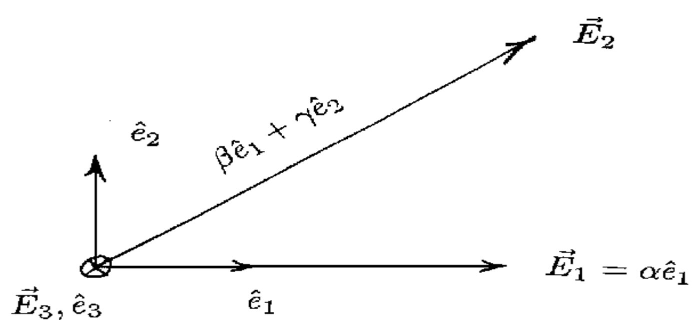

Figure (2.17).physical components

IntheCartesiancoordinatesystem wecanrepresentavector $\overrightarrow{A}$ as:

$\vec{A} = A_x\hat{e}_1 + A_y\hat{e}_2 + A_z\hat{e}_3$

Where $(\hat{e}_1, \hat{e}_{2,}\hat{e}_3)$ are the basic vector.

Consider acoordinate transformation to amore genral coordinate system say (x^1, x^2, x^3).The vector $\vec{A}$ can be represented with contravariant components as

$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3$(3.1)

withrespecttothetangentialbasisvectors$(\vec{E}_1, \vec{E}_2, \vec{E}_3)$.Alternatively,thesamevector$\vec{A}$canberepresented intheform

$$\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3 \qquad (3.2)$$

These equations are just differentwaysofrepresentingthesamevector. Inthe aboverepresentationsthebasisvectors neednotbeorthogonal andtheyneednotbeunit vectors. Ingeneral,thephysicaldimensions ofthecomponents A^i and A_j are not the same.

Definition (3.1)

The physical components of the vector $\vec{A}$ in adirection is defined as [The projection of $\overrightarrow{A]}$ upon aunit vector in the desired direction.For example, the physical components of $\vec{A}$in the direction $\vec{E}_1$ is :-

$\vec{A} \cdot \frac{\vec{E}_1}{|\vec{E}^1|} \cdot \frac{A_1}{|\vec{E}_1|}$ = projection of $\vec{A}$ on $\vec{E}_1$ (3.3)

Similarly, thephysicalcomponentof$\vec{A}$inthedirection $\vec{E}^1$is:-

$\vec{A} \cdot \frac{\vec{E}^1}{|\vec{E}^1|} = \frac{A^1}{|\vec{E}^1|}$ =projection of $\vec{A}$ on $\vec{E}^1$ (3.4)

Example (3.2): physical components

Letα, β, γdenotenonzeropositiveconstantssuchthatthe product relation $\alpha\gamma = 1$ issatisfied. Consider thenonorthogonalbasisvectors

$\vec{E}_1 = \alpha\hat{e}_1$,$\vec{E}_2 = \beta\hat{e}_1 + \gamma\hat{e}_2$,$\vec{E}_3 = \hat{e}_3$

illustratedinthefigure (2.17) Itisreadily verifiedthatthereciprocal basic

$\vec{E}^1 = \gamma\hat{e}_1 - \beta\hat{e}_2$,$\vec{E}^2 = \alpha\hat{e}_2$,$\vec{E}^3 = \hat{e}_3$

Consider the problem of representing the vector $\vec{A} = A_x\hat{e}_1 + A_y\hat{e}_2$ in the contravariant vector form:-

$\vec{A} = A^1E_1 + A^2E_2$ or tensor form $A^i, i = 1,2, \ldots..$

The vector has the contravariant components

$A^1 = \vec{A}.\vec{E}^1 = \gamma A_x - \beta A_y$ and $A^2 = \vec{A}.\vec{E}^2 = \alpha A_y$

Alternatively,thissamevectorcanberepresentedasthecovariantvector $\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2$ which has tensor form $A^i, i = 1,2,\dots..$

Thecovariantcomponentsarefoundfromtherelations

$$A_1 = \vec{A}.\vec{E}_1 = \alpha A_x \quad ,A_2 = \vec{A}\,.\vec{E}_2 = \beta A_x + \gamma A_y$$

The physical components of $\vec{A}$ in the directions $\vec{E}^1$ and $\overrightarrow{E}^2$ are found to be:

$\vec{A}.\frac{\vec{E}^1}{|\vec{E}^1|} = \frac{A^1}{|\vec{E}^1|} = \frac{\gamma A_x - \beta A_y}{\sqrt{\gamma^2+\beta^2}} = A(1)$

$\vec{A}.\frac{\vec{E}^2}{|\vec{E}^2|} = \frac{A^2}{|\vec{E}^2|} = \alpha\frac{A_y}{\alpha} = A_y = \text{A (2)}$

