Chapter 1
Nevanlinna and Hardy classes of vectors

The purpose of this chapter is to set down the most basic concerning Nevanlinna and Hardy classis of vector and operator valued holomorphic function. The emphasis in the chapter is on characterizations of the classes and boundary behavior. After these ideas have been worked out, generalizations; of many familiar results from the scalar theory follow in a routine way. Example of such results is given in Great generality and completeness is not objectives of the chapter. In our choice of material we are guided mainly by what is needed for subsequ8ent applications.

We assume familiarity with the scalar theory of Nevalinna and Hardy classes on a disk or half-plane. What we need may be found, for example, in Daren, Hoffman, and Krylov. Other prerequisites from the theory of subharomonic functions are collected .These are given in Hille and Phillips.

Let X denotes a complex Banach space with, norm $|.|_{x}$.

We write $\mathscr{G}$ for a separable Hilbert space and $|.|_{\mathscr{G}}$ for the norm and inner product on $\langle .,. \rangle_{\mathscr{G}}$. The norm on $\mathscr{G}$. The space. Of bounded linear operator on $\mathscr{G}$, is written $|.|_{(\mathscr{G})}$.

Sec (1.1):Nevanlinna Hardy_orliez classes and their characterization:

The key definitions are conveniently made in terms of harmonic major ants for sub harmonic functions. To begin we show how sub harmonic function arises in the study of holomorphic functions with values in a Banach space.

Theorem (1.1.1):

If $f(z)$ is a zoomorphic X-valued function on a region $\Omega \subseteq C$,then each of the functions listed below is sub harmonic on Ω:

(i) $\log|f(z)|_{x}$,

(ii) $\log^{+}|f(z)|_{x}$,

(iii) $|f(z)|_{x}^{p}\ for 0<p<\infty$, and

(iv) $\phi\left(\log^{+}|f(z)|_{x}\right)$,, where ϕ is any no decreasing convex function $(-\infty,\infty)$

Here $\log^{*} t=\max(\log t,0)\ for t>0 and \log 0=-\infty$

Proof:

We show that $\log\left|f(z)\right|_x$ is sub harmonic on Ω. Let $\bar{D}(a,r)\subseteq\Omega$, and let $p(z)$ any polynomial such that $\log\left|f(z)\right|_x \leq Rep(t)$ on $\partial D(a,r)$.Then $\left|\exp(-p(z))f(z)\right|_x \leq 1$ on $\partial D(a,r)$.By the maximum principle (Hille and Phillips), the same inequality extends to $D(a,r)$, and therefore $\log\left|f(z)\right|_x \leq Rep(t)$ on $D(a,r)$. Hence $\log\left|f(z)\right|_x$ is sub harmonic on Ω.

Once it is known that $\log\left|f(z)\right|_x$ is sub harmonic, it follows by standard properties of sub harmonic functions that all of the functions listed in the theorem are sub harmonic.

Definition (1.1.2):

Let $\Omega\subseteq\mathbb{C}$ be any region.

(i) A holomorphic X-valued function $f(z)$ on Ω is of bounded type on Ω if $\log^{+}\left|f(z)\right|_x$ has a harmonic major ant on Ω. The class of all such functions is denoted $N_x(\Omega)$.

(ii) If ϕ is any strongly convex function then by $\mathscr{K}_{\phi.x}(\Omega)$ we mean the class of all holomorphic X-valued functions $f(z)$ on Ω such that $\phi\left(\log^{+}\left|f(z)\right|_x\right)$ has a harmonic major on Ω.

(iii) We define $N_x^{+}(\Omega)\bigcup\mathscr{K}_{\phi,x}(\Omega)$, where the union is over all strongly convex function ϕ.

(iv) By $H_x^{\infty}(\Omega)$ we mean the set of all bounded holomorphic X-valued functions on Ω.

The sets $N_X(\Omega)$ and $N_x^{+}(\Omega)$ are called Nevanlinna classes, and $\mathscr{K}_{\phi.x}(\Omega)$ is aHardy-Orlicz class. The term ‘bounded type' comes from the property expressed in Theorem (1.1.5) below.

When $X=C$ in the absolute valued norm, we droop the subscript X, and write simply $N(\Omega), N^{+}(\Omega), \mathscr{K}_{\phi}(\Omega), H^{\infty}(\Omega)$ for the classes. We refer to this as the scalar case

Theorem (1.1.3):

For any region Ω and strongly convex function $\phi, H_x^{\infty}(\Omega)$, $\mathscr{K}_{\phi,x}(\Omega) N_x^{+}$ and $N_x(\Omega)$ are linear spaces and

$$H_x^{\infty}(\Omega) \subseteq \mathscr{K}_{\phi,x}(\Omega) \subseteq N_x^{+}(\Omega) \subseteq N_x(\Omega). \tag{1}$$

Proof:

We use the elementary inequalities

$$\log^{+}(xy) \leq \log^{+} + \log^{+} y،$$

$$\log^{+}(x+y) \leq \max\left(\log^{+}(2x)\right) \leq \log^{+} x \log^{+} y + \log 2$$

In the proof

It is clear that $H_x^{\infty}(\Omega)$ is a linear space.

Let $f, g \in \mathscr{K}_{\phi,x}$ and let $a \in \mathbb{C}$.Then

$$\phi\left(\log^{+}|\alpha f|_x\right) \leq \phi\left(\log^{+}|f|_x + \log^{+}|\alpha|\right) \leq M\left(\log^{+}|f|_x\right) + K$$

For constants $M \geq 0$ and $K \geq 0$by properties of a strongly convex function since $f \in \mathscr{K}_{\phi,x}(\Omega)$, the right side has a harmonic majorant is Ω .Hence $\alpha f \in \mathscr{K}_{\phi,x}(\Omega)$.Examining separately the cases $|f(z)|_x \leq |p(z)|_x$ and $|f(z)|_x > |g(z)|_x$. We see that

$$\phi\left(\log^{+}|f+g|_x\right) \leq \phi\left(\log^{+}|2f|_x\right) + \phi\left(\log^{+}|2g|_x\right).$$

It follows easily that $f + g \in \mathscr{K}_{\phi,x}(\Omega)$ and so $\mathscr{K}_{\phi,x}(\Omega)$ is linear space.

The proof that $N_x^{+}(\Omega)$ is a linear space is straightforward once it is known that for any two strongly convex functions ψ_1, andψ_2 there exists a strongly convex function ψ such that $\psi \leq \psi_1$, and $\psi \leq \psi_2$.To see this ,result that a convex function is the integral of a no decreasing function. By the properties of a strongly convex function we can write

$$\psi_2(x) = \int_{-\infty}^{x} g_j(t)\,dt + c_p \qquad -\infty < x < \infty,$$

Where g_j is nonnegative and no decreasing on $(-\infty,\infty), g(t) \to \infty$ as $t \to \infty$, and$c_j \geq 0, f = 1,2$. Constant in any way a nonnegative and no decreasing function$g o(-\infty,\infty)$such that $Q(t) \to \infty$ as $t \to \infty, g \leq g_1$ and $g \leq g_2$ on$(-\infty,\infty)$, and$g(t+1) \leq 2g(t)$ for all real$t$. These conditions. Then in a straightforward way we see that

$$\psi(x)=\int_{-\infty}^{x} g(t)dt, \quad -\infty<x<\infty,$$

Have the required properties. It follows that $N_x^+(\Omega)$ is a linear space.

The fact that $N_x(t)$ is a linear space follows from the inequalities

$$\log^+|\alpha f|_x \le \log^+|f|_x + \log^+|x|,$$

$$\log^+|f+g|_x \le \log^+|f|_x + \log^+|g|_+ \log 2,$$

Which hold for any $f_, g \in N_2(\Omega)$ and $\alpha \in \mathbb{C}$.

The first two inclusions in are obvious. If $f \in n_x^+(\Omega)$ then $f \in \mathscr{K}_{\phi,x}(\Omega)$ for some strongly convex function ϕ. Choose $\alpha>0$ such that $\psi(t)/t \ge 1$ for $t>a$. Then $\log^+|f|_x \le \psi\left(\log^+|f|_x\right)+a$ .Therefore $f \in N_x(\Omega)$ and the third inclusion of follows.

Theorem (1.1.4):

Let f belong to one of classes $H\ (\Omega), \mathscr{K}_{\phi,}(\Omega), N_x^+(\Omega)$ or $N_x(\Omega)$ for some region Ω.

(i) If h is holomorphic on a region Ω' and $h(\Omega') \subseteq \Omega$, then $f \in h$ belong to the corresponding class on Ω'

(ii)If Ω'' is a region contained is Ω.then $f|\Omega''$ belongs to the corresponding class on Ω''.

Proof:

The assertions are immediate from the definition of the classes.

Theorem (1.1.5):

Let f be a holomorphic X-valued function on Ω.

(i) A sufficient condition for f to belong to $N_x(\Omega)$ is that $f=g/u$, where $g \in H_x^\infty(\Omega)$ and u is a scalar valued holomorphic function $0<|u|\le 1$ on Ω.

(ii) If Ω is simply connected, then the sufficient condition of (t) is also necessary.

Proof:

(i)Let $f=g/u$ in (i).We can assume without loss of generality that $|g|_x \leq 1$ on Ω.Then

$$\log^+|f|_x \leq \log^+|g|_x + \log(1/|u|) = -\log|u|$$

On Ω. Since $-\log|u|$ is harmonic, $f \in N_x(\Omega)$. This proves (i).

(ii) Assume that Ω is simply connected and $f \in N_x(\Omega)$.Let (i) be a harmonic Majorant for $\log^+|f|_x$. For each disk $D(a,y) \subseteq \Omega$ there is a holomorphic function k , on $D(a,r)$ such that $k_{a,r} = h$ on $D(a,r)$.By the monodromy theorem (Rudin), there a holomorphic function k on Ω such that $\mathrm{Re}k = k$ on Ω.

Then $f = g/u$.where $g = fe^{-k}$ and $u = e^{-k}$ have the required properties. For example,

$$\log|f|_x \leq \log^+|f|_x \leq h = \mathrm{Re}k$$

And so $|g|_x = |fe^{-k}|_x \leq 1$. The theorem follows.

When the region Ω is a disk or half-plane, the defining properties for the Nevanlinna and Hardy-Ortiz classes have useful equivalent forms.

Theorem (1.1.6):

Let $\Omega = D$ or Π, and let X be a holomorphic Ω.-valued function on Ω. The following are equivalent:

(i) f is of bounded type, that is, $f \in N_x(\Omega)$;

(ii) $\log^+|f|_x$ has a harmonic major on Ω;

(iii) According as $\Omega = D$or Π,

$$\sup_{0<\to 1} \int_t \log^+\left|f\left(re^{i\theta}\right)\right|_x d\sigma < \infty \qquad (2)$$

Or

$$\sup_{p>0} \int_{-\infty}^{\infty} \frac{\log^+|f(x+iy)|_x}{x^2+(y+1)^2} dx < \infty; \qquad (3)$$

(iv) $f = g/u$Where g is a bounded holomorphic X-valued function on Ω and u is a scalar valued holomorphic function such that $0 < |u| \leq 1$ on Ω.

Proof:

The equivalence of (i) and (ii) is by the definition of the classes $N_x(\Omega)$ concerning (iii) see the criteria for the existence of harmonic major ants and for (iv) use Theorem (1.1.5).

Theorem (1.1.7):

Let $\Omega = D$ or Π, and let Ω be a strongly convex function. If f is a holomorphic X-valued function on Ω,then the following are equivalent:

(i) $f \in \mathscr{K}_{z,x}(\Omega)$;

(ii) $\phi\left(\log^+ |f|_x\right)$ has a harmonic major ant on Ω;

(iii) According as $\Omega = D$ or Π,

$$\sup_{0<r<1} \int_{\Gamma} \phi\left(\log^+ \left|f\left(re^{i\theta}\right)\right|_x\right) d\sigma < \infty \quad (4)$$

Or

$$\sup_{y>0} \int_{-\infty}^{\infty} \frac{\phi\left(\log^+ |f(x+iy)|_x\right)}{x^2 + (y+1)^2} dx < \infty; \quad (5)$$

(iv) Same as (iii), but with ‘ $\log^+$ ’replaced by ‘ $\log$"

Proof:

The equivalence of (i),(ii) and (iii) follows from the definition.and the criteria for the existence of harmonic major ants since

$$\phi\left(\log |f|_x\right) \le \phi\left(\log^+ |f|_x\right) \le \phi\left(\log |f|_x\right) + \phi(0)$$

(iii) Is equivalent to (iv).

Theorem (1.1.8):

Let $\Omega = D$ or Π, and let f be a holomorphic X-valued function on Ω .The following are equivalent:

(i) $f \in N_x^+(\Omega)$;

(ii) $f \in \mathscr{K}_{\phi,x}(\Omega)$ for some strongly convex function ϕ;

(iii) $f = h/v$, where h: is a bounded holomorphic X-valued function on Ω and v is a scalar valued outer function such that $0 < |v| \le 1$ on Ω.

Moreover, in the case $\Omega = D$, $(i)-(iii)$, are equivalent to:

(iv) The function $\left\{\log^{+}\left|f\left(re^{i\theta}\right)\right|_{x}\right\}_{0<r<1}$ are uniformly integrable with respect to normalized Lebesgue measure σ on Γ.

See the definition of a uniformly integrable family of function.

Proof:

The equivalence. Of (i) and (ii) is by the definition When $\Omega = D$, the equivalence of (ii) and (iv) follows from a theorem of de la Vallce Poussin and Nagumo. It remains to show that (ii) and (iii) is Equivalent. It is sufficient to treat the case Ω since the other case then follows is by conformal mapping.

Assume (iii) Without loss of generality we can further assume that $|h_x| \le 1$ on D. Since v is an outer function,

$$\log^{+}|f(z)|_{x} \le \log^{+}|h(z)|_{x} + \log^{+}\left(1/|v(z)|\right)$$
$$= -\log|v(z)| = -\int_{\Gamma} p\left(z, e^{it}\right)\log\left|v\left(e^{it}\right)\right| d\sigma$$

On D. The family consisting of the single function $-\log\left|v\left(e^{it}\right)\right|_{x}$. In $L^{1}(\sigma)$ is uniformly integrable, so by the theorem of de la. Vallee Poussin and Nagumo, there is a strongly convex function Ω such that

$$\int_{\Gamma} \phi\left(-\log\left|v\left(e^{it}\right)\right|\right) d\sigma < \infty$$

By Jensen's inequality (Rudin),

$$\phi\left(\log^{+}|f(z)|_{x}\right) \le \phi\left(-\int_{\Gamma} P\left(z, e^{it}\right)\log\left|v\left(e^{it}\right)\right| d\sigma\right)$$
$$\le \int_{\Gamma} P\left(z, e^{it}\right)\phi\left(-\log\left|v\left(e^{it}\right)\right|\right) d\sigma$$

And hence

$$\int_{\Gamma} \phi\left(\log^{+}\left|f\left(re^{i\theta}\right)\right|_{x}\right) d\sigma\left(e^{i\theta}\right)$$
$$\le \int_{\Gamma}\int_{\Gamma} P\left(re^{i\theta}, e^{it}\right) d\sigma\left(e^{i\theta}\right)\phi\left(-\log\left|v\left(e^{it}\right)\right|\right) d\sigma\left(e^{it}\right)$$
$$= \int_{\Gamma} \phi\left(-\log\left|v\left(e^{it}\right)\right|\right) d\sigma\left(e^{it}\right).$$

It follows; that $\phi\left(\log^{+}\left|f(x)\right|\right)$ has a harmonic Major ant. Therefore $f \in \mathscr{K}_{\phi,x}(D)$ and (ii) holds.

Assume (ii) trivial case $f \equiv 0$.It is easy to see that the function $u(z) = \left|f(z)\right|_x$ Satisfies the hypotheses of the Szegö-Solomentsev theorem .The inequality in the second part of that theorem may be written

$$\left|f\right|_x \leq \left|gS_+ / S_-\right|,$$

Where g is a scalar valued outer function and S_+, S_- are scalar valued singular inner functions. By our assumption. (ii) And the third part of the Szegö-Solomentsev theorem, S_+. Is a constant of modulus 1. Choose an outer function v such that $0 < |v| \leq 1$ and $|vg| \leq 1$ on D. Setting $h = vf$, we obtain $f = h/v$ as required in (iii).

Sec (1.2): Hardy classes and Fatou's lemma:

We define if $H_x^p(D)=\mathscr{K}_{\phi,x}(D)$, where $\phi(t)=e^{pt}, 0<p<\infty$. The class $H_x^\infty(D)$ has previously been defined

Theorem (1.2.1):

Let $0<p<\infty$ is a holomorphic f-valued function on D,

Then the following are equivalent:

(i) $f\in H_x^p(D)$;

(ii) $|f|_x^p$ has harmonic major ant on D;

(iii) $\sup_{0<r<1}\int_\Gamma |f(re^{i\theta})|_X^P \, d\sigma < \infty$

Proof:

By Theorem (1.1.1), $|f|_x^p$ is subharmonic on D. It is easy to see that $\{f\}_X^p$ harmonic majorant on D. Hence the result follows from the definition of $H_X^p(D)$ and the condition for the existence of a harmonic majorant for subharmonic function on D.

For each $f\in H_X^p(D), 0<p<\infty$, set

$$\|f\|_p=\sup_{0<r<1}\left(\int_\Gamma |f(re^{i\theta})|_X^P \, d\sigma\right)^{1/p}$$

Since $|f(z)|_X^p$ is sub harmonic on D we can see also write this as

$$\|f\|_p=\lim_{r\uparrow 1}\left(\int_\Gamma |f(re^{i\theta})|_X^d \, d\sigma\right)^{1/p}.$$

For $f\in H_X^\infty(D)$ set.

$$\|f\|_\infty=\sup_{zD}|f(z)|_x.$$

As in the scalar case (Duren), we define two kinds of Hardy classes or. The upper half-plane Π.

