Sudan University of Science and Technology

College of science

Department of Mathematics

A project Submitted in Fulfillment for the Degree of B.Sc. (Honor) in Mathematics

A survey to Analytical Mechanics

By

Fatima Bakhet said

Maab Yousuf Abd elnabi

Supervisor

Dr. Emad eldeen AbdAllah Abd elrahim

الايه

بِ ٍ سم الله الرحمن الرحيم

قال تعالى:

(اقرأ باسم ربك الذي خلق (1) خلق الانسان من علق(2)اقرأ وربك الاكرم (3) الذي علم بالقلم (4) علم الانسان مالم يعلم (5))

صدق الله العظيم

(سوره العلق)

Dedication

I dedicate my research work to my family and many friends. A special feeling of gratitude to my loving parents, my mother and father

Whose words of encouragement and push for tenacity ring in my ears .My brother and sister, have never left my side and are very special.

I also dedicate this research to my many friends who have supported me throughout the process. I will always appreciate all they have done.

I dedicate this work and give special thanks to my best friend Uncle Magdi for being there for me throughout my work.

Acknowledgements

I would like to express my sincere gratitude to my advisor teacher Dr.Emad eldeen AbdAllah for the continuous support to my research, for his patience, motivation, enthusiasm, and immense knowledge.

His guidance helped me in preparing of this research and other useful life advices.

Besides, I would like to thanks all my teachers form the early years of my school studies up to now for their participating making a good student and person for the community and Sudan.

Abstract

The scope and Aim of Mechanics is the science of motion. It has a twofold object:

First, to describe the motions of bodies an interpret, them by means of a few laws and principles, which are generalizations derived from observation and experience.

Second ,to predict the motion of bodies for all times when the circumstances of the motion for any one instant are given ,in addition to the special laws which govern the motion .

This research aims to cover some of these concepts, and it is organized and divided in two chapters:-

In chapter one, we discuss the Newton Mechanic's, and the equations of motion according to Newton laws of motion and their solution. Also we study the Lagrangian mechanics and law, and we explain the concepts' of the action and symmetries with some applications and examples.

In the last chapter, we illustrate the concept of the conic section; also we study the central force two body problems with some applications and examples.

ملخص

نطاق وهدف المكنيكا أنها علم الحركه. وهي تحتوي على قسمين :-

اولا, لوصف حركات الاجسام وتفسيرها عن طريق عدد قليل من القوانين والمبادئ التي تستمد تعميماتها من التجربه والملاحظه.

ثانيا,للتنبؤ بحركه الاجسام لجميع الازمان عندما تكون ظروف الحركه لاجل لحظه ما معطاه , بالاضافه الى القوانين الخاصه التي تحكم الحركه ,

يهدف هذا البحث الى تغطيه بعض هذه المفاهيم, وقد نظم وقسم الى اثنين من الابواب:-

في الباب الاول ناقشنا المكنيكا النيوتينيه لمعادلات الحركه وفقا لقانون نيوتن للحركه وحلولها. ايضا درسنا مكنيكا وقوانين لاجرانج, ووضحنا مفاهيم العمل وتماثلات مع بعض التطبيقات والامثله.

في الباب الاخير , وضحنا مفهوم القسم المخروطي , وايضا درسنا القوه المركزيه لمساله جسمين مع بعض التطبيقات والامثله,

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Chapter (1)

Newton and lagrangian mechanics

Section (1.1): Newton Mechanic's

Sir Isaac Newton was a Master of the Mint. He also formulated three celebrated Laws of mechanics, which we can paraphrase as follows:

The First law:-

A particle not subject to any force moves on a straight line at constant Speed.

The second law:-

In the presence of a force, the position of a particle obeys the equations of Motion

$$m\ddot{x}_i = F_i(x, \dot{x}) . \qquad \textbf{(1.1)}$$

The Third law:-

The force exerted by a particle on another is equal in magnitude, but op posited in direction, to the force exerted by the other particle on the first.

A "particle" is here thought of as an entity characterized by its mass m, its location in space $x(t)$, and by nothing else, the aim of Newton's mechanics is to predict the location at arbitrary times, given the position and velocity at some initial time, this is done by means of a solution of the differential equations above.

$I \in \{1,2,3\}$ eq. (1.1) stands for three separate equations. Throughout we employ Einstein's summation convention, which means that whenever a certain Index occurs twice in a particular term, a sum over all its allowed values is understood.

$$x_i y_i = \sum_{i=1}^{3} x_i y_i = \sum_{i=1}^{3} x_j y_j = x_j y_j \quad \textbf{(1.2)}$$

Now we discuss Analytical and Hamiltonian mechanics a physicist might follow Newton in using them to predict the position of the planets as they go around the sun, and will conclude that they are very meaningful.

Analytical mechanics is at once more general and more special than Newton's Theory; it is more general because it is more abstract. Its equations do not necessarily describe the positions of particles, but may be applied to much more general physical systems (such as field theories, including Einstein's general relativity theory), in the version we will study it is more special because only a restricted set of forces are allowed in eq. (1.1). It can be reformulated as the statement that the total momentum of a system composed of several particles is conserved:

$$\frac{dp_i}{dt} = \frac{d}{dt}\sum\nolimits_{particles} m\dot{x}_i = 0 \quad \textbf{(1.3)}$$

Where the sum is over all the particles in the system, now consider the function

$$E = T + V = \frac{m\dot{x}^2}{2} + V(x) \quad (1.4)$$

Where V is some function of x, known as the potential energy. The function

Is called the kinetic energy, while E itself is the energy of the system, clearly

$$E = \dot{x}(m\ddot{x}_i + \partial_i V(x)) \quad \textbf{(1.5)}$$

It follows that if the force is given by

$$F_i(x, \dot{x}) = F_i(x) = -\partial_i V(x) \quad \textbf{(1.6)}$$

Then the energy of the system is conserved. Systems for which a conserved Energy function exists are called conservative. In our example, and indeed in many interesting cases, the energy can be divided into kinetic and potential

Parts, and the equation of motion is given by

$$m\ddot{x}_i = -\partial_i V(x) \quad \textbf{(1.7)}$$

This move is typical of analytical mechanics, where vectors are usually derived from scalar functions analytical mechanics devises methods to derive the differential equations de scribing a given system, strategies for solving them, and ways of describing the solutions if they cannot be obtained in explicit form. We will tentatively restrict ourselves to conservative systems only. If you like, this is a strengthening of the third law. Indeed, we will deal with a special form of analytical mechanics called Hamiltonian mechanics, which is believed to apply to all isolated systems in Nature.

What we are trying to do is to find some properties that all the Laws of Physics, and in particular all allowed equations of motion, have in common. Now the philosopher Leibnitz—who was the other of the two inventors of Differential Calculus—argued that we live in the best of all possible worlds. as was first realized Half a century after the publication of Newton's Principia, The inspiration Came from optics, and the laws of reflection and refraction, It is known that The angle of reflection is equal to the angle of incidence, and it was observed By the Greeks that this implies that light always travels on the shortest path available between two points A and B, subject to the restriction that it should be reflected against the surface. If the angle of reflection were to differ from the angle of incidence, the distance covered by light in going from A to B would be greater than it has to be, for refraction, we have Snell's Law. Any Medium can be assigned an index of refraction n, and the angle of refraction is related to the angle of incidence through

$$n1 \sin \theta_i = n2 \sin \theta_r . \qquad \textbf{(1.8)}$$

Fermat noted that if

$$n = \frac{c}{v} \qquad \textbf{(1.9)}$$

Where v is the velocity of light in the medium and c is a constant, then Snell's law can be derived from what is now known as Fermat's principle,

Namely that the time taken for light to go from A to B is a minimum, Fermat's principle unifies the laws of refraction and reflection, since it also implies the equality between the angles of incidence and reflection.

Consider two pointsA and B, and suppose that a particle starts out at A at time $t = t_1$, and then moves along an arbitrary path from A to B with whatever speed that is consistent with the requirement that it should arrive at B at the time $t = t_2$, in mathematical terms we are dealing with a function x (t)such that

$$x(t_1) = x_A \qquad x(t_2) = x_B , \qquad \textbf{(1.10)}$$

But otherwise arbitrary , for any such function x (t) we can evaluate the integral

$$s[x(t)] = \int_{t_1}^{t_2}(t - v)dt = \int_{t_1}^{t_2}(\frac{m\dot{x}^2}{2} - v(x))] \quad \textbf{(1.11)}$$

S is known as the action. It is a functional, i.e. a function of a function—the

Functional $S[x\ (t)]$ assigns a real number to any function $x\ (t)$. Note that $S[x\ (t)]$ is not a function of t, it is a function ofx_A,x_B, t_1 andt_2, but this is rarely written out explicitly.

The statement, to be verified in the next section, is that the action functional (1.11) has extremism (not necessarily a minimum) for precisely that function x (t) which obeys the differential equation (1.7). This is known as Hamilton's Principle, or—with less than perfect historical accuracy—as the principle of Least Action, Hamiltonian mechanics deals with those, and only those. This is a much more general class than that given by eq. (1.7), but it does exclude some cases of physical interest, and forms only a part of analytical Mechanics. Note once again what is going on. The original task of mechanics was to predict the trajectory of a particle, given a small set of data concerning its state at some initial time$t$. We claim that there exists another formulation of the problem, where we can deduce the trajectory given half, as much data at each of two different times. So there seems to be a local causal way of looking at things, And at a first sight quite different global Teleological viewpoint, the claim begins to look reasonable when we observe that the amount of "free data" in the two formulations are the same. Moreover, if the two times$t1$, and t_2 approach each other infinitesimally closely, then what we are in effect, specifying is the position and the velocity at time$t1$, just as in the causal Approach, we are eternizing a quantity evaluated along a path, and the path actually taken By matter in nature is the one which makes the quantity in question assume an extremely value, the point about extreme—not only minima—is that if the path is varied slightly away from the extremely path, to a path which differs to Order ϵ from the extremely one. Then the value of the path dependent quantity suffers a change, which is of order ϵ squared at extremism the first derivative Vanishes. In the case of optics, we know that the description of light as a bundle of rays is valid only in the approximation, where the wavelength of light is much less, than the distance between $A\ and\ B$, in the wave theory, in a way, every path between $A\ and\ B$ is allowed. If we vary the path slightly, the time taken by light to arrive from $A\ to\ B$ changes, and this means that it arrives out of phase with the light arriving along the first path, If the wavelength Is very small, phases from light arriving by different paths will be randomly distributed, and will cancel each other out through destructive interference.

This argument fails precisely for the extremely paths: for them, neighboring paths take approximately the same time, light from all neighboring paths will arrive with the same phase, and constructive interference takes place. Thus, whenever the wavelength is negligibly small, it will appear that light always travels along extremely paths. Only in the twentieth century was it realized that Hamilton's Principle works for the same reason, that Fermat's Principle works, Classical mechanics is a kind of geometrical optics limit of a "wave mechanics" of matter, operating in configuration space, but that is another story.

Now we study the calculus of variations, let us now verify the claim made in the first section, namely that Newton's differential equations for suitable choices of the dynamical system are mathematically equivalent to the requirement that a certain functional of all possible paths of the particles should assume an extremism value at the actual trajectory.