For example we can write

$\vec{A}.\frac{\vec{E}^1}{|\vec{E}^1|} = \frac{A_1(\vec{E}^1.\vec{E}^1)+A_2(\vec{E}^2.\vec{E}^2)}{|\vec{E}^1|} = \text{A (1)}$

And

$$\vec{A}.\frac{\vec{E}^2}{|\vec{E}^2|} = \frac{A_1(\vec{E}^1.\vec{E}^2) + A_2(\vec{E}^2.\vec{E}^2)}{|\vec{E}^2|} = A(2)$$

Ingeneral, thephysical componentsofavector $\vec{A}$inadirection ofaunit vector λ^iisthegeneralized dotproductinV_N.Thisdotproductisaninvariantandcanbeexpressed:-

$g_{ij}A^i\lambda^i = A^i\lambda_i = A_i\lambda^i = projection\ of\,\vec{A}$ in direction λ^i

In the other side we want to show physical components for orthogonal coordinate observe the element of arc length squared V_3 is

$$\text{d}S^2 = g_{ij}dx^i dx^j = (h_1)^2(dx^1)^2 + (h_2)^2 + (h_3)^2(dx^3)^2$$

Where

$$g_{ij} = \begin{pmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{pmatrix} \quad (3.5)$$

Inthiscasethecurvilinear coordinates areorthogonal and $h^2{}_{(i)} = g_{(i)(i)}$ i , is not summed and $g_{ij} = 0, i \neq j$. Atanarbitrarypointinthiscoordinate system wetake $\lambda^i, i = 1,2,3$asaunit vector inthedirection ofthecoordinatex^1.

Wethen obtain

$\lambda^1 = \frac{dx^1}{ds}$,$\lambda^2 = 0$, $\lambda^3 = 0$.

This aunit vector since

$1 = g_{ij}\lambda^i\lambda^j = g_{11}\lambda^1\lambda^1 = h_1^2(\lambda^1)^2$ or $\lambda^1 = \frac{1}{h_1}$..Herethecurvilinear coordinate system isorthogonal and inthiscasethephysicalcomponentofavector A^i,inthedirection x^i,istheprojection ofA^ionλ^iinV_3.

The projectioninthex^1direction isdeterminedfrom

$$A(1) = g_{ij}A^i\lambda^j = g_{11}A^1\lambda^1 = h_1^2A^1\frac{1}{h_1} = h_1A_1$$

Similary, we choose unit vectors M^i *and* V^i*,i=1,2,3 in the* x^2 *and* x^3 directions.these unit vectors can be represented

$\mu^1 = 0$, $\mu^2 = \frac{dx^2}{ds} = \frac{1}{h^2}$, $\mu^3 = 0$

$V^1 = 0$, $V^2 = 0$,$V^3 = \frac{dx^3}{ds} = \frac{1}{h^3}$

andthephysicalcomponentsofthevectorA^iinthesedirections arecalculatedas

$$A(2) = h_2 = A^2 \quad \text{and} \quad A(3) = h_3A^3.$$

Insummary,wecansaythatinanorthogonal coordinate systemthephysicalcomponentsof acontravarianttensoroforderonecanbedeterminedfromtheequations

A(i)=$h_{(i)}A^{(i)}$=$\sqrt{g_{(i)(i)}}A^i$,$i = 1{,}2\ or\ 3$ no summation on i.

Inanorthogonal coordinate system thenonzeroconjugate metric componentsare

$g^{(i)(i)} = \frac{1}{g_{(i)(i)}}$,$i = 1{,}2\ or\ 3$,no summation on i.

These components are needed to calculate the physical components associated with acovarianttensorof orderone. Forexample, inthex^1−direction,wehavethecovariantcomponents

λ_1=$g_{11}\lambda^1 = h_1^2 \frac{1}{h_1}$ =h_1 ,$\lambda_2 = 0$,λ_3= 0 . andconsequently theprojectioninV_3 canberepresented

$$g_{ij}A^i\lambda^j = g_{ij}A^i g^{jm} = A_j g^{jm}\lambda_m = A_1\lambda_1 g^{11} = A_1 h_1 \frac{1}{h_1^2} = \frac{A_1}{h_1} = A(1)$$

In a similar manner we calculate the relation

$$\text{A}(2)=\frac{A_2}{h_2} \quad \text{and} \quad \text{A}(3)=\frac{A_3}{h_3}$$

fortheotherphysicalcomponentsinthedirections x^2andx^3.Thesephysicalcomponents canberepresented intheshorthand notation

A(i)=$\frac{A_i}{h_i} = \frac{A_i}{\sqrt{g_{(i)(i)}}}$,$i = 1{,}2, or\ 3$.no summation on i.

Inan orthogonal coordinate system thephysical components associated with boththecontravariantand covariantcomponentsarethesame. Toshowthiswenotethatwhen$A^i g_{ij} = A_j$issummed on$i$weobtain$A^1 g_{1j}$ +$A^2 g_{2j}$ +$A^3 g_{3j} = A_j$.