First kind. Let $H_X^p, 0<p<\infty$ be the set of all holomorphic X.-valued functions F on Π such that

$$\|F\|_p = \sup_{p>0}\left(\int_{-\infty}^{\infty}|F(x+iy)|_X^p\right)^{1/p} < \infty.$$

The class $H_X^\infty(\Pi)$ is as previously defined. For any $F \in H_X^\infty(\Pi)$

$$\|F\|_\infty = \sup_{p>0}|F(z)|_x.$$

Second kind. Let $\mathscr{K}_x^p(\Pi), 0 < p \le \infty$ be the set of all holomorphic X-valued - functions F on Π such that $F \circ \alpha \in H_x^p$ where α is the mapping of D on Π given by

$$\alpha : w \to i(1+w)/(1-w).$$

For each $F \in \mathscr{K}_x^p(\Pi)$ set

$$|||F|||_p = \|F \circ \alpha\|_p.$$

Theorem (1.2.2):

Let $0 < p < \infty$. If F is a holomorphic X-valued function on Π then the following are equivalent:

(i) $F \in \mathscr{K}_X^p(\Pi)$;

(ii) $|F|_X^p$ has a harmonic majorant on Π;

(iii) $\sup_{y>0}\int_{-\infty}^{\infty}\frac{|F(x+iy)|_X^p}{x^2+(y+1)^2}dx < \infty.$

Proof:

The equivalence, of (i) and (ii) follows from Theorem $\mathscr{K}_X^p(\Pi)$ and the definition of (ii) and the equivalence of D and is by theorem of Flett and Kuran.

Theorem (1.2.3):

If $0 < p < \infty$, then

$$H_X^\infty(D) \subseteq N_X^+(D) \text{ And } H_X^p(\Pi) \subseteq \mathscr{K}_X^p(\Pi) \subseteq N_X^+(\Pi).$$

Proof:

The inclusion follows easily from definitions and characterizations of the classes given above.

From now on we assume that X either Hilbert space $\mathscr{E}$ or the space $\mathscr{B}(\mathscr{E})$ of bounded linear operators on $\mathscr{E}$.

In this section we prove version of Fatou's theorem on the existence of nontangential limits at boundary points. Boundary behavior in general is discussed latter.

FATou's Theorem(i) each f in $H_{\mathscr{E}}^{\infty}(D)$ has a nonangertial limit

$$f\left(e^{i\theta}\right)=\lim_{z\to e^{i\theta}} f(z).$$

$\sigma.a.e$. In the strong topology of$\mathscr{E}$.

(ii) Each F in $H_{\mathscr{R}(\mathscr{E})}^{\infty}(D)$ a nontangential limit

$$F\left(e^{i\theta}\right)=\lim_{z\to e^{i\theta}} F(z)$$

$\sigma.a.e$ In the strong operator topology on$\mathscr{R}(\mathscr{E})$.

We digress briefly to review some notions from measure theory. As this material is widely used and well generally known, we merely give and state the facts, we need 'without proof.

Let $(A,\mathscr{F},u)$ be a measure space. A $\mathscr{E}$-valued function f on A is weakly measurable if $\langle f(.),a\rangle_{\mathscr{E}}$ is measurable for each$a\in\mathscr{E}$. A $\mathscr{R}(\mathscr{E})$-valued function F on A is weakly measurable if $\langle F(.)a,b\rangle_{\mathscr{E}}$ is measurable for all$a,b\in\mathscr{E}$.

Let fg be weakly measurable $\mathscr{E}$ -valued function, and let F,G be weakly measurable $\mathscr{B}(\mathscr{E})$-valued function on$A$. Then Ff is a weekly measurable $\mathscr{E}$-valued function, $Fg$is a weekly measurable $\mathscr{R}(\mathscr{E})$-valued function, and $\langle f(.),g(.)\rangle_{\mathscr{E},} |f(.)|_{\mathscr{E}}$ and $|F(.)|_{\mathscr{R}(\mathscr{E})}$ are measurable scalar valued functions on$A$.

Let $f$be a weakly measurable $\mathscr{E}$-valued function and F a weakly measurable $\mathscr{B}(\mathscr{E})$-valued function on A such that

$$\int_A |f|_{\mathscr{E}}d\mu<\infty \qquad \text{And} \qquad \int_A |f|_{\mathscr{B}(\mathscr{E})}d\mu<\infty.$$

We define

$$\int_A fd\mu=e \qquad \text{And} \qquad \int_A Fd\,\mu=C,$$

And so

$$f(z)=\int_{\Gamma} p\left(z,e^{it}\right)\phi\left(e^{it}\right)d\sigma. \tag{6}$$

Let $\mathscr{E}$ be a countable dence in $\mathscr{E}$. By the scalar version of Fatou's theorem, there is a σ -null set $N\subseteq L$ 'such that

$$\lim_{z\to e^{i\theta}}\int_\Gamma P\left(z,e^{it}\right)\left|\phi\left(e^{it}\right)-a\right|_{\mathscr{E}}d\sigma=\left|\phi\left(e^{i\theta}\right)-a\right|_{\mathscr{E}} \qquad (7)$$

Nontangentially for each $e^{i\theta}\in\Gamma\setminus N$ and $a\in\mathscr{E}$.

Fix $e^{i\theta}\in\Gamma\setminus N$. Let D be an open triangular sector in D with vertex $e^{i\theta}$ given

‹‹ $<D$,choose $a\in\mathscr{E}$ such that $\left|\phi\left(e^{i\theta}\right)-a\right|_{\mathscr{E}}<\varepsilon/2$. By (6),

$$\left|f(z)-\left(e^{i\theta}\right)\right|_{\mathscr{E}}=\left|\int_\Gamma p\left(z,e^{it}\right)\left[\phi\left(e^{it}\right)-a+-\phi\left(e^{i\theta}\right)\right]d\sigma\right|_{\mathscr{E}}.$$

$$\leq\int_\Gamma P\left(z,e^{it}\right)\left|\phi\left(e^{it}\right)-a\right|_{\mathscr{E}}d\sigma+\left|\phi\left(e^{i\theta}\right)-a\right|_{\mathscr{E}}..$$

Hence by (7),

$$\limsup_{\substack{z\to e^{i\theta}\\ z\in S}}\left|f(z)-\phi\left(e^{i\theta}\right)\right|_{\mathscr{E}}\leq 2\left|\phi\left(e^{i\theta}\right)-a\right|_{\mathscr{E}}<\varepsilon.$$

By the arbitrariness of $\varepsilon,\left|f(z)-\phi\left(e^{i\theta}\right)\right|_{\mathscr{E}}\to 0$ for $z\in S,z\to e^{i\theta}$.

We thus obtain (i) with $f\left(e^{i\theta}\right)=\phi\left(e^{i\theta}\right)$ for all $e^{i\theta}\in\Gamma\setminus N$

(i) Let $\mathscr{E}$ be as above, and apply (i) to $F(z)$a for each fixed$\mathscr{E}$. Since $\mathscr{E}$ is' countable, there , is a σ-null set $N\subseteq\Gamma$ such that

$$\lim_{z\to e^{i\theta}}F(z)a=\phi_a\left(e^{i\theta}\right)$$

Exists nontangentially for all $e^{i\theta}\in I\setminus Nanda\in\mathscr{E}$.

Fix$e^{i\theta}\in\Gamma\setminus N$. Define $s_0\left(e^{i\theta}\right)$ on $\mathscr{E}\times\mathscr{E}$ by

$$s_0\left(e^{i\theta,},a,b\right)=\left\langle\phi_a\left(e^{i\theta}\right),b\right\rangle_{\mathscr{E}.}, \quad a,b\in\mathscr{E}$$

For any$a_1,a_2,b_1,b_2\in\mathscr{E}$,

$$\left|s_0\left(e^{i\theta},a_1,b_1\right)-s_0\left(e^{i\theta,},a_2,b_2\right)\right|$$

$$=\lim_{s\to e^{i\theta}}\left|\left\langle F(z)a_1,b_1-b_2\right\rangle_{\mathscr{E}}+\left\langle F(z)\left(a_1-_2\right),b_2\right\rangle_{\mathscr{E}}\right|$$

$$\leq\|F\|_\infty|a_1|_{\mathscr{E}}|b_1-b_2|_{\mathscr{E}}+\|F\|_\infty|a_1-a_2|_{\mathscr{E}}|b_2|_{\mathscr{E}}.$$

Therefore $s_0\left(e^{i\theta,},\cdot,\cdot\right)$ has unique extension by continuity $s\left(e^{i\theta,},\cdot,\cdot\right)$ to $\mathscr{E}\times\mathscr{E}$. By construction.

$$s\left(e^{i\theta};a,b\right)=\lim_{z\to e^{i\theta}}\left\langle F(z)a,b\right\rangle_{\mathscr{E}}$$

Nontangentially for all$a,b\in\mathscr{E}$. Routine arguments now show that

$s\left(e^{i\theta},\cdot,\cdot\right)$ is a bounded sesquilinear form on $\mathscr{E}$ with $\|s\| \le \|F\|_\infty$. Hence there is an operator $F\left(e^{i\theta}\right) \in \mathscr{B}\left(\mathscr{E}\right)$ such that $\left|F\left(e^{i\theta}\right)\right|_{\mathscr{B}(\mathscr{E})} \le \|F\|_\infty$ and

$$s\left(e^{i\theta}a,b\right) = \left\langle F\left(e^{i\theta}\right)a,b\right\rangle_{\mathscr{E}}, \qquad a,b \in \mathscr{E}.$$

It follows that $F\left(z\right)a \to F\left(e^{i\theta}\right)a$ nontangentially in the norm of $\mathscr{E}$ for all $a \in \mathscr{E}$. Since $F\left(z\right)$ is bounded on D . The same holds for all $a \in \mathscr{E}$. The result follows.

Chapter-2
Bounded functions and Hardy classes

Sec (2.1): Boundary Behavior of bounded functions:

The boundary properties of vector and operator valued functions of bounded type are very similar to the scalar theory. The most serious loss is that in the case of operator valued functions, Fatou's theorem fails in the norm topology However, as we have seen Fatou'stheorem holds relative to the strong operator topology, and this is an dequate substitute.

Theorem (2.1.1):

Let $X=\mathscr{E}$ or $\mathscr{R}(\mathscr{E})$. For each $f\in N_X(D)$ a nontangenrial limit

$$f\left(e^{i\theta}\right)=\lim_{z\to e^{i\theta}} f(z) \qquad (1)$$

Exists $\sigma.a.e$. On Γ in the strong topology if $X=\mathscr{E}$ and strong operator topology if $X=\mathscr{R}(\mathscr{E})$.Also, for $X=\mathscr{E}$ or $\mathscr{R}(\mathscr{E})$,

$$\left|f\left(e^{i\theta}\right)\right|_X=\lim_{z\to e^{i\theta}}\left|f(z)\right|_X \qquad (2)$$

Nontangentially $\sigma.a.e$. On Γ. Moreover, if $f\not\equiv 0$ on D, then

$$\log\left|f\left(e^{i\theta}\right)\right|_X\in L^1(\sigma) \qquad (3)$$

Proof:

Assume $f\not\equiv 0$. By, Theorem (1.1.6)(iv), we may further assume that f is bounded on D. Then the existence of a nontangential boundary function (1) follows from Fatou's theorem. We obtain (3) from the Szergo-Solomentsev theorem applied to the function $u(z)=\left|f(z)\right|_X$.(When $X=\mathscr{E}$(2) is clear. Ii remains to prow; (2) when $X=\mathscr{R}(\mathscr{E})$ .Choose a σ-null set $N\subseteq\Gamma$ such that for each $e^{i\theta}\in\Gamma\setminus N, f\left(e^{i\theta}\right)$ exists, and is nonzero and

$$\lim_{z\to e^{i\theta}}\int_r P\left(z,e^{i\theta}\right)\log\left|f\left(e^{i\theta}\right)\right|_{\mathscr{R}(\mathscr{E})} d\sigma=\log\left|f\left(e^{i\theta}\right)\right|_{\mathscr{R}(\mathscr{E})} \qquad (4)$$

nontangentially.Fix $e^{i\theta}\in\Gamma\setminus N$, and let S be a triangular sector in D with vertex $e^{i\theta}$. For all $z\in D$,

$$\log\left|f(z)\right|_X \le \int_r P\left(z,e^{i\theta}\right)\log\left|f\left(e^{i\theta}\right)\right|_X d\sigma$$

Hence by (4),

$$\limsup_{\substack{z\to e^{i\theta}\\ z\in S}}\left|f(z)\right|_{\mathscr{R}(\mathscr{E})} \le \left|f\left(e^{i\theta}\right)\right|_{\mathscr{R}(\mathscr{E})}.$$

For any $a,b\in\mathscr{E}\,|a|\mathscr{E}=|b|_{\mathscr{E}}=1$,

$$\left\langle f\left(e^{i\theta}\right)a,b\right\rangle_{\mathscr{E}} = \lim_{\substack{z\to e^{i\theta}\\ z\in S}}\left|\left\langle f(z)a,b\right\rangle_{\mathscr{E}}\right| \le \liminf_{\substack{z\to e^{i\theta}\\ z\in S}}\left|f(z)\right|_{\mathscr{R}(\mathscr{E})}$$

By the arbitrariness of a and b,

$$\left|f\left(e^{i\theta}\right)\right|_{\mathscr{R}(\mathscr{E})} \le \liminf_{\substack{z\to e^{i\theta}\\ z\in S}}\left|f(z)\right|_{\mathscr{R}(\mathscr{E})}.$$

Therefore,

$$\left|f\left(e^{i\theta}\right)\right|_{\mathscr{R}(\mathscr{E})} = \lim_{\substack{z\to e^{i\theta}\\ z\in S}}\left|f(z)\right|_{\mathscr{R}(\mathscr{E})}.$$

Since the sector S is arbitrary, the result follows.

Theorem (2.1.2):

Let $X=\mathscr{E}$ or $\mathscr{R}(\mathscr{E})$. For each $f\in N_X(\Pi)$, a nontangential limit

$$f(x)=\lim_{z\to x} f(z) \qquad (5)$$

Exists $a.e.$ On $(-\infty,\infty)$ in the strong topology if $X=\mathscr{E}$, and strong operator topology if $X=\mathscr{R}(\mathscr{E})$.Also, for $X=\mathscr{E}$ or $\mathscr{R}(\mathscr{E})$.

$$\left|f(x)\right|_X = \lim_{z\to x}\left|f(z)\right|_X \qquad (6)$$

Nontangentially $a.e.$ On $(-\infty,\infty)$. Moreover, either $f\equiv 0$ on Π or

$$\int_{-\infty}^{\infty}\frac{\left|\log\left|f(t)\right)}{1+t^2}dt<\infty. \qquad (7)$$

Proof:

The mapping $\alpha: w\to i(1+w)/(1-w)$ takes D onto Π and $\Gamma\setminus\{1\}$ onto $(-\infty,\infty)$.It is conformal at all points of $\bar{D}\setminus\{1\}$. We thus Obtain Theorem (2.1.2) from Theorem (2.1.1) by a routine change of variables.

Theorem (2.1.3):

Let $X=\mathscr{E}$ or $\mathscr{R}(\mathscr{E})$, and let ϕ cf.) be a strongly convex function.

(i) If $f\in N_X^+(D)$, then $f\in\mathscr{K}_{\phi,X}(D)$ if and only if

$$\int_\Gamma \phi\left(\log\left|f\left(e^{i\theta}\right)\right|_X\right)d\sigma<\infty. \tag{8}$$

In this case,

$$\lim_{r\uparrow 1}\int_\Gamma \phi\left(\log\left|f\left(re^{i\theta}\right)\right|_X\right)d\sigma=\int_\Gamma \phi\left(\log\left|f\left(e^{i\theta}\right)\right|_X\right)d\sigma. \tag{9}$$

(i) If $X=\mathscr{E}$ and $\lim_{t\to\infty}\phi(t)=0$, then also for each $f\in\mathscr{K}_{\phi,\mathscr{E}}(D)$

$$\lim_{r\to 1}\int_\Gamma \phi\left(\log\left|f\left(e^{i\theta}\right)-f\left(re^{i\theta}\right)\right|_{\mathscr{E}}\right)d\sigma=0. \tag{10}$$

(ii) Assertion fails if $\mathscr{E}$ is replaced by $\mathscr{R}(\mathscr{E})$.

Proof:

(i) Let $f\in N_X^+(D)$ and $f\not\equiv 0$. If $f\in\mathscr{K}_{\phi,X}(D)$, then by Theorem (2.1.2), and Fatous's lemms.

$$\int_\Gamma \phi\left(\log\left|f\left(e^{it}\right)\right|_X\right)d\sigma\le\lim_{r\to 1}\int_\Gamma \phi\left(\log\left|f\left(re^{it}\right)\right|_X\right)d\sigma$$

$$\le\lim_{r\to 1}\int_\Gamma \phi\left(\log^+\left|f\left(re^{it}\right)\right|_X\right)d\sigma<\infty \tag{11}$$

Thus (8) holds.

Conversely suppose that (8) holds. For all $z\in D$ ،

$$\log\left|f(z)\right|_X\le\int_\Gamma P\left(z,e^{it}\right)\log\left|f\left(e^{it}\right)\right|_X d\sigma.$$

By Jensen's inequality (Rudin)

$$\phi\left(\log\left|f(z)\right|_X\right)\le\int_\Gamma P\left(z.e^{it}\right)\phi\left(\log\left|f\left(e^{it}\right)\right|_X\right)d\sigma.$$

Hence for $0<r<1$,

$$\int_\Gamma \phi\left(\log\left|f\left(re^{i\theta}\right)\right|_X\right)d\sigma\left(e^{i\theta}\right)$$

$$\le\int_\Gamma\int_\Gamma P\left(re^{i\theta},e^{it}\right)d\sigma\left(e^{i\theta}\right)\phi\left(\log\left|f\left(e^{it}\right)\right|_X\right)d\sigma\left(e^{it}\right)=\int_\Gamma \phi\left(\log\left|f\left(e^{it}\right)\right|_X\right)d\sigma\left(e^{it}\right). \tag{12}$$

A similar argument with "$\log$" replaced by "$\log^+$" yields

$$\int_\Gamma \phi\left(\log^+\left|f\left(re^{i\theta}\right)\right|_X\right)d\sigma\le\int_\Gamma \phi\left(\log^+\left|f\left(e^{it}\right)\right|_X\right)d\sigma,$$

Whereby (8) the integral on the right is finite .In particular, $f\in\mathscr{K}_{\phi,X}(D)$ by Theorem (2.1.2). Moreover.we obtain (9) by combining (11) and (12).

