

Chapter 1 Shift Operators

In this chapter are assumed to be Hilbert. We write $\mathcal{B}(\mathbf{X})$ for the set of bounded linear operators on \mathbf{X} and $\mathcal{B}(\mathbf{X}, \mathbf{X})$ for the set of bounded linear operators on \mathbf{X} to \mathbf{X} . Triangular brackets $\langle \cdot, \cdot \rangle$ denote an inner Product

Sec (1.1): Partial isometries and Wold Decomposition with inner operators Hilbert spaces:

Definition (1.1.1):

An operator S in $\mathcal{B}(\mathbf{X})$ is a shift operator if S is an isometric and $S^{*n} \rightarrow 0$ strongly, that is, $\|S^{*n} f\| \rightarrow 0$ for all f in \mathbf{X} .

It is convenient to present the general theory of shift operators as a chapter in pure theory. The central structure theorem is the World decomposition, which shows, in particular, that a shift operator is determined up to unitary equivalence by its multiplicity operators that commute with a shift operators play a special role in both theory and applications. A universal model for linear operators-on a Hilbert space the Beurling-Lax theorem which characterizes the invariant subspace of a shift operator the lifting theorem a concrete realization for an arbitrary shift operator.

At the same time, the study of shift operators should not be separated from the study of examples. The operator multiplication by z on $H^2(D)$, defined by $S : f(z) \rightarrow zf(z)$ for all $f(z)$ in $H^2(D)$ a shift operator with adjoint

$$S^* : f(z) \rightarrow z[f(z) - f(0)]Iz$$

For any Hilbert space ξ the operators $s : (c_0, c_1, c_2, \dots) \rightarrow (0, c_0, \dots)$ on $I\xi^2 = \xi \oplus \xi \oplus \dots$ is a shift operator. $I + s$ Ad joint is

$\zeta^* : (c_0, c_1, c_2, \dots) \rightarrow (c_1, c_2, c_3, \dots)$. These examples are sufficient for illustrating the results in the chapter.

Additional examples of shift operators are given in the Examples.

An operator W in $\mathcal{B}(\mathbf{X})$ is a Partial isometric if W is isometric on the orthogonal complement of its kernel. In this case we call $M = (kew)^\perp$ the initial space and $N = wM$ the final space of w operator version of the Would Decomposition.

Theorem (1.1.2):

Let $V \in \mathcal{B}(\xi)$ be an isometric then:

- (i) $P_o = 1 - VV^*$ is the projection of ξ on $\xi \ominus V\xi$;
- (ii) as $n \rightarrow \infty, V^n V^{*n}$ converges strongly to a projection operator P ;
- (iii) $P\xi = \bigcap_o^\infty V^{\bar{o}} \xi$;
- (iv) $\sum_o^\infty V_j P_o V^{*j}$ converges strongly to $Q : 1 - P$;
- (v) $Q\xi = \{g \in \xi ; \lim_{A \rightarrow \infty} \|V^{*n} g\| = o\}$;
- (vi) $Q\xi$ and $P\xi$ reduce V ;
- (vii) $V|_{P\xi}$ is unitary;
- (viii) $V|_{Q\xi}$ is a shift operator;
- (ix) $1 = P + \sum_o^\infty V_j P_o V^{*j}$

$$\xi = P\xi \oplus \sum_0^\infty \oplus V^j P_o \xi$$

Either version can be proved directly or deduced from the other. We prove the operator version only.

Proof:

The projection of ξ on $V\xi$ so $P_o = 1 - VV^*$ is the projection of ξ on $\xi \ominus V - \xi$ on $V^n \xi$. This proves $s(i)$.

For any $n, 1, 2, \dots, V^n V^{*n}$ is the projection of ξ on $V^n \xi$ hand $V^{n+1} V^{*n+1} = V^n (VV^*) V^{*n} \leq V^n V^{*n}$.

Therefore (ii) and (iii) follow. The identity.

$$\begin{aligned}\sum_0^n V_j P_o V^{*j} &= \sum_0^n V^j (1 - VV^*)^{*j} \\ &= 1 - V^{n+1} V^{*n+1}\end{aligned}$$

Implies

For any (ii)

Hence (v) holds. For any $n = 0, 1, 2, \dots$, $VV^n V^{*n} = V^{n+1} V$. Letting $n \rightarrow \infty$, we get $VP = PV$, so (VI) holds similarly, $VPV^* = P$ and (vii) follows.

Let $\delta = V|_{Q\xi} : \text{Then } \delta^{*n} = V^{*n}|_{Q\xi} \rightarrow 0$ strongly by (v). Thus δ is a shift operator and (viii) holds.

The first relation in (ix) follows from (IV). Arguing as in the proof of (me), we see that

$$V^j P_o V^{*j} = V^j V^{*j} - V^{j+1} V^{*j+1}$$

Is the projection of ξ on $V^j \xi \ominus V^{j+1} \xi = V^j P_o \xi$? Thus

$P\xi P_o \xi, VP_o \xi, V^2 P_o \xi, \dots$ are orthogonal subspaces of ξ with associated projections $P, P_o, VP_o V^*, V^2 P_o V^{*2}, \dots$. Hence the second relation in (ix) follows, and this completes the proof.

Corollary (1.1.3):

An isometry $V \in \mathcal{B}(\xi)$ is a shift operator if and only if $\bigcap_0^* VI\xi = \{0\}$. Specializing the Wold decomposition the case of a shift operator we obtain.

Corollary (1.1.4):

If $\delta \mathcal{B}(\mathbf{X})$ is a shift operator and $\mathbf{X} = \ker \delta^*$, then $\mathbf{X} = f = \sum_0^\infty \delta^{jk\delta}$,

Where $k_j \in \mathbf{X}, j \geq 0$. In this case $\|f\|^2 = \sum_0^\infty \|k_j\|^2$ and

$$K_j = P_o S^{*j} f, \quad j \geq 0;$$

Where $P_o = 1 - SS^*$ the projection of \mathbf{X} on \mathbf{X} .

When S is multiplication by Z on $H^2(D)$, $\mathbf{X} = \ker s^*$ is the set of constant function in $H^2(D)$. If we identify \mathbf{X} with C in the obvious way, then the expansion of any function in $H^2(D)$ takes the form

$$f(Z) = \sum_0^{\infty} a_j Z^j \text{ and coincides with the Taylor series representation.}$$

Corollary (1.1.5):

Let $S \in \mathcal{B}(\mathbf{X})$ be a shift operator. A subspace M of \mathbf{X} reduces S if and only if.

$$M = \sum_0^{\infty} \oplus S^j M_0;$$

Where M_0 is a subspace of? $\mathbf{X} = \ker S^*$

Proof:

If $\langle A_0^* A_0 K, K \rangle$ reduces S , then $S_m = S/M$ is a shift operator on M and $S_m^* = S^*|_M$. Let $M_0 = \ker S_m^*$. Then $M_0 \subseteq \ker S^* = \mathbf{X}$.

Let \mathbf{X} be a Hilbert space. A subspace \mathbf{E} of \mathbf{X} is called cyclic for an operator $A \in \mathcal{B}(\mathbf{X})$ if $V_o^\infty A^j \mathbf{E} = \mathbf{X}$.

Theorem (1.1.6):

If $S \in \mathcal{B}(\mathbf{X})$ is a shift operator, then $\mathbf{X} = \ker S^*$ is cyclic for S , and $\dim \mathbf{X} \leq \dim \mathbf{E}$ for every cyclic subspace \mathbf{E} for S .

Proof:

Corollary (1.1.4) implies that \mathbf{X} is cyclic for S . Let \mathbf{E} be any cyclic subspace for S . If P_0 is the Projection of \mathbf{X} on \mathbf{X} then $T = P_0|_{\mathbf{E}}$ is in $\mathcal{B}(\mathbf{E}, \mathbf{X})$ claim $T\mathbf{E} = \mathbf{X}$. To see this, consider any $k \in \mathbf{X} \ominus T\mathbf{E}$ for all $e \in \mathbf{E}$,

$$\langle e, k \rangle = \langle e, P_0 k \rangle = \langle P_0 e, k \rangle = 0$$

Because $k \perp TE = P_oE$. Since $\mathbf{X} = \mathbf{X} \oplus S\mathbf{X}$ we also have

$S^{je} \perp k, j = 1, 2, 3, \dots$. Thus $k \perp \delta^j E$ for all $j = 0, 1, 2, \dots$ and since E is cyclic $k = o$ Therefore $\overline{TE} = \mathbf{X}$.

Now $T^* \in \mathcal{B}(\mathbf{X}, E)$ and $\text{Ker } T^* = \mathbf{X} \ominus E$ Hence T^* is one-to one, and by lemma (1.1.7), $\dim \mathbf{X} \leq \dim E$.

We define the multiplicity of a shift operator $S \in \mathcal{B}(\mathbf{X})$ to be the minimum dimension of a cyclic subspace for S . By Theorem (1.1.6). The multiplicity of S is $\dim \mathbf{X}$, where $\mathbf{X} = \text{ker } S^*$. For any Hilbert space ξ the multiplicity of $S : (c_0, c_1, c_2, \dots) \rightarrow (0, c_0, c_1, \dots)$ on $I^2 \xi$ is to the dimension of ξ .

Lemma (1.1.7):

Let \mathbf{X}_1 and \mathbf{X}_2 be two Hilbert spaces. If there exists a one-to one operator $A \in \mathcal{B}(\mathbf{X}_1, \mathbf{X}_2)$ then $\dim \mathbf{X}_1 \leq \dim \mathbf{X}_2$

Proof:

Let $A = WB$ be the Polar decomposition of A , so that B is a nonnegative operator in $\mathcal{B}(\mathbf{X}_1)$ and w is a Partial isometry on \mathbf{X}_1 to \mathbf{X}_2 with initial space $\overline{B\mathbf{X}_1}$ and Final space $A\mathbf{X}_1$.

Since $\text{ker } B = \text{ker } A = \{0\}$, we have $(B\mathbf{X}_1)^\perp = \text{ker } B^* = \text{ker } B = \{0\}$ and $\overline{B\mathbf{X}_1} = \mathbf{X}_1$. Thus w is an isometry on \mathbf{X}_1 to \mathbf{X}_2 . If $\{e_j\}_{j \in J}$ is an orthogonal basis for \mathbf{X}_1 , then $\{we_j\}_{j \in J}$ is an orthonormal set in \mathbf{X}_2 . Hence $\dim \mathbf{X}_1 \leq \dim \mathbf{X}_2$.

Shift operators have the following remarkable property:

Up to unitary equivalence and multiplication constants, the classes of operators $T = \delta^* | M$, where S is a shift bounded and M is invariant subspace for δ^* include every bounded Linear on a Hilbert space.

Theorem (1.1.8):

Let T be a bounded linear operator on a Hilbert space \mathbf{X} such that $\|T\| \leq 1$ and $\|T^n f\| \rightarrow 0$ for each $f \in \mathbf{X}$. Let S be a shift operator on a Hilbert space ξ of multiplicity $\geq \dim((1-T^*T)\mathbf{X})^-$. Then there exists an invariant subspace M of S^* such that T is unitarily equivalent to $S^*|_M$.

If $T \in \mathcal{B}(\mathbf{X})$ and T does not satisfy the hypotheses of the theorem, then cT will satisfy the hypotheses for any scalar $c \neq 0$ such that $\|cT\| < 1$. In this case, it is necessary to choose a shift operator S whose multiplicity is \mathbf{X} .

Proof:

Let $\mathbf{X} = \ker S^*$. Our assumptions imply that

$$\dim\left(\left(1-T^*T\right)^{1/2}\mathbf{X}\right)^- = \dim\left(\left(1-T^*T\right)^{1/2}\mathbf{X}\right)^- \text{ Into}$$

\mathbf{X} Define $W : \mathbf{X} \rightarrow \xi$ by

$$Wf = \sum_0^{\infty} S^j J(1-T^*T)^{1/2} T^j f, f \in \mathbf{X}$$

By corollary (1.1.5) for any $f \in \mathbf{X}$

$$\begin{aligned} \|Wf\|^2 &= \sum_0^{\infty} \left\| J(1-T^*T)^{1/2} T^j f \right\|_{\xi}^2 \\ &= \sum_0^{\infty} \left\| 1-T^*T^{1/2} T^j f \right\|^2 \\ &= \lim_{n \rightarrow \infty} \sum_0^{\infty} \left\langle T^{*j} (1-T^*T) T^j f, f \right\rangle \\ &= \lim_{n \rightarrow \infty} \left(\|f\|^{1/2} \mathbf{X} - \|T^{n+1}\|^2 \mathbf{X} \right) = \|f\|^2 \mathbf{X} \end{aligned}$$

Hence W is an isometry on \mathbf{X} to ξ . Let $M = W\mathbf{X}$.

Then W is a Hilbert space isomorphism of \mathbf{X} onto M . For each

$$f \in \mathbf{X} \quad S^*Wf = \sum_0^{\infty} S^j J(1-T^*T)^{1/2} T^j (Tf) = WTf$$

It follows that M is invariant under S^* , and T is unitarily equivalent to $S^*|_M$.

Let $S \in \mathcal{B}(\mathbf{X})$ be a shift operator. An operator $A \in \mathcal{B}(\mathbf{X})$

- (i) is S analytic if $AS = SA$,
- (ii) S – inner if A is analytic and partially isometric, and
- (iii) S – Outer if A is analytic and $\overline{A\mathbf{X}}$ reduces S .

An analytic operator $A \in \mathcal{B}(\mathbf{X})$ is said to be Constant if is A^* also analytic. The terminology analytic, inner, and outer is also used when there is no possibility of confusion. To justify the terminology, consider the example where S is multiplication by z on $H^2(D)$.

Let $S \in \mathcal{B}(\mathbf{X})$ be a shift operator, and let \mathbf{X}^\perp

Theorem (1.1.9):

The initial space of any inner operator $B \in \mathcal{B}(\mathbf{X})$ reduces S

Proof:

The initial space of B is given by $M = \{f \in \mathbf{X} : \|Bf\| = \|f\|\}$. If $f \in \mathbf{X}$ and

$$\|Bf\| = \|f\|, \text{ then } \|BSf\| = \|SBf\| = \|f\| = \|Sf\|$$

Hence M is invariant under S . Since $M^\perp = \ker B$ and $BS = SB$, M^\perp is also invariant under S . Thus M reduces S .

We next describe all of the S constant inner operators on \mathbf{X} . To construct an example, choose a partial isometry $B_0 \in \mathcal{B}(\mathbf{X})$. By corollary (1.1.5),

each $f \in \mathbf{X}$ has the form $f = \sum_0^\infty S^j k_j$, where $\{K_j\}_0^\infty \subseteq \mathbf{X}$. Define an

operator $B \in \mathcal{B}(\mathbf{X})$ by setting

$$Bf = \sum_0^\infty S^j B_0 k_j$$

In this situation. It is easy to see that B is inner. Moreover,

$$B^* f = \sum_0^\infty S_j B_0^* k_j$$

If $f = \sum_0^\infty S^j k_j$ as above. Hence B^* is also inner, and B is Constant. This

example is general.

Theorem (1.1.10):

Every Constant inner operator $B \in \mathcal{B}(\mathbf{X})$ has the form just described for Some partial isometry $B_o \in \mathcal{B}(\mathbf{X})$.

Proof:

First note that \mathbf{X} reduces B . For since $B^*S = S^*B$, $B\mathbf{X} \subseteq \mathbf{X}$, and since $BS=SB$, $\mathbf{X}^\perp = S\mathbf{X}$ is also invariant under B . Therefore the projection P_o of \mathbf{X} on \mathbf{X} commutes with B , and hence P_o also commutes with B^* .

Theorem (1.1.11):

The final space of an inner operator $B \in \mathcal{B}(\mathbf{X})$ reduces S if and only if B is Constant.

Proof:

If B is Constant, then B^* is also inner. The final space for B is the initial space for B^* . Hence the sufficiency part follows from Theorem (1.1.9).

Conversely, suppose that the final space N of B reduces S . By BB^* Is the projection of \mathbf{X} on N ? Since N reduces S , $S(BB^*) = (BB^*)S$

.Therefore $B(SB^* - B^*S) = 0$ and $(SB^* - B^*S)\mathbf{X} \subseteq \ker B$.

Claim : $(SB^* - B^*S)\mathbf{X} \perp \ker B$. For if $u \in \ker B$, then $S^*u \in \ker B$ by Theorem (1.1.9). Hence for any

$$f \in \mathbf{X} \langle (SB^* - B^*S)f, u \rangle = \langle f, BS^*u \rangle - \langle Sf, Bu \rangle = 0.$$

The Claim follows. Then

$$(SB^* - B^*S)\mathbf{X} \subseteq \ker B \cap (\ker B)^\perp = \{0\},$$

So $SB^* = B^*S$. Thus B^* analytic and so B is S-constant.

Theorem (1.1.12):

Let $S \in \mathcal{B}(\mathbf{X})$ be a shift operator, and let $\mathbf{X} = \ker S^*$. If $T \in \mathcal{B}(\mathbf{X})$ then the following are equivalent:

- (i) $T=AA^*$ for some S-analytic operator $A \in \mathcal{B}(\mathbf{X})$;
- (ii) $T - STS^* = J^*J$ for some operator $J \in \mathcal{B}(\mathbf{X}, \mathbf{X})$;
- (iii) $T - STS^* \geq 0$ And the rank of $T - STS^*$ does not exceed the multiplicity of S .

The rank of an operator is the dimension of closure of its range.

Proof:

(i) \Leftrightarrow (ii) If $T = AA^*$ where A is S-analytic, then

$$T - STS^* = A(I - SS^*)A^* = AP_0A^* = J^*J ,$$

Where $P_0 = I - SS^*$ is the projection of \mathbf{X} on \mathbf{X} and

$J = P_0A^* \in \mathcal{B}(\mathbf{X}, \mathbf{X})$ conversely, where $J \in \mathcal{B}(\mathbf{X}, \mathbf{X})$? Repeated application of the equation

$T - STS^* = J^*J$ Yields.

$$T - S^{n+1}TS^{*n+1} = \sum_0^\infty S^j J^* J S^{*j}$$

$n = 0, 1, 2, \dots$.viewing J an operator on \mathbf{X} to \mathbf{X} ,we obtain

$$\begin{aligned} \langle Tf ,g \rangle - \langle S^{n+1}TS^{*n+1}f ,g \rangle &= \sum_0^n \langle JS^{*j}f ,JS^{*j}g \rangle \\ &= \left\langle \sum_0^n S^j JS^{*j}f , \sum_0^n S^j JS^{*j}g \right\rangle \end{aligned}$$

For all $f, g \in \mathbf{X}$ and $n = 0, 1, 2, \dots$ Define $A \in \mathcal{B}(\mathbf{X})$

So that $A^* = \sum_0^\infty S^j JS^{*j}$.It is easy to see that the series for A^* converges

strongly and A is S -analytic. Letting $n \rightarrow \infty$ in the preceding identity, we obtain

$$\langle Tf ,g \rangle = \langle A^*f ,A^*g \rangle \text{ For all } f, g \in \mathbf{X} ,\text{so } T = AA^* .$$

(ii) \Leftrightarrow (iii) Let $T - STS^* = J^*J$, where $J \in \mathcal{B}(\mathbf{X}, \mathbf{X})$.

