Chapter 1
Integral and Fractional Differential Equations

We aim to extend the use of the Riemann-Liouville definition of fractional calculus to solve a differintegral equation of Volterra's type of the form $D^{\mu} f(x) + aD^{-\nu} f(x) = g(x)$, with positive $\Re(\mu)$ and $\Re(\nu)$, $a \in C$ and $g(x)$ being a given function.

Sec (1.1) :The Integral Equations and Operators of Arbitrary Order

An integral of the form

$$\frac{1}{\Gamma(\nu)} \int_{c}^{x} (x - t)^{\nu-1} f(t) dt, \qquad \mathrm{Re}(\nu) \geqq 0 \tag{1}$$

is known as the Riemann-Liouville integral. It defines integration and differentiation to an arbitrary order. The **R-L** integral can be denoted by ${}_cD_x^{-\nu} f(x)$. The subscripts on the operator D denote the terminals of integration. The fractional calculus is not limit derived; hence, the words "terminals of integration" are more appropriate than "limits of integration''. When $\nu = n$ an integer, the operators ${}_oD_x^{-\nu}$ and ${}_oD_x^{\nu}$ are, respectively, ordinary integration and differentiation.

The idea of differentiation to an arbitrary order started in 1695 when L′Hôpital asked Leibniz what would happen with $d^n y/dx^n$ when $n = 1/2$. Subsequently, the topic started with the misnomer fractional calculus because ν can be rational, irrational or complex. After a half century of controversy due to the lack of a precise definition of fractional derivatives, the matter was finally settled in 1886 by Laurent who developed (1) starting with Cauchy's integral formula in the complex plane.

One of the applications of the fractional calculus is the simplification of the solution of certain integral equations as shown in [1]. Abel, in 1823, was the first to apply the fractional calculus in the solution of the integral equation

$$\int_{0}^{x} (x - t)^{\nu-1} f(t) dt = k,$$

where k is a constant, which arises in the formulation of the tautochrone (isochrone) problem and other physical problems [2].

The solution of the integral equation

$$f(x) + \lambda \int_{c}^{x} K(x,t) f(t)\, dt = \mathrm{g}(x)$$

was originally attained by Volterra in 1896. When $K(x,t)$ is a difference kernel of the form $K(x - t)$, the standard technique for solving the above when $c = 0$ is the use of Laplace transforms provided that the Laplace transform (one-sided) of the functions $f(x), K(x),$ and $\mathrm{g}(x)$ exist. Volterra's method which stemmed from the work of J. C. F. Sturm (1836) was attained as the limiting form of a set of algebraic equations in which differences of equally spaced points on (c, x) were used.

This chapter has a dual purpose. One purpose is to exemplify the power and elegance of the Riemann-Liouville operators in the solution of a Volterra type equation of the form

$$f\,(x)\ +\frac{1}{\Gamma(\nu)}\int_0^x (x - t)^{\nu-1} f(t)\,dt\ =\ \mathrm{g}(x), \qquad (2)$$

where $\mathrm{g}(x)$ is a known function and where $f(0)$ and $f'(0)$ are assumed to be finite. The exponent on the kernel can be arbitrary with the exception that $\nu - 1 \neq a$ negative integer.

A second purpose is to introduce a method of solving (2) which is conceptually different than that of Volterra, Fredholm, and Laplace transforms. This method consists of repeatedly applying **R-L** operators of appropriate order until (2) is transformed into an ordinary differential equation. This transformed equation is called the rational equivalent. The nonhomogeneous term $\mathrm{g}(x)$, as a result of repeated operations, gives rise to generalized integrals and derivatives. Thus, $\mathrm{g}(x)$ is in a wide class of functions such that $D^{\pm q}\,\mathrm{g}(x)$ is calculable. A table of integrals and derivatives to an arbitrary order is given in [3]. As an example to illustrate the procedure, we will specify $\nu = 2/3$ and $\mathrm{g}(x) = x^2$.

$Eq.(2)$ is written in operator form:

$$f(x) +\ {}_oD_x^{-2/3} f(x)\ =\ x^2. \qquad (3)$$

Operate on (3) with $D^{-1/3}$ and $D^{1/3}$. Omitting subscripts on D for conve- nience, we get the two equations

$$D^{-1/3} f(x)\ + D^{-1} f(x)\ = D^{-1/3} x^2 \qquad (4)$$

and

$$D^{1/3} f(x)\ + D^{-1/3} f(x)\ = D^{1/3} x^2. \qquad (5)$$

Operate on (4) with $D^{4/3}$ which yields

$$Df(x) + D^{1/3} f(x)\ =\ Dx^2. \qquad (6)$$

In (6) substitute for $D^{1/3} f(x)$ which is obtained from (5) getting

$$Df(x) - D^{-1/3} f(x) +\ D^{1/3} x^2\ = Dx^2. \qquad (7)$$

In (7) substitute for $D^{-1/3} f(x)$ from Eq. (4) and we will have

$$Df(x) - \left(-D^{-1} f(x) +\ D^{-1/3} x^2\right) + D^{1/3} x^2 =\ Dx^2, \qquad (8)$$

or

$$Df(x) + D^{-1} f(x) = Dx^2 - D^{1/3} x^2 + D^{-1/3} x^2. \qquad (9)$$

Finally, operate on each term above with D:

$$D^2 f(x) + f\,(x) = D^2 x^2 - D^{4/3} x^2 + D^{2/3} x^2$$

which we write as

$$f''(x) + f(x) = 2 - D^{4/3}x^2 + D^{2/3}x^2. \qquad (10)$$

In developing Eq. (10) which is the rational equivalent of (2), it is important to note the use of the index laws for integration and differentiation to an arbitrary order. In the Riemann-Liouville fractional calculus the index law for integration to an arbitrary order

$${}_0D_x^{-p}\left({}_0D_x^{-q}f(x)\right) = {}_0D_x^{-(p+q)}f(x)$$

generally holds true; however, the index law for differentiation to an arbitrary order

$${}_0D_x^{p}\left({}_0D_x^{q}f(x)\right) = {}_0D_x^{p+q}f(x)$$

holds only for certain $p + q$, and $f(x)$, [2]. This is one of the subtle perils in dealing with fractional operators. It is worth examining the operation we did earlier with $D^{4/3}$. We assumed $D^{4/3}\left(D^{-1/3}x^2\right) = Dx^2 = 2x$.

The above result can be readily verified by first computing ${}_0D_x^{-1/3}x^2$. From the list of formulas for integration and differentiation to an arbitrary order, we have the formula for integration to an arbitrary order

$${}_0D_x^{-\nu}x^a = \frac{\Gamma(a+1)}{\Gamma(a+\nu+1)}x^{a+\nu} \quad \mathrm{Re}(\nu) \geqq 0, \quad a \geqq 0.$$

$${}_0D_x^{-1/3}x^2 = \frac{\Gamma(3)}{\Gamma(10/3)}x^{7/3}.$$

$$D^{-1/3}x^2 = \frac{\Gamma(3)}{\Gamma(10/3)}x^{7/3}.$$

For differentiation to an arbitrary order we have the formula

$${}_0D_x^{\nu}x^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}x^{a-\nu}, \quad \mathrm{Re}(\nu) \geqq 0, \quad a \geqq 0.$$

$${}_0\mathrm{D}_\mathrm{x}^{4/3}\left(\frac{\Gamma(3)}{\Gamma(10/3)}\mathrm{x}^{7/3}\right) = \frac{\Gamma(3)}{\Gamma(10/3)} \cdot \frac{\Gamma(10/3)}{\Gamma((7/3)-(4/3)+1)}x = 2x. \quad (11)$$

We compute the second and third terms of the right side of (10) with the above formula (11).

$$D^{4/3}x^2 = \frac{2}{\Gamma(5/3)}x^{2/3} = K_1x^{2/3} \qquad (12)$$

$$D^{2/3}x^2 = \frac{2}{\Gamma(7/3)}x^{4/3} = K_2 x^{4/3}\,. \tag{13}$$

The differential equation (10) can now be written as

$$y'' + y = 2 - K_1 x^{2/3} + K_2 x^{4/3}. \tag{14}$$

The solution to (14) is the complementary solution

$$c_1 \sin x + c_2 \cos x$$

plus the particular solutions. The particular solution for 2 is clearly 2. To obtain y_λ for $-K_1 x^{2/3}$ the method of variation of parameters is used. The Wronskian

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Identifying $y_1 = \sin x$ and $y_2 = \cos x$, we have

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-(\cos x)(-K_1 x^{2/3})}{-1} = -K_1 x^{2/3} \cos x,$$

and

$$u_1 = -K_1 \int_0^x x^{2/3} \cos x dx.$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{(\sin x)(-K_1 x^{2/3})}{-1} = K_1 x^{2/3} \sin x,$$

$$u_2 = K_1 \int_0^x x^{2/3} \sin x dx.$$

Thus, the particular solution for $-K_1 x^{2/3}$ is

$$-K_1 \sin x \int_0^x x^{2/3} \cos x dx + K_1 \cos x \int_0^x x^{2/3} \sin x dx.$$

Similarly, we find the particular solution for $K_2 x^{4/3}$ to be

$$K_2 \sin x \int_0^x x^{4/3} \cos x dx - K_2 \cos x \int_0^x x^{4/3} \sin x dx.$$

The solution to (14) is thus

$$y = f(x) = c_1 \sin x + c_2 \cos x + 2 + \sin x \int_0^x \left(K_2 x^{4/3} - K_1 x^{2/3}\right) \cos x dx$$

$$+\cos x \int_0^x \left(K_1 x^{2/3} - K_2 x^{4/3}\right) \sin x dx. \quad (15)$$

We can determine the constants c_1 and c_2 from the original integral equation which we rewrite for convenience:

$$f(x) + \frac{1}{\Gamma(2/3)} \int_0^x (x-t)^{-1/3} f(t)\, dt = x^2 \quad (16)$$

We note in the above that $f(0) = 0$. Thus, we take $c_2 = -2$ to meet this condition. Thus, the solution is

$$f(x) = c_1 \sin x - 2\cos x + 2 + \sin x \int_0 \ldots + \cos x \int_0 \ldots . \quad (17)$$

To determine c_1 we take the derivative of the original integral equation (1.1.16) and determine $f'(0)$:

$$f'(x) + \frac{d}{dx} \frac{1}{\Gamma(2/3)} \int_0^x (x-t)^{-1/3} f(t)\, dt = 2x. \quad (18)$$

Leibniz's rule for differentiating an integral with respect to a parameter is

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t,x)\, dt = \int_{a(x)}^{b(x)} \frac{\partial f(t,x) dt}{\partial x} + f[b(x),x] b'(x) - f[a(x),x] a'(x).$$

The derivative of the integral in (18) is then

$$-\frac{1}{3} \int_0^x (x-t)^{-4/3} f(x) dx + f[x,x](1) - 0.$$

For $f'(0)$ Eq. (1.1.18) is

$$f'(0) = 0 + \left[\lim_{t \to x} \frac{f(t)}{(x-t)^{1/3}}\right]_{x=0} - 0. \quad (19)$$

Passing to the limit and then letting $x = 0$ yields the indeterminate form $0/0$ because $f(0) = 0$. The application of L'Hôpital's rule yields

$$\lim_{x\longrightarrow 0}\left[\lim_{t\longrightarrow x} -3(x-t)^{\frac{2}{3}}f'(t)\right]$$

Obtaining

$$f'(0) = 0 \cdot f'(0) = 0. \tag{20}$$

To determine the constant c_1 in (17), we take the derivative of (17) and set $x = 0$ which gives

$$f'(0) = c_1 \cos 0.$$

Thus $c_1 = 0$ and the solution is

$$f(x) = 2 - 2\cos x + \sin x \int_0^x \left(K_2 x^{\frac{4}{3}} - K_1 x^{\frac{2}{3}}\right) \cos x dx$$

$$+\cos x \int_0^x \left(K_1 x^{2/3} - K_2 x^{4/3}\right) \sin x dx, \tag{21}$$

where $K_1 = 2/\Gamma(5/3)$and $K_2 = 2/\Gamma(7/3)$.

To get a good feel for the efficiency of the method of fractional operators, we can compare it to the method of Laplace transforms. Our original integral equation is

$$f(x) + \frac{1}{\Gamma(2/3)} \int_0^x (x-t)^{-1/3} f(t) dt = x^2.$$

From the theory of convolution integrals we can write

$$L\{f(x)\} + \frac{1}{\Gamma(2/3)} L\left\{x^{-1/3} * f(x)\right\} = L\{x^2\},$$

$$f(s) + \frac{1}{\Gamma(2/3)} L\left\{x^{-1/3}\right\} \cdot L\{f(x)\} = \frac{2}{s^3}.$$

Then

$$f(s) + \frac{1}{\Gamma(2/3)} \cdot \frac{\Gamma(2/3)}{s^{2/3}} f(s) = \frac{2}{s^3},$$

$$f(s) = \frac{2}{s^3} \cdot \frac{s^{2/3}}{s^{2/3}+1} = \frac{2}{s^{7/3}\left(s^{2/3}+1\right)}.$$

The inverse Laplace transform of $F(s)$ is $f(x)$. However, the inverse of the function on the right above requires a substantial effort to obtain. This effort supports the assertion that the method of fractional operators is indeed an efficient one.

Sec (1.2) : Fractional Differintegral Equation

Among several definitions of fractional calculus, see, for example, Oldham and Spanier [4]; Nishimoto [5]; Samko, Kilbas and Marichev [6], and Miller and Ross [7], the Riemann-Liouville definition is the most widely used.

The Riemann-Liouville (R.-L.) fractional integral operator is defined by

$$D^{-\delta} f(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t)dt, \quad \Re(\delta), x > 0, \tag{22}$$

whereas the R.-L. fractional differential operator is defined by

$$D^{\delta} f(x) = D^n [D^{\alpha-n} f(x)]\,,\ \Re(\delta) \geq 0,\ x > 0, n = [\alpha] + 1 \tag{23}$$

Our aim is to solve a general differintegral equation of Volterra's type of the form

$$D^{\mu} f(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt = g(x), \tag{24}$$

that can be rewritten in the R.-L. notations

$$D^{\mu} f(x) + aD^{-\nu} f(x) = g(x), \tag{25}$$

where $\Re(\mu)$ and $\Re(\nu)$ are positive, $g(x)$ is any given integrable function on the finite interval $[0, b]$ and $a \in C$.

Ross and Sachdeva[8], Suarez [9] have considered equation (24) for $\mu = 0, a = 1$ and ν being a positive rational number by applying successively the R.-L. operators until the integral equation reduced to an ordinary differential equation. Al-Saqabi] has generalized the method used in [9] and showed its efficiency for solving a differintegral equation (25) with $a = 1$ and $\mu + \nu$ being a positive rational number. For the case $\Re(\mu)$ and $\Re(\nu)$ having different signs, equation (25) reduces to purely fractional integral or differential equations which were studied earlier by many others.

Applying the fractional integral operator $D^{-\mu}$ to the both sides of (25), we have

$$D^{-\mu} D^{\mu} f(x) + aD^{-\mu-\nu} f(x) = D^{-\mu} g(x). \tag{26}$$

Let n be an integer such that $n = [\mu] + 1$. Then according to formula from [6], we have

$$D^{-\mu} D^{\mu} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{x^{\mu-k-1}}{\Gamma(\mu-k)} f_{n-\mu}^{(n-k-1)}(0), \tag{27}$$

where

$$f_{n-\mu}^{(n-k-1)}(x) = D^{n-k-1} D^{\mu-n} f(x) = D^{\mu-k-1} f(x) \tag{28}$$

Therefore (26) becomes

$$f(x) = D^{-\mu} g(x) - aD^{-\mu-\nu} f(x) + \sum_{k=0}^{n-1} \frac{x^{\mu-k-1}}{\Gamma(\mu-k)} D^{-\mu-k-1} f(0). \tag{29}$$

Applying to the both sides of (29) operator $(-a)^m D^{-m(\mu+\nu)}$ we get

$$(-a)^m D^{-m(\mu+\nu)} f(x) = (-a)^m D^{-\mu-m(\mu+\nu)} g(x) + (-a)^{m+1} D^{-(m+1)(\mu+\nu)} f(x)$$

$$+\sum_{k=0}^{n-1} D^{\mu-k-1} f(0)(-a)^m D^{-m(\mu+\nu)} \frac{x^{\mu-k-1}}{\Gamma(\mu-k)}, \qquad m=0,1,2,\dots \quad . \tag{30}$$

Using equation from [6]

$$D^{-m(\mu+\nu)} \frac{x^{\mu-k-1}}{\Gamma(\mu-k)} = \frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)} \tag{31}$$

we have

$$(-a)^m D^{-m(\mu+\nu)} f(x) = (-a)^m D^{-\mu-m(\mu+\nu)} g(x) + (-a)^{m+1} D^{-(m+1)(\mu+\nu)} f(x)$$
$$+\sum_{k=0}^{n-1} D^{\mu-k-1} f(0)(-a)^m \frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)}. \tag{32}$$

Summing up (32) from $m=0$ to ∞, we get

$$\sum_{m=0}^{\infty} (-a)^m D^{-m(\mu+\nu)} f(x) = \sum_{m=0}^{\infty} (-a)^m D^{-\mu-m(\mu+\nu)} g(x)$$
$$+\sum_{m=1}^{\infty} (-a)^m D^{-m(\mu+\nu)} f(x) + \sum_{k=0}^{n-1} D^{\mu-k-1} f(0) x^{\mu-k-1} \sum_{m=0}^{\infty} \frac{(-a x^{\mu+\nu})^m}{\Gamma(\mu+m(\mu+\nu)-k)}. \tag{33}$$

The inner series in formula (33) can be expressed via the Mittag-Leffler function. The Mittag-Leffler function

$$E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} \frac{(x)^m}{\Gamma(\alpha m+\beta)} \tag{34}$$

is defined usually under the restriction that α and β are real numbers and $\alpha > 0$ (see Erdelyi [10]). But using the Stirling's asymptotic formula for the gamma function it is not difficult to see that the series in the right-hand side of formula (34) converges even when α, β are complex numbers and $\Re(\alpha) > 0$. Furthermore, the resulting function is also an entire function and has many properties of the Mittag-Leffler function (in particular, formula (43) and (45)) are still valid.

Canceling all common terms in the left and right-hand sides of (33) and rewriting the inner series as the Mittag-Leffler function in the general meaning we get

$$f(x) = \sum_{m=0}^{\infty} (-a)^m D^{-\mu-m(\mu+\nu)} g(x)$$
$$+\sum_{k=0}^{n-1} D^{\mu-k-1} f(0) x^{\mu-k-1} E_{\mu+\nu,\mu-k}(-a x^{\mu+\nu}). \tag{35}$$

We have

$$\sum_{m=0}^{\infty} (-a)^m D^{-\mu-m(\mu+\nu)} g(x) = \sum_{m=0}^{\infty} \int_0^x \frac{(x-t)^{\mu+m(\mu+\nu)-1}}{\Gamma\big(\mu+m(\mu+\nu)\big)} g(t)\,dt$$

$$= \int_0^x (x-t)^{\mu-1} E_{\mu+\nu,\mu}(-a(x-t)^{\mu+\nu}) g(t)dt,$$

where the interchange of order of summation and integration is possible, since

$$\sum_{m=0}^{\infty} \left| \frac{(x-t)^{m(\mu+\nu)}}{\Gamma\big(\mu + m(\mu+\nu)\big)} \right|$$

is uniformly bounded in the domain $0 \le t \le x \le b$. Therefore, formula (35) now becomes

$$f(x) = \int_0^x (x-t)^{\mu-1} E_{\mu+\nu,\mu}(-a(x-t)^{\mu+\nu}) g(t)dt$$
$$+ \sum_{k=0}^{n-1} \alpha_k \, x^{\mu-k-1} E_{\mu+\nu,\mu-k}(-a x^{\mu+\nu}). \quad (36)$$

We shall prove that formula (36) gives indeed a general solution of equation (25) when $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are arbitrary numbers. Applying operator $aD^{-\nu}$ to the both sides of formula (36), or equivalently, to (35), we have

$$aD^{-\nu} f(x) = a \sum_{m=0}^{\infty} (-a)^m D^{-(\mu+\nu)(m+1)} g(x)$$
$$+ a \sum_{k=0}^{n-1} \alpha_k \sum_{m=0}^{\infty} (-a)^m D^{-\nu} \frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu + m(\mu+\nu) - k)}$$
$$= -\sum_{m=1}^{\infty} (-a)^m D^{-(\mu+\nu)m} g(x) - \sum_{k=0}^{n-1} \alpha_k \sum_{m=0}^{\infty} (-a)^{m+1} \frac{x^{(\mu+\nu)(m+1)-k-1}}{\Gamma((\mu+\nu)(m+1)-k)}$$
$$= -\sum_{m=1}^{\infty} (-a)^m D^{-(\mu+\nu)m} g(x) - \sum_{k=0}^{n-1} \alpha_k \sum_{m=1}^{\infty} (-a)^m \frac{x^{(\mu+\nu)m-k-1}}{\Gamma\big((\mu+\nu)m-k\big)}. \quad (37)$$

On the other hand,

$$D^{\mu} f(x) = \sum_{m=1}^{\infty} (-a)^m D^{\mu} D^{-\mu-m(\mu+\nu)} g(x)$$
$$+ \sum_{k=0}^{n-1} \alpha_k \sum_{m=0}^{\infty} (-a)^m D^{\mu} \frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu + m(\mu+\nu) - k)}. \quad (38)$$

When $m = 0$, we get

$$D^{\mu} x^{\mu-k-1} = 0 \quad \text{for} \quad k = 0,1,\dots,n-1, \quad (39)$$

whereas when $m > 0$

$$D^{\mu}\,\frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)} = \frac{x^{m(\mu+\nu)-k-1}}{\Gamma(m(\mu+\nu)-k)} \tag{40}$$

Consequently

$$D^{\mu}f(x) = \sum_{m=0}^{\infty}(-a)^m D^{-m(\mu+\nu)}g(x) + \sum_{k=0}^{n-1}\alpha_k \sum_{m=1}^{\infty}(-a)^m \frac{x^{m(\mu+\nu)-k-1}}{\Gamma(m(\mu+\nu)-k)}. \tag{41}$$

Now summing up formulae (37) and (41), we obtain

$$D^{\mu}f(x) + aD^{-\nu}f(x) = g(x), \tag{42}$$

that means (36) is a solution of equation (25). Consequently, the homogeneous equation (25) has $[\mu]+1$ independent solutions. If we consider a Cauchy problem

$$D^{\mu}f(x) + aD^{-\nu}f(x) = g(x), \quad D^{\mu-k-1}f(0) = \alpha_k, \;\; k = 0,1,\dots,[\mu], \tag{43}$$

then the Cauchy problem (42) has the unique solution (36).

PARTICULAR CASES

(i) Let $g(t) = \; t^{\alpha-1},\; \alpha > 0$. By [11]:

$$\int_0^x (x-t)^{\alpha-1}\,E_{\mu+\nu,\mu}(-at^{\mu+\nu})t^{\mu-1}dt = \Gamma(\alpha)x^{\mu+\alpha-1}E_{\mu+\nu,\mu+\alpha}(-ax^{\mu+\nu}), \tag{44}$$

the solution $f(x)$ will be given in this case as follows:

$$f(x) = \Gamma(\alpha)x^{\mu+\alpha-1}E_{\mu+\nu,\mu+\alpha}(-ax^{\mu+\nu}) + \sum_{k=0}^{n-1}\alpha_k\, x^{\mu-k-1}E_{\mu+\nu,\mu-k}(-ax^{\mu+\nu}). \tag{45}$$

(ii) Put $g(t) = E_{\mu+\nu,\alpha+\nu}(-bt^{\mu+\nu})x^{\alpha+\nu-1}$. By [11]:

$$\int_0^x E_{\mu+\nu,\alpha+\nu}(zt^{\mu+\nu})t^{\alpha+\mu-1}(-a(x-t)^{\mu+\nu})(x-t)^{\mu-1}\,dt = \frac{E_{\mu+\nu,\alpha}(zx^{\mu+\nu})-E_{\mu+\nu,\alpha}(-ax^{\mu+\nu})}{z+a}x^{\alpha-1} \tag{46}$$

the solution $f(x)$ in this case becomes

$$f(x) = \frac{E_{\mu+\nu,\alpha}(-bx^{\mu+\nu})-E_{\mu+\nu,\alpha}(-ax^{\mu+\nu})}{a-b}x^{\alpha-1} + \sum_{k=0}^{n-1}\alpha_k\, x^{\mu-k-1}E_{\mu+\nu,\mu-k}(-ax^{\mu+\nu}). \tag{47}$$

Chapter 2

Analysis of Bounded Variation and Autoconvolution Equation with Determination of A density Function

For $Au = z$. Let

$$T(u) = \|Au - z\|^2 + \alpha J(u)$$

where the penalty, or 'regularization', parameter $\alpha > 0$ and the functional $J(u)$ is the bounded variation norm or semi-norm of u, also known as the total variation of u. Under mild restrictions on the operator A and the functional $J(u)$, it is shown that the functional $T(u)$ has a unique minimizer which is stable with respect to certain perturbations in the data z, the operator A, the panmeter α, and the functional $J(u)$. We concerned with the numerical analysis of the autoconvolution equation $x * x = y$ restricted to the interval [0,1]. We give results on existence and make notes on uniqueness and stability. We show the ill-posedness of the equation by an example and make assertions on its regularization by Tikhonov's method. We show the weak closedness of the forward operator for some appropriate domain. We show that the autoconvolution coefficients of a monotone sequence of functions is a continuous function.

Sec(2.1) : Bounded Variation Methods for Ill-posed Problems

Consider the equation

$$Au = \mathcal{Z} \qquad (1)$$

where A is a linear operator from $L^p(\Omega)$ into a Hilbert space Z containing the data vector $\mathcal{Z}$. Of particular interest is the case where problem (1) is ill-posed, e.g. when A is compact. The data $\mathcal{Z}$ and the operator A are assumed to be inexact, and approximate solutions to (1) are desired which minimize the undesirable effects of perturbations in $\mathcal{Z}$ and A. Of practical interest are Fredholm integral operators of the first kind

$$Au(x) = \int_\Omega k(x, y) \qquad (2)$$

For example, certain blurring effects in image processing may be described by convolution operators, in which case $k(x, y) = k(x - y)$.

Problem (1) is ill-posed and discretizations of it are highly ill-conditioned. To deal with ill-posedness, one should apply methods which impose stability while retaining certain desired features of the solution. Historically, these have come to be known as 'regularization' methods, since stability was typically obtained by imposing smoothness constraints on the approximate solutions. In many applications, particularly in image processing (see [20,14]) and parameter identification (see [16]), a serious shortcoming of standard regularization methods is that they do not allow discontinuous solutions. This difficulty can be overcome by achieving stability with the requirement that the solution be of bounded variation rather than smooth. For problem (1), this requirement may be enforced in several ways. One approach is to solve a constrained minimization problem like

$$\min_u J(u) \quad \text{subject to } \|Au - \mathcal{Z}\| = \sigma^2 \tag{3}$$

where σ^2 is an estimate of the size of the error in the data and $J(u)$ is the bounded variation (BV) norm or semi-norm of u (see [15] for definitions and background). This is essentially the approach taken by Rudin et al [19, 20]. A closely related approach is taken by Dobson and Santosa [14], where the constraint in (3) is replaced by the operator equation (1). In the application considered, discretizations of (1) are severely underdetermined. An earlier reference on the use of BV functions in a parameter identification setting (where a constraint on $J(u)$ is imposed instead) is by Gutman ([16]).

Another closely related approach, which is taken by Santosa and Symes [21] and Vogel [24], is to solve the unconstrained minimization problem

$$\min_u \|Au - \mathcal{Z}\|^2 + \alpha J(u) \tag{4}$$

This can be viewed as a penalty method approach to solving the constrained minimization problem (3). Here the penalty parameter $a > 0$ controls the trade- off between goodness of fit to the data, as measured by $\|Au - \mathcal{Z}\|^2$, and the variability of the approximate solution, as measured by $J(u)$. This penalty approach is widely known in the inverse problems community as Tikhonov regularization, although the term 'regularization' seems inappropriate here since discontinuous minimizers may be obtained.

A slightly more general penalty functional than the BV semi-norm will be considered. For sufficiently smooth , define

$$J_\beta(u) = \int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx \tag{5}$$

where $\beta \geq 0$. When $\beta = 0$, this reduces to the usual BV semi-norm (the BV norm is given by $\|u\|_{BV} = \|u\|_{L^1(\Omega)} + J_0(u)$). $J_0(u)$ is also commonly referred to as the total variation of u. A variational definition of J_β is presented below which extends (5) to (non-smooth) functions u. Taking $\beta > 0$ offers certain computational advantages, such as differentiability of the functional J_β when $\nabla u = 0$.

A number of important questions arise in the implementation of numerical methods to solve the minimization problem (4). For instance,

i. Is problem (4) really well-posed?
ii. In what function space does the solution to (4) lie, and what norm is appropriate to measure convergence? These questions are of more than academic interest, since they should influence the choice of approximation schemes and the

selection of stopping criteria. For instance, the analysis below shows that the choice of L^2 to measure convergence in an iterative solution of (4) may be inappropriate if the solution is a function of two or more (spatial) variables.

iii. What is the effect of taking small $\beta > 0$ in (5) rather than taking $\beta = 0$?

iv. As perturbations in the data z and the operator A vanish (say, as discrete approximations become more accurate), what conditions on the regularization parameter a are necessary in order to obtain convergence to an underlying exact solution (to an unperturbed problem)?

This section contains an overview of functions of bounded variation. Most of the results in this section are standard extensions to $L^p(\Omega)$ for $p > 1$ of results found in Giusti. Included in this section is a variational definition of J_β and a discussion of important properties such as convexity, semicontinuity, and compactness.

Let Ω be a bounded convex region in $R^d, d = 1,2,$ or 3, whose boundary $\partial\Omega$ is Lipschitz continuous. Let $|x| = \sqrt{\sum_{i=1}^{d} x_i^2}$ denote the Euclidean norm on R^d. Denote the norm on the Banach spaces $L^p(\Omega)$ by $\|\cdot\|_{L^p(\Omega)}, 1 \le p \le \infty$. Let $|\Omega|$ denote the (Lebesgue) measure of Ω, and unless otherwise specified, let χ_S denote the indicator function for a set $S \subset Q$.

As in [15],define the BV semi-norm, or total variation,

$$J_0(u) \overset{\text{def}}{=} \min_{v \in \mathcal{V}} \int_\Omega (-u \operatorname{div} v)\, dx \tag{6}$$

where the set of test functions

$$\mathcal{V} = \{v\ C_0^1(\Omega; \mathrm{R^d}) : |v(x)| \le 1 \quad for\ all \quad x \in \Omega\}. \tag{7}$$

If $u \in C^1(\Omega)$, one can show using integration by parts that

$$J_0(u) = \int_\Omega |\nabla u|\, dx\,. \tag{8}$$

By a standard denseness argument, this also applies for u in the Sobolev space $W^{1,1}(\Omega)$. The space of functions of bounded variation on Ω is defined by

$$BV(\Omega) = \{u \in L^1(\Omega) : J_o(u) < \infty\}. \tag{9}$$

The BV norm is given by

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + J_0(u). \tag{10}$$

BV(Ω) is complete, and hence a Banach space, with respect to this nom. The Sobolev space $W^{1,1}(\Omega)$ is a proper subset of BV(Ω), as is shown by the example in [15]. Note that for Ω bounded, $L^p(\Omega) \subset L^1(\Omega)$f or $p > 1$. From the definition, BV(Ω) $\subset L^1(\Omega)$. It is shown below that BV(Ω) $\subset L^p(\Omega)$ for $1 < p < d/\,(d-1)$.

Next, define an extension of (5) which is analogous to (6) .Identifying the convex

Functional $f(x) = \sqrt{|x|^2 + \beta}$ with its second conjugate, or Fenchel transform see [13] ,

$$\sqrt{|x|^2 + \beta} = \sup\left\{\mathrm{x} \cdot \mathrm{y} + \sqrt{\beta(1 - |y|^2)} : \mathrm{y} \in R^d |y| \leq 1\right\}, \quad (11)$$

the supremum being attained for $y = x\sqrt{|x|^2 + \beta}$. Motivated by this and (6), define

$$J_\beta(u) \stackrel{\text{def}}{=} \min_{v \in \mathcal{V}} \int_\Omega \left(-u \operatorname{div} v + \sqrt{\beta(1 - |v(x)|^2)}\right) dx. \quad (12)$$

Note that for $\beta > 0$, J_β is not a semi-norm.

Theorem $(\mathbf{2.1.1})[\mathbf{174}]$ **.** If $u \in W^{1,1}(\Omega)$, then (5) holds.

Proof. Since $C^1(\Omega)$ is dense in $W^{1,1}(\Omega)$, it suffices to show (5) for $u \in W^{1,1}(\Omega)$. In this case, for any $v \in \mathcal{V}$, Green's theorem (integration by parts) gives

$$\int_\Omega \left(-u \operatorname{div} v + \sqrt{\beta(1 - |v|^2)}\right) dx = \int_\Omega \left(\nabla u \cdot v + \sqrt{\beta(1 - |v|^2)}\right) dx$$

$$\leq \int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx. \quad (13)$$

The inequality above follows from (11). Consequently, $J_\beta(u) \leq \int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx$. To show the reverse inequality, take $\bar{v} = -\nabla u \sqrt{|\nabla u|^2 + \beta}$, and observe that

$$\int_\Omega \left(\nabla u \cdot \bar{v} + \sqrt{\beta(1 - |\bar{v}|^2)}\right) dx = \int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx$$

and $\bar{v} \in C(\Omega; R^d)$ with $|\bar{v}(x)| < 1$ for all $x \in \Omega$. By multiplying $\bar{v}$ by a suitable characteristic function compactly supported in Ω and then mollifying, one can obtain $v \in \mathcal{V} \cap C_0^\infty(\Omega)$ for which the left-hand side of (13) is arbitrarily close to $\int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx$.

The next theorem shows that both J_0 and J_β have $\mathrm{BV}(\Omega)$ for their effective domain, and that J_0 is the pointwise limit of J_β.

Theorem $(\mathbf{2.1.2})[\mathbf{174}]$**.** (i) For any $\beta > 0$ and $u \in L^1(\Omega)$, $J_0(u) < \infty$ if and only if $J_\beta(u) < \infty$; (ii) For any $u \in \mathrm{BV}(\Omega)$,

$$\lim_{\beta \to 0} J_\beta(u) = J_0(u). \quad (14)$$

Proof. For any $v \in \mathcal{V}$ and $u \subset L^1(\Omega)$,

$$\int_\Omega (-u \operatorname{div} v)\, dx \leq \int_\Omega \left(-u \operatorname{div} v + \sqrt{\beta(1 - |v|^2)}\right) dx$$

$$\leq \int_\Omega \left(-u \operatorname{div} v + \sqrt{\beta}\right) dx$$

Taking the sup over $v \in \mathcal{V}$,

$$J_0(u) \leq J_\beta(u) \leq J_0(u) + \sqrt{\beta}|\Omega|. \tag{15}$$

The results follow from the boundedness of Ω.

Theorem (2.1.3)[174]. For any $\beta \geq 0, J_\beta$ is weakly lower semicontinuous with respect to the L^p topology for $1 \leq p < \infty$.

Proof. Let $u_n \rightharpoonup \bar{u}$ (weak convergence in $L^p(\Omega)$). For any $v \in \mathcal{V}$, $\operatorname{div} v \in C(\Omega)$, and hence,

$$\int_\Omega \left((-\bar{u}\operatorname{div} v) + \sqrt{\beta(1-|v|^2)}\right)dx = \lim_{n\to\infty}\int_\Omega \left((-u_n \operatorname{div} v) + \sqrt{\beta(1-|v|^2)}\right)dx$$

$$= \lim_{n\to\infty}\inf\int_\Omega \left((-u_n \operatorname{div} v) + \sqrt{\beta(1-|v|^2)}\right)dx$$

$$\leq \lim_{n\to\infty}\inf J_\beta(u_n).$$

Taking the supremum over $v \in \mathcal{V}$ gives $J_\beta(\bar{u}) \leq \lim_{n\to\infty}\inf J_\beta(u_n)$.

Theorem (2.1.4)[174]. For any $\beta \geq 0$, J_β is convex.

Proof. Let $0 \leq \gamma \leq 1$ and $u_1, u_2 \in L^p(\Omega)$. For any $v \in \mathcal{V}$,

$$\int_\Omega \left(-(\gamma u_1 + (1-\gamma)u_2)\operatorname{div} v + \sqrt{\beta(1-|v|^2)}\right)dx$$

$$= \gamma\int_\Omega \left((-u_1 \operatorname{div} v) + \sqrt{\beta(1-|v|^2)}\right)dx$$

$$+(1-\gamma)\int_\Omega \left((-u_2 \operatorname{div} v) + \sqrt{\beta(1-|v|^2)}\right)dx$$

$$\leq \gamma J_\beta(u_1) + (1-\gamma)J_\beta(u_2).$$

Taking the supremum in the top line over $v \in \mathcal{V}$ gives the convexity of J_β.

A set of functions S is defined to be BV-bounded if there exists a constant $\beta > 0$for which $\|u\|_{BV} \leq B$ for all $u \in S$. The relative compactness of BV-bounded sets in $L^p(\Omega)$ follows from the next lemma (see [12] and [15]).

Lemma (2.1.5)[174]. If $u \in \text{BV}(\Omega)$, then there exists a sequence $\{u_n\}$ in $C^\infty(\Omega)$ such that
$\lim\|u_n - u\|_{L^p(\Omega)} = 0$ and $\lim J_0(u_n) = J_0(u)$.

Theorem (2.1.6)[174]. Let S be a BV-bounded set of functions. Then S is relatively compact in $L^p(\Omega)$ for $1 \leq p \leq d/(d-1)$. S is bounded, and hence relatively weakly compact for dimensions $d \geq 2$, in $L^p(\Omega)$ for $p = d/(d-1)$.

Proof. See [15]. Note that $d/(d-1)$ is the Sobolev conjugate of 1 in dimension d, the Sobolev conjugate of p, where $1 \le p < d$, being defined by $1/p^* = 1/p - 1/d$. For $1 \le p \le d/(d-1)$, the Rellich-Kondrachov compact embedding theorem holds. A sequence u_n, in S may then be approximated by a sequence of functions $\widetilde{u_n}$ in $C^\infty(\Omega)$, themselves uniformly bounded in BV(Ω) and in $L^p(\Omega)$, so that their sequence must have a subsequence converging in $L^p(\Omega)$ to some u. By semicontinuity of J_0 and lemma (1), $u \in$ BV(Ω) and is the limit (in L^p) of a subsequence extracted from u_n.

For $p = d/(d-1)$, one can similarly use lemma (2.1.1) to extend to BV -functions the Poincaré-Wirtinger inequality: if

$$\mu = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$$

then there exists C such that

$$\|u-\mu\|_{L^p(\Omega)} \le CJ_0(u-\mu) = CJ_0(u). \tag{16}$$

Hence, if, say, $\|u\|_{BV} \le M$, then $J_0(u-\mu)$ is also bounded by M, and, by the Poincaré- Wirtinger inequality, $\|u_n - u\|_{L^p} \le$ CM. Consequently,

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\le \|\mu\chi_\Omega\|_{L^p(\Omega)} + \|u-\mu\|_{L^p(\Omega)}\\ &\le |\Omega|\,|\Omega|^{1/p} + CM\\ &\le \|u\|_{L^1(\Omega)}|\Omega|^{1/p-1} + CM\\ &= \left(|\Omega|^{1/p-1} + C\right)M.\end{aligned}$$

Relative weak compactness in dimensions $d \ge 2$ follows from the Banach-Alaoglu theorem [17].

The following example shows that the above result is sharp. otherwise.

Example (2.1.7)[174]. Let $\Omega = \{x \in R^d : |x| < 2\}$ and $u_n = n^{d-1}\chi_n$, where

$$\chi_n(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{n} \\ 0 & \text{otherwise.}\end{cases}$$

Let ω_d denote the volume of the unit ball in R^d. Then

$$\|u_n\|_{L^p(\Omega)} = \begin{cases} n^{d-1}\left(\int_\Omega \chi_n\,dx\right)^{\frac{1}{p}} = n^{d-1-\frac{d}{p}}\omega_d^{1/p} & \text{if } 1 \le p < \infty \\ n^{d-1} & \text{if } p = \infty.\end{cases} \tag{17}$$

Hence the sequence $\{u_n\}$ is unbounded in $L^p(\Omega)$ whenever $p > d/d-1$. Similarly, if $p > d/d-1,\ d > 1$, and $m > n$, then

$$\|u_n - u_m\|_{L^p(\Omega)} \geq n^{d-1}\|\chi_n - \chi_m\|_{L^p(\Omega)}$$
$$= \omega_d^{1/p}\left(1 - \left(\frac{m}{n}\right)^{d-1}\right).$$

On the other hand, if $d = 1$ and $m > n$, then $\|u_n - u_m\|_{L^p(\Omega)} = 1$. In either case, the sequence $\{u_n\}$ is bounded but not Cauchy in $L^p(\Omega)$ for $p = d/(d-1)$.

Now let σ_d denote the area of the unit sphere S_{d-1} in R^d. From (17) and (10),

$$\|u_n\|_{\text{BV}} = \frac{1}{n}\omega_d + \sigma_d. \tag{18}$$

Hence, the sequence is BV -bounded but has no convergent subsequence in $L^p(\Omega)$ whenever $p \geq d/(d-1)$.

Recall that a functional J is strictly convex if

$$J(\gamma\, u_1 + (1-\gamma)u_2) < \gamma J(u_1) + (1-\gamma)J(u_2), \tag{19}$$

whenever $u_1 \neq u_2$ and $0 < \gamma < 1$. The following example shows that J_β fails to be strictly convex on BV(Ω).

Example $(\mathbf{2.1.8})[\mathbf{174}]$. Take $\Omega = (0,1), u_1 = \chi_{[a,b]}$, where $0 < a < b < c < d <$ 1. For

any $\beta \geq 0$, a direct computation shows that $J_\beta(u_1) = J_\beta\big((u_1 + u_2)/2\big) = 2 + \sqrt{\beta}$. Since $u_1 \neq u_2, J_\beta$ cannot be strictly convex.

A problem is said to be well-posed in the sense of Hadamard if (i) it has a solution, (ii) the solution is unique, and (iii) the solution is stable. Let T be a functional defined on $L^p(\Omega)$ with values in the extended reals. Theorems (2.1.6) and (2.1.7)below, guarantee the well-posedness of the unconstrained minimization problem

$$\min_{u \in L^p(\Omega)} T(u). \tag{20}$$

These theorems are followed by some illustrative examples pertaining to problem (4).

In order to use the compactness results of section 1 while still dealing with unconstrained minimization problems, we introduce the following property: define T to be BV-coercive if

$$T(u) \longrightarrow +\infty \quad \text{whenever} \quad \|u\|_{\text{BV}} \longrightarrow +\infty. \tag{21}$$

Note that 'lower level sets' $\{u \in L^p(\Omega): T(u) \leq a\}$, where $a \geq 0$, are BV -bounded.

Theorem $(\mathbf{2.1.9})[\mathbf{174}]$ (Existence a d uniqueness of minimizers). Suppose that T is BV -coercive. If $1 \leq p < d(d-1)$ and T is lower semicontinuous, then problem (20) has a solution. If in addition $p = d/(d-1)$, dimension $d \geq 2$, and T is weakly lower semicontinuous, then a solution also exists. In either case, the solution is unique if T is strictly convex.

Proof. The following argument is standard (see [13]): Let u_n, be a minimizing sequence for T ; in other words,

$$T(u_n) \longrightarrow +\infty \inf_{u \in L^p(\Omega)} T(u) \stackrel{\text{def}}{=} T_{\min} . \tag{22}$$

By hypothesis (21), the $u_n s$ are BV -bounded. As a consequence of theorem $(2.1.6)$, there exists a subsequence u_{n_j} which converges to some $\bar{u} \in L^p(\Omega)$. Convergence is weak if $p = d/(d-1)$. By the (weak) lower semicontinuity of T,

$$T(\bar{u}) \leq \liminf T\left(u_{n_j}\right) = T_{\min} .$$

Uniqueness of minimizers follows immediately from strict convexity.

Next consider a sequence of perturbed problems

$$\min_{u \in L^p(\Omega)} T_n(u) \tag{23}$$

Theorem $(\mathbf{2.1.10})[\mathbf{174}]$. Assume that $1 \leq p < d(d-1)$ and that T and each of the $T_n s$ are BV-coercive, lower semicontinuous, and have a unique minimizer. Assume in addition:

(i) Uniform BV -Coercivity: For any sequence $v_n \in L^p(\Omega)$,

$$\lim T_n(v_n) = +\infty \quad \text{whenever} \quad \lim \|v_n\|_{\text{BV}} \tag{24}$$

(ii) Consistency: $T_n \longrightarrow T$ uniformly on BV -bounded sets, i.e. given $B > 0$ and $\epsilon > 0$, there exists N such that

$$|T_n(u) - T(u)| < \epsilon \quad \text{whenever} \quad n \geq N, \|u\|_{\text{BV}} \leq B. \tag{25}$$

Then problem (20) is stable with respect to the perturbations (23), i.e. if $\bar{u}$ minimizes T and u_n minimizes T_n, then

$$\|u_n - \bar{u}\|_{L^p(\Omega)} \longrightarrow 0 \tag{26}$$

If $p = d/(d-1)$, $d \geq 2$, and one replaces the lower semicontinuity assumption on T and each T_n by weak lower semicontinuity, then convergence is weak:

$$u_n - \bar{u} \rightharpoonup 0 \tag{27}$$

Proof. Note that $T_n(u_n) \leq T_n(\bar{u})$. From this and (25),

$$\liminf T_n(u_n) \leq \limsup T_n(u_n) \leq T(\bar{u}) < \infty \tag{28}$$

and hence by (24), the $u_n s$ are BV-hounded. Now suppose (26) (or (27) if $p = d/(d-1)$) does not hold. By Theorem (2.1.6) there exists a subsequence u_{n_j}, which converges in $L^p(\Omega)$ (weak L^p) to some $\hat{u} \neq \bar{u}$. By the (weak) lower semicontinuity of T, (27), and (25),

$$\begin{aligned} T(\hat{u}) &\leq \liminf T\left(u_{n_j}\right) \\ &= \lim\left(T\left(u_{n_j}\right) - T_{n_j}\left(u_{n_j}\right)\right) + \liminf T_{n_j}\left(u_{n_j}\right) \\ &\leq T(\bar{u}). \end{aligned}$$

But this contradicts the uniqueness of the minimizer $\bar{u}$ of T.

Example $(\mathbf{2.1.11})[\mathbf{174}]$. Consider the problem of minimizing

$$T(u) = \|Au - z\|_{\mathcal{Z}}^2 + \alpha\|u\|_{\mathrm{BV}} \tag{29}$$

for $u \in L^p(\Omega)$. where the restrictions on p in theorem (2.1.9) apply. Here ($\alpha > 0$ and $z \in \mathcal{Z}$ are fixed, and $A: L^p(\Omega) \longrightarrow \mathcal{Z}$ is bounded and linear. Then

$$\|u\|_{\mathrm{BV}} \leq \frac{1}{\alpha} T(u) \tag{30}$$

and hence, the coercivity condition (21) holds. Weak lower semicontinuity of T follows from the boundedness of A, the weak lower semicontinuity of the norms on Banach spaces, and theorem (2.1.3). By theorem (2.1.4), the linearity of A, and convexity of norms, T is convex. By theorem (2.1.9) a minimizer exists. T is strictly convex if A is injective, in which case the minimizer is unique.

The following examples deal with stability. In the next three examples, assume again that the restrictions on p of theorem (2.1.9) apply.

Example $(\mathbf{2.1.12})[\mathbf{174}]$. (Perturbations in the data z). Let

$$T_n(u) \stackrel{\text{def}}{=} \|Au - z\|_{\mathcal{Z}}^2 + \alpha\|u\|_{\mathrm{BV}} \tag{31}$$

where $z_n z + \eta_n$ and $\|\eta_n\|_{\mathcal{Z}} \longrightarrow 0$ as $n \longrightarrow \infty$. Then

$$\begin{aligned} |T_n(u) - T(u)| &= \|\eta_n\|_{\mathcal{Z}}^2 + |2\langle Au - z, \eta_n\rangle \mathcal{Z}| \\ &\leq \|\eta_n\|_{\mathcal{Z}}\big(\|\eta_n\|_{\mathcal{Z}} + 2\|A\|\|u\|_{L^p(\Omega)} + 2\|z\|_{\mathcal{Z}}\big). \end{aligned}$$

Here $\langle .,. \rangle \mathcal{Z}$ denotes the inner product on the Hilbert space $\mathcal{Z}$, and the above inequality follows from Cauchy-Schwarz. Note that if u is BV-bounded, then it is norm bounded in $L^p(\Omega)$ by theorem (2.1.6), and hence (25) holds. (24) holds because for each n,

$$\|u\|_{\mathrm{BV}} \leq \frac{T_n(u)}{\alpha}. \tag{32}$$

Example $(\mathbf{2.1.13})[\mathbf{174}]$. Take

$$T_n(u) \stackrel{\text{def}}{=} \|Au - z\|_{\mathcal{Z}}^2 + \alpha\big(\|u\|_{L^p(\Omega)} + J_{\beta_n}(u)\big) \tag{33}$$

where $\beta_n \longrightarrow 0$. In this case,

$$|T_n(u) - T(u)| = \alpha\big|J_{\beta_n}(u) - J_0(u)\big| \leq \alpha\sqrt{\beta_n}|\Omega|. \tag{34}$$

The above inequality follows from (15). This verifies (2.1.25). Similarly, (2.1.24) holds because

$$\frac{1}{\alpha} T_n(u) \geq \|u\|_{L^p(\Omega)} + J_\beta(u) \geq \|u\|_{L^p(\Omega)} + J_0(u) = \|u\|_{\mathrm{BV}} \tag{35}$$

Example $(\mathbf{2.1.14})[\mathbf{174}]$ (Perturbations in the penalty parameter α). Let

$$T_n(u) = \|Au - z\|_{\mathcal{Z}}^2 + \alpha\|u\|_{\mathrm{BV}} \tag{36}$$

where the $\alpha_n s$ are bounded below by $\alpha_{\min} > 0$ and converge to α. Stability follows from the facts that

$$\|u\|_{\mathrm{BV}} \leq \frac{T_n(u)}{\alpha_{\min}}$$

and

$$|T_n(u) - T(u)| \leq |\alpha_n - \alpha| \, \|u\|_{\mathrm{BV}}.$$

Example $(\mathbf{2.1.15})[\mathbf{174}]$ (Perturbations of the operator A). Assume $1 \leq p < d/(d-1)$, and let

$$T_n(u)) \overset{\text{def}}{=} \|A_n u - z\|_{\mathcal{Z}}^2 + \alpha \, \|u\|_{\mathrm{BV}} \tag{37}$$

where the $A_n s$ converge strongly (i.e. pointwise) in $L^p(\Omega)$ to A. Note that strong operator convergence is a reasonable assumption. It holds for consistent Galerkin approximations, e.g. finite element approximations as the mesh spacing $h \longrightarrow 0$. Then

$$\begin{aligned} |T_n(u) - T(u)| &= \left| \|A_n u\|_{\mathcal{Z}}^2 - \|Au\|_{\mathcal{Z}}^2 - 2\langle (A_n - A)u, z\rangle z \right| \\ &\leq (\|A_n u\|_z + \|Au\|_z + 2\|z\|_z)\|(A_n - A)u\|_z. \end{aligned}$$

Note that pointwise convergence of bounded linear operators becomes uniform on compact sets. Since BV -boundedness implies relative compactness in $L^p(\Omega)$, (25) holds. Uniform coercivity (24) again holds because of (32).

The BV norm in the penalty term is replaced by the BV semi-norm J_o, or more generally, by J_β. Consider the following functional defined on $L^p(\Omega)$:

$$T(u) = \|A_n u - z\|_{\mathcal{Z}}^2 + \alpha J_\beta(u) \tag{38}$$

again taking on values in the extended reals. From a computational standpoint, for positive β the penalty functional $J_\beta(u)$ is Gateaux differentiable with respect to u , and hence much easier to deal with than $\|u\|_{\mathrm{BV}}$. However, the analysis becomes much more complicated. Certain conditions on A are clearly needed to guarantee BV - coercivity. For example from (5), T cannot be BV -coercive if A annihilates constant functions. Conversely,

Lemma$(\mathbf{2.1.16})[\mathbf{174}]$**.** Assume that $1 \leq p \leq d/(d-1)$, and that A does not annihilate constant functions. Equivalently, since A is linear, assume

$$A\chi_\Omega \neq 0. \tag{39}$$

Then T in (38) is BV–coercive.

Proof. From the inequalities (15), it suffices to consider the case of $\beta = 0$. Any $u \in \mathrm{BV}(\Omega)$ has decomposition

$$u = v + w \tag{40}$$

where

$$w = \left(\frac{\int_\Omega u dx}{|\Omega|} \right) \chi_\Omega \qquad \int_\Omega u dx = 0. \tag{41}$$

By equation (16) and Hölders inequality, there exists a positive constant C such that for any p such that $1 \leq p \leq \frac{d}{d-1} = q$,

$$\begin{aligned} \|v\|_{L^p(\Omega)} &\leq |\Omega|^{1/p - 1/q} \|v\|_{L^p(\Omega)} \\ &\leq (|\Omega| + 1)^{1-1/q} C J_0(v) \end{aligned}$$

$$= C_1 J_0(\nu) \tag{42}$$

where $C_1 \overset{\text{def}}{=} (|\Omega| + 1)^{1/d} C$. Using (42) and the decomposition (40),

$$\|u\|_{\text{BV}} \leq \|w\|_{L^1(\Omega)} + (C_1 + 1) J_0(\nu). \tag{43}$$

From (39) there exists $C_2 > 0$ such that

$$\|Aw\|_z = C_2 \|w\|_{L^1(\Omega)}. \tag{44}$$

On the other hand, the decomposition (40) yields

$$\begin{aligned} T(u) &= \|(A\nu - z) + Aw\|_Z^2 + \alpha J_0(\nu) \\ &\geq (\|A\nu - z\|_z - \|Aw\|_z)^2 + \alpha J_0(\nu) \\ &\geq \|Aw\|_z(\|Aw\|_z - 2\|A\nu - z\|_z) + \alpha J_0(\nu). \end{aligned} \tag{45}$$

But by (42),

$$\|A\nu - z\|_z \leq \|A\| C_1 J_0(\nu) + \|z\|_z. \tag{46}$$

Combining (45) with this and (44)yields

$$T(u) \geq C_2 \|w\|_{L^1(\Omega)} \left(C_2 \|w\|_{L^1(\Omega)} - 2(\|A\| C_1 J_0(\nu) + \|z\|_z) \right) + \alpha J_0(\nu). \tag{47}$$

Now if

$$C_2 \|w\|_{L^1(\Omega)} - 2(\|A\| C_1 J_0(\nu) + \|z\|_z) \geq 1 \tag{48}$$

Then from (47)

$$\|w\|_{L^1(\Omega)} \leq \frac{1}{C_2} T(u). \tag{49}$$

As consequence of this and

$$J_0(\nu) \leq \frac{1}{\alpha} T(u) \tag{50}$$

one obtains from (43)

$$\|u\|_{\text{BV}} \leq \left(\frac{1}{C_2} + \frac{C_1 + 1}{\alpha} \right) T(u). \tag{51}$$

But if (48) does not hold, then

$$\|w\|_{L^1(\Omega)} < \frac{1 + 2(\|A\| C_1 J_0(\nu) + \|z\|_z)}{C_2}. \tag{52}$$

and hence from (43) and (50),

$$\|u\|_{\text{BV}} - \frac{1 + 2\|z\|_z}{C_2} \leq \left(\frac{2\|A\| C_1}{C_2} + C_1 + 1 \right) \frac{1}{\alpha} T(u). \tag{53}$$

From (51) and (53), one obtains BV -Coercivity.

One now obtains the following from theorem (2.1.6).

Theorem (2.1.17)[174]. Suppose p satisfies the restrictions of theorem (2.1.9)., $p \geq 0$, and A is bounded linear and satisfies (39). Then the functional T in (38) has a minimizer.

The following example illustrates that a condition stronger than (39) may be necessary to guarantee uniqueness of minimizers of T in (38).

Example $(\mathbf{2.1.18})[\mathbf{174}]$. Define $A : L^1(-2,2) \longrightarrow\ R^2$ by

$$[Au]_1 = \int_{-2}^{-1} u(x)dx \qquad\qquad [Au]_2 = \int_{1}^{2} u(x)dx.$$

Let $z = [z_1, z_2]^T\ = [-1,1]^T \in R^2$. Define

$$T_\beta(u) = \sum_{i=1}^{2}([Au]_i - z_i)^2 + J_\beta(u). \tag{54}$$

For any $\beta > 0$, the unique minimizer of T_β over $L^1(\Omega)$ is

$$u(x) = \begin{cases} -1 & \text{if } \ x \le -1 \\ x & \text{if } \ -1 < x < 1 \\ 1 & \text{if } \ x \ge -1. \end{cases} \tag{55}$$

On the other hand, for $,\beta = 0$ one obtains a minimizer by defining u on the subinterval $-1 < x < 1$ to be any monotonic increasing function taking on values between -1 and 1.

This next theorem addresses the stability of minimizers to functionals of the form (38). Consider perturbed functionals

$$T_n(u) = \|A_n u - z_n\|_{\mathcal{Z}}^2 + \alpha J_\beta(u). \tag{56}$$

Theorem $(\mathbf{2.1.19})[\mathbf{174}]$. Assume $1 \le p < d/(d-1)$, $\|z_n - z\|_z \longrightarrow 0$, the $A_n s$ are each bounded linear and converge pointwise to A, and for each n,

$$\|A_n \chi_\Omega\|_{\mathcal{Z}}^2 \ge \gamma > 0. \tag{57}$$

Also assume each T_n has a unique minimizer u_n, and that T has a unique minimizer $\bar{u}$. Then

$$\|u_n - \bar{u}\|_{L^p(\Omega)} \longrightarrow 0. \tag{58}$$

Proof. It suffices to show that conditions (i) and (ii) of theorem (2.1.10) hold. For condition (i) (uniform BV-coercivity), put $u_n = v_n + w_n$, as in (40) and (41), and repeat the proof of lemma (2.1.16). Since $\|A_n w_n\|_z \ge \gamma \|w_n\|_{L^1(\Omega)}$, letting M be an upper bound on $\|A_n w_n\|$ and each $\|A_n\|$ (such a bound exists by the Banach-Steinhaus theorem, also known as the uniform boundedness principle), and m be an upper bound on $\|z\|_z$ and each l$\|z_n\|_z$, one obtains

$$T_n(u_n) \ge \gamma \|w_n\|_{L^1(\Omega)}\big(-2(MC_1 J_0(v_n) + m)\big) + \alpha J_0(v_n). \tag{59}$$

This yields uniform coercivity as in the proof of Lemma (2.1.2).

Condition (ii) (consistency) follows as in Example (2.1.12)and (2.1.15).

Assume an exact problem

$$Au = z \tag{60}$$

which has a unique solution $u_{\text{exact}} \in \mathrm{BV}(\Omega)$. Assume a sequence of perturbed problems

$$A_n u = z_n \tag{61}$$

having approximate solutions u_n (not necessarily unique) obtained by minimizing the functional

$$T_n(u) = \|A_n u - z_n\|_{\mathcal{Z}}^2 + \alpha_n \|u\|_{\mathrm{BV}}. \tag{62}$$

The following theorem provides conditions which guarantee convergence of the $u_n s$ to u_{exact} .

Theorem $(\mathbf{2.1.20})[\mathbf{174}]$. Let $1 \leq p \leq d/(d-1)$. Suppose $\|z_n - z\|_z \longrightarrow 0$, $A_n \longrightarrow A$. A pointwise in $L^p(\Omega)$, and $\alpha_n \longrightarrow 0$ at a rate for which $\|A_n u_{\text{exact}} - z_n\|^2/\alpha_n$, remains bounded. Then $u_n \longrightarrow u_{\text{exact}}$ strongly in $L^p(\Omega)$ if $1 \leq p < d/(d-1)$. Convergence is weak in $L^p(\Omega)$ if $p = d/(d-1)$.

Proof. Note that

$$\begin{aligned}\|A_n u_n - z_n\|_{\mathcal{Z}}^2 &\leq T_n(u_n) \\ &\leq T_n(u_{\text{exact}}) \\ &= \|A_n u_{\text{exact}} - z_n\|_{\mathcal{Z}}^2 + \alpha_n \|u_{\text{exact}}\|_{\mathrm{BV}}.\end{aligned}$$

Thus from the assumption that $\|A_n u_{\text{exact}} - z_n\|^2/\alpha_n$ remains bounded and the fact that an $\alpha_n \longrightarrow 0$,

$$\|A_n u_n - z_n\|_{\mathcal{Z}}^2 \longrightarrow 0. \tag{63}$$

Similarly,

$$\begin{aligned}\|u_n\|_{\mathrm{BV}} &\leq \frac{T_n(u_n)}{\alpha_n} \leq \frac{T_n(u_{\text{exact}})}{\alpha_n} \\ &= \frac{\|A_n u_{\text{exact}} - z_n\|^2}{\alpha_n} + \|u_{\text{exact}}\|_{\mathrm{BV}}\end{aligned}$$

and hence, the $u_n s$ are BV -bounded. Suppose they do not converge strongly (weakly, if $= d/(d-1)$) to u_{exact} . By Theorem (2.1.6) there is a subsequence u_{n_j}, which converges strongly (weakly, respectively) in $L^p(\Omega)$ to some $\hat{u} \neq u_{\text{exact}}$. For any $\nu \in \mathcal{Z}$,

$$\begin{aligned}|\langle A\hat{u} - z, \nu\rangle z| \leq &\left|\langle A\left(\hat{u} - u_{n_j}\right), \nu\rangle z\right| + \left|\langle\left(A - A_{n_j}\right) u_{n_j}, \nu\rangle z\right| \\ &+ \left|\langle A_{n_j} u_{n_j} - z_{n_j}, \nu\rangle z\right| + \left|\langle z_{n_j} - z, \nu\rangle z\right|.\end{aligned} \tag{64}$$

The third and fourth terms on the right-hand side vanish as $j \longrightarrow \infty$ because of (63) and the assumption $z_n \longrightarrow z$. The second term also vanishes, since

$$\left|\langle\left(A - A_{n_j}\right) u_{n_j}, \nu\rangle z\right| \leq \left\|u_{n_j}\right\|_{L^p(\Omega)} \left\|\left(A^* - A_{n_j}^*\right)\nu\right\|_{L^p(\Omega)} \longrightarrow 0$$

by the pointwise convergence of the $A_n s$ (and hence, their adjoints) and the norm boundedness of the $u_n s$ in $L^p(\Omega)$. The first term vanishes as well, taking adjoints and using the (weak) convergence of u_{n_j} to $\hat{u}$. Consequently, $\langle A\hat{u} - z, \nu\rangle z = 0$ for any $\nu \in \mathcal{Z}$, and hence, $A\hat{u} = z$. But this violates the uniqueness of the solution u_{exact} of (60).

As previous , one can consider instead the functional

$$T_n(u) = \|A_n u - z_n\|_Z^2 + \alpha J_\beta(u) \tag{65}$$

and obtain the same results as in the previous theorem.

Theorem $(\mathbf{2.1.21})[\mathbf{174}]$. In Theorem (2.1.20), replace T_n by (65), and make the same assumptions on A_n, α_n, z_n and p . Furthermore, assume that $|A_n \chi_\Omega| \geq \gamma > 0$. Then the conclusions of theorem (2.1.20) follow.

Proof. From the inequalities (15) one can assume $\beta = 0$. As in the proof of Theorem (2.1.20), one obtains that llAnun$\|A_n u_n - z_n\|^2 \leq \|A_n u_{\text{exact}} - z_n\|^2 + \alpha J_0(u_{\text{exact}})$ which implies (63). On the other hand, putting $u_n = v_n + w_n$ and referring again to the proofs of lemma (2.1.16) and theorem (2.1.19), the present assumptions also imply that (59) holds. As in limma (2.1.16), this implies that the u_n are uniformly BV-bounded. The last part of the proof is then the same as that of theorem (2.1.20).

Sec(2.2) : On the Autoconvolution Equation and Total Variation Constraints

Gorenflo and Hofmann [32] the nonlinear ill-posed autoconvolution equation

$$\int_0^s x(s-t)x(t)dt = y(s), \qquad 0 \leq s \leq 1, \tag{66}$$

on the finite interval [0,1]. This autoconvolution problem can be written as an operator equation

$$F(x) = y \tag{67}$$

with the continuous nonlinear operator $F: D(F) \subset X \to Y$ defined by

$$[F(x)](s) := [x * x](s) := \int_0^s x(s-t)x(t)dt, \qquad 0 \leq s \leq 1, \tag{68}$$

and mapping between Banach spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, containing real functions on the interval [0,1]. In [32] there have been discussed intrinsic properties of the autoconvolution operator F from (68) and conditions for its compactness, injectivity and weak closedness, in particular for the Hilbert space $X = Y = L^2(0,1)$. As a consequence the general theory of Tikhonov regularization became applicable to equation (66). The character of ill-posedness in this equation strongly depends on the solution point x and its local degree of ill-posedness. Applications of the autoconvolution equation arising in physics and in stochastics are also mentioned in [32].

Therefore, we are going to investigate stable approximate discretized solutions to (66), where both the function x to be determined and the data function y that can be measured are restricted to arguments from the interval [0,1].

The approximate solution of the autoconvolution equation (66) will be based for $Y := L^2(0,1)$ on the restriction of admissible solutions x to compact subsets of the domain $D(F)$ with prescribed properties. Provided that F is injective the inverse operator F^{-1} becomes continuous. We will show in that a compactification of the autoconvolution equation in $X := L^p(0,1)$ can be based on a prescribed upper bound c for the total variation $T(x)$ of solutions x, which are in addition uniformly bounded below and above by positive constants a and b, respectively. This allows us to construct convergent discretized solutions also in the case of non-smooth solutions possessing jumps. We generalize the well-known descriptive regularization approach using the set of monotone functions uniformly bounded below and above as a compact subset in $L^p(0,1)$, $1 \le p < \infty$ ([26]). The total variation bound c plays in our consideration the role of a regularization parameter. The ideas are extended to the Sobolev space case $X := H^1(0,1)$. A brief reference to the case of monotone functions is given in this section. We study the behaviour of discretized least-squares solutions to the autoconvolution equation subject to uniform bounds of the total variation is investigated, where both the case of a smooth and of a non-smooth solution are reflected.

Let us consider the autoconvolution operator (68) between the Banach spaces $X := L^p(0,1)$ for fixed $2 \le p < \infty$ with norm $\|x\|_{L^p(0,1)} = \left(\int_0^1 |x(t)|^p\,dt\right)^{1/p}$ and $Y := L^2(0,1)$. In this context, we define the sets

$$D_\varepsilon^+ := \{x \in L^p(0,1) : x(t) \ge 0 \ a.e. \text{ in}[0,1], \varepsilon = \sup\{\tau : x(t) = 0 \ a.e. \ in\ [0,\tau]\}\} \qquad (69)$$

and

$$R_\varepsilon^+ := \{y \in L^2(0,1) : y(s) \ge 0 \ a.e. \text{ in}[0,1], \varepsilon = \sup\{\chi : y(s) = 0 \ a.e. \ in\ [0,\chi]\}\} \qquad (70)$$

Then we have the following proposition which, because of $L^p(0,1)$ being densely embedded in $L^2(0,1)$, follows from [32] , [29]:

Proposition $(\mathbf{2.2.1})[\mathbf{40}]$ The autoconvolution operator $F: L^p(0,1) \to L^2(0,1)$ from (2.2.3) is a continuous nonlinear operator for all $2 \le p < \infty$. In the restricted case $F: D_0^+ \subset L^p(0,1) \to R_0^+ \subset L^2(0,1)$ the operator is injective, but the autoconvolution equation (67) is locally ill- posed in the sense of Definition (2.2.2) in all points $x \in D_\varepsilon^+$.

Definition $(\mathbf{2.2.2})[\mathbf{40}]$. We call the equation (67) locally ill-posed in $x \in D(F)$ if, for arbitrarily small $r > 0$ and balls $B_r := \{\tilde{x} \in X : \|\tilde{x} - x\|_X \le r\}$, there is an infinite sequence $\{x_k\} \subset D(F) \cap B_r(x)$ with

$$\|F(x_k) - F(x)\|_Y \to 0, \text{ but } \|x_k - x\|_X \nrightarrow 0 \text{ as } k \to \infty. \qquad (71)$$

Otherwise the equation is called locally well-posed in $x \in D(F)$.

To overcome the difficulties of ill-posedness of a problem under consideration one can restrict the domain $D(F)$ to a subset, which is compact in the Banach space X.

For a real function $x(t)$ $(0 \le t \le 1)$ we denote by

$$T(x) := \sup_{0\le t_0<t_1<\cdots<t_{k-1}<t_k\le 1} \sum_{i=1}^{k} |x(t_i) - x(t_{i-1})| \qquad (72)$$

the total variation of the function x on [0,1] and by $T_S(x)$ the analogously defined total variation of x on a closed subinterval $S \subset [0,1]$. Note that the supremum in formula (72) is to be taken over all possible finite grids of the form $0 \le t_0 < t_1 < \cdots < t_{k-1} < t_k \le 1$ with an arbitrarily chosen integer k. We consider, for given positive constants a, b and c, where

$$0 < a < b, \qquad (73)$$

the domain

$$D := \left\{ x : [0,1] \to [\mathrm{a,b}],\ \ \mathrm{T(x)} \le \mathrm{c},\ \ \begin{matrix} \text{x left}-\text{continuous} & \text{for } t \in (0,1], \\ \text{x left}-\text{continuous} & \text{for } t=0 \end{matrix} \right\}. \qquad (74)$$

For technical reasons we assume that the lower bound a is strictly positive. Obviously we have $D\, L^p(0,1)$ for all $1 \le p < \infty$. The requirement of the left- and right-continuity for the functions $x \in D$ is reasonable, since a function of bounded variation has due to only a countable set of discontinuity points, namely jumps. Therefore, the left limit $\lim_{t\to t_0-0} x(t)$ exists in all points of the interval (0,1]. In the continuity points t_0 this limit coincides with the value $x(t_0)$. In all other points let be the values of x defined by $x(t_0) := \lim_{t\to t_0-0} x(t)$. That means, with respect to $L^p(0,1)$-elements we consider the representative, which is left-continuous in every point $t \in (0,1]$. Moreover let $x(0) := \lim \mathrm{t} \to 0 + 0\, x(t)$, i.e. we consider no jumps at $t = 0$.

Lemma (2.2.3)[40]. The domain D from (73) – (74) is a compact subset of $L^p(0,1), 1 \le \mathrm{p} < \infty$, and we have $D \subset D_0^+$.

The proof of compactness of D is based on Helly's theorem [40]. For the proof ideas we refer to [29]. On the other hand, note that Lemma (2.2.3) is a corollary of Theorem in [25] of Acar and Vogel. Namely, the set D from (73) – (74) is bounded with respect to the BV-norm

$$\|x\|_{BV[0,1]} := \|x\|_{L^1[0,1]} + T(x). \qquad (75)$$

Based on Lemma (2.2.3) providing compactness the following well-known Lemma of Tikhonov will allow us to prove stability results.

Lemma (2.2.4)[40]. Let $F: D(F) \subset X \to Y$ be a continuous and injective operator between the Banach spaces X and Y with a compact domain $D(F)$. We denote by x^*, for given right-hand side $y^* \in F(D(F))$, the unique solution of the operator equation (67). Then for a family of approximate solutions $x_\eta \in D(F)$ the convergence of residual norms

$$\left\|F(x_\eta) - F(x^*)\right\|_Y \to 0 \quad \text{as} \quad \eta \to 0 \qquad (76)$$

implies the convergence of the approximate solutions

$$\left\|x_\eta - x^*\right\|_X \to 0 \quad \text{as} \quad \eta \to 0. \qquad (77)$$

In order to obtain numerical approximate solutions, in the sequel we are going to discretize the autoconvolution equation (66) – (68), where the restriction of F to the compact subset D from (73) – (74),

$$F: D \subset L^p(0,1) \to L^2(0,1), \tag{78}$$

is used. Similar to the discretization methods in [31], where also a total variation constraint is essential, we subdivide the interval [0,1] into n subintervals I_i of the uniform length $h := 1/n$, where

$$I_i := \big((i-1)h, ih\big] \qquad (i = 1, \dots, n).$$

For simplicity we set $T_i(x) := T_{[(i-1)h,ih]}(x)$ for $x \in D$. Moreover, let

$$t_j := \frac{h}{2} + (j-1)h \quad (j = 1, \dots, n)$$

denote the midpoints and

$$s_i := ih \qquad (i = 1, \dots, n)$$

the right endpoints of such intervals.

To discretize the nonlinear integral equation (66), for all $i, j = 1, 2, \dots, n$ the values $x(t_j)$ and $y(s_i)$ will be approximated by some t_j and y_i, respectively. A discrete autoconvolution operator

$$\underline{F} : \mathbb{R}^n \to \mathbb{R}^n \tag{79}$$

can be defined by

$$\underline{F}(\underline{x}) := \left(\sum_{j=1}^{i} h x_{i-j+1}\, x_j \right)_{i=1}^{n}, \quad \underline{x} = (x_1, \dots, x_n)^T. \tag{80}$$

In its discrete form the autoconvolution equation then reads as

$$\underline{F}(\underline{x}) = \underline{y}, \quad \underline{y} = (y_1, \dots, y_n)^T, \tag{81}$$

or as

$$\sum_{j=1}^{i} h x_{i-j+1}\, x_j = y_i, \qquad (i = 1, 2, \dots, n). \tag{82}$$

The realistic situation that the given data are noisy can be included. Instead of the exact data y_i for the right-hand side we will use perturbed data $\hat{y}_i$, where

$$\left\| \underline{\hat{y}} - \underline{y} \right\|_2 \le \delta \tag{83}$$

and δ is a fixed upper bound for the noise of the data vector $\underline{\hat{y}} = (\hat{y}_1, \dots, \hat{y}_n)^T$. Here we have used the scaled Euclidean norm

$$\|\underline{z}\|_2 := \left(\sum_{j=1}^{i} h z_i^2 \right)^{\frac{1}{2}}$$

for $\underline{z} \in \mathbb{R}^n$. For our further investigations we introduce the restriction operators

$$R: D \subset L^p(0,1) \to \mathbb{R}^n \text{ and } Q: F(D) \subset L^2(0,1) \to \mathbb{R}^n$$

by

$$(R(x))_j := x(t_j) \quad (j = 1, \dots, n) \tag{84}$$

and

$$(Q(y))_i := y(s_i) \quad (i = 1, \dots, n), \tag{85}$$

as well as the extension operators $E_1: \mathbb{R}^n \to L^p(0,1)$ and $E_2 \coloneqq \mathbb{R}^n \to L^2(0,1)$ by

$$\left(E_1(\underline{x})\right)(t) := x_j \quad \left(t \in I_j, j = 1, \dots, n\right), \quad \left(E_1(\underline{x})\right)(0) := x_1 \tag{86}$$

and

$$\left(E_2\left(\underline{y}\right)\right)(s) \coloneqq y_i \quad \left(s \in I_j, i = 1, \dots, n\right), \quad \left(E_2\left(\underline{y}\right)\right)(0) := y_1. \tag{87}$$

We are searching now for an optimal solution vector

$$\underline{x}^{opt} = (x_1{}^{opt}, \dots x_n{}^{opt})^T$$

solving the discrete least-squares problem

$$\left\|\underline{F}(\underline{x}) - \underline{\hat{y}}\right\|_2 \to \min, \text{ subject to } \underline{x} \in M, \tag{88}$$

where M is defined as

$$M \coloneqq \left\{\underline{x} \in \mathbb{R}^n : 0 < a \le x_i \le b \ (i = 1, \dots, n), \sum_{i=1}^{n-1} |x_{i+1} - x_i| \le c\right\}. \tag{89}$$

There exist solutions of (88), since M is compact in $\mathbb{R}^n$ and $\left\|\underline{F}(\underline{x}) - \underline{\hat{y}}\right\|_2 : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous functional possessing a minimum over M. The condition $0 < a \le x_i \le b$ is more restrictive than the discretized version of $x \in D_0^+$. We require this stronger condition, because we want M to be a compact subset of $\mathbb{R}^n$.

For the vectors $\eta \coloneqq (\delta, h)^T$, $\underline{x}^{opt} \in M$ and $\underline{\hat{y}}$ we define the piecewise constant function $x_\eta \in D$ *by*

$$x_\eta(t) \coloneqq E_1(\underline{x}^{opt})(t) \ (0 \le t \le 1). \tag{90}$$

and the piecewise constant function y_δ by

$$y_\delta(s) := E_2\left(\underline{\hat{y}}\right)(s) \quad (0 \le s \le 1).$$

Lemma $(2.2.5)[40]$. If we define the operator $F_\eta: L^p(0,1) \to L^2(0,1)$ by the formula

$$\left[F_\eta(x)\right](s) \coloneqq \sum_{j=1}^{i} \int_{I_j} x(s_i - t)\, x(t) dt \quad (s \in I_i), \tag{91}$$

then we have the equation

$$\|F_\eta(\xi)-\zeta\|^2_{L^2(0,1)} = \left\|\underline{F}\left(\underline{\xi}\right)-\underline{\zeta}\right\|_2^2 \tag{92}$$

for all $\xi := E_1(\underline{\xi})$, where $\underline{\xi} := (\xi_1,..,\xi_n)^T \in \mathbb{R}^n$ and all $\zeta := E_2(\underline{\zeta}) \in L^2(0,1)$, where $\underline{\zeta} := (\zeta_1,..,\zeta_n)^T \in \mathbb{R}^n$.

Proof:

$$\begin{aligned}\|F_\eta(\xi)-\zeta\|^2_{L^2(0,1)} &= \int_0^1 \left([F_\eta(\xi)](s)-\xi(s)\right)^2 ds \\ &= \sum_{i=1}^n \int_{I_i} \left(\sum_{j=1}^i \int_{I_i} \xi(s_i-t)\xi(t)dt-\xi(s)\right)^2 ds \\ &= \sum_{i=1}^n h\left(\sum_{j=1}^i \xi_{i-j+1}\xi_j - \zeta_i\right)^2 ds \\ &= \left\|\underline{F}\left(\underline{\xi}\right)-\underline{\zeta}\right\|_2^2.\end{aligned}$$

This proves the lemma

Lemma $(\mathbf{2.2.6})[\mathbf{40}]$. Let $x \in D$ from (73) – (74). Then we have the estimation

$$\|F(x) - F_\eta(x)\|_{L^2(0,1)} \le 2hb^2 + 2hbc.$$

Proof: We write

$$\|F(x) - F_\eta(x)\|_{L^2(0,1)}$$

$$= \left(\sum_{i=1}^n \int_{I_i} \left(\int_0^s x(s-t)x(t)dt - \int_0^{s_i} x(s_i-t)x(t)dt\right)^2 ds\right)^{\frac{1}{2}}. \tag{93}$$

Then we can estimate the expression in the inner parentheses by

$$\begin{aligned}&\left|\int_0^s x(s-t)x(t)dt - \int_0^{s_i} x(s_i-t)x(t)dt\right| \\ &\le \left|\int_{s_{i-1}}^s x(s-t)x(t)dt\right| + \left|\int_0^{s_{i-1}} x(s-t)x(t)dt - x(s_i-t)x(t)dt\right| \\ &\qquad + \left|\int_{s_{i-1}}^{s_i} x(s_i-t)x(t)dt\right| \\ &\le hb^2 + \sum_{j=1}^{i-1} \int_{I_j} |x(s-t)-x(s_i-t)|\,|x(t)|dt + hb^2\end{aligned}$$

$$\leq b\sum_{j=1}^{i-1}\int_{I_j} |x(s-t)-x(s_i-t)|\,dt+2hb^2 \tag{94}$$

Now we substitute $u := s_i - t,\ du := -dt$. For a fixed point $t \in (s_{j-1}, s_j\,] = I_j$ we obtain $u \in (s_{i-j}, s_{i-j+1}] = I_{i-j+1}$ and in view of $-h \leq s - s_i \leq 0$

$$s - s_i + u \in (s_{i-j-1}, s_{i-j+1}] = I_{i-j} \cup I_{i-j+1}.$$

Moreover, we can estimate (94) by

$$\begin{aligned}
&b\sum_{j=1}^{i-1}\int_{I_j} |x(s-t)-x(s_i-t)|\,dt+2hb^2\\
&=\sum_{j=1}^{i-1}\int_{I_{i-j+1}} |x(s-s_i+u)-x(u)|\,du+2hb^2\\
&\leq hb\sum_{j=1}^{i-1}\left(T_{i-j}(x)+T_{i-j+1}(x)\right)du+2hb^2\\
&\leq hbT(x)+hbt(x)+2hb^2\\
&\leq 2hbc+2hb^2.
\end{aligned}$$

Finally we substitute this estimation into equation (93). This yields the assertion of the lemma.

$\mathbf{Lemma\ (2.2.7)[40]}$. Under the assumptions stated above we have

$$\left\|F(x_\eta)-F(x^*)\right\|_{L^2(0,1)} \leq\ 4hb^2+6hbc+2\delta \to 0\ \ as\ \ \eta \to 0. \tag{95}$$

Proof: From the triangle inequality we obtain

$$\begin{aligned}
\left\|F(x_\eta)-F(x^*)\right\|_{L^2(0,1)} \leq \left\|F(x_\eta)-F(x_\eta)\right\| + \left\|F_\eta(x_\eta)-y_\delta\right\|_{L^2(0,1)}\\
+\|y_\delta - y\|_{L^2(0,1)}.
\end{aligned} \tag{96}$$

The right-hand side of (96) consists of three terms which we want to estimate one by one: Due to Lemma (2.2.6) for the first term it holds

$$\left\|F(x_\eta)-F_\eta(x_\eta)\right\|_{L^2(0,1)} \leq\ 2hb^2+2hbc \quad (x_\eta \in D).$$

To estimate the second term of (96) we define $\underline{x}^* := R(x^*)$ as the vector of the function values of the exact solution x^* of the autoconvolution equation (66) in the midpoints of the intervals I_i. Since we have $\underline{x}^{opt}$ as the least-squares solution of (88), the residual norm of $\underline{x}^*$ cannot be smaller than the residual norm of $\underline{x}^{opt}$. Furthermore, we can apply Lemma (2.2.5) with $\xi := x_\eta\ \ and\ \ \zeta := y_\delta$. This yields

$$\left\|F_\eta(x_\eta)-y_\delta\right\|_{L^2(0,1)} = \left\|\underline{F}(\underline{x}^{opt})-\underline{\hat{y}}\right\|_2 \leq \left\|\underline{F}(\underline{x}^*)-\underline{\hat{y}}\right\|_2.$$

Using the identity

$$F_\eta(x) = E_2\big(Q(F(x))\big) \qquad (x \in D),$$

this allows us to estimate further as follows:

$$\left\|\underline{F}(\underline{x}^*) - \underline{\hat{y}}\right\|_2 \le \left\|\underline{F}(\underline{x}^*) - Q(F(x^*))\right\|_2 + \left\|Q(F(x^*)) - \underline{\hat{y}}\right\|_2$$

$$= \left\|F_\eta\left(E_1(R(x^*))\right) - E_2\left(Q(F(x^*))\right)\right\|_{L^2(0,1)} + \left\|\underline{y} - \underline{\hat{y}}\right\|_2$$

$$= \left(\sum_{i=1}^{n}\int_{I_i}\left(\sum_{j=1}^{i}\int_{I_j}(\tilde{x}(s_i-\mathrm{t})\tilde{x}(t) - x^*(s_i-\mathrm{t})x^*(t))dt\right)^2 ds\right)^{\frac{1}{2}} + \delta$$

$$\le \left(\sum_{i=1}^{n}\int_{I_i}\left(\sum_{j=1}^{i}\int_{I_j}|\tilde{x}(s_i-\mathrm{t})||\tilde{x}(t) - x^*(t)|\right.\right.$$

$$\left.\left. + |x^*(s_i-\mathrm{t}) - \tilde{x}(s_i-\mathrm{t})||x^*(t)|dt\right)^2 ds\right)^{\frac{1}{2}} + \delta$$

$$\le \left(\sum_{i=1}^{n}\int_{I_i}\left(\sum_{j=1}^{i}\int_{I_j}2bT_j\, x^* dt\right)^2 ds\right)^{\frac{1}{2}} + \delta \le 2hbc + \delta,$$

where $\tilde{x} := E_1(R(x^*))$. The last inequalities essentially used Lemma (2.2.5) with $\xi = E_1(R(x^*)) = \tilde{x}$ and $\zeta = E_2\left(Q(F(x^*))\right)$, respectively. Note that we have $\tilde{x}(t) = x^*(t_j)$ for $t \in I_j$ and thereby $|\tilde{x}(t) - x^*(t)| \le T_j(x^*)$. Taking into account $|y_i - \hat{y}_i| \le \delta$ and the identity

$$\left\|E_2(\underline{y})\right\|_{L^2(0,1)} = \left\|\underline{y}\right\|_2,$$

which can easily be proved, we hence can estimate the third term of (2.2.31) as follows (cf. Lemma (2.2.6)):

$$\|y_\delta - y\|_{L^2(0,1)} \le \left\|y - E_2(Q(y))\right\|_{L^2(0,1)} + \left\|E_2(Q(y)) - y_\delta\right\|_{L^2(0,1)}$$

$$= \left\|F(x^*) - E_2\left(Q(F(x^*))\right)\right\|_{L^2(0,1)} + \left\|E_2(Q(y)) - E_2(Q(y_\delta))\right\|_{L^2(0,1)}$$

$$= \left\|F(x^*) - F_\eta(x^*)\right\|_{L^2(0,1)} + \|Q(y) - Q(y_\delta)\|_{L^2(0,1)} \le 2hb^2 + 2hbc + \delta.$$

Finally we can add the three terms and obtain by (96) the inequality (95). Evidently, the right-hand side of (95) tends to zero as h and δ both tend to zero. This proves the lemma.

By the result of Lemma (2.2.7) we can apply Lemma (2.2.4) to prove in L^p-spaces the convergence of approximate solutions to the autoconvolution equation under total variation constraints.