First, we stare at the definition of the action functional:

$$S[x(t)] = \int_{t_1}^{t_2} dt(\frac{m}{2}\dot{x}^2 \_ v(x)) \qquad \textbf{(1.12)}$$

For a function f of an ordinary number$x$, it

Is easy enough to find the extreme, we consider how the function values $f(x)$

Change as we change the number$x$:

$$\delta f(x) \equiv f(x + \delta x) - f(x) = \delta x \partial x f(x) \qquad \textbf{(1.13)}$$

If the derivative is zero at the point x, the function has a minimum, or a maximum, or at least an inflection point there, for a function of several variables, the condition for extremism (a minimum, a maximum, or a saddle point) is that

$$\delta f(x) = \sum_i \delta x_i \frac{\partial f}{\partial x_i}(x_1, \dots, x_N) = 0 \qquad \textbf{(1.14)}$$

For arbitrary choices of theδx_i, which means that all the N partial derivatives? Have to vanish at the extremely points. Now a functional of a function $x\,(t)$ can be regarded as a function of an infinite number of variables, say of the Fourier coefficients of the original function. You can also regard t as a label of the infinite number of variables on which the functional depends—a kind of continuous index—and then what we have to do is to replace the sum in eq. (1.14) with an integral, like this:

$$\delta s = S[x(t) + \delta x(t)] - S[x(t)] = \int_{t_1}^{t_2} dt \delta x(t) \frac{\delta s}{\delta x}(t) \qquad (1.15)$$

We assume that it is possible to bring as to this form. Then the functional

Derivative of S[x] will be defined as the very expression that occurs to the

Right in the integrand, the equations of motion, as obtained from Hamilton's principle, then state that the functional derivative of the action is zero, since the form of the function $\delta x(t)$ is arbitrary.

It remains to be seen if we really can bring as to this form—if not, we would

Have to conclude that $S[x(t)]$ is "not differentiable". First of all, note that we are all the time evaluating the action between definite integration limits? Then the extremism, if it exists, will be given by that particular trajectory which starts at the point $x(t_1)$ at time$t1$, and ends at the point $x(t2)$ at time $t2$, and for which the functional derivative vanishes. We can make this work for the action functional (1.12), Imagine that we know its value for a particular function $x(t)$, and ask how this value changes if we evaluate it for a slightly different function

$$\tilde{x}(t) = x(t) + \delta x(t), \delta(t) = \in f(t) \qquad (1.16)$$

Where $f(t)$ is, for the time being, an arbitrary function while ϵ is an infinitesimally small constant, it is important for the following argument that $f(t)$ is arbitrary, or nearly so, that ϵ is "infinitesimally small" simply means that we will neglect terms of quadratic and higher orders in ϵin the calculation, which follows:

$$\delta S = S[x(t] - S[x(t)] = \int_{t1}^{t2} dt(\frac{m}{2}\left(\dot{x} + \delta\dot{x^2}\right) - V(x + \delta x) - S[x(t)] =$$

$$\int_{t1}^{t2} dt(\frac{m}{2}\dot{x^2} + m\dot{x}\delta\dot{x} - V(x) - \delta x \partial x v(x) + O(\epsilon^2)) =$$

$$S[x\ (t)]\ = \int_{t_1}^{t_2} dt(m\dot{x}\delta\dot{x} - \delta x \partial x V(x) + O(\epsilon^2)) \qquad (1.17)$$

The action functional has an extremism at the particular function x (t) for

Which this expression vanishes to first order in Q, what we want to see is

What kind of restrictions this requirement sets on the function. To see this, we perform a partial integration

$$\delta S = \int_{t_1}^{t_2} dt(\_\delta x(m\ddot{x} + \partial x V(x)) + \frac{d}{dt}(m\delta x\dot{x})) \qquad (1.18)$$

Unfortunately this is not quite of the form (1.15), due to the presence of the total derivative in the integrand, Therefore we impose a restriction on the so far arbitrary function f (t) that went into the definition of ax, so that

$$\delta x(t1)\ =\ \delta x(t2)\ =\ 0 \qquad (1.19)$$

This is a way of saying that. We are interested only in functions x (t) that

Have certain reassigned starting and end points at specified times. With this restriction, the total derivative in eq. (1.18) goes away. The first term has to vanish for all allowed choices of the functions ax (t). After a moment's reflection, we see that this can happen only if the factor-multiplying ax in the integrand is zero! Hence, we have proved that the action functional has extremism, among all possible functions obeying

$$x\,(t_1) = x_A \;,\; x(t_2) \;=\; x_B \qquad (1.20)$$

For those and only those functions which obey

$$m\ddot{x} \;=\; -\partial_x V(x) \qquad (1.21)$$

That Newton's equations of motion can be derived from the condition. That a certain action functional to obtain a definite Trajectory, it is not enough to impose the equations of motion. It is also necessary to set some initial values, for differential equations of Second order. It is natural to make a choice of$x\;(0)\;and\;\dot{x}\;(0)$, from the point of view of the action. It is natural to impose the value of $x\;(t)$ at two different times, that whatever values of $x(0)\;and\;\dot{x}\;(0)$ we choose, there is always a unique solution for some range of T, while it is perfectly possible that the equation of motion is such that there is no solution, or several solutions. For a given Pair of$x\;(t_1)\;and\;x\;(t_2)$, the rest of this course is an elaboration of the contents of this section. If you have not understood everything perfectly yet there is still time.

It is one thing to be able to set up equations for a physical system, and perhaps to prove theorems to the effect that a solution always exists and is unique, given suitable initial conditions, another issue of obvious interest is how to solve these equations, or at least how to extract information from them. What precisely do we mean when we say that a differential equation is "soluble"? Consider, as an exercise, a first order differential equation for a single variable:

$$\dot{x} = f(x) \qquad (1.22)$$

Where f is some function. This can be solved by means of separation of variables:

$$\boldsymbol{dt = \frac{dx}{f(x)} \Rightarrow t(x) = \int_c^x \frac{dx\prime}{f(x\prime)}} \qquad \textbf{(1.23)}$$

Where c is a constant determined by the initial condition. If we do this integral, and then invert the resulting function t(x) to obtain the function x (t), we have solved the equation, we will regard eq. (1.23) as an implicit definition of x (t), and eq. (1.22) is soluble in this sense. This is reasonable, since the manipulations required to extract t(x) can be easily done on a computer, to any desired accuracy, even if we cannot express the integral in terms of elementary functions, but there are some limitations here: It may not be possible to invert the function t(x) except for small times, Next, consider a second order equation, such as the equation of motion for A harmonic oscillator:

$$\boldsymbol{m\ddot{x} = -kx} \qquad \textbf{(1.24)}$$

This is a linear equation, and we know how to express the solution in terms of trigonometric functions, but our third example—a pendulum of length l—is already somewhat worse:

$$\boldsymbol{ml^2\ddot{\theta} = -gml\,sin\,\theta} \qquad \textbf{(1.25)}$$

Let us therefore approach eq. (1.24) in a systematic fashion, which might

Yield results also for the pendulum. As a first step, note that any second order Differential equation can be rewritten as a pair of coupled first order equations:

$$\dot{p} = -kx \qquad m\dot{x} = p \tag{1.26}$$

The second equation defines the new variable p. unfortunately coupled first order equations are difficult to solve, except in the linear case when they can be decoupled through a Fourier transformation.

The number of degrees of freedom of a dynamical system is defined to be one half times the number of first order differential equations, needed to describe the evolution. It will turn out that, for systems whose equations of motion are derivable from the action principle, the number of first order equations will always be even. Therefore, the number of degrees of freedom is always an integer for such systems. A system with n degrees of freedom will, in general, be described by a set of 2n coupled first order equations, and the difficulties one encounters in trying to solve them will rapidly become severe. In the cases at hand, with one degree of freedom only, one uses the fact that these are conservative systems, which will enable us to reduce the problem to that of solving a single first order equation, for the harmonic oscillator the conserved quantity is

$$E = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} \tag{1.27}$$

The number E does not depend on t. equivalently

$$\dot{x}^2 = \frac{2E}{m} - \frac{k}{m}x^2 \tag{1.28}$$

Taking a square root we are back to the situation we know, and we proceed

As before:

$$\boldsymbol{dt = dx\sqrt{\frac{m}{2E-kx^2}} \Leftrightarrow t(x) = \int_c^x dx'\sqrt{\frac{m}{2E-kx'^2}}} \qquad \textbf{(1.29)}$$

Inverting the function defined by the integral, we find the solution x (t). The

Answer is a trigonometric function, with two arbitrary constants E and c, which determine the phase and the amplitude. For our purposes the trigonometric function is defined by this procedure!

We can play the same trick with the non-linear equation for the pendulum,

Moreover, we end up with

$$t(\theta) = \int_c^\theta \frac{d\theta'}{\sqrt{\frac{2}{ml^2}}(E+gmlcos\theta')} \qquad \textbf{(1.30)}$$

We integrate, and we invert. This defines the function θ (t). We could leave it at that, but since our example is a famous one, we manipulate the integral a bit further for the fun of it. Make the substitution

$$\boldsymbol{sin\frac{\theta'}{2} = k\,sin\,\emptyset' \Rightarrow d\theta' = \frac{2k\,cos\,\emptyset' d\emptyset'}{\sqrt{1-k^2sin^2\emptyset'}}} \qquad \textbf{(1.31)}$$

This converts the integral to

$$t(\theta) = \sqrt{\frac{l}{2g}}\int_c^{\emptyset(\theta)} \frac{2k\,cos\,\emptyset' d\emptyset'}{\sqrt{1-k^2sin^2\emptyset'}\sqrt{\frac{E}{gml}+1-2sin^2\frac{\theta'}{2}}} \qquad \textbf{(1.32)}$$

Now we choose the so far undetermined constant k by

$$2k^2 \equiv \frac{E}{gml} + 1 \qquad \textbf{(1.33)}$$

The integrand then simplifies, and one further substitution takes us to our

Desired standard form;

$$\boldsymbol{t(\theta)} = \sqrt{\frac{\boldsymbol{l}}{\boldsymbol{g}}} \int_{\boldsymbol{c}}^{\emptyset(\boldsymbol{\theta})} \frac{\boldsymbol{d\emptyset\prime}}{\sqrt{\boldsymbol{1-k^2 sin^2\emptyset\prime}}} = |\boldsymbol{sin}\,\emptyset' \equiv \boldsymbol{x}'| = \sqrt{\frac{\boldsymbol{l}}{\boldsymbol{g}}} \int_{\boldsymbol{c}}^{\boldsymbol{x(\theta)}} \frac{\boldsymbol{dx\prime}}{\sqrt{\left(\boldsymbol{1-x'^2}\right)(\boldsymbol{1-k^2x\prime^2})}}$$

(1.34)

Just as eq. (1.29) can be taken as an implicit definition of a trigonometric Function, this integral implicitly defines the function θ (t) as an elliptic function.

If you compare it with the previous integral (1.29), you see that an elliptic

Function is a natural generalization of a trigonometric, i.e. “circular”, function. Since elliptic functions turn up in many contexts, they have been

Studied in depth by mathematicians, anyway, the above examples were some of the simplest examples of completely soluble dynamical system; just wait until we get to the insoluble ones!