(i) Let $f \in \mathscr{K}_{\phi,\mathscr{E}}(D), f \not\equiv 0$, and assume that $\phi(-\infty)=0$, where $\phi(-\infty)=\lim_{x\to\alpha}\phi(x)$. Fix a sequence $r_t \uparrow 1$ and set $f_t(e^{i\theta})=f(r_t e^{i\theta})$ on Γ for each $n=1,2,3,\ldots$Let $t>0$ be given. By (8) there is a $\delta>0$ such that $\delta<\varepsilon$ and

$$\int_{\Delta}\phi\left(\log|f|_{\mathscr{E}}\right)d\sigma<\varepsilon \qquad (13)$$

For every Borel set $\Delta\subseteq\Gamma$ with $\sigma(\Delta)<b$.By Egoroff's theorem we can choose Δ such that $\sigma(\Delta)<\delta$ and $|f-f_n|_{\mathscr{E}}\to 0$ uniformly on $\Gamma\setminus\Delta$ .Since $\phi(-\infty)=0$, $\phi\left(\log|f-f_n|_{\mathscr{E}}\to 0\right)$ uniformly on $\Gamma\setminus\Delta$, and so

$$\lim_{n\to\infty}\int_{\Gamma\setminus\Delta}\phi\left(\log|f-f_n|_{\mathscr{E}}\right)d\sigma=0. \qquad (14)$$

By the definition of a strongly convex function there exist constants $M\geq 0$ and $K\geq 0$ such that $\phi(t+\log 2)\leq M\phi(t)+K$ for ll real t. Then for all $n=1,2,3,\ldots$,

$$\begin{aligned}\phi\left(\log|f-f_n|_{\mathscr{E}}\right)&=\phi\left(\log\left(\frac{1}{2}|f-f_n|_{\mathscr{E}}\right)+\log 2\right)\\&\leq M\phi\left(\log\left(\frac{1}{2}|f-f_n|_{\mathscr{E}}\right)\right)+K\\&\leq M\max\left(\phi\left(\log|f|_{\mathscr{E}}\right),\phi\left(\log|f_n|_{\mathscr{E}}\right)\right)+K\\&\leq M\left[\phi\left(\log|f|_{\mathscr{E}}\right)+\phi\left(\log|f_n|_{\mathscr{E}}\right)\right]+K.\end{aligned}$$

σ-a.e.on Δ.Since $\delta<\varepsilon$.

$$\int_{\Delta}\phi\left(\log|f-f_n|_{\mathscr{E}}\right)d\sigma<(M+K)\varepsilon+M\int_{\Delta}\phi\left(\log|f_n|_{\mathscr{E}}\right)d\sigma$$

By (13). By (9) and (13),

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\Delta}\phi\left(\log|f_n|_{\mathscr{E}}\right)d\sigma\\&=\lim_{n\to\infty}\int_{\Gamma}\phi\left(\log|f_n|_{\mathscr{E}}\right)d\sigma-\liminf_{n\to\infty}\int_{\Gamma\setminus\Delta}\phi\left(\log|f_n|_{\mathscr{E}}\right)d\sigma\\&\leq\int_{\Gamma}\phi\left(\log|f|_{\mathscr{E}}\right)d\sigma-\int_{\Gamma\setminus\Delta}\phi\left(\log|f|_{\mathscr{E}}\right)d\sigma\\&=\int_{\Delta}\phi\left(\log|f|_{\mathscr{E}}\right)d\sigma<\varepsilon.\end{aligned}$$

Therefore for all sufficiently large n,

$$\int_{\Delta} \phi\left(\log\left|f - f_n\right|_{\mathscr{E}}\right) d\sigma < (2M + K)\varepsilon.$$

By the arbitrariness of s,,

$$\lim_{n\to\infty} \int_{\Delta} \phi\left(\log\left|f \_ f_n\right|_{\mathscr{E}}\right) d\sigma = 0. \qquad (15)$$

Combining (14) and (15), we obtain (10) first for r tending to 1 through the sequence $r_n \uparrow I$ and then as asserted by the arbitrariness of the sequence $r_n \uparrow I$.

This completes the proof.

Definition (2.1.4):

Let $X = \mathscr{E}$ or $\mathscr{R}(\mathscr{E})$. We write $N_X(\Gamma), H_X^p(\Gamma)$. For the classes of boundary functions of functions in $N_X(D), H_X^p(D)$. We define $N_X(R), H_X^p(R), \ldots$ similarly from $N_X(\Pi), H_X^p(\Pi), \ldots$

The mapping $f(z) \to f(e^{i\theta})$ from $N_X(D)$ to $N_X(\Gamma)$ is one-to-one and linear. If $F(z), G(z) \in N_{\mathscr{R}(\mathscr{E})}(D)$ and $f(z) \in N_{\mathscr{E}}(D)$, then $F(z)G(z)$ is in $N_{\mathscr{R}(\mathscr{E})}(D)$ and has boundary function $F(e^{i\theta})G(e^{i\theta})$, and $F(z)f(z)$ in $N_{\mathscr{E}}(D)$ and has boundary function $F(e^{i\theta})f(e^{i\theta})$.we caution that there is more. Here than meets the eyes since multiplication is not continuous in the strong operator topology. However; Theorem (2.1.1), we can reduce the assertions to the case of bounded then they are easily proved. The situation for $N_X(\Pi)$ and $N_X(R)$ is similar.

It is not easy to derive-the main facts concerning boundary behavior for the Hardy classes.

Theorem (2.1.5):

Let $X = \mathscr{E}$ or $\mathscr{R}(\mathscr{E})$

(i) For $0 < p \le \infty, H_X^p(\Gamma) = N_X^+(\Gamma) \cap L_X^p(\sigma)$,

(ii) If $0 < p < \infty$, and $f \in H_X^p(D)$ then

$$\|f\|_P^p = \lim_{r\uparrow I} \int_{\Gamma} \left|f\left(re^{i\theta}\right)\right|_X^P d\sigma = \int_{\Gamma} \left|f\left(e^{i\theta}\right)\right|_X^p d\sigma.$$

(iii) If $f \in H_X^\infty(D)$, then

$$\|f\|_\infty = \lim_{r\uparrow 1}\max_{|z|=r}|f(z)|_X = ess\sup_\Gamma |f(e^{i\theta})|_X$$

Proof:

(i) For $0<p<\infty$ this follows from, Theorem (2.1.3).with $\phi(t)=e^{pt}$. The inclusion $H_X^\infty(\Gamma)\subseteq N_X^+(\Gamma)\cap L_X^p(\sigma)$ is clear, and we obtain the reverse inclusion, from the result.

(i) Apply Theorem (2.1.3) $\phi(t)=e^{pt}$.

(ii) The first equality follows from the definition of $\|f\|_\infty$ and the maximum modulus principle (Hille and Phillips). Another application of the results yields

$$\|f\|_\alpha \le ess\sup_\Gamma |f(e^{i\theta})|_X .$$

This inequality is also a consequence of the Poisson representation in Theorem (2.1.3) below. The reverse inequality follows from (2).

Theorem (2.1.6):

If $f\in H_{\mathscr{E}}^p(D), 0<p<\infty$, then

$$\lim_{r\uparrow 1}\int_\Gamma |f(e^{i\theta})-f(re^{i\theta})|_{\mathscr{E}}^p d\sigma = 0.$$

Proof:-

Apply Theorem (2.1.3) with $\phi(t)=e^{pt}$.

Theorem (2.1.7):

Let $X=\mathscr{E}$ or $\mathscr{R}(\mathscr{E})$.

(i) If $1\le P\le\infty$, then $H_X^\infty(D)$ is the subspace of $L_X^p(\sigma)$ consisting of all functions $f(e^{i\theta})$ in $L_X^p(\sigma)$ such that

$$\int_\Gamma f(e^{i\theta})e^{i\theta}d\sigma = 0, \qquad j=1,2,3,\ldots \qquad (16)$$

(ii) If $f\in H_X^p(D), 1\le p\le\infty$, then for all $z\in D$,

$$f(z)=\int_\Gamma \frac{f(e^{i\theta})}{1-ze^{-it}}d\sigma = \int_\Gamma P(z,e^{it})f(e^{it})d\sigma. \quad (17)$$

The integrals in (16) and (17) re taken in the weak sense.

The two formulates for $f(z)$ in (17) are called the Cauchy and Poisson representations, respectively.

Proof:

We take the result as known in the scalar case (Duren; Hoffman) If $f(z)$ belongs to $H_X^p(D), I \le p \le \infty$, then the boundary function $f(e^{i\theta})$ belongs to $L_X^p(\sigma)$ by Theorem (2.1.1) .Then (16)and (17) follow by applying the scalar version of the theorem to the functions $\langle f(z), c \rangle_{\mathscr{E}}$ when $X = \mathscr{E}$ and $\langle f(z)a, b \rangle_{\mathscr{E}}$ when $X = \mathscr{R}(\mathscr{E})$ for arbitrary $a, b, c \in \mathscr{E}$.

Conversely, let $f(e^{i\theta})$ be a given function in $L_X^p(\sigma)$ satisfying (16). Then since

$$\frac{1}{1 - ze^{-it}} = P(z, e^{it}) - \frac{\overline{z}e^{it}}{1 - \overline{z}e^{it}} = P(z, e).$$

There is a function $f(z)$ on D satisfying (17) .The first representation of $f(z)$ in (17) shows that $f(z)$ is holomorphic, and by familiar properties of the Poisson kernel the second implies that $f(z)$ is in $H_X^p(D)$. It remains to show that the boundary function of $f(z)$ is the given function $f(e^{i\theta})$.For definiteness suppose that $X = \mathscr{E}$. It is enough to show that for a countable dense set of vectors $c \in \mathscr{E}$.the boundary function of $\langle f(z), c \rangle_{\mathscr{E}}$ is equal to $\langle f(e^{i\theta}), c \rangle_{\mathscr{E}}$ σ-a.e . This follows from the scalar version of the theorem. The case $X = \mathscr{R}(\mathscr{E})$ is treated similarly.

Theorem (2.1.8):

Let $X = \mathscr{E}$ or $\mathscr{R}(\mathscr{E})$. If $1 \le p \le \infty$, then $H_X^p(D)$ and $H_X^p(\Gamma)$ are Banach spaces (the norm in $H_X^p(\Gamma)$ is that of $L_X^p(\sigma)$) . The mapping $f(z) \to f(e^{i\theta})$ is an isometry from $H_X^p(D)$ on to $H_X^p(\Gamma)$.

Proof:-

This results may be obtained as a corollary of Theorem (2.1.3).

Section (2.2): Hardy classes on the disk and half plane:
The scalar theory for the half-plane is given in Duren [1970], Day and McKean [1972], Hoffman [1962], and Krylov [1939]. As in the disk case, vector and operator generalizations of many theorems follows in a straightforward way from the classical theory and results. The results stated below present no unusual difficulties, and the proofs can be safely omitted.

Theorem (2.2.1):-
.let $X=\mathscr{E}$ or $\mathscr{B}(\mathscr{E})$. If $g\in\mathscr{H}_X^p(\Pi), 1\le p\le\infty$, then

$$g(z)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(t-x)^2+y^2}dt, \quad y>0. \quad (18)$$

Theorem (2.2.2):-
Let $X=\mathscr{E}$ or $\mathscr{B}(\mathscr{E})$ and $fix p\in[1,\infty)$'

(i) If $g(z)$ is $H_X^p(\Pi)$, then the boundary function $g(x)$ to $L_X^p(-\infty,\infty)$,

$$g(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{t-z}dt, \quad y>0, \quad (19)$$

And

$$0=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{t-z}dt, \quad y<0 \quad (20)$$

(ii) Conversely let $g(x)$ be a given function in $L_X^p(-\infty,\infty)$ which satisfies (4-27).
Then (18) and (19) define one and the same function $g(z)$ on $\Pi, g(z)$, belongs to $H_X^p(\Pi)$, and its boundary function is the given function.

Theorem (2.2.3):
Let $X=\mathscr{E}$ and $\mathscr{B}(\mathscr{E})$ fix $p\in[1,\infty]$. Then $H_X^p(\Pi)$ and $H_X^p(R)$ are Banach spaces, and the mapping $g(z)\to g(x)$ is an isometry from $H_X^p(\Pi)$ onto $H_X^p(R)$.

The norm in $H_X^p(R)$ is that $L_X^p(-\infty,\infty)$. A similar result holds for $\mathscr{K}_X^p(\Pi)$ and $\mathscr{H}_X^p(R)$, provided that norm in $\mathscr{H}_X^p(R)$ is defined by

$$\|g\|_p = \left(\frac{1}{z} \int_{-\infty}^{\infty} \frac{|g(x)|_X^p}{1+x^2} dx \right)^{1/p},$$

If $1 \le p < \infty$ and

$$\|g\|_\infty = \operatorname*{ess\,sup}_{-\infty<x<\infty} |g(x)|_x .$$

If $p = \infty$.

The case $p = 2$ and $X = \mathscr{E}$ (tractable to the Plancherel theorem and Paley-Wiener representation.

Theorem (2.2.4):

(Planchearel Theorem).There is an isometry $\mathscr{F} : F \to \hat{F}$ of $L_{\mathscr{E}}^2(-\infty,\infty)$ onto itself such that for each $F \in L_{\mathscr{E}}^2(-\infty,\infty)$,

$$\hat{F}(x) = \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-i\infty} F(t)\, dt$$

And

$$F(x) = \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-i\infty} \hat{F}(t)\, dt$$

With convergence in the metric of $L_{\mathscr{E}}^2(-\infty,\infty)$.

Theorem (2.2.5):

(Palley-Wiener Representation). Given $f \in L_{\phi,X}(0,\infty)$,define

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{itz} f(t)\, dt, \quad y > 0.$$

Then the mapping $U : f \to F$ is an isometry from $L_{\mathscr{E}}^p(0,\infty)$ onto $H_{\mathscr{E}}^p(\Pi)$. If f and F.

Are related in this way, then for each $y \ge 0$,

$$\lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-ixt} F(t+iy)\, dt = \begin{cases} e^{-yx} f(x) & \cdot x > 0, \\ 0, & x < 0, \end{cases}$$

With convergence in the metric of $L_{\mathscr{E}}^2(-\infty,\infty)$.

In particular, $H^p_{\mathscr{E}}(\Pi)$ is a Hilbert space which is naturally isomorphic with $L^2_{\mathscr{E}}(0,\infty)$.

Theorem (2.2.6):

For any F$\in H^p_{\mathscr{E}}(\Pi)\int_{-\infty}^{\infty}|F(x+iy)|^2_{\mathscr{E}}dx$, is a nonincreasing function of $y>0$. For any $F,G\in H^p_{\mathscr{E}}(\Pi)$.

$$\langle F,G\rangle_2=\lim_{r\downarrow 0}\int_{-\infty}^{\infty}\langle F(x+iy),G(x+iy)\rangle_{\mathscr{E}}dx$$

And

$$\lim_{r\downarrow 0}\int_{-\infty}^{\infty}|F(x+iy)-F(x)|^2_{\mathscr{E}}dx=0.$$

Chapter-3
Operator Valued Inner and Outer Functions

In this Chapter we construct an inner outer factorization theory for operator valued functions that are of bounded type on a disk or half-plane. The theory is less complete than in the scalar case, but it retains many of the characteristic features of the classical situation. In the case of bounded functions we obtain our results from the factorization theory for Toeplitz operator. Unbounded functions are handled with the aid of scalar mollifies. We characterize outer functions in terms of extremal properties. In particular, $\mathscr{E}$ always denotes a. separable Hilbert space, and D and Π arc the open unit disk and open upper half-plane.
Let $\Omega = D \text{or} \Pi$.

Section (3.1)Inner and outer functions with Beur ling|-lax theorem and canonical factorization functions

Definition (3.1.1):

If $A \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Omega)$,then:

(i) A is inner function if the operator

$$T(A): f \to Af, \quad f \in H^2_{\mathscr{E}}(\Omega). \tag{1}$$

Is a partial isometry on $H^2_{\mathscr{E}}(\Omega)$;

(ii) A an outer function if

$$\vee\left\{Af : f \in H^2_{\mathscr{E}}(\Omega)\right\} = H^2_M(\Omega) \tag{2}$$

For some subspace M of $\mathscr{E}$.

We use a scalar mollifier to extend the definition of an outer function to allow for unbounded functions.

Definition (3.1.2):

A holomorphic $\mathscr{B}(\mathscr{E})$-valued function F on Ω is an outer function if there is a bounded scalar valued outer function $\phi \not\equiv 0$ on Ω such that ϕF is bounded in the sense of Definition(3.1.1).

The boundary function of an inner (resp. outer) function is also called an inner (resp.outer) function. The function identically zero is both inner and outer by Definition C.

Theorem (3.1.3):

In this scalr case except for the function identically zero, the classes of inner and outer functions obtained from Definitions (3.1.1) and (3.1.2) coincide with the classes obtained from the classical definitions.

The classical definitions and properties of inner and outer function are given in Duren [1970] and Hoffman [1962].