Since$g_{ij} = 0\ for\ i = j$thisequation reducesto$A^i g_{(i)(i)}$ =$A(i)$, i is not summed.

Anotherformforthisequation is

$$A(i)=A^{(i)}\sqrt{g(i)(i)} = \frac{A(i)}{\sqrt{g(i)(i)}}, \quad i \text{ is not summed}$$

whichdemonstratesthatthephysical componentsassociated withthecontravariantand covariantcomponentsareidentical.

NOTATION: Thephysicalcomponentsaresometimesexpressedbysymbolswithsubscriptswhichrepresent the coordinatecurve along which theprojectionistaken.For example, let H^idenotethecontravariantcomponentsofafirstorder tensor.The followingaresomeexamples oftherepresentationofthephysicalcomponents ofH^iinvariouscoordinatesystems:

System coordinates	components	Coordinate orthogonal	Physical Tensor
general	(x^1, x^2, x^3)	H^i	H(1),H(2),H(3)
rectangular	(x,y,z)	H^i	H_x, H_y, H_z
cylindrical	(r,θ,z)	H^i	$H_r, H_{\theta,} H_z$
spherical	$(\rho, \theta, \emptyset)$	H^i	$H_\rho, H_\theta, H_{,\emptyset}$
general	(u,v,w)	H^i	H_u, H_v, H_w

Higher OrderTensors:

The physical componentsassociatedwith higher ordered tensors aredefinedbyprojectionsin$V_N$just likethecasewithfirstordertensors.Forannthorderedtensor$T_{ij....k}$wecanselect$n$unitvectors$\lambda^i, u^i, ..., v^i$andformtheinnerproduct(projection) $T_{ij...K}\lambda^i M^j ....V^k$.

Whenprojectingthetensorcomponentsontothecoordinatecurves,thereare$N$choicesforeachofthe unit vectors. Thisproduces N^nphysical components.

Theaboveinnerproductrepresentsthephysicalcomponentofthetensor $T_{ij....k}$alongthedirections of the unit vectors $\lambda^i, u^i, ..., v^i$.

The selectedunit vectors mayormaynotbeorthogonal.Inthecaseswheretheselected unit vectors are allorthogonal tooneanother,thecalculation ofthephysical components is greatlysimplified. Byrelabeling the unit vectors $\lambda^{i}{}_{(m)}, \lambda^{i}{}_{(n)}$ ....$\lambda^{i}{}_{p}$ where$(m), (n), \ldots, (p)$representoneof the$N$directions, thephysical components ofageneralnthordertensorisrepresented

$$T(m\,n\ \ldots.p) = T_{ij}\ldots{}_{K}\lambda_{m}^{i}\lambda_{n}^{j}\ \ldots.\lambda_{p}^{k}$$

Example(3.3):Physicalcomponent

Inanorthogonal curvilinear coordinate system $V_3$with metric $g_{ij},\ i,j = 1,2,3$findthe physical componentsof

(i)thesecondordertensor A_{ij}

(ii)thesecondordertensor A^{ij}

(iii)thesecondordertensor A_{j}^{i}

Solution:

The physical components of$A_{mn}, m, n = 1,2,3$alongthe directions oftwounit vectors λ^{i}andu^{i}isdefinedastheinnerproductinV_3.Thesephysical components canbeexpressed$A_{(ij)} = A_{mn}\lambda_{(i)}^{m}M_{(j)}^{n}$, $i,j = 1,2,3$..

wherethesubscripts$(i)$and(j)representoneofthecoordinatedirections. Dropping thesubscripts$(i)$and (j),wemaketheobservationthatinanorthogonal curvilinear coordinate systemtherearethreechoicesfor thedirection oftheunit vectorλ^{i}andalsothreechoicesforthedirection oftheunit vector u^{i}.

Thesethree choicesrepresentthedirectionsalongthe$x^1, x^2, or\ x^3$coordinatecurves whichemanatefromapointofthecurvilinear coordinatesystem. Thisproducesatotalofninepossiblephysical componentsassociatedwiththetensorA_{mn}.

Forexample, wecanobtain thecomponentsoftheunit vector$\lambda^i, i = 1,2,3$inthex^1direction directly fromanexaminationoftheelementofarclengthsquared

$$ds^2 = (h_1)^2(dx^1)^2 + {h_2}^2(dx^2)^2 + (h_3)^2(dx^3)^2$$

Bysetting $dx^2 = dx^3 = 0$,wefind$\frac{dx^1}{ds} = \frac{1}{h_1} = \lambda^1,\ \lambda^2 = 0, \lambda^3 = 0$

This isthe vector $\lambda^i_1,\ i = 1,2,3$ .Similarly, ifwechooseto selectthe unit vector $\lambda^i, i = 1,2,3$inthex^2direction, weset$dx^1 = dx^3 = 0$intheelementofarclength squared andfindthecomponents

$$\lambda^1 = 0, \quad \lambda^2 = \frac{dx^2}{ds} = \frac{1}{h^2}, \quad \lambda^3 = 0$$