Why did this work at all? The answer is that we had one degree of freedom,

In addition, one constant of the motion, namely Ethos reduced the problem to that of solving a single uncoupled equation. This suggests a general strategy for solving the equations of motion for a system containing n degrees of freedom, i.e. solving 2n coupled first order equations: One must find a set of n constants of the motion with suitable properties, so that the problem reduces to that of computing n integrals, this idea forms the core of

Hamilton-Jacobi theory. It works sometimes, but not very often: a typical Hamiltonian system will exhibit an amount of "chaotic" behavior, as a result the notion of what it means to "solve" a set of differential equations evolved somewhat: a solution might consist, say, of a convergent power series in t. But frequently this strategy also fails, and there may not exist any effective procedure to generate the long term behavior of the solutions on a computer, what one has to do then is to find out which questions one can reasonably ask concerning such systems. Even in situations where one can solve the equations, things may not be altogether simple, Consider two harmonic oscillators, with the explicit solution

$$x = a\,cos(\omega_1 t + \delta_1) \qquad y = b\,cos(\omega_{2t} + \delta_2) \qquad \textbf{(1.35)}$$

The trajectory in the x-y-plane is a Lissajous-figure, Examples are readily produced with a computer, If ω1 = ω2 the trajectory is an ellipse, with circles and straight lines as special cases. More generally, if there exist integers m and n such that

$$m\,w_1 = nw_2 \qquad \textbf{(1.36)}$$

The trajectory is a closed curve. If there are no such integers the trajectory

Eventually fills a rectangle densely, and never closes on itself. Put yourself into the position of an experimentalist trying to determine by means of measurements whether the trajectory will be closed or not!

Now we discuss Phase space it is worthwhile formalizing things a bit further. With the understanding that every set of ordinary differential equations can be written in first order form, We write down the general form of N coupled first order equations for N real Variables zip:

$$\dot{z}_i = f_i\left(z_{1,\dots,}z_N; t\right), 1 \leq \boldsymbol{i} \leq \boldsymbol{N} \qquad \textbf{(1.37)}$$

Where the N functions if is arbitrary. We simplify things by assuming that

There is no explicit dependence on time,

$$\dot{\boldsymbol{z}}_{\boldsymbol{i}} = \boldsymbol{f}_{\boldsymbol{i}}(\boldsymbol{z}_{\boldsymbol{i}}, \dots, \boldsymbol{z}_{\boldsymbol{N}}) \qquad \textbf{(1.38)}$$

There are theorems that guarantee the existence and uniqueness of such systems for some range of the parameter t, thus

$$\boldsymbol{z}_{\boldsymbol{i}} = \boldsymbol{z}_{\boldsymbol{i}}(\boldsymbol{z}_{\boldsymbol{01}}, \dots, \boldsymbol{z}_{\boldsymbol{0N}}, \boldsymbol{t}) \qquad \textbf{(1.39)}$$

Where z0i are the initial values of zip (There is no guarantee that such solution can be obtained in any explicit form) we assume that the physical systems.

We are interested in—as far as we attempt to describe them—can be fully characterized by the N real numbers zip, we imagine a space whose points these numbers label in a one-to-one fashion, and call it phase space.

The set of all possible states of a physical system is in one-to-one correspond-Dance with the points of an N dimensional phase space. The time development of a system is uniquely determined by its position in phase space.

This is the first of several abstract spaces that we will encounter, and you must get used to the idea of abstract spaces, a particle moving in space has a 6 dimensional phase space, because its position (3 numbers) and its velocity (3 numbers) at a given time determine its position at all times, given Newton's laws. Anything else can either be computed from these numbers—this is true for its acceleration—or else it can be ignored—this would be true for how it smells, if it does. The particle also has a mass, but this number is not included in phase space because it is given once and for all, two particles moving in space have a 12 dimensional phase Space, so high dimensional phase spaces are often encountered. We will have to picture them as best we may now consider time evolution according to eq. (1.38). Because of the theorems I alluded to; we know that through any point z0 there passes a unique curve Z (t), with a unique tangent vector $\dot{z}$, these curves never cross each other.

When the system is at a definite point in phase space, it knows where it is going, the curves are called trajectories, and their tangent vectors define a Vector field on phase space called the phase space flow. Imagine that we can see such a flow. Then there are some interesting things to be observed. We Say that the flow has a fixed point wherever the tangent vectors vanish. If the System starts out at a fixed point at t = 0, it stays there forever. There is an important distinction to be made between stable and unstable fixed points, If You start out a system close to an unstable fixed point it starts to move away From it, while in the stable case it will stay close forever. The stable fixed point May be an attractor, in which case a system that starts out close to the fixed?

Point will start moving towards it. The region of phase space which is close enough for this to happen is called the basin of attraction for the attractor. Consider a one dimensional phase space, with the first order system

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) \qquad \textbf{(1.40)}$$

For generic choices of the function f all fixed points are either stable attractor, or unstable resellers, but for special choices of f we can have fixed a point that is approached by the flow only on one side. The latter are structurally Unstable, in the sense that the smallest change in f will either turn them into Pairs of attractors and resellers, or cause them to disappear altogether.

In two dimensions there are more possibilities. We can have sources and Sinks, as well as stable elliptic and unstable hyperbolic fixed points, to see what the latter two look like, we return to the examples given in section 1.3. The phase space of the harmonic oscillator is R2, and it contains one elliptic fixed point, it is elliptic because it is surrounded by closed trajectories, and hence it is stable. In the case of the pendulum phase space has a non-trivial Topology: since one of the coordinates is a periodic angle phase space is the Surface of an infinitely long cylinder, it contains two fixed points. One of them Is elliptic, and the other—the state where the pendulum is pointing upwards— Is hyperbolic, what is special about the hyperbolic fixed point is that there Are two trajectories leading into it, and two leading out of it? The length of the Tangent vectors $\dot{\theta}$ decrease as the fixed point is approached. Taking the global Structure of phase space into account we see that a trajectory leaving the fixed Point is in fact identical to one of the incoming ones. Hence there are really only Two special trajectories, a striking fact about them is that they divide phase Space into regions with qualitatively different behavior, One region where The trajectories go around the elliptic fixed point, and two regions where the Trajectories go around the cylinder. For this reason the special trajectories are called separatrices, and the regions into which they divide phase space are called invariant sets —by definition an invariant set in phase space is a region that one cannot leave by following the phase space flow.

It is very important that you see how to relate this abstract discussion of the phase space of the pendulum to known facts about real pendulum, Do this!

It is not by accident that the phase space of the pendulum is free of sources

And sinks, the reason is, as we will see in section 7.1, that only elliptic or hyperbolic fixed points can occur in Hamiltonian mechanics. Real pendulum Tend to have some amount of dissipation present (because they are imperfectly Isolated from the environment), and then the situation changes; see exercise 6. Speaking of Hamiltonian systems it is worthwhile to point out that the Example of the two harmonic oscillators in eq. (1.35) is less frivolous than it 14 The Best of all Possible Worlds May appear, the phase space is four dimensional, but there are two conserved Quantities

$$2E_1 = p_1{}^2 + \omega_1{}^2 x_1{}^2 \qquad 2E_2 = p_2{}^2 + \omega_2{}^2 x_2{}^2 \qquad \textbf{(1.41)}$$

This means that any given trajectory will be confined to a two dimensional Surface in phase space, labeled by E1 and E2. This surface is a torus, with Topology S1 × S1, in a sense to be made clear later, non-chaotic motion in a Hamiltonian system always takes place on a torus in phase space.

Finally we observe that we have the beginnings of a strategy to understand any given dynamical system. We begin by locating the fixed points of the Phase space flow. Then we try to determine the nature of these fixed points. If the equations are linear this is straightforward. If not, we can try linearization of the equations around the fixed points, there is a theorem that we can lean on here:

The Hartman-Gorman theorem: The nature of the fixed points is unchanged by linearization, as long as the fixed points are isolated and as long as no Elliptic fixed points occur. Thus, consider the pendulum. You know that its phase space is a cylinder Described by the coordinates (θ, p_), to see if the phase space flow has any fixed points, you set

$$\dot{\theta} = \frac{1}{ml^2} p_\theta = 0 \qquad \dot{p}_\theta = -gml \sin\theta = 0 \qquad \textbf{(1.42)}$$

Hence there are fixed points at (θ, p_) = (0, 0) and (π, 0). Linear zing around them you find the former to be elliptic and the latter to be hyperbolic. If this remains true for the non-linear equations you can easily draw a qualitatively correct picture of the phase space flow. No integration is needed. Were we justified in assuming that the fixed points are elliptic? To see what can go wrong, consider the non-linear equation

$$\ddot{x} + \epsilon x^2 \dot{x} + x = 0 \qquad \textbf{(1.43)}$$

In the linearised case (ϙ = 0) there is a single elliptic fixed point. In the Non-linear system the flow will actually spiral in or out from the fixed point, Depending on the sign of ϙ, so this is an example where the exception to The Hartman-Gorman theorem is important. But in the case of the “pure” Pendulum we know that the non-linear system is Hamiltonian, and therefore Sources and sinks cannot appear—our analysis of the pendulum was therefore Accurate, Our tentative strategy works very well when the phase space is two dimensional, But if the dimension of phase space exceeds two things can get very Complicated indeed, A famous example is the at first sight innocent looking Lorenz equations

$$\begin{cases} \dot{z}_1 = -az_1 + az_2 \\ \dot{z}_2 = bz_1 - z_2 - z_1z_3 \\ \dot{z}_3 = -cz_3 + z_1z_2 \end{cases} \tag{1.44}$$

They capture some aspects of thermal convection in a fluid. Lorenz was a Meteorologist interested in the long term accuracy of weather prediction, but He found himself unable to do long term prediction even in this simple model.

The non-linear terms have a dramatic effect, and the slightest change in the Initial data will cause the trajectory to go to completely different regions of the three dimensional phase space.

$$E = \int_0^l dx\left(\frac{k}{2}y''y'' - p(x)y\right) \tag{1.45}$$

Where the slash denotes differentiation with respect to x and k is a constant, the Bar will minimize its energy. Analyze the variation problem to see what equation determines the equilibrium position, and what conditions one must impose on the end of the bar in order to obtain a unique solution, Archers want their bows to bend like Circles. Conclude that bows must have a value of k that depends on x.

Section (1.2):-

Lagrangian mechanics

With the agreement that the action integral is an important object, we give a name also to its integrand, and call it the Lagrangian, in the examples that we considered so far, and in fact in most cases of interest, the Lagrangian is a Function of a set of n variables quid and their n first order derivatives quid:

$$\boldsymbol{s}[\boldsymbol{q}(\boldsymbol{t})] = \int_{t_1}^{t_2} \boldsymbol{dtL}(\boldsymbol{q_i}, \dot{\boldsymbol{q}}_i) \qquad \textbf{(1.46)}$$

We use "q" to denote the coordinates because the Lagrangian formalism is very general, and can be applied to all sorts of systems where the interpretation of the variables may differ from the interpretation of "x" as the position of some particle. The space on which q_i are the coordinates is called the con- figuration space, while $\dot{q}_i$ are its the tangent vectors. Its dimension is one half that of phase space, it is an intrinsic property of the physical system we are Studying, and is a very useful concept. You should try to think as much as Possible in terms of the configuration space itself, and not in terms of the Particular coordinates that we happen to use (the as), since the latter can be Changed by coordinate transformations, In fact one of the advantages of the Lagrangian formalism is that it is easy to perform coordinate transformations directly in the Lagrangian. We will see examples of this later on. Moreover there are situations—such as that of a particle moving on a sphere—when one needs several coordinate systems to cover the whole configuration space.

In this case one sees clearly that the important thing is the sphere itself, not the coordinates that are being used to describe it, which is not to say that Coordinates are not useful in calculations—they definitely are!

Now we Study the scope of Lagrangian mechanics among all those functions q_i(t) for which $q_i(t_1)$ and $q_i(t_2)$ are equal to some arbitrarily Prescribed values, the action functional has extremism for precisely those functions quid (t) which obey the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad \textbf{(1.47)}$$

Provided such functions exist. This is straightforward to verify by means of

The calculus of variations; indeed (suppressing indices)

$$\boldsymbol{\partial s = \int_{t_1}^{t_2} dt\left(\partial q \frac{\partial L}{\partial q} + \partial \dot{q}\frac{\partial L}{\partial \dot{q}}\right) = \int_{t_1}^{t_2} dt\partial q\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}(\partial q\frac{\partial L}{\partial \dot{q}})} \qquad \textbf{(1.48)}$$

The total derivative term gives rise to a boundary term that vanishes because

We are only varying the functions whose values at t1 and t2 are kept fixed, so that δq is zero at the boundary. The Euler-Lagrange equations follow as advertized.