Proof:

For definitions take $\Omega = D$. In the scalar case, a function A in $H^{\infty}(D), \not\equiv 0$ Is inner in the sense of definition (3.1.1) if and only if $\langle f, Ag\rangle_2 = \langle f, g\rangle_2$. Or what is the same thing,

$$\int_{\Gamma} \left|A\left(e^{i\theta}\right)\right|^2 f\left(e^{i\theta}\right)\bar{g}\left(e^{i\theta}\right)d\sigma = \int_{\Gamma} f\left(e^{i\theta}\right)\bar{g}\left(e^{i\theta}\right)d\sigma.$$

For all $f, g \in H^2(D)$.This holds if and only if $\left|A\left(e^{i\theta}\right)\right| = 1\sigma$-a.e, that is, A is inner in the classical sense.

The equivalence of the two definitions of an outer Lunction follows from Beurling's theorem (Duren) in the case of bounded functions. The general case is easily reduced to this case.

The canonical shift operators on $H^2_{\mathscr{E}}(D)$ and $H^2_{\mathscr{E}}(\Pi)$ are defined by

$$S: f(z) \to zf(z) \quad on\ H^2_{\mathscr{E}}(D) \tag{3}$$

And

$$S: f(z) \to \frac{z-i}{z+i} f(z) \quad on\ H^2_{\mathscr{E}}(\Pi). \tag{4}$$

These operators are unitarily equivalent by means of the isomorphism. Therefore the results for the operator

(3) Transfer to analogous results for (4). In either case, $\Omega = D$. Or Π, we set

$$\mathscr{K} = \ker S^* \qquad \text{and} \qquad P_0 = 1 - SS^*.$$

By the Wold decomposition, P_0 is the projection of $H^2_{\mathscr{E}}(\Omega)$ on $\mathscr{K}$ each $H^2_{\mathscr{E}}(\Omega)$ has an expansion

$$f = \sum_{0}^{\infty} S^j P_0 S^{*j} f \tag{5}$$

Which converges in the metric of $H^2_{\mathscr{E}}(\Omega)$.The expansion (5) also converges pointwise on Ω in the norm of $\Omega$and is easily identified -in classical terms.

Theorem (3.1.4):

(i)When $\Omega = D, \mathscr{K}$ is the space of constant function in $H^2_{\mathscr{E}}(\Omega)$, that is with an obvious identification, $\mathscr{K} = \mathscr{E}$. For each $f \in H^2_{\mathscr{E}}(D)$, (5) coincides with the Taylor expansion

$$f(z)=\sum_{0}^{\infty} a_j z^i, \qquad z\in D.$$

(ii) When $\Omega=\Pi,\mathscr{K}$ is the space of function of the form $\pi^{-1/2}c/(z+i)$, where $c\in\mathscr{E}$. For each $f\in H^2_{\mathscr{E}}(\Pi)$ coincides with the expansion

$$f(z)=\sum_{0}^{\infty}\frac{\pi^{-1/2}a_j}{z+i}\left(\frac{z-i}{z+i}\right)^j, \quad z\in\Pi,$$

Where $a_0,a_1,\ldots$ are the Taylor coefficients of the function $g\in H^2_{\mathscr{E}}(D)$ such that $f(z)=\pi^{-1/2}(z+i)^{-1}g((z-i)/(z+i))$ for all $z\in\Pi$.

Corollary (3.1.5):

A subspace $\mathscr{M}$ of $H^2_{\mathscr{E}}(\Omega)$ reduces the canonical shift operator S if and only $\mathscr{M}=H^2_M(\Omega)$ for some subspace M of $\mathscr{E}$.

The next result enables us to translate many theorems on operators, such as the theorems for Toephtz. Operator::, into analogous theorems on operator valued function.

Theorem (3.1.6):

Let S be the canonical shift operator on $H^2_{\mathscr{E}}(\Omega)$. Let $A\in H^\infty_{\mathscr{B}(\mathscr{E})}(D)$ and define $T(A)$ on $H^2_{\mathscr{E}}(\Omega)$ by (1). Then:

(i) $T(A)$ is Sanalytic in this sense of 1.6, and every S-analytic operator on $H^2_{\mathscr{E}}(\Omega)$ has this form؛

(ii) $T(A)$is S-constant in the sense of 1.6 if and only if $A(z)\equiv$ const. on Ω;

(iii) $T(A)$ is s-inner (rep. S-outer) in the sense of 1.6 if and only if A' is an inner (resp. outer) function؛

(iv) $T(A)$ Both S-inner and C-constant if and only if $A(z)\equiv A_0$ on $\Omega$where $A_0\in\mathscr{B}(\mathscr{E})$ is a partial isometry.

These results are straightforward, and we omit the proofs.

It turns out that. Any. $\mathscr{B}(\mathscr{E})$-valued inner function A does all of its work on a subspace M of $\mathscr{E}$ and is trivial on the orthogonal complement of M. This subspace if. Denoted $M_{in}(A)$. The formal definition and key properties are. Given below.

Theorem (3.1.7):

Let A be a $\mathscr{B}(\mathscr{E})$-valued inner function on $\Omega = D$ or Π. There exists a subspace M of $\mathscr{E}$ such that

$$\left\{f : f \in H^2(\Omega) \quad \text{and} \quad \|Af\|_2 = \|f\|_2\right\} = H_M^2(\Omega)$$

And $Ag = 0$ for all $g \in H_N^2(\Omega)$, where $N = M^{\perp}$.

Proof:

The set, of functions f in. $H_{\mathscr{E}}^2(\Omega)$ such that $\|Af\|_2 = \|f\|_2$ is a reducing subspace for the canonical shift operator S by Theorem (3.1.3). By (corollary to Theorem (3.1.4)), this subspace has the form $H_M^2(\Omega)$ for some subspace M of $\mathscr{E}$. Moreover, $Ag = 0$ for every $g \in H_M^2(\Omega)^{\perp} = H_N^2(\Omega)$, where $N = M^{\perp}$.

Definition (3.1.8):

We write $M_{in}(A)$ for the subspace M in the situation of Theorem (3.1.3).

Theorem (3.1.9):

For any $\mathscr{B}(\mathscr{E})$-valued inner function A on $\Omega = D$ or Π.

$$M_{in}(A) = \bigvee_{w \in \Omega_0} A(w)^{\mathscr{B}} \mathscr{E}. \qquad (6)$$

Where Ω_0 is any subset of Ω that has an accumulation point in Ω.

Proof:

For definition let $\Omega = D$.If $M = M_{in}(A)$, then $H_M^2(D)$ is the initial space of the partial $T(A)$ defined by (5-1). Equivalently, $H_M^2(D)$ is the range of $T(A)^*$.Function of the form $c/(1 - z\bar{w}), c \in \mathscr{E}$ span a dense subset of $H_{\mathscr{E}}^2(D)$, and

$$T(A)^* : c/(1 - z\bar{w}) \to A(w)^* c/(1 - z\bar{w}).$$

Thus (6) follows.

Theorem (3.1.10):

Let A be a $\mathscr{B}(\mathscr{E})$-valued inner function $M = D$ or Π. Then the values of the nontangential boundary function of A are partial isometrics on with-initial space $M = M_{in}(A)\sigma$-a.e.on Γ or $a.e$ on $(-\infty, \infty)$

Depending on the case.A converse result is given.

Proof:

It is sufficient to prove this when $\Omega = D$. Let P_M be the projection of $\mathscr{E}$ on M. Since the operators (1) is a partial isometry with initial space $H_M^2(D)$,

$$\int_{\Gamma}\left\langle A\left(e^{i\theta}\right)^{*} A\left(e^{i\theta}\right)a,b\right\rangle_{\mathscr{E}} e^{itj-kjB}d\sigma$$

$$=\left\langle z^{j}A(z)a,z^{k}A(z)b\right\rangle_{2}=\left\langle z^{j}P_{M}a,z^{k}P_{M}b\right\rangle_{2}$$

$$=\int_{\Gamma}\left\langle P_{M}a,b\right\rangle_{\mathscr{E}}e^{i(j-k)\theta}d\sigma$$

For all $a,b\in\mathscr{E}, j,k=0,1,2,\ldots$.Hence $A\left(e^{i\theta}\right)^{*} A\left(e^{i\theta}\right)=P_M\sigma$-a.e. On Γ ,and the result follows.

We show that values of any $\mathscr{B}(\mathscr{E})$-valued outer; function F- on D have ranges that are dence in a constant subspace M of $\mathscr{E}$. This subspace denoted $M_{out}(F)$.The formal definition follows a prelimiminary result. Except where otherwise stated. We assume that $\Omega = D$ or Π.

Theorem (3.1.11):

Let $\mathscr{B}(\mathscr{E})$ be a C-valued holomorphic function on Ω, and let $\phi\not\equiv 0$ and $\psi\not\equiv 0$ be bounded scalar valued holomorphic function such that ϕF and ψF are bounded on Ω. If ϕ and ψ are outers, then

$$\left(\phi FH_{\mathscr{E}}^{2}(\Omega)\right)^{-}=\left(\psi FH_{\mathscr{E}}^{2}(\Omega)\right)^{-}.$$

By Theorem (3.1.10), this is a special case of 1.12. Similarly, the assertions following 1.12 yields companion uniqueness result.

Theorem (3.1.12):

If $AH_{\mathscr{E}}^{2}(\Omega)=CH_{\mathscr{E}}^{2}(\Omega)$ for two function A,C on $\Omega(\Omega=D)$ or Π , then

$$C(z)\equiv A(z)B_0,\quad A(z)B_0^{*} \text{ On}\,\Omega.$$

Where $B_0\in\mathscr{B}(\mathscr{E})$ is apartial isometry with initial space $M_{in}(C)$ and finial space $M_{in}(A)$. Conversal, if two inner functions A,C on Ω are so related, then $AH_{\mathscr{E}}^{2}(\Omega)=CH_{\mathscr{E}}^{2}(\Omega)$.

Let $\Omega = D$ or Π. Every $F\in N_{\mathscr{B}(\mathscr{E})}(\Omega)$ has a representation

$$F = AG/b \qquad (7)$$

Where A is a $\mathscr{B}(\mathscr{E})$-valued inner function, G is $a\,\mathscr{B}(\mathscr{E})$ -valued outer function with $M_{out}(G)=M_{in}(A)$, and b scalar valued singular -inner function. For any representation (7), either

$$F\left(e^{i\theta}\right)^{*}\left(e^{i\theta}\right)=G\left(e^{i\theta}\right)^{*}G\left(e^{i\theta}\right) \quad \sigma\text{-a.e On } \Gamma \qquad (8)$$

Or

$$F(x)^{*}F(x)=G(x)^{*}G(x) \quad a.\text{e} \quad \text{on}(-\infty,\infty) \qquad (9)$$

Depending on the case. For any $w_0 \in \Omega$, there exists a representation (7) such that $G(w_0)\geq 0$

Proof:

We take $\Omega=D$ and $w_0=0$. The general case follows by conformal mapping for any $C\in H^{\infty}_{\mathscr{B}(\mathscr{E})}(D)$,let $T(C)$ be the operator multiplication by C on $H^2_{\mathscr{E}}(D)$.

Suppose first that $F\in H^{\infty}_{\mathscr{B}(\mathscr{E})}(D)$. By 3.6 and 5.2, Theorem (3.1.10),

$$T(F)=T(A)T(G)$$

For some inner function A and some bounded outer function G such that in $M_{in}(A)=M_{out}(G)$. Then $F=AG$, so there exists a factorization (7) with $b\equiv 1$. Moreover by 3.6 we can choose the factorization so that for all $c\in\mathscr{E}$

$$\langle G(0)c,c\rangle_{\mathscr{E}}=\langle G(z)c,c\rangle_2\geq 0,$$

That $G(0)\geq 0$

Now any $F\in N_{\mathscr{B}(\mathscr{E})}(D)$. By 4.3. Theorem (3.1.3), $F=F_0/u$, where $F_0\in H^{\infty}_{\mathscr{B}(\mathscr{E})}(D)$ and u is a scalar valued holomorphic function such that $0<|u|\leq 1$ on D. Factor $F_0=A_0G_0$ as above with $G_0(0)\geq 0$. Factor $u=bv$, where b is inner and v is outer with $v(0)>0$. Since $u\neq 0$ on D,b is singular inner function. The required factorizations (7) is then obtained with $A=A_0, G=G_0/v$, and the singular inner b.

We prove (8) for any factorization (7). First let $F,G\in H^{\infty}_{\mathscr{B}(\mathscr{E})}(D)$ and $b\equiv 1$.By 3.6,

$$T(F)^{*}T(F)=T(G)^{*}T(G). \qquad (10)$$

Hence for any $f_1,f_2\in H^2_{\mathscr{E}}(D)$,

$$\langle Ff_1,Ff_2\rangle_2=\langle Gf_1,Gf_2\rangle_2. \qquad (11)$$

And so

$$\int_{\Gamma}\left\langle F\left(e^{i\theta}\right)\right\rangle^{*} F\left(e^{i\theta}\right) f_1\left(e^{i\theta}\right), f_2\left(e^{i\theta}\right)_{\mathscr{E}}\, d\sigma$$

$$=\int_{\Gamma}\left\langle G\left(e^{i\theta}\right)^{*} G\left(e^{i\theta}\right) f_1\left(e^{i\theta}\right), f_2\left(e^{i\theta}\right)\right\rangle_{\mathscr{E}} d\sigma \qquad (12)$$

By the arbitrariness of f_1, f_2 (8) follows.

The general case of (8) can be reduced to the special case Consider any factorization (7). By 4.3, Theorem (3.1.3) ,$F = F_0/(u,u)$ where $F_0 \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(D), u_i, u_0$ are scalar valued functions, u_i is a singular inner function, and u_0 is an outer function such that $0<|u_0|\leq 1$ on D. By5-2, $G = G_0 / v$,where G_0 is a bounded outer function and v is a bounded scalar valued outer function on D. By (7),

$$b\, vF_0 = (u_1 A)(u_0 G_0).$$

Applying, the special case to this factorization, we get

$$\left[b\left(e^{i\theta}\right)v\left(e^{i\theta}\right)F_0\left(e^{i\theta}\right)\right]^{*}\left[b\left(e^{i\theta}\right)v\left(e^{i\theta}\right)F_0\left(e^{i\theta}\right)\right]$$

$$=\left[u_0\left(e^{i\theta}\right)G_0\left(e^{i\theta}\right)\right]^{*}\left[u_0\left(e^{i\theta}\right)G_0\left(e^{i\theta}\right)\right],$$

$\sigma.a.e$ On Γ, which implies (8).

Let $\Omega = D$ or Π every $F \in N^{+}_{\mathscr{B}(\mathscr{E})}(\Omega)$ has representation •

$$F = AG\,, \qquad (13)$$

Where A is $\mathscr{B}(\mathscr{E})$-valued inner function and G is a $\mathscr{B}(\mathscr{E})$-valued outer function such that $M_{out}(G) = M_{in}(A)$.For any representation (13) either (8) or (9) holds, depending on the case. For any $w_0 \in \Omega$ there a representation (13) such that $G(w_0) \geq 0$.

Proof:

The argument is essentially the same as for 5.6.In place of Section 4.3 Theorem (3.1.3), use Theorem (3.1.10).

Corollary (3.1.13):

If $\Omega = D$ or $\Pi, N^{+}_{\mathscr{B}(\mathscr{E})}(\Omega)$ is the smallest algebra containing an ll $\mathscr{B}(\mathscr{E})$-valued inner and outer functions on Ω.

Sec(3.2):Uniqueness of inner –Outer factorization and outer functions on the disk and half _plane:

Let $F,G\in N^{+}_{\mathscr{B}(\mathscr{E})}(\Omega),\Omega=D$ or Π.

Theorem (3.2.1):

If G is outer, the following are equivalent:

(i) The boundary function of F and G satisfy (8) or (9), depending on the case;

(ii) $F=G$ for some $\mathscr{B}(\mathscr{E})$-valued inner function A such that $M_{in}(A)=M_{out}(G)$.

Proof:

$(i)\Rightarrow(ii)$ For definition take $\Omega=D$. Using 4.3, Theorem (3.1.10), we easily reduce to the case in which F and G are bounded. In this multiplication by F and multiplication by G are bounded operators $T(F)$ and $T(G)$ on $H^2_{\mathscr{E}}(D)$. Our hypothesis (8) implies (12) for arbitrary $f_1,f_2\in h^2_{\mathscr{E}}(D)$. Hence (11) and (10) hold, and (ii) follows from 3.5, Theorem (3.1.3)

$(ii)\Rightarrow(i)$ This follows from 5.7.

Theorem (3.2.2):

If F,G are both outer, the following are equivalent:

(i) The boundary function of F and G satisfy (8) or (9). Depending on the case;

(ii) $G(z)\equiv CF(z)$ And $F(z)\equiv C^{*}g(z)$ on Ω where $C\in\mathscr{B}(\mathscr{E})$ is a partial isometry with initial space $M_{out}(F)$ and final space $M_{out}(G)$.

Proof:

Argue is in the proof of Theorem (3.2.1) but in place of 3.5, Theorem A use the corollary to 3.5, Theorem (3.2.1).

Theorem (3.2.3):

Let F,G both be outer, and let their boundary function satisfy (8) or (9), depending on the case. If $F(w_0)\geq 0$ and $G(w_0)\geq 0$ for some $w_0\in\Omega$, then $F\equiv G$ on Ω.

Proof:

Without loss of generality we can take $\Omega=D$ and $w_0=0$. It is easy to reduce the assertion to the case where F and G are bounded, and then the result follows from 3.5, Theorem (3.2.2).