Theorem $(\mathbf{2.2.8})[\mathbf{40}]$. Consider the autoconvolution problem (66) – (68) with $D(F) := D$ from (73)- (74) and denote by $x^* \in D$, for given right-hand side $y^* \in F(D(F))$, the unique solution of the autoconvolution equation. Then the family of approximate solutions x_η according to (90) converges to the solution x^* of (67):

$$\left\| x_\eta - x^* \right\|_{L^p(0,1)} \to 0 \quad as \quad \eta \to 0 \quad \text{for all } 1 \le p < \infty. \qquad (97)$$

Proof: In the case $p \ge 2$ based on Lemma (2.2.7) the Lemma (2.2.4) immediately yields the convergence property (97), since the autoconvolution operator $F : D \subset L^p(0,1) \to L^2(0,1)$ is continuous and injective. Furthermore, D is a compact subset in $L^p(0,1)$ because of Lemma (2.2.3). For $1 \le p < 2$ the norm $\|\cdot\|_{L^p(0,1)}$ is 'weaker' than the norm $\|\cdot\|_{L^2(0,1)}$. This ensures the convergence condition (97) also in this case

By using the method of Tikhonov regularization in Hilbert spaces X and Y the minimizers x_α of the auxiliary extremal problems

$$\|F(x) - y\|_Y^2 + \alpha \|x\|_X^2 \longrightarrow \min, \text{ subject to } D(F) \qquad (98)$$

with the regularization parameter $\alpha > 0$ are exploited to find stable approximate solutions of an ill-posed operator equation (67). The smaller the regularization parameter α is chosen, the 'closer' the original and the auxiliary problem are related, but the more instable and highly oscillating the solution of the auxiliary problem will become. In general, α has to be selected such that an appropriate trade-off between stability and approximation is realized. In our compactification approach using upper bounds c of the total variation the inverse value $\frac{1}{c}$ plays a comparable role. In fact, if we consider small values $\frac{1}{c}$, then highly oscillating functions with large total variation values are admissible. On the other hand, for small values c the solutions obtained cannot oscillate very much, and the approximate solutions will be computed in a more stable way. However, if c is selected too small, then it may occur that the (unknown) exact solution is not an element of the set D. In such a case we would 'overregularize' the autoconvolution equation. By controlling the upper bound c of total variation we are able to suppress oscillations. Compared to the frequently used compactification in L^p by using monotonicity constraints and lower and upper bounds for the function values the approach of this section allows us to handle a more comprehensive class of (also non-monotone) functions. A numerical case study presented in this section will illustrate the theoretical results of this section and some specific effects of the discretized solution of the autoconvolution equation under total variation constraints.

In the case $p = \infty$ we cannot assert convergence under our assumption of bounded total variation. If the solution x^* has a jump point, then $\left\| x_\eta - x^* \right\|_{L^\infty(0,1)} \rightarrow 0 \quad as \quad \eta \rightarrow 0$ is not true in general.

In [32] it was already mentioned that the operator F of autoconvolution according to (68) mapping from $X := L^2(0,1)$ into the space $Y := L^2(0,1)$ is non-compact, but it becomes a compact operator if we change the problem to the Sobolev space $X := H^1(0,1) \cong W_2^1(0,1)$ of functions x with a quadratically integrable generalized derivative x' and norm

$$\|x\|_{H^1(0,1)} = \left(\int_0^1 |x(t)|^2 \, dt + \int_0^1 |x'(t)|^2 \, dt \right)^{1/2}. \tag{99}$$

In both cases the autoconvolution equation is locally ill-posed everywhere. But for compact operators F, we have in general a stronger form of ill-posedness. If our pairs of spaces X and Y are Hilbert spaces, following the concept of [32] (see also [33]) we can express the local degree of ill-posedness $\mu \, (0 \leq \mu \leq \infty)$ of the autoconvolution equation in a solution point x^* by the decay rate of the singular value sequence $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_i \geq \ldots > 0$ tending to zero as $i \rightarrow \infty$ of the $Fréchet$ derivative $F'(x^*)$ in the form

$$\mu := \sup\{\nu : \sigma_i = O(i^{-\nu}) \text{ as } i \rightarrow \infty\}, \tag{100}$$

where this linear operator given by $F^{'}(x^*)h = 2h * x^*$ is compact. Since the compact embedding operator from $H^1(0,1)$ into $L^2(0,1)$ has a sequence of singular values $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_i \geq \ldots > 0$ tending to zero with a rate $\kappa_i \sim 1/i$ as $i \rightarrow \infty$, for the Sobolev space $X := H^1(0,1)$ under consideration in this section the ill-posedness degree grows at least by one [34] compared to the $L^2(0,1)$. Thus, for a compactification in $H^1(0,1)$ 'stronger' restrictions on the admissible solutions x are necessary. However, our aim in this section is also stronger, namely to obtain convergence of approximate solutions x_η to x^* in the $H^1(0,1)$-norm (99).

Here we consider, for given constants a_1, a_2, b_1, b_2 and c with

$$0 < a_1 < b_1, a_2 < b_2, \tag{101}$$

the domain

$$D := \left\{ x: [0,1] \rightarrow [a_1, b_1], \begin{array}{c} \exists x^{'}: [0,1] \rightarrow [a_2, b_2], x^{'} \text{left-continuous for } t \in (0,1] \\ \mathrm{T}(x^{'}) \leq \mathrm{c}, \qquad x^{'} \text{right-continuous for } t = 0 \end{array} \right\}, \tag{102}$$

where the function $x'(t)$ $(0 \leq t \leq 1)$ a.e. in $[0,1]$ coincides with a derivative of $x(t)$ in the classical sense. Obviously we get $D \subset H^1(0,1)$ and hence every function $x \in D$ with D from (101) – (102) is continuous. In analogy to Lemma (2.2.3) we have in the Sobolev space case:

Lemma (2.2.9)[40]. The domain D from (101) – (102) is a compact subset of $H^1(0,1)$ with $D \subset D_0^+$.

In contrast to the L^p-case the restriction of the total variation, here $T(x') \leq c$, is only needed to show the compactness of the domain D. It has no relevance for the convergence of the images $F(x^*)$ of approximate solutions x_η to $F(x^*)$ in $L^2(0,1)$ as η tends to zero.

The discretization of the autoconvolution problem (66) – (68), where the operator F from (68) maps in the form

$$F : D \subset H^1(0,1) \rightarrow L^2(0,1) \tag{103}$$

and where the domain D is defined by (101) – (102) will be performed similar to the $L^p(0,1)$ case. However, piecewise constant functions are not in $H^1(0,1)$. Therefore, we use continuous piecewise linear approximate functions. Here, let

$$t_j := jh \quad (j = 0, \dots, n)$$

denote the $n+1$ nodes subdividing the interval $[0,1]$, and again $I_j = ((j-1)h, jh]$. Furthermore, the x_j again denote approximate values of $x(t_j)$. As the discrete autoconvolution operator we introduce here:

$$\underline{F} : \mathrm{IR}^{n+1} \rightarrow \mathrm{IR}^n, \tag{104}$$

where $\underline{F}(\underline{x}) = (z_1, \dots, z_1)^T$ and for $i = 1, 2, \dots, n$:

$$z_i = \int_0^{ih} \left(E_1(\underline{x})\right)(ih - t) \left(E_1(\underline{x})\right)(t)dt$$

$$= \sum_{j=1}^{i} \frac{h}{6}\left(2x_{i-j}x_j + x_{i-j+1}x_j + x_{i-j}x_{j-1} + 2x_{i-j+1}x_{j-1}\right). \tag{105}$$

By $E_1 : \mathrm{IR}^{n+1} \rightarrow H^1(0,1)$ we denote in contrast to this section the operator of piecewise linear interpolation according to

$$\left(E_1(\underline{x})\right)(t) := \frac{t - jh}{h}\left(x_j - x_{j-1}\right) + x_j \ (t \in I_j\ ,\ j = 1, \dots, n). \tag{106}$$

For noisy data (see (83)) we search for a minimizer

$$\underline{x}^{opt} = (x_0^{opt}, x_1^{opt}, \dots, x_n^{opt})^T$$

of the least-squares problem (88) with M from

$$M := \left\{\underline{x} : \in \mathrm{IR}^{n+1} : \begin{matrix} 0 < \mathrm{a}_1 \leq x_i \leq \mathrm{b}_1 (i = 0, \dots, n), \\ h\mathrm{a}_2 \leq x_{i-1} \leq h\mathrm{b}_2 (i = 0, \dots, n), \end{matrix} \sum_{i=1}^{n-1} |x_{i+1} - 2x_i + x_{i-1}| \leq \mathrm{hc}\right\}, \tag{107}$$

With the same arguments as before it follows that (88) is solvable. The choice of F is due to the fact that we have to guarantee the validity of formula (92) with F_η from (91).

By setting for the approximate solution

$$x_\eta := E_1(\underline{x}^{opt}), \tag{108}$$

where $\eta = (\delta, h)^T$, we also have $x_\eta \in D$with D according to (101)-(102). Moreover, it can be shown that as in Lemma (2.2.7) we have $\|F(x_\eta) - F(x^*)\|_{L^2(0,1)} \to 0$ for $\eta \to 0$. The proof dealing with the $H^1(0,1)$ approximation of functions by linear splines is omitted here. Using again Lemma (2.2.4) with $X := H^1(0,1)$ and $Y := L^2(0,1)$ we obtain:

Theorem (2.2.10)[40]. Consider the autoconvolution problem (66) – (68) with $D(F) := D$ from (101)-(102) and denote by $x^* \in D$, for given right-hand side $y^* \in F(D(F))$, the unique solution of the autoconvolution equation. Then the family of approximate solutions x_η converges to the solution x^* of (67):

$$\|x_\eta - x^*\|_{H^1(0,1)} \to 0 \ \ as \ \ \eta \to 0. \tag{109}$$

In this section we deal with solutions of the autoconvolution equation subject to the set of monotone and uniformly bounded functions considered as a particular subset of the functions possessing a bounded total variation.

First we consider the domain

$$D := \{x : 0 \le x(t) \le b, \ t \in [0,1], \ x \text{ non} - \text{increasing}\} \tag{110}$$

forming a compact subset in $L^p(0,1), 1 \le p < \infty$. Then the operator F from (78) is also injective, since $D \subset D_0^+$ and $x(t) = 0$ $(0 \le t \le 1)$ is the only function of D according to (100) with $x(0) = 0$. The discretization of this monotonicity case is completely the same as given in this section for the total variation case with the exception of the fact that we have to introduce

$$M := \{\underline{x} \in \mathrm{IR}^n : 0 \le x_n \le \cdots \le x_1 \le b\}. \tag{111}$$

replacing (89). Since each monotone function is of bounded variation, we obtain the convergence results of this section with c = b and $a = 0$.

Now we change to the case of non-decreasing solutions, where

$$D := \{x : 0 \le x(t) \le b, t \in [0,1], x \text{ non} - \text{decreasing}\} \tag{112}$$

and

$$M := \{x \in \mathrm{IR}^n : 0 \le x_1 \le \cdots \le x_n \le b\}. \tag{113}$$

The set D from (112) is also compact in $L^p(0,1)$, but the injectivity of F fails [32]. Because of that we have to distinguish two cases:

On the one hand let $y \in R_0^+$, i.e. $y(s) > 0$ if $s > 0$. Then the corresponding solution $x^*(t)$ is uniquely determined from y a.e. in $[0,1]$ and $\|F(x_\eta) - F(x^*)\|_{L^2(0,1)} \to 0$ for $\eta \to 0$ also implies $\|x_\eta - x^*\|_{L^p(0,1)} \to 0$, since Tikhonov's lemma (see Lemma (2.2.4)) in fact only needs the local injectivity condition $F(x) = F(x^*)$ $(x \in D) \Rightarrow x = x^*$.

On the other hand, let $y \in R_0^+$ for $\varepsilon > 0$, i.e. $y(s) = 0$ if $s \in [0, \varepsilon]$. As shown in [32], in such a case the autoconvolution operator F is non-injective and it holds:

$$x^*(t) = \begin{cases} 0 & a.e \text{ in} \left[0, \frac{\varepsilon}{2}\right] \\ \text{uniquely determined} & \text{a.e. in} \left[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right] \\ \text{arbitrarily non} - \text{negative} & \text{in} \left[1 - \frac{\varepsilon}{2}, 1\right] \end{cases} \quad (114)$$

Consequently, we have $x^* \in D^+_{\frac{\varepsilon}{2}}$. Since the values $x^*(t)$ do not depend on y for $t \in \left[1 - \frac{\varepsilon}{2}, 1\right]$, we cannot expect any information about the solution in this subinterval from the data.Therefore, it makes sense to solve the equation (66) only on the interval $\left[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right]$. We will show that this case is reducible to the already treated case $y \in R^+_0$. Because of this we define the operator $F_\varepsilon : L^p\left(\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right) \longrightarrow L^2(\varepsilon, 1)$ as

$$[F_\varepsilon(x)](s) \coloneqq \int_{\frac{\varepsilon}{2}}^{s-\frac{\varepsilon}{2}} x(s-t)x(t)dt. \quad (115)$$

Then we have $[F(x)](s) = [F_\varepsilon(x)](s)$ for $\frac{\varepsilon}{2} \leq s \leq 1 - \frac{\varepsilon}{2}$. By using the transformations

$$\tilde{t} \coloneqq \frac{t - \frac{\varepsilon}{2}}{1-\varepsilon}, \quad \tilde{s} \coloneqq \frac{s-\varepsilon}{1-\varepsilon}, \quad (116)$$

and

$\tilde{x}(\tilde{t}) := x\left((1-\varepsilon)\tilde{t} + \frac{\varepsilon}{2}\right) = x(t),\ \tilde{y}(\tilde{s}) = y\big((1-\varepsilon)\tilde{s} + \varepsilon\big) = y(s), \widetilde{F_\varepsilon}(\tilde{x}) = F_\varepsilon(x),$

we obtain an operator $\widetilde{F_\varepsilon}: L^p(0,1) \rightarrow L^2(0,1)$ defined by

$$\left[\widetilde{F_\varepsilon}(\tilde{x})\right](\tilde{s}) := (1-\varepsilon)\int_0^{\tilde{s}} \tilde{x}(\tilde{s}-\tilde{t})\,\tilde{x}(\tilde{t})d\tilde{t}. \quad (117)$$

Then we get $\tilde{x} \in L^p(0,1)$ if $x \in L^p(0,1)$, and instead of (67) we have to solve the equation $\widetilde{F_\varepsilon}(\tilde{x}) = \tilde{y}$ now. From $y \in R^+_\varepsilon$and $x \in R^+_{\frac{\varepsilon}{2}}$ it follows that $\tilde{y} \in R^+_0$and $\tilde{x} \in D^+_0$, respectively. Hence we have $\widetilde{F_\varepsilon}(\tilde{x}) = (1-\varepsilon)F_\varepsilon(\tilde{x})$ for all $\tilde{x} \in D^+_0$. Therefore, we can proceed as in the injective case and compute converging approximate solutions $\tilde{x}_\eta$. Then we transform back to the interval $\left[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right]$ and obtain approximate solutions with satisfactory properties on this interval, where the performed linear transformation retains the monotonicity. Finally we extend the solution by zero on the interval $\left[0, \frac{\varepsilon}{2}\right)$. On the other remaining subinterval $\left(1 - \frac{\varepsilon}{2}, 1\right]$ the solution can be extended arbitrarily provided that the monotonicity requirement is satisfied. Unfortunately, the value of ε is unknown if only discrete noisy data are given. In some situations, however, this value can be estimated and the transformation procedure becomes applicable.

Sec (2.3): On the Determination of A density Function by its Autoconvolution Coefficient

Solving the auto convolution equation

$$\int_0^t x(t-s)x(s)ds = y(t) \qquad (0 \le t \le T), \tag{118}$$

on a fnite interval $[0,T]$ is a simply written ill-posed nonlinear inverse problem which is interesting with respect to inverse problem theory and has a couple of applications. So the equation (118) arises in spectroscopy [35] and in some problems of probability theory mentioned in [42]. Also it serves as a benchmark example for an ill-posed nonlinear integral equation, where as a characteristic property the local degree of ill-posedness can rapidly vary for neighbouring solutions [42],[44].

We study a modifed autoconvolution problem aimed at fnding a non-negative integrable function x with support on the non-negative real half-axis. Instead of data for the autoconvolution function $y := x * x$ on $[0,T]$ we assume to have data for the pointwise quotient $k := y/x$ that we call autoconvolution coefficient, i.e., we solve the generalized autoconvolution equation of the third kind

$$k(t)x(t) - \int_0^t x(t-s)x(s)ds = 0 \quad (0 \le t \le T), \tag{119}$$

which has been studied recently by L. Berg, J. Janno and in [36] and [48]. More precisely, we focus on the case that the function x is a probability density function of a non-negative absolutely continuous random variable, i.e., $x(t) = 0$ ($- \infty < t < 0$), $x(t) \ge 0$ $(0 \le t < \infty)$ and $\int_0^\infty x(t)dt = 1$. We are going to recover this density function x on the finite interval $[0,T]$ from data of k on the same interval. We assume to know the value $0 < \kappa \le 1$ of the cumulative density function of this random variable at the right end $T > 0$ of the interval under consideration, i.e.,

$$\int_0^T x(t)dt = \kappa\,. \tag{120}$$

Moreover, we assume positivity

$$x(t) \ge 0 \quad (0 < t \le T) \tag{121}$$

for that interval.

We present a new existence theorem for those third kind equations and discuss questions of uniqueness and stability. Since ill-posedness emerges, a regularization approach is required for the stable approximate solution of such problems. We focus on the method of Tikhonov regularization for the nonlinear integral equation under consideration and formulate a couple of related open problems.

Definition $(\mathbf{2.3.1})[\mathbf{134}]$. Let $x(t)$ $(0 < t \leq T)$ be an integrable function satisfying the positivity condition (121). As the autoconvolution coefficient of x we call the function $k(t)$ $(0 < t \leq T)$ defined by

$$k(t) := \frac{1}{x(t)} \int_0^t x(t-s)x(s)ds > 0 \quad (0 < t \leq T), k(0) := 0. \quad (122)$$

If x is a density function with support $[0, \infty)$ satisfying (121) for all $T > 0$, the autoconvolution coefficient $k(t)$ $(0 \leq t \leq T)$ is well-defined for all $T > 0$. Since then the Rautoconvolution function $y = x * x$ is also a density function with support $[0, \infty)$, we have $\int_0^\infty k(t)x(t)dt = 1$, i.e., the product function kx is also a density function.

Example $(\mathbf{2.3.2})[\mathbf{134}]$. (Power-type functions) Let us consider the functions

$$x(t) = t^{\rho-1} \qquad (0 < t \leq T),$$

which are integrable for every parameter $\rho > 0$ and satisfy (121) for all $T > 0$. In terms of the Beta function $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ $(a, b > 0)$ we easily find $y(s) = B(\rho, \rho)s^{2\rho-1}(0 < t \leq T)$ for the autoconvolution function $y = x * x$ and hence the continuous autoconvolution coefficient

$$k(t) = t^{\rho} \qquad (0 \leq t \leq T)$$

for all $\rho > 0$.

Based on the result of this example we suggest by the following two propositions two different classes of functions $x(t)$ $(0 < t \leq t)$, for which the autoconvolution coefficient k is well-defined.

Proposition $(\mathbf{2.3.3})[\mathbf{134}]$. The autoconvolution coefficient k according to Definition (2.3.1) is a well-defined positive function on $(0, T]$ with

$$\lim_{t \to +0} k(t) = 0 \quad (123)$$

if the function x is measurable and satisfies inequalities of type

$$A_1 t^{p_1} \leq x(t) \leq A_2 t^{p_2} \ (0 < t \leq T;\ A_2 \geq A_1 > 0) \quad (124)$$

for some real exponents p_1 and p_2 with

$$-\frac{1}{2} < p_2 \leq p_1 < 2p_2 + 1. \quad (125)$$

If moreover x is continuous on $(0, T]$, then with $k(0) := 0$ the function k is continuous on $[0, T]$.

Proof: Measurable functions x with (124) have the properties $x \in L^2(0, T)$ and $x * x \in C[0, T]$ with $[x * x](0) := 0$. Since for integrable functions x_1 and x_2 the inequalities $0 < x_1 \leq x_2$ on $(0, T]$ imply inequalities $x_1 * x_2 \leq x_2 * x_1$ on $(0, T]$, due to Example (2.3.2) and because of (124) $k = x * x/x$ is a well-defined function on $(0, T]$ with $0 < k(t) \leq A_0 t^{2p_2 - p_1 + 1}(0 < t \leq T)$ for some constant $A_0 > 0$: Moreover, (125) yields $2p_2 - p_1 + 1 > 0$ and provides us with the limit (123).

Hence, for $k(0) := 0$ the continuity of k on $[0,T]$ holds whenever x is continuous on $(0,T]$.

On the other hand, we can prove:

Corollary $(\mathbf{2.3.3})'$ **[140].** The autoconvolution coefficients k of the matrix sequence of function $x_n(t)$ is a well defined positive function on $(0,T)$ with

$$\lim_{t\longrightarrow+0} k(t) = 0$$

If the x_n is measurable and satisfy

$$A_1 t^{p_1} \leq x_n(t) \leq A_2 t^{p_2}$$

where $0 < t \leq T$; $A_2 \geq A_1 > 0$

For two exponents p_1 and p_2 with

$$2p_2 \leq p_1 < 2p_2 + 1 \quad \text{and} \quad p_2 > -\frac{1}{2}$$

If $x_n(t)$ is continuous on $(0,T)$, then $k(0) = 0$ when $t \longrightarrow +0$ and hence k is continuous.

Proof: Since $x_n(t)$ is bounded then $x_n(t) \in L^2(0,T)$ and $x_n * x_n \in C[0,T]$ with $[x_n * x_n](0) = 0$ and since if x_n is monotonic increasing and integrable $0 < x_n \leq x_{n+1}$ on $(0,T]$ since $n \geq 1$, which gives $x_n * x_{n+1} \leq x_{n+1} * x_n$ on $(0,T]$. Due to proposition (2.3.14) we have $k(t) = x_n * x_n$ is well defined on $(0,T]$ then we get

$$0 < k(t) \leq A_0 t^{2p_2 - p_1 + 1}$$

For $0 < t \leq T$ and a constant $A_0 > 0$, then $k(t)$ is bounded and hence continuous for $k(0) = 0$ on $[0,T]$ wherever $x_n(t)$ is continuous in the half open interval $(0,T]$.

Proposition $(\mathbf{2.3.4})[\mathbf{134}]$. The autoconvolution coefficient k according to Definition (2.3.1) is a well- defined positive function on $(0,T]$ with the limit condition (123) if the function x is measurable, non-decreasing on some interval $(0,\varepsilon]$ with $0 < \varepsilon \leq T$, i.e., we have

$$0 < x(s) \leq x(t) \; (0 < s < t \leq \varepsilon) \qquad (126$$

and satisfies inequalities of type

$$0 < A_3 \leq x(t) \leq A_4 < \infty \; (\varepsilon \leq t \leq T). \qquad (127)$$

If moreover x is continuous on $(0,T]$, then with $k(0) :=$ the function k is continuous on $[0,T]$.

Proof: For measurable functions x from (126)-(127) we again have the properties $x \in L^2(0,T)$ and $x * x \in C[0,T]$ with $[x * x](0) := 0$. Moreover, k is a well-defined positive function on $(0,T]$. To prove the limit condition (123), here we formulate as a consequence of the monotonicity of x the estimates

$$0 < k(x) = \int_0^t \frac{x(t-s)x(s)}{x(t)} ds \leq \int_0^t x(t-s)\, ds \leq \sqrt{t}\|x\|_{L^2(0,T)} \; (0 < t \leq \varepsilon)$$

Obviously, $\sqrt{t}\|x\|_{L^2(0,T)}$ and hence $k(t)$ tend to zero as $t \longrightarrow 0$. Again, for $k(0) := 0$ the continuity of k on $[0,T]$ holds whenever x is continuous on $(0,T]$.

We consider the inverse problem of determining the density function x satisfying (119) - (121) from data of its autoconvolution coefficient k. This problem can be written as an operator equation

$$F(x) = k \quad \left(x \in \widetilde{D}(F) \subset X, k \in Y\right) \qquad (128)$$

with the nonlinear forward operator

$$[F(x)](t) := \int_0^t \frac{x(t-s)x(s)}{x(t)} ds \quad (0 < t \le T), \quad [F(x)](0) := 0 \quad (129)$$

mapping between the Banach spaces of real functions $X \subset L^2(0,T)$ and $Y \subset L^\infty(0,T)$, with domain

$$\widetilde{D}(F) := \{x \in X \subset L^2(0,T): x \text{ satiisfies } (124) - (125) \text{ or } (126) - (127)\}. \quad (130)$$

Note that every solution $x \in \widetilde{D}(F)$ of (128) fulfills both the integral equation (119) and the positivity condition (121). Moreover we have the following proposition.

Proposition $(\mathbf{2.3.5})[\mathbf{134}]$. If for given $k \in L^\infty(0,T)$ the operator equation (128) has a solution $x_0 \in \widetilde{D}(F)$, then for all $a \in \mathbb{R}$ the functions $x_a(t) = e^{at}\, x_0(t)$ $(0 < t \le T)$ belong to $\widetilde{D}(F)$ and are also solutions of (128). Moreover, for given $\kappa \in (0,1]$, there is a uniquely determined value $a_\kappa \in \mathbb{R}$ such that $x = x_{a_\kappa}$ satisfies (120).

Proof: Obviously, with x_0 the whole family of functions x_a satisfies (119) and belongs to the domain $\widetilde{D}(F)$. For given k any solution $x_0 \in \widetilde{D}(F)$ of (128) is positive on $(0,T]$ and integrable. Hence the integral $\int_0^T x_0(t)dt$ attains a finite positive value. Then the assertion of this proposition is a consequence of the fact that $\int_0^T x_a(t)dt$ is a continuous and increasing function of the real variable a tending to zero as $a \longrightarrow -\infty$ and tending to infinity as $a \longrightarrow \infty$.

We conclude this section with some examples of autoconvolution coefficients k and associated density functions x satisfying the equation (119) (resp. (128)). For further details concerning underlying well-known probability distributions we refer, e.g., to [39].

Example $(\mathbf{2.3.6})[\mathbf{134}]$. (Gamma distribution on $(0,\infty)$)

$$x(t) = \frac{\lambda^p}{\Gamma(p)} t^{p-1} e^{-\lambda t} \qquad (\lambda > 0, p > 0)$$

with $x(t) \sim \frac{\lambda^p}{\Gamma(p)} t^{p-1}$ as $t \longrightarrow +0$ and

$$y(t) = \frac{\lambda^{2p}}{\Gamma(2p)} t^{2p-1} e^{-\lambda t}, \; k(t) = \lambda^p \frac{\Gamma(p)}{\Gamma(2p)} t^p .$$

Special cases occur for

$p = 1$: Exponential distribution with $k(t) = \lambda t$

and

$$p = n \in \mathbb{N} := \{1,2,\dots\} : k(t) = \frac{(n-1)!}{(2n-1)!}\lambda^n t^n.$$

Example $(\mathbf{2.3.7})[\mathbf{134}]$. (Cauchy distribution on $(0,\infty)$)

$$x(t) = \frac{2}{\pi}\frac{\lambda}{\lambda^2+t^2} \qquad (\lambda > 0)$$

with $\quad x(t) \sim \frac{2}{\pi\lambda} - \frac{2}{\pi}\frac{t^2}{\lambda^3}\quad$ as $t \longrightarrow +0$, where

$$y(t) = \frac{8}{\pi^3}\frac{\lambda}{t^2+4\lambda^2}\left[\frac{\lambda}{t}\ln\left(1+\frac{t^2}{\lambda^2}\right) + \arctan\left(\frac{t}{\lambda}\right)\right]$$

and

$$k(t) = \frac{4}{\lambda}\frac{t^2+\lambda^2}{t^2+4\lambda^2}\left[\frac{\lambda}{t}\ln\left(1+\frac{t^2}{\lambda^2}\right) + \arctan\left(\frac{t}{\lambda}\right)\right] \sim \frac{2}{\pi\lambda}t + \frac{7}{6\pi}\frac{t^3}{\lambda^3} \quad \text{as } t \longrightarrow +0.$$

Example $(\mathbf{2.3.8})[\mathbf{134}]$. (Beta distribution on $(0,1)$)

$$x(t) = \frac{1}{B(p,q)}t^{p-1}(1-t)^{q-1} \qquad (p,q > 0)$$

with $x(t) \sim \frac{1}{B(p,q)}t^{p-1}$ as $t \longrightarrow +0$. It holds

$$y(t) = \frac{B(p,p)}{B^2(p,q)}t^{2p-1}(1-t)^{q-1}\, F_1\left(p, 1-q, 1-q, 2p; t, \frac{t}{1-t}\right)$$

and

$$k(t) = \frac{B(p,p)}{B(p,q)}t^p F_1\left(p, 1-q, 1-q, 2p; t, \frac{t}{1-t}\right) \sim \frac{B(p,p)}{B(p,q)}t^p \quad \text{as } t \longrightarrow +0,$$

where B is again the Beta function and F_1 is Horn's hypergeometric function of two variables [4].

We state the existence of continuous solutions to the integral equation (119) for $k(t) \sim At^n$ $(n \in \mathbb{N})$ as $t \longrightarrow +0$.

Theorem $(\mathbf{2.3.9})[\mathbf{134}]$. For fixed $0 < T < 1$, let $k \in C[0,T]$ with $k(t) > 0$ $(0 < t \le T]$ have the finite asymptotic expansion

$$k(t) = At^n + B(t) \qquad (A > 0, n \in \mathbb{N}) \tag{131}$$

with $B \in C[0,T]$ satisfying $B(t) = o(t^{n+1})$ as $t \longrightarrow +0$ and $\int_0^T \frac{|B(t)|}{t^{n+2}}dt < \infty$. Then the equation (119) has a one-parametric family of solutions $x_K \in C[0,T]$ $(K \in \mathbb{R})$ of the form

$$x_K(t) = A\gamma_n\, t^{n-1} + t^n\, z_K(t), \qquad \gamma_n = \frac{(2n-1)!}{[(n-1)!]^2} \tag{132}$$

with functions $z_K \in C[0,T]$ and the parameter $K = z_K(0) \in \mathbb{R}$. These solutions are the unique ones in the class of functions of type (132). The family of solutions $x_K (K \in \mathbb{R})$ can also be parametrized by the parameter $a = \frac{K}{A\gamma_n}$ through the relation $x_K(t) = e^{at}x_0(t)$.

With respect to uniqueness of the solutions to (119) there remains the difficult question if there exist in addition to x_K from Theorem (2.3.9) further (not identically vanishing) continuous solutions. In [36] Berg showed for $n = 1$ and a power-type function B that this is not the case. It is conjectured that this holds in general under the assumptions of Theorem (2.3.9). But this has not been proven at present. However, the following uniqueness assertion for equation (119) can be easily shown by the relation

$$x_1(t) = \int_0^t x_2(s)\,x_2(t-s)ds = x_2(t)\int_0^t x_1(s)\,x_1(t-s)ds \qquad (133)$$

holding for two solutions x_1, x_2 of (2.3.2). Namely, if for $k = 1,2$

$$x_k(t) \sim A_k t^{p_k}\,|\ln t|^{n_k} \text{ as } t \longrightarrow +0$$

with $A_k > 0, p_k > 0$ and $n_k \in \mathbb{N}$ then we have $p_1 = p_2, n_1 = n_2$ and $A_1 = A_2$. Furthermore, (133) leads to the following simple uniqueness proposition:

Proposition $(\mathbf{2.3.10})[\mathbf{134}]$. Let $x_1(t), x_2(t)$ $(0 < t \leq T)$ be positive, integrable and continuous solutions to equation (119), for which the quotient $q = x_2/x_1$ has the asymptotic expansion

$$q(t) = 1 + \eta t + o(t) \qquad \text{as } t \longrightarrow +0 \qquad (134)$$

with some $\eta \in \mathbb{N}$. Moreover, let the function $f_t(s) = q(s)q(t-s)$ be monotone both in$\left[0,\frac{t}{2}\right]$ *and* $\left[\frac{t}{2},t\right]$. Then $q(t) = e^{\eta t}, i.e., x_2(t) = e^{\eta t}\,x_1(t)$ $(0 \leq t \leq T)$.

Proof: In view of (133) the quotient q satisfies the relation

$$q(t) = \int_0^t x_1(s)\,x_1(t-s)ds = \int_0^t q(s)q(t-s)x_1(s)\,x_1(t-s)ds \;(0 \leq t \leq T).$$

By the second mean value theorem and $q(0) = 1$ we have

$$\int_0^{t/2} q(s)q(t-s)x_1(s)\,x_1(t-s)ds$$

$$= q(t)\int_0^{\xi_1(t)} x_1(s)\,x_1(t-s)ds + q^2(t/2)\int_{\xi_1(t)}^{t/2} x_1(s)\,x_1(t-s)ds,$$

$$\int_{t/2}^{t} q(s)q(t-s)x_1(s)\,x_1(t-s)ds$$

$$= q^2(t/2)\int_{t/20}^{\xi_2(t)} x_1(s)\,x_1(t-s)ds + q(t/2)\int_{\xi_2(t)}^{t} x_1(s)\,x_1(t-s)ds,$$

where $0 < \xi_1(t) < t/2$ and $t/2 < \xi_2(t) < t$. Hence the equality

$$q(t)\int_{\xi_1(t)}^{\xi_2(t)} x_1(s)\,x_1(t-s)ds = q^2(t/2)\int_{\xi_1(t)}^{\xi_2(t)} x_1(s)\,x_1(t-s)ds$$

or the functional equation

$$q(t) = q^2\left(\frac{t}{2}\right) \qquad (0 \le t \le T) \tag{135}$$

follows. Applying [49] to $p = \ln q$ or [49] directly to (135) with (134), we obtain $q(t) = e^{\eta t}$.

Recalling the solution $x = x_K$ to equation (119) constructed in Theorem (2.3.9) depends continuously on the scalar parameter A and on the parameter function B. But there is a strong restriction on the character of perturbations for the coefficient k which are allowed in this context. Namely, for this stability assertion the data $\tilde{k}$ of k must have the same form (131) only with perturbed $\tilde{A}$ and $\tilde{B} \in C[0,T]$. Under rather weak assumptions the nonlinear integral equation of the third kind (119) and the operator equation (128), respectively, are ill-posed if the forward operator F maps between appropriate spaces X and Y of continuous and square integrable functions. Namely, on the one hand equation (119) has no square integrable solutions if $k(t) \nrightarrow 0$ as $t \longrightarrow +0$ or if the decay rate of $k(t) \longrightarrow 0$ as $t \longrightarrow +0$ is too slow. Moreover, as will be shown by the following counterexample, we have a lack of stability if we measure deviations in X and Y with the maximum norm.

In this section we assume in the sequel $T = 1$ without loss of generality.

Example $(\mathbf{2.3.11})[\mathbf{134}]$. We consider the sequence of continuous coefficients

$$k_n(t) = \begin{cases} et & \text{if } \ 0 \le t \le \frac{1}{n} \\ \frac{2}{n}e - te^{2-nt} & \text{if } \ \frac{1}{n} \le t \le \frac{2}{n}, \\ \frac{2}{n}(e-2) + t & \text{if } \ \frac{2}{n} < t \le 1 \end{cases}$$

which as $n \longrightarrow \infty$ converges to the function $k_\infty(t) = t$ $(0 \le t \le 1)$ in $Y = C[0,1]$ with the maximum norm, i.e., $\lim_{n\to\infty} \|k_n - k_\infty\|_Y = 0$. But the corresponding continuous solutions to (119)

$$x_n(t) = \begin{cases} e^{1+a_n t} & \text{if } \ 0 \le \mathrm{t} < \frac{1}{n} \\ e^{(n+a_n)t} & \text{if } \ \frac{1}{n} \le \mathrm{t} \le 1 \end{cases}$$

with $a_n \in \mathbb{R}$ do not converge in the maximum norm ($X = C[0,1]$) to one of the solutions $x_\infty(t) = e^{at}$ $(0 \le t \le 1)$ to (119) associated with k_∞ for any $a \in \mathbb{R}$. Namely, we have

$$\|x_n - x_\infty\|_X \ge |x_n(0) - x_\infty(0)| = e - 1 > 0$$

independently of the choice of a_n and a. In particular, this assertion holds for the density function $x_\infty = 1$ $(0 \le t \le 1)$ with $a = 0$ and the solutions

$$x_n(t) = \begin{cases} e^{1-nt} & \text{if } \ 0 \le \mathrm{t} < \frac{1}{n} \\ 1 & \text{if } \ \frac{1}{n} \le \mathrm{t} \le 1 \end{cases},$$

where $a_n = -n$ and we have $x_n(t) \longrightarrow x_\infty(t)$ as $n \longrightarrow \infty$ pointwise in (0,1] with

$$\int_0^1 x_n(t)dt = 1 + \frac{e-2}{n} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

Because of such ill-posedness effects, a regularization is required for the stable approx- imate solution of the inverse problem (128) of finding the density functions x from noisy data $\tilde{k}$ of the autoconvolution coefficient k.

The most important method of regularization is Tikhonov's method, which is discussed with respect to autoconvolution. For the problem of determining the autocovolution coefficient the equation (119) written as operator equation (128) is approximated by the extremal problem

$$\left\|F(x) - \tilde{k}\right\|_Y^2 + \alpha\|x - \bar{x}\|_X^2 \longrightarrow \min, \text{ subject to } \quad x \in D(F), \qquad (136)$$

with domain $D(F) \subset \widetilde{D}(F)$ (see ((130))), where $F: D(F) \subset X \longrightarrow Y$ maps between Hilbert spaces X and Y, $\bar{x} \in X$ is an initial guess and $\alpha > 0$ denotes the regularization parameter.

We focus on the Sobolev space $X = Y = H^1(0,1)$ with norm

$$\|x\|_{H^1(0,1)} = \left(\|x\|_{L^2(0,1)}^2 + \|x'\|_{L^2(0,1)}^2\right)^{1/2}$$

and on the domain

$$D(F) = \{x \in H^1(0,1): K_1 t^{p_1} \le x(t) \le K_2 t^{p_2} \ (0 \le t \le 1;\ K_2 \ge K_1 > 0)\}, \ (137)$$

where stronger than (2.3.8)

$$0 \le p_2 \le p_1 \le p_2 + \frac{1}{2} \qquad (138)$$

is assumed for the operator F from (129) with $T = 1$. We prove for the domain (137) that the extremal problem (136) has a solution for any $k \in H^1(0,1)$, $\bar{x} \in H^1(0,1)$ and $\alpha > 0$. As is well-known in regularization theory it is sufficient to prove the weak closedness of F. This will be done in the following. We start with a lemma.

Lemma (2.3.12)[134]. The operator $F : D_0 \subset C[0,1] \longrightarrow C[0,1]$ with F defined by formula (129), where the domain is

$$D_0(F) = \{x \in C[0,1] : K_1 t^{p_1} \le x(t) \le K_2 t^{p_2} \ \ (0 \le t \le 1;\ K_2 \ge K_1 > 0)\}$$

with p_1, p_2 satisfying (138), maps into the subset $C_0[0,1] = \{v \in C[0,1] : v(0) = 0\}$ of $C[0,1]$ and is continuous.

Proof: The property $F(x) \in C_0[0,1]$ for $x \in D_0(F)$ follows directly from Proposition (2.3.14). Let further be $x_0, x_n \in D_0(F)$ with $x_n \longrightarrow x_0$ in $C[0,1]$. We put $v_0 = F(x_0), v_n = F(x_n)$ and obtain

$$|v_n(t) - v_0(t)| \le \frac{1}{x_n(t)} \left| \int_0^t [x_n(t-s)x_n(s) - x_0(t-s)x_0(s)]ds \right|$$

$$+\frac{1}{x_n(t)x_0(t)}|x_n(t)-x_0(t)|\int_0^t x_0(t-s)x_0(s)ds = I_1(t)+I_2(t).$$

Using (137) and (138) the first integral $I_1(t)$ is estimated by

$$I_1(t)=\frac{1}{x_n(t)}\left|\int_0^t [x_n(t-s)+x_0(t-s)][x_n(s)-x_0(s)]ds\right|$$

$$\leq \frac{1}{K_1 t^{p_1}}\max_{0\leq s\leq t}|x_n(s)-x_0(s)|\int_0^t [x_n(s)+x_0(s)]ds$$

$$\leq \frac{2K_2}{K_1}\frac{t^{p_2-p_1+1}}{p_2+1}\max_{t\in[0,1]}|x_n(t)-x_0(t)|$$

and the second integral $I_2(t)$ can be estimated by

$$I_2(t)=\frac{|x_n(t)-x_0(t)|}{x_n(t)x_0(t)}\int_0^t x_0(t-s)x_0(s)ds$$

$$\leq \frac{K_2^2}{K_1^2}B(p_2+1,p_2+1)t^{2p_2-2p_1+1}\max_{t\in[0,1]}|x_n(t)-x_0(t)|,$$

where $B(a,b)$ again denotes the Beta function. This shows that $\nu_n \longrightarrow \nu_0$ in $C[0,1]$.

We further prove

Proposition $(\mathbf{2.3.13})[\mathbf{134}]$. The operator $F: D(F)\subset H^1(0,1)\longrightarrow H^1(0,1)$ is bounded in the sense that subsets of $D(F)$ bounded in $H^1(0,1)$ are transformed into bounded subsets of $H^1(0,1)$.

Proof: Let $x\in D(F)$ and

$$\nu(t)=[F(x)](t)=\frac{1}{x(t)}\int_0^t x(t-s)x(s)ds\in C_0[0,1].$$

Again by (137), (138) we have

$$\nu(t)\leq \frac{K_2^2}{K_1^2}t^{2p_2-p_1+1}B(p_2+1,p_2+1)\leq C_1 := \frac{K_2^2}{K_1^2}B(p_2+1,p_2+1)$$

implying

$$\|\nu\|_{L^2(0,1)}\leq C_1, \tag{139}$$

where the constant C_1 does not depend on x.

Further, the derivative ν' of ν is given by

$$\nu'(t)=x(0)+\frac{1}{x(t)}\int_0^t x'(t-s)x(s)ds-\frac{x'(t)}{x^2(t)}\int_0^t x(t-s)x(s)ds$$

and we get the estimate

$$\|v'\|_{L^2(0,1)} \leq x(0) + \left(\int_0^1 \frac{1}{x^2(t)}\left[\int_0^t x'(t-s)x(s)ds\right]^2 dt\right)^{1/2} + \left(\int_0^1 \frac{(x')^2(t)}{x^4(t)}\left[\int_0^t x(t-s)x(s)\right]^2 dt\right)^{1/2} \quad (140)$$

By (137), (138) it holds

$$\left|\int_0^t x'(t-s)x(s)ds\right| \leq \left(\int_0^t x^2(s)ds\right)^{1/2} \|x'\|_{L^2(0,1)} \leq \frac{K_2}{\sqrt{2p_2+1}} t^{p_2+\frac{1}{2}} \|x'\|_{L^2(0,1)}$$

and

$$\int_0^1 \frac{1}{x^2(t)}\left[\int_0^t x'(t-s)x(s)ds\right]^2 dt \leq \frac{K_2^2}{K_1^2} \cdot \frac{1}{2p_2+1} \cdot \frac{1}{2(p_2-p_1+1)} \|x'\|_{L^2(0,1)}^2.$$

Finally, again due to (137), (138), we have

$$\int_0^t x(t-s)x(s)ds \leq K_2^2 t^{2p_2+1} B(p_2+1, p_2+1)\,,$$

hence

$$\frac{1}{x^4(t)}\left[\int_0^t x(t-s)x(s)ds\right]^2 \leq \frac{K_2^4}{K_1^4}[B(p_2+1,p_2+1)]^2\, t^{4p_2-4p_1+2}$$

and

$$\int_0^t \frac{(x')^2(t)}{x^4(t)}\left[\int_0^t x(t-s)x(s)ds\right]^2 dt \leq \frac{K_2^4}{K_1^4}[B(p_2+1,p_2+1)]^2 \|x'\|_{L^2(0,1)}^2.$$

Therefore, from (140) we obtain the estimation

$$\|v'\|_{L^2(0,1)} \leq x(0) + C_0\|x'\|_{L^2(0,1)}, \quad (141)$$

where the constant $C_0 = \frac{K_2}{K_1}\frac{1}{\sqrt{2p_2+1}}\frac{1}{\sqrt{2(p_2-p_1+1)}} + \frac{K_2^2}{K_1^2}B(p_2+1,p_2+1)$ does not depend on x. Then the inequalities (139) and (141) show that the operator $F : D(F) \subset H^1(0,1) \longrightarrow H^1(0,1)$ is bounded.

Now we can prove the main theorem of this section.

Theorem $(\mathbf{2.3.14})[\mathbf{134}]$. The operator $F : D(F) \subset H^1(0,1) \longrightarrow H^1(0,1)$ is weakly continuous.

Proof: Let $x_n, x_0 \in D(F)$ with $x_n \rightharpoonup x_0$ in $H^1(0,1)$, i.e.,

$$(x_n, \varphi) + (x_n', \varphi) \longrightarrow (x_0, \varphi) + (x_0', \varphi) \text{ for any } \varphi \in H^1(0,1),$$

where $(\cdot,\cdot)$ denotes the generic inner product in $L^2(0,1)$. We have to prove that

$$v_n = F(x_n) \rightharpoonup v_0 = F(x_0) \text{ in } H^1(0,1).$$

For this it is sufficient to show that both variants of weak convergence $v_n \rightharpoonup v_0$ in $L^2(0,1)$ and $v_n' \rightharpoonup v_0'$ in $L^2(0,1)$ hold. The first weak convergence follows directly from Lemma (2.3.12) since weak convergence $x_n \rightharpoonup x_0$ in $H^1(0,1)$ implies strong convergence $x_n \rightharpoonup x_0$ in $C[0,1]$ as a consequence of the compact embedding of $H^1(0,1)$ into $C[0,1]$ and then we have $v_n \rightharpoonup v_0$ in $L^2(0,1)$ from $v_n \rightharpoonup v_0$ in $C[0,1]$.

So it remains to prove that $v_n' \rightharpoonup v_0'$ in $L^2(0,1)$. This is equivalent to the following both conditions: (a) The sequence v_n' is bounded in $L^2(0,1)$; (b) $\int_0^t v_n'(s)ds \longrightarrow \int_0^t v_0'(s)ds$ pointwise for all $t \in [0,1]$. The boundedness (a) of the v_n' in $L^2(0,1)$ follows from the estimation (141), because the sequence x_n is bounded in $H^1(0,1)$ due to the its weak convergence in $H^1(0,1)$. The condition (b) is equivalent to $v_n(t) \longrightarrow v_0(t)$ pointwise for all $t \in [0,1]$ since $v_n(0) = v_0(0)$. But from Lemma (2.3.23) we know that even $v_n(t) \longrightarrow v_0(t)$ uniformly in $[0,1]$.

Based on well-known assertions of Hilbert space and regularization theory we obtain the following corollary as an immediate consequence of Theorem (2.3.25) taking into account that the domain (137) is a closed and convex, hence a weakly closed subset of $H^1(0,1)$ and so the weak continuity of F implies the weak closedness of this operator.

Corollary $(\mathbf{2.3.15})[\mathbf{134}]$. The operator $F : D(F) \subset H^1(0,1) \longrightarrow H^1(0,1)$ is weakly closed and the extremal problem (136) has always a solution.

Remark $(\mathbf{2.3.16})[\mathbf{134}]$. Under the assumption $0 \le p_2 \le p_1 \le \frac{p_2}{2} + \frac{1}{4}$, which is stronger than (138), one can show that the operator $F : D(F) \subset H^1(0,1) \longrightarrow H^1(0,1)$ is also continuous.

Another approach to approximate solutions of (119) for given noisy data $\tilde{k}\ 2 \in [0,T]$ of k is using the differintegral equation

$$\varepsilon x_\varepsilon'(t) + \tilde{k} x_\varepsilon(t) = \int_0^t x_\varepsilon(t-s) x_\varepsilon(s) ds \qquad (0 < t \le T) \tag{142}$$

for small $\varepsilon > 0$ with the initial condition $\varepsilon(0) = \tilde{A} > 0$. This is a variant of Lavrentiev's regularization method to (119). The initial value $\tilde{A}$ should be chosen as $\tilde{A} = \lim_{t \longrightarrow +0} \tilde{k}(t)/t^p$ $(p > 0)$ if this limit exists for some p > 0 or in general as $\tilde{A} = \tilde{A}_\varepsilon = (\tilde{k}(\varepsilon) - \tilde{k}(0))/\varepsilon^p$. In [48] the corresponding singular perturbation problem with $\varepsilon \longrightarrow +0$ for exact $k \in C^2[0,T]$ was investigated. A study under the weaker assumption $k \in C^1[0,T]$ seems to be difficult and is still missing.

Chapter 3
Multiplicity Free Theorem and Measurable Proper Actions with Conformal Geometry

We give a new approach for constructing examples of a unitary highest weight module of a reductive Lie group and of homogeneous spaces G/H with no compact quotients where G is a Lie group and H is a closed noncompact subgroup. This approach is based on the study of the restriction to H of matrix coefficients of unitary representations of G. A similar method also gives a criterion when the restriction to H of an action of G on a locally compact space X with a G-invariant infinite measure is measurably proper in the sense that, for almost all $x \in X$, the natural map $h \longmapsto hx$ of H onto Hx is proper. We calculate the minimal nilpotent coadjoint orbit to certain natural dual pairs in the unitary representation of $O(p,q)$. We show that an estimate for the $\theta -$ measure and it is a locally integrable for any bounded Borel subset.

Sec(3.1) : Branching Problems of Unitary Highest Weight Mmodules

Let G be a reductive Lie group, and $\hat{G}$ the unitary dual. Suppose H is a reductive subgroup of G. If $\pi \in \hat{G}$, then the restriction $\pi|_H$ is no more irreducible as a representation of H in general. The irreducible decomposition formula of $\pi|_H$ is called the branching law (breaking symmetry in physics) and is written in terms of the direct integral of unitary representations of H:

$$\pi|_H \simeq \int_{\hat{H}}^{\oplus} m_H(\tau : \pi|_H)\, \tau d\mu(\tau) \qquad (1)$$

where $d\mu$ is a Borel measure on $\hat{H}$ and $m_H(\cdot : \pi|_H): \hat{H} \longrightarrow \mathbb{N} \cup \{\infty\}$ is the multiplicity defined almost everywhere with respect to $d\mu$.

One expects a simple and detailed study for the branching problem when no continuous spectrum arises in the decomposition (1) (discrete branching law), and the general theory for discrete branching laws has been studied in [56],[57],[58],[59],[60]. A very special and simple setting of the discrete branching laws is when the following (a) and (b) hold:

a) $\pi \in \hat{G}$ is an irreducible unitary highest weight module, and
b) (G, H) is a semisimple symmetric pair satisfying (2) .

The purpose of this section is to investigate the restriction $\pi|_H$ in this special setting
(a) and (b).

Let G be a non-compact simple Lie group of finite center, θ a Cartan involution of G, and $K := \{g \in G : \theta g = g\}$. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition of the Lie algebra g of G, corresponding to the Cartan involution θ. We assume that G

is of Hermitian type, that is, the center $c(\mathfrak{k})$ of k is non-trivial. Then, it is well-known that $c(\mathfrak{k})$ is one dimensional and that there exists $Z \in c(\mathfrak{k})$ so that

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

is the direct sum decomposition of eigenspaces of $\mathrm{ad}(Z)$ with eigenvalues $0, \sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Definition $(\mathbf{3.1.1})[\mathbf{62}]$. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G, and $\mathcal{H}_K$ the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module. $(\pi, \mathcal{H})$ is called an irreducible unitary highest weight module if $\mathcal{H}_K^{\mathfrak{p}^+} \neq \{0\}$, where we put

$$\mathcal{H}_K^{\mathfrak{p}^+} := \{v \in \mathcal{H}_K : d\pi(Y)v = 0 \text{ for any } Y \in \mathfrak{p}^+\}.$$

Then, $\mathcal{H}_K^{\mathfrak{p}^+}$ is an irreducible representation of K. We say that π is of scalar type (or of scalar minimal K-type) if $\mathcal{H}_K^{\mathfrak{p}^+}$ is one dimensional. By a holomorphic discrete series representation for G, we mean that π is a unitary highest weight module that can be realized as a closed G-invariant subspace of $L^2(G)$ (if G has an infinite center, then we need a slight modification as usual).

Lowest weight modules and anti-holomorphic discrete series are defined similarly with $\mathfrak{p}^+$ replaced by $\mathfrak{p}^-$.

Suppose τ is an involutive automorphism of G commuting with θ. Because $\tau c(\mathfrak{k}) = c(\mathfrak{k}) = \mathbb{R}Z$ and $\tau^2 = \mathrm{id}$, there are two exclusive possibilities:

$$\tau Z = Z, \qquad (2)$$

$$\tau Z = -Z\,. \qquad (3)$$

Let $G^\tau = \{ g \in G : \tau g = g\}$ and $K^\tau := G^\tau \cap K$.

Geometrically, (2) implies:

i) τ acts holomorphically on the Hermitian symmetric space G/K,

ii) $G^\tau/K^\tau \hookrightarrow G/K$ is a complex submanifold.

On the other hand, (3) implies:

i) τ acts anti-holomorphically on the Hermitian symmetric space G/K,

ii) $G^\tau/K^\tau \hookrightarrow G/K$ is a totally real submanifold.

Let G be a non-compact simple Lie group of Hermitian type. Here are our main results:

Theorem $(\mathbf{3.1.2})[\mathbf{62}]$. Let π_1 and π_2 be unitary highest weight modules of G. Then, thereis a constant $C(\pi_1, \pi_2) < \infty$ with the following properties:

i) The tensor product $\pi_1 \widehat{\otimes} \pi_2$ splits into a discrete Hilbert sum of irreducible unitary representations of G:

$$\pi_1 \widehat{\otimes} \pi_2 \simeq \sum_{\mu \in \widehat{G}}^{\otimes} m_{\pi_1, \pi_2}(\mu)\mu\,, \quad (\textit{Hilbert direct sum}),$$

with the multiplicity satisfying

$$m_{\pi_1,\pi_2}(\mu) \leq C(\pi_1, \pi_2) \quad for\ all\ \ \mu \in \hat{G}. \tag{4}$$

ii) $C(\pi_1, \pi_2) = 1$ if both π_1 and π_2 are of scalar minimal K-types. Namely, the tensor product $\pi_1 \widehat{\otimes} \pi_2$ is decomposed discretely into irreducible unitary representations of G with multiplicity free, for any unitary highest weight modules π_1 and π_2 of scalar minimal K-types.

Theorem $(3.1.3)$[62]. Let π be a unitary highest weight module of G. Then, there is a constant $C(\pi) < \infty$ with the following properties: Suppose that τ is an involutive automorphism of G satisfying (2). Let H be an open subgroup of G^{τ} .

i) The restriction $\pi|_H$ splits into a discrete Hilbert sum of irreducible unitary representations of H:

$$\pi|_H \simeq \sum_{\mu \in \widehat{H}}^{\otimes} m_{\pi}(\mu)\mu \qquad (Hilbert\ direct\ sum),$$

with the multiplicity satisfying

$$m_{\pi}(\mu) \leq C(\pi) \ \ for\ all\ \ \mu \in \widehat{H}. \tag{5}$$

ii) $C(\pi) = 1$ if π is of scalar minimal K-type. Namely, the restriction $\pi|_H$ is decomposed discretely into irreducible unitary representations of H with multiplicity free, for any unitary highest weight module π of G having scalar minimal K-type.

The infinitesimal classification of irreducible symmetric pairs was achieved by M. Berger [50]. We give a list of the infinitesimal classification of irreducible symmetric pair (G, H) satisfying the condition (2) (see Theorem (3.1.2)).

$(\mathfrak{g}, \mathfrak{g}^{\tau})$ satisfying (2) $\tau Z = Z$	
$\mathfrak{g}$	$\mathfrak{g}^{\tau}$
$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(i,j) + \mathfrak{u}(p-i, q-j))$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{su}(n,n)$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2p) + \mathfrak{so}^*(2n-2p)$
$\mathfrak{so}^*(2n)$	$\mathfrak{u}(p, n-q)$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,p) + \mathfrak{so}(n-p)$
$\mathfrak{so}(2,2n)$	$\mathfrak{u}(1,n)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(p, n-p)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(p,\mathbb{R}) + \mathfrak{sp}(n-p,\mathbb{R})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}^*(10) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(8,2) + \mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(5,1) + \mathfrak{sl}(2,\mathbb{R})$

$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(4,2)+\mathfrak{su}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6+\mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)}+\mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(10,2)+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}(6,2)$

Table (3.1.1)

Here are simplest examples of Theorem (3.1.2) and Theorem (3.1.3), respectively:

Example $(\mathbf{3.1.4})$**[62].** We denote by $\pi_n$the holomorphic discrete series representation of $SL(2,\mathbb{R})$ with minimal K-type χ_n $(n\geq 2)$, where $\chi_n(n\in\mathbb{Z})$ stands for a character of $SO(2)$. Then, the following branching formulae are well-known:

$$\pi_m\widehat{\otimes}\pi_n \simeq \sum_{k\in\mathbb{N}}^{\oplus}\pi_{m+n+2k},$$

$$\pi_n|_{SO(2)} \simeq \sum_{k\in\mathbb{N}}^{\oplus}\pi_{m+n+2k}\,.$$

Here, $\mathbb{N}=\{0,1,2,\dots\}$. We note that any holomorphic discrete series representation of $SL(2,\mathbb{R})$ is of scalar minimal K-type.

The conditions ‘‘highest weight modules’’, ‘‘discrete branching’’, ‘‘scalar minimal K-type’’ are crucial in the multiplicity free, uniformly bounded, or bounded theorems in Theorem (3.1.2) and Theorem (3.1.3). Here are related remarks:

Remark $(\mathbf{3.1.5})$**[62].**

- **i)** The discrete decomposability in Theorems (3.1.2) and (3.1.3) was previously known ([67],[66] and [54]) . The novelty of Theorems (3.1.2) and (3.1.3) is the estimate of multiplicities (4) and (5).
- **ii)** The Cartan involution θ automatically satisfies (2). In this case, we have $H = K$ and the multiplicity free result in Theorem (3.1.3) is known by B. Kostant, W. Schmid and K. Johnson ([71],[55]) by explicit branching laws in the case where π is a holomorphic discrete series representation of scalar type. Theirformula will be generalized to a non-compact H.
- **iii)** If $\pi=A_q(\lambda)$ in the sense of Vogan-Zuckerman (e. g. a discrete series representation) and if (G,H) is a semisimple symmetric pair such that $\pi|_H$ is discrete decomposable, then the multiplicity always satisfies

  $$m_\pi(\tau)<\infty \quad \text{for any} \quad \tau\in\widehat{H}$$

  However, there is an example with

$$\sup_{\tau\in\widehat{H}} m_\pi(\tau) = \infty$$

Namely, the multiplicity is always finite but not necessarily uniformly bounded in the discrete branching laws of non-highest weight modules with respect to a reductive symmetric pair.

iv) The multiplicity can be infinite in the continuous spectrum if $\pi = A_q(\lambda)$ is not a highest weight module and if (G, H) is a symmetric pair [51].

v) It follows from R. Howe [53] and J. Repka [70] that the irreducible decomposition of the tensor product $\pi_1 \widehat{\otimes} \pi_2$ always involves a continuous spectrum, if π_1 is a holomorphic discrete series representation and π_2 is an anti-holomorphic discrete series representation. This is regarded as an opposite extremal case to Theorem (3.1.2). Likewise, if π is a highest weight module of scalar minimal K- type and if τ satisfies (3) instead of (2), then $\acute{O}$lafsson and B. ϕrsted proved that $\pi|_H$ is decomposed into only continuous spectrum with multiplicity free [69]. This is an opposite extremal case to Theorem (3.1.3) (2).

vi) If we drop the assumption of the scalar minimal K-type in Theorem (3.1.2) or Theorem (3.1.3), then there is a counter example for multiplicity free [62]. Namely, $C(\pi_1, \pi_2)$ in Theorem (3.1.2) (also $C(\pi)$ in Theorem (3.1.3)) cannot be always taken to be 1.

vii) Finally, we mention the case where $\dim\pi < \infty$. Our method here also gives a suficient condition for the multiplicity free branching laws for finite dimensional representations of compact groups,which is analogous to the second part of Theorems (3.1.2) and (3.1.3). A complete list of the multiplicity free cases that can be obtained by our method is given. Some of them could be also proved by using so called the Littlewood-Richardson rule and the algorithm of K. Koike and I. Terada. S. Okada recently obtained a number of multiplicity free branching laws by combinatorial arguments of character formulae for classical compact Lie groups. It might be interesting from combinatorial view point to obtain explicit branching laws for the remaining cases (many of them are exceptional cases) for which the multiplicity is proved to be free by our method.

Let $\mathcal{L} \longrightarrow D$ be a holomorphic line bundle over a complex manifold D. We denote by $\mathcal{O}(\mathcal{L})$ the space of holomorphic sections of $\mathcal{L} \longrightarrow D$. Then $\mathcal{O}(\mathcal{L})$ carries a Fr$\acute{e}$echet topology by the uniform convergence on compact sets. If a Lie group H acts holomorphically and equivariantly on the holomorphic line bundle $\mathcal{L} \longrightarrow D$, then H defines a (continuous) representation on $\mathcal{O}(\mathcal{L})$ by the pull-back of sections.

Let $\{U_\alpha\}$ be a trivializing neighbourhood of D, and $g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$ the transition functions of the holomorphic line bundle $\mathcal{L} \longrightarrow D$.Then an anti holomorphic line bundle $\bar{\mathcal{L}} \longrightarrow D$ is a complex line bundle with the transition functions $\overline{g_{\alpha\beta}}$. We denote by $\bar{\mathcal{O}}(\bar{\mathcal{L}})$ the space of anti-holomorphic sections of $\bar{\mathcal{L}} \longrightarrow D$.

Suppose σ is an anti-holomorphic diffeomorphism of D. Then the pull-back $\sigma^*\mathcal{L} \longrightarrow D$ is an anti-holomorphic line bundle over D. In turn, $\overline{\sigma^*\mathcal{L}} \longrightarrow D$ is a holomorphic line bundle over D.

A main machinery for the proof of Theorem (3.1.2) and Theorem (3.1.3) is the commutativity of the commutant algebra

$$\mathrm{End}_H(\mathcal{H}) := \{T \in \mathrm{End}(\mathcal{H}) : T \text{ is continuous}, T_\pi(h) = \pi(h)T \text{ for any } h \in H\},$$

if a unitary representation $(\pi, \mathcal{H})$ of the group H is realized on holomorphic functions (or holomorphic sections) on a complex manifold D.

Faraut and Thomas [51], in the case of trivial twisting parameter, gives a sufficient condition for the commutativity of $\mathrm{End}_H(\mathcal{H})$ by using the theory of reproducing kernels, which we extend to the general, twisted case below.

Lemma (3.1.6)[62]. Let $(\pi, \mathcal{H})$ be a unitary representation of a Lie group H. Assume that there exist an H-equivariant holomorphic line bundle $\mathcal{L} \longrightarrow D$ and an antiholomorphic involutive diffeomorphism σ of D with the following three conditions:

There is an injective (continuous) H-intertwining map $\mathcal{H} \longrightarrow \mathcal{O}(\mathcal{L})$ (6)

There exists an isomorphism of H-equivariant holomorphic line bundles $\Psi : \mathcal{L} \xrightarrow{\sim} \overline{\sigma^*\mathcal{L}}$. (7)

Given $x \in D$, there exists $g \in H$ such that $\sigma x = g \cdot x$. (8)

Then, $\mathrm{End}_H(\mathcal{H})$ is a commutative algebra.

The idea of Lemma (3.1.6) parallels to [51], which goes back to a lemma due to I. M. Gelfand:

Lemma (3.1.7)[62]. Let G be a locally compact unimodular group, and K a compact subgroup. Assume that there exists an antiinvolutive automorphism σ of G such that given $x \in g$ there exist $k_1, k_2 \in K$ satisfying $\sigma x = k_1 x k_2$. Then, the Hecke algebra $L^1(K \backslash G / K)$ is a commutative ring.

The following is a key lemma to apply Lemma (3.1.6)by supplying a sufficient condition for (8) in the setting where $D = G/K$ is a Riemannian symmetric space.

Lemma (3.1.8)[62]. Let G be a non-compact semisimple Lie group of finite center, K a maximal compact subgroup of G corresponding to a Cartan involution θ. Let σ and τ are involutive automorphisms of G. We assume the following two conditions:

σ, τ and θ commute with one another. (9)

$\mathbb{R} - \mathrm{rank}\ \mathfrak{g} / \mathfrak{g}^\tau = \mathbb{R} - \mathrm{rank}\ \mathfrak{g}^\sigma / \mathfrak{g}^{\sigma,\tau}$. (10)

Then for any $x \in G/K$, there exists $g \in G_0^\tau$ such that $\sigma(x) = g \cdot x$.

The proof of Theorem (3.1.3) (similar, but easier for Theorem (3.1.2)) completes by showing the existence of $\sigma \in \mathrm{Aut}(G)$ satisfying (3), (9) and (10), for each $\tau \in \mathrm{Aut}(G)$ satisfying (2).

Once we obtain (abstract) results on free multiplicities, then we with to obtain explicit formulae of such branching problems as a second stage. Theorem (3.1.3) asserts the multiplicity freeness of the branching law $\pi|_H$, especially in the case where

$\pi \in \widehat{G}$: holomorphic discrete series of scalar minimal K-type

$H \coloneqq G_0^\tau$: satisfies the condition (2):

This section presents an explicit branching law of $\pi|_H$ in this setting. In particular, we generalize the Kostant-Schmid formula ([71],[55]) which corresponds to the case $\tau = \theta$ (Cartan involution), namely $H = K$.

Let us fix notation. Suppose that G is a simple non-compact connected Lie group of Hermitian type, and that $\tau \in \mathrm{Aut}(G)$ satisfies (2). We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ such that $\mathfrak{t}^\tau \coloneqq \{X \in \mathfrak{t} : \tau X = X\}$ is also a Cartan subalgebra of $\mathfrak{k}^\tau \coloneqq \{X \in \mathfrak{k} : \tau X = X\}$. We fix positive systems $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$ and $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Because τ satisfies (2), the direct sum decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

is stable under τ (complex linear extension). Then we have a direct sum decomposition $\mathfrak{p}^+ = (\mathfrak{p}^+)^\tau \oplus (\mathfrak{p}^+)^{-\tau}$. Let $\Delta((\mathfrak{p}^+)^{-\tau}, \mathfrak{t}^\tau)\left(\subset \sqrt{-1}(\mathfrak{t}^\tau)^*\right)$ be the set of weights of $(\mathfrak{p}^+)^{-\tau}$ with respect to $\mathfrak{t}^\tau$.

The roots α and β are called strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots, say $\{\nu_1, \nu_2, \ldots, \nu_k\}$, such that

i) ν_1 is the highest root among $\Delta((\mathfrak{p}^+)^{-\tau}, \mathfrak{t}^\tau)$,

ii) ν_{j+1} is the highest root in $\Delta((\mathfrak{p}^+)^{-\tau}, \mathfrak{t}^\tau)$ strongly orthogonal to$\nu_1, \ldots, \nu_j$.

We note that

$$k = \mathbb{R} - \mathrm{rank} G/G^\tau.$$

We denote by $V^G(\mu)$ the irreducible highest weight module of G if $(V^G(\mu))^{\mathfrak{p}^+}$ is an irreducible representation of K with highest weight $\mu \in \sqrt{-1}\mathfrak{t}^*$ with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Likewise, $V^H(\nu)$ denotes the irreducible highest weight module of $H = G_0^\tau$ if $(V^H(\nu))^{(\mathfrak{p}^+)^\tau}$ is an irreducible representation of K_0^τ with highest weight $\mu \in \sqrt{-1}(\mathfrak{t}^\tau)^*$ with respect to $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$.

Clearly, $V^G(\mu)$ is of scalar minimal K-type if and only if μ vanishes on the maximal semisimple ideal of $\mathfrak{k}$.

Now we are ready to state an explicitly branching formula:

Theorem (3.1.9)[62]. Let G be a connected non-compact simple Lie group of Hermitian type, and $H \coloneqq G_0^\tau$ the connected component of the fixed point group G^τ of an involution $\tau \in \mathrm{Aut}(G)$ satisfying (2). If $V^G(\mu) \in \widehat{G}$ is a holomorphic discrete series representation of scalar minimal K-type, then

$$V^G(\mu)|_H \simeq \sum_{\substack{a_1 \geq \cdots \geq a_k \geq 0 \\ a_j \in \mathbb{N}}}^{\oplus} V^H\left(\mu|_{\mathfrak{t}^\tau} + \sum_{j=1}^{k} a_j \nu_j\right). \tag{11}$$

If $\tau = \theta$ then $H = K$ and $\dim V^H\left(\mu|_{\mathfrak{t}^\tau} + \sum_{j=1}^{k} a_j \nu_j\right) < \infty$ in this case, (11) coincides with the formula.

Sec(3.2) : Compact quotients of homogeneous spaces and decay of matrix coefficients

G is a locally compact group, K is a compact subgroup of G, and H is a closed subgroup of G. Let θ denote a (left invariant) Haar measure on H.

Let G act continuously by measure preserving transformations on a (noncompact) locally compact space X with an infinite regular Borel measure μ. Consider the regular unitary representation ρ of G on $L^2(X,\mu)$:

$$(\rho(g)f)(x) = f(g^{-1}x); \quad g \in G,\ x \in X,\ f \in L^2(X,\mu).$$

Definition $(\mathbf{3.2.1})[\mathbf{169}]$. We say that the action of G on X is (G,K,H)- *tempered* if there exists a (positive) function $q \in L^1(H,\theta)$ such that

$$|\langle \rho(h)f_1, f_2\rangle| \leq q(h)\|f_1\| \cdot \|f_2\| \tag{12}$$

for any $h \in H$ and any $\rho(K)$-invariant functions $f_1, f_2 \in L^2(X,\mu)$.

Proposition$(\mathbf{3.2.2})[\mathbf{169}]$. *If the action of G on X is (G,K,H)-tempered then $\mu(X - HM) > 0$ and, consequently, $HM \neq X$ for any compact subset M of X.*

Proof. Let M be a compact subset of X. Then there exists a nonnegative K-invariant continuous function f on X with compact support such that $f(x) > 1$ for any $x \in M$. Consider a function

$$\phi = \int_H \rho(h) f d\theta(h), \quad \varphi(x) = \int_H f(h^{-1}x)d\theta(h).$$

(The function φ can be infinite, and if HM is not compact then usually φ is not in $L^2(X,\mu)$.) Since f is continuous, M is compact and $f(x) > 1$ for any $x \in M$, there exists a neighborhood$W$ of e in H such that $f(w^{-1}x) > \frac{1}{2}$ for all $x \in M$ and $w \in W$. Now if $x \in HM$ then $\varphi(x) > \frac{1}{2}\,\theta(W)$ (because if $h^{-1}x \in M$ then $f((hw)^{-1}x) > \frac{1}{2}$ for any $w \in W$). Thus

$$\varphi(x) > \frac{1}{2}\,\theta(W) \text{ for any } x \in HM. \tag{13}$$

Take a compact subset L of H such that

$$\int_{H-L} q(h)d\mu(h) < \frac{1}{2\|f\|}\theta(W) \tag{14}$$

where $\|f\| = sup\{f(x)|\ x \in X\}$. Since the measure μ is Borel and infinite and the support $\operatorname{supp} f$ of f is not compact, there exists a K-invariant set $A \subset X$ such that

$\mu(A) = 1$ and $(L \cdot suppf) \cap A = \emptyset$. Let χ_A denote the characteristic function of A. Then using (12) and (13) we get

$$\int_A \varphi(x)\, d\mu(x) = \int_X (\varphi \cdot \chi_A)(x)\, d\mu(x)$$
$$= \int_H \left(\int_X \big((\rho(h)f)\chi_A\big)(x) d\mu(x) \right) d\theta(h)$$
$$= \int_H \langle \rho(h)f, \chi_A \rangle d\theta(h) = \int_{H-L} \langle \rho(h)f, \chi_A \rangle d\theta(h)$$
$$\le \int_{H-L} q(h) \|f\| \cdot \|\chi_A\| d\theta(h) = \|f\| \int_{H-L} q(h) d\theta(h)$$
$$< \frac{1}{2}\, \theta(W). \qquad (15)$$

The equality $\mu(A) = 1$ and the inequalities (13) and (15) imply that $\mu(A - HM) > 0$ and, consequently $\mu(X - HM) > 0$.

Proposition $(\mathbf{3.2.3})[\mathbf{169}]$**.** Let A be a bounded Borel subset of X. For any x ∈ X, let $\psi_A(x)$ denote the θ-measure of the set $\{h \in H \mid hx \in A\}$. Suppose that the action of G on X is (G, K, H)-tempered.

(a) The function ψ_A is locally integrable, that is

$$\int_B \psi_A(x)\, d\mu(x) < \infty$$

for any bounded Borel subset B of X.

(b) If X is σ-compact then $\psi_A(x) < \infty$ for almost all $x \in X$.

Proof. Clearly (a) implies (b). Let us prove (a). Replacing A by KA and B by KB, we can assume that A and B are K-invariant. Let χ_A and χ_B denote the characteristic functions of A and B. It is easy to see that

$$\psi_A = \int_H \rho(h)\chi_A d\theta(h).$$

Then using (12) we get

$$\int_B \psi_A(x)\, d\mu(x) = \langle \psi_A, \psi_B \rangle = \int_H \langle \rho(h)\chi_A, \chi_B \rangle d\theta(h)$$
$$\le \int_H q(h) \langle \chi_A, \chi_B \rangle d\theta(h) < \infty.$$

In this section G, K, H and θ denote the same as.

corollary $(\mathbf{3.2.3})'[\mathbf{140}]$**.** Show that

(i) $\|\Psi_A\| \leq M$.
(ii)Verify that

$$\int_H \psi_A(x)\, d\mu(x) < M < \infty$$

Proof : Since

$$|\psi_A| = \left|\int_H \rho(h)\, \chi_A d\theta(h)\right| \leq \int_H |\rho(h)||\chi_A|\, d\theta(h) \leq \int_H q(h)|\chi_A|\, d\theta(h)$$

$$\leq \int_H q(h)\|\chi_A\|\, d\theta(h) = \int_H q(h)\, d\theta(h) = M$$

$$|\psi_A| \leq M$$

Hence we can show that

$$|\psi_A(x) d\mu(x)| \leq \left|\int_H q(h)\right| < \chi_A \quad , \quad \chi_B > d\theta(h)$$

$$\leq \int_H q(h)\|\chi_A\|\|\chi_B\|\ d\theta(h) = \int_H q(h)\, d\theta\ (h) = M < \infty$$

Proposition (3.2.4) [169]. We say that H is (G, K)-tempered if there exists a function $q \in L^1(H, \theta)$ such that

$$|\langle \pi(h) w_1, w_2 \rangle| \leq q(h)\|w_1\| \cdot \|w_2\| \qquad (16)$$

for any $h \in H$, any $\pi(K)$-invariant vectors w_1 and w_2 and any unitary representation π of G without non-trivial $\pi(G)$-invariant vectors.

Let us consider a continuous action of G by measure preserving transformations on a locally compact space X with an infinite regular Borel measure μ, and let us denote by ρ the regular representation of G on $L^2(X, \mu)$. If $a > 0, f \in L^2(X, \mu)$ and $\rho(G)f = f$, then the sets

$$\{x \in X \mid f(x) > a\} \quad \text{and} \quad \{x \in X \mid f(x) < -a\}$$

have finite measure and they are G-invariant (modulo sets of measure 0). Hence if X has no G-invariant subsets of finite nonzero measure and the subgroup H is (G, K)-tempered then the action of G on X is (G, K, H)- tempered.

Remark (3.2.5) [169]. Let

$$\pi = \int_y \pi_y\, d\sigma(y)$$

be a decomposition of π into a continuous sum of irreducible unitary representations, and let

$$W = \int_Y W_y\ d\sigma(y), \qquad w_1 = \int_Y w_{1y}\, d\sigma(y), \qquad w_2 = \int_Y w_{2y}\, d\sigma(y),$$

$w_{1y}, w_{2y} \in W_y$, be corresponding decompositions of the space W of the representation π and of vectors $w_1, w_2 \in W$. Suppose that for all $y \in Y$

$$|\langle \pi_y(h) w_{1y}, w_{2y} \rangle| \leq q(h) \|w_{1y}\| \cdot \|w_{2y}\|.$$

Then using Cauchy-Schwartz inequality we get

$$\begin{aligned}
|\langle \pi(h) w_1, w_2 \rangle| &= \left| \int_Y \langle \pi_y(h) w_{1y}, w_{2y} \rangle \, d\sigma(y) \right| \\
&\leq q(h) \int_Y \|w_{1y}\| \cdot \|w_{2y}\| \, d\sigma(y) \\
&\leq q(h) \sqrt{\int_Y \|w_{1y}\|^2 d\sigma(y) \int_Y \|w_{2y}\|^2 \, d\sigma(y)} \\
&= q(h) \|w_1\| \cdot \|w_2\|.
\end{aligned}$$

Thus H is (G, K)-tempered if and only if the inequality (16) is true for any $h \in H$, any $\pi(K)$-invariant vectors w_1 and w_2 and any non-trivial irreducible unitary representation π of G.

Let us now give some examples of (G, K)-tempered subgroups.We give only indications of the proofs because more precise and general results are obtained by Hee Oh ([74]).

Examples (3.2.6) [169].

(a) Let G be a connected semisimple Lie group having Kazhdan's property (T) and K a maximal compact subgroup of G. Then any commutative diagonalizable subgroup H of G is (G, K)-tempered. To show this it is enough to use Howe-Moore estimates which provide uniform exponential decay for matrix coefficients corresponding to K-invariant vectors and irreducible nontrivial unitary representations of semisimple groups with property (T).

(b) Let $G = \mathrm{SL_n}(\mathbb{R})$, $K = \mathrm{SO}(n)$, and α_n the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Suppose that $n \geq 4$. Then the subgroup $H = \alpha_n(\mathrm{SL}(2, \mathbb{R}))$ is (G, K)-tempered. Let us show this in the case where $n = 4$ and

$$\alpha_4 = (d_t) = r_t, \;\; d_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \;\; r_t = \begin{pmatrix} e^{3t} & & & 0 \\ & e^t & & \\ & & e^{-t} & \\ 0 & & & e^{-3t} \end{pmatrix}$$

It is well known that the restriction of any nontrivial irreducible unitary representation π of $\mathrm{SL}_4(\mathbb{R})$ to the subgroup

$$F = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| \, A \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

does not contain complementary series. But r_t belongs to the subgroup

$$\left\{\begin{pmatrix} a & 0 & 0 & 0 & b \\ 0 & & & & 0 \\ 0 & & A & & 0 \\ c & & & & d \end{pmatrix} \middle| A \in \mathrm{SL}_2(\mathbb{R}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

which is the direct product of two conjugates of F.

Using these facts and formulas for matrix coefficients of the principal series of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ we easily get that for some $c > 0$

$$|\langle \pi(r_t)w, w\rangle| \leq ce^{-4t}t^2 \cdot |\langle w, w\rangle|,\ \ t \geq 0,$$

for any $\pi(K)$-invariant vector w. Now it remains to notice that the function

$$f(k_1 d_t k_2) = e^{-4t}t^2,\ k_1, k_2 \in \mathrm{SO}(2),\ t \geq 0,$$

is integrable on $\mathrm{SL}_2(\mathbb{R})$ because the Haar measure of the set

$$\{k_1 d_t k_2 | k_1, k_2 \in \mathrm{SO}(2), \qquad 0 \leq t \leq T,\}$$

is asymptotically ce^{2T} when $T \to +\infty$.

(c) Let L be a connected simple Lie group, $n \geq 3$, $\varphi : L \to \mathrm{SL_n}(\mathbb{R})$ an n-dimensional representation of L, and $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_i$ a decomposition of φ into the sum of irreducible representations of L. Let us denote by β the sum of the positive roots of L with respect to a maximal $\mathbb{R}$ -split torus $S \subset L$ and an ordering on the character group $X(S)$ of S, and by χ_j the highest weight of the representation $\varphi_j, 1 \leq j \leq i$. Then using arguments similar to those from the example (b) one can prove that the subgroup $\varphi(L)$ is $(\mathrm{SL_n}(\mathbb{R}), \mathrm{SO}(n))$-tempered whenever

$$\sum_{j \in \mathcal{J}} \chi_j > \beta(1 + \varepsilon) \text{for some } \varepsilon > 0, \qquad \text{where } \mathcal{J} = \{j | \dim \varphi_j \geq 2\}.$$

From this we easily deduce the existence of $N > 0$ such that if

$$\sum_{j \in \mathcal{J}} \dim \varphi_j > N$$

then $\varphi(L)$ is $(\mathrm{SL_n}(\mathbb{R}), \mathrm{SO}(n))$-tempered. (Let us note that $\sum_{j \in \mathcal{J}} \dim \varphi_j$ is the codimension in $\mathbb{R}\ \ n$ of the subspace of $\varphi(L)$-invariant vectors.)

As usual we say that a continuous action of a locally compact group G on a locally compact space X is proper if, for every compact subset L ⊂ X, the set {g ∈ G | gL ∩ L ≠ ∅} is compact. If G acts properly on X then the quotient space G\X is Hausdorff. We say that the action of G on X is compact if there exists a compact subset L of X such that $X = GL$. For proper actions this property is equivalent to the compactness of $G\backslash X$.

It is well known and easy to check that, for any locally compact group G and any closed subgroups P and Q of G, the following conditions are equivalent:

(I) the action of P on G/Q by left translations is proper (resp. cocompact);
(II) the action of Q on $P\backslash G$ by right translations is proper (resp. cocompact);
(III) the action $(p,q)g = pgq^{-1}, p \in P, q \in Q, g \in G,$ of $P \times Q$ on G is proper (resp. cocompact).

It is natural to call the equivalence $(I) \Leftrightarrow (II)$ the duality principle.

Theorem $(\mathbf{3.2.7})$[$\mathbf{169}$]. Let G be a unimodular locally compact group, H a closed subgroup of G, and F a closed subgroup of H. Suppose that H is (G,K)-tempered for some compact subgroup K of **G**.

(a) If Γ is a discrete subgroup of G such that the volume of $\Gamma\backslash G$ with respect to Haar measure is infinite then the action of Γ on G/F by left translations is not cocompact.

(b) If F is not compact then there are no discrete subgroups Γ of G such that Γ acts properly on G/F by left translations and the quotient $\Gamma\backslash(G/F)$ is compact.

Proof.

(a) The group G is unimodular. Therefore the action of G on $\Gamma\backslash G$ by right translations preserves Haar measure μ. Since $\mu(\Gamma\backslash G) = \infty$ there are no G-invariant subsets in $\Gamma\backslash G$ of finite measure. Hence the action of G on $\Gamma\backslash G$ is (G,K,H)-tempered. Now applying Proposition (3.2.2) we get that the action of H on $\Gamma\backslash G$ and, consequently, the action of F on $\Gamma\backslash G$ are not cocompact. From this, using the above mentioned duality principle, we deduce that the action of Γ on G/F is not cocompact.

(b) In view of (a) it is enough to consider the case where $\mu(\Gamma\backslash G) < \infty$, but in this case F can not act properly on $\Gamma\backslash G$ because any continuous action of a noncompact group by transformations preserving a finite nonzero regular Borel measure is not proper.

Combining Theorem (3.2.7) with examples (b) and (c) from [2] we get the following two corollaries.

Corollary $(\mathbf{3.2.8})$[$\mathbf{169}$]. Let α_n denote the n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Let $G = \mathrm{SL}_n(\mathbb{R}), H = \alpha_n(\mathrm{SL}_2(\mathbb{R})) \subset G$, and F a closed subgroup of H. Suppose that $n \geq 4$. Then for G, H and F the statements (a) and (b) in Theorem (3.2.1) are true. In particular G/H has no compact quotients by discrete subgroups.

Corollary $(\mathbf{3.2.9})$[$\mathbf{169}$]. Let L be a connected simple Lie group, $n \geq 3$, and let $\varphi: L \to \mathrm{SL}_n(\mathbb{R})$ be an n-dimensional representation of L such that the condition from example (c) of 2 is satisfied. Then the statements (a) and (b) of Theorem (3.2.7) are true for $G = \mathrm{SL}_n(\mathbb{R}), H = \varphi(L)$ and a closed subgroup F of H.

Let H be a locally compact second countable group acting continuously on a locally compact second countable space X with an H-quasi-invariant Borel measure μ. Let θ be a left invariant Haar measure on H. Then the following conditions are equivalent:

(a) for almost all (with respect to μ) points $x \in X$, the orbit Hx is closed in X and the stabilizer $H_x = \{h \in H \mid hx = x\}$ is compact;

(b) for almost all $x \in X$, the stabilizer H_x is compact and the natural map $hH_x \mapsto hx$ of H/H_x onto Hx is a homeomorphism;

(c) for almost all $x \in X$, the natural map $h \mapsto hx$ of H onto Hx is proper or, in other words, the set $\{h \in H \mid H_x \in A\}$ is bounded in H for any bounded subset A of X;

(d) for almost all $x \in X$ and any bounded subset A of X, the θ- measure of the set $\{h \in H \mid H_x \in A\}$ is finite.

The equivalences $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (c)$ are standard facts about group actions. The implication $(c) \Rightarrow (d)$ is trivial. To prove $(d) \Rightarrow (c)$ let us consider a bounded neighborhood U of e in H. Then

$$\{h \in H \mid hx \in UA\} = U\{h \in H \mid hx \in A\}.$$

Therefore if $\{h \in H \mid Hx \in A\}$ is unbounded then $\{h \in H \mid hx \in UA\}$ has infinite measure. It remains to notice that if A is bounded then UA is also bounded.

If the conditions (a)–(d) are satisfied then we say the action of H on X is measurable proper. It is easy to see that if the action of H on X is measurably proper then almost all components in the decomposition of μ into H-ergodic measures are supported on closed H-orbits Hx with compact stabilizers H_x. In particular if the measure μ is H-ergodic then there exists $x \in X$ such that $\mu(X - Hx) = 0$, Hx is closed in X and H_x is compact. Let us also note that if H acts measurably proper on X and F is a closed subgroup of H then the action of F is also measurably proper.

Sec(3.3) : Branching Laws for Unitary Representations and Minimal Nilpotent Orbits

Let G be a reductive Lie group and G' a reductive subgroup of G. We denote $\widehat{G}$ the unitary dual of G, the equivalence classes of irreducible unitary representations of G. Likewise $\widehat{G'}$ for G'. If $\pi \in G'$, then the restriction $\pi|_{G'}$ is not necessarily irreducible decomposition formula:

$$\pi|_{G'} = \int_{\widehat{G'}}^{\oplus} m_\pi(\tau)\tau d(\mu)\tau \qquad (\text{ direct integral }), \tag{17}$$

where $m_\pi \in \mathbb{N} \cup \{\infty\}$ and μ is Borel measure on G'.

We denote by $\mathfrak{g}_0$ the Lie algebra of G. The orbit method due Kirillov-Kostant in the unitary representation theory of Lie group indicates that the coadhoint representation $Ad^*\colon \mathrm{G} \longrightarrow \mathrm{GL}(\mathfrak{g}_0^*)$ often has a surprising intimate relation with the unitary dual G'. It works perfectly for simply connected nilpotent Lie groups. For real reductive Lie groups G, known examples suggest that the set of coadjoint orbits $\sqrt{-1}\mathfrak{g}_0^*/\mathrm{G}$ (with certain integral conditions) still gives a fairly good approximation of the unitary dual $\widehat{G}$.

Here is a rough sketch of a unitary representation π_λ of G, attached to an elliptic element $\lambda \in \sqrt{-1}\mathfrak{g}_0^*$: The elliptic coadjoint orbit $\mathcal{O}_\lambda = Ad^*(\mathrm{G})\lambda$ carries a G-invariant complex structure, and a holomorphic equivariant line $\widetilde{\mathcal{L}_\lambda} \coloneqq \mathcal{L}_\lambda \otimes (\wedge^{top}\, T^*\mathcal{O}_\lambda)^{\frac{1}{2}}$ is defined if λ satisfies some integral condition. Then, we have a Fr$\acute{e}$chet representation of G on the Dolbeault cohomology group $H_{\bar{\partial}}^{S}(\mathcal{O}_\lambda, \widetilde{\mathcal{L}_\lambda})$, where $S \coloneqq \dim_{\mathbb{C}} Ad^*(\mathrm{K})\lambda$ ([80],[84]), and of which a unique dense subspace we can define a unitary representation π_λ of G ([81]) if λ satisfies certain positivity. The unitary representation π_λ is irreducible and non-zero if λ is sufficiently regular. The underlying $(\mathfrak{g}, K)$-module is so called " $A_q(\lambda)$ '' in the sense of Zuckerman-Vogan after certain ρ-shift.