The question is to what extent the equations of motion that actually Occur in physics are of this form.

There are some that cannot be brought to this form by any means, including some of considerable physical interest; most of them involve dissipation of Energy of some sort, the archetypical example is that of white elephant sliding down a hillside covered with flowers, and is described by the equation of motion

$$m\ddot{x} = g - \gamma\dot{x} \qquad (1.49)$$

Due to the form of its right hand side this equation cannot be derived from a

Lagrangian, However, in some sense the frictional force involved here is Not a fundamental force. A complete description of the motion of the elephant Would involve the motion of the atoms in the elephant and in the flowers, Both being "heated" by friction, it is believed that all complete, fundamental Equations are derivable from Hamilton's principle, and hence that they fall within the scope of Lagrangian mechanics (or quantum mechanics, which is structurally similar in this regard).

Generally speaking we expect Lagrangian mechanics to be applicable whenever there is no dissipation of energy. For many simple mechanical systems the Lagrangian equals the difference between the kinetic and the potential energy,

$$\boldsymbol{L(x,\dot{x}) = T(\dot{x}) - V(x)} \qquad \mathbf{(1.50)}$$

For instance,

$$\boldsymbol{L = \frac{m\dot{x}^2}{2} - V(x) \Rightarrow 0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial V}{\partial x} - m\ddot{x}} \qquad \mathbf{(1.51)}$$

Even in some situations where there is no conservation of energy, analytical

Mechanics applies. The simplest examples involve Lagrangians which depend explicitly on the time t. Dissipation is not involved because we keep careful track of the way that energy is entering or leaving the system.

Now for an example where the Lagrangian formalism is useful. Suppose we

Wish to describe a free particle in spherical polar coordinates

$$\boldsymbol{x = r\cos\emptyset\sin\theta \qquad y = r\sin\emptyset\sin\theta \qquad z = r\cos\theta} \qquad \textbf{(1.52)}$$

That is to say, we wish to derive the equations for $\ddot{r}, \ddot{\theta}, and\ \ddot{\emptyset}$. This requires a certain amount of calculation. The amount shrinks if we perform the change of variables directly in the Lagrangian:

$$\boldsymbol{L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\emptyset}^2\right)} \qquad \textbf{(1.53)}$$

Then we obtain the answer as the Euler-Lagrange equations from this Lagrangian.

(Do the calculation both ways, and see!) This is often the simplest way to perform a coordinate transformation even if the Lagrangian is not Known, so that one first has to spend some time in deriving it, A famous example for which $L \neq T - V$is that of an electrically charged Particle moving in an external electromagnetic field, this example is so important That we will give it in some detail, First of all, “external” signifies that We are dealing with an approximation, in which we ignore that, the presence of The electrically charged particle will affect the electromagnetic field in which it

Moves, in many situations, this is an excellent approximation. The equations

Of motion to be derived are the Lorentz equations

$$\boldsymbol{m\ddot{x}_i = e\left(E_i(x,t) + \epsilon_{ijk}\dot{x}_j B_k(x,t)\right)} \qquad \textbf{(1.54)}$$

The epsilon tensor occurring here may be unfamiliar .For the moment let me just say that the second term on the right hand side means. The cross product of the velocity and the magnetic field, with this hint you Should be able to follow the argument at least in outline, so we proceed, this Example is trickier than the previous ones, since the force depends not only. On the position but also on the velocity of the particle (as well as explicitly on Time, but this is no big deal). It turns out that in order to derive the Lorentz equation from a Lagrangian , we need not only one but four potentials, as Follows:

$$\boldsymbol{E_i(x,t) = -\partial_i \emptyset(x,t) - \partial_t A_i(x,t) \quad B_i(x,t) = \epsilon_{ijk}\partial_j A_k(x,t)} \quad \textbf{(1.55)}$$

Here φ is known as the scalar potential and Ai as the vector potential. It is possible to show that the Following action yields the Lorentz equation when varied with respect to x:

$$\boldsymbol{s[x(t)] = \int dt(\frac{m\dot{x}^2}{2} + e\dot{x}_i A_i(x,t) - e\emptyset(x,t))} \quad \textbf{(1.56)}$$

Please verify this! If we consider a time independent electric field with no magnetic field Present, the Lorentz equation reduces to the more familiar form

$$\boldsymbol{m\ddot{x}_i = -e\partial_i \emptyset(x)} \quad \textbf{(1.57)}$$

This has the same form as Newton's Law of Gravity, if the potential is specified Correctly, The reason why the full Lorentz equation is much more complicated Has to do with the special relativity theory, the magnetic field is a relativistic Effect, The relativistic version of Newton's law of gravity is yet more complicated, And is given by Einstein's general relativity theory, An important difference between gravity and electricity, also in the no relativistic Case, is that particles couple to gravity through the mass, and all

Particles have mass while only some have electric charge. Moreover the mass Serving as “charge” for gravitational forces is the same as the mass occurring On the left hand side of Newton’s equations, Now we illustrate constrained systems There is a delightful trick, due to Lagrange, which illustrates the suppleness of Variation calculus very nicely, the type of problem to be considered is this: A Particle is moving in space but, one way or another; it is constrained to move on A two dimensional surface, The Lagrangian formalism is well suited to derive Equations of motion consistent with this requirement, for definiteness, let the Surface be a sphere defined by the equation

$$\boldsymbol{x^2{}_1 x^2{}_2 x^2{}_3 = 1} \qquad \textbf{(1.58)}$$

The first idea that springs to mind is that one should solve this for

$$\boldsymbol{x_3 = x_3(x_1, x_2) = \pm\sqrt{1 - {x_1}^2 - {x_2}^2}} \qquad \textbf{(1.59)}$$

And insert the result back into the action that describes the free particle, i.e.

$$s[x_1, x_2] = \int dt \frac{m}{2}({\dot{x}_1}^2 + {\dot{x}_2}^2 + {\dot{x}_3}^2(x_1, x_2)^2) \qquad \textbf{(1.60)}$$

This can be done, and will indeed result in a set of Euler-Lagrange equations describing a particle confined to the surface of a sphere, but there are drawbacks.

From eq. (1.60) it appears at first sight as if the configuration space is the unit disk in the plane, since x1 and x2 are not allowed to take values?

Outside this disk, at second sight it appears that the configuration space is two copies of the unit disk, since there are two branches of the square root. But the true configurations space is a sphere. What we see is a reflection of the known fact that it is impossible to cover a sphere with a single coordinate System—our equations have only a “local” validity. This is in a way an Unavoidable problem, but the procedure contains some arbitrariness, and it is rather clumsy.

A better way is to transform to polar coordinates, after which the constraint Will inform us to keep the value of r fixed, But this procedure has some of the Same drawbacks, and moreover such a “natural” coordinate system exists only In very special cases (such as the sphere), while we are heading for the general problem: Consider a Lagrangian L0 defined on an n dimensional configuration Space, with coordinates q1, an, and suppose that the system is confined to Live in the (n − m) dimensional sub manifold defined by the m conditions

$$\boldsymbol{\Phi_I(q_1, \dots, q_2) = 0 \qquad 1 \leq I \leq m} \qquad \textbf{(1.61)}$$

Derive equations of motion consistent with this requirement. One way to do this is to solve for m of the as, q1, am say, by means of the m conditions (1.61), and insert the result in the action. In general this will be a lot of hard work, and the difficulties we had with coordinatizing the sphere will recur with a vengeance, the method of Lagrange multipliers is more convenient. It does not avoid the difficulties with coordinatizing, but postpones them to a later stage. The claim that we will verify is that the following action is equivalent to the previous One:

$$\boldsymbol{s[q, A] = \int dt L_0(q, \dot{q}) + A_1 \Phi_1(q) + \cdots + A_m \Phi_m(q)} \qquad \textbf{(1.62)}$$

The _s are the Lagrange multipliers, to be treated as new dynamical variables.

When the action (1.61) is varied with respect to the _s we obtain the constraints (1.60) as equations of motion, When we vary with respect to the as the Resulting equations will contain the otherwise undetermined Lagrange multipliers, and it not obvious that these equations have anything to do with the Problem we wanted to consider. But they do. Consider the analogous problem Encountered in trying to find the extreme of an ordinary function f (q) of the N variables q, subject to the m conditions _ (q) = 0. (Remember suppression of indices) First suppose that we use the constraints to solve for m of the as— It will not matter which ones—and call them y, leaving n − m independent Variables x, the extreme of f (q) may be found through the equations

$$0 = \partial f = \delta_x \partial_x f + \delta_y \partial_y f \quad \textbf{(1.63)}$$

Where, however, the variations δy are not independent variations, but have to be consistent with the constraints. In fact they are linear function of the axes, given by the conditions

$$0 = \delta\Phi = \delta x \partial_x \Phi + \delta y \partial_y \Phi \quad \textbf{(1.64)}$$

This equation has to be solved for δy and the result inserted into eq. (1.63),

Which is therefore really an expression of the form? $\delta x(\partial x f + \mathit{something\ else}) = 0$. It does not imply $\partial i f \ = \ 0$.

Since δ_ = 0 for the variations we consider, nothing prevents us from rewriting eq. (1.63) in the form

$$\mathbf{0} = \boldsymbol{\delta f} = \boldsymbol{\delta f} + \boldsymbol{A\delta\Phi} = \boldsymbol{\delta x}(\boldsymbol{\partial_x f} + \boldsymbol{A\partial_x \Phi}) + \boldsymbol{\delta y}(\boldsymbol{\partial_y f} + \boldsymbol{A\partial_y \Phi})$$

(1.65)

Where the _s are arbitrary functions, the yes are still given in terms of the axes, So it would seem at first sight that we cannot conclude that ∂if + _∂x_ = 0, But and here comes the punch line—in fact we can, provided we choose the So far arbitrary functions _ in such a way that ∂if + _∂y_ = 0. Since the

Division of the as into as and yes was arbitrary, we see that the "restricted" way Of finding the extreme—making variations consistent with the constraints—is Equivalent to solving the n + m equations

$$\Phi(q) = 0 \qquad \partial_q f + A\partial_q \Phi = 0 \qquad \textbf{(1.66)}$$

For q and, But these are precisely the equations that we obtain from the Lagrange Multiplier method, in which we do not care about the constraints while Varying the action! In all fairness though, we have not solved the equations, we have just derived them in a convenient way.

As long as the constraints depend only on q (and not on q˙) it is straightforward to generalize the argument from functions to functional. From the Action

$$\boldsymbol{s}[\boldsymbol{q}, \boldsymbol{A}] = \boldsymbol{s_0}[\boldsymbol{q}] + \int \boldsymbol{dtA\Phi}(\boldsymbol{q}) \qquad \textbf{(1.67)}$$

We redeliver the constraints, together with the equations of motion

$$\frac{\partial s}{\partial q} = \frac{\partial s_0}{\partial q} + A\partial_q \Phi = 0 \qquad \textbf{(1.68)}$$

This is the analogue of the second equation (1.66).