Clearly $T - STS^* \geq 0$.Let $J = WB$ be the polar decomposition of J .Thus $B = (J^*J)^{1/2}$ and W is a Partial isometry on \mathbf{X} To \mathbf{X} With initial space

$\overline{B\mathbf{X}}$. The range of $T - STS^*$ is contained in $B\mathbf{X}$ since $T - STS^* = J^*J = B^2$. Since W maps $\overline{B\mathbf{X}}$ isometrically into \mathbf{X} , the rank of $T - STS^*$ does not exceed $\dim \mathbf{X}$, which is the multiplicity of S . Hence (ii) implies (iii).

Conversely, Let (iii) hold, and set $B = (T - STS^*)^{1/2}$. Since the range of B and the range of $B^2 = T - STS^*$ have the same closure $\dim \overline{B\mathbf{X}} \leq \dim \mathbf{X}$. Therefore there is an isometry W in $B(\overline{B\mathbf{X}}, \mathbf{X})$. Then $J = WB \in B(\mathbf{X}, \mathbf{X})$ and $T - STS^* = J^*J$; that is, (ii) holds.

Theorem (1.1.13):

Let $S \in \mathcal{B}(\mathbf{X})$ be a shift operator, let $\mathbf{X} = \ker S^*$, and let $P_0 = 1 - SS^*$ be the projection of \mathbf{X} on \mathbf{X} .

By the support of an S -analytic operator $A \in \mathcal{B}(\mathbf{X})$ we mean the smallest reducing subspace $M(A)$ for S containing $\overline{A^*\mathbf{X}} = (\ker A)^\perp$.

Equivalently, $M(A)$ is the smallest reducing subspace N for S such that $A|_N^\perp = 0$

Thus $M(A)$ reduces S , show that

$$M(A) = \sum_0^\infty \oplus S^j M_0(A);$$

Where $M_0(A) = \overline{P_0 A^* \mathbf{X}}$. Indeed, $M(A)$ contains

$$(I - SS^*) A^* \mathbf{X} = P_0 A^* \mathbf{X}, \text{ and}$$

$$M(A) \supseteq \sum_0^\infty \oplus S^j M_0(A).$$

The direct sum on the right reduces S and contains $A^*\mathbf{X}$. For if $f \in \mathbf{X}$ and $A^*f = \sum_0^\infty S^j k_j$

$$k_j = P_0 S^{*j} A^* f = P_0 A^* S^{*j} f \in M_0(A).$$

Lemma (1.1.14):

For any projection P on a separable Hilbert \mathbf{X} and any orthonormal basis $\{e_j\}_{j \in J}$ for \mathbf{X} ,

$$\dim P\mathbf{X} = \sum_{j \in J} \|Pe_j\|^2$$

Proof:

Let $\{f_k\}_{k \in K}$ be an orthonormal basis for $P\mathbf{X}$. Then

$$\begin{aligned} \sum_{j \in J} \|Pe_j\|^2 &= \sum_{j \in J} \sum_{k \in K} |\langle Pe_j, f_k \rangle|^2 \\ &= \sum_{k \in K} \sum_{j \in J} |\langle e_j, f_k \rangle|^2 \\ &= \sum_{k \in K} \|f_k\|^2, \\ &= \dim P\mathbf{X}. \end{aligned}$$

Sec (1-2): Beurling-Lax and lifting Theorems with concrete Realization of a shift operator:

Theorem (1.2.1):

Let S be a shift operator on a Hilbert space \mathfrak{X} . A subspace M of \mathfrak{X} is invariant under S if and only if $M = A\mathfrak{X}$ for some S -inner operator A on \mathfrak{X} .

This representation of an invariant subspace is essentially unique. Suppose that an invariant subspace M of S is represented as $M = A\mathfrak{X}$ and $M = C\mathfrak{X}$ for two S -inner operators A and C . Then $AA^* = CC^*$, so

$$C = AB \quad \text{And} \quad A = CB^*$$

Where B is an S -constant inner operator whose initial space the support is of C and whose final space is the support of A . Conversely,

$$A\mathfrak{X} = C\mathfrak{X}$$

Whenever A and C are S -inner operators related in this way.

Proof:

If $M = A\mathfrak{X}$, where A is S -inner, then $SM = SA\mathfrak{X} = AS\mathfrak{X} \subseteq A\mathfrak{X} = M$.

Conversely, assume that M is invariant under S . Let P be the projection of \mathfrak{X} on M . Then SPS^* is the projection of \mathfrak{X} on SM and

$Q = P - SPS^*$ is the projection of \mathfrak{X} on $M \ominus SM$.

We show that the dimension of $Q\mathfrak{X}$ does not exceed the multiplicity of S , that is, $\dim Q\mathfrak{X} \leq \dim \mathfrak{X}$, where $\mathfrak{X} = \ker S^*$. If \mathfrak{X} is infinite dimensional of any cardinality, then $\dim Q\mathfrak{X} \leq \dim \mathfrak{X} = \dim \mathfrak{X}$ let \mathfrak{X} be finite dimensional with orthonormal basis $\{e_k\}_{k \in K}$. Then $\{S^j e_k : k \in K, j = 0, 1, 2, \dots\}$ is an orthonormal basis for \mathfrak{X} . In this case \mathfrak{X} is separable,

$$\begin{aligned} \dim Q\mathfrak{X} &= \sum_{k \in K} \sum_{j=0}^{\infty} \langle QS^j e_k, S^j e_k \rangle \\ &= \lim_{n \rightarrow \infty} \sum_{k \in K} \sum_{j=0}^n \langle (P - SPS^*) S^j e_k, S^j e_k \rangle \\ &= \lim_{n \rightarrow \infty} \sum_{k \in K} \langle PS^n e_k, S^n e_k \rangle \\ &\leq \sum_{k \in K} \|e_k\|^2 = \dim \mathfrak{X} \end{aligned}$$

As required

It follows from what we have shown that P satisfies condition (iii) $P = AA^*$ for some S -analytic operator A . Since P is a projection, A is partially isometric and hence S -inner. By construction, $M = P\mathfrak{X} = A\mathfrak{X}$, and this completes the proof.

The commutant of an operator $T \in \mathcal{B}(\mathfrak{X})$ is the set $C(T)$ of all $X \in \mathcal{B}(\mathfrak{X})$ such that $XT = TX$. More generally, if $T_1 \in \mathcal{B}(\mathfrak{X}_1)$ any $T_2 \in \mathcal{B}(\mathfrak{X}_2)$, let $C(T_1, T_2)$ be the set of all $X \in \mathcal{B}(\mathfrak{X}_1, \mathfrak{X}_2)$ such that $XT_1 = T_2X$. The lifting characterizes $C(T_1, T_2)$ when T_1 and T_2 is represented as in the universal model.

Theorem (1.2.2):

Let M be an invariant sub space for S^* , and let P be can the projection of \mathfrak{X} on M . Let M' be an invariant sub space for R^* . Let $X \in \mathcal{B}(\mathfrak{X}, \mathfrak{X})$ satisfy

(i) $XM' \subseteq M$ and $XM'^{\perp} = \{0\}$,

(ii) $PSX = XR$,

(iii) $X^*X \leq T$

Then there exists an operator $Y \in \mathcal{B}(\mathfrak{X}, \mathfrak{X})$ such that

(i') $X = PY$,

(ii') $\delta Y = YR$,

(iii') $Y^*Y \leq T$.

Proof:

Let $Q = I - P$. Then $SQ\mathfrak{X} \subseteq Q\mathfrak{X}$, and $S|_{Q\mathfrak{X}}$ is an isometry. Let P_0 be the projection of \mathfrak{X} on $\text{Ker}((S|_{Q\mathfrak{X}})^*) = Q\mathfrak{X} \ominus SQ\mathfrak{X}$. Thus

$P_0 = Q - SQS^*$, and for $j, k = 0, 1, 2, \dots$,

$$P_0 S^{*k} S^j P_0 = \begin{cases} 0 & \text{if } j \neq k, \\ P_0 & \text{if } j = k, \end{cases}$$

And

$$P_0 S^{*j} X = 0 \quad \text{And} \quad X^* P_0 S^j P_0 = 0$$

We inductively construct sequences $\{B_n\}_{-1}^{\infty}$ and $\{Y_n\}_{-1}^{\infty} \subseteq \mathcal{B}(\mathfrak{X}, \mathfrak{X})$ such that

$$Y_{-1} = X,$$

$$Y_n = X + \sum_0^n S^j B_j, \quad n = 0, 1, 2, \dots,$$

We require for $n = -1, 0, 1, 2, \dots$, that

$$\begin{aligned} \alpha(n): & B_n \in \mathcal{B}(\mathfrak{X}, P_0 \mathfrak{X}), \\ \beta(n): & Y_n^* Y_n - R^* Y_n^* Y_n R = B_n^* B_n, \\ \gamma(n): & Y_n^* Y_n \leq T \leq R^* T R, \end{aligned}$$

And for $n = 0, 1, 2, \dots$ that

$$\delta(n): B_{n-1} = B_n R.$$

Let $B_{-1} = Q S X$. Since $Q S^* Q S X = Q S^* (1 - P) S X = Q X - Q S^* X R = 0 - 0 = 0$,

$$P_0 B_{-1} = (Q - S Q S^*) Q S X = Q S X = B_{-1}$$

Thus $\alpha(-1)$ holds. Also,

$$X^* X - R^* X^* X R = X^* X - X^* S^* P S X = X^* S^* Q S X = B_{-1}^* B_{-1},$$

So $\beta(-1)$ holds. The two inequalities in $\gamma(-1)$ hold by assumption.

Suppose that B_{-1}, B_0, \dots, B_n , have been constructed for some $n \geq -1$. Then

$$B_n^* B_n = Y_n^* Y_n - R^* Y_n^* Y_n R \leq R^* (T - Y_n^* Y_n) R.$$

By the lemma there exists an operator $C_{n+1} \in \mathcal{B}(\mathfrak{X}, P_0 \mathfrak{X})$ such that

$$\|C_{n+1}\| \leq 1 \text{ and}$$

$$B_n = C_{n+1} (T - Y_n^* Y_n)^{1/2} R.$$

Let

$$B_{n+1} = C_{n+1} (T - Y_n^* Y_n)^{\frac{1}{2}}$$

Clearly $\alpha(n+1)$ and $\delta(n+1)$ hold. Hence.

$$B_{n+1} = P_0 B_{n+1},$$

$$B_{n+1}^* \delta^{*n+1} Y_n = B_{n+1}^* P_0 \delta^{*n+1} \left(X + \sum_0^n \delta^j B \right) = 0$$

Therefore

$$Y_{n+1}^* Y_{n+1} - R^* Y_{n+1}^* Y_{n+1} R$$

$$= (Y_n^* + B_{n+1}^* S^{*n+1})(Y_n + S^{n+1} B_{n+1}) - R^* (Y_n^* + B_{n+1}^* S^{*n+1})(Y_n + S^{n+1} B_{n+1}) R$$

$$\begin{aligned}
&= Y_n^* Y_n + B_{n+1}^* B_{n+1} \\
&\quad - R^* (Y_n^* Y_n + B_{n+1}^* B_{n+1}) R \\
&= (Y_n^* Y_n - R^* Y_n^* Y_n R) \\
&\quad + (B_{n+1}^* B_{n+1} - R^* B_{n+1}^* B_{n+1} R) \\
&= B_n^* B_n + (B_{n+1}^* B_{n+1} \\
&\quad - B_n^* B_n) \\
&= B_{n+1}^* B_{n+1},
\end{aligned}$$

So $B_{(n+1)}$ holds. Similarly,

$$\begin{aligned}
Y_{n+1}^* Y_{n+1} &= Y_n^* Y_n + (T - Y_n^* Y_n)^{\frac{1}{2}} C_{n+1}^* C_{n+1} (T - Y_n^* Y_n)^{\frac{1}{2}} \\
&\leq Y_n^* Y_n + (T - Y_n^* Y_n) \\
&= T,
\end{aligned}$$

And $Y(n+1)$ follows. This completes the inductive construction.

It follows that $\{Y_n\}_0^\infty$ converges strongly to an operator $Y \in \mathcal{B}(\mathfrak{X}, \mathfrak{X})$. Thus

$$Y = X + \sum_0^\infty S^j B_j$$

Where the series converges strongly. The assertions (i') (and (iii')) are immediate. For each $1, 2, \dots$,

$$\begin{aligned}
Y_n R &= X R + \sum_0^\infty S^j B_j R \\
&= X R + B_{-1} + \sum_0^{n-1} S^{j+1} B_j \\
&= P S X + Q S X + \sum_0^{n-1} S^{j+1} B_j \\
&= S X + \sum_0^{n-1} S^{j+1} B_j = S Y_{n-1}
\end{aligned}$$

Thus (ii') holds and the results follow.

Lemma (1.2.3):

Let $A \in \mathcal{B}(\mathfrak{X}_1, \mathfrak{X}_3)$, $C \in \mathcal{B}(\mathfrak{X}_1, \mathfrak{X}_2)$, and $\beta > 0$ be given. The following are equivalent:

- (i) $A = BC$ for some $B \in \mathcal{B}(\mathfrak{X}_1, \mathfrak{X}_3)$ such that $\|B\| \leq \beta$;
- (ii) $A^*A \leq \beta^2 C^*C$.

Proof:

Assume (ii). For each $f \in \mathfrak{X}_1$,

$$\|Af\|_3^2 = \langle A^*Af, f \rangle_1 \leq \beta^2 \langle C^*Cf, f \rangle_1 = \beta^2 \|Cf\|_2^2$$

Hence we may define $B_0: \overline{C\mathfrak{X}_1} \rightarrow \mathfrak{X}_3$ by $B_0(Cf) = Af$, $f \in \mathfrak{X}_1$. we have $\|B_0\| \leq \beta$. Extend B_0 to an operator $B \in \mathcal{B}(\mathfrak{X}_2, \mathfrak{X}_3)$ such that B is zero on $\mathfrak{X}_2 \ominus C\mathfrak{X}_1$. Then $A = BC$ and $\|B\| \leq \beta$; that is, (i) follows.

Let ℓ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\ell$ and norm $\|\cdot\|_\ell$. The norm on $\mathcal{B}(\ell)$ is denoted $\|\cdot\|_{\mathcal{B}(\ell)}$.

Definition (1.2.4):

By $H_\ell^2(D)$ we mean the space of all ℓ -valued holomorphic function on D for which the quantity

$$\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_\ell^2 d\theta = \sum_0^\infty |a_j|_\ell^2 r^{2j}$$

Remains bounded for $0 \leq r < 1$.

For the rudiments of the theory of vector and operator valued holomorphic functions. It is easy to see that $H_\ell^2(D)$ is a Hilbert space with inner product

$$\langle f, g \rangle_2 = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta}) \rangle_\ell d\theta = \sum_0^\infty \langle a_j, b_j \rangle_\ell$$

For any $f(z) = \sum_0^\infty a_j z^j$ and $g(z) = \sum_0^\infty b_j z^j \ell$ in the space. Thus $H_\ell^2(D)$ is isomorphic with $L^2 \ell$ the correspondence between a function and its Taylor coefficients. As a consequence of this isomorphism, we obtain:

Theorem (1.2.5):

The operator multiplication by Z on $H_\ell^2(D)$, defined by $\delta : f(z) \rightarrow zf(z)$: for all $f(z)$ in $H_\ell^2(D)$, is a shift operator of multiplicity $\dim \ell$. The adjoint of S is $S^* = f(z) \rightarrow [f(z) - f(0)]/z$.

Corollary (1.2.6):

Every shift operator on a Hilbert space is unitarily equivalent to multiplication by z on $H_\ell^2(D)$ for some choice of ℓ .

By $H_{\mathcal{B}(\ell)}^\infty$ we mean the Banach algebra of bounded $\mathcal{B}(\ell)$ -valued holomorphic function A on D in the norm $\|A\|_\infty = \sup\{|A(z)|\}_{\mathcal{B}(\ell)}$. Each $A \in H_{\mathcal{B}(\ell)}^\infty(D)$ induces an operator $T(A)$ on $H_\ell^2(D)$ called multiplication by A defined by

$$T(A) : f \rightarrow Af, \quad f \in H_\ell^2(D).$$

Theorem (1.2.7):

Let S be multiplication by z on $H_\ell^2(D)$, A bounded linear operator T on $H_\ell^2(D)$ is S -analytic if and only if $T = T(A)$ for some $A \in H_{\mathcal{B}(\ell)}^\infty(D)$. In this case, $\|T\| = \|A\|_\infty$, and T is S -constant if and only if $A \equiv \text{const}$.

Proof:

If $T = T(A)$ where $A \in H_{\mathcal{B}(\ell)}^\infty(D)$, then it is clear that T is S -analytic and $\|T\| \leq \|A\|_\infty$.

Conversely, assume that T is S -analytic. We may view any $c \in \ell$ a constant function in $H_\ell^2(D)$. If $T : c \rightarrow f_c$, Then for any $w \in D$ the mapping $A(w) : c \rightarrow f_c(w)$ on ℓ to ℓ belongs to $\mathcal{B}(\ell)$ as a function of z , $A(z)$ is holomorphic on D . By construction, $T : c \rightarrow A(z)c$ for all $c \in \ell$. Since $TS = ST$,

$$T : cz^j \rightarrow z^j A(z)c$$

For all $c \in \ell$ and $j = 0, 1, 2, \dots$ every $f(z)$ in $H_\ell^2(D)$ has a representation $f(z) = \sum_0^\infty a_j z^j$ that converges both point wise on D and in the metric of $H_\ell^2(D)$. Since T is continuous

$$\begin{aligned} (Tf)(z) &= \sum_0^\infty T\{a_j z^j\} = \sum_0^\infty z^j A(z) a_j \\ &= A(z) f(z). \end{aligned}$$

$$T^* : c/(1 - \bar{w}z) \rightarrow A(w)^* c/(1 - \bar{w}z)$$

For each $c \in \ell$ and $w \in D$. Hence

$$\begin{aligned} \left| A(w)^* c \right|_\ell / (1 - |w|^2) &= \left\| A(w)^* c / (1 - \bar{w}z) \right\|_2^2 \\ &\leq \|T^*\|^2 \left\| C / (1 - \bar{w}z) \right\|_2^2 \\ &= \|T\|^2 |C|_\ell^2 / (1 - |w|^2). \end{aligned}$$

Thus $A \in H_{\mathcal{B}(\ell)}^\infty(D)$ And $\|A\|_\infty \leq \|T\|$. By $T = T(A)$.

Hence $\|T\| \leq \|A\|_\infty$, and so $\|T\| = \|A\|_\infty$.

Suppose $T = T(A)$ is S-constant. Then $T^* = T(C)$ for some

$C \in H_{\mathcal{B}(\ell)}^\infty(D)$ by what we just proved. $C(z) = A(w)^*$ For all $z, w \in D$. Hence $A(z) \equiv \text{const}$. Or D , conversely, it is clear that if $A(z) \equiv \text{const}$, on D , then $T = T(A)$ is S-constant.

Lemma (1.2.8):

For all $f(z)$ in $H_\ell^2(D)$, $c \in \ell$, and $w \in D$,

$$\left\langle f(z), c / (1 - \bar{w}z) \right\rangle_2 = \left\langle f(w), c \right\rangle_\ell.$$

Proof:

Compute the left and right sides of the identity in terms of the Taylor explosions of $f(z)$ and $c/(1 - \bar{w}z)$.

1. The only reducing sub spaces of a shift operator $S \in \mathcal{B}(\mathfrak{X})$ of multiplicity 1 are $\{0\}$ and \mathfrak{X} .