In general, the decomposition (17) contains both discrete and continuous spectrum. The condition for the discrete decomposition (without continuous spectrum) has been studied in [77],[78], especially for π_λ attached to elliptic orbits $\mathcal{O}_\lambda$. It is likely that if $\pi \in \hat{G}$ " attached to '' a nilpotent orbit, which is contained in the limit set of $\mathcal{O}_\lambda$, (Definition (3.3.4))then the discrete decomposability of $\pi|_{G'}$ should be inherited from that of the elliptic case $\pi_\lambda|_{G'}$. We shall see in Theorem (3.3.3) and Theorem (3.3.5) that this is the case in our situation.

There have been a number of attempts to construct representations attached to nilpotent orbits. Among all, rhe Segal-Shale-Wiel representation (or the oscillator representation) of $\widetilde{Sp}(n, \mathbb{R})$, denoted by ϖ', has been best studied, which is supposed to be attached to the minimal nilpotent orbit of $\mathfrak{sp}(n, \mathbb{R})$. The restriction of ϖ' to a reductive dual pair $G' = G_1' G_2'$ gives Howe's correspondence ([76]).

The group $\widetilde{Sp}(n, \mathbb{R})$ is a split group of type C_n, and analogous to ϖ', Kostant constructed a minimal representation of $SO(n, n)$, a split group of type D_n, and then Binegar-Zierau generalized it for $SO(p, q)$ with $p + q \in 2\mathbb{N}$. This representation (precisely, of (p, q)) will be denoted by $\varpi^{p,q}$.

Let $G' = G_1' G_2' = O(p', q') \times O(p'', q''), (p' + p'' = p, q' + q'' = q)$, be a subgroup of $G = O(p, q)$. Our object of study is the branching law $\varpi^{p,q}|_{G'}$. We note that G_1' and G_2' form a mutually centralizing pair of subgroups in G.

It is interesting to compare the feature of the following two cases:
(i)the restriction $\varpi' | G_1' G_2'$ (the Segal-Shale-Wiel representation for type C_n),
(ii)the restriction $\varpi^{p,q} | G_1' G_2'$(the Kostant-Binegar-Zierau representation for type D_n).

The reductive dual pair $(G, G') = (G, G_1', G_2')$ is of the $\otimes$-type in (i), that is, induced from $GL(V) \times GL(W) \longrightarrow GL(V \otimes W)$; is of the $\oplus$-type in (ii), that is, induced from $GL(V) \times GL(W) \longrightarrow GL(V \oplus W)$. On the other hand, both of the restriction in (i) and (ii) are discretely decomposable if one factor G_2' is compact.
On the other hand ϖ' is a highest weight module in (i), while $\varpi^{p,q}$ is not if $p, q > 2$ in (ii).

Let M be an n-dimensional manifold with pseudo-Riemannian metric g_M. We denote by Δ_M the Laplacian on M, and K_M the scalar curvature of M. We set

$$\widetilde{\Delta_M} := \Delta_M - \frac{n-2}{4(n-1)} K_M.$$

Suppose (M, g_M) and (N, g_N) are pseudo-Riemannian manifolds of dimension n. A local diffeomorphism $\Phi: M \longrightarrow N$ is called a conformal map if there exists a positive valued function Ω on M such that $\Phi^* g_N = \Omega^2 g_M$. If Φ is conformal, then we have the following formula ([82]):

$$\Omega^{\frac{n+2}{2}}(\Phi^* \widetilde{\Delta_N} f) = \widetilde{\Delta_M} \left(\Omega^{\frac{n-2}{2}} \Phi^* f\right). \quad (18)$$

Let G be a Lie group acting conformally on M. The action is denoted by $x \mapsto L_h x$ $(h \in G,\ x \in M)$. Then, we have a positive function $\Omega(h, x) \in C^\infty(G \times M)$ such that

$$L_h^* g_M = \Omega(h, \cdot)^2\, g_M, \quad (h \in G).$$

We form a representation ϖ_λ of G, with parameter $\lambda \in \mathbb{C}$, on $C^\infty(M)$ as follows:

$$\varpi_\lambda = (h^{-1}) f(x) = \Omega(h, x)^\lambda f(L_h, x), \ \ (h \in G, f \in C^\infty(M), x \in M). \quad (19)$$

Then formula (18) implies that $\widetilde{\Delta_M} := C^\infty(M) \longrightarrow C^\infty(M)$ is intertwining operator from $\varpi_{\frac{n-2}{2}}$ to $\varpi_{\frac{n+2}{2}}$. Thus, we have constructed a representation of the group G:

Theorem $(\mathbf{3.3.1})[\mathbf{170}]$. Let (M, g_M) be a pseudo-Riemannian manifold, on which a Lie group G acts conformally. Then $\ker \widetilde{\Delta_M}$ is a representation space of G through $\varpi_{\frac{n-2}{2}}$.

We construct irreducible representation of the indefinite orthogonal group $O(p, q)$ $(p, q \geq 2)$, denoted by $\varpi^{p,q}$ and $\pi_{+,\lambda}^{p,q}$, $\pi_{-,\lambda}^{p,q}$, which are supposed to be attached to the minimal nilpotent orbit, and minimal elliptic orbits, respectively.

Let $\mathbb{R}^{p,q}$ be a manifold $\mathbb{R}^{p,q}$ equipped pseudo-Riemannian metric

$$ds^2 = {dx_1}^2 + \cdots + {dx_p}^2 - {dy_1}^2 - \cdots - {dy_q}^2.$$

We define submanifolds of $\mathbb{R}^{p,q}$ by

$$X(p, q) := \{(x, y) \in \mathbb{R}^{p,q} : |x|^2 - |y|^2 = 1\},$$

$$\Xi \quad := \{(x, y) \in \mathbb{R}^{p,q} : |x| = |y|\} \backslash \{0\},$$

$$M \quad := \{(x, y) \in \mathbb{R}^{p,q} : |x| = |x| = 1\} \simeq S^{p-1} \times S^{q-1}.$$

The indefinite orthogonal grpup $G := O(p, q)$ acts naturally on $\mathbb{R}^{p,q}, X(p, q)$, and Ξ. The action is denoted by $z \mapsto g \cdot z$ $(g \in G, \in \mathbb{R}^{p,q})$. G also acts on M. In fact, the dilation action of $\mathbb{R}_+^\times := \{r \in \mathbb{R} : r > 0\}$ on Ξ commutes with that of G. The induced action of G on $M \simeq \Xi / \mathbb{R}_+^\times$ will be denoted by $x \mapsto L_h\, x$ $(h \in M,\ h \in G)$.

We start with the conformal construction of the " minimal unipotent " representation $\varpi^{p,q}$ for $p + q \in 2\mathbb{N}$ by applying Theorem (3.3.1) to $M = S^{p-1} \times S^{q-1}$ and $G = O(p, q)$.

We equip M with pseudo-Riemannian metric induced from $\mathbb{R}^{p,q}$. Then G acts conformally on M, and $\widetilde{\Delta_M} := \Delta_{S^{p-1}} - \Delta_{S^{q-1}} - \left(\frac{p-2}{2}\right)^2 + \left(\frac{q-2}{2}\right)^2$, with notation in conformal geometry. We set $\nu: \Xi \longrightarrow \mathbb{R}, (x,y) \mapsto |x|$, and define

$$(\varpi^{p,q}(h^{-1})f)(z) := \nu(h \cdot z)^{-\frac{p+q-4}{2}} f(L_h, z),$$

for $h \in G = O(p,q), z \in M = S^{p-1} \times S^{q-1}$, $f \in V := \{f \in C^\infty(S^{p-1} \times S^{q-1}) : \widetilde{\Delta_M} f = o\}$. Then $(\varpi^{p,q}, V)$ is a representation of $G = O(p,q)$ by Theorem (3.3.1). Moreover, $(\varpi^{p,q}, V)$ is a non-zero irreducible representation of G if $p + q \in 2\mathbb{N}$. By comparing the construction of [75], the underlying $(\mathfrak{g}, K)$-moduleV_K is unitarizable with the inner product

$$(f_1, f_2) := \int_M (Df_1)\, \overline{f_2} d\omega, \qquad f_1, f_2 \in V_K$$

Where $D := \sqrt{-\Delta_{S^{p-1}} + \frac{(p-2)^2}{4}}$, and $d\omega$ is the standard measure on M. We use the same notation $\varpi^{p,q}$ to denote the irreducible unitary representation of G.

Next, we consider minimal elliptic coadjoint orbits of $G = O(p,q)$. According to the notation , we write $\pi^{p,q}_{+,\lambda}$ for $\pi_{\lambda f_1}$, and $\pi^{p,q}_{-,\lambda}$ for $\pi_{\lambda f_2}$ attached to elliptic orbits $Ad^*(\mathrm{G})\lambda f_i$ $(i = 1,2)$ for $\lambda \in \mathbb{Z} + \frac{1}{2}(p+q)$ with $\lambda \geq 0$, where $f_1, f_2 \in \sqrt{-1}\mathfrak{g}_0^*$ is defined by

$\langle f_1, X\rangle := -\sqrt{-1}X_{12}$, $\langle f_2, X\rangle := -\sqrt{-1}X_{p+q-1,p+q}$, for $X = (X_{ij})_{1 \leq i,j \leq p+q} \in \mathfrak{g}_0$, where $\mathfrak{g}_0 = \mathfrak{o}(p,q)$. It is convenient to put $\pi^{1,0}_{+,-\frac{1}{2}} = 1$, the trivial representation of $O(1,0) = O(1)$; and $\pi^{1,0}_{+,\frac{1}{2}} = \mathrm{sgn}$, the signature representation of $O(1)$. Then, $\{\pi^{p,q}_{+,\lambda} : \lambda \in A(p,q)\}$ $(\subset \widehat{O(p,q)})$ is the totality of discrete series representations for $X(p,q)$, where we put

$$A(p,q) := \begin{cases} \left\{\lambda \in \mathbb{Z} + \frac{p+q}{2} : \lambda > 0\right\} & (p > 1, q \neq 0), \\ \left\{\lambda \in \mathbb{Z} + \frac{p+q}{2} : \lambda \geq \frac{p}{2} - 1\right\} & (p > 1, q = 0), \\ \emptyset & (p = 1, q \neq 0) \text{or } (p = 0) \\ \left\{-\frac{1}{2}, \frac{1}{2}\right\} & (p = 1, q = 0). \end{cases}$$

We consider the discrete spectrum in the branching law of the minimal unipotent representation $\varpi^{p,q} \in \widehat{G}$ with respect to the reductive dual pair $(G, G') = (O(p,q), O(p',q') \times O(p'',q''))$, where $p' + p'' = p(\geq 2), q' + q'' = q(\geq 2), p + q \in 2\mathbb{N}$, and $(p,q) \neq (2,2)$. We set $A'(p,q) := A(p,q) \cap \{\lambda \in \mathbb{R} : \lambda > 1\}$.

Theorem $(3.3.2)$[170] . The restriction $\varpi^{p,q}|_{G'}$ contains

$$\sum_{\lambda \in A'(p',q') \cap A'(q'',p'')}^{\oplus} \pi_{+,\lambda}^{p',q'} \boxtimes \pi_{-,\lambda}^{p'',q''} \oplus \sum_{\lambda \in A'(q',p') \cap A'(p'',q'')}^{\oplus} \pi_{-,\lambda}^{p',q'} \boxtimes \pi_{+,\lambda}^{p'',q''}$$

as a discrete specrum.

The idea of the proof is based on the conformal embedding

$$X'(p',q') \times X(q'',p'') \hookrightarrow M = S^{p-1} \times S^{q-1}$$

Together with the conformal construction of $\varpi^{p,q}$.

If one of p',q',p'' or q'' is zero, then the restriction $\varpi^{p,q}|_{G'}$ is decomposed discretely into irreducible representations of $G' = O(p',q') \times O(p'',q'')$ by a general criterion. Then, we can determine the branching law $\varpi^{p,q}|_{G'}$ as follows:

Theorem (3.3.3)[170]. Let $p+q \in 2\mathbb{N}$. If $q'' \geq 2$ *and* $q' + q'' = q$, then

$$\varpi^{p,q}|_{O(p,q') \times O(q'')} \simeq \sum_{l=0}^{\infty \; \oplus} \pi_{+,l+\frac{q''}{2}-1}^{p,q'} \boxtimes \pi_{-,l+\frac{q''}{2}-1}^{0,q''}.$$

Suppose $p+q \in 2\mathbb{N}$, $p,q \geq 2$. The annihilator of the representation $\varpi^{p,q}$ is the Joseeph idea. In this sense, $\varpi^{p,q}$ is supposed to be attached to the unique minimal nilpotent coadjoint orbit, denoted by $\mathcal{O}_{min}$ $\left(\subset \sqrt{-1}\mathfrak{g}_0^*\right)$, whose dimension is $2(p+q-3)$.

Definition (3.3.4)[170]. Let M_ν be a family of subsets, parametrized by $\nu \in \mathbb{R}_+$, of a topological space. We denote by $\overline{M}$ the closure of a subset M. Then, "limit set" is defined by

$$\lim_{\nu \downarrow 0} M_\nu \coloneqq \bigcap_{\varepsilon > 0} \overline{\bigcup_{\varepsilon > \nu > 0} M_\nu}.$$

Here is the limit behavior as the elliptic orbits tend to the nilpotent orbits:

Theorem (3.3.5)[170]. If $p \geq 2$ *and* $q \geq 2$, then we have the $Ad^*(G)$-orbit decomposition:

$$\lim_{\nu \downarrow 0} \mathcal{O}_\nu = \mathcal{O}_0 \cup \mathcal{O}_{min} \cup \{0\},$$

where $\mathcal{O}_0$ is a nilpotent orbit of dimension $2(p+q-2)$.

The construction of the representations $\pi_{+,\nu}^{p,q}$, attached to elliptic orbits $Ad(G)\nu f_1$ is built on the polarization (or, equivalently, the θ-stable parabolic subalgebra if we employ Zuckerman-Vogan's construction) depending on the signature of ν.

Let us consider the Dolbeault cohomology group with $\nu = -1$ where the polarization is kept the same as that of positive ν. It turns out that the Dolbeault cohomology group in the same degree (i,e. $p-2$) is still non-zero in this special setting. We write $\lim_{\nu \to -1} \pi_{+,\nu}^{p,q}$ for the underlying $(\mathfrak{g}, K)$-module. Then, the following theorem may be regarded as a quantization of Theorem (3.3.5).

Theorem (3.3.6)[170]. Suppose $p+q \in 2\mathbb{N},\ and\ \ p,q \geq 2$. Then we have

$$\lim_{\nu \longrightarrow -1} \pi_{+,\nu}^{p,q} = \pi_{+,1}^{p,q} + \varpi^{p,q}$$

In the Grothendieck group of $(\mathfrak{g}, K)$-modules.

Let $G^{'} = O(p,q^{'}) \times O(q^{''})(q^{'} + q^{''} = q,\ p+q \in 2\mathbb{N})$.

Theorem (3.3.7)[170]. Let $pr_{\mathfrak{g}\rightarrow\mathfrak{g}'}: \mathfrak{g}^* \longrightarrow \mathfrak{g}'^*$ be the projection dual to $\mathfrak{g}^{'} \longrightarrow \mathfrak{g}$. Then we have

$$pr_{\mathfrak{g}\rightarrow\mathfrak{g}'}(\mathcal{O}_{min}) = Ad^*(G^{'})\{\lambda f_1 + \lambda f_2 : \lambda > 0\}.$$

The branching law in Theorem (3.3.3) may be regarded as a quantization of Theorem (3.3.7).

Chapter 4
Gronwall Inequality with Bounds for Weakly Singular Inequalities Applied to Fractional Differential Equations

We use the transmutation method to reduce the solutions of Volterra of second kind to known solutions of simpler (Riemann-Liouville) equations of the same type. Some examples are given. Using the inequality of Gronwall, we study the dependence of the solution on the order and the initial condition of a fractional differential equation. Some applications to fractional differential and integral equations are also indicated by using singular integral inequalities of Gronwall-Bellman type. Applications to fractional differential and integral equations with sharp bounds are show.

Sec(4.1) : Fractional integral and differential equations involving Erdélyi-Kober operators

Fractional integral, differential and differintegral equations (FIE, FDE, FDIE), involving the Riemann-Liouville ($R-L$) integrals of order $\delta > 0$

$$R^{\delta} y(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1}\, y(x) dt = \frac{x^{\delta}}{\Gamma(\delta)} \int_0^1 (x-\sigma)^{\delta-1}\, y(x\sigma) d\sigma \quad (1)$$

and $\mathrm{R-L}$ derivatives $R^{-\delta} := D^{\delta}$, $\delta > 0$:

$$D^{\delta} y(x) = \begin{cases} \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^{\mathrm{n}} R^{\delta-n} y(x) \text{ if } \delta \text{ noninteger}, n = [\delta]+1 \\ y^{(n)}(x) \qquad \text{if } \delta = n \text{ integer} \end{cases} \quad (2)$$

have been recently solved (see [85-96]). The solutions of the first kind equations $R^{\pm\delta} y(x) = f(x)$ are well known. Abel (1823) was the first to solve effectively such an equation with $\delta = \frac{1}{2}$ (called now Abel integral equation) by means of fractional calculus, thus giving a good motivation for further development of this topic.

Definition $(4.1.1)[125]$. The equations

$$y(x) - \lambda \int_a^x K(x,t) y(t)\, dt = f(x)\ , \quad (3)$$

where $f(x), K(x,t)$ are given functions, λ is aparameter and $y(x)$ is the saught solution, are called Volterra integral equations of second kind.

Solutions of the $\mathbf{R-L}$ fractional integral equation of second kind

$$\tilde{y}(x) - \lambda R^{\delta} \tilde{y}(x) = \tilde{f}(x) \quad (4)$$

have been found, respectively, by Hille and Tamarkin [85] by means of the Laplace transform, Ross and Sachdeva [86] by the techniques of fractional calculus, and many others by means of operational calculus. These solutions can be put in the form

$$\tilde{y}(x) = \tilde{f}(x) + \lambda \int_0^x (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda (x-t)^{\delta}\right] \tilde{f}(t) dt. \quad (5)$$

Analogously, the Cauchy problem for the $\mathbf{R-L}$ fractional differential equation

$$D^{\delta}\tilde{y}(x) - \lambda\tilde{y}(x) = \tilde{f}(x),$$
$$D^{\delta-j}\tilde{y}(x)\big|_{x=0} = b_j, \ \ j = 1,2,\dots,n, n-1 < \delta \le n \quad (6)$$

has a solution:

$$\tilde{y}(x) = \sum_{j=1}^{n} b_j \ x^{\delta-j} E_{\delta,1+\delta-j}\left(\lambda x^{\delta}\right) + \int_0^x (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda (x-t)^{\delta}\right] \tilde{f}(t) dt. \quad (7)$$

Recently AI-Saqabi [91], Tuan and A1-Saqabi [92], using techniques similar to those in [86], have found solutions to more general Volterra equations of second kind, involving both $\mathbf{R-L}$ fractional integrals and derivatives. Thus, the solutions of the equation

$$D^{\mu}\tilde{y}(x) - \lambda R^{v}\tilde{y}(x) = \tilde{f}(x), \ \mu > 0, \ v > 0 \quad (8)$$

have been given by

$$\tilde{y}(x) = \sum_{j=0}^{n-1} \alpha_j \ x^{\mu-j-1} E_{\mu+v,\mu-k}(\lambda x^{\mu+v})$$
$$+ \int_0^x (x-t)^{\mu-1} E_{\mu+v,\mu}[\lambda (x-t)^{\mu+v}] \tilde{f}(t) dt. \quad (9)$$

All the above solutions involve the Mittag-Leffler ($\mathbf{M-L}$)functions [(13), (14), (8), (10)]

$$E_{x,\beta}(x) = \sum_{jk=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0, \qquad \beta > 0 \quad (10)$$

We consider integral and differential equations involving more general operators of fractional integration and differentiation, called $Erdélyi-Kober$ ($\mathbf{E-K}$) fractional *integrals* and derivatives, respectively:

$$I_{\beta}^{\gamma,\delta} \ y(x) = \left[x^{-(\gamma,\delta)} R^{\delta} x^{\gamma} y\left(x^{1/\beta}\right)\right]\Big|_{x\to x^{\beta}}$$
$$= \frac{x^{-\beta(\gamma,\delta)}}{\Gamma(\delta)} \int_0^x \left(x^{\beta} - t^{\beta}\right)^{\delta-1} t^{\beta\gamma} \ y(t) d\left(t^{\beta}\right)$$
$$= \frac{1}{\Gamma(\delta)} \int_0^x (1-\delta)^{\delta-1} \sigma^{\gamma} y\left(x\sigma^{1/\beta}\right) d\sigma \quad (11)$$

and

$$D_{\beta}^{\gamma,\delta} y(x) = \left[x^{-\gamma} D^{\delta} x^{\gamma+\delta} y\left(x^{1/\beta}\right)\right]_{x\longrightarrow x^{\beta}} \quad (12)$$

with real $\delta > 0$, γ and $\beta > 0$. Evidently, for $\gamma = 0, \beta = 1$ Eqs. (11) and (12) turn into Eqs. (1) and (2). The additional parameters $\gamma,\ \beta$ allow more generality and these operators have found a large number of applications in analysis, mathematical physics and other disciplines [99]. Volterra type integral equations of second kind and fractional differential equations, involving **E-K** operators

$$y(x) - \lambda x^{\beta\delta}\ I_{\beta}^{\gamma,\delta} y(x) = f(x),\ x^{-\beta\delta}\ D_{\beta}^{\alpha,\delta} - \lambda y(x) = f(x),\ \delta > 0 \quad (13)$$

arise very often in various problems, especially describing physical processes with aftereffects. However, solutions of Eq. (13) have not been found explicitly by now.

To solve equations of form (13), we apply the transmutation method. It consists in applying suitable transformation operators that allow to find solutions of new and more complicated problems (like Eq. (13)) via their "translation" to known solutions of simpler old problems (in our case - to solutions (5) and (.7) of Eqs. (4) and (6) with **R-L** operators. The transmutation operators used here are fractional calculus operators.

We look for solutions in spaces of weighted continuous Junctions of the form

$$C_{\mu}^{(k)} := \left\{f(x) = x^{p} \tilde{f}(x);\ p > \mu, \tilde{f} \in C^{(k)}\,[0,\infty)\right\},$$

$$C_{\mu} := C_{\mu}^{(0)} \text{ with real } \mu, \quad (14)$$

although spaces L_P, L_P^{μ} can be considered too.

By analogy with Eq. (4), we consider a Volterra integral equation of second kind, involving an **E-K** fractional integral (11) with arbitrary parameters $\delta > 0, \gamma \in \mathbb{R}, \beta > 0$:

$$y(x) - \lambda x^{\beta\delta}\ I_{\beta}^{\gamma,\delta} y(x) = f(x). \quad (15)$$

Similarly to the operational method applied in [85] to Eq. (4), one can apply to Eq. (15) a suitable Laplace type integral transform. In this case it is the so called Borel – Dzrbashjan transform, corresponding to the E-K operator $L = x^{\beta\delta}\ I_{\beta}^{\gamma,\delta}$ and studied by Dimovski and Kiryakova in 1981, and Kiryakova in 1987 . Eq. (15) can be also reduced to Eq. (4) by means of a suitable substitution. However, we apply the transmutation method since it turns to be more effective and gives an idea how Eq. (3) with more complicated kernels $K(x,t)$ can be attacked.

Theorem $(\mathbf{4.1.2})[\mathbf{125}]$**.** The unique solution $y(x) \in C_{\beta\mu}, \mu \geq \max\{0, -\gamma\} - 1$ of the FIE of the second kind Eq. (4.1.15), i.e.

$$y(x) - \lambda x^{-\beta\gamma} = \int_0^x \frac{\left(x^{\beta} - t^{\beta}\right)^{\delta-1}}{\Gamma(\delta)} t^{\beta\gamma}\ y(t) d\left(t^{\beta}\right) = f(x) \quad (16)$$

with f $\in C_{\beta\mu}$, $\mu \geq \max\{0,-\gamma\}-1$, has the explicitjbrm of a convolutional type integral:

$$y(x) = f(x) + \lambda x^{-\beta\gamma}\int_0^x \left(x^\beta - t^\beta\right)^{\delta-1} E_{\delta,\delta}\left[\lambda\left(x^\beta - t^\beta\right)^\delta\right] t^{\beta\gamma} f(t) d\left(t^\beta\right). \quad (17)$$

Proof. The homogeneous equations (15) and (16) ($f \equiv 0$) have the only trivial solution $y \equiv 0$ and this yields the uniqueness of the solution $\in C_{\beta\mu}$ in the nonhomogeneous case. In fact, if we suppose that Eq. (15) and (16) have two different solutions $y_1(x)$ and $y_2(x)$ in $C_{\beta\alpha}$, then $\tilde{y}(x) = y_1(x) - y_2(x) \equiv 0$ is the solution of the corresponding homogeneous equation.

To use the transmutation method we need transformations, relating the operators

$$L_{\delta,\beta} = x^{\beta\gamma} I_\beta^{\gamma,\delta}, \;\; L_{\delta,1} = x^\beta I_1^{\gamma,\delta}, \;\; R^\delta = x^\delta I_1^{0,\delta}.$$

They are given by the **E-K** fractional integral

$$T = I_1^{0,\gamma} \text{ in the form } T\Phi(x) := x^{-\gamma}\int_0^x \frac{(x-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\tau)d\tau. \quad (18)$$

and by the mapping

$$\Omega^{-1}: C_\mu \mapsto C_{\beta\mu}, \quad \Omega^{-1}: f(x) \mapsto f\left(x^\beta\right), \; \beta > 0. \quad (19)$$

Let us have in mind that **E-K** fractional integrals (11) preserve spaces (14) , in a sense that

$$I_\beta^{\gamma,\delta}: C_{\beta\mu} \mapsto C_{\beta\mu}^{(n)} \subset C_{\beta\mu} \text{ for } \mu \geq -(\gamma+1), \;\; n-1 < \delta \leq n$$

and whence for the considered operators we have

$$R^\delta = x^\delta I_1^{0,\delta}: C_\mu \mapsto C_{\mu+\delta}^{(n)} \subset C_\mu \text{ with } \mu \geq -1;$$
$$T = I_1^{0,\gamma}: C_\mu \mapsto C_\mu^{(n)} \subset C_\mu \text{ with } \mu \geq -1;$$
$$L_{\delta,1} = x^\delta I_1^{\gamma,\delta}: C_\mu \mapsto C_{\mu+\delta}^{(n)} \subset C_\mu \text{ with } \mu \geq -\gamma-1;$$
$$L_{\delta,\beta} = x^{\beta\gamma} I_\beta^{\gamma,\delta}: C_{\beta\mu} \mapsto C_{\beta(\mu+\delta)}^{(n)} \subset C_{\beta\mu} \text{ with } \mu \geq -\gamma-1.$$

To encompass the basic spaces C_μ, $C_{\beta\mu}$ in both cases, depending on whether $\gamma \geq 0$ or $\gamma \leq 0$, we deal with the denotations C_μ, $C_{\beta\mu}$, with $\mu \geq \max\{0,-\gamma\}-1$.

The techniques of fractional calculus allow to establish that

$$TR^\delta = I_1^{0,\gamma}\left(x^\delta I_1^{0,\delta}\right) = x^\delta I_1^{\delta,\gamma} I_1^{0,\delta} = x^\delta I_1^{0,\gamma+\delta},$$
$$L_{\delta,1}T = x^\delta I_1^{\gamma,\delta} I_1^{0,\gamma} = x^\delta I_1^{0,\gamma+\delta},$$

i.e. the following similarity relations in C_μ, $\mu \geq \max\{0,-\gamma\}-1$:

$$TR^{\delta} = L_{\delta,1}T, \text{ i.e. } T: R^{\delta} = x^{\delta} I_1^{0,\delta} \mapsto L_{\delta,1} = x^{\delta} I_1^{\gamma,\delta}, \gamma \in \mathbb{R}, \tag{20}$$

$$\Omega^{-1}L_{\delta,1} = L_{\delta,\beta}\Omega^{-1}, \text{ i.e. } \Omega^{-1}: L_{\delta,1} \mapsto L_{\delta,\beta}. \tag{21}$$

This means that the transmutation operator from Eqs. $(4)-(15)$ in C_{μ} is

$$T^{*} = \Omega^{-1}T: R^{\delta} \longrightarrow L_{\delta,\beta}, \text{ since } T^{*}R^{\delta} = L_{\delta,\beta}T^{*}. \tag{22}$$

The above similarity relations can be illustrated also by the following commutative diagram, where we have denoted $\mu \geq \max\{0, -\gamma\} - 1$:

$$\begin{array}{ccc} C_{\mu} & \overset{R^{\delta}}{\rightarrow} & C_{\mu} \\ T \big\downarrow & & \big\downarrow T \\ C_{\mu} & \underset{L_{\delta,1}}{\longrightarrow} & C_{\mu} \\ \Omega^{-1} \big\downarrow & & \big\downarrow \quad \Omega^{-1} \\ C_{\beta\mu} & \underset{L_{\delta,\beta}}{\longrightarrow} & C_{\beta\mu} \end{array}$$

For simplicity, let us consider first equation (15) and (16) with $\beta = 1$. Denoting $\tilde{y}(x) := y(x)$, $T\tilde{f}(x) := f(x)$, from relation (20) we observe that T transforms the simpler Eq. (4.1.4) into the **E-K** equation, namely

$$T: \tilde{y}(x) - \lambda R^{\delta}\tilde{y}(x) = \tilde{f}(x) \mapsto y(x) - \lambda L_{\delta,1}y(x) = f(x),$$

that is, it transforms also the known solution (5) into the saught solution of Eq. (15)):

$$\begin{aligned} y(x) &= T\,\tilde{y}(x) \\ &= T\left\{\tilde{f}(x) + \lambda\int_0^x (x-t)^{\delta-1}\,\mathrm{E}_{\delta,\delta}\left[\lambda(x-t)^{\delta}\right]\tilde{f}(x)dt\right\} \\ &= f(x) + \lambda x^{-\gamma}\int_0^x \frac{(x-\tau)^{\gamma-1}}{\Gamma(\gamma)}\left[\int_0^x (\tau-t)^{\delta-1}\,\mathrm{E}_{\delta,\delta}\left[\lambda(\tau-t)^{\delta}\right](T^{-1}f)(t)dt\right]d\tau \\ &= f(x) + \lambda x^{-\gamma}\int_0^x T^{-1}f(t)dt\left[\int_t^x \frac{(x-\tau)^{\gamma-1}}{\Gamma(\gamma)}(\tau-t)^{\delta-1}E_{\delta,\delta}\left[\lambda(\tau-t)^{\delta}\right]d\tau\right] \\ &:= f(x) + \lambda x^{-\gamma}\int_0^x T^{-1}f(t)\,F(x-t)dt \end{aligned} \tag{23}$$

To evaluate the inner integral, denoted by $F(x-t)$, we substitute $x - t := y$, $\tau - t := \theta|_0^y$, $d\tau = d\theta$ and obtain

$$F(y) = \int_0^y \frac{(y-\theta)^{\gamma-1}}{\Gamma(\gamma)}\theta^{\delta-1}E_{\delta,\delta}\left[\lambda\theta^{\delta}\right]d\theta = y^{\gamma+\delta-1}E_{\delta,\gamma+\delta}\left[\lambda y^{\delta}\right]$$

$$= (x-t)^{\gamma+\delta-1} E_{\delta,\gamma+\delta}\left[\lambda (x-t)^{\delta}\right] = F(x-t),$$

according to a known formula for fractional integrals of **M-L** functions.

$$R^{\gamma}\{y^{\mu-1} E_{\delta,\mu}\left[\lambda y^{\delta}\right]\} = y^{\mu+\gamma-1} E_{\delta,\mu+\gamma}\left(\lambda y^{\delta}\right). \tag{24}$$

Further, to deal with the term $T^{-1}f(t)$ in Eq. (23), we observe that $(T^{-1}f(t) = D_1^{0,\gamma} f(t) = D_1^{\gamma}\, t^{\gamma} f(t)$. Then, denoting $G(t) = t^{\gamma} f(\,t\,)$, we get

$$y(x) = f(x) + \lambda x^{-\gamma} \int_0^x F(x-t)\, D_t^{\gamma}[t^{\gamma} f(t)]dt$$

$$= f(x) + \lambda x^{-\gamma} \int_0^x F(x-t)\, D_t^{\gamma}[G(t)]dt$$

$$= f(x) + \lambda x^{-\gamma} \int_0^x D_{x-t}^{\gamma}\, [F(x-t)]G(t)dt$$

$$= f(x) + \lambda x^{-\gamma} \int_0^x D_{x-t}^{\gamma} \left[(x-t)^{\gamma+\delta-1} E_{\delta,\gamma+\delta}\left[\lambda (x-t)^{\delta}\right]\right] t^{\gamma} f(t)dt,$$

where the following auxiliary assertion, valid for the Duhamel convolution

$$F \star G(x) = \int_0^x F(x-t)\, G(t)dt = \int_0^x F(t)\, G(x-t)dt$$

has been used:

$$F \star (R^{\gamma} G) = (R^{\gamma} F) \star G \quad \text{and} \quad F \star (D^{\gamma} G) = (D^{\gamma} F) \star G.$$

Formula (4.1.24) with $\mu = \delta$, $y = x - t$, gives

$$D_{x-t}^{\gamma}[F(x-t)] = (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda (x-t)^{\delta}\right]$$

and finally the solution $y(x)$ of Eqs. (15) and (16) takes the form

$$y(x) = f(x) + \lambda x^{-\gamma} \int_0^x (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda (x-t)^{\delta}\right] t^{\gamma} f(t)dt, \tag{25}$$

i.e. expression (17) with $\beta = 1$.

To transfer this result to the case of arbitrary $\beta > 0$, we apply mapping (19):

$$\Omega^{-1}: y(x) \mapsto y\left(x^{\beta}\right) \coloneqq \tilde{y}(x), \quad f(x) \mapsto f\left(x^{\beta}\right) \coloneqq \tilde{f}(x)$$

and use relation (4.1.21). Thus,

$$\Omega^{-1} y(x) + \lambda \Omega^{-1} L_{\delta,1} y(x) = \tilde{y}(x) + \lambda \mathrm{L}_{\delta,\beta} \tilde{y}(x) = \tilde{f}(t),$$

that is, the image $\tilde{y}(x) = \Omega^{-1} y(x) \in C_{\beta\mu}$, of $y(x)$ given by Eq. (25), is the unique solution of **FIE** (15) and (16) with arbitrary $\beta > 0$:

$$\tilde{y}(x) = \tilde{f}(x) + \lambda x^{-\beta\gamma} \int_0^x \left(x^{\beta} - t^{\beta}\right)^{\delta-1} E_{\delta,\delta}\left[\lambda \left(x^{\beta} - t^{\beta}\right)^{\delta}\right] t^{\beta\gamma} \tilde{f}(t) d\left(t^{\beta}\right).$$

Similarly, by means of a transmutation operator T we can find the solutions of **FDEs** involving Erdélyi − Kober derivatives of the form (see Eq. (12),):

$$\mathfrak{D} = x^{-\beta\delta} D_{\beta}^{\alpha,\beta}\ ,\ \delta > 0 \tag{26}$$

as transformations $T\tilde{y}(x)$ of the known solutions (47) of Eq. (6).

First we need an auxiliary result.

Lemma $(\mathbf{4.3.3})[\mathbf{125}]$. In $C_\mu^{(n)},\ \mu \geq -1$ the following relation between fractional derivatives D^δ and $\mathfrak{D}$, $\beta = 1$ via the transmutation operator

$$T = I_1^{0,\alpha+\delta} : C_\mu^{(i)} \mapsto C_\mu^{(i)},\ \mu \geq -1\ , i = 0, 1, 2, \dots \tag{27}$$

holds:

$$TD^\delta \tilde{y}(x) = \mathfrak{D}\, \tilde{y}(x) - \sum_{k=1}^{n} b_k \frac{x^{-k}}{\Gamma(\alpha+\delta-k+1)}. \tag{28}$$

In fact, if we put $y(x) := D^\delta \tilde{y}(x), \tilde{y} \in C_\mu^{(n)}$in relation (20) with $\gamma = \alpha + \delta$ and apply the **E-K** derivative $\mathfrak{D} = x^{-\delta} D_1^{\alpha,\delta} = \left(x^\delta I_1^{\alpha+\delta,\delta}\right)^{-1}$, we obtain

$$\left(x^{-\delta} D_1^{\alpha,\delta}\right) T\left(R^\delta D^\delta\right) \tilde{y}(x) = \left(x^\delta I_1^{\alpha+\delta,\delta}\right)^{-1} TD^\delta \tilde{y}(x),$$

that means

$$TD^\delta \tilde{y}(x) = \mathfrak{D}\, T\left(R^\delta D^\delta\right) \tilde{y}(x)\ .$$

Next we have in mind that $D^\delta R^\delta \tilde{y}(x) = \tilde{y}(x)$, but

$$R^\delta D^\delta \tilde{y}(x) = \tilde{y}(x) - \sum_{k=1}^{n} b_k \frac{x^{\delta-k}}{\Gamma(\delta-k+1)} \text{ with } b_k = D^{\delta-k}\, \tilde{y}(0), k = 1, \dots, n$$

and also,

$$\mathfrak{D}\, T\left\{x^{\delta-k}\right\} = \frac{\Gamma(\delta-k+1)}{\Gamma(\alpha+\delta-k+1)} \{x^{-k}\},$$

to obtain Eq. (28). Especially, if all $b_k = 0, k = 1, \dots, n$, i.e. if all the initial data are zero, then T is a similarity relation: $TD^\delta = \mathfrak{D}T$.

Theorem $(\mathbf{4.1.4})[\mathbf{125}]$**.** The general solution of the **E-K** fractional differential equation of second kind with $f \in C_{\beta\mu}, \mu \geq \max\{0, -\alpha-\delta\} - 1$,

$$x^{-\beta\delta} D_\beta^{\alpha,\delta} y(x) - \lambda y(x) = f(x) \tag{29}$$

in the space $C_{\beta\mu}^{(n)}$ $, \mu \geq \max\{0, -\alpha-\delta\} - 1, n \in \mathbb{N}, n-1 < \delta \leq n,$ has the form

$$y(x) = \sum_{j=1}^{n} b_j\, x^{\beta(\delta-j)} E_{\delta,\alpha+2\delta-j+1}\left(\lambda x^{\beta\delta}\right)$$
$$+ x^{-\beta(\alpha+\delta)} \int_0^x \left(x^\beta - t^\beta\right)^{\delta-1} t^{\beta(\alpha+\delta)} E_{\delta,\delta}\left[\lambda\left(x^\beta - t^\beta\right)^\delta\right] f(t) d\left(t^\beta\right) \tag{30}$$

with arbitrary constants $b_j,\ j = 1, \dots, n$ depending on the initial value data.

Proof. For simplicity we assume that $\beta = 1$.

Consider **R-L** fractional differential equation (6) and apply to both sides **E-K** transmutation operator (27) , denoting $y(x) = T\tilde{y}(x), F(x) = T\tilde{F}(x)$ .According to Lemma (4.1.1), Eq. (28), we obtain

$$\mathfrak{D}y(x) - \lambda y(x) = F(x) + \sum_{k=1}^{n} b_k \frac{x-k}{\Gamma(\alpha+\delta-k+1)} \coloneqq f(x),$$

that is, Eq. (29). Thus, its solution is the T-image of the solution (7) with

$$\tilde{F}(x) = T^{-1}F(x) = T^{-1}\left\{f(x) - \sum_{k=1}^{n} b_k \frac{x^{-k}}{\Gamma(\alpha+\delta-k+1)}\right\},$$

namely

$$y(x) = T\left\{\sum_{j=1}^{n} b_j x^{\delta-j} E_{\delta,1+\delta-j}\left(\lambda x^{\delta}\right) + \int_0^x (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda(x-t)^{\delta}\right]\tilde{F}(t)dt\right\}$$

$$= \left\{\sum_{j=1}^{n} b_j T\left[E_{\delta,1+\delta-j}\left(\lambda x^{\delta}\right)\right]\right\}$$

$$+T\left\{\int_0^x (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda(x-t)^{\delta}\right]\tilde{F}(t)dt\right\} \coloneqq A + B.$$

To find

$$T\left[x^{\delta-j} E_{\delta,1+\delta-j}\left(\lambda x^{\delta}\right)\right] = x^{-(\alpha+\delta)} \int_0^x \frac{(x-\tau)^{\alpha+\delta-1}}{\Gamma(\alpha+\delta)} \tau^{\delta-1} E_{\delta,1+\delta-j}\left(\lambda x^{\delta}\right)d\tau,$$

we apply again formula (4.1.24) (with $\gamma = \alpha+\delta, \mu = \delta - j + 1$) and thus,

$$A = \sum_{j=1}^{n} b_j x^{\delta-j} E_{\delta,\alpha+2\delta-j+1}\left(\lambda x^{\delta}\right).$$

To evaluate B, we observe that this repeated integral is the same as in the second term of Eq. (23) (see the proof of Theorem (4.1.2)), i.e.

$$B = x^{-(\alpha+\delta)} \int_0^x \tilde{F}(x)\left\{(x-t)^{\alpha+2\delta-1} E_{\delta,\alpha+2\delta}\left[\lambda(x-t)^{\delta}\right]\right\}dt$$

$$= x^{-(\alpha+\delta)} \int_0^x T^{-1}\{f(t)\}\left\{(x-t)^{\alpha+2\delta-1} E_{\delta,\alpha+2\delta}\left[\lambda(x-t)^{\delta}\right]\right\}$$

$$-x^{-(\alpha+\delta)} \int_0^x T^{-1}\left\{\sum_{k=1}^{n} b_k \frac{t^{-k}}{\Gamma(\alpha+\delta-k+1)}\right\}$$

$$\times\left\{(x-t)^{\alpha+2\delta-1} E_{\delta,\alpha+2\delta}\left[\lambda(x-t)^{\delta}\right]\right\}dt \coloneqq C - D,$$

where the first term C is the same as in Eq. (23) and further, by the same manipulations as when dealing with $T^{-1}f(t) = D_t^{\gamma}\, t^{\gamma}\, f(t)$ (here $\gamma \longrightarrow \alpha+\delta$), we find similarly to Eq. (25):

$$C = x^{-(\alpha+\delta)} \int_0^x (x-t)^{\delta-1} E_{\delta,\delta}\left[\lambda(x-t)^{\delta}\right]t^{\alpha+\delta} f(t)dt.$$

It remains to work out only the term D involving

$$T^{-1}\left\{\sum_{k=1}^{n} b_k \frac{t^{-k}}{\Gamma(\alpha+\delta-k+1)}\right\} = \sum_{k=1}^{n} b_k \frac{D^{\alpha+\delta}\{t^{\alpha+\delta-k}\}}{\Gamma(\alpha+\delta-k+1)}.$$

Since $D^{\alpha+\delta}\{t^{\alpha+\delta-k}\} = [\Gamma(\alpha+\delta-k+1)/\Gamma(1-k)]t^{-k}$ yields

$$T^{-1}\left\{\sum_{k=1}^{n} b_k \frac{t^{-k}}{\Gamma(\alpha+\delta-k+1)}\right\} = \sum_{k=1}^{n} \frac{b_k t^{-k}}{\Gamma(1-k)} \equiv 0,$$

it follows that $D = 0$ and finally, $y(x) = A + B = A + C$ which gives the form of solution (30) with $\beta = 1$. The conditions $\delta > 0, \mu > -\alpha - \delta - 1$ ensure the convergence of the latter integral.

The solution in the case of arbitrary $\beta > 0$ follows by the transformation Ω^{-1}.

To demonstrate the efficiency of the solutions from Theorems (4.1.2) and (4.1.4) we give some examples. First we take particular right-hand sides (x) .

Example $(\mathbf{4.1.5})[\mathbf{125}]$. Solution (17) of **E-K** integral equation (16) of second kind takes the form:

(i) for $f(x) = x^{\beta p}, p > -\gamma - 1$:

$$y(x) = x^{\beta p}\left[1 + \lambda\Gamma(p+\gamma+1)x^{\beta\delta} E_{\delta,\delta+p+\gamma+1}\left(\lambda x^{\beta\delta}\right)\right]; \tag{31}$$

(ii) for $f(x) = E_{\mu,\nu}\left(\alpha x^{\beta}\right)$, arbitrary $\mu, \nu, \alpha \in \mathbb{R}$:

$$y(x) = E_{\mu,\nu}\left(\alpha x^{\beta}\right) + \lambda x^{\beta(\delta-\gamma)} \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\nu+k\mu)} \left(\alpha x^{\beta}\right) E_{\delta,\delta+k+1}\left(\lambda x^{\beta\delta}\right); \tag{32}$$

(iii) if in the above case $\mu = \nu = 1$, we obtain the solution for $f(x) = exp\left(\alpha x^{\beta}\right)$:

$$y(x) = exp\left(\alpha x^{\beta}\right) + \lambda x^{\beta(\delta-\gamma)} \sum_{k=0}^{\infty} \left(\alpha x^{\beta}\right) E_{\delta,\delta+k+1}\left(\lambda x^{\beta\delta}\right). \tag{33}$$

Example $(\mathbf{4.1.6})[\mathbf{125}]$. FDE (29) with $f(x) = x^{\beta p}, p > -\alpha - \delta - 1$ has solutions (30) of the form

$$y(x) = \sum_{j=1}^{n} b_j\, x^{\beta(\delta-j)} E_{\delta,\alpha+2\delta-j+1}\left(\lambda x^{\beta\delta}\right)$$
$$+\Gamma(\alpha+\delta+p+1) x^{\beta(\delta+p)} E_{\delta,\alpha+2\delta+p+1}\left(\lambda x^{\beta\delta}\right). \tag{34}$$

It is interesting to consider also special cases of Erdélyi – Kober operators. Of course, if $\delta = \beta = 1, \gamma = 0, \alpha = -1$, Eqs. (16) and (29) turn into simplest ones Eqs. (4) and (6) used here as a base.

Example $(\mathbf{4.1.7})[\mathbf{125}]$. Consider now the so-called Dzrbashjan-Gelfond-Leontiev (D-G-L) integrals and derivatives, special cases of the E-K operators (11) , (12) , studied by Dimovski and Kiryakova (1981):

$$l_{\rho,\nu} := x\, I_{\rho}^{\nu-1,1/\rho} = x^{\rho(1/\rho)}\, I_{\rho}^{\nu-1,1/\rho},$$

$$d_{\rho,\nu} := D_{\rho}^{\nu-1,1/\rho} x^{-1} = x^{-\rho(1/\rho)} D_{\rho}^{\nu-\frac{1}{\rho}-1,1/\rho}. \qquad (35)$$

For analytic functions $y(x) = \sum_{k=0}^{\infty} a_k x^k$ these operators have also series representations:

$$l_{\rho,\nu} y(x) = \sum_{k=0}^{\infty} a_k \frac{\Gamma\left(\nu + \frac{k}{\rho}\right)}{\Gamma\left(\nu + \frac{k+1}{\rho}\right)} x^{k+1},$$

$$d_{\rho,\nu} y(x) = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\nu + \frac{k}{\rho}\right)}{\Gamma\left(\nu + \frac{k-1}{\rho}\right)} x^{k-1}. \qquad (36)$$

The corresponding **D-G-L** integral and differential equations of second kind

$$y(x) - \lambda l_{\rho,\nu} y(x) = f(x), \; d_{\rho,\nu} y(x) - \lambda y(x) = f(x)$$

are special cases of Eqs. (16) and (29) with $\delta = \frac{1}{\rho}$, $\beta = \rho$, $\gamma = \nu - 1$, $\alpha = \nu - (1/\rho) - 1$. Then Theorems (4.1.2) and (4.1.4) give their solutions:

$$y(x) = f(x) + \lambda x^{-\rho(\nu-1)} \int_0^x (x^{\rho} - t^{\rho})^{(1/\rho)-1} E_{(1/\rho),(1/\rho)} \times \left[\lambda (x^{\rho} - t^{\rho})^{(1/\rho)}\right] t^{\rho(\nu-1)} f(t) d(t^{\rho}) \qquad (37)$$

and, respectively,

$$y(x) = \sum_{j=1}^{n} b_j \, x^{1-\rho j} E_{(1/\rho),\nu+(1/\rho)-j}(\lambda x)$$

$$+ x^{-\rho(\nu-1)} \int_0^x (x^{\rho} - t^{\rho})^{(1/\rho)-1} \, t^{\rho(\nu-1)} E_{(1/\rho),(1/\rho)} \left[\lambda (x^{\rho} - t^{\rho})^{(1/\rho)}\right] f(t) d(t^{\rho}). \qquad (38)$$

Example $(\mathbf{4.1.8})[\mathbf{125}]$. The so-called Rusheweyh derivatives, defined by means of the Hadamard product (convolution)

$$\mathfrak{D}^{\alpha} y(x) = \left\{\frac{x}{(1-x)^{1+x}}\right\} \circ y(x) = \frac{1}{\Gamma(\alpha+1)} D_1^{-1,\alpha} y(x), \quad \alpha > 0 \qquad (39)$$

are often used in analytic functions theory. The corresponding "differential" equation of second kind

$$x^{-\alpha} \mathfrak{D}^{\alpha} y(x) - \lambda y(x) = f(x) \quad \text{for} \quad y(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad f(x) = \sum_{i=0}^{\infty} c_i x^i \, (40)$$

has the form

$$\sum_{k=0}^{\infty} a_k \left[\frac{\Gamma(\alpha + k + 1)x^{-\alpha}}{\Gamma(\alpha + 1)k!} - \lambda\right] x^k = \sum_{i=0}^{\infty} c_i\, x^i$$

and according to Theorem (4.1.4), solutions

$$y(x) = \sum_{j=1}^{n} b_j\, x^{\alpha-j} E_{\alpha,2\alpha-j}(\lambda x^\alpha) + \sum_{i=0}^{\infty} c_i\, \Gamma(\alpha + i) x^{\alpha+i} E_{\alpha,2\alpha+i}(\lambda x^\alpha), \qquad (41)$$

from where interesting relations between coefficients a_k, a_i ; $k, i = 0,1,2,\ldots$ follow.

Especially, for $\alpha = 1$ we obtain the following solution of the 1st order differential equation $x^{-1} D_1^{-1,1}\, y(x) - \lambda y(x) = y'(x) - \lambda y(x) = f(x)$, $f(x) = \sum_{i=0}^{\infty} c_i\, x^i$:

$$y(x) = y(0) exp(\lambda x) + x \sum_{i=0}^{\infty} c_i\, i!\, x^i E_{1,i+2}(\lambda x). \qquad (42)$$

It follows easily also from the well-known solution $y(x) = \exp(\lambda x) \star f(x)$ if we replace $f(x)$ by its series and evaluate the integrals under the summation sign as Mittg-Leffler functions.

The expressions in the sample formulas (31) (34), (41) -(42) allow efficient numerical procedures for their evaluation.

Sec(4.2) : Gronwall Inequality and Applications to Fractional Differential Equation

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations. The celebrated Gronwall inequality known now as Gronwall–Bellman–Raid inequality provided explicit bounds on solutions of a class of linear integral inequalities.

Theorem (3.2.1) [116]. If

$$x(t) \le h(t) + \int_{t_0}^{t} k(s)x(s)ds, t \in [t_0, T),$$

where all the functions involved are continuous on $[t_0, T\,), T \le +\infty$, and $k(t) \ge 0$, then $x(t)$ satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)\, k(s) exp\left[\int_{s}^{t} k(u)du\right] ds, \qquad t \in [t_0, T).$$

If, in addition, $h(t)$ is nondecreasing, then

$$x(t) \leq h(t) exp\left(\int_{t_0}^{t} k(s)\, ds\right), \qquad t \in [t_0, T).$$

However, sometimes we need a different form, to discuss the weakly singular Volterra integral equations encountered in fractional differential equations. In this section we present a slight generalization of the Gronwall inequality which can be used in a fractional differential equation. Using the inequality, we study the dependence of the solution on the order and the initial condition for a fractional differential equations with Riemann–Liouville fractional derivatives.

We wish to establish an integral inequality which can be used in a fractional differential equation. The proof is based on an iteration argument.

Theorem (4.2.2) [116]. Suppose $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$) and $\mathrm{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $\mathrm{g}(t) \leq M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + \mathrm{g}(t) \int_{0}^{t} (t-s)^{\beta-1} u(s) ds$$

on this interval. Then

$$u(t) \leq a(t) + \int_{0}^{t} \left[\sum_{n=1}^{\infty} \frac{\left(\mathrm{g}(t)\Gamma(\beta)\right)^{\mathrm{n}}}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s)\right] ds, \quad 0 \leq t < T$$

Proof. Let $B\phi(t) = \mathrm{g}(t) \int_0^t (t-s)^{\beta-1} \phi(s) ds,\ t \geq 0$, for locally integrable functions ϕ. Then

$$u(t) \leq a(t) + B(t)$$

Implies

$$u(t) \leq \sum_{k=0}^{n-1} B^k\, a(t) + B^n u(t).$$

Let us prove that

$$B^n u(t) \leq \int_{0}^{t} \frac{\left(\mathrm{g}(t)\Gamma(\beta)\right)^{\mathrm{n}}}{\Gamma(n\beta)} (t-s)^{n\beta-1} u(s) ds \tag{43}$$

and $B^n u(t) \longrightarrow 0$ as $n \longrightarrow +\infty$ for each t in $0 \leq t < T$.

We know this relation (43) is true for $n = 1$. Assume that it is true for some $n = k$. If $n = k + 1$, then the induction hypothesis implies

$$B^{k+1}u(t) = B\left(B^k u(t)\right)$$

$$\leq \mathrm{g}(t)\int_0^t (t-s)^{\beta-1}\left[\int_0^t \frac{\left(\mathrm{g}(s)\Gamma(\beta)\right)^{\mathrm{k}}}{\Gamma(n\beta)}(t-\tau)^{k\beta-1}u(\tau)d\tau\right]ds.$$

Since $\mathrm{g}(t)$ is nondecreasing, it follows that

$$B^{k+1}u(t) \leq (\mathrm{g}(t))^{k+1}\int_0^t (t-s)^{\beta-1}\left[\int_0^t \frac{\left(\Gamma(\beta)\right)^{\mathrm{k}}}{\Gamma(k\beta)}(t-\tau)^{k\beta-1}u(\tau)d\tau\right]ds.$$

By interchanging the order of integration, we have

$$B^{k+1}u(t) \leq (\mathrm{g}(t))^{k+1}\int_0^t \left[\int_\tau^t \frac{\left(\Gamma(\beta)\right)^{\mathrm{k}}}{\Gamma(k\beta)}\ (t-s)^{\beta-1}(s-\tau)^{k\beta-1}ds\right]u(\tau)d\tau$$

$$= \int_0^t \frac{\left(\mathrm{g}(t)\Gamma(\beta)\right)^{\mathrm{k+1}}}{\Gamma\left((k+1)\beta\right)}\ (t-s)^{(k+1)\beta-1}u(s)ds,$$

where the integral

$$\int_\tau^t (t-s)^{\beta-1}\,(s-\tau)^{k\beta-1}ds = (t-\tau)^{k\beta+\beta-1}\int_0^t (1-z)^{\beta-1}\ z^{k\beta-1}dz$$

$$= (t-\tau)^{(k+1)\beta-1}B(k\beta,\beta)$$

$$= \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}(t-\tau)^{(k+1)\beta-1}$$

is evaluated with the help of the substitution $s = \tau + z(t-\tau)$ and the definition of the beta function.

The relation (43) is proved.

Since $\quad B^k u(T) \leq \int_0^t \frac{\left(M\Gamma(\beta)\right)^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}u(s)ds \to 0$ as $n \to +\infty$ for $\mathrm{t} \in [0,\mathrm{T})$, the theorem is proved.

For $\mathrm{g}(t) \equiv b$ in the theorem we obtain the following inequality.

Corollary $(\mathbf{4.2.3})[\mathbf{116}]$. Suppose $b \geq 0,\ \beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds$$

on this interval; then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{\left(b\Gamma(\beta)\right)^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T$$

Corollary (4.2.4)[116]. Under the hypothesis of Theorem (4.2.2), let $a(t)$ be a nondecreasing function on $[0, T)$. Then

$$u(t) \leq a(t) E_\beta\left(g(t)\Gamma(\beta) t^\beta\right),$$

where E_β is the Mittag-Leffler function defined by $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

Proof. The hypotheses imply

$$u(t) \leq a(t) \left[1 + \int_0^t \sum_{n=1}^{\infty} \frac{\left(g(t)\Gamma(\beta)\right)^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} ds \right] = a(t) \sum_{n=0}^{\infty} \frac{\left(g(t)\Gamma(\beta)\right)^n}{\Gamma(n\beta+1)}$$
$$= a(t) E_\beta\left(g(t)\Gamma(\beta) t^\beta\right).$$

The proof is complete.

We will show that our result is useful in investigating the dependence of the solution on the order and the initial condition to a certain fractional differential equation with Riemann–Liouville fractional derivatives.

Let us consider the following initial value problem in terms of the Riemann–Liouville fractional derivatives:

$$D^\alpha y(t) = f(t, y(t)), \tag{44}$$
$$D^{\alpha-1} y(t)|_{t=0} = \eta, \tag{45}$$

where $0 < \alpha < 1$, $0 \leq t < T \leq +\infty$, $f: [0, T) \times R \to R$ and D^α denotes Riemann–Liouville derivative operator.

Riemann–Liouville derivative and integral are defined below [105–107].

Definition (4.2.5)[116]. The fractional derivative of order $0 < \alpha < 1$ of a continuous function $f: R^+ \to R$ is given by

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t) dt$$

provided that the right side is point wise defined on R^+.

Definition (4.2.6)[116]. The fractional primitive of order $\alpha > 0$ of a function $f: R^+ \to R$ is given by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

provided the right side is point wise defined on R^{+}.

The existence and uniqueness of the initial value problem (44) –(45) have been studied in [6]. Also the dependence of a solution on initial conditions has been discussed in [6]. We present the dependence of the solution on the order and the initial condition. We shall consider the solutions of two initial value problems with neighboring orders and neighboring initial values. It is important to note that here we are considering a question which does not arise in the solution of differential equations of integer order.

First, let us reduce the problem (44) –(45) to a fractional integral equation. We obtain

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (46)$$

It is clear that Eq. (46) is equivalent to the initial value problem (44) –(45) .

Theorem (4.2.7) [116]. Let $\alpha > 0$ and $\delta > 0$ such that $0 < \alpha - \delta < \alpha \leq 1$. Let the function f be continuous and fulfill a Lipschitz condition with respect to the second variable; i.e.,

$$|f(t,y) - f(t,z)| \leq L|y-z|$$

for a constant L independent of t, y, z in R. For $0 \leq t \leq h < T$, assume that y and z are the solutions of the initial value problems (44) –(45) and

$$D^{\alpha-\delta} z(t) = f(t, z(t)), \quad (47)$$

$$D^{\alpha-\delta-1} z(t)|_{t=0} = \bar{\eta}, \quad (48)$$

respectively. Then, for $0 < t \leq h$ the following holds:

$$|z(t) - \leq y(t)| \leq A(t) + \int_0^t \left[\sum_{n=1}^{\infty} \left(\frac{\mathrm{L}}{\Gamma(\alpha)} \Gamma(\alpha-1) \right)^n \frac{(t-s)^{n(\alpha-\delta)-1}}{\Gamma(\mathrm{n}(\alpha-\delta))} A(s) \right] ds,$$

where

$$A(t) = \left| \frac{\bar{\eta}}{\Gamma(\alpha-\delta)} t^{\alpha-\delta-1} - \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} \right| + \left| \frac{t^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right| . \|f\|$$
$$+ \left| \frac{t^{\alpha-\delta}}{\alpha-\delta} \left[\frac{1}{\Gamma(\alpha-\delta)} - \frac{1}{\Gamma(\alpha)} \right] \right| . \|f\|$$

and

$$\|f\| = \max_{0 \leq t \leq h} |f(t,y)|.$$

Proof. The solutions of the initial value problem (44) –(45) and (47) –(48) are given by

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, z(\tau)) d\tau$$

and

$$z(t)=\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}+\frac{1}{\Gamma(\alpha-\delta)}\int_0^t(t-\tau)^{\alpha-\delta-1}f(\tau,z(\tau))d\tau,$$

respectively. It follows that

$$\begin{aligned}
&|z(t)-y(t)|\\
&\le\left|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}t^{\alpha-1}\right|\\
&+\left|\frac{1}{\Gamma(\alpha-\delta)}\int_0^t(t-\tau)^{\alpha-\delta-1}f(\tau,z(\tau))d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-\delta-1}f(\tau,z(\tau))d\tau\right|\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-\delta-1}f(\tau,z(\tau))d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-\delta-1}f(\tau,y(\tau))d\tau\right|\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-\delta-1}f\big(\tau,y(\tau)\big)d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\big(\tau,y(\tau)\big)d\tau\right|\\
&\le A(t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-\delta-1}L\,|z(t)-y(t)|d\tau,
\end{aligned}$$

where

$$\begin{aligned}A(t)=\left|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}t^{\alpha-1}\right|+\left|\frac{t^{\alpha-1}}{(\alpha-\delta)\Gamma(\alpha)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right|\cdot\|f\|.\\
+\left|\frac{t^{\alpha-\delta}}{\alpha-\delta}\left[\frac{1}{\Gamma(\alpha-\delta)}-\frac{1}{\Gamma(\alpha)}\right]\right|\cdot\|f\|\end{aligned}$$

An application of Theorem (4.2.2) yields

$$|z(t)-y(t)|\le A(t)+\int_0^t\left[\sum_{n=1}^{\infty}\left(\frac{L}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\right)^n\frac{(t-s)^{n(\alpha-\delta)-1}}{\Gamma\big(n(\alpha-\delta)\big)}A(s)\right]ds$$

and the theorem is proved.

A general theorem of existence and uniqueness for the nonautonomous case $f(t,y)$ can be found in.

Corollary (4.2.8) [116]. Under the hypothesis of Theorem (4.2.7), if $\delta=0$, then

$$|z(t)-y(t)|\le|\bar{\eta}-\eta|t^{\alpha-1}E_{\alpha,\alpha}(Lt^{\alpha}),$$

for $0<t\le h$, where $E_{\alpha,\alpha}$ is the Mittag-Leffler function defined by $E_{\alpha,\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\alpha)}$ $(\alpha>0)$.

Proof. If $\delta = 0$, then

$$A(t) = \left|\frac{t^{\alpha-1}}{\Gamma(\alpha)}(\bar{\eta} - \eta)\right|.$$

By Theorem (4.2.7), we obtain

$$|z(t) - y(t)| \leq A(t) + \int_0^t \left[\sum_{n=1}^{\infty} L^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)} A(s)\right] ds$$

$$= (\bar{\eta} - \eta)t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(Lt^{\alpha})^n}{\Gamma(n\alpha + \alpha)} = |\bar{\eta} - \eta| t^{\alpha-1} E_{\alpha,\alpha}(Lt^{\alpha}),$$

for $0 < t \leq h$. The proof is complete.

Sec(4.3) : Weakly Singular Integral Inequalities with Applications to Fractional Differential and Integral Equations

It is well known that Gronwall type integral inequalities play a dominant role in the study quantitative properties of solutions of differential and integral equations.. Usually, the integrals concerning this type inequalities have regular or continuous kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, D. Henry [114] used this type integral inequalities to prove a global existence and an exponential decay result for a parabolic Cauchy problem; Sano and Kunimatsu [115] gave a sufficient condition for stabilization of semilinear parabolic distributed systems by making use of a modification of Henry's type inequality. Very recently, Ye, Gao and Ding [116] also proved a generalized this type inequality and used it to study the dependence of the solution on the order and the initial condition of a fractional differential equation. All this type inequalities are proved by an iteration argument and the estimation formulas are expressed by a complicated power series which are sometimes not very convenient for applications. To avoid the weakness, Medveď [117] presented a new method to solve Henry's type inequalities and got the explicit bounds with a quite simple formulas which are similar to the classic Gronwall–Bellman inequalities.

In this section, we use the modification of Medveď's method to study a certain class of nonlinear inequalities of Henry's type, which generalizes some known results and can be used as handy and effective tools in the study of differential equations and integral equations.

In what follows, R denotes the set of real numbers, $R_+ = [0, +\infty)$; $C^i(M, S)$ denotes the class of all i-times continuously differentiable defined on set M with range in the set S $(i = 1, 2, \ldots)$ and $C^0(M, S) = C(M, S)$.

Lemma $(\mathbf{4.3.1})[\mathbf{171}]$. (See [118].) Let $a \geq 0, p \geq q \geq 0$ and $p \neq 0$, then

$$a^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} + \frac{p-q}{p} k^{\frac{q}{p}}$$

For any $k > 0$.

Definition $(\mathbf{4.3.2})$**[171]** (See [119].) Let $[x, y, z]$ be an ordered parameter group of nonnegative real numbers. The group is said to belong to the first class distribution and denoted by $[x, y, z] \in I$ if conditions $x \in (0,1], y \in \left(\frac{1}{2}, 1\right)$ and $z \geq \frac{3}{2} - y$ are satisfied; The group is said to belong to the second class distribution and denoted by $[x, y, z] \in II$ if conditions $x \in (0,1], y \in (0, \frac{1}{2}]$ and $z > (1 - 2y^2)/(1 - y^2)$ are satisfied.

Lemma $(\mathbf{4.3.3})[\mathbf{171}]$. (See [120]). Let α, β, γ and p be positive constants. Then

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \qquad t \in R_+,$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds\ (\Re\xi > 0,\ \Re\eta > 0)$ is the well-known B-function and $\theta = p[\alpha(\beta-1) + \gamma - 1] + 1$.

Lemma $(\mathbf{4.3.4})[\mathbf{171}]$**.** (See [119]). Suppose that the positive constants $\alpha, \beta, \gamma, p_1$ and p_2 satisfy conditions:

$(a)\, if\ [\alpha, \beta, \gamma] \in I, p_1 = \frac{1}{\beta}$;

$(b)\ if\ [\alpha, \beta, \gamma] \in II, p_2 = \frac{1+4\beta}{1+3\beta},\ then$

$$B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] \in (0, +\infty)$$

and

$$\theta_i = p_i[\alpha(\beta-1) + \gamma - 1] + 1 \geq 0$$

are valid for $i = 1,2$.

Lemma $(\mathbf{4.3.5})[\mathbf{171}]$**.** (See [121].) Let $u(t), f(t), g(t)$ and $h(t)$ be nonnegative continuous functions on R_+, and let $r \geq 1$ be a real number. If

$$u(t) \leq u_0(t) + w(t)\left[\int_0^t v(s) u^r(s) ds\right]^{1/r}, \qquad t \in R_+,$$

then

$$\int_0^t v(s) u^r(s) ds \leq \left[1 - (1 - W(t)^{1/r}\right]^{-r} \int_0^t v(s) u_0^r(s) W(s) ds, \quad t \in R_+,$$

where

$$W(t) = exp\left(-\int_0^t v(s)w^r(s)ds\right).$$

Theorem $(4.3.6)$[171]. Let $u(t), a(t), b(t)$ and $f(t)$ be nonnegative continuous functions for $t \in R_+$. Let p and q be constants with $p \geq q \geq 0$. If $u(t)$ satisfies

$$u^p \leq a(t) + b(t)\int_0^t (t^\alpha - s^\alpha)^{(\beta-1)} s^{(\gamma-1)} ds, \qquad t \in R_+, \tag{49}$$

then for any $K > 0$ we have

(i) $if\ [\alpha, \beta, \gamma] \in I$,

$$u(t) \leq \left\{a(t) + M_1^\beta t^{(\alpha+1)(\beta-1)+\gamma} b(t)\left[\mathcal{A}_1^{1-\beta}(t) + K^{\frac{q-p}{p}} M_1^\beta \left[1 - \left(1 - V_1(t)\right)^{1-\beta}\right]^{-1} \times \left(\int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}} f^{\frac{1}{1-\beta}}(s)\mathcal{A}_1(s)V_1(s)ds\right)^{1-\beta}\right]\right\}^{\frac{1}{p}} \tag{50}$$

where

$$M_1 = \frac{1}{\alpha}\beta\left[\frac{\beta+\gamma-1}{\alpha\beta}, \frac{2\beta-1}{\beta}\right], \quad A(t) = \frac{q}{p} K^{\frac{q-p}{p}} a(t) + \frac{p-q}{p} K^{\frac{q}{p}},$$

$$\mathcal{A}_1(t) = \int_0^1 f^{\frac{1}{1-\beta}}(s)A^{\frac{1}{1-\beta}}(s)ds$$

and

$$V_1(t) = exp\left(-K^{\frac{p-q}{p(1-\beta)}} M_1^{\frac{\beta}{1-\beta}} \int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}} f^{\frac{1}{1-\beta}}(s)\, b^{\frac{1}{1-\beta}}(s)ds\right);$$

(ii) $if\ [\alpha, \beta, \gamma] \in II$,

$$u(t) \leq \left\{a(t) + M_2^{\frac{1+3\beta}{1+4\beta}} t^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{1+4\beta}} b(t)\left[\mathcal{A}_2^{\frac{\beta}{1+4\beta}}(t) + K^{\frac{q-p}{p}} M_2^{\frac{1+3\beta}{1+4\beta}}\left[1 - \left(1 - V_2(t)\right)^{\frac{\beta}{1+4\beta}}\right]^{-1} \times \left(\int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}} \mathcal{A}_2(s)V_2(s)ds\right)^{\frac{\beta}{1+4\beta}}\right]\right\}^{\frac{1}{p}}, \tag{51}$$

where

$$M_2 = \frac{1}{\alpha}\beta\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)}, \frac{4\beta^2}{1+3\beta}\right], \quad \mathcal{A}_2(t) = \int_0^t f^{\frac{1+4\beta}{\beta}}(s) A^{\frac{1+4\beta}{\beta}}(s)ds$$

and

$$V_2(t) = exp\left(-K^{\frac{(q-p)(1+4\beta)}{\alpha\beta}} M_2^{\frac{1+3\beta}{\beta}} \int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} f^{\frac{1}{1-\beta}}(s)\, b^{\frac{1+4\beta}{\beta}}(s)ds\right).$$

Proof. Define a function $v(t)$ by

$$v(t) = b(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s)u^q(s)ds, \quad t \in R_+, \tag{52}$$

then

$$u^p(t) \le a(t) + v(t)$$

or

$$u(t) \le \left(a(t)+v(t)\right)^{\frac{1}{p}}. \tag{53}$$

By Lemma (4.3.1) and (53), for any $K > 0$, we have

$$u^q(t) \le \left(a(t)+v(t)\right)^{\frac{q}{p}} \le \frac{q}{p}\,\mathrm{K}^{\frac{q-p}{p}}\left(\mathrm{a(t)+v(t)}\right) + \frac{\mathrm{p-q}}{\mathrm{p}}\,K^{\frac{q}{p}}.$$

Substituting the last relations into (52) we get

$$u(t) \le b(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s)ds\left[\frac{q}{p}\mathrm{K}^{\frac{q-p}{p}}(\mathrm{a(s)+v(s)}) + \frac{\mathrm{p-q}}{\mathrm{p}}\,K^{\frac{q}{p}}\right]ds$$

$$= b(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s)A(s)ds$$

$$+\frac{q}{p}\mathrm{K}^{\frac{q-p}{p}}\mathrm{b(t)}\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s)v(s)ds, \tag{54}$$

where $A(t) = \frac{q}{p}\mathrm{K}^{\frac{q-p}{p}}\mathrm{a(t)} + \frac{\mathrm{p-q}}{\mathrm{p}}\,K^{\frac{q}{p}}$.

If $[\alpha,\beta,\gamma] \in I$, let $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$; $if\ [\alpha,\beta,\gamma] \in II$, let $p_2 = (1+4\beta)/(1+3\beta), q_2 = (1+4\beta)/\beta$, then $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1,2$,and then using Hölder's inequality with indexes p_i , q_i to (54) we get

$$u(t) \le b(t)\left[\int_0^t (t^\alpha - s^\alpha)^{p_i(\beta-1)} s^{p_i(\gamma-1)}ds\right]^{1/p_i}\left[\int_0^t f^{q_i}(s)A^{q_i}(s)ds\right]^{1/q_i}$$

$$+K^{\frac{q-p}{p}}b(t)\left[\int_0^t (t^\alpha - s^\alpha)^{p_i(\beta-1)} s^{p_i(\gamma-1)} ds\right]^{1/p_i} \left[\int_0^t f^{q_i}(s) v^{q_i}(s) ds\right]^{1/q_i}.$$

By Lemmas (4.3.3) and (4.3.4), the last inequality can be rewritten as

$$v(t) \le \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} \mathcal{A}_i^{\frac{1}{q_i}}(t) b(t) + K^{\frac{q-p}{p}} \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} b(t) \left[\int_0^t f^{q_i}(s) v^{q_i}(s) ds\right]^{\frac{1}{q_i}} \qquad (55)$$

for $t \in R_+$, where

$$M_i = \frac{1}{\alpha}\beta\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right], \quad \mathcal{A}_i(t) = \int_0^t f^{q_i}(s) A^{q_i}(s) ds$$

and θ_i is given as in Lemma (4.3.4) for $i = 1, 2$.
Using Lemma (4.3.5)to (55), we get

$$v(t) \le \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} \mathcal{A}_i^{\frac{1}{q_i}}(t) b(t) + K^{\frac{q-p}{p}} \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} b(t) \left[1 - (1 - V_i(t))^{\frac{1}{q_i}}\right]^{-1}$$

$$\times\left(\int_0^1 f^{q_i}(s)\left(M_i t^{\theta_i}\right)^{\frac{q_i}{p_i}} b^{q_i}(s) \mathcal{A}_i(s) V_i(s) ds\right)^{\frac{1}{q_i}}, \qquad (56)$$

where

$$V_i(t) = exp\left(-K^{\frac{q_i(q-p)}{p}} \int_0^t f^{q_i}(s)\left(M_i s^{\theta_i}\right)^{\frac{q_i}{p_i}} b^{q_i}(s) ds\right).$$

Finally, substituting (56) into (53), considering two situations for $i = 1, 2$ and using parameters α, β *and* γ to denote p_i, q_i *and* θ_i *in* (56), we can get the desired estimations (50) and (51), respectively.

Theorem (4.3.7)[171]. *Let* $u(t), a(t), b(t), f(t), p$ *and* q be defined as in Theorem (4.3.6), $u(t)$ satisfy (49). Then for any $K > 0$ we have

(i) *if* $[\alpha, \beta, \gamma] \in I$,

$$u(t) \le \left\{a(t) + M_1^\beta t^{(\alpha+1)(\beta-1)+\gamma} b(t)\left[\mathcal{A}_1^{1-\beta}(t) + K^{\frac{q-p}{p}} M_1^\beta \frac{M_1^\beta}{1-\beta} V_1^{-1}(t)\right.\right.$$

$$\left.\left.\times\left(\int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) \mathcal{A}_1(s) V_1(s) ds\right)^{1-\beta}\right]\right\}^{\frac{1}{p}}, \qquad (57)$$

where $M_1, \mathcal{A}_1(t) and\ V_1(t)$ are defined as in Theorem (4.3.6) for $t \in R_+$;
$(ii)\ if\ [\alpha, \beta, \gamma] \in II$,

$$u(t) \leq \left\{ a(t) + M_2^{\frac{1+3\beta}{1+4\beta}} t^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{1+4\beta}} b(t) \left[\mathcal{A}_2^{\frac{\beta}{1+4\beta}}(t) + K^{\frac{q-p}{p}} M_2^{\frac{1+3\beta}{1+4\beta}} \right.\right.$$

$$\left.\left. \times \left(\frac{1+4\beta}{\beta}\right) V_2^{-1}(t) \left(\int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} f^{\frac{1+4\beta}{\beta}}(s) \mathcal{A}_2(s) V_2(s) ds \right)^{\frac{\beta}{1+4\beta}} \right] \right\}^{\frac{1}{p}}, \quad (58)$$

where $M_2, A_2(t)$ and $V_2(t)$ are defined as in Theorem (4.3.6) for $t \in R_+$.
Proof. By the generalized Bernoulli inequality [122], we have

$$(1 - V_i(t))^{\frac{1}{q_i}} < 1 - \frac{1}{q_i} V_i(t)$$

or

$$\left[1 - (1 - V_i(t))^{\frac{1}{q_i}}\right]^{-1} < q_i V_i^{-1}(t)$$

for $i = 1, 2$, where $V_i(t)$ is defined as in Theorem (4.3.6). Substituting the last inequalities into (50) and (51) we can obtain (57)and (58), respectively.

Corollary $(\mathbf{4.3.8})[\mathbf{171}]$. Let functions $u(t), a(t), b(t)$ and $f(t)$ be defined as in Theorem (4.3.6). Suppose that

$$u(t) \leq a(t) + b(t) \int_0^t (t - s)^{\beta-1} f(s) u(s) ds, \qquad t \in R_+. \quad (59)$$

Then we have
$(i)\ if\ \beta \in \left(\frac{1}{2}, 1\right)$,

$$u(t) \leq a(t) + M_{11}^{\beta} t^{2\beta-1} b(t) \left[\mathcal{A}_{11}^{1-\beta}(t) \right.$$

$$\left. + \frac{M_{11}^{\beta}}{1-\beta} V_{11}^{-1}(t) \int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) \mathcal{A}_{11}(s) V_{11}(s) ds \right], \quad (60)$$

where

$$M_{11} = B\left[1, \frac{2\beta - 1}{\beta}\right], \qquad \mathcal{A}_{11}(t) = \int_0^t f^{\frac{1}{1-\beta}}(s) a^{\frac{1}{1-\beta}}(s)\, ds$$

and

$$V_{11}(t) = exp\left(-M_{11}^{\frac{\beta}{1-\beta}} \int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) ds \right)$$

for $t \in R_+$;

(ii) $if\ \beta \in (0, \frac{1}{2}]$,

$$u(t) \leq a(t) + M_{12}^{\frac{1+3\beta}{1+4\beta}} t^{4\beta} b(t) \left[\mathcal{A}_{12}^{\frac{\beta}{1+4\beta}}(t) + \frac{1+4\beta}{\beta} M_{12}^{\frac{1+3\beta}{1+4\beta}} V_{12}^{-1}(t) \times \int_0^t s^{4\beta} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s) \mathcal{A}_{12}(s) V_{12}(s) ds \right], \qquad (61)$$

where

$$M_{12} = B\left[1, \frac{4\beta^2}{1+3\beta}\right], \qquad \mathcal{A}_{12}(t) = \int_0^t f^{\frac{1+4\beta}{\beta}}(s) a^{\frac{1+4\beta}{\beta}}(s)\, ds$$

and

$$V_{12}(t) = exp\left(-M_{12}^{\frac{1+3\beta}{\beta}} \int_0^t s^{4\beta} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s) ds \right)$$

for $t \in R_+$;

Corollary $(\mathbf{4.3.9})[\mathbf{171}]$**.** Let functions $u(t), a(t), b(t)\ and\ f(t)$ be defined as in Theorem (4.3.6). Suppose that

$$u^2(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} f(s) u(s) ds, \qquad t \in R_+. \qquad (62)$$

Then for any $K > 0$ we have

(i) $if\ \beta \in \left(\frac{1}{2}, 1\right)$,

$$u(t) \leq \Big\{ a(t) + M_{11}^{\beta} t^{2\beta-1} b(t) \Big[\tilde{\mathcal{A}}_{11}^{1-\beta}(t) K^{-\frac{1}{2}} \frac{M_{11}^{\beta}}{1-\beta} \tilde{V}_{11}^{-1}(t)$$

$$\times \int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) \widetilde{\mathcal{A}}_{11}(s) \tilde{V}_{11}(s) ds \Bigg] \Bigg\}^{\frac{1}{2}} \qquad (63)$$

where

$$\tilde{\mathcal{A}}_{11}(t) = \left(\frac{1}{2} K^{\frac{1}{2}}\right)^{\frac{1}{1-\beta}} \int_0^t f^{\frac{1}{1-\beta}}(s) \left(\frac{a(s)}{K} + 1\right)^{\frac{1}{1-\beta}} ds,$$

$$\tilde{V}_{11}(t) = exp\left[-\left(\frac{M_{11}^{\beta}}{K^{\frac{1}{2}}}\right)^{\frac{1}{1-\beta}} \int_0^t s^{\frac{2\beta-1}{1-\beta}}(s) f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s)\, ds\right]$$

and M_{11} is defined as in Corollary (4.3.8) for $t \in R_+$;

(ii) $if\ \beta \in (0, \frac{1}{2}]$,

$$u(t) \le \left\{ a(t) + M_{12}^{\frac{1+3\beta}{1+4\beta}} t^{\frac{4\beta^2}{1+4\beta}} b(t) \left[\tilde{\mathcal{A}}_{12}^{\frac{\beta}{1+4\beta}}(t) K^{-\frac{1}{2}} M_{12}^{\frac{1+3\beta}{1+4\beta}} \left(\frac{1+4\beta}{\beta}\right) \tilde{V}_{12}^{-1}(t) \right.\right.$$

$$\left.\left. \times \left(\int_0^t s^{\frac{4\beta^2}{1+4\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s) \tilde{\mathcal{A}}_{12}(s) \tilde{V}_{12}(s) ds \right)^{\frac{\beta}{1+4\beta}} \right] \right\}^{\frac{1}{2}} \qquad (64)$$

where

$$\tilde{\mathcal{A}}_{12}(t) = \left(\frac{1}{2} K^{\frac{1}{2}}\right)^{\frac{1+4\beta}{\beta}} \int_0^t f^{\frac{1+4\beta}{\beta}}(s) \left(\frac{a(s)}{K} + 1\right)^{\frac{1+4\beta}{\beta}} (s)\, ds,$$

$$\tilde{V}_{12}(t) = exp\left[-\left(\frac{M_{12}^{1+3\beta}}{K^{\frac{1+4\beta}{2}}}\right)^{\frac{1}{\beta}} \int_0^t s^{\frac{4\beta^2}{1+\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s)\, ds\right]$$

and M_{12} is defined as in Corollary (4.3.8) for $t \in R_+$;

In this section, we will indicate the usefulness of our results in the study of the boundedness of certain fractional differential equations with Riemann–Liouville (R–L) fractional operator and Erdélyi–Kober (E–K) operator.

Riemann–Liouville derivative and integral, and Erdélyi–Kober (E–K) operator are defined as below, respectively:

Definition $(\mathbf{4.3.10})[\mathbf{171}]$. (See [123]). The fractional derivative of order $0 < \alpha < 1$ of a function $f(x) \in C(R_+, R)$ is given by

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t)dt$$

provided that the right side is pointwise defined on R_+.

Definition $(\mathbf{4.3.11})[\mathbf{171}]$. (See [123]). The fractional primitive of order $\alpha > 0$ of a function $f : R_+ \to R$ is given by

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

provided the right side is pointwise defined on R_+.

Definition $(\mathbf{4.3.12})[\mathbf{171}]$. (See [124,125]). The Erdélyi–Kober fractional integral of a continuous $f : R_+ \to R$ is defined by

$$I_{\beta}^{\gamma,\delta} f(x) = \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x \left(x^{\beta} - t^{\beta}\right)^{\delta-1} t^{\beta\gamma} f(t)\, d(t^{\beta})$$

with real δ, γ and $\beta > 0$, provided the right side is pointwise defined on R_+.

(I)Consider the following initial value problem of Podlubny [124] in terms of the Riemann–Liouville fractional derivatives:

$$D^{\alpha} y(t) = f\big(t, y(t)\big), \tag{65}$$

$$D^{\alpha-1} y(t)|_{t=0} = \eta, \tag{66}$$

where $0 < \alpha < 1,\ 0 \le t < T \le +\infty, f : [0, T) \times R \to R;$ and D^{α} denotes **R–L** derivative operator.

From the problem (65)–(66) we can get a fractional integral equation

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(\tau, y(\tau)\big)\, d\tau, \tag{67}$$

which is equivalent to the initial value problem (65)–(66) (cf. [123]).

Theorem $(4.3.13)$[171]. Let $0 < \alpha \le 1$ *and* f be continuous and satisfy the condition

$$|f(t,y)| \le g(t)|y|^q , \qquad (68)$$

where $0 < q \le 1$ is a constant, g(t) is nonnegative continuous function for $0 \le t < T \le +\infty$. Then for any solutions $y(t)$ of the initial value problem (65)–(66)

(i) *if* $\alpha \in \left(\frac{1}{2}, 1\right)$,

$$|y(t)| \le \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\widetilde{M}_{11}^{\alpha}\, t^{2\alpha-1}}{\Gamma(\alpha)} \left[\mathcal{A}_{1q}^{1-\alpha}(t) + \frac{K^{q-1}\widetilde{M}_{11}^{\alpha}}{(1-\alpha)\Gamma(\alpha)} \widetilde{V}_{1q}^{-1}(t) \times \left(\int_0^t s^{\frac{2\alpha-1}{1-\alpha}} g^{\frac{1}{1-\alpha}}(s)\, \mathcal{A}_{1q}(s) \widetilde{V}_{1q}(s) ds\right)^{\frac{\alpha}{1+4\alpha}}\right], 0 < t < T \le +\infty, (69)$$

where

$$A_q(t) = \frac{q|\eta|}{K^{1-q}\ \Gamma(\alpha)} t^{\alpha-1} + (1-\ q)K^q,$$

$$\widetilde{M}_{11} = B\left[1, \frac{2\alpha-1}{\alpha}\right], \qquad \mathcal{A}_{1q}(t) = \int_0^1 g^{\frac{1}{1-\alpha}}(s) A_q^{\frac{1}{1-\alpha}}(s) ds$$

and

$$\widetilde{V}_{1q}(t) = exp\left[-\left(\frac{K^{1-q}\ \widetilde{M}_{11}^{\alpha}}{\Gamma(\alpha)}\right)^{\frac{1}{1-\alpha}} \int_0^t s^{\frac{2\alpha-1}{1-\alpha}} g^{\frac{1}{1-\alpha}}(s)\, ds\right];$$

(ii) *if* $\alpha \in (0, \frac{1}{2}]$,

$$|y(t)| \le \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}}\ t^{4\alpha}}{\Gamma(\alpha)} \left[\mathcal{A}_{2q}^{\frac{\alpha}{1+4\alpha}}(t) + \frac{K^{q-1}\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}}(1+4\alpha)}{\alpha\Gamma(\alpha)} \widetilde{V}_{2q}^{-1}(t) \times \left(\int_0^t s^{A\alpha} g^{\frac{1+4\alpha}{\alpha}}(s)\, \mathcal{A}_{2q}(s) \widetilde{V}_{2q}(s) ds\right)^{\frac{\alpha}{1+4\alpha}}\right], \qquad 0 < t < T \le +\infty, \qquad (70)$$

where

$$\widetilde{M}_{12} = B\left[1, \frac{4\alpha^2}{1+3\alpha}\right], \qquad \mathcal{A}_{2q}(t) = \int_0^t g^{\frac{1+4\alpha}{\alpha}}(s) A_q^{\frac{1+4\alpha}{\alpha}}(s)ds$$

and

$$\widetilde{V}_{2q}(t) = exp\left[-\left(\frac{K^{1-q}}{\Gamma(\alpha)}\right)^{\frac{1+3\alpha}{\alpha}} \widetilde{M}_{12}^{\frac{1+3\alpha}{\alpha}} \int_0^t s^{4\alpha} g^{\frac{1+4\alpha}{\alpha}}(s)\, ds\right].$$

Proof. From (67) and (68) we have

$$|y(t)| \le \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|f\big(\tau, y(\tau)\big)\right| d\tau$$
$$\le \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)|y(\tau)|^q d\tau.$$

An application of Theorem (4.3.7) $\left(with\ a(t) = \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}, b(t) = \frac{1}{\Gamma(\alpha)}, f(t) = g(t), p = 1, \alpha = \gamma = 1\ and\ \beta = \alpha\right)$ to the last inequality yields the desired estimations (69) and (70).