The generalization from functions to functional is not straightforward in all

Cases: the Lagrange multiplier method works only and for homonymic constraint, that is constraints that involve the configuration space variables only. But consider a ball moving without friction across a plane. The configuration space Have five dimensions: the position (x, y) of the center of mass, and three angular coordinates describing the orientation of the ball, now suppose instead that the ball rolls without slipping. If we are given the position of the center of Mass as a function of time then the motion of the ball is fully determined, Which suggests that x and y are the "true" degrees of freedom, and that the Constrained configuration space is two dimensional. But the situation is more complicated than that (and cannot be described by homonymic constraints). It is impossible to solve for the angular coordinates in terms of x and y. Indeed From our experience with such things we know that the orientation of the Ball at a given point depends on how it got there. Mathematically there is a Constraint relating the velocity of the center of mass to the angular velocity; the Ball spins in a way determined by the motion of its center of mass. Constraints that cannot be expressed as conditions on the configuration space are called a homonymic; we take the simple way out and restrict ourselves to homonymic Constraints only,

We do a simple example in detail. We choose a free particle in three dimensions, with the unconstrained Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 - \dot{z}^2) \tag{1.69}$$

The minus sign here is unusual, indeed in its unconstrained form this Lagrangian is pathological. (Why?) We constrain the particle to the hyperboloid

$$x^2 + Y^2 - Z^2 = -1, \quad Z > 0 \tag{1.70}$$

There are two physical degrees of freedom. One way, among many, to see

This is to observe that any point on the hyperboloid can be described by the

Coordinates r and φ, where

$$X = \sinh r \cos \emptyset \quad Y = \sinh r \sin \emptyset \quad Z = \cosh r \tag{1.71}$$

You can convince yourself that there is a one-to-one correspondence between the pairs (r, φ) and the points (X, Y, Z) on the hyperboloid. Actually this is an exceptional situation: it is often impossible to find a set of coordinates that cover the entire constraint surface. If we use Esq. (2.26) in the Lagrangian we obtain

$$L = \frac{m}{2}(\dot{r}^2 + \sinh^2 r \dot{\emptyset}^2) \tag{1.72}$$

The Lagrangian is equal to the kinetic energy T of the particle. It can be

Checked that it is a conserved quantity, here we see an advantage with using

The physical degrees of freedom only, because it is now evident that the kinetic Energy is bounded from below, a physically important property that was not at all obvious in the original Lagrangian.

The advantages of the Lagrange multiplier method make themselves felt when we look for further constants of the motion, from

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 - \dot{z}^2) + A(x^2 + y^2 + z^2 + 1) \qquad \textbf{(1.73)}$$

We get the equations of motion

$$m\ddot{X} = 2AX \quad m\ddot{y} = 2AY \quad m\ddot{Z} = 2AZ \qquad \textbf{(1.74)}$$

It is then evident that we have three constants of the motion

$$J_x = Z\dot{Y} - Y\dot{Z} \quad J_y = Z\dot{X} - X\dot{Z} \quad J_z = X\dot{Y} - Y\dot{X} \qquad \textbf{(1.75)}$$

Using Esq. (2.26) we can express these constants of the motion

$$\begin{cases} J_x = \dot{r}\sin\emptyset + \emptyset\cosh r \sinh r \cos\emptyset \\ J_y = \dot{r}\cos\emptyset - \emptyset\cosh r \sinh r \sin\emptyset \\ \qquad J_z = \emptyset\sinh^2 r \end{cases} \qquad \textbf{(1.76)}$$

It is possible to check directly, using the equations of motion for r and φ, that these are constants of the motion—but only JZ is "obviously" conserved. The Coordinate system (r, φ) somehow "hides" the others. By the way, the kinetic Energy can be expressed as

$$T = \frac{m}{2}({J_x}^2 + {J_y}^2 + {J_z}^2) \qquad \textbf{(1.77)}$$

This example gives a little bit of the flavor of constrained systems. It is not

Taken quite out of the blue, since it describes motion of a particle on a surface with constant negative curvature, as opposed to a sphere which has constant Positive curvature, but this is by the way.

Now we discuss Symmetries. Let us return to Newton's Third Law. It amounts to a restriction on the kind of forces that are allowed in the second law, and implies that there exist a Set of constants of the motion, namely the moment. (The terminology is a Little unfortunate, since we will soon introduce something called "canonical Moment", they are indeed identical with the conserved moment in simple Cases, but logically there need be no connection.) Constants of the motion are useful when trying to solve the equations of motion, and Emmy Nether Proved a theorem explaining when and why they exist. We present the proof for a Lagrangian of the general form L = L (q, q˙), and afterwards we discuss a Simple example, Let us say at the outset that the argument is quite subtle.

Consider first an arbitrary variation of the action. According to eq. (2.3) the result is

$$\boldsymbol{\delta S} = \int_{t_1}^{t_2} \boldsymbol{\delta q} \left(\frac{\partial l}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) + [\boldsymbol{\delta q}\,\frac{\partial l}{\partial \dot{q}}]_{t_1}^{t_2} \qquad \textbf{(1.80)}$$

In deriving the equations of motion the variations δq (t) were restricted in such a way that the boundary terms vanish. This time we do something different.

The variations are left unrestricted, but we assume that the function q (t) that we vary around obeys the Euler-Lagrange equations. Then the only no vanishing Term is the boundary term, and

$$\delta s = \epsilon\big(Q(t_1) - Q(t_2)\big),\ \epsilon Q(t) \equiv \delta q\,\frac{\partial L}{\partial q} \qquad \textbf{(1.81)}$$

Here and in the following Q is the constant occurring in $\delta q \,=\, \epsilon f$, where f is an Arbitrary function of t, the point is to ensure that there is nothing infinitesimal About Q, So far nothing has been assumed about the variations. Now suppose that, for the given Lagrangian, there exists a set of variations δq of some specified Form

$$\delta q = \delta q(q, \dot{q}) , \qquad \textbf{(1.82)}$$

Such that for these special variations

$$\delta S = 0 . \qquad \textbf{(1.83)}$$

It is understood that the Lagrangian is such that eq. (1.83) holds as an identity, Regardless of the choice of $q(t)$, for the special variationsδq. (Note that, given a Lagrange an, it is not always the case that such variations exist. But sometimes they do.)

Next comes the crux of the argument. Consider variations of the particular kind that makes eq. (1.83) hold as an identity—so that $\delta q = \epsilon f$ is a known Function—and restrict attention to q (t) s that obeys the equations of motion. With both these restrictions in force, we can combine Esq. (1.83) and (1.81) to conclude that

$$0 = \delta S = \text{Q}\,(Q(t_2) - Q(t_1)) . \qquad \textbf{(1.84)}$$

The times t1 and t2 are arbitrary, and therefore we can conclude that $Q =$

$Q\ (q, \dot{q})\ is$ A constant of the motion.

What this theorem does for us is to transform the problem of looking for Constants of the motion to the problem of looking for variations under which the variation of the action is identically zero. Before we turn to examples we generalize the argument slightly, and state the theorem properly. Thus, suppose that there exists a special form of δq, such that

$$\delta S = \int_{t_1}^{t_2} dt \ \frac{d}{dt} \Lambda(q, \dot{q}) \qquad \textbf{(1.85)}$$

Here Λ can be any function—the important and unusual thing is that the Integrand is a total time derivative. Then the quantity Q, defined by

$$\epsilon Q(q, \dot{q}) = \delta q_i \frac{\partial L}{\partial q_i} - \Lambda(q, \dot{q}) \qquad \textbf{(1.86)}$$

Is a constant of the motion? This is easy to see along the lines we followed Above, The theorem can now be stated as follows:

No ether's theorem: To any variation for which as takes the form (2.38), there Corresponds a constant of the motion given by eq. (1.86), we will have to investigate whether Lagrangians can be found for which such variations exist, otherwise the theorem is empty. Fortunately it is by no means empty, indeed eventually we will see that all useful constants of the motion Arise in this way.

For now, one example—but one that has much symmetry—will have to suffice. Consider a free particle described by

$$L = \frac{m}{2} \dot{x}_i \dot{x}_i \qquad \textbf{(1.87)}$$

Since only x˙ appears in the Lagrangian, we can choose

$$\delta xi = \epsilon_i \qquad \textbf{(1.88)}$$

Whereϵ_i is independent of time. Then the variation of the action is automatically Zero, No ether's theorem applies, and we obtain a vector's worth of Conserved charges

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \qquad \textbf{(1.89)}$$

We use the letter P rather than Q because this is the familiar conserved Momentum vector whose presence is postulated in Newton's Third Law. Another set of three conserved charges can be found easily, since

$$\delta xi = \text{Q}ijk\text{Q}jxk \Rightarrow \delta S = 0\,. \quad \textbf{(1.90)}$$

Here if is again independent of t, and ink is the totally anti-symmetric epsilon Tensor, No ether's theorem now implies the existence of another conserved Vector, namely

$$Li = \text{Q}ijkxjx^{\cdot}\, k \qquad \textbf{(1.91)}$$

This is the angular momentum vector.

We know that there is at least one more conserved quantity, namely the Kinetic energy, actually there are several, but the story now becomes a bit more complicated because we have to deal with variations for which the variation of the Lagrangian is a total derivative, rather than zero. Thus

$$\delta x_i = \epsilon \dot{x}_i \Longrightarrow \delta S = \int dt \frac{d}{dt}\left(\frac{\epsilon m}{2}\dot{x}^2\right) \qquad \textbf{(1.92)}$$

Using eq. (1.91) we obtain the constant of the motion

$$\boldsymbol{E = \frac{m}{2}\dot{x}_i\dot{x}_i} \qquad \textbf{(1.93)}$$

This is the conserved energy of the particle. There is yet another conserved Quantity that differs from the others in being an explicit function of time—but its total time derivative vanishes since it also depends on the time dependent Dynamical variables, thus

$$\delta x_i = -\epsilon_i t \Longrightarrow \delta S = \int dt \frac{d}{dt}(-m\epsilon_i x_i) \qquad \textbf{(1.94)}$$

Give the conserved charge

$$Q_i = mx_i - tm\dot{x}_i \,, \qquad \textbf{(1.95)}$$

And it is easy to check that its total time derivative vanishes as a consequence Of the equations of motion, our analysis of the free particle ends here, but us Will return to it in a moment, to show that the conserved quantities have a Clear physical meaning. To see this, select a solution q (t) of the equations of motion. We know that this gives an extremum of the action. Then consider $q'(t) = q(t) + \delta q(t),$ where the variation is of the special kind that leaves the value of the action unchanged. Obviously then $S[q'(t)] = S[q(t)]$, so that the extremum is not an isolated point in the space Of allqs, but rather occurs for a set of qs that can be reached from each other By means of iteration of the special variation $\delta q(t),$ In other words, given a Particular solution of the equations of motion, we can get a whole set of new Solutions if we apply the special variation, without going through the work of Solving the equations of motion again, this leads to an important definition:

A symmetry transformation is any transformation of the space of functions q (T) having the property that it maps solutions of the equations of motion to other solutions, this is not a property of the individual solutions, but of the set of all solutions. The special variations occurring in the statement of No ether's theorem are Examples of symmetry transformations, given the converse of the statement That we proved (which is also true), namely that any constant of the motion Gives rise to a special variation of the kind considered by Nether, we observe That any constant of the motion arises because of the presence of symmetry, Let us interpret the symmetry transformations that we found for the free Particle, beginning with eq. (1.95), this is clearly a translation in space. Therefore Momentum conservation is a consequence of translation invariance. It is Immediate that we can iterate the infinitesimal translations used in No ether's Theorem to obtain finite translations and the statement is that given a solution To the equations of motion all trajectories that can be obtained by translating This solution is solutions, too. To be definite, given that (vat, 0, 0) is a solution for constant $v, (a + vt, b, c)$ is a solution too, for all real values of (a, b, c).