Thisisthevector$\lambda^i_{(2)}, i = 1,2,3.$,ifweselect$\lambda^i_1,\ i = 1,2,3$inthex^3direction,weset$dx^1 = dx^2 = 0$ intheelementofarclength squared anddeterminetheunit vector.

$$\lambda^1 = 0\ , \lambda^2 = 0\ , \lambda^3 = \frac{dx^3}{ds} = \frac{1}{h^3}$$

this the vector $\lambda^i_{(3)}\ , i = 1,2,3$. Similarly, the unit vector u^icanbeselected asoneoftheabove three directions. Examining allninepossiblecombinationsforselectingtheunit vectors,wecalculate thephysical componentsinanorthogonal coordinate system as:

$$A_{(11)} = \frac{A_{11}}{h_1 h_1} \qquad ,A_{(12)} = \frac{A_{12}}{h_1 h_2} \qquad ,A_{(13)} = \frac{A_{13}}{h_1 h_3}$$

$$A_{21} = \frac{A_{21}}{h_2 h_1} \qquad , A_{(22)} = \frac{A_{22}}{h_1 h_2} \qquad ,A_{(23)} = \frac{A_{23}}{h_2 h_3}$$

$$A_{(31)} = \frac{A_{31}}{h_3 h_1} \qquad , A_{(32)} = \frac{A_{32}}{h_3 h_2} \qquad , A_{(33)} = \frac{A_{33}}{h_3 h_3}$$

Theseresultscanbewritteninthemorecompact form

$$A_{ij} = \frac{A_{(i)(j)}}{h_{(i)} h_{(j)}} = no\ summation\ on\ i\ or\ j\ \rightarrow \qquad (3.6)$$

Formixedtensors wehave

$$A_j^i = g^{im} A_{mj} = g^{i1} A_{1j} + g^{i2} A_{2j} + g^{i3} A_{3j} \rightarrow \qquad (3.7)$$

From thefact $g^{ij} = 0$ for $i = j$,togetherwiththephysicalcomponentsfromequation(2.16),theequation (2.17)reduces to

$$A_{(j)}^{(i)} = g^{(i)(j)} A_{(i)(j)} = \frac{1}{h_{(i)}^2} . h_{(i)} h_{(j)} A_{(ij)}\ no\ summation\ on\ i, j$$
$$= 1,2, or\ 3$$

Thiscanalsobewritteninheform

$$A(ij) = A_j^i \frac{h_i}{h_j} \text{ no summation on i or j} \quad \rightarrow \qquad (3.8)$$

Hence, the physical components associated with the mixed tensor A_j^i In the an orthogonal coordinate system can be expressed as

$$A(11) = A_1^1 \quad A(12) = A_2^1 \frac{h^1}{h_2} A(13) = A_3^1 \frac{h^1}{h_3}$$

$$A(21) = A_1^2 \frac{h^2}{h_1} \quad A(22) = A_2^2 \quad A(23) = A_3^2 \frac{h^2}{h_3}$$

$$A(31) = A_1^3 \frac{h^3}{h_1} A(32) = A_2^3 \frac{h^3}{h_2} A(33) = A_3^3$$

Forsecondordercontravarianttensorswemaywrite

$A^{ij}g_{im} = A^i_m = A^{i1}g_{1m} + A^{i^2}g_{2m} + A^{i^3}g_{3m}$

Weusethefact $g_{ij} = 0$for$i = j$togetherwiththephysicalcomponentsfromequation(3.8) to reduces the above equation to the form $A^i_m = A^{(i)(m)}g_{(m)(m)}$)nosummationon$m$.Intermsofphysicalcomponents we have

$$\frac{h_{(m)}}{h_{(i)}}A(im) = A^{(i)(m)}h^2_{(m)}\ or\ A_{(im)}$$

$$= A^{(i)(m)}h_{(i)}h_{(m)}, no\ summation\ on\ i, i = i, m$$

$$= 1,2,3\ (3.9)$$

Examining the results from equation (3.9) wefind thatthe physical components associatedwith thecontravarianttensorA^{ij},inanorthogonal coordinate system, canbewrittenas:

$$A(11) = A^{11}h_1h_1 \quad A(12) = A^{12}h_1h_2 \quad A(13) = A^{13}h_1h_{13}$$

$$A(21) = A^{21}h_2h_1 \quad A(22) = A^{22}h_2h_2 \quad A(23) = A^{23}h_2h_{23}$$

$$A(13) = A^{31}h_3h_1 \quad A(32) = A^{32}h_3h_2 \quad A(33) = A^{33}h_3h_{33}$$

Definition (3.4) PhysicalComponentsinGeneral

Inanorthogonal curvilinear coordinate system, thephysicalcomponentsassociated withthenthordertensor$T_{ij\ldots kl}$alongthecurvilinear coordinatedirections canberepresented:

$$T(ij \dots \dots k) = \frac{T_{(i)(j).....(k)(l)}}{h_{(i)}h_{(j)} \dots \dots h_{(k)}h_{(l)}} no\ summation$$