(II) Consider the following Volterra type integral equations of second kind, involving an E–K fractional integral with parameters $\delta, \gamma\ and\ \beta$,

$$y^p(t) - \lambda t^{-\beta\gamma} \int_0^t \frac{\left(t^\beta - \tau^\beta\right)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta(1+\gamma)-1} y^q(\tau) d(\tau) = f(t), \tag{71}$$

which arises very often in various problems, especial describing physical processes with after effects. When (71) is a linear equation, i.e., $p = q = 1$, the other parameters satisfy some conditions and $y(t)$ belong to a space of weighted continuous functions, Al-Saqabi and Kiryakova [125] have found the solutions of (71) in the explicit form with convolutional type integral involving Mittag–Leffler function. Here we give the explicit bound of the solutions of nonlinear equation (71) under some suitable conditions.

Theorem $(\mathbf{4.3.14})[\mathbf{171}]$. Let $y(t), f(t) \in C[0, +\infty), p \ge q > 0$ be constants and $y(t)$ satisfy (71). Then for any constant $K > 0$ we have

$(i)\ if\ [\beta, \delta, \beta(1+\gamma)] \in I$,

$$|y(t)| \leq \left\{ |f(t)| + \frac{|\lambda| \bar{M}_1^{\delta}}{\Gamma(\delta)} t^{\delta(\beta+1)-1} \left[\bar{\mathcal{A}}_1^{1-\delta}(t) + K^{\frac{q-p}{p}} \frac{|\lambda| \bar{M}_1^{\delta}}{(1-\delta)\Gamma(\delta)} \bar{\mathrm{V}}_1^{-1}(\mathrm{t}) \right.\right.$$

$$\left.\left. \times \left(\int_0^t s^{\frac{\delta(\beta+1)-1}{1-\delta}}(s)\, \bar{\mathcal{A}}_1(s) \bar{V}_1(s) ds \right)^{1-\delta} \right] \right\}^{\frac{1}{p}} \quad 0 > t \,, \qquad (72)$$

where

$$\bar{M}_1 = \frac{1}{\beta} B \left[\frac{\delta + \beta(1+\boldsymbol{\gamma}) - 1}{\beta\delta}, \frac{\beta\delta - 1}{\delta} \right],$$

$$\bar{A}(t) = \frac{q}{p} K^{\frac{q-p}{p}} |f(t)| + \frac{p-q}{p} K^{\frac{q}{p}}, \qquad \bar{\mathcal{A}}_1(t) \int_0^t \bar{\mathcal{A}}^{\frac{1}{1-\delta}}(s) ds$$

and

$$\bar{V}_1(t) = exp \left[- \frac{(1-\delta) K^{\frac{p-q}{p(1-\delta)}}}{\beta\delta} \left(\frac{\bar{M}_1^{\delta} |\lambda|}{\Gamma(\delta)} \right)^{\frac{1}{1-\delta}} t^{\frac{\delta\beta}{1-\delta}} \right];$$

$(ii)\ if\ [\beta, \delta, \beta(1+\gamma)] \in II,$

$$|y(t)| \leq \left\{ |f(t)| + \frac{|\lambda| \bar{M}_2^{\frac{1+3\delta}{1+4\delta}}}{\Gamma(\delta)} t^{\frac{\beta(\delta+\gamma+4\delta^2+3\delta\gamma)-\delta}{\delta}} \left[\bar{\mathcal{A}}_2^{\frac{\delta}{1+4\delta}}(t) + \frac{K^{\frac{q-p}{p}} \bar{M}_2^{\frac{1+3\delta}{1+4\delta}} (1+4\delta)|\lambda|}{\delta\Gamma(\delta)} \right.\right.$$

$$\left.\left. \times \bar{\mathrm{V}}_2^{-1}(\mathrm{t}) \left(\int_0^t s^{\beta(4\delta+1)-1}(s)\, \bar{\mathcal{A}}_2(s) \bar{V}_2(s) ds \right)^{\frac{\delta}{1+4\delta}} \right] \right\}^{\frac{1}{p}} \quad 0 > t \,, \qquad (73)$$

where

$$\bar{M}_2 = \frac{1}{\beta} B \left[\frac{\beta(1+\boldsymbol{\gamma})(1+4\delta) - \delta}{\beta(1+4\delta)}, \frac{4\delta^2}{1+3\delta^2} \right], \qquad \bar{\mathcal{A}}_2(t) \int_0^t \bar{\mathcal{A}}^{\frac{1+4\delta}{\delta}}(s) ds$$

and

$$\bar{V}_2(t) = exp \left[- \frac{K^{\frac{(q-p)(1+4\delta)}{p\delta}} \bar{M}_2^{\frac{1+3\delta}{\delta}}}{\beta(1+4\delta)} \left(\frac{|\lambda|}{\Gamma(\delta)} \right)^{\frac{1+4\delta}{\delta}} t^{\beta(1+4\delta)} \right].$$

Proof. From (71) we have

$$|y|^p(t) \leq |f(t)| + \frac{|\lambda|}{\Gamma(\delta)} t^{-\beta\gamma} \int_0^t \left(t^\beta - \tau^\beta\right)^{\delta-1} \tau^{\beta(1+\gamma)-1} |y|^q(\tau) d(\tau).$$

An application of Theorem (4.3.7) $(with\ a(t) = f(t), b(t) = \frac{|\lambda|}{\Gamma(\alpha)} t^{-\beta\gamma}, \alpha = \beta, \beta = \delta$ and $\gamma = \beta(1+\gamma))$ to the last inequality yields the desired estimations (72) and (73).

Letting $p = q = 1$ in Theorem (4.3.14), we can obtain an interesting result as follows.

Corollary $(4.3.15)$[171]. Let $y(t), f(t) \in C[0, +\infty)$ and $y(t)$ satisfy the equation

$$y(t) - \lambda t^{-\beta\gamma} \int_0^t \frac{\left(t^\beta - \tau^\beta\right)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta(1+\gamma)-1} y(\tau) d(\tau) = f(t), \tag{74}$$

Then we have

$$(i)\ \ if\ [\beta, \delta, \beta(1+\gamma)] \in I,$$

$$|y(t)| \leq |f(t)| + \frac{|\lambda| \bar{M}_1^\delta}{\Gamma(\delta)} t^{\delta(\beta+1)-1} \left[\bar{\mathcal{A}}_1^{*1-\delta}(t) + \frac{|\lambda| \bar{M}_1^\delta}{(1-\delta)\Gamma(\delta)} \bar{\mathrm{V}}_1^{*-1}(\mathrm{t}) \times \left(\int_0^t s^{\frac{\delta(\beta+1)-1}{1-\delta}}(s)\, \bar{\mathcal{A}}_1^*(s) \bar{V}_1^*(s) ds\right)^{1-\delta}\right] 0 > t\,, \tag{75}$$

where

$$\bar{M}_1 = \frac{1}{\beta} B\left[\frac{\delta + \beta(1+\boldsymbol{\gamma}) - 1}{\beta\delta}, \frac{\beta\delta - 1}{\delta}\right], \quad \bar{\mathcal{A}}_1^*(t) = \int_0^t |f(s)|^{\frac{1}{1-\delta}} ds$$

and

$$\bar{V}_1^*(t) = exp\left[-\frac{1-\delta}{\beta\delta}\left(\frac{\bar{M}_1^\delta |\lambda|}{(\Gamma(\delta))}\right)^{\frac{1}{1-\delta}} t^{\frac{\delta\beta}{1-\delta}}\right],$$

(ii) $if\ [\beta, \delta, \beta(1+\gamma)] \in II,$

$$|y(t)| \leq |f(t)| + \frac{|\lambda| \bar{M}_2^{\frac{1+3\delta}{1+4\delta}}}{\Gamma(\delta)} t^{\frac{\beta(\delta+\gamma+4\delta^2+3\delta\gamma)-\delta}{\delta}} \left[\bar{\mathcal{A}}_2^{*\frac{\delta}{1+4\delta}}(t) + \frac{\bar{M}_2^{\frac{1+3\delta}{1+4\delta}}(1+4\delta)|\lambda|}{\delta\Gamma(\delta)} \times \bar{\mathrm{V}}_2^{*-1}(\mathrm{t}) \left(\int_0^t s^{\beta(4\delta+1)-1}(s)\, \bar{\mathcal{A}}_2^*(s) \bar{V}_2^*(s) ds\right)^{\frac{\delta}{1+4\delta}}\right], \quad t > 0\,, \tag{76}$$

where

$$\bar{M}_2=\frac{1}{\beta}B\left[\frac{\beta(1+\gamma)(1+4\delta)-\delta}{\beta(1+4\delta)},\frac{4\delta^2}{1+3\delta^2}\right],\qquad \bar{\mathcal{A}}_2^*(t)\ \int_0^t |f(s)|^{\frac{1+4\delta}{\delta}}(s)ds$$

and

$$\bar{V}_2^*(t)=exp\left[-\frac{\bar{M}_2^{\frac{1+3\delta}{\delta}}}{\beta(1+4\delta)}\left(\frac{|\lambda|}{\Gamma(\delta)}\right)^{\frac{1+4\delta}{\delta}}t^{\beta(1+4\delta)}\right].$$

Corollary (4.3.16)[140]. Let $u(t), a(t), b(t)$ and $f(t)$ be nonnegative continuous functions for $t\in R_+$. Let $p=q+\epsilon\geq 0$. If $u(t)$ satisfies

$$u^p\leq a(t)+b(t)\int_0^t (t^\alpha-s^\alpha)^{(2\alpha-1)}s^{(3\alpha-1)}ds,\qquad t\in R_+,\quad (77)$$

then for any $K>0$ we have

$(i)\ if\ [\alpha,2\alpha,3\alpha]\in I$,

$$u(t)\leq\left\{a(t)+M_1^{2\alpha}t^{2\alpha(\alpha+2)-1}b(t)\left[\mathcal{A}_1^{1-2\alpha}(t)+K^{\frac{\epsilon_1}{p}}M_1^\beta\left[1-\left(1-V_1(t)\right)^{1-2\alpha}\right]^{-1}\right.\right.$$
$$\left.\left.\times\left(\int_0^t s^{\frac{2\alpha(\alpha+2)-1}{1-2\alpha}}f^{\frac{1}{1-2\alpha}}(s)\mathcal{A}_1(s)V_1(s)ds\right)^{1-2\alpha}\right]\right\}^{\frac{1}{p}}\qquad (78)$$

where

$$M_1=\left[\frac{5\alpha-1}{\alpha^2},\frac{4\alpha-1}{\alpha}\right],\qquad A(t)=\left(1+\frac{\epsilon_1}{p}\right)K^{\frac{\epsilon_1}{p}}a(t)-\frac{\epsilon_1}{p}K^{1+\frac{\epsilon_1}{p}},$$

$$\mathcal{A}_1(\mathrm{t})=\int_0^1 f^{\frac{1}{1-2\alpha}}(\mathrm{s})A^{\frac{1}{1-2\alpha}}(s)ds$$

and

$$V_1(t)=exp\left(-K^{\frac{-\epsilon_1}{p(1-2\alpha)}}M_1^{\frac{2\alpha}{1-2\alpha}}\int_0^t s^{\frac{2\alpha(\alpha+2)-1}{1-2\alpha}}f^{\frac{1}{1-2\alpha}}(s)\,b^{\frac{1}{1-2\alpha}}(s)ds\right);$$

$(ii)\ if\ [\alpha,2\alpha,3\alpha]\in II$,

$$u(t) \leq \left\{ a(t) + M_2^{\frac{1+6\alpha}{1+8\alpha}} t^{\frac{2\alpha^2(8\alpha+9)}{1+16\alpha}} b(t) \left[\mathcal{A}_2^{\frac{2\alpha}{1+8\alpha}}(t) + K^{\frac{\epsilon_1}{p}} M_2^{\frac{1+6\alpha}{1+8\alpha}} \left[1 - \left(1 - V_2(t)\right)^{\frac{2\alpha}{1+8\alpha}} \right]^{-1} \right.\right.$$

$$\left.\left. \times \left(\int_0^t s^{\alpha(8\alpha+9)} f^{\frac{1+8\alpha}{2\alpha}}(s) b^{\frac{1+8\alpha}{2\alpha}} \mathcal{A}_2(s) V_2(s) ds \right)^{\frac{2\alpha}{1+8\alpha}} \right] \right\}^{\frac{1}{p}}, \qquad (79)$$

where

$$M_2 = 2\left[\frac{1+24\alpha}{1+6\alpha}, \frac{16\alpha^2}{1+6\alpha}\right], \quad \mathcal{A}_2(t) = \int_0^t f^{\frac{1+8\alpha}{2\alpha}}(s) A^{\frac{1+8\alpha}{2\alpha}}(s) ds$$

and

$$V_2(t) = exp\left(-K^{\frac{\epsilon_1(1+8\alpha)}{2p\alpha}} M_2^{\frac{1+6\alpha}{2\alpha}} \int_0^t s^{\alpha(8\alpha+9)} f^{\frac{1+8\alpha}{2\alpha}}(s)\, b^{\frac{1+8\alpha}{2\alpha}}(s) ds \right).$$

Proof. Define a function $v(t)$ by

$$v(t) = b(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{3\alpha-1} f(s) u^{p+\epsilon_1}(s) ds, \quad t \in R_+, \qquad (80)$$

then

$$u^p(t) \leq a(t) + v(t)$$

or

$$u(t) \leq \left(a(t) + v(t)\right)^{\frac{1}{p}}. \qquad (81)$$

By Lemma (4.3.1) and (81), for any $K > 0$, we have

$$u^{p+\epsilon_1}(t) \leq \left(a(t) + v(t)\right)^{1+\frac{\epsilon_1}{p}} \leq \left(1 + \frac{\epsilon_1}{p}\right) K^{\frac{\epsilon_1}{p}} \left(a(t) + v(t)\right) - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}}.$$

Substituting the last relations into (80) we get

$$u(t) \leq b(t) \int_0^t (t^\alpha - s^\alpha)^{2\alpha-1} s^{3\alpha-1} f(s) ds \left[\left(1 + \frac{\epsilon_1}{p}\right) K^{\frac{\epsilon_1}{p}} (a(s) + v(s)) \right.$$

$$\left. - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}} \right] ds$$

$$= b(t) \int_0^t (t^\alpha - s^\alpha)^{2\alpha-1} s^{3\alpha-1} f(s) A(s) ds$$

$$+\left(1+\frac{\epsilon_1}{p}\right)\mathrm{K}^{\frac{\epsilon_1}{p}}\mathrm{b(t)}\int_0^t (t^\alpha - s^\alpha)^{2\alpha-1}\, s^{3\alpha-1} f(s)v(s)ds\,, (82)$$

where $A(t) = \left(1+\frac{\epsilon_1}{p}\right)\mathrm{K}^{\frac{\epsilon_1}{p}}\mathrm{a(t)} - \frac{\epsilon_1}{p}\, K^{1+\frac{\epsilon_1}{p}}$.

If $[\alpha, 2\alpha, 3\alpha] \in I$, let $p_1 = \frac{1}{2\alpha}$, $q_1 = \frac{1}{1-2\alpha}$; if $[\alpha, 2\alpha, 3\alpha] \in II$, let $p_2 = (1+8\alpha)/(1+6\alpha), q_2 = (1+8\alpha)/2\alpha$, then $\frac{1}{p_i} + \frac{1}{q_i} = 1\; for\; i = 1,2$,and then using Hölder's inequality with indexes p_i , q_i to (82) we get

$$u(t) \le b(t)\left[\int_0^t (t^\alpha - s^\alpha)^{p_i\,(2\alpha-1)}\, s^{p_i\,(3\alpha-1)} ds\right]^{1/p_i} \left[\int_0^t f^{q_i}\,(s)A^{q_i}(s)ds\right]^{1/q_i}$$

$$+\mathrm{K}^{\frac{-\epsilon}{p}}\mathrm{b(t)}\left[\int_0^t (t^\alpha - s^\alpha)^{p_i\,(2\alpha-1)}\, s^{p_i\,(3\alpha-1)} ds\right]^{1/p_i} \left[\int_0^t f^{q_i}\,(s)v^{q_i}(s)ds\right]^{1/q_i}.$$

By Lemmas (4.3.3) and (4.3.4), the last inequality can be rewritten as

$$v(t) \le \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} \mathcal{A}_i^{\frac{1}{q_i}}(t)b(t) + \mathrm{K}^{\frac{\epsilon_1}{p}}\left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} b(t)\left[\int_0^t f^{q_i}\,(s)v^{q_i}(s)ds\right]^{\frac{1}{q_i}} \qquad (83)$$

for t $\in R_+$, where

$$M_i = 2\left[\frac{p_i\,(3\alpha-1)+1}{\alpha}, p_i\,(2\alpha-1)+1\right], \quad \mathcal{A}_{\mathrm{i}}(\mathrm{t}) = \int_0^t f^{q_i}\,(s)A^{q_i}(s)ds$$

and θ_i is given as in Lemma 4 for $i = 1, 2$.

Using Lemma (4.3.5) to (83), we get

$$v(t) \le \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} \mathcal{A}_i^{\frac{1}{q_i}}(t)b(t) + \mathrm{K}^{\frac{\epsilon_1}{p}}\left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}} b(t)\left[1 - (1 - V_i(t))^{\frac{1}{q_i}}\right]^{-1}$$

$$\times\left(\int_0^1 f^{q_i}(s)\left(M_i t^{\theta_i}\right)^{\frac{q_i}{p_i}} b^{q_i}(s)\mathcal{A}_i(s)V_i(s)ds\right)^{\frac{1}{q_i}}, \qquad (84)$$

where

$$V_i(t) = exp\left(-\mathrm{K}^{\frac{q_i\epsilon_1}{\mathrm{p}}}\int_0^t f^{q_i}(s)\left(M_i s^{\theta_i}\right)^{\frac{q_i}{p_i}} b^{q_i}(s)ds\right).$$

Finally, substituting (84) into (81), considering two situations for $i = 1, 2$ and using parameters $\alpha, 2\alpha$ *and* 3α to denote p_i, q_i *and* θ_i *in* (84), we can get the desired estimations (78) and (79), respectively.

Corollary (4.3.17) [140]. *Let* $u(t), a(t), b(t), f(t)$, $p = q + \epsilon$ be defined as in Theorem 6, $u(t)$ satisfy (77). Then for any $K > 0$ we have

(i) *if* $[\alpha, 2\alpha, 3\alpha] \in I$,

$$u(t) \le \left\{ a(t) + M_1^{2\alpha} t^{\alpha(2\alpha+1)+2} b(t) \left[\mathcal{A}_1^{1-2\alpha}(t) + K^{\frac{\epsilon_1}{p}} M_1^{2\alpha} \frac{M_1^{2\alpha}}{1-2\alpha} V_1^{-1}(t) \right.\right.$$
$$\left.\left. \times \left(\int_0^t s^{\frac{\alpha(2\alpha+1)+2}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) b^{\frac{1}{1-2\alpha}}(s) \mathcal{A}_1(s) V_1(s) ds \right)^{1-2\alpha} \right] \right\}^{\frac{1}{p}}, \quad (85)$$

where $M_1, \mathcal{A}_1(t)$ *and* $V_1(t)$ are defined as in Theorem (4.3.6) for $t \in R_+$;

(ii) *if* $[\alpha, 2\alpha, 3\alpha] \in II$,

$$u(t) \le \left\{ a(t) + M_2^{\frac{1+6\alpha}{1+4\alpha}} t^{\frac{2\alpha^2(8\alpha+9)}{1+8\alpha}} b(t) \left[\mathcal{A}_2^{\frac{2\alpha}{1+8\alpha}}(t) + K^{\frac{\epsilon_1}{p}} M_2^{\frac{1+6\alpha}{1+8\alpha}} \right.\right.$$
$$\left.\left. \times \left(\frac{1+8\alpha}{2\alpha}\right) V_2^{-1}(t) \left(\int_0^t s^{\frac{2\alpha^2(8\alpha+9)}{1+8\alpha}} f^{\frac{1+8\alpha}{2\alpha}}(s) \mathcal{A}_2(s) V_2(s) ds \right)^{\frac{2\alpha}{1+8\alpha}} \right] \right\}^{\frac{1}{p}}, \quad (86)$$

where $M_2, A_2(t)$ and $V_2(t)$ are defined as in Theorem (4.3.6) for $t \in R_+$.

Proof. By the generalized Bernoulli inequality [122], we have

$$(1 - V_i(t))^{\frac{1}{q_i}} < 1 - \frac{1}{q_i} V_i(t)$$

or

$$\left[1 - (1 - V_i(t))^{\frac{1}{q_i}}\right]^{-1} < q_i V_i^{-1}(t)$$

for $i = 1, 2$, where $V_i(t)$ is defined as in Theorem (4.3.6). Substituting the last inequalities into (78) and (79) we can obtain (85) and (86) respectively.

Corollary (4.3.18) [140]. Let $0 < \alpha \le 1$ *and* f be continuous and satisfy the condition

$$|f(t, y)| \le g(t)|y|^{p+\epsilon_1}, \quad (87)$$

where $0 < p + \epsilon_1 \le 1$ is a constant, $g(t)$ is nonnegative continuous function for $0 \le t < T \le +\infty$. Then for any solutions $y(t)$ of the initial value problem (65)–(66)

$(i) if\ \alpha \in \left(\frac{1}{2}, 1\right),$

$$|y(t)| \le \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\widetilde{M}_{11}^{\alpha}\, t^{2\alpha-1}}{\Gamma(\alpha)} \left[\mathcal{A}_{1(p+\epsilon_1)}^{1-\alpha}(t) + \frac{K^{p+\epsilon_1-1}\widetilde{M}_{11}^{\alpha}}{(1-\alpha)\Gamma(\alpha)} \tilde{V}_{1(p+\epsilon_1)}^{-1}(t)\right.$$

$$\left. \times \left(\int_0^t s^{\frac{2\alpha-1}{1-\alpha}} g^{\frac{1}{1-\alpha}}(s)\, \mathcal{A}_{1(p+\epsilon_1)}(s) \tilde{V}_{1(p+\epsilon_1)}(s) ds\right)^{\frac{\alpha}{1+4\alpha}}\right], 0 < t < T \le +\infty, \quad (88)$$

where

$$A_{p+\epsilon_1}(t) = \frac{(p+\epsilon_1)|\eta|}{K^{1-(p+\epsilon_1)}\ \Gamma(\alpha)} t^{\alpha-1} + \left(1 - (p+\epsilon_1)\right) K^{p+\epsilon_1},$$

$$\widetilde{M}_{11} = B\left[1, \frac{2\alpha-1}{\alpha}\right], \qquad \mathcal{A}_{1(p+\epsilon_1)}(t) = \int_0^1 g^{\frac{1}{1-\alpha}}(s) A_{p+\epsilon_1}^{\frac{1}{1-\alpha}}(s) ds$$

and

$$\tilde{V}_{1(p+\epsilon_1)}(t) = exp\left[-\left(\frac{K^{1-(p+\epsilon_1)}\ \widetilde{M}_{11}^{\alpha}}{\Gamma(\alpha)}\right)^{\frac{1}{1-\alpha}} \int_0^t s^{\frac{2\alpha-1}{1-\alpha}} g^{\frac{1}{1-\alpha}}(s)\, ds\right];$$

$(ii)\ if\ \alpha \in (0, \frac{1}{2}],$

$$|y(t)| \le \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}}\, t^{4\alpha}}{\Gamma(\alpha)} \left[\mathcal{A}_{2(p+\epsilon_1)}^{\frac{\alpha}{1+4\alpha}}(t) + \frac{K^{p+\epsilon_1-1}\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}}(1+4\alpha)}{\alpha\Gamma(\alpha)} \tilde{V}_{2(p+\epsilon_1)}^{-1}(t)\right.$$

$$\left. \times \left(\int_0^t s^{A\alpha} g^{\frac{1+4\alpha}{\alpha}}(s)\, \mathcal{A}_{2(p+\epsilon_1)}(s) \tilde{V}_{2(p+\epsilon_1)}(s) ds\right)^{\frac{\alpha}{1+4\alpha}}\right], \qquad 0 < t < T \le +\infty, \qquad (89)$$

where

$$\widetilde{M}_{12} = B\left[1, \frac{4\alpha^2}{1+3\alpha}\right], \qquad \mathcal{A}_{2(p+\epsilon_1)}(t) = \int_0^t g^{\frac{1+4\alpha}{\alpha}}(s) A_{p+\epsilon_1}^{\frac{1+4\alpha}{\alpha}}(s) ds$$

and

$$\tilde{V}_{2(p+\epsilon_1)}(t) = exp\left[-\left(\frac{K^{p+\epsilon_1}}{\Gamma(\alpha)}\right)^{\frac{1+3\alpha}{\alpha}} \widetilde{M}_{12}^{\frac{1+3\alpha}{\alpha}} \int_0^t s^{4\alpha} g^{\frac{1+4\alpha}{\alpha}}(s)\, ds\right].$$

Proof. From (67) and (87) we have

$$|y(t)| \leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left| f\big(\tau, y(\tau)\big) \right| d\tau$$
$$\leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, \mathrm{g}(\tau) |y(\tau)\,|^{p+\epsilon_1} d\tau.$$

An application of Theorem 8 ($with\ a(t) = \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}, b(t) = \frac{1}{\Gamma(\alpha)}, f\ (t) = \mathrm{g}(t), p = 1, \alpha = 3\alpha = 1\ and\ \alpha = 0$) to the last inequality yields the desired estimations (21) and (22).

(II) Consider the following Volterra type integral equations of second kind, involving an E–K fractional integral with parameters $\alpha, 3\alpha\ and\ 2\alpha$,

$$y^p(t) - \lambda t^{-6\alpha^2} \int_0^t \frac{(t^{2\alpha} - \tau^{2\alpha})^{\alpha-1}}{\Gamma(\alpha)} \tau^{2\alpha(\alpha+1)-1} y^{p+\epsilon_1}(\tau) d(\tau) = f(t), \tag{90}$$

which arises very often in various problems. When (90) is a linear equation, i.e., $p = 1$, the other parameters satisfy some conditions and $y(t)$ belong to a space of weighted continuous functions, Al-Saqabi and Kiryakova [125] have found the solutions of (90) in the explicit form with convolutional type integral involving Mittag–Leffler function. Here we give the explicit bound of the solutions of nonlinear equation (90) under some suitable conditions (see[171]).

Corollary (4.3.19) [140]. Let $y(t), f\ (t) \in C[0, +\infty)$, $p = q + \epsilon > 0$ be constants and $y(t)$ satisfy (90). Then for any constant $K > 0$ we have

$(i)\ if\ [\alpha, 2\alpha, 2\alpha(1\ + 3\alpha\ )] \in I$,

$$|y(t)| \leq \left\{ |f(t)| + \frac{|\lambda| \bar{M}_1^{\alpha}}{\Gamma(\alpha)} t^{\alpha(2\alpha+1)-1} \left[\bar{\mathcal{A}}_1^{1-\alpha}(t) + K^{\frac{\epsilon_1}{p}} \frac{|\lambda| \bar{M}_1^{\alpha}}{(1-\alpha)\Gamma(\alpha)} \bar{\mathrm{V}}_1^{-1}(\mathrm{t}) \right.\right.$$
$$\left.\left. \times \left(\int_0^t s^{\frac{\alpha(2\alpha+1)-1}{1-\alpha}}(s)\, \bar{\mathcal{A}}_1(s) \bar{V}_1(s) ds \right)^{1-\alpha} \right] \right\}^{\frac{1}{p}} 0 > t\ , \tag{91}$$

where

$$\bar{M}_1 = \frac{1}{2\alpha} B\left[\frac{3\alpha(2\alpha+1) - 1}{2\alpha^2}, \frac{2\alpha^2 - 1}{\alpha} \right],$$

$$\bar{A}(t) = \left(1 + \frac{\epsilon_1}{p}\right) K^{\frac{\epsilon_1}{p}} |f(t)| - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}}\ , \qquad \bar{\mathcal{A}}_1(t) \int_0^t \bar{\mathcal{A}}^{\frac{1}{1-\alpha}}(s) ds$$

and

$$\bar{V}_1(t) = exp\left[-\frac{(1-\alpha)K^{\frac{-\epsilon_1}{p(1-\alpha)}}}{2\alpha^2}\left(\frac{\bar{M}_1^{\alpha}|\lambda|}{\Gamma(\alpha)}\right)^{\frac{1}{1-\alpha}} t^{\frac{2\alpha^2}{1-\alpha}}\right];$$

(ii) if $[\alpha, 2\alpha, 2\alpha(1+3\alpha)] \in II$,

$$|y(t)| \leq \left\{|f(t)| + \frac{|\lambda|\bar{M}_2^{\frac{1+3\alpha}{1+4\alpha}}}{\Gamma(\alpha)} t^{4\alpha(5\delta+2)-1}\left[\bar{\mathcal{A}}_2^{\frac{\alpha}{1+4\alpha}}(t) + \frac{K^{\frac{\epsilon_1}{p}}\bar{M}_2^{\frac{1+3\alpha}{1+4\alpha}}(1+4\alpha)|\lambda|}{\alpha\Gamma(\alpha)}\right.\right.$$

$$\left.\left.\times \bar{V}_2^{-1}(t)\left(\int_0^t s^{2\alpha(4\delta+1)-1}(s)\,\bar{\mathcal{A}}_2(s)\bar{V}_2(s)ds\right)^{\frac{\alpha}{1+4\alpha}}\right]\right\}^{\frac{1}{p}} \quad 0 > t\,, \tag{92}$$

where

$$\bar{M}_2 = \frac{1}{2\alpha}B\left[\frac{2\alpha(12\alpha+7)+1}{2(1+4\alpha)}, \frac{4\alpha^2}{1+3\alpha^2}\right], \qquad \bar{\mathcal{A}}_2(t) \int_0^t \bar{\mathcal{A}}^{\frac{1+4\alpha}{\alpha}}(s)ds$$

and

$$\bar{V}_2(t) = exp\left[-\frac{K^{\frac{\epsilon_1(1+4\alpha)}{p\alpha}}\bar{M}_2^{\frac{1+3\alpha}{\alpha}}}{2\alpha(1+4\alpha)}\left(\frac{|\lambda|}{\Gamma(\alpha)}\right)^{\frac{1+4\alpha}{\alpha}} t^{\,2\alpha(1+4\alpha)}\right].$$

Proof. From (90) we have

$$|y|^p(t) \leq |f(t)| + \frac{|\lambda|}{\Gamma(\alpha)}t^{-6\alpha^2}\int_0^t (t^{2\alpha}-\tau^{2\alpha})^{\alpha-1}\,\tau^{2\alpha(3\alpha+1)-1}|y|^{p+\epsilon_1}(\tau)d(\tau).$$

An application of Corollary (4.3.16) ($with$ $a(t) = f(t), b(t) = \frac{|\lambda|}{\Gamma(\alpha)}t^{-6\alpha^2}, \alpha = 0$ and $\alpha = \frac{1}{6}$.

to the last inequality yields the desired estimations (91) and (92).

Chapter 5
Differintegral Equations with Theory and Class of Autoconvolution Equations of the Third Kind

An existence theorem is shown for the model equation with data and solutions of a general logarithmic form. Moreover, a singular perturbation problem for a related differintegral equation of first order to the model equation is studied which could serve as a basis for its regularization by the Lavrentiev method. Also uniqueness results for the linear convolution equations are extended to more general function spaces. Further, a special class of differintegral equations with autoconvolution integral and two classes of the linear singular Abel–Volterra equations are dealt with. We find by a change of variable specific verification estimates. We deduce a determination of the eigenvalues.

Sec(5.1) : Differintegral Equations with Autoconvolution Integral

J. M. Burgers [128] (for Burgers' turbulence see also [131, 132, 137]) studied an differintegral equation which can be reduced to the equation

$$y'(x)+\left(\frac{1}{2x}-\frac{1}{16}x^2\right)y(x)=\int_0^x y(\xi)\,y(x-\xi)d\xi,x>0, \tag{1}$$

with autoconvolution integral $I(y)=\int_0^x y(\xi)\,y(x-\xi)d\xi$ and derived a solution of this equation by series expansions in powers and exponentials.

In this section we deal with a general first order differintegral equation of the form

$$y'(x)+k(x)y(x) = \int_0^x a(x,\xi)\,y(\xi)y(x-\xi)d\xi+\int_0^x b(x,\xi)\,y(\xi)d\xi+g(x),x\in(0,T) \tag{2}$$

with given numbers $T\in(0,\infty)$ and given functions $k;a;b$ and g. Equation (2) comprises equations with singular coefficients of y at $x=0$ like (1) (Type I) as well as related equations with singular coefficients of y and $I(y)$ (Type II). For both types of equations we prove general existence and stability theorems applying the iteration method with weighted norms in the form of our chapter. We state the corresponding theorem from below in this Introduction. The theorem was formerly used in [127, 134, 136, 139] to study integral equations of the third kind with autoconvolution integral. In this way we obtain a nearly complete picture about the solvability of the integro-diferential equations on a fnite interval [0,T].

Further, following Burgers, we derive some solutions by power and exponential series expansions as a basis for a discussion of the asymptotics of the solutions at infnity and state basic asymptotic solutions of generalized Burgers' equation and a related equation of type II.

The existence proofs in this section are based on an existence theorem from [135] for operator equations of the form

$$y = f + G[y] + L[y, y] \qquad (3)$$

with a linear operator G and a bilinear operator L in a Banach space X endowed with the scale of norms $\|\mathcal{Z}\|_\sigma, \ \sigma \geq 0$ satisfying the condition

$$\lambda(\sigma)\|\mathcal{Z}\|_0 \leq \|\mathcal{Z}\|_0 \ \text{ for any } \mathcal{Z} \in X \ \text{ and } \ \sigma \geq \sigma_0 \geq 0 \qquad (4)$$

where $\lambda \in C(\mathbb{R}_+ \longrightarrow \mathbb{R}_+), \lambda > 0$, which we cite here as Lemma (5.1.1).

Lemma $(\mathbf{5.1.1})[\mathbf{150}]$. Let the linear operator $G : X \longrightarrow X$ and the bilinear operator $L : X \times X \longrightarrow X$ fulfill the inequalities

$$\|G[\mathcal{Z}]\|_\sigma \leq M(\sigma)\|\mathcal{Z}\|_\sigma, \qquad \sigma \geq \sigma_0 \qquad (5)$$

for any $\mathcal{Z} \in X$ with a continuous function M satisfying $M(\sigma) \longrightarrow 0$ as $\sigma \longrightarrow \infty$, and

$$\|L[\mathcal{Z}_1, \mathcal{Z}_2]\|_\sigma \leq N\|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma, \qquad \sigma \geq \sigma_0 \qquad (6)$$

with a constant N and

$$\|L[\mathcal{Z}_1, \mathcal{Z}_2]\|_\sigma \leq \begin{cases} v_1(\sigma)\|\mathcal{Z}_1\|\|\mathcal{Z}_2\|_\sigma \\ v_2(\sigma)\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\| \end{cases} \qquad (7)$$

with continuous functions $v_k, \ k = 1,2$, satisfying $v_k(\sigma) \longrightarrow 0$ as $\sigma \longrightarrow \infty$, for any pair $\mathcal{Z}_1, \mathcal{Z}_2 \in X$.

Then equation (3) has a uniquely determined solution $y \in X$. Moreover, for solutions y_1 and y_2, corresponding to functions $f = f_1$ and $f = f_2$, respectively, the stability estimate

$$\|y_1 - y_2\| \leq \Lambda(Q_1, Q_2)\|f_1 - f_2\| \qquad (8)$$

holds, where $Q_k = (\|f_k\|, \|G|f_k|\|), k = 1,2,$ and $\Lambda \in C(\mathbb{R}_+^4 \longrightarrow \mathbb{R})$, $\Lambda > 0$, and $\Lambda\,(x_1, \dots, x_4)$ is increasing in $x_1, \dots, x_4$.

We are going to study the equation (2) in several weighted functional spaces where the weight is defned by the main part of the coefficient k. In the sequel we will always assume that

$$k = \varkappa + B \ \text{ with } \ \varkappa \in \mathfrak{T}, B \in L^1(0, T) \qquad (9)$$

where

$$\mathfrak{T} = \bigcap_{\epsilon \in (0,T)} L^1(\epsilon, T). \qquad (10)$$

Important examples are $\varkappa(x) = \gamma x^{-\alpha}, \alpha > 0, \gamma \in \mathbb{R}$ and (with $T < 1$) $\varkappa(x) = \frac{\gamma}{x|lnx|}$ and $(x) = \frac{\gamma|lnx|}{x}, \gamma \in \mathbb{R}$. For Burger's equation (1) we have $\varkappa(x) = \frac{1}{2x}$ and $B(x) = \frac{1}{16}x^2, a \equiv 1, b \equiv 0$.

We remark that instead of $B \in L^1(0,T)$ we also can assume that B is (improperly) Riemann integrable with finite integral $\int_0^T B\,(x)dx$.

Let us defne the following basic functional spaces related to the coefficient $\varkappa \in \mathfrak{T}$:

$$L_\varkappa^p := \{u : \ e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \in L^p(0,T)\} \text{ with the norm } \|u\|_{L_\varkappa^p} = \left\| e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \right\|_{L^p(0,T)}, \tag{11}$$

$$C_\varkappa := \{u : \ e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \in C(0,T)\} \text{ with the norm } \|u\|_{C_\varkappa} = \left\| e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \right\|_{C[0,T]}, \tag{12}$$

$$W_\varkappa^1 := \{y : \ y \in C_\varkappa, y' \in \mathfrak{T}\} \tag{13}$$

Note that

$$W_0^1 = \{y : \ y \in C(0,T), y' \in \mathfrak{T}\}. \tag{14}$$

We will treat the case when equation (2) can be reduced to a family of integral equations of the second kind by means of solving it with respect to the left-hand side. For such a reduction we need the following lemmas.

Lemma $(\mathbf{5.1.2})[\mathbf{150}]$. If $\varphi \in L_\varkappa^1$, then the family of solutions of the equation $y'(x) + k(x)y(x) = \varphi(x),\ x \in (0,T)$, is in the space

$$W := \{y : \ y \in C(0,T], y' \in \mathfrak{T} \tag{15}$$

given by the formula

$$y(x) = Ke^{\int_x^T k(\xi)d\xi} + \int_0^x e^{\int_\xi^x k(\eta)d\eta}\, \varphi(\xi)d\xi, \qquad x \in (0,T), K \in \mathbb{R}. \tag{16}$$

Proof uses well-known arguments from the theory of linear ordinary differential equations.

Lemma $(\mathbf{5.1.3})[\mathbf{150}]$. If $g \in L_\varkappa^1$, then the equation (2) is in the space

$$S_{\varkappa,a,b} = \left\{ y : y \in W, \int_0^\cdot a(\cdot,\tau)\, y(\cdot - \tau) y(\tau) d\tau \in L_\varkappa^1, \qquad \int_0^\cdot b(\cdot,\tau)\, y(\tau) d\tau \in L_\varkappa^1 \right\} \tag{17}$$

equivalent to the following family of integral equations

$$y(x) = Ke^{\int_x^T k(\xi)d\xi} + \int_0^x e^{-\int_\xi^x k(\eta)d\eta}\, g(\xi)d\xi + \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi a(\xi,\tau)\, y(\tau)y(\xi-\tau)d\tau d\xi$$

$$+ \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi b(\xi,\tau)\, y(\tau)d\tau d\xi, \qquad x \in (0,T), \qquad (18)$$

where $K \in \mathbb{R}$ is an arbitrary parameter.

Proof. Denoting

$$\varphi[y](x) := g(x) + \int_0^x a(x,\tau)\, y(x-\tau)y(\tau)d\tau + \int_0^x b(x,\tau)\, y(\tau)d\tau,$$

the equation (2) can be rewritten in the form $y'(x) + k(x)y(x) = \varphi[y](x)$, $x \in (0,T)$. Further, by the assumptions of Lemma (3) the function $\varphi[y]$ belongs to $L^1_\varkappa$ for any $y \in S_{\varkappa,a,b}$. Now we observe that the assertion of Lemma (3) immediately follows from Lemma (5.1.2).

We can establish the behavior of the solution of (18) at $x \longrightarrow 0^+$, as well. Namely, the following lemma is valid:

Lemma $(\mathbf{5.1.4})[\mathbf{150}]$. Let $g \in L^1_\varkappa$, K be some real number and $y \in S_{\varkappa,a,b}$ solve (18). Then $y \in W^1_\varkappa$. Moreover, y has the property

$$\lim_{x \longrightarrow 0^+} e^{-\int_x^T \varkappa(\xi)d\xi}\, y(x) = A := Ke^{\int_0^T B(\xi)d\xi}. \qquad (19)$$

Proof. Multiplying (18) by $e^{-\int_x^T \varkappa(\xi)d\xi}$ and observing that $k = \varkappa + B$ we obtain

$$e^{-\int_x^T \varkappa(\eta)d\eta} y(x) = Ke^{\int_x^T B(\xi)d\xi} + \int_0^x e^{\int_\xi^x B(\xi)d\xi}\, e^{-\int_\xi^T \varkappa(\eta)d\eta} g(\xi)d\xi$$

$$+ \int_0^x e^{\int_\xi^x B(\xi)d\xi}\, e^{-\int_\xi^T \varkappa(\eta)d\eta} \int_0^\xi a(\xi,\tau)\, y(\tau)(\xi-\tau)d\tau d\xi$$

$$+ \int_0^x e^{-\int_\xi^x B(\xi)d\xi}\, e^{-\int_\xi^T \varkappa(\eta)d\eta} \int_0^\xi b(\xi,\tau)\, y(\tau)d\tau d\xi, \qquad x \in (0,T). \quad (20)$$

Observing that $B \in L^1(0,T)$, $g \in L^1_\varkappa$, $\int_0^{\cdot} a(\cdot,\tau)\, y(\cdot-\tau)y(\tau)d\tau \in L^1_\varkappa$, $\int_0^{\cdot} b(\cdot,\tau)\, y(\tau)d\tau \in L^1_\varkappa$ (cf. defnition of $S_{\varkappa,a,b}$), and the defnition of $L^1_\varkappa$ we see that the right-hand side of (5.1.20) belongs to $C[0,T]$. Thus, $y \in C_\varkappa$. Further, since $y' \in \mathfrak{T}$ for $y \in S_{\varkappa,a}$, we obtain $y \in W^1_\varkappa$. Finally, taking the limit $x \longrightarrow 0^+$in (20) we deduce (19).

We note that in case of positive $\varkappa(x)$ the solution space $W_{\varkappa}^{1}$ may contain functions that are singular at $x=0$. However, this singularity may be only integrable provided either a or b is bounded away from zero. This follows form the next lemma.

Lemma $(5.1.5)[150]$. Let there exist $\delta>0$ such that either the inequality

$$|a(x,\xi)|\geq\delta \quad for\ \ a.e.\ 0<\xi<x<\ T \tag{21}$$

or the inequality

$$|b(x,\xi)|\geq\delta \quad for\ \ a.e.\ 0<\xi<x<\ T \tag{22}$$

is fulfilled. Then any solution $y\in W_{\varkappa}^{1}$ of (18) belongs to the space $L^{1}(0,T)$.

Proof. Let $y\in W_{\varkappa}^{1}$ solve (18). If $y\equiv 0$ then $y\in L^{1}(0,T)$ trivially. Thus, let $y\not\equiv 0$. This in view of $W_{\varkappa}^{1}\subset C(0,T]$ implies that there exist $x_0\in(0,T), s\in(0,x_0)$ and $q>0$ such that

$$|y(x_0-\xi)|\geq q \quad for\ \ any\ \ \xi\in(0,s) \tag{23}$$

Let us prove the assertion of lemma in the case (21). From (18) we see that $a(\xi,\cdot)y(\cdot)y(\xi-\cdot)\in L^{1}(0,T)$ for a.e. $\xi\in(0,T)$. Evidently, we can choose x_0 so that $a(x_0,\cdot)y(\cdot)y(x_0-\cdot)y(\cdot)\in L^{1}(0,s)$. Due to (21) and (23). we have $|a(x_0,\xi)y(x_0-\xi)|\geq\delta q>0$ for any $\xi\in(0,s)$. Therefore, $y\in L^{1}(0,s)$. This with $y\in C(0,T]$ implies $y\in L^{1}(0,T)$. In case (22) the proof is similar.

In this Sections we will study the solvability of the family of equations (18) or, equivalently, the equation (2) mainly in the largest possible solution space $W_{\varkappa}^{1}$.

We are going to deal with two main types of the differintegral equation (2) :

Type I. The kernels $a(x,\xi)$ and $b(x,\xi)$ are integrable with respect to x and ξ.

Type II. The kernels a and b are representable in the form

$$a(x,\xi)=\varkappa(x)a_0(x,\xi),\ \ b(x,\xi)=\varkappa(x)b_0(x,\xi)\ , \tag{24}$$

where $a_0(x,\xi)$ and $b_0(x,\xi)$ are integrable with respect to x and ξ. We remark that equation (2) with a and b of the form (24) can be obtained from the differintegral equation of the third kind

$$\nu(x)y'(x)+\big(1+B_0(x)\big)y(x)$$
$$=\int_0^x a_0(x,\xi)\,y(\xi)y(x-\xi)d\xi+\int_0^x b_0(x,\xi)\,y(\xi)d\xi+h(x), x\in(0,T), \tag{25}$$

where $\nu\in C[0,T]$ with $\nu(0)=0, \nu(x)\neq 0$ for $0<x\leq T$ if we set $\varkappa(x)=\frac{1}{\nu(x)}$, $B(x)=\frac{B_0(x)}{\nu(x)}$, $g(x)=\frac{h(x)}{\nu(x)}$ and assume $\frac{B_0}{\nu}\in L^{1}(0,T]$.

The cases of non-positive and integrable $\varkappa$

Here we will study the equation (2) of type I in the cases when either $\varkappa$ is non-positive having possibly non-integrable singularity at $x = 0$ or $\varkappa \in L^1(0, T)$.

We start by proving a technical lemma.

Lemma$(\mathbf{5.1.6})[\mathbf{150}]$ .Let $l \in L^1(0,T)$, $l(x) \geq 0$ and $u_\sigma(x) = \int_0^x e^{-\sigma(x-\xi)}\, l(\xi) d\xi$. Then $u_\sigma \longrightarrow 0$ in $C[0,T]$ as $\sigma \longrightarrow \infty$.

Proof. To prove Lemma (5.1.6), we make use of the following general result (see[133]):

Lemma $(\mathbf{5.1.7})[\mathbf{150}]$. Let u_σ , $\sigma \geq 0$, be an equicontinuous family of functions in $C[0,T]$ such that $u_\sigma(x) \longrightarrow u(x)$ as $\sigma \longrightarrow \infty$ for any $x \in [0,T]$ where $u \in C[0,T]$. Then $u_\sigma \longrightarrow u$ in $C[0,T]$ as $\sigma \longrightarrow \infty$.

Let ω be the modulus of continuity of the continuous function $\nu(x) = \int_0^x l(\xi) d\xi$. Then, for any $\sigma \geq 0$ and $x_1 \leq x_2$ from $[0,T]$ we have

$$|u_\sigma(x_1) - u_\sigma(x_2)| = \int_{x_1}^{x_2} e^{-\sigma(x_2-\xi)}\, l(\xi) d\xi + \int_0^{x_1} \left(e^{-\sigma(x_2-\xi)} - e^{-\sigma(x_1-\xi)}\right) l(\xi) d\xi$$

$$\leq \int_{x_1}^{x_2} l(\xi) d\xi = \omega(|x_1 - x_2|)$$

because $l(\xi) \geq 0$ and $e^{-\sigma(x_2-\xi)} \leq 1$ for $0 \leq \xi \leq x_2$ and $e^{-\sigma(x_2-\xi)} - e^{-\sigma(x_1-\xi)} \leq 0$ for $0 \leq \xi \leq x_1 \leq x_2$. This implies that the family u_σ , $\sigma \geq 0$, is equicontinuous. Furthermore, since $e^{-\sigma(x-\xi)} l(\xi) \longrightarrow 0$ as $\sigma \longrightarrow \infty$ $a.e.\, \xi \in (0,x)$ for any $x \in [0,T]$ we have $u_\sigma(x) \longrightarrow 0$ as $\sigma \longrightarrow \infty$ for any $x \in [0,T]$. Consequently, by Lemma (5.1.6a) we obtain $u_\sigma \longrightarrow 0$ in $C[0,T]$ as $\sigma \longrightarrow \infty$.

Now we prove a theorem concerning the equation (2) in the case of non-positive $\varkappa$.

Theorem $(\mathbf{5.1.8})[\mathbf{150}]$ **.** Let $g \in L^1_\varkappa$ and $\varkappa(x) \leq 0, x \in (0,T)$. Assume that

$$\int_0^{\cdot} |a(\cdot,\xi)|\, d\xi, \quad \int_0^{\cdot} |b(\cdot,\xi)|\, d\xi \in L^1(0,T). \tag{26}$$

Then the equation (2) has a one-parametric family of solutions in the space $W^1_\varkappa$ with the parameter

$$A = \lim_{x \longrightarrow 0^+} e^{-\int_x^T \varkappa(\xi) d\xi}\, y(x) \in \mathbb{R}. \tag{27}$$

Any solution of (2) belongs to this family. Moreover, for solutions y_1 and y_2, corresponding to the functions $g = g_1$and $g = g_2$, respectively, and satisfying the initial condition

$$A = \lim_{x \to 0^+} e^{-\int_x^T \varkappa(\xi)d\xi}\, y_1(x) = \lim_{x \to 0^+} e^{-\int_x^T \varkappa(\xi)d\xi}\, y_2(x)$$

the stability estimate

$$\|y_1 - y_2\|_{C_\varkappa} \le \Lambda\big(|A|, \|f_1\|_{C_\varkappa}, \|f_2\|_{C_\varkappa}\big)\|f_1 - f_2\|_{C_\varkappa} \tag{28}$$

holds where

$$f_k(x) = \int_0^x e^{-\int_\xi^x k(\eta)d\eta}\, g_k(\xi)d\xi, \quad k = 1,2, \tag{29}$$

and

$$\Lambda \in C(\mathbb{R}_+^3 \longrightarrow \mathbb{R}), \Lambda > 0, \Lambda(x_1, x_2, x_3) - \text{increasing in } x_1, x_2, x_3. \tag{30}$$

Proof. Let us fix $k \in \mathbb{R}$ and rewrite the equation (18) in the form $y = f + G[y] + L[y, y]$, where

$$f(x) = Ke^{\int_x^T k(\xi)d\xi} + \int_0^x e^{-\int_\xi^x k(\eta)d\eta}\, g(\xi)d\xi, \tag{31}$$

$$G[y](x) = \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi b(\xi,\tau)\, y(\tau)d\tau d\xi, \tag{32}$$

$$L[y, z](x) = \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi a(\xi,\tau)\, y(\tau)z(\xi - \tau)d\tau d\xi. \tag{33}$$

Observing the decomposition $k = \varkappa + B$ from (31) we have

$$e^{-\int_x^T \varkappa(\xi)d\xi} f(x) = Ke^{\int_x^T B(\xi)d\xi} + \int_0^x e^{-\int_\xi^x B(\eta)d\eta}\, e^{-\int_\xi^T \varkappa(\eta)d\eta} g(\xi)d\xi. \tag{34}$$

Note that

$$e^{\int_\cdot^T B(\xi)d\xi} \in C[0,T],\ e^{-\int_\xi^x B(\eta)d\eta} \in C(\Delta_T) \text{ where } \Delta_T = \{(x,\xi){:}\, 0 \le \xi \le T, 0 \le \xi \le x \tag{35}$$

because $B \in L^1(0,T)$. Moreover, $e^{-\int_\cdot^T \varkappa(\eta)d\eta} g \in L^1(0,T)$. due to the assumption $g \in L_\varkappa^1$. Consequently, from (34) we see that $e^{-\int_\cdot^T \varkappa(\eta)d\eta} f \in C[0,T]$, hence

$$f \in C_\varkappa. \tag{36}$$

Further, we introduce the scale of norms

$$\|u\|_\sigma = \|e^{-\sigma}u\|_{C_\varkappa} = \left\|e^{-\sigma\cdot}e^{-\int_\cdot^T \varkappa(\eta)d\eta}u\right\|_{C[0,T]}, \sigma \geq 0, \qquad (37)$$

in the space $C_\varkappa$. This scale satisfies the condition (4) with $\lambda(\sigma) = e^{-\sigma T}$. Using the relation $k = \varkappa + B$ we compute

$$e^{-\sigma x}e^{-\int_x^T \varkappa(\eta)d\eta}L[\mathcal{Z}_1,\mathcal{Z}_2](x)$$

$$= e^{-\sigma x}e^{-\int_x^T \varkappa(\eta)d\eta}\int_0^x e^{-\int_\xi^x k(\eta)d\eta}\int_0^\xi a(\xi,\tau)\,\mathcal{Z}_1(\tau)\mathcal{Z}_2(\xi-\tau)d\tau d\xi$$

$$= \int_0^x e^{-\sigma(x-\xi)}\,e^{-\int_\xi^x B(\eta)d\eta}\Psi(\xi)d\xi \qquad (38)$$

where

$$\Psi(\xi) = \int_0^\xi a(\xi,\tau)\,e^{\int_\tau^\xi \varkappa(\eta)d\eta}\,e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta}$$

$$\times\, e^{-\sigma\tau}e^{-\int_\tau^T \varkappa(\eta)d\eta}\mathcal{Z}_1(\tau)e^{-\sigma(\xi-\tau)}e^{-\int_{\xi-\tau}^T \varkappa(\eta)d\eta}\mathcal{Z}_2(\xi-\tau)d\tau. \qquad (39)$$

Due to the assumption $\varkappa(\eta) \leq 0, \eta \in (0,T)$, the functions $e^{\int_\tau^\xi \varkappa(\eta)d\eta}$ and $e^{\int_{\xi-\tau}^\xi \varkappa(\eta)d\eta}$ are bounded by 1 for $0 < \tau < \xi < T$. This due to (26) implies

$$|\Psi(\xi)| \leq l_1(\xi)\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\|_\sigma, \qquad l_1(\xi)\int_0^\xi |a(\xi,\tau)|\,d\tau \in L^1(0,T). \qquad (40)$$

The relation (40) with (35) implies that the second row of 38) belongs to $C[0,T]$ provided $\mathcal{Z}_1,\mathcal{Z}_2 \in C_\varkappa$. Thus,

$$L[\mathcal{Z}_1,\mathcal{Z}_2] \in C_\varkappa \text{ for any } \mathcal{Z}_1,\mathcal{Z}_2 \in C_\varkappa. \qquad (41)$$

Similarly, from (32) we have

$$e^{-\sigma x}e^{-\int_x^T \varkappa(\eta)d\eta}G|\mathcal{Z}|(x) = \int_0^x e^{-\sigma(x-\xi)}\,e^{-\int_\xi^x B(\eta)d\eta}\Phi(\xi)d\xi \qquad (42)$$

where

$$\Phi(\xi) = \int_0^\xi b(\xi,\tau)\,e^{\int_\tau^\xi \varkappa(\eta)d\eta}\,e^{-\sigma(\xi-\tau)}e^{-\sigma\tau}\,e^{-\int_\tau^T \varkappa(\eta)d\eta}\mathcal{Z}(\tau)d\tau. \qquad (43)$$

Due to (26) and the boundedness of $e^{\int_\tau^\xi \varkappa(\eta)d\eta}$ we immediately obtain

$$|\Phi(\xi)| \le l_2(\xi)\|\mathcal{Z}\|_\sigma, \qquad l_2(\xi) = \int_0^\xi |b(\xi,\tau)|\,d\tau \in L^1(0,T). \tag{44}$$

The relation (44) with (35) implies that (42) belongs to $C[0,T]$ provided $y \in C_\varkappa$. Thus,

$$G[\mathcal{Z}] \in C_\varkappa \text{ for any } \mathcal{Z} \in C_\varkappa. \tag{45}$$

The relations (36), (41) and (45) show that the equation $y = f + G[y] + L[y,y]$ is well-defined in the space $C_\varkappa$.

Taking in (38) and (42) maximum over $\varkappa \in [0,T]$ and observing (40), (44) we obtain

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le \nu(\sigma)\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\|_\sigma, \ \ \|G[\mathcal{Z}]\|_\sigma \le \nu(\sigma)\|\mathcal{Z}\|_\sigma, \sigma \ge 0 \tag{46}$$

with

$$\nu(\sigma) = \max_{0\le\xi\le x\le T}\left|e^{-\int_\xi^x B(\eta)d\eta}\right| \|u_\sigma\|_{C[0,T]} \text{ and } u_\sigma(x) = \int_0^x e^{-\sigma(x-\xi)}\, l(\xi)d\xi \tag{47}$$

where $l = \max\{l_1, l_2\} \in L^1(0,T)$. Lemma (5.1.6) yields

$$\nu(\sigma) \to 0 \ \text{ as } \sigma \to \infty$$

Taking this relation and (46) into account and using Lemma (5.1.1) we come to the conclusion that the equation $y = f + G[y] + L[y,y]$ or, equivalently, (18) with given K has a unique solution in the space $C_\varkappa$. This solution is differentiable for any $x \in (0,T)$. Thus, $y \in W_\varkappa^1$.

Due to the assertion (19) of Lemma (5.1.4) the solution y corresponding to $K \in \mathbb{R}$ satisfies the condition (27) with $A = Ke^{\int_0^T B(\xi)d\xi}$.

Summing up, we have shown the existence of a one-parametric family of solutions of (18) in $W_\varkappa^1$ with the parameter (27). From the uniqueness of the solution for fixed $K \in \mathbb{R}$ and Lemma (5.1.4) we deduce that any solution of (18) belongs to the constructed family. Since by Lemma (5.1.3) equation (2) and the family of equations (18) are equivalent in $W_\varkappa^1$ all these statements remain valid for the equation (2), too. This proves the solvability assertions of Theorem (5.1.8).

Finally, the stability estimate (28) follows from the estimate (8) of Lemma (5.1.1) in view of the relation (31) for the function f in the equation $y = f + G[y] + L[y,y]$ and (19). Theorem is completely proved.

At the end we deal with the case $\varkappa \in L^1(0,T)$ without any assumptions about the sign of $\varkappa$. Note that in this case according to the definitions (11), (13). and (14). we have $L_\varkappa^1 = L^1(0,T)$ and $W_\varkappa^1 = W_0^1 = \{y : y \in C[0,T], y' \in \mathfrak{T}\}$.

Corollary $(\mathbf{5.1.9})[\mathbf{150}]$. Let $\varkappa \in L^1(0,T)$. Assume that (26) holds. Then (2) has a one-parametric family of solutions in the space W_0^1 with the parameter $A = y(0) \in \mathbb{R}$. Any solution of (2) belongs to this family and its derivative satisfies $y' \in L^1(0,T)$. Moreover, for solutions y_1 and y_2, corresponding to the functions $g = g_1$ and $g = g_2$, respectively, and satisfying the initial condition $A = y_1(0) = y_2(0)$ the stability estimate

$$\|y_1 - y_2\|_{C[0,T]} \leq \Lambda\big(|A|, \|f_1\|_{C[0,T]}, \|f_2\|_{C[0,T]}\big)\|f_1 - f_2\|_{C[0,T]} \qquad (48)$$

holds, where f_k are given in terms g_k by (29) and Λ is a function with properties (30).

Proof. Since $\varkappa \in L^1(0,T)$. we can decompose $k = \varkappa_1 + B_1$ with $\varkappa_1 = 0$, $B_1 = \varkappa + B \in L^1(0,T)$ and apply Theorem (5.1.8). with $\varkappa_1$ instead of $\varkappa$. This yields that (2) has a one-parametric family of solutions in the space $W_0^1 = \{y : y \in C[0,T],\ y' \in \mathfrak{T}\}$. with the parameter $A = y(0) \in \mathbb{R}$ and that any solution of (2) belongs to this family. Further, if y is the solution of (2) then due to the relations $g, k \in L^1(0,T)$, $\int_0^{\cdot} |a(\cdot,\xi)|\, d\xi$, $\int_0^{\cdot} |b(\cdot,\xi)|\, d\xi \in L^1(0,T)$ and $y \in C[0,T]$ all terms except for y' in (2) belong to $L^1(0,T)$. Thus, we have $y' \in L^1(0,T)$, too. Finally, the estimate (48) follows from (28).

Here we will study the equation (2) of type I provided $\varkappa$ is positive.

In case $e^{\int_x^T \varkappa(\eta)d\eta}$. is integrable at $x = 0$, we can prove a result that is similar to Theorem (5.1.1).

Theorem $(\mathbf{5.1.10})[\mathbf{150}]$. Let $g \in L_\varkappa^1$ and $\varkappa(x) > 0$, $x \in (0,T)$. Assume that

$$e^{\int_{\cdot}^T \varkappa(\eta)d\eta} \in L^1(0,T) \qquad (49)$$

and

$$\sup_{\xi \in (0,\cdot)} |a(\cdot,\xi)| \in L_\varkappa^\infty, \quad \sup_{\xi \in (0,\cdot)} |b(\cdot,\xi)| \in L_\varkappa^1 \qquad (50)$$

Then the assertions of Theorem (5.1.1) are valid.

Proof of this theorem repeats the proof of Theorem (5.1.8). The only difference is the way of deriving the relations (41), (45) and (46) for the operators L and G.

Let us start by deducing (41) for the operator L. We have the formula (38) with the function ψ that we rewrite in the following form:

$$\Psi(\xi) = e^{-\int_\xi^T \varkappa(\eta)d\eta} \int_0^\xi a(\xi,\tau)\, e^{\int_\tau^T \varkappa(\eta)d\eta}\, e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta}$$

$$\times\, e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta)d\eta} \mathcal{Z}_1(\tau) e^{-\sigma(\xi-\tau)} e^{-\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \mathcal{Z}_2(\xi-\tau)d\tau. \qquad (51)$$

By virtue of the assumptions (49) and (50), the definition of $L^{\infty}_{\varkappa}$ and the well-known relation for convolutions

$$\varphi_1, \varphi_2 \in L^1(0,T) \Longrightarrow \int_0^{\cdot} \varphi_1(\tau)\, \varphi_2(\cdot - \tau) d\tau \in L^1(0,T) \qquad (52)$$

we obtain

$$|\Psi(\xi)| \leq l_3(\xi) \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma ,$$

$$l_3(\xi) = \left\| \sup_{\tau \in (0,\cdot)} |a(\cdot,\tau)| \right\|_{L^{\infty}_{\varkappa}} \int_0^{\xi} e^{\int_\tau^T \varkappa(\eta) d\eta}\, e^{\int_{\xi-\tau}^T \varkappa(\eta) d\eta} d\tau \in L^1(0,T). \qquad (53)$$

In view of this relation and (35) the right hand side of (38) belongs to $C[0,T]$ provided $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{\varkappa}$. Thus, we obtain (41).

Next we show (45) for the quantity $G[z]$. We make use of the formula (42) that holds with the function Φ of the form

$$\Phi(\xi) = e^{-\int_\xi^T \varkappa(\eta) d\eta} \int_0^{\xi} b(\xi,\tau)\, e^{\int_\tau^T \varkappa(\eta) d\eta}\, e^{-\sigma(\xi-\tau)} e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta) d\eta} \mathcal{Z}(\tau) d\tau. \quad (54)$$

Due to the assumptions (49), (50) and the definition of $L^1_{\varkappa}$ we obtain

$$|\Phi(\xi)| \leq l_4(\xi) \|\mathcal{Z}\|_\sigma, l_4(\xi)$$

$$= \int_0^T e^{\int_\tau^T \varkappa(\eta) d\eta} d\tau\, e^{-\int_\xi^T \varkappa(\eta) d\eta} \sup_{\tau(0,\xi)} |b(\xi,\tau)| \in L^1(0,T). \qquad (55)$$

In view of this relation and (35) the right hand side of (42) belongs to $C[0,T]$ provided $\mathcal{Z} \in C_{\varkappa}$. Consequently, we get (45).

Finally, taking in (38) and (42) maximum over $x \in [0,T]$ and observing (53), (55) we obtain the estimates (46) with (47) where $l = \max\{l_3, l_4\} \in L^1(0,T)$ and $\nu(\sigma) \longrightarrow 0$ as $\sigma \longrightarrow \infty$ due to Lemma (5.1.6).

Before we continue investigating the set of solutions of equation (2) in $W^1_{\varkappa}$ in case $e^{\int_{\cdot}^T \varkappa(\eta) d\eta} \notin L^1(0,T)$, we study this equation in the space W^1_0 that is a subspace of $W^1_{\varkappa}$ provided $\varkappa(x) > 0$. (Indeed, according to the definition of $C_{\varkappa}$, $W^1_{\varkappa}$ and $\varkappa(x) > 0$ we have $\{u : u \in C_0 = C[0,T], u' \in \mathfrak{X}\} = W^1_0 \subseteq W^1_{\varkappa} = \{u : u \in C_{\varkappa}, u' \in \mathfrak{X}\}$.) Since the case $\varkappa \in L^1(0,T)$ was completely covered by Corollary (5.1.9), we treat only the case $\varkappa \notin L^1(0,T)$.

Theorem (5.1.11)[150]. Let $g \in L^1(0,T)$, $\varkappa(x) > 0$, $x \in (0,T)$ and $\varkappa \notin L^1(0,T)$. Assume that a and b satisfy (5.1.26). Then the equation (2) has a unique solution in the space W^1_0 . This solution has the initial value $y(0) = 0$. Moreover, for solutions y_1 and y_2, corresponding to the functions $g = g_1$ and $g = g_2$, respectively, the stability estimate

$$\|y_1 - y_2\|_{C[0,T]} \leq \Lambda\big(\|f_1\|_{C[0,T]}, \|f_2\|_{C[0,T]}\big)\|f_1 - f_2\|_{C[0,T]} \tag{56}$$

holds where f_k are given in terms g_k by (29) and

$$\Lambda \in C(\mathbb{R}_+^2 \longrightarrow \mathbb{R}), \Lambda > 0, \Lambda(x_1, x_2) - \text{increasing in } x_1, x_2. \tag{57}$$

Proof. Again, we rewrite the equation (18) in the form $y = f + G[y] + L[y, y]$, where f, G and L are given by $(31) - (33)$. Observing (.9), (35) and the assumptions of Theorem (5.1.11) we have

$$e^{\int_\xi^x k(\eta)d\eta} \in C(\Delta_T \backslash \{(0,0)\}) \cap L^\infty(\Delta_T)\,, \tag{58}$$

$$e^{\int_x^T k(\xi)d\xi} \longrightarrow \infty \text{ as } x \longrightarrow 0^+. \tag{59}$$

Thus, in view of $g \in L^1(0,T)$ from (31) we get

$$f \in C[0,T] \text{ in case } K = 0\,, \qquad f \notin C[0,T] \text{ in case } K \notin 0. \tag{60}$$

Introduce the scale of norms

$$\|u\|_\sigma = \|e^{-\sigma} u\|_{C[0,T]}\,, \sigma \geq 0\,, \tag{61}$$

in the space $C[0,T]$. This scale satisfies (4) with $(\sigma) = e^{-\sigma T}$. From (33) we obtain

$$e^{-\sigma x} L[\mathcal{Z}_1, \mathcal{Z}_2](x) = \int_0^x e^{-\sigma(x-\xi)} \; e^{-\int_\xi^x k(\eta)d\eta} \Psi(\xi) d\xi \tag{62}$$

where

$$\Psi(\xi) = \int_0^\xi a(\xi,\tau)\, e^{-\sigma\tau} \mathcal{Z}_1(\tau) e^{-\sigma(\xi-\tau)} \mathcal{Z}_2(\xi - \tau) d\tau. \tag{63}$$

Observing the assumption (26) we have

$$|\Psi(\xi)| \leq l_5(\xi) \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma, \;\; l_5(\xi) = \int_0^\xi |a(\xi,\tau)|\, d \in L^1(0,T). \tag{64}$$

In view of this relation and (58) the right hand side of (62) belongs to $C[0,T]$ provided $\mathcal{Z}_1, \mathcal{Z}_2 \in C[0,T]$. Thus,

$$L[\mathcal{Z}_1, \mathcal{Z}_2] \in C[0,T] \text{ for any } \mathcal{Z}_1, \mathcal{Z}_2 \in C[0,T]. \tag{65}$$

Similarly for the quantity $G[\mathcal{Z}]$ from (32) we have

$$e^{-\sigma x} G[\mathcal{Z}](x) = \int_0^x e^{-\sigma(x-\tau)}\, e^{-\int_\xi^x k(\eta)d\eta}\, \Phi(\xi) d\xi \tag{66}$$

where

$$\Phi(\xi) = \int_0^{\xi} b(\xi,\tau)\, e^{-\sigma(x-\tau)} e^{-\sigma\tau} \mathcal{Z}(\tau) d\tau. \tag{67}$$

Due to the assumption (26) we obtain

$$|\Phi(\xi)| \le l_6(\xi)\|\mathcal{Z}\|_\sigma, \quad l_6(\xi) = \int_0^{\xi} |b(\xi,\tau)|\, d\tau \in L^1(0,T). \tag{68}$$

Again, due to this relation and (58) the right hand side of (66) belongs to $C[0,T]$ provided $\mathcal{Z} \in C[0,T]$. Therefore,

$$G[\mathcal{Z}] \in C[0,T] \text{ for any } \mathcal{Z} \in C[0,T]. \tag{69}$$

Summing up, the relations (60), (65) and (69) show that the equation $y = f + G[y] + L[y,y]$ is well-defined in the space $C[0,T]$ in case $K = 0$ and has no solution in the space $C[0,T]$ in case $K \ne 0$. Therefore, we continue studying this equation in case $K = 0$.

Taking in (62) and (66) maximum over $x \in [0,T]$ and observing (64), (68) we obtain the estimates (46) with (47) where $l = \max\{l_5, l_6\} \in L^1(0,T)$. Lemma (5.1.6) yields

$$\nu(\sigma) \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow \infty.$$

Thus, by Lemma (5.1.1) the equation $y = f + G[y] + L[y,y]$ or, equivalently, (18) with $K = 0$ has a unique solution y in the space $C[0,T]$. This solution is differentiable for any $x \in (0,T)$, which implies $y \in W_0^1$. By Lemma (5.1.3) also y is the unique solution of (2) in W_0^1 . The property $y(0) = 0$ follows from the equality $y(t) = f(t) + G[y](t) + L[y,y](t)$, because its right-hand side equals 0 at $t = 0$ (cp. (31) with $K = 0$, (32), (33)) . Finally, the stability estimate (56) follows from the estimate (8) of Lemma (5.1.1) in view of the relation (31) for the function f in the equation $y = f + G[y] + L[y,y]$ and $K = 0$. Theorem is proved.

Now we return to the study of (18) in $W_\varkappa^1$.

Lemma $(\mathbf{5.1.12})[\mathbf{150}]$. Let $g \in L^1(0,T)$ and $\varkappa(x) > 0,\ x \in (0,T)$ Assume that

$$e^{\int_\cdot^T \varkappa(\eta)d\eta} \notin L^1(0,T) \tag{70}$$

and a, b satisfy the conditions

$$\sup_{\xi\in(0,\cdot)} |a(\cdot,\xi)| \in L^\infty(0,T), \qquad \sup_{\xi\in(0,\cdot)} |b(\cdot,\xi)| \in L^1(0,T). \tag{71}$$

Moreover, let either (21) or (22) holds with some $\delta > 0$. Then any solution $y \in W_\varkappa^1$ of (2) is a solution of (18) with $K = 0$ and belongs to $C[0T]$.

Proof. As in the proof of Theorem (5.1.11) we have the relation (58).Furthermore, Lemma (5.1.5) implies $y \in L^1(0,T)$. This means that the right-hand side of (18) must belong to $L^1(0,T)$. The relation $y \in L^1(0,T)$ with the assumptions (71) yields $\int_0^x a(x,\xi)\, y(\xi) y(x-\xi) d\xi$, $\int_0^x b(x,\xi)\, y(\xi) d\xi \in L^1(0,T)$. In view of these relations, the assumption $g \in L^1(0,T)$ and (58) the right-hand side of (18) besides the term $Ke^{\int_.^T \varkappa(\eta)d\eta}$ belongs to $C[0,T]$. Due to the assumption (70) the term $Ke^{\int_.^T \varkappa(\eta)d\eta} \in L^1(0,T)$ if and only if $K=0$. Consequently, we must have $K=0$. But then the whole right-hand side of (18) belongs to $C[0,T]$. This proves $y \in C[0,T]$.

As a corollary of Theorem (5.1.11) and Lemma (5.1.12) we can formulate the following theorem.

Theorem $(\mathbf{5.1.13})[\mathbf{150}]$**.** Let $g \in L^1(0,T), \varkappa(x) > 0\,, x \in (0,T)$ and (70) hold. Moreover, assume that a, b satisfy (71) and either (21) or (22) holds with some $\delta > 0$. Then the equation (2) has a unique solution in the space $W_\varkappa^1$. This solution belongs to W_0^1 and has the initial value $y(0) = 0$. Moreover, for solutions y_1 and y_2, corresponding to the functions $g = g_1$and $g = g_2$, respectively, the stability estimate (56) is valid where f_k are given in terms g_k by (29) and Λ is a function with the properties (57).

Examples $(\mathbf{5.1.14})[\mathbf{150}]$ **.**

Here we apply results to the equation (2) with the function $k = \varkappa + B$ where $B \in L^1(0,T)$ and

$$\varkappa(x) = \frac{\gamma}{x^\alpha |lnx|^\beta} \quad \text{in} \quad (0,T] \tag{72}$$

where $0 < T < 1$ and $\gamma \neq 0,\ \alpha \geq 0,\ \beta \in \mathbb{R}$ are constants. Using results of this section we can formulate the following statements concerning this equation.

i) Assume that $\gamma < 0,\ \alpha \geq 0,\ \beta \in \mathbb{R}$. Moreover, let $g \in L_\varkappa^1$ and a, b satisfy (26). Then Theorem (5.1.1) and in case $\varkappa \in L^1(0,T)$ Corollary (5.1.1) hold.

ii) Assume that either $\gamma < 0,\ 0 \leq \alpha < 1,\ \beta \in \mathbb{R}$ or $\gamma > 0,\ \alpha = 1,\ \beta > 0$ or $0 < \gamma < 1, \alpha = 1, \beta = 0$. Moreover, let again $g \in L_\varkappa^1$ and a, b satisfy (50). Then Theorem (5.1.10) and in case $\varkappa \in L^1(0,T)$Corollary (5.1.9) hold.

iii) Assume that either $\gamma > 0,\ \alpha = 1,\ \beta \leq 1$ or $\gamma > 0,\ \alpha > 1,\ \beta \in \mathbb{R}$. Moreover, let $g \in L^1(0,T)$ and a, b satisfy (26). Then Theorem (5.1.11) holds.

iv) Assume that either $\gamma > 0,\ \alpha > 1,\ \beta \in \mathbb{R}$ or $\gamma \geq 1,\ \alpha = 1,\ \beta \leq 0\ or.$ $0 < \gamma < 1, \alpha = 1, \beta < 0$ .Moreover, let $g \in L^1(0,T)$ and a, b satisfy

(5.1.71) and either (21) or (22) with some $\delta > 0$. Then Theorem (5.1.13) holds.

Let us describe more precisely the solution sets in particular cases under suitable assumptions on g, a and b.

If either $0 \le \alpha < 1, \beta \in \mathbb{R}$ or $\alpha = 1, \beta > 1$ then $\varkappa \in L^1(0,T)$, hence (2) has a one-parametric family of solutions in the space W_0^1 with the parameter $A = y(0) \in \mathbb{R}$,

if $\alpha = \beta = 1$then (2) has a one-parametric family of solutions in the space $\{y: |\ln x|^{-\gamma} y(x) \in C[0,T], y' \in \mathfrak{T}\}$ with the parameter

$$A = \lim_{x \to 0^+} |\ln x|^{-\gamma}\, y(x) \in \mathbb{R},$$

if either $\gamma < 0$, $\alpha = 1$, $\beta < 1$ or $\gamma > 0$, $\alpha = 1$, $0 < \beta < 1$ or $0 < \gamma < 1$, $\alpha = 1, \beta = 0$ then (2) has a one-parametric family of solutions in the space

$$\left\{y:\ x^{\frac{\gamma}{(1-\beta)|\ln x|^{\beta}}} y(x) \in C[0,T], y' \in \mathfrak{T}\right\}$$

with the parameter $A = \lim_{x \to 0^+} x^{\frac{\gamma}{(1-\beta)|\ln x|^{\beta}}}\, y(x) \in \mathbb{R}$

(in particular, if $\gamma < 1$, $\alpha = 1$, $\beta = 0$ then (2) has a one-parametric family of solutions in the space $\{y:\ x^{\gamma} y(x) \in C[0,T], y' \in \mathfrak{T}$ with the parameter $A = \lim_{x \to 0^+} x^{\gamma}\, y(x) \in \mathbb{R}$,

if $\gamma < 0$, $\alpha > 1$, $\beta \in \mathbb{R}$ then (2) has a one-parametric family of solutions in the space

$$\left\{y:\ e^{-\gamma(1-\alpha)^{\beta-1}\Gamma(1-\beta,(1-\alpha)|\ln x|)} y(x) \in C[0,T], y' \in \mathfrak{T}\right\}$$

with the parameter $A = \lim_{x \to 0^+} e^{-\gamma(1-\alpha)^{\beta-1}\Gamma(1-\beta,(1-\alpha)\ln|x|)} y(x) \in \mathbb{R}$

(in particular, in case $\gamma < 0$, $\alpha > 1$, $\beta = 0$ equation (2) has a one-parametric family of solutions in the space

$$\left\{y:\ e^{-\frac{\gamma}{(\alpha-1)x^{\alpha-1}}} y(x) \in C[0,T], y' \in \mathfrak{T}\right\}$$

with the parameter $A = \lim_{x \to 0^+} e^{-\frac{\gamma}{(\alpha-1)x^{\alpha-1}}} y(x) \in \mathbb{R}$.

Here $\Gamma(a,x) = \int_x^{\infty} t^{\alpha-1}\, e^{-t} dt$ is the (complementary) incomplete Gamma function.

Differintegral equation of type II

We deal with the equation (2) of type II in the cases of non-positive and integrable $\varkappa$. Let us start with the former case.

Theorem (5.1.15)[150]. Let $g \in L^1_\varkappa$ and $\varkappa(x) \le 0,\ x \in (0,T)$. Moreover, let

$$\varkappa(\tau) \le \varkappa(\xi) \quad for \quad 0 < \tau < \xi < T. \tag{73}$$

Assume that a and b have the form (24) where a_0 and b_0 satisfy

$$\sup_{\xi \in (0,\cdot)} |a_0(\cdot,\xi)| \in L^1(0,T), \qquad \sup_{\xi \in (0,\cdot)} |b_0(\cdot,\xi)| \in L^1(0,T). \tag{74}$$

Then assertions of Theorem (5.1.8) are valid.

Proof repeats the proof of Theorem (5.1.8). Only it is necessary to deduce again the relations (41), (45) and (46) for the operators L and G under the assumptions of Theorem (5.1.15).

We start with the relation (38) for L with the function Ψ given by (39). Due to the assumptions $\varkappa(\tau) \le 0$ and (73) we have $|\varkappa(\xi)| = -\varkappa(\xi) \le -\varkappa(\tau)$

for $0 < \tau \le \xi$. Thus,

$$|\varkappa(\xi)|\, e^{\int_\tau^\xi \varkappa(\eta)d\eta} \le \frac{d}{d\tau}\, e^{\int_\tau^\xi \varkappa(\eta)d\eta} \tag{75}$$

and from (39) in view of (24), (74) and the inequality $e^{\int_{\xi-\tau}^{\xi} \varkappa(\eta)d\eta} \le 1$

following from the assumption $\varkappa(\tau) \le 0$ we obtain

$$\begin{aligned}
|\Psi(\xi)| &\le \int_0^\xi |\varkappa(\xi)|\, |a_0(\xi,\tau)|\, e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{\int_{\xi-\tau}^{\xi} \varkappa(\eta)d\eta}\, d\tau \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma \\
&\le \sup_{\xi \in (0,\xi)} |a_0(\xi,\tau)| \int_0^\xi |\varkappa(\xi)| \frac{d}{d\tau}\, e^{\int_\tau^\xi \varkappa(\eta)d\eta}\, d\tau \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma \\
&= \sup_{\tau \in (0,\xi)} |a_0(\xi,\tau)| \left(1 - \lim_{\tau \to 0^+} e^{\int_\tau^\xi \varkappa(\eta)d\eta}\right) \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma \\
&\le l_7(\xi) \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma,\ \ l_7(\xi) = \sup_{\tau \in (0,\xi)} |a_0(\xi,\tau)| \in L^1(0,T).
\end{aligned} \tag{76}$$

The relation (76) with $B \in L^1(0,T)$ implies that the second row of (38) belongs to $C[0,T]$ provided $\mathcal{Z}_1, \mathcal{Z}_2 \in C_\varkappa$. Consequently, we obtain the relation (41).

Next we proceed to the formula (42) with $\Phi(x)$ given by (43). From (43) due to (41), (74), (75) and the inequality $e^{\int_{\xi-\tau}^{\xi} \varkappa(\eta)d\eta} \le 1$ we deduce

$$\Phi(\xi) \leq \sup_{\tau\in(0,\xi)} |b_0(\xi,\tau)| \int_0^{\xi} |\varkappa(\xi)| \frac{d}{d\tau}\, e^{\int_\tau^\xi \varkappa(\eta)d\eta}\, d\tau \|\mathcal{Z}\|_\sigma$$

$$\leq l_8(\xi)\|\mathcal{Z}\|_\sigma, \;\; l_8(\xi) = \sup_{\tau\in(0,\xi)} |b_0(\xi,\tau)| \,\epsilon L^1(0,T). \qquad (77)$$

The relation (77) with $B \in L^1(0,T)$ implies that the right-hand side of (42) belongs to $C[0,T]$ provided $y \in C_\varkappa$.. Thus, we get (45).

Finally, taking in (38) and (42) maximum over $x \in [0,T]$ and observing (76), (77) we obtain the estimates (46) with (47) where $l = \max\{l_7, l_8\} \in L^1(0,T)$ and $v(\sigma) \to 0$ as $\sigma \to \infty$.

Next we state a result concerning the case of integrable $\varkappa$.

Corollary (5.1.16)[150]. Let $g, \varkappa \in L^1(0,T)$. Assume that (24) and (74) hold. Then assertions of Corollary (5.1.9) are valid.

Proof is analogous to the proof of Corollary (5.1.9). We set $k = \varkappa_1 + B_1$ where $\varkappa_1 = 0$ and $B_1 = \varkappa + B$. Then $\varkappa_1$ satisfies the conditions $\varkappa_1(x) \leq 0$, (73) and the function B_1 belongs to $L^1(0,T)$, hence we can apply Theorem (5.1.13).

We deal with the equation (2) of type II in the case of positive $\varkappa$.Here we didn't succeed to prove the solvability in $W_\varkappa^1$ in case of arbitrary integrable $e^{\int_x^\xi \varkappa(\eta)d\eta}$ as in the case of type I. We were able to prove such a result only in the particular case when $\varkappa$ satisfies the condition

$$0 < \varkappa(x) \leq \frac{\gamma}{x}, x \in (0,T) \quad \text{with some } \gamma \in (0,1). \qquad (78)$$

Note that in this case the function $e^{\int_x^\xi \varkappa(\eta)d\eta}$ can have maximally a power-type integrable singularity at $t = 0$, i.e.

$$0 < e^{\int_x^\xi \varkappa(\eta)d\eta} \leq \frac{T^\gamma}{x^\gamma}, \qquad x \in (0,T).$$

Theorem (5.1.17)[150]. Let $g \in L_\varkappa^1$. and $\varkappa$ satisfies (78). Assume that a and b have the form (24) where a_0 and b_0 satisfy

$$x^{-\gamma} \sup_{\xi\in(0,x)} |a_0(x,\xi)| \,\epsilon L^1(0,T), \qquad \sup_{\xi\in(0,\cdot)} |b_0(x,\xi)| \,\epsilon L^1(0,T). \qquad (79)$$

Then assertions of Theorem (5.1.8) are valid.

Proof. Again, the proof repeats proof of Theorem (5.1.8). The only difference is the deduction of (41), (45) and (46).

To derive (41) we follow the equality (8) and rewrite the involved function Ψ in the form

$$\Psi(\xi) = \varkappa(\xi) \int_0^\xi a_0(\xi,\tau)\, e^{\int_\tau^\xi \varkappa(\eta)d\eta}\, e^{\int_{\xi-\tau}^\xi \varkappa(\eta)d\eta}$$

$$\times\, e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta)d\eta} \mathcal{Z}_1(\tau) e^{-\sigma(\xi-\tau)} e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \mathcal{Z}_2(\xi-\tau)d\tau. \quad (80)$$

Owing to the assumption (78) we have

$$\int_0^\xi e^{\int_\tau^\xi \varkappa(\eta)d\eta}\, e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} d\tau \le \xi^\gamma T^\gamma \int_0^\xi \tau^{-\gamma}\,(\xi-\tau)^{-\gamma} d\tau = B(1-\gamma,1-\gamma)T^\gamma \xi^{1-\gamma}$$

where B is the Beta function. Using this estimate and $\varkappa(\xi) \le \frac{\gamma}{\xi}$ as well as the assumption (79) in (80) we obtain

$$|\Psi(\xi)| \le l_9(\xi)\|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma\,,$$

$$l_9(\xi) = \gamma T^\gamma B(1-\gamma,1-\gamma)\xi^{-\gamma} \sup_{\tau\in(0,\xi)} |a_0(\xi,\tau)| \in L^1(0,T) \quad (81)$$

In view of this relation and (35) the right hand side of (38) belongs to $C[0,T]$ provided $z_1, z_2 \in C_\varkappa$. Thus, we have deduced (41).

Next we consider (42) where we rewrite Φ as follows.

$$\Phi(\xi) = \varkappa(\xi) \int_0^\xi b_0(\xi,\tau)\, e^{\int_\tau^\xi \varkappa(\eta)d\eta}\, e^{-\sigma(\xi-\tau)} e^{-\sigma\tau}\, e^{-\int_\tau^T \varkappa(\eta)d\eta} \mathcal{Z}(\tau)d\tau. \quad (82)$$

Due to the assumption (78) we have $\int_0^\xi e^{\int_\tau^\xi \varkappa(\eta)d\eta}\, d\tau \le \frac{\xi}{1-\gamma}$. By this relation, $\varkappa(\xi) \le \frac{\gamma}{\xi}$ and (79) from (82) we deduce

$$|\Phi(\xi)| \le l_{10}(\xi)\|\mathcal{Z}\|_\sigma, \qquad l_{10}(\xi) = \frac{\gamma}{1-\gamma} \sup_{\tau\in(0,\xi)} |b_0(\xi,\tau)| \in L^1(0,T). \quad (83)$$

In view of this relation and (35) the right-hand side of (42) belongs to $C[0,T]$ provided $z \in C_\varkappa$. This means that we have proved (45).

