

APPENDIX A

NUMERICAL EXAMPLES FOR DATUM DEFECT SOLUTION

A.1 Adjustment with Datum Constraints

Observations in Table (A.1) are considered for a levelling network of four stations, Fig.(A.1).

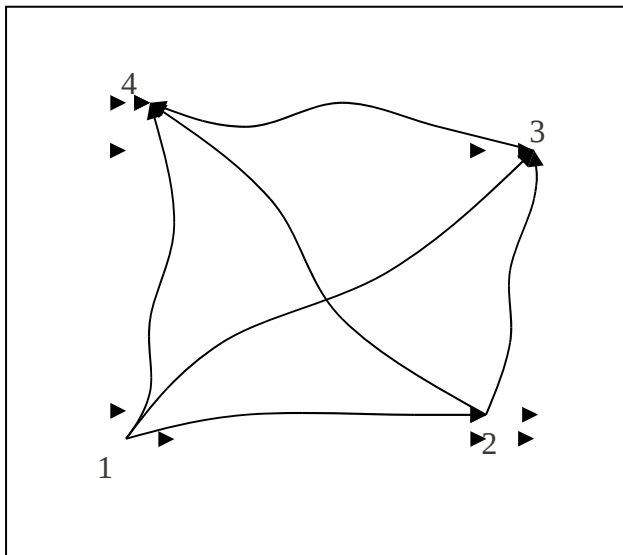


Fig (A.1)- Vertical Control Network

From point	To point	Observation(m)
1	2	1.503
1	3	2.005
1	4	2.492
2	3	0.510
2	4	1.002
3	4	0.495

Table (A.1)

Observations of the Vertical Network

The following observation equations can be established,

$$x_2 - x_1 = 1.503 + v_1$$

$$x_3 - x_1 = 2.005 + v_2$$

$$x_4 - x_1 = 2.492 + v_3$$

$$x_3 - x_2 = 0.510 + v_4$$

$$x_4 - x_2 = 1.002 + v_5$$

$$x_4 - x_3 = 0.495 + v_6$$

and, the following matrices are constructed:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1.503 \\ 2.005 \\ 2.492 \\ 0.510 \\ 1.002 \\ 0.495 \end{bmatrix}$$

Assuming that, observations are taken with the same accuracy i.e. identity weight matrix, then the normal matrix N is computed from:

$$N = A^T W A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

To obtain the solution for the network we must define the datum using one constraint, fixing the height of one point. In this example three solutions are obtained using three different constraints.

A.1.1 Fixed Datum Adjustment

The matrix N is a singular matrix i.e. the regular inverse can not be obtained unless one reduced level is known. The datum can be defined by fixation of one point (say point 1). Two approaches for fixed datum solution can be applied:

- **Classical Approach:** the coefficient matrix A is reduced after eliminating its first column as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

The inverse of the normal matrix N can be obtained directly as follows:

$$N^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Assuming the height of point (1) is equal to zero, then:

$$A^T W b = \begin{bmatrix} -0.009 \\ 2.020 \\ 3.989 \end{bmatrix}$$

The vector of unknown parameters (heights) and its covariance matrix are computed as:

$$\hat{x}_1 = N^{-1} A^T W b = \begin{bmatrix} 1.498 \\ 2.005 \\ 2.497 \end{bmatrix} \quad \text{and} \quad C_{\hat{x}} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

- **Alternative Approach:** the same solution can be obtained using the following datum constraint with the general solution model in (Chapter Two):

$$D^T X = 0$$

where

$$D = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From the general solution model, in Chapter Two, using the full matrix N and matrix D, the following adjustment can be obtained:

$$C_{\hat{x}} = (N + DD^T)^{-1} N (N + DD^T)^{-1}$$

$$\hat{x} = C_{\hat{x}} A^T W b$$

$$N + DD^T = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$(N + DD^T)^{-1} = \frac{1}{4} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 6 & 5 & 5 \\ 4 & 5 & 6 & 5 \\ 4 & 5 & 5 & 6 \end{bmatrix}$$

$$C_{\hat{x}_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{bmatrix} \text{ and } \hat{x}_1 = \begin{bmatrix} 0 \\ 1.498 \\ 2.005 \\ 2.497 \end{bmatrix}$$

- The norm of the parameters $\|\hat{x}_1\| = 3.53$
- The trace of the covariance matrix $tr C_{\hat{x}_1} = 1.5$