Translation invariance acquires more content when used in the fashion of Newton's Third law, which we can restate as "the action for a set of particles has Translation symmetry", for free particles this is automatic. When interactions between two particles are added, the law becomes a restriction on the kind of Potentials that is admitted in

$$L = \frac{m_2}{2}\dot{x}_1{}^2 + \frac{m_2}{2}\dot{x}_2{}^2 - V(x_1, x_2) \qquad \textbf{(1.96)}$$

Indeed invariance under (1.95) requires that V (x_1,x_2) = V ($x_1 - x_2$), which is a strong restriction, expresses the fact that the Lagrangian has rotation symmetry, While, is an infinitesimal translation in time: Given a solution $x\ (t)$,the function

$$\acute{x}(t) = x(t + t_0) = x(t) + t_0\dot{x}(t) + o(t_0{}^2) \quad \textbf{(1.97)}$$

Is a solution too, so we can make the elegant summary that conservation of Momentum, angular momentum and energy are consequences of symmetries Under translations and rotations in space, together with translations in time, Eq. (1.97) expresses invariance under "boosts", since it changes all velocities By a constant amount, the free particle is exceptional because we can reach Any solution by a symmetry transformation, starting from any given solution, Having said all this, it is not true that every symmetry gives rise to a Constant of the motion, discrete symmetries like reflections, that does not arise By iterating an infinitesimal symmetry, To sum up, symmetries are important from two quite different points of View, given the equations they facilitate the search for solutions, but they Also facilitate the search for the correct equations (if we believe that they Should exhibit certain symmetry), No ether's theorem is a tool for discovering Symmetries, as well as for deducing their corresponding

Chapter 2

Conic section and central force two body problem

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Section (2.1): Conic sections

The theory of conic sections was one of the crowning achievements of the Greeks, Their results will be important in the gravitational two body problem, But the theory is no longer as well known as it deserves to be, so here is a brief Account, By definition a conic section is the intersection of a circular cone with a plane, the straight lines running through the apex of the cone are called Its generators and we will consider a cone that extends in both directions From its apex, It is the set of one dimensional subspace in a three dimensional vector space, Generically, a plane will intersect the cone in such a Way that every generator crosses the plane once, or in such a way that exactly Two of the generators miss the plane. Apollonius proved that the intersection is an ellipse in the first case and a hyperbola in the second, there is a borderline Case when exactly one generator is missing. Then the intersection is a parabola. This is all very easy if we use the machinery of analytic geometry. For Simplicity, choose a cone with circular base, symmetry axis orthogonal to the Base and opening angle 90 degrees. It consists of all points obeying

$$x^2 + y^2 - z^2 = 0 . \qquad \textbf{(2.1)}$$

Without loss of generality, the plane can be described by

$$cx + z = d . \qquad \textbf{(2.2)}$$

Inserting the solution for z in the equation it is easy to see that the intersection is either an ellipse, a hyperbola, or a parabola—provided you recognize their Equations, as I assume. The section is a circle if c = 0 and a parabola if c = ±1.

It is an interesting exercise to prove this in the style of Apollonius. Let the Cone have arbitrary opening angle. Take the case when the plane intersects

Every generator once, let us says in the upper half of the cone. Place two Spheres inside the cone, one above and one below the cone, and let them grow Until each touches the plane in a point and the cone in a circle, this clearly Defines the spheres uniquely, Denote the points by F1 and F2, and the circles By C1 and C2, Now consider a point P in the intersection of the cone and The plane, the generator passing through P intersects the circles C1 and C2 In the points Q1 and Q2, Now the trick is to prove that the distance PF1 Interlude: Conic sections 31

Equals the distance PQ1, and similarly the distance PF2 equals the distance PQ2. This is true because the distances measure the lengths of two tangents to the sphere, meeting at the same point, it then follows that the sum of the Distances PF1 and PF2 is constant and equal to the length of the segment of

The generator between the circles C_1 and C_2, independently of which point P On the intersection we choose, this property defines the ellipse. This is the Proof that the intersection between the cone and the plane is an ellipse with its foci at F_1 andF_2, if you are unable to see this, consult an old fashioned Geometry book, for our purposes it is convenient to define the ellipse somewhat differently.

An ellipse of eccentricity e < 1 can be defined as the set of points whose Distance from a given point, called a focus, is e times the distance to a straight Line, called a directory, for the circle e = 0, and the directory is at infinity.

The lotus rectum of an ellipse is a chord through the focus parallel to the Directory, and has length 2p. Now place the origin of a coordinate system at that focuses, with the x-axis pointing towards the directory. The distance of a Point on the ellipse to the focus is

$$r = e(distance\ to\ the\ directrix) = e\left(\frac{p}{e} - x\right) \qquad \textbf{(2.3)}$$

(To see this, note that the distance from the focus to the directory is P/E.) Otherwise expressed

$r = p - \text{err coos } \varphi \Leftrightarrow \frac{p}{r} = 1 + e \text{ coos } \varphi$, **(2.4)**

Where φ = 0 gives the point closest to the directory. For a general point on the Ellipse we find

$$x^2 + y^2 = r^2 = (p - ex)^2 \Leftrightarrow \frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \textbf{(2.5)}$$

Where

$$A \equiv \frac{p}{1-e^2} \qquad b^2 = pa(1 - e^2)a^2 \qquad \textbf{(2.6)}$$

The major axis of the ellipse is 2a, and the minor axis is 2b. Finally the Distance between the center and the focus equals ea. To see this we set φ = 0

And φ = π in eq. (3.4), and calculate

$$\frac{r(\pi)-r(0)}{2} = \frac{p}{2}\left(\frac{1}{1-e} - \frac{1}{1+e}\right) = ea \qquad \textbf{(2.7)}$$

For some further information consult exercise 1.

Similar treatments can be given for the hyperbola, for which e > 1, and for the parabola, for which e = 1.

In our study of the two-body problem we will find it interesting to relate an ellipse centered at a focus to an ellipse centered at the origin. The latter is described in Cartesian coordinates by the complex trajectory

$$w(t) \; = \; a\,cos\,t \; + \; ib\,sin\,t\;. \qquad \textbf{(2.8)}$$

The parameter t must not be confused with the angle between the radiuses Vector and the x-axis, surprisingly; if we square this ellipse we obtain an ellipse Centered at the origin:

$$\boldsymbol{Z(t) = W^2 = \frac{a^2+b^2}{2} cos\,2t + iab\,sin\,2t + \frac{a^2-b^2}{2} \equiv A\,cos\,2t + iB\,sin\,2t + EA} \quad \textbf{(2.9)}$$

Where

$$\boldsymbol{EA \; = \; \sqrt{A^2 - B^2}} \qquad \textbf{(2.10)}$$

E is the eccentricity of an ellipse with semi-major axis A and semi-minor axis B, and consequently EA is the distance between its focus and its center. This is again an ellipse, but centered at one of its foci and traversed twice as the Original ellipse is traversed once. This trick was introduced by Karl Bohlen, working at the Pullover observatory In Russia in 1911, His point was that the transformation w = √Z is not

Analytic at the origin, this enabled him to deal with collisions between points Particles thought of as limiting cases of elliptical orbits whose eccentricity Approaches 1, at the collision the particle presumably reverse its direction, But this is a rather singular occurrence. In terms of the variable w it is an Unromantic event.

Section (2.2): The central force two-body problem

Johannes Keller spent his life pondering the observations of the solar system Made by Tyco Brahe, and found that the motion of the planets around the Sun follows three simple rules:

1. **A planet moves along an ellipse with the sun in one of the foci.**
2. **The radius vector covers equal areas in equal times.**
3. **The square of the period of all the planets is proportional to the cube of their major axes.**

To appreciate Keller's work fully, note that there are important facts about

The solar system (such as what the distances are) that do not follow simple

Rules, Moreover the observational data gave the planetary orbits projected on A sphere centered at a point which itself moves along an ellipse around the Sun, so it was not obvious that they admitted of a simple description at all. Newton derived Keller's laws from his own Laws, with the additional assumption That the force between the planets and the sun is directed along The radius vector (the force is central) and is inversely proportional to the Square of the distance, this remains the number one success story of physics, So we should be clear about why this is so. Naively Keller's laws may seem Simpler than Newton's, but this is not so, for at least two reasons. One is that Newton's laws unify a large body of phenomena, from the motion of planets To the falling of stones close to the Earth, The other reason is

that improved Observations reveal that Keller's laws are not quite exact, and the corrections Can be worked out mathematically from Newton's laws, For Mercury (which is hard to observe) the eccentricity e = 0.21, for the Earth e = 0.02, and for Mars e = 0.09. Keller's main concern was with Mars. If e = 0 the ellipse becomes a circle. Unbound motion through the solar system is described by hyperbolas with e > 1, but Keller did not know this.

Now we Study the problem and its formal solution. We want to derive Keller's laws. As an approximation we assume that it suffices to treat the planets independently of each other, moreover we asset that the precise shapes of the sun and the planets are unimportant and that they can be approximated as being point like. (In the Principia, Newton proved from properties of the inverse square law that this approximation is exact for Spherical bodies) Later on, we can go back to these assumptions and see if we can relax them—this will give the corrections referred to above. So we have decided that the configuration space of our problem has six Dimensions, spanned by the positions of the sun (X) and one planet (up), and we try the Lagrangian

$$L = \frac{m}{2}\dot{X}_i\dot{X}_i + \frac{m_p}{2}\dot{x}_{p_i}\dot{x}_{p_i} - V(X, x_p) \qquad \textbf{(2.11)}$$

Depending on the form of the function V we may have to exclude the points

X = up from the configuration space—we insist that the function V takes Finite values only as a function on configuration space, anyway this will give six Coupled second order differential equations. In general they will not admit any Simple solutions, Keller's work implies that the solutions should be simple, so we try to build some symmetry into the problem. We aim for at least six conserved quantitative, one for each degree of freedom, since this should Result in a soluble problem. Conservation of momentum and energy is already Postulated, so we need two more, the answer is rotational symmetry. It is not An objection that an ellipse is not symmetric under rotations, all we need is That if a particular ellipse is a solution, then any ellipse which can be obtained From it by means of rotations is a solution too, even if there is no planet moving Along it due to the choice of initial conditions, it might seem that rotational Symmetry is overdoing it, since it will yield three conserved quantitative, but— For reasons fully explained by Hamilton-Jacobi theory—only two of these are Really useful, With translational and rotational symmetry in place, we find

$$V\,(X, xP\,) \;=\; V\,(X \;-\; xP\,) \;=\; V\,(|X \;-\; xP\,|) \qquad \textbf{(2.12)}$$

This works for any function V of one variable. Next we introduce coordinates Xi = pie−Xi which are invariant under translations, together with coordinates Describing the center of mass, then the center of mass coordinates decouple, And their equations can be solved and set aside. There remains a Lagrangian for a one-body problem, involving only three degrees of freedom:

$$L = \frac{m}{2}\dot{x}_i{}^2 - V(|x|) \qquad \textbf{(2.13)}$$

Here m is the reduced mass, almost equal to the mass of the planet since the sun is very heavy in comparison, the coordinate xi vanishes at the center of mass of the system, which is well inside the sun, and can be approximately identified? With the center of the sun, this maneuver should be familiar from elementary Mechanics, I just want to emphasize that it is translational symmetry in action. Rotational symmetry implies the existence of a conserved vector

$$Li \ = \ m\epsilon_{ijk}xj\dot{x}_k. \qquad \textbf{(2.14)}$$

This vector is orthogonal to both x and x˙, which means that the motion Is confined to a plane orthogonal to the angular momentum vector, we are Down to a two dimensional configuration space, to take maximal advantage of Spherical symmetry we introduce spherical polar coordinates, chosen so that The plane containing the orbit is at θ = π/2. The Lagrangian simplifies to

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\emptyset}^2 - V(r) \qquad \textbf{(2.15)}$$

We have used the constant direction of the angular momentum vector. But

Its magnitude is constant too. This happens because the Lagrangian (2.14) is

Invariant under translations in the angle φ. Using No ether's theorem we find the constant of the motion

$$\boldsymbol{l} = \frac{\partial L}{\partial \dot{\emptyset}} = \boldsymbol{mr^2}\dot{\emptyset} \qquad \textbf{(2.16)}$$

This equation is of considerable interest in itself. It says that

$$\dot{A} = \frac{r^2\dot{\emptyset}}{2} = \frac{1}{2m} = \boldsymbol{constant} \qquad \textbf{(2.17)}$$

Where ˙A is the area covered by the radius vector per unit time. But this is Keller's Second Law, which therefore holds for all central forces. We are on the right track!