Finally, from (38) and (42) with the help of (81) and (83) we deduce (46) with (47) where $l = \max\{l_9, l_{10}\} \in L^1(0,T)$ and $\nu(\sigma) \to 0$ as $\sigma \to \infty$.

The analogue of Theorem (5.1.11) in the case of equation of type II is as follows.

Theorem (5.1.18)[150] . Let $g \in L^1(0,T)$, $\varkappa(x) > 0$, $x \in (0,T)$ and $\varkappa \notin L^1(0,T)$. Assume that a and b have the form (35) where a_0 and b_0 satisfy the conditions

$$\sup_{x\in(\cdot,T)} |a_0(x,\cdot)| \epsilon L^1(0,T), \qquad \sup_{x\in(\cdot,T)} |b_0(x,\cdot)| \epsilon L^1(0,T). \tag{84}$$

Then the assertions of Theorem (5.1.11) are valid.

Proof is partially similar to proof of Theorem (5.1.11). We write the equation (18) in the form $y = f + G[y] + L[y,y]$, where f, G and L are given by (31) - (33). Then have the relations (58), (59) and the function f given by (31) satisfies (60). To analyze the equation in the space $C[0,T]$ we make use of the scale of norms (61).

From (33) using the relations $k = \varkappa + B$ and $a_0(\xi,\eta) = \varkappa(\xi)a_0(\xi,\eta)$ we obtain

$$e^{-\sigma x} L[\mathcal{Z}_1,\mathcal{Z}_2](x)$$
$$= \int_0^x e^{-\sigma(x-\xi)}\, e^{\int_\xi^x B(\eta)d\eta} \varkappa(\xi) e^{-\int_\xi^x \varkappa(\eta)d\eta} \int_0^\xi a_0(\xi,\tau)\, e^{-\sigma\tau} \mathcal{Z}_1(\tau) e^{-\sigma(x-\xi)} \mathcal{Z}_2(\xi-\tau)d\tau d\xi \tag{85}$$

Observing the equality $\varkappa(\xi)e^{-\int_\xi^x \varkappa(\eta)d\eta} = \frac{d}{d\xi} e^{-\int_\xi^x \varkappa(\eta)d\eta}$, the assumption (84), the positivity of $\varkappa$ and the equality $\lim_{\xi\to 0^+} e^{-\int_\xi^x \varkappa(\eta)d\eta} = 0$ that holds in view of (9) and $\varkappa \notin L^1(0,T)$) from (85) we deduce the estimate

$$|e^{-\sigma x} L[\mathcal{Z}_1,\mathcal{Z}_2](x)| \le C_B \int_0^x \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)| \int_0^x d\left(e^{-\int_\xi^x \varkappa(\eta)d\eta}\right) \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma$$
$$= C_B \int_0^x \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)|\, d\tau \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma \tag{86}$$

where $C_B = \max_{0\le\xi\le x\le T} e^{-\int_\xi^x B(\eta)d\eta}$. This implies

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le N \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma, \qquad \sigma \ge 0, \tag{87}$$

with the positive constant $N = C_B \int_0^x \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)|\, d\tau$. Moreover, (86) yields the continuity of $L[\mathcal{Z}_1,\mathcal{Z}_2](x)$ at $x = 0$ and the equality $L[\mathcal{Z}_1,\mathcal{Z}_2](0) = 0$. The continuity of $L[\mathcal{Z}_1,\mathcal{Z}_2](x)$ in $(0,T]$ follows from the continuity in the domain $\Delta_T\backslash\{(0,0)\}$ of the function of arguments (x,ξ) under the integral $\int_0^x$ in the right-hand side of the formula (85). Thus,

$$L[\mathcal{Z}_1,\mathcal{Z}_2] \in C[0,T] \text{ for any } \mathcal{Z}_1,\mathcal{Z}_2 \in C[0,T]. \tag{88}$$

Analogously to (86) from (85) we derive the estimates

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le \begin{cases} \nu_1(\sigma)\|\mathcal{Z}_1\|_0\|\mathcal{Z}_2\|_\sigma \\ \nu_2(\sigma)\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\|_0 \end{cases}, \qquad \sigma \ge 0, \tag{89}$$

where

$$\nu_1(\sigma) = C_B \max_{x\in[0,T]} \int_0^x e^{-\sigma(x-\xi)} \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)|\, d\tau \ ,$$

$$\nu_2(\sigma) = C_B \int_0^T e^{-\sigma\tau} \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)|\, d\tau.$$

Due to the assumption (84) and Lemma (5.1.6) we have

$$\nu_k(\sigma) \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow \infty, \quad k = 1,2. \tag{90}$$

For the quantity $G[\mathcal{Z}]$ in view of (24) from (32) we get

$$e^{-\sigma x} G[\mathcal{Z}](x)$$

$$= \int_0^x e^{-\sigma(x-\xi)}\, e^{-\int_\xi^x B(\eta)d\eta} \varkappa(\xi) e^{-\int_\xi^x \varkappa(\eta)d\eta} \int_0^\xi b_0(\xi,\tau)\, e^{-\sigma(\xi-\tau)} e^{-\sigma\tau} \mathcal{Z}(\tau) d\tau d\xi. \tag{91}$$

Similarly as above we deduce the estimate

$$|e^{-\sigma x} G[\mathcal{Z}](x)| \le C_B \max_{0\le y\le x} \int_0^y e^{-\sigma(y-\tau)} \sup_{\xi\in(\tau,T)} |b_0(\xi,\tau)|\, d\tau \|\mathcal{Z}\|_\sigma \tag{92}$$

Thus, we obtain

$$\|G[\mathcal{Z}]\|_\sigma \le M(\sigma)\|\mathcal{Z}\|_\sigma \ , \ \ \sigma \ge 0, \tag{93}$$

with

$$M(\sigma) = C_B \max_{0\le y\le T} \int_0^y e^{-\sigma(y-\tau)} \sup_{\xi\in(\tau,T)} |b_0(\xi,\tau)|\, d\tau.$$

Due to Lemma (5.1.6) we have

$$M(\sigma) \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow \infty. \tag{94}$$

Moreover, (2) implies that $G[\mathcal{Z}](x)$ is continuous at $x = 0$ and $G[\mathcal{Z}](0) = 0$. The continuity of $G[\mathcal{Z}](x)$ in $(0,T]$ follows from the continuity in the domain $\Delta_T\backslash\{(0,0)\}$ of the function of arguments (x,ξ) under the integral $\int_0^x$ in the right-hand side of the formula (91). Summing up,

$$G[\mathcal{Z}] \in C[0,T] \quad \text{for any} \quad \mathcal{Z} \in C[0,T] \tag{95}$$

The relations (60), (88) and (95) show that the equation $y = f + G[y] + L[y,y]$ is well-defined in the space $C[0,T]$ in case $K = 0$ and has no solution in the space $C[0,T]$ in case $K \neq 0$. Therefore, we take into consideration only the case $K = 0$. Observing (87), (89), (90), (93), (94) and Lemma (5.1.1) we see that the equation $y = f + G[y] + L[y,y]$ or, equivalently, (18) with $K = 0$ has a unique solution in the space $C[0,T]$. The rest of the proof is identical to the proof of Theorem (5.1.11).

We emphasize that Theorem (5.1.13) cannot be extended to the equation of type II. This means that in case $e^{\int_x^T \varkappa(\eta)d\eta} \notin L^1(0,T)$ solutions may exist in $W_\varkappa^1 \setminus W_0^1$,too. We provide a corresponding example.

Examples (5.1.19)[150]

Firstly, let us analyse the equation (2) of type II with $\varkappa$ of the form (72). For the sake of shortness we consider only the case $\beta = 0$, i.e.

$$\varkappa(x) = \frac{\gamma}{x^\alpha} \quad \text{in} \quad (0,\mathrm{T}] \tag{96}$$

with $\gamma \neq 0$ and $\alpha \geq 0$. We can formulate the following statements for this equation.

i) Assume that $\gamma < 0$ ¸ $\alpha \geq 0$. Moreover, let $g \in L_\varkappa^1$ and a_0, b_0 satisfy (74). Then Theorem (5.1.15) and in case $\varkappa \in L^1(0,T)$ Corollary (5.1.16) hold.

ii) Assume that either $\gamma > 0, 0 \leq \alpha < 1$ *or* $0 < \gamma < 1,\ \alpha = 1$. Moreover, let $g \in L_\varkappa^1$ and a_0, b_0 satisfy (79). Then Theorem (5.1.17). and in case $\varkappa \in L^1(0,T)$ Corollary (5.1.16). hold.

iii) Assume that > 0 , $\alpha \geq 1$. Moreover, let $g \in L^1(0,T)$ and a_0, b_0 satisfy (84). Then Theorem (5.1.18) holds.

More precisely, the solution sets in particular cases under suitable assumptions on g, a_0 and b_0 are as follows.

If $0 \leq \alpha < 1$ then $\varkappa \in L^1(0,T)$, hence (2) has a one-parametric family of solutions in the space W_0^1 with the parameter $A = y(0) \in \mathbb{R}$,

if $\gamma < 1,\ \alpha = 1$ then (2) has a one-parametric family of solutions in the space $\{y :\ x^\gamma y(x) \in C[0,T], y' \in \mathfrak{T}$ with the parameter $A = \lim_{x\to 0^+} x^\gamma\, y(x) \in \mathbb{R}$,

if $\gamma < 0,\ \alpha = 1$ then (2) has a one-parametric family of solutions in the space

$$\{y :\ e^{-\frac{\gamma}{(\alpha-1)x^{\alpha-1}}} y(x) \in C[0,T], y' \in \mathfrak{T}\}$$

with the parameter $A = \lim\limits_{x\to 0^+} e^{-\frac{\gamma}{(\alpha-1)x^{\alpha-1}}} y(x) \in \mathbb{R}$.

Further, let us analyse the equation (2) of type II with $\varkappa$ of the form (96) more closely in the case $\gamma = \alpha = 1$. Then $e^{\int_x^T \varkappa(\eta)d\eta} = \frac{T}{x} \notin L^1(0,T)$. We show that there exist non-continuous solutions to this equation in $W_\varkappa^1 \backslash W_0^1$.

The basic equation of this form is

$$y'(x) + \frac{y(x)}{x} = \frac{1}{x}\int_0^x y(x-\xi)\, y(\xi)d\xi, \quad x > 0. \tag{97}$$

It is easy to check (using Laplace transform) that this equation has besides the solution $y_0 \equiv W_0^1$ the one-parametric family of solutions

$$y_k(x) = \frac{1}{K}\nu'\left(\frac{x}{K}\right), \quad K > 0 \tag{98}$$

with the Volterra's function

$$\nu(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)}dt.$$

The function ν and its derivatives ν', ν'' have the asymptotic expansions

$$\nu(x) \sim -\frac{1}{\ln x} + \frac{C}{\ln^2 x} \quad \text{as} \quad x \to 0^+$$

where $C \sim \Gamma'(1)$ is the Euler constant and

$$\nu'(x) \sim \frac{1}{x\ln^2 x} - \frac{2C}{x\ln^3 x} \quad \text{as} \quad x \to 0^+, \tag{99}$$

$$\nu''(x) \sim \frac{1}{x^2\ln^2 x} - \frac{2(1-C)}{x^2\ln^3 x} \quad \text{as} \quad x \to 0^+. \tag{100}$$

The sign $\sim$ denotes the asymptotic equality. In particular, for the solutions (98) the asymptotic expansion

$$y_K(x) \sim \frac{1}{x\ln^2 x} + \frac{2(\ln K - C)}{x\ln^3 x} \quad \text{as} \quad x \to 0^+. \tag{101}$$

holds. Thus, $y_K \in W_\varkappa^1 \backslash W_0^1$ for any $K > 0$ in every finite interval $(0,T)$.

Finally, we study a more general equation

$$y'(x) + \left(\frac{1}{x} + B(x)\right)y(x) = \frac{1}{x}\int_0^x a_0(x,\xi)y(x-\xi)\, y(\xi)d\xi + g(x), \quad x \in (0,T), \tag{102}$$

where $0 < T < 1$ and

$$B(x) \in L^1(0,T), \qquad |\ln x|^{\delta} \sup_{\xi \in (0,x)} |a_0(x,\xi)| \in L^{\infty}(0,T), \qquad x\ln^2 x\, g(x) \in L^1(0,T) \tag{103}$$

with some $\delta > 0$. We seek a solution in the form $y(x) = \rho(x)\, w(x)$ where $\rho(x) = \nu'(x)$ is the above solution $y_1(x)$ of equation (97) and $w \in W_0^1$ is the unknown function. The function w obeys the differintegral equation

$$w'(x) + k_1(x) w(x) = \int_0^x a_1(x,\xi)\, w(x-\xi) w(\xi) d\xi + g_1(x)\,, x \in (0,T), \tag{104}$$

where

$$k_1(x) = \varkappa_1(x) + B(x) \quad \text{with} \quad \varkappa_1(x) = \frac{1}{x} + \frac{\nu''(x)}{\nu'(x)},$$

$$a_1(x,\xi) = \frac{1}{x\nu'(x)} a_0(x,\xi)\nu'(\xi)\nu'(x-\xi) \quad \text{and} \quad g_1(x) = \frac{g(x)}{\nu'(x)}.$$

We are going to show that under the assumptions (103) the conditions of Theorem (5.1.11) for equation (104) are fulfilled. By (99) we have $g_1 \in L^1(0,T)$. Further, by (99) and (100) the asymptotic relation

$$\varkappa_1(x) \backsim \frac{1}{x} - \frac{\frac{1}{x^2\ln^2 x} + \frac{2(1-C)}{x^2\ln^3 x}}{\frac{1}{x^2\ln^2 x} - \frac{2C}{x\ln^3 x}} \backsim -\frac{2}{x\ln x} \quad \text{as} \quad x \to 0^+$$

holds implying $\varkappa \notin L^1(0,T)$ together with $B \in L^1(0,T)$. Finally, observing the solution $y_1 = \nu'$ of (97) and using its positivity we again have

$$\int_0^x |a_1(x,\xi)|\, d\xi \le \frac{C_{a_1}}{x\nu'(x)|\ln x|^{\delta}} \int_0^x \nu'(\xi)\, \nu'(x-\xi) d\xi$$

$$= \frac{C_{a_1}}{x\nu'(x)|\ln x|^{\delta}} [x\nu''(x) + \nu'(x)] = C_{a_1} \frac{\varkappa_1(x)}{|\ln x|^{\delta}} \in L^1(0,T)$$

with the constant $C_{a_1} = \sup_{0<\xi<x<T} |\ln x|^{\delta}\, |a_0(x,\xi)| \in L^{\infty}(0,T)$. Hence, applying Theorem (5.1.3), we prove

Theorem (5.1.20)[150]. Let the assumptions (103) hold. Then equation (102) has a solution of the form $y(x) = \nu'(x)\, w(x)$ where $w \in W_0^1$ with $w(0) = 0$.

Series expansion of solutions and asymptotics at infinity

Exponential series

Burgers' equation

$$y'(x) + \left(\frac{1}{2x} + \beta_0 x^2\right) y(x) = \int_0^x y(\xi) y(x-\xi)\, d\xi, \quad (\beta_0 > 0) \quad (105)$$

has a unique solution $y = y_A$ in $(0, \infty)$ which fulfills the initial condition

$$x^{\frac{1}{2}} y(x)\Big|_{x=0} = A \quad (106)$$

for prescribed A. Following Burgers we are looking for a solution in form of the (for $x > 0$ convergent) series

$$y(x) = Kx \sum_{n=1}^{\infty} e^{-\alpha_n x} \quad (107)$$

with $K \in \mathbb{R}$ and $0 < \alpha_1 < \alpha_2 < \cdots$ for some $A \in \mathbb{R}$ in (106).

Inserting the ansatz (107) into equation (105), we get the equation for K and $\alpha_n,\ n = 1,2,\ldots$

$$\frac{3}{2} K \sum_{n=1}^{\infty} e^{-\alpha_n x} - Kx \sum_{n=1}^{\infty} \alpha_n e^{-\alpha_n x} - K\beta_0 x^3 \sum_{n=1}^{\infty} e^{-\alpha_n x}$$
$$= \frac{K^2}{6} x^3 \sum_{n=1}^{\infty} e^{-\alpha_n x} + 2K^2 x \sum_{n=1}^{\infty} \left(\sum_{m \neq n} \frac{1}{(\alpha_n - \alpha_m)^2} \right) e^{-\alpha_n x}$$
$$+ 4K^2 \sum_{n=1}^{\infty} \left(\sum_{m \neq n} \frac{1}{(\alpha_n - \alpha_m)^3} \right) e^{-\alpha_n x}$$

which is satisfied if $K = -6\beta_0$ and

$$\sum_{m \neq n} \frac{1}{(\alpha_m - \alpha_n)^2} = \frac{\alpha_n}{12\beta_0}, \quad \sum_{m \neq n} \frac{1}{(\alpha_m - \alpha_n)^3} = \frac{\alpha_n}{16\beta_0}. \quad (108)$$

Comparison with (129) shows that the relations (108) are fulfilled if ¸ $\lambda = \frac{1}{3\sqrt{\beta_0}}$ in (129) i.e. if α_n are chosen as the zeros of the entire function $q_0 = z^{1/2} K_{1/3}\left(\frac{1}{3\sqrt{\beta_0}} z^{3/2}\right)$.

It remains to calculate the initial condition (106) for this solution (107) of equation (105). In view of (123) we have

$$\varphi(x)=\sum_{n=1}^{\infty}e^{-\alpha_n x}\sim\varphi_0(x)=\sum_{n=1}^{\infty}\exp\left[-\left(\frac{\pi}{\lambda}\right)^{2/3}\ xn^{2/3}\right]$$

and by

$$\varphi_0(x)\sim\frac{3}{4}\frac{\lambda}{\sqrt{\pi}}x^{3/2}\quad\text{as}\quad x\to 0^+,$$

hence

$$\varphi_0(x)\sim Ax^{-1/2}\,,\qquad A=\frac{3}{4}\,\lambda\frac{K}{\sqrt{\pi}}=-\frac{3}{2}\sqrt{\frac{\beta_0}{\pi}}\,. \tag{109}$$

Power series

The generalized Burgers' equation

$$y^{'}(x)+\left(\frac{1}{2x}+wx^{\frac{1}{2}}-\beta_0x^2\right)y(x)=\int_0^x y(\xi)\,y(x-\xi)d\xi \tag{110}$$

where $w,\beta_0\in\mathbb{R}$ also has a unique continuous solution $y=y_A$ on $(0,\infty)$ satisfying the initial condition (106) for prescribed $A\in\mathbb{R}$. The solution is of the form

$$y(x)=Ax^{-1/2}+x^{-1/2}\,z(x)$$

with a continuous function z on $[0,\infty)$ where $z(0)=0$. Like Burgers for $w=0$ we are looking for the solution in form of a power series

$$y(x)=\sum_{n=0}^{\infty}a_n\,x^{-1/2+3/2n},\qquad a_0=A. \tag{111}$$

Inserting the ansatz (111) into equation (110), we obtain $a_1=\frac{2}{3}(\pi A^2-wA)$

and the recurrence system for $a_2,a_3,\ldots$

$$\frac{3}{2}(n+1)a_{n+1}+wa_n-\beta_0a_{n-1}=\sum_{m=0}^{n}a_m\,a_{n-m}B_{m,n-m},\quad n=1,2,\ldots$$

where $B_{m,k}=\left(\frac{3}{2}m+\frac{1}{2},\frac{3}{2}k+\frac{1}{2}\right)$ with the Euler Beta function $B(p,q)$.

We ask for a solution of the simple form

$$y(x)=Ax^{-1/2}+a_1x\,,\ \ a_1=\frac{2}{3}(\pi A^2-wA). \tag{112}$$

Such a solution exists if

$$\text{either } w = \pi A,\ \beta_0 = 0 \quad \text{or } w = \frac{5}{2}A,\ \beta_0 = -\frac{1}{9}\left(\pi - \frac{5}{2}\right)A^2 \tag{113}$$

where $a_1 = 0$ or $a_1 = \frac{2}{3}\left(\pi - \frac{5}{2}\right)A^2$, respectively.

Asymptotics at infinity

The representation (107) of the solution y_A to equation (105) with (106) for $A = -\frac{3}{2}\sqrt{\frac{\beta_0}{\pi}}$ is an asymptotic expansion for y_A as $x \to +\infty$ and yields the asymptotic representation

$$y_A(x) \sim Kxe^{-\alpha_1 x} \quad \text{as} \quad x \to +\infty. \tag{114}$$

Formula (114) shows that y_A tends (exponentially) to zero at infinity.

The function $\tilde{y}(x) = e^{\epsilon x} y_A(x), \epsilon \in \mathbb{R}$, satisfies the differintegral equation

$$\tilde{y}'(x) + \left(\frac{1}{2x} - \beta_0 x^2 - \epsilon\right)\tilde{y}(x) \int_0^x \tilde{y}(\xi)\, \tilde{y}(x-\xi) d\xi \tag{115}$$

and the same initial condition (106) as y_A. But although the coefficient of $\tilde{y}$ in (115) has analogous asymptotic behavior like the coeffcient of y_A in (105) as $x \to +\infty$ we have a different asymptotic behavior of $\tilde{y}$ and y_A as $x \to +\infty$ for $\epsilon \neq 0$. In particular, for $\epsilon > \alpha_1$ the function $\tilde{y}$ tends (exponentially) to infinity as $x \to +\infty$.

Further, the solution (112) of equation (110) under the conditions (113) with $\beta_0 < 0$ behaves like $a_1 x$ as $x \to +\infty$, i.e. also tends to infinity and the solutions (8) of equation (97) are asymptotically equal to $\frac{1}{K}e^{\frac{x}{K}}$ as $x \to +\infty$. Since it seems difficult to obtain a more or less complete picture about the asymptotic behavior of the exact solutions as $x \to +\infty$ we shall study asymptotic solutions for two classes of differintegral equations in the this section. This means the left-hand side of equation (2) is asymptotically equal to the right-hand side but not necessarily it is the asymptotic representation of an exact solution.

Asymptotic solutions

Differintegral equations of type I

We consider the class of equations

$$y'(x) + k(x)y(x) = \int_0^x y(\xi)\, y(x-\xi) d\xi \tag{116}$$

where

$$k(x) = \frac{1}{x^{\alpha}}\left[\gamma + \delta x^{\beta} + b(x)\right] \tag{117}$$

with $0 < \alpha < \beta, \gamma \neq 0$ and $B(x) = o\left(x^{\beta}\right)$ as $x \to +\infty$, $B(x) = o(1)$ as $x \to 0^{+}$. Equation (116) with (117) has the asymptotic solutions as $x \to +\infty$,

$$\bar{y}(x) = \lambda x^{\mu} e^{\nu x} \tag{118}$$

where $\lambda B(\beta - \alpha, \beta - \alpha) = \delta,\ \mu = \beta - \alpha - 1 > -1$ and arbitrary ν. The asymptotic solutions (118) are the solutions of the approximate equation

$$\delta x^{\beta-\alpha} \bar{y}(x) = \int_0^x \bar{y}(\xi)\, \bar{y}(x-\xi) d\xi.$$

To fix the parameter ν the equation (116) with $\bar{k}(x) = \gamma x^{-\alpha} + \delta x^{\beta-\alpha}$ and the corresponding initial condition

$$x^{\gamma} y(x)|_{x=0} = A \tag{119}$$

has to be taken into account. An approximate value for ν in case of continuous B can be obtained by the simple approximation

$$\hat{y}(x) = \begin{cases} Ax^{-\gamma} & \text{in } (0,\rho) \\ \lambda x^{\mu} e^{\nu x} & \text{in } (\rho,\infty) \end{cases}$$

for the exact y with some $\rho \in (0,\infty)$. The values of ν and ρ are determined by the continuity of $\hat{y}$ and $\hat{y}'$ at $x = \rho$. In the case of Burgers' equation (5.1.105) with $\beta_0 = \frac{1}{16}$ where $\alpha = 1, \beta = 3, \gamma = \frac{1}{2},\ \delta = -\frac{1}{16}$ and $B(x) = 0$ we get $\rho \approx 0.5388$ and $\nu \approx -2.784$ in this way where the exact value is $\nu = -2.920$. Further, for non-vanishing B the shifting of ν by the substitution $\tilde{y} = e^{\epsilon x} y$ should be observed.

We remark that in the case $\alpha = 1$with $\gamma < 1$ if $k(x) \sim \delta x^{\beta-1} \ln x$ as $x \to +\infty$ asymptotic solutions of the form

$$\bar{y}(x) = \lambda x^{\mu} e^{\nu x} \ln x\,, \qquad \mu = -\gamma > -1$$

exist.

Differintegralequations of type II

Finally, the class of equations

$$x^{\alpha} y'(x) + k_0(x) y(x) = \int_0^x y(\xi)\, y(x-\xi) d\xi \tag{120}$$

where

$$k_0(x) = \gamma + \delta x^{\beta} + B_0(x) \tag{121}$$

with $\alpha > 0,\ \beta > 0,\ \gamma \neq 0, \delta \neq 0$ and $B_0(x) = o\left(x^{\beta}\right)$ as $x \to +\infty$, $B(x) = o(1)$ as $x \to 0^+$ is dealt with. Equation (120) with (121) has the asymptotic solutions (118) where for arbitrary $\nu \neq 0$

$$\mu = \mu_0 - 1, \mu_0 = \max\{\alpha, \beta\} > 0 \ and \ \lambda B(\mu_0, \mu_0) = \begin{cases} \delta & \text{if } \beta > \alpha \\ \delta + \nu & \text{if } \beta = \alpha \\ \nu & \text{if } \beta < \alpha \end{cases}$$

and for $\nu = 0$

$$\mu = \mu_1 - 1, \mu_1 = \max\{\alpha - 1, \beta\} > 0 \ and \ \lambda B(\mu_1, \mu_1) = \begin{cases} \delta & \text{if } \beta > \alpha - 1 \\ \delta + \alpha - 2 & \text{if } \beta = \alpha - 1 \\ \alpha - 2 & \text{if } \beta < \alpha - 1 \end{cases}$$

Sec(5.2) : Autoconvolution Equations of the Third Kind

We deal with a general class of such autoconvolution equations of the form

$$k(x)y(x) = \int_0^x m(x,\xi)\, y(\xi)y(x-\xi)d\xi + \int_0^x n(x,\xi)\, y(\xi)d\xi + p(x) \qquad (122)$$

for $0 \leq x \leq T$, with given continuous functions k, m, n, p, where $k(0) = 0$. For $m(x,\xi) = a(\xi)$and $n(x,\xi) = p(x) = 0$ this equation is the model equation considered in [141]. The general class of equations (122) contains the well-known integral equations of F. Bernstein and F. Bernstein and G. Doetsch for the elliptic theta zero function and for the Mittag-Leffler function, but under our assumptions unfortunately only the latter equation can be treated. Further, following [141] we restrict ourselves to basic existence theorems for solutions of (122) with power or logarithm behaviour at $x = 0$. But we expect that also theorems on the smoothness of the solutions for the model equation [141] could be extended to equation (122). Moreover, we add to the existence theorems in [141] a such one for a class of model equations with data k, a and solution y containing general logarithmic terms. Finally, as a new aspect a singular perturbation problem for a related integrodifferential equation of first order to the model equation in the superlinear case of [141] is investigated. The results of this investigation are basic for a regularization of the model integral equation of the third kind by a neighbouring integrodifferential equation (a kind of Lavrentiev regularization. [143]).

We remark that with a solution y also the function $e^{Cx}y$, where C is an arbitrary constant, is a solution to equation (122) if $n = p = 0$ as in the case of the model equation. In the general case we have to expect a more complex structure of a general solution of (122).

we deal with the singular perturbation problem and the general logarithmic case of the model equation in this section, respectively. The general class of equations (130) is then treated in this section.

The existence proofs are based on (a simplified version of) an existence theorem from operator equations of the form

$$\mathcal{Z}(x) = f(x) + G[\mathcal{Z}](x) + L[\mathcal{Z},\mathcal{Z}](x) \tag{123}$$

with a linear operator G and a bilinear operator L in $C[0,T]$, $0 < T < \infty$, with the exponentially weighted norms

$$\|\mathcal{Z}\|_\sigma = \|e^{-\sigma x}\mathcal{Z}(x)\| = \max_{0\le x\le T}|e^{-\sigma x}\mathcal{Z}(x)| \ , \ \sigma > 1 ,$$

where $\|\mathcal{Z}\| = \|\mathcal{Z}\|_0$, which we cite here as Lemma (5.2.1).

Lemma (5.2.1)[48]. Let the linear operator $G : C[0,T] \to C[0,T]$ and the bilinear operator $L : C[0,T] \times C[0,T] \to C[0,T]$ fulfill the inequalities

$$\|G[\mathcal{Z}]\|_\sigma \le M(\sigma)\|\mathcal{Z}\|_\sigma , \qquad \sigma \ge \sigma_0 > 1 \tag{124}$$

for any $\mathcal{Z} \in C[0,T]$ with a continuous function M satisfying $M(\sigma) \to 0$ as $\sigma \to \infty$,and

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le N\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\|_\sigma , \qquad \sigma \ge \sigma_0 > 1 \tag{125}$$

with a constant N and

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le \begin{cases} \nu_1(\sigma)\|\mathcal{Z}_1\| \ \|\mathcal{Z}_2\|_\sigma \\ \nu_2(\sigma)\|\mathcal{Z}_1\|_\sigma \ \|\mathcal{Z}_2\| \end{cases} \tag{126}$$

with continuous functions $\nu_k, k = 1,2$, satisfying $\nu_k(\sigma) \to o$ as $\sigma \to \infty$ for any pair $\mathcal{Z}_1,\mathcal{Z}_2 \in C[0,T]$. Then equation (123) has a uniquely determined solution $\mathcal{Z} \in C[0,T]$. Moreover, for solutions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ corresponding to functions $f = f_1$ and $f = f_2$, respectively, the stability estimate

$$\|\mathcal{Z}_1 - \mathcal{Z}_2\| \le \wedge (Q_1,Q_2)\|f_1 - f_2\| \tag{127}$$

holds, where $Q_k = (\| f_k\|, \|G\,[f_k]\|), k = 1,2$, and $\wedge \in (\mathbb{R}_+^4 \to \mathbb{R})$, $\wedge > 0$ with $\wedge\,(x_1, \dots, x_4)$ increasing in $x_1, \dots, x_4$.

Singular perturbation problem

Let us consider the model equation

$$k(x)y(x) = \int_0^x a(\xi)\, y(x-\xi)y(\xi)d\xi. \tag{128}$$

If $k(x){\sim}Ax, A > 0$ and $a(x){\sim}1$ as $x \to 0$ then the continuous solutions y of (128) have at $x = 0$ either the value $y(0) = 0$ or the value $y(0) = A$. With interest in the second case, in this section we study the initial value problem for the related integrodifferential equation of the first order

$$\epsilon y_\epsilon'(x) + k(x)y_\epsilon(x) = \int_0^x a(\xi)\, y_\epsilon(x-\xi)y_\epsilon(\xi)d\xi, \quad y_\epsilon(0) = A \qquad (129)$$

with $\epsilon \neq 0$. We remark that $y(0) = 0$ for a continuous solution y of (136) is only fulfilled for the trivial solution $y(x) \equiv 0$ if, in addition to the above asymptotic relations, there holds $k, a \in C[0,T]$ with $k(x) > 0$ in $(0,T]$.

Theorem $(\mathbf{5.2.2})[\mathbf{48}]$**.** Let $\epsilon \neq 0$, $k \in C[0,T]$ and $a \in L^1(0,T)$. Then problem (130) has a unique solution in $C^1[0,T]$.

Proof. The initial value problem (37) is in $C^1[0,T]$ equivalent to the equation

$$y_\epsilon(x) = L[y_\epsilon, y_\epsilon](x) + f(x) \qquad (130)$$

where

$$L[\mathcal{Z}_1,\mathcal{Z}_2](x) = \int_0^x \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} \int_0^\eta a(\xi)\, \mathcal{Z}_1(\eta-\xi)\mathcal{Z}_2(\xi)d\xi d\eta$$

$$f(x) = Ae^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau}.$$

Let us show that (130) has a unique solution in $C[0,T]$. We have

$$e^{-\sigma x} L[\mathcal{Z}_1,\mathcal{Z}_2](x)$$
$$= \int_0^x e^{-\sigma(x-\eta)} \frac{1}{\epsilon} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} \int_0^\eta a(\xi)\, e^{-\sigma(x-\eta)}\mathcal{Z}_1(\eta-\xi)\mathcal{Z}_2(\xi)d\xi d\eta.$$

Thus,

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \leq \frac{1}{|\epsilon|} e^{\frac{1}{|\epsilon|}\int_0^T |k(\tau)|d\tau} \int_0^T |a(\xi)|d\xi \int_0^T e^{-\sigma(T-\eta)}\, d\eta \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma$$
$$\leq \text{Const}\, \frac{1}{\sigma} \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma.$$

This estimate shows that the assumptions of Lemma (5.2.1) are satisfied for equation (130). Consequently, (130) has a unique solution y_ϵ in $C[0,T]$. Finally, since the right-hand side of (130) is continuously differentiable for $y_\epsilon \in C[0,T]$, we obtain $y_\epsilon \in C^1[0,T]$. Theorem (5.2.2) is proved.

Lemma $(\mathbf{5.2.3})[\mathbf{48}]$**.** Let $\epsilon \neq 0$, g be a measurable function such that $|g(x)| \leq x^{\delta-1}$ with $C \geq 0, \delta > 0, k \in W^{2,1}(0,T)$ and $A_0 x \leq k(x) \leq A_1 x$ with $0 < A_0 \leq A_1$. Then the function

$$y(x) = k(x)\int_0^x v(\eta)\,d\eta \qquad (131)$$

with

$$v(x) = \frac{1}{\epsilon k^2(x)}\int_0^x k(\eta)\, e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} g(\eta)d\eta \qquad (132)$$

belongs to $W^{2,1}(0,T)$ and solves the problem

$$\epsilon y''(x) + k(x)y'(x) - \left[k'(x) + \frac{\epsilon k''(x)}{k(x)}\right]y(x) = g(x), y(0) = y'(0) = 0 \quad (133)$$

Proof. Due to the assumptions of the lemma, the function y, defined in (132) with v given by (142), belongs to $W^{2,1}(0,T)$. One can immediately check that v is a solution to the equation

$$\epsilon k(x)v'(x) + [2\epsilon k'(x) + k^2(x)]v(x) = g(x). \qquad (134)$$

Further, from (131) we see that $v = \left(\frac{y}{k}\right)'$. Substituting $\left(\frac{y}{k}\right)'$ for v in (134) we derive the equation (133). Finally, the conditions $y(0) = y'(0) = 0$ follow from (131) with (132) by the assumptions on k.

Lemma (5.2.4)[48]. Let $\epsilon > 0$. Then

$$\max_{0\le x\le T}\int_0^x \eta^\beta\, e^{-\frac{1}{\epsilon}(x^2-\eta^2)}d\eta \le \frac{1}{\beta+1}\epsilon^{\frac{\beta+1}{2}} \qquad \text{if} \quad -1<\beta\le 1 \qquad (135)$$

$$\max_{0\le x\le T}\int_0^x \eta^\beta\, e^{-\frac{1}{\epsilon}(x^2-\eta^2)}d\eta \le \frac{T^{\beta-1}}{2}\epsilon \qquad \text{if} \quad \beta>1. \qquad (136)$$

Proof. Changing the variable of integration $\mathcal{Z} = \frac{\eta^2}{\epsilon}$ we obtain

$$\int_0^x \eta^\beta\, e^{-\frac{1}{\epsilon}(x^2-\eta^2)}d\eta = \frac{1}{2}e^{\frac{\beta+1}{2}}\int_0^t \mathcal{Z}^{\frac{\beta-1}{2}}\, e^{-(t-x)}d\mathcal{Z}\,, \qquad (137)$$

where $t = \frac{x^2}{\epsilon}$. Let $-1 < \beta \le 1$. Then in case $t \ge 1$ we have

$$\int_0^t \mathcal{Z}^{\frac{\beta-1}{2}}\, e^{-(t-x)}d\mathcal{Z} = \int_0^1 \mathcal{Z}^{\frac{\beta-1}{2}}\, e^{-(t-x)}d\mathcal{Z} + \int_1^t \mathcal{Z}^{\frac{\beta-1}{2}}\, e^{-(t-x)}d\mathcal{Z}$$

$$\le e^{1-t}\int_0^1 \mathcal{Z}^{\frac{\beta-1}{2}}\,d\mathcal{Z} + \int_1^t e^{-(t-\mathcal{Z})}d\mathcal{Z}$$

$$= 1 + \left(\frac{2}{\beta + 1} - 1\right) e^{1-t}.$$

Thus,

$$\sup_{1 \le t \le \infty} \int_0^t \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(t-x)} d\mathcal{Z} \le \frac{2}{\beta + 1}. \tag{138}$$

In case $0 \le t < 1$ we obtain

$$\int_0^t \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(t-x)} d\mathcal{Z} \le \int_0^t \mathcal{Z}^{\frac{\beta-1}{2}} d\mathcal{Z} = \frac{2t^{\frac{\beta+1}{2}}}{\beta + 1}.$$

This implies

$$\sup_{0 \le t < 1} \int_0^t \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(t-x)} d\mathcal{Z} \le \frac{2}{\beta + 1}. \tag{139}$$

Applying (138) and (139) in (137) we deduce (138). If $\beta > 1$ then we obtain $\int_0^x \eta^\beta \, e^{-\frac{1}{\epsilon}(x^2-\eta^2)} d\eta = T^{\beta-1} \int_0^x \eta \, e^{-\frac{1}{\epsilon}(x^2-\eta^2)} d\eta$. Estimate (136) follows using here (135) with $\beta = 1$.

Corollary $(\mathbf{5.2.4})'[\mathbf{140}]$. Let $\epsilon > 0$ and $\theta = \frac{1}{\delta^n}$ $, n \ge 1$. Then

(i) $\max\limits_{0 \le x \le T} \int_0^x \eta^\beta \, e^{-\frac{1}{\epsilon}(x^2-\eta^2)} d\eta \le M(\epsilon)$ if $\beta \in (-1,1]$

(ii) $\max\limits_{0 \le x \le T} \int_0^x \eta^\beta \, e^{-\frac{1}{\epsilon}(x^2-\eta^2)} d\eta \le M_1(\epsilon)$

Proof: Let $\mathcal{Z} = \frac{\eta^2}{\epsilon}$ we have

$$\int_0^x \eta^\beta \, e^{-\frac{1}{\epsilon}(x^2-\eta^2)} d\eta = \frac{1}{2} e^{\frac{\beta+1}{2}} \int_0^t \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(t-x)} d\mathcal{Z}$$

Where $t = \frac{x^2}{\epsilon}$. Let $\beta \in (-1,1]$ then $t = 1 + \theta, n \ge 1$.

We have

$$\int_0^{1+\theta} \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(1+\theta-x)} d\mathcal{Z} = \int_0^1 \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(1+\theta-x)} d\mathcal{Z} + \int_1^{1+\theta} \mathcal{Z}^{\frac{\beta-1}{2}} e^{-(1+\theta-x)} d\mathcal{Z}$$

$$\leq e^{-\theta}\int_1^1 \mathcal{Z}^{\frac{\beta-1}{2}}\,d\mathcal{Z}+\int_1^{1+\theta} e^{-(1+\theta-x)}\,d\mathcal{Z}=1+\left(\frac{2}{\beta+2}-1\right)e^{-\theta}$$

Hence

$$\sup_{0\leq\theta\leq\infty}\int_0^{1+\theta} \mathcal{Z}^{\frac{\beta-1}{2}}e^{-(1+\theta-x)}d\mathcal{Z}\leq\frac{2}{\beta+1}$$

If $0\leq\theta\leq T-1$ we have

$$\int_0^{1+\theta} \mathcal{Z}^{\frac{\beta-1}{2}}e^{-(1+\theta-x)}d\mathcal{Z}\leq\int_0^{1+\theta} \mathcal{Z}^{\frac{\beta-1}{2}}\,d\mathcal{Z}=\frac{2(1+\theta)^{\frac{\beta+1}{2}}}{(\beta+1)}$$

Which implies that

$$\sup_{0\leq\theta\leq T-1}\int_0^{1+\theta} \mathcal{Z}^{\frac{\beta-1}{2}}e^{-(1+\theta-x)}d\mathcal{Z}\leq\frac{2}{\beta+1}$$

Hence proposition Lemma (5.2.4)

Show that $M(\epsilon)=\frac{2}{\beta+1}e^{\frac{\beta+1}{2}}$ *and* $M_1(\epsilon)=\frac{T^{\beta-1}}{2}\epsilon$ wherever $t=1+\theta$

Theorem (5.2.5)[48]. Let a and k fulfill the assumptions of Theorem (7) in [141], i.e.,

$k\in C^2[0,T], k>0$ in $(0,T]$, $a\in C^1[0,T]$, where

$$\left.\begin{aligned} k(x)&=Ax+B^{x^2+\delta}+o\left(x^{2+\delta}\right)\\ k'(x)&=A+B(2+\delta)x^{1+\delta}+o\left(x^{1+\delta}\right)\\ k''(x)&=B(1+\delta)(2+\delta)x^{\delta}+o\left(x^{\delta}\right)\end{aligned}\right\}\tag{140}$$

$$\left.\begin{aligned} a(x)&=1+\lambda x^{1+\delta}+o\left(x^{1+\delta}\right)\\ a'(x)&=\lambda(1+\delta)x^{\delta}+o\left(x^{\delta}\right)\end{aligned}\right\}\tag{141}$$

as $x\to 0$ with $A,\delta>0$ and $B,\lambda\in\mathbb{R}$. Further, let y_0 be the solution of equation (128) satisfying $y_0\in c^1[0,T]\cap c^2(0,T]$ and

$$y_0(x)=ACx^{1+\delta}+o\left(x^{1+\delta}\right),\ \ y_0'(x)=C(1+\delta)x^{\delta}+o\left(x^{\delta}\right)\tag{142}$$

as $x\to 0$ with $C\in\mathbb{R}$ and

$$|y_0''(x)|=\leq Const\ \ x^{\delta-1}\tag{143}$$

Then for any $q\in(1-\delta,1)\cap(0,1)$ the estimates

$$\left.\begin{array}{l}\max\limits_{0\le x\le T} x^{q-2}|y_\epsilon(x)-y_0(x)| \\ \max\limits_{0\le x\le T} x^{q-1}|y_\epsilon'(x)-y_0'(x)|\end{array}\right\}\le M\mu(\epsilon) \tag{144}$$

are valid for the solution y_ϵ of problem (129) with $\epsilon > 0$. Here

$$\mu(\epsilon)=\begin{cases}\epsilon^{\frac{\delta+q-1}{2}} & if\ \ \delta+q\le 3\\ \epsilon & if\ \ \delta+q>3,\end{cases} \tag{145}$$

and M is a constant depending on $T, k, a,$ and q.

Remark $(\mathbf{5.2.6})[\mathbf{48}]$**.** Existence of a solution y_0 of equation (128) with properties $y_0 \in c^1[0,T] \cap c^2(0,T]$, (142) and (143) follows from Theorems (1) and (7) in [141].

Proof. Let $\epsilon > 0$. Denote $y = y_\epsilon - y_0$ and subtract (128) from (129). We obtain

$$\epsilon y'(x)+k(x)y(x)=\int_0^x [a(x-\eta)a(\eta)]\, y_0(x-\eta)y(\eta)d\eta$$
$$+\int_0^x a(\eta)y(x-\eta)\, y(\eta)d\eta-\epsilon y_0'(x) \tag{146}$$
$$y(0)=0$$

By Theorem (5.2.2) this problem admits a unique solution in $C^1[0,T]$. Due to the assumptions of k, a and the properties of y_0 this solution even belongs to $W^{2,1}(0,T)$ and $y'(0)=0$. Consequently, differentiating equation (146), the equation is equivalent to the problem of the second order

$$\epsilon y''(x)+k(x)y'(x)-\left[k'(x)+\frac{\epsilon k''(x)}{k(x)}\right]y(x)=g[y](x), y(0)=y'(0)=0, \tag{147}$$

where

$$g[y](x)=\int_0^x [a'(x-\eta)(y_0(x-\eta)-A)+(a(x-\eta)+a(\eta))y_0'(x-\eta)]\, y(\eta)d\eta$$
$$+\int_0^x a(\eta)y'(x-\eta)y(\eta)d\eta + A\int_0^x a'(x-\eta)y(\eta)d\eta \tag{148}$$
$$+\left[A\big(1+a(x)\big)-2k'(x)-\frac{\epsilon k''(x)}{k(x)}\right]y(x)-\epsilon y_0''(x).$$

Let us consider the related equation

$$\nu(x) = \int_0^x \frac{k(\eta)}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} g\left[k\int_0^{\cdot} \nu(\xi)\,d\xi\right](\eta)d\eta \qquad (149)$$

and define a function y by means of the solution of this equation using as in (131) the formula

$$y(x) = k(x)\int_0^x \nu(\eta)\,d\eta. \qquad (150)$$

It follows from the assumptions on k that there exist $0 < A_0 \leq A_1$ such that

$$A_0 x \leq k(x) \leq A_1 x\,, \quad x \in [0,T]. \qquad (151)$$

Further, in case the solution v of (149) satisfies the conditions

$$\nu \in [0,T]\,, |\nu(x)| \leq Const\ x^{-q}\,, \qquad (152)$$

where $q < 1$ by assumption, then, as we can easily check, the function $g[y] = g\left[k\int_0^{\cdot} \nu(\xi)\,d\xi\right]$ satisfies the relation $|g(x)| \leq Const\ x^{p-1}$ with $p > 0$. Consequently, by Lemma (5.2.3), the function y given by (150) belongs to $W^{2,1}(0,T)$ and solves (147), hence (146). In the following we will show the existence of a solution ν with the property (152).

Let us define

$$w(x) \;=\; x^q \nu(x). \qquad (153)$$

The solution v of (149) satisfies (152) if and only if $w \in C(0,T] \cap L^\infty(0,T)$.

The corresponding equation for w writes

$$w(x) \;=\; G[w](x) + L[w,w](x) + f(x)\,, \qquad (154)$$

where

$$\begin{aligned} G[w](x) = \int_0^x \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} \Bigg\{ \int_0^x & [a'(\eta-\xi)(y_0(\eta-\xi) - A) \\ & + \big(a(\eta-\xi) + a(\xi)\big)y_0'(\eta-\xi)]k(\xi)\int_0^\xi \tau^{-q}\ w(\tau)d\tau d\xi \\ & + A\int_0^x a'(\eta-\xi)\,k(\xi)\int_0^\xi \tau^{-q}\ w(\tau)d\tau d\xi \\ & + \left[\Big(A\big(1+a(\eta)\big) - 2k'(\eta)\Big)k(\eta) - \epsilon k''(\eta)\right]\int_0^\xi \tau^{-q}\ w(\tau)d\tau \Bigg\} d\eta \end{aligned} \qquad (155)$$

and

$$L[w_1,w_2](x) = \int_0^x \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} \int_0^\eta a(\xi)\left[k^{'}(\eta-\xi) \times \int_0^{\eta-1} \tau^{-q}\ w_1(\tau)d\tau \right.$$
$$\left. + k(\eta-\xi)(\eta-\xi)^{-q} w_1(\eta-\xi)\right] \times k(x)\int_0^\xi \tau^{-q}\ w_2(\tau)d\tau d\xi d\eta \quad (156)$$

and

$$f(x) = -\int_0^x \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} y_0^{''}(\eta)d\eta. \quad (157)$$

We will prove that (154) has a unique solution in $C[0,T]$ and this solution satisfies a proper estimate implying (144).

In view of the assumption $q > 1-\delta$ by $k(x) \geq 0, \epsilon > 0$, (140) and (143) it follows that $f \in C[0,T]$. Further, multiplying by $e^{-\sigma x}$ in (155), (156)we have

$$e^{-\sigma x} G[w](x) = \int_0^x e^{-\sigma(x-\eta)} \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau} \left\{ \int_0^\eta e^{-\sigma(x-\xi)} \right.$$
$$\times \left[a^{'}(\eta-\xi)(y_0(\eta-\xi)-A) + \big(a(\eta-\xi)+a(\xi)\big)y_0^{'}(\eta-\xi)\right]$$
$$\times k(\xi)\int_0^\xi e^{-\sigma(x-\tau)} \tau^{-q} e^{-\sigma\tau} w(\tau)d\tau d\xi \quad (158)$$
$$+A\int_0^\eta e^{-\sigma(x-\xi)} a^{'}(\eta-\xi)k(\xi)\int_0^\xi e^{-\sigma(x-\tau)} \tau^{-q} e^{-\sigma\tau} w(\tau)d\tau d\xi$$
$$+\left[\Big(A\big(1+a(\eta)\big) - 2k^{'}(\eta)\Big)k(\eta) - \epsilon k^{''}(\eta)\right]$$
$$\left. \times \int_0^\eta e^{-\sigma(x-\tau)} \tau^{-q} e^{-\sigma\tau} w(\tau)d\tau \right\} d\eta,$$

$$e^{-\sigma x} L[w_1,w_2](x) = \int_0^x e^{-\sigma(x-\eta)} \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)d\tau}$$
$$\times \int_0^\eta a(\xi)\left[k^{'}(\eta-\xi)\int_0^{\eta-\xi} e^{-\sigma(\eta-\xi-\tau)} \tau^{-q} e^{-\sigma\tau} w_1(\tau)d\tau \right. \quad (159)$$
$$\left. +k(\eta-\xi)(\eta-\xi)^{-q} e^{-\sigma(\eta-\xi)} w_1(\eta-\xi)d\tau\right]$$

$$\times k(\xi) \int_0^{\xi} e^{-\sigma(\xi-\tau)} \tau^{-q} e^{-\sigma\tau} w_2(\tau) d\tau d\eta.$$

In the estimations of G and L we apply the inequality

$$\int_0^x e^{-\sigma(x-\tau)} \tau^{-q} d\tau = \frac{1}{\sigma^{1-q}} \int_0^{\sigma x} z^{-q} e^{-(\sigma x - z)} dz \leq \frac{Const}{\sigma^{1-q}} \qquad (160)$$

following from Lemma (5.2.4).

We now estimate (158) making use of the assumptions of the theorem, Lemma (5.2.4), (151) and (160). We obtain

$$\|G[w]\|_\sigma \leq Const \max_{0 \leq x \leq T} \int_0^x \frac{\eta}{\epsilon x^{2-q}} e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)}$$

$$\times \left\{ \int_0^{\eta} \left[(\eta-\xi)^{1+2\delta} + (\eta-\xi)^{\delta} \right] \xi \frac{1}{\sigma^{1-q}} d\xi \|w\|_\sigma \right.$$

$$\left. + \int_0^{\eta} (\eta-\xi)^{\delta} \xi \frac{1}{\sigma^{1-q}} d\xi \|w\|_\sigma + \left(\eta^{1+\delta} + \epsilon \eta^{\delta}\right) \frac{1}{\sigma^{1-q}} \|w\|_\sigma \right\} d\eta \qquad (161)$$

$$\leq Const \max_{0 \leq x \leq T} \left\{ \frac{1}{\epsilon} \int_0^x \eta \, e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)} d\eta + \int_0^x e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)} d\eta \right\} \frac{1}{\sigma^{1-q}} \|w\|_\sigma$$

$$\leq \frac{Const}{\sigma^{1-q}} \|w\|_\sigma .$$

Similarly, for $L[w_1, w_2]$ in (159) we derive

$$\|L[w_1, w_2]\|_\sigma \leq Const \max_{0 \leq x \leq T} \int_0^x \frac{\eta}{\epsilon x^{2-q}} e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)}$$

$$\times \int_0^{\eta} (\eta-\xi)^{1-q} \|w_1\|_\sigma \xi \frac{1}{\sigma^{1-q}} d\xi \|w_2\|_\sigma d\xi d\eta \qquad (162)$$

$$\leq \frac{Const}{\sigma^{1-q}} \|w_1\|_\sigma \|w_2\|_\sigma .$$

The estimates (161)and (162) imply the assumptions (124) – (126) of Lemma (5.2.1). Thus, by Lemma (5.2.1), equation (154) has a unique solution w in $C[0,T]$. In particular, the equation (154) has a unique solution $w = 0$ in $C[0,T]$ if $f = 0$. Consequently, the stability estimate (127) in Lemma (5.2.1) with $z_1 = w$, $z_2 = 0$ and $f_1 = f$, $f_2 = 0$ yields $\|w\| \leq Const \, \|f\|$. Further, estimating (157) by means of the assumption $q > 1 - \delta$, (143), (151)and Lemma (5.2.4) we have $\|f\| \leq Const \, \mu(\epsilon)$ with $\mu(\epsilon)$ defined in (145). Thus,

$$\|w\| \leq Const\ \mu(\epsilon). \tag{163}$$

Finally, by (150) and (153) we have the formula for $y = y_\epsilon - y_0$ in terms of w

$$y(x) = k(x) \int_0^x \xi^{-q}\ w(\xi) d\xi. \tag{164}$$

From (140), (163) and (164) we obtain the first estimate in (144). Using the differentiated formula (164) also the second estimate in (144) follows. Theorem (5.2.5) is proved.

The assertion (144) of Theorem (5.2.5) implies the following corollary.

Corollary (5.2.7)[48]. Under the assumptions of Theorem (5.2.5) the uniform convergence

$$y_\epsilon \longrightarrow y_0, \qquad y_\epsilon' \longrightarrow y_0' \ \ in \ \ C[0,T]$$

$as\ \epsilon \longrightarrow 0^+ holds.$

General logarithmic case

In the following we study the existence of solutions for two types of generalized autoconvolution equations. We start with equation (128) where

$$k(x) = Ax + x^2 \sum_{n=0}^{N} B_n\ ln^2 x + C(x) \quad (A > 0, B_n \epsilon \mathbb{R}\ ) \tag{165}$$

with $C(x) = o(x^2)\ \ as\ \ x \longrightarrow 0$ and $\int_0^T \frac{|C(x)|}{x^3} dx < \infty$,

$$a(x) = 1 + x \sum_{n=0}^{N} \beta_n\ ln^n x + \gamma(x) \qquad (\beta_n \epsilon \mathbb{R}) \tag{166}$$

with $\gamma(x) = o(x)\ \ as\ \ x \longrightarrow 0$ and $\int_0^T \frac{|C(x)|}{x^3} dx < \infty$.

Theorem (5.2.8)[48]. Let k with $1/k \in C(0,T]$ and $a \in [0,T]$ have the finite asymptotic expansions (165) and (166), respectively. Then equation (128) has a solution $y \in C[0,T]$ of the form

$$y(x) = A + x \sum_{n=1}^{N+1} \mu_n\ ln^n x + x\mathcal{Z}(x) \tag{167}$$

with $\mathcal{Z} \in C[0,T]$ and $\mathcal{Z}(0) = 0$, where the $\mu_n, n = 0,1, \ldots, N+1$, are the solutions of the equations

$$(-1)^n \sum_{j=n+1}^{N+1} (-1)^j \frac{j!}{2^j} \mu_j = \beta_n - A(-1)^n \frac{2^{n-1}}{n!} \sum_{j=n}^{N} (-1)^j \frac{j!}{2^j} \beta_j \qquad (168)$$

for $n = 0, \ldots, N$. This solution is unique in the class of functions of type (175).

Proof. Inserting the ansatz (167) into equation (128) we get the equation for $\mathcal{Z}$

$$\mathcal{Z}(x) = f_0(x) + G_0[\mathcal{Z}](x) + L_0[\mathcal{Z}, \mathcal{Z}](x)\,, \qquad (169)$$

where

$$f_0(x) = \frac{1}{xk(x)} \left\{ A^2 \int_0^x a(\xi) d\xi + A \int_0^x \xi[a(\xi) + a(x-\xi)] \sum_{n=1}^{N+1} \mu_n \, ln^n \xi d\xi \right.$$
$$-k(x) \left[A + x \sum_{n=1}^{N+1} \mu_n \, ln^n x \right]$$
$$\left. + \int_0^x \xi(x-\xi) a(\xi) \sum_{n=1}^{N+1} \mu_n \, ln^n \xi \sum_{m=1}^{N+1} \mu_m \, ln^m (x-\xi) d\xi \right\}$$

and

$$G_0[\mathcal{Z}](x) = \frac{1}{xk(x)} \int_0^x \xi[a(\xi) + a(x-\xi)]$$
$$\times \left[A + (x-\xi) \sum_{n=1}^{N+1} \mu_n \, ln^n (x-\xi) \right] \mathcal{Z}(\xi) d\xi$$

$$L[\mathcal{Z}_1, \mathcal{Z}_2](x) = \frac{1}{xk(x)} \int_0^x \xi(x-\xi) a(\xi) \mathcal{Z}_1(\xi) \mathcal{Z}_2(x-\xi) d\xi\,.$$

In view of assumptions (165) and (166) we have

$$xk(x) f_0(x) = A^2 x + A^2 \sum_{n=0}^{N} \beta_n \int_0^x \xi \, ln^n(\xi)\, d\xi + 2A \sum_{n=1}^{N+1} \mu_n \int_0^x \xi \, ln^n(\xi)\, d\xi$$
$$- \left[Ax + x^2 \sum_{m=0}^{N} \mu_m \, ln^m x \right] \left[A + x \sum_{n=1}^{N} \mu_n \, ln^n x \right] + F_0(x)\,, \qquad (170)$$

where $F_0 \in C[0,T]$ with $F_0(x) = o(x^2)$ as $x \to 0$ and $\int_0^T \frac{|F_0(x)|}{x^3} dx < \infty$. Calculating the coefficients of the functions $x^2\ln^n x\,, n = 0,1, \dots, N, N+1$, in the right-hand side of (170), we see that the coefficient of the highest term $x^2\ln^{N+1}x$ automatically vanishes as well as the coefficient of x. Putting the $N+1$ coefficients of $x^2\ln^n x\,, n = 0, \dots, N$ equal to zero, we obtain the linear system (176) for the $N+1$ parameters $\mu_n\,,\ n = 1, \dots, N+1$. This system is regular since it has upper triangular matrix with nonvanishing elements in the main diagonal. So, (168) has a unique solution $(\mu_1, \dots, \mu_{N+1})$, and for these parameters μ_n we have the relation $xk(x)f_0(x) = F_1(x)$, where F_1 has the same properties mentioned above as F_0. Therefore, by (165) and $1/k \in C(0,T]$ then $f_0 \in C[0,T]$ with $f_0(0) = 0$ and $\int_0^T \frac{|f_0(x)|}{x} dx < \infty$ holds.

We decompose

$$G_0[\mathcal{Z}](x) = \frac{2}{x^2}\int_0^x \xi \mathcal{Z}(\xi)\, d\xi + G_1[\mathcal{Z}](x)$$

where

$$\begin{aligned} G_1[\mathcal{Z}](x) = {} & \frac{2\big(Ax - k(x)\big)}{x^2 k(x)}\int_0^x \xi \mathcal{Z}(\xi)\, d\xi \\ & + \frac{A}{xk(x)}\int_0^x \xi[a(\xi) + a(x-\xi) - 2]\,\mathcal{Z}(\xi) d\xi \\ & + \frac{A}{xk(x)}\int_0^x \xi(x-\xi)[a(\xi) - a(x-\xi) - 2]\sum_{n=1}^{N+1} \mu_n \, ln^n(x-\xi)\mathcal{Z}(\xi) d\xi \end{aligned} \tag{171}$$

and write equation (169) in the form

$$\mathcal{Z}(x) - \frac{2}{x^2}\int_0^x \xi \mathcal{Z}(\xi)\, d\xi = g(x),$$

where $g(x) = f_0(x) + G_1[z](x) + \ L_0[\mathcal{Z},\mathcal{Z}](x)$. On account of (165) and (166) we obtain the estimates

$$|L_0[\mathcal{Z},\mathcal{Z}](x)| \le Const\ x\|\mathcal{Z}\|^2$$

$$|G_1[z](x)| \le Const\ x\,[1 + |\ln x|^{N+1}]\|\mathcal{Z}\|$$

which imply $L_0[\mathcal{Z},\mathcal{Z}], G_1[z] \in C[0,T]$ for any $\mathcal{Z} \in C[0,T]$ with $L_0[\mathcal{Z},\mathcal{Z}](0) = G_1[z](0)$ and $\int_0^T \frac{|L_0[\mathcal{Z},\mathcal{Z}](x)|}{x} dx < \infty$, $\int_0^T \frac{|G_1[\,\mathcal{Z}](x)|}{x} dx < \infty$. Observing the above relations for f_0 we therefore obtain that also $g \in C[0,T]$ with $g(0) = 0$ and $\int_0^T \frac{|g(x)|}{x} dx < \infty$.

Solving (171), equation (169) with $\mathcal{Z}(0) = 0$ is reduced to the equation

$$\mathcal{Z}(x) = f(x) + G[z](x) + L[\mathcal{Z},\mathcal{Z}](x)\,, \tag{172}$$

where

$$f(x) = f_0(x) + 2\int_0^x \frac{f_0(\xi)}{\xi}\,d\xi \in C[0,T]$$

with $f(0) = 0$ and

$$G[z](x) = G_1[\,\mathcal{Z}](x) + 2\int_0^x \frac{G_1[\,\mathcal{Z}](\xi)}{\xi}\,d\xi \tag{173}$$

$$L_0[\mathcal{Z}_1,\mathcal{Z}_2](x) = L_1[\mathcal{Z}_1,\mathcal{Z}_2](x) + 2\int_0^x \frac{L_0[\mathcal{Z}_1,\mathcal{Z}_2](\xi)}{\xi}\,d\xi.$$

Again, for any $\mathcal{Z} \in C[0,T]$ we have $G[z] \in C[0,T]$ with $G[z](0) = 0$ and for any pair $\mathcal{Z}_1,\mathcal{Z}_2 \in C[0,T]$ also $L[\mathcal{Z}_1,\mathcal{Z}_2] \in C[0,T]$ with $L[\mathcal{Z}_1,\mathcal{Z}_2](0) = 0$. Hence $\mathcal{Z}(0) = f(0) = 0$ for the solution $\mathcal{Z}$ of (172).

Applying Lemma (5.2.1) we have to verify inequalities (124) – (126) for $G[z]$ and $L[\mathcal{Z}_1,\mathcal{Z}_2]$. At first by (165) and (166) we estimate in (171) and get from (173)

$$\|G[z]\|_\sigma \le Const\,\frac{1}{\sigma}[1+\ln^{N+1}\sigma]\|z\|_\sigma \qquad (\sigma > 1)$$

which proves (132). Further, as in the proof of Theorem (5.2.3) in [141] we have the estimates $\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le Const\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\|_\sigma$ and $\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le Const\,\frac{1}{\sigma}\|\mathcal{Z}_1\|_\sigma\|\mathcal{Z}_2\|_\sigma$

and analogously with $\mathcal{Z}_1$ and $\mathcal{Z}_2$ interchanged. This shows (125) and (126). Theorem (5.2.8) is proved.

Now we deal with equation (122) under the assumptions that $1/k \in C(0,T]$ and

$$k(x) = Ax + Bx^{1+\alpha} + C(x) \;\; (A > 0,\; B \in \mathbb{R}), \tag{174}$$

where $\alpha > 0\,, C(x) = o(x^{1+\alpha})\; as \;\; x \longrightarrow 0$ with $\int_0^T \frac{|C(x)|}{x^{2+\alpha}}dx < \infty\,, m \in C([0,T] \times [0,T])$ and

$$m(x,\xi) = 1 + M_1x^\alpha + M_2\xi^\alpha + \gamma(x,\xi) \quad \left(M_j \in \mathbb{R}\right) \tag{175}$$

where $\gamma(x,\xi) = o(x^\alpha + \xi^\alpha)\;\; as \;\; x^2+\xi^2 \longrightarrow 0$ with $\int_0^T \frac{1}{x^{2+\alpha}}\int_0^x |\gamma(x,\xi)|\,d\xi dx < \infty$,

$n \in C([0,T] \times [0,T])$ and

$$n(x,\xi) = N_0 + N_1x^\alpha + N_2\xi^\alpha + \delta(x,\xi) \quad \left(N_j \in \mathbb{R}\right), \tag{176}$$

where $\delta(x,\xi) = o(x^\alpha + \xi^\alpha)\ \ as\ \ x^2 + \xi^2 \longrightarrow 0$ with $\int_0^T \frac{1}{x^{2+\alpha}} \int_0^x |\delta(x,\xi)|\, d\xi dx < \infty$, $p \in C[0,T]$ and

$$p(x) = cx + dx^{1+\alpha} + \epsilon(x) \qquad (c, d \in \mathbb{R})\,, \tag{177}$$

where $\epsilon(x) = o(x^{1+\alpha})\ as\ x \longrightarrow 0$ with $\int_0^T \frac{|\epsilon(x)|}{x^{2+\alpha}} dx < \infty$.

At first we are looking for solutions to (130) of the form

$$y(x) = \lambda + \sum_{j=1}^{\nu} \mu_j\, x^{k_j} + x^\alpha \mathcal{Z}(x),\ \ \mathcal{Z} \in C[0,T], \tag{178}$$

where $\lambda \in \mathbb{R},\ \nu \in \{1,2,\dots\},\ 0 < k_1 < k_2 < \cdots < k_\nu < \alpha$ and without loss of generality $\mu_j \neq 0,\ j = 1,\dots,\nu$. Plugging the ansatz (178) and the asymptotic expansions (174) – (177) into equation (122) and comparing the coefficients of x, we obtain the possible values for λ

$$\lambda_{1,2} = \frac{1}{2}\left[A - N_0 \pm \sqrt{(A - N_0)^2 - 4c}\right]\ . \tag{179}$$

We remark that for $c = 0$ as in the model equation (128) we have the values $A - N_0$ and zero for λ. In case $p(x) = 0$ as in equation (128) the value zero of λ yields the trivial solution $y = 0$ of the equation. In dealing with real solutions of (122) only, we assume the inequality

$$4c \leq (A - N_0)^2 \tag{180}$$

in the following.

In view of (178) equation (122) reduces to the following equation for $\mathcal{Z}$

$$\mathcal{Z}(x) = f_0(x) + G_0[z](x) + L_0[\mathcal{Z},\mathcal{Z}](x)\,, \tag{181}$$

where

$$f_0(x) = \frac{1}{x^\alpha k(x)}\Bigg\{p(x) - \left[\lambda + \sum_{j=1}^{\nu} \mu_j\, x^{k_j}\right] k(x) + \lambda \int_0^x n(x,\xi)\, d\xi$$

$$+ \sum_{j=1}^{\nu} \mu_j \int_0^x n(x,\xi)\, \xi^{k_j} d\xi + \lambda^2 \int_0^x m(x,\xi)\, d\xi$$

$$+ \lambda \sum_{j=1}^{\nu} \mu_j \int_0^x m(x,\xi)\left[\xi^{k_j} + (x-\xi)^{k_j}\right] d\xi$$

$$+\sum_{j=1}^{\nu}\mu_j\sum_{i=1}^{\nu}\mu_i\int_0^x m(x,\xi)\,\xi^{k_j}(x-\xi)^{k_i}d\xi\Bigg\} \qquad (182)$$

and

$$G_0[z](x)$$

$$=\frac{1}{x^\alpha k(x)}\int_0^x\Bigg\{n(x,\xi)\xi^\alpha\mathcal{Z}(\xi)+\lambda m(x,\xi)[\xi^\alpha\mathcal{Z}(\xi)+(x-\xi)^\alpha\mathcal{Z}(x-\xi)]$$

$$+m(x,\xi)\left[\sum_{j=1}^{\nu}\mu_j\left(\xi^{k_j}(x-\xi)^\alpha\mathcal{Z}(x-\xi)\right.\right.$$

$$\left.\left.+(x-\xi)^{k_i}\xi^\alpha\mathcal{Z}(\xi)\right)\right]\Bigg\}d\xi \qquad (183)$$

and

$$L_0[\mathcal{Z}_1,\mathcal{Z}_2](x)=\frac{1}{x^\alpha k(x)}\int_0^x m(x,\xi)\xi^\alpha(x-\xi)^\alpha\mathcal{Z}_1(\xi)\mathcal{Z}_2(x-\xi)d\xi\ . \qquad (184)$$

Since $k(x)\sim Ax\ \ as\ \ x\longrightarrow 0$, for obtaining $f_0\in C[0,T]$ in (182) we have to put the coefficients of the powers x and $x^{1+k_j}, j=1,\dots,\nu$ in the brackets to zero. For the power x we obtain the relation $c=\lambda(A-N_0)-\lambda^2$ already used in the determination of λ by (179). For the power x^{1+k_1} the relation

$$\lambda=\tfrac{1}{2}[(1+k_1)A-N_0] \qquad (185)$$

between λ and k_1 follows. This gives a positive value

$$k_1=\tfrac{1}{A}\sqrt{(A-N_0)^2-4c} \qquad (186)$$

only for $\lambda=\lambda_1$, i.e.,

$$\lambda=\tfrac{1}{2}\left[A-N_0+\sqrt{(A-N_0)^2-4c}\right]. \qquad (187)$$

In case $\nu\geq 2$ for the power $x^{1+k_j},\ j=2,\dots,\nu$, it must be $k_j=jk_1, j=2,\dots,\nu$, which by $k_\nu<\alpha$ yields the inequality $k_1<\frac{\alpha}{\nu}\ \ for\ k_1$. Under the further inequality $k_1\geq\frac{\alpha}{\nu+1}$ we get the recursive equations for $\mu_j,\ j=2,\dots,\nu$,

$$\left(A - \frac{N_0 + 2\lambda}{1 + k_m}\right)\mu_m = \sum_{j=1}^{m-1} \mu_j\, \mu_{m-j}\, B\left(k_j + 1, k_{m-j} + 1\right), \qquad (188)$$

where $m = 2, \dots, \nu$, the letter B denotes the Beta function, and μ_1 is arbitrary.

In view of (185) we have

$$A - \frac{N_0+2\lambda}{1+k_m} = \frac{(m-1)Ak_1}{1+mk_1} \neq 0 \quad \text{for} \quad m = 2, \dots, \nu$$

so that (188) determines μ_j, $j = 2, \dots, \nu$ uniquely for prescribed μ_1. Further, the value of $f_0(0)$ is given by the formulas

$$Af_0(0) = d - \lambda B + \lambda N_1 + \lambda \frac{N_2}{1+\alpha} + \lambda^2 M_1 + \lambda^2 \frac{M_2}{1+\alpha} \qquad (189)$$

in case $k_1 > \frac{\alpha}{\nu+1}$ and with the additional term

$$\sum_{j=1}^{\nu} \mu_j\, \mu_{\nu+1-j}\, B\left(k_j + 1, k_{\nu+1-j} + 1\right)$$

on the right-hand side of (189) in case $k_1 = \frac{\alpha}{\nu+1}$.

We are now ready to formulate the first existence theorem.

Corollary $(\mathbf{5.2.8})'[\mathbf{140}]$. Show that

$$\lambda_1 = \frac{k_1 A \pm \sqrt{k_1^2 A^2 + 4C}}{2}$$

So that gives

$\lambda_{1,2} = \frac{1}{2}\left[A - N_0 + \sqrt{(A - N_0)^2 - 4C}\right] = \frac{1}{2}[A - N_0 + k_1 A]$

By substituting (186) thus take

$$\lambda_1 = \frac{1}{2}[A - N_0 + k_1 A]$$

Since we have $(2\lambda_1 - k_1 A)^2 = (A - N_0)^2$ and $(2\lambda_1 - k_1 A)^2 = 4C + k_1^2 A^2$

Hence

$$\lambda_1{}^2 = \lambda_1 k_1 A - C = 0$$

Which give

$$\lambda_1 = \frac{k_1 A \pm \sqrt{k_1^2 A^2 + 4C}}{2}$$

Similarly we can find the value of λ_2.

Theorem $(\mathbf{5.2.9})[\mathbf{48}]$. Let the assumptions (174) – (177) be fulfilled and let the inequality

$$(A - N_0)^2 - \frac{\alpha^2}{\nu^2} A^2 < 4c \leq (A - N_0)^2 - \frac{\alpha^2}{(\nu+1)^2} A^2 \tag{190}$$

hold for $\nu \in \{1, 2, \dots\}$. Then equation (122) has a one-parametric family of solutions of the form (178), where λ is given by (187), k_1 by (186), $\mu_1 \in \mathbb{R}$ is an arbitrary non-vanishing parameter, and for $\nu \geq 2$ there holds $k_j = jk_1,\ j = 2, \dots, \nu$, and the $\mu_j,\ j = 2, \dots, \nu$ are determined by μ_1 via relations (188).

Proof. Due to (186) we have the relation

$$4c = (A - N_0)^2 - k_1^2 A^2$$

between c and k_1. Therefore, inequality (198) is equivalent to the above inequality $\frac{\alpha}{\nu+1} \leq k_1 < \frac{\alpha}{\nu}$. Further, (190) implies assumption (180).

We split the linear operator G_0 in (183)

$$G_0[z](x) = \frac{\beta}{x^{1+\alpha}} \int_0^x \xi^\alpha\, \mathcal{Z}(\xi) d\xi + G_1[z](x),$$

where $\beta = \frac{1}{A}[N_0 + 2\lambda] = 1 + k_1$ observing (185) and

$$\begin{aligned} G_1[z](x) = {} & \frac{\beta}{x^{1+\alpha}} \frac{Ax - k(x)}{k(x)} \int_0^x \xi^\alpha\, \mathcal{Z}(\xi) d\xi \\ & + \frac{1}{x^\alpha k(x)} \int_0^x \{[n(x,\xi) - N_0] \xi^\alpha \mathcal{Z}(\xi) \\ & + \lambda [m(x,\xi) - 1][\xi^\alpha \mathcal{Z}(\xi) + (x-\xi)^\alpha \mathcal{Z}(x-\xi)]\} d\xi \\ & + \frac{1}{x^\alpha k(x)} \int_0^x m(x,\xi) \\ & \times \sum_{j=1}^{\nu} \mu_j \left[\xi^{k_j} (x-\xi)^\alpha \mathcal{Z}(x-\xi) + (x-\xi)^{k_j} \xi^\alpha \mathcal{Z}(\xi)\right] d\xi \end{aligned} \tag{191}$$

and write equation (181) in the form

$$\mathcal{Z}(\xi) - \frac{\beta}{x^{1+\alpha}} \int_0^x \xi^\alpha \mathcal{Z}(\xi) d\xi = g(x), \tag{192}$$

where $g(x) = f_0(x) + G_1[\mathcal{Z}](x) + L_0[\mathcal{Z}, \mathcal{Z}](x)$. Estimating (191) and (184) we get

$$|G_1[\mathcal{Z}](x)| \leq Const\ x^{k_1} \|\mathcal{Z}\| \text{ and } |L_0[\mathcal{Z}_1, \mathcal{Z}_2](x)| \leq Const\ x^\alpha \|\mathcal{Z}_1\| \|\mathcal{Z}_2\| \ .$$

So, for any $\mathcal{Z} \in C[0,T]$ we have $G_1[\mathcal{Z}] \in C[0,T]$ with $G_1[\mathcal{Z}](0) = 0$ and $L_0[\mathcal{Z}, \mathcal{Z}] \in C[0,T]$ with $L_0[\mathcal{Z}, \mathcal{Z}](0) = 0$, therefore $g \in C[0,T]$ with $g(0) = f_0(0)$.

The auxiliary equation (192) for known $g \in C[0,T]$ has the unique continuous solution

$$\mathcal{Z}(x) = g(x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} g(\xi) d\xi, \tag{193}$$

where $\alpha - \beta = \alpha - k_1 > -1$. Hence we obtain instead of (181) the equivalent equation

$$\mathcal{Z}(x) = f(x) + G[\mathcal{Z}](x) + L[\mathcal{Z}, \mathcal{Z}](x) \tag{194}$$

where

$$f(x) = f_0(x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} f_0(\xi) d\xi$$

with $f(0) = \frac{1+\alpha}{1+\alpha-\beta} f_0(0) = \frac{\alpha+1}{\alpha-k_1} f_0(0)$ and

$$G[\mathcal{Z}](x) = G_1[\mathcal{Z}](x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} G_1[\mathcal{Z}](\xi) d\xi$$

$$L[\mathcal{Z}_1, \mathcal{Z}_2](x) = L_0[\mathcal{Z}_1, \mathcal{Z}_2](x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} L_0[\mathcal{Z}_1, \mathcal{Z}_2](\xi) d\xi$$

We have the estimations

$$\|G[\mathcal{Z}](x)\|_\sigma \leq \frac{\alpha+1}{\alpha-k_1} \|G_1[\mathcal{Z}]\|_\sigma \text{ and } \|L[\mathcal{Z}_1, \mathcal{Z}_2]\|_\sigma \leq \frac{\alpha+1}{\alpha-k_1} \|L_0[\mathcal{Z}_1, \mathcal{Z}_2]\|_\sigma \ .$$

So, for any $\mathcal{Z} \in C[0,T]$ we have $G[\mathcal{Z}] \in C[0,T]$ with $G[\mathcal{Z}](0) = 0$ and for any pair $\mathcal{Z}_1, \mathcal{Z}_2 \in C[0,T]$ also $L[\mathcal{Z}_1, \mathcal{Z}_2] \in C[0,T]$ with $L[\mathcal{Z}_1, \mathcal{Z}_2](0) = 0$. Hence $\mathcal{Z}(0) = f(0)$ for the solution $\mathcal{Z}$ of (194).

To apply Lemma (5.2.1) to equation (194) we have to prove the inequalities (124) – (126). We can show that

$$\|G[\mathcal{Z}]\|_\sigma \le \begin{cases} \text{const}\frac{1}{\sigma}\|\mathcal{Z}\|_\sigma & \text{if} \quad k_1 \ge 1 \\ \text{const}\frac{1}{\sigma^{k_1}}\|\mathcal{Z}\|_\sigma & \text{if} \quad 0 < k_1 < 1 \end{cases}$$

and further $\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le \text{Const}\ \|\mathcal{Z}_1\|_\sigma \|\mathcal{Z}_2\|_\sigma$ and

$$\|L[\mathcal{Z}_1,\mathcal{Z}_2]\|_\sigma \le \begin{cases} \text{const}\frac{1}{\sigma}\|\mathcal{Z}_1\|\|\mathcal{Z}_2\|_\sigma & \text{if} \quad \alpha \ge 1 \\ \text{const}\frac{1}{\sigma^{k_1}}\|\mathcal{Z}_1\|\|\mathcal{Z}_2\|_\sigma & \text{if} \quad 0 < \alpha < 1 \end{cases}$$

and also with $\mathcal{Z}_1$ and $\mathcal{Z}_2$ interchanged. These estimates verify (124) – (126) and by Lemma (5.2.1) the theorem is proved.

In the next theorem we prove the existence of solutions to equation (122) of the simpler form

$$y(x) = \lambda + x^\alpha \mathcal{Z}(x)\,,\ \mathcal{Z} \in C[0,T]\,, \tag{195}$$

where $\lambda \in \mathbb{R}$. We again have the possible values $\lambda_{1,2}$ from (179) for λ assuming the assumption (180) for real solutions, too. In equation (181) for $\mathcal{Z}$ the functions f_0 and G_0 are now defined by the formulas (182) and (183) without the terms with sums whereas the formula (184) for L_0 remains.

In contrast to the former case we now obtain solutions for both values $\lambda_{1,2}$ of λ. The value of $f_0(0)$ is given by (189). In the proof of existence of solutions we again split G_0 introducing G_1 by (191) without the last integral with sums. In the auxiliary equation (192) the parameter $\beta = \frac{1}{A}[N_0 + 2\lambda]$ now has the two possible values

$$\beta_{1,2} = 1 \pm \gamma_0\,, \quad \gamma_0 = \frac{1}{A}\sqrt{(A - N_0)^2 - 4c}\,. \tag{196}$$

In the following we distinguish the three cases $0 \le \gamma_0 < \alpha$, $\gamma_0 = \alpha$ and $\gamma_0 > \alpha$. In the case $0 \le \gamma_0 < \alpha$ we have $\beta_1 \in [1, 1+\alpha)$, $\beta_2 \in (1-\alpha, 1]$. For both $\beta = \beta_{1,2}$ the inversion formula (193) holds and we can proceed as above to obtain two solutions $\mathcal{Z}_{1,2}$ of equation (194) and hence solutions $y_{1,2}$ of form (195) to equation (122). Only if $\gamma_0 = 0$, the values β_1 and β_2 are equal (to 1) and the solutions y_1 and y_2 coincide.

In the case $\gamma_0 = \alpha$ we have $\beta_1 = 1 + \alpha, \beta_2 = 1 - \alpha$. For $\beta = \beta_2$, again the inversion formula (193) holds and we get a solution y of form (195). For $\beta = \beta_1$ instead of (193) the inversion formula

$$\mathcal{Z}_k(x) = K + g(x) + \beta_1 \int_0^x \frac{g(\xi)}{\xi}\, d\xi$$

is valid with an arbitrary $K \in \mathbb{R}$ if $g \in C[0,T]$ satisfies $g(0) = 0$ and $\int_0^T \frac{|g(x)|}{x} dx < \infty$. In view of (189) and the assumptions on the integrals of $C, \gamma, \delta, \epsilon$ this is fulfilled if the condition

$$d = \lambda_1 \left(B - N_1 - \frac{N_2}{\alpha+1}\right) - \lambda_1^2 \left(M_1 + \frac{M_2}{\alpha+1}\right) \tag{197}$$

holds. Then as in the logarithmic case in Theorem (5.2.8), for any $K \in \mathbb{R}$ we obtain a solution of the form (195), this means we have a one-parametric family of solutions y_k with parameter $K = \mathcal{Z}_k(0) \in \mathbb{R}$. If (197) does not hold, also as in Theorem (5.2.8) we can prove the existence of a family of solutions y_k of the form

$$y_k(x) = \lambda_1 + \mu x^{\alpha} \ln x + x^{\alpha} \mathcal{Z}_k(x), \mathcal{Z}_k \in C[0,T] \tag{198}$$

with $\lambda_1 = \frac{1}{2}[(\alpha+1)A - N_0]$,

$$\mu = \frac{\alpha+1}{A}\left[d + \lambda_1 \left(N_1 + \frac{N_2}{\alpha+1} - B\right) + \lambda_1^2 \left(M_1 + \frac{M_2}{\alpha+1}\right)\right]$$

and arbitrary $K = \mathcal{Z}_k(0) \in \mathbb{R}$. Under the condition (197) we have $\mu = 0$ and the solutions (198) take the form (195).

In the remaining case $\gamma_0 > \alpha$ we have $\beta_1 > 1 + \alpha,\ \beta_2 < 1 + \alpha$. For $\beta = \beta_2$ again the inversion formula (193) holds leading to a solution y of form (195).

For $\beta = \beta_1$ we take the inversion formula as follows:

$$\mathcal{Z}_k(x) = Kx^{\beta_1-\alpha-1} + g(x) - \beta_1 x^{\beta_1-\alpha-1} \int_x^T \xi^{\alpha-\beta_1} f_0(\xi) d\xi$$

$$+\beta_1 x^{\beta_1-\alpha-1} \int_0^x \xi^{\alpha-\beta_1} g_0(\xi) d\xi$$

with an arbitrary $K \in \mathbb{R}$ and $g_0(x) = G_1[\mathcal{Z}](x) + L_0[\mathcal{Z}, \mathcal{Z}](x)$. Under the restriction $1 + \alpha < \beta_1 < 1 + 2\alpha$ we can proceed as in the proof of Theorem (5.2.2) to obtain a family of solutions y_k of form (195) with parameter $K = \lim_{x\to 0} x^{\alpha+1-\beta_1} \in \mathbb{R}$.

So we have the following second existence theorem.

Theorem $(\mathbf{5.2.10})$[48]. Let the assumptions (174) – (177) and the inequality (180) be satisfied. Then equation (122) has the following solutions, where γ_0 is given by (196):

i) In case $\gamma_0 = 0$, i.e., $4c = (A - N_0)^2$: a unique solution y_0 of form (195) with $\lambda = \lambda_0 = \frac{1}{2}(A - N_0)$.

ii) In case $0 < \gamma_0 < \alpha$, i.e., $0 < (A - N_0)^2 - 4c < \alpha^2 A^2$: two solutions $y_{1,2}$ of form (195) with $\lambda = \lambda_{1,2}$ given by (179).

iii) In case $\gamma_0 = \alpha$, i.e., $(A - N_0)^2 - 4c = \alpha^2 A^2$: for $\lambda = \lambda_2$ one solution y_2 of form(195) and for $\lambda = \lambda_1$ a one-parametric family of solutions y_k of form(198) with parameter $K = \mathcal{Z}_k(0) \in \mathbb{R}$.

iv) In case $\gamma_0 > \alpha$, i.e., $(A - N_0)^2 - 4c > \alpha^2 A^2$ for $\lambda = \lambda_2$one solution y_2 of form (195) and in case $\alpha < \gamma_0 < 2\alpha$, i.e., $\alpha^2 A^2 < (A - N_0)^2 - 4c < 4\alpha^2 A^2$, for $\lambda = \lambda_1$ a one-parametric family of solutions y_k of form (195) with parameter $K = \lim_{x \to 0} x^{\alpha - \gamma_0} \mathcal{Z}_k(x)$.

Summarizing the results of Theorem (5.2.9) and (5.2.10) we get the following picture of solvability of equation (122), where we take the solution in case (ii) of Theorem (5.2.10) for $\lambda = \lambda_1$ as the member of the family of solutions in Theorem (5.2.9) with parameter $\mu_1 = 0$.

Corollary (5.2.11)[48]. Under the assumptions (174) – (177) and the inequality (5.2.59) the following solutions to equation (122) exist.

a) In case $4c = (A - N_0)^2$: a solution of form (195).
b) In case $(A - N_0)^2 - \alpha^2 A^2 < 4c < (A - N_0)^2$: a one-parametric family ofsolutions of form (206) with parameter $\mu = \mu_1 \in \mathbb{R}$ for $\lambda = \lambda_1$ choosing a corresponding $\nu \in \{1,2, \dots\}$ in (180) and an additional solution of form (195) for $\lambda = \lambda_2$.
c) In case $4c = (A - N_0)^2 - \alpha^2 A^2$: a one-parametric family of solutions y_k of form (198) with parameter $K \in \mathbb{R}$ for $\lambda = \lambda_1$ and an additional solution of form (195) for $\lambda = \lambda_2$.
d) In case $(A - N_0)^2 - \alpha^2 A^2 < 4c < (A - N_0)^2 - 4\alpha^2 A^2$: a one-parametric family of solutions y_k of form (195) with parameter $K \in \mathbb{R}$ for $\lambda = \lambda_1$ and in case $4c < (A - N_0)^2 - \alpha^2 A^2$ a solution of form (195) for $\lambda = \lambda_2$.

Remark (5.2.12)[48]. In case (a) of Corollary (5.2.11) there may exist other continuous solutions of equation (122) which are not of form (187), (195) or (198). So the equation

$$xy(x) = \int_0^x y(\xi)y(x-\xi)d\xi + \int_0^x y(\xi)d\xi \tag{199}$$

has besides $y_0(x) \equiv 0$ the family of solutions $y(x) = \nu\left(\frac{x}{\gamma}\right)$, $\gamma > 0$ with Volterra's function

$$\nu(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt \sim \frac{1}{-\ln x} \quad \text{as} \quad x \longrightarrow 0 .$$

This follows applying the method of Laplace transform to the equation. We remark that any equation of the form

$$Ax w(x) = \int_0^x w(\xi) w(x-\xi) d\xi + N_0 \int_0^x w(\xi) d\xi + c \qquad (200)$$

with $4c = (A - N_0)^2$ can be reduced to equation (199) substituting $y(x) = \frac{1}{A}\left[w(x) - \frac{1}{2}(A - N_0)\right]$. The general equation (122) in case 1 of Corollary (5.2.11) can be treated as a perturbation of equation (200)

If $4c > (A - N_0)^2$ we have the conjugate complex values

$$\lambda_{1,2} = \frac{1}{2}[A - N_0 \pm iAw_0], \ \ \beta_{1,2} = 1 \pm iw_0$$

where $w_0 = \frac{1}{A}\sqrt{4c - (A - N_0)^2}$. From Re $\beta_{1,2} = 1$ it follows that there exist two complex solutions of form (195) now as in case 2 of Theorem (5.2.10).