These physical components can berelatedtothevarious tensorsassociated with$T_{ij.....kl}$Forexample, in anorthogonal coordinate system, thephysical componentsassociated withthemixedtensorcanbe expressedas:

$$T(ij \dots .mn \dots .kl) = T^{(i)(j)....(m)}_{(n)......(k)(l)} \frac{h_{(i)}h_{(j)} \dots \dots h_{(k)}h_{(l)}}{h_{(n)}.. \dots \dots h_{(k)}h_{(l)}} \quad (3.10)$$

Example(3.5):Physicalcomponents

Let$x^i = x^i(t), i = 1,2,3$denote thepositionvector ofa particlewhichmovesasafunction oftime*t.*Assumethere existsacoordinatetransformation$\bar{x}^i = \bar{x}^i(x)$for $i = 1,2,3$. Thepositionof theparticle whenreferencedwithrespect tothebarred system ofcoordinates canbefoundbysubstitution.The generalized velocity oftheparticle intheunbarredsystem isavectorwithcomponents

$$v^i = \frac{dx^i}{dt}, i = 1,2,3 .$$

The generalized velocitycomponentsofthesameparticle inthebarred system isobtainedfromthechain.rule. Wefindthisvelocityisrepresentedby

$$\bar{v}^i = \frac{d\bar{x}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} v^j$$

Thisequation impliesthatthecontravariantquantities

$$(v^1, v^2, v^3) = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right)$$

aretensorquantities.Thesequantitiesarecalledthecomponents ofthegeneralizedvelocity. Thecoordinates x^1, x^2, x^3aregeneralized coordinates. This means wecanselectanysetofthree independentvariables for therepresentationofthemotion.

Thevariablesselectedmightnothavethesamedimensions. Forexample, incylindricalcoordinateswelet$(x^1 = r, x^2 = \theta, x^3 = z)$. Here$x^1andx^3$havedimensions ofdistance butx^2hasdimensions ofangular displacement.Thegeneralized velocitiesa

$$v^1 = \frac{dx^1}{dt} = \frac{dr}{dt} \ , v^2 = \frac{dx^2}{dt} = \frac{d\theta}{dt} \ , v^3 = \frac{dx^3}{dt} = \frac{dz}{dt}$$

Here$v^1$and v^3haveunits oflengthdivided bytimewhilev^2hastheunits ofangular velocity orangular change divided by time. Clearly, these dimensions are notall the same. Let us examine the physical components ofthe generalized velocities. Wefindincylindrical coordinates$h_1 = 1, h_2 = r, h_3 = 1$andthe physicalcomponentsofthevelocityhavetheforms:

$$v_r = v(1) = v^1 h_1 = \frac{dr}{dt} \, , v_\theta = v(2) = v^2 h_2 = r\frac{d\theta}{dt} \ , v_z = v(3) = v^3 h_3 = \frac{dz}{dt}$$

Nowthephysical components ofthevelocityallhavethesameunits oflengthdivided bytime. Additionalexamples oftheuseofphysical componentsareconsidered later. For the timebeing, just remember thatwhen tensor equations are derived, the equationsare valid inany generalized coordinate system. Inparticular,weareinterestedintherepresentationof physicallawswhicharetobeinvariantand independentofthe coordinatesystem usedtorepresenttheselaws. Onceatensor equation isderived, we canchoseanytypeofgeneralized coordinatesandexpand thetensorequations.

Beforeusinganyexpanded tensorequations wemust replaceallthetensorcomponentsbytheircorresponding physical componentsin orderthattheequationsaredimensionally homogeneous. Itistheseexpanded equations,expressedinterms ofthephysicalcomponents,whichareusedtosolveapplied problems.

TensorsandMultilinearForms:

Tensors canbethoughtofasbeingcreated bymultilinearformsdefinedonsomevector space$V$.Let usdefineonavector space$V$alinear form,abilinear formand ageneral multilinearform. Wecanthen illustratehowtensors arecreated fromtheseforms.

Definition (3.6): Linearform

Let$V$denoteavectorspacewhich containsvectors$\vec{x}, \vec{x}_1, \vec{x}_2$Alinearformin $\vec{x}$,isascalarfunction $\varphi(\vec{x})$havingasinglevectorargument$\tilde{x}$whichsatisfiesthelinearity properties:

$$(i) \quad \varphi(\vec{x}_1 + \vec{x}_2) = \varphi(\vec{x}_1) + \varphi(\vec{x}_2)$$

$$(ii) \quad \varphi(\mu\vec{x}_1) = \mu\varphi(\vec{x}_1)$$

forallarbitraryvectors$\vec{x}_1, \vec{x}_2$in$V$andallrealnumbers μ.