Together with Keller's second law, energy conservation is enough to solve the problem, using eq. (2.16) the conserved energy is

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\emptyset}^2\right) + V(r) = \frac{m\dot{r}^2}{2} + \frac{l^2}{2mr^2} + V(r) = const$$

(2.18)

This gives the formal solution

$$dt = \frac{dr}{\sqrt{\frac{2}{m}(E - \frac{l^2}{2mr^2} - V(r))}} \qquad \textbf{(2.19)}$$

It may or may not, depending on our choice of V (r), be possible to do the

Integral in terms of elementary functions, but anyway this equation determines the function r (t), and hence solves the problem. To find φ (t) we combine Esq. (2.16) and (2.19), and get

$$d\emptyset = \frac{ldt}{mr^2} = \frac{ldr}{r^2\sqrt{2m(E - \frac{l^2}{2mr^2} - V(r))}} \qquad \textbf{(2.20)}$$

This equation determines φ(r (t)), and the central force problem is thereby

Fully solved at the formal level, If we are only interested in the form of the

Orbits, and not the time development, eq. (2.20) is all we need—it will give us Φ(r), and after inversion r (φ), which is the equation for the form of the orbit.

Now we study the existence and stability of circular orbits. There is a special kind of solution that we can look for directly, with no great Expense of effort, namely circular orbits, from the expression (4.8) we see that the two body central force problem has been reduced to one dimensional Motion in the effective potential

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r) \qquad \textbf{(2.21)}$$

A simple case is V (r) = 0, i.e. no force at all. In the effective one dimensional Problem this corresponds to a repulsive Jeff—the particle comes in from infinity, Reaches a minimum value of r, and then disappears to infinity again. If we want a bound orbit the potential V (r) must be attractive. A circular orbit is one for which $\dot{r} = 0$ identically, which means that the Particle is sitting at the bottom of the effective potential—if it does have a Bottom, The radius r of the circular orbit must obey

$$V'_{eff}(r) = 0 \qquad \textbf{(2.22)}$$

If this happens at a local maximum of Jeff the solution instable, and Unlikely to be realized in Nature, The orbit is stable under small perturbations If and only if

$$\boldsymbol{V''_{eff} > 0} \qquad \textbf{(2.23)}$$

At the value of r for which V ′

Off = 0.

We look into these equations for the special choice

$$V(r) = -kr^{\alpha} \Longrightarrow V_{eff}(r) = \frac{l^2}{2mr^2} - kr^{\alpha} \qquad \textbf{(2.24)}$$

With α arbitrary, the radius of the circular orbit is found to be

$$r = (-\frac{l^2}{\alpha km})^{\frac{1}{\alpha+2}} \qquad (2.25)$$

This makes sense only if l2 6= 0—a question of initial conditions—and eke < 0. (The case α = −2 is special.) Stability of the circular orbit requires in addition that

$$\alpha > -2\,. \qquad (2.26)$$

There is still the question whether small departures from the circular orbit will give rise to ellipses, or to something more complicated. This is really a

Question about the ratio between the times it takes for the planet to complete

A full revolution in φ, to the time it takes to complete a full oscillation in r. If the orbit is an ellipse centered at a focus these times must be equal, and this.

Is likely to happen only for a very special V (r), for general bounded motion

The amount by which the perihelion processes during one period of the radial Motion follows from eq. (4.10). It is

$$\Delta\emptyset = 2\int_{T_{min}}^{T_{max}} \frac{ldr}{r^2\sqrt{2m(E-V_{eff}(r))}} \qquad (2.27)$$

For the planets, Keller's first law requires that _φ = 2π.

A final comment: if we do choose α = −1 we have the problem that the Energy is unbounded from below. We now see that this problem cannot be too Serious, because

$$V_{eff}(r) = \frac{l^2}{2mr^2} - \frac{k}{r} \qquad (2.28)$$

Is in fact bounded from below whenever l 6= 0. The case when l = 0 is indeed Troublesome from a physical point of view, because then the two bodies will Collide, and we do not have a prescription for what is to happen after the Collision, The case of coinciding particles is not included in our configuration Space,

Now we discuss the Keller's First Law. What force laws are consistent with Keller's First Law? Hooke's law does give Elliptical orbits. This is most easily seen by transforming the Lagrangian back to Cartesian coordinates:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\emptyset}^2\right) - kr^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - k(x^2 + y^2) \qquad \textbf{(2.29)}$$

This is two harmonic oscillators of equal frequencies, and the corresponding

Lissajous figures are indeed ellipses. They are not the right kind of ellipses However, since they are centered at the origin. Let us call them Hooke ellipses. They can be related to Keller ellipses (centered at a focus) by means

Of Bohlen's trick If

$$w = a \text{ coos } t + \text{if } \sin t \qquad \textbf{(2.30)}$$

Is a Hooke ellipse, then?

$$Z = w^2 = \frac{a^2-b^2}{2} + \frac{a^2+b^2}{2}\cos 2t + iab\sin 2t \qquad \textbf{(2.31)}$$

This is indeed an ellipse centered at a focus, as we saw in chapter 3. For the Hooke ellipse w (t) we know that the force law is

$$\ddot{w} = -w \Rightarrow |\dot{w}|^2 + |w|^2 = 2\epsilon \qquad \textbf{(2.32)}$$

Moreover Keller's second law holds, so then

$$\frac{|w|^2 d\emptyset}{dt} = constant \qquad \textbf{(2.33)}$$

This relates the parameter t to the angle φ between the radius vector and the Major axis, the idea now is to introduce a new time τ, related to t in such a Way that Keller's second law holds also for the ellipse we get when we square The Hooke ellipse, thus we require

$$\frac{2|Z|^2 d\emptyset}{dr} = constant \qquad \textbf{(2.34)}$$

A suitable choice of the two constants gives the desired relation

$$\frac{dr}{dt} = \frac{|Z|^2}{|w|^2} = |w|^2 \qquad \textbf{(2.35)}$$

The rest is a simple calculation, using eq. (4.22):

$$\frac{d^2 Z}{dr^2} = \frac{1}{|w|^2}\frac{d}{dt}\left(\frac{1}{|w|^2}\frac{dw^2}{dt}\right) = \frac{2}{|w|^2}\frac{d}{dt}\left(\frac{\dot{w}}{\bar{w}}\right) = \cdots = -4\epsilon\frac{Z}{|Z|^3} \qquad \textbf{(2.36)}$$

Where Q is the constant energy of the Hooke ellipse, but this is precisely Newton's Force law for gravity. So we conclude that Keller's First and Second Laws Together imply the inverse square law, with the potential

$$V(r) = -\frac{k}{r} \qquad \textbf{(2.37)}$$

There is no other solution. The argument is water tight because every Keller Ellipse can be obtained from a Hooke ellipse using Bohlen's trick, and a Hooke Ellipse arises only in the harmonic oscillator potential.

To confirm our conclusion, let us go back to eq. (2.20), which gives a formal

Solution for the form of the orbit, we choose eq. (2.37) for V (r), and we also Perform the substitution

$$\boldsymbol{u = \frac{1}{r} \Longrightarrow du = -\frac{dr}{r^2}} \qquad \textbf{(2.38)}$$

The result is

$$d\emptyset = -\frac{ldu}{\sqrt{2mE - l^2u^2 + 2mku}} \qquad \textbf{(2.39)}$$

This defines u as a trigonometric function of φ, with the energy E < 0 and the phase φ0 as undetermined integration constants, in fact

$$\emptyset = \emptyset_0 - \boldsymbol{ar\cos\frac{\frac{l^2u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}} \qquad \textbf{(2.40)}$$

Inverting this, and cleaning up the answer a little, we obtain

$$\boldsymbol{\frac{l^2}{mkr} = \frac{l^2u}{mk} = 1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos(\emptyset - \emptyset_0)} \qquad \textbf{(2.41)}$$

The constant φ0 is the value of the coordinate φ for which the planet is at its perihelion that is when it is closest to the sun, Comparing to eq. (2.4) we Read off that the eccentricity of the ellipse is

$$\boldsymbol{e = \sqrt{1 + \frac{2El^2}{mk^2}}} \qquad \textbf{(2.42)}$$

The semi-major axis of the ellipse is

$$\boldsymbol{a = p\frac{1}{1 - e^2} = \frac{l^2}{mk}\frac{mk^2}{2|E|l^2} = \frac{k}{2|E|}} \qquad \textbf{(2.43)}$$

The solution remains valid also for E > 0, in which case it describes a hyperbola With e > 1, physically this is an unbound trajectory, like that of a

Spaceship heading for the stars,

Now we discuss the Keller's Third Law. Keller's Third Law awaits proof. Here is a simple one: since the areal velocity is constant, the period T is simply related to the area of the ellipse. For the Special case we are looking at eq. (4.7), together with exercise 3.1, implies that

$$\boldsymbol{A = \frac{l}{2m}T = \pi ab} \qquad \textbf{(2.44)}$$

Remembering that b2 = ape, and using eq. (2.41) to identify the lotus rectum

P gives

$$\boldsymbol{T^2 = \frac{4\pi^2 m^2 a^3 p}{l^2} = 4\pi^2 \frac{m}{k} a^3} \qquad \textbf{(2.45)}$$

This is Keller's Third Law. We see that it holds only to the extent that we

Can regard the reduced masses m of all the planets to be the same, here is a more involved proof, using the full force of our solution (2.9). We Rewrite it using our expressions for a and e:

$$\boldsymbol{dt = \sqrt{\frac{ma}{k}} \frac{rdr}{\sqrt{-\frac{l^2}{2m|E|}+2ar-r^2}} = \sqrt{\frac{ma}{k}} \frac{rdr}{\sqrt{a^2e^2-(r-a)^2}}} \qquad \textbf{(2.46)}$$

4.5 Self-similarity and the viral theorem 41

If we use the substitution

$$r = a + ae\cos\sigma \qquad \textbf{(2.47)}$$

We can do the integral, and d with a suitable choice of the integration constant we obtain

$$t = \sqrt{\frac{ma^3}{k}}(\sigma + e \sin \sigma) \qquad \textbf{(2.48)}$$

Taken together, Esq. (2.47-2.48) provide a parametric representation of the

Orbit and we can read off its period

$$T = 2\pi\sqrt{\frac{ma^3}{k}} \qquad \textbf{(2.49)}$$

This is Keller's Third Law once again.