Throughout this section S denotes the canonical shift operator on $H^2_{\mathscr{E}}(D)$. We identify $\mathscr{K}=\ker S^*$ with $\mathscr{E}$ in the obvious way. The Taylor coefficients of $\mathscr{B}(\mathscr{E})$ valued holographic functions $A, B, \ldots$ on the disk are denoted $\{A_j\}_0^\infty, \{B_j\}_0^\infty, \ldots$. If $A \in H^\infty_{\mathscr{B}(\mathscr{E})}(D)$. Then $T(A)$ is the operator multiplication by A on $H^2_{\mathscr{E}}(D)$. The matrix of $T(A)$ as definied in 3.2 is given by

$$T(A) \sim \begin{bmatrix} A_0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & \ldots \\ A_2 & A_1 & A_0 & \ldots \\ & \cdots & & \end{bmatrix}$$

Theorem (3.2.4):

Let $C \in N^+_{\mathscr{B}(\mathscr{E})}(D)$. Then C is outer if and only if

$$C_0^* C_0 \geq A_0^* A_0 \qquad (14)$$

For every $A \in N^+_{\mathscr{B}(\mathscr{E})}(D)$ such that

$$A(e^{i\theta})^* A(e^{i\theta}) = C(e^{i\theta})^* C(e^{i\theta}) \qquad \sigma\text{-a.e} \qquad (15)$$

In this case, for every $A \in N^+_{\mathscr{B}(\mathscr{E})}(D)$ satisfying (15), we have

$$\sum_0^n C_j^* C_j \geq \sum_0^n A_j^* A_j, \qquad n = 0,1,2,\ldots \qquad (16)$$

Proof:

If we replace $N^+_{\mathscr{B}(\mathscr{E})}(D)$ by $H^+_{\mathscr{B}(\mathscr{E})}(D)$, the theorem follows from 3.10(see Theorem (3.2.4) and the corollary to Theorem (3.2.4)). We deduce the general result from the bounded version.

Consider any $C \in N^+_{\mathscr{B}(\mathscr{E})}(D)$. By4.3, Theorem (3.1.10), there is a scalar value, outer function v such that $v(0) > 0, 0 < |v| \leq 1$, and $C = vc$ is bounded on D . For each $k = 1,2,3\ldots,$, the function defined on D by

$$v_k(z) \exp\left(\frac{1}{2\pi} \int_0^{2n} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\left(\min\left(k \left| v\left(e^{i\theta}\right) \right|, 1 \right) \right) d\sigma \right)$$

Is outer, $0 < |v_k| \leq 1$,and $C(k) = v_k C$ is bounded on D. Moreover,

$v_k(z) \to 1$

Uniformly on all compact subsets of D.Thus،

$$\frac{1}{v_k(z)} = \sum_{j=0}^{\infty} \alpha_{jk} z^j, \quad z \in D,$$

Then،

$$\lim_{k\to\infty} \alpha_{jk} = \begin{cases} 1, & j = 0, \\ 0, & j \geq 1. \end{cases} \tag{17}$$

Suppose that A is outer. Or any $A \in N^{+}_{\mathscr{B}(\mathscr{E})}(D)$ satisfying (15), set $A^k = v_k A, k \geq 1$, for $k \geq 1$،

$$A^{(k)}(e^{i\theta})^* A^{(k)}(e^{i\theta}) = C^{(k)}(e^{i\theta})^* C^{(k)}(e^{i\theta}).$$

σ-a.e. On Γ. Since $C^{(k)}$ is bounded on D so is $A^{(k)}$ by 4,7, Theorem(3.2.4). By the special case of the theorem not A above،

$$\sum_{j=0}^{n} C_j^{(k)*} C_j^{(k)} \geq \sum_{j=0}^{n} A_j^{(k)*} A_j^{(k)}, \quad n = 0,1,2,\ldots.$$

By (17), (16) follows on letting $k \to \infty$.In particular. (14) Holds.

Conversely, suppose that (14) holds for every $A \in N^{+}_{\mathscr{B}(\mathscr{E})}(D)$ that satisfies (15). Consider any $A' \in N^{+}_{\mathscr{B}(\mathscr{E})}(D)$ such that

$$A'(e^{i\theta})^* A'(e^{i\theta}) = C'(e^{i\theta})^* C'(e^{i\theta})$$

σ-a.e. On Γ. By 4.7, Theorem (3.2.4), $A' \in N^{\infty}_{\mathscr{B}(\mathscr{E})}(D)$. If $A = A'/v$, then $A \in N^{+}_{\mathscr{B}(\mathscr{E})}(D)$ and (15) holds. Then assumption (14) holds so

$$C_0'^* C_0' = |v(0)|^2 C_0^* C_0 \geq |v(0)|^2 A_0'^* A_0'$$

Since the result is knows for bounded function, C' is outer. Hence C is outer.

This completes the proof.

Let $\mathscr{P}_{\mathscr{E}}$ be the set of polynomials $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ with coefficients in $\mathscr{E}$.

Theorem (3.2.5):

Let $C \in N^{+}_{\mathscr{B}(\mathscr{E})}(D)$. And assume that $C(z)$ belong to $H^2_{\mathscr{E}}(D)$ for each $a \in \mathscr{E}$. Then C is outer if and only if for all $a \in \mathscr{E}$,

$$\left\langle C_0^* C_0 a,a\right\rangle_{\mathscr{E}} = \inf_{p\in\mathscr{P}_{\mathscr{E}}}\int_{\Gamma}\left\langle C\left(e^{i\theta}\right)^* C\left(e^{i\theta}\right)\left[a-e^{i\theta}p\left(e^{i\theta}\right)\right], a-e^{i\theta}p\left(e^{i\theta}\right)\right\rangle_{\mathscr{E}} d\sigma \quad (18)$$

In this case, for all A and A

$$\left\langle \sum_0^n C_j^* C_j a,a\right\rangle_{\mathscr{E}}$$

$$= \inf_{p\in\mathscr{P}_{\mathscr{E}}}\int_{\Gamma}\left\langle C\left(e^{i\theta}\right)^* C\left(e^{i\theta}\right)\left[a-e^{i(n+1)\theta}p\left(e^{i\theta}\right)\right], a-e^{i(n+1)\theta}p\left(e^{i\theta}\right)\right\rangle_{\mathscr{E}} d\sigma \quad (19)$$

The infimum in (18) may he viewed as a form of Szego's infimum (Ahiezer [1056]. Grenander and Szego [19581 and Dyrn and McKean [1972]

Proof:

Assume that C is outer. The proof of (19) is similar to the Proof .of 3.10, Theorem (3.1.10) ,$(i)\Rightarrow(iii)$.Fix $a\in\mathscr{E}$ and $n\geq 0$.Let $M=M_{out}(C)$, so$\{Cp: p\in\mathscr{P}_{\mathscr{E}}\}^- = H_M^2$. The infimum of$\left\|Ca-S^{n+1}g\right\|_2^2$. Over all $H_M^2(D)$is attained with$g=S^{*n+1}$. Thus

$$\inf_{p\in\mathscr{P}_{\mathscr{E}}}\int_{\Gamma}\left\langle C\left(e^{i\theta}\right)^* C\left(e^{i\theta}\right)\left[a-e^{i(n+1)\theta}p\left(e^{i\theta}\right)\right], a-e^{i(n+1)\theta}p\left(e^{i\theta}\right)\right\rangle_{\mathscr{E}} d\sigma$$

$$= \inf_{p\in\mathscr{P}_{\mathscr{E}}}\left\|C_a - S^{n+1}Cp\right\|_2^2$$

$$= \inf_{n\in H_M^2(D)}\left\|C_a - S^{n+1}h\right\|_2^2$$

$$= \left\|C_a - S^{n+1}S^{*n+1}Ca\right\|_2^2$$

$$= \left\langle \sum_0^n C_j^* C_j a,a\right\rangle_{\mathscr{E}}$$

This proves (19), and (18) follows as a special case.

Conversely, assume that$C$ satisfies (18) for all$a\in\mathscr{E}$. We apply Theorem (3.2.4) to show that C is outer. Let$A\in N^+_{\mathscr{B}(\mathscr{E})}(D)$, and suppose that (15) holds. For any$Aa\in\mathscr{E}$, by (18) and (15)

$$\left\langle C_0^* C_0 a,a\right\rangle_{\mathscr{E}} = \inf_{p\in\mathscr{P}_{\mathscr{E}}}\int_{\Gamma}\left\langle A\left(e^{i\theta}\right)^* A\left(e^{i\theta}\right)\left[a-e^{i\theta}p\left(e^{i\theta}\right)\right], a-e^{i\theta}\right\rangle_{\mathscr{E}} d\sigma$$

$$= \inf_{p\in\mathscr{P}_{\mathscr{E}}}\left\|A(z)\left[a-zp(z)\right]_1^1\right\|_2^2$$

$$\geq \left|A(0)_a\right|_{\mathscr{E}}^2$$

$$=\left\langle A_0^* A_0 a, e\right\rangle_{\mathscr{E}}$$

Thus (5-18) holds, and C is outer by Theorem (3.2.4).• .‘

Let S be the canonical shift operator on $H_{\mathscr{E}}^2(D)$,that is, S is multiplication by, $(z-i)/(z+i)$. For each $t \geq 0$, define V_t,on $H_{\mathscr{E}}^2(D)$ by $V_p^* f(z) \rightarrow e^{itz} f(z)$

The identity

$$\frac{z-i}{z+i}=1-2\int_0^{\infty} e^{-1} e^{itz} di \qquad (20)$$

Holds for each $z \in \Pi$. Weshow that it also holds in an operator theoretic sence.

Theorem (3.2.4) we have

$$S=1-2\int_0^{\infty} e^{-i} V, dt, \qquad (21)$$

Where the integral is taken in the weak sense defined in 4.5.

Proof it is enough to show that

$$\langle Sf, g\rangle_2=\langle f, g\rangle_2-2\int_0^{\infty} e^{-1} e^{-1}\langle V, f, g\rangle_2 dt \qquad (22)$$

For all $f \in H_{\mathscr{E}}^2(D)$ and all Ain some set whose linear span is dense in $H_{\mathscr{E}}^2(\Pi)$,

Choose g of the form

$$g(z)=\frac{1}{2\pi i} \frac{c}{\bar{w}-z}, \quad z \in \Pi. \qquad (23)$$

Where $w \in \Pi$ and $c \in \mathscr{E}$. In this case, reduces to (20) with $z=w$, and the result follows.

Theorem (3.2.5).The clouser .In the weak operator topology of the linear span of $\left(V_t\right)_{t \geq 0}$ contains S.

Proof**.** By theorem (3.2.4),

$$\left\|S-1+2\int_0^a e^{-t} V, dt\right\|=\left\|2\int_a^{\infty} e^{-t} v, dt\right\| \leq 2\int_a^{\infty} e^{-t} dt=2e^{-a}.$$

The two integrals involving $e^{-t} V$, are taken in the weak sense defined before it is easy to see that Riemann sums for $\int_0^a e^{-t} v, dt$, converge to the integral in the weak operator topology as well, and so the result follows.

Theorem (3.2.6):

Let $C \in N^{+}_{\mathscr{B}(\mathscr{E})}(\Pi)$. For C to be outer it is necessary and sufficient that

$$\int_0^t |v(s)|^2_{\mathscr{E}} ds \geq \int_0^t |u(s)|^2_{\mathscr{E}} ds, \quad t>0, \tag{24}$$

Whenever

$$\begin{cases} C(z)a(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^\infty e^{isz} v(s)ds, & z \in \Pi, \\ A(z)a(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^\infty e^{isz} u(s)ds, & z \in \Pi, \end{cases} \tag{25}$$

For some $A \in N^{+}_{\mathscr{B}(\mathscr{E})}(\Pi)$ such that $A(x)^* A(x) = C(x)^* C(x) a.ee$ on $(-\infty,\infty)$ and some $a \in N^{\infty}_{\mathscr{E}}(\Pi)$ such that $C_a, Aa \in H^2_{\mathscr{E}}(\Pi)$.

The integrals in (25) give the Paley-Wiener representation of the functions $Ca, Aa \in H^2_{\mathscr{E}}(\Pi)$ Thus $u,v \in L^2_{\mathscr{E}}(0,\infty)$. Note that by the Plancherel theorem, since

$$|A(x)a(x)|^2_{\mathscr{E}} = |C(x)a(x)|^2_{\mathscr{E}}$$

$a.e$ On $(-\infty,\infty)$, we have

$$\int_0^\infty |v(s)|^2_{\mathscr{E}} ds = \int_0^\infty |u(s)|^2_{\mathscr{E}} ds \tag{26}$$

In the sufficiency direction, the proof can easily be made to show more. Assume only that the condition holds when A is outer, and, for any fixed outer A for a set of $a's$ such that the span of vectors $\mathscr{E}$, is dense in A .Then A is outer,

Proof:

We begin with some preliminary remarks, concerning the Paley-Wiener representation. Every $f \in N^2_{\mathscr{E}}(\Pi)$ has a representation

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{t_z}^\infty e^{itz} h(s)ds, \quad z \in \Pi,$$

Where $h \in L^2_{\mathscr{E}}(0,\infty)$. It is easy to see that for any $t>0$,

$$(V_t^* f)(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{isz} h(s+t)ds, \quad z \in \Pi.$$

Hence by the Plancherel theorem,

$$\left\|V_t^* f\right\|_2^2 = \int_t^\infty \left|h(s)\right|_{\mathscr{G}}^2 ds.$$

Assume that is outer. Let u,v satisfy (25) for some A and a as in the theorem. By, theorem (3.2.4) A=BC for some inner function B. Such that $M_{in}(B)=M_{out}(C)$.Appling, to the operator multiplication by B on $H_{\mathscr{G}}^2(\Pi)$ we obtain

$$\left\|V_t^* ca\right\|_2^2 \le \left\|V_t^* BCa\right\|_2^2 = \left\|V_t^* Aa\right\|_2^2$$

For all $t>0$.By (25) and the remarks at the beginning of the proof, this yields

$$\int_t^\infty \left|v(s)\right|_{\mathscr{G}}^2 ds \le \int_t^\infty \left|u(s)\right|_{\mathscr{G}}^2 ds, \quad t>0.$$

Then (24) follows from (26).

Conversely, assume that (24) holds whenever u,v are related -as in the theorem. $C=BA$, where A is outer, B is inner $M_{out}(A)$, and $A(x)^* A(x)=C(x)^* C(x)$ a.e. On $(-\infty,\infty)$. For this choice of A and any a as in the theorem, there exist $u,v\in L_{\mathscr{G}}^2(0,\infty)$.satisfying (25), and then (24) holds by assumption. By what we proved above with the roles of A and C interchanged now A is outer), equality holds in (24). Arguing as in tilt proof necessity, we obtain

$$\left\|V_i^* Aa\right\| = \left\|V_t^* Ba\right\|_2; \qquad t\ge 0 \tag{27}$$

We now apply. To the operator $T(B)$ of multiplication by B on $H_{\mathscr{G}}^2(\Pi)$. , choose $\{V_j\}_{j\in J}$, to be the family $\{V_t\}_{t\ge 0}$, and let $\{g_k\}_{k\in K}$ be the set of functions Aa with a as in the theorem. Notice that holds by (27). By Theorem (3.2.5) above, the hypothesis Win. Satisfied. Hypothesis (ii) in 3.11 Lemma B, requires that the vectors $A(i)a(i)$ span a dense set in $M_{in}(B)$, and this holds. Hence T (B) is an A-constant inner operator, and hence B is a constant inner function. Since $C=BA$ is outer and A outer and $M_{in}(B)=M_{out}(A)$, C is outer.

Chapter 4
Factorization of Nonnegative Operator Valued functions

If $Q(x)=Q_1x+\cdots+Q_nx^n$ is a polynomial with operator coefficients, then $P(x)=Q(x)^*(x)$ is polynomial such that $P(x)\geq 0$ for all real x. We shall, conversely, every polynomial $P(x)$ with operator coefficients such that $P(x)\geq 0$ for all real has this form.

'More generally, We study the operator analogue Szegő's problem.this is interpreted as the problem of giving condition on an operator valued function $F(\cdot)$ on the circle Γ on line R which imply that

$$F(\cdot)=G(\cdot)^*G(\cdot),$$

Where $G(\cdot)$ is the boundary function of a holomorphic operator valued function of class N^+ on the unit disk. DOr upper half-plane Πinspired, respectively. Our results are inspired principally by three theorems in classical function theory to Feier. Riesz. Alliezer [1948] and Szego [1921].

FeJER- RIESZ, Theorem. Any trigonometric polynomial, $f\left(e^{i\theta}\right)=\sum_{-n}^{n}a_je^{ij\theta}$ that is nonnegative on the joint circle Γ ,has the form $f\left(e^{i\theta}\right)=\left|g\left(e^{i\theta}\right)\right|^2$, where $g\left(e^{i\theta}\right)=\sum_0^n b_je^{ij\theta}$ is an analytic trigonometric polynomial such that $g(z)=\sum_0^n b_jz^j$ has no zero on the disk D.

An elementary proof can be based on the fundamental theorem of algebra. See Riesz and Sz.-Nagy [1955].

AHIEZER'S Theorem. Let $f(z)$ be an entire function of exponential type that is nonnegative on the real axis and satisfies

$$\int_{-\infty}^{\infty}\frac{\log^+ f(x)}{1+x^2}dx<\infty$$

Then there exists an entire function $g(z)$of exponential type $\tau/2$ hunting no zeros for $y>0$ such that $f(x)=\left|g(x)\right|^2$ on the real axis.

Ahjezer; s theorem is a $g(z)$ generalization of the Fejer-Riesz theorem (Boas [1954]).

SZBGO'S Theorem. Let $f\left(e^{i\theta}\right)$ be a nonnegative function in $L^1(\sigma)$. For the existence a function $g(z)$ in $H^2(D)$ having no zeros on D, such that

$f\left(e^{i\theta}\right)=\left|g\left(e^{i\theta}\right)\right|^2$. On Γ, it is necessary and sufficient that

$$\int_{\Gamma} \log f\left(e^{i\theta}\right) d\sigma > -\infty.$$

We obtain extension of these results to operator valued function. The operator versions of the Fejer-Riesz and Ahiezer theorems follow as special cases of a general factorization theorem for till pseudoincromm functions. We also prove a generalization of Krein; s theorem for operator valued functions.

Thus $\mathscr{E}$ denotes a separable Hilbert space.