A.1.2 Free Datum Adjustment

Free datum or minimum trace solution in (Section 2.2.3) can be obtained from the following datum constraints i.e D is substituted by S:

$$S^T x = 0$$

where

$$S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Using the solution models of in Eq (2.32)

$$C_{\hat{x}_2} = (N + SS^T)^{-1} N - S(S^T S)^{-1} (S^T S)^{-1} S^T$$

$$\hat{x} = C_{\hat{x}} A^T W b$$

$$N + SS^T = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$(N + SS^T)^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_{\hat{x}_2} = \frac{1}{16} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \text{ and } \hat{x}_2 = \begin{bmatrix} -1.500 \\ -0.002 \\ 0.505 \\ 0.997 \end{bmatrix}$$

- The norm of the parameters $\|\hat{x}_2\| = 1.87$
- The trace of the covariance matrix $trC_{\hat{x}_2} = 0.75$
- Summation of parameters $\sum \hat{x}_2 = 0$

A.1.3 Partial Free Datum Adjustment

In this solution the datum can be defined by using the first three points, point four is removed from datum constraints, as follows:

$$D^T x = 0$$

where:

$$D = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Then, the solution can be obtained using the same models in previous sections as follows:

$$N + DD^T = \begin{bmatrix} 4 & 0 & 0 & -1 \\ 0 & 4 & 0 & -1 \\ 0 & 0 & 4 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$C_{\hat{x}_3} = (N + DD^T)^{-1} N (N + DD^T)^{-1} = \begin{bmatrix} 0.167 & -0.083 & -0.083 & 0 \\ -0.083 & 0.167 & -0.083 & 0 \\ -0.083 & -0.083 & 0.167 & 0 \\ 0 & 0 & 0 & 0.333 \end{bmatrix}$$

$$\text{and } \hat{x}_3 = \begin{bmatrix} -1.168 \\ 0.330 \\ 0.837 \\ 1.329 \end{bmatrix}$$

- The norm of the parameters $\|\hat{x}_3\| = 1.98$
- The norm of parameters of datum $\|\hat{x}_3\| = 1.47$
- The trace of the covariance matrix $tr C_{\hat{x}_3} = 0.83$
- Summation of parameters defining the datum $\sum_{i=1}^3 \hat{x}_{i3} = 0$

A.2 Datum Transformation

Using transformation models in Chapter Two, Section 2.3, then the following applications are carried out.

Firstly, to obtain the minimum trace solution from any other solution, the transformation factor k is established as follows:

$$k = I - S(S^T S)^{-1} S^T$$

where $S^T = [1 \quad 1 \quad 1 \quad 1]$

$$k = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

- Using the first solution result

$$\hat{x}_2 = k_2 \hat{x}_1$$

$$C_{\hat{x}_2} = k_2 C_{\hat{x}_1} k_2^T$$

$$\hat{x}_2 = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1.498 \\ 2.005 \\ 2.497 \end{bmatrix} = \begin{bmatrix} -1.500 \\ -0.002 \\ 0.505 \\ 0.997 \end{bmatrix}$$

$$C_{\hat{x}_2} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$C_{\hat{x}_2} = \frac{1}{16} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

- Using the third solution result

$$\hat{x}_2 = k_2 \hat{x}_3$$

$$C_{\hat{x}_2} = k_2 C_{\hat{x}_3} k_2^T$$

$$\hat{x}_2 = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1.168 \\ 0.330 \\ 0.837 \\ 1.329 \end{bmatrix} = \begin{bmatrix} -1.500 \\ -0.002 \\ 0.505 \\ 0.997 \end{bmatrix}$$

$$C_{\hat{x}_2} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0.167 & -0.082 & -0.082 & 0 \\ -0.083 & 0.167 & -0.083 & 0 \\ -0.083 & 0.083 & 0.167 & 0 \\ 0 & 0 & 0 & 0.333 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$C_{\hat{x}_2} = \frac{1}{16} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Secondly, to obtain the third solution from other solution, the transformation factor k is established as follows.

$$k_3 = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

- Using the first solution result

$$\hat{x}_3 = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1.498 \\ 2.005 \\ 2.497 \end{bmatrix} = \begin{bmatrix} -1.168 \\ 0.330 \\ 0.837 \\ 1.329 \end{bmatrix}$$

$$C_{\hat{x}_3} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

- If the second solution is used also, the same results of the third step (\hat{x}_3) can be obtained.