Example (3.7)

$$\varphi(\vec{x}) = \vec{A} \cdot \vec{x}$$

Where $\vec{A}$ isaconstantvectorand$\vec{x}$isanarbitraryvectorbelongingtothevector space$V$.

Notethatalinearformin$\vec{x}$canbeexpressedintermsofthecomponentsofthe vector$\vec{x}$andthebase vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$usedtorepresent$\vec{x}$Toshowthis, wewritethevector$\vec{x}$inthecomponentform

$$\vec{x} = x^i \hat{e}_i = x^1 \hat{e}_1 + x^2 \hat{e}_2 + x^3 \hat{e}_3$$

Where $x^i, i, 1,2,3$ are the components of $\vec{x}$with respect to the basis vector $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. by linearity property of φ we can write

$$\varphi(\vec{x}) = \varphi\left(x^i \hat{e}_i\right) = \varphi(x^1 \hat{e}_1 + x^2 \hat{e}_2 + x^3 \hat{e}_3)$$

$$= \varphi(x^1 \hat{e}_1) + \varphi(x^2 \hat{e}_2) + \varphi(x^3 \hat{e}_3)$$

$$= x^1 \varphi(\hat{e}_1) + x^2 \varphi(\hat{e}_2) + x^3 \varphi(\hat{e}_3) + x^i \varphi(\hat{e}_i)$$

Thus we can write $\varphi(\vec{x}) = x^i \varphi(\hat{e}_i)$ and by the definition the quantity $\varphi(\hat{e}_i) = ai$ as tensor we obtain $\varphi(\vec{x}) = x^i a_i$.

Note that if we change basis from $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ to $\left(\vec{E}_1, \vec{E}_2, \vec{E}_3\right)$ then the component of $\vec{x}$must change.

Lobbing $\bar{x}^i$ denote the component of $\vec{x}$ respect to the new basis , we would have

$$\vec{x} = \bar{x}^i \vec{E}_i \; and \;\; \varphi(\vec{x}) = \left(\bar{x}^i \vec{E}_i\right) = \bar{x}^i \varphi\left(\vec{E}_i\right)$$

Thelinearformφdefinesanewtensor$\overline{ai} = \varphi\left(\vec{E}_i\right)$sothat $\varphi(\vec{x}) = \bar{x}^i \bar{a}_i$..

Whenever there isadefiniterelation between the basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3) \; and \; \left(\vec{E}_1, \vec{E}_2, \vec{E}_3\right)$ say

$$\vec{E}_i = \frac{\partial x^j}{\partial \bar{x}^i} \hat{e}_j \ .$$

then there existsadefiniterelationbetween thetensors$a_i$and$\bar{a}_i$..Thisrelation is

$$\bar{a}_i = \varphi(\vec{E}_i) = \varphi\left(\frac{\partial x^j}{\partial \bar{x}^i} \hat{e}_j\right) = \frac{\partial x^j}{\partial \bar{x}^i} (\hat{e}_j) = \frac{\partial x^j}{\partial \bar{x}^i} a_j$$

Thisisthetransformationlawforanabsolutecovarianttensorofrank ororderone.

Definition(3.8): Bilinearform

Abilinear formin$\vec{x}$and$\vec{y}$isa scalar function$\varphi(\vec{x},\vec{y})$withtwovector arguments,whichsatisfies thelinearityproperties:

(i) $\varphi(\vec{x}_1 + \vec{x}_1, \vec{y}_1) = \varphi(\vec{x}_1, \vec{y}_1) + \varphi(\vec{x}_2, \vec{y}_1)$

(ii) $\varphi(\vec{x}_1, \vec{y}_1 + \vec{y}_2) = \varphi(\vec{x}_1, \vec{y}_1) + \varphi(\vec{x}_1, \vec{y}_2)$

(iii) $\varphi(\mu\vec{x}_1, \vec{y}_1) = \mu\varphi(\vec{x}_1, \vec{y}_1)$

(iv) $\varphi(\vec{x}_1, \mu\vec{y}_1) = \mu\varphi(\vec{x}_1, \vec{y}_1)$ (3.11)

forarbitraryvectors$\vec{x}_1, \vec{x}_{2,}\vec{y}_{1,}\vec{y}_2$ inthevectorspace$V$andforall realnumbers μ.Noteinthedefinitionof abilinearformthatthescalarfunctionφislinearinboththearguments$\vec{x}$ and $\vec{y}$.

Example (3.9)
abilinear formisthedotproductrelation

$$\varphi(\vec{x},\vec{y}) = \vec{x} \cdot \vec{y} \qquad (3.12)$$

whereboth$\vec{x}$ and $\vec{y}$belongtothesamevectorspace$V$.