By S we always mean the canonical shift operator defined on $H^2_{\mathscr{E}}(D)$ or $H^2_{\mathscr{E}}(\Pi)$.Equivalently, we may view S as acting on boundary function, so that either

$$S: f\left(e^{i\theta}\right) \to e^{i\theta} f\left(e^{i\theta}\right) \quad on \ H^2_{\mathscr{E}}(\Gamma)$$

Or

$$S: f(x) \to \frac{x-i}{x+i} f(x) \quad on \ H^2_{\mathscr{E}}(R)$$

Depending on the case.

Sec(4.1):Toilets operators with operators valued and Pseudomeromorphic Functions Analyticity:

Consider the disk case we describe the class of S-Toilets operators on $H^2_{\mathscr{E}}(\Gamma)$nd relate the factorization properties of these operators to the problem at hand.

Let P be the projection of $L^2_{\mathscr{E}}(\sigma)$ on $H^2_{\mathscr{E}}(\Gamma)$.

Theorem (4.1.1):

Abounded linear operator T on $H^2_{\mathscr{E}}(\Gamma)$ is S -Toilets in the sense, if and only if $T=t(W)$, where _

$$T(W): f \to PWf, \quad f \in H^2_{\mathscr{E}}(\Gamma), \tag{1}$$

For some $W \in N^{\infty}_{\mathscr{B}(\mathscr{E})}(\sigma)$. In this case, $\|T\|=\|W\|_{\infty}$,

When $W \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)$,$T(W)$ is S-analytic, and (1) may be-written

$$T(W): f \to Wf, \quad f \in H^2_{\mathscr{E}}(\Gamma). \tag{2}$$

Conversely. Theorem (4.1.2), every S-analytic operator has this form.

Proof:

Let $T(W)$be defined by (1) for some $W \in L^{\infty}_{\mathscr{B}(\mathscr{E})}(\sigma)$. Cleary $\|T(W)\|_{\infty}$.

For any $f,g\in H^2_{\mathscr{E}}(\Gamma)$,

$$\left\langle T(W)Sf,Sg\right\rangle_2=\int_{\Gamma}\left\langle W\left(e^{i\theta}\right)e^{i\theta}f\left(e^{i\theta}\right),e^{i\theta}g\left(e^{i\theta}\right)\right\rangle_{\mathscr{E}}d\sigma=\left\langle T(W)f,g\right\rangle_2.$$

Hence $S^*T(W)S=T(W)$.That is $T(W)$ is S-Toeplitz.Sufficiengy follows.

Conversely, let T be an S-Toeplitz operator on $H^2_{\mathscr{E}}(\Gamma)$. Set $\mathscr{K}_n=U^{-n}H^2_{\mathscr{E}}(\Gamma),n=0,1,2,\ldots$, where U is the operator on $L^2_{\mathscr{E}}(\sigma)$ defined by (3).Then $\bigcup_0^\infty\mathscr{K}_{\mathrm{n}}$ is dense in $H^2_{\mathscr{E}}(\sigma)$ and

$$H^2_{\mathscr{E}}(\Gamma)=\mathscr{K}_0\subseteq\mathscr{K}_1\subseteq\mathscr{K}_2\subseteq\ldots L^2_{\mathscr{E}}(\sigma).$$

For each $n\geq 0$, set

$$s_n(f,g)=\left\langle TU^nf,U^ng\right\rangle_2,\quad f,g\in\mathscr{K}_{\mathrm{n}}.$$

Then $s_n(\cdot,\cdot)$ is a bounded sequillinear form on $\mathscr{K}_n$ with $\|S_n\|\leq\|T\|$ .Since $S^*TS=T$ by assumption and $S=U\mathscr{K}_0$,for any $tf,g\in\mathscr{K}_n$

$$s_{n+1}(f,g)=\left\langle TU^{n+1}f,U^{n+1}g\right\rangle_2=\left\langle TSU^*f_tSU^ng\right\rangle_2$$

$$=\left\langle TU^nf,U^*g\right\rangle_2=s_n(f,g).$$

Hence there is a bounded sequilinear form $s(\cdot,\cdot)$ on $L^2_{\mathscr{K}}(\sigma)$ that extends each $s_n(\cdot,\cdot)$and satisfies $\|s\|\leq\|T\|$.Let L be the unique operator on $L^2_{\mathscr{E}}(\sigma)$ such that

$$s(f,g)=\left\langle Lf,g\right\rangle_2,\quad f,g\in L^2_{\mathscr{E}}(\sigma)$$

And $\|L\|=\|s\|\leq\|T\|$ .For $f,g\in\mathscr{K}_{n+1},n\geq 0$,

$$\left\langle LUf,Ug\right\rangle_2=s(Uf,Ug)=s_n(Uf,Ug)$$

$$=\left\langle TU^*uf,U^*Ug\right\rangle_2=s_{n+1}(f,g)=s(f,g)=\left\langle Lf,g\right\rangle_2.$$

Therefore $U^*LU=L$ and $LU=UL$. By the lemma. L Has form (4) for some $W\in L^\infty_{\mathscr{B}(\mathscr{E})}(\sigma)$with $\|W\|_\infty=\|L\|\leq\|T\|$.For any $f,g\in H^2_{\mathscr{E}}(\Gamma)$.

$$\left\langle Tf,g\right\rangle_2=s_0(f,g)=\left\langle Lf,g\right\rangle_2$$

$$=\int_1\left\langle W\left(e^{i\theta}\right)f\left(e^{i\theta}\right),g\left(e^{i\theta}\right)\right\rangle_{\mathscr{E}}d\sigma=\left\langle T(W)f,g\right\rangle_2,$$

It follows that $T=T(W)$ .By construction $\|W\|_\infty\leq\|T\|$. The reverse inequality holds automatically as in the proof. Of sufficiency, so the result follows.

Theorem (4.1.2):

Let $W \in L^{\infty}_{\mathscr{B}(\mathscr{E})}(\sigma)$ **and** $A \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)$.

(i) $T(W) \geq 0$ if and only if $W(e^{i\theta}) \geq 0$ on $T(W) \geq 0$;

(ii) $T(W) = T(A)^{*} T(A)$. If and only if $W(e^{i\theta}) = A(e^{i\theta})^{*} A(e^{i\theta}) \sigma$-*a.e* on Γ

Proof:

(i) If $W(e^{i\theta}) \geq 0 \sigma$-a.e. On Γ, then

$$\langle T(W) f, f \rangle_2 = \int_{\Gamma} \langle W(e^{i\theta}), f(e^{i\theta})_{\mathscr{E}} \, d\sigma \geq 0 \rangle$$

For every $f \in H^{2}_{\mathscr{B}(\mathscr{E})}(\Gamma)$, so $T(W) \geq 0$.

Conversely, Let $T = T(W) \geq 0$.Construct Lon $L^{2}_{\mathscr{E}}(\sigma)$ as in the proof of Theorem(4.1.1). Then $L \geq 0$, so

$$\int_{\Gamma} \langle W(e^{i\theta}) f(e^{i\theta}), f(e^{i\theta}) \rangle_{\mathscr{E}} \, d\sigma \geq 0$$

For every $f \in H^{2}_{\mathscr{B}(\mathscr{E})}(\Gamma)$.It follows that $W(e^{i\theta}) \geq 0 \sigma$-a.e on Γ.

(ii)Since $A \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)_{\infty} t(A)$. A Is multiplication by A on $H^{2}_{\mathscr{E}}(\Gamma)$ Thus

$T(W) = T(A)$ If and only if for all $f, g \in H^{2}_{\mathscr{E}}(\Gamma)$.

$$\langle T(W)f, g \rangle_2 = \langle Af, Ag \rangle_2$$

That is,

,

Of equivalently $W(e^{i\theta}) = A(e^{i\theta})^{*} A(e^{i\theta}) \sigma$-a.e on Γ.

The exterior of the unit circle denoted $\tilde{D}$; the lower half-plane, $\tilde{\Pi}$;that is,

$$\tilde{D} = \{ z : |z| > 1 \} \quad \text{And} \quad \tilde{\Pi} = \{ z : \operatorname{Im} z < 0 \}, \tag{8}$$

If F is $F \, \mathscr{B}(\mathscr{E})$-valued function on a set $\Omega \subseteq \mathbb{C}$, the reflection of F with respect to Γ is the function

$$\tilde{F}(z) = F(1/\bar{z})^{*} \quad \text{On} \quad \tilde{\Omega} = \{ z : 1/\tilde{z} \in \Omega \}. \tag{9}$$

The reflection of F with respect to R is

$$\tilde{F}(z) = F(\bar{z})^{*} \quad \text{On} \quad \tilde{\Omega} = \{ z : \tilde{z} \in \Omega \}. \tag{10}$$

Whether (9) or (10) is intended will. Either is clear from context or indicated.

The notion of a Laurent .expansion has a routine extension to $\mathscr{B}(\mathscr{E})$-

valued functions. Removable singularities, essential singularities, poles, and principal parts are then defined in the usual wav. We assume that removable singularities have been removed. $A\mathscr{B}(\mathscr{E})$-va1ued function F is meromorphic on an open set $\Omega \text{in} C_\infty = C \cup \{\infty\}$ if it is holornorphic on. Ω Except for poles.

Lemma (4.1.3):

Bounded linear operator L on $L^2_{\mathscr{E}}(\sigma)$ commutes with the operator

$$U : f(e^{i\theta}) \to e^{i\theta} f(e^{i\theta}) \tag{3}$$

On $L^2_{\mathscr{E}}(\sigma)$ if and only if

$$L : f \to Wf \tag{4}$$

For some $W \in L^\infty_{\mathscr{B}(\mathscr{E})}(\sigma)$. In this case $\|W\|_\infty = \|L\|$

Proof:

Cleary any operator of the form (4) comutes with U and $\|L\| \le \|W\|_\infty$.

Conversely, suppose that $LU = UL$ Then $LU^j = U^j L$.for all $j = 0, \pm 1, \pm 2, \ldots$.

Hence

$$L(\phi f) = \phi \cdot (Lf), \qquad f \in L^2_{\mathscr{B}(\mathscr{E})}(\sigma). \tag{5}$$

For any trigonometric polynomial ϕ with scalar coefficients. By a routine approximation argument, (5) holds for all continuous complex valued functions $\phi \,\text{on}\, \Gamma$.

Let $\mathscr{E}$ be a countable dense set in $\mathscr{E}$.For each $c \in \mathscr{E}$, let $g_n(e^{i\theta})$ be a represntatative in the coset Lc.Four any fixed $a, b \in \mathscr{E}$.

$$\lim_{w \to e^{i\theta}} \int_\Gamma P(w, e^{i\theta}) \langle g_n(e^{i\theta}), b \rangle_{\mathscr{E}} \, d\sigma = \langle g_n(e^{i\theta}), b \rangle_{\mathscr{E}} \tag{6}$$

Nontangentially σ-a.e. On Γ by Fatou; s theorem. Since $\mathscr{E}$ is countable, we can choose σ-null set $N \subseteq \Gamma$ such that (6) holds for all $a, b \in \mathscr{E}$ and $e^{i\theta} \in \Gamma \setminus N$.

Fix $e^{i\theta} \in \Gamma \setminus N$. Define $s_0(e^{i\theta}, \cdot, \cdot)$ on $\mathscr{E} \times \mathscr{E}$ by

$$s_0(e^{i\theta}, a, b) = \langle g_n(e^{i\theta}), b \rangle_{\mathscr{E}}, \qquad , b \in \mathscr{E}.$$

Define $\phi_w(e^{it}) = P(w, e^{i\theta})^{1/2}$ for $w \in D, e^{it} \in \Gamma$. By (5).

$$\int_{\Gamma} P\left(w,e^{it}\right)\left\langle g_a\left(e^{it}\right),b\right\rangle_{\mathscr{E}} d\sigma = \left\langle \phi_w\left(La\right),\phi_w b\right\rangle_2 = \left\langle L\left(\phi_w a\right),b\right\rangle_2$$

So by (6)

$$s_0\left(e^{i\theta};a,b\right) = \lim_{w\to e^{i\theta}} \left\langle L\left(\phi_w a\right),\phi_w b\right\rangle_2 \qquad (7)$$

Nontangentially for all $a,b\in\mathscr{E}$. It is.easy to see that the limit on the right of (7) exists for all $a,b\in\mathscr{E}$ and defines a bounded sesquilinear form $s\left(e^{i\theta},\cdot,\cdot\right)$on $s_0\left(e^{i\theta},\cdot,\cdot\right)$ and satisfies$\|s\|\le\|L\|$. Hence there is an operator $W\left(e^{i\theta}\right)\in\mathscr{B}\left(\mathscr{E}\right)$. Such that $\left|W\left(e^{i\theta}\right)\right|_{\mathscr{B}(\mathscr{E})}\le\|L\|$ and

$$s\left(e^{i\theta},a,b\right) = \left\langle W\left(e^{i\theta}\right)a,b\right\rangle_{\mathscr{E}}, \quad a,b\in\mathscr{E}$$

Now consider.$W\left(e^{i\theta}\right)$ As a function of$e^{i\theta}$. By construction, $W\in L^{\infty}_{\mathscr{B}(\mathscr{E})}\left(\sigma\right)$ and$\|W\|_{\infty}\le\|L\|$. For$c\in\mathscr{E}, L: c\to W\left(e^{i\theta}\right)c$. In a straightforward way we obtain (4)
And the result follows .

Definition (4.1.4):

(i) Let u,v be nonzero scalar valued functions in$N^{+}\left(\Gamma\right), A\mathscr{B}\left(\mathscr{E}\right)$. Valued function F . On Γ is of class$\mathscr{M}\left(u,v\right)$ if$uF, vF^{*}\in N^{+}_{\mathscr{B}(\mathscr{E})}\left(\Gamma\right)$.

(ii)Let u, v nonzero scalar valued functions in$N^{+}\left(R\right)A\mathscr{B}\left(\mathscr{E}\right)$ valued functions$F$ on $R$is of class $\mathscr{M}\left(u,v\right)$ if $uF, vF^{*}\in N^{+}_{\mathscr{B}(\mathscr{E})}\left(R\right)$

.Functions of class $\mathscr{M}\left(u,v\right)$.are called. Pseudomerornorphic because of the –characterizations in Theorems (4.1.1) and (4.1.2) below. The class $\mathscr{M}\left(u,v\right)$. Does not depend on outer factors in u and$v: F$ is of class. $\mathscr{M}\left(u,v\right)$If and only if it is of class$\mathscr{M}\left(u_i,v_i\right)$, where $u_i,v_i$are the inner factors ofu,v , respectively.

Example (4.1.5):

Let$u\left(e^{i\theta}\right)=e^{im\theta}, v\left(e^{i\theta}\right)=e^{in\theta}$, where $m,n$are nonnegative integers. Every trigonometric polynormial

$$F\left(e^{i\theta}\right)=\sum_{-m}^{n} A_j e^{ij\theta} \qquad (11)$$

With coefficients in$\mathscr{B}\left(\mathscr{E}\right)$ is of class$\mathscr{M}\left(u,v\right)$.

(i) If $F \in L^1_{\mathscr{B}(\mathscr{E})}(\sigma)$ and F is of class $\mathscr{M}(u,v)$,then F has the form (11). For, Theorem $A, uF, vF^* \in N^+_{\mathscr{B}(\mathscr{E})}(\Gamma) \cap L^2_{\mathscr{B}(\mathscr{E})}(\Gamma) = H^1_{(\mathscr{B})\mathscr{E}}(\Gamma)$, .Hence, for all,
j>1·
$\int_\Gamma e^{ij\theta} u(e^{i\theta}) F(e^{i\theta}) d\sigma = \int_\Gamma e^{ij\theta} v(e^{i\theta}) F(e^{i\theta})^* d\sigma = 0$
There for j>m or j<-n
$\int_\Gamma e^{ij\theta} F(e^{i\theta}) d\sigma = 0$
By the Cauchy representation (17), $t > 0$ has the form (11)
(ii) For each $p \in (0,1)$, there is a function $F \in L^p_{\mathscr{B}(\mathscr{E})}(\sigma)$ of class $M(u,v)$ and not of the form(11)
An example $F_0(e^{i\theta}) = F_0 / (e^{i\theta} - 1)$ for any nonzero $F_0 \in \mathscr{B}(\mathscr{E})$.

Theorem(4.1.6):
Let $u_0(e^{i\theta}), v_0(e^{i\theta})$ be nonzero scalar valued function in $N^+(\Gamma)$, and let u(z),v(z) be thecorresponding functions in $N^+(D)$,let F(z) be a $\mathscr{B}(\mathscr{E})$-valued meromorphic functiom on $D \cup \widetilde{D}$ such that
(i) the restrictions of uF an $v\widetilde{F}$ to D are in $N^+_{\mathscr{B}(\mathscr{E})}(D)$;
(ii) $F(re^{i\theta})$ has the same strong limit $F_0(e^{i\theta})$ for r↑any1 and r↑1 $\sigma.a.e$ on Γ, Then $F_0(e^{i\theta})$ is of class $M(u_0,v_0)$. Conversely, every function of class $M(u_0,v_0)$ has this form.
Proof:
The restrictions of uF and $v\widetilde{F}$ to D have boundary functions u_0F_0 and $v_0F_0^*$. These functions therefore belong to $N^+_{\mathscr{B}(\mathscr{E})}(\Gamma)$, and hence F_0 is of class. $\mathcal{M}(u_0, v_0)$.
Conversely ,let G_0 be any $\mathscr{B}(\mathscr{E})$-valued function of class $M(u_0,v_0)$. Then $u_0G_0, v_0G_0^* \in N^+_{\mathscr{B}(\mathscr{E})}(\Gamma)$ and so $u_0G_0, v_0G_0^*$ are the boundary functions of some functions G_+, G_- in $N^+_{\mathscr{B}(\mathscr{E})}(D)$. Set

$$G(z) = \begin{cases} G_+(z)/u(z), & z \in D, \\ \tilde{G}_-(z)/\tilde{v}(z), & z \in \widetilde{D}, \end{cases}$$

A routine check shows that G_0 and G are related in the required manner

Theorem(4.1.7):
Let $u_0(x), v_0(x)$ be nonzero scalar valued functions in $N^+(R)$, and let u(z),v(z) be the corresponding functions in $N^+(\Pi)$. Let F(z) be $\mathscr{B}(\mathscr{E})$-valued meromorphic function on $\Pi \cup \widetilde{\Pi}$ such that:

(i) the restr ictions of uF an $v\widetilde{F}$ to Π are in $N^{+}_{\mathscr{B}(\mathscr{E})}(\Pi)$;

(ii) F(x+iy) has the same strong limit $F_0(x)$ for y↓0 and y↑0 $a,e.$ on $(-\infty,\infty)$.