2. Operators $A \in \mathcal{B}(\mathbf{X})$ and $B \in \mathcal{B}(\mathfrak{X})$ are called similar if $A = X^{-1}BX$ some invertible operator $X \in \mathcal{B}(\mathfrak{X}, \mathfrak{X})$. Two isometries that are similar are unitarily equivalent

3. (I) If M is an invariant sub space of a shift operators S , then $S|M$ is a shift operator of multiplicity not greater than the multiplicity of S .

(ii) Let $S \in \mathcal{B}(\mathfrak{X})$ be a shift operator. If N is a subspace of \mathbf{X} such that $S^j N \perp S^k N$ whenever $j \neq k, j, k = 0, 1, 2, \dots$ then the dimension of N does not exceed the multiplicity of S .

4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ viewed as an operator on C^2 . Then

$$\|A\| = \frac{1}{2}N + \frac{1}{2}\left(N^2 - 4|D|^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

Where $N = |a|^2 + |b|^2 + |c|^2 + |d|^2$ and $D = ad - bc$.

In particular, $\|A\| \leq 1$ if and only if $N \leq 1 + |D|^2$.

5. Inequality of Neumann and the invariant form of schraos lemma.

(I) prove Non Neumann's inequality: if $T \in \mathcal{B}(\mathbf{X})$ any $\|T\| \leq 1$, then

$\|p(T)\| \leq 1$ for every polynomial $p(z)$ such that $|p(z)| \leq 1$ for $|z| \leq 1$.

(ii) Let $T = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ on $\mathfrak{X} = C^2$, where $a, c \in D$ and

$$|b|^2 = (1 - |a|^2)(1 - |c|^2).$$

Then $\|T\| = 1$ and for any polynomial $p(z)$,

$$p(T) = \begin{bmatrix} p(a) & \frac{b[p(a)-p(c)]}{a-c} \\ 0 & p(c) \end{bmatrix}$$

Hence if $|P(z)| \leq 1$ for $|z| \leq 1$, then $\|p(T)\| \leq 1$ and so

$$\left| \frac{p(z) - p(w)}{z - w} \right|^2 \leq \frac{1 - |p(z)|^2}{1 - |z|^2} \frac{1 - |p(w)|^2}{1 - |w|^2}, \quad z, w \in D.$$

(iii) Let $f(z)$ be holomorphic and satisfy $|f(z)| < 1$ on D . Use (ii) and an approximation argument to show that

$$\left| \frac{f(z) - f(w)}{z - w} \right|^2 \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \frac{1 - |f(w)|^2}{1 - |w|^2}, z, w \in D$$

Then use the identity $|1 - u\bar{v}|^2 L^2 = (1 - |u|^2)(1 - |v|^2) + |u - v|^2$ to deduce

$$\left| \frac{f(z) - f(w)}{1 - \bar{f}(w)f(z)} \right| \leq \left| \frac{z - w}{1 - \bar{w}z} \right|, z, w \in D$$

Similar results are given by Williams

A connection between von Neumann's inequality and the Pick-Nevanlinna theorem is shown in Rovnyak

Has shown that von Neumann's inequality is false in general for every Banach space

That is not a Hilbert space. For the classical view of Schwarz's lemma.

6. Laguerre shift. The laguerre polynomials of order 0 can be defined by either of the relations

$$e^{-t} L_n(t) = \frac{1}{n!} \left(\frac{d}{dt} \right)^n (t^n e^{-t}), \quad n = 0, 1, 2, \dots,$$

$$(1 - z)^{-1} e^{-tz/(1-z)} = \sum_0^\infty L_n(t) z^n, \quad |z| < 1.$$

For each

$$\int_0^x L_n(t) dt = L_n(x) - L_{n+1}(x),$$

$$\int_0^\infty e^{-st} e^{-\frac{1}{2}t} L_n(t) dt = \left(\delta - \frac{1}{2} \right)^n / \left(\delta + \frac{1}{2} \right)^{n+1}, \text{Res} > -\frac{1}{2}$$

The function $\left\{ e^{-\frac{1}{2}t} L_n(t) \right\}_0^\infty$ form an orthonormal basis for $L^2(0, \infty)$.

Theorem (1.2.9):

(i) Let be the shift operator on $L^2(0, \infty)$ such that

$$S : e^{-\frac{1}{2}t} L_n(t) \rightarrow e^{-\frac{1}{2}t} L_{n+1}(t), n \geq 0$$

Then for each $f \in L^2(0, \infty)$,

$$S: f(x) \rightarrow f(x) - \int_0^x e^{-\frac{1}{2}(x-t)} f(t) dt.$$

(ii) Let T be the symmetric operator id/dx on $L^2(0, \infty)$, where the domain of T is taken as the set of (locally) absolutely continuous functions f on $(0, \infty)$ such that $f, f' \in L^2(0, \infty)$ and $f(x) \rightarrow 0$ as $x \downarrow 0$.

Then. $S = (T - \frac{1}{2}iI)(T - \frac{1}{2}iI)^{-1}$

We call S the leaguered shift on $L^2(0, \infty)$.

Proof:

(i) By holds if $f(b) = \bar{e}^{\frac{1}{2}L_n(t)}$ for some $n \geq 0$. The general case of follows by linearty and approximation.

(ii) By the elementary theory of symmetric operators, δ are the clayey Trans for of the symmetric operators to with graph.

$$\mathcal{G}(T_0) = \left\{ \left(f - Sf, \frac{1}{2}i(f + Sf) \right) : f \in L^2(0, \infty) \right\}.$$

Thus $(P, q) \in \mathbf{KG}(T_0)$ if and only if

$$p(x) = \bar{e}^{\frac{1}{2}x} \int_0^x e^{\frac{1}{2}t} f(t) dt$$

$$q(x) = if(x) - \frac{1}{2}ie^{-\frac{1}{2}x} \int_0^x e^{\frac{1}{2}t} f(t) dt$$

For some $f \in L^2(0, \infty)$. A straight forward argument then shows that $\mathbf{G}(T_0)$ coincides with the graph of T , and (ii) follows.

7. Shift operators and the Chebychev polynomials. The Chebychev polynomials $\{T_n(x)\}_0^\infty$ and $\{Un(x)\}_0^\infty$ can be defined by formal expansions

$$\frac{1-xt}{1-2xt+t^2} = \sum_0^\infty T_n(x)t^n, \frac{1}{1-2xt+t^2} = \sum_0^\infty U_n(x)t^n.$$

For each $n \geq 0$, $T_n(\cos \theta) = \cos n\theta$, $U_n(\cos \theta) = \sin((n+1)\theta) / \sin \theta$.

Theorem (1.2.10):

Let $S \in \mathcal{B}(\mathfrak{X})$ be a shift operator. Write $S = X + iY$, where $X = \operatorname{Re} S$, $Y = \operatorname{Im} S$, and let P_0 be the projection on $\mathfrak{X} = \ker S^*$. Then $S^n P_0 = U_n(X) P_0$, $i Y S^n P_0 = T_{n+1}(X)$

For all $n \geq 0$, and

$$1 = \sum_0^\infty U_n(X) P_0 U_n(X)$$

With convergence in the strong operator topology.

Proof:

By induction using the identities

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x), n \geq 0, \text{ and}$$

$$T_{n+1}(x) = \frac{1}{2} [U_{n+1}(x) - U_{n-1}(x)], n \geq 1. \text{ Then}$$

Follows by the world decomposition.

Theorem (1.2.11):

There is unique shift operator S_0 on $L^2(-1, 1)$ such that $S_0: (1-x^2)^{1/4} U_n(x) \rightarrow (1-x^2)^{1/4} U_{n+1}(x)$, $N \geq 0$.

For each $f \in L^2(-1, 1)$,

$$S_0: f(x) \rightarrow xf(x) - PV \int_{-1}^1 \frac{(1-x^2)^{1/4} (1-t^2)^{1/4}}{t-x} f(t) dt.$$

The real part of S_0 , $X_0 = \operatorname{Re} S_0$, is multiplication by x on $L^2(-1, 1)$, that is

$$X_0: f(x) \rightarrow xf(x).$$

Proof:

The existence of S_0 follows from the fact that the functions

$$\left\{ (2/\pi)^{1/2} (1-x^2)^{1/4} U_n(x) \right\}_0^\infty \text{ form an orthonormal basis for } L^2(-1, 1).$$

First check on basis elements using the identity

$$PV \int_{-1}^1 \frac{(1-t^2)^{1/2} U_n(t)}{t-x} dt = -T_{n+1}(x), \quad |x| < 1$$

Theorem (1.2.12):

The general form of a shift operator S on $L^2(-1,1)$ whose real part $X = \text{Re } S$ coincides with the real part $X_0 = \text{Re } S_0$ of the operator is

$$S_0 : f(x) \rightarrow x f(x) - PV \frac{1}{\pi} \int_{-1}^1 \frac{(1-x^2)^{\frac{1}{4}}(1-t^2)^{\frac{1}{4}}}{t-x} C(x)C(t) f(t) dt,$$

Where $C(x)$ is a measurable function such that

$$|C(x)| = 1 \text{ a.e. on } (-1,1).$$

8. The functional equation $g(x) - g(2x) = f(x)$.

We follow Rochberg Let \mathfrak{X}_0 be the Hilbert space of measurable complex valued functions $f(x)$ on $(-\infty, \infty)$ such that $f(x+1) = f(x)$ a.e.

$$\|f\|^2 = \int_0^1 |f(x)|^2 dx < \infty, \text{ and } \int_0^1 f(x) dx = 0.$$

(i) The operator $S_0 : f(x) \rightarrow f(2x)$ on \mathfrak{X}_0 is a shift operator with ad joint

$$S_0^* : f(x) \rightarrow \frac{1}{2} f\left(\frac{1}{2}x\right) + \frac{1}{2} f\left(\frac{1}{2}x + \frac{1}{2}\right). \text{ If } \mathfrak{X}_0 = \ker S_0^*, \text{ then for each } n =$$

$0, 1, 2, \dots,$

$$S^n \mathfrak{X}_0 = V\{\exp(2\pi i(2j+1)2^n x) : j = 0, \pm 1, \pm 2, \dots\}.$$

(ii) Let $S \in \mathcal{B}(\mathfrak{X})$ be any shift operator, and let f be a vector in \mathfrak{X} for which the coefficients in the expansion $f = \sum_0^\infty S^j k_j$ of satisfy

$$\|k_j\| \leq M r^j, \quad j \geq 0,$$

For some constant $r \in (0,1)$ and $M \in (0, \infty)$. then the equation $g - Sg = f$

has a solution $g \in \mathfrak{X}$ if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_0^n S^j f \right\|^2 = 0.$$

(iii) Call a function $f(x)$ in \mathfrak{X}_0 smooth if the coefficient is in the expansion

$$f(X) = \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} a_j e^{2\pi i j x} \text{ Satisfy } \sum_{j=-\infty}^{\infty} \left| a_{(2j+1)2^n} \right|^2 \leq M r^n, n \geq 0, \text{ for}$$

some constants $r \in (0,1)$ and $M \in (0, \infty)$. For $f(x)$ to be smooth, it is sufficient that satisfy a Holder condition of order $> \frac{1}{2}$

Theorem (1.2.13):

If $f(x)$ is a smooth function in \mathbf{X}_0 , then a necessary and sufficient condition for the existence of a $g(x)$ in \mathbf{X}_0 such that

$$g(x) - g(2x) = f(x) \text{ a.e. on } (-\infty, \infty)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \left| \sum_0^n f(2^j x) \right|^2 dx = 0$$

When a solution $g(x)$, it is unique and also smooth.

9. A shift operator $S \in \mathcal{B}(\mathfrak{X})$ of multiplicity 1 has no square root in $\mathcal{B}(\mathfrak{X})$

10. If C_1 is defined on $L^2(0,1)$ by

$$(C_1 f)(x) = x^{-1} \int_0^x f(t) dt, \quad 0 < x < 1,$$

Then $1 - C_1^*$ is a shift operator of multiplicity 1.

Chapter 2
Pick-Nevanlinna and Loewner types with Interpolation

For $\Omega = D$ or II , let $B(\Omega)$ be the set of all function $W(Z)$ that are holomorphic and bounded by 1 on Ω we are concerned with interpolation theorems for $B(\Omega)$ that is characterizations of functions in $B(\Omega)$ in terms of data on subset of Ω or $\partial\Omega$ when the data are prescribed in Ω two classical theorems serve as prototypes: the pick-Nevanlinna theorem. And the Caratheodory-Fejer theorem the prototypes for the situation in which data are prescribed on $\partial\Omega$ is Loewner's theorem we use an operator method based on the lifting. We also ketch the theory of monotone operator functions.

Sec (2-1): Generalization of the pick- Nevanlinna and Caretheodory-Fejer Theorems Restrictions Boundary Functions Pick class:

For any complex vector space r , let r' be the space fall linear functional on r , and let $\xi(r)$ is the space of all linear operators on r to r' . The value of functional $x' \in r'$ on a vector $x \in r$ is written (x, x') . Each $A \in \xi(r)$ induces an operator $A' \in \xi(r')$ such that.

$$(Ax, x') = (x, A'x')$$

For all $x \in r$ and $x' \in r'$. We use no topology on r , so questions concerning continuity do not arise.

Theorem (2.1.1):

Let $A \in \xi(r)$ and $b, C \in r$ be given let $\mathcal{D} \subseteq r'$ be linear sub space such that $A'\mathcal{D} \subseteq \mathcal{D}$ and

$$\sum_0^{\infty} |(A^j c, x')|^2 < \infty$$

For every $x' \in \mathcal{D}$. The following are equivalent:

- (i) There exist $w(z) = \sum_0^{\infty} w_j z^j$ in $B(D)$ such that

$$(b, x') = \sum_0^{\infty} w_j (A^j c, x'), \quad x' \in \mathcal{D}$$
- (ii) For al $x' \in \mathcal{D}$

$$\sum_0^{\infty} \left| (A^j b, x') \right|^2 \leq \sum_0^{\infty} \left| (A^j c, x') \right|^2 .$$

Proof:

Think of $H^2(D)$ as the space of power series $\sum_0^{\infty} a_j Z^j$ with square summable coefficient. Let S be the shift operator.

$$S: f(z) \rightarrow zf(z)$$

On $H^2(D)$.

Assume (ii). We apply the lifting theorem with $\xi_j = H^2(D)$ and

$S_j = S, j = 1, 2$. for each $x' \in \mathcal{D}$, $\sum_0^{\infty} (A^j c, x') Z^j$, is in $H^2(D)$, and

$$S^* \left\{ \sum_0^{\infty} (A^j c, x') z^j \right\} = \sum_0^{\infty} (A^{j+1} c, x') z^j = \sum_0^{\infty} (A^j c, A' x') z^j .$$

Let \mathbf{X}_1 be the closure in $H^2(D)$ of all series $\sum_0^{\infty} (A^j c, x^1) Z^j$, where

$x^1 \in \mathcal{D}$. Since $A' \mathcal{D} \subseteq \mathcal{D}$, it follows that \mathbf{X}_1 is invariant under

$\delta \mathbf{X}_2 = H^2(D)$. Let $T_j = S^*|_{\mathbf{X}_j}, j = 1, 2$, by (ii) there is a unique operator $X \in \mathcal{B}(\mathbf{X}_1, \mathbf{X}_2)$ such that $\|X\| \leq 1$ and for each $x' \in \mathcal{D}$,

$$X \left\{ \sum_0^{\infty} (A^j c, x') z^j \right\} = \sum_0^{\infty} (A^j b, x') z^j .$$

For each $x' \in \mathcal{D}$,

$$\begin{aligned} XS^* \left\{ \sum_0^{\infty} (A^j c, x') z^j \right\} &= \sum_0^{\infty} (A^j b, x') z^j \\ S^* X &= \left\{ \sum_0^{\infty} (A^j c, x') z^j \right\} . \end{aligned}$$

Thus $XT_1 = T_2 X$, and the hypotheses of the lifting are satisfied. By the

lifting theorem, $X = Y|_{\mathbf{X}_1}$ for some operator on $H^2(D)$ such that

$YS^* = S^*Y$ and $\|Y\| = \|X\| \leq 1$. we obtain

$$Y^*f = \tilde{w}f, \quad f \in H^2(D),$$

For some function $\bar{w}(z) = \sum_0^\infty \bar{w}_j Z^j$ in $B(D)$. For any $x' \in \mathcal{D}$

$$\begin{aligned} (b, x') &= \left(\sum_0^\infty (A^j c, x') z^j, 1 \right)_2 \\ &= \left(X \left\{ \sum_0^\infty (A^j c, x') z^j \right\}, 1 \right)_2 \\ &= \left(\sum_0^\infty (A^j c, x') z^j, Y^* \{1\} \right)_2 \\ &= \left(\sum_0^\infty (A^j c, x') z^j, \sum_0^\infty \bar{w}_j z^j \right)_2 \\ &= \sum_0^\infty (A^j c, x') w_j. \end{aligned}$$

Thus $w(Z) = \sum_0^\infty w_j Z^j$ is in $B(D)$ and (i) holds.

Conversely, assume (i). Let Y be the operator on $H^2(D)$ such that Y^* is multiplication by $\bar{w}(Z) = \sum_0^\infty \bar{w}_j Z^j$. Then $\|Y\| \leq 1$. For each $x' \in \mathcal{D}$ and $k = 0, 1, 2, \dots$,

$$(A^k b, x') = (b, A'^k x') = \sum_0^\infty (A^j c, A'^k x') w_j = \sum_0^\infty (A^{j+k} c, x') w_j$$

Hence if $Y \left\{ \sum_0^\infty (A^j c, x') Z^j \right\} = \sum_0^\infty g_k Z^k$, Then

$$\begin{aligned} g_k &= \left(Y \left\{ \sum_0^\infty (A^j c, x') z^j \right\}, S^k \{1\} \right)_2 \\ &= \left(\sum_0^\infty (A^j c, x') z^j, Y^* S^k \{1\} \right)_2 \end{aligned}$$

$$\begin{aligned}
A \in \xi(r) &= \left(\sum_0^\infty (A^j c, x') Z^j, \sum_0^\infty \bar{w}_j Z^{j+k} \right)_2 \\
&= \sum_0^\infty (A^{1+k} c, x') w_1 \\
&= (A^k b, x')
\end{aligned}$$

We have shown that for all $x' \in \mathcal{D}$

$$Y \left\{ \sum_0^\infty (A^j c, x') Z^j \right\} = \sum_0^\infty (A^j b, x') Z^j$$

Since $\|Y\| \leq 1$.

Theorem (2.1.2):

Let $r = C^{card J}$ be the space of all indexed sets $x = \{x_j\}_{j \in J}$ in C with coordinate wise addition and scalar multiplication. Define and $b, c \in r$ by

$$A: \{x_j\}_{j \in J} \rightarrow \{z_j x_j\}_{j \in J},$$

$b = \{w_j\}_{j \in J}$, And $c = \{c_j\}_{j \in J}$ where $C_j = 1, j \in J$. Let \mathcal{D} be the set all linear functional x' one r of the form

where $\{a_j\}_{j \in J} \subseteq C$ and $\{j: a_j \neq 0\}$ Is finite.

Theorem (2.1.3):

$$T(a_0, \dots, a_n) = \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 \\ a_1 & a_0 & 0 & \dots & 0 \\ a_2 & a_1 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix}_{(n+1) \times (n+1)}$$

The norm $\|M\|$ of a $(n+1) \times (n+1)$ matrix M is its norm as an operator on C^{n+1} in the usual inner product.