The assumptions (174) – (177) on the data of equation (122) allow to handle as a special case the equation of Bernstein and Doetsch

$$xy(x) = \gamma \int_0^x y(\xi) y(x-\xi) d\xi + (1-\gamma) \int_0^x y(\xi) d\xi \qquad (0 < \gamma < 1)$$

with the solutions $y(x) = E_\gamma(Cx^\gamma)$, where $C \in \mathbb{R}$ is an arbitrary parameter and E_γ denotes the Mittag-Leffler function. But the integral equation for the elliptic theta zero function

$$2xy(x) = \int_0^x y(\xi) y(x-\xi) d\xi + \int_0^x y(\xi) d\xi - 1$$

cannot be dealt with by the present method because of the free term $p(x) \equiv -1$ and requires further investigation.

Sec(5.3) : Convolution Equations of the Third Kind

We deal with two types of autoconvolution equations of the third kind whose free terms possess nonzero values at $x = 0$. The first type of equations has a coefficient $k(x)$ of the unknown function with asymptotics $k(x) \sim Ax$ as $x \to 0$. It comprises the well-known equation of Bernstein–Doetsch [149] as an important special case. The second type of equations has a coefficient $k(x)$ with $k(x) \sim Ax^{1/2}$ as $x \to 0$. We derive existence theorems for a one-parametric family of solutions and an additional solitary solution for both types of equations.

Further, we complete our recent investigation on differintegral equations of autoconvolution type by [150] proving an existence theorem for a one-parametric family of solutions to the special class of differintegral equations with solutions given by Volterra type functions. Also we extend the uniqueness theorems in the first part to more general function spaces. Finally, we study the basic linear singular Abel–Volterra integral equations of the third kind which are needed for our investigations.

We derive the existence theorems for the equations of first and second type, respectively.

Autoconvolution equation of the first type

We deal with the equation

$$k(x)y(x)=\int_0^x y(\xi)y(x-\xi)d\xi+\lambda\int_0^x y(\xi)d\xi+p(x),\qquad 0<x<1, \quad (201)$$

where $k\in C[0,1] with\ k(x)>0\ in\ (0,1]\ and$

$$k(x)=Ax+B(x)\quad (A>0),\ B(x)=o(x)\quad as\ x\to 0, \quad (202)$$

$\lambda\in\mathbb{R},\qquad p\in C[0,1]\ with$

$$p(x)=-\gamma^2+\omega\sqrt{x}+r(x)\quad (\gamma>0),\ \ r(x)=o\left(x^{\frac{1}{2}}\right)\ as\ x\to 0. \quad (203)$$

Eq. (201) in the case $\gamma=0\ (with\ \omega=0\ and\ r(x)=O(x)\ as\ x\to 0)$ has been considered in [149].

We make the first ansatz in Eq. (201)

$$y(x)=\frac{\gamma}{\sqrt{\pi x}}+c+z(x),\ z\in C[0,1] with\ z(0)=0 \quad (204)$$

where $c\in\mathbb{R}$ is given by

$$c=\frac{1}{4}(A-2\lambda)-\frac{\sqrt{\pi}\omega}{4\gamma} \quad (205)$$

Then z obeys the equation

$$k(x)z(x)=\int_0^x z(\xi)z(x-\xi)d\xi+\mu_0\int_0^x \frac{z(\xi)}{\sqrt{x-\xi}}d\xi+\lambda_0+\int_0^x z(\xi)d\xi+q(x) \quad (206)$$

with $\mu_0=\frac{2\gamma}{\sqrt{\pi}}\ (>0),\ \lambda_0=\lambda+2c=\frac{A}{2}-\frac{\sqrt{\pi}\omega}{2\gamma}\ and$

$$q(x)=\rho(x)-\left(\frac{\gamma}{\sqrt{\pi x}}+c\right)B(x),\ \rho(x)=r(x)+c(\lambda+c-A)x. \quad (207)$$

Splitting up, we write Eq. (206) as follows

$$xz(x)=\mu_1\int_0^x \frac{z(\xi)}{\sqrt{x-\xi}}d\xi+\lambda_1\int_0^x z(\xi)\,d\xi+g(x) \quad (208)$$

where $\mu_1=\frac{\mu_0}{A}=\frac{2\gamma}{A\sqrt{\pi}}\ (>0),\ \lambda_1=\frac{\lambda_0}{A}=\frac{1}{2}-\frac{\sqrt{\pi}\omega}{2A\gamma}\ and$

$$g(x)\equiv g[z](x)=g_0(x)+G_0[z](x)+L_0[z,z](x) \quad (209)$$

$with\ g_0(x)=\frac{x}{k(x)}q(x),$

$$G_0[z](x)=-\frac{\mu_1B(x)}{k(x)}\int_0^x \frac{z(\xi)}{\sqrt{x-\xi}}d\xi-\frac{\lambda_1B(x)}{k(x)}\int_0^x z(\xi)\,d\xi \quad (210)$$

$$L_0[z_1,z_2](x) = \frac{x}{k(x)}\int_0^x z_1(\xi) z_2(x-\xi)\,d\xi. \tag{211}$$

Eq. (208) has the form of Eq. (274). Under the condition

$$|g(x)| \le C\, x^{\delta} e^{-\frac{v_1^2}{x}}, \quad \delta > \Lambda_{\mathbf{1}} + \frac{1}{2}, \tag{212}$$

where $\Lambda_{\mathbf{1}} = |\lambda_1| N$ and N is defined in (295) with $\nu = v_1 = \sqrt{\pi}\,\mu_1$, Eq. (208) is equivalent to

$$z(x) = K f_0(x) + \frac{g(x)}{x} + \int_0^x M(x,\xi) g(\xi)\,d\xi$$

where the kernel $M(x,\xi)$ is given by (293) with $\lambda = \lambda_1$, or

$$z(x) = f(x) + G[z](x) + L[z,z](x) \tag{213}$$

where

$$f_0(x) = x^{\lambda_1 - 1} exp\left[-\frac{v_1^2}{2x}\right] D_1 - 2\lambda_1 \left(\frac{\sqrt{2}v_1}{\sqrt{x}}\right),$$

$$f(x) = K f_0(x) + \frac{1}{x} g_0(x) + \int_0^x M(x,\xi) g_0(\xi)\,d\xi,$$

$$G[z](x) = \frac{1}{x} G_0[z](x) + \int_0^x M(x,\xi) G_0[z](\xi)\,d\xi,$$

$$L[z_1,z_2](x) = \frac{1}{x} L_0[z_1,z_2](x) + \int_0^x M(x,\xi) L_0[z_1,z_2](\xi)\,d\xi.$$

The kernel $M(x,\xi)$ fulfills the estimation (299) with $\Lambda = \Lambda_1$. We make the assumptions

$$|B(x)| \le C_1\, x^{\delta+\frac{1}{2}} e^{-\frac{v_1^2}{x}}, \quad |\rho(x)| \le C_2\, x^{\delta}\, e^{-\frac{v_1^2}{x}} \tag{214}$$

with $\delta > \Lambda_1 + \frac{1}{2}$ as in (212) and look for solutions z of Eq. (213) of the form

$$z(x) = x^{\beta} e^{-\frac{v_1^2}{x}} w(x), \quad w \in C[0,1], \tag{215}$$

with $= 2\lambda_1 - \frac{3}{2}$. From (215) the estimations

$$|q(x)| \le \hat{C}\, x^{\delta}\, e^{-\frac{v_1^2}{x}} \quad and \quad |g_0(x)| \le C_0\, x^{\delta}\, e^{-\frac{v_1^2}{x}} \tag{216}$$

follow. Observing the second inequality in (216), for functions z of the form (215) we obtain the estimate (212) for the function g defined in (209).

The function $w \in C[0,1]$ in (215) satisfies the operator equation

$$w(x) = \varphi(x) + \hat{G}[w](x) + \hat{L}[w,w](x), \tag{217}$$

where

$$\varphi(x) = K\varphi_0(x) + x^{-\beta-1}e^{\frac{v_1^2}{x}}g_0(x) + x^{-\beta}e^{\frac{v_1^2}{x}}\int_0^x M(x,\xi)g_0(\xi)d\xi \qquad (218)$$

with

$$\varphi_0(x) = x^{-\beta}e^{\frac{v_1^2}{x}}f_0(x) \sim (\sqrt{2}v_1)^{1-2\lambda_1} \quad as\ x \to 0 \qquad (219)$$

by (275), (276), and

$$\hat{G}[w](x) = x^{-\beta}e^{\frac{v_1^2}{x}}\left(\frac{1}{x}G_0[z](x) + \int_0^x M(x,\xi)G_0[z](\xi)d\xi\right), \qquad (220)$$

$$\hat{L}[w_1,w_2](x) = x^{-\beta}e^{\frac{v_1^2}{x}}\left(\frac{1}{x}L_0[z_1,z_2](x) + \int_0^x M(x,\xi)L_0[z_1,z_2](\xi)d\xi\right) \qquad (221)$$

where z_1, z_2 like z are connected with w_1, w_2 and w by (215).

From (219), (216) with $\delta > \Lambda_1 + \frac{1}{2}$ and the estimation (299) for $M(x,\xi)$ one easily gets that $\varphi \in C[0,1]$ with $(0) = K\varphi_0(x)(0) = K(\sqrt{2}v_1)^{1-2\lambda_1}$. Further, in view of (210) with (214), (211) and (299) the expressions (220), (221) are lying in $C[0,1]$ for any $w \in C[0,1]$ and $w_1, w_2 \in C[0,1]$, respectively, satisfying $\hat{G}[w](0) = \hat{L}[w_1,w_2](0) = 0$.

To prove the existence of a unique solution $w = w_K \in C[0,1]$ to Eq. (217) we use the method of exponentially weighted norms $\|w\|_\sigma = max_{0\le x\le 1}|e^{-\sigma x}w(x)|, \sigma > 0$, in $C[0,1]$ equivalent to $\|w\| = \|w\|_0$. By Theorem for this is sufficient to show the following inequalities for $\sigma > 0$:

$$\left\|\hat{G}[w]\right\|_\sigma \le u(\sigma)\|w\|_\sigma \qquad (222)$$

with a continuous function u satisfying $u(\sigma) \to 0\ as\ \sigma \to \infty$for any $w \in C[0, 1]$, and

$$\left\|\hat{L}[w_1,w_2]\right\|_\sigma \le \begin{cases} v_0\|w_1\|_\sigma\|w_2\|_\sigma, \\ min(v_1(\sigma)\|w_1\|\|w_2\|_\sigma, v_2(\sigma)\|w_1\|_\sigma\|w_2\|) \end{cases} \qquad (223)$$

with a constant v_0 and continuous functions v_k , $k = 1,2,$ satisfying $v_k(\sigma) \to 0$ as $\sigma \to \infty$ for any pair $w_1, w_2 \in C[0,1]$.

Observing assumptions (202) and (214) we estimate in (210) (with generic positive constants α_k)

$$|e^{-\sigma x}G_0[z](x)| \le \alpha_1 x^{\delta-\frac{1}{2}}e^{-\frac{v_1^2}{x}}\int_0^x \frac{\xi^\beta}{\sqrt{x-\xi}}e^{-\frac{v_1^2}{\xi}}e^{-\sigma(x-\xi)}d\xi \cdot \|w\|_\sigma$$

$$\le \alpha_2 x^{\beta+1}e^{-\frac{v_1^2}{x}}\frac{\|w\|_\sigma}{\sigma^{1/2}}$$

where we used that

$$e^{-\frac{v_1^2}{\xi}} \le \alpha_0\, \xi^p \quad \text{with arbitrary } p > 0 \;\; in \;\; [0,1],$$

and

$$\int_0^x \frac{e^{-\sigma\eta}}{\sqrt{\eta}} d\eta \le \frac{\sqrt{\pi}}{\sigma^{1/2}}.$$

Hence, from (220) we have

$$\left|e^{-\sigma x}\; \hat{G}[w](x)\right|$$

$$\le \alpha_2 \frac{\|w\|_\sigma}{\sigma^{1/2}} + \alpha_1 x^{-\beta}\, e^{\frac{v_1^2}{x}} \int_0^x |M(x,\xi)|\, e^{-\sigma(x-\xi)}\, \xi^{\delta-\frac{1}{2}} \int_0^x \frac{\eta^\beta}{\sqrt{\xi-\eta}} e^{\frac{v_1^2}{\eta}}\, e^{-\sigma(\xi-\eta)} d\eta d\xi \cdot \|w\|_\sigma$$

$$\le \alpha_2 \frac{\|w\|_\sigma}{\sigma^{1/2}} + \alpha_3 N_p(x) \frac{\|w\|_\sigma}{\sigma^{\frac{1}{2}}},$$

$$N_p(x) = x^{-\beta}\, e^{\frac{v_1^2}{x}} \int_0^x |M(x,\xi)|\; \xi^p e^{-\frac{v_1^2}{\xi}} d\xi$$

with sufficiently large p. Using the estimation (299) for $M(x,\xi)$ and the obvious relation $e^{-\frac{v_1^2}{\xi}} \le e^{-\frac{v_1^2}{x}}$ for the first part in (299) , we finally get (222) with $u(\sigma) = \frac{\alpha_4}{\sigma^{1/2}}$.

Further, we estimate in (211)

$$|e^{-\sigma x}\; L_0[z_1,z_2](x)| \le \alpha_5 \int_0^x \xi^\beta\, (x-\xi)^\beta e^{-\frac{v_1^2}{\xi}-\frac{v_1^2}{x-\xi}} d\xi \|w_1\|_\sigma \|w_2\|_\sigma$$

$$\le \alpha_5 e^{-\frac{v_1^2}{x}} \int_0^x \xi^\beta\, (x-\xi)^\beta e^{-\frac{v_1^2}{2\xi}-\frac{v_1^2}{2(x-\xi)}} d\xi \|w_1\|_\sigma \|w_2\|_\sigma$$

$$\le \alpha_6\, x^{\beta+1}\, e^{-\frac{v_1^2}{x}} \|w_1\|_\sigma \|w_2\|_\sigma$$

and analogously

$$|e^{-\sigma x}\; L_0[z_1,z_2](x)| \le \alpha_5 \int_0^x \xi^\beta\, (x-\xi)^\beta\; e^{-\sigma\xi}\; e^{-\frac{v_1^2}{\xi}-\frac{v_1^2}{x-\xi}} d\xi \|w_1\|_\sigma \|w_2\|_\sigma$$

$$\le \alpha_7\, x^{\beta+1}\, e^{-\frac{v_1^2}{x}}\, \frac{1}{\sigma} \|w_1\|_\sigma \|w_2\|_\sigma$$

where we used that $\int_0^x e^{-\sigma\xi}\; d\xi \le \frac{1}{\sigma}$. From (221) we then obtain

$$\left|e^{-\sigma x}\; \hat{L}[w_1,w_2](x)\right|$$

$$\le \|w_1\|_\sigma \|w_2\|_\sigma \left[\alpha_6 + \alpha_5 x^{-\beta} e^{\frac{v_1^2}{x}} \int_0^x |M(x,\xi)| \int_0^\xi \eta^\beta \, (\xi - \eta)^\beta \; e^{-\frac{v_1^2}{\xi} - \frac{v_1^2}{x-\xi}} d\eta d\xi\right]$$
$$\le \|w_1\|_\sigma \|w_2\|_\sigma \left[\alpha_6 + \alpha_8 \, N_p(x)\right] \le \alpha_9 \|w_1\|_\sigma \|w_2\|_\sigma$$
and
$$\left|e^{-\sigma x} \, \hat{L}[w_1, w_2](x)\right|$$
$$\le \|w_1\| \|w_2\|_\sigma \left[\frac{\alpha_7}{\sigma} + \alpha_5 x^{-\beta} e^{\frac{v_1^2}{x}} \int_0^x |M(x,\xi)| \int_0^\xi \eta^\beta \, (\xi - \eta)^\beta \, e^{-\sigma\eta} \; e^{-\frac{v_1^2}{\xi} - \frac{v_1^2}{x-\xi}} d\eta d\xi\right]$$
$$\le \|w_1\| \|w_2\|_\sigma \left[\frac{\alpha_7}{\sigma} + \alpha_{10} \, N_p(x) \frac{1}{\sigma}\right]$$
$$\le \frac{\alpha_{11}}{\sigma} \|w_1\| \|w_2\|_\sigma$$
This yields the inequalities (223) with $v_0 = \alpha_9 \; and \; v_1(\sigma) = v_2(\sigma\,) = \frac{\alpha_{11}}{\sigma}$.

In view of Theorem in [148] we get the existence and uniqueness of a solution $w = w_k \in C[0,1] \; with \; w_k(0) = K_0, \;\; K_0 = K(\sqrt{2}v_1)^{1-2\lambda_1}$ to Eq. (217). That means, there exists a family of solutions $y_K, K \in \mathbb{R}$, to Eq. (201) of the form (204), (215).

Further, we are looking for an additional solution to Eq. (208) making the second ansatz

$$y(x) = -\frac{\gamma}{\sqrt{\pi x}} + \bar{c} + \bar{z}(x) \tag{224}$$

with

$$\bar{c} = \frac{1}{4}(A - \lambda) + \frac{\sqrt{\pi}\,\omega}{4\gamma}$$

yielding Eq. (213) for $\bar{z}$ with coefficients $\bar{\mu}_0 = -\frac{2\gamma}{\sqrt{\pi}}(< 0),\; \bar{\lambda}_0 = \frac{A}{2} + \frac{\sqrt{\pi}\,\omega}{2\gamma}$ and the free term

$$\bar{q}(x) = \bar{p}(x) + \left(\frac{\gamma}{\sqrt{\pi x}} - \bar{c}\right) B(x), \quad \bar{p}(x) = r(x) + \bar{c}(\lambda + \bar{c} - A)x.$$

The representation

$$z(x) = x^{\bar{\beta}} \, e^{-\frac{v_1^2}{x}} \bar{w}(x) \tag{225}$$

where $\bar{\beta} = 2\bar{\lambda}_1 - \frac{3}{2}, \bar{\lambda}_1 = \frac{\bar{\lambda}_0}{A}, \bar{v}_1 = \sqrt{\pi}\,\bar{\mu}_1, \; \bar{\mu}_1 = \bar{\mu}_0/A$ then leads to a corresponding Eq. (217) but with $K = 0$. The same reasoning as for (206) by the first ansatz then proves the existence of a solution *y* to Eq. (201) of the form (224), (225) where $\bar{w}(0) = 0$ under the corresponding assumptions (214).

Summing up, we obtained the following existence theorem.

Theorem (5.3.1)[139]. Let the assumptions (202), (203) with (214) be fulfilled. Then Eq. (201) has a one-parametric family of solutions $y_K, K \in \mathbb{R}$, of the form (204),

(215) with $\varpi_K(0) = K(\sqrt{2}v_1)^{1-2\lambda_1}$. Under the corresponding assumptions (214), where v_1 is replaced by $\bar{v}_1$ and Λ_1 by $\bar{\Lambda}_1 = |\bar{\lambda}_1|\bar{N},\ \bar{N}$ defined by (295) for $\bar{v}_1$, Eq. (201) has an additional single solution $\bar{y}$ of the form (224), (225) with $\bar{\varpi}(0) = 0$.

Now we consider Eq. (201) under more general assumptions on the free term $p(x)$. We assume that the function $q(x)$ in (207) has a decomposition of the form

$$q(x) = q_1(x) + q_2(x) \qquad (226)$$

where $q_1(x) = o(x^{1/2})$ *and* $q_2(x)$ satisfies the estimation (216). Let us put analogously in Eq. (206)

$$z(x) = z_1(x) + z_2(x) \qquad (227)$$

where $z_1 \in C[0,1]$ with $z_1(0) = 0$ fulfills the equation

$$Axz_1(x) = \int_0^x z_1(\xi)z_1(x-\xi)d\xi + \mu_0 \int_0^x \frac{z_1(\xi)}{\sqrt{x-\xi}}d\xi + q_1(x). \qquad (228)$$

Then z_2 satisfies the equation

$$kxz_2(x) = \int_0^x z_2(\xi)z_2(x-\xi)d\xi + \mu_0 \int_0^x \frac{z_2(\xi)}{\sqrt{x-\xi}}d\xi + \int_0^x x(\xi)z_2(\xi)d\xi + \hat{q}_2(x). \qquad (229)$$

where $(x) = \lambda_0 + 2z_1(x) \in C[0,1]$ *and* $\hat{q}_2(x) = q_2(x) - B(x)z_1(x)$. The function $\hat{q}_2$ fulfills the inequality (216). Therefore, based on Corollary (5.3.12), Eq. (201) has also a one-parametric family of solutions $y_K, K \in \mathbb{R}$, of the form (204), (215). The same procedure yields an additional single solution $\bar{y}$. This proves the following corollaryto Theorem (5.3.1).

Corollary (5.3.2)[139]. Let assumptions (202), (203) with (214) for B and (226) with (5.3.16) for q_2 be fulfilled. Further, let there exist a solution $z_1 \in C[0,1]$ with $z_1(0) = 0$ for Eq. (228) with $\mu_0 > 0$. Then Eq. (201) has a family of solutions y_K of the form (204) with $z_1 + z_{2,k}$, $K \in \mathbb{R}$, where $\{z_{2,k}\}$ is a family of solutions of the form (215) to Eq. (229). Under corresponding assumptions the second ansatz (224) yields an additional solution y of the form (204) with $\bar{z} = \bar{z}_1 + \bar{z}_2$ where $\bar{z}_1$ is a solution of Eq. (228) with $\bar{\mu}_0 < 0$ and $\bar{z}_2$ a solution of (229) for $\bar{\mu}_0, \bar{x} = \bar{\lambda}_0 + 2\bar{z}_1$, and corresponding free term.

Furthermore, we discuss the methods of finding one solution of Eq. (228) in the case that

$$q_1(x) = \sum_{n=2}^{\infty} a_n\ x^{\frac{n}{2}}, \quad 0 \le x \le 1. \qquad (230)$$

At first, the ansatz of z_1 as a related series

$$z_1(x) = \sum_{n=1}^{\infty} c_n \ x^{\frac{n}{2}}, \quad 0 \le x \le 1. \tag{231}$$

yields a recurrent system of equations for the c_n with unique solutions. So, in case of convergence of the series (231) we obtain exactly one solution of this kind for Eq. (228).

Secondly, if the series (230) holds in $\mathbb{R}_+$, application of the Laplace transform to Eq. (228) yields the Riccati equation for the Laplace transform $Z_1(p)$ *of* z_1

$$Z_1'(p) + \frac{1}{A} Z_1^2(p) + \left(\frac{v_1}{p^{1/2}} + \frac{\lambda_1}{p}\right) Z_1(p) + Q(p) = 0 \tag{232}$$

with

$$Q(p) = \frac{1}{A} \sum_{n=2}^{\infty} b_n \ p^{-1-\frac{n}{2}}, \quad b_n = a_n \Gamma\left(\frac{n}{2} + 1\right) \tag{233}$$

Putting $s = p^{1/2}$ *and* $\Phi(s) = Z_1(p)$, Eq. (232) takes the form of a Riccati equation with holomorphic coefficients

$$\Phi'(s) + \frac{2}{A} \Phi^2(s) + \left(2v_1 + \frac{2\lambda_1}{s}\right) \Phi(s) = F(s) \tag{234}$$

where

$$F(s) = 2sQ(s^2) = \frac{2}{A} \sum_{k=2}^{\infty} b_{k-1} \ s^{-k}. \tag{235}$$

As a Riccati equation, Eq. (234) has only fixed algebraic branching solutions, so the solutions of (234) can have only a finite number of poles in $Re\ s > 0$ (precisely, one simple pole at $-\frac{1+2\lambda_1}{2v_1}$). Therefore, the solutions are regular analytic functions of *s* in some half-plane $Re\ s > s_0 > 0$. Further, at infinity the solutions behave like a power of $s^{-m}, m \ge 3$, where b_{m-1} is the first non-vanishing coefficient of (235). For construction of a corresponding solution $\Phi(s)$ of (234), we transform Eq. (234) in usual way to a linear second-order equation

$$W''(s) + \left[2v_1 - \frac{1-2\lambda_1}{s}\right] W'(s) - \frac{2}{A} sF(s)W(s) = 0$$

for the function

$$\Phi(s) = \frac{AW'(s)}{2sW(s)}$$

and apply the theory of Thomé’s normal series (cf. [144], [147], [152]). From the known asymptotic behavior $p^{\frac{-m}{2}}$ of the solution $Z_1(p)$ at infinity, then one obtains a continuous solution $z_1(x)$ of Eq. (228). We also mention that the method of Laplace transformation to Eq. (228) is not restricted to free terms q_1 of the series form (230).

Finally, we deal with the special class of Eqs. (201) where

$$k(x) = Ax, \ \lambda \in \mathbb{R}, \ p(x) = -1 + \omega\sqrt{x} \quad (A > 0, \ \omega \in \mathbb{R}) \tag{236}$$

as an example, which for $A = 2,\ \lambda = 1,\ \omega = 0$ is the Bernstein–Doetsch equation. In case (236) we have $q(x) = c(\lambda + c - A)x$ with c given by (205).

For the Bernstein–Doetsch equation it is $c = 0$ implying $q(x) = 0$. We get a family of solutions $y_K(x) = \frac{1}{\sqrt{\pi x}} + x^{\frac{1}{2}} \exp\left(-\frac{1}{2}\right) w_K(x), w_K(x) \in C[0,1] with$ $w_K(0) = K \in \mathbb{R}$ and the additional solution $\bar{y}(x) = -\frac{1}{\sqrt{\pi x}}$. The functions y_K have the Laplace transforms

$$Y_K(p) = \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}} \frac{K\sqrt{\pi}}{exp\,[2\sqrt{p}] - \frac{K}{2}\sqrt{\pi}},$$

for $K = 0$ we have $y(x) = \frac{1}{\sqrt{\pi x}}$ and for $K = \pm\frac{2}{\sqrt{\pi}}$ one gets the Jacobian theta zero functions $\vartheta_3(x),\ \vartheta_2(x)$, respectively (cf. [154]).

In the general case (236) the Laplace transformation can be applied, too. In the four cases $A = 1\, with\, c = 0,\ c = 1 - \lambda,\ c = \frac{1}{2}$ the integration of the Riccati equation for the Laplace transform $Y(p)$ of the solution y can be performed by quadratures but the back transformation can be done explicitly in some special cases only.

Therefore, we only consider the equation for $A = 1,\ \lambda = 0,\ \omega = \frac{1}{\sqrt{\pi}}$ with $c = 0, \bar{c} = \frac{1}{2}$ and $q(x) = 0,\ \bar{q}(x) = -\frac{x}{4}$ yielding the solutions $y_K(x) = \frac{1}{\sqrt{\pi x}} + x^{-3/2} \exp\left(-\frac{4}{x}\right) w_K(x), w_K(x) \in C[0,1] with\, w_K(0) = K \in \mathbb{R}$, which possess the Laplace transforms

$$Y_K(p) = \frac{1}{\sqrt{p}} + \frac{\frac{\pi}{2}K}{exp\,[4\sqrt{p}] - \frac{\pi}{16}K(1 + 4\sqrt{p})}.$$

For $K = 0$ we have the solution $y(x) = \frac{1}{\sqrt{\pi x}}$ again and for $K \to \infty$ the Laplace transform

$$\bar{Y}(p) = \frac{1}{\sqrt{p}} - \frac{2}{\sqrt{p} + \frac{1}{4}}$$

of the solution

$$\bar{y}(x) = -\frac{1}{\sqrt{\pi x}} + \frac{1}{2} + \bar{z}(x),\ \bar{z}(x) = \frac{1}{2} e^{\frac{x}{16}}\, erfc\left(\frac{\sqrt{x}}{4}\right) - \frac{1}{2} \qquad (237)$$

with the complementary error function erfc (cf. [147]). The function $\bar{z}$ in (237) is the solution $\bar{z}_1$ of Eq. (228) for the second ansatz whereas the solution $\bar{z}_2$ of corresponding Eq. (229) is zero.

Autoconvolution equation of the second type

In the context of equations with continuous free terms having a nonzero value at $x = 0$ we further deal with the equation

$$k(x)y(x) = \int_0^x y(\xi)y(x-\xi)d\xi \ + p(x), \quad 0 < x \ < 1, \tag{238}$$

where $k \in C[0,1]$ $with$ $k(x) > 0$ in $(0,1]$ and

$$k(x) = \ Ax^{\frac{1}{2}} + B(x) \qquad (A > 0),\ B(x) = o\left(x^{\frac{1}{2}}\right) \quad as\ \ x \to 0, \tag{239}$$

$p \in C[0,1]$ $with$

$$p(x) = -\gamma^2 \ + q(x) \ \ (\gamma > 0), \quad q(x) \ = o(1) \quad as\ x \to 0. \tag{240}$$

The case $\gamma = 0$ has been considered in the first part.

We have the following existence theorem.

Theorem $(\mathbf{5.3.3})[\mathbf{139}]$. Let $k \in C[0,1]$ $with$ $k(x) > 0$ in $(0,1]$ have the form (239) and $p \in C[0,1]$ $the\ form$ (240) $with\ \gamma > 0$. Then Eq. (338) has the following solutions:

Under assumptions (274):

$$q(x) = o(x) \ \ as \ \ x \to 0 \ \ with \ \ \tfrac{q(x)}{x^2} \in L^1(0,1), \tag{241}$$

$$B(x) = o\left(x^{3/2}\right) \ as \ \ x \to 0 \ \ with \ \ \frac{B(x)}{x^{5/2}} \in L^1(0,1) \tag{242}$$

a family of solutions $y_K, K \in \mathbb{R}$, of the form

$$y(x) = E_1 x^{-\frac{1}{2}} \ + x^{\frac{1}{2}} z(x) \tag{243}$$

$where\ \ E_1 = \frac{1}{2}\ (\frac{A}{\pi} + \sqrt{\frac{A^2}{\pi^2} + \frac{4}{\pi}\gamma^2}\ ) > 0,\ \ with\ \ z = z_K \in C[0,1],\ \ z_K(0) = K.$

Under assumptions (275):

$$q(x) = dx^{\delta+\frac{1}{2}} + e(x) \ \ (d \in \mathbb{R}), \quad e(x) = \ o\left(x^{\delta+\frac{1}{2}}\right) \ as\ x \to 0, \tag{244}$$

$$B(x) = bx^{\delta+1} + c(x) \ \ (b \in \mathbb{R}), \quad c(x) = \ o\left(x^{\delta+1}\right) \ \ as\ x \to 0 \tag{245}$$

where $\delta > -\frac{1}{2}$ a solution y of the form

$$y(x) = E_2 x^{-1/2} \ \ + x^{\delta}\zeta(x) \tag{246}$$

where $E_2 = \frac{1}{2}\left(\frac{A}{\pi} - \sqrt{\frac{A^2}{\pi^2} + \frac{4}{\pi}\gamma^2}\right) < 0,\ \ with\ \ \zeta \in C[0,1],\ \ \zeta(0) = \frac{d - E_2 b}{A - 2E_2 B(\delta+1,\frac{1}{2})}.$

Proof. We only prove the second assertion of the theorem, the first one can be shown like in the proof of Theorem (5.3.3) in [139]. Inserting the ansatz (246) into Eq. (238) we obtain the equation for ζ

$$\zeta(x) = \ f_0(x) + G_0[\zeta\,](x) \ + L_0[\zeta,\zeta\,](x) \tag{247}$$

where

$$f_0(x) = \frac{h(x)}{x^{\delta}k(x)}, \quad h(x) = q(x) - E_2 x^{-1/2}\, B(x),$$

$$G_0[\zeta](x) = 2E_2 \frac{1}{x^\delta k(x)} \int_0^x \frac{\xi^\delta \zeta(\xi)}{\sqrt{x - \xi}} \, d\xi$$

$$L_0[\zeta_1, \zeta_2](x) = \frac{1}{x^\delta k(x)} \int_0^x \xi^\delta (x - \xi)^\delta \, \zeta_1(\xi)\zeta_2(x - \xi) d\xi. \tag{248}$$

In view of (239)with (245), (240)with (241) we have in (247) that $f_0 \in C[0,1]$ with $f_0(0) = \frac{d}{A} - \frac{E_2 b}{A}$. We split up

$$G_0[\zeta](x) = -\frac{\lambda}{x^{\delta+\frac{1}{2}}} \int_0^x \frac{\xi^\delta \zeta(\xi)}{\sqrt{x - \xi}} \, d\xi + G_1[\zeta](x)$$

where $\lambda = -2E_2/A > 0$ and

$$G_1[\zeta](x) = \frac{\lambda B(x)}{x^{\delta+\frac{1}{2}} k(x)} \int_0^x \frac{\xi^\delta \zeta(\xi)}{\sqrt{x - \xi}} \, d\xi \tag{249}$$

and write Eq. (5.3.47)for ζ in the form

$$\zeta(x) + \frac{\lambda}{x^{\delta+\frac{1}{2}}} \int_0^x \frac{\xi^\delta \zeta(\xi)}{\sqrt{x - \xi}} \, d\xi = g(x) \tag{250}$$

where

$$g(x) \equiv g[\zeta](x) = f_0(x) + G_1[\zeta](x) + L_0[\zeta, \zeta](x).$$

Due to (245) it holds $G_1[\zeta](x) \in C[0,1]$ with $G_1[\zeta](0) = 0$ for any $\zeta \in C[0,1]$. Also by (239) we obtain $L_0[\zeta_1, \zeta_2] \in C[0,1]$with $L_0[\zeta, \zeta](0) = 0$ for any $\zeta, \zeta_1, \zeta_2 \in C[0,1]$. Hence we have $g \in C[0,1] \; with \; g(0) = \; f_0(0) = \frac{d}{A} - \frac{E_2 b}{A}$.

Finally, solving Eq. (250) for fixed g, we get the equation for $\zeta \in C[0,1]$

$$\zeta(x) = f(x) + G[\zeta](x) + L[\zeta, \zeta](x) \tag{251}$$

where

$$f(x) = f_0(x) - \frac{\lambda}{x} \int_0^x \tilde{r}\left(\frac{\xi}{x}\right) f_0(\xi) d\xi \in C[0,1],$$

$$G[\zeta](x) = G_1[\zeta](x) - \frac{\lambda}{x} \int_0^x \tilde{r}\left(\frac{\xi}{x}\right) G_1[\zeta](\xi) d\xi \in C[0,1], \qquad \zeta_1, \zeta_2 \in C[0,1],$$

$$L[\zeta_1, \zeta_2](x) = L_0[\zeta_1, \zeta_2](x) - \frac{\lambda}{x} \int_0^x \tilde{r}\left(\frac{\xi}{x}\right) L_0[\zeta_1, \zeta_2](\xi) d\xi \in C[0,1], \zeta_1, \zeta_2 \in C[0,1].$$

The function_$\tilde{r}(u)$, $0 \leq u \leq 1$, is the resolvent function of Eq. (250). It is given by_$\tilde{r}(u) = u^{\delta} r(u)$ where r denotes the resolvent function of (250) for $\delta = 0$. Hence, it satisfies the estimation

$$|\tilde{r}(u)| \leq C\left[u^{\delta} + (1-u)^{-1/2}\right], \delta > -\frac{1}{2}. \tag{252}$$

We estimate by (245)in (249) for $\sigma > 0$

$$|e^{-\sigma x} G_1[\zeta](x)| \leq b_0 \int_0^x e^{-\sigma(x-\xi)} \xi^{\delta} (x-\xi)^{-\frac{1}{2}} d\xi \cdot \|\zeta\|_{\sigma}$$

with a positive constant b_0 which yields (with generic positive constants C_k)

$$\|G_1[\zeta]\|_{\sigma} \leq C_1 \frac{1}{\sigma^x} \|\zeta\|_{\sigma}, \quad x = \begin{cases} \frac{1}{2} & if\ \delta \geq 0, \\ \delta + \frac{1}{2} & if\ \delta < 0 \end{cases}$$

where for $\delta \geq 0$ we used the inequality (275) [139] and for $\delta < 0$. This immediately implies the related estimation

$$\|G[\zeta]\|_{\sigma} \leq C_1 \frac{1}{\sigma^x} \|\zeta\|_{\sigma}, \quad x > 0. \tag{253}$$

Further, it holds by (239) in (248)

$$|e^{-\sigma x} L_0[\zeta_1, \zeta_2](x)| \leq C_0 x^{-\delta-\frac{1}{2}} \int_0^x \xi^{\delta} (x-\xi)^{-\frac{1}{2}} d\xi \cdot \|\zeta_1\|_{\sigma} \|\zeta_2\|_{\sigma}$$

which gives $\|L_0[\zeta_1, \zeta_2]\|_{\sigma}$ implying

$$\|L_0[\zeta_1, \zeta_2]\|_{\sigma} \leq C_3 \|\zeta_1\|_{\sigma} \|\zeta_2\|_{\sigma}. \tag{254}$$

Finally,

$$\begin{aligned} |e^{-\sigma x} L_0[\zeta_1, \zeta_2](x)| &\leq C_4 x^{-\delta-\frac{1}{2}} \int_0^x e^{-\sigma\delta} \xi^{\delta} (x-\xi)^{\delta} d\xi \cdot \|\zeta_1\| \|\zeta_2\|_{\sigma} \\ &\leq C_4 \int_0^x e^{-\sigma\xi} \xi^{-\frac{1}{2}} (x-\xi)^{\delta} d\xi \cdot \|\zeta_1\| \|\zeta_2\|_{\sigma} \\ &\leq C_5 \frac{1}{\sigma^x} \|\zeta_1\| \|\zeta_2\|_{\sigma} \end{aligned}$$

with $x > 0$ as above. This gives in the same way

$$\|L[\zeta_1, \zeta_2]\|_{\sigma} \leq C_6 \frac{1}{\sigma^x} min(\|\zeta_1\| \|\zeta_2\|_{\sigma}, \|\zeta_1\|_{\sigma} \|\zeta_2\|). \tag{255}$$

From $(253) - (255)$ by means of Theorem in [148] the assertion of Theorem $(5.3.3)$ follows, where the value of $\zeta(0)$ results from Eq. (250) with $g(0) = f_0(0) = \frac{d - E_2 b}{A}$.

An differintegral equation

We further extend our recent investigations on differintegral equations with autoconvolution integral [150] and study (slightly generalized) again. The equation writes

$$y'(x)+\left(\frac{1}{2}+B(x)\right)y(x)=\frac{1}{x}\int_0^x a_0\,(x,\xi)y(x-\xi)y(\xi)d\xi$$

$$+\frac{1}{x}\int_0^x b_0\,(x,\xi)y(\xi)d\xi\ +\mathrm{g}(x),\qquad \in(0,T\,),\qquad (256)$$

where we choose $T\in(0,1)$. Assuming that $xy(x)\to 0\ as\ x\to 0$, from (256) by integration we obtain the equivalent integral equation

$$y(x)=\frac{1}{x}\int_0^x \mathrm{g}_0\,(\xi)d\xi-\frac{1}{x}\int_0^x B_0\,(\,\xi)y(\xi)d\xi+\frac{1}{x}\int_0^x\int_0^\xi a_0(x,\eta)y(x-\eta)y(\eta)d\eta d\xi$$

$$+\frac{1}{x}\int_0^x\int_0^\xi b_0(\xi,\eta)y(\eta)d\eta d\xi \qquad (257)$$

where $\mathrm{g}_0(x)=x\mathrm{g}(x),\ \mathrm{B}_0(x)=xB(x)$. To this equation we are looking for solutions of the form

$$y(x)=\tau(x)+w(x) where\ \tau(x)=\frac{1}{K}v'\left(\frac{x}{K}\right),\ K>0,$$
$$w(x)=x^{-\gamma}\,z(x),\ 0\ \le\gamma<1\ \ and\ z\in C[0,T\,] \qquad (258)$$

and ν is Volterra's function (see [146])

$$\nu(x)=\int_0^\infty \frac{x^t}{\Gamma(t+1)}dt,\quad x>0.$$

The function ν and its derivative possess the asymptotic expansions

$$\nu(x)\sim-\frac{1}{lnx}+\frac{C}{ln^2x}\ \ as\ x\to 0$$

where $C=-\Gamma'(1)$ is the Euler constant, and

$$v'(x)\sim\frac{1}{x\,ln^2x}-\frac{2C}{x\,ln^3x}\ \ as\ \ x\to 0$$

so that for the positive function τ we have

$$\tau(x)\sim\frac{1}{x\,ln^2x},\qquad \int_0^x\tau(\xi)\,d\xi=v\left(\frac{x}{K}\right)\sim\frac{1}{\left|ln\frac{x}{K}\right|}\ as\ \ x\to 0. \qquad (259)$$

We make the following *Assumptions*:

For some $\gamma\in[0,1)$,

$A3.\ x^\gamma\ \mathrm{g}(x)\in L^1(0,T\,)$,

$A4.\ x^\gamma\ \mathrm{B}(x)\in L^\infty(0,T\,)$,

$A5.\, x^{\gamma}\, |b_0(x,\xi)| \le C_1 \; for \;\; 0 \le \xi \le x \le T,$
$A6.\, |a_0(x,\xi)| \le \; C_2 \; for \; 0 \le \xi \le x \le T \;, where \;\; a_0(x,\xi) = 1 + a_1(x,\xi)$
$with \; |a_1(x,\xi)| \le \; C_3 x^{\alpha}, \alpha = 1-\gamma > 0.$

Then we prove

Theorem $(\mathbf{5.3.4})[\mathbf{139}]$**.** Under Assumptions (276)– (279) Eq. (256) has a family of solutions of the form

$$y(x) = \frac{1}{K} v'\left(\frac{x}{K}\right) + x^{-\gamma} z(x)\,, \;\; where \;\; z \in C[0,T] \; with \;\; z(0) = 0 \tag{260}$$

and $K \in \mathbb{R}_{+}$.

Proof. Inserting the ansatz (265) into Eq. (264), we obtain the equation for $z \in C[0,T\,]$

$$z(x) = f(x) + G[z](x) + L[z,z](x) \tag{261}$$

where

$$f(x) = x^{\gamma-1}\int_0^x g_0(\xi)d\xi - x^{\gamma-1}\int_0^x B_0(\xi)\tau(\xi)d\xi$$

$$+ x^{\gamma-1}\int_0^x\int_0^{\xi} b_0(\xi,\eta)\tau(\eta)d\eta d\xi + x^{\gamma-1}\int_0^x\int_0^{\xi} a_1(\xi,\eta)\tau(\xi-\eta)\tau(\eta)d\eta\, d\xi,$$

$$G[z](x) = - x^{\gamma-1}\int_0^x B_0(\xi)\xi^{-\gamma}z(\xi)d\xi + x^{\gamma-1}\int_0^x\int_0^{\xi} b_0(\xi,\eta)\eta^{-\gamma}z(\eta)d\eta d\xi$$

$$+ x^{\gamma-1}\int_0^x\int_0^{\xi} a_0(\xi,\eta)[\eta^{-\gamma}\tau(\xi-\eta) + (\xi-\eta)^{-\gamma}\tau(\eta)]d\eta d\xi,$$

$$L[z_1,z_2](x) = x^{\gamma-1}\int_0^x\int_0^{\xi} a_0(\xi,\eta)\eta^{-\gamma}(\xi-\eta)^{-\gamma} z_1(\xi-\eta)z_2(\eta)d\eta d\xi\,. \tag{262}$$

Here we have used the relation

$$\tau(x) = \frac{1}{x}\int_0^x\int_0^{\xi} \tau(\xi-\eta)\tau(\eta)d\eta\, d\xi$$

following from (cf. [149]) $(x\tau\,(x))' = \int_0^x \tau(x-\xi)\tau(\xi)d\xi \; and \; x\tau(x) \to 0 \;\; as \; x \to 0.$

From (276) it follows that

$$g_1(x) = x^{\gamma-1}\int_0^x g_0(\xi)d\xi = x^{\gamma-1}\int_0^x \xi g(\xi)d\xi \in C[0,T]$$

with $g_1(0) = 0$. Further, by (277) and (259) we have

$$g_2(x) = x^{\gamma-1}\int_0^x B_0(\xi)\tau(\xi)d\xi = x^{\gamma-1}\int_0^x \xi B(\xi)\tau(\xi)d\xi \in C[0,T]$$

with $g_2(0) = 0$ since

$$\int_0^x \xi^\gamma |B(\xi)|\,\tau(\xi)d\xi \le\ Const\int_0^x \tau(\xi)d\xi \sim\ Const\,\frac{1}{\left|ln\frac{x}{K}\right|}\quad as\ x\to 0.$$

Next, using (278) we obtain

$$g_3(x) = x^{\gamma-1}\int_0^x\int_0^\xi b_0(\xi,\eta)\tau(\eta)d\eta\, d\xi \in C[0,T\,]$$

with $g_3(0) = 0$ since

$$x^{\gamma-1}\int_0^x \xi^{-\gamma}\int_0^\xi \tau(\eta)d\eta\, d\xi \le Const\, x^{\gamma-1}\int_0^x \xi^{-\gamma}\frac{d\xi}{\left|ln\frac{\xi}{K}\right|}$$

and $\int_0^x \xi^{-\gamma}\frac{d\xi}{ln\xi} \sim \frac{1}{1-\gamma}\frac{1}{lnx}x^{1-\gamma}\ as\ \ x\to 0.\, Lastly,\ \ A6(279)\ and\ (259) yield$

$$g_4(x) = x^{\gamma-1}\int_0^x\int_0^\xi a_1(\xi,\eta)\tau(\xi-\eta)\tau(\eta)d\eta d\xi \in C[0,T\,]$$

with $g_4(0) = 0$ since

$$x^{\gamma-1}\int_0^x \xi^\alpha\int_0^\xi \tau(\xi-\eta)\tau(\eta)d\eta\, d\xi$$

$$= x^{\gamma-1}\int_0^x \xi^\alpha\big(\xi\tau(\xi)\big)' d\xi = x^{\gamma-1}\left[x^{1+\alpha}\tau(x) - \alpha\int_0^x \xi^\alpha\tau(\xi)d\xi\right]$$

$$\le Const\left[x\tau(x) + \alpha\int_0^x \tau(\xi)d\xi\right] \le Const\left[\frac{1}{ln^2 x} + \frac{\alpha}{\left|ln\frac{x}{K}\right|}\right].$$

Summing up, we have $f \in C[0,T\,]\ \ with\ \ f(0) = 0.$

Further, we estimate for

$$G_1[z](x)\ = x^{\gamma-1}\int_0^x B_0(\xi)\xi^{-\gamma}z(\xi)d\xi = x^{\gamma-1}\int_0^x \xi^{1-\gamma}B(\xi)z(\xi)d\xi$$

that

$$|e^{-\sigma x} G_1[z\,](x)| \leq \int_0^x |B(\xi)|\ e^{-\sigma(x-\xi)} d\xi \cdot \|z\|_\sigma$$

where as before

$$\|z\|_\sigma = \max_{0 \leq x \leq T} |e^{-\sigma x} z(x)|\ , \qquad \sigma \geq 0.$$

Since by (277)

$$\int_0^x |B(\xi)|\, e^{-\sigma(x-\xi)} d\xi \leq Const \int_0^x \xi^{-\gamma}\ e^{-\sigma(x-\xi)} d\xi \leq Const\, \frac{\Gamma(1-\gamma\,)}{\sigma^{1-\gamma}}$$

(cf. [155]) we obtain the estimation $\|G_1[z]\|_\sigma \leq C_1 \frac{1}{\sigma^{1-\gamma}}$ $(c_1 > 0)$ besides $G_1[z] \in C[0,T]$ with $G_1[z](0) = 0$ for any $z \in C[0,T]$. Analogously, for

$$G_2[z\,](x) = x^{\gamma-1} \int_0^x \int_0^\xi b_0(\xi,\eta)\eta^{-\gamma}\tau(\eta) d\eta d\xi$$

in view of (248) we estimate

$$|e^{-\sigma x} G_2[z\,](x)| \leq x^{\gamma-1} \int_0^x \int_0^\xi e^{-\sigma(\xi-\eta)}\, \eta^{-\gamma} |b_0(\xi,\eta)| d\eta d\xi \cdot \|z\|_\sigma$$

$$\leq C_1 x^{\gamma-1} \int_0^x \xi^{-\gamma} \int_0^\xi e^{-\sigma(\xi-\eta)}\, \eta^{-\gamma} d\eta d\xi \cdot \|z\|_\sigma$$

$$\leq \frac{C_1\Gamma(1-\gamma\,)}{\sigma^{1-\gamma}} x^{\gamma-1} \int_0^x \xi^{-\gamma}\, d\xi \cdot \|z\|_\sigma = c_2\ \frac{1}{\sigma^{1-\gamma}}\ \|z\|_\sigma\ \ (c_2 > 0)$$

implying $\|G_2[z\,]\|_\sigma \leq c_2\ \frac{1}{\sigma^{1-\gamma}}\ \|z\|_\sigma$ besides $G_2[z\,] \in C[0,T]$ with $G_2[z\,](0) = 0$ for any $z \in C[0,T\,]$. Furthermore, for

$$G_3[z\,](x) = x^{\gamma-1} \int_0^x \int_0^\xi a_0\,(\xi,\eta)[\eta^{-\gamma}\tau(\xi-\eta)z(\eta) + (\xi-\eta)^{-\gamma}\tau(\eta)z(\xi-\eta)] d\eta d\xi$$

we have

$$|e^{-\sigma x} G_2[z\,](x)| \leq x^{\gamma-1} \int_0^x e^{-\sigma(\xi-\eta)} \int_0^\xi |a_0(\xi,\eta)|\ \ [\eta^{-\gamma}\tau(\xi-\eta)e^{-\sigma\eta}\,|z(\eta)|$$

$$+(\xi-\eta)^{-\gamma} + \tau(\eta)e^{-\sigma(\xi-\eta)}|z(\xi-\eta)|\big]\ d\eta d\xi$$

and using (279)

$$|e^{-\sigma x} G_2[z](x)| \le 2C_2 x^{\gamma-1} \|z\|_\sigma \int_0^x \eta^{-\gamma} \int_\eta^\xi e^{-\sigma(\xi-\eta)} \tau(\xi-\eta)\, d\xi d\eta$$

$$\le 2C_2 x^{\gamma-1} \|z\|_\sigma \int_0^x \eta^{-\gamma}\, d\eta \int_0^\infty e^{-\sigma\rho} \tau(\rho)\, d\rho = \frac{2C_2}{1-\gamma} \frac{1}{\ln(K\sigma)} \|z\|_\sigma$$

for $K\sigma > 1$ where we have used the Laplace integral [147] $\int_0^\infty e^{-pt}\, v'(t) dt = \int_0^\infty v(t,-1)\, dt = \frac{1}{lnp}, Re\, p > 1$. Hence, it holds $\|G_3[z]\|_\sigma \le \frac{c_3}{\ln(K\sigma)} \|z\|_\sigma$ $(c_3 > 0)$ $for\ \sigma > 1/K$ besides $G_3[z] \in C[0,T]\ with\ G_3[z](0) = 0$ for any $z \in C[0,T]$. Therefore, we have $G[z] \in C[0,T]$ with $G[z](0) = 0$ satisfying the estimate

$$\|G[z]\|_\sigma \le \frac{c}{\ln(K\sigma)} \|z\|_\sigma \qquad (c > 0) \tag{263}$$

for any $z \in C[0,T]\ and\ \sigma > 1/K$.

Finally, we estimate (262), using (279) again, by

$$|e^{-\sigma x} L[z_1, z_2](x)| \le C_2 \|z_1\|_\sigma \|z_2\|_\sigma x^{\gamma-1} \int_0^x e^{-\sigma(\xi-\eta)} \int_0^\xi \eta^{-\gamma} (\xi-\eta)^{-\gamma}\, d\eta d\xi$$

$$\le C_2 \|z_1\|_\sigma \|z_2\|_\sigma B(1-\gamma, 1-\gamma)\, x^{\gamma-1} \int_0^x \xi^{1-2\gamma}\, d\xi$$

$$= \frac{C_2 B(1-\gamma, 1-\gamma)}{2(1-\gamma)}\, x^{\gamma-1} \|z_1\|_\sigma \|z_2\|_\sigma$$

implying the estimation

$$\|L[z_1, z_2]\|_\sigma \le d_1 \|z_1\|_\sigma \|z_2\|_\sigma \qquad (d_1 > 0) \tag{264}$$

Besides $L[z_1, z_2] \in C[0,T]\ with\ L[z,z](0) = 0\ for\ any\ z,\ z_1,\ z_2 \in C[0,T]$. Moreover, we have $(with\ \|z\| = \|z\|_0)$

$$|e^{-\sigma x} L[z_1, z_2](x)| \le C_2 \|z_1\|_\sigma \|z_2\|_\sigma x^{\gamma-1} \int_0^x \int_0^\xi e^{-\sigma\eta} \eta^{-\gamma} (\xi-\eta)^{-\gamma}\, d\eta d\xi$$

$$= C_2 \|z_1\|_\sigma \|z_2\|_\sigma \frac{x^{\gamma-1}}{1-\gamma} \int_0^0 e^{-\sigma\eta} \eta^{-\gamma} (\xi-\eta)^{1-\gamma}\, d\eta$$

$$\le C_2 \frac{D}{1-\gamma} \frac{1}{\sigma^{1-\gamma}} \|z_1\|_\sigma \|z_2\|$$

with some constant $D > 0$ by [155]. Hence the estimation

$$\|L[z_1, z_2]\|_\sigma \le \frac{d_2}{\sigma^{1-\gamma}} \|z_1\|_\sigma \|z_2\| \qquad (d_2 > 0) \tag{265}$$

is valid (and the analogous one with interchanging z_1 and z_2).

Applying again the existence theorem [148] to Eq. (261), observing the estimations (263), (264) and the relations $f, G[z], L[z_1, z_2] \in C[0,T]$ $with$ $f(0) = G[z](0) = L[z,z](0,0) = 0$ $for\ any$ $z, z_1, z_2 \in C[0,T]$.

Looking for the derivative z' of z in (260)we take w' from (256), (258) and use the relation $w' = x^{-\gamma} z' - \gamma x^{-\gamma-1} z$ to obtain

$$xz'(x) = (\gamma-1)z(x) + x^{\gamma+1}g(x) - x^{\gamma+1}B(x)\tau(x) - xB(x)z(x)$$

$$+x^{\gamma}\int_0^x b_0(x,\xi)\tau(\xi)d\xi + x^{\gamma}\int_0^x b_0(x,\xi)\xi^{-\gamma}z(\xi)d\xi + x^{\gamma}\int_0^x a_1(x,\xi)\tau(x-\xi)\tau(\xi)d\xi$$

$$+x^{\gamma}\int_0^x a_0(x,\xi)[\tau(x-\xi)\xi^{-\gamma}z(\xi) + \tau(\xi)(x-\xi)^{-\gamma}z(x-\xi)]d\xi$$

$$+x^{\gamma}\int_0^x a_0(x,\xi)\xi^{-\gamma}(x-\xi)^{-\gamma}z(\xi)z(x-\xi)d\xi.$$

From this we have $xz'(x) \in C[0,T] with\, xz'(x) \to 0\ as\, x \to 0$ if additionally to (276)–(279) we assume

$A7.$ $x^{\gamma+1}g(x) \in C[0,T]$ with $x^{\gamma+1}g(x) \to 0$ as $x \to 0$,

$A8.$ $B(x) \in C(0,T]$ $with$ $\frac{x^{\gamma}}{ln^2 x}$ $B(x) \to 0$ as $x \to 0$,

$A9.$ x^{γ} $b_0(x,\xi) \in C(\Delta_T) for$ where $\Delta_T = \{(x,\xi): 0 \le \xi \le x \le T\}$,

$A10.$ $a_0(x,\xi) \in C(\Delta_T)$ and x^{α} $a_1(x,\xi) \in C(\Delta_T)$.

We point out that in general z' does not belong to $L^1(0,T)$.

Uniqueness theorems for linear equations

In the first part two uniqueness theorems for continuous solutions of two linear singular integral equations are given. Reducing these equations to equations of Wiener–Hopf type and applying the theory of M.G. Krein [151] for such equations (cf. [152]), we derive more general uniqueness theorems in different function spaces for these equations. We start with the equation

$$w(x) = \frac{x^{1-\alpha-\beta}}{B(\alpha,\beta+1)}\int_0^x (x-\xi)^{\alpha-1}\xi^{\beta-1}\, w(\xi)d\xi, \qquad 0 < x < 1, \qquad (266)$$

where $\alpha > 0, \beta > 0$. We now prove

Theorem (5.3.5)[139]. Eq. (266)with $\alpha > 0, \beta > 0$ has in the spaces $C[0,1]$, $M(0,1)$ and $\hat{L}_p(0,1), p \ge 1$ only the solutions $w(x) = Kx, K \in \mathbb{R}$.

Here $M(0,1)$ denotes the space of bounded measurable functions on $(0,1)$ and $\hat{L}_p(0,1)$ the space of measurable functions on $(0,1)$ with finite integral $\int_0^1 |w(x)|^p\, dx/x$.

Proof. We put $\lambda = 1/B(\alpha, \beta+1)$ and substitute $t = ln\,\frac{1}{x}, s = ln\,\frac{1}{\xi}$ in (266). Then the function $\varphi(t) = w(x)$ satisfies the Wiener–Hopf type equation

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)ds = 0, \qquad t \in \mathbb{R}_+, \tag{267}$$

with the kernel

$$k(u) = \begin{cases} 0 & for\ u > 0, \\ \lambda e^{\beta u}(1-e^u)^{\alpha-1} & for\ u < 0. \end{cases} \tag{268}$$

The kernel k is summable and has the Fourier transform

$$K(y) = \int_{-\infty}^{\infty} e^{iyt}\, k(t)dt = \lambda B(\alpha, \beta+iy), \qquad -\infty < y < \infty.$$

The related function

$$D(y) = 1 - K(y) = 1 - \frac{B(\alpha,\beta+iy)}{B(\alpha,\beta+1)} \tag{269}$$

obeys the relations $D(-y) = \overline{D(y)}, D(0) = -\alpha/\beta < 0$ and $D(\infty) = \lim_{y\to\infty} D(y) = 1$.

Further, $D(y)$ is not real-valued for $0 < y < \infty$, in particular $D(y) \neq 0\ on\ \mathbb{R}$ so that the theory of M.G. Krein [151] applies. Namely, real-valued $D(y)$ means that $A(y) = A(-y)$ holds for the function

$$A(y) := \frac{\Gamma(\beta+iy)}{\Gamma(\beta)} \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta+iy)}. \tag{270}$$

By [146] this is equivalent to the relation

$$\frac{\alpha+\beta+iy}{\beta+iy} \prod_{n=1}^{\infty} \frac{1+\frac{iy}{n+\alpha+\beta}}{1+\frac{iy}{n+\beta}} = \frac{\alpha+\beta-iy}{\beta-iy} \prod_{n=1}^{\infty} \frac{1-\frac{iy}{n+\alpha+\beta}}{1-\frac{iy}{n+\beta}}$$

or

$$\frac{(\alpha+\beta)\beta+y^2-i\alpha y}{(\alpha+\beta)\beta+y^2+i\alpha y} = \prod_{n=1}^{\infty} \frac{\left[1+\frac{y^2+i\alpha y}{(n+\beta)(n+\alpha+\beta)}\right]}{\left[1+\frac{y^2-i\alpha y}{(n+\beta)(n+\alpha+\beta)}\right]}.$$

Taking the argument of both sides of this relation we obtain the equation for y

$$-arctan\frac{\alpha y}{(\alpha+\beta)\beta+y^2} = \sum_{n=1}^{\infty} \arctan\frac{\alpha y}{(n+\beta)(n+\alpha+\beta)+y^2}$$

in which for arctan the principal value has to be chosen since it must be fulfilled for $y = 0$. But then the equation cannot be true for $0 < y < \infty$ and we indeed have $A(y) \neq A(-y)$, $0 < y < \infty$for the function (270). Hence $D(y) \neq 0\ on\ \mathbb{R}$, the contour $z = D(y)$, $-\infty < y < \infty$, meets the real axis only for $y = 0$ at the point

$-\frac{\alpha}{\beta}$ *and for* $y \to \pm\infty\}$ at the point 1, and $arg\, D(0) = \pm\pi,\ arg\, D(\infty) = \lim_{y\to\infty} arg\, D(y) = 0$.

We calculate the index of ν of Eq. (267):

$\nu = -indD = -\frac{1}{2\pi}[argD(y)]_{-\infty}^{\infty} = -\frac{1}{\pi}[argD(y)]_0^{\infty} = \frac{1}{\pi}[argD(0) - argD(\infty)] = \pm 1$.

If $\nu = -1$ no continuous solution of Eq. (267) would exist, but we already know the evident solution $= e^{-t}$. Hence $\nu = 1$ and by [151] we have exactly this one linearly independent solution of Eq. (267) in the spaces C_+ of continuous functions f on $\mathbb{R}_+$ with $\lim_{t\to\infty} f(t) = f(\infty), M_+$ of bounded measurable functions on $\mathbb{R}_+$, and $L_p(0,\infty), p \geq 1$. This proves Theorem (5.3.4).

Since we have

$$\int_{-\infty}^{\infty} e^{-ht}\, |k(t)|\, dt = \int_{-\infty}^{0} e^{-ht}\, k(t) dt < \infty \quad for\ \ 0 < h < \beta$$

for the kernel (268), a substitution of $\varphi = e^{ht}\hat{\varphi}$ in Eq. (267) is possible. Observing that for the corresponding function $D_h(y) = D(y + ih)$ *we have* $D_h(0) < D(0) < 0$, we obtain

Corollary (5.3.6)[139]. The uniqueness assertion in Theorem (5.3.5) also holds true for the solution spaces $C_{-h}[0,1] = \{w\colon x^h w \in C[0,1]\}, M_{-h}(0,1) = \{w\colon x^h w \in M(0,1)\}$ and $\hat{L}_p(\rho)(0,1) = \{w\colon \int_0^1 |\rho(x) w(x)|^p\, dx/x < \infty\ with\ \rho(x) = x^h,\ 0 < h < \beta$.

Further, we consider the equation

$$z(x) = \frac{x^{-\alpha-\beta}}{B(\alpha, \beta+1)} \int_0^x (x-\xi)^{\alpha-1} \xi^{\beta}\, z(\xi) d\xi, \qquad 0 < x < 1, \tag{271}$$

where $\alpha > 0,\ \beta > -1$. For this equation we prove

Theorem (5.3.7)[139]. Eq. (271) with $\alpha > 0,\ \beta > -1$ has in the spaces $C_{-h}[0,1]$, $M_{-h}(0,1)$ and $\hat{L}_p(\rho)(0,1),\ p \geq 1$ with $\rho(x) = x^h, 0 < h < \beta + 1$, only the solutions $z(x) = K, K \in \mathbb{R}$.

Proof. We again put $\lambda = 1/B(\alpha, \beta+1)$ and substitute $t = ln\,\frac{1}{x}\,, s = ln\,\frac{1}{\xi}$ in Eq. (271). Then for the function $\psi(t) = z(x)$ the Wiener–Hopf type equation

$$\psi(t) - \int_0^{\infty} k_0(t-s)\psi(s) ds = 0, \qquad t \in \mathbb{R}_+, \tag{272}$$

with the kernel

$$k_0(u) = \begin{cases} 0 & for\ u > 0, \\ \lambda e^{(\beta+1)u}(1-e^u)^{\alpha-1} & for\ u < 0. \end{cases} \tag{273}$$

follows. We have

$$\int_{-\infty}^{\infty} e^{-ht}\,|k_0(t)|dt = \int_{-\infty}^{0} e^{-ht}\,|k_0(t)|dt < \infty \quad for\ 0 < h < \beta + 1$$

and

$$D_{0,h}(y) \equiv D_0(y + ih) = 1 - k_0(y\ + ih) = 1 - \frac{B(\alpha,\gamma + iy)}{B(\alpha,\beta + 1)},$$

where $\gamma = \beta + 1 - h > 0\ and\ k_0$ denotes the Fourier transform of k_0. As above for the function $D(y)$ in (269) we have for $D_{0,h}$ that $D_{0,h}(-y) = \overline{D_{0,h}(y)}$,

$$D_{0,h}(0) = 1 - \frac{B(\alpha,\gamma\ )}{B(\alpha,\beta + 1)} < 0$$

since $\gamma = \beta + 1 - h < \beta + 1\ \ for\ \ h > 0$ and $\lim_{y\to+\infty} D_{0,h}(y) = 1$. Further, as above, $D_{0,h}(y)$ is not real-valued for $0 < y < \infty$. Applying [151], we get the assertion of Theorem (5.3.7).

In case $\alpha > 0, \beta > 0$ Eq. (271) for z can be reduced to Eq. (266) for the function $w(x) = \int_0^x z(\xi)d\xi$ as follows writing Eqs. (266) and (271)in the form

$$w(x) = \frac{1}{B(\alpha,\beta + 1)}\int_0^1 (1 - t)^{\alpha - 1}\, t^{\beta - 1} w(xt)dt$$

and

$$z(x) = \frac{1}{B(\alpha,\beta + 1)}\int_0^1 (1 - t)^{\alpha - 1}\, t^{\beta} z(xt)dt,$$

respectively, and integrate the last equation. From Theorem (5.3.5) for the spaces C and $\hat{L}_p$ and from Corollary (5.3.6) for the spaces $\hat{L}_p(\rho),\ \rho = x^h,\ 0 < h < \beta$, we therefore obtain the following statement.

Theorem (5.3.8)[139]. *Eq.* (271)with $\alpha > 0,\ \beta > 0$ has in the space $L_1(0,1)$ and in the classes of functions $\{z\!: \int_0^x z(\xi)d\xi\ \in \hat{L}_p(\rho),\ \rho = x^h\}\ \ with\ \ p\ \geq 1,\ 0\ \leq h < \beta,$

in particular in the space $L_1(\rho)(0,1)(0,1),\ \rho = ln\,\frac{1}{x}$, only the solutions $z(x) = K, K \in \mathbb{R}$.

The first linear auxiliary integral equation

We consider the integral equation

$$xz(x) = \mu\int_0^x \frac{z(\xi)}{\sqrt{x - \xi}}\,d\xi + \lambda\int_0^x z(\xi)\,d\xi + \mathrm{g}(x), \qquad 0 \leq\ x \leq 1, \qquad (274)$$

for $\mu,\lambda \in \mathbb{R}\ with\ \mu = 0$ looking for solutions $z \in C[0,1]$. For $\mu > 0$ the homogeneous equation (274) has the solutions

$$z_0(x) = z_0(x, K) = Kx^{\lambda-1} \, exp\left[-\frac{\nu^2}{2x}\right] D_{1-2\lambda}\left(\frac{\sqrt{2}\nu}{\sqrt{x}}\right) \tag{275}$$

where $K \in \mathbb{R}, \nu = \sqrt{\pi}\, \mu$ *and* D_α is the parabolic cylinder function with index α (cf. as follows applying the Laplace transform to (274). Since $D_\alpha(z) \sim z^\alpha \, exp\left(\frac{z^2}{4}\right)$ *as* $z \to +\infty$, it holds

$$z_0(x) \sim K_0 x^{2\lambda-\frac{3}{2}} \, exp\left[-\frac{\nu^2}{x}\right] \; as \;\; x \to 0 \tag{276}$$

with a constant $K_0 = (\sqrt{2}\nu)^{1-2\lambda} K$. Hence we have $z_0 \in C[0,1]$ with $z_0(0) = 0$. For $\mu < 0$ the homogeneous equation (274) has no non-trivial solution from $C[0,1]$ (and even from $L^1(0,1)$). We still mention the special solutions for $\mu > 0$:

$$x^{-3/2} \, exp\left[-\frac{\nu^2}{x}\right] \quad for \;\; \lambda = 0, \qquad x^{-1/2} \, exp\left[-\frac{\nu^2}{x}\right] \quad for \;\; \lambda = \frac{1}{2}, \qquad erfc\left(\frac{\nu}{\sqrt{x}}\right)$$

In constructing a particular solution of the nonhomogeneous equation (274), at first we deal with the *special case* $\lambda = 0$, i.e. with the equation

$$xz(x) = \mu \int_0^x \frac{z(\xi)}{\sqrt{x-\xi}} \, d\xi + \mathrm{g}(x), \qquad 0 \le \; x \le 1. \tag{277}$$

Extending an idea of Nakhushev [153] for the homogeneous equation (277) to the nonhomogeneous equation, we apply the differential operator

$$(Pu)(x) = \; xu^{'}(x) + \mu \frac{d}{dx}\left[\int_0^x \frac{u(\xi)}{\sqrt{x-\xi}} d\xi\right] + \frac{1}{2} u(x) \tag{278}$$

to $(277)$and obtain the linear differential equation of first order

$$x^2 z^{'}(x) + \left[\frac{3}{2}x - \nu^2\right] z(x) = h(x) \tag{279}$$

where

$$h(x) = \; x\mathrm{g}^{'}(x) + \; \mu \frac{d}{dx}\left[\int_0^x \frac{\mathrm{g}(\xi)}{\sqrt{x-\xi}} d\xi\right] + \frac{1}{2}\mathrm{g}(x) \, .$$

For $\mu > 0$ the equation $Pu = 0$ has no non-trivial generalized solutionsfrom $C[0,1]$ as follows applying again the Laplace transformation to it. Therefore, for $\mu > 0$ Eqs. (277) and (279) are equivalent under suitable assumptions on g.

Solving Eq. (279) with the method of variation of constants and performing two integrations by parts in the occurring integrals, under the assumption that $\mathrm{g} \in C[0,1]$ with

$$x^{1/2} \, e^{\frac{\nu^2}{x}} \, \mathrm{g}(x) \to 0, \quad x^{-\frac{1}{2}} \, e^{\frac{\nu^2}{x}} \int_0^x \frac{\mathrm{g}(\xi) d\xi}{\sqrt{x-\xi}} \to 0 \;\; as \;\; x \to 0 \tag{280}$$

we obtain the particular solution of Eq. (277)

$$z(x) = \frac{\mathrm{g}(\xi)}{x} + \int_0^x M_0(x,\xi)\mathrm{g}(\xi)\, d\xi, \tag{281}$$

where

$$M_0(x,\xi) = \frac{\nu}{\sqrt{\pi}}\frac{x^{-2}}{\sqrt{x-\xi}} + \nu^2\, x^{-\frac{3}{2}}\xi^{-\frac{3}{2}}\, exp\left[\nu^2\left(\frac{1}{\xi} - \frac{1}{x}\right)\right]$$
$$+ \frac{\nu}{2\sqrt{\pi}} x^{-3/2} e^{-\frac{\nu^2}{x}}\, \mathrm{J}_1(x,\xi) + \frac{\nu^3}{2\sqrt{\pi}} x^{-3/2} e^{-\frac{\nu^2}{x}}\, \mathrm{J}_2(x,\xi) \tag{282}$$

with the integrals

$$\mathrm{J}_1(x,\xi) = \int_\xi^x \eta^{-3/2}\, e^{\frac{\nu^2}{\eta}} \frac{d\eta}{\sqrt{\eta-\xi}}, \qquad \mathrm{J}_2(x,\xi) = \int_\xi^x \eta^{-5/2}\, e^{\frac{\nu^2}{\eta}} \frac{d\eta}{\sqrt{\eta-\xi}}.$$

The particular solution (281) of Eq. (277) holds true for $\mu > 0$ because of the equivalence of Eqs. (277) and (279) and for ($\mu = 0$ and) $\mu < 0$ by analytic continuation with respect to ν. It could be shown also by inserting (281) directly in (277). Further, we make the *assumption* on g that g $\in C[0,1]$ satisfies the inequality

$$|\mathrm{g}(x)| \le\; C x^{\delta} e^{-\frac{\nu^2}{x}},\; \delta > \frac{1}{2}, \tag{283}$$

with a positive constant C which is sufficient for the limiting relations (280).

It remains to *estimate* the kernel (282) and the solution (281). Obviously, we have $0 \;\le \mathrm{J}_1(x,\xi) \;\le\; \mathrm{J}_2(x,\xi)$. Further, putting $\rho = \frac{\nu^2}{x}\; and\;\; t = \frac{\nu^2}{\eta}$ we obtain

$$\mathrm{J}_2(x,\xi) = \xi^{-2}\rho^{-3/2} \int_{\frac{\xi}{x}\rho}^{\rho} t(\rho-t)^{-1/2}\, e^t dt \le \xi^{-2}\rho^{-\frac{3}{2}} I_0(\rho)$$

with

$$I_0(\rho) = \int_0^\rho t(\rho-t)^{-1/2}\, e^t dt = e^\rho\left[\int_0^\rho s^{-1/2}\, e^{-s} ds - \int_0^\rho s^{1/2}\, e^{-s} ds\right]$$
$$\le e^\rho \rho \int_0^\rho s^{-1/2}\, e^{-s} ds = \sqrt{\pi}\rho e^\rho.$$

This gives $\mathrm{J}_2(x,\xi) \le \frac{\sqrt{\pi}}{|\nu|}\xi^{-3/2} e^{\frac{\nu^2}{\xi}}$ implying the estimation in (282)

$$|M_0(x,\xi)| \le D_0 \frac{x^{-2}}{\sqrt{x-\xi}} + D_1 (x\,\xi)^{-3/2} exp\left[\nu^2\left(\frac{1}{\xi} - \frac{1}{x}\right)\right] \tag{284}$$

where $D_0 = \frac{|\nu|}{\sqrt{\pi}}, D_1 = \frac{1}{2} + 2\nu^2$. Using (283) and (284) we estimate the particular solution (281) of Eq. (277):

$$|z(x)| \le Cx^{\delta-1} e^{-\frac{\nu^2}{x}} + CD_0 B\left(\delta+1,\frac{1}{2}\right) x^{\delta-\frac{3}{2}} e^{-\frac{\nu^2}{x}} + CD_1 \frac{1}{\delta-\frac{1}{2}} x^{\delta-2} e^{-\frac{\nu^2}{x}}$$

leading to the estimation

$$|z(x)| \le Ex^{\gamma} e^{-\frac{\nu^2}{x}}, \ \ \gamma = \delta - 2 > -\frac{3}{2}, \tag{285}$$

with a positive constant E. From (285) the limiting relation $\lim_{x\to 0}\left[x^{3/2} e^{\frac{\nu^2}{x}} z(x)\right] = 0$ follows in comparison to the solution $x^{3/2} exp\left[-\frac{\nu^2}{x}\right]$ of the homogeneous equation (277) for $\mu > 0$. We summarize the results for Eq. (277).

Theorem $(\mathbf{5.3.9})[\mathbf{139}]$. Let g $\in C[0,1]$ fulfill assumption (283). Then Eq. (277) for $\mu \neq 0$ has in $C[0,1]$ the solution (281) satisfying the estimation (285). The homogeneous equation (277) has in $C[0,1]$ for $\mu > 0$ the solutions $z_0(x) = Kx^{-3/2} exp\left[-\frac{\nu^2}{x}\right], K \in \mathbb{R}$, and for $\mu < 0$ only the trivial solution.

For general $\lambda \in \mathbb{R}$ we make the ansatz in Eq. (277)

$$z(x) = \sum_{n=0}^{\infty} \lambda^n w_n(x) \tag{286}$$

where (see ((281))

$$w_0(x) = \frac{g(x)}{x} + \int_0^x M_0(x,\xi) g(\xi) d\xi \tag{287}$$

and

$$x w_n(x) = \mu \int_0^x \frac{w_n(\xi)}{\sqrt{x-\xi}} d\xi + h_n(x), \quad n = 1,2,\dots, \tag{288}$$

with

$$h_n(x) = \int_0^x w_{n-1}(\xi) d\xi, \qquad n = 1,2,\dots.$$

From (288), taking the particular solution (281) of this equation, we have

$$w_n(x) = \frac{h_n(x)}{x} + \int_0^x M_0(x,\xi) h_n(\xi) d\xi = \int_0^x w_{n-1}(\xi) m_0(x,\xi) d\xi$$

where

$$m_0(x,\xi) = \frac{1}{x} + \int_0^x M_0(x,\eta)d\eta \tag{289}$$

or

$$w_n(x) = \int_0^x m_0^{[n]}(x,\xi)w_0(\xi)d\xi\,, \quad n = 1,2,\ldots, \tag{290}$$

with the *n*th iterated kernel of m_0

$$m_0^{[n]}(x,\xi) = \int_0^x m_0(x,r)m_0^{[n-1]}(r,\xi)dr. \tag{291}$$

The relations (286), (287) and (290) yield the integral representation of z

$$z(x) = \frac{\mathrm{g}(x)}{x} + \int_0^x M(x,\xi)\mathrm{g}(\xi)d\xi \tag{292}$$

with the kernel

$$M(x,\xi) = M_0(x,\xi) + \frac{1}{\xi}\sum_{n=1}^{\infty} \lambda^n\, m_0^{[n]}(x,\xi) + \sum_{n=1}^{\infty} \lambda^n \int_\xi^x m_0^{[n]}(x,\eta)\, M_0(\eta,\xi)d\eta. \tag{293}$$

We have to prove (uniform) convergence of the two series in (293). Let us start with the estimation of the function m_0. Observing (284), we have

$$\left|\int_\xi^x M_0(x,\eta)d\eta\right| \le 2D_0x^{-2}\sqrt{x-\xi} + D_1x^{-3/2}e^{-\frac{\nu^2}{x}}\, I_1(x,\xi)$$

where, putting as above $\rho = \frac{\nu^2}{\xi}$,

$$I_1(x,\xi) = \int_\xi^x \eta^{-\frac{3}{2}}\, e^{\frac{\nu^2}{\eta}} d\eta = \xi^{-\frac{1}{2}}\, \frac{1}{\sqrt{\rho}} \int_{\frac{\xi}{x}\rho}^{\rho} \rho^{-\frac{1}{2}}\, e^{\rho}\, d\rho$$

$$\le \frac{1}{|\nu|}\int_0^\rho t^{-\frac{1}{2}}\, e^t dt = \frac{2}{|\nu|}\int_0^{\sqrt{\rho}} e^{s^2}\, ds \le \frac{2}{|\nu|}\frac{1}{\sqrt{\rho}}e^{\rho} = \frac{2}{\nu^2}\xi^{\frac{1}{2}}\, e^{\frac{\nu^2}{\xi}}.$$

Hence, for the function m_0 in (289) we obtain the estimation

$$|m_0(x,\xi)| \le \frac{1}{x} + 2D_0\frac{\sqrt{x-\xi}}{x^2} + \frac{2D_1}{\nu^2}\frac{\xi^{1/2}}{x^{3/2}}exp\left[\nu^2\left(\frac{1}{\xi} - \frac{1}{x}\right)\right] \tag{294}$$

where as above $D_0 = \frac{|\nu|}{\sqrt{\pi}}, D_1 = \frac{1}{2} + 2\nu^2$. Further, from the elementary inequality $u \leq \frac{1}{2} e^{u^2}$ the estimate $\sqrt{x-\xi} \leq \frac{1}{2|\nu|} x^{1/2} \xi^{1/2} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right]$ follows, and since the function g$(x) = x^{1/2} e^{\frac{\nu^2}{x}}$ has the maximum at $x = 2\nu^2$ the estimate

$$\frac{1}{x} \leq b \frac{\xi^{1/2}}{x^{3/2}} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right], \quad b = max\left(1; \frac{1}{\sqrt{2}|\nu|} e^{\nu^2 - \frac{1}{2}}\right)$$

is valid. Therefore, (294)implies the simpler estimation for m_0:

$$|m_0(x,\xi)| \leq N \frac{\xi^{1/2}}{x^{3/2}} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right], \quad N = b + \frac{1}{\sqrt{\pi}} + 4 + \frac{1}{\sqrt{2}}. \tag{295}$$

From (295) by induction we further prove the estimations for $m_0^{[n]}$, $n = 1, 2, \ldots$:

$$\left|m_0^{[n]}(x,\xi)\right| \leq \frac{N^n}{(n-1)!} x^{-3/2} \xi^{1/2} \left(ln \frac{x}{\xi}\right)^{n-1} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right]. \tag{296}$$

Namely, by (295) and (296) it holds

$$\left|m_0^{[n+1]}(x,\xi)\right| = \left|\int_\xi^x m_0(x,r) m_0^{[n]}(r,\xi) dr\right|$$

$$\leq \frac{N^{n+1}\xi^{1/2}}{(n-1)!\, x^{3/2}} e^{\nu^2\left(\frac{1}{\xi} - \frac{1}{x}\right)} \int_\xi^x \frac{1}{r}\left(ln \frac{r}{\xi}\right)^{n-1} dr = \frac{N^{n+1}\xi^{1/2}}{(n)!\, x^{3/2}} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right]\left(ln \frac{x}{\xi}\right)^n$$

Further, from (296) and (284) we have

$$\left|\sum_{n=1}^\infty \lambda^n m_0^{[n]}(x,\xi)\right| \leq \sum_{n=1}^\infty |\lambda|^n \frac{N^n}{(n-1)!}\left(ln \frac{x}{\xi}\right)^{n-1} \frac{\xi^{1/2}}{x^{3/2}} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right]$$

$$= \Lambda x^{-1} \left(\frac{x}{\xi}\right)^{\Lambda - \frac{1}{2}} exp\left[\nu^2 \left(\frac{1}{\xi} - \frac{1}{x}\right)\right], \quad \Lambda := |\lambda| N, \tag{297}$$

and

$$\left|\sum_{n=1}^\infty \lambda^n \int_\xi^x m_0^{[n]}(x,\eta) M_0(\eta,\xi) d\eta\right|$$

$$\leq \Lambda x^{\Lambda - \frac{3}{2}} e^{-\frac{\nu^2}{x}} \left[D_0 \int_\xi^x \eta^{\Lambda - \frac{3}{2}} \frac{e^{\frac{\nu^2}{\eta}}}{\sqrt{x-\xi}} d\eta + D_1 \xi^{-\frac{3}{2}} \int_\xi^x \eta^{-\Lambda - 1} d\eta e^{\frac{\nu^2}{\xi}}\right].$$

But $\int_\xi^x \eta^{-\Lambda-1} d\eta = \frac{1}{\Lambda}\left[\xi^{-\Lambda} - x^{-\Lambda}\right]$ and

$$\int_{\xi}^{x} \eta^{\wedge-\frac{3}{2}} \frac{e^{\frac{\nu^2}{\eta}}}{\sqrt{x-\xi}} d\eta \le 2e^{-\wedge-\frac{3}{2}} \sqrt{x-\xi}\, e^{\frac{\nu^2}{\xi}}$$

so that

$$\left|\sum_{n=1}^{\infty} \lambda^n \int_{\xi}^{x} m_0^{[n]}(x,\eta) M_0(\eta,\xi) d\eta\right| \le \wedge_0\, (x\xi)^{-\frac{3}{2}} \left(\frac{x}{\xi}\right)^{\wedge} exp\left[\nu^2\left(\frac{1}{\xi}-\frac{1}{x}\right)\right], \quad (298)$$

where $\wedge_0 = D_1 + 2D_0\Lambda$. The estimations (284), (297) and (298) yield the desired estimation for the kernel M in (293)

$$|M(x,\xi)| \le D_0 \frac{x^{-2}}{\sqrt{x-\xi}} + D_2 \left(\frac{x}{\xi}\right)^{\wedge} (x\xi)^{-3/2}\, exp\left[\nu^2\left(\frac{1}{\xi}-\frac{1}{x}\right)\right] \quad (299)$$

where we have put $D_2 = D_1 + \Lambda + \wedge_0$.

Finally, we estimate the solution z in (292) under the assumption (283) with suitable $\delta > 1/2$. Observing the estimation (284) for the kernel M_0 it remains to consider the integral

$$J(x) = \int_0^x (x\xi)^{-\frac{3}{2}} \left(\frac{x}{\xi}\right)^{\wedge} \xi^{\delta} d\xi = \frac{1}{\delta - \wedge - \frac{1}{2}} x^{\delta-2}$$

if $> \Lambda + \frac{1}{2}$. Hence if g satisfies the assumption (283) with $\delta > \Lambda + \frac{1}{2}$ then the particular solution z obeys the estimation (285) again with $= \delta - 2 > \Lambda - \frac{3}{2}$.

So we proved the following theorem:

Theorem (5.3.10)[139]. Let $g \in C[0,1]$ fulfill assumption (283) with $\delta > \Lambda + \frac{1}{2}, \Lambda = |\lambda| N, N$ defined in (295). Then Eq. (274) for $\mu \ne 0$ has in $C[0,1]$ the solution (292) satisfying the estimation (285) with $\gamma > \Lambda - \frac{3}{2}$. The homogeneous equation (274) has in $C[0,1]$ for $\mu > 0$ the solutions (275) and for $\mu < 0$ only the trivial solution.

Corollary (5.3.11)[139]. The assumption(283) on g is fulfilled if

$$|g(x)| \le C_1 \exp[-\epsilon x^{-\omega} - v^2 x^{-1}] \quad (300)$$

with positive constants C_1, ϵ, ω.

We state the following corollary.

Corollary (5.3.12)[139]. The equation

$$xz(x) = \mu \int_0^x \frac{z(x)}{\sqrt{x-\xi}} d\xi + \int_0^x \varkappa(\xi) z(\xi) d\xi + g(x), \quad 0 \le x \le 1, \quad (301)$$

with $\mu \ne 0$ and a bounded measurable function $\varkappa$ has under the assumptions of Theorem (5.3.10) on g, where $|\lambda|$ is replaced by $sup_{0<x<1} |\varkappa(x)|$, the solution (292) with the kernel M which satisfies the estimation (299). The solution again fulfills the estimation $(285)$with $\gamma > \Lambda - \frac{3}{2}$. The homogeneous equation (301) has in $C[0,1]$ for $\mu > 0$ a family of solutions $z_0(x) = z_0(x, K)$ with the parameter $K \in \mathbb{R}$ satisfying $z_0(0) = 0$.

Proof. The proof of this corollary follows analogously as the proof of Theorem $(5.3.10)$by means of the iteration procedure for a solution $z(x) = \lim_{n\to\infty} W_n(x)$ to Eq. (301)

$$xW_n(x) = \mu \int_0^x \frac{W_n}{\sqrt{x-\xi}} d\xi + \int_0^x x(\xi) W_{n-1}(\xi) d\xi + \mathrm{g}(x) \quad (n = 1,2,\dots)$$

where $W_0 = w_0$ given by (287) for the nonhomogeneous equation and W_0 given by z_0 in (275) with $\lambda = 0, i.e.\ W_0(x) = K_0 x^{-\frac{3}{2}} exp\left[-\frac{\pi\mu^2}{x}\right], K_0 = \sqrt{2\pi}\,\mu K, K \in \mathbb{R}$,for the homogeneous equation.