Then $F_0(x)$ is of class $M(u_0,v_0)$. Conversely, every function of class $M(u_0,v_0)$ has this form.

Proof:

This follows from Theorem(4.1.6) by a change of variables.

Let $\Omega = D$ or Π, and Let Δ be an open subset of $\partial\Omega$. Define $\tilde{\Omega} = \tilde{D}$ *or* $\tilde{\prod}$ or M as in (8) .

Definition(4.1.8):

In the situation of either Theorem(4.1.6) or Theorem(4.1.7) , we say that F is of class M(u,v). We refer to F_0 as the boundary function of F.

Definition (4.1.9):

Memorphic $\mathscr{B}(\mathscr{E})$-valued function F on $\Omega\cup\tilde{\Omega}$ is analytic across Δ if F can be defined on Δ so that when viewed as a function on $\Omega\cup\tilde{\Omega}\cup\Delta$ F, M is holomorphic at each point of Δ.

A holomorphic scalar valued function f on Ω is said to have an analytic continuation across Δ if $f = g|\Omega$, where g is holomorphic on some open set G containing $\Omega\cup\Delta$. The following result generalized a theorem of Carleman [1944],

Theorem (4.1.10):

Let F be a meromorphic $\mathscr{B}(\mathscr{E})$-valued function on $\Omega\cup\tilde{\Omega}$ of class M for some nonzero functions $u,\Omega\cup\tilde{\Omega}$. Assume that:

(i) M have analytic continuations across Δ;

(ii) If F_o is the boundary function of F, then for each $a \in$ L, the scalar valued function M is integrable over every compact subset of Δ.

Then F is analytic across Δ.

The function in the lemma below are scalar valued.

Proof:

We give the proof for the case $\Omega = \prod$. Then the other case follows by a change of variables.

We may assume that $\Delta = (\alpha,\beta), -\infty < \alpha < \beta < \infty$, and that the analytic continuations of u and M across Δ have no zeroes on Δ. For if there are zeroes, we can contract the interval slightly and reduce to the case of a finite number of zeros; dividing out factors to remove these zeros does

not change M because the factors are outer functions.
Consider an interval $[c,d]\subseteq(\alpha,\beta)$ such that $F(x+iy)$ has a strong limit for $x=c,d$ as $|y|\to 0$. We can choose such an interval with c arbitrary near α and d arbitrary near β. Choose a rectangle $Q=(c,d)\times(-\delta,\delta)$, where $\delta>0$ is small enough that the analytic continuations of M across Δ are defined and nonnegative on $\bar{Q}$. Define

$$G(z)=\frac{1}{2\pi i}\int_{\partial Q}\frac{F(t)}{t-z}dt, \quad z\in Q.$$

Our assumptions imply that F is sufficiently regular on ∂Q for the integral to exists in the weak sense: $F(x+iy)$ remains bounded for $x=c,d,|y|\to 0$, by the uniform boundedness principle. The function $G(z)$ is holomorphic on Q. To complete the proof, we show that F coincides with G on $Q\cap\left(\Pi\cup\tilde{\Pi}\right)$. By considering the scalar valued functions $\langle F(.)a,a\rangle_{\mathscr{G}}$ and $\langle G(.)a,a\rangle_{\mathscr{G}}$ for arbitrary $a\in\mathscr{G}$, we can assume without loss of generality that we are in the scalar case, that is, F and G are themselves scalar valued functions.
Set $Q(\varepsilon+)=(c,d)\times(\varepsilon,\delta)$ for any $\varepsilon\in(0,\delta)$. By Cauchy's theorem,

$$F(z)=\frac{1}{2\pi i}\int_{\partial Q(\varepsilon+)}\frac{F(t)}{t-z}dt, \quad z\in Q(\varepsilon+).$$

We show that

$$\lim_{y\downarrow 0}\int_c^d\left|F(+iyx)-F(x)\right|dx=0 \qquad (12).$$

By assumption, F is of class M, so $uF\in N^+(\Pi)$. By the lemma, (1) holds with F replaced by uF. The assumptions on u imply that we can drop the factor u , and (12) follows. Letting $\varepsilon\downarrow 0$, we obtain

$$F(z)=\frac{1}{2\pi i}\int_{\partial Q^+}\frac{F(t)}{t-z}dt \quad ,z\in Q^+$$

Where $Q+=Q\cap\Pi$ and the boundary function of F is used on the lower edge. Combining this formula with an analogous formula for $Q-=Q\cap\tilde{\Pi}$, we obtain

$$F(z)=\frac{1}{2\pi i}\int_{\partial Q-}\frac{F(t)}{t-z}dt, \quad z\in Q\cap\left(\Pi\cup\tilde{\Pi}\right).$$

Thus $F = G$ on $Q \cap \left(\Pi \cup \tilde{\Pi}\right)$, and the result follows.

Lemma (4.1.11):

Let $f \in N^+(\Pi)$ and suppose that

$\int_a^b |f(x)|^p \, dx < \infty$, where $-\infty < a < b < \infty$ and $0 < b < \infty$. Then

$$\lim_{y \downarrow 0} \int_c^d |f(x) - f(x + iy)|^p \, dx = 0$$

For every closed subinterval $[c,d]$ of (a,b).

Proof:

Choose q such that $pq > 1$. Let g be an outer function such that $|g(x)| = 1$ on (a,b) and $|g(x)| = (|x| + 1)^q |f(x)|$ otherwise. Then $h = \int / g \in N^+(\Pi)$ and $\int_{-\infty}^{\infty} |h(x)|^p \, dx < \infty$. Therefore $h \in H^p(\Pi)$ and

$$\lim_{y \downarrow 0} \int_{-\infty}^{\infty} |h(x) - h(x + iy)|^p \, dx = 0$$

(See Krylov [1939]). The function g has an analytic continuation across (a,b), and so $\lim_{y \downarrow 0} g(x + iy) = g(x)$ uniformly on every closed subinterval $[c,d]$ of (a,b). In view of the elementary inequality

$$|u + v|^p \le \left(\max\left(2|u|, 2|v|\right)\right)^p \le 2^p \left(|u|^p, |v|^p\right),$$

This is sufficient to simply the lemma.

Theorem (4.1.12):

Let v be any nonzero scalar valued function in $N^+(\Gamma)$ or $N^+(R)$. If M is any nonnegative $\mathscr{R}(\mathscr{E})$-valued function of class $\mathscr{M}(u,v)$ on Γ or R, then

$$F = G^* G \qquad (13)$$

σ-a.e. on Γ or a.e. on R, where G is an outer function of class $\mathscr{M}(1,v)$ on Γ or R, respectively.

The factorization is essentially unique, Theorem (4.1.2)

Proof:

We give the proof in the circle case. The other case then follows by a change of variables. Since $\mathscr{M}(u,v)$ does not depend on the outer factor in v, we may that v Inner

Since F of class $\mathscr{M}(u,v), uF \in N^{+}_{\mathscr{B}(\mathscr{E})}(\Gamma)$. Theorem (4.1.7), there a bounded scalar valued outer function (1). On Γ such the $\phi v f \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)$. Since $|v| = 1c\text{-}a.e$,on Γ, the function $W = \bar{\phi} F \phi$ belongs to $L^{\infty}_{\mathscr{B}(\mathscr{E})}(\sigma)$. Let P be the projection of $L^2_{\mathscr{E}}(\sigma)$ on $H^2_{\mathscr{E}}(\Gamma)$, and define the Toeplitz operator $T(W)$., Theorem (4.1.2), and thus is applicable. We show that condition (iii) of is satisfied. For each $c \in \mathscr{E}$ and $n = 0,1,2\ldots$.

$$I_n(c) = \sup\left\{\left|\langle T(W)_{c,} S^n f\rangle_2\right| : f \in H^2_{\mathscr{E}}(\Gamma), \langle T(W) f, f\rangle_2 = 1\right\}$$

Here e is viewed as a constant function in $H^2_{\mathscr{E}}(\Gamma)$. Set `

$$\chi\left(e^{i\theta}\right) = e^{i\theta}, \quad e^{i\theta} \in \Gamma$$

For any $f \in H^2_{\mathscr{E}}(\Gamma)$,

$$\left|\langle T(W)_c, S^n f\rangle\rangle_{\partial 2}\right| = \left|\int_\Gamma \langle W_c, X^n f\rangle_{\mathscr{E}} \, d\sigma\right|$$

$$= \left|\int_r \left\langle (\phi/\bar{\phi}) vc, \chi^n P \phi^2 v F f\right\rangle_{\mathscr{E}} d\sigma\right|$$

$$= \left|\int_r \left\langle P\chi^{-n} P(\phi/\bar{\phi}) vc, v\left(F^{1/2}\phi\right)^2 f\right\rangle_{\mathscr{E}} d\sigma\right|$$

$$= \left|\int_r \left\langle F^{1/2}\bar{\phi} P\chi^{-n} P(\phi/\bar{\phi}) vc, vF^{1/2}\phi f\right\rangle_{\mathscr{E}} d\sigma\right|$$

$$\leq \left(\int_r \left|F^{1/2}\bar{\phi} P\chi^{-n} P(\phi/\bar{\phi}) vc\right|^2_{\mathscr{E}} d\sigma\right)^{1/2} \langle T(W) f, f\rangle_2$$

By the choice of $\phi, F^{1/2}\phi$ is bounded

$$I_n(c) \leq const\left(\int_r \left|P\chi^{-n} P(\phi/\bar{\phi}) vc\right|^2_{\mathscr{E}} d\sigma\right)^{1/2}$$

$$\leq const\left\|S^* g_r\right\|_2$$

Where $g_c = P(\phi/\bar{\phi}) vc$. Since S is a shift operator, $I_n(c) \to 0$ as $n \to \infty$. Thus condition (iii) is satisfied. There is an outer function A such that $W = A^* A \ \sigma$ -a.e. on Γ . Therefore

$$F = W/\left(\bar{\phi}/\phi\right) = A^* A/\left(\bar{\phi}/\phi\right) = G^* G$$

σ -a.e. on Γ, where $G = A/\phi$ is outer.

We show that G is of class $\mathscr{M}(1,v)$. Consider the function $C = vA^*\left(\phi/\overline{\phi}\right)$ in $L^{\infty}_{\mathscr{B}(\mathscr{E})}(\sigma)$. By the choice of $\phi, CA = \phi^2 vF \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)$. Thus

$$C\left(AH^2_{\mathscr{E}}(\Gamma)\right) \subseteq H^2_{\mathscr{E}}(\Gamma)$$

Since A is outer, $\left(AH^2_{\mathscr{E}}(\Gamma)\right)^-$ reduces S. Hence if g is in $H^2_{\mathscr{E}}(\Gamma)$ and orthogonal to $\left(AH^2_{\mathscr{E}}(\Gamma)\right)^-$, so is $S^j g, j = 0,1,2,\ldots$. Then for any $u \in H^2_{\mathscr{E}}(\Gamma)$ and $j \geq 0$,

$$\int_{\Gamma} \left\langle A^* g, \chi^{-j} \right\rangle_{\mathscr{E}} d\sigma = \int_{\Gamma} \left\langle \chi g, Au \right\rangle_{\mathscr{E}} d\sigma = \left\langle S^j g, Au \right\rangle_2 = 0$$

Therefore $A^* g = 0$ σ-a.e. on Γ, and so $C_g = 0$ σ-a.e. on Γ. It follows that

$$C \in H^2_{\mathscr{E}}(\Gamma) \subseteq H^2_{\mathscr{E}}(\Gamma)$$

And hence $C \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)$. Therefore $\phi v G^* = C \in H^{\infty}_{\mathscr{B}(\mathscr{E})}(\Gamma)$, and so $vG \in N^+_{\mathscr{B}(\mathscr{E})}(\Gamma)$. Trivially $G \in N^+_{\mathscr{B}(\mathscr{E})}(\Gamma)$, and hence G is of class $\mathscr{M}(1,v)$. This completes the proof.

Sec (4.2):Fejér-Riesz Operator and Rational Functions with Entire Functions of Exponential Type:

Theorem (4.2.1):

Let $F\left(e^{i\theta}\right)=\sum_{-n}^{n} A_j e^{ij\theta}$ be trigonometric polynomial with coefficients in $\mathscr{B}\left(\mathscr{E}\right)$ such that $F\left(e^{i\theta}\right)\geq 0$ on Γ. Then

$F\left(e^{i\theta}\right)=G\left(e^{i\theta}\right)^{*} G\left(e^{i\theta}\right), \qquad e^{i\theta}\in\Gamma$ (14)

Where $G\left(e^{i\theta}\right)$ is an outer function of the form $G\left(e^{i\theta}\right)=\sum_{0}^{n} B_j e^{ij\theta}$ with coefficients in $\mathscr{B}\left(\mathscr{E}\right)$.

Proof:

In view of the example this follows as a special case

Theorem (4.2.2):

Let $P\left(x\right)=\sum_{0}^{2n} P_j x^j$ be a polynomial with coefficients in $\mathscr{B}\left(\mathscr{E}\right)$ such that $P\left(x\right)\geq 0$ for all x, then

$$P\left(x\right)=Q\left(x\right)^{*} Q\left(x\right), \quad x\in R, \qquad (15)$$

Where $Q\left(x\right)$ is outer function of the form $Q\left(x\right)=\sum_{0}^{n} Q_j x^j$.

Proof:

Set $v\left(x\right)=\left(x-i\right)^{n}/\left(x+i\right)^{n}$ and $F\left(x\right)=\left(x^2+1\right)^{-n} P\left(x\right)$. Then $v\left(x\right)f\left(x\right)$ and $v\left(x\right)F^{n}$ are the boundary functions of functions that are bounded and holomorphic Π. Hence F is of class $\mathscr{M}\left(u,v\right)$, so $F\left(x\right)=G\left(x\right)^{*} G\left(x\right) a.e.$ On $\left(-\infty,\infty\right)$, where $G\left(x\right)$ is an outer function of class $\mathscr{M}\left(u,v\right)$. $F\left(x\right)=G\left(x\right)^{*} G\left(x\right) a.e$ On $\left(-\infty,\infty\right)$, where $G\left(x\right)$ is an outer function of class $\mathscr{M}\left(1,v\right)$.Consider the associated meromorphic function $G\left(z\right)$ as, Theorem (4.2.2). Since $G\left(x\right)$ and $v\left(x\right)G\left(x\right)^{*}$ are bounded functions in $N^{+}_{\mathscr{B}\left(\mathscr{E}\right)}\left(R\right), G\left(z\right)$ and $v\left(z\right)\tilde{G}\left(z\right)$ are bounded on Π Hence

$$\left\|(z+i)^n G(z)\right\| \leq M(|z|+1)^n, \quad z \in \Pi \cup \tilde{\Pi},$$

For some constant $M > 0$.We have $(z+i)^n G(z)$ is an entire function. Therefore by the Cauchy estimates, $Q(z) = (z+i)^n G(z)$ is a polynomial of degree at most n . Since $G(x)$ is outer as a function on $(-\infty,\infty)$, so is $Q(x)$. By construction, (15) holds.

A $\mathscr{B}(\mathscr{E})$- Valued function $F(x)$ is called relation rational if it is meromorphic on $C_n = C \cup (\infty)$. It is not hard to see that $F(z)$ is rational if and only if $F(z) = P(z)/q(z)$, where $P(z)$ is a $\mathscr{B}(\mathscr{E})$-valued polynomial and $q(z)$ is a scalar valued polynomial.

Theorem (4.2.3):

Let $F(z)$ be a $\mathscr{B}(\mathscr{E})$- valued relational function that is either non-negative at all points $e^{i\theta} \in \Gamma$ that are not poles, or nonnegative at all points $x \in R$ that are one poles. In the circle case,

$$F(z) = \tilde{G}(z) G(z),$$

Where $G(z)$ is a $\mathscr{B}(\mathscr{E})$-valued relational function that is holomorphic on D and whose restriction to D is outer. In the line case there is a similar factorization with respect to the half-plane Π.

The tide notation. Thus $\tilde{G}(z) = G(1/\bar{z})^*$ or $\tilde{G}(z) = G(\bar{z})^*$, depending on the case.

Proof:

In the disk case, choose a scalar polynomial $q(z)$ whose restriction to D is outer such that $P(z) = \tilde{q}(z) F(z)$ is a polynomial is z and $1/z$. Then $P(e^{i\theta}) \geq 0$ on Γ, so by 6.6,

$$P(e^{i\theta}) = Q(e^{i\theta})^* Q(e^{i\theta})$$

Where $Q(z)$ is a polynomial whose restriction to D is outer. The required factorization is obtained with $G(z) = Q(e^{i\theta})^* Q(e^{i\theta})$.

In the half-plane case, choose a scalar polynomial $q(z)$ whose restriction to D is outer such that $P(z) = \bar{q}(z) f(z)$ is a polynomial in z. Argue as above using 6.7 instead of 6.6.