Theorem (2.1.4):

Given $b_0, \dots, b_n \in C$, there exists $w \in B(D)$ such that.

$w(Z) = b_0 + b_1Z + \dots + b_nZ^n$ +higher powers if and only if

$$\|T(b_0, \dots, b_n)\| \leq 1.$$

With no extra effort we obtain a more general result.

Theorem (2.1.5):

Given $b_0, \dots, b_n, c_0, \dots, c_n \in C$, there exists $w(Z) = \sum_0^\infty w_j z^j$ in $B(D)$

such that

$$\begin{cases} b_0 = c_0 w_0 \\ b_1 = c_0 w_1 \\ \vdots \\ b_n = c_0 w_n + c_1 w_{n+1} + \dots + c_n w_0 \end{cases}$$

If and only if

$$T(b_0, \dots, b_n)T(b_0, \dots, b_n)^* \leq T(c_0, \dots, c_n)T(c_0, \dots, c_n)^*$$

Proof:

In let $r = C^{n+1}$, $A = T(0, 1, 0, \dots, 0)$, $b = [b_0, \dots, b_n]^t$, $c = [c_0, \dots, c_n]^t$, and $D = r'$ since $A^j = 0$ for $j > n$, condition (i). asserts existence of

$$w(z) = \sum_0^\infty w_j Z^j \text{ in } B(D) \text{ such that for all } x' \in r',$$

$$(b, x') = w_0(c, x') + w_1(Ac, x') + \dots + w_n(A^n c, x'),$$

That is holds. Any $x' \in r'$ can be represented as an inner product

$$(x, x') = \langle x, a \rangle = a^* x \text{ for some } a = [a_0, \dots, a_n]^t \in C^{n+1} \text{ then}$$

$$\begin{aligned} \sum_0^\infty |(A^j b, x')|^2 &= |a^* b|^2 + |a^* A b|^2 + \dots + |a^* A^n b|^2 \\ &= |\bar{b}_0 a_0 + \dots + \bar{b}_n a_n|^2 + |\bar{b}_0 a_1 \\ &+ \dots + \bar{b}_{n-1} a_n|^2 + \dots + |\bar{b}_0 a_n|^2 = \|T(b_0, \dots, b_n)^* a\|^2 \end{aligned}$$

Similarly with b replaced by c thus condition (ii) of 2.3 asserts that for all $a \in \mathcal{C}^{n+1}$

$$\|T(b_0, \dots, b_n)^* a\|^2 \leq \|T(c_0, \dots, c_n)^* a\|^2,$$

Theorem (2.1.6):

For $\alpha, \beta \in D$, set

$$\frac{1}{1 - (\alpha + s)(\bar{\beta} + \bar{t})} = \sum_{p,q=0}^\infty K_{pq}(\alpha, \beta) s^{p\bar{t}q}$$

For $|s|, |t|$ sufficiently small. Differentiation of the identity

$$\frac{1}{1 - \alpha\bar{\beta}} = \int_r \frac{d\sigma}{(1 - \alpha e^{i\theta})(1 - \bar{\beta} e^{-i\theta})}$$

Yields

$$\begin{aligned} K_{pq}(\alpha, \beta) &= \frac{1}{p! q!} \left(\frac{\partial}{\partial \alpha} \right)^p \left(\frac{\partial}{\partial \bar{\beta}} \right)^q \frac{1}{1 - \alpha\bar{\beta}} \\ &= \int_r \frac{e^{i(p-q)\theta}}{(1 - \alpha e^{i\theta})^{p+1} (1 - \bar{\beta} e^{-i\theta})^{q+1}} d\sigma \end{aligned}$$

For all $P, q = 0, 1, 2, \dots$

For simplicity, consider first the Pick and caratheodory- Fejer themes.

Define $T(a_0, \dots, a_n)$ for $a_0, \dots, a_n \in \mathcal{C}$ As

Theorem (2.1.7):

Let $Z = \{Z_j\}_j^n = 1 \subseteq D$. For each $j = 1, \dots, n$, Let b_{j_0}, \dots, b_{j_r} (i) be

given complex numbers. There exists $w \in B(D)$ such that for each

$j = 1, \dots, n$, $w(Z) = b_{j_0} + b_{j_1} (Z - Z_j) + \dots + b_{j,(j)} (Z - Z_j)_{-+}^{r(j)}$ higher

if and only if

$$T(b)P(Z)T(b)^* \leq P(Z)$$

Where $T(b) = [T_{jk}(b)]_{j,k=1}^n$ and $P(Z) = [P_{jk}(Z)]^n$ are block matrices defined as follows: for each $j, k = 1, \dots, n$, $T_{jk}(b)$ and $P_{jk}(Z)$ have order $(r(j)+1) \times (r(k)+1)$, and

$$\begin{aligned} T_{jk}(b) &= T(b_{j0}, \dots, b_{j,r(j)}) && \text{if } j = k; \\ T_{jk}(b) &= 0 && \text{if } j \neq k \\ P_{jk}(Z) &= [k_{pq}(Z_j, Z_k)]_{p=0, \dots, r(j), q=0, \dots, r(k)} \end{aligned}$$

Theorem (2.1.8):

Let $Z = [Z_j]_{j \in J} \subseteq D$, and for each $j \in J$ let $b_{j0}, \dots, b_{j,r(j)}$ and $c_{j0}, \dots, c_{j,r(j)}$, be given complex numbers. There exists $w \in B(D)$ such that for each $j \in J$ the coefficients in the expansion

$$w(Z) = w_{j0} + w_{j1}(Z - Z_j) + w_{j2}(Z - Z_j)^2 + \dots$$

Satisfy

$$\begin{cases} b_{j0} = c_{j0}w_{j0} \\ b_{j1} = c_{j0}w_{j1} + c_{j1}w_{j0} \\ \vdots \\ b_{j,r(j)} = c_{j0}w_{j,r(j)} + c_{j1}w_{j,r(j)-1} + \dots + c_{j,r(j)}w_{j0} \end{cases}$$

If and only if

$$T(b)P(Z)T(b)^* \leq T(c)P(Z)T(c)^*$$

Similar to that used in Theorem. Define $T(b) = \{T_{jk}(b)\}_{j,k \in J}$ and

$P(Z) = \{P_{jk}(Z)\}_{j,k \in J}$ as *cxceP* + that the index set J is used in place of $[1, \dots, n]$. Define $T(c)$ similarly by replacing all the b, s by c, s the meaning of is that

$$\sum_{j,k \in J} a_j^* T_{jj}(b) P_{jk}(Z) T_{jk}(b)^* a_k \leq \sum_{j,k \in J} a_j^* T_{jj}(c) P_{jk}(Z) T_{kk}(c)^* a_k$$

Where ever $a_j \in C^{r(j)+1}$, $j \in J$, and $\{j : a_j \neq 0\}$ is finite.

Proof:

Let r be the set of all indexed sets $X = \{X_j\}_{j \in J}$, where

$X_j \in C^{r(j)+1}$, $j \in J$, with linear operations coordinate wise. Let

$$A: \{X_j\}_{j \in J} \rightarrow \{A_j \times j\}_{j \in J}$$

Where

$$A_j = \begin{bmatrix} z_j & & & & \\ 1 & z_j & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & z_j \end{bmatrix}_{(r(j)+1) \times (r(j)+1)}, j \in J$$

Let $b = \{b_j\}_{j \in J}$, $c = \{c_j\}_{j \in J}$, where for all $j \in J$

$$b_j = [b_{j0}, \dots, b_{j,r}(j)] \quad \text{and} \quad c_j = [c_{j0}, \dots, c_{j,r}(j)]^t.$$

Let \mathcal{D} be the set of all linear functional x' one r of the form

$$(x, x') = \sum_{j \in J} a_j^* x_j$$

Where $a_j \in C^{r(j)+1}$, $j \in J$ and $\{j : a_j \neq 0\}$ is finite.

It is convenient to introduce a functional calculus for A . For any holomorphic function $f(Z)$ on D , define $f(A)$ on r by

$$f(A): \{x_j\}_{j \in J} \rightarrow \{f(A_j)x_j\}_{j \in J}$$

Where for each j , $f(A_j)$ is defined by the standard matrix calculus. The main fact concerning the matrix calculus that we need this: for any square matrix M with eigenvalues in D if

$$f(z) = f_0 + f_1 z + f_2 z^2 + \dots, \text{ Then } f(M) = f_0 I + f_1 M + f_2 M^2 + \dots,$$

Condition of assets the existence of $w \in B(D)$ such that $b = w(A)c$ — is

$b_j = w(A_j)c_j$ for all $j \in J$ — that is coefficients satisfy for all $j \in J$.

We interpret condition. Let $u = e^{i\theta}$ and let $x' \in \mathcal{D}$ be given by Then

$$\begin{aligned} \sum_0^\infty |(A^p b, x')|^2 &= \int_\Gamma \left| \sum_0^\infty (A^p b, x') u^p \right|^2 d\sigma(u) \\ &= \int_\Gamma |(I - Au)^{-1} b, x'|^2 d\sigma(u) \\ &= \int_\Gamma \left| \sum_{j \in J} a_j^* (I_j - A_j u)^{-1} b_j \right|^2 d\sigma(u), \end{aligned}$$

Where I is the identity operator on r and I_j is the identity matrix on $C^{r(i)+1}$, $j \in J$. For any $j \in J$ and $u \in \Gamma$, set

$$h_j(u) = [1/(1 - z_j u), u/(1 - z_j u)^2, \dots, u^{r(j)}/(1 - z_j u)^{r(j)+1}]^t$$

Then $(I_j - A_j u) T_{jj}(b) h_j(u) = b_j$

And so

$$(I_j - A_j u)^{-1} b_j = T_{jj}(b) h_j(u)$$

Hence

$$\begin{aligned} \sum_0^\infty |(A^p b, x')|^2 &= \int_\Gamma \left| \sum_{j \in J} a_j^* T_{jj}(b) h_j(u) \right|^2 d\sigma(u), \\ &= \int_\Gamma \sum_{j, k \in J} a_j^* T_{jj}(b) h_j(u) h_k(u)^* T_{kk}(b)^* a_k d\sigma(u) \\ &= \sum_{j, k \in J} a_j^* T_{jj}(b) p_{jk}(z) T_{kk}(b)^* a_k \end{aligned}$$

For the last equality we used the identity

$$P_{JK}(z) = \int_\Gamma h_j(u) h_k(u)^* d\sigma(u), \quad j, k \in J$$

Similarly

$$\sum_0^\infty |(A^p c, x')|^2 = \sum_{j, k \in J} a_j^* T_{jj}(c) p_{jk}(z) T_{kk}(c)^* a_k$$

The result follows

We characterize the restrictions of boundary functions of function in $B(\Omega), \Omega = D$ or H , to an arbitrary Borel sub set Δ of $\partial\Omega$.

To shorten formulas in the disk case, we write $u = e^{i\theta}, v = e^{i\psi}$, for typical points on $L = \partial D$, measure theoretic notions are relative to normalized Lebesgue measure b on L .

Theorem. Let b, c be measurable complex valued functions on a Borel set $\Delta \subseteq I$. There exists $w \in B(D)$ such that

$$b(u) = w(u)c(u) \quad \sigma - a. e \text{ on } \Delta$$

If and only if

$$\lim_{r \uparrow 1} \int_{\Delta} \int_{\Delta} \frac{c(u)\bar{c}(v) - b(u)\bar{b}(v)}{1 - r^2 u\bar{v}} \phi(u)\bar{\phi}(v) d\sigma(u) d\sigma(v) \geq 0$$

For every measurable complex valued function ϕ on Δ such that

$$b\phi, c\phi \in L^2(\Delta)$$

Proof: Let r be the space of complex valued functions on Δ of the form $f = Pb + qc$ where P, q are Poly nominal's. Thus $b, c \in r$. Define

$$A: f(u) \rightarrow uf(u)$$

On r let \mathcal{D} be the set of linear functional x' on r the form

$$(f, x') = \int_{\Delta} f\phi d\sigma, \quad f \in r$$

Where ϕ is a measurable function such that $b\phi, c\phi \in L^2(\Delta)$. Then

$A'\mathcal{D} \subseteq \mathcal{D}$. For every functional. $\sum_0^\infty |(A^j c, x')|^2 < \infty$ Because the Fourier coefficient of a square summable function are square summable. Thus the hypotheses of satisfied. Condition holds if and onle if there exists $w(z) = \sum_0^\infty w_j z^j$ in $B(D)$ such that

$$\begin{aligned} \int_{\Delta} b(u)\phi(u)d\sigma(u) &= \sum_0^\infty w_j \int_{\Delta} u^j c(u)\phi(u)d\sigma(u) \\ &= \lim_{r \uparrow 1} \sum_0^\infty r^j w_j \int_{\Delta} u^j c(u)\phi(u)d\sigma(u) \\ &= \lim_{r \uparrow 1} \int_{\Delta} w(ru)c(u)\phi(u)d\sigma(u) \\ &= \int_{\Delta} w(u)c(u)\phi(u)d\sigma(u) \end{aligned}$$

For all ϕ such that $b\phi, c\phi \in L^2(\Delta)$, that is holds.

$$\sum_0^\infty \left| \int_{\Delta} u^j b\phi d\sigma \right|^2 \leq \sum_0^\infty \left| \int_{\Delta} u^j c\phi d\sigma \right|^2.$$

Now

$$\begin{aligned}
& \sum_0^\infty \left| \int_\Delta u^j b \phi db \right|^2 = \lim_{r \uparrow 1} \int_L \left| \sum_0^\infty \int_\Delta u^j b \phi db(t) \right|^2 \\
& = \lim_{r \uparrow 1} \int_L \left| \int_\Delta u^j b \phi db \right|^2 = \lim_{r \uparrow 1} \int_L \left| \sum_0^\infty \left(\int_\Delta u^\delta b \phi db \right) r^j t^j \right|^2 db(t) \\
& = \lim_{r \uparrow 1} \int_L \left| \int_\Delta \frac{b(u) \phi(u)}{1 - rut} db(u) \right|^2 db(t) \\
& = \lim_{r \uparrow 1} \int_\Delta \int_\Delta \int_\Delta \frac{db(t)}{(1 - rut)(1 - r\bar{u}t)} b(u) \bar{b}(u) \phi(u) \phi(u) db(u) db(V) \\
& = \lim_{r \uparrow 1} \int_\Delta \int_\Delta \frac{b(u) \bar{b}(V)}{1 - u\bar{v}} \phi(u) \bar{\phi}(u) db(u) db(U)
\end{aligned}$$

We define the Pick class \mathcal{D} as the set of holomorphic functions f on II such that $f(Z) \geq 0, Z \in II$

There is a one-to-one correspondence between \mathcal{D} and $B(II) \setminus \{1\}$ is the function identically 1 on II : if $w \in B(II) \setminus \{1\}$: if we $(-\infty)$ then

$$f = i(1 + w)/(1 - w)$$

Is in \mathcal{D} in the Pick class \mathcal{D} that is not identically zero is an over function in particular each fin \mathcal{D} has on tangential boundary function defined a.e. on $(-\infty, \infty)$.

Section (2.2): Generalized Loewner Thermos and Hilbert Transform with Imaginary functions:

Theorem (2.2.1):

Let $f_0(x)$ be a measurable complex valued function on a Borel subset Δ of $(-\infty, \infty)$. There exists $f \in \mathcal{D}$ such that

$$f(x) = f_0(x) \quad a.e. \text{ on } \Delta$$

If and only if

$$\lim_{\varepsilon \uparrow 0} \int_{\Delta} \int_{\Delta} \frac{f_0(s) - \bar{f}_0(t)}{s - t + i\varepsilon} \phi(s) \bar{\phi}(t) ds dt \geq 0$$

Whenever $\phi, f_0\phi \in L^2(\Delta)$,

Proof:

Apply with $b = f_0 - i, c = f_0 + i$ and use the correspondence. Between \mathcal{D} and $B(H/\{1\})$.

The L^2 theory of Hilbert transforms is sufficient for our purposes. Although this is well known, we include statements of the principal results for the convenience of the reader and later reference.

If $\phi(x) \in L^2(-\infty, \infty)$, it's Hilbert transform is defined by

$$(H\phi)(x) = PV \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(t)}{t - x} dt,$$

Where PV a Cauchy principal indicates value integral:

$$PV \frac{1}{\pi} \int_{-\infty}^{\infty} = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|t-x|>\varepsilon}$$

Theorem (2.2.2):

If $\phi(x) \in L^2(-\infty, \infty)$, then the limit exists *a.e.* And in the metric of $L^2(-\infty, \infty)$. If

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(t)}{t - z} dt, \quad z \neq \bar{z},$$

Then

$$\begin{aligned}\Phi(x + i0) - \Phi(x - i0) &= \phi(x), \\ \Phi(x + i0) + \Phi(x - i0) &= -i(H\phi)(x)\end{aligned}$$

a.e. On $(-\infty, \infty)$, where $\Phi(x \pm i0) = \lim_{y \downarrow 0} \Phi(x \pm iy)$, whenever the limit exists.

Let Δ be a fixed Borel subset of $(-\infty, \infty)$. If $\phi \in L^2(\Delta)$, set

$$(H_\Delta \phi)(x) = PV \frac{1}{\pi} \int_\Delta \frac{\phi(t)}{t-x} dt \quad \text{a.e. on } \Delta.$$

This defines a bounded linear operator on $L^2(\Delta)$ with ad joint $H_\Delta^* = -H_\Delta$

Theorem (2.2.3):

Let $f_0(x)$ be a measurable complex valued function on Δ . There exists $f \in$ such that

$$f(x) = f_0(x) \quad \text{a.e. on } \Delta$$

If and only if

$$\operatorname{Re} \langle (H_\Delta - iI) f_0 \phi, \phi \rangle_2 \geq 0$$

Whenever $\phi, f_0 \in L^2(\Delta)$.