Definition (3.10) :Multilinearforms

Amultilinearformofdegree Mora$M$degree linearforminthevectorarguments $\vec{x}_1, \vec{x}_2, \dots \dots ., \vec{x}_M$inascalarfunction

$\varphi(\vec{x}_1, \vec{x}_2, \dots \dots ., \vec{x}_M)ofM$vectorargumentswhichsatisfiesthepropertythatitisalinearformineachofits arguments.Thatis,φmust satisfyforeach$j = 1,2, \dots, M$theproperties:

$$(i)\varphi\left(\vec{x}_1, \dots .., \vec{x}_{j1}, \vec{x}_{j2+} \dots \dots ., \vec{x}_M\right)$$
$$= \varphi\left(\vec{x}_1, \dots .., \vec{x}_{j1} \dots ., \vec{x}_M\right) + \varphi\left(\vec{x}_1, \dots .., \vec{x}_{j2} \dots ., \vec{x}_M\right)$$

$$(ii)\ \varphi\left(\vec{x}_1, \dots .., \mu\vec{x}_j \dots ., \vec{x}_M\right) = \mu\varphi\left(\vec{x}_1, \dots .., \vec{x}_j \dots ., \vec{x}_M\right) \qquad (3.13)$$

forallarbitraryvectors$\vec{x}_1, \dots ., \vec{x}_M$inthevectorspace$V$andallrealnumbers μ .

Example (3.11)

Thirddegreemultilinearformortrilinearformisthetriplescalarproduct

$$\varphi(\vec{x}, \vec{y}, \vec{z}) = \vec{x} \cdot (\vec{y}, \vec{z})(3.14)$$

Note thatmultilinearformsareindependentofthe coordinatesystem selected and depend onlyupon the allvectors withrespect tothis basis set. For example, if$\vec{x}, \overline{y}, \vec{z}$are threevectors wecan representthese vectorsinthecomponentforms

$$\vec{x} = x^i \hat{e}_i \ , \vec{y} = y^j \hat{e}_j, \vec{z} = x^k \hat{e}_k(3.15)$$

wherewehaveemployedthesummationconvention ontherepeatedindices$i, j$and$k$.Substituting equations(3.15)intoequation (3.14)weobtain

$$\varphi\left(x^i \hat{e}_i \ , y^j \hat{e}_j, x^k \hat{e}_k\right) = x^i y^j z^k \varphi\left(\hat{e}_i \ , \hat{e}_j, \hat{e}_k\right)(3.16)$$

Since φislinearinallitsarguments.Bydefiningthetensorquantity

$$\varphi\left(\hat{e}_i\ ,\hat{e}_j,\hat{e}_k\right) = e_{ijk} \quad (3.17)$$

The trilinearform, givenbyequation (3.14), with vectors fromequations (3.15),canbeexpressedas

$$\varphi\left(x^i y^j z^k\right) = e_{ijk}\ x^i y^j z^k, i,j,k = 1,2,3 \qquad (3.18)$$

Thecoefficients$e_{ijk}$of thetrilinearformiscalledathird ordertensor. Itisthefamiliarpermutationsymbol considered earlier.

Inamultilinearformofdegree$M =$ $\varphi(\vec{x},\vec{y},\ldots\vec{z})theM$argumentscanberepresentedinacomponent formwithrespecttoasetofbasisvectors$(\hat{e}_1,\hat{e}_2,\hat{e}_3)$ Letthesevectorshaveco mponents $x^i,y^i,z^i,\ i = 1,2,3$withrespect totheselectedbasisvectors. Wethencanwrite

$$\vec{x} = x^i\hat{e}_i\ ,\vec{y} = y^j\hat{e}_j,\vec{z} = x^k\hat{e}_k$$

Substitutingthesevectorsintothe$M$degreemultilinearformproduces

$$\varphi\left(x^i\hat{e}_i\ ,y^j\hat{e}_j\ ,\ldots\ldots,x^k\hat{e}_k\right)$$
$$= x^i y^j,\ldots\ldots,z^k\varphi(\left(\hat{e}_i\ ,\hat{e}_j,\ldots\ldots,\hat{e}_k\right) \qquad (3.19)$$

Consequently,themultilinearformdefinesasetofcoefficients

$$\bar{a}_{ij\ldots\ldots k} = \left(\vec{E}_i,\vec{E}_j,\ldots.\vec{E}_k\right)(3.20)$$

Thisnewtensorhasabaroverittodistinguishitfromtheprevioustensor.Ade finiterelation existsbetweenthe newand oldbasisvectors and consequently thereexists adefinite relationbetween thecomponentsofthebarred andunbarredtensorscomponents.Recallthatifwearegivenasetoftransformationequations

$$y^i = y^i(x^1, x^2, x^3), i = 1,2,3 \qquad (3.21)$$

fromrectangulartogeneralized curvilinear coordinates,wecanexpressthebasisvectors inthenewsystem bytheequations

$$\vec{E}_i = \frac{\partial y^j}{\partial x^i} \hat{e}_j \,, i = 1,2,3 \;. \qquad (3.22)$$