The notion of mean type for scalar valued function in $N(\Pi)$ is defined in the Appendix, Section 6. We now extend this notion to functions F in $N_{\mathscr{B}(\mathscr{E})}(\Pi)$. For each $c \in \mathscr{E}$, define F by

$$F(z) = \langle F(z)c, 0\rangle_{\mathscr{E}} \qquad z \in \Pi.$$

Definition (4.2.4):

The mean type of a function F in $N_{\mathscr{B}(\mathscr{E})}(\Pi)$ is the number $\tau = \sup_{c\in\mathscr{E}} \tau_c$,where τ_c is the mean type of F_c for any $c \in \mathscr{E}$.

Theorem (4.2.5):

The mean type τ of any F in $N_{\mathscr{B}(\mathscr{E})}(\Pi)$ satisfies $-\infty \le \tau < \infty$, with $\tau = -\infty$ only if $\Gamma \equiv 0$.

Proof:

Let τ_c be the mean type of F_c for any $c \in \mathscr{E}$.

$$\tau_c = \limsup_{y\to\infty} y^{-1} \log\left|F_c(iy)\right| \le \limsup_{y\to\infty} y^{-1} \log\left|F(iy)\right|_{\mathscr{B}(\mathscr{E})}.$$

By Theorem (4.2.1), there is a scalar valued holomorphic function u such that $0 < |u| \le 1$ and $|uF|_{\mathscr{B}(\mathscr{E})} \le 1$ on Π. Then

$$\begin{aligned}
&\limsup_{y\to\infty} y^{-1} \log\left|F(iy)\right|_{\mathscr{B}(\mathscr{E})} \\
&\quad = \limsup_{y\to\infty} \left[y^{-1} \log\left|u(iy)F(iy)\right|_{\mathscr{B}(\mathscr{E})} - y^{-1} \log\left|u(iy)\right| \right] \\
&\quad \le -\lim_{y\to\infty} y^{-1} \log\left|u(iy)\right| \\
&\quad = m.
\end{aligned}$$

Where m is finite real constant. Thus $\tau_c \le m < \infty$ for every $c \in \mathscr{E}$ and $\tau < \infty$.If $\tau = -\infty$, then $\tau_c = -\infty$ for every $c \in \mathscr{E}$.Hence $F_c \equiv 0$ for every $c \in \mathscr{E}$ and $F \equiv 0$.

If F is $\mathscr{B}(\mathscr{E})$-valued entire function, define F_c, for any $c \in \mathscr{E}$ by

$$F_c(z)\langle F(z)c, c\rangle_{\mathscr{E}}, \qquad z \in \mathbb{C}$$

Definition (4.2.6):

$\mathscr{B}(\mathscr{E})$-valued entire function F is of expontial type if there is a real constant m such that for each $c \in \mathscr{E}$,

$$\left|F_c(z)\right| \le M_c e^{m|z|}, \quad z \in \mathbb{C},$$

For some $M_c > 0$.In this case the exact type τ of F is the infimum of all such m.

$$\tau_F = \sup_{c\in\mathscr{G}}\left(\limsup_{|z|\to\infty}\log\frac{\left|\langle F(z)c,c\rangle_{\mathscr{G}}\right|}{|z|}\right)$$

We say that F is of exponential type τ if F is of exponential type and $\tau_F\tau$.

It is easy to see that F is of exponential type τ and only if F_c is of exponential type τ for each $c\in\mathscr{G}$.The exact type τ_F is the supermum of the exact types of all functions $F_\infty c\in\mathscr{G}$.As in the scalar case, either $F\equiv 0$

And $\tau_F = -\infty$,or $F\not\equiv 0$ and $\tau_F \ge 0$.

If F is a $\mathscr{R}(\mathscr{E})$-valued entire function, let $\tilde{F}$ be the reflection F with respect to the real line: $\tilde{F}(z) = F(\tilde{z}), z\in\mathbb{C}$.The following result generalizes Kreĭn's theorem.

Theorem (4.2.7):

If F is a $\mathscr{B}(\mathscr{E})$-valued entire function, the following are equivalent;

(i) F is of exponential type and

$$\int_{-\infty}^{\infty}\frac{\log^+|F(x)|_{\mathscr{B}(\mathscr{E})}}{1+x^2}dx < \infty; \qquad (16)$$

(ii) The restrictions of F and $\tilde{F}$ to Π are of bounded type, that is they belong to $N_{\mathscr{B}(\mathscr{E})}(\Pi)$.

Let F satisfy these conditions, and let $\tau_{+,}\tau_-$ be the mean types of the restriction of $F,\tilde{F}$ to Π respectively. Then

$$\tau_+ + \tau_- \ge 0 \qquad (17)$$

And

$$|\tau_+| \le \max(\tau_{+,}\tau_-) = \tau_F \qquad (18)$$

Where τ_F is the exact type of F.

For $\tilde{a}\mathscr{B}(\mathscr{E})$-valued function F on Π or $\mathbb{C}$, define F,for each $c\in\mathscr{G}$ as in (10)

Proof:

Let F satisfy

(i) Let τ_F be the exact type of F and choose $m > \tau_F$.For each $c \in \mathcal{S}$ there is a constant $M_c > 0$ such that $\left|F_c(t)\right| \le M_c e^{m|z|}, z \in \mathbb{C}$.

Claim: the restrictions of $e^{imc}F_c$ and $e^{imz}\tilde{F}_c$ to Π belong to $N^+(\Pi)$ .This is trivial if $F_c \equiv 0$, so suppose $F_c \not\equiv 0$. By the scalar version of Kreln's theorem the mean type of these restrictions do not exceed m.Since F_c, and $\tilde{F}_c$ are entire the restriction of F_c and $\tilde{F}_c$ to Π have no singular inner function in their canonical factorizations Thus

$$F_c = e^{-ipz} B_1 g \quad \text{And} \quad \tilde{F}_c = e^{-iqz} B_2 g \tag{19}$$

On Π, where $p \le m, q \le m, B_1$ and B_2,are Blaschke product, and g is outer (the outer factor may be chosen the same in each case since F_c and $\tilde{F}_c$ have the same modulus on R).Therefore the restrictions of $e^{imc}F_c$,and $e^{imc}\tilde{F}_c$ to Π belong to $N^+(\Pi)$ this proves the claim.

In view of the claim just proved and (16), the lemma implies that the restrictions of $e^{imz}F$ and $c^{imz}\tilde{F}$ to Π belong to $N^+_{\mathcal{B}(\mathcal{E})}(\Pi)$.Hence the restrictions of F and $\tilde{F}$ to Π are of bounded type that is (ii) holds.

Conversely, let (ii) holds. By theorem B, f satisfied (16).Let τ_+, τ_- be the mean type the restrictons of $F, \tilde{F}$ to Π. For any $c \in \mathcal{S}$, let $\tau_{c+,} \tau_{c-}$ be the mean type of the restrictions of $F_c, \tilde{F}_c$, to. Π By the scalar version of Kren's theorem F_c,is of exponential type and exct type equal to max $(\tau_{c+,} \tau_{c-})$. Since

$$\tau_\pm = sup_{c \in l}\, \tau_{c\pm} \tag{20}$$

Max$(\tau_{c+,} \tau_{c-})\le$ max$(\tau_{c+,} \tau_{c-})$ ($\tau_\pm$are finite). Hence F is of exponential type and exact type $\tau_F \le \max(\tau_+, \tau_-)$ .In particular, (i) holds.

If $\tau_F < \max(\tau_+, \tau_-)$ then by the first part of the proof, $\tau_{c\pm} \le \tau_F$ for all $c \in \mathcal{S}$ and so $\tau_\pm \le \tau_F < \max(\tau_+, \tau_-)$ a contradiction. Hence $\tau_F = \max(\tau_+, \tau_-)$.Since $\tau_{c+} + \tau_{c-} \ge 0$ for all $c \in \mathcal{E}$,by (21), $\tau_+ + \tau_- \ge 0$ .By an elementary argument this implies $|\tau_\pm| \le \max(\tau_+, \tau_-)$,and the proof is complete.

Let $\tau_1 \ge 0$ and $\tau_2 \ge 0$. $A\mathcal{R}(\mathcal{E})$-valued entire function F is of class $\mathcal{K}(\tau_1, \tau_2)$ if the restriction of F to Π is of bounded type and mean type $\leq \tau_1$ and the restriction of $\tilde{F}$ to $\tilde{F}$is of bounded type and mean type $\leq \tau_2$.The classes $\mathscr{K}(\tau_1, \tau_2)$ re called Krein classes.

Lemma (4.2.8):

$A\mathscr{B}(\mathscr{E})$-valued holomorphic function F on Π belongs to $N^+_{\mathscr{B}(\mathscr{E})}(\Pi)$ if and only if:

(i) $F_\tau \in N^+(\Pi)$ for each $c \in \mathscr{E}$, and

(ii) The limit $F_0(x) = \lim_{y \downarrow 0} f(x+iy)$ exists in the weak operator topology $a.e$ on R, and

$$\int_{-\infty}^{\infty} \frac{\log^+ |F_0(z)|_{\mathscr{B}(\mathscr{E})}}{1+x^2} dx < \infty. \tag{21}$$

Proof:

Necessary follows.

Conversely, assume that (i) and (ii) holds. By (19) there is scalar valued outer function v_0 on R such that $1/|v_0| = \max\left(1 |F_0|_{\mathscr{B}(\mathscr{E})}\right)$a.$e$. On R then $|v_0|$ and $|v_0 F_0|_{\mathscr{B}(\mathscr{E})} \leq 1z.e$ on R. Multiplying $v_0(t)$ by $1/(t+i)$ if necessary, we can assume that

$$\int_{-\infty}^{\infty} |v_0(t) F_v(t)|^2_{\mathscr{B}(\mathscr{E})} dt < \infty \text{ ‘}$$

Let v be the outer function on Π whose boundary function is v_0. By $(i), vF_c \in N^+(II)$ for each $c \in \mathscr{E}$. Since vF_c has a square summable boundary function, it belongs to $H^2(\Pi)$ and

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\langle v_0(t) F_v(t) c, c \rangle}{t - \bar{z}} dt = 0 \qquad z \in \Pi.$$

By the arbitrariness of c,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{v_0(t) F_0(t)}{t - \bar{z}} dt = 0, \qquad z \in \Pi.$$

By Theorem $B, v_0F_0 \in H^2_{\mathscr{B}(\mathscr{E})}(R) \subseteq N^+_{\mathscr{B}(\mathscr{E})}(R)$.it follows that v_0F_0 is the boundary function of some $G \in N^+_{\mathscr{B}(\mathscr{E})}(\Pi)$.For each $c \in \mathscr{E}$, the scalar valued function vF_c, and G_c belong to $N^+(\Pi)$ and have the same

boundary function.

Hence $vF_c \equiv G_\tau$, on Π. Therefore $vF \equiv G$.Since $G \in N^+_{\mathscr{B}(\mathscr{E})}(\Pi)$ and v is outer, $F \in N^+_{\mathscr{B}(\mathscr{E})}(\Pi)$

Theorem (4.2.4):
Let $u(x) = e^{it_1x}, v(x) = e^{it_2x}$ for all real x. Let F_0 be a weakly measurable $\mathscr{B}(\mathscr{E})$-valued function on R such that $|F_0|_{\mathscr{B}(\mathscr{E})}$ is integrable over every bounded interval. Then F_0 is of class $\mathscr{M}(u,v)$ if and only if F_0 is equal a.e. To the restriction to R of $\mathscr{B}(\mathscr{E})$ -valued entire function F of class $\mathscr{K}(\tau_1,\tau_2)$.
Proof:
Let F_0 be class $\mathscr{M}(u,v)$. F_0 Is equal $a.e$. To the restriction to R of an entire function F such that the restriction to Π of $e^{it_1z}F$ and $e^{it_2z}\tilde{F}$ belong to $N^+_{\mathscr{B}(\mathscr{E})}(\Pi)$. It follows that the restrictions to Π of F and $\tilde{F}$ are of bounded type and mean type at most τ_1 and τ_2 respectively, that is F is of class $\mathscr{K}(\tau_1,\tau_2)$.
Conversely, let $F_0(x) = F(x) a.e$. On R where F is $\mathscr{B}(\mathscr{E})$–valued entire function of class $\mathscr{K}(\tau_1,\tau_2)$.For c in $\mathscr{E}$,let F_c and $\tilde{F}_c$ be definied By

$F_c(z) = \langle F(z)c, c\rangle_{\mathscr{E}}$ And $\tilde{F}_c(z) = \langle \tilde{F}(z)c, c\rangle_{\mathscr{E}}$

For $z \in \mathbb{C}$. Then F_c is entire and the restrictions of F_c and $\tilde{F}_c$ to Π have canonical factorizations of the form (20), where $p \le \tau_1, q \le \tau_2$.Hence the restrictions of $e^{it_1z}F_c$ and $e^{it^2z}\tilde{F}_c$ to Π belong to $N^+(\Pi)$. Since F satisfies (16) the implies that the restriction of $e^{it_1z}F$ and $e^{it_2z}\tilde{F}$ to Π

belong to $N^{+}_{\mathscr{B}(\mathscr{E})}(\Pi)$.The boundary function of these restrictions are uF_0 and uF_0^{*}.Hence $uF_0, uF_0^{*} \in N^{+}_{\mathscr{B}(\mathscr{E})}(R)$ and F_0 is of class $\mathscr{M}(u,v)$
We now apply the preceding results to generalize Ahezer's theorem.

Theorem (4.2.5):
Let F be $\mathscr{B}(\mathscr{E})$ -valued centre function of exponential type $\tau, \tau \geq 0$ such that $F(x) \geq 0$ for all real x, and

$$\int_{-\infty}^{\infty} \frac{\log^{+}|F(x)|_{\mathscr{B}(\mathscr{E})}}{1+x^2} dx < \infty.$$

Then $F = \tilde{G}G$ for some $\mathscr{B}(\mathscr{E})$-valued entire function G such that $e^{itz/2}G$ is of exponential type $\tau/2$ and the restriction of G two Π is an outer function
Here $\tilde{G}(z) = G(\tilde{z}), z \in \mathbb{C}$
Proof:
The function F is of class $\mathscr{K}(\tau,\tau)$.The restriction F_0 of F to R be of class $\mathscr{M}(u,v)$, where $v(x) = e^{itx}$ for all real x. Hence by (5), $F_0 = G_0^{*}G_0 a.e.$ on R , where G_0 is outer and of class $\mathscr{M}(1,v)$ on R.By since (12), G_0 is the restriction to R of an entire function G of class $\mathscr{K}(0,\tau)$.Since $F_0 = G_0^{*}G_0 a.e.$ on, $R, F = \tilde{G}G$ .Since G is of class $\mathscr{K}(0,\tau), e^{-itz/2}G$ is of class $\mathscr{K}(\tau/2, \tau/2)$ and hence of exponential type $\tau/2$.The restriction of G two Π is of bounded type and has boundary function G_0.Since is outer on G_0, the restriction of G to Π is an outer function on Π.
The following result generalization Szego's theorem.

Theorem (4.2.6):
Let F be a weakly measurable nonnegative $\mathscr{B}(\mathscr{E})$-valued function that has invertible values $\sigma.a.e.$ On Γ or $a.e.$ On R. In the circle case assume that

$$\log^{+}\left|F\left(e^{i\theta}\right)\right|_{\mathscr{B}(\mathscr{E})} \quad and \quad \log^{+}\left|F\left(e^{i\theta}\right)^{-1}\right|_{\mathscr{B}(\mathscr{E})} \in L^{1}(\sigma),$$

And in the case of the real line assume that

$$\left(1+x^{2}\right)^{-1}\log^{+}\left|F(x)\right|\mathscr{B}(\mathscr{E}) \quad and \quad \left(1+x^{2}\right)^{-1}\log\left|F(x)^{-1}\right|_{\mathscr{B}(\mathscr{E})} \in L^{1}(-\infty,\infty)$$

Then $F = G^{*}G\sigma - a.e.$.on Γ or $a.e$ on R for some $\mathscr{B}(\mathscr{E})$-valid outer function G on Γ or R respectively.

Proof:

We first reduce to the case in which F is bounded Introduce $F_1 = F/f$, where $f = \max\left(1_1 \left|F\right|_{\mathscr{B}(\mathscr{E})}\right)$ on Γ.Since $\log f \in L^{1}(\sigma), f = |g|^{2}$ for some outer function g on Γ. If $F_1 = G_1^{*}G_1\sigma.a.e.$ On Γ for some $\mathscr{B}(\mathscr{E})$-valued outer function G_1, then $F = G^{*}G\sigma - a.e.$ On Γ, where $G = gG_1$ is outer. Since F_1 satisfies the hypotheses of the theorem and is bounded, we may assume without loss of general that F is bounded

We apply to the Toeplitz operator $T_2 = T(F)$ induced on $H^{2}_{\mathscr{E}}(\Gamma)$ by F .Choose $T_1 = T(\phi I_{\mathscr{E}})$.where $\phi\left(c^{it}\right) = 1/\left|F\left(e^{i\theta}\right)^{-1}\right|_{\mathscr{B}(\mathscr{E})}$ $\sigma - a.e.$ On Γ. By an elementary argument $\phi/_{\pm\mathscr{E}} \le F\sigma\text{-}a.e.$ On Γ. Therefore $T_1 \le T_2$ .We check the hypotheses (i) and (ii).

Our assumptions imply that $\phi \in L^{1}(\sigma)$, so $\phi = |\psi|^{2}$ for some $\psi \in H^{\infty}(\Gamma)$. If A_1 is multiplication by $\psi I_{\mathscr{E}}$ on $H^{2}_{\mathscr{E}}(\Gamma)$, then A_1 is analytic and $T_1 = A_1^{*}A_1$.

(ii) Let $\{f_n\}_1^{\infty}$ is sequence in $H^{2}_{\mathscr{E}}(\Gamma)$ such that

$$\lim_{n,k\to\infty}\left\langle T_2\left(f_n - f_k\right)f_n - f_k\right\rangle_2 = 0 \qquad (22)$$

And

$$\lim_{n\to\infty}\left\langle T_1 f_n f_n\right\rangle_1 = 0 \qquad (23)$$

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