Proof:

We show that for any $\phi, \psi \in L^2(\Delta)$,

$$\lim_{\varepsilon \downarrow 0} \int_\Delta \int_\Delta \frac{\phi(s) \tilde{\psi}(t)}{s-t+i\varepsilon} ds dt = \pi \langle (H_\Delta - iI) \phi, \psi \rangle_2$$

To this end, set

$$\Phi(z) = \frac{1}{2\pi i} \int_\Delta \frac{\phi(t)}{t-z} dt, \quad y \neq 0$$

$$\lim_{\varepsilon \downarrow 0} \Phi(x \pm i\varepsilon) = \frac{1}{2} [\pm \phi(x) - i(H_\Delta \phi)(x)]$$

a.e. On Δ in the metric of $L^2(\Delta)$. Therefore,

$$\lim_{\varepsilon \downarrow 0} \int_\Delta \int_\Delta \frac{\phi(s) \bar{\psi}(t)}{s-t+i\varepsilon} ds dt = \int_\Delta \left(\lim_{\varepsilon \downarrow 0} \int_\Delta \frac{\phi(s) ds}{s-t+i\varepsilon} \right) \bar{\psi}(t) dt$$

$$= \langle \pi i(-\phi - iH_\Delta \phi, \psi) \rangle_2$$

If $\phi, f_0 \phi \in L^2(\Delta)$, then

$$\begin{aligned} &= \lim_{\varepsilon \downarrow 0} \int_{\Delta} \int_{\Delta} \frac{f_0(s) - f_0(t)}{s - t + i\varepsilon} \phi(x) \bar{\phi}(t) ds dt, \\ &= \pi \langle (H_\Delta - iI) f_0 \phi, \phi \rangle_2 - \pi \langle (H_\Delta - iI) \phi, f_0 \phi \rangle_2 \\ &= 2\pi \operatorname{Re} \langle (H_\Delta - iI) f_0 \phi, \phi \rangle_2. \end{aligned}$$

Theorem (2.2.4):

Let $f_0(x)$ be a real valued measurable function on a Borel subset Δ of $(-\infty, \infty)$. There exists $f \in \mathcal{D}$ such that

$$f(x) = f_0(x) \quad \text{a.e. on } \Delta$$

If and only if

$$\lim_{\varepsilon \downarrow 0} \int_{E(\varepsilon)} \int \frac{f_0(s) - f_0(t)}{s - t} \phi(x) \bar{\phi}(t) ds dt,$$

Whenever $\phi, f_0 \phi \in L^2(\Delta)$ where

$$E(\varepsilon) = \{(s, t) : s, t \in \Delta \text{ and } |s - t| > \varepsilon\} \text{ For each } \varepsilon > 0.$$

Proof:

By Theorem (2.2.2) if $\phi, f_0 \phi \in L^2(\Delta)$, then

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int_{E(\varepsilon)} \int \frac{f_0(s) - f_0(t)}{s - t} \phi(x) \bar{\phi}(t) ds dt, \\ &= \lim_{\varepsilon \downarrow 0} \int_{\Delta} \left(\int_{\Delta} X_{E(\varepsilon)}(s, t) \frac{f_0(s) \phi(s)}{s - t} \right) \bar{\phi}(t) dt \\ &\quad - \lim_{\varepsilon \downarrow 0} \int_{\Delta} \left(\int_{\Delta} X_{E(\varepsilon)}(s, t) \frac{\phi(s)}{s - t} ds \right) f_0(t) \bar{\phi}(t) dt \\ &= \pi \langle H_\Delta f_0 \phi, \phi \rangle_2 - \pi \langle H_\Delta \phi, f_0 \phi \rangle_2 \\ &= 2\pi \operatorname{Re} \langle H_\Delta f_0 \phi, \phi \rangle_2 \\ &= 2\pi \operatorname{Re} \langle (H_\Delta - iI) f_0 \phi, \phi \rangle_2. \end{aligned}$$

Here $X_{E(\varepsilon)}$ the characteristic function of $E(\varepsilon)$ the last equality holds because f_0 is real valued. Thus the result follows from the.

A holomorphic function f on Π is said to have an analytic continuation across an open subset Δ of $(-\infty, \infty)$ if $f = g|_{\Pi}$, where g is holomorphic on a set G containing $\Pi \cup \Delta$

Suppose that $f \in \mathcal{D}$ and f has an analytic continuation across an open subset Δ of $(-\infty, \infty)$. Suppose further that continuation is real valued on Δ . Then we may extend f to a holomorphic function on $\Pi \cup \tilde{\Pi} \cup \Delta$ such

$$\bar{f}(\bar{z}) = f(z) \quad , z \in \Pi \cup \tilde{\Pi} \cup \Delta.$$

Where $\tilde{\Pi} = \{z: \text{Im } z < 0\}$. The next result characterizes this class of functions.

Theorem (2.2.5):

Let g_0 be a nonnegative measurable function on a Borel subset Δ of $(-\infty, \infty)$. The following are equivalent:

- (i) There exists $f \in \mathcal{D}$ such that

$$f(x) = i g_0(x) \quad \text{a. e. On } \Delta$$
- (ii) Whenever $\phi, g_0(\phi) \in L^2(\Delta)$

$$\int_{\Delta} |H_{\Delta}\phi|^2 g_0(x) dx \leq \int_{\Delta} |\phi|^2 g_0(x) dx$$

Proof:

We first prove theorem under the hypothesis that $g_0 \in L^1(\Delta) \cap L^{\infty}(\Delta)$.

We show that in this case (i) and (ii) are equivalent to.

- (iii) There exists $h \in \mathcal{D}$ such that

$$h(x) = -(H_{\Delta}g_0)(x) \quad \text{a. e. On } \Delta.$$

(ii) \Leftrightarrow (iii) Assume (ii), and set $f_0 = -H_{\Delta}g_0$ for any $\phi \in L^2(\Delta) \cap L^{\infty}(\Delta)$

we have $\phi, g_0 \in L^2(\Delta)$, so

$$\begin{aligned} \text{Re}\langle (H_{\Delta} - iI)f_0\phi, \phi \rangle_2 &= \frac{1}{2}\langle H_{\Delta}(f_0\phi), \phi \rangle_2 + \frac{1}{2}\langle \phi, H_{\Delta}(f_0\phi) \rangle_2 \\ &= \frac{1}{2} \int_{\Delta} (\phi(H_{\Delta}\bar{\phi}) + (H_{\Delta}\phi)\bar{\phi})H_{\Delta}g_0 dx \\ &= -\frac{1}{2} \int_{\Delta} \left\{ H_{\Delta} \left(\phi(H_{\Delta}\bar{\phi}) + (H_{\Delta}\phi)\bar{\phi} \right) \right\} g_0 dx \\ &= -\frac{1}{2} \int_{\Delta} (|H_{\Delta}\phi|^2 - |\phi|^2) g_0 dx \geq 0 \end{aligned}$$

(i) \Leftrightarrow (iii) Assume (iii), and choose $h \in \mathcal{D}$ such that holds then.

$$f(z) = \frac{1}{\pi} \int_{\Delta} \frac{g_0(t)}{t-z} dt + h(z) \quad , z \in \Pi$$

Defines a function in \mathcal{D} .

$$\begin{aligned} f(x) &= \frac{1}{2} \cdot 2i \{g_0(x) - i(H_{\Delta}g_0)(x)\} + h(x) \\ &= ig_0(x) \quad a.e \text{ on } \Delta \end{aligned}$$

So (I) hold.

Conversely .Let (i) holds and chose $f \in \mathcal{D}$ such that

$$f(x) = ig_0(x) \quad a.e \text{ on } \Delta$$

Define h on that Π so that holds since g_0 is essentially bounded on Δ his bounded below on Δ Im and $\exp(ih) \in h^{\infty}(\Pi)$. But

$$\text{Im } h(Z) = \text{Im } f(Z) - \frac{Y}{\pi} \int_{\Delta} \frac{g_0(t)}{(t-x)^2 + Y^2} dt \quad , Y > 0$$

Where

$$\lim_{Y \downarrow 0} \frac{Y}{\pi} \int_{\Delta} \frac{g_0(t)}{(t-x)^2 + Y^2} dt = \begin{cases} g_0(x) & a.e \text{ on } \Delta \\ 0 & a.e \text{ on } (-\infty, \infty) \Big|_{\Delta} \end{cases}$$

And thus $h(x) \geq 0$ a.e on $(-\infty, \infty)$. Therefore the boundary function of $\exp(ih)$ is bounded by 1 a.e on $(-\infty, \infty)$ and so $|\exp(ih)| \leq 1$ on Π .

Hence

$h \in \mathcal{D}$. Since.

$$\begin{aligned} h(x) &= f(x) - \frac{1}{2} \cdot 2i \{g_0(x) - i(H_{\Delta}g_0)(x)\} \\ &= -(H_{\Delta}g_0)(x) \quad , \quad 0.e \text{ on } \Delta, (iii) \text{ follows.} \end{aligned}$$

Lemma (2.2.6):

Let f be a locally bound Borel function on $\Delta = (a, b)$ let $\phi \geq 0$ be a C^∞ function with compact support in $(-1, 1)$ such that $\int_{-1}^1 \phi dt = 1$. For each ε , $0 < \varepsilon < (b - a)/2$, set $f_\varepsilon(x) = \int_{-1}^1 \phi(t)(x + \varepsilon t) dt$ On $\Delta_\varepsilon = (a + \varepsilon, b - \varepsilon)$. Then $f_\varepsilon \in C^\infty(\Delta_\varepsilon)$ and $f_\varepsilon(x) \rightarrow f(x)$ at every point $x \in \Delta$ where f is continuous.

Proof:

Extend ϕ, f to $(-\infty, \infty)$. By setting both equal to zero off their domains. Then

$$f_\varepsilon(x) = \int_{-\infty}^{\infty} \varepsilon^{-1} \phi((s - x)/\varepsilon) f(s) ds, x \in \Delta_\varepsilon$$

With this representation the proof become a pleasant exercise in real analysis.

Theorem (2.2.7):

Let f be a monotone matrix function on Δ as a first case suppose that f is continuously differentiable on Δ then.

$$k(x, y) = \begin{cases} [f(x) - f(y)] / (x - y), & x \neq y, \\ f(x), & x = y, \end{cases}$$

Is continuous on $\Delta \times \Delta$. We show that

$$\sum_{j, k=1}^m k(\lambda_j, \lambda_k) C_j \bar{C}_k \geq 0$$

Whenever $\{\lambda_1, \dots, \lambda_n\} \subseteq \Delta$ and $\{c_1, \dots, c_n\} \subseteq C$. Without loss of generality we can assume that $\lambda_1, \dots, \lambda_n$ are distinct.

Let X be a Hilbert space with $\dim X = n$ and let e_1, \dots, e_n be an orthonormal basis for X . Let $A = \sum_j^n \lambda_j P_j$ where $P_j = \langle \cdot, e_j \rangle e_{jj}$ the projection of X on the span of $e_j, j = 1, \dots, n$. Then A a self-adjoint operator with $\sigma(A) \subseteq \Delta$ set $B = A + \varepsilon Q$, where

$$Q = \langle \cdot, v \rangle v, v = e_1 + \dots + e_n \text{ And } \varepsilon > 0$$

For all sufficiently ε , $SP(B) \subseteq \Delta$. Since $A \leq B$.

$$f(A) \leq f(B)$$

$$\sum_1^n \lambda_j^{(\varepsilon)} P_j^{(\varepsilon)} \text{ that } \lambda_j^{(\varepsilon)} \rightarrow \lambda_j \text{ and } P_j^{(\varepsilon)} \rightarrow P_j \text{ as } \varepsilon \downarrow 0.$$

$$0 \leq f(B) - f(A) = \sum_{j,k=1}^m k(\lambda_j, \lambda_k^{(\varepsilon)}) P_j^\varepsilon Q P_k^\varepsilon$$

Dividing by ε , and then letting T , we obtain

$$\sum_{j,k}^m k(\lambda_j, \lambda_k) P_j Q P_k \geq 0$$

$$Pv + u = \sum_1^n \bar{c}_j e_j.$$

Then the inequality

$$\sum_{j,k=1}^m k(\lambda_j, \lambda_k) \langle P_j Q P_k u, u \rangle \geq 0.$$

Chapter 3
Factorization of Toeplitz Operators

Sec (3-1): Factorization of Non negative Invertible Toeplitz operators

Assume that S a shift operator on a Hilbert space. We write $P_0 = 1 - SS^*$ for the projection of \mathfrak{X} on $k = \ker S^*$. Analytic, Inner, over, and S -Constance operators are defined relative.

Definition (3.1.1):

An operator $T \in \mathcal{B}(\mathfrak{X})$ Toeplitz or more precisely- Toeplitz if $S^*TS = T$.

Example (3.1.2):

Let $S_1 : (C_0, C_1, C_2, \dots) \rightarrow (0, C_0, C_1, \dots)$ On L^2 , and let T is a bounded linear operator on L^2 with matrix $[w_{jk}]_{j,k}^\infty = 0$. Thus

$T : (a_0, a_1, a_2) \rightarrow (b_0, b_1, b_2)$ If and only if

$$b_j = \sum_{k=0}^{\infty} w_{jk} a_k, \quad j = 0, 1, 2, \dots$$

The matrix of S^*TS is $[w_{j+1, k+1}]_{j,k}^\infty = 0$. Hence T is an S_1 -Toilets operator if and only if its matrix has the form $[C_j - k]_{j,k}^\infty = 0$ for some sequence

$$\{C_u\}_{-\infty}^\infty$$

Such a matrix is called a Toeplitz matrix.

Example (3.1.3):

In general Examples of Toeplitz operators are easily constructed from analytic operators. If $A, C \in \mathcal{B}(\mathfrak{X})$ are analytic, then the operators A, C^*, C^*A are Toeplitz.

For if $T = C^*A$, then $S^*TS = S^*C^*AS = C^*S^*SA = C^*A = T$.

To each $T \in \mathcal{B}(\mathfrak{X})$ we associative a matrix of operators in

$$\mathcal{B}(\mathfrak{X}): T \sim [A_{jk}]_{j,k}^\infty = 0,$$

Where

$$A_{jk} = P_0 S^{*j} T S^k P_0 \mathfrak{X} \quad j, k \geq 0.$$

Theorem (3.1.4):

Let $T \in \mathcal{B}(\mathfrak{X})$ and $T \square \left\{ A_{jk} \right\}_{j,k=0}^{\infty}$.

For any, $f \in \mathfrak{X}$ if $\sum_0^{\infty} S^j a_j$, then, where

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} & \dots \\ A_{10} & A_{11} & A_{12} & \dots \\ A_{20} & A_{21} & A_{22} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

Proof:

If $Tf = \sum_0^{\infty} S^j b_j$ where

$$b_j = P_0 S^{*j} T f = P_0 S^{*j} T \sum_0^{\infty} S^k a_k = \sum_{k=0}^{\infty} A_{jk} a_k, \quad j \geq 0.$$

The some are strongly convergent, and the sequence of relation is equivalent to the correspondence $T \sim [A_{jk}]_{j,k=0}^{\infty}$ is clearly linear and well behaved with respect to a djoints: if, c then, $T^* \sim [A_{jk}]_{j,k=0}^{\infty}$ where $B_{jk} = A_{kj}^*$ for all $j, k \geq 0$

Theorem (3.1.5):

If $T_1 \sim [A_{jk}]_{j,k=0}^{\infty}$ and $T_2 \sim [B_{jk}]_{j,k=0}^{\infty}$, then, $T_1 T_2 \sim [C_{jk}]_{j,k=0}^{\infty}$ where

$$C_{jk} = \sum_{l=0}^{\infty} A_{jl} B_{lk}, \quad j, k \geq 0.$$

With convergence of the series in the strong operator topology.

Proof:

By the word decomposition

$$I = \sum_{k=0}^{\infty} S^1 P_0 S^{*1}.$$

With convergence in the strong operator topology. Hence for any $j, k \geq 0$,

$$\begin{aligned} C_{jk} &= P_0 S^{*j} T_1 T_2 S^k \mathfrak{X} \\ &= P_0 S^{*j} T_1 \sum_{l=0}^{\infty} S^1 P_0 S^{*1} T_2 S^k P_0 \mathfrak{X} \\ &\quad \sum_{k=0}^{\infty} A_{jl} B_{lk} \end{aligned}$$

An operator $T \in \mathcal{B}(\mathfrak{X})$ is if and only if its matrix has the form

$$[A_{j-1}]_{j,k=0}^{\infty} = \begin{bmatrix} A_0 & A_{-1} & A_{-2} & \dots \\ A_1 & A_0 & A_{-1} & \dots \\ A_2 & A_1 & A_0 & \dots \end{bmatrix} \quad (*)$$

$$\text{In this case } A_j = \begin{cases} P_0 S^{*j} T P_0 \mathfrak{X}, & \text{if } j \geq 0 \\ P_0 T S^{|j|} P_0 \mathfrak{X}, & \text{if } j < 0 \end{cases} \quad (**)$$

A matrix of the form **(X)** is called Toeplitz matrix $[A_{jk}]_{j,k}^{\infty} = 0$. Then

$S^* T S^k = T$ for all $k \geq 0$. Hence

$$A_j = \begin{cases} P_0 S^{*j-k} T P_0 \mathfrak{X}, & \text{if } j \geq k \\ P_0 T S^{k-j} P_0 \mathfrak{X}, & \text{if } j < k \end{cases}$$

Thus that the matrix **(*)** where the entries are defined by **(**)**.

Conversely, let the matrix of T have the form **(*)** then by the operators T and $S^* T S$ have the same matrix. Hence $T = S^* T S$ and T is Toeplitz.

Corollary (3.1.6):

An operator $A \in \mathcal{B}(\mathfrak{X})$ is analytic if and only if its matrix has the form.

$$\begin{bmatrix} A_0 & 0 & 0 & \dots \\ A_1 & A_0 & 0 & \dots \\ A_2 & A_1 & A_0 & \dots \end{bmatrix}$$

In this case

$$A_j = P_0 S^{*j} A P_0 \mathfrak{X}, \quad j \geq 0.$$

Moreover, A is S -constant if and only if $A_j = 0$ for all $j \geq 1$, that is, the matrix of A has the

$$\text{diag}\{A_0, A_0, A_0, \dots\}$$

Let $T \in \mathcal{B}(\mathfrak{X})$ be a non negative Toeplitz operator. Since $T \geq 0$. There exists a unique non negative. Square root $T^{1/2}$. Since T is Toeplitz, $S^* T S = T$ and so for any $f \in (\mathfrak{X})$.

$$\|T^{1/2} S f\|^2 = \langle S^* T S f, f \rangle = \langle T f, f \rangle = \|T^{1/2} f\|^2$$

It follows that there exists a unique isometry on $T^{1/2} \mathfrak{X}$ to $T^{1/2} \mathfrak{X}$ that maps $T^{1/2} f$ to $T^{1/2} S f$ for each $f \in (\mathfrak{X})$. The extension by continuity of this isometry to $\overline{T^{1/2} \mathfrak{X}}$ plays a central role in what follows.

Definition (3.1.7):

Let $T \in \mathcal{B}(\mathfrak{X})$ be a non-negative Toeplitz operator. We set

$$\mathfrak{X}_T = \overline{T^{1/2}\mathfrak{X}}$$

And view \mathfrak{X}_T as a Hilbert space in the inner product of \mathfrak{X} .

Theorem (3.1.8):

Let $T \in \mathcal{B}(\mathfrak{X})$ be a non-negative Toeplitz operator. The following assertions are equivalent:

- (i) $T = A^*A$ for some analytic operator $A \in \mathcal{B}(\mathfrak{X})$
- (ii) Lowdenslager's isometry S_T is a shift operator;
- (iii) For all vectors C in some dense subset D of \mathfrak{X} .

$$\lim_{n \rightarrow \infty} (\sup \{ |\langle Tc, S^n f \rangle| : f \in \mathfrak{X}, \langle Tf, f \rangle = 1 \}) = 0$$

Proof:

(i) \Rightarrow (ii) Let $T = A^*A$, where $A \in \mathcal{B}(\mathfrak{X})$ is analytic. If $c \in \mathfrak{X}$, $f \in \mathfrak{X}$, and $\langle Tf, f \rangle = 1$, then

$$\begin{aligned} \left| \langle Tc, S^n f \rangle \right|^2 &= \left| \langle A^*Ac, S^n f \rangle \right|^2 = \left| \langle S^{*n}Ac, Af \rangle \right|^2 \\ &\leq \|S^{*n}Ac\|^2 \|Af\|^2 = \|S^{*n}Ac\|^2 \langle Tf, f \rangle = \|S^{*n}Ac\|^2 \end{aligned}$$

For each $n = 0, 1, 2, \dots$ since S is a shift operator, $\|S^{*n}Ac\| \rightarrow 0$ so (iii) holds with $D = \mathfrak{X}$.