Chapter 6
Sec(6.1) : Weyl Calculus and Composition Formulas

No symbolic calculus of operators is more popular or better known than the Weyl calculus. It is the one that associates to a function $\mathfrak{S} = \mathfrak{S}(\mathrm{x}, \xi)$ of $n + n$ variables, lying in $S(\mathbb{R}^n \times \mathbb{R}^n)$, the operator $Op(\mathfrak{S})$, called the operator with symbol $\mathfrak{S}$, defined by the equation

$$(\mathrm{Op}(\mathfrak{S})\mathrm{u})(\mathrm{x}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{S}\left(\frac{\mathrm{x}+\mathrm{y}}{2}, \eta\right) \mathrm{e}^{2i\pi\langle \mathrm{x}-\mathrm{y}, \eta\rangle}\, u(y)dyd\eta : \qquad (1)$$

such a linear operator extends as a continuous operator from $S^{'}(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ while, in the case when $\mathfrak{S} \in S^{'}(\mathbb{R}^n \times \mathbb{R}^n)$, one can still define $\mathrm{Op}(\mathfrak{S})$ as a linear operator from $S^{'}(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$;also, Op sets up an isometry from L^2 $(\mathbb{R}^n \times \mathbb{R}^n)$ onto the space of Hilbert–Schmidt operators on L^2 $(\mathbb{R}^n)$. The sharp composition $\mathfrak{S}_1 \# \mathfrak{S}_2$ of two symbols, say lying in $S(\mathbb{R}^n \times \mathbb{R}^n)$, is that which makes the formula

$$\mathrm{Op}(\mathfrak{S}_1)\, \mathrm{Op}(\mathfrak{S}_2) = \mathrm{Op}(\mathfrak{S}_1 \# \mathfrak{S}_2), \qquad (2)$$

in which the left-hand side denotes the usual composition of operators, valid.

The image of the Heisenberg representation is the group of unitary transformations $exp(2i\pi(\langle \eta, \mathrm{Q}\rangle - \langle y, P\rangle - t))$ of $\mathrm{L}^2(\mathbb{R}^n)$, as made meaningful by Stone's theorem, where the j th component of the vector $Q = (Q_1, \dots, Q_n)$ is the multiplication by the j th coordinate x_j $, P = (P_1, \dots, P_n)$ with $P_j = \frac{1}{2i\pi} \frac{\partial}{\partial x_j}$, and $y, \eta \in \mathbb{R}^n$, $\mathrm{t} \in \mathbb{R}$. Introducing on $(\mathbb{R}^n \times \mathbb{R}^n)^2$, the symplectic form [,] such that

$$[(x, \xi), (\mathrm{y}, \eta)] = -\langle \mathrm{x}, \eta\rangle + \langle y, \xi\rangle\, , \qquad (3)$$

let us use on $\mathbb{R}^n \times \mathbb{R}^n$ the simplistic Fourier transformation $\mathcal{F}$ defined by the equation

$$(\mathcal{F}\mathfrak{S})(\mathrm{x}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{S}(\mathrm{Y})\mathrm{e}^{-2i\pi[\mathrm{X},\mathrm{Y}]}\, dY\, , \qquad (4)$$

which commutes with all symplectic linear transformations of the variable in $\mathbb{R}^n \times \mathbb{R}^n$. Another,fully equivalent, way to define the Weyl calculus is by means of the equation

$$Op(\mathfrak{S}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\mathcal{F}\mathfrak{S})(y, \eta)\exp(2i\pi(\langle \eta, \mathrm{Q}\rangle - \langle \mathrm{y}, \mathrm{P}\rangle))\, dyd\eta \qquad (5)$$

The first covariance rule of the Weyl calculus is the observation that

$$exp\big(2i\pi(\langle \eta, \mathrm{Q}\rangle - \langle \mathrm{y}, \mathrm{p}\rangle)\big)\, Op(\mathfrak{S})\, exp\big(-2i\pi(\langle \eta, \mathrm{Q}\rangle - \langle \mathrm{y}, \mathrm{p}\rangle)\big)$$

$$= Op\big((x,\xi) \mapsto \mathfrak{S}(x-y, \xi-\eta)\big). \tag{6}$$

One way to emphasize this action on symbols of the group of translations of $\mathbb{R}^{2n}$ is to decompose in a systematic way the space of symbols $L^2(\mathbb{R}^{2n})$ with respect to this action. Now, the operators which commute with it are just the partial differential operators with constant coefficients: the generalized joint eigenfunctions of these are exactly the exponentials $X = (x, \xi) \mapsto e^{2i\pi[A,X]}$ with $A \in \mathbb{R}^{2n}$, and the sought-after decomposition of a symbol is provided by the symplectic Fourier transformation. On the other hand, if $A = (y, \eta)$, the operator with symbol $e^{2i\pi[A,X]}$is none other than the operator $exp\big(2i\pi(\langle\eta, Q\rangle - \langle y, P\rangle)\big)$, so that Heisenberg's commutation relation, expressed in Weyl's exponential version, takes the form

$$e^{2i\pi[A^1,X]} \# e^{2i\pi[A^2,X]} = e^{i\pi[A^1,A^2]}\, e^{2i\pi[A^1+A^2,X]}. \tag{7}$$

Let us briefly recall a few immediate consequences of this relation. First, one has (say, when $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ lie in $S(\mathbb{R}^{2n})$, using (5), the integral composition formula

$$(\mathfrak{S}_1 \# \mathfrak{S}_2)(x) = 2^{2n} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \mathfrak{S}_1(Y)\mathfrak{S}_2(Z) e^{-4i\pi[Y-X,Z-X]}\, dY dZ \tag{8}$$

or (a fully equivalent one)

$$(\mathfrak{S}_1 \# \mathfrak{S}_2)(x) = [\exp(i\pi L)\mathfrak{S}_1(Y)\mathfrak{S}_2(Z)] \quad (Y = Z = X) \tag{9}$$

with (setting $Y = (y, \eta), Z = (z, \zeta)$)

$$i\pi L = \frac{1}{4i\pi} \sum_{j=1}^{n} \left(-\frac{\partial^2}{\partial y_j \partial \zeta_j} + \frac{\partial^2}{\partial z_j \partial \eta_j}\right) \tag{10}$$

Expanding the exponential into a series, one obtains the so-called Moyal formula

$$(\mathfrak{S}_1 \# \mathfrak{S}_2)(x,\xi) = \sum \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \left(\frac{1}{4i\pi}\right)^{|\alpha|+|\beta|} \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\beta} \mathfrak{S}_1(x,\xi) \left(\frac{\partial}{\partial x}\right)^{\beta} \left(\frac{\partial}{\partial \xi}\right)^{\alpha} \mathfrak{S}_2(x,\xi). \tag{11}$$

This formula is an exact one in the case when the two operators under consideration are differential operators, which means exactly that their symbols (of course, not in $S(\mathbb{R}^{2n})$) are polynomial with respect to the variables ξ , with coefficients depending on x in a smooth, but otherwise fairly arbitrary way; it is also exact when one of the two symbols is a polynomial in (x, ξ).

As it turns out, this version of the composition formula is the only universally known one. Indeed, it has considerable importance in applications of pseudodifferential analysis to partial differential equations.

Our derivation of (8) was obtained as the result of pairing the concept of sharp composition of symbols with the decomposition of symbols according to the action by translations of the group $\mathbb{R}^{2n}$: the success of this point of view was essentially dependent on the fact that this action is an ingredient of the covariance formula (6). This takes us to the aim of the present chapter: to take advantage of the other covariance property of the Weyl calculus—to be recalled now—and follow the same policy.

Recall that the metaplectic representation Met in $L^2(\mathbb{R}^n)$ is a certain unitary representation of the twofold cover of the symplectic group Sp $(n,\mathbb{R})$, which consists of all linear transformations g of $\mathbb{R}^n \times \mathbb{R}^n$ such that $[gX, gY] = [X, Y]$ for every pair (X, Y) of points of $\mathbb{R}^n \times \mathbb{R}^n$: it acts irreducibly on each of the two subspaces of $L^2(\mathbb{R}^n)$ consisting of functions with a given parity. Unitary transformations in the image of the metaplectic representation also act as automorphisms of the space $S(\mathbb{R}^n)$ or of the space $S'(\mathbb{R}^n)$: moreover, if such a unitary transformation U lies above $g \in Sp(n,\mathbb{R})$, and if $\mathfrak{S} \in S'(\mathbb{R}^{2n})$: , one has the covariance formula

$$U\mathrm{Op}(\mathfrak{S})U^{-1} = \mathrm{Op}(\mathfrak{S} \circ \mathrm{g}^{-1}). \tag{12}$$

In full analogy with the procedure adopted above in connection with the Heisenberg representation, we now start from a decomposition of the phase space representation $(\mathrm{g}, \mathfrak{S}) \mapsto \mathfrak{S} \circ \mathrm{g}^{-1}$ of $Sp(n,\mathbb{R})$, in $L^2(\mathbb{R}^{2n})$ into irreducibles: this is just the same as decomposing functions in $L^2(\mathbb{R}^{2n})$ as integral superpositions of functions homogeneous of a given degree, and with a given parity.

The main result is the formula which takes the place of (7): it decomposes the sharp product of two symbols h_1 *and* h_2 , homogeneous of degrees $-n - i\lambda_1$ *and* $-n - i\lambda_2$ and with parities characterized by indices δ_1 *and* δ_2, as an integral superposition of functions homogeneous of degrees $-n - i\lambda$, with the parity $\delta \equiv \delta_1 + \delta_2$. It involves the integral kernel

$$|[X,Y]|_{\varepsilon_2}^{\frac{-n-i\lambda+i\lambda_1-i\lambda_2}{2}} \; |[X,Z]|_{\varepsilon_1}^{\frac{-n-i\lambda-i\lambda_1+i\lambda_2}{2}} \; |[Z,Y]|_{\varepsilon}^{\frac{-n+i\lambda+i\lambda_1+i\lambda_2}{2}}, \tag{13}$$

a product of three signed powers, obtained from the decomposition into homogeneous components with respect to the three variables of the integral kernel which occurs in the composition formula (8). Some preparation is needed in order to give this kernel a genuine meaning as adistribution, not only as a partially defined function. The principle of the proof of the new composition formula is simple, and relies on the decomposition of symbols into hyperplane waves, and the dual notion of rays. Its main difficulty lies in the singular nature of such distributions, which are nevertheless the only ones, sufficiently general, for which explicit computations are possible.

In the one-dimensional case, the integral kernel above reduces to a function

$$J(x,y,z) = |x-y|_{\varepsilon_2}^{\frac{-1-i\lambda+i\lambda_1-i\lambda_2}{2}} \; |z-x|_{\varepsilon_1}^{\frac{-1-i\lambda-i\lambda_1+i\lambda_2}{2}} \; |y-z|_{\varepsilon}^{\frac{-1+i\lambda+i\lambda_1+i\lambda_2}{2}} \tag{14}$$

of three real variables, and the composition formula was treated along these lines in [166]. It is true that the proof, in the higher-dimensional case, is actually, for the main part,a reduction to the one-dimensional case: but signed powers of linear forms with exponents lying on the line $-n + i\mathbb{R}$, the consideration of which is necessary for spectral-theoretic reasons, are more singular distributions when $n \geq 2$, which has made some technical improvements necessary. It may be interesting to recall briefly what can be done in the one-dimensional case in relation to automorphic distribution theory.

In the automorphic situation, the integral kernel (14) enables one to build new nonholomorphic modular forms from given pairs of such. [11], introduced the notion of automorphic distribution: this is a distribution in $\mathbb{R}^2$ invariant under linear changes of coordinates associated to elements of some arithmetic subgroup of $SL(2,\mathbb{R})$, for instance $SL(2,\mathbb{Z})$. This concept is equivalent—in a non-trivial way—to the Lax–Phillips notion of *pairs* of non-holomorphic modular forms, as introduced in their scattering theory [162] for the automorphic wave equation. Automorphic distributions can be taken as symbols in the Weyl calculus and, at the price of important difficulties, the one-dimensional case of the analysis of sharp-products in the present chapter can be developed in the automorphic environment. Things are more interesting, in some sense, since besides a continuous part, in which Eisenstein distributions serve as generalized eigenfunctions, the automorphic Euler operator has a discrete spectrum, and the corresponding eigendistributions are cusp-distributions. Finding the appropriate composition formulas calls for the explicit computation of integrals of $J(x,y,z)$ against three non-holomorphic modular forms, in the realization of these as distributions on the line invariant under representations taken from the principal series of the arithmetic subgroup of $SL(2,\mathbb{R})$, under consideration: this has been completed up to some large extent, for the case of the full modular group, in [166], and it provides a pseudodifferential theoretic approach to such notions as L-functions, convolution L-functions, etc. As a preparation for automorphic pseudodifferential analysis, and in view of other applications as well, either to arithmetic or to quantization theory, a study of the integral kernel (14) had been made in [165]. It has also been considered recently in [163], in the automorphic case, and we take it from the references there that, outside the automorphic environment, it had already appeared in [164]: note that the objects called automorphic distributions in [163] are not the same as those in [165,166] (they are close to what was called modular distributions in [165]).

Obviously, it would be of great interest to push the present composition formula for *n*-dimensional pseudodifferential analysis up to an automorphic environment, despite the great difficulties experienced with automorphic pseudodifferential analysis in the one-dimensional case. In any case, linking pseudodifferential analysis to harmonic analysis, then to modular form theory (also the subject of [167], though the connection between these domains is different there) is certain to bring rewards in the future. In a non-automorphic environment, the basic idea put forward in [172], namely that of building composition formulas from the pairing of covariance with the decomposition of representations into irreducibles, may also [166] be of use whenever some symbolic calculus of operators is examined, thus finding its place within quantization theory in general.

Decomposing the action of the symplectic group on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$

Consider the linear space $(\mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}})$ with its canonical symplectic form (3) and measure $dx\, d\xi$: we also set, when convenient, $X = (x, \xi)$. The symplectic group $\mathrm{G} = \mathrm{Sp}(\mathrm{n}, \mathbb{R})$ is the group of linear transformations g of $\mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}}$ which preserve the symplectic form, *i.e.*, satisfy the identity $[\mathrm{gX}, \mathrm{gY}] = [\mathrm{X}, \mathrm{Y}]$ for any pair X, Y of points of $\mathbb{R}^{\mathbf{2n}}$. The phase space representation of G in $L^2(\mathbb{R}^n)$ is defined by the action $(\mathrm{g}, \mathrm{h}) \longmapsto \mathrm{g.h}$ such that $(\mathrm{g.h})(\mathrm{X}) = \mathrm{h}(\mathrm{g}^{-1}\mathrm{X})$. It is unitary, and since all linear transformations on $\mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}}$ preserve the parity of functions and commute with the Euler operator

$$2i\pi\varepsilon = \sum \left[x_j \, \frac{\partial}{\partial x_j} + \xi_j \, \frac{\partial}{\partial \xi_j} \right] + n \tag{15}$$

(the additional constant turns ε into a formally self-adjoint operator on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$), the (extension of the) phase space representation under study preserves the linear space of functions on $\mathbb{R}^{2n} \backslash \{0\}$ homogeneous of a given degree, and with a given parity.

Given $h \in L^2(\mathbb{R}^{2n})$, we first decompose it into its even and odd parts. Then, setting for every real number $s \neq 0$ *and* $\propto \in \mathbb{C}$

$$|s|_0^{\propto} = |s|^{\propto}, \quad |s|_1^{\propto} = \langle s \rangle^{\propto} = |s|^{\propto} signs, \tag{16}$$

we may write

$$h = \sum_{\delta = 0,1} \int_{-\infty}^{\infty} h_{i\lambda,\delta} \, d\lambda \,, \tag{17}$$

provided we set

$$h_{i\lambda,\delta}(X) = \frac{1}{4\pi} \int_{-\infty}^{\infty} |t|_\delta^{n-1+i\lambda} \; h(tX) dt. \tag{18}$$

Then, $h_{i\lambda,\delta}$ is homogeneous of degree $-n - i\lambda$ and has the parity associated to δ: we shall refer to the pair $(-n - i\lambda, \delta)$ as the *type* of $h_{i\lambda,\delta}$. More generally, we may consider on $\mathbb{R}^{2n} \backslash \{0\}$ functions of type $(-n - v, \delta)$ for an arbitrary complex parameter *v*.

So as to cut down, as is needed, the dimension by 1, one may realize functions of a given type as sections of some appropriate line bundle over the projective space $p_{2n-1}(\mathbb{R})$. We first need to introduce the so-called tautological bundle $E_{\mathbb{C}}$ over $p_{2n-1}(\mathbb{R})$, the fibre of which above a point $p(\theta)$ (p being the canonical map: $\mathbb{R}^{2n} \backslash \{0\} \to p_{2n-1}(\mathbb{R})$) is the complex line $\mathbb{C}\theta$ in $\mathbb{C}^{2n}$. Incidentally, note that the total space of the real line analogue $E_{\mathbb{R}}$ of this bundle is just the blown up space $\widehat{\mathbb{R}^{2n}}$ which is used consistently for desingularization purposes, as will be the case.

A canonical set of charts of $p_{2n-1}(\mathbb{R})$ is obtained in the following way: given a vector $S \in \mathbb{R}^{2n}\{\theta\}$, set $\Omega_S = \{\theta \in \mathbb{R}^{2n} : [\theta, S] \neq 0\}$ and, in $\omega_S = p(\Omega_S)$, take the chart

$p(\theta) \longmapsto \frac{\theta}{[\theta,S]}$, which identifies ω_S with the affine hyperplane $M_S = \{X \in \mathbb{R}^{2n} : [X,S] = 1$. Above M_S , a section of $E_{\mathbb{C}}$ can be identified with a complex-valued function f_S , associating to such a function the section$X \longmapsto f_S(X)X$. Note that,$X \in M_S$satisfies $[X,T] \neq 0$ for some new vector $T \in \mathbb{R}^{2n}\backslash\{0\}$, the points $X \in M_S$ and $\frac{X}{[X,T]} \in M_T$ are truly the images, under the charts associated with S *and* T , of the same point in $p_{2n-1}(\mathbb{R})$. Identifying $f_S(X)X$ with $f_T(Y)Y$, where we have set $Y = \frac{X}{[X,T]}$, leads to the compatibility condition

$$f_T\left(\frac{X}{[X,T]}\right) = [X,T]f_S(X)\,, \tag{19}$$

which defines the transition functions of the line bundle $E_{\mathbb{C}}$.

More generally, given (μ,δ) with $\mu \in \mathbb{C}$ *and* $\delta = 0$ *or* 1 , define the signed power $|E_{\mathbb{C}}|_\delta^\mu$ of $E_{\mathbb{C}}$ by taking the corresponding signed powers of the transition functions: then, a section of the line bundle $|E_{\mathbb{C}}|_\delta^\mu$ is associated to a set (f_S) of functions, f_S defined in M_S , satisfying the requirement that

$$f_T\left(\frac{X}{[X,T]}\right) = |[X,T]|_\delta^\mu f_S(X) \tag{20}$$

whenever $X \in M_0$ *and* $[X,T] \neq 0$. Then, a function h of type $(-n-\nu,\delta)$ can be identified with the section of $|(E_{\mathbb{C}})|_\delta^{n+\nu}$ characterized by the fact that, for every $S \in \mathbb{R}^{2n} \setminus \{0\}, f_S$ is the restriction of h to M_S . Conversely, any function f in M_S uniquely lifts as a function $f^\sharp$ in the part of $\mathbb{R}^{2n} \setminus \{0\}$ consisting of vectors θ such that $[\theta,S] \neq 0$, to wit the one defined by the equation

$$f^\sharp(\theta) = |[\theta,S]|_\delta^{-n-\nu} f\left(\frac{\theta}{[\theta,S]}\right). \tag{21}$$

The representation $\pi_{\nu,\delta}$ from the full, non-unitary principal series of $Sp(n,\mathbb{R})$ is by definition the restriction of the phase space representation of $Sp(n,\mathbb{R})$ (again, this is defined by the assignment $(g,h) \mapsto h \circ g^{-1}$) to the space of functions in $\mathbb{R}^{2n}\backslash\{0\}$ of type $(-n-\nu,\delta)$. It will be convenient—but there is a price to pay—not to have to change the hyperplane M_S consistently, and we denote as M_0 the one which should really be denoted as M_{e_1} (where e_1 is the first vector from the canonical basis of $\mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}}$) , *i.e.* , the one consisting of vectors $X = (x;\xi) \in \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}}$ such that $\xi_1 = 1$. Starting from (21) and using the fact that $f^\sharp$ is of type $(-n-\nu,\delta)$, together with the relation $[g^{-1}X,e_1] = [X,ge_1]$, one obtains the relation

$$\left(\pi_{\nu,\delta}(g)f\right)(X) = |[X,ge_1]|_\delta^{-n-\nu} f\left(\frac{g^{-1}X}{[X,ge_1]}\right). \tag{22}$$

As an example, when $n = 1$ *and* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, starting from $X = \begin{pmatrix} x \\ 1 \end{pmatrix}$, so that $g^{-1}X = \begin{pmatrix} dx-b \\ -cx+a \end{pmatrix}$, one obtains, after one has abbreviated $f\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right)$ as $f^b(x)$, the relation

$$\left(\pi_{v,\delta},(g)f\right)^{b}(x) = |-cx+a|_{\delta}^{-1-v} f^{b}\left(\frac{dx-b}{-cx+a}\right) \tag{23}$$

Still specializing, for the time being, in the hyperplane M_0, we set

$$X = (x;\xi) = (x_1, x_*; \xi_1, \xi_*), \tag{24}$$

and denote as $h_{i\lambda,\delta}^{b}$ the restriction of $h_{i\lambda,\delta}$ toM_0 (it is the same as the function which would have been denoted as $(h_{i\lambda,\delta})e_1$ in the less specialized setting above). One has the reciprocal equations

$$h_{i\lambda,\delta}^{b}(x;\xi_*) = h_{i\lambda,\delta}(x;1,\xi_*)$$
$$h_{i\lambda,\delta}(x;\xi) = |\xi_1|_{\delta}^{-n-i\lambda}\ h_{i\lambda,\delta}^{b}\left(\frac{x}{\xi_1};\frac{\xi_*}{\xi_1}\right). \tag{25}$$

Proposition $(\mathbf{6.1})[\mathbf{172}]$**.** The space $L^2(\mathbb{R}^{2n})$ can be decomposed as the Hilbert direct integral

$$L^2(\mathbb{R}^{2n}) \sim \bigoplus_{\delta=0,1} \int^{\oplus} \mathcal{H}_{i\lambda,\delta} d\lambda\,, \tag{26}$$

if one denotes as $\mathcal{H}_{i\lambda,\delta}$ the inverse image under the map $h_{i\lambda,\delta} \mapsto h_{i\lambda,\delta}^{b}$ of the space $L^2(M_0; dxd\xi_*)$: the decomposition is provided by (17), and it commutes with the phase space representation of G in $L^2(\mathbb{R}^{2n})$.

Proof. What remains to be done is proving the equation

$$\|h\|_{L^2(\mathbb{R}^{2n})}^2 = 4\pi \sum_{\delta=0,1} \int_{-\infty}^{\infty} \left\|h_{i\lambda,\delta}^{b}\right\|_{L^2 M_0}^2 d\lambda, \tag{27}$$

using onM_0 the measure $dxd\xi_*$. Indeed, with $h_{(\delta)} = h_{even}\ or\ h_{odd}$ according to the parity of δ, set

$$\phi_X(s) = e^{2\pi ns} h_{(\delta)}(e^{2\pi s}X), \quad s \in \mathbb{R},\ X \in \mathbb{R}^{2n}\backslash\{0\}, \tag{28}$$

So that

$$\hat{\phi}_X(\lambda) = h_{i\lambda,\delta}(X). \tag{29}$$

The one-dimensional Fourier inversion formula then yields (17) (of course, using the Mellin transform rather than coupling a Fourier transform with the change of variable $t = e^{2\pi s}$ would be more natural: the choice really depends on your familiarity with the inversion formula in both cases). Next, using (25) and the Plancherel formula for the Fourier transformation,

$$\left\|h_{(\delta)}\right\|_{L^2(\mathbb{R}^{2n})}^2 = 4\pi \int_{-\infty}^{\infty} e^{2\pi s}\, ds \int_{\mathbb{R}^{2n-1}} \left|h_{(\delta)}(x; e^{2\pi s}, \xi_*)\right|^2 dxd\xi_*$$

$$= 4\pi \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^{2n-1}} \left|\phi_{(x;1,\xi_*)}(s)\right|^2 dxd\xi_*$$

$$= 4\pi \int_{\mathbb{R}^{2n-1}} dxd\xi_* \int_{-\infty}^{\infty} \left|\hat{\phi}_{(x,1,\xi_*)}(s)\right|^2 ds$$

$$= 4\pi \int_{\mathbb{R}^{2n-1}} dxd\xi_* \int_{-\infty}^{\infty} \left|h_{i\lambda,\delta}(x;\xi_*)\right|^2 ds \tag{30}$$

which proves (27).

The decomposition above gives right to the series $(\pi_{i\lambda,\delta})\,\lambda \in \mathbb{R}, \delta = 0{,}1$ of representations of G in $L^2(\mathrm{M}_0)$, a special case of the representations $\pi_{v,\delta}$ already considered; it suffices to set

$$\pi_{i\lambda,\delta}(\mathrm{g})\, h^b_{i\lambda,\delta} = f^b_{i\lambda,\delta} \tag{31}$$

if $h \in L^2(\mathbb{R}^{2n})$, $g \in G$, $f = h\, og^{-1}$. Each representation $\pi_{i\lambda,\delta}(g)$ is unitary as a consequence of Proposition 2.1: to show that $\left\|\pi_{i\lambda,\delta}(g)\, h^b_{i\lambda,\delta}\right\| = \left\|f^b_{i\lambda,\delta}\right\|$ for every λ such that $h^b_{i\lambda,\delta} \in L^2(\mathrm{M}_0)$, not only almost every λ, it suffices to start from a dense space of functions h such that $h^b_{i\lambda,\delta}(g)\, h^b_{i\lambda,\delta} = f^b_{i\lambda,\delta}$ depends in a continuous way on λ, which is ensured for instance when h lies in $S(\mathbb{R}^{2n})$. Recall that we also set $\pi_{i\lambda,\delta}$.

It will be proved that most representations $\pi_{i\lambda,\delta}$ are irreducible.

The (symplectic) Fourier transform of a function homogeneous of degree $-n - i\lambda$ with a given parity is homogeneous of degree $-n + i\lambda$, and has the same parity, so that, given $h \in L^2(\mathbb{R}^{2n})$, one has

$$\mathcal{F}h_{i\lambda,\delta} = (\mathcal{F}h)_{-i\lambda,\delta} : \tag{32}$$

consequently, the representations $\pi_{i\lambda,\delta}$ *and* $\pi_{-i\lambda,\delta}$ are unitarily equivalent.

Corollary $(6.1)'$[140]. The Hilbert space $L^2(\mathbb{R}^{2n})$ can be decomposed as integral

$$L^2(\mathbb{R}^{2n}) \sim \bigoplus_{\delta_1,\delta_2,\ldots,\delta_n = 0,1} \int^{\oplus} \mathcal{H}_{\mathrm{i}\lambda,(\delta_1,\delta_2,\ldots,\delta_n)}\mathrm{d}\lambda$$

If $\mathcal{H}_{\mathrm{i}\lambda,(\delta_1,\delta_2,\ldots,\delta_n)}$ the inverse image under the map $\mathrm{h}_{\mathrm{i}\lambda,(\delta_1,\delta_2,\ldots,\delta_n)} \mapsto h^b_{i\lambda,(\delta_1,\delta_2,\ldots,\delta_n)}$ of the Hilbert space $L^2(\mathrm{M}_0; dxd\xi_*)$. The decomposition is provided by equation (17) which commutes with the phase space representation of G in $L^2(\mathbb{R}^{2n})$.

Proof: To prove equation

$$\|h\|^2_{L^2(\mathbb{R}^{2n})} = 4\pi \sum_{\delta_1,\delta_2,\dots,\delta_n=0,1} \int_{-\infty}^{\infty} \left\|h^b_{i\lambda,\delta_1,\delta_2,\dots,\delta_n}\right\|^2_{L^2 \mathrm{M}_0} d\lambda$$

upon using the measure $dxd\xi$. With $h_{(\delta_1,\delta_2,\dots,\delta_n)} = h_{even}$ or h_{odd} according to the parity of $(\delta_1, \delta_2, \dots, \delta_n)$, now set

$$\phi_X(s_1 + s_2 + \dots + s_2) = \prod_{j=1}^{n} e^{2\pi n s_j}\, h_{(\delta_1,\delta_2,\dots,\delta_n)}(e^{2\pi s_j} X)$$

$$s_1 + s_2 + \dots + s_2 \in \mathbb{R} \ , \ \mathrm{X} \in \mathbb{R}^{2n} \backslash \{0\}$$

Hence $\hat{\phi}_X(\lambda) = h_{i\lambda,(\delta_1,\delta_2,\dots,\delta_n)}(X)$.

Upon proposition (6.1) we can yields (17) and using (24) and the plancherel formula the Fourier transformation

$$\left\|h_{(\delta_1,\delta_2,\dots,\delta_n)}\right\|^2_{L^2(\mathbb{R}^{2n})} = 4\pi \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{2\pi n s_j}\, ds_j \int_{\mathbb{R}^{2n-1}} \left|h_{(\delta_1,\delta_2,\dots,\delta_n)}(x; e^{2\pi s_j}, \xi_*)\right|^2 dxd\xi_*$$

$$= 4\pi \int_{-\infty}^{\infty} ds_j \int_{\mathbb{R}^{2n-1}} \left|\phi_{(x;1,\xi_*)}(s_1 + s_2 + \dots + s_2)\right|^2 dxd\xi_*$$

$$= 4\pi \int_{\mathbb{R}^{2n-1}} dxd\xi_* \int_{-\infty}^{\infty} \left|\hat{\phi}_{(x,1,\xi_*)}(s_1 + s_2 + \dots + s_2)\right|^2 ds_j$$

$$= 4\pi \int_{\mathbb{R}^{2n-1}} dxd\xi_* \int_{-\infty}^{\infty} \left|h_{i\lambda,(\delta_1,\delta_2,\dots,\delta_n)}(x; \xi_*)\right|^2 ds_j$$

Hence gives (26)

Definition $(\mathbf{6.2})$[172]. The (unitary) intertwining operator $\theta_{i\lambda,\delta}$ is the one characterized by the validity of the equation

$$\theta_{i\lambda,\delta} h_{i\lambda,\delta} = (\mathcal{F}h)_{-i\lambda,\delta} \tag{33}$$

for every $h \in L^2(\mathbb{R}^{2n})$. We also set

$$\theta_{i\lambda,\delta} h^b_{i\lambda,\delta} = (\mathcal{F}h)^b_{-i\lambda,\delta}. \tag{34}$$

The proof that $\theta_{i\lambda,\delta}$ preserves the L^2-norm for every λ, not only almost every λ, is the same as the one which, in connection with the definition of $\pi_{i\lambda,\delta}$, followed (31). It is easy to make the unitary intertwining operator $\theta_{i\lambda,\delta}$ associated to (32) explicit in terms of the coordinates on (M_0). Indeed, starting from (25), one can write

$$(\mathcal{F}h)^b_{-i\lambda,\delta}\ (x; \xi_*)$$

$$= \left(\mathcal{F}h_{i\lambda,\delta}\right) (x; 1, \xi_*)$$

$$= \int |\eta_1|_\delta^{-n-i\lambda} h^b_{i\lambda,\delta}\left(\frac{y}{\eta_1};\frac{\eta_*}{\eta_1}\right) exp(2i\pi[x_1\eta_1 + \langle x_*, \eta_*\rangle - y_1 - \langle y_*, \xi_*\rangle]) dy d\eta_1 \eta_*$$

$$= \int |\eta_1|_\delta^{n-1-i\lambda} h^b_{i\lambda,\delta}(y;\eta_*) exp(2i\pi\eta_1[x_1 + \langle x_*, \eta_*\rangle - y_1 - \langle y_*, \xi_*\rangle]) dy d\eta_1 \eta_* \quad (35)$$

Making a one-dimensional Fourier transformation explicit, this gives another approach to the intertwining operator $\theta_{i\lambda,\delta}$ from $\pi_{i\lambda,\delta}$ to $\pi_{-i\lambda,\delta}$: the operator $\theta_{i\lambda,\delta}$ is defined formally as the operator with integral kernel

$$k_{i\lambda,\delta}(x,\xi_*;y,\eta_*) = i^\delta \pi^{\frac{1}{2}-n+i\lambda} \frac{\Gamma\left(\frac{n-i\lambda+\delta}{2}\right)}{\Gamma\left(\frac{1-n+i\lambda+\delta}{2}\right)} |x_1 - y_1 + \langle x_*, \eta_*\rangle - \langle y_*, \xi_*\rangle|_\delta^{-n+i\lambda}. \quad (36)$$

Note that, while Definition (6.2.2) is a rigorous definition of the intertwining operator, (4536) can only be used after some preparation.

While $X = (x;\xi)$ $(or\ \ Y = (y;\eta), \dots)$ will always denote a generic point in $\mathbb{R}^{2n}$, we shall draw attention to points $(x;1,\xi_*) = (x_1, x_*; 1, \xi_*)$ of M_0 by denoting them as X_*: similarly, $Y_* = (y;1,\eta_*)$. Given $X_* \in \mathrm{M}_0$, we set $X_{**} = (x_*;\xi_*)$, so that one can also identify X_* with (x_1, X_{**}). We abbreviate the measure $dxd\xi_*$ on M_0 as $dmdm(X_*)$. On $\mathbb{R}^{2n-2}$, one can also consider the symplectic form obtained from an appropriate restriction of the one available on $\mathbb{R}^{2n}$, i.e.,set

$$[X_{**}, Y_{**}] = -\langle x_*, \eta_*\rangle + \langle y_*, \xi_*\rangle, \quad (37)$$

while, on M_0, one must define

$$[X_*, Y_*] = \left[\left((x_1, x_*);(1,\xi_*)\right), \left((y_1, y_*);(1,\eta_*)\right)\right] = -x_1 + y_1 - \langle x_*, \eta_*\rangle + \langle y_*, \xi_*\rangle. \quad (38)$$

One may then rewrite (36) as

$$\theta_{i\lambda,\delta} f(X_*) = i^\delta \pi^{\frac{1}{2}-n+i\lambda} \frac{\Gamma\left(\frac{n-i\lambda+\delta}{2}\right)}{\Gamma\left(\frac{1-n+i\lambda+\delta}{2}\right)} \int_{\mathrm{M}_0} |[X_*, Y_*]|_\delta^{-n+i\lambda} f(Y_*) dm(Y_*). \quad (39)$$

The intertwining operator may be better understood after some transformation. Denote as $\mathcal{F}_1$ the usual Fourier transformation as applied when emphasis is set on the first variable only of a function of several variables. Given a function f on M_0, write it as $h^b_{i\lambda,\delta}$, which, according to (25), is possible in a unique way for a given pair $(i\lambda,\delta)$, so that the left-hand side of (35) is just $\left(\theta_{i\lambda,\delta} f\right)(x;\xi_*)$ according to (32). Starting from (35), one can then write, if $n \geq 2$,

$$\left(\mathcal{F}_1 \theta_{i\lambda,\delta} f\right)(t, x_*;\xi_*) = \left(\mathcal{F}_1 \theta_{i\lambda,\delta} f\right)(t, X_{**})$$

$$= |t|_\delta^{n-1-i\lambda} \int_{\mathrm{M}_0} f(y_1, Y_{**}) \exp\big(-2i\pi t(y_1 + [X_{**}, Y_{**}])\big)\, dy_1 dY_{**}$$

$$= |t|_\delta^{n-1-i\lambda} \int_{\mathbb{R}^{2n-2}} (\mathcal{F}_1 f)\,(t, Y_{**}) exp(-2i\pi t[X_{**}, Y_{**}]) dY_{**}. \qquad (40)$$

In this definition of the intertwining operator, $\theta_{i\lambda,\delta}$ appears as the "product" of a one-dimensional intertwining operator with respect to the first variable and of a Fourier transformation in $\mathbb{R}^{2n-2}$: only, some rescaling, by the variable dual to the first one, is performed with respect to the last $2n - 2$ variables. As a straightforward application of this equation, note the formula, in which $\delta_2 := \delta_1 + \delta$,

$$\big(\mathcal{F}_1 \theta_{i\lambda_1,\delta_1} \theta_{i\lambda,\delta} f\,\big)(t, X\, **) = |t\,|_{\delta_2}^{-i(\lambda_1+\lambda)} (\mathcal{F}_1 f\,)(t, X\, **): \qquad (41)$$

hence, the composition of the two intertwining operators under consideration reduces to an intertwining operator with respect to the first variable, with integral kernel

$$\big((x_1, X_{**}), (y_1, X_{**})\big)$$

$$\mapsto i^{\delta_2} \pi^{-\frac{1}{2}+i(\lambda_1+\lambda)} \frac{\Gamma\left(\frac{1-i(\lambda_1+\lambda)+\delta_2}{2}\right)}{\Gamma\left(\frac{i(\lambda_1+\lambda)+\delta_2}{2}\right)} |x_1 - y_1|_{\delta_2}^{-1+i(\lambda_1+\lambda)} \delta(X_{**} - Y_{**}). \qquad (42)$$

At this point, it may be useful to clarify the respective roles of the coordinates ξ_1 and x_1, as they occur in what precedes. Isolating the coordinate ξ_1 is tantamount to singling out the affine hyperplane M_0, the equation of which is $[X, e_1] = 1$, while $[X, e_1] = \xi_1$ generally. The expression $\frac{\partial f}{\partial x_1}, for\ f \in C^\infty(\mathrm{M}_0)$, is then the image of f under a canonical operator on M_0, since it may be thought of as the Poisson bracket of the function $X \longmapsto \xi_1$ with an arbitrary smooth extension of f to the whole of $\mathbb{R}^{2n}$. One may interpret the convolution operator the integral kernel of which is given in (42) as a function, in the sense of functional calculus, of the operator $\frac{1}{2i\pi}\frac{\partial}{\partial x_1}$. On the other hand, the coordinate x_1 is not intrinsically attached to M_0: with the help of a well-chosen symplectic transformation preserving the coordinate ξ_1, it can be transformed to the sum of x_1 and of an arbitrary linear combination of $x_2, \dots, x_n, \xi_1, \dots, \xi_n$.

Note if $f \in L^2(\mathrm{M}_0)$ the relation

$$\overline{\pi_{i\lambda,\delta}(g) f} \;=\; \pi_{-i\lambda,\delta}(g)\bar{f} \qquad (43)$$

from which, polarizing the identity which expresses that $\pi_{i\lambda,\delta}$ is unitary, we obtain the identity

$$\int_{\mathrm{M}_0} f_2(X) f_1(X_*)\, dm(X_*)$$

$$= \int_{\mathrm{M}_0} \left(\pi_{-i\lambda,\delta}(g) f_2\right)\left(\pi_{i\lambda,\delta}(g) f_1\right)(X_*)\, dm(X_*) \qquad (44)$$

involving a pair (f_1, f_2) of functions in $L^2(\mathrm{M}_0)$: this can also be regarded as a particular case of (41), to the effect that the inverse of the isometry $\theta_{i\lambda,\delta}$ is $\theta_{-i\lambda,\delta}$. Assuming convergence, one can extend (44) as

$$\int_{\mathrm{M}_0} f_2(X_*) f_1(X_*)\, dm(X_*)$$

$$= \int_{\mathrm{M}_0} \left(\pi_{-v,\delta}(g) f_2\right)(X_*)\left(\pi_{v,\delta}(g) f_1\right) dm(X_*). \qquad (45)$$

We now introduce the integral kernel obtained from the decomposition into homogeneous components of the integral kernel $e^{4i\pi[Y,X]} e^{4i\pi[X,Z]} e^{4i\pi[Z,Y]}$ which occurs in the composition formula (8). Consider on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ the (almost everywhere defined only) function

$$(Y, Z; X) \longmapsto |[Y,X]|_{\varepsilon_2}^{\alpha_1} |[X,Z]|_{\varepsilon_1}^{\alpha_2} |[Z,Y]|_{\varepsilon}^{\alpha_3}, \qquad (46)$$

where the exponents and indices of parity are given. It is of type $(\alpha_1 + \alpha_3, \varepsilon + \varepsilon_2 \ mod\ 2), resp.\ (\alpha_2 + \alpha_3, \varepsilon + \varepsilon_1\ mod\ 2), resp.\ (\alpha_1 + \alpha_2, \varepsilon_1 + \varepsilon_2\ mod\ 2)$ with respect to Y, $resp.\ Z, resp.\ X$.

Given a triple (v_1, v_2, v) of complex numbers, and a triple $(\delta_1, \delta_2, \delta)$ of numbers equal to 0 or 1, satisfying the relation $\delta \equiv \delta_1 + \delta_2\ mod\ 2$, the system of equations

$$\varepsilon_2 + \varepsilon \equiv \delta_1,\ \ \varepsilon_1 + \varepsilon \equiv \delta_2,\ \ \varepsilon_1 + \varepsilon_2 \equiv \delta \qquad (47)$$

for $\varepsilon, \varepsilon_1, \varepsilon_2\ mod\ 2$ has two solutions, obtained as

$$\varepsilon \equiv j + \delta, \varepsilon_1 \equiv j + \delta_1, \varepsilon_2 \equiv j + \delta_2 \qquad (48)$$

with $j = 0\ or\ 1$. Then, the types of the function above with respect to Y, Z, X will be $(-n + v_1, \delta_1), (-n + v_2, \delta_2)\ and\ (-n - v, \delta)$ if and only if

$$\alpha_1 = \frac{-n-v+v_1-v_2}{2}, \alpha_2 = \frac{-n-v-v_1+v_2}{2}, \alpha_3 = \frac{-n+v+v_1+v_2}{2} \qquad (49)$$

Hence, provided that (47) is satisfied, the integral kernel

$$J_{v_1,v_2;v}^{\varepsilon_1,\varepsilon_2;\varepsilon}(Y, Z; X) = |[Y,X]|_{\varepsilon_2}^{\frac{-n-v+v_1-v_2}{2}} |[X,Z]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} |[Z,Y]|_{\varepsilon}^{\frac{-n+v+v_1+v_2}{2}} \qquad (50)$$

in $(\mathbb{R}^{2n} \setminus \{0\}) \times (\mathbb{R}^{2n} \setminus \{0\}) \times (\mathbb{R}^{2n} \setminus \{0\})$ satisfies the covariance relation

$$\pi_{v,\delta}(g)\left(X \longmapsto J_{v_1,v_2;v}^{\varepsilon_1,\varepsilon_2;\varepsilon}(Y, Z; X)\right)$$

$$= \left[\pi_{-v_1,\delta_1}(g^{-1}) \otimes \pi_{-v_2,\delta_2}(g^{-1})\right]\left((Y, Z) \longmapsto \mathrm{J}_{v_1,v_2;v}^{\varepsilon_1,\varepsilon_2;\varepsilon}(Y, Z; X)\right). \qquad (51)$$

We may also restrict this integral kernel to $\mathrm{M}_0 \times \mathrm{M}_0 \times \mathrm{M}_0$: the relation of covariance is preserved, though with a slightly different understanding (cf. 31)). In next section, we shall see, after we have given the integral kernel so obtained a meaning in an appropriate distribution sense, not only as a partially defined function, that if one denotes as $J^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}$ the associated operator, thought of as being defined by the equation

$$\left(\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(f_1,f_2)\right)(X_*) = \int_{\mathrm{M}_0\times\mathrm{M}_0} \mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(Y_*,Z_*;X_*)\, f_1(Y_*)f_2(Z_*)dm(Y_*)dm(Z_*). \quad (52)$$

one has the covariance identity

$$\pi_{v,\delta}(\mathrm{g})\left(\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(f_1,f_2)\right) = \mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}\left(\pi_{v_1,\delta_1}(\mathrm{g})f_1,\pi_{v_2,\delta_2}(\mathrm{g})f_2\right), \quad (53)$$

formally immediate from (51) and (45). In the case when $f_1 = (h_1)^b_{v_1,\delta_1}$ *and* $f_2 = (h_2)^b_{v_2,\delta_2}$, we can, and shall sometimes, write $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}\left((h_1)_{v_1,\delta_1},(h_2)_{v_2,\delta_2}\right)$ for $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(f_1,f_2)$. Also, the result can be regarded as a function in $\mathbb{R}^{2n}\setminus\{0\}$ of type $(-n-v,\delta)$ rather than, again, as being defined only onM_0. The integral kernel $\mathrm{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(Y,Z;X)$

In all this section, we deal with functions of a given type in their realizations as functions on M_0. Rather than trying to define $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(f_1,f_2)$, as in (52), as a function of X_*, we lower our requirements, only trying to define the expression

$$\left(\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(f_1,f_2),f\right)$$

$$= \int_{\mathrm{M}_0\times\mathrm{M}_0\times\mathrm{M}_0} \mathbf{J}^{\boldsymbol{\varepsilon_1},\boldsymbol{\varepsilon_2};\boldsymbol{\varepsilon}}_{\boldsymbol{v_1},\boldsymbol{v_2};\boldsymbol{v}}(\mathrm{Y}_*,\mathrm{Z}_*;\mathrm{X}_*)\, f_1(\mathrm{Y}_*)f_2(\mathrm{Z}_*)f(\mathrm{X}_*)dm(\mathrm{Y}_*)dm(\mathrm{Z}_*)dm(\mathrm{Z}_*) \quad (54)$$

for appropriate triples (f_1,f_2,f). This is of course tantamount to a reinterpretation of $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}$ as a distribution of some kind, a notion dependent on that of C^∞-vectors of the representations $\pi_{v_1,\delta_1},\pi_{v_2,\delta_2},\pi_{-v,\delta}$ involved (the sign change in the last subscript is an effect of duality(*cf.* (44)).

First, we observe that, though the representation $\pi_{v,\delta}$ is not unitary unless v is pure imaginary, it is still useful to regard it as a representation in some Hilbert space, to wit the one defined by the equation

$$\|f\|^2_c = \int_{\mathrm{M}_0} |f(\mathrm{X}_*)|^2\, |\mathrm{X}_*|^{2Re\ v}\, dm(\mathrm{X}_*): \quad (55)$$

here, $|X_*|^2 = |x|^2 + 1 + |\xi_*|^2$ *when* $X_* = (x;1,\xi_*)$. We now show that, for any given $g \in Sp(n,\mathbb{R})$, the transformation $\pi_{v,\delta}(g)$ is a bounded endomorphism of the Hilbert space $\mathcal{H}_v$ thus defined. First,

$$Y := \frac{\mathrm{g}^{-1}X}{[X,\mathrm{g}e_1]}\ lies\ in\ \mathrm{M}_0\ if\ X \in \mathbb{R}^{2\mathrm{n}}\ and\ [X,\mathrm{g}e_1] \neq 0; \quad (56)$$

indeed, recall that $\xi_1 = [X, e_1]$ if $X = (x; \xi)$ and that $[X, ge_1] = [g^{-1}X, e_1]$. Recalling the recipe, just before (44), which served as a definition of $\pi_{v,\delta}(g)$, we first extend f , initially defined on M_0, as a function $f^{\#}$ in $\mathbb{R}^{2\mathrm{n}} \setminus \{0\}$, setting

$$f^{\#}(x; \xi_1, \xi_*) = |\xi_1|_\delta^{-n-v} f\left(\frac{x}{\xi_1}; 1, \frac{\xi_*}{\xi_1}\right), \tag{57}$$

so that

$$f^{\#}(g^{-1}.(x; \xi_1, \xi_*)) = |[X, ge_1]\,|_\delta^{-n-v} f\left(\frac{g^{-1}X}{[X, ge_1]}\right), \tag{58}$$

and

$$\left(\pi_{v,\delta}(g)f\right)(\mathrm{X}_*) = |[X, ge_1]\,|_\delta^{-n-v} f(\mathrm{Y}_*) \tag{59}$$

with $\mathrm{Y}_* = \frac{\mathrm{g}^{-1}\mathrm{X}_*}{[\mathrm{X}_*, \mathrm{g}e_1]}$. The next thing to do is to compute the Jacobian $\frac{dm\,(\mathrm{Y}_*)}{dm\,(\mathrm{X}_*)}$ when X_* lies inM_0: to this effect, the simplest way is to use the unitarity of $\pi_{0,\delta}$, to wit the relation

$$\int_{\mathrm{M}_0} |[\mathrm{X}_*, ge_1]|^{-2n}\, |f(\mathrm{Y}_*)|^2 dm(\mathrm{X}_*) = \int_{\mathrm{M}_0} |f(\mathrm{X}_*)|^2\, dm(\mathrm{X}_*), \tag{60}$$

finding

$$dm(\mathrm{Y}_*) = |[\mathrm{X}_*, ge_1]|^{-2n} dm(\mathrm{X}_*). \tag{61}$$

Then, with the help of the same change of variables, one has more generally

$$\begin{aligned} \left\|\pi_{v,\delta}(g)f\right\|_v^2 &= \int_{\mathrm{M}_0} |[\mathrm{X}_*, ge_1]|^{-2n-2Re\ v}\, |f(\mathrm{Y}_*)|^2 |\mathrm{X}_*|^{2Re\ v} dm(\mathrm{X}_*) \\ &= \int_{\mathrm{M}_0} |[\mathrm{X}_*, ge_1]|^{-2Re\ v}\, |f(\mathrm{Y}_*)|^2 |\mathrm{X}_*|^{2Re\ v} dm(\mathrm{Y}_*) \\ &= \int_{\mathrm{M}_0} \left(\frac{|\mathrm{X}_*|}{|g^{-1}\mathrm{X}_*|}\right)^{2Re\ v} |f(\mathrm{Y}_*)|^2 |\mathrm{Y}_*|^{2Re\ v} dm(\mathrm{Y}_*), \end{aligned} \tag{62}$$

an expression which we want to bound in terms of $\|f\|_v^2$. It suffices to observe that the ratio $\left(\frac{|\mathrm{X}_*|}{|g^{-1}\mathrm{X}_*|}\right)^{2Re\ v}$ is bounded for $\mathrm{X}_* \in \mathrm{M}_0$, the bound depending of course on g. Hence, $\pi_{v,\delta}$ is a representation by means of bounded operators in $\mathcal{H}_v$.

This makes it possible, in the usual way, to define the space of C^∞ vectors of the given representation. Recalling that the Lie algebra of the symplectic group consists of block matrices $\begin{pmatrix} A & B \\ C & -A' \end{pmatrix}$ with B and C symmetric, one sees that the space of infinitesimal operators of the phase space representation of $Sp(n, \mathbb{R})$ in $L^2(\mathbb{R}^{2\mathrm{n}})$ is generated by the vector fields $\xi_j \frac{\partial}{\partial x_k} + \xi_k \frac{\partial}{\partial x_j}$, $x_j \frac{\partial}{\partial x_k} - \xi_k \frac{\partial}{\partial \xi_j}$, $x_j \frac{\partial}{\partial \xi_k} + x_k \frac{\partial}{\partial \xi_j}$, the values of which at each point $(x;\xi)$ with $\xi_1 = 1$ generate the linear subspace of $\mathbb{R}^{2n}$ tangent toM_0. It follows that the space of C^∞-vectors of the representation $\pi_{\nu,\delta}$ consists of C^∞ functions in the usual sense. This condition is of course not sufficient: there are conditions "at infinity" best rephrased by simply changing the hyperplane M_0 to an appropriate finite collection of hyperplanes M_S , as will be seen for instance in the proof of Lemma (6.6).

Proposition (6.3)[172]. When $\operatorname{Re} v_1 = \operatorname{Re} v_2 = n$ and $\operatorname{Re} v = -n$, the function $J^{\varepsilon_1,\varepsilon_2;\varepsilon}_{v_1,v_2;v}(Y_*, Z_*; X_*)$ as defined in (50) is a bounded function. One can extend its meaning as a distribution in $M_0 \times M_0 \times M_0$, holomorphic with respect to $v_1, v_2; v$ in the open subset of $\mathbb{C}^3$ defined, recalling (47) and (48), by the conditions

$$\frac{n+\nu-\nu_1+\nu_2}{2} \neq \varepsilon_3 + 1, \varepsilon_2 + 3, \ldots; \quad \frac{n+\nu+\nu_1-\nu_2}{2} \neq \varepsilon_1 + 1, \varepsilon_1 + 3, \ldots;$$
$$\frac{n-\nu-\nu_1-\nu_2}{2} \neq \varepsilon + 1, \varepsilon + 3, \ldots \qquad (63)$$

together with the fact that at least one of three following conditions should hold:

$$3n + \nu - \nu_1 - \nu_2 \neq \begin{cases} 1,3,\ldots \\ 2j+2, 2j+6,\ldots \end{cases} \textit{and } n + \nu \neq \delta + 1, \delta + 3, \ldots \quad (64)$$

or any of the conditions obtained from (64) by changing $(\nu, \nu_1, \nu_2; \delta, \delta_1, \delta_2)$ to$(-\nu_1, -\nu, \nu_2; \delta_1, \delta, \delta_2)$or to$(-\nu_2, \nu_1, -\nu; \delta_2, \delta_1, \delta)$. When n = 1, one can delete the condition $3 + \nu - \nu_1 - \nu_2 \neq 1,3, \ldots$ from (64).

Something entirely similar holds after one has replaced M_0 by M_S for an arbitrary $S \in \mathbb{R}^{2n} \setminus \{0\}$. In view of the inclusion $C^\infty(\pi_{\nu,\delta}) \subset C^\infty(M_0)$ and this will automatically make it a continuous trilinear form on the space of $(f_1, f_2, f) \in C^\infty(\pi_{\nu_1,\delta_1}) \times C^\infty(\pi_{\nu_2,\delta_2}) \times C^\infty(\pi_{-\nu,\delta})$. Setting, when ν_1, ν_2, ν satisfy (63) and (64), and f_1, f_2, f are C^∞ functions with compact support in M_0,

$$\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{\nu_1,\nu_2;\nu}(f_1, f_2; f)$$
$$= \int_{M_0 \times M_0 \times M_0} J^{\boldsymbol{\varepsilon_1,\varepsilon_2;\varepsilon}}_{\boldsymbol{\nu_1,\nu_2;\nu}}(Y_*, Z_*; X_*)\, f_1(Y_*) f_2(Z_*) f(X_*) dm(Y_*) dm(Z_*) dm(X_*) \quad (65)$$

one has the covariance relation

$$\mathbf{J}^{\boldsymbol{\varepsilon_1,\varepsilon_2;\varepsilon}}_{\boldsymbol{\nu_1,\nu_2;\nu}}\left(\pi_{\nu_1,\delta_1}(g) f_1, \pi_{\nu_2,\delta_2}(g) f_2; \pi_{-\nu,\delta}(g) f\right) = \mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{\nu_1,\nu_2;\nu}(f_1, f_2; f) \quad (66)$$

for every symplectic transformation g such that the transformed versions of f_1, f_2, f also have compact support in M_0.

Proof. The "integral" on the right-hand side of (65)is of course a usual notation for what is in effect the result of testing a certain distribution on the function $f_1 \otimes f_2 \otimes$

f. Before coming to the proof, let us indicate that one should not worry about the condition of compact support: one can dispense with it, only replacing the domain of integration $\mathrm{M}_0 \times \mathrm{M}_0 \times \mathrm{M}_0$ by a finite collection of domains $\mathrm{M}_S \times \mathrm{M}_S \times \mathrm{M}_S$.

When $\mathrm{Re}\, \mathrm{v}_1 = \mathrm{Re}\, \mathrm{v}_2 = \mathrm{n}$ and $\mathrm{Re}\, \mathrm{v} = -\mathrm{n}$, all exponents in definition (50) of $\mathbf{J}^{\boldsymbol{\varepsilon_1,\varepsilon_2;\varepsilon}}_{\boldsymbol{v_1,v_{2;v}}}(\mathrm{Y}_*, \mathrm{Z}_*; \mathrm{X}_*)$ have real part zero, so that the first point is obvious. To define when possible, in the distribution sense, complex powers of possibly vanishing functions can often be done by using Hironaka's desingularization theorem [159], in particular, when necessary (this will be the case here because we wish to find the poles as they appear in conditions (63) and (64) explicit blow-up transformations: the idea was used in general, and applied toward a shorter proof of a classical theorem in partial differential equations, in [156,158]. We shall use it here, following its use in the one-dimensional case in [163]. Recall that one can define the direct image of a distribution under any C^∞ proper map. Our point is to give products of signed powers of the three functions

$$\begin{aligned} \ell_1 &:= [Y_*, X_*] = x_1 - y_1 + \langle x_*, \eta_* \rangle - \langle y_*, \xi_* \rangle, \\ \ell_2 &:= [X_*, Z_*] = z_1 - x_1 + \langle z_*, \xi_* \rangle - \langle x_*, \zeta_* \rangle, \\ \ell_3 &:= [Z_*, Y_*] = y_1 - z_1 + \langle y_*, \zeta_* \rangle - \langle z_*, \eta_* \rangle \end{aligned} \tag{67}$$

a meaning for generic values of the parameters. Note that it is not necessary to desingularize fully the variety of zeros of the product $\ell_1 \ell_2 \ell_3$, only to reach a situation in which we are dealing locally with products of signed powers of functions with linearly independent differentials at common zeros.

Considering only the partial derivatives with respect to x_1, y_1, z_1, one observes that a linear relation between the differentials of these three functions cannot hold unless it consists in the fact that the sum of the three differentials is zero: computing then the partial derivatives with respect to ξ_*, η_*, ζ_*, finally with respect to x_*, y_*, z_*, one sees that the three differentials are linearly dependent if and only if $X_{**} = Y_{**} = Z_{**}$ with the notation given before

In the open set where this condition is not satisfied, one can complete the set of three functions under consideration into a local coordinate system in $\mathbb{R}^{2\mathrm{n}}$, and the proposition follows in this case from the following well-known fact from the theory of distributions in one variable: the function $\nu \longmapsto |x|_\delta^{-1-\nu}$, a locally summable function if $Re\, \nu < 0$, extends as a distribution valued holomorphic function of ν for $\nu \neq \delta, \delta + 2, ...$ This gives the distribution $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{\nu_1,\nu_2;\nu}$ a (local) meaning provided that $\frac{n+\nu+\nu_1-\nu_2}{2} \neq \varepsilon_1 + 1,\ \varepsilon_1 + 3, \dots$, $\frac{n+\nu-\nu_1+\nu_2}{2} \neq \varepsilon_2 + 1, \varepsilon_2 + 3, \dots and$ $\frac{n-\nu-\nu_1-\nu_2}{2} \neq \varepsilon + 1, \varepsilon + 3, ...$

When the condition $X_{**} = Y_{**} = Z_{**}$ is satisfied, saying that $[Z_*, Y_*]$ is zero is the same as saying that $y_1 = z_1$, and there are two analogous statements related to the last two equations. At points where none of the three functions under consideration

vanishes, there is of course no problem. Near points where only, say, the first function $[Z_*, Y_*]$vanishes, it can be taken as one of a set of local coordinates, and the distribution under examination makes sense whenever $\frac{n-\nu-\nu_1-\nu_2}{2} \neq \varepsilon+1, \varepsilon+3, \ldots$.
The only problem remains near points at which $X_{**} = Y_{**} = Z_{**}$ and $x_1 = y_1 = z_1$ $i.e., X_{**} = Y_{**} = Z_{**}$. We thus need to tame the three functions under consideration near a point such as(X_*^0, X_*^0, X_*^0), and there is no loss of generality in assuming that $X_*^0 = e_{n+1}th$ vector from the canonical basis of $\mathbb{R}^n \times \mathbb{R}^n$, since a symplectic transformation
preserving the linear form $X \longmapsto \xi_1$ can take us to this case.

We first replace the triple$(Y_*, Z_*, X_*) \in \mathrm{M}_0 \times \mathrm{M}_0 \times \mathrm{M}_0$ by the set of points $(T_1, T_2; x_1; Y_{**}, Z_{**}, X_{**})$ in $\mathbb{R}^2 \times \mathbb{R} \times (\mathbb{R}^{2n-2})^3$, with

$$T_1 = \ell_1(Y_*, Z_*, X_*), \quad T_2 = \ell_2(Y_*, Z_*, X_*). \tag{68}$$

That these equations define, near (X_*^0, X_*^0, X_*^0), an admissible new set of coordinates, follows the fact thatℓ_1 and ℓ_2 have linearly independent partial differentials with respect to the pair (y_1, z_1). Next, we blow up the (T_1, T_2)-plane around 0, replacing it by the subspace $\widehat{\mathbb{R}^2}$ of $P_1(\mathbb{R}) \times \mathbb{R}^2$ consisting of pairs (τ, T) such that, in the case when $T \neq 0, \tau$ is the image of T under the canonical projection map $p: \mathbb{R}^2 \backslash \{0\} \to P_1(\mathbb{R})$. Generally setting $\tau = p(\theta)$, the domain ω_j of $P_1(\mathbb{R})$ characterized by the condition $\theta_j \neq 0$ gives rise to the domain Ω_j of $\widehat{\mathbb{R}^2}$ consisting of pairs (τ, T) such that either $T_j \neq 0$ and $p(T) = \tau$ or $T = 0$ and $\tau \in \omega_j$. The domains Ω_1 and Ω_2 cover $\widehat{\mathbb{R}^2}$ and taking in Ω_1 the set of coordinates

$$(\tau_2, T_1) = \left(\frac{\theta_2}{\theta_1}, T_1\right), \tag{69}$$

and in Ω_2 the set of coordinates

$$(\tau_1, T_2) = \left(\frac{\theta_1}{\theta_2}, T_2\right), \tag{70}$$

one turns $\widehat{\mathbb{R}^2}$ into a smooth manifold. The projection map $\emptyset: (\tau, T) \mapsto T$ is proper since the inverse image of a point $T \neq 0$ reduces to the point $(p(T), T)$, while that of 0 is $\Sigma = P_1(\mathbb{R}) \times \{0\}$.

In Ω_1, one has $\ell_1 = T_1, \ell_2 = \tau_2 T_1$, so that the pullbacks in $\widehat{\mathbb{R}^2} \times \mathbb{R} \times (\mathbb{R}^{2n-2})^3$ of the three functions under consideration express themselves as

$$\ell_1^\# = T_1,$$
$$\ell_2^\# = \tau_2 T_1,$$
$$\ell_3^\# = -(1+\tau_2)T_1 + [X_{**}, Y_{**} - Z_{**}] - [Y_{**}, Z_{**}]. \tag{71}$$

The differentials of $\ell_1^\# 1$ and $\ell_2^\#$ are not linearly independent when $T_1 = 0$, but the differentials of T_1 and τ_2 are, which is sufficient as a start. We must now insert a lemma, in order to take care of the extra terms in $\ell_3^\#$.

Lemma $(\mathbf{6.4})[\mathbf{172}]$. Consider on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ the function

$$F(Y, Z, X) = [X, Y - Z] - [Y, Z], \tag{72}$$

which is critical exactly at points $(-X^0, -X^0, -X^0)$, where it vanishes. Consider the blow-up $\widehat{\mathbb{R}^{6n}}$ of $\mathbb{R}^{6n}$ at such a point, and the pullback $\widetilde{\boldsymbol{F}}$ in $\widehat{\mathbb{R}^{6n}}$ of the function $\boldsymbol{F}$. Locally around any point lying in the inverse image of $(-X^0, -X^0, -X^0)$,, one can find two smooth real-valued functions R and S such that $\widetilde{\boldsymbol{F}}$ expresses itself as RS^2.

Proof. First, observe the identity

$$F(-X^0 + Y, -X^0 + Z, X^0 + X) = F(Y, Z, X), \tag{73}$$

so that there is no loss of generality in assuming that $X^0 = 0$. The space $\widehat{\mathbb{R}^{6n}}$ obtained as the result of blowing up $\mathbb{R}^{6n}$ around 0 is covered by a family $(\Omega_j)1 \leq j \leq 6n$ of open sets with the following properties: for each j, there is a function S_j taken from the set of canonical coordinates of one of the three vectors Y, Z, X such that, within Ω_j, the equation $S_j = 0$ defines the inverse image $P_{6n-1}(\mathbb{R}) \times \{0\} of$ $0 \in \mathbb{R}^{6n}$; next, there is a set of smooth vector-valued functions $\dot{Y}, \dot{Z}, \dot{X}$, each of which has $2n$ components, such that the identities $Y = S_j \dot{Y}$, $Z = S_j \dot{Z}$, $X = S_j \dot{X}$ hold, and such that, deleting from the set of components of the vectors $\dot{Y}, \dot{Z}, \dot{X}$ the coordinate which, of necessity, is the constant 1, one obtains a family of functions which, when completed by the function S_j, constitutes an admissible set of coordinates in Ω_j. Then, one may write

$$\tilde{F}(S_j, \dot{Y}, \dot{Z}, \dot{X}) = S_j^2([\dot{X}, \dot{Y} - \dot{Z}] - [\dot{Y}, \dot{Z}]), \tag{74}$$

and it suffices to observe that the second factor is a function without critical point. Indeed, assuming for instance that the coordinate S_j has been taken from the components of Y (it would be fully similar if it had been taken from any of the other two remaining vectors), the equation$(\dot{Y})_j = 1$ shows that the partial derivatives of $\tilde{\phi}$ with respect to the coordinates in $\dot{X}$ or $\dot{Z}$ "conjugate with respect to the symplectic form" to $(\dot{Y})_j$ are not zero.

End of proof of Proposition (6.1.3). Applying Lemma (6.1.4) with n-1 substituted for n, we may rewrite (71), more precisely the pullbacks of the three functions there to a new blown-up space, as

$$\begin{aligned} \ell_1^{\#\#} &= T_1, \\ \ell_2^{\#\#} &= \tau_2 T_1, \\ \ell_3^{\#\#} &= -(1 + \tau_2)T_1 + RS^2, \end{aligned} \tag{75}$$

where the four functions T_1, τ_2, R, S have linearly independent differentials.

The differential $d\ell_3^{\#\#}$ is a linear combination of $d\ell_1^{\#\#}$ and $d\ell_3^{\#\#}$exactly at points where $S = 0$, but let us not forget the origin (69) of the coordinate T_1, which implies that there is no loss of generality in assuming that we are near a point where $T_1 = 0$ as well.

In the open set where $1+\tau_2$ does not vanish, we may take $\ell_3^{\#\#}$ to the form $-T_1+RS^2$, and we blow up the plane of the variables T_1, S around 0: this amounts, with new variables, to setting in appropriate domains either $S = T_1 S^{'}\, or\, T_1 = ST_1^{'}$ finding either $-T_1 + RS^2 = T_1(-1+RT_1 S^{'2})\, or\, -T_1+RS^2 = S(-T_1^{'}+RS)$. In the first case we are dealing with a pair of functions, the first of which is T_1 and the second is the product of T_1 by a function which, at points where it vanishes, has a differential linearly independent from dT_1. In the second case, we still have to desingularize the pair of functions $(ST_1^{'}, S(-T_1^{'}+RS))$ or, setting aside the factors S in the product of signed powers to be analyzed, the triple of functions $(S, T_1^{'}, -T_1^{'} + RS)$. Again, we blow up the $(T_1^{'}, S)$-space, which amounts to setting either $S = T_1^{'}S''$, in which case the triple becomes $(T_1^{'}S^{''}, T_1^{'}, T_1^{'}(-1+RS^{''}))$, or $T_1^{'} = ST_1^{''}$, in which case the triple becomes $(S, ST_1^{''}, S(-T_1^{''}+R))$, a satisfactory situation.

Finally, we must place ourselves near a point where T_1 and $1+\tau_2$ vanish.We may then forget about $\ell_2^{\#\#}$ entirely, and we blow up the variables $T_1, 1+\tau_2, S$ near 0. In local charts, this makes up one of the three following possibilities:

$$\begin{aligned} 1+\tau_2 = T_1\sigma_2, S = T_1 S^{'}, \qquad & \ell_3^{\#\#\#} = T_1^2(-\sigma_2 + RS^{'2}), \\ T_1 = (1+\tau_2)T_1^{'}, S = (1+\tau_2)S^{'}, \qquad & \ell_3^{\#\#\#} = (1+\tau_2)^2(-T_1^{'} + RS^{'2}), \\ T_1 = ST_1^{'}, 1+\tau_2 = S\sigma_2, \qquad & \ell_3^{\#\#\#} = S^2(-\sigma_2 T_1^{'} + R). \end{aligned} \tag{76}$$

In the first (resp. third) case, a product of signed powers of T_1 and $\ell_3^{\#\#\#}$ becomes a product of signed powers of T_1 and $-\sigma_2 + RS^{'2}$ (resp. a product of signed powers of S, of $T_1^{'}$ and $-\sigma_2 T_1^{'} + R$, a satisfactory situation since we are dealing in each case with two functions with linearly independent differentials. This is not the case on the second line, in which, after leaving the factors $1+\tau_2$ aside, we have to consider the pair of functions $T_1^{'}$ and $-T_1^{'} + RS^{'2}$: these do not have linearly independent differentials; however, this pair can be desingularized since we are back to the situation examined above, relative to the pair $(T_1, -T_1 + RS^2)$.

We are now in a position to define locally the distribution $\mathbf{J}_{\nu_1,\nu_2;\nu}^{\varepsilon_1,\varepsilon_2;\varepsilon}$ as the direct image, under a proper map, of a distribution of the kind

$$|\ell_1^{\#}|_{\varepsilon_2}^{\frac{-n-\nu+\nu_1-\nu_2}{2}}\ |\ell_2^{\#}|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\ |\ell_3^{\#}|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\ , \tag{77}$$

where the factors $\ell_1^{\#}, \ell_2^{\#}, \ell_3^{\#}$ really denote the initial functions ℓ_1, ℓ_2, ℓ_3 after they have been pulled back in one of the appropriate ways just described: only, we here dispense with the collection of $\#$ superscripts which has been used before in order to keep track of the number of blow-ups needed. In case the reader should worry about it, the fact that the subscript ε_2 should be associated to ℓ_1, not ℓ_2, is not a blunder: the index δ_1 is actually that which must be associated to ℓ_1, and we recall (47). The important fact is that, in local charts, the functions $\ell_1^{\#}, \ell_2^{\#}, \ell_3^{\#}$ are all built as powers of the same set of functions with linearly independent differentials. Recall from (49) that

$$\alpha_1 = \frac{-n-v+v_1-v_2}{2}, \quad \alpha_2 = \frac{-n-v-v_1+v_2}{2}, \quad \alpha_3 = \frac{-n+v+v_1+v_2}{2}. \qquad (78)$$

To find the poles, as a distribution-valued function of $v_1, v_2; v$, of the distribution (78), we must go back to the desingularizing operations and keep track of the signed powers involved in each case, starting from the fact that $|f|_\delta^{-1-\mu}$ makes sense as a distribution, assuming that f has no critical zero, when $\mu \neq \delta, \delta+2, \ldots$. As already said, when none of the three functions ℓ_1, ℓ_2, ℓ_3 vanishes, there is of course no condition on the exponents involved, and when just one of them vanishes (the case discussed between (67) and (68)), we must assume

$$-\alpha_1 \neq \varepsilon_2 + 1, \varepsilon_2 + 3, \ldots; \quad -\alpha_2 \neq \varepsilon_1 + 1, \varepsilon_1 + 3, \ldots;$$
$$-\alpha_3 \neq \varepsilon + 1, \varepsilon + 3, \ldots \qquad (79)$$

Next, we go to our discussion following (75). Forgetting the factors without zeros, the product of signed powers we are led to is of one of the following species, in which we introduce the new letter $V, S'', T_1'', \ldots$ for each of the functions, with differentials independent from the other ones at points where they vanish, such as $-1 + RT_1S'^2$, which have appeared in the discussion:

$$|T_1|_{\varepsilon_2}^{\alpha_1} \ |\tau_2 T_1|_{\varepsilon_1}^{\alpha_2} \ |T_1 V|_{\varepsilon}^{\alpha_3} \quad or$$
$$|T_1' S'' T_1'|_{\varepsilon_2}^{\alpha_1} \ |\tau_2 T_1' S'' T_1'|_{\varepsilon_1}^{\alpha_2} \ |T_1' S'' T_1' V|_{\varepsilon}^{\alpha_3} \quad or$$
$$|S^2 T_1''|_{\varepsilon_2}^{\alpha_1} \ |\tau_2 S^2 T_1''|_{\varepsilon_1}^{\alpha_2} \ |S^2 V|_{\varepsilon}^{\alpha_3} \quad or$$
$$|T_1|_{\varepsilon_2}^{\alpha_1} \ |\tau_2 T_1|_{\varepsilon_1}^{\alpha_2} \ |T_1^2 V|_{\varepsilon}^{\alpha_3} \quad or$$
$$|S T_1'|_{\varepsilon_2}^{\alpha_1} \ |S T_1'|_{\varepsilon_1}^{\alpha_2} \ |S^2 V|_{\varepsilon}^{\alpha_3} \quad or$$
$$|1+\tau_2|_{\varepsilon_2}^{\alpha_1} \ |1+\tau_2|_{\varepsilon_1}^{\alpha_2} \ |(1+\tau_2)^2|_{\varepsilon}^{\alpha_3} \ |T_1'' S'' T_1''|_{\varepsilon_2}^{\alpha_1} \ |T_1'' S'' T_1''|_{\varepsilon_1}^{\alpha_2} \ |T_1'' S'' T_1'' V|_{\varepsilon}^{\alpha_3} \ or$$
$$|1+\tau_2|_{\varepsilon_2}^{\alpha_1} \ |1+\tau_2|_{\varepsilon_1}^{\alpha_2} \ |(1+\tau_2)^2|_{\varepsilon}^{\alpha_3} \ |S'^2 T_1''|_{\varepsilon_2}^{\alpha_1} \ |S'^2 T_1''|_{\varepsilon_1}^{\alpha_2} \ |S'^2 V|_{\varepsilon}^{\alpha_3} \quad (80)$$

Besides, we must not forget that all these local forms are only available in some domains above parts of Ω_1, not Ω_2 (cf. (69)), so we must complete the preceding list with the one obtained from it by exchanging the two pairs (ε_2, v_1) and (ε_1, v_2). All lines are treated in the same way: let us consider the last one, which happens to make all possible demands on the exponents, and let us rewrite it as

$$|1+\tau_2|_{\varepsilon_1+\varepsilon_2 mod\,2}^{\alpha_1+\alpha_2} \, |1+\tau_2|^{\alpha_3} \, |S'|^{2(\alpha_1+\alpha_2+\alpha_3)} \, |T_1''|_{\varepsilon_1+\varepsilon_2 mod\,2}^{\alpha_1+\alpha_2} \, |V|_{\varepsilon}^{\alpha_3}. \qquad (81)$$

Since $\varepsilon_1 + \varepsilon_2 + \varepsilon \equiv j \ mod\ 2$, this can be written as

$$|1+\tau_2|_{j}^{\alpha_1+\alpha_2+\alpha_3} \, |1+\tau_2|_{\varepsilon}^{\alpha_3} \, |S'|^{2(\alpha_1+\alpha_2+\alpha_3)} \, |T_1''|_{\varepsilon_1+\varepsilon_2 mod\,2}^{\alpha_1+\alpha_2} \, |V|_{\varepsilon}^{\alpha_3}. \qquad (82)$$

Now, one has

$$\alpha_1+\alpha_2+\alpha_3 = \frac{-3n-v+v_1+v_2}{2}, \quad \alpha_1+\alpha_2 = -n-v,$$
$$\varepsilon_1+\varepsilon_2 \equiv j+\varepsilon \equiv \delta \ mod\ 2, \qquad (83)$$

so that, besides the conditions (79), it suffices to assume moreover that

$$\frac{3n+v-v_1-v_2}{2} \neq j+1\,, j+3, \ldots, \quad 3n+v-v_1-v_2 \neq 1+3, \ldots, \tag{84}$$

and that $n+v \neq \delta+1, \delta+3, \ldots$.

These conditions are clearly invariant under the exchange of pairs (ε_2, ν_1) and (ε_1, ν_2). They are not fully necessary: the reason for this is that, in our desingularization procedure, we have started with giving the pair (ℓ_1, ℓ_2)special consideration, while we might just as well startedfrom giving the pair (ℓ_2, ℓ_3) or (ℓ_3, ℓ_1) special consideration. This takes us to the assumptions in Proposition (6.1.3), not forgetting that in the one-dimensional case, the desingularization process stops at (71).

The rest of the proof is trivial.

We shall also need the following result, in the same spirit as Proposition (6.3), though of course its proof presents no difficulty.

Proposition (6.5)[172]. Set, assuming

$$-\rho \neq \delta+1, \delta+3, \ldots \text{and } \rho \neq \delta, \delta+2, \ldots,$$

$$c(\rho,\delta) = (-i)^{\delta} \pi^{-\frac{1}{2}-\rho} \, \frac{\Gamma\left(\frac{\rho+1+\delta}{2}\right)}{\Gamma\left(\frac{-\rho+\delta}{2}\right)}\,, \tag{85}$$

so that one should have, in one dimension,

$$\left(\mathcal{F}\left(|S|_{\delta}^{\rho}\right)\right)(\sigma) = c(\rho,\delta)|\sigma|_{\delta}^{-\rho-1} \tag{86}$$

(of course, we are using here the usual Fourier transformation, with integral kernel $e^{-2i\pi s\sigma}$: there is no symplectic Fourier transformation on an odd-dimensional space). Recalling (36), consider the integral kernel

$$k_{\nu,\delta}(x,\xi_*;y,\eta_*) = (-1)^{\delta} c(n-1-v,\delta)|x_1-y_1+\langle x_*,\eta_*\rangle - \langle y_*,\xi_*\rangle|_{\delta}^{-n+v}. \tag{87}$$

When $-n < Re\, \nu < 1-n$, this is the integral kernel of an operator $\theta_{\nu,\delta}$ well defined, in the weak sense, from the space of C^{∞} vectors of the representation $\pi_{\nu,\delta}$ to the dual of that space (which contains the space of C^{∞} vectors of the representation $\pi_{-\nu,\delta}$). As an operator-valued function of $\nu, \theta_{\nu,\delta}$ extends as a holomorphic function in $\mathbb{C}\backslash P$, where the set P consists of the values ν such that $-n+\nu = \delta, \delta+2, \ldots$ or $n-\nu = \delta+1, \delta+3, \ldots$. The operator $\theta_{\nu,\delta}$ is an intertwiner from the representation $\pi_{\nu,\delta}$ to the representation $\pi_{-\nu,\delta}$. When $\nu \in i\mathbb{R}$, it coincides with the one introduced in another way in Definition (6.2).

The latter way to define the operator $\theta_{i\lambda,\delta}$ has the advantages, especially in the version (33), that on one hand it continues to be meaningful after $v \in \mathbb{C}$ has been substituted for $i\lambda$, on the other hand that it extends to a (tempered) distribution setting: but this requires that the homogeneous functions, or distributions, under consideration, should have a well-defined meaning as distributions in $\mathbb{R}^{2n}$, not only as functions, or distributions, in $\mathbb{R}^{2n} \setminus \{0\}$.

Hyperplane waves and rays

We decompose here symbols as integral superpositions of homogeneous hyperplane waves, also of homogeneous rays, by which we mean homogeneous measures carried by straight lines through the origin of $\mathbb{R}^{2n}$. With the help of such decompositions, we shall transform, the triple product studied in a way crucial towards the proof of the main theorem.

Consider the transformation $\mathcal{G}$, a rescaled version of the symplectic Fourier transformation
(also a unitary involution of $L^2(\mathbb{R}^{2n})$) defined as

$$(\mathcal{G}h)(X) = 2^n \int_{\mathbb{R}^{2n}} h(Y)\ e^{-4i\pi[X,Y]}\, dy: \tag{88}$$

part of our interest in this transformation [165] is that, for every $\mathfrak{S} \in S'(\mathbb{R}^{2n})$, the distribution $\mathcal{G}\mathfrak{S}$ is the Weyl symbol of the operator $u \mapsto Op(\mathfrak{S})\check{u}$, where $\check{u}(x) = u(-x)$. If a symbol $h = h(x;\xi)$ depends only on ξ_1, say $h(x;\xi) = \phi(\xi_1)$, it is immediate that $(\mathcal{G}h)(x;\xi) = 2\hat{\phi}(-2x_1)\delta(x_*)\delta(\xi\,)$: in other words, $\mathcal{G}h$ is the measure carried by the line $\{te_1 \colon t \in \mathbb{R}\}$, with density $2\hat{\phi}(-2t)dt$. More generally, if $S \in \mathbb{R}^{2n}\setminus\{0\}$, setting $S = ge_1$ with $g \in Sp(n,\mathbb{R})$, the $\mathcal{G}$ transform of the hyperplane wave $X \mapsto \phi([X,S])$ is the measure carried by the line $\{tS \colon t \in \mathbb{R}\}$, with density $2\hat{\phi}(-2t)dt$.

In particular, for any $\rho \in \mathbb{C}, -\rho\ \neq \delta +\ 1, \delta+3, \ldots$, we shall denote as $\mu s(\rho,\delta)$ the measure carried by the line $\{tS \colon t \in \mathbb{R}\}$, with density $|t|_\delta^\rho\, dt$. Recalling the definition (85) of $c(\rho,\delta)$, we have, provided that $n + v \neq \delta + 1, \delta + 3, \ldots$ and $-n - v \neq \delta, \delta + 2, \ldots$,

$$\mathcal{G}(X \mapsto |[X,S]|_\delta^{-n-v}) = (-1)^\delta\, 2^{\mathrm{v}} \mathrm{c}(-\mathrm{n} - \mathrm{v}, \delta) \mu s(n - 1 + v, \delta). \tag{89}$$

Note that the measure $\mu s(\rho,\delta)$ is a homogeneous distribution of type $(\rho + 1 - 2n, \delta)$ (do not forget that, in $\mathbb{R}^{2n-1}$, the Dirac mass at the origin is homogeneous of degree $1 - 2n$).

Let us first decompose functions in $S(\mathbb{R}^{2n})$ into homogeneous hyperplane waves. Start from the continuation of (18), to wit

$$h_{v,\delta}(X) = \frac{1}{4\pi} \int_{-\infty}^{\infty} |t|_\delta^{n-1+v}\ h(tX)\, dt\,, \tag{90}$$

where the integral converges for every $X \neq 0$ provided that $Re\ v > -n$. In this case, the function $h_{v,\delta}$ is, as we now show, a C^∞ vector of the representation $\pi_{v,\delta}$. With $X_* = (x;1,\xi_*)$, one has for every N the inequality $|h(tX_*)| \le C(1 + |t|)^{-N}(1 +$

$|x| + |\xi_*|)^{-N}$ for some constant C: then, with the norm defined in (55), one has $\|X_* \mapsto h(tX_*)\|_v \leq C(1+|t|)^{-N}$, from which one obtains, since $Re(n-1+v) > -1$, that the function $h_{v,\delta}$ lies in the Hilbert space $\mathcal{H}_v$ defined in association with this norm. That it is a C^∞ vector of the representation $\pi_{v,\delta}$ follows from the fact that this representation corresponds, under the transformation (90) from $h \; to \; h_{v,\delta}$, to the phase space representation of $Sp(n,\mathbb{R}) \; in \; S(\mathbb{R}^{2n})$.

In the case when, moreover, $Re\ v < 1-n$, one may write

$$\pi_{v,\delta} = 2^n \int_{-\infty}^{\infty} |t|_\delta^{n-1+v}\ dt \int_{\mathbb{R}^{2n}} e^{-4i\pi t\,[X,S]}\ (\mathcal{G}h)(S)dS$$

$$= \frac{2^{-v}}{4\pi}\, c(n-1+v,\delta) \int_{\mathbb{R}^{2n}} |[X,S]|_\delta^{-n-v}\,(\mathcal{G}h)(S)dS\,, \qquad (91)$$

which leads to the decomposition of h into homogeneous hyperplane waves if coupled with the equation

$$h = \sum_{\delta=0,1} \int_{Re v=a} h_{v,\delta}\,\frac{dv}{i}, \qquad (92)$$

in which $-n < a < 1-n$. From (17), however, the line of integration we are particularly interested in is the pure imaginary line, for which this decomposition is just the spectral decomposition of h relative to the (self-adjoint) operator $\boldsymbol{\varepsilon}\ in\ L^2(\mathbb{R}^{2n})$. Starting from (91) and moving the set of values of v, we certainly reach, for fixed S, poles of the distribution-valued function $v \mapsto |[X,S]|_\delta^{-n-v}$, at points $v = -n+\delta+1, v = -n+\delta+3, \ldots$, but these poles are simple, and disappear after multiplication by the factor $c(n-1+v,\delta)$, as seen from (85). This makes it possible to continue the decomposition of h into homogeneous hyperplane waves up to the spectral line.

Starting from $\mathcal{G}h$ in place of h and noting that $(\mathcal{G}h)_{-v,\delta} = \mathcal{G}h_{v,\delta}$, one obtains also, if $Re\ v < n$,

$$h_{v,\delta} = \frac{2^v}{4\pi} c(n-1-v,\delta) \int_{\mathbb{R}^{2n}} h(S)\,\mathcal{G}(X_* \mapsto |[X,S]|_\delta^{-n+v})dS$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^{2n}} h(S)\mu_S\,(n-1-v,\delta)dS, \qquad (93)$$

after one has used the equation

$$(-1)^\delta c(\rho,\delta)c(-\rho-1,\delta) = 1: \qquad (94)$$

this leads to a decomposition of h into rays if coupled with the equation

$$h = \sum_{\delta=0,1} \int_{Re\ a} h_{-\nu,\delta} \frac{d\nu}{i}, \qquad (95)$$

in which, starting from a value of a between $-n$ and $1-n$, we can actually take $a = 0$ when so desired.

The following lemma will enable us to deal with multipliers of the species which occurs consistently in the present work.

Lemma $(\mathbf{6.6})[\mathbf{172}]$. Let $S \in \mathbb{R}^{2n}\backslash\{0\}. If\ \varepsilon, \delta = 0\ or\ 1\ and\ \alpha, \nu \in \mathbb{C}$ satisfy the condition $-\frac{1}{2} < Re\ \nu < \frac{1}{2} + Re\ \nu$, the multiplication by the function $X_* \mapsto |[S, X_*]|_{\varepsilon}^{\alpha}$ sends the space $C^{\infty}(\pi_{\nu,\delta})\ of\ C^{\infty}$ vectors of the representation $\pi_{\nu,\delta}$ to the space $L^2(\mathrm{M}_0)$.

Proof. It is no loss of generality to assume that $S = e_{n+1}, i.e., [S, X_*] = x_1$. Given $f \in C^{\infty}(\pi_{\nu,\delta})$ extending to $\mathbb{R}^{2n} \setminus \{0\}$ as a function $f^{\#} of\ type\ (-n - \nu, \delta)$, the function

$$k(x;\xi) = |x_1|_{\varepsilon}^{\alpha}\ |\xi_1|_{\varepsilon+\delta mod\ 2}^{\nu-\alpha}\ f^{\#}(x;\xi) \qquad (96)$$

is of type $(-n, 0)$. Since the corresponding representation $\pi_{0,0}$ preserves the Hilbert space $L^2(\mathrm{M}_0)$, it suffices, to check that the restriction of the function k, to M_0 lies in the space $L^2_{loc}(\mathrm{M}_0)$which leads to the two conditions indicated.

We now come back to a study of the bilinear operator $(f_1, f_2) \mapsto \mathbf{J}_{\nu_1,\nu_2;\nu}^{\varepsilon_1,\varepsilon_2;\varepsilon}(f_1, f_2)$, or of the associated triple product obtained when testing this distribution against $f \in C^{\infty}(\pi_{-\nu,\delta})$. Recall that such expressions can also use as arguments objects with the proper type defined in $\mathbb{R}^{2n}\backslash\{0\}$ rather than their restrictions to M_0, the distinction being purely notational. We shall eventually assume, but not at one stroke, that

$$f_1 = (h_1)_{\nu_1,\delta_1}, \quad f_2 = (h_2)_{\nu_2,\delta_2}, \quad f = h_{-\nu,\delta} \qquad (97)$$

for a triple of functions $h_1, h_2, h \in S(\mathbb{R}^{2n})$.