For example, seeequations (2.11) with$y^1 = x, y^2 = y,\ y^3 = z, x^1 = u, x^2 = v, x^3 = w$.Substituting equations(3.22)intoequations(3.20)weobtain

$$\bar{a}_{ij......k} = \varphi\left(\frac{\partial y^{\alpha\alpha}}{\partial x^i}\hat{e}_{\alpha\alpha}, \frac{\partial y^{\beta\beta}}{\partial x^j}\hat{e}_{\beta\beta}, \frac{\partial y^{\gamma\gamma}}{\partial x^k}\hat{e}_{\gamma\gamma}\right)$$

Bythelinearitypropertyof ,thisequation isexpressible intheform

$$\bar{a}_{ij......k} = \frac{\partial y^{\alpha\alpha}}{\partial x^i}\frac{\partial y^{\beta\beta}}{\partial x^j}\dots..\frac{\partial y^{\gamma\gamma}}{\partial x^k}\ \varphi\left(\hat{e}_{\alpha\alpha}, \hat{e}_{\beta\beta} \dots, \hat{e}_{\gamma\gamma}\right)$$

$$\bar{a}_{ij......k} = \frac{\partial y^{\alpha\alpha}}{\partial x^i}\frac{\partial y^{\beta\beta}}{\partial x^j}\dots..\frac{\partial y^{\gamma\gamma}}{\partial x^k} \quad \alpha, \beta, \gamma.$$

This isthe familiar transformationlawforacovarianttensorofdegree *M*.By selecting reciprocal basis vectorsthecorresponding transformationlawsforcontravariantvectorscanbedetermined.

The aboveexamples illustratethattensorscanbeconsidered asquantitiesderivable frommultilinearformsdefinedonsomevector space.

Definition (3.12): DualTensors

The e-permutationsymbol isoftenused togenerate newtensorsfromgiventensors.For $T_{i_1,i_2,....,i_m}$ a skew-symmetric tensor, wedefinethetensor

$$\acute{T}^{j_1 j_2 \ldots\ldots j_{n-m}} = \frac{1}{m!} e^{j_1 j_2 \ldots\ldots\ldots\ldots j_{n-m}} i_1 i_2 \ldots \ldots . i_m T_{i_1 i_2 \ldots\ldots i_m} \quad m \le n \quad (3.23)$$

asthe dual tensorassociated with $T_{i_1,i_2,....,i_m}$. Notethatthee-permutationsymboloralternatingtensor has aweightof+1andconsequentlythedualtensorwillhaveahigherweighttha ntheoriginaltensor.

Thee-permutationsymbolhasthefollowingproperties

$$e^{i_1 i_2 \ldots iN} e_{i_1 i_2} \ldots\ldots iN = N!_{i_1, i_2, \ldots i_N}$$

$$e^{i_1 i_2 \ldots iN} e_{i_1 i_2} e_{j_1, j_2, \ldots j_N} = \delta^{i_1 i_2 \ldots\ldots i_n}_{j_1 j_2 \ldots\ldots j_n}$$

$$e_{k_1 k_2 \ldots\ldots\ldots k_m} i_1 i_2 \ldots \ldots . i_{N-m} e^{j_1 j_2 \ldots\ldots j_m i_1 i_2 i_{N-m}} = (N-m)!\, \delta^{j_1 j_2 \ldots\ldots j_m}_{k_1 k_2 \ldots\ldots k_m}$$

$$\delta^{j_1 j_2 \ldots\ldots j_m}_{k_1 k_2 \ldots\ldots k_m} T_{j_1 j_2 \ldots\ldots j_m} = m!\, T_{k_1 k_2 \ldots\ldots k_m} \quad (3.24)$$

Usingtheabovepropertieswecansolvefortheskew-symmetric tensor interms ofthedualtensor.Wefind

$$T_{i_1 i_2 \ldots\ldots i_m} = \frac{1}{(n-m)!} e_{i_1 i_2 \ldots\ldots i_m j_1 j_2 j_{n-m}} \acute{T}^{j_1 j_2 \ldots\ldots\ldots j_{n-m}} (3.25)$$

whichisafirstordertensororvector. NotethatA_{ij}hasthecomponents

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} (3.26)$$

Notethatthe vector components haveacyclicordertothe indiceswhichcomesfromthe cyclicproperties ofthee-permutationsymbol.

Asanotherexample,considerthefourth orderskew-symmetric tensorA_{ijkl},$i, j, k, l = 1, \ldots, n$.Wecanassociatewiththistensor anyofthedualtensorquantities

$$v = \frac{1}{4!} e^{ijkl} A_{ijkl}$$

$$v^i = \frac{1}{4!} e^{ijklm} A_{ijklm}$$

$$v^{ij} = \frac{1}{4!} e^{ijklmn} A_{ijklmn}$$

$$v^{ijk} = \frac{1}{4!} e^{ijklmnp} A_{ijklmnp}$$

$$v^{ijkl} = \frac{1}{4!} e^{ijklmnpr} A_{ijklmnpr}$$