(iii) \Rightarrow (ii) Assume (iii). Claim: for each $c \in \mathcal{D}$ and $n \geq 0$.

$$\|S_T^{*n}T^{1/2}c\| = \sup \{ |\langle Tc, S^n f \rangle| : f \in \mathfrak{X}, \langle Tf, f \rangle = 1 \}.$$

To see this, note that for any $f \in \mathfrak{X}$.

$$\begin{aligned} \langle Tc, S^n f \rangle &= \langle T^{1/2}c, T^{1/2}S^n f \rangle = \langle T^{1/2}c, S_T^n T^{1/2} \rangle \\ &= \langle S_T^{*n}T^{1/2}c, T^{1/2}f \rangle \end{aligned}$$

The claim then follows from the fact that the set of vectors $T^{1/2}f$, where $f \in \mathfrak{X}$ and $\langle Tf, f \rangle = 1$, is dense in the unit sphere of \mathfrak{X}_T .

Since we assume $\|S_T^{*n}g\| \rightarrow 0$ for all $g \in \mathfrak{X}_T$ of the form $g = T^{1/2}c$, $c \in \mathcal{D}$.

Suppose next that $g = T^{1/2}S^k c$ for some $c \in \mathcal{D}$ and $K \geq 0$. Then for $n \geq K$,

$$S_T^{*n}g = S_T^{*n}T^{1/2}S^k c = S_T^{*n}S_T^k T^{1/2}c = S_T^{*(n-k)}T^{1/2}c$$

And a gain $\|S_T^{*n} g\| \rightarrow 0$. A routine an approximation argument shows that $\|S_T^{*n} g\| \rightarrow 0$ for every $g \in \mathfrak{X}_T$, so (ii) follows.

(ii) \rightarrow (i) Let S_T be a shift operator. By the definitions of S_T , for all.

$$f \in \mathfrak{X}, \quad g \in \mathfrak{X}_T$$

$$\langle T^{1/2} S_T^* g - S^* T^{1/2} g, f \rangle = \langle g, S_T T^{1/2} f \rangle - \langle g, T^{1/2} f S f \rangle = 0$$

Hence

$$T^{\frac{1}{2}} S_T^* g = S^* T^{\frac{1}{2}} g, \quad g \in \mathfrak{X}_T$$

And $T^{\frac{1}{2}}(\ker S_T^*) \subseteq S^*$. Set $\mathfrak{X}_T = \ker S_T^*$. Then $J = T^{\frac{1}{2}} \mathfrak{X}_T \in \mathcal{B}(\mathfrak{X}_T, \mathfrak{X})$. By the polar decomposition of an operator, $J^* = V^* R$ where $(JJ^*)^{\frac{1}{2}} \in \mathcal{B}(\mathfrak{X})$ and $V \in \mathcal{B}(\mathfrak{X}_T, \mathfrak{X})$ is a partial isometry with initial space $\overline{J^* \mathfrak{X}}$. A actually V is and isometry. For,

$$\ker V = \mathfrak{X}_T \ominus J^* \mathfrak{X} = \ker J \subseteq \ker T^{\frac{1}{2}},$$

And at the same time

$$\ker V \subseteq \mathfrak{X}_T = \mathfrak{X} \ominus \ker T^{\frac{1}{2}},$$

So $\ker V = \{0\}$

Since S_T is a shift operator, each $g \in \mathfrak{X}_T$ has a unique representation

$$g = \sum_0^{\infty} S_T^n K_j \quad \text{Where} \quad \{k_j\}_0^{\infty} \subseteq \mathfrak{X}_T$$

Set

$$K_g = \sum_0^{\infty} S^j V K_j$$

This defines an isometry K that maps \mathfrak{X}_T into \mathfrak{X} and satisfies $KS_T = SK$.

Define $A \in \mathcal{B}(\mathfrak{X})$ by

$$Af = K T^{1/2} f, \quad f \in \mathfrak{X}$$

For any $f \in \mathfrak{X}$

$$ASf = K T^{1/2} S f = K S_T T^{1/2} f = SK T^{1/2} f = SAf,$$

And

$$\langle Tf, f \rangle = \langle T^{1/2} f, T^{1/2} f \rangle =$$

$$\langle KT^{1/2} f, KT^{1/2} f \rangle = \langle Af, Af \rangle$$

Thus $AS = SA$ and $T = A^*A$; that is, (i) holds.

We complete the proof by showing that the operator A constructed above is outer and $A_0 = P_0AP_0$ is nonnegative. Setting $M_0 = V\mathfrak{X}_T$, we obtain

$$\overline{A\mathfrak{X}} = \overline{KT^{1/2}\mathfrak{X}} = K\mathfrak{X}_T = \sum_0^\infty \oplus S^j M_0,$$

And so $\overline{A\mathfrak{X}}$ reduces S . Thus A is outer. The operator R constructed above is non negative. We show that $A_0 = R$. If $c \in \mathfrak{X}$, then.

$$A_0c = P_0AP_0c = P_0KT^{1/2}c$$

Let $T^{\frac{1}{2}}c = \sum_0^\infty S_T^j k_j$ as in so $KT^{\frac{1}{2}}c = \sum_0^\infty S^j V k_j$. Hence

$$A_0c = P_0KT^{1/2}c = P_0 \sum_0^\infty S^j V k_j = V k_0$$

If P_T is the projection of \mathfrak{X}_T onto \mathfrak{X}_T , then $k_0 = P_T T^{1/2}c = J^*c$, and so

$$A_0c = V k_0 = V J^*c = Rc$$

Therefore $A_0 = R \geq 0$, and the proof is complete.

Theorem (3.1.9):

Let $A \in \mathcal{B}(\mathfrak{X})$ be analytic, and let $C \in \mathcal{B}(\mathfrak{X})$ be outer. Then $A^*A = C^*C$ if and only if

$$A = BC,$$

Where $B \in \mathcal{B}(\mathfrak{X})$ is inner and has initial space $\overline{C\mathfrak{X}}$.

Proof:

Assume that $A^*A = C^*C$ Then for any $f \in \mathfrak{X}$,

$$\|A\|^2 = \langle A^*Af, f \rangle = \langle C^*Cf, f \rangle = \|Cf\|^2$$

Hence there is a unique partial isometry $B \in \mathcal{B}(\mathfrak{X})$ with initial space $\overline{C\mathfrak{X}}$ such that $A = BC$. For any $f \in \mathfrak{X}$,

$$(SB - BS)Cf = SAf - ASf = 0$$

Thus SB and BS coincide on $\overline{C\mathfrak{X}}$. Since C is outer, $\overline{C\mathfrak{X}}$ reduces S . Therefore $g \perp C\mathfrak{X}$ implies $Sg \perp C\mathfrak{X}$.

Since B is zero on $(C\mathfrak{X})^\perp$, $SBg = 0 = BSg$ for all $g \in (C\mathfrak{X})^\perp$ hence $SB = BS$, so B is inner.

Conversely, let $A = BC$, where $B \in \mathcal{B}(\mathfrak{X})$ is inner with initial space $\overline{C\mathfrak{X}}$.

Then BB^* is the projection of \mathfrak{X} on $\overline{C\mathfrak{X}}$. Hence $A^*A = C^*B^*BC = C^*C$.

Corollary (3.1.10):

Let $A, C \in \mathcal{B}(\mathfrak{X})$, be two outer operators. Then $A^*A = C^*C$ if and only if

$$A = BC$$

Where $B \in \mathcal{B}(\mathfrak{X})$ is an S -constant inner operator with initial space $\overline{C\mathfrak{X}}$ and final space $\overline{A\mathfrak{X}}$?

Theorem (3.1.11):

Let $A, C \in \mathcal{B}(\mathfrak{X})$ be outer operators, and let A_0, C_0 be the diagonal entries in their matrices. If $A^*A = C^*C$, $A_0 \geq 0$, and $C_0 \geq 0$, then $A = C$.

Proof:

If $A^*A = C^*C$, then by the corollary to theorem (3.1.6), where B is an S -constant inner operator with initial space $\overline{C\mathfrak{X}}$ and final space $\overline{A\mathfrak{X}}$ the diagonal entries A_0, B_0, C_0 of A, B, C satisfy

$$A_0 = B_0C_0 \text{ Hence}$$

$$A_0^2 = C_0B_0^*B_0C_0 \leq C_0^2$$

Inter changing the roles of A and C , we obtain also $C_0^2 \leq A_0^2$, and hence

$A_0^2 = C_0^2$. Since A_0 and C_0 are non negative, $A_0 = C_0$. Since $A_0 = B_0C_0$ coincides with the identity operator on $\overline{C_0\mathfrak{X}}$. Since $\overline{C_0\mathfrak{X}} = \overline{P_0C\mathfrak{X}}$.

$$\overline{C\mathfrak{X}} = \sum_0^\infty \oplus S^j(\overline{C_0\mathfrak{X}}).$$

Thus each $f \in \overline{C\mathfrak{X}}$ has the form $f = \sum_0^\infty S^j K_j$, where $k_j \in \overline{C_0\mathfrak{X}}$. Then

$$Bf = \sum_0^\infty S^j B_0 K_j = \sum_0^\infty S^j K_j f$$

It follows that B coincides with the identity operator on $\overline{C\mathfrak{X}}$. Since $A = BC$, we therefore have $A = C$.

Lemma (3.1.12):

If $A \in \mathcal{B}(\mathfrak{X})$ is outer, then

$$\overline{A\mathfrak{X}} = \sum_0^\infty \oplus S^j M_0(A^*),$$

Where $M_0(A^*) = \overline{P_0 A\mathfrak{X}}$.

Proof:

Since A is outer, $\overline{A\mathfrak{X}}$ reduces S .

$$\overline{A\mathfrak{X}} = \sum_0^\infty \oplus S^j M$$

Where $M = P_0(\overline{A\mathfrak{X}})$. Since $P_0(\overline{A\mathfrak{X}})$ is closed and $P_0 A\mathfrak{X} \subseteq P_0(\overline{A\mathfrak{X}})$ we have $\overline{P_0 A\mathfrak{X}} \subseteq P_0(\overline{A\mathfrak{X}}) = M$.

Conversely, if $g \in M$, then $g \in \overline{A\mathfrak{X}}$ so $g = \lim_{n \rightarrow \infty} Af_n$ for some sequence $\{f_n\}_1^\infty$ in. Then $g = P_0g = \lim_{n \rightarrow \infty} P_0Af_n \in \overline{P_0A\mathfrak{X}}$. Thus $M = \overline{P_0A\mathfrak{X}}$, and the results follows.

Theorem (3.1.13):

If $A \in \mathcal{B}(\mathfrak{X})$ is analytic, then

$$A = BC$$

Where $C \in \mathcal{B}(\mathfrak{X})$ is outer and $B \in \mathcal{B}(\mathfrak{X})$ is inner with initials space $\overline{C\mathfrak{X}}$.

For any such factorization, $A^*A = C^*C$

Moreover, we may choose the factorization so that the diagonal entry C_0 in the matrix for C satisfies $C_0 \geq 0$, and then the factors B and C are unique

Proof:

Applying to the operator $T = A^*A$, we obtain an outer operator $C \in \mathcal{B}(\mathfrak{X})$ such that $A^*A = C^*C$ and $C_0 = P_0CP_0\mathfrak{X} \geq 0$, there is an inner operator $B \in \mathcal{B}(\mathfrak{X})$ with initial space $\overline{C\mathfrak{X}}$ such that $A = BC$

Theorem (3.1.14):

Let $A = BC$ be any factorization where $C \in \mathcal{B}(\mathfrak{X})$ is outer and $B \in \mathcal{B}(\mathfrak{X})$ is inner with initial space $\overline{C\mathfrak{X}}$. B^*B Is the projection of \mathfrak{X} on $\overline{C\mathfrak{X}}$... There Fore?

$$A^*A = C^*B^*BC = C^*C$$

Uniqueness of the outer factor, C when $C_0 \geq 0$.

It remains to show that the inner factor B also unique. If B_1, B_2 are two inner operators with initial space $\overline{C\mathfrak{X}}$. Such that $B_1C = B_2C$, then B_1 and B_2 coincide on $\overline{C\mathfrak{X}}$. Since B_1 and B_2 are both zero on the orthogonal complement of $C\mathfrak{X}$,

$B_1 = B_2$. The result follows.

We give sufficient conditions on a nonnegative Toeplitz operator $T \in \mathcal{B}(\mathfrak{X})$ for the existence of an analytic operator $A \in \mathcal{B}(\mathfrak{X})$ such that $T = A^*A$.

Theorem (3.1.15):

Let $T \in \mathcal{B}(\mathfrak{X})$ be nonnegative Toeplit operator .If $T \geq \delta I$ for some number $\delta > 0$, then $T = A^*A$ for some analytic operator $A \in \mathcal{B}(\mathfrak{X})$.

Proof:

The hypotheses imply that T is invertible $\mathfrak{X}_T = \mathfrak{X}$, and $S_T = T^{1/2}ST^{-1/2}$. It follows that $S_T^{*n} \rightarrow 0$ strongly. So S_T is a shift operators, and the results follows.

Theorem (3.1.16):

Let $T_1, T_2 \in \mathcal{B}(\mathfrak{X})$ be two nonnegative Toeplitz operators with $T_1 \leq T_2$.

Assume that

- (i) $T_1 = A_1^*A_1$ for some analytic operator $A_1 \in \mathcal{B}(\mathfrak{X})$
- (ii) $\lim_{n \rightarrow \infty} \langle T_2 f_n, f_n \rangle = 0$ for every sequence $\{f_n\}_2^\infty$ in \mathfrak{X} such that $\lim_{n,k \rightarrow \infty} \langle T_2(f_n - f_k), f_n - f_k \rangle = 0$ and $\lim_{n \rightarrow \infty} \langle T_1 f, f_n \rangle = 0$
- (i) Then $T_2 = A_2^*A_2$ for some analytic operator $A_2 \in \mathcal{B}(\mathfrak{X})$

Proof:

The Lowdenslager is isometry S_{T_1} is a shift operator on \mathfrak{X}_{T_1} . We show that

S_{T_2} is a shift operator on \mathfrak{X}_{T_2} .

Since $T_1 \leq T_2$, for each $f \in \mathfrak{X}$

$$\|T_1^{1/2}f\|^2 = \langle T_1 f, f \rangle \leq \langle T_2 f, f \rangle = \|T_2^{1/2}f\|^2$$

Hence there is a unique $C \in \mathcal{B}(\mathfrak{X}_{T_1}, \mathfrak{X}_{T_2})$ such that

$$CT_2^{1/2}f = T_1^{1/2}f, f \in \mathfrak{X}.$$

The assumption implies that $\ker C = \{0\}$, and hence the range of C^* is dense in \mathfrak{X}_{T_2} . For each $f \in \mathfrak{X}$ and $n \geq 0$,

$$CS_{T_2}^n T_2^{1/2}f = CT_2^{1/2}S_T^n f = T_1^{1/2}S_T^n f = S_{T_1}^n T_1^{1/2}f = S_{T_1}^n CT_1^{1/2}f.$$

Thus $CS_{T_2}^n = S_{T_1}^n C$ and $S_{T_2}^{*n} C^* = C^* S_{T_1}^{*n}$ for all $n = 0, 1, 2, \dots$

Since S_{T_1} is a shift operator, $S_{T_1}^{*n} \rightarrow 0$ strongly on \mathfrak{X}_{T_1} . Hence $S_{T_2}^{*n} g \rightarrow 0$

for each $g \in C^* \mathfrak{X}_{T_1}$. Since the range of C^* is dense in \mathfrak{X}_{T_2} , $S_{T_2}^{*n} \rightarrow 0$ on \mathfrak{X}_{T_2} .

Sec (3.2): Scalar Analytic operators and Extremely Properties of outer operators

We call $V \in \mathcal{B}(\mathfrak{X})$ Scalar analytic if $AV = VA$ for every analytic A in $\mathcal{B}(\mathfrak{X})$.

Theorem (3.2.1):

An operator $V \in \mathcal{B}(\mathfrak{X})$ is scalar analytic if and only if its matrix. Where each entry is a scalar multiple of the identity operator on \mathfrak{X} .

Proof:

If V is scalar analytic, then V is analytic, so its matrix must further commute with $\text{diag}\{B_0, B_0, B_0, \dots\}$ for every $B_0 \in \mathcal{B}(\mathfrak{X})$ and hence the commute with all operators in $\mathcal{B}(\mathfrak{X})$. Therefore the entries of scalar multiples of the indent operator on \mathfrak{X} .

For the other direction it is enough to check that two matrices of commute if all entries of one commute with all entries of the other. The calculation is routine.

We state our result here but. We understand that whenever $A, C \in \mathcal{B}(\mathfrak{X})$ are analytic, their matrices are

$$\begin{bmatrix} A_0 & 0 & 0 & \dots \\ A_1 & A_0 & 0 & \dots \\ A_2 & A_1 & A_0 & \dots \\ \dots & & & \end{bmatrix}, \begin{bmatrix} C_0 & 0 & 0 & \dots \\ C_1 & C_0 & 0 & \dots \\ C_2 & C_1 & C_0 & \dots \\ \dots & & & \end{bmatrix}$$

Respectively, so for all $j = 0, 1, 2, \dots$,

$$A_j = P_0 S^{*j} A P_0 |_{\mathfrak{X}}$$

Theorem (3.2.2):

Let $C \in \mathcal{B}(\mathfrak{X})$ be outer, and let $V \in \mathcal{B}(\mathfrak{X})$ be scalar analytic. Then

$$\|V^* C f\| \leq \|V^* A f\|$$

For every analytic operator $A \in \mathcal{B}(\mathfrak{X})$ such that $A^* A = C^* C$ and every $f \in \mathfrak{X}$.

Proof:

If $A \in \mathcal{B}(\mathfrak{X})$ is analytic with $A^* A = C^* C$ $A=BC$ for some inner operator $B \in \mathcal{B}(\mathfrak{X})$ with initial space $\overline{C\mathfrak{X}}$. Applying lemma (3.2.6) with $g=Cf$, we obtain

Corollary (3.2.3):

If $C \in \mathcal{B}(\mathfrak{X})$ is outer, then

$$\sum_0^n C_j^* C_j \geq \sum_0^n A_j^* A_j, \quad n = 0, 1, 2, \dots$$

For every analytic operator $A \in \mathcal{B}(\mathfrak{X})$ such that $A^* A = C^* C$.

Proof:

By theorem (3.2.2). If $A \in \mathcal{B}(\mathfrak{X})$ be analytic with $A^* A = C^* C$. In theorem choose $V = S^{n+1}$ for a fixed $n \geq 0$ Set $f = P_0 h$ for an arbitrary $h \in \mathfrak{X}$.