Now we study Self-similarity and the viral theorem. Keller's Third Law says that, given a solution, one can simply enlarge it to get Another solution—provided one also slows down the rate at which things are Happening, it is really a consequence of mechanical similarity or self-similarity, A kind of symmetry not covered by No ether's theorem, it arises as follows. Take the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - V(q) \qquad \textbf{(2.50)}$$

And assume that the potential is homogeneous of degree β, meaning that there exists a real number β such that for any real non-zero number λ

$$V\,(\lambda q) \;=\; \lambda^{\beta} V\,(q)\,. \qquad \textbf{(2.51)}$$

There could be several variables quid. For simplicity I write only q. Let us also Change the time scale, and define a new function q′ by

$$\boldsymbol{q}(\boldsymbol{t}) \rightarrow \boldsymbol{q}'(\boldsymbol{t}') = \boldsymbol{\lambda q}(\mathrm{T}), \boldsymbol{t}' = \boldsymbol{\lambda}^{\frac{2-\beta}{2}} \boldsymbol{t} \qquad \textbf{(2.52)}$$

It follows that

$$\dot{\boldsymbol{q}}' \equiv \frac{\boldsymbol{dq'}}{\boldsymbol{dt'}} = \frac{\boldsymbol{dt}}{\boldsymbol{dt'}} \frac{\boldsymbol{d}}{\boldsymbol{dt}} \left(\boldsymbol{\lambda_q}(\boldsymbol{t}) \right) = \boldsymbol{\lambda}^{\frac{\beta}{2}} \dot{\boldsymbol{q}} \qquad \textbf{(2.53)}$$

(This is a slight abuse of the dot notation.) We can now check that our rescaling Represent a symmetry because, under this transformation,

$$L(q, \dot{q}) \rightarrow L(q', \dot{q}') = \lambda^{\beta} L(q, q^{\cdot}) . \qquad \textbf{(2.54)}$$

This has the effect of changing the value of the action with a constant factor, and it follows that q′ (t′) is an extremism of S [q′ (t′)] if q (t) is an extremism of S [q (t)], in this sense it is a symmetry of the action. (Compare problem 2.5. If you find the argument difficult, you can check directly that q′ (t′) is a solution whenever q (t) is)

The harmonic oscillator has a potential V ∼ q2, homogeneous with β = 2. Scaling symmetry is present with t′ = t. Given a solution q (t) there is another Solution that is a blown up version of this, with amplitude a factor of λ larger, Because t = t′ the period of the oscillations are unaffected by the scaling, And we see—without looking at any explicit solutions—that the period of The oscillations are independent of their amplitudes. Galilee first made this Observation while celebrating mass in the cathedral of Pisa, Newton's law of gravity uses a homogeneous potential with β = −1, so Similarity holds with t → t′ = λ3/2t. Two ellipses with the same shape (and

The planetary orbits are all close to circular) will therefore have their periods and their axes related by

$$R \to R' = \lambda R \qquad T \to T' = \lambda^{3/2} T \Longrightarrow \frac{T'^2}{T^2} = \frac{R'^3}{R^3} \qquad \textbf{(2.55)}$$

This is Keller's Third Law for the third time. Another dramatic theorem can be proved for self-similar systems. It is called The Viral Theorem, and relates the time averages of the kinetic and potential Energies to each other, if it exists, the time average of a function f (t) is defined

By

$$\langle f \rangle \equiv lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} dt' \, f(t') \qquad \textbf{(2.56)}$$

For the argument to follow it is important that the time average of the derivative of a bounded function is zero, i.e.

$$\langle \frac{df}{dt} \rangle = lim_{t\to\infty} \frac{1}{t} \big(f(t) - f(t_0) \big) = 0 \qquad \textbf{(2.57)}$$

Whenever f (t) < ∞ for all t. We are ready to study the time average of the kinetic energy, given the Assumptions that the system obeys Newton's law

$$m\ddot{x}_i \;=\; -\partial iV\,(x)\,, \qquad \textbf{(2.58)}$$

That the potential is homogeneous of degree β that the motion is bounded in

Space, and that the velocities are everywhere finite, On the other hand we are Not restricting the index I. It could run between 1 ≤ I ≤ 3N, in which case we Are actually studying an N-body problem; this could be a cluster of galaxies Under the tentative assumption that the cluster is a bound system or it could Are 1023 atoms confined in a box? Regardless of the number of variables Euler's Theorem on homogeneous functions states that

$$V\ (\lambda x) = \lambda^{\beta} V\ (x) \Rightarrow xi\partial iV\ (x) = \beta V\ (x) \qquad \textbf{(2.59)}$$

Then

$$\mathbf{2\langle T\rangle = \langle m\dot{x}^2\rangle = \langle \frac{d}{dt}(mx_i\dot{x}_i) - mx_i\ddot{x}_i\rangle = -(x_i m\ddot{x}_i) = \langle x_i\partial_i V(x)\rangle =}$$

$$\boldsymbol{\beta}\langle \mathbf{V(x)}\rangle \qquad \textbf{(2.60)}$$

This is the conclusion we were after. For bounded motion in homogeneous potentials

$$2\ \langle T\rangle\ =\ \beta\langle V\rangle \qquad \textbf{(2.61)}$$

Where β is the degree of homogeneity of V (x), for the inverse square law the viral theorem implies that

$$\langle 2T + V\rangle = 0 \Rightarrow \langle E\rangle = \langle T + V\rangle = -\langle T\rangle \leq 0\ . \qquad \textbf{(2.62)}$$

This is the familiar fact that motion bounded by gravity can take place only If the total energy is negative, for the harmonic oscillator we deduce that the Time averages hit and he I are equal.

The calculation in eq. (2.60) is of interest even for non-potential forces, if we break it off after the first line:

$$2\langle T\rangle = -\langle x_i F_i\rangle\ . \qquad \textbf{(2.63)}$$

If the forces are the constraint forces keeping an ideal gas contained inside a Box, we can use this relation to deduce the ideal gas law. We turn the sum into an integral, recall the definition of the pressure P as force per unit area, and apply Gauss' law to the result:

$$\mathbf{2\langle T\rangle = p\int dA_i x_i = p\int dV\partial_i x_i = 3PV} \qquad \textbf{(2.64)}$$

If we are willing to identify hit with (a factor times) the temperature T we

Have Boyle's Law,

$$PV = RT . \qquad \textbf{(2.65)}$$

Now we Study the three-body problem, the three-body problem—three masses interacting according to Newton's Law of Gravity—is not soluble in the sense that the two-body problem is. The Number of conserved quantities is the same in both problems, and for the Nine degrees of freedom in the three-body problem this is not enough. But The three-body problem is also a very important one, and in fact it motivated Many of the developments that we will come to later on, A natural first step is to look for special exact solutions, which may be used As starting points for perturbation theory, or in other ways, an interesting example Was found by Lagrange, Let us begin by assuming that the motion takes Place in a plane, and use complex numbers zip (t) to denote the trajectories. The Equations are

$$\begin{cases} \ddot{Z}_1 = -m_2 \frac{Z_1 - Z_2}{|Z_1 - Z_2|^3} - m_3 \frac{Z_1 - Z_3}{|Z_1 - Z_3|^3} \\ \ddot{Z}_2 = -m_3 \frac{Z_2 - Z_3}{|Z_2 - Z_3|^3} - m_1 \frac{Z_2 - Z_1}{|Z_2 - Z_1|^3} \\ \ddot{Z}_3 = -m_1 \frac{Z_3 - Z_1}{|Z_3 - Z_1|^3} - m_2 \frac{Z_3 - Z_2}{|Z_3 - Z_2|^3} \end{cases} \qquad \textbf{(2.66)}$$

We assume that the center of mass is at rest,

$$m1z1 + m2z2 + m3z3 = 0 . \qquad \textbf{(2.67)}$$

The particles form a triangle, with sides represented by

$$w1 = z3 - z2 , w2 = z1 - z3 , w3 = z2 - z1 . \qquad \textbf{(2.68)}$$

In terms of these variables the equations of motion take the form

$$\begin{cases} \ddot{W}_1 = -m\frac{w_1}{|w_1|^3} + m_1 a \\ \ddot{W}_2 = -m\frac{w_2}{|w_2|^3} + m_2 a \\ \ddot{W}_3 = -m\frac{w_3}{|w_3|^3} + m_3 a \end{cases} \qquad \textbf{(2.69)}$$

$$\mathbf{m = m_1 + m_2 + m_3} \qquad \textbf{(2.70)}$$

$$\mathbf{a = \frac{w_1}{|w_1|^3} + \frac{w_2}{|w_2|^3} + \frac{w_3}{|w_3|^3}} \qquad \textbf{(2.71)}$$

This time we are looking for a special solution, not at the general case. So let we assume that the triangle is an equilateral one,

$$w2 = \mathbf{e}^{2\pi i/3} w1\,, w3 = e^{4\pi i/3} w1\,. \ \textbf{(2.72)}$$

Then a = 0, and the only equation we need to solve is

$$\mathbf{\ddot{w}_1 = -m\frac{w_1}{|w_1|^3}} \qquad \textbf{(2.73)}$$

This we know how to do.

To interpret the solution, solve for

$$mz1 = m3w2 - m2w3 \Rightarrow m^2|z_1|^2 = (m_2{}^2 + m2m3 + m_3{}^2)|w_1|^2$$
(2.74)

And so on. We can use this to show that

$$\ddot{z}_1 = -\frac{(m_2{}^2 + m_2 m_3 + m_3{}^2)^{3/2}}{m^3}\frac{z_1}{|z_1|^3} \qquad \textbf{(2.75)}$$

And similarly for the other two particles, hence the particles are all being

Accelerated towards their common center of mass, with “effective masses” that take an unexpected form. Each particle travels on an ellipse, but they do so in unison, in such a way that they always span an equilateral triangle,

A special case of this solution is of considerable physical interest. Let one of the particles have legibly small mass, Then the remaining pair trace out

The same orbits that they would follow in the absence of the third member.

Nevertheless the three particles span an equilateral triangle. This is the origin of the two Lagrange points on the orbit of a planet, where small bodies may sit, to draw this conclusion we should also investigate whether the exact solution Is stable under small perturbations, this turns out to be the case. The Lagrange points we have found are called L4 and L5, since there is Another set of three unstable equilibrium on the axis through the two bodies About a thousand asteroids have in fact been found close to the Lagrange Points L4 and L5 on the orbit of Jupiter, They are known as the Trojan Asteroids, it has been observed that the Earth’s Lagrange points are suitable Places where an alien civilization could place a satellite surveying the Earth;

However, when the STEREO spacecrafts passed through (in 2009) they found nothing of the sort, but what can we say about the three-body problem in general? To celebrate The sixtieth anniversary of King Oscar II of Sweden and Norway a large prize Was offered for a solution to the following problem: “For a system of arbitrarily Many mass points that attract each other according to Newton’s laws, Assuming that no two points ever collide, give the coordinates of the individual Points for all time as the sum of a uniformly convergent series whose terms Are made up of known functions” The prize was awarded to Herne Poincare, Who did not solve the problem but whose contribution laid the foundations of? The modern theory of possibly chaotic dynamical systems, for the three-body Problem a solution was in fact found by Karl F. Sandman in 1912. He did express a generic solution as a uniformly convergent power series in $t^{1/3}$. The Catch is that the series converges very slowly. It is estimated that, in order to get useful information, one would have to sum the first $10^{8000000}$ terms. Hence the interest in the exact general solution dwindled from that point in.

With the advent of the computer it has become possible to follow a large Number of solutions to the three body problem on the screen, with no special Effort, The zoo of solutions include ones where the third body escapes from The system, leaving the remaining pair more tightly bound than before.

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