Lemma $(\mathbf{6.7})[\mathbf{172}]$. Assume that $h_2 \in S(\mathbb{R}^{2n})$ and that all hypotheses of Proposition (6. 3) are valid. Moreover, assume that $Re\ \nu_2 < n$ and that

$$Re\ (\nu - \nu_1 + \nu_2) = n, \quad Re\ \nu_1 > -\frac{1}{2}, \quad Re\ \nu < \frac{1}{2}. \qquad (98)$$

If $f_1 \in C^{\infty}(\pi_{\nu_1,\delta_1})$, one has in the weak sense, i.e., when integrated against $f(X_*)dm(X_*)$ for some $f \in C^{\infty}(\pi_{-\nu,\delta})$,

$$\mathbf{J}_{\nu_1,\nu_2;\nu}^{\varepsilon_1,\varepsilon_2;\varepsilon}\left(f_1, (h_2)_{\nu_2,\delta_2}\right)(X_*)$$

$$= \frac{1}{4\pi} \frac{(-1)^{\varepsilon_2}}{C\left(\frac{n-2+\nu-\nu_1+\nu_2}{2}, \varepsilon_2\right)} \int_{\mathbb{R}^{2n}} h_2(S)dS$$

$$\times |[X_*, S]|_{\varepsilon_1}^{-n-\nu-\nu_1+\nu_2} \left[\theta_{\frac{n-\nu+\nu_1-\nu_2}{2},\varepsilon_2}\left(Y_* \mapsto |[S, Y_*]|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}} f_1(Y_*)\right)\right](X_*). \qquad (99)$$

Proof. First, we observe, as a consequence of Lemma (6.6), that, under the conditions (98), the multiplication by the function $Y_* \longmapsto |[S,Y_*]|_{\varepsilon}^{\frac{-n+v+v_1+v_2}{2}}$ sends the space $C^\infty(\pi_{v_1,\delta_1})$ to the space $L^2(\mathrm{M}_0)$ and that the multiplication by the function $X_* \longmapsto |[X_*,S]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}}$ sends the space $L^2(\mathrm{M}_0)$ to the space of distributions $C^{-\infty}(\pi_{v,\delta})$, the topological dual of $C^\infty(\pi_{-v,\delta})$ (*i.e.*, the linear space of continuous linear forms on that space). On the other hand, the first condition (98) gives the intertwining operator $\theta_{\frac{n-v+v_1-v_2}{2},\varepsilon_2}$ a meaning as a unitary operator in $L^2(\mathrm{M}_0)$, so that the right-hand side of the equation to be proved is meaningful.

If one makes there the integral kernel of the operator $\theta_{\frac{n-v+v_1-v_2}{2},\varepsilon_2}$ explicit, as $(-1)^{\varepsilon_2} C\left(\frac{n-2+v-v_1+v_2}{2},\varepsilon_2\right) |[Y_*,X_*]|_{\varepsilon_2}^{\frac{-n-v+v_1-v_2}{2}}$, then if one sets $\mathrm{S}=sZ_*$, *so that*

$$|S|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}}\ |S|_{\varepsilon}^{\frac{-n+v+v_1+v_2}{2}}\ dS = |S|_{\delta_2}^{n-1+v_2}\ ds\ dm(Z_*), \tag{100}$$

and if one uses the equation

$$(h_2)_{v_2,\delta_2}(X) = \frac{1}{4\pi}\int_{-\infty}^{\infty} |S|_{\delta}^{n-1+v_2}\ h_2(sX)ds, \tag{101}$$

one transforms the right-hand side of (99) into the left-hand side. However, the operator on the left-hand side has been defined with the help of the desingularization of its integral kernel as done before, while on the right-hand side, the claimed unitarity of the intertwining operator into consideration is a consequence of Definition (6.2): to identify the two ways to introduce it, one must use again the connection between (35) and (36).

Let us rewrite (99), as tested against , with

$$f(X_*)h_{-v,\delta}\ (X_*) = \frac{1}{4\pi}\int_{-\infty}^{\infty} |t|_{\delta}^{n-1-v}\ h(t\,X_*)dt. \tag{102}$$

One has

$$\langle \mathbf{J}_{v_1,v_2;v}^{\varepsilon_1,\varepsilon_2;\varepsilon}\ (f_1,(h_2)_{v_2,\delta_2}),h_{-v,\delta}\rangle = \frac{1}{(4\pi)^2}\,\frac{(-1)^{\varepsilon_2}}{C\left(\frac{n-2+v-v_1+v_2}{2},\varepsilon_2\right)}$$

$$\times \int_{\mathbb{R}^{2n}} h_2(S)\ \langle\, \mathcal{F}\left(\mathrm{Y}\longmapsto |[S,Y]|_{\varepsilon}^{\frac{-n+v+v_1+v_2}{2}}\ f_1(Y)\right), T\longmapsto |[T,S]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} h(T)\rangle\,\mathrm{dS}: \tag{103}$$

note that the two pairs of brackets ⟨, ⟩do not denote the same pairings: on the left-hand side, it corresponds to the duality between $C^{-\infty}(\pi_{v,\delta})$ and $C^\infty(\pi_{-v,\delta})$; within the integrand on the right-hand side, it corresponds to the one between $S^{'}(\mathbb{R}^{2n})$ and $S(\mathbb{R}^{2n})$. To prove this, we start from the right-hand side, expressing the intertwining operator there as a Fourier transformation. The function

$$T \mapsto |[T,S]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} \mathcal{F}\left(\mathrm{Y} \mapsto |[S,T]|_{\varepsilon}^{\frac{-n+v+v_1+v_2}{2}} f_1(Y)\right)(\mathrm{T}) \quad (104)$$

is of type (recalling (47))

$$\left(\frac{n-v-v_1+v_2}{2}, \varepsilon_1\right) + (-2\mathrm{n}, 0) + \left(\frac{n-v-v_1-v_2}{2}, \varepsilon\right) + (\mathrm{n} + v_1, \delta) = (-n-v, \delta). \quad (105)$$

Set $T = tX_*$, so that $dT = |t|^{2n-1} dt\, dm(X_*)$: then, the right-hand side of (103) transforms into the left-hand side in view of (105) and (102).

As a last step, we now use the decomposition

$$(h_1)_{v_1,\delta_1}(Y) = \frac{2^{-\nu_1}}{4\pi} \mathrm{c}(\mathrm{n}-1+\nu_1, \delta_1) \int_{\mathbb{R}^{2n}} (\mathcal{G}\mathrm{h}_1)(\mathrm{R}) |[\mathrm{Y},\mathrm{R}]|_{\delta_1}^{-\mathrm{n}-\nu_1} \mathrm{dR} \quad (106)$$

of $f_1 = (h_1)_{v_1,\delta_1}$, as provided by (100).

Proposition $(\mathbf{6.8})[\mathbf{172}]$**.** Assume that all hypotheses from Proposition (6.3) are satisfied and that, moreover,

$$v + \nu_1 \neq \delta_2, \delta_2 + 2, \ldots, \frac{-n+v+v_1+v_2}{2} \neq \varepsilon, \varepsilon + 2, \ldots, \frac{2-n-v+v_1+v_2}{2} \neq \varepsilon_2 + 1, \varepsilon_2 + 3, \ldots \quad (107)$$

and

$$Re\, v_1 > -n,\ Re\, v_2 < n,\ Re\, v < n. \quad (108)$$

Then,

$$\langle \mathbf{J}_{v_1,v_2;v}^{\varepsilon_1,\varepsilon_2;\varepsilon} ((h_1)_{v_1,\delta_1}, (h_2)_{v_2,\delta_2}), h_{-v,\delta} \rangle = \frac{(-1)^{\varepsilon_2} 2^{-\boldsymbol{v_1}}}{(4\pi)^3} \frac{C\left(\frac{-n+v+v_1+v_2}{2}, \varepsilon_2\right)}{C\left(\frac{n-2+v-v_1+v_2}{2}, \varepsilon_2\right)} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} (\mathcal{G}h_1)(R) h_2(S) |[\mathrm{R},\mathrm{S}]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} \mathrm{dRdS}$$

$$\times \int_{\mathbb{R}^2} |\mathrm{r}|_j^{\frac{n-2-v+v_1+v_2}{2}} |\mathrm{s}|_{\varepsilon}^{\frac{n-2-v-v_1-v_2}{2}} \mathrm{h}(\mathrm{rR}+\mathrm{sS}) \mathrm{drds}, \quad (109)$$

where the last integral must be understood in the distribution sense: recall that j was defined in (48).

Proof. First, write the equation, of immediate verification,

$$\mathcal{F}\left((\mathrm{y};\eta) \mapsto |-\mathrm{y}_1|_{\delta_1}^{-\mathrm{n}-\mathrm{v}_1} |-\eta_1|_{\varepsilon}^{\frac{-n+v+v_1+v_2}{2}}\right)(\mathrm{t}_1, \mathrm{t}_*; \tau_1, \tau_*)$$

$$= (-1)^{\delta_1} c(-n - v_1, \delta_1) c\left(\frac{-n+v+v_1+v_2}{2}, \varepsilon\right) |t_1|_\varepsilon^{\frac{n-2-v-v_1-v_2}{2}} |\tau_1|_{\delta_1}^{n-1+v_1} \delta(t_*) \delta(\tau_*). \quad (110)$$

Next, under the generic condition $[R, S] \neq 0$, one can find $g \in Sp(n, \mathbb{R})$ such that

$$S = ge_1, \quad R = [R, S] ge_{n+1}: \quad (111)$$

it follows that

$$\langle \mathcal{F}\left(Y \longmapsto |[S, Y]|_\varepsilon^{\frac{-n+v+v_1+v_2}{2}} |[Y, R]|_{\delta_1}^{-\boldsymbol{n}-v_1}\right), T \longmapsto |[T, S]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} h(T) \rangle$$

$$= (-1)^{\delta_1} c(-n - v_1, \delta_1) c\left(\frac{-n+v+v_1+v_2}{2}, \varepsilon\right) |[R, S]|_{\delta_1}^{-\boldsymbol{n}-v_1}$$

$$\times \langle |t_1|_\varepsilon^{\frac{n-2-v-v_1-v_2}{2}} |\tau_1|_{\delta_1}^{n-1+v_1} \delta(t_*) \delta(\tau_*), |\tau_1|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} (h \circ g)(t_1, t_*; \tau_1, \tau_*) \rangle \quad (112)$$

Since

$$(h \circ g)(t_1, 0; \tau_1, 0) = h\left(t_1 S + \tau_1 \frac{R}{[R,S]}\right), \quad (113)$$

we set $\tau_1 = [R, S] r$ and, for clarity, $t_1 = s$, getting

$$\langle \mathcal{F}\left(Y \longmapsto |[S, Y]|_\varepsilon^{\frac{-n+v+v_1+v_2}{2}} |[Y, R]|_{\delta_1}^{-\boldsymbol{n}-v_1}\right), T \longmapsto |[T, S]|_{\varepsilon_1}^{\frac{-n-v-v_1+v_2}{2}} h(T) \rangle$$

$$= (-1)^{\delta_1} c(-n - v_1, \delta_1) c\left(\frac{-n+v+v_1+v_2}{2}, \varepsilon\right) |[R, S]|_{\delta_1}^{-\boldsymbol{n}-v_1}$$

$$\times \int_{\mathbb{R}^2} |r|_j^{\frac{n-2-v+v_1+v_2}{2}} |s|_\varepsilon^{\frac{n-2-v-v_1-v_2}{2}} h(rR + sS) dr ds, \quad (114)$$

as a result.

Then, using (103) and (106) together with Eq. (94)

$$(-1)^{\delta_1} c(n - 1 + v_1, \delta_1) c(-n - v_1, \delta_1) = 1, \quad (115)$$

we obtain (109) under the conditions which made Lemma (6.7), and (103) as a consequence, valid. Analytic continuation is possible, the hypotheses from Proposition (6. 3) giving a meaning to the left-hand side. The conditions (108) make it possible to extract $(h_1)_{v_1,\delta_1}, (h_2)_{v_2,\delta_2}$ *and* $h_{-v,\delta}$ from h_1, h_2, h; the first condition (107) gives a meaning to $|s|_{\delta_2}^{-1-v-v_1}$ as a distribution (the factor depending on r is already locally summable from the previous condition), and the other two inequalities (107) make up half the conditions needed in order that the ratio $\frac{c\left(\frac{-n+v+v_1+v_2}{2}, \varepsilon\right)}{c\left(\frac{n-2+v-v_1+v_2}{2}, \varepsilon_2\right)}$ be well defined and nonzero while, as it turns out, the other two conditions necessary for that have already been taken care of by the assumptions of Proposition (6.3).

Some one-dimensional preparation

Let us briefly recall the spectral decomposition of the one-dimensional Euler operator in $L^2(\mathbb{R})$. Given a function $h_{i\lambda,\delta}$ on $\mathbb{R}^2$, homogeneous of degree $-1-i\lambda$ and with a given parity specified by the index $\delta = 0 \;\; or \;\; 1$, we set

$$h^b_{i\lambda,\delta}(s) = h_{i\lambda,\delta}(s,1) \tag{116}$$

so that

$$h_{i\lambda,\delta}(x,\xi) = |\xi|_\delta^{-1-i\lambda} \; h^b_{i\lambda,\delta} \; \left(\frac{x}{\xi}\right). \tag{117}$$

Then, every function $h \in L^2(\mathbb{R}^2)$, can be decomposed as

$$h = \sum_{\delta=0,1} \int_{-\infty}^{\infty} h_{i\lambda,\delta} \, d\lambda \tag{118}$$

with

$$h_{i\lambda,\delta}(x,\xi) = \frac{1}{2\pi} \int_0^\infty t^{i\lambda} \;\; h_\delta(tx,t\xi)dt, \tag{119}$$

where h_δ denotes the even, or odd, part of h, according to whether $\delta = 0$ or 1. Note that we denote here as $h^b_{i\lambda,\delta}$ the function denoted as $h^b_{\lambda,\delta}$.

Using the equations (in which signed powers such as $|s|_\delta^\alpha$ have been defined in (16))

$$\frac{d}{dx}|x|_\delta^{-1-v} = -(1+v)|x|_{1-\delta}^{-v-2} \quad and \quad \frac{d}{dx} \, log|x| = x^{-1}, \tag{120}$$

one obtains the well-known fact, already used, that the function $v \longmapsto |x|_\delta^{-1-v}$, a locally summable function if $Rev < 0$, extends as a distribution-valued holomorphic function of v for $v \neq \delta, \delta+2, \ldots$.

If $|x|_{\delta_1}^{-1-v_1}$ and $|\xi|_{\delta_2}^{-1-v_2}$ make sense as distributions as just defined, the symbol $h(x,\xi) = |x|_{\delta_1}^{-1-v_1} \# |\xi|_{\delta_2}^{-1-v_2}$ makes sense as a tempered distribution on $\mathbb{R}^2$: in other words, the composition of the two operators, the first of which is the convolution by the inverse Fourier transform of $|\xi|_{\delta_2}^{-1-v_2}$, and the second is the multiplication by $|x|_{\delta_1}^{-1-v_1}$, is well defined as an operator from $S(\mathbb{R})$ to $S'(\mathbb{R})$. To see this, one may use as an intermediary space the space $\mathcal{O}_M$ of C^∞ functions on the line each derivative of which is bounded by some polynomial.

Under the lift from $h^b_{i\lambda,\delta}$ to $h_{i\lambda,\delta}$ provided by (117), the distribution associated to the function $|s|^{\frac{-1-v_1+v_2-i\lambda}{2}}$ is given as

$$(x,\xi) \longmapsto |x|^{\frac{-1-v_1+v_2-i\lambda}{2}} \, |\xi|_{\delta_2}^{\frac{-1+v_1-v_2-i\lambda}{2}} \tag{121}$$

and the distribution associated to the function $\langle s\rangle^{\frac{-1-v_1+v_2-i\lambda}{2}}$ is given as

$$(x,\xi) \mapsto \langle s\rangle^{\frac{-1-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2}} |\xi|_{1-\delta}^{\frac{-1+\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}}. \tag{122}$$

Both distributions make sense if $\frac{-1\pm(\mathrm{v}_1-\mathrm{v}_2)-i\lambda}{2} \neq -1,-2,\ldots$, which is the case whenever $\lambda \in \mathbb{R}$ if one assumes that $|Re(\mathrm{v}_1-\mathrm{v}_2)| < 1$.

We may then recall Lemma (6.4) from [165] as follows:

Lemma (6.9)[172]. *Let* $\nu_1, \nu_2 \in \mathbb{C}$ *and* $\delta_1, \delta_2 = 0$ *or* 1: assume that $\nu_1 \neq \delta_1, \nu_2 \neq \delta_2$ and that $|Re(\nu_1 \pm \nu_2)| < 1$ which implies that $|Re\,\mathrm{v}_1| < 1$, $|Re\mathrm{v}_2| < 1$. *Let* $\delta = 0$ *or* 1 be such that $\delta \equiv \delta_1 + \delta_2 mod\ 2. set\ \mathrm{h}_1(x,\xi) = |x|_{\delta_1}^{-1-\mathrm{v}_1}, \mathrm{h}_2(x,\xi) = |\xi|_{\delta_2}^{-1-\mathrm{v}_2}$ *and* $h = \mathrm{h}_1\#\mathrm{h}_2$, a tempered distribution in $\mathbb{R}^2$. It admits the weak decomposition in $S'(\mathbb{R}^2)$ given as

$$h = \int_{-\infty}^{\infty} h_{i\lambda,\delta}^{(n)}\, d\lambda \tag{123}$$

with

$$h_{i\lambda,\delta}(x,\xi) = 2^{\frac{\mathrm{v}_1+\mathrm{v}_2-i\lambda-5}{2}} \pi^{\frac{\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2}} \frac{\Gamma\left(\frac{-\mathrm{v}_1+\delta_1}{2}\right)\Gamma\left(\frac{-\mathrm{v}_2+\delta_2}{2}\right)}{\Gamma\left(\frac{\mathrm{v}_1+\delta_1+1}{2}\right)\Gamma\left(\frac{\mathrm{v}_2+\delta_2+1}{2}\right)}$$
$$\times \left[i^{\delta_2-\delta} \frac{\Gamma\left(\frac{1+\mathrm{v}_1-\mathrm{v}_2+i\lambda}{4}\right)\Gamma\left(\frac{1+\mathrm{v}_1+\mathrm{v}_2-i\lambda+2\delta_1}{4}\right)\Gamma\left(\frac{1-\mathrm{v}_1+\mathrm{v}_2+i\lambda+2\delta}{4}\right)}{\Gamma\left(\frac{1-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{4}\right)\Gamma\left(\frac{1-\mathrm{v}_1-\mathrm{v}_2+i\lambda+2\delta_1}{4}\right)\Gamma\left(\frac{1+\mathrm{v}_1-\mathrm{v}_2-i\lambda+2\delta}{4}\right)} \times \right.$$
$$|x|^{\frac{-1-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2}} |\xi|_{\delta}^{\frac{-1+\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}}$$
$$+ i^{-\delta_2-\delta+1} \frac{\Gamma\left(\frac{3+\mathrm{v}_1-\mathrm{v}_2+i\lambda}{4}\right)\Gamma\left(\frac{3+\mathrm{v}_1+\mathrm{v}_2-i\lambda-2\delta_1}{4}\right)\Gamma\left(\frac{3-\mathrm{v}_1+\mathrm{v}_2+i\lambda-2\delta}{4}\right)}{\Gamma\left(\frac{3-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{4}\right)\Gamma\left(\frac{3-\mathrm{v}_1-\mathrm{v}_2+i\lambda-2\delta_1}{4}\right)\Gamma\left(\frac{3+\mathrm{v}_1-\mathrm{v}_2-i\lambda-2\delta}{4}\right)}$$
$$\left. \times \langle x\rangle^{\frac{-1-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2}} |\xi|_{1-\delta}^{\frac{-1+\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}} \right]. \tag{124}$$

Note that the integrand, as a distribution-valued function of λ, has no singularity on the real line. Also, as a consequence of Stirling's formula, the coefficient is bounded, for large $|\lambda|$, by some power of $|\lambda|$: since our claim is that the integral decomposition (123) is valid in a weak sense in $S'(\mathbb{R}^2)$, we may ensure convergence by means of the equation

$$|x|^{\frac{-1-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2}} |\xi|_{\delta}^{\frac{-1+\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}}$$
$$= (1+\lambda^2)^{-N} (1+4\pi^2\varepsilon^2)^N \left(|x|^{\frac{-1-\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2}}\right) |\xi|_{\delta}^{\frac{-1+\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}}, \tag{125}$$

in which $2i\pi\varepsilon = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}$, and of a similar one involving the second term on the right-hand side of (124).

We now need to consider the case of two symbols $|x|_{\delta_1}^{-n-\nu_1}$ *and* $|\xi|_{\delta_2}^{-n-\nu_2}$, in which $n = 1, 2, \ldots$ is given, the same in both functions. The reason is that, even though the proof of the main theorem depends on the decomposition of symbols into homogeneous hyperplane waves, which are essentially one-dimensional objects, the spectral decomposition of the Euler operator in $L^2(\mathbb{R}^2)$ demands that we consider decompositions of the same species as (118) in which, however, the degrees of homogeneity of the functions in the decomposition lie on the complex line with real part $-n$ rather than -1.

Let Q and P be the basic infinitesimal operators of Heisenberg's representation, where Q is the operator of multiplication by the variable x on the real line, and $P = \frac{1}{2i\pi}\frac{d}{dx}$. Then, in the one-dimensional Weyl calculus, one has the commutation relations

$$[Q, Op(h)] = -\frac{1}{2i\pi} Op\left(\frac{\partial h}{\partial \xi}\right), \quad [P, Op(h)] = \frac{1}{2i\pi} Op\left(\frac{\partial h}{\partial x}\right). \quad (126)$$

Also, $POp(h) = Op\left(\xi h + \frac{1}{4i\pi}\frac{\partial h}{\partial x}\right)$. If $h_1 (resp. h_2)$ is a tempered distribution depending only on $(resp. \xi)$, and if one sets $A_1 = Op(h_1)$, $A_2 = Op(h_2)$, one has (using the facts that A_1 commutes with Q, A_2 commutes with P and the Heisenberg relation $[P, Q] = \frac{1}{2i\pi}$

$$[P, A_1][Q, A_2] = P[Q, A_1A_2] - [Q, A_1A_2P] - \frac{1}{2i\pi} A_1A_2: \quad (127)$$

it follows that if $h = h_1 \# h_2$, the symbol of the operator $[P, Op(h_1)][Q, Op(h_2)]$is the function

$$\left(\xi + \frac{1}{4i\pi}\frac{\partial}{\partial x}\right)\left(-\frac{1}{2i\pi}\frac{\partial h}{\partial \xi}\right) + \frac{1}{2i\pi}\frac{\partial}{\partial \xi}\left(\xi h - \frac{1}{4i\pi}\frac{\partial h}{\partial x}\right) - \frac{1}{2i\pi} h = \frac{1}{4\pi^2}\frac{\partial^2 h}{\partial x \partial \xi}. \, (128)$$

In other words, under the present assumptions,

$$\frac{\partial h_1}{\partial x} \# \frac{\partial h_2}{\partial \xi} = \frac{\partial^2 \mathrm{h}}{\partial \mathrm{x}\, \partial \xi}. \quad (129)$$

Introduce, for $k = 0, 1, \ldots$ and $a \in \mathbb{C}$, the Pochhammer symbols $(a)_k = a(a+1)\ldots(a+k-1)$, and extend the definition of $|s|_\delta^\alpha$beyond the case when $\delta = 0 \; or \; 1$, setting $|s|_p^\alpha = |s|_{p \; mod \; 2}^\alpha$. With the same assumptions about $\nu_1, \nu_2, \delta_1, \delta_2$ as in Lemma (6. 3), one has for $n = 1,2, \ldots$ (using (120)) the equation

$$(1+\nu_1)_{n-1}(1+\nu_2)_{n-1}|x|_{n-1-\delta_1}^{n-\nu_1} \# |\xi|_{n-1-\delta_2}^{n-\nu_2}$$

$$= \int_{-\infty}^{\infty} \left(\frac{1+\nu_1-\nu_2+i\lambda}{2}\right)_{n-1} \left(\frac{1-\nu_1+\nu_2+i\lambda}{2}\right)_{n-1}$$

$$\times 2^{\frac{\nu_1+\nu_2-i\lambda-5}{2}} \pi^{\frac{\nu_1+\nu_2-i\lambda}{2}} \frac{\Gamma\left(\frac{-\nu_1+\delta_1}{2}\right)\Gamma\left(\frac{-\nu_2+\delta_2}{2}\right)}{\Gamma\left(\frac{\nu_1+\delta_1+1}{2}\right)\Gamma\left(\frac{\nu_2+\delta_2+1}{2}\right)}$$

$$\times \left[i^{\delta_2-\delta} \frac{\Gamma\left(\frac{1+v_1-v_2+i\lambda}{4}\right)\Gamma\left(\frac{1+v_1+v_2-i\lambda+2\delta_1}{4}\right)\Gamma\left(\frac{1-v_1+v_2+i\lambda+2\delta}{4}\right)}{\Gamma\left(\frac{1-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{1-v_1-v_2+i\lambda+2\delta_1}{4}\right)\Gamma\left(\frac{1+v_1-v_2-i\lambda+2\delta}{4}\right)} \times \right.$$

$$|x|_{n-1}^{\frac{1-2n-v_1+v_2-i\lambda}{2}} |\xi|_{n-1+\delta}^{\frac{1-2n+v_1-v_2-i\lambda}{2}}$$

$$+ i^{-\delta_2-\delta+1} \frac{\Gamma\left(\frac{3+v_1-v_2+i\lambda}{4}\right)\Gamma\left(\frac{3+v_1+v_2-i\lambda-2\delta_1}{4}\right)\Gamma\left(\frac{3-v_1+v_2+i\lambda-2\delta}{4}\right)}{\Gamma\left(\frac{3-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{3-v_1-v_2+i\lambda-2\delta_1}{4}\right)\Gamma\left(\frac{3+v_1-v_2-i\lambda-2\delta}{4}\right)}$$

$$\left. \times |x|_{n}^{\frac{1-2n-v_1+v_2-i\lambda}{2}} |\xi|_{n-\delta}^{\frac{1-2n+v_1-v_2-i\lambda}{2}} \right] d\lambda. \tag{130}$$

Note that the degree of homogeneity of each of the two terms under the integral sign is $1-2n-i\lambda$, not $-n-i\lambda$ as we would wish it to be: we must thus perform a deformation of contour. We substitute $z \in \mathbb{C}$ for $i\lambda$ and we must move z from the pure imaginary line to the line with real part $1-n$. There is no convergence problem at infinity in the process, in view of (125). We must then chase for possible poles, setting $\mu = \frac{v_1-v_2+z}{2}$ *and* $\mu' = \frac{v_1-v_2-z}{2}$. The only singularities can arise from the factors depending on *or* ξ , or from the first and third Gamma functions in the numerator of each of the two major coefficients. We make a group of each of the expressions

$$\left(\frac{1}{2}+\mu\right)_{n-1} \Gamma\left(\frac{1}{4}+\frac{\mu}{2}\right) |x|_{n-1}^{\frac{1}{2}-n-\mu},$$

$$\left(\frac{1}{2}+\mu\right)_{n-1} \Gamma\left(\frac{3}{4}+\frac{\mu}{2}\right) |x|_{n}^{\frac{1}{2}-n-\mu},$$

$$\left(\frac{1}{2}-\mu'\right)_{n-1} \Gamma\left(\frac{1}{4}+\frac{\delta}{2}-\frac{\mu'}{2}\right) |\xi|_{n-1+\delta}^{\frac{1}{2}-n-\mu'},$$

$$\left(\frac{1}{2}-\mu'\right)_{n-1} \Gamma\left(\frac{3}{4}-\frac{\delta}{2}-\frac{\mu'}{2}\right) |\xi|_{n-\delta}^{\frac{1}{2}-n+\mu'}. \tag{131}$$

We now show that each of the four functions under consideration remains a holomorphic function of z in a neighbourhood of the closed strip $1-n \leq Re z \leq 0$. First we show that the Gamma factor and the distribution ($in\ x\ or\ \xi$) on any of the four lines have disjoint sets of singularities as functions of z. This is a consequence of the fact, noted just after (120), that $|x|_{\delta}^{-\alpha}$ a well-defined distribution in x provided that $\neq \delta+1, \delta+3, \ldots$. For, as a consequence, the singularities of the factor depending on $x\ or\ \xi$ on the four lines are reached when $\mu \in \frac{1}{2}+2\mathbb{N}, resp. \mu \in \frac{3}{2}+2\mathbb{N}, resp. \mu \in -\delta-\frac{1}{2}-2\mathbb{N}, resp. \mu \in \delta-\frac{3}{2}+2\mathbb{N}$, while the singularities of the corresponding Gamma factors are reached when $\mu \in -\frac{1}{2} - 2\mathbb{N}, resp. \mu \in -\frac{3}{2} - 2\mathbb{N}. resp. \mu \in -\delta - \frac{1}{2} + 2\mathbb{N}, resp.\ \ \mu \in -\delta + \frac{3}{2} + 2\mathbb{N}$.

Since the two sets of singularities under consideration are disjoint, what remains to be proved is that each of the eight expressions

$$\begin{aligned} &\left(\frac{1}{2}+\mu\right)_{n-1} \Gamma\left(\frac{1}{4}+\frac{\mu}{2}\right), && \left(\frac{1}{2}+\mu\right)_{n-1} |x|_{n-1}^{\frac{1}{2}-n-\mu}, \\ &\left(\frac{1}{2}+\mu\right)_{n-1} \Gamma\left(\frac{3}{4}+\frac{\mu}{2}\right), && \left(\frac{1}{2}+\mu\right)_{n-1} |x|_{n}^{\frac{1}{2}-n-\mu}, \\ &\left(\frac{1}{2}-\mu^{'}\right)_{n-1} \Gamma\left(\frac{1}{4}+\frac{\delta}{2}-\frac{\mu^{'}}{2}\right), && \left(\frac{1}{2}-\mu^{'}\right)_{n-1} |\xi|_{n-1+\delta}^{\frac{1}{2}-n+\mu^{'}}, \\ &\left(\frac{1}{2}-\mu^{'}\right)_{n-1} \Gamma\left(\frac{3}{4}-\frac{\delta}{2}-\frac{\mu^{'}}{2}\right), && \left(\frac{1}{2}-\mu^{'}\right)_{n-1} |\xi|_{n-\delta}^{\frac{1}{2}-n+\mu^{'}} \end{aligned} \tag{132}$$

is regular for z lying in the strip $1 - n \leq Re z \leq 0$. So far as the distribution on the right of each line is concerned, we write it as$(-1)^{n-1}$ times the $\left(\frac{d}{dx}\right)^{n-1}, or \left(\frac{d}{d\xi}\right)^{n-1}$ - derivative of the distribution $|x|^{-\frac{1}{2}-\mu}, resp. \langle x\rangle^{-\frac{1}{2}-\mu}, resp. |\xi|_{\delta}^{\frac{1}{2}+\mu^{'}}, resp. |\xi|_{1-\delta}^{-\frac{1}{2}+\mu^{'}}$. Now, the condition $Re\, z \leq 0$, together with the assumption $|Re(v_1 - v_2)| < 1$, implies that $Re\mu < \frac{1}{2}\ and\ Re\mu^{'} > -\frac{1}{2}$, which gives the four distributions under consideration a meaning as locally summable functions. So far as the Gamma factors are concerned, every other term in the product

$$\begin{aligned} \left(\frac{1}{2}+\mu\right)_{n-1} &= \left(\frac{1}{2}+\mu\right)\left(\frac{3}{2}+\mu\right)\ldots\left(n-\frac{1}{2}+\mu\right)\ or \\ \left(\frac{1}{2}-\mu^{'}\right)_{n-1} &= \left(\frac{1}{2}-\mu^{'}\right)\left(\frac{3}{2}-\mu^{'}\right)\ldots\left(n-\frac{1}{2}-\mu^{'}\right) \end{aligned} \tag{133}$$

will help in killing the relevant poles of the corresponding Gamma factor. Indeed, with $p = 1, 2, \ldots,$ each of the two expressions $\left(\frac{1}{2}+\mu\right)_{2p-1} \Gamma\left(\frac{1}{4}+\frac{\mu}{2}\right)\ and\ \left(\frac{1}{2}+\mu\right)_{2p-2} \Gamma\left(\frac{1}{4}+\frac{\mu}{2}\right)$ is the product of a polynomial in μ by the function $\Gamma\left(p+\frac{1}{4}+\frac{\mu}{2}\right)$, while each of the two expressions $\left(\frac{1}{2}+\mu\right)_{2p-1} \Gamma\left(\frac{3}{4}+\frac{\mu}{2}\right)\ and\ \left(\frac{1}{2}+\mu\right)_{2p-2} \Gamma\left(\frac{3}{4}+\frac{\mu}{2}\right)$ is the product of a polynomial in μ by the function $\Gamma\left(p-\frac{1}{4}+\frac{\mu}{2}\right)$. The last two expressions to be analyzed are $\left(\frac{1}{2}-\mu^{'}\right)_{n-1} \Gamma\left(\frac{1}{4}-\frac{\mu^{'}}{2}\right)\ and\ \left(\frac{1}{2}-\mu^{'}\right)_{n-1} \Gamma\left(\frac{3}{4}-\frac{\mu^{'}}{2}\right)$. We use this time the inequality $Re\mu^{'} < \frac{n}{2}$ and observe that each of the two expressions $\left(\frac{1}{2}-\mu^{'}\right)_{2p-1} \Gamma\left(\frac{1}{4}-\frac{\mu^{'}}{2}\right)\ and\ \left(\frac{1}{2}-\mu^{'}\right)_{2p-2} \Gamma\left(\frac{1}{4}-\frac{\mu^{'}}{2}\right)$ is the product of a polynomial by $\Gamma\left(p+\frac{1}{4}-\frac{\mu^{'}}{2}\right)$, while each of the two expressions $\left(\frac{1}{2}-\mu^{'}\right)_{2p-1}\ \Gamma\left(\frac{3}{4}-\frac{\mu^{'}}{2}\right)\ and\ \left(\frac{1}{2}-\mu^{'}\right)_{2p-2}\Gamma\left(\frac{3}{4}-\frac{\mu^{'}}{2}\right)$ is the product of a polynomial by $\Gamma\left(p-\frac{1}{4}-\frac{\mu^{'}}{2}\right)$.

Performing the change of contour which was the aim of the lengthy preparation just made, we finally obtain the following.

Lemma $(\mathbf{6.10})[\mathbf{172}]$**.** Let $v_1, v_2\ and\ \delta_1, \delta_2 = 0\ or\ 1$: assume that $v_1 \neq \delta_1, v_2 \neq \delta_2$ and that $|Re(v_1 \pm v_2)| < 1$. Let $n = 1,2 \ldots$, and let $\delta, \delta_1, \delta_2$ be the numbers, all equal to $0\ or\ 1$, characterized by the congruences mod 2

$$\delta \equiv \delta_1 + \delta_2, \quad \delta_1^{'} \equiv n-1-\delta_1, \quad \delta_2^{'} \equiv n-1-\delta_2. \qquad (134)$$

Set $h_1(x,\xi) = |x|_{\delta_1}^{-n-v_1}$, $h_2(x,\xi) = |\xi|_{\delta_2}^{-n-v_2}$ and let $h = h_1 \# h_2$, a tempered distribution in $\mathbb{R}^2$. It admits the weak decomposition in $S^{'}(\mathbb{R}^2)$ given as

$$h = \int_{-\infty}^{\infty} h_{i\lambda,\delta}^{(n)}\, d\lambda \qquad (135)$$

with

$$\begin{aligned}
h_{i\lambda,\delta}^{(n)}&(x,\xi) \\
&= (1+v_1)_{n-1}^{-1}(1+v_2)_{n-1}^{-1}\left(\frac{2-n+v_1-v_2+i\lambda}{2}\right)_{n-1}\left(\frac{2-n-v_1+v_2+i\lambda}{2}\right)_{n-1} \\
&\times 2^{\frac{v_1+v_2-i\lambda-6}{2}}\pi^{\frac{n-1+v_1+v_2-i\lambda}{2}}\frac{\Gamma\left(\frac{-v_1+\delta_1^{'}}{2}\right)\Gamma\left(\frac{-v_2+\delta_2^{'}}{2}\right)}{\Gamma\left(\frac{v_1+\delta_1^{'}+1}{2}\right)\Gamma\left(\frac{v_2+\delta_2^{'}+1}{2}\right)} \\
&\times\left[i^{\delta_2-\delta}\frac{\Gamma\left(\frac{2-n+v_1-v_2+i\lambda}{4}\right)\Gamma\left(\frac{n+v_1+v_2-i\lambda+2\delta_1^{'}}{4}\right)\Gamma\left(\frac{2-n-v_1+v_2+i\lambda+2\delta}{4}\right)}{\Gamma\left(\frac{n-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{2-n-v_1-v_2+i\lambda+2\delta_1^{'}}{4}\right)\Gamma\left(\frac{n+v_1-v_2-i\lambda+2\delta}{4}\right)}\right. \\
&\times|x|_{n-1}^{\frac{-n-v_1+v_2-i\lambda}{2}}|\xi|_{n-1+\delta}^{\frac{-n+v_1-v_2-i\lambda}{2}} \\
&+i^{-\delta_2-\delta+1}\frac{\Gamma\left(\frac{4-n+v_1-v_2+i\lambda}{4}\right)\Gamma\left(\frac{n+2+v_1+v_2-i\lambda-2\delta_1^{'}}{4}\right)\Gamma\left(\frac{4-n-v_1+v_2+i\lambda-2\delta}{4}\right)}{\Gamma\left(\frac{n+2-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{4-n-v_1-v_2+i\lambda-2\delta_1^{'}}{4}\right)\Gamma\left(\frac{n+2+v_1-v_2-i\lambda-2\delta}{4}\right)} \\
&\left.\qquad\times|x|_{n}^{\frac{-n-v_1+v_2-i\lambda}{2}}|\xi|_{n-\delta}^{\frac{-n+v_1-v_2-i\lambda}{2}}\right], \qquad (136)
\end{aligned}$$

where we recall our convention that $|s|_{p^{'}}^{\alpha,}\ with\ p^{'} = 0\ or\ 1\ and\ p = p^{'} mod\ 2$.

In the proof of Lemma (6.10), we have avoided moving v_1 and v_2, which would have complicated the pole chasing even more. It is, however, necessary to check that analytic continuation with respect to v_1 and v_2 is possible up to some point, in the sense of the following lemma.

Lemma (6.11)[172] . Set $v_1^{'} = n-1+v_1, v_2^{'} = n-2+v_2$, so that $|x|_{\delta_1}^{1-v_1^{'}} = |x|_{\delta_1}^{-n-v_1^{'}}$ *and* $|x|_{\delta_2}^{1-v_2^{'}} = |\xi|_{\delta_2}^{-n-v_2^{'}}$. To obtain the term $h_{i\lambda,\delta}^{(n)}$ from the decomposition (135) of $h_1 \# h_2$ (same notation as in Lemma (6.10), it suffices to perform the substitutions $v_1 \longmapsto v_1^{'},\ v_2 \longmapsto v_2^{'}$ and $i\lambda \longmapsto v^{'} = i\lambda + n - 1$ on the right-hand side of (124).

Proof. The proof, based on the duplication formula and on the formula of complements for the Gamma function, is perfectly ugly, though one can take solace in the fact that it offers a means of verification. Starting from the right-hand side of (124) and making the substitution$(v_1, v_2, i\lambda) \longmapsto (v_1^{'}, v_2^{'}, i\lambda + n - 1)$, we want to show that we just obtain the right-hand side of (145). We shall limit ourselves to the case when n is odd. One has

$$(1+v_1)_{n-1}^{-1} = \frac{\Gamma(1-n-v_1)}{\Gamma(-v_1)} = 2^{1-n} \frac{\Gamma\left(\frac{1-n-v_1+\delta_1^{'}}{2}\right)\Gamma\left(\frac{2-n-v_1-\delta_1^{'}}{2}\right)}{\Gamma\left(\frac{-v_1+\delta_1^{'}}{2}\right)\Gamma\left(\frac{1-v_1-\delta_1^{'}}{2}\right)}, \quad (137)$$

so that

$$(1+v_1)_{n-1}^{-1} \frac{\Gamma\left(\frac{-v_1+\delta_1^{'}}{2}\right)}{\Gamma\left(\frac{1+v_1+\delta_1^{'}}{2}\right)} 2^{1-n} \frac{\Gamma\left(\frac{1-n-v_1+\delta_1^{'}}{2}\right)\Gamma\left(\frac{2-n-v_1-\delta_1^{'}}{2}\right)}{\Gamma\left(\frac{1+v_1+\delta_1^{'}}{2}\right)\Gamma\left(\frac{1-v_1-\delta_1^{'}}{2}\right)} = 2^{1-n} \frac{\Gamma\left(\frac{1-n-v_1+\delta_1^{'}}{2}\right)}{\Gamma\left(\frac{n+v_1+\delta_1^{'}}{2}\right)} \quad (138)$$

2^{1-n} times the corresponding coefficient $\frac{\Gamma\left(\frac{-v_1^{'}+\delta_1}{2}\right)}{\Gamma\left(\frac{1+v_1^{'}+\delta_1}{2}\right)}$ arising after the shift $v_1 \longmapsto v_1^{'}$ from a factor in (124). The same goes so far as the comparable coefficient depending on v_2 is concerned.The powers of 2 and π, as well as the Gamma factors in the middle of the coefficients we are interested in, transform in an immediately satisfactory way. The remaining headache arises from the coefficient, obtained from (124). and the required shift,

$$B := \frac{\Gamma\left(\frac{n+v_1-v_2+i\lambda}{4}\right)\Gamma\left(\frac{n-v_1+v_2+i\lambda+2\delta}{4}\right)}{\Gamma\left(\frac{2-n-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{2-n+v_1-v_2-i\lambda+2\delta}{4}\right)}: \quad (139)$$

multiplying by $\Gamma\left(\frac{4-n-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{4-n+v_1+v_2-i\lambda-2\delta}{4}\right)$ up and down, using the formula of complements upstairs and the duplication formula downstairs, we obtain

$$B = \frac{\pi}{2^{n+i\lambda}}\left[sin\pi\left(\frac{n+v_1-v_2+i\lambda}{4}\right) sin\pi\left(\frac{n-v_1+v_2+i\lambda+2\delta}{4}\right)\right]^{-1} \times \left[\Gamma\left(\frac{2-n-v_1+v_2-i\lambda}{2}\right)\Gamma\left(\frac{2-n+v_1-v_2-i\lambda}{2}\right)\right]^{-1}. \quad (140)$$

This must be compared to the similar coefficient from (136), which must be accompanied, as a factor, by the product of the two remaining Pochhammer symbols. This is

$$A := \frac{\Gamma\left(\frac{n-v_1+v_2-i\lambda}{2}\right)\Gamma\left(\frac{n+v_1-v_2-i\lambda}{2}\right)}{\Gamma\left(\frac{2-n+v_1+v_2-i\lambda}{2}\right)\Gamma\left(\frac{2-n+v_1-v_2-i\lambda}{2}\right)} \times \frac{\Gamma\left(\frac{2-n+v_1-v_2+i\lambda}{2}\right)\Gamma\left(\frac{2-n-v_1+v_2+i\lambda+2\delta}{2}\right)}{\Gamma\left(\frac{n-v_1+v_2-i\lambda}{2}\right)\Gamma\left(\frac{n+v_1-v_2-i\lambda+2\delta}{2}\right)}: \qquad (141)$$

if we multiply the product of fractions on the second line, up and down, by $\Gamma\left(\frac{2+n-v_1+v_2-i\lambda}{4}\right)\,\Gamma\left(\frac{2+n+v_1-v_2-i\lambda-2\delta}{4}\right)$, if we apply again the formula of complements upstairs and the duplication formula downstairs, it becomes

$$\frac{\pi}{2^{2-n+i\lambda}}\left[sin\pi\left(\frac{2-n+v_1-v_2+i\lambda}{4}\right)sin\pi\left(\frac{2-n-v_1+v_2+i\lambda+2\delta}{4}\right)\right]^{-1} \times\left[\Gamma\left(\frac{n-v_1+v_2-i\lambda}{2}\right)\Gamma\left(\frac{n+v_1-v_2-i\lambda}{2}\right)\right]^{-1}. \qquad (142)$$

It follows that $A = 2^{2n-2}B$, which completes our verification, in the case when n is odd, so far as the coefficient of the first term on the right-hand side of (124) *or* (136) is concerned. We shall not write down everything in the case when (still with *n* odd) the coefficient of the second term is concerned. The trick is, this time, to multiply the fraction B' which takes the place of B, up and down, by $\Gamma\left(\frac{2-n-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{2-n+v_1-v_2-i\lambda+2\delta}{4}\right)$; next, the fraction on the second line of the expression A' which takes the place of A is to be multiplied, up and down, by $\Gamma\left(\frac{n-v_1+v_2-i\lambda}{4}\right)\Gamma\left(\frac{n+v_1-v_2-i\lambda+2\delta}{4}\right)$: again, we find that $A' = 2^{2n-2}B'$.The lemma is thus proved in the case when n is odd. The proof is of course similar in the case when it is even: only, one should not forget that, in this case, $\delta_1' = 1-\delta_1$ *and* $\delta_2' = 1-\delta_2$. Also, the right-hand side of (124). will yield, after transformation, the two terms on the right-hand side of (136) in reverse order.

Making all Gamma factors apparent has been necessary for the discussion of the change of complex contour. Using the shorthand provided by (85), *i.e.*, making the substitution

$$\frac{\Gamma\left(\frac{\rho+1+\delta}{2}\right)}{\Gamma\left(\frac{-\rho+\delta}{2}\right)} = i^{\delta}\,\pi^{\rho+\frac{1}{2}}\,c(\rho,\delta), \qquad (143)$$

one obtains the following.

Proposition (6.12)[172]. Under the assumptions of Lemma (6.10), one has

$$h_{i\lambda,\delta}^{(n)}(x,\xi) = C_0(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)|x|^{\frac{-n-v_1+v_2-i\lambda}{2}}|\xi|_{\delta}^{\frac{-n+v_1-v_2-i\lambda}{2}} + C_1(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)\langle x\rangle^{\frac{-n-v_1+v_2-i\lambda}{2}}|\xi|_{1-\delta}^{\frac{-n+v_1-v_2-i\lambda}{2}}, \qquad (144)$$

with

$C_0(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)$

$$= 2^{\frac{\mathrm{v}_1+\mathrm{v}_2-i\lambda+n-6}{2}}\pi^{-1}(-1)^{\delta}c(-n-\mathrm{v}_1,\delta_1)c(-n-\mathrm{v}_2,\delta_2)c\left(\frac{n-2+\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},0\right)$$
$$\times c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},\delta_1\right)c\left(\frac{n-2-\mathrm{v}_1+\mathrm{v}_2+i\lambda}{2},\delta\right) \quad (145)$$

and

$C_1(\mathrm{v}_1,\mathrm{v}_2,i\lambda;\delta_1,\delta_2,\delta)$
$$= 2^{\frac{\mathrm{v}_1+\mathrm{v}_2-i\lambda+n-6}{2}}\pi^{-1}(-1)^{\delta}c(-n-\mathrm{v}_1,\delta_1)c(-n-\mathrm{v}_2,\delta_2)c\left(\frac{n-2+\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},1\right)$$
$$\times c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},1-\delta_1\right)c\left(\frac{n-2-\mathrm{v}_1+\mathrm{v}_2+i\lambda}{2},1-\delta\right). \quad (146)$$

In view of the proof of the main theorem, we compute the $\mathcal{G}$-transform (88) of the symbol $|x_1|_{\delta_1}^{-n-\mathrm{v}_1}\#|\xi_1|_{\delta_2}^{-n+\mathrm{v}_2}$, considered as a distribution in $(\mathbb{R}^{2n})$: we still set $x=(x_1,x_*),\xi=(\xi_1,\xi_*)$. The change $\mathrm{v}_2\mapsto-\mathrm{v}_2$ is needed for the application: at the same time, we change the variable of integration λ to $-\lambda$ below so as to decompose the result as an integral superposition of distributions of type $(-n-i\lambda,\delta)$; we denote as $k_{-i\lambda,\delta}^{(n)}$ the function obtained from $h_{i\lambda,\delta}^{(n)}$ after these two sign changes.

Proposition (6.13)[172]. Assume that $\mathrm{v}_1\neq\delta_1,-\mathrm{v}_2\neq\delta_2$ *and* $|Re(\mathrm{v}_1\pm\mathrm{v}_2)|<1$. One has the weak decomposition in $S'(\mathbb{R}^{2n})$, given by the equation
$$\left[\mathcal{G}\left(Y\mapsto|y_1|_{\delta_1}^{-n-\mathrm{v}_1}\#|\eta_1|_{\delta_2}^{-n+\mathrm{v}_2}\right)\right](x,\xi)=\int_{-\infty}^{\infty}\left(\mathcal{G}k_{-i\lambda,\delta}^{(n)}\right)(x,\xi)d\lambda \quad (147)$$
with
$\left(\mathcal{G}k_{-i\lambda,\delta}^{(n)}\right)(x,\xi)$
$$=B_0(\mathrm{v}_1,\mathrm{v}_2,i\lambda;\delta_1,\delta_2,\delta)|x_1|_{\delta}^{\frac{n-2-\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}}|\xi_1|^{\frac{n-2+\mathrm{v}_1-\mathrm{v}_2-i\lambda}{2}}\delta(x_*)\delta(\xi_*) \quad (148)$$
where

$B_0(\mathrm{v}_1,\mathrm{v}_2,i\lambda;\delta_1,\delta_2,\delta)=$
$$2^{\frac{\mathrm{v}_1-\mathrm{v}_2-i\lambda+n-6}{2}}\pi^{-1}c(-n-\mathrm{v}_1,\delta_1)c(-n+\mathrm{v}_2,\delta_2)c\left(\frac{n-2+\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},\delta_1\right) \quad (149)$$

and

$B_1(\mathrm{v}_1,\mathrm{v}_2,i\lambda;\delta_1,\delta_2,\delta)$
$$=-2^{\frac{\mathrm{v}_1-\mathrm{v}_2-i\lambda+n-6}{2}}\pi^{-1}c(-n-\mathrm{v}_1,\delta_1)c(-n+\mathrm{v}_2,\delta_2)c\left(\frac{n-2+\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},1-\delta_1\right). \quad (150)$$

Proof. This is a consequence of the preceding proposition, together with the equation
$\left(\mathcal{G}\left(Y\mapsto|y_1|_{\omega_1}^{\alpha}|\xi_1|_{\omega_2}^{\beta}\right)\right)(x,\xi)$
$$=2^{-n-\alpha-\beta}(-1)^{\omega_2}c(\alpha,\omega_1)c(\beta,\omega_2)|x_1|_{\omega_2}^{-1-\beta}|\xi_1|_{\omega_1}^{-1-\alpha}\delta(x_*)\delta(\xi_*). \quad (151)$$
A simplification occurs from the use of Eqs. (94)
$$c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},0\right)c\left(\frac{-n-\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},0\right)=1,$$

$$c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},\delta\right)c\left(\frac{-n-\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},\delta\right)=(-1)^{\delta},$$
$$c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},1\right)c\left(\frac{-n-\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},1\right)=-1,$$
$$c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},1-\delta\right)c\left(\frac{-n-\mathrm{v}_1-\mathrm{v}_2+i\lambda}{2},1-\delta\right)=(-1)^{1-\delta}. \quad (152)$$

Another composition of Weyl symbols

Corollary $(\mathbf{6.13})'$[140]. If $\mathrm{v}_1\neq\delta_1$, $-\mathrm{v}_2\neq\delta_2$ and $\delta_1+\delta_2=\theta$ where $\mathrm{v}_1\pm\mathrm{v}_2\approx\epsilon$ such that $|Re(\epsilon)|<1$.

Then the weak decomposition in $S'(\mathbb{R}^{2n})$, given by

$$\left[\mathcal{G}\left(Y\mapsto|y_1|_{\delta_1}^{-n-\mathrm{v}_1}\#|\eta_1|_{\theta-\delta_1}^{-n+\mathrm{v}_1-\epsilon}\right)\right](x,\xi)=\int_{-\infty}^{\infty}\left(\mathcal{G}k_{-i\lambda,\theta}^{(n)}\right)(x,\xi)d\lambda$$

with

$$\left(\mathcal{G}k_{-i\lambda,\theta}^{(n)}\right)(x,\xi)$$
$$=B_0(\mathrm{v}_1,\mathrm{v}_1-\epsilon,i\lambda;\delta_1,\theta-\delta_1,\theta)|x_1|_{\theta}^{\frac{n-2-\epsilon-i\lambda}{2}}|\xi_1|^{\frac{n-2+\epsilon-i\lambda}{2}}\theta(x_*)\theta(\xi_*)$$

where

$$B_0(\mathrm{v}_1,\mathrm{v}_1-\epsilon,i\lambda;\delta_1,\theta-\delta_1,\theta)=$$
$$2^{\frac{\epsilon-i\lambda+n-6}{2}}\pi^{-1}c(-n-\mathrm{v}_1,\delta_1)c(-n+\theta-\mathrm{v}_1,\theta-\delta_1)c\left(\frac{n-2+\epsilon+i\lambda}{2},\delta_1\right)$$

and

$$B_1(\mathrm{v}_1,\mathrm{v}_1-\epsilon,i\lambda;\delta_1,\theta-\delta_1,\theta)$$
$$=-2^{\frac{\epsilon-i\lambda+n-6}{2}}\pi^{-1}c(-n-\mathrm{v}_1,\delta_1)c(-n+\mathrm{v}_1-\epsilon,\theta-\delta_1)c\left(\frac{n-2+\epsilon+i\lambda}{2},1-\delta_1\right)$$

Proof: proposition (6.13) implies that

$$\left(\mathcal{G}\left(Y\mapsto|y_1|_{\omega_1}^{\alpha}|\xi_1|_{\omega_2}^{\beta}\right)\right)(x,\xi)$$
$$=2^{-n-\alpha-\beta}(-1)^{\omega_2}c(\alpha,\omega_1)c(\beta,\omega_2)|x_1|_{\omega_2}^{-1-\beta}|\xi_1|_{\omega_1}^{-1-\alpha}\delta(x_*)\delta(\xi_*)$$

using (94) we have

$$c\left(\frac{n-2+\epsilon-i\lambda}{2},0\right)c\left(\frac{-n-\epsilon+i\lambda}{2},0\right)=1,$$
$$c\left(\frac{n-2+\epsilon-i\lambda}{2},\delta\right)c\left(\frac{-n-\epsilon+i\lambda}{2},\delta\right)=(-1)^{\theta},$$
$$c\left(\frac{n-2+\mathrm{v}_1+\mathrm{v}_2-i\lambda}{2},1\right)c\left(\frac{-n-\epsilon+i\lambda}{2},1\right)=-1,$$
$$c\left(\frac{n-2+\epsilon-i\lambda}{2},1-\theta\right)c\left(\frac{-n-\epsilon+i\lambda}{2},1-\theta\right)=(-1)^{1-\theta}$$

Theorem $(\mathbf{6.14})$[172]. Given δ_1,δ_2 *and* $\delta=0$ *or* 1 with $\delta\equiv\delta_1+\delta_2\ mod\ 2$, and $j=0$ *or* 1, define $\varepsilon_1,\varepsilon_2,\varepsilon$ by means of (48), and set, for real $\lambda_1,\lambda_2,\lambda$,

$a_{\delta_1,\delta_2;\delta}^{(j)}(i\lambda_1,i\lambda_2;i\lambda)$

$$= 2^{\frac{n-6+i(\lambda_1+\lambda_2-\lambda)}{2}} i^{\varepsilon-\varepsilon_1-\varepsilon_2} \pi^{\frac{3(1-n)-2+i(\lambda_1+\lambda_2-\lambda)}{2}}$$
$$\times \frac{\Gamma\left(\frac{n+i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2}\right)}{\Gamma\left(\frac{2-n-i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2}\right)} \frac{\Gamma\left(\frac{n+i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2}\right)}{\Gamma\left(\frac{2-n-i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2}\right)} \frac{\Gamma\left(\frac{n-i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2}\right)}{\Gamma\left(\frac{2-n+i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2}\right)}. \quad (153)$$

Given two symbols h_1 and h_2 in the space $S(\mathbb{R}^{2n})$, one has, in the weak sense in $S'(\mathbb{R}^{2n})$,

$$h_1 \# h_2 = \int_{-\infty}^{\infty} (h_1 \# h_2)_{i\lambda} d\lambda, \quad (154)$$

with

$$(h_1 \# h_2)_{i\lambda} = \sum_{\delta_1=0,1} \sum_{\delta_2=0,1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=0,1} a^{(j)}_{\delta_1,\delta_2;\delta}(i\lambda_1, i\lambda_2; i\lambda)$$
$$\times \mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{i\lambda_1,i\lambda_2;i\lambda}\left((h_1)_{i\lambda_1,\delta_1}, (h_2)_{i\lambda_2,\delta_2}\right) d\lambda_1 d\lambda_2 , \quad (155)$$

where $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{i\lambda_1,i\lambda_2;i\lambda}$ is the bilinear operator from $C^\infty(\pi_{i\lambda_1,\delta_1}) \times C^\infty(\pi_{i\lambda_2,\delta_2})\ to\ C^{-\infty}(\pi_{i\lambda,\delta})$ formally introduced in (52) and discussed before.

Proof. One has $h_1 \# h_2 = \mathcal{G}(h_1 \# \mathcal{G} h_2)$, as it follows from the interpretation of the transformation $\mathcal{G}$ of symbols recalled at the beginning. Next, we decompose h_1 into hyperplane waves with the help of (91), and h_2 into rays with the help of (93), recalling that one can move the line of integration up to the spectral line and writing

$$h_1 = \sum_{\delta_1=0,1} \int_{-\infty}^{\infty} (h_1)_{i\lambda_1,\delta_1} d\lambda_1, \ \ \mathcal{G} h_2 = \sum_{\delta_2=0,1} \int_{-\infty}^{\infty} (\mathcal{G} h_2)_{-i\lambda_2,\delta_2} d\lambda_2 \quad (156)$$

with

$$(h_1)_{i\lambda_1,\delta_1}(X) = \frac{2^{-i\lambda_1}}{4\pi} c(n-1+i\lambda_1,\delta_1) \int_{\mathbb{R}^{2n}} (\mathcal{G} h_1)(R) |[X,R]|_{\delta_1}^{-n-i\lambda_1} dR,$$

$$(\mathcal{G} h_2)_{-i\lambda_2,\delta_2}(X) = \frac{2^{i\lambda_2}}{4\pi} c(n-1-i\lambda_2,\delta_2) \int_{\mathbb{R}^{2n}} h_2(S) |[X,S]|_{\delta_2}^{-n+i\lambda_2} dS: \quad (157)$$

recall that the product $c(n-1+\nu_1,\delta_1)|[X,R]|_{\delta_1}^{-n-\nu_1}$, can be continued analytically with respect to ν_1, as a distribution in X. Then,

$$(h_1 \# h_2)(X) = \sum_{\delta_1=0,1} \sum_{\delta_2=0,1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2}(X) d\lambda_1 d\lambda_2 \quad (158)$$

with

$$F^{\delta_1,\delta_2}_{v_1,v_2}(X) = \frac{2^{-v_1+v_2}}{(4\pi)^2} c(n-1+v_1,\delta_1) c(n-1-v_2,\delta_2)$$

$$\times \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R) h_2(S) \left(\mathcal{G}\left(|[X,R]|_{\delta_1}^{-n-v_1} \# |[X,S]|_{\delta_2}^{-n+v_2}\right)\right) dRdS, \quad (159)$$

the two signed powers under the sharp product of which appears under the integral sign being regarded as functions of X. Actually, so as to obtain the last equation, we have changed the order of the bilinear operation # and of the integration with respect to $dR\, dS$. Though not completely trivial, the justification is fully similar to that, based on the consideration of the domains of powers of the harmonic oscillator, which occurred, in the one-dimensional case, in [166]: we shall not reproduce it here.

Generically, one has $[R,S] \neq 0$ and, as noticed in (111), there exists $g \in Sp(n,\mathbb{R})$ such that

$$\mathrm{g}^{-1}S = e_1, \quad \mathrm{g}^{-1}R = [R,S]e_{n+1} \quad (160)$$

in terms of the canonical basis of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Then, using the covariance of the Weyl calculus, and the fact that the transformation $\mathcal{G}$ commutes with symplectic changes of coordinates, we obtain

$$F^{\delta_1,\delta_2}_{v_1,v_2}(X) = \frac{2^{-v_1+v_2}}{(4\pi)^2} c(n-1+v_1,\delta_1) c(n-1-v_2,\delta_2)$$

$$\times \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R) h_2(S) |[X,R]|_{\delta_1}^{-n-v_1} \mathcal{G}\left(Y \mapsto |y_1|_{\delta_1}^{-n-v_1} \# |\eta_1|_{\delta_2}^{-n+v_2}\right)(\mathrm{g}^{-1}X)\, dRdS. \quad (161)$$

The function $F^{\delta_1,\delta_2}_{v_1,v_2}$ can then be made explicit, starting from (161), with the help of Proposition (6.13). Rewrite the result of this proposition, tested against $h \in S(\mathbb{R}^{2n})$, as

$$\langle \mathcal{G}\left(Y \mapsto |y_1|_{\delta_1}^{-n-v_1} \# |\eta_1|_{\delta_2}^{-n+v_2}\right), h\rangle = \int_{-\infty}^{\infty} d\lambda \int_{\mathbb{R}^2} h(se_1 + re_{n+1})$$

$$\Bigg[B_0(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)|r|^{\frac{n-2+v_1+v_2-i\lambda}{2}} |s|_{\delta}^{\frac{n-2-v_1-v_2-i\lambda}{2}}$$

$$+ B_1(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)\langle r_1\rangle^{\frac{n-2+v_1+v_2-i\lambda}{2}} |s|_{1-\delta}^{\frac{n-2-v_1-v_2-i\lambda}{2}}\Bigg] drds. \quad (162)$$

Then,

$$\langle\left(\mathcal{G}\left(Y \mapsto |y_1|_{\delta_1}^{-n-v_1} \# |\eta_1|_{\delta_2}^{-n+v_2}\right)\right) \mathrm{o}\mathrm{g}^{-1}, h\rangle = \int_{-\infty}^{\infty} d\lambda \int_{\mathbb{R}^2} h(Ss + rR)$$

$$\Big[B_0(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)|[R,S]|^{\frac{n+v_1+v_2-i\lambda}{2}}|r|^{\frac{n-2+v_1+v_2-i\lambda}{2}}|s|_\delta^{\frac{n-2-v_1-v_2-i\lambda}{2}}$$

$$+B_1(v_1,v_2,i\lambda;\delta_1,\delta_2,\delta)$$

$$\langle[R,S]\rangle^{\frac{n-2+v_1+v_2-i\lambda}{2}}\langle r\rangle^{\frac{n-2+v_1+v_2-i\lambda}{2}}|s|_{1-\delta}^{\frac{n-2-v_1-v_2-i\lambda}{2}}\Big]drds, \quad (163)$$

as seen after one has used (160) and the change of variable $r \longmapsto [R,S]r$, and

$$F_{i\lambda_1,i\lambda_2}^{\delta_1,\delta_2} = \int_{-\infty}^{\infty} F_{i\lambda_1,i\lambda_2;i\lambda}^{\delta_1,\delta_2}\, d\lambda \quad (164)$$

with

$$\langle F_{i\lambda_1,i\lambda_2;i\lambda}^{\delta_1,\delta_2}, h\rangle = (-1)^{\delta_1}\frac{2^{i(-\lambda_1+\lambda_2)}}{(4\pi)^2}c(n-1+i\lambda_1+\delta_1)c(n-1-i\lambda_2+\delta_2)$$

$$\times \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)h_2(S)dRdS \int_{\mathbb{R}^2} h(rR+sS)$$

$$\times\,[B_0(i\lambda_1,i\lambda_2,i\lambda;\delta_1,\delta_2,\delta)$$

$$\times\ |[R,S]|_{\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2-\lambda)}{2}}|r|^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}|s|_\delta^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}$$

$$+B_1(i\lambda_1,i\lambda_2,i\lambda;\delta_1,\delta_2,\delta)$$

$$\times\,|[R,S]|_{1-\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2+\lambda)}{2}}\langle r\rangle^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}|s|_{1-\delta}^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}]drds. \quad (165)$$

Finally, making the coefficients B_0 and B_1 explicit with the help of Proposition (6.13) and using (94) again,

$$\frac{1}{4\pi}\langle F_{i\lambda_1,i\lambda_2;i\lambda}^{\delta_1,\delta_2}, h\rangle$$

$$= \frac{(-1)^{\delta_1}2^{i(-\lambda_1+\lambda_2)}}{(4\pi)^2}\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)h_2(S)dRdS \int_{\mathbb{R}^2} h(rR+sS)$$

$$\times\Big[c\left(\frac{n-2+i(\lambda_1-\lambda_2+\lambda)}{2},\delta_1\right)$$

$$\times\,|[R,S]|_{\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2-\lambda)}{2}}|r|^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}|s|_\delta^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}$$

$$-c\left(\frac{n-2+i(\lambda_1-\lambda_2+\lambda)}{2},1-\delta_1\right)$$
$$\times |[R,S]|_{1-\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2+\lambda)}{2}} \langle r\rangle^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}} |s|_{1-\delta}^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}\Bigg] drds. \quad (166)$$

The distribution $F_{i\lambda_1,i\lambda_2;i\lambda}^{\delta_1,\delta_2} \epsilon S'(\mathbb{R}^{2n})$ is of type $(-n-i\lambda,\delta)$. Now, given any element $\mathfrak{S}$ of $C^{-\infty}(\pi_{i\lambda,\delta})$ extended as a distribution in $\mathbb{R}^{2n}$ of type $(-n-i\lambda,\delta)$ with the same name, and any function $h \in S(\mathbb{R}^{2n})$, one has the equation

$$\langle \mathfrak{S},h\rangle_{S'(\mathbb{R}^{2n})\times S(\mathbb{R}^{2n})} = 4\pi\langle \mathfrak{S},h_{-i\lambda,\delta}\rangle_{C^{-\infty}(\pi_{i\lambda,\delta})\times C^{\infty}(\pi_{-i\lambda,\delta})} \quad (167)$$

linking the two kinds of pairings. Starting from the case when $\mathfrak{S}$ is a function, one obtains (167) from the equation $\mathfrak{S}(tX_*) = |t|_\delta^{-n-i\lambda}\mathfrak{S}(X_*)$ and (18) or, if preferred, from a polarization of (27). The left-hand side of (166) can thus also be regarded as being $\langle F_{i\lambda_1,i\lambda_2;i\lambda}^{\delta_1,\delta_2}, h_{-i\lambda,\delta}\rangle$, the pairing now denoting that between $C^{-\infty}(\pi_{i\lambda,\delta})$ and $C^{\infty}(\pi_{-i\lambda,\delta})$. The comparison with (109) is now easy.

With another look at (48), one sees that $\mathbf{J}_{v_1,v_2;v}^{1-\delta_1,1-\delta_2;1-\delta}$ coincides with $\mathbf{J}_{v_1,v_2;v}^{\delta_1,\delta_2;\delta}$ when $j=0$, and with $\mathbf{J}_{v_1,v_2;v}^{1-\delta_1,1-\delta_2;1-\delta}$ when $j=0$. Then, the first or second term on the right-hand side of (166) is a multiple of the right-hand side of (109) taken with $j = 0 \; or \; 1$, as it follows from a comparison of the exponents and subscripts in (109) and in each of the two terms of (166) of the signed powers of $[R,S], r$ and s. The coefficient by which one must multiply the expression on right-hand side of (109) to obtain the corresponding term in right-hand side of (166) is

$$\frac{1}{4\pi}2^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}} c\left(\frac{n-2+i(\lambda_1-\lambda_2+\lambda)}{2},\varepsilon_1\right)\frac{c\left(\frac{n-2+i(-\lambda_1+\lambda_2+\lambda)}{2}\right)}{c\left(\frac{-n+i(\lambda_1+\lambda_2+\lambda)}{2},\varepsilon\right)}. \quad (168)$$

Expanding, we can write this as

$$2^{\frac{n-6+i(\lambda_1+\lambda_2-\lambda)}{2}} i^{\varepsilon-\varepsilon_1-\varepsilon_2}\pi^{\frac{3(n-1)-2+i(\lambda_1+\lambda_2-\lambda)}{2}}$$
$$\times \frac{\Gamma\left(\frac{n+i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2}\right)}{\Gamma\left(\frac{2-n-i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2}\right)}\frac{\Gamma\left(\frac{n+i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2}\right)}{\Gamma\left(\frac{2-n-i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2}\right)}\frac{\Gamma\left(\frac{n-i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2}\right)}{\Gamma\left(\frac{2-n+i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2}\right)}. \quad (169)$$

This concludes the proof of Theorem (6.3.1).

As an example, let us consider the harmonic oscillator $L = Op(\pi\ell)$ with $\ell(x,\xi) = |x|^2 + |\xi|^2$, and sharp products of fractional powers of ℓ.

Proposition (6.15)[172]. Let $v_1, v_2 \in \mathbb{C}$ satisfy the conditions $-n < Rev_1 < n$, $-n < Rev_2 < n$. Then, the decomposition into homogeneous components $h_{i\lambda}$ of the symbol $h = \ell^{\frac{-n-v_1}{2}} \# \ell^{\frac{-n-v_2}{2}}$ is given by the equation

$$h_{i\lambda} = \frac{1}{4}(2\pi)^{\frac{n-2+v_1+v_2-i\lambda}{2}} \ell^{\frac{-n-i\lambda}{2}}$$

$$\times \frac{\Gamma\left(\frac{n+v_1+v_2-i\lambda}{2}\right)\Gamma\left(\frac{n+v_1-v_2+i\lambda}{2}\right)\Gamma\left(\frac{n-v_1+v_2+i\lambda}{2}\right)\Gamma\left(\frac{n-v_1-v_2-i\lambda}{2}\right)}{\Gamma\left(\frac{n+v_1}{2}\right)\Gamma\left(\frac{n+v_2}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)}. \quad (170)$$

Proof. It is identical to that of the one-dimensional case. Only, one starts this time from the equation

$$\mathrm{Op}\left(\mathrm{e}^{-2\pi s \ell}\right) = (1-\mathrm{s}^2)^{-\frac{n}{2}}\left(\frac{1-s}{1+s}\right)^{L} \quad (171)$$

(same reference as in the one-dimensional case), leading rapidly to the equation

$$h = \frac{(2\pi)^{\frac{v_1+v_2+2n}{2}}}{\Gamma\left(\frac{n+v_1}{2}\right)\Gamma\left(\frac{n+v_2}{2}\right)}\int_0^\infty\int_0^\infty s_1^{\frac{n+v_1-2}{2}}\, s_2^{\frac{n+v_2-2}{2}}\, e^{-2\pi\frac{s_1+s_2}{1+s_1s_2}\ell}\,\frac{ds_1ds_2}{(1+s_1s_2)^n}, \quad (172)$$

Then

$$h_{i\lambda} = \frac{1}{2}(2\pi)^{\frac{v_1+v_2+n-2-i\lambda}{2}}\frac{\Gamma\left(\frac{n+i\lambda}{2}\right)}{\Gamma\left(\frac{n+v_1}{2}\right)\Gamma\left(\frac{n+v_2}{2}\right)}\ell^{\frac{-n-i\lambda}{2}}$$

$$\times\int_0^\infty\int_0^\infty s_1^{\frac{n+v_1-2}{2}}\, s_2^{\frac{n+v_2-2}{2}}(s_1+s_2)^{\frac{-n-i\lambda}{2}}(1+s_1s_2)^{\frac{-n+i\lambda}{2}}ds_1ds_2, \quad (173)$$

from which it is easy to conclude.

In general we can show

Corollary $(6.15)'$[140]. Let the squence $\{v_j\}_{j=1}^{n} \in \mathbb{C}$ satisfy the conditions $-n < Re(v_j) < n$ for each $j = 1, \dots, r$. The decomposition in to homogenous components $h_{i\lambda}$ of symbols $h = \ell^{\frac{-n-v_1}{2}} \# \ell^{\frac{-n-v_2}{2}} \# \dots \# \ell^{\frac{-n-v_j}{2}}$ is given by

$$h_{i\lambda} = \frac{1}{4}(2\pi)^{\frac{n-2+\sum_{j=1}^r v_j - i\lambda}{2}}\ell^{\frac{-n-i\lambda}{2}}$$

$$\times\frac{\Gamma\left(\frac{n+\sum_{j=1}^r v_j-i\lambda}{2}\right)\Gamma\left(\frac{n+v_1-v_2+v_3-v_4+\cdots+v_n+i\lambda}{2}\right)\Gamma\left(\frac{n-v_1+v_2-v_3+v_4+\cdots+v_n+i\lambda}{2}\right)\Gamma\left(\frac{n-v_1-v_2-v_3-\cdots-v_n-i\lambda}{2}\right)}{\Gamma\left(\frac{n+v_1}{2}\right)\Gamma\left(\frac{n+v_2}{2}\right)\dots\Gamma\left(\frac{n+v_n}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)}$$

Proof. It is identical to that of the two-dimensional. We start from the equation

$$\mathrm{Op}\left(\mathrm{e}^{-2\pi s \ell}\right) = (1-\mathrm{s}^2)^{-\frac{n}{2}}\left(\frac{1-s}{1+s}\right)^{L}$$

which leads to

$$h = \frac{(2\pi)^{\frac{\sum_{j=1}^r v_j+2n}{2}}}{\prod_{j=1}^{r}\Gamma\left(\frac{n+v_j}{2}\right)}\int_0^\infty\int_0^\infty\dots\int_0^\infty\prod_{j=1}^{r}s_j^{\frac{n+v_j-2}{2}}\; e^{-2\pi\frac{\sum_{j=1}^r s_j}{1+\prod_{j=1}^r s_j}\ell}\,\frac{ds_1ds_2\dots ds_r}{\left(1+\prod_{j=1}^r s_j\right)^n}$$

$$h_{i\lambda} = \frac{1}{2}(2\pi)^{\frac{\sum_{j=1}^{r} v_j + n - 2 - i\lambda}{2}} \frac{\Gamma\left(\frac{n+i\lambda}{2}\right)}{\prod_{j=1}^{r} \Gamma\left(\frac{n+v_j}{2}\right)} \ell^{\frac{-(n+i\lambda)}{2}}$$

$$\times \int_0^\infty \int_0^\infty \dots \int_0^\infty s_1^{\frac{n+v_1-2}{2}} \dots s_r^{\frac{n+v_r-2}{2}} (s_1 + s_2 + \cdots + s_r)^{\frac{-(n+i\lambda)}{2}}$$

$$(1 + s_1 s_2 \dots s_r)^{\frac{-n+i\lambda}{2}} ds_1 ds_2 \dots ds_2$$

Hence gives the result.

Let us observe that, if not dealing with differential operators (*i.e.*, when $\frac{-n-v_1}{2}$ and $\frac{-n-v_2}{2}$ are not both non-negative integers),Moyal's expansion (20) would lead in this example to a sum of terms with increasing singularities at 0, without significance, even asymptotic, as a distribution in $\mathbb{R}^{2n}$:however, let us hasten to say that microlocal analysis does not attach much significance to *points* of the phase space.

Irreducibility of the decomposition of $L^2(R^{2n})$

We prove here the irreducibility of most unitary representations appearing in the spectral decomposition of Proposition (6.1). In the last decades, general irreducibility results such as Kostant's irreducibility theorem for spherical (minimal) principal series representations [161] and Vogan–Wallach's irreducibility theorem for generic parameters [168] have been developed. Also, many specific cases have been studied in detail by R. Howe, E.-T. Tan, S.-T. Lee, S. Sahi and many other. by algebraic and combinatorial methods. However, to the best of our knowledge, neither the general theory nor the known special results contain Theorem (6.18) below, the proof of which is based on the extension of the idea of branching laws to non-compact subgroups [160] and on properties of the Weyl calculus in $\mathbb{R}^{n-1}$.

Lemma (6.16)[172]. Let $M_0^{\text{vect}} = \{S = (s_1, s_*; 0, \sigma_*)\}$ denote the linear space of translations of the affine hyperplaneM_0. Given $S \in M_0$, define the linear automorphism $\mathcal{T}_S$ *of* $\mathbb{R}^{2n}$ by the equation

$$\mathcal{T}_S X = X + [S, X]e_1 + [e_1,]S. \qquad (174)$$

For every $S \in M_0^{\text{vect}}$, $\mathcal{T}_S$ is a symplectic transformation of $\mathbb{R}^{2n}$ preserving M_0. The group of all such symplectic transformations is generated by the group N of transformations $\mathcal{T}_S, S \in M_0^{\text{vect}}$, together with the group M of transformations $(x_1, x_*; \xi_1, \xi_*) \mapsto (x_1, y_*; \xi_1, \eta_*)$, where the map $(x_*; \xi_*) \mapsto (y_*; \eta_*)$ is a symplectic transformation in the $2n-2$ variables involved; the latter normalizes the first within $Sp(n, \mathbb{R})$.

Proof. That $[\mathcal{T}_S X, \mathcal{T}_S Y = [X, Y]]$ for every pair X, Y is an immediate consequence of the relations $[e_1, e_1] = [e_1, S] = [S, S] = 0$. That the group MN generates the stabilizer of M_0 is a consequence of the observation following (42).

Eq. (22) reduces when $g \in MN$ to

$$\left(\pi_{\nu,\delta}(\mathrm{g})f\right)(X) = f(\mathrm{g}^{-1}X), X \in \mathrm{M}_0. \quad (175)$$

If one sets $S_{**} = (s_*;\sigma_*)$, $X_{**} = (x_*;\xi_*)$, the transformation $\mathcal{T}_{-S}$ expresses itself when considered on M_0 as

$$\mathcal{T}_{-S}(x_1, x_*; 1, \xi_*) = (x_1 - 2s_1 + [S_{**}, X_{**}], x_* - s_*; 1, \xi_* - \sigma_*): \quad (176)$$

it follows in particular that, given $(i\lambda,\delta) \in i\mathbb{R} \times \{0,1\}$, all transformations $\pi_{i\lambda,\delta}(\mathrm{g})$ with $\mathrm{g} \in \mathrm{MN}$, when regarded as unitary transformations of $L^2(\mathrm{M}_0)$, commute with the differential operator $\frac{1}{2i\pi}\frac{\partial}{\partial x_1}$.

Let us first decompose the restriction of the representation $\pi_{i\lambda,\delta}$ to MN: from what has just been said, it can be analyzed when coupled with the spectral decomposition of the operator $\frac{1}{2i\pi}\frac{\partial}{\partial x_1}$, in other words when fixing the first variable t in the partial Fourier transform $\mathcal{F}_1 f$ of $f \in L^2(\mathrm{M}_0)$, as already done. From (175), one has if $n \geq 2$ the identity

$$\left(\mathcal{F}_1\left(\pi_{i\lambda,\delta}(\mathcal{T}_S)f\right)\right)(t, x_*; \xi_*)$$
$$= e^{-2i\pi t\,(2s_1 - [S_{**}, X_{**}])}(\mathcal{F}_1 f\,)(t, x_* - s_*; \xi_* - \sigma_*), \quad (177)$$

a group of transformations in which we may regard $t \neq 0$ as a parameter by specializing to $s_1 = 0$, getting a projective representation $\pi_{i\lambda,\delta}^{(t)}$ of $\mathbb{R}^{2n-2}$, actually independent of $(i\lambda,\delta)$, as a result; the same is true when considering transformations $\mathcal{F}_1\left(\pi_{i\lambda,\delta}^{(t)}(\mathrm{g})\right)\mathcal{F}_1^{-1}$ with $\mathrm{g} \in \mathrm{M}$.

Lemma (6.17)[172]. Assume that $n \geq 2$. For fixed $t \neq 0$, the linear space of bounded operators in $L^2(\mathbb{R}^{2n-2})$, which commute with all transformations $\mathcal{F}_1\left(\pi_{i\lambda,\delta}^{(t)}(\mathrm{g})\right)\mathcal{F}_1^{-1}$ with $\mathrm{g} \in \mathrm{MN}$ is generated by the identity and the transformation $\mathcal{F}_1\sum_{\mathrm{t}}\mathcal{F}_1^{-1}$ characterized by the equation

$$(\mathcal{F}_1\textstyle\sum_{\mathrm{t}} f)(t, X_{**}) = |t|^{n-1}\displaystyle\int_{\mathbb{R}^{2n-2}} e^{-2i\pi t\,[X_{**}, Y_{**}]}\,(\mathcal{F}_1 f)(t, Y_{**})dY_{**}. \quad (178)$$

Proof. First assume that $t = 2$. Looking at (177), one sees that the linear space of infinitesimal operators of the representation of N under consideration is generated by the following operators, where $j, k \geq 2$: (i) the operators $\xi_j + \frac{1}{4i\pi}\frac{\partial}{\partial x_1}$, where ξ_j denotes the operator of multiplication by ξ_j ; (ii) the operators $x_k - \frac{1}{4i\pi}\frac{\partial}{\partial \xi_k}$. From (20), these are just the operators $h \longmapsto \xi_j \# h$ and $h \longmapsto x_k \# h$. Taking advantage of the Weyl calculus in $\mathbb{R}^{n-1}$, set

$$\varpi^2(\mathrm{g})Op = \left(\mathcal{F}_1\left(\pi_{i\lambda,\delta}^{(2)}(\mathrm{g})\right)\mathcal{F}_1^{-1}\mathrm{h}\right), \ \mathrm{g} \in \mathrm{MN}, \quad (179)$$

defining in this way a unitary representation ϖ^2 of MN in the space of Hilbert–Schmidt operators in $L^2(\mathbb{R}^{n-1})$. From what has just been seen, the image $\varpi^2(N)$ consists of the automorphisms

$$A \longmapsto exp(2i\pi(\langle\eta, Q\rangle - \langle y, P\rangle))A \tag{180}$$

On the other hand, in view of (20), the image under ϖ^2 of M consists of the maps $A \longmapsto UAU^{-1}$ with U in the image of the metaplectic representation. Since the Heisenberg representation in $L^2(\mathbb{R}^{n-1})$. is irreducible, while that of the metaplectic representation decomposes into its restrictions to spaces of functions with a given parity, it follows that the commutant of the representation ϖ^2 of MN is the linear space generated by the identity together with the automorphism $A \longmapsto ACh$, where Ch is the parity map $u \longmapsto \check{u}$, of the space of Hilbert–Schmidt operators in $L^2(\mathbb{R}^{n-1})$. Going back to symbols and using what immediately follows (88), one obtains the case $t = 2$ of Lemma (6.17), from which one obtains the general case by a simple rescaling of coordinates of S.

Consider now any bounded operator $\mathcal{K}$in the commutant of the representation $\pi_{i\lambda,\delta}$. Restricting the representation to MN, it follows from Lemma (6.17) that the operator $\mathcal{F}_1\mathcal{K}\mathcal{F}_1^{-1}$ is a linear combination, with coefficients depending on t (the variable used in the definition of the partial Fourier transform), of the operators I and $\mathcal{F}_1\sum_{\mathrm{t}}\mathcal{F}_1^{-1}$. Introduce the group A of symplectic transformations of $\mathbb{R}^{2n}$ defined as

$$\mathrm{ga}: (\mathrm{x}, \xi) \longmapsto (ax, a^{-1}\xi), \quad a > 0. \tag{181}$$

From (22), one has

$$\big(\pi_{i\lambda,\delta}(\mathrm{ga})f\big)(x_1, x_*; 1, \xi_*) = a^{-n-i\lambda} f(a^2x_1, a^{-2}x_*; 1, \xi_*). \tag{182}$$

Then, the operator $\mathcal{K}$ must also commute with the Euler operator$\sum_{\mathrm{j}\geq 1} \mathrm{x_j} \frac{\partial}{\partial \mathrm{x_j}}$, and the operator $\mathcal{F}_1\mathcal{K}\mathcal{F}_1^{-1}$ must commute with the operator $-t\frac{\partial}{\partial t} + \sum_{\mathrm{j}\geq 2} \mathrm{x_j} \frac{\partial}{\partial \mathrm{x_j}}$: after a change of variables in (187), it follows that the above-referred coefficients depend only on sign t .

Theorem $(\mathbf{6.18})[\mathbf{172}]$. Given any $n \geq 1$, and any pair $(i\lambda, \delta) \in i\mathbb{R} \times \{0, 1\}$ such that $(i\lambda, \delta) \neq (0, 1)$ and $(i\lambda, \delta) \neq (0, 0)$, the representation $\pi_{i\lambda,\delta}$ is irreducible; if $(i\lambda, \delta) = (0, 1)$, it decomposes as the direct sum of two irreducible representations, and such is the case if $(i\lambda, \delta) = (0, 0)$ and $n \geq 2$.

Proof. We may assume that $n \geq 2$, since the one-dimensional case is classical [157]. From the considerations that precede in this section, any operator commuting with the representation $\pi_{i\lambda,\delta}$ must lie in the algebra generated by the following two involutions:

(i) the transformation Σ defined by

$$(\mathcal{F}_1 \Sigma f)(t, X_{**}) = |t|^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-2i\pi t\,[X_{**}, Y_{**}]} (\mathcal{F}_1 f)(t, Y_{**}) dY_{**}; \quad (183)$$

(ii) the transformation $\Psi = sign\left(\frac{1}{2i\pi}\frac{\partial}{\partial x_1}\right)$ defined by

$$(\mathcal{F}_1(\Psi f))(t, X_{**}) = (sign t)(\mathcal{F}_1 f)(t, X_{**}). \quad (184)$$

Looking at (40), one may note that $\Sigma = \theta_{0,0}$ and that the composition $\Sigma\Psi = \Psi\Sigma$ coincides with the intertwining operator $\theta_{0,1}$. Now, $\theta_{0,1}$ is a non-trivial (i.e., distinct from a scalar) intertwiningoperator of the representation $\pi_{0,1}$ with itself, and $\theta_{0,0}$ is an intertwining operator of the representation $\pi_{0,0}$ with itself, non-trivial as soon as $n \geq 2$.

What remains to be seen, fixing $n \geq 2$, is that the operator $\theta_{0,1}$ cannot commute with the representation $\pi_{i\lambda,\delta}$ unless $(i\lambda, \delta) = (0, 1)$ and that the operator $\theta_{0,0}$ cannot commute with the representation $\pi_{i\lambda,\delta}$ unless $(i\lambda, \delta) = (0, 0)$, finally that Ψ can never (if $n \geq 2$) commute with
a representation $\pi_{i\lambda,\delta}$. Given $(i\lambda, \delta)$, set

$$\Theta_j = \theta_{i\lambda,\delta}\, \theta_{0,j} \quad (185)$$

so that, from (50),

$$(\mathcal{F}_1 \Theta_j f)(t, X_{**}) = |t|_{j-\delta}^{-i\lambda} (\mathcal{F}_1 f)(t, X_{**}). \quad (186)$$

If $\theta_{0,j}$ happens to be an intertwining operator from the representation $\pi_{i\lambda,\delta}$ to itself, the operator Θ_j is an intertwining operator from $\pi_{i\lambda,\delta}$ to $\pi_{-i\lambda,\delta}$. This operator, in its realization on $L^2(\mathrm{M}_0)$, has an integral kernel which, evaluated at some pair $((x_1, X_{**}), (y_1, Y_{**}))$, is the product of some distribution in $x_1 - y_1$ by $\delta(X_{**} - Y_{**})$: as $n \geq 2$, it is obvious that such an integral kernel, unless it is that of a scalar operator, cannot satisfy the covariance property that would make it an intertwining operator between two representations of the species under consideration. The same applies to the operator Ψ.

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