Then yields

$$\|S^{*n+1} C P_0 h\| \leq \|S^{*n+1} A P_0 h\|.$$

By the arbitrariness of h ,

$$P_0 C^* S^{n+1} S^{*n+1} C P_0 \leq P_0 A^* S^{n+1} S^{*n+1} A P_0$$

Since $P_0 = 1 - S S^*$, we get

$$P_0 C^* \left(1 - \sum_0^n S^j P_0 S^{*j} \right) C P_0 \leq P_0 A^* \left(1 - \sum_0^n S^j P_0 S^{*j} \right) A P_0.$$

Since $C^* C = A^* A$, this is the same as

$$\sum_0^n P_0 C^* S^j P_0 S^{*j} C P_0 \geq \sum_0^n P_0 A^* S^j P_0 S^{*j} A P_0,$$

Theorem (3.2.4):

Let $C \in \mathcal{B}(\mathfrak{X})$ be analytic. Let $\{V_j\}_{j \in J} \subseteq \mathcal{B}(c)$ be scalar analytic operators, and let $\{f_k\}_{k \in K} \subseteq \mathfrak{X}$ be vectors such that

$$\|V_j^* C f_k\| \leq \|V_j^* A f_k\|, \quad j \in J, k \in K$$

For every analytic operator $A \in \mathcal{B}(\mathfrak{X})$ such that

$A^* A = C^* C$. Assume that:

- (i) The closure in the weak operator topology of the linear span of $\{V_j\}_{j \in J}$ contains S .
- (ii) The closed linear span of $\{P_0 f_k\}_{k \in K}$ is \mathfrak{X} .
- (iii) Then C is outer

Proof:

By there is factorization $C = BA$, where A is outer, B is inner with initial space $\overline{A\mathfrak{X}}$ and $A^*A = C^*C$ claim: B is S -constant we apply Lemma (3.2.7) with $g_k = Af_k$. For all $j \in J$, $k \in K$,

$$\|V_j^* Bg_k\| = \|V_j^* Cf_k\| \leq \|V_j^* Af_k\| = \|V_j^* g_k\|.$$

Since the reverse inequality is automatic by Lemma (3.2.6) holds. The only hypothesis in Lemma(3.2.7) that is not Cleary met is that the closed linear span of the vectors $P_0 g_k, k \in K$, is $P_0 M$, where $M = \overline{A\mathfrak{X}}$ the initial space of is B . To see this, note that $P_0 A = P_0 A P_0$, and hence

$P_0 g_k = P_0 Af_k = P_0 A P_0 f_k, k \in K$. Since by hypothesis the vectors $P_0 f_k, k \in K$ span a dense subset K , we have

$$\bigvee_{k \in K} \{P_0 g_k\} = \overline{P_0 A\mathfrak{X}} = \overline{P_0 A\mathfrak{X}} = P_0 M$$

The hypotheses of Lemma (3.2.7) are thus satisfied. By Lemma (3.2.7) B is S -constant.

Since B is S -constant and A is outer, $C = BA$ is also outer.

Theorem (3.2.5):

Let $C \in \mathcal{B}(\mathfrak{X})$ be analytic. The following are equivalent:

- (i) C is outer
- (ii) $C_0^* C_0 \geq A_0^* A_0$ for every analytic operator A such that $A^* A = C^* C$;
- (iii) for each $k \in \mathfrak{X}$,

$$\langle C_0^* C_0 k, k \rangle = \inf_{f \in \mathfrak{X}} \langle C^* C(k - Sf), k - Sf \rangle.$$

Moreover, if C is outer, then

$$\langle \sum_0^n C_j^* C_j k, k \rangle = \inf_{f \in \mathfrak{X}} \langle C^* C(k - S^{n+1}f), k - S^{n+1}f \rangle$$

For all $k \in \mathfrak{X}$ and $n = 0, 1, 2, \dots$

Proof:

(i) \Rightarrow (iii) Let C be outer, so $\overline{C\mathfrak{X}}$ reduces S . Fix $k \in \mathfrak{X}$ and $n \geq 0$, the infimum of $\|ck - S^{n+1}g\|$ overall $g \in \overline{C\mathfrak{X}}$ is attainable with $g = S^{*n+1}ck$.

Hence

$$\begin{aligned} & \inf_{f \in \mathfrak{X}} \langle C^* C(k - S^{n+1}f), k - S^{n+1}f \rangle \\ &= \inf_{f \in \mathfrak{X}} \|Ck - S^{n+1}Cf\|^2 \\ &= \inf_{g \in \mathfrak{X}} \|Ck - S^{n+1}g\|^2 = \|Ck - S^{n+1}S^{*n+1}Ck\|^2 \\ &= \|\sum_0^n S^j(1 - SS^*)S^{*j}Ck\|^2 = \sum_0^n \|P_0 S^{*j}Ck\|^2 \end{aligned}$$

$$= \sum_0^n \|C_j k\|^2 = \langle \sum_0^n C_j^* C_j k, k \rangle.$$

Thus holds and (iii) follows.

(iii) \Rightarrow (ii) Assume (iii), and let A be analytic with $A^* A = C^* C$. Then for any $k \in \mathfrak{X}$,

$$\begin{aligned} \langle C_0^* C_0 k, k \rangle &= \inf_{f \in \mathfrak{X}} \langle A^* A(k - Sf), k - Sf \rangle \\ &= \inf_{f \in \mathfrak{X}} \|Ak - SAf\|^2 \\ &\geq \inf_{g \in \mathfrak{X}} \|Ak - Sg\|^2 \\ &= \langle A_0^* A_0 K, K \rangle. \end{aligned}$$

Hence (ii) holds.

(ii) \Rightarrow (i), Assume (ii) $C = BA$ where A is outer and B is inner with initial space $\overline{A\mathfrak{X}}$ and $C^* C = A^* A$. By (ii), $C_0^* C_0 \geq A_0^* A_0$. Since $C = BA$, $C_0 = B_0 A_0$, Where $B_0 = P_0 B P_0|_{\mathfrak{X}}$ satisfies $\|B_0\| \leq 1$ Hence $C_0^* C_0 \leq A_0^* A_0$ so $C_0^* C_0 = A_0^* A_0$ and B_0 is isometric on $\overline{A\mathfrak{X}}$. for any $k \in \mathfrak{X}$,

$$\|A_0 k\| = \|B_0 A_0 k\| = \|P_0 B A_0 k\| \leq \|B A_0 k\| \leq \|A_0 k\|$$

Therefore equality holds throughout, and so

$$P_0 A_0 k = P_0 B A_0 k = B A_0 k.$$

Thus B and B_0 coincide on $\overline{A_0 \mathfrak{X}}$ Then BS^j and $S^j B_0$ coincide on $\overline{A_0 \mathfrak{X}}$ for any $j \geq 0$. Hence for any $j \geq 0$,

$$BS^j \overline{A_0 \mathfrak{X}} = S^j B_0 \overline{A_0 \mathfrak{X}} = S^j \overline{C_0 \mathfrak{X}}.$$

$$\overline{C\mathfrak{X}} = B \overline{A\mathfrak{X}} = \left(\sum_0^\infty \oplus S^j \overline{A_0 \mathfrak{X}} \right) = \sum_0^\infty \oplus S^j \overline{C_0 \mathfrak{X}}.$$

It follows that $\overline{C\mathfrak{X}}$ reduces S ; that is (i) holds. This completes the proof.

Lemma (3.2.6):

Let $B \in \mathcal{B}(\mathfrak{X})$ be a partial isometry with initial space M . If $V \in \mathcal{B}(\mathfrak{X})$, $VB = BV$, and $g \in M$, then

$$\|V^* g\| \leq \|V^* Bg\|$$

With equality if and only if $BV^* g = V^* Bg$.

Proof:

B^*B is the projection of \mathfrak{X} on M ? Hence $B^*Bg = g$ and

$$\|V^*g\| = \|V^*B^*Bg\| = \|B^*V^*Bg\| \leq \|V^*Bg\|.$$

If equality holds, then V^*Bg in the initial space of B^* , hence $V^*Bg = Bh$

for some $h \in M$. Then $B^*V^*Bg = B^*Bh, V^*BBg = B^*Bh$ and so

$V^*g = h$. Thus $V^*Bg = Bh = BV^*g$. Conversely, if $BV^*g = V^*Bg$, then

$$\|V^*Bg\| = \|BV^*g\| \leq \|V^*g\|$$

And so equality holds.

Lemma (3.2.7):

Let $B \in \mathcal{B}(\mathfrak{X})$ be inner with initial space M . let $\{V_j\}_{j \in J} \subseteq \mathcal{B}(\mathfrak{X})$ with

$V_jB = BV_j, j \in J$ and let $\{g_k\}_{k \in K} \subseteq M$ be vectors such that

$$\|V_j^*gk\| = \|V_j^*Bgk\|, \quad j \in J, \quad k \in K$$

Assume that:

- (i) The closure in the weak operator topology of the linear span of $\{V_j\}_{j \in J}$ contains S .
- (ii) The closed linear span of $\{P_0gk\}_{k \in K}$ is P_0M .

Then B is S -constant.

Proof:

By Lemma (3.2.6)

$$BV_j^*gk = V_j^*Bgk, \quad j \in J, \quad k \in K.$$

Hence by (I),

$$BS^*gk = S^*Bgk, \quad k \in K$$

Act on both sides with S and use $P_0 = 1 - S^*S$ to get

$$BP_0gk = P_0Bgk, \quad k \in K$$

Hence by (ii), $B\mathfrak{X} \subseteq \mathfrak{X}$, and so B has a diagonal matrix. Therefore B is

S -constant

These operators are unitarily equivalent by meant of the isomorphism

$U_{jk} \in \mathcal{B}(\mathfrak{X}_j, \mathfrak{X}_k)$ Such that

$$U_{jk}: \phi_j^{(n)} \rightarrow \phi_k^{(n)}, \quad n = 0, 1, 2, \dots$$

The isomorphism U_{34} is the Paley- wiener representation of $H^2(\mathbb{R})$. That is $U_{34} = F^{-1} \setminus L^2(0, \infty)$ where

$$F: f(x) \rightarrow \lim_{T \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^T e^{-ixt} f(t) dt$$

Is the Fourier-Plancherel operator on $L^2(-\infty, \infty)$ [$L^2(0, \infty)$ is viewed as a subspace of $L^2(-\infty, \infty)$].

Theorem (3.2.8):

Let $w \in L^\infty(\sigma)$ and $w \in L^\infty(-\infty, \infty)$ be related by

$$W(x) = w\left(\frac{x - \frac{1}{2}i}{x + \frac{1}{2}i}\right), \quad x \text{ real,}$$

Define operators T_1, T_2, T_3, T_4 as above. Then for each $j = 1, 2, 3, 4$

$$c_{m-n} = \langle T_j \phi_j^{(n)}, \phi_j^{(m)} \rangle_{\mathfrak{X}_j}, \quad m, n = 0, 1, 2, \dots$$

Therefore the operators T_1, T_2, T_3, T_4 are unitarily equivalent by means of the Isomorphism $U_{jk}, j, k = 1, 2, 3, 4$.

Proof:

By the definition of T_1 , for any $m, n = 0, 1, 2, \dots$

$$\begin{aligned} \langle T_1 \phi_1^{(n)}, \phi_1^{(m)} \rangle_{\mathfrak{X}_1} &= c_{m-n} \\ \int_{\Gamma} w(e^{it}) e^{it(n-m)} d\sigma(e^{it}) &= \langle T_2 \phi_2^{(n)}, \phi_2^{(m)} \rangle_{\mathfrak{X}_2}. \end{aligned}$$

Changing variables with the substitution $e^{it} = \left(x - \frac{1}{2}i\right) / \left(x + \frac{1}{2}i\right)$ we

obtain also

$$\begin{aligned} c_{m-n} &= \int_{\Gamma} w(e^{it}) e^{it(n-m)} d\sigma(e^{it}). \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{w(x)}{x^2 + \frac{1}{4}} \left(\frac{x - \frac{1}{2}i}{x + \frac{1}{2}i}\right)^{n-m} dx \end{aligned}$$

$$\begin{aligned}
&= \langle T_4 \phi_4^{(n)}, \phi_4^{(m)} \rangle_{\mathfrak{X}_4} \\
&= \langle QWF^{-1} \phi_3^{(n)}, F^{-1} \phi_3^{(m)} \rangle_{\mathfrak{X}_4} \\
&= \langle FQF^{-1} FWF^{-1} \phi_3^{(n)}, \phi_3^{(m)} \rangle_{\mathfrak{X}_3} \\
&= \langle P_+ FWF^{-1} \phi_3^{(n)}, \phi_3^{(m)} \rangle_{\mathfrak{X}_3} \\
&= \langle T_3 \phi_3^{(n)}, \phi_3^{(m)} \rangle_{\mathfrak{X}_3}.
\end{aligned}$$

For the next to last equality we used the relation $Q=F^{-1}PF$, which is a consequence of the Paley-Wiener representation of $H^2(R)$. The result follows.

If $K \in L^1(-\infty, \infty)$, then the operator

$$T : f(x) \rightarrow \int_0^{\infty} k(x-t)f(t)dt$$

Is everywhere defined and bounded on $L^2(0, \infty)$. where

$$w(x) = \int_0^{\infty} e^{ixt}k(t)dt .$$

Examples (3.2.9):

- (i) For $w(e^{it}) = Cost = \frac{1}{2}(e^{it} + e^{-it})$, we have

$$C_n : \begin{cases} 1/2 & \text{if } n = \pm 1 \\ 0 & \text{other wise} \end{cases}$$

And

$$W(x) = w\left(\frac{x - \frac{1}{2}i}{x + \frac{1}{2}i}\right) = \frac{x^2 - \frac{1}{4}}{x^2 + \frac{1}{4}} = 1 - \frac{1}{2} \int_{-\infty}^{\infty} e^{ixt} e^{-\frac{1}{2}|t|} dt.$$

Thus

$$T_1 \square \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots \\ & & \dots & & \end{bmatrix} \quad \text{On } \mathfrak{X}_1 = l^2$$

$$T_2 f = P(\cos t) f, \quad f \in \mathfrak{X}_2 = H^2(\Gamma)$$

$$T_3: f(x) \rightarrow f(x) - \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}|x-t|} f(t) dt, \quad f \in \mathfrak{X}_3 = L^2(0, \infty),$$

$$T_4 = P + \frac{x^2 - \frac{1}{4}}{x^2 + \frac{1}{4}} f, \quad f \in H_4 = H^2(R)$$

(ii) For any complex part meter P , $|P| < 1$, set

$$w(e^{it}) = \frac{1 - P^2}{1 - 2P \cos t + P^2}$$

Then $C_n = P^{|n|}$, $n = 0, \pm 1, \pm 2, \dots$, Setting $\mu = (1 + P)/(1 - P)$, we find that $\text{Re } \mu > 0$ and

$$w(x) = w\left(\frac{x - \frac{1}{2}i}{x + \frac{1}{2}i}\right) = \mu \frac{x^2 + \frac{1}{4}}{x^2 + \frac{1}{4}\mu^2} = \mu - \frac{1}{4}(\mu^2 - 1) \int_{-\infty}^\infty e^{ixt} e^{-\frac{1}{2}|t|} dt.$$

Thus

$$T_1 \square \begin{bmatrix} 1 & P & P^2 & P^3 & \dots \\ P & 1 & P & P^2 & \dots \\ P^2 & P & 1 & P & \dots \\ & & \dots & & \end{bmatrix} \quad \text{On } \mathfrak{X}_1 = l^2$$

$$T_2 f = P \frac{1 - P^2}{1 - 2P \cos t + P^2} f, \quad f \in \mathfrak{X}_2 = H^2(\Gamma),$$

$$T_3: f(x) \rightarrow \mu f(x) - \frac{1}{4} (\mu^2 - 1) \int_0^\infty e^{-\frac{1}{2}\mu|x-t|} f(t) dt, \quad f \in \mathfrak{X}_3 = L^2(0, \infty)$$

$$T_4 f = P + \mu \frac{x^2 - \frac{1}{4}}{x^2 + \frac{1}{4}\mu^2} f, \quad f \in \mathfrak{X}_4 = H^2(\mathbb{R}).$$

Any number of similar examples can in principle be constructed. However, it is typically the case that an operator is simple and 'natural' in new scheme and complicated or unrecognizable in another. We invite the reader to try the example

$$T: f(x) \rightarrow \int_0^\infty K(x-t) f(t) dt, \quad f \in L^2(0, \infty) \quad \text{Where}$$

$$K(x) = \int_{|x|}^\infty y^{-1} e^{-y} dy.$$

Chapter 4
Concrete Spectral Theory

In this chapter we sketch the explicit diagonalization of a self-ad joint Toeplitz operator when the underlying shift has multiplicity 1.

1. Notion and Preliminaries. Let S be multiplication by $e^{i\theta}$ on $H^2(\Gamma)$, and Let P be the Projection $L^2(\sigma)$ of on $H^2(\Gamma)$. For any $w \in L^\infty(\sigma)$ define $T(w)$ on $H^2(\Gamma)$ by

$$T(w)f = Pwf \quad , \quad f \in H^2(\Gamma)$$

$T(w)$ is S -Toeplitz and $\|T(w)\| = \|w\|_\infty$. Every S -Toeplitz operator has this for. Moreover:

- (i) $T(w)$ is self-ad joint if and only if w is essentially real valued;
- (ii) $T(w) \geq 0$ if and only if $w \geq 0 \sigma - a.e$;
- (iii) If $a \in H^\infty(\Gamma)$, then $T(w) = T(a)^*T(a)$ if and only if $w = |a|^2 \sigma - a.e$.

2. Let w be a real valued function in $L^\infty(\sigma)$. Then $T(w)$ is self- ad joint with spectrum

$$SP(T(w)) = [c, d], \text{ where } c = \text{ess inf } w \text{ and } d = \text{ess sup } w.$$

If w is not equal $\sigma - a.e$ to a constant. Then $T(w)$ has no point spectrum.

Theorem (4.1):

Let w be a real valued function in $L^\infty(\sigma)$, and set $c = \text{ess inf } w$. For any $\alpha, \beta \in D$ and $Z \in C \setminus [c, \infty)$

$$\langle (T(w) - zI)^{-1}(1 - \bar{\alpha}e^{it})^{-1}, (1 - \bar{\beta}e^{it})^{-1} \rangle_2 \\ \bar{\alpha}(\alpha, \bar{z})a(\beta, z)/(1 - \bar{\alpha}\beta)$$

Proof:

Where for $\alpha \in D$ and $z \in c \setminus [c, \infty)$

By no analyticity it is enough to Prove for $z = x, x < c$.

By the Lemma (4.2) for such x ,

$$(T(w) - xI)^{-1} = T(a(\cdot, x))T(a(\cdot, x))^*$$

$$a(\alpha, z) = \exp\left(-\frac{1}{2} \int_{\Gamma} \frac{e^{it} + \alpha}{e^{it} - \alpha} \log[w(e^{it}) - z] d\sigma\right)$$

With the principal branch of the logarithm,

Lemma (4.2):

If $\delta \geq \sigma - a.e.$ For some $\delta > 0$, then $T(w)$ is invertible and

$T(w)^{-1} = T(a)T(a)^*$ where $a \in H^\infty(\Gamma)$ is any function such that

$$1/w = |a|^2 \quad \sigma - a.e. .$$

Proof:

By no. 1(iii) $T(w) = T(1/a)^*T(1/a)$.

Theorem (4.3):

Let w a real valued function in $L^\infty(\sigma)$. If w is not equal $\sigma - a.e.$ To a constant, then $T(w)$ is absolutely continuous.

Proof:

Let $T(w) = \int f(t) dE(t)$. For $\alpha \in D$ set $K_\alpha(e^{i\theta}) = (1 - \bar{\alpha}e^{i\theta})^{-1}$, the

function $t \rightarrow \langle E(t)_{K_\alpha}, K_\alpha \rangle_z$ is continuous on $(-\infty, \infty)$ and constant on

$(-\infty, c]$, $[d, \infty)$ ($c = \text{ess inf } w$, $d = \text{ess sup } w$)

$z \in \mathbb{C} \setminus [c, \infty)$

$$\begin{aligned} & \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\langle E(t)k_\alpha, k_\alpha \rangle_2}{(t-x)^2 + y^2} \\ &= \frac{1}{\pi} \text{Im} \left\langle [T(w) - zI]^{-1} K_\alpha, K_\alpha \right\rangle_2 \\ &= \pi^{-1} (1 - |\alpha|^2)^{-1} \text{Im} \bar{a}(\alpha, \bar{z}) a(\alpha, z) \\ &= \pi^{-1} (1 - |\alpha|^2)^{-1} \text{Im} \exp\left(-\int_{\Gamma} P(\alpha, e^{i\theta}) \log[w(e^{i\theta}) - z] d\sigma\right) \end{aligned}$$

$$\begin{aligned}
&= \pi^{-1}(1 - |\alpha|^2)^{-1} \exp \left(- \int_{\Gamma} P(\alpha, e^{i\theta}) \log |w(e^{i\theta}) - z| d\sigma \right) \cdot \sin \left(- \int_{\Gamma} P(\alpha, e^{i\theta}) \arg [w(e^{i\theta}) - z] d\sigma \right) \\
&\leq \pi^{-1}(1 - |\alpha|^2)^{-1} \exp \left(- \int_{\Gamma} P(\alpha, e^{i\theta}) \log |w(e^{i\theta}) - x| d\sigma \right)
\end{aligned}$$

By Lemma (4.5) and the Stieltjes inversion formulas the function $t \rightarrow \langle E(t)K_\alpha, K_\alpha \rangle_2$ is absolutely continuous on (c, d) . Hence K_α belongs to the absolutely continuous sub space \mathfrak{X}_{ac} for $T(w)$. Since $\alpha \in D$ is arbitrary, $\mathfrak{X}_{ac} = H^2(\Gamma)$ and the result follows

Lemma (4.4):

For almost all x ,

$$\int_{\Gamma} \log \|w(e^{i\theta}) - x\| d\sigma < \infty.$$

Proof:

For $t > \|w\|_{\infty}$ and $e^{i\theta} \in \Gamma$,

$$\int_{-t}^t \log |w(e^{i\theta}) - x| dx = \int_0^{t-w(e^{i\theta})} \log y dy + \int_0^{t+w(e^{i\theta})} \log y dy$$

$[t - w(e^{i\theta})] \log [t - w(e^{i\theta})] + [t + w(e^{i\theta})] \log [t + w(e^{i\theta})] - 2t \geq K_t$,
Where K_t , is a constant, $K_t > -\infty$? Hence

$$\int_{\Gamma} \int_{-t}^t \log |w(e^{i\theta}) - x| dx d\sigma > -\infty$$

And the result follows.

Lemma (4.5):

If $ess \inf w < s < t < ess \sup w$ And $\alpha \in D$,

$$\int_s^t \exp \left(- \int_{\Gamma} P(\alpha, e^{i\theta}) \log |w(e^{i\theta}) - x| d\sigma \right) dx < \infty$$

Proof:

The function $V_\alpha(z) = \text{Im } \bar{\alpha}(\alpha, \bar{z})\alpha(\alpha, z)$ is positive and harmonic on Π , so

$$\int_{-\infty}^{\infty} V_{\alpha}(x)(1+x^2)^{-1}dx < \infty$$

For any fixed $e^{i\theta} \in \Gamma$.

$$\lim_{y \downarrow 0} \log[w(e^{i\theta}) - x - iy] = \log|w(e^{i\theta}) - x| - i\pi\chi_{y(x)}(e^{i\theta}),$$

Where $y(x) = \{e^{it} : w(e^{i\theta}) < x\}$. Thus

$$\begin{aligned} V_{\alpha}(z) &= \lim_{y \downarrow 0} \operatorname{Im} \exp \left(-\frac{1}{2} \int_{\Gamma} \frac{e^{-i\theta} + \bar{\alpha}}{e^{-i\theta} - \bar{\alpha}} \log[w(e^{i\theta}) - x - iy] d\sigma \right. \\ &\quad \left. - \frac{1}{2} \int_{\Gamma} \frac{e^{i\theta} + \alpha}{e^{i\theta} - \alpha} \log[w(e^{i\theta}) - x - iy] d\sigma \right) \\ &= \operatorname{Im} \exp \left(- \int_{\Gamma} P(\alpha, e^{i\theta}) [\log|w(e^{i\theta}) - x| - i\pi\chi_{y(x)}(e^{i\theta})] d\sigma \right) \\ &= \exp \left(- \int_{\Gamma} P(\alpha, e^{i\theta}) \log|w(e^{i\theta}) - x| d\sigma \right) \sin \left(\pi \int_{y(x)} P(\alpha, e^{i\theta}) d\sigma \right) \end{aligned}$$

Theorem (4.6):

There is a direct integral Hilbert space

$$\mathfrak{X} = \int_X \oplus \mathfrak{X}(x) d\mu$$

Where $X = SP(T)$, such that T is unitarily equivalent to multiplication by x on \mathfrak{X} .

We shall not prove this theorem, but we include the definition of the space.

X Is a compact sub set $(-\infty, \infty)$, and μ is a finite non negative Borel measure on X .

For μ -a. e. $x \in X$, $\mathfrak{X}(x)$. Is a separable Hilbert space.

A class M of ‘measurable ‘function is assumed given such that:

(M1) He elements of M are function for X such that $f(x) \in \mathfrak{X}(x)\mu$ -a. e

(M2) For any $f, g \in M$ the scalar valued function $x \rightarrow \langle f(x), g(x) \rangle_{\mathfrak{X}(x)}$ μ is- measurable

(M3) If g is a function on X such that $g(x) \in \mathfrak{X}(x)\mu$ -a. e. And $x \rightarrow \langle f(x), g(x) \rangle_{\mathfrak{X}(x)}$ is -measurable for all $f \in M$, then $g \in M$.

(M4) There is a sequence $\{p_j\}_1^{\infty} \subseteq M$ such that $\mathfrak{X}(x) = \vee \{p_j(x) : j = 1, 2, \dots\}$ μ -a. e

There is a sequence B_1 such that B_1 .

Function that are equal μ -a. e. are identified. The Hilbert space is defined as the

Space of all $f \in \mathcal{M}$ such that $\int_x \|f(x)\|_{\mathfrak{X}(x)}^2 d\mu < \infty$ in the inner product

$$\langle f, g \rangle = \int_x \langle f(x), g(x) \rangle_{\mathfrak{X}(x)} d\mu$$

In order to define the space it is thus necessary to specify a class of functions satisfying $(M_1) - (M_4)$.

In applications, the following result is helpful for this purpose

Lemma (4.7):

Let μ be a finite nonnegative Borel measure on a compact set $X \subseteq (-\infty, \infty)$.

For μ -a. e. $x \in X$, let $\mathfrak{X}(x)$ be a separable Hilbert space. Assume given a

sequence $\{q_j\}_1^\infty$ of function on X such $q_j(x) \in \mathfrak{X}(x)$. For each $j \geq 1$,

and

(i) for each $j, k \geq 1$, $x \rightarrow \langle q_j(x), q_k(x) \rangle_{\mathfrak{X}(x)}$ is μ -measurable, and

(ii) $\bigvee \{q_j(x) : j = 1, 2, \dots\} = \mathfrak{X}(x)$ μ -a. e.

. Define M to be class of all function f on X such the

. And $f(x) \in \mathfrak{X}(x)$ μ -a. e and $x \rightarrow \langle f(x), q_j(x) \rangle_{\mathfrak{X}(x)}$ is μ measurable for

Each $j \geq 1$. Then μ satisfies $(M1) - (M4)$, and M is the only such

containing $\{q_i\}_1^\infty$

In the situation of theorem A, we define $m(x) = \dim \mathfrak{X}(x)$ μ -a. e on

$SP(T)$ we call a multiplicity function for T . The quantities

$(sp(T), m, \mu)$ are called the unitary invariants for T . The terminology is justified by the following result.

Theorem (4.8):

Let T_j be a bounded self-adjoint operator on a separable Hilbert space H_j

write associated triple $(sp(T_j), m_j, \mu_j)$ as above $j = 1, 2$. Then T_1 and

T_2 are unitarily equivalent if and only if (i) $SP(T_1) = SP(T_2)$,

(ii) μ_1 and μ_2 are mutually absolutely continuous, that is, they have the same class of null sets, and (iii) $m_1 = m_2$ μ_j -a. e. ($j = 1, 2$).

Proof:

T is absolutely continuous and $sp(T) = [c, d]$. Hence if T has spectral representation

$T = \int x dE(x)$, then for any $f, g \in H^2(\Gamma)$,

$$\langle Tf, g \rangle_2 = \int_c^d x \frac{d}{dx} \langle E(x)f, g \rangle_2 dx.$$

The strategy of the proof is to use the generation's formula for resolvents to compute this for, $f = k_\alpha, g = k_\beta, \alpha, \beta \in D$ where for any $\alpha \in D$

$$k_\alpha(e^{i\theta}) = (1 - \alpha e^{-i\theta})^{-1} \text{ On } \Gamma.$$

$$\begin{aligned} \frac{d}{dx} \langle E(x)K_\alpha, K_\beta \rangle &= \lim_{y \downarrow 0} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d \langle E(t)K_\alpha, K_\beta \rangle_2}{(t-x)^2 + y^2} \\ &= \lim_{y \downarrow 0} \frac{1}{2\pi i} (\langle (T - zI)^{-1}k_\alpha, k_\beta \rangle_2 - \langle (T - \bar{z}I)^{-1}k_\alpha, k_\beta \rangle_2) \\ &= \lim_{y \downarrow 0} \frac{1}{2\pi i} \frac{\bar{a}(\alpha, \bar{z})a(\beta, z) - \bar{a}(\alpha, z)a(\beta, \bar{z})}{1 - \bar{\alpha}\beta} \\ &= \frac{1}{2\pi i} \frac{\bar{a}(\alpha, x-i0)a(\beta, x+i0) - \bar{a}(\alpha, x+i0)a(\beta, x-i0)}{1 - \bar{\alpha}\beta} \\ &= \xi_\alpha(x) \bar{\xi}_\beta(x) \frac{\phi_x(\beta) + \bar{\phi}_x(\alpha)}{1 - \bar{\alpha}\beta} \end{aligned}$$

Hence, for any $\alpha \in D$ $\xi_\alpha(x) = \bar{a}(\alpha, x+i0)$ and $\phi_x(\alpha) = -\frac{1}{2\pi i} \frac{a(\alpha, x-i0)}{a(\alpha, x+i0)}$.

The limits $a(\alpha, x \pm i0) = \lim_{y \downarrow 0} a(\alpha, x \pm iy)$ exist for all x satisfying

$$\begin{aligned} a(\alpha, x \pm i0) &= \exp\left(-\frac{1}{2} \int_\Gamma \frac{e^{it+\alpha}}{e^{it-\alpha}} \log|w(e^{it}) - x| d\sigma\right) \\ &\cdot \exp\left(\pm \frac{1}{2\pi i} \int_{\Gamma/E_x} \frac{e^{it+\alpha}}{e^{it-\alpha}} d\sigma\right), \end{aligned}$$

Where $E_x = \{e^{it} : w(e^{it}) \geq x\}$ as in (ii) Thus

$$\begin{aligned} \phi_x(\alpha) &= -\frac{1}{2\pi i} \exp\left(-\pi i \int_{\Gamma/E_x} \frac{e^{it+\alpha}}{e^{it-\alpha}} d\sigma\right), \\ &= \frac{1}{2\pi i} \exp\left(\pi i \int_{\Gamma/E_x} \frac{e^{it+\alpha}}{e^{it-\alpha}} d\sigma\right). \end{aligned}$$

By the lemma (4.9),

$$\frac{\phi_x(\beta) + \bar{\phi}_x(\alpha)}{1 - \bar{\alpha}\beta} = \int_\Gamma \frac{dv_x}{(1 - \bar{\alpha}e^{it})(1 - \beta e^{-it})} = \langle k_\alpha, k_\beta \rangle_{L^2(v_x)},$$

Where V_x is nonnegative singular Boral measure on Γ such that $\dim L^2(V_x) =$

Index of E_x

Hence

$$\frac{d}{dx} \langle E(x)k_\alpha, k_\beta \rangle_2 = \xi_\alpha(x) \overline{\xi_\beta(x)} \langle k_\alpha, k_\beta \rangle_{L^2(v_x)},$$

And

$$\langle T \langle k_\alpha, k_\beta \rangle_2 = \int_c^d x \xi_\alpha(x) \overline{\xi_\beta(x)} \langle k_\alpha, k_\beta \rangle_{L^2(v_x)} dx.$$

Lemma (4.9):

Let $E \subseteq \Gamma$. Be a Boral set, and let

$$\phi(z) = \frac{1}{2\pi i} \exp \left(\pi i \int_E \frac{e^{it} + z}{e^{it} - z} d\sigma \right), \quad z \in D$$

The $\text{Re}\phi(z) > 0$ none D , and $\text{Re}\phi(e^{i\theta}) = 0$ σ -a. e on Γ . There is a nonnegative singular Borel measure V on Γ such that

$$\frac{\phi(\beta) + \overline{\phi(\alpha)}}{1 - \beta\overline{\alpha}} = \int_\Gamma \frac{dv}{(1 - \overline{\alpha}e^{it})(1 - \beta e^{-it})}, \quad \alpha, \beta \in D$$

We have $L^2(v) = \text{index of } E$

Proof:

For any $z \in D$

$$\text{Re}\phi(z) = (2\pi)^{-1} \exp \left(\text{Re} \pi i \int_E \frac{e^{it} + z}{e^{it} - z} d\sigma \right) \sin \left(\pi \int_E P(z, e^{it}) d\sigma \right)$$

Hence $\text{Re}\phi(z) > 0$ on D . By Fatou's theorem. The sine factor tends none

tangentially to $\sin(\pi \chi_E(e^{i\theta})) = 0$ σ -a. e , and hence

$\text{Re}\phi(e^{i\theta}) = 0$ σ -a. e . By the Riesz Herglotz theorem, there nonnegative Borel measure V on Γ such that holds. Since, $\text{Re}\phi(e^{i\theta}) = 0$ σ -a. e is singular by Fatou's theorem.

Suppose that the index of E is a positive integer n . Then modulo $a\sigma$ null set,

$E = E_1 \cup \dots \cup E_n$ where, $E_j = \{e^{i\theta} : a_j \leq \theta \leq b_j\}, j = 1, \dots, n$ and the arcs proper and disjoint. By direct calculation,

$$\phi(z) = \frac{1}{2\pi i} e^{-i\pi\sigma(E)} \prod_1^n \frac{e^{ib_j} - z}{e^{ia_j} - z}.$$

Hence ϕ is rational with n simple poles, which all lie on Γ . Therefore n consists of n point masses, and so.

Conversely, let $\dim L^2(v)$ be positive integer n . Then V consists of n point masses, ϕ is a rational function with n simple poles, all on Γ .

For $z \in D$,

$$\operatorname{arg} i\phi(Z) = \pi \int_E p(z, e^{i\theta}) d\sigma$$

Where the argument is chosen in $[0, \pi]$. Passing to the boundary, we obtain

$$\operatorname{arg} i\phi(e^{i\theta}) = \pi \chi_E(e^{i\theta}) \quad \sigma\text{-a. e}$$

Since $i\phi(e^{i\theta})$ is contains except at the zero and Poles of ϕ, E is anion of intervals modulo σ -null set. Hence by the argument of the Preceding paragraph, $L^2(V) = n = \text{index of } E$.

Theorem (4.10):

There is a unique isometry V mapping $H^2(\Gamma)$ onto $L^2(p)$ such that for all $\alpha \in D$

$$(Vk_\alpha)(x) = [\bar{\psi}(\alpha, x)(1 - \bar{\alpha}e^{ia(x)})^{\frac{1}{2}}((1 - \bar{\alpha}e^{ib(x)})^{\frac{1}{2}})^{-1}],$$

Where $\{K_\alpha\}_{\alpha \in D}$ is as Further more:

- (i) VTV^{-1} is multiplication by x on $L^2(p)$, and
- (ii) if $f \in L^2(P)$ then for all $\alpha \in D$,

$$(V^{-1}f)(\alpha) = \int_c^d f(x) \overline{(Vk_\alpha)(x)} dp(x)$$

Proof:

This can be deduced from the constructions for all $\alpha, \beta \in D$,

$$\phi_x(\alpha) = \frac{1}{2\pi} e^{\frac{1}{2}i(b-a)} \frac{1 - \alpha e^{-ib}}{1 - \alpha e^{-ia}}$$

And so

$$\frac{\phi_x(\beta) + \overline{\phi_x(\alpha)}}{1 - \bar{\alpha}\beta} = p'(x)(1 - \bar{\alpha}e^{ia})^{-1}(1 - \beta e^{ia})^{-1}$$

Then we obtain

$$\frac{d}{dx} \langle E(x)k_\alpha, k_\beta \rangle_2 = p'(x)[\bar{\psi}(\alpha, x)(1 - \bar{\alpha}e^{ia})^{\frac{1}{2}}(1 - \beta e^{ib})^{\frac{1}{2}}]^{-1} [\psi(\beta, x)(1 - \bar{\alpha}e^{ia})^{\frac{1}{2}}(1 - \beta e^{ib})^{\frac{1}{2}}]^{-1}$$

Example (4.11):

Let $w(e^{it}) = \cos t$ then $[c, d] = [-1, 1]$, and we can choose

$$b(x) = -a(x) = \arccos x \text{ for } -1 < x < 1$$

Thus

$$p'(x) = \pi^{-1}(1 - x^2)^{\frac{1}{2}}, -1 < x < 1.$$

Since

$$|\cos t - x| = \frac{1}{2} |1 - 2xe^{it} + e^{2it}|$$

We see by in section that

$$|\cos t - x| = \frac{1}{2} |1 - 2xe^{it} + e^{2it}|,$$

$$\psi(\alpha, x) = 2^{-\frac{1}{2}} (1 - 2x\alpha + \alpha^2)^{\frac{1}{2}}$$

It follows that

$$(VK_\alpha)(x) = 2^{\frac{1}{2}} (1 - 2x\bar{\alpha} + \bar{\alpha}^2)^{-1}$$

$$= \sum_{n=0}^{\infty} 2^{\frac{1}{2}} U_n(x) \alpha^{-n}$$

For all $\alpha \in D$, where $\{U_n(x)\}_0^\infty$ are Chebychev polynomials. If $f \in L^2(p)$ then

$$(V^{-1})(\alpha) = \frac{2^{\frac{1}{2}}}{\pi} \int_{-1}^1 f(x) (1 - 2x\alpha + \alpha^2)^{-1} (1 - x^2)^{\frac{1}{2}} dx,$$

$\alpha \in D$. Operator V diagonalizes $T(w)$.

Example (4.12):

For fixed $K > 0$ consider the Wiener-Hopf operator

$$(T_k f)(x) = \int_0^\infty e^{-k|x-t|} f(t) dt, \quad f \in L^2(0, \infty).$$

Then T_K is diagonalized by the isometric operator U_k mapping $L^2(0, \infty)$ onto $L^2(v_k)$ such that

$$dv_k(\omega) = \frac{2}{\pi} \frac{d\omega}{\omega^2 + k^2} \quad \text{on}(0, \infty),$$

With

$$(U_k f)(\omega) = \int_0^\infty (\omega \cos \omega t + K \sin \omega t) f(t) dt$$

For each $f \in L^2(0, \infty) \cap L^1(0, \infty)$, and

$$(U_k^{-1} g)(t) = \int_0^\infty (\omega \cos \omega t + K \sin \omega t) g(\omega) dv_k(\omega)$$

For each $g \in L^2(v_k) \cap L^1(v_k)$, we find that

$$(U_k T_k U_k^{-1} g)(\omega) = \frac{2k}{\omega^2 + k^2} g(\omega)$$

For each $g \in L^2(v_k